Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

Arbeiten aus dem
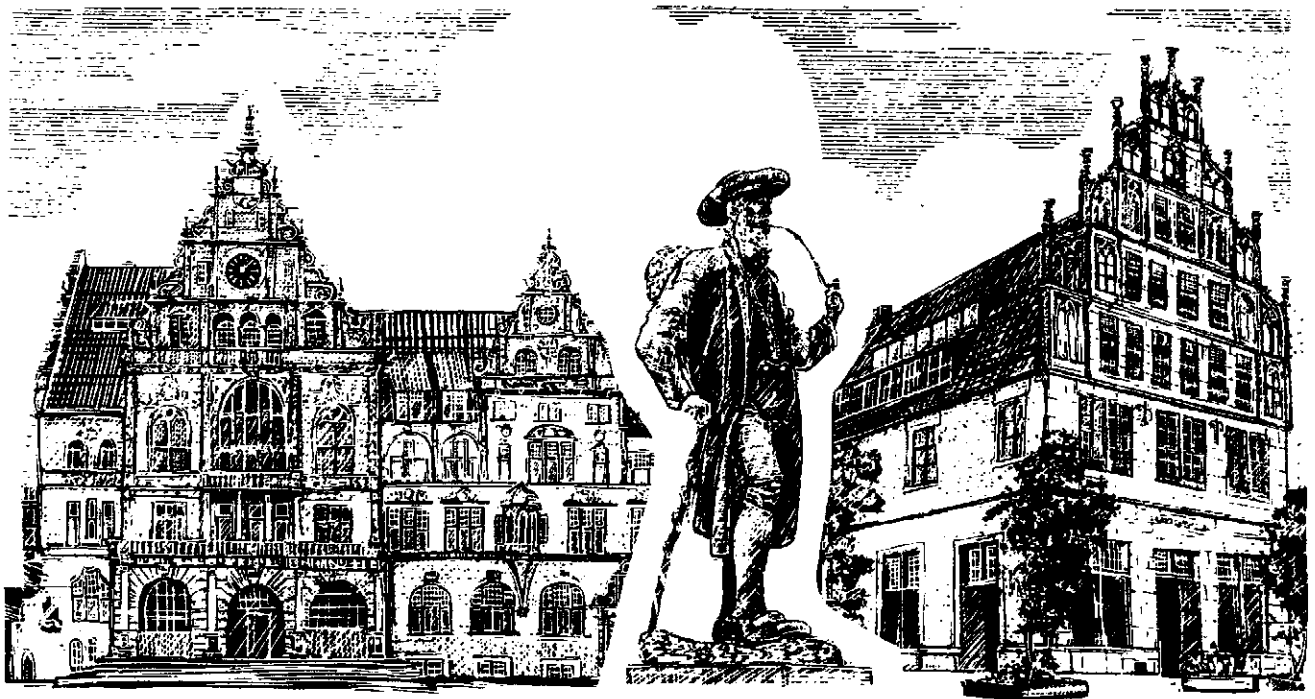# Institut für Mathematische Wirtschaftsforschung

No. 92

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A Non-Cooperative Solution Theory with
Cooperative Applications

Chapter 2
Consequences of Desirable Properties

March 1980

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Chapter 2.   Consequences of Desirable Properties

The nature of the problem of equilibrium point selection in
non-cooperative games does not seem to permit a satisfactory
solution concept which can be characterized by a set of simple
axioms. Nevertheless, it is useful to look at desirable pro-
perties which one might want to require and to explore their
consequences.

Even if full scale axiomization cannot be achieved, important
conclusions can be drawn from axiomatic considerations of li-
mited scope. The simplest class of games where the equili-
brium point selection problem occurs is that of all 2x2-games
with two strong pure strategy equilibrium points; here the
word "strong" is to be understood in the sense that a player
loses by a deviation from his equilibrium strategy if the
other players stick to their equilibrium strategies. A central
notion of our theory, namely that of risk dominance can be
fully axiomatized for this admittedly very restricted class
of games.

It is also important to see that certain properties which
may seem to be desirable at first glance cannot be achieved.
As we shall see, it is impossible to define a continuous so-
lution. Another impossibility result to be derived below con-
cerns a way of subdividing one information set into two which
we call "sequential agent splitting". An agent who has to
choose between a, b, c is subdivided into two agents, one who
first chooses between a and bc and another who then, if neces-
sary, decides between b and c. Unfortunately, it is not pos-
sible to require that this kind of agent splitting should not
essentially change the solution of the game without violating
other axioms which we judge to be intuitively more compelling.

We shall also look at substructures of agent normal forms
which are closely connected to subgames in the extensive form.
These substructures, called cells, give rise to powerful re-
quirements which reduce the task of finding a solution for
general games to the task simpler one of finding a solution

for games without such substructures.

1. Continuity

Consider the class of all 2x1-games of the form shown in figure II-1. For t ≠ 0 the game has only one equilibrium point, namely

Figure II-1: a class of 2x1-games

A for $t > 0$ and B for $t < 0$.

For $t = 0$ every mixed strategy of player 1 is an equilibrium strategy. Clearly, no solution concept can assign a unique equilibrium point to every game in the class in a continuous way. Not only player 1's strategy but also player 2's payoff must behave discontinuously as a function of t at $t = 0$.

If a payoff parameter is varied continuously, some equilibrium points may suddenly disappear and others which have not been there before may suddenly appear. In order to show how this problem may arise in a less trivial way we add a further example. Consider the class of games given by figure II-2. Here for $t < -1$ the strategy combination Aa is the only equilibrium point

Figure II-2: A class of 2x2

of the game. For $-1 \leq t \leq +1$ both Aa and Bb are equilibrium points. Moreover, for $-1 < t < +1$ the game has a third equilibrium point in mixed strategies where player 1 uses A with probability $2/(3-t)$ and player 2 uses a with probability $(1+t)/(3+t)$. For $-1 < t < +1$ the game has no further equilibrium point. For $t = -1$ and $t = +1$ there are infinitely many equilibrium points, but this does not matter as far as our argument is concerned. Any function which assigns a unique equilibrium point to every game in the class must behave discontinuously with respect to t at some point in the interval $-1 \leq t \leq +1$. This example differs from the preceding one inasmuch as for every t the game has only a finite number of equilibrium points.

It is now clear that a certain amount of discontinuity cannot be avoided in a theory of equilibrium point selection. Continuity considerations seem to be of little relevance for the problem.

2. Definitions and notations

Before we can go on to investigate further desirable properties of a non-cooperative solution theory we must introduce some definitions and notations.

Normal forms: A game in normal form $G = (\Phi, H)$ consists of a set of pure strategy continuations

$$(2.1) \qquad \Phi = \underset{i \in N}{X} \Phi_i$$

and a payoff function H which assigns a payoff vector

$$(2.2) \qquad H(\varphi) = (H_i(\varphi))_N$$

to every pure strategy combination $\varphi = (\varphi_i)_N \in \Phi$. The lower index N indicates that φ contains one φ_i for every $i \in N$ and $H(\varphi)$ contains one $H_i(\varphi)$ for every $i \in N$. Player i's payoff $H_i(\varphi)$ for φ is a real number. The elements φ_i of Φ_i are player i's pure strategies. The sets Φ_i of pure strategies are finite.

In many cases the player set N will simply be set $\{1, \ldots, n\}$ of

the first n integers but since we must look at substructures
of games which are games with fewer players it is convenient
to define a normal form in such a way that the player set
can be any non-empty finite subset of the set of positive
integers. For $N = \{1,\ldots,n\}$ we can write $\varphi = (\varphi_1,\ldots,\varphi_n)$
and $H(\varphi) = (H_1(\varphi),\ldots,H_n(\varphi))$.

Often a game in normal form will simply be called a game
where this can be done without risk of confusion. We shall
mainly be concerned with such games even if extensive forms
will be looked at occasionally in order to clarify conceptu-
ally important points. It must be kept in mind that in the
framework of our theory a normal form must be interpreted
as a perturbed agent normal form of an extensive game.

Further definitions in this section will refer to a fixed
game $G = (\Phi,H)$.

<u>Mixed strategies</u>: A mixed strategy q_i of player i is a proba-
bility distribution over player i's set Φ_i of pure strategies.
$q_i(\varphi_i)$ denotes the probability assigned to φ_i. No distinction
is made between a pure strategy φ_i and that mixed strategy which
assigns probability 1 to φ_i and 0 to all other pure strategies.
The set of all mixed strategies q_i of player i is denoted by
Q_i.

A combination $q = (q_i)_N$ of mixed strategies contains a mixed
strategy q_i for every $i \in N$. The set of all combinations q of
this kind is denoted by Q. For $q = (q_i)_N \in Q$ and $\varphi = (\varphi_i)_N \in \Phi$ it is
convenient to introduce the notation

$$(2.3) \qquad q(\varphi) = \prod_{i \in N} q_i(\varphi_i)$$

In other words, $q(\varphi)$ is the product of all $q_j(\varphi_j)$ with $j \in N$.
The product $q(\varphi)$ is called the realization probability of φ
under q. The definition of the payoff function H is extended
from Φ to Q in the usual way:

$$(2.4) \qquad H(q) = \sum_{\varphi \in \Phi} q(\varphi)H(\varphi)$$

It will be necessary to look at combinations of the form
$q_{-i} = (q_j)_{N \setminus \{i\}}$ which contain one strategy for every player
with the exception of i. The index -i is used in order to de-
signate such combinations which are called i-incomplete.

Φ_{-i} denotes the set of all i-incomplete pure combinations and the symbol Q_{-i} is used for the set of all i-incomplete mixed combinations. We use the notation $q_i q_{-i}$ in order to describe that q which contains q_i and the components of q_{-i}. If for all players with the exception of player i the strategies in q_{-i} agree with that in q we call q_{-i} the i-incomplete combination <u>derived</u> from q.

<u>Best replies and equilibrium points</u>: $r_i \in Q_i$ is called a <u>best reply</u> to $q_{-i} \in Q_{-i}$ if we have

$$(2.5) \qquad H_i(r_i q_{-i}) = \max_{q_i \in Q_i} H_i(q_i q_{-i})$$

It is a well-known fact of game theory that r_i is a best reply to q_{-i} if and only if every pure strategy φ_i with $r_i(\varphi_i) > 0$ is a best reply to q_{-i}.

We say that r_i is a best reply to $q \in Q$ if r_i is a best reply to the i-incomplete combination q_{-i} derived from q. A combination $r \in Q$ is called a <u>vector best reply</u> or shortly a <u>best reply</u> to $q \in Q$ if every r_i in r is a best reply to q.

r_i is called a <u>strong</u> best reply to q_{-i} if r_i is the only best reply to q_{-i}. In view of what has been said above, a strong best reply is always a pure strategy.

<u>An equilibrium point</u> is a strategy combination $r \in Q$ which is a best reply to itself.

An equilibrium point r is called <u>strong for i</u> if player i's strategy r_i in r is a strong best reply to r. A strong equilibrium point is an equilibtrium point which is strong for every $i \in N$.

<u>Solution function</u>: Let $\mathcal{G}$ be a class of games in normal form. A solution function L for $\mathcal{G}$ is a function which assigns one of its equilibrium points r = L (G) to every $G \in \mathcal{G}$.

In the following we shall look at desirable properties of solution functions and of concepts which are used in the definition of solution functions. We have already discussed

continuity in the previous section.

3. Invariance with respect to positive linear payoff transformation

The payoff of the players are von-Neumann-Morgenstern utilities. Interpersonal comparisons may be possible but they should not be considered as relevant for a non-cooperative solution theory where each player is assumed to be motivated by his own payoff exclusively.

Interpersonal utility comparisons are important for ethical theory but they have no room in a solution concept which is exclusively based on individualistic rationality assumptions.

Since von-Neumann-Morgenstern utilities are determined only up to positive linear transformations and since interpersonal comparisons are considered irrelevant, a game remains essentially unchanged if each player's payoff is subjected to a different positive linear transformation. This leads to the following definition of equivalence between games.

Equivalence: Two games $G = (\Phi, H)$ and $G' = (\Phi, H')$ with the same set $\Phi = \underset{i \in N}{X} \Phi_i$ of pure strategy combinations are equivalent if constants $\alpha_i > 0$ and β_i can be found for every $i \in N$ such that

$$(2.6) \qquad H_i' (\varphi) = \alpha_i H_i(\varphi) + \beta_i$$

holds for every $\varphi \in \Phi$ and every $i \in N$.

Invariance with respect to positive linear payoff transformation: A solution function L for a class $\mathcal{G}$ of normal form games is invariant with respect to positive linear payoff transformations if for two equivalent games G and G' in $\mathcal{G}$ we always have $L(G) = L(G')$.

Invariance with respect to positive linear payoff transformations is a very important requirement. It is more than a

desirable property. In our judgement it is indispensable.

4. Symmetry

A rational theory of equilibrium point selection must determine a solution which is independent of strategically irrelevant features of the game. Names and numbers used to distinguish players and strategies should not matter [1]. Games which do not differ in other ways must be considered as isomorphic and should not be treated differently.

Invariance with respect to renaming of players and strategies may be looked upon as a symmetry property since its most important implication can be seen in the fact that the solution must reflect the symmetries of the game.

A $\underline{\text{renaming}}$ of players and strategies in a game $G = (\Phi, H)$ may be thought of as a system of one-to-one mappings $f = (f_i)_N$ where f_i maps player i's pure strategy set Φ_i onto a new strategy set $\Psi_{\sigma(i)}$. Here $\sigma(i)$ is a one-to-one mapping from the player set N onto a new player set N' and $\sigma(i)$ is the new name of player i. In this way a new game $G' = (\Psi, H')$ with $\Psi = \underset{i \in N'}{X} \Psi_i$ arises.

