Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

Nr. 105

John C. Harsanyi and Reinhard Selten

A GENERAL THEORY OF EQUILIBRIUM SELECTION IN GAMES

Chapter 2
Games in Standard Form

December 1980

H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Universitätsstraße
4800 Bielefeld 1
Bundesrepublik Deutschland
Federal Republic of Germany

A GENERAL THEORY OF EQUILIBRIUM SELECTION IN GAMES

Chapter 2
Games in Standard Form

John C. Harsanyi and Reinhard Selten

Note: This chapter belongs to a revised version of the earlier manuscript "A non-cooperative solution theory with cooperative applications" by John C. Harsanyi and Reinhard Selten. The new chapter 2 does not cover the same material as the old chapter 2. The discussion of desirable properties will be the subject of a new chapter 3. Chapter 4, 5 and 6 will be revised versions of the former chapters 3, 4 and 5.

Chapter 2

Chapter 2. Games in Standard Form

Our theory will be based on a game form which is intermediate between the extensive and the normal form. We call it the standard form. The standard form shows how the strategies are made up of choices at information sets without preserving more than the dependence of payoffs on choices. There are two reasons for using the standard form instead of the normal form. First, it is important to identify certain substructures called cells which correspond to subgames in the extensive form and are invisible in the normal form. Second, we want to select a perfect equilibrium point and perfectness cannot be satisfactorily defined in the framework of the normal form.

We restrict our attention to games with perfect recall and substructures of such games. The reasons for this will be explained in section 1. In section 2 we shall introduce basic notations and definitions concerning the standard form. Section 3 will explore the special properties of standard forms derived from extensive games with perfect recall. This will lead us to a definition of perfect recall in terms of the standard form. Important properties of standard forms with perfect recall will be derived in section 4. Our solution concept is recursive in the sense that it is necessary to look at certain substructures of a game in order to solve it. A general definition of a substructure is given in section 5. Perturbed games and their substructures are of special importance for the theory. They belong to a class of substructures of the original game which are called interior substructures. Interior substructures of games with perfect recall have special properties which permit the decentralization of certain aspects of a player's strategy choice. These decentralization properties will be discussed in section 6.

Uniformly perturbed games will be introduced in section 7 and uniformly perfect equilibrium points will be defined in section 8. On the basis of this,section 9 will discuss the way in which our theory deals with the perfectness problem. The concepts of a solution function and a limit solution function will be defined. Solution functions select equilibrium points for perturbed games and limit solution functions are obtained from solution functions by letting the perturbance parameter go to zero. In this way one selects a perfect equilibrium point for the unperturbed game. Formally, our solution concept is a limit solution function. The solution function on which it is based will be defined in chapter 5. In section 9 of chapter 2 we are only concerned with the connection between solution functions and limit solution functions.

1. Reasons for the exclusion of imperfect recall

Our solution theory will be in terms of the standard form but it aims at the selection of a unique perfect equilibrium point for every extensive game with perfect recall. Perfect equilibrium points have been defined for such games only (Selten 1975).

An extensive game has perfect recall if every player at each of his information sets knows all his previous choices. We shall not give a precise definition since we want to avoid the formalism of extensive games. Instead of this we shall later give a definition of perfect recall in the framework of the standard form. A precise definition of perfect recall in extensive games can be found elsewhere (Kuhn 1953, Selten 1975).

Clearly, an absolutely rational player who is a single person should never forget what he has done before. Imperfect recall becomes important only if one wants to examine games where some players are teams.

We take the point of view that games with teams as players are misspecified models. Each team member should be modelled as a separate player whose payoff is that of the team. A team is not really different from any other group of players who happen to have identical payoffs. We think of a player as an entity with completely integrated mental processes. This means that individual rationality alone is sufficient to enforce consistency of expectations within one player. Typically, a team consists of several members whose expectations cannot be coordinated exclusively by individual rationality. Consider the game of figure 1.

	A	B	C
A	2 2	1 1	0 0
B	1 1	0 0	2 2

Figure 2.1: A 2-person team problem

In this game both players have the same payoffs. They face a typical team problem. Obviously, both (A,A) and (B,C) realize the maximal payoff 2 for both of them. Individual rationality alone does not provide a criterion how to select among these two possibilities of receiving the maximal team payoff. The team members must either communicate in order to reach a common decision or they must apply some kind of game theoretical reasoning which permits them to select one of the two pure strategy equilibrium points (A,A) and (B,C) of the game in figure 1. As we shall show in chapter 5, section our theory selects (B,C).

There are, of course, groups of persons which in a sense can be modelled as one player. If for example a group consists of one leader and several subordinates who must follow his orders, then only one person, namely the leader, is the real player. The orders to be given to his subordinates are choices he has to make.

Less trivial teams are groups of players with identical interests and special preplay communication opportunities. Here it must be admitted that preplay communication and especially differential opportunities of preplay communication pose modelling problems which are not automatically solved by our theory. We shall come back to this point at the end of the book. The mere inclusion of formal communication moves into the game model proves to be insufficient.

Our theory mainly aims at extremely non-cooperative games without any communication. Such games are more basic than those with communication. In order to solve a game with communication it is necessary to know what would happen if communication broke down. The breakdown of communication furnishes a threat point which may be of great importance. It is at least necessary to know whether a player should prefer the breakdown of communication or not.

In many cases it is not necessary to consider communication explicitely since as in the example of figure 2.1 game theoretical reasoning often is a good substitute for communication. Where communication is not really needed it is reasonable to suppose that its presence or absence does not influence the solution of the game.

2. Games in standard form

The normal form of an extensive game describes the dependence of expected payoffs on pure strategies and abstracts from everything else. A pure strategy of a player is a function which assigns to everyone of his information sets a choice at this information set. The standard form does not only look at pure strategies and payoffs but also at the structure of pure strategies. It tells us of which choices a pure strategy is made up.

The standard form is derived from the extensive form by thinking of each information set of player i as administrated by a separate agent. Therefore, the standard form does not only have a player set N but also an agent set M_i for every player $i \in N$. Every agent has a choice set which represents the choices at his information sets. A pure strategy of a player is conceptualized as a collection of choices for his agents. Otherwise the standard form is not different form the normal form.

Notations and definitions: We shall use positive integers as names for the players. The player set N will be a non-empty finite set of positive integers. In many cases N will simply be the set $\{1,\ldots,n\}$ of the first n integers but since we must look at substructurers of games which are games with fewer players it is convenient to define player sets in a more general way.

We use pairs of positive integers ij in order to identify agents. The first of both integers is the number of the player to whom the agent belongs,the second is the number of the agent. Player i's agent set M_i is a non-empty finite set of pairs of integers of the form ij. The union of all M_i with $i\in N$ is denoted by M.

Each agent ij has a non-empty finite choice set Φ_{ij} of choices φ_{ij}. A pure strategy φ_i of player i may be thought of as a collection of choices for his agents.

(2.1) $$\varphi_i = (\varphi_{ij})_{M_i}$$

The lower index M_i indicates that φ_i contains one element φ_{ij} for every $ij\in M_i$. The same system of notation will be employed at other occasions, too. Player i's pure strategy set Φ_i is the set of all these collections.

(2.2) $$\Phi_i = \underset{ij\in M_i}{\times} \Phi_{ij}$$

A pure strategy combination or shortly a pure combination is a collection of pure strategies $\varphi = (\varphi_i)_N$ containing one for each player $i\in N$. Alternatively, we may look at a pure strategy combination as a collection of choices $\varphi = (\varphi_{ij})_M$ containing one choice $\varphi_{ij}\in\Phi_{ij}$ for each agent $ij\in M$. We shall make no distinction between $(\varphi_i)_N$ and $(\varphi_{ij})_M$ if both prescribe the same choices to the agents in M. The pure strategy combination set Φ is the set of all these collections φ:

(2.3) $$\Phi = \underset{i\in N}{\times} \Phi_i = \underset{ij\in M}{\times} \Phi_{ij}$$

Payoffs are defined only for players not for agents. A payoff function H on Φ assigns a payoff vector

(2.4) $$H(\varphi) = (H_i(\varphi))_N$$

to each $\varphi \in \Phi$. As before, the lower index N indicates that $H(\varphi)$ contains one component $H_i(\varphi)$ for every $i \in N$.

We look at Φ as a structure endowed with all the information on the sets N, M_i and Φ_{ij}. We say that Φ is admissable if all these sets are finite and non-empty. We now can give a formal definition of a game in standard form.

DEF: Standard form : A game in standard form or shortly a standard form $G = (\Phi, H)$ consists of an admissable set of pure strategy combinations Φ with the structure indicated by (2.3) together with a payoff function H on Φ.