We may look at f as a mapping from Φ to Ψ. This suggests the notation $f(\varphi)$ for that combination $\Phi \in \Psi$ whose components are related by $\Phi_{\sigma(i)} = f_i(\varphi_i)$ to those of φ. The new payoff function H' satisfies the condition

$$(2.7) \quad H'_{\sigma(i)}(f(\varphi)) = H_i(\varphi)$$

for every $i \in N$ and every $\phi \in \Phi$

An example is given in figure II-3.

It is convenient to adopt a notion of isomorphism which permits us to say that equivalent games are isomorphic. Therefore our definition of an isomorphism will involve a combination of a renaming with a system of positive linear payoff transformations.

$G = (\Phi, H)$ $G' = (\Psi, H')$

$$
\begin{array}{c|c|c|}
 & a & b \\
\hline
A & 0 \quad\quad 0 & 2 \quad\quad 3 \\
\hline
B & 4 \quad\quad 1 & 0 \quad\quad 0 \\
\hline
\end{array}
\qquad
\begin{array}{c|c|c|}
 & a & b \\
\hline
A & 2 \quad\quad 3 & 0 \quad\quad 0 \\
\hline
B & 0 \quad\quad 0 & 4 \quad\quad 1 \\
\hline
\end{array}
\qquad
\begin{array}{c|c|c|}
 & a & b \\
\hline
A & 3 \quad\quad 2 & 0 \quad\quad 0 \\
\hline
B & 0 \quad\quad 0 & 1 \quad\quad 4 \\
\hline
\end{array}
$$

exchange of columns exchange of transposed matrices

$\sigma(1) = 2$ $\sigma(2) = 1$

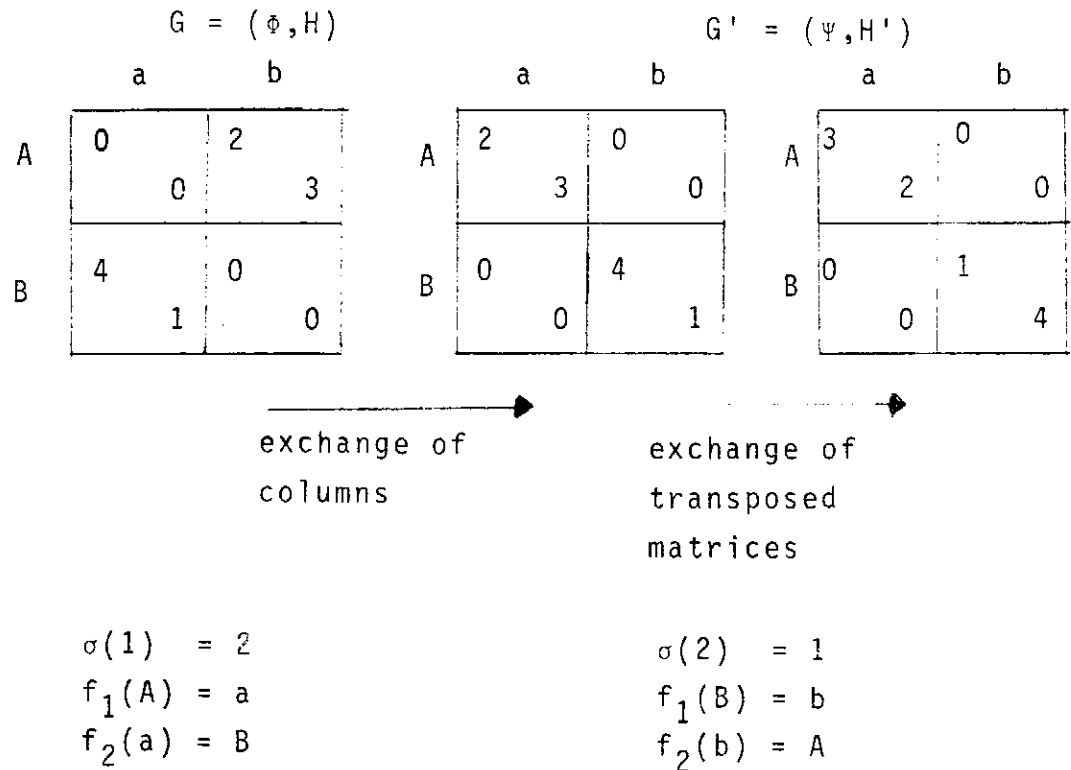$f_1(A) = a$ $f_1(B) = b$

$f_2(a) = B$ $f_2(b) = A$

Figure II-3: A renaming of players and strategies. - The renaming may be thought of as performed in two steps. The first one is a renaming of player 2's strategies which corresponds to an exchange of columns in the bimatrix representation. The second one is the renumbering of the players; the payoff matrices are transposed and exchanged.

Isomorphism: An isomorphism from $G = (\Phi, H)$ to $G' = (\Psi, H')$ is a system of mappings $f = (f_i)_N$ where f_i is a one-to-one mapping from player i's pure strategy set Φ_i in G onto $\Psi_{\sigma(i)}$, the pure strategy set of player $\sigma(i)$ in G', such that σ is a one-to-one mapping of the player set N in G onto the player set N' in G' and H' satisfies conditions of the form

$$(2.8) \quad H'_{\sigma(i)}(f(\varphi)) = \alpha_i H_i(\varphi) + \beta_i \quad \text{for all } \varphi \in \Phi$$

for every $i \in N$ with constants $\alpha_i > 0$ and β_i. (The notation $f(\varphi)$ has been explained above.)

Extension to mixed strategies: Consider an isomorphism $f = (f_i)_N$ from $G = (\Phi, H)$ to $G' = (\Psi, H')$. We write $q'_{\sigma(i)} = f_i(q_i)$ if we have

(2.9) $q'_{\sigma(i)}(f_i(\varphi_i)) = q_i(\varphi_i)$

for every $\varphi_i \in \Phi_i$. In this way f_i is extended from Φ_i to Q_i. We write $q' = f(q)$ if the components of q' are related to those of q as in (2.9). Obviously (2.8) and (2.9) imply

(2.10) $H_{\sigma(i)}(f(q)) = \alpha_i H_i(q) + \beta_i$

for every $q \in Q$ and every $i \in N$. It is clear that an isomorphism f looked upon as a mapping defined on Q preserves best reply relationships and carries equilibrium points into equilibrium points.

Two games G and G' are called <u>isomorphic</u> if at least one isomorphism from G to G' exists.

<u>Invariance with respect to isomorphisms</u>: A solution function L for a class of normal form games $\nearrow$ is invariant with respect to isomorphisms if for every isomorphism f from a game G $\in$ to a game G' $\in$ $\mathcal{I}$ (which may or may not be different from G) we have

(2.11) $L(G') = f(L(G))$

<u>Interpretation</u>:Equation (2.11) is the formal expression of what is meant by saying that isomorphic games should not be treated differently. Invariance with respect to isomorphisms includes invariance with respect to positive linear utility transformations to which it adds an invariance with respect to renaming. A formal description of this latter invariance need not be given here. In our judgement invariance with respect to isomorphisms is an indispensable requirement for any rational theory of equilibrium point selection which is based on strategic considerations exclusively.

With the help of the notion of an isomorphism we can give a precise meaning to the idea that the solution should correctly reflect the symmetries of a game.

<u>Symmetries</u>: A symmetry of a game $G = (\Phi,H)$ is an isomorphism from G to itself.

Symmetry invariant equilibrium points: An equilibrium point
r of G = (Φ,H) is called <u>symmetry</u> <u>invariant</u> if for every
symmetry f of G we have r = f(r).

Nash has shown that every finite game in normal form has
a symmetry invariant equilibrium point [Nash   ].

A solution function L which is invariant with respect to
isomorphisms must assign a symmetry invariant equilibtium
point to every game in the class where it is defined:

	a	b
A	2 2	0 0
B	0 0	1 4

	a	b
A	4 1	1 0
B	0 0	2 2

	a	b
A	2 2	0 0
B	0 0	1 4

$\longrightarrow$ $\longrightarrow$

renaming   linear transformations
of players   of payoffs
and strategies   player 1's payoff is
                             divided by 2 and

$\sigma(1)$ = 2 $\sigma(2)$ = 1   player 2's payoff
$f_1(A)$ = b $f_1(B)$ = a   is multiplied by 2
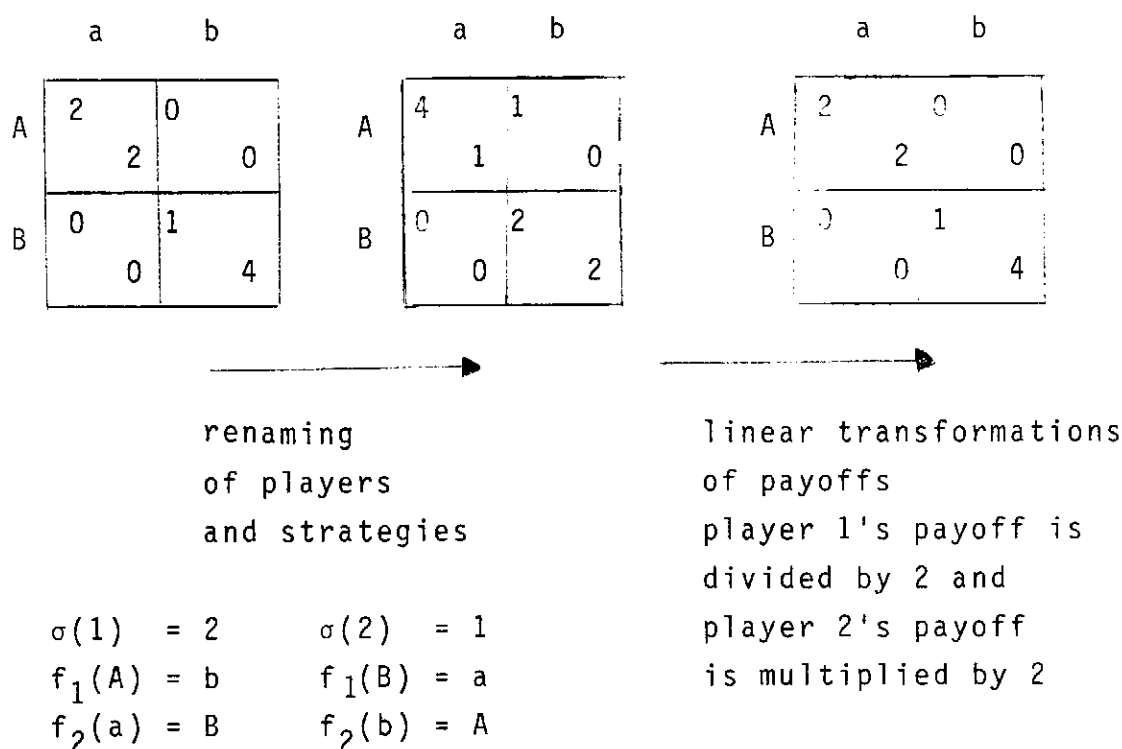$f_2(a)$ = B $f_2(b)$ = A

Figure II-4: An example of a symmetry

An example of a game with a symmetry is given in figure II-4.
The game has three equilibrium points, two in pure strate-
gies, namely Aa and Bb and a mixed one r = (r_1,r_2) with
$r_1(A)$ = 2/3 and $r_2(s)$ = 1/3. The symmetry f carries Aa to
Bb and vice versa. The mixed equilibrium point r is the
only one which is symmetry invariant. Any solution function
L which is invariant with respect to isomorphism cannot
assign anything else but L(G) = r to this game.

The payoff vector of r is $H(r) = (\frac{2}{3}, 1\frac{1}{3})$. Note that both players receive more at each of both pure strategy equilibrium points. Nevertheless, invariance with respect to isomorphism forces us to adopt r as the solution. [2/]

5. Best reply structure

In the last section we have argued that invariance with respect to positive linear payoff transformations has to be supplemented by invariance with respect to renamings of players and strategies. In this way, we obtained the stronger notion of invariance with respect to isomorphisms.

As we have seen, isomorphisms preserve best reply relationships. One may take the point of view that these relationships contain the essence of a non-cooperative game since no other information is needed in order to determine the set of all equilibrium points. This suggests the idea that two games should be treated in the same way if they do not differ with respect to their best reply relationships. Unfortunately, invariance requirements of this kind turn out to be too strong if they are imposed on the solution function. As we shall see in a later section, one would have to accept counter-intuitive consequences.

Our solution concept is composed of a number of different parts which interact in a process of equilibrium point selection. One of the most important notions which enter the definition of the solution as a building block is that of risk dominance. The concept will be explained in later sections. There we shall argue that an invariance requirement based on best reply considerations is very natural with respect to risk dominance even if it cannot be imposed on the solution concept as a whole.

In order to obtain a clear picture of what constitutes the best reply relationships of a game we shall introduce the definition of a best reply structure. A best reply structure may be thought of as a game form which is even more condensed than the normal form.

For reasons which will become apparent later we must be
careful not to eliminate too much in our definition of
the best reply structure of a game $G = (\Phi, H)$. It is not
sufficient to preserve the information on best replies
to mixed strategy combinations. It is true that nothing
else but a mixed combination can be played in a game but we
must also look at all possible expectations a player i
may have on the other players' behavior. Player i's expecta-
tions may not necessarily take the form of an i-incomplete
mixed strategy combination.

Player i may think that one of two equilibrium points
$U = (U_1, \ldots, U_n)$ and $V = (V_1, \ldots, V_n)$ is known to be the
solution by all other players. Either all of them will
play their strategies U_j or all of them will play V_j.
Player i may have subjective probabilitiy z for the first
alternative and $1-z$ for the second. [3] His expectation
can be described as a joint mixture of U_{-i} and V_{-i}, sym-
bolically expressed by $zU_{-i}+(1-z)V_{-i}$. If player i holds
this expectation he must play a strategy which optimizes
against it or, in other words, he must select a best reply
to $zU_{-i}+(1-z)V_{-i}$. Therefore, it is necessary to introduce
formal definitions of joint mixtures and best replies to
them before we can go on to define a best reply structure
which covers all possible subjective probability distri-
butions a player may have on the behavior of other players.

All definitions will refer to a fixed game $G = (\Phi, H)$ with
$\Phi = \underset{i \in N}{X} \Phi_i$.

Joint mixtures: A joint mixture over the i-incomplete
combinations is a probability distribution $q_{\cdot i}$ over Φ_{-i}.
The probability assigned to φ_{-i} by $q_{\cdot i}$ is denoted by $q_{\cdot i}(\varphi_{-i})$.
We use a dot as a lower index in order to distinguish
joint mixtures from combinations of mixed strategies. The
set of all joint mixtures over Φ_{-i} is denoted by $Q_{\cdot i}$.

It is clear that probability distributions over Q_{-i} would yield nothing new. We need not consider more general joint mixtures than those in the sets $Q._i$.

We say that the i-incomplete mixed combination q_{-i} generates the joint mixture $q._i$ if for every $\varphi_{-i} = (\varphi_j)_{N \setminus \{i\}}$ we have

$$(2.12) \qquad q._i(\varphi_{-i}) = \prod_{j \in N \setminus \{i\}} q_j(\varphi_j)$$

Every $q_{-i} \in Q_{-i}$ generates a joint mixture $q._i$ but not every $q._i \in Q._i$ is generated by a q_{-i}. In this respect 2-person games are special cases since here both $Q._i$ and Q_{-i} coincide with the mixed strategy set Q_j of the other player.

Hybrid combinations: A hybrid combination $q_i q._i$ consists of a mixed strategy $q_i \in Q_i$ and a joint mixture $q._i \in Q._i$. Player i's payoff for $q_i q._i$ is defined as follows:

$$(2.13) \quad H_i(q_i q._i) = \sum_{\varphi_i \in \Phi_i} \sum_{\varphi_{-i} \in \Phi_{-i}} q_i(\varphi_i) q._i(\varphi_{-i}) H_i(\varphi_i \varphi_{-i})$$

This is player i's subjectively expected payoff if he uses q_i and $q._i$ is his subjective probability distribution on the behavior of the other players. It is clear that $H_i(q_i q._i)$ agrees with $H_i(q_i q_{-i})$ if $q._i$ is generated by q_{-i}. Instead of (2.13) we can also write

$$(2.14) \quad H_i(q_i q._i) = \sum_{\varphi_{-i} \in \Phi_{-i}} q._i(\varphi_{-i}) H_i(q_i \varphi_{-i})$$

Payoffs of the other players for $q_i q._i$ could be defined in an analogous way but these payoffs have no theoretical significance.

Best replies: r_i is a best reply to $q._i$ if we have

$$(2.15) \quad H_i(r_i q._i) = \max_{q_i \in Q_i} H_i(q_i q._i)$$

The well-known fact that r_i is a best reply if and only if every pure strategy φ_i with $r_i(\varphi_i) > 0$ is a best reply holds

here, too. If we know which are the pure best replies we
have a full overview over all best replies.

r_i is called a strong best reply to $q._i$ if r_i is the only
best reply to $q._i$. A strong best reply must be a pure stra-
tegy.

It is clear that r_i is a best reply to q_{-i} if and only if r_i
is a best reply to the joint mixture $q._i$ generated by q_{-i}.

Best reply structure: The set of all pure best replies of
player i to $q._i$ is denoted by $A_i(q._i)$. The correspondence A_i
which assigns the set $A_i(q._i)$ to $q._i \in Q._i$ is called player i's
best reply correspondence. $A = (A_i)_N$ is the system of best
reply correspondences.

The best reply structure $B = (\Phi, A)$ of $G = (\Phi, H)$ consists of
the set of pure strategy combinations $\Phi = X_{i \in N} \Phi_i$ and the system
$A = (A_i)_N$ of best reply correspondences.

It is clear that an isomorphism f from G to G' carries the
best reply structure of G to that of G'.

Stability sets: The set of all $q._i \in Q_{-i}$ such that a given pure
strategy φ_i is a best reply to $q._i$ is denoted by $S(\varphi_i)$. The
set $S(\varphi_i)$ is called the stability set of φ_i. Obviously $S(\varphi_i)$
is the set of all $q._i$ with $\varphi_i \in A_i(q._i)$. One may look upon S as
a correspondence from the union of all Φ_i to the union of all
$Q._i$. In a sense the correspondence S is the inverse of the
system A of best reply correspondences. The pair (Φ, S) could
also serve as a formal description of the best reply structure.

Graphical representation for 2x2-games: The best reply structure
of 2x2-games can be visualized with the help of a simple gra-
phical representation. Consider the class of 2x2-games describ-
ed by figure II-5. These games have strong equilibrium points
in the upper left and lower right corners. It is convenient
to introduce the notation u_i and v_i for the losses faced by
player i if he deviates from the equilibrium point $U = U_1 U_2$
and $V = V_1 V_2$, respectively, whereas the other player plays

his equilibrium strategy (see figure II-5).

	U_2	V_2
U_1	a_{11} b_{11}	a_{12} b_{12}
V_1	a_{21} b_{21}	a_{22} b_{22}

$$u_1 = a_{11} - a_{21} > 0$$
$$u_2 = b_{11} - b_{12} > 0$$
$$v_1 = a_{22} - a_{12} > 0$$
$$v_2 = b_{22} - b_{21} > 0$$

Figure II-5:   2x2-games with strong equilibrium point in north-west and south-east corners

A mixed strategy q_i in a 2x2-game is fully described by one of both probabilities. We shall use the notation

(2.16) $p_i = q_i(V_i)$

Player 1's strategy U_1 is a best reply for

(2.17) $a_{11}p_2 + a_{12}(1-p_2) \geq a_{21}p_2 + a_{22}(1-p_2)$

and V_1 is his best reply for

(2.18) $a_{11}p_2 + a_{12}(1-p_2) \leq a_{21}p_2 + a_{22}(1-p_2)$

This yields

(2.19) $U_1 \in A_1(q_2)$ for $0 \leq p_2 \leq \dfrac{u_1}{u_1+v_1}$

(2.20) $V_1 \in A_1(q_2)$ for $\dfrac{u_1}{u_1+v_1} \leq p_2 \leq 1$

Similarily we obtain

(2.21) $U_2 \in A_2(q_1)$ for $0 \leq p_1 \leq \dfrac{u_2}{u_2 + v_2}$

(2.22) $V_2 \in A_2(q_1)$ for $\dfrac{u_2}{u_2 + v_2} \leq p_1 \leq 1$

We can draw a diagram which represents all mixed strategy combinations as points (p_1, p_2) in a rectangular coordinate system. This is done in figure II-6 for a special case $(u_1 = 2, u_2 = 6, v_1 = 8, v_2 = 4)$. The diagram will be called the <u>stability</u> <u>diagram</u> of the game.
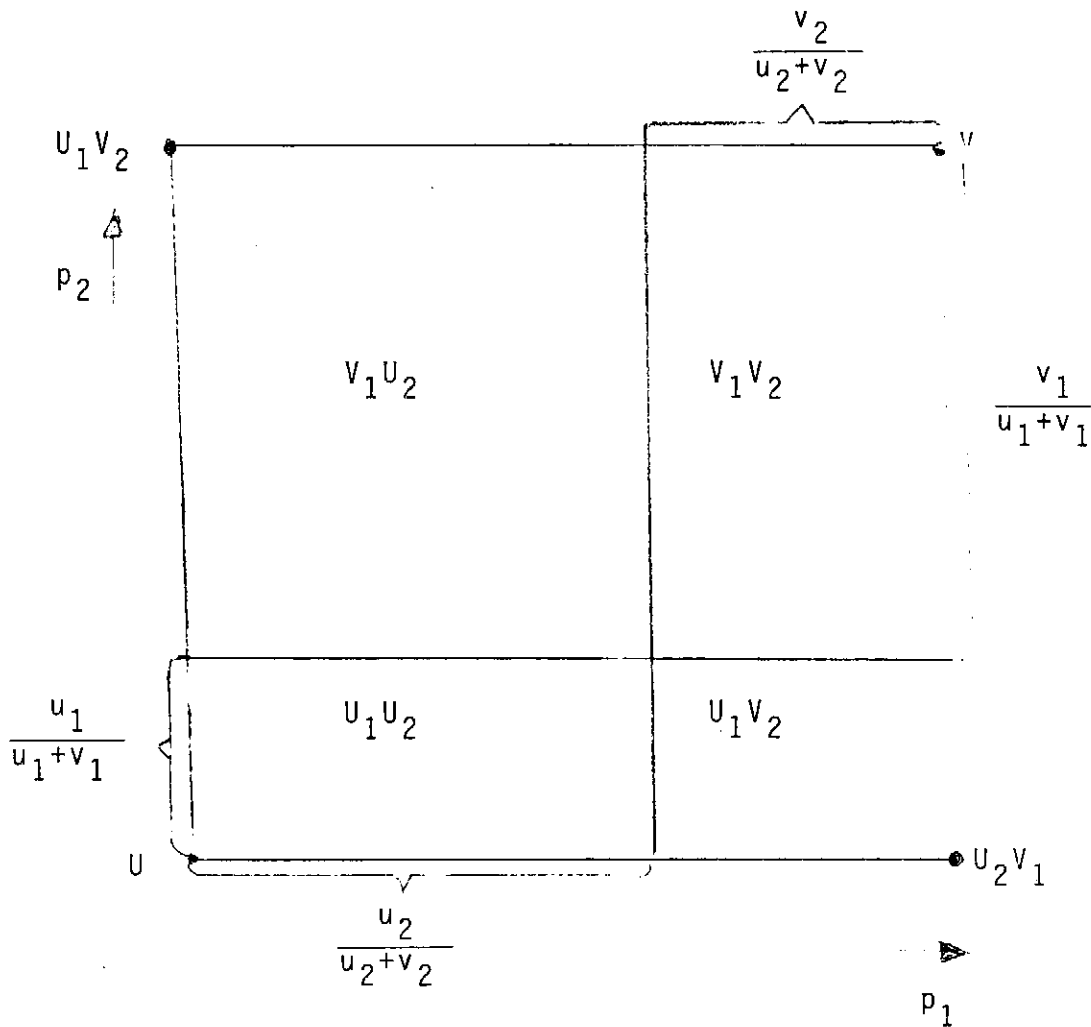

Figure II-6:   Stability diagram of the game of figure II-5

The regions where the four pure strategy combinations are best replies are indicated in figure II-6. We call these regions the stability regions of the respective pure strategy combinations.

The stability regions are closed rectangles, all of which have one corner in common, the mixed equilibrium point with $p_1 = u_2/(u_2 + v_2)$ and $p_2 = u_1/(u_1 + v_1)$. The equilibrium points U and V belong to their stability region but the "cross combinations" U_1V_2 and V_1U_2 belong to the stability region of the opposite cross combination.

It is interesting to note that the best reply structure of a game in the class of figure II-5 does not depend on anything else but the ratios u_1/v_1 and u_2/v_2 of the players' deviation losses at both strong equilibrium points. Absolute payoff levels do not matter. Only ratios of payoff differences are important.

Payoff transformations which preserve the best reply structure: Let $G = (\Phi, H)$ be a game and let ψ_{-j} be a fixed j-incomplete pure strategy combination for G. We construct a new game $G' = (\Phi, H')$ with the same set Φ of pure strategy combinations. For $i \neq j$ define

(2.23) $\qquad H_i'(\varphi) = H_i(\varphi) \qquad$ for every $\varphi \in \Phi$

Let λ be a constant. Player j's payoff is defined as follows:

(2.24) $\qquad H_j'(\varphi_j(\varphi_j \psi_{-j})) = H_j(\varphi_j \psi_{-j}) + \lambda$

(2.25) $\qquad H_j'(\varphi_j \varphi_{-j}) = H_j(\varphi_j \varphi_{-j}) \qquad$ for $\varphi_{-j} \neq \psi_{-j}$

We say that G' results from G by adding λ to player j's payoff at ψ_{-j}.

A look at (2.14) shows that the same amount $\lambda q_{\cdot j}(\psi_{-j})$ is added to every payoff of the form $H_j(q_j q_{\cdot j})$ in the transition from H_j to H_j'. Therefore (2.15) holds in G' if and only if it holds in G. We obtain the following

result: Adding λ to player j's payoff at ψ_{-j} does not change the best reply structure.

Consider the games of figure II-5. We receive the game of figure II-7 if we make the following changes one after the other:

1. We add $-a_{21}$ to player 1's payoffs at U_2
2. We add $-b_{12}$ to player 2's payoffs at U_1
3. We add $-a_{12}$ to player 1's payoffs at V_2
4. We add $-b_{21}$ to player 2's payoff at V_1

This confirms once more what we already know from the investigation of the best reply structure of the games of figure II-5: Every game in this class has the same best reply structure as the corresponding game of figure II-7.

It may be worth-while to point out that not every payoff transformation which preserves the best reply structure can be obtained by a combination of positive linear payoff transformations with the repeated application of the operation of adding a constant to player j's payoffs at ψ_{-j}. 2x2-games are exceptional in this respect. Already in 2x3-games other best reply structure preserving payoff transformations are possible.

	U_2	V_2
U_1	u_1 $\quad u_2$	0 $\quad 0$
V_1	0 $\quad 0$	v_1 $\quad v_2$

Figure II-7: Games received by best reply structure preserving transformations from those of figure II-5

An example is the class of games in figure II-8. A positive linear transformation or adding a constant at player 2'payoffs at a or b cannot change the quotient

$$(2.26) \qquad \frac{H_2(bd) - H_2(bc)}{H_2(be) - H_2(bd)} = 3 \, \frac{1+t}{1-t}$$

which clearly depends on t. Therefore, a combination of such transformations cannot yield the same result as a transition from one t to another.

	c	d	e
a	2 1	0 1-t	0
b	0 0	0 1+t	1 $\frac{4}{3} + \frac{2}{3}t$

$0 < t < 1$

c is best reply for $0 \le q_1(b) \le \frac{1}{2}$

d is best reply for $\frac{1}{2} \le q_1(b) \le \frac{3}{4}$

e is best reply for $\frac{3}{4} \le q_1(b) \le 1$

Figure II-8: A class of 2x3-games with the same best reply structure

Invariance with respect to payoff transformations which preserve the best reply structure: A solution function L for a class of games $\mathcal{G}$ is called invariant with respect to payoff transformations which preserve the best reply structure or shortly best reply invariant if for any two games $G = (\Phi, H)$ and $G = (\Phi, H')$ in $\mathcal{G}$ with the same best reply structure we have $L(G) = L(G')$.