Comment: A game in standard form differs from a normal form game by the additional information on the structure of the pure strategy sets Φ_i. It also differs from the agent normal form which has been introduced for the purpose of defining perfect equilibrium points (Selten 1975). In the agent normal form each agent becomes a separate player with the payoff of the player to whom he belongs in the original game. The agent normal form keeps the information on the agents' choices but it neglects the information on the relationship between agents and players.

We feel that both players and agents must be identifiable for the purposes of our theory. One cannot define perfect equilibrium points without looking at agents (Selten 1975). For the purposes of perfectness it would be sufficient to work with the agent normal form. However, it is natural to look at players as centers of expectation formation. This will be important for the definition of risk dominance in chapter 5. We shall make use of the idea that different agents of the same player should have the same expectations on other players. Therefore, we need a game form with both agents and players.

A game in standard form combines the information of the normal form and the agent normal form. We may think of these two game forms as different aspects of the standard form.

Normal form: The normal form of a standard form $G=(\Phi,H)$ has the same structure as G except that the information on the internal structure of the pure strategy sets given by (2.2) is suppressed. Notationally, we need not make any distinction between a standard form and its normal form.

Sometimes we shall look at games in standard from where each player has only one agent. In such cases we need not distinguish between a player and his agent. The pure strategy set of a player coincides with his agent's choice set. We refer to such games as games with normal form structure.

Agent normal form: The agent normal form $G' = (\Phi,H')$ of a standard form $G = (\Phi,H)$ is a game with normal form structure whose players are the agents of G with their choice sets Φ_{ij} as pure strategy sets. The payoff $H'_{ij}(\varphi)$ of $ij \in M_i$ is defined as $H_i(\varphi)$. Sometimes it will be convenient to think of the agents as renumbered by positive integers instead of pairs of positive integers if we look at the agent normal form.

Use of the word "game": Often a game in standard form will simply be called a game where this can be done without risk of confusion. We shall mainly be concerned with such games even if extensive forms will be looked at occasionally in order to clarify conceptually important points.

Further definitions of this section will always refer to a fixed game in standard form $G = (\Phi,H)$.

Mixed strategies: A mixed strategy q_i of player i is a probability distribution over player i's set of pure strategies. $q_i(\varphi_i)$ denotes the probability assigned to φ_i. A mixed strategy q_i is called completely mixed if $q_i(\varphi_i)$ is positive for every pure strategy $\varphi_i \in \Phi_i$.

No distinction is made between a pure strategy φ_i and that mixed strategy which assigns probability 1 to φ_i and 0 to all other pure strategies. The set of all mixed strategies q_i of player i is denoted by Q_i.

A combination $q = (q_i)_N$ of mixed strategies or shortly a mixed combination contains a mixed strategy q_i for every $i \in N$.

The set of all combinations of this kind is denoted by Q. For $q = (q_i)_N$ and $\varphi = (\varphi_i)_N$ it is convenient to introduce the notation.

$$(2.5) \qquad q(\varphi) = \prod_{i \in N} q_i(\varphi_i)$$

In other words, $q(\varphi)$ is the product of all $q_i(\varphi)$ with $i \in N$. The product $q(\varphi)$ is called the realization probability of φ under q. The definition of the payoff function H is extended from Φ to Q in the usual way:

$$(2.6) \qquad H(q) = \sum_{\varphi \in \Phi} q(\varphi) H(\varphi)$$

Equations for H are to be understood as vector equations which hold for every component H_i of H.

i-incomplete combinations: It will be necessary to look at combinations of the type $q_{-i} = (q_i)_{N \setminus \{i\}}$ which contain one mixed strategy for every player with the exception of i. Such combinations are called i-incomplete. The index -i is used in order to designate i-incomplete combinations. -i may be thought of as an abbreviation of $N \setminus \{i\}$.

Φ_{-i} denotes the set of all i-incomplete combinations of pure strategies and the symbol Q_{-i} is used for the set of all i-incomplete mixed combinations. We use the notation $q_i q_{-i}$ in order to describe that $q \in Q$ which contains q_i and the components of q_{-i}. If for all players with the exception of player i the strategies in q_{-i} agree with those in q we call q_{-i} prescribed by q. Similarily, we say that each of the components of a combination q or an i-incomplete combination q_{-i} is prescribed by q or q_{-i}, respectively.