Comment: As has been said before we do not insist on best reply invariance as a desirable property of a solution function. Nevertheless, it is an intuitively attractive requirement which should not be violated without a good reason. We want to keep as much of it as possible.

6. Payoff dominance

Consider the game of figure II-9. The equilibrium point $U = U_1U_2$ yields higher payoffs for both players than the other pure strategy equilibrium point $V = V_1V_2$. The mixed equilibrium point with probabilities of .4 and .8 for U_1 and U_2, respectively, yields even worse payoffs, namely 7.2 for player 1 and 4 for player 2. Clearly, among the three equilibrium points of the game V_1V_2 is the most attractive one for both players. This suggests that they should not have any trouble to coordinate their expectations at the commonly preferred equilibrium point V_1V_2. The solution of the game should be V_1V_2. The idea that equilibrium points with greater payoffs for all players should be given preference in problems of equilibrium point selection leads to the following definition.

Payoff dominance: Let r and s be two equilibrium points of $G = (\Phi, H)$ with $\Phi = \underset{i \in N}{X} \Phi_i$. We say that r payoff dominates s if we have

(2.27) $H_i(r) > H_i(s)$ for every $i \in N$

In (2.27) we require strict inequality since we want to restrict considerations of payoff dominance to cases where the interest of all players unambigously points in the same direction.

The idea of payoff dominance must be handled with care. We cannot require that L(G) should never be payoff dominated by any other equilibrium point. As we have seen in section 4 invariances with respect to isomorphisms forces us to accept the mixed equilibrium point as the solutions of the game in figure II-4 even if it is payoff dominated by both pure strategy equilibrium points.

	U_2	V_2
U_1	9 7	0 1
V_1	7 2	8 6

Figure II-9: Example of a 2x2-game with payoff dominance

	U_2	V_2
U_1	2 6	0 0
V_1	0 0	8 4

Figure II-10: Game with the best reply structure of the
game in figure II-9

The example of figure II-4 shows that we should not pay any attention to payoff dominance relationships where the dominating equilibrium point fails to be symmetry invariant. This leads to the following definitions.

Payoff efficiency: A symmetry invariant equilibrium point r of a game $G = (\Phi, H)$ is called payoff efficient if G has no other symmetry invariant equilibrium point s which payoff dominates r.

A solution function L for a class of games $\mathcal{Y}$ is payoff efficient if L(G) is payoff efficient for every $G \in \mathcal{Y}$.

Unfortunately, payoff efficiency is a very strong requirement which cannot be easily satisfied by a solution concept such as ours. Moreover, there are reasons why it should not be satisfied in general. One of these reasons will be discussed in the section on cells.

Another reason is connected to the fact that a situation similar to that in figure II-4 may arise without any lack of symmetry invariance. Two equilibrium points which both payoff dominate a third one but not each other may be equally strong in the sense that the theory does not yield a sufficient reason to select one rather than the other. In such situations it may be unavoidable to select an equilibrium point which fails to be payoff efficient.

In spite of the difficulties arising with this notion, payoff dominance is an important criterion of equilibrium point selection which cannot be completely ignored.

Payoff dominance relationships can easily be reversed by repeated additions of constants to a player j's payoff at some ψ_{-j}. Any strong equilibrium point ψ can be made the only payoff efficient one by performing the operations of adding a sufficiently great constant λ_j to the payoffs of every player j at his j-incomplete ψ_{-j} derived from ψ. This shows that best reply invariance and payoff efficiency are in conflict.

In the construction of our solution concept we have rejected full best reply invariance in favor of keeping the possibility of giving some room to considerations of payoff dominance without going as far as imposing the requirement of payoff efficiency.

7. The intuitive notion of risk dominance

Consider the game of figure II-11. There is no payoff dominance relationship between both pure strategy equilibrium points $U = (U_1, U_2)$ and $V = (V_1, V_2)$. Player 1 has higher payoffs at U and player 2 has higher payoffs at V.

Suppose that the players are in a state of mind where they
think that either U or V must be the solution of the game.
What is the risk of deciding one way or the other? If player 1
expects that player 2 will choose U_2 with a probability of
more than .01 it is better for him to choose U_1. Only if
player 2 chooses V_2 with a probability of at least .99
player 1's strategy V_1 will be the more profitable one. In
this sense U_1 is much less risky than V_1.

	U_2	V_2
U_1	99 49	0 0
V_1	0 0	1 51

Figure II-11:   An extreme example of risk dominance

Now let us look at the situation of player 2. His strategy V_2
is the better one if he expects player 1 to select V_1 with
a probability of more than .49 and U_2 is preferable if he
expects U_1 with a probability greater than .51. In terms
of those numbers V_2 seems to be slightly less risky than U_2.

It is obvious that player 1's reason to select U_1 rather
than V_1 is much stronger than player 2's reason to select V_2
rather than U_2. The players must take this into account when
they try to form subjective probabilities on the other player's
behavior. Presumably player 1 will select U_1 with high proba-
bility and since player 2 knows this he is likely to think
that it is better for him to choose U_2 rather than V_2. It is
plausible to assume that at the end both players will come to
the conclusion that both of them will play the equilibrium
point U.

The same line of reasoning can be followed for less extreme situations. Consider a game of the form of figure II-7 with $u_1 > v_1$ and $v_2 > u_2$. Player 1's risk situation is connected to the ratio u_1/v_1 and player 2's risk situation to the ratio v_2/u_2. Player 1 is more strongly attracted to U than player 2 to V if u_1/v_1 is greater than v_2/u_2. This is the case if and only if we have $u_1u_2 > v_1v_2$.

These considerations suggest the following notion of risk dominance for the games under consideration. U risk dominates V for $u_1u_2 > v_1v_2$ and V risk dominates U for $v_1v_2 > u_1u_2$.

The heuristic arguments which lead to this conclusion are fully in terms of the best reply structure. We have compared probabilities of the form $u_i/(u_1+v_i)$ and $v_i/(u_i+v_i)$. The probabilities which must be compared are the same in the more general situation of figure II-5. These probabilities depend only on the best reply structure.

Since similar products appear in Nash's cooperative bargaining theory we call u_1u_2 and v_1v_2 the Nash-products of U and V, respectively.

It is interesting to note that the areas of the stability regions of U and V (see figure II-5) are proportional to the Nash-products of U and V. This is a further argument for a notion of risk dominance based on the comparison of Nash-products.

Risk dominance and payoff dominance may point in different directions. An example is the game of figure II-9 where U payoff dominates V but V has the greater Nash-product (the Nash-products are the same as in figure II-10).

The notion of risk dominance between strong equilibrium points which has been obtained heuristically can be characterized by a set of simple axioms. This will be done in a later section.

8. Payoff Monotonicity

Consider a game $G = (\Phi, H)$ with $\Phi = \underset{i \in N}{X} \Phi_i$ and let ψ be a pure strategy equilibrium point of G. We construct a new game $G' = (\Phi, H')$ with the same set Φ of pure strategy combinations. Let λ_i with $i \in N$ be non-negative constants at least one of which is positive. Define

(2.28) $H'(\varphi) = H(\varphi)$ for $\varphi \neq \psi$

(2.29) $H'_i(\psi) = H_i(\psi) + \lambda_i$ for every $i \in N$

If G and G' are related in this way we say that G' results from G by strengthening ψ. The only difference between G and G' consists in the fact that some players receive more at ψ.

Payoff monotonicity: A solution function L for a class Γ of normal form games is called payoff monotonous if the following is true: If the solution L(G) of a game $G \in \Gamma$ is a pure strategy equilibrium point and if G' results from G by strengthening L(G) then we have L(G') = L(G).

Interpretation: The requirement of payoff monotonicity is a very appealing one. Why should an equilibrium point become less attractive if some of its payoffs are increased? Nevertheless, an objection can be raised which makes it doubtful whether one should insist on payoff monotonicity as a general property.

In order to explain the nature of the counter-argument we look at the example of the three-person games of figure II-12 and of figure II-13. The game of figure II-13 results from that of figure II-12 by strengthening $U = U_1 U_2 U_3$. In the second game player 3 receives 1 unit more at U than in the first one. Otherwise both games agree in all payoffs.

It is reasonable to start a crude analysis of the risk situation in both games with the assumption that player 3 is more

	U_2	V_2	
U_1	7 7 1	0 0 0	
V_1	0 0 0	15 15 0	

U_3

	U_2	V_2	
U_1	3 3 0	0 0 0	
V_1	0 0 0	4 4 3	

V_3

Figure II-12: A three-person game. Player 3 chooses between the left and the right matrix

	U_2	V_2
U_1	7 7 2	0 0 0
V_1	0 0 0	15 15 0

U_3

	U_2	V_2
U_1	3 3 0	0 0 0
V_1	0 0 0	4 4 3

V_3

Figure II-13: A game which results from that of figure II-12 by strengthening U

likely to choose U_3 in the second game. But does this strengthen U more than $V = V_1V_2V_3$?

Suppose that each of the players 1 and 2 expects the other to behave in the same way in both games. Then an increase of their subjective probability for U_3 will increase their incentive to use their strategies V_1 and V_2. The numbers are chosen in such a way that it is not unreasonable to expect that the change from the first game to the second one enhances the stability of V more than that of U.

The solution concept which we shall propose here actually assigns the solution U to the first game and the solution V to the second. It does not have the payoff monotonicity property.

In spite of the fact that we reject payoff monotonicity as a general property we think that it is a very reasonable requirement for 2x2-games. There we cannot find any reason to suppose that one of two strong equilibrium points can be made more attractive than the other. The nature of the example seems to indicate that at least three players are needed in order to produce an example where payoff monotonicity fails to be convincing.

9. Axiomatic characterization of risk dominance between strong equilibrium points in 2x2-games

Let $\mathcal{R}$ be the class of all 2x2-games with 2 strong equilibrium points. We shall axiomatise a <u>risk</u> <u>dominance</u> <u>relationship</u> which is defined between the two strong equilibrium points of any game in $\mathcal{R}$. The notation U $\sim$ V is used in order to indicate that U risk dominates V. We also permit that neither U risk dominates V nor V risk dominates U and we write U|V if this is the case. For any game G $\in \mathcal{R}$ with strong equilibrium points U and V exactly one of the following statements must hold:

1.   U $\sim$ V   U risk dominates V in G
2.   V $\sim$ U   V risk dominates U in G
3.   U|V   There is no risk dominance between
                   U and V in G.

This is part of the definition of the concept of a risk dominance relationship and not yet a requirement to be imposed on it.

The axioms are stated below. It will always be understood that U and V are the strong equilibrium points of a game $G = (\Phi, H) \in \mathcal{R}$.

(I). Invariance with respect to isomorphisms: Let f be an isomorphism from G to G'. Then we have $f(U) \succ f(V)$ in G' if and only if we have $U \succ V$ in G.

(II). Best reply invariance: Let $G' = (\Phi, H')$ be a game which has the same best reply structure as $G = (\Phi, H)$. Then $U \succ V$ holds in G' if and only if it holds in G.

(III). Payoff monotonicity: Let $G' = (\Phi, H')$ be a game which results from $G = (\Phi, H)$ by strengthening U. If $U \succ V$ or $U I V$ holds in G then $U \succ V$ holds in G'.

Interpretation: It is clear that we must require invariance with respect to isomorphisms. The reasons are the same as those discussed in section 4. As we have seen in section 7, the intuitive arguments which we have used in order to compare risks attached to different equilibrium points run in terms of the best reply structure. Imposing axiom (II) means that we look for a concept of this kind without specifying a precise way in which risk comparisons should be made.

Payoff monotonicity has been discussed in section 8. As far as 2x2-games are concerned it seems to be a very desirable property even if for more complicated games the situation is less clear.

Theorem: There is one and only one risk dominance relationship for $\mathcal{R}$ which satisfies (I), (II) and (III). As in figure II-5 let u_i and v_i with $i = 1,2$ be the deviation losses of player i at the strong equilibrium points U and V of a game $G \in \mathcal{R}$. Then we have

(2.30)　　$U \succ V$　　for $u_1 u_2 > v_1 v_2$

(2.31)　　$V \succ U$　　for $v_1 v_2 > u_1 u_2$

(2.32)　　$U I V$　　for $u_1 u_2 = v_1 v_2$

Proof: Up to renamings of the strategies every game $G \in \mathcal{R}$ is in the class of games of figure II-5. Any such game has the same best reply structure as the corresponding game of figure II-7 (see section 5). Multiplication of player 1's payoff by $1/v_1$ and player 2's payoff by $1/u_2$ transforms a game of figure II-7 into a game of figure II-14.

$$
\begin{array}{c|c|c}
 & V_1 & V_2 \\
\hline
U_1 & \begin{matrix} u & \\ & 1 \end{matrix} & \begin{matrix} 0 & \\ & 0 \end{matrix} \\
\hline
U_2 & \begin{matrix} 0 & \\ & 0 \end{matrix} & \begin{matrix} 1 & \\ & v \end{matrix} \\
\end{array}
\qquad
\begin{aligned}
u &= \frac{u_1}{v_1} \\[2ex]
v &= \frac{v_2}{u_2}
\end{aligned}
$$

Figure II-14:   Games equivalent to those of figure II-7

For $u = v$ the game of figure II-14 has a symmetry which carries U to V (renaming of strategies and exchanging the players). Therefore, in view of (I) for $u = v$ we must have U|V.

A game of figure II-14 with $u > v$ results from a game with $u = v$ from strengthening U. Therefore, in view of (III) we must have $U \succ V$ for every game of figure II-14 with $u > v$ and similarily $V \succ U$ for every game of figure II-14 with $v > u$.

Since the best reply structure of a game of figure II-5 is the same as that of the corresponding game of figure II-14 we must have $U \succ V$ for $u > v$ there, too. We have $u > v$ if and only if $u_1 u_2 > v_1 v_2$. Analogously, we have $V \succ U$ if and only if $v_1 v_2 > u_1 u_2$. This proves the theorem.

Comment: The theorem gives a firm basis to our intuitive considerations on risk dominance between strong equilibrium points in 2x2-games. The only notion of risk dominance which

agrees with the axioms can be described as a comparison of Nash-products of deviation losses.

It is interesting that our result supports Nash's bargaining theory under fixed threats without relying on anything similar to the axiom of irrelevant alternatives which plays a crucial role in his axiomization.

On the basis of the risk dominance relationship characterized by the theorem one can define a solution function which will be called pure risk dominance solution function since it completely ignores the aspect of payoff dominance.

<u>The pure risk dominance solution</u>: The pure risk dominance <u>solution function</u> $\bar{L}$ on $\mathcal{A}$ is defined as follows: Let U and V be the strong equilibrium points of $G = (\Phi, H)$ and let u_i and v_i for $i = 1,2$ be the deviation losses at U and V (as in figure II-5). Let $r = (r_1, r_2)$ with

$$(2.33) \qquad r_1(U_1) = \frac{v_2}{u_2 + v_2} \quad , \quad r_2(U_2) = \frac{v_1}{u_1 + v_1}$$

be the third equilibrium point of G. Then we have:

$$(2.34) \qquad \bar{L}(G) = \begin{cases} U & \text{for } u_1 u_2 > v_1 v_2 \\ V & \text{for } v_1 v_2 > u_1 u_2 \\ r & \text{for } u_1 u_2 = v_1 v_2 \end{cases}$$

<u>Conflict between risk dominance and payoff dominance</u>: We have already pointed out in section 7 that a risk dominance relationship in one direction is compatible with a payoff dominance relationship in the other direction. It is maybe useful to look at the extreme example of figure II-15.

Here U payoff dominates V but V strongly risk dominates U. It is reasonable to expect that most players would prefer to play V_i rather than U if the game is played for a considerable amount of money (say $ 1000,- per unit) without preplay communication. On the other hand, with preplay com-

	U_2	V_1
U_1	9 9	0 8
V_2	8 0	8 8

Figure II-15: Example of payoff dominance and risk dominance in opposite directions

munication they may very well come to the conclusion that they can trust each other to choose $U = (U_1, U_2)$. An agreement to do so is selfstabilizing and does not need any commitment power.

If it is common knowledge of both players that both are fully rational then there should not be any need to enter preplay communication before the beginning of this game since the outcome can be predicted easily anyhow. Therefore, even under conditions which do not permit preplay communication they should trust each other to play U.

The pure risk dominance solution involves a certain lack of rationality. Nevertheless, under certain circumstances distrust may be justified. Suppose for example that in the game under consideration preplay communication has taken place and for some mysterious reason the players could not agree on U. Then, after the breakdown of communication, it is certainly justified not to look at payoff dominance and to rely on risk dominance only.

For a long time the authors took the point of view that everything which goes beyond pure risk dominance should be captured by formal models of preplay communication which explicitly describe how trust is developed rationally under the threat of conflict. In a theory of this type the pure

risk dominance solution would serve as a threat point of pre-play bargaining. Preplay bargaining itself would be described as a game where an equilibrium point has to be selected. Hopefully in this bargaining game the conflict between risk dominance and payoff dominance may not occur. Otherwise one would meet the difficulty that bargaining on bargaining is required before the beginning of the bargaining game. In spite of the difficulties involved in this approach it may still be worth trying.

It is our impression that a theory which gives room to both payoff dominance and risk dominance is more in agreement with the usual image of what constitutes rational behavior. Moreover, it avoids some of the difficulties of the approach outlined above even if models of preplay communication may still be necessary for some purposes.

10. The proposed solution function for 2x2-games with two strong equilibrium points

The solution function L for $\mathcal{R}$ which results from the application of our general concept to this class gives absolute priority to payoff dominance. It can be described as follows. Let U and V be the strong equilibrium points of G = (Φ, H). Then we have:

(2.35)
$$L(G) = \begin{cases} U & \text{if } H_i(U) > H_i(V) \text{ for } i = 1,2 \\ V & \text{if } H_i(U) < H_i(V) \text{ for } i = 1,2 \\ \bar{L}(G) & \text{else} \end{cases}$$

where $\bar{L}(G)$ is the pure risk dominance solution function introduced in section 9. We call this solution function L the proposed solution function for $\mathcal{R}$.

One may ask how the solution function L should be extended to the class of all 2x2-games. Obviously, those games which have only one equilibrium point raise no difficulties. Some degenerate cases with an infinity of equilibrium points like the example of figure II-16 cannot be fully discussed before the introduction of further basic concepts. An important de-

	U_2	V_2
U_1	1 1	0 1
V_i	1 0	0 0

Figure II-16: A degenerate 2x2-game

finition, namely that of a cell will be introduced in the next section in order to prepare the impossibility result on sequential player splitting which has been mentioned in the introduction of the chapter.

11. Cells

It is natural to require that a solution function for extensive games is subgame consistent in the sense that the behavior prescribed on a subgame is nothing else than the solution of the subgame. After all, once the subgame has been reached all other parts of the game are strategically irrelevant.

It is not immediately clear how subgame consistency can be achieved in the framework of the agent normal form. The definition of a subgame depends on the tree structure of the extensive form. The agent normal form abstracts from the information on the sequential order in which choices are made.

Nevertheless, the essential features of a subgame are not lost in the transition to the agent normal form. In order to capture these essential features we shall define substructures of the agent normal form which will be called cells. As we shall see, a subgame always corresponds to a cell but it is also possible that an agent normal form has

a cell which does not arise from a subgame of the exten-
sive form. Concentration on the essential features of a
subgame leads us to the more general notion of a cell.

Before we procede to the definition of a cell we shall
investigate the question what happens to a subgame in the
transition to the agent normal form. This shall be done
with the help of an example. After the discussion of the
example it will be easy to see the general picture.

Subgames and the agent normal form: Let Γ be the extensive
game of figure II-17. The game Γ has two proper subgames,
one following the choice ℓ of player 1 and another after
his choice r. The former subgame will be called Γ_ℓ and the
latter Γ_r.

What happens to Γ_ℓ and Γ_r in the agent normal form of Γ?
In the agent normal form each information set u_i belongs
to a separate player i. Thus the agent normal form has two
new players 2 and 4 which control u_2 and u_4 instead of
the old player 2. Similarily the old player 3 is split
into two new players 3 and 5.

Obviously, a subgame can be looked upon as a game which
is played by a subset of all agents. Thus Γ_r is played by
the new players 4 and 5. Once the subgame has been reached,
the payoffs depend only on the behavior of the agents in
the subgame. In this sense the agents of a subgame depend
only on each other and not on the other agents of the game.
As far the agents 4 and 5 are concerned the strategic si-
tuation is that of the game of figure II-18.

The fact that the subgame agents are independent of outside
agents is less obvious if one looks at the agent normal
form without knowing from which extensive form it is derived.
Since an agent always receives the payoff of the correspond-
ing player in the original game, he receives payoffs not
only inside the subgame but also at other endpoints. More-
over, outside agents decide with which probability the sub-
game is reached. In the agent normal form of the game of
figure II-17 let p_1 be the probability with which r is
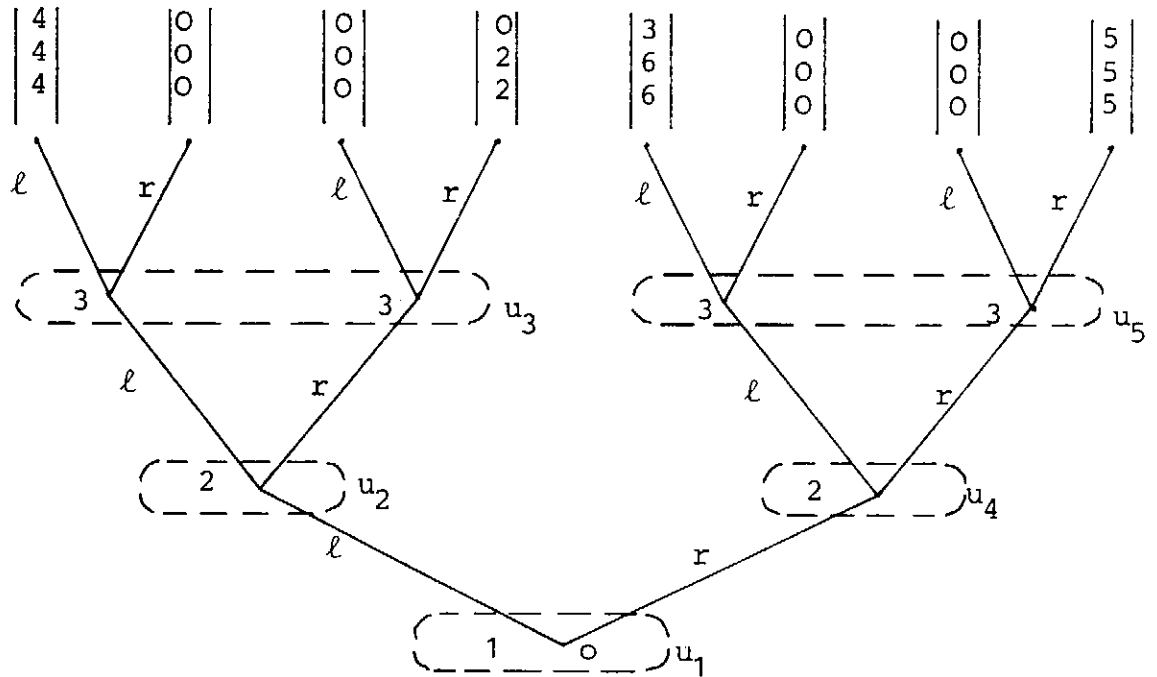chosen by player 1. Then each of the agents 4 and 5 re-

Figure II-17: Example of an extensive game with two proper subgames. Information sets are represented by dashed lines. Choices are indicated by the letters ℓ and r (standing for left and right). Payoff vectors are indicated by column vectors above the corresponding endpoints

ceives p_1 times his payoff in the subgame Γ_r plus $1-p_1$ times his payoff in the subgame Γ_ℓ. For fixed choices of agents 1,2 and 3 the payoffs of players 4 and 5 are non-negative linear transformations of the payoffs of figure II-18. The transformation is non-negative but not necessarily positive since the coefficient p_1 may be zero.

agent 5

r ℓ

agent 4	ℓ	6 6	0 0
	r	0 0	5 5

Figure II-18: The strategic situation of agents 4 and 5
in the game of figure II-17

If p_1 is zero then Γ_r is not reached. In the perturbed
agent normal form p_1 is constrained by a positive lower
bound and the transformation is always positive.

It can be seen easily that the emerging picture holds for
subgames in general. Let C be the set of agents in a sub-
game and let N be the set of all agents. Then the agents in
C are independent of outside agents in the sense that up
to non-negative linear transformations their payoffs de-
pend on their choices only. Moreover, in a perturbed agent
normal form these transformations are always strictly po-
sitve since the multiplicative coefficient is the proba-
bility that the subgame is reached. The additive constant
is due to payoffs at outside endpoints.

One may say that in the perturbed game a change of the
outside agent's strategies has essentially the same effect on
the agents in C as a transition to an equivalent game (see
section 3). This is the distinguishing feature of a sub-
game which shall be captured by our defintion of a cell.

In the case of a subgame, the multiplicative coefficient
of the linear transformation connected to a strategy change
by outside agents is always the same for all agents of the
subgame. There seems to be no good reason why this peculia-

rity should be reproduced by the cell notion. Therefore, in
our definition of a cell we shall permit different multi-
plicative constants for different agents.

Suppose that two players 1 and 2 with linear utilities in
money are involved in a bimatrix game whose entries are
in terms of unknown currencies. Before they make their
choices a third player secretly selects between two alter-
native possibilities (a) player 1 receives dollars and
player 2 reveices Israeli poinds or (b) player 1 receives
French francs and player 2 receives German marks. - It is
reasonable to define cells in such a way that in this example
players 1 and 2 form a cell. Obviously, their strategic si-
tuation is not influenced by the currency assignment.

We may change the currency example by giving an additional
choice (c) to the third player. If he selects (c) then
player 1 and 2 will receive zero payoffs no matter what
they do. We shall take the point of view that 1 and 2 form
a cell in this case, too. In the perturbed games 1 and 2
would form a cell anyhow, even if a definition were adopted
which would be based on positive transformations only. It
seems to be preferable to work with a cell concept which
does not give different results for an agent normal form
and its perturbed agent normal forms. Therefore, we shall
permit zero as a multiplicative coefficient.

In order to prepare the definition of a cell we must intro-
duce some auxiliary definitions and notations which all re-
fer to a fixed game $G = (\Phi, H)$ with $\Phi = \underset{i \in N}{X} \Phi_i$.

Subset strategy combinations: Let C be a subset of N. A sub-
set strategy combination for C or shortly a strategy combina-
tion for C is a collection $q_C = (q_i)_C$ which contains a stra-
tegy $q_i \in Q_i$ for every $i \in C$. A pure strategy combination $\varphi_C = (\varphi_i)_C$
for C contains a pure strategy $\varphi_i \in \Phi_i$ for every $i \in C$. The set
of all pure strategy combinations for C is denoted by Φ_C and
Q_C stands for the set of all mixed strategy combinations for
C. Instead of $\varphi_{N \smallsetminus C}$ and $q_{N \smallsetminus C}$ we also use the shorter notation

φ_{-C} and q_{-C}. In Φ_{-C} and Q_{-C}, too, $-C$ stands for $N \setminus C$.