Behavior strategies: A local strategy b_{ij} of an agent ij is a probability distribution over his choice set Φ_{ij}. The probability assigned to $\varphi_{ij} \in \Phi_{ij}$ by b_{ij} is denoted by $b_{ij}(\varphi_{ij})$. No distinction is made between a choice φ_{ij} and that local strategy b_{ij} which selects φ_{ij} with probability 1. The set of all local strategies of ij is denoted by B_{ij}.

A <u>behavior strategy</u>

$$b_i = (b_{ij})_{M_i} \tag{2.7}$$

is a collection of local strategies b_{ij} containing one for each player i's agents. The set of all behavior strategies of player i is denoted by B_i. It is convenient to introduce the following notation:

$$b_i(\varphi_i) = \prod_{ij \in M_i} b_{ij}(\varphi_{ij}) \tag{2.8}$$

for $b_i = (b_{ij})_{M_i}$ and $\varphi_i = (\varphi_{ij})_{M_i}$

We call $b_i(\varphi_i)$ the <u>realization probability</u> of φ_i under b_i.

Obviously, the realization probabilities $b_i(\varphi_i)$ for all $\varphi_i \in \Phi_i$ are non-negative and sum up to one. b_i can be looked upon as a mixed strategy. If we know only the realization probabilities $b_i(\varphi_i)$ we can easily reconstruct the local strategies b_{ij}. The probability $b_i(\varphi_{ij})$ is the sum of all $b_i(\varphi_i)$ for pure strategies φ_i which contain φ_{ij}. Therefore, a behavior strategy is uniquely determined by its realization probabilities. This permits us to look at behavior strategies as special mixed strategies. We shall make no distinction between a behavior strategy b_i and that mixed strategy which assigns the realization probabilities $b_i(\varphi_i)$ to the pure strategies. This has the consequence that B_i is identified with a subset of Q_i. If a player has at least two agents then B_i is a proper subset of Q_i. Not every mixed strategy permits a representation of the form (2.8).

A <u>behavior strategy combination</u> $b = (b_i)_N$ or shortly a <u>behavioral combination</u> contains a behavior strategy b_i for each player. The set of all such combinations is denoted by B. The definition of payoffs for mixed combinations automatically applies to behavioral combinations since they are special mixed combinations. A behavioral combination can also be looked upon as a collection $b = (b_{ij})_M$ of local strategies for all agents.

Payoff equivalence: Two mixed strategies r_i and s_i of player i are called payoff equivalent if we have:

$$H(r_i\varphi_{-i}) = H(s_i\varphi_{-i}) \qquad (2.9)$$

for every $\varphi_{-i} \in \Phi_{-i}$. Equation (2.9) implies that $H(r_i q_{-i})$ and $H(s_i q_{-i})$ are equal for all $q_{-i} \in Q_{-i}$. Note that (2.9) does not only concern player i's payoff but the whole payoff vector.

Comment: Our solution theory is restricted to games with perfect recall. Such games have special properties which shall be examined in sections 3 and 4. Here we only want to explain why for some purposes one can restrict one's attention to behavior strategies. A very important special property of games with perfect recall is expressed by Kuhn's theorem: For every mixed strategy $q_i \in Q_i$ a behavior strategy b_i can be found such that q_i and b_i are payoff equivalent (Kuhn 1953, Selten 1975).

Kuhn's theorem shows that in games with perfect recall a player does not lose anything of his range of strategic opportunities if he restricts his strategy selection to the set B_i of behavior strategies.

Since sometimes we have to look at substructures of games which are games with fewer players we have to introduce the notion of a subcombination which specifies local strategies only for a subset of agents.

Subcombinations: Let $C \subseteq M$ be a non-empty subset of the set M of all agents. A collection $b_C = (b_{ij})_C$ of local strategies $b_{ij} \in B_{ij}$ containing one for each agent $ij \in C$ is called a subcombination for C. The set of all subcombinations for C is denoted by B_C.

Suppose that b_C and b_D are subcombinations for two non-intersecting subsets C and D of M. We use the notation $b_C b_D$ for that combination $b_{C\cup D}$ which contains the components of b_C and b_D. A pure subcombination φ_C is a subcombination which contains a choice for every member of C. The set of all pure subcombinations for C is denoted by Φ_C.