Fixed players: Let r_{-C} be a subset strategy combination for $N \setminus C$. We construct a game $G' = (\Phi', H')$ with

$$(2.36) \qquad \Phi' = \underset{i \in C}{X} \Phi_i$$

The payoff function H' is defined as follows:

$$(2.37) \qquad H'_i(\varphi_i) = H_i(r_{-C}\varphi_C) \quad \text{for } i \in C$$

We say that G' is the game which results from G by fixing the players in $N \setminus C$ at r_{-C} .

Centroids: Let D be a non-empty subset of N and for every $i \in D$ let ψ_i be a non-empty subset of this player's pure strategy set Φ_i. The centroid of $\psi = \underset{i \in D}{X} \psi_i$ is a strategy combination $r_D = (r_i)_D$ for D which is defined as follows:

$$(2.38) \qquad r_i(\varphi_i) = \begin{cases} 1/|\psi| & \text{for } \varphi_i \in \psi_i \\ 0 & \text{for } \varphi_i \notin \psi_i \end{cases}$$

for every $i \in D$ where $|\psi_i|$ is the number of elements in ψ_i. The centroid of ψ is denoted by $c(\psi)$.

Cells: Let C be a non-empty proper subset of N and let $G^C = (\Phi^C, H^C)$ be the game which results from G by fixing the players in $N \setminus C$ at the centroid $c(\Phi_{-C})$ of the cartesian product of their pure strategy sets. The game G^C is a cell of G if for every $\psi_{-C} \in \Phi_{-C}$ and for every $i \in C$ a number $\alpha_i(\psi_{-C}) \geq 0$ and number $\beta_i(\psi_{-C})$ can be found such that we have:

$$(2.39) \qquad H_i(\psi_{-C}\varphi_C) = \alpha_i(\psi_{-C})H^C_i(\varphi_C) + \beta_i(\psi_{-C})$$

We say that C forms a cell if G^C is a cell. If this is the case G^C is the cell formed by C.

Remark: If G is the agent normal form of an extensive game and C is the set of all agents in a subgame then C forms a cell in G. This is clear from the discussion in the sub-

section on subgames and the agent normal form. On the other hand, a cell may not necessarily arise from a subgame.

In the case of a cell arising from a subgame the subgame is always reached with positive probability by the centroid $c(\Phi_{-C})$. Therefore, the definition of G^C with the help of the centroid always results in a normal form which is equivalent to the agent normal form of the subgame.

The centroid $c(\Phi_{-C})$ serves to pick a specific game as a representation of the class of all games G' which result from G by fixing the players in $N \smallsetminus C$ at a completely mixed strategy combination q_{-C}. Here completely mixed means that every player $j \in N \smallsetminus C$ selects each of his pure strategies with positive probability. All these games are equivalent. Therefore, the use of the centroid in the definition of a cell is non-arbitrary.

Lemma on cells: Let C and C' with $C \cap C' \neq \emptyset$ be two proper subsets of N which both from cells of $G = (\Phi, H)$ with $\Phi = \underset{i \in N}{X} \Phi_i$. Then $D = C \cap C'$ forms a cell of G, too.

Proof: Any change of the strategy combination q_{-D} for $N \smallsetminus D$ can be achieved by two successive changes, such that first only the strategies of players in $N \smallsetminus C$ are changed and then only those of players in $C \smallsetminus D$. Both changes are connected with non-negative linear payoff transformations for the payoffs of players in D, in the first case since C forms a cell, in the second since C' forms a cell. Two successive non-negative linear transformations performed one after another are equivalent to non-negative linear transformation. In this way we receive the non-negative linear transformations whose existence is required by the definition of a cell as applied to D.

Counterexample: One may think that the union of two subsets C and C' form a cell if C and C' form cells and if the union is a proper subset of N. The example of the game exhibited in figure II-19 shows that this is not necessarily true. There both $\{1\}$ and $\{2\}$ form cells since for fixed strategies of the other players the difference between the payoffs for V_1 and U_1 is always 1.

$$U_3: \qquad
\begin{array}{c|cc}
 & U_2 & V_2 \\
\hline
U_1 & 3 \;\; 3 \;\; 1 & 0 \;\; 4 \;\; 0 \\
V_1 & 4 \;\; 0 \;\; 0 & 1 \;\; 1 \;\; 0
\end{array}
\qquad\qquad
V_3: \qquad
\begin{array}{c|cc}
 & U_2 & V_2 \\
\hline
U_1 & 0 \;\; 0 \;\; 0 & 2 \;\; 1 \;\; 0 \\
V_1 & 1 \;\; 2 \;\; 0 & 3 \;\; 3 \;\; 1
\end{array}$$

Figure II-19: A counterexample. {1} and {2} form cells but {1,2} does not form a cell. Player 1, 2 and 3 choose rows, columns and matrices, respectively.

The situation is the same one for player 2. Nevertheless, {1,2} is not a cell since a shift from U_3 to V_3 reverses the sign of the payoff differences between U_1U_2 and V_1V_2. No nonnegative payoff transformation can produce a result like this.

Elementary cells: Let C be a non-empty subset of N which forms a cell G^C. The cell G^C is called elementary if no proper subset of C forms a cell in G. It follows by the lemma on cells that subsets which form elementary cells do not intersect.

Comment: The fact that elementary cells do not intersect is an important one since it enables us to define a solution function which is based on the idea that a game with cells should be solved by first solving the elementary cells and then solving the game which results by fixing the players of the elementary cells at the strategies prescribed by the solutions of these cells.

12. Cell consistency and truncation consistency

In this section we shall look at two additional desirable properties of solution functions. Roughly speaking,cell consistency requires that the solution of the whole game agrees

with that of its cells as far as the cell players are concerned. Truncation consistency concerns the "truncated" game which results if the players in a cell are fixed at the solution of this cell. The requirement postulates that the solution of this game should agree with the solution of the original one, as far as its players are concerned.

Completeness: A class $\mathcal{Y}$ of normal form games is called <u>complete</u> if a game G' , which results from a game $G \in \mathcal{Y}$ by fixing some but not all of the players at arbitrary strategies, also belongs to $\mathcal{Y}$.

Truncations: Let L be a solution function for a complete class $\mathcal{Y}$ of games in normal form. Let $G \in \mathcal{Y}$ be a game with a cell G^C. The <u>truncation</u> <u>of</u> G <u>with</u> <u>respect</u> <u>to</u> G^C <u>and</u> L is the game which results from G by fixing the players of G^C at their strategies in the solution $L(G^C)$ of G^C.

Remark: The completeness of $\mathcal{Y}$ is important since it guarantuees that both G^C and the truncation G' of G with respect to G^C and L will be in G, if G^C is a cell of G. Both G^C and G' result from G by fixing some of the players but not all of them.

Cell consistency: A solution function L for a complete class $\mathcal{Y}$ of normal form games is called <u>cell</u> <u>consistent</u> if for a cell G^C of a game $G \in \mathcal{Y}$ the solutions $L(G^C)$ and $L(G)$ of G^C and G always prescribe the same strategies to the players of G^C.

Truncation consistency: A solution function L for a complete class $\mathcal{Y}$ of normal form games is called <u>truncation</u> <u>consistent</u> if for a truncation G' of a game $G \in \mathcal{Y}$ with respect to a cell G^C of G and L the solutions $L(G')$ and $L(G)$ always prescribe the same strategies for all players of G'.

Interpretation: As far as their strategic situation is concerned the players in a cell do not depend on outside players. This has been discussed in section 11. Obviously, cell consistency is a very natural requirement.

Truncation consistency is a very natural requirement, too, since the outside players know that the cell players do not depend on them. It is rational to expect that the cell

players will play the cell solution. Therefore, the outside
players find themselves in the situation of the truncated
game.

As we shall see cell consistency and truncation consistency
have the consequence that it is sufficient to know the solu-
tions of the games without cells in order to compute the
solutions of all games in a complete class.

Decomposibility: A game G is called decomposable if it has
at least one cell. Games without cells are called indecom-
posable. We say that G is fully decomposable if every player
belongs to an elementary cell. Decomposable games which
are not fully decomposable are called partially decomposable.

Main truncation: Let L be a solution function for a complete
class $\mathcal{G}$. For every partially decomposable game G $\in$ $\mathcal{G}$ we
construct a game G' which is called the main truncation of G.
Let $G^1,\ldots,G^k$ be the elementary cells of G. The game G' results
from by fixing the players in the elementary cells at their stra-
tegies in the solutions $L(G^1),\ldots,L(G^k)$ of the elementary cells.

Composition: Let L be a solution function for a complete class
of games $\mathcal{G}$ and let G $\in$ $\mathcal{G}$ be a fully decomposable game. Let r
be the strategy combination for G which contains for every
player i his strategy prescribed by the solution $L(G^j)$ of
the elementary cell to which he belongs. This strategy com-
bination r is called the composition of the elementary cell
solutions. Now consider a partially decomposable game G $\in$ $\mathcal{G}$.
Let r be the strategy combination which (a) for every player i
in an elementary cell G^j of G contains his strategy in $L(G^j)$
and (b) for every player in the main truncation G' of G con-
tains his strategy in L(G'). This strategy combination r is
called the composition of the main truncation and elementary
cell solutions.

Extension: Let $\mathcal{G}$ be a complete class of normal form games
and let $\mathcal{G}_0$ be the subclass of all indecomposable games in $\mathcal{G}$.
Moreover, let L_0 be a solution function for $\mathcal{G}_0$. On the basis
of L_0 we shall construct a solution function L for $\mathcal{G}$

which will be called the _extension_ _of_ L_0 _to_ $\mathcal{O}_l$. The extension L is recursively defined by the following properties (A), (B) and (C).

(A) For $G \in \mathcal{O}_{l_0}$ we have $L(G) = L_0(G)$

(B) If $G \in \mathcal{O}_l$ is fully decomposable then $L(G)$ is the composition of the elementary cell solutions.

(C) If $G \in \mathcal{O}_l$ is partially decomposable then $L(G)$ is the composition of the main truncation and the elementary cell solutions.

It is clear that in this way a solution $L(G)$ is uniquely defined for every game $G \in \mathcal{O}_l$. Property (C) may have to be applied several times first to the game itself, then to its main truncation, etc. but finally a truncation will arise which is either indecomposable or fully decomposable.

Extension theorem: Let $\mathcal{O}_l$ be a complete class of games in normal form, let $\mathcal{O}_{l_0}$ be the subclass of indecomposable games in $\mathcal{O}_l$ and let L_0 be a solution function for $\mathcal{O}_{l_0}$. There is one and only one cell consistent and truncation consistent solution function for $\mathcal{O}_l$ which agrees with L_0 on $\mathcal{O}_{l_0}$, namely the extension L of L_0 to $\mathcal{O}_l$.

Proof: It is clear that a cell consistent and truncation consistent solution function must agree with the extension L of L_0 since these two properties permit us to compute the solution with the help of (A), (B) and (C).

It remains to show that the extension L of L_0 has the properties of cell consistency and truncation consistency. This will be done by induction on the number of players in C. Both properties trivially hold for 1-person games. Assume that they hold for games in $\mathcal{O}_l$ with at most n-1 players. Consider a decomposable game $G \in \mathcal{O}_l$ with n players; let G^C be a cell of G and let G' be the truncation of G with respect to G^C and L. We have to show that $L(G)$ prescribes the same strategies as $L(G)$ and $L(G')$

Let D be the set of all players in G^C which do not belong to elementary cells. Let E be the set of all players who belong to elementary cells outside G^C. Let G" be the main truncation of G.

If D and E are both empty, then G' agrees with G" and the assertion is an immediate consequence of the definition of L.

Suppose that we have $D \neq \emptyset$. Let G^D be the main truncation of G^C. It is clear that G^D is a cell of G" since the non-negative linear transformations which establish the cell property of G^D in G' can easily be constructed from those which establish the cell property of G^C in G.

Suppose that $E \neq 0$. Let $G^1,\ldots,G^m$ be the elementary cells with players in E. For $j = 1,\ldots,m$ let $\hat{G}^j$ be the game which results from G' by fixing the players in the cells $G^1,\ldots,G^j$ at their strategies in $L(G^1),\ldots,L(G^j)$. Let $N_1,\ldots,N_m$ be the player sets of $G^1,\ldots,G^m$, respectively. It can be seen easily that for $j = 2,\ldots,m$ the set N_j forms a cell in G^{j-1} even if this cell may not be an elementary one. The non-negative linear transformations which establish the cell property of N_j in G^{j-1} can easily be constructed from those which establish the cell property of N_j in G.

Since G' has fewer than n players we can repeatedly apply cell consistency and truncation consistency to the games G', $\hat{G}^1,\ldots,\hat{G}^m$ in order to conclude that L(G') prescribes the strategies in $L(G^1),\ldots,L(G^m)$ and $L(\hat{G}^m)$.

Consider the case $D = \emptyset$ and $E \neq \emptyset$. In this case we have $\hat{G}^m = G"$. This together with our conclusion on G' immediately yields the assertion.

Assume $D \neq \emptyset$ and $E = \emptyset$. In this case G is the truncation of G" with respect to G^D and the assertion follows by the application of cell conistency and subgame consistency to G", G^D and G'.

Finally consider the case $D \neq \emptyset$ and $E \neq \emptyset$. Here the assertion follows by the fact that on the one hand $\hat{G}^m$ is related

to G' in the way which has been explained above and that,
on the other hand, $\tilde{G}^m$ is the truncation of G" with respect
to G^D. The arguments for the cases D = ∅, E ≠ ∅ and D ≠ ∅,
E = ∅ can be combined in order to obtain the result.

Comment: The extension theorem shows that cell consistency
and truncation consistency are powerful properties which
reduce the task of defining a solution concept to the task
of defining a solution concept for indecomposable games.

Cell consistency and truncation consistency require that
all considerations which may influence the selection of equi-
librium points are applied strictly locally, i.e. only to
those indecomposable games which appear in the process of
computing the solution with the help of (A), (B) and (C)
on the basis of a solution concept for indecomposable games.
These indecomposable games shall be called the bricks of
the original game.

Local and global payoff efficiency: Payoff efficiency is an
example of a selection criterion which cannot be applied
to the game as a whole but only locally to its bricks. Fi-
gure II-17 shows an example of a conflict between global
and local payoff dominance. The subgame Γ_ℓ after player 1's
choice ℓ has two strong equilibrium points, namely (ℓ,ℓ) and
(r,r). The same is true for the subgame Γ_r after player 1's
choice r. The agent normal form of the game lies in no other
cells than those corresponding to the subgames.

(ℓ,ℓ) is the only payoff efficient equilibrium point of the
cell corresponding to Γ_ℓ and also the only payoff efficient
equilibrium point in the cell corresponding to Γ_r. (Player 1's
agent is not a player in these cells.) If payoff efficiency
is applied locally as a selection criterion we must select (ℓ,ℓ)
in both cells. Obviously, if this is done player 1 faces a choice
between payoff 4 for ℓ and payoff 3 for r. He has to choose ℓ in
the truncated game. Local application of the payoff efficiency
criterion yields the equilibrium point where all five agents
choose ℓ. The payoffs are 4 for everybody.

Consider the strategy combination where all agents choose r. This is also an equilibrium point. It yields a payoff of 5 for everybody. It payoff dominates the equilibrium point, where all agents choose ℓ. It is in the interest of everybody to play this equilibrium point rather than the other one. Unfortunately, this is true only at the beginning of the game. After the subgame Γ_r has been reached the interests of player 1 do not count anymore and it is now in the interest of all others to play (ℓ,ℓ).

Both equilibrium points, that one where all agents choose ℓ and that one where all agents choose r, are uniformly perfect. It can be seen easily that sufficiently small perturbances do not matter.

13. Sequential agent splitting

Figure II-20 shows what sequential agent splitting means in the extensive form. An agent of player j who has to choose between a, b and c is split into two agents such that first one has to select either a or bc and then in case of bc the other decides between b and c. In the graphical representation of the extensive form the upper substructure shown by figure II-20 is taken out and the lower one is put in.

At least,at first glance it is hard to imagine why sequential agent splitting should in any way change the strategic situation. Nevertheless, as we shall see, one cannot avoid the conclusion that sequential agent splitting does have a considerable influence on risk comparisons between equilibrium points in some games.

For the purposes of our theory sequential agent splitting must be formally defined in the framework of the agent normal form. Since there the agents are players we shall use the term player splitting instead of agent splitting. Moreover, we shall drop the word sequential since the order in which the decisions are made is not really important in the agent normal form.
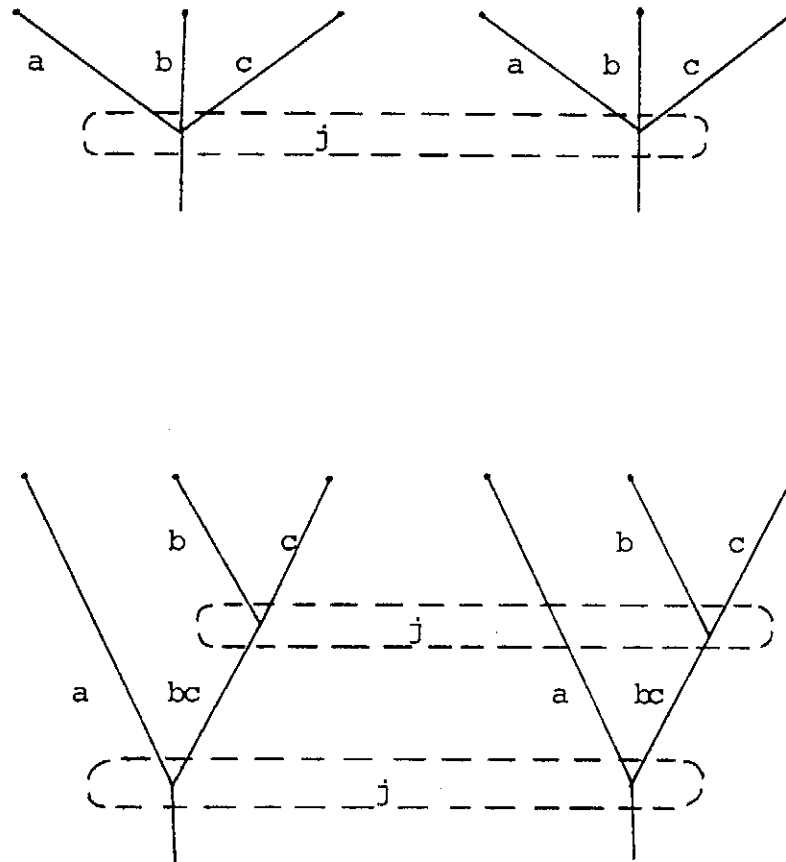
Figure II-20:   An example of sequential agent splitting
                in the extensive form

Player splitting: Let G = (Φ,H) with Φ = $\underset{i \in N}{\text{X}}$ Φ_i be a game
in normal form; let j$\in$N be one of the players and let
ψ_j $\in$ Φ_j be one of his pure strategies. Morevoer, let k be
a positive integer with k $\notin$ N and define N' = N $\cup$ {k}. We
construct a game G' = (Φ,H') with

(2.40) Φ' = $\underset{i \in N'}{\text{X}}$ Φ'_j

(2.41) Φ'_i = Φ_i for i$\in$N$\setminus${j}

(2.42) $\Phi'_k = \{\psi_j, -\psi_j\}$

(2.43) $\Phi'_j = \Phi_j \smallsetminus \{\psi_j\}$

(In (2.42) the alternative of not choosing ψ_j is symbolized by $-\psi_j$.) We say that $q' = (q'_i)_{N'}$ corresponds to $q = (q_i)_N$ and write $q' \to q$ if we have

(2.44) $q_i = q'_i$ for $i \in N \smallsetminus \{j\}$

(2.45) $q_j(\psi_j) = q'_k(\psi_j)$

(2.46) $q_j(\varphi_j) = q'_n(-\psi_j)q'_j(\varphi_j)$ for $\varphi_j = \psi_j$

The payoffs for G' are defined as follows:

(2.47) $H'_i(\varphi') = H_i(\varphi)$ with $\varphi' \to \varphi$ for $i \in N$

(2.48) $H'_k(\varphi') = H'_j(\varphi')$ for every $\varphi' \in \Phi$

The game $G' = (\Phi, H')$ is called the game which results from $G = (\Phi, H)$ by __splitting off a player__ k __for__ ψ_j .

__Interpretation:__ Even if the formal definition may appear to be somewhat complicated it can be seen easily that it is the correct translation of the idea of sequential agent splitting into the language of the agent normal form. In figure II-20 strategy ψ_j corresponds to the choice a and player k corresponds to the agent who chooses between a and bc. Players j and k in G' receive the same payoff since in the extensive form they are agents of the same player. It can also be seen that any pair q'_k, q'_j of strategies for the new players j and k has the same effect as the strategy q_j defined by (2.45) and (2.46).

__Remark:__ In the light of the interpretation it is clear that (2.47) and (2.48) also hold for mixed combinations instead of pure ones.

Invariance with respect to player splitting: Let L be a
solution function for a class $\mathcal{G}$ of normal form games and
let G = (Φ,H) and G' = (Φ',H') be games in $\mathcal{G}$ such that G'
results from G by splitting off a player k at ψ_j. Then we have
L(G') → L(G).

Interpretation: Invariance with repsect to player splitting
requires that L(G') and L(G) should prescribe essentially the
same behavior. In the case that L(G) prescribes ψ_j with pro-
bability 1, player j's behavior in L(G') is not restricted
by the requirement since it does not matter what he does
if player k selects ψ_j.

Impossibility theorem: Let $\mathcal{G}$ be a complete class of normal
form games which contains all games with at most 4 players
and at most 3 strategies for every player. Let L be a so-
lution function for $\mathcal{G}$ which satisfies the requirements of
cell consistency and truncation consistency and which for
2x2-games with two strong equilibrium points either agrees
with the proposed solution function (section 10) or with
the pure risk dominance solution function (section 9). Then
L does not satisfy the requirement of invariance with respect
to player splitting.

Proof: Assume that L has all properties mentioned in the
theorem, including invariance with respect to player split-
ting. It will be shown that two different ways of finding
the solution of a 3x3-game lead to a contradiction. This game
G is shown on the top of figure II-21.

Consider the game G^1 which results from G by first splitting
off a player 3 at player 1's strategy b and then splitting off
a player 4 at player 2's strategy b. The result is found in
figure II-21 if one follows the left arrow leading away from
G.

The game G^1 has a cell formed by players 1 and 2. It can be
seen immediately that for fixed strategies of 3 and 4 the
payoffs of players 1 and 2 are non-negative linear trans-
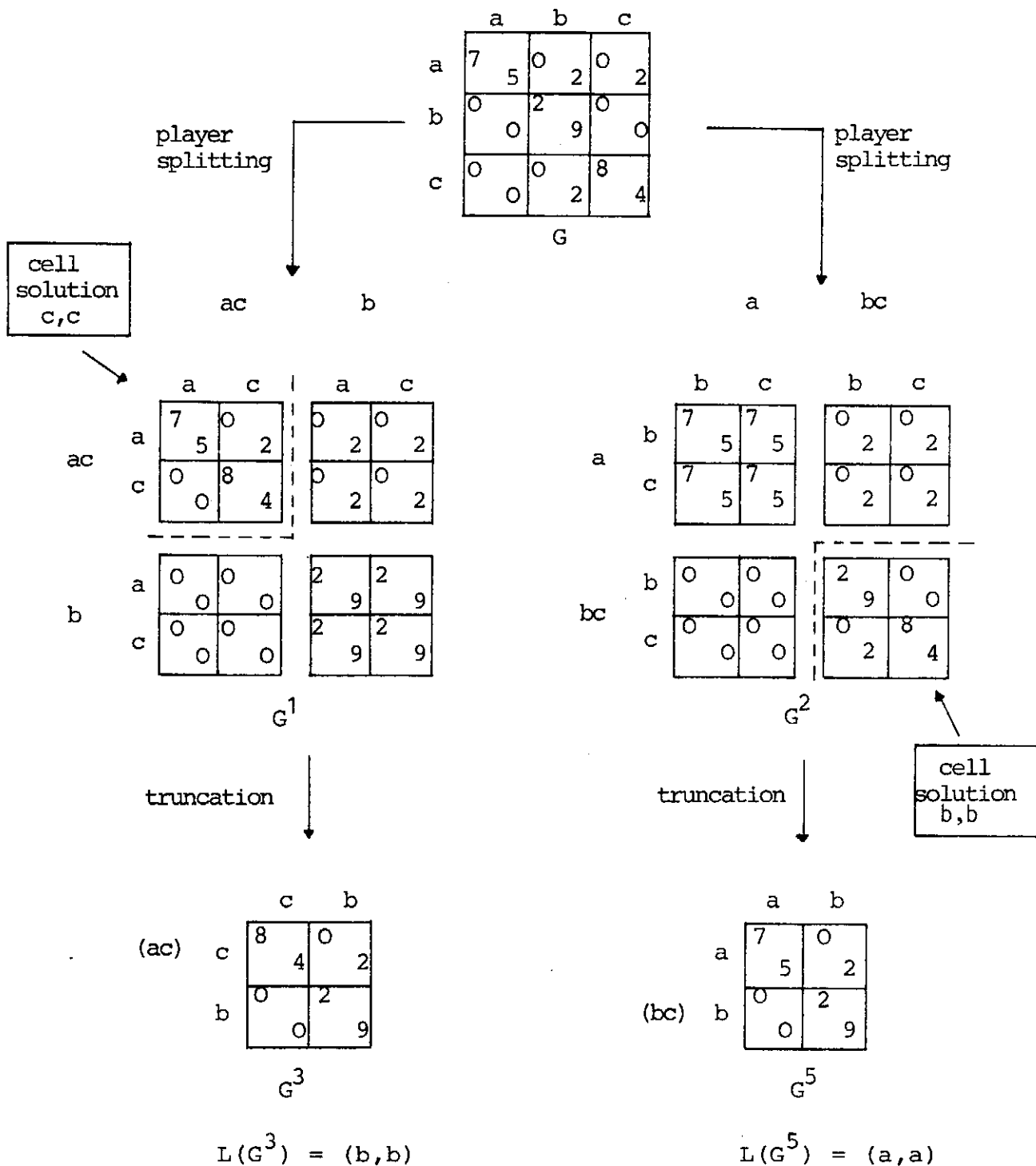forms of the payoffs obtained in the upper left bimatrix.