Subcombinations for $M\setminus\{ij\}$ are called ij-incomplete combinations. The notation b_{-ij} is used for such subcombinations. Accordingly, B_{-ij} and Φ_{-ij} are the sets of mixed and pure ij-incomplete combinations, respectively. A subcombination for $M_i\setminus\{ij\}$ is called an ij-incomplete behavior strategy. We use the notation $b_{i\setminus ij}$ for ij-incomplete behavior strategies. $B_{i\setminus ij}$ and $\Phi_{i\setminus ij}$ are the sets of mixed and pure ij-incomplete behavior strategies, respectively. For $D\subseteq C$ we say that b_D is prescribed by b_C if b_D and b_C contain the same local strategies for the agents in D.

Comment: In our theory we shall sometimes have to look at expectations which one player can form on the behavior of the other players. In a disequilibrium situation such expectations may take the form of i-incomplete mixed combinations but this is not the most general case. A player may for example expect that with probability z all other players will behave according to φ_{-i} and with probability 1-z all other players will behave according to ϕ_{-i} where φ_{-i} and ϕ_{-i} are two i-incomplete combinations prescribed by two different solution theories between whom the player cannot decide. In order to describe such expectations we need the concept of i-incomplete joint mixtures which are probability distributions over the i-incomplete pure combinations.

Joint mixtures: An i-incomplete joint mixture $q_{.i}$ is a probability distribution over Φ_{-i}. The probability assigned to an i-incomplete pure combination by $q_{.i}$ is denoted by $q_{.i}(\varphi_{-i})$. We use a dot before i as a lower index in order to distinguish i-incomplete joint mixtures from i-incomplete combinations. Wherever this can be done without risk of confusion we shall drop the adjective "i-incomplete" and simply speak of joint mixtures. The set of all i-incomplete joint mixtures is denoted by $Q_{.i}$.

Realization probabilities: Let b_C be a mixed subcombination for $C\subseteq M$ We use the following notation:

$$b_C(\varphi_C) = \prod_{ij\in C} b_{ij}(\varphi_{ij}) \tag{2.10}$$

where φ_{ij} is prescribed by φ_C. We call $b_C(\varphi_C)$ the realization probability of φ_C under b_C.

Player subcombinations: Let $D \subseteq N$ be a non-empty set of players. A player subcombination for D is a collection of mixed strategies containing one for each member of D. The set of all player subcombinations for D is denoted by Q_D. The realization probability of $\varphi_D \in \Phi_D$ under q_D is defined as follows:

$$(2.11) \qquad q_D(\varphi_D) = \prod_{i \in D} q_i(\varphi_i)$$

where φ_i is prescribed by φ_D. Obviously, an i-incomplete mixed combination q_{-i} is a special player subcombination. The realization probabilities $q_{-i}(\varphi_{-i})$ defined for q_{-i} by (2.11) form a probability distribution over Φ_{-i} or, in other words, a joint mixture. We shall make no distinction between q_{-i} and that i-incomplete joint mixture which assigns the realization probabilities under q_{-i} to the i-incomplete pure combinations. In this way, Q_{-i} is identified with a subset of $Q_{.i}$.

Realization probabilities of pure subcombinations: Let q_i be a mixed strategy of player i and let φ_C be a pure subcombination for a subset $C \subseteq M_i$ of player i's agent set. The realization probability $q_i(\varphi_C)$ of φ_C under q_i is defined as follows:

$$(2.12) \qquad q_i(\varphi_C) = \sum_{\varphi_{M_i \setminus C} \in \Phi_{M_i \setminus C}} q_i(\varphi_C \varphi_{M_i \setminus C})$$

Obviously, $q_i(\varphi_C)$ is the probability that q_i selects a pure strategy φ_i which prescribes φ_C.

Hybrid combinations: A hybrid combination $q_i q_{.i}$ for player i consists of a mixed strategy $q_i \in \Omega_i$ and a joint mixture $q_{.i} \in Q_{.i}$. The set of all hybrid combinations is the cartesian product $\Omega_i X Q_{.i}$. The definition of the payoff function H is extended to hybrid combinations in the obvious way:

$$(2.13) \qquad H(q_i q_{.i}) = \sum_{\varphi_{-i} \in \Phi_{-i}} q_{.i}(\varphi_{-i}) H(q_i \; \varphi_{-i})$$

The payoff vector described by (2.13) can be interpreted as player i's expected value of the payoff vector if he uses q_i and his expectations on the other players are described by $q_{.i}$.