a b c

	a	b	c
a	7 5	0 2	0 2
b	0 0	2 9	0 0
c	0 0	0 2	8 4

G

player splitting

player splitting

cell solution c,c

ac   b

a   bc

ac

	a	c		a	c
a	7 5	0 2		0 2	0 2
c	0 0	8 4		0 2	0 2

b

	a	c		a	c
a	0 0	0 0		2 9	2 9
c	0 0	0 0		2 9	2 9

G^1

a

	b	c		b	c
b	7 5	7 5		0 2	0 2
c	7 5	7 5		0 2	0 2

bc

	b	c		b	c
b	0 0	0 0		2 9	0 0
c	0 0	0 0		0 2	8 4

G^2

cell solution b,b

truncation

truncation

(ac)

	c	b
c	8 4	0 2
b	0 0	2 9

G^3

(bc)

	a	b
a	7 5	0 2
b	0 0	2 9

G^5

$L(G^3) = (b,b)$

$L(G^5) = (a,a)$

<u>Figure II-21</u>: Proof of the impossibility theorem. In G^1 and G^2 the common payoffs of players 1 and 3 are shown in the upper left corner and the common payoffs of players 2 and 4 are shown in the lower right corner. Players 1 and 2 choose between a and c in G^1 and between b and c in G^2. Players 3 and 4 choose between ac and b in G' and between a and bc in G^2.
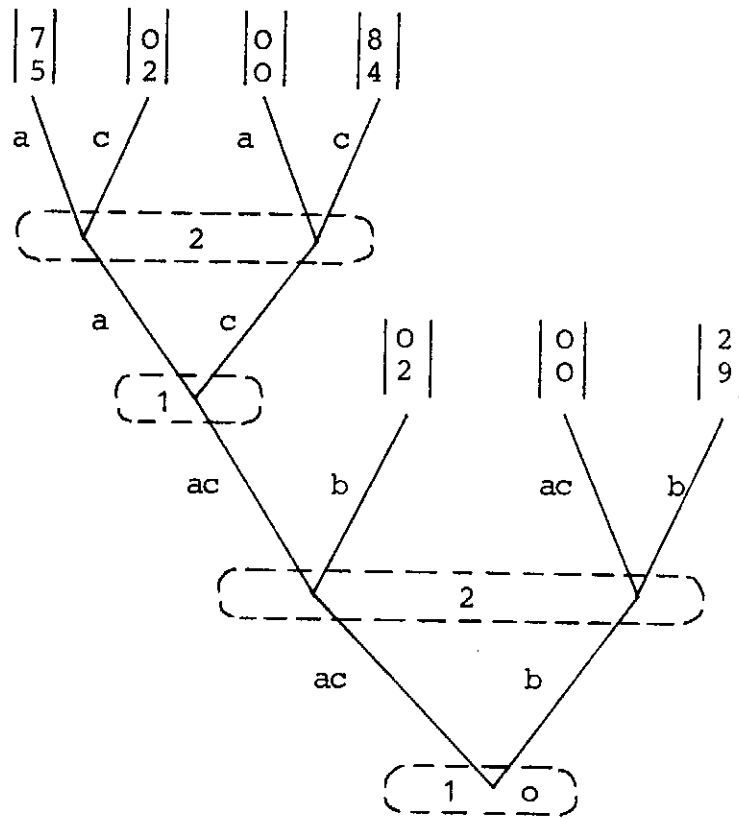
Figure II-22: Extensive form whose agent normal form agrees with the game G^1 in figure II-21
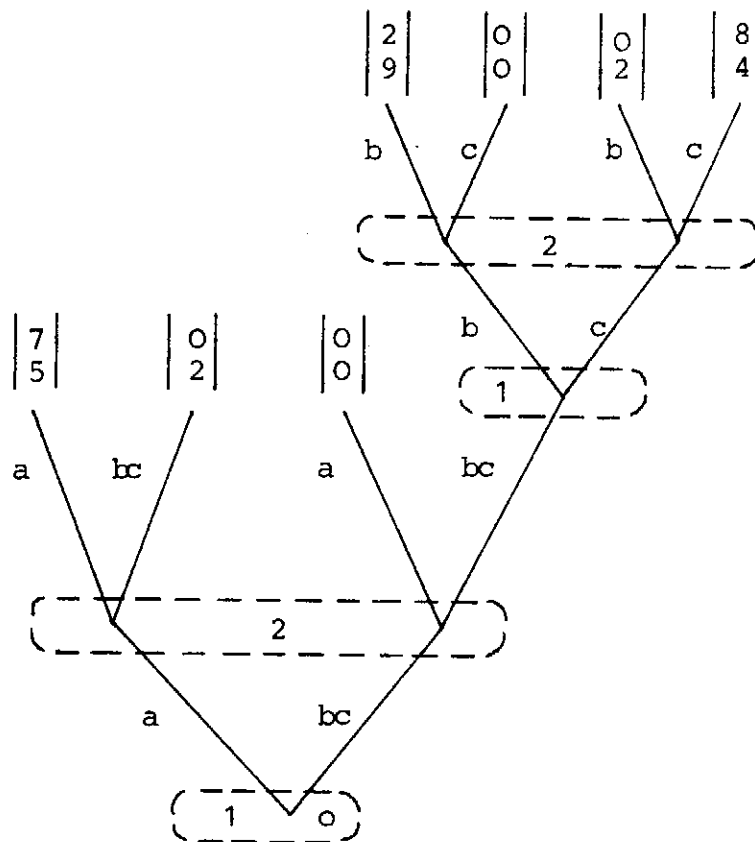
Figure II-23: Extensive form whose agent normal form agrees with the game G^2 in figure II-21

The Nash-product criterion immediately shows that (c,c) is the solution of the cell.

Note that the issue of payoff dominance does not arise since there is no payoff dominance between the three strong equilibrium points (a,a) (b,b) and (c,c).

G^3 is the truncation of G^1 with respect to the cell formed by 1 and 2. The Nash-product criterion shows that (L,L) is the solution of this game. It follows that players 3 and 4 both must choose b in $L(G^1)$. Obviously, (b,b) corresponds to $L(G^1)$ in G. Consequently, we must have $L(G) = (b,b)$.

A similar argument is shown on the right side of figure II-21. The game G^2 results from G by splitting off players 3 and 4 at the strategies a of both players. In G^1 player 1 and 2 form a cell whose solution is (b,b). The truncated game G^5 has the solution $L(G^5) = (a,a)$. Consequently, we must have $L(G) = (a,a)$. This is a contradiction to $L(G) = (b,b)$.

Remark: It is interesting to ask the question whether the result could be avoided by a more restrictive definition of a cell which would narrow down the applicability of the cell and truncation consistency requirements. In any case, a more restrictive definition of a cell would have to cover the case of a subgame of an extensive form. In this connection, it is worth pointing out that extensive forms can be found whose agent normal forms agree with G^1 and G^2, where the cells formed by 1 and 2 correspond to subgames. These extensive forms are shown in figures II-22 and II-23. Consequently, the impossibility result cannot be avoided by a more restrictive definition of cells.

Interpretation: We must draw the conclusion that it is by no means irrelevant whether a choice between a, b and c has a sequential structure or not. Games where a simultaneous choice has to be made can be different from others where the decision is split into two steps involving choices between ab and c and between a and b. If we do not want to give up the idea of a solution function altogether we must abolish one of the

properties which lead to the impossibility result. Among those properties invariance with respect to player splitting seems to be the least compelling one. Upon reflection it does not appear to be an unreasonable idea that risk comparisons between three alternatives may be changed by the imposition of a sequential structure.

After all, one must think of the fact that after a decision between ab and c has been made, in favor of ab, alternative c has become irrelevant and the risk comparisons may look quite different from those which would arise in a simultaneous choice situation. Different sequential orders may require different ways of looking at the situation. Even if it is not easy to understand why this should be so, it is reasonable to suppose that the basic reason for the impossibility result must be searched in this direction.

The proof of the impossibility theorem makes use of the fact that both ways of sequential player splitting in figure II-21 reduce the risk dominance comparisons to comparisons in 2x2-games which result from G by removing either a or b or c from the strategy sets of both players. The three comparisons which can be made in this way result in an intransitive pattern: (a,a) dominates (b,b) and (b,b) dominates (c,c) but (c,c) dominates (a,a). Moreover, each of both ways of sequential player splitting removes one of the three comparisons, namely that between (a,a) and (b,b) in the case of G^1, and that between (c,c) and (a,a) in the case of G^2. In this way we can see already here that the impossibility result is connected to intransitivities of risk dominance. We shall return to the phenomenon in chapter 4, section 3, after the introduction of our general definition of risk dominance.

14. Splitting into identical types

In the last section we came to the conclusion that we have
to reject invariance with respect to sequential agent split-
ting as a desirable property. In the following we shall
look at another way of substituting two players for one
player. This kind of player splitting has a natural inter-
pretation in terms of games of incomplete information. In
such games a player may have different types which differ
with respect to hidden variables known to the player him-
self but not to the other players. The information on
the other players' types takes the form of a probability
distribution over type combinations (Harsanyi   ).

In the extensive form different types have different informa-
tion sets. Therefore, the agent normal form will treat them as
either different players if each type has only one infor-
mation set or as non-intersecting groups of players in the
more general case.

Suppose that one of the players, say player 1, has two types I
and II whereas all other players have only one type. Assume
that type I occurs with probability α and type II with
probability $1-\alpha$. The other players do not know which of
both types will make the decisions of player 1 but they
know the probabilities. In general the interest of such
game models lies in the fact that different types may
have different payoff functions or strategy sets, but
for us the possibility that both types I and II are iden-
tical in every relevant respect is of special theoretical
importance. We may think of a situation where the hidden
variable known only to player 1 is his hair colour which
does not have any strategic significance for the game.
In such cases it should not make any difference whether
a type distinction is made or not. The solution should
remain essentially the same if a player is split into
two identical types.

<u>Splitting a player into identical types</u> : Let $G = (\Phi, H)$ be a game with $\Phi = \underset{i \in N}{X} \Phi_i$ and let $j \in N$ be a player. We construct a game $G' = (\Phi', H')$ whose player set $N' = N \cup \{k\}$ contains an additional player $k \in N$. The pure strategy sets Φ'_i and the payoff function H' of G' are as follows:

(2.49) $\Phi'_i = \Phi_i$ for $i \in N$

(2.50) $\Phi'_k = \Phi_j$

(2.51) $H'_j(\varphi'_j \varphi'_k \varphi_{-i}) = H_j(\varphi'_j \varphi_{-i})$

(2.52) $H'_k(\varphi'_j \varphi'_k \varphi_{-i}) = H_j(\varphi'_k \varphi_{-i})$

(2.53) $H'_i(\varphi'_j \varphi'_k \varphi_{-i}) = \alpha H_i(\varphi'_j \varphi_{-i}) + (1-\alpha) H_i(\varphi'_k \varphi_{-i})$
$\qquad$ for every $i \in N \smallsetminus \{j\}$

where α is number with $0 < \alpha < 1$. This game G' is called the game which results from G by splitting <u>player</u> j into identical types j and k with probabilities α and $1-\alpha$.

Let $q' = (q'_i)_{N'}$ be a mixed strategy combination for G'. The strategy combination $q = (q_i)_N$ for G which <u>corresponds</u> to q' is defined as follows:

(2.54) $q_j(\varphi_j) = \alpha q'_j(\varphi_j) + (1-\alpha) q'_k(\varphi_j)$
$\qquad$ for every $\varphi_j \in \Phi_j$

(2.55) $q_i = q'_i$ for every $i \in N \smallsetminus \{j\}$

It is clear from (2.54) that we have

(2.56) $H_i(q) = H'_i(q')$ for every $i \in N$

if q corresponds to q'. We say that $q' = (q'_i)_{N'}$ results from $q = (q_i)_N$ by <u>splitting</u> j <u>into</u> j <u>and</u> k if in addition to (2.55) we have $q'_j = q'_k = q_j$.

<u>Identical type invariance</u>: Let L be a solution function for a class of games $\mathcal{G}$. The solution function L is called invariant with respect to <u>splitting</u> <u>into</u> <u>identical</u> <u>types</u> if the following is true for any two games G and G' in $\mathcal{G}$ such that G' results from G by splitting player j into two identical types j and k with probabilities α and $1-\alpha$ where $0<\alpha<1$. The solution L(G') of G' results from L(G) by splitting into j and k.

<u>Theorem</u>: Let $\mathcal{G}$ be a complete class of games, let $\mathcal{G}_0$ be the subclass of indecomposable games in $\mathcal{G}$ and let L_0 be the solution function for $\mathcal{G}_0$. If L_0 is invariant with respect to splitting into identical types, then the extension L of L_0 to $\mathcal{G}$ is invariant with respect to splitting into indentical types.

<u>Proof</u>: Let G' be the game which results from G by splitting player j into identical types j and k with probabilities α and $1-\alpha$. Consider a group C of players which forms a cell in G. The way in which H' is linearily related to H has the immediate consequence that C U {k} forms a cell in G'. Moreover, it can be seen that G' has no other cells than those which arise in this way. This has the consequence that G' is indecomposable if and only if G is indecomposable. Therefore L(G') results from L(G) by splitting j into j and k if G is indecomposable. On the basis of this fact we prove the theorem by induction on the number k of players in G.

Since 1-person games are indecomposable the assertion holds for n = 1. Assume that it holds for any number of players up to n-1. We have to show that L(G') results from L'(G) by splitting into j and k if G is a decomposable game with n players. Obviously, the elementary cells of G' are either identical to elementary cells of G or they result from such cells by splitting j into two types j and k with probabilities α and $1-\alpha$. The solution of each elementary cell of G' results from that of the corresponding cell of G by splitting j into j and k. The assertion follows if G is fully decomposable. If G is not fully decom-

posable then it follows that the main truncation of G'
either is identical to that of G or it results from it by
splitting j into j and k with probabilities α and 1-α. In
the latter case the solution of the main truncation of G'
results from that of G by splitting into j and k since
the main truncation of G has fewer than n players. It fol-
lows that L(G') results from L(G) by splitting j into j and
k.

Comment: As we shall see, the solution function proposed
in this book is invariant with respect to splitting into
identical types. Certainly, this is a desirable property.
As a tool of axiomatic characterization the requirement is
probably not a strong one. Nevertheless, it has proved to
be quite useful in the search for a reasonable equilibrium
point selection theory since it excludes many ideas which
would otherwise suggest themselves. It is perhaps sufficient
to mention just one example. On the basis of our axiomatic
characterization of risk dominance in 2x2-games the fol-
lowing generalization of the definition obtained there
suggests itself. Let $U = (U_i)_N$ and $V = (V_i)_N$ be two equi-
librium points of a game $G = (\Phi, H)$. Define

(2.57) $u_i = H_i(U) - H_i(U_iV_{-i})$

(2.58) $v_i = H_i(V) - H_i(V_iU_{-i})$

One is tempted to define risk dominance as follows:
U risk dominates V if the product $u_1 \cdot u_2 \cdot \ldots \cdot u_n$ is greater
than the product $v_1 \cdot v_2 \cdot \ldots \cdot v_n$. This definition fails to
be invariant with respect to splitting into identical types
since after a splitting of player j the factors u_j and v_j
will appear twice in the deviation loss products.

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