We shall also look at <u>hybrid</u> <u>combinations</u> of the form $b_C\, q_{.i}$ where $C \subseteq M_i$ is a subset of agents of player i. The cartesian product $B_C \times Q_{.i}$ is the set of all such combinations.

<u>Best reply</u>: $r_i \in Q_i$ is called a <u>best</u> <u>reply</u> to $q_{.i} \in Q_{.i}$ if we have

$$(2.14) \qquad H_i(r_i q_{.i}) = \max_{q_i \in Q_i} H_i(q_i q_{.i})$$

An important fact on best replies deserves to be expressed as a lemma but it need not be proved here for it is a well-known result of game theory:

> <u>Lemma on best replies</u>: r_i is a best reply to $q_{.i}$ if and only if every $\varphi_i \in \Phi_i$ with $r_i(\varphi_i) > 0$ is a best reply to $q_{.i}$.

Since i-incomplete mixed combinations are special i-incomplete joint mixtures (2.14) also defines best replies to i-incomplete mixed combinations. We say that r_i is a <u>best</u> <u>reply</u> to $q \in Q$ if r_i is a best reply to the i-incomplete combination q_{-i} prescribed by q. A combination $r \in Q$ is called a <u>vector</u> <u>best</u> <u>reply</u> or shortly a <u>best</u> <u>reply</u> to $q \in Q$ if every r_i in r is a best reply to q.

<u>Strong best reply</u>: r_i is called a <u>strong</u> <u>best</u> <u>reply</u> to $q_{.i}$ if r_i is the only best reply to $q_{.i}$. This means that player i receives a smaller payoff against $q_{.i}$ if he uses any other strategy than r_i. In view of the lemma on best replies it is clear that a strong best reply must be a pure strategy.

<u>Local best reply</u>: A <u>local</u> <u>best</u> <u>reply</u> r_{ij} of an agent $ij \in M$ to a hybrid combination $b_{i \setminus ij} q_{.i}$ is a local strategy $r_{ij} \in B_{ij}$ with the following property:

$$(2.15) \qquad H_i(r_{ij} b_{i \setminus ij} q_{.i}) = \max_{b_{ij} \in B_{ij}} H_i(b_{ij} b_{i \setminus ij} q_{.i})$$

A statement analogous to the lemma on best replies also holds for local best replies:

> <u>Lemma on local best replies</u>: r_{ij} is a local best reply to $b_{i\setminus ij}q_{\cdot i}$ if and only if every choice φ_{ij} with $r_{ij}(\varphi_{ij}) > 0$ is a local best reply to $b_{i\setminus ij}q_{\cdot i}$.

The proof is analogous to that of the lemma on best replies and, therefore, need not be given here.

We say that r_{ij} is a <u>local best reply</u> to $b_i q_{\cdot i}$ if r_{ij} is a local best reply to $b_{i\setminus ij}q_{\cdot i}$ where $b_{i\setminus ij}$ is prescribed by b_i. A behavior strategy b_i is called a <u>local best reply of player</u> i to $q_{\cdot i}$ if every b_{ij} prescribed by b_i is a local best reply to $b_i q_{\cdot i}$. Since ij-incomplete combinations b_{-ij} can be interpreted as special hybrid mixtures, equation (2.15) also defines local best replies to ij-incomplete combinations. The combination $r \in B$ is a <u>local vector best reply</u> or shortly a local best reply to $b \in B$ if every r_{ij} in r is a local best reply to the corresponding b_{-ij} prescribed by b.

<u>Strong local best reply</u>: r_{ij} is called a <u>strong local best reply</u> to $b_{i\setminus ij}q_{\cdot i}$ if r_{ij} is the only best reply to this hybrid combination. In view of the lemma on local best replies it is clear that a strong local best reply must be a choice.

<u>Equilibrium points</u>: An <u>equilibrium point in mixed</u> strategies is a mixed combination $r \in Q$ which is a best reply to itself. Similarily, an <u>equilibrium point in behavior strategies</u> $b \in B$ and an <u>equilibrium point</u> $\varphi \in \Phi$ <u>in pure strategies</u> are defined by the property of being a best reply to itself. Obviously, an equilibrium point in pure strategies is also an equilibrium point in behavior strategies, and an equilibrium point in behavior strategies is also an equilibrium point in mixed strategies.

Nash's theorem on the existence of equilibrium points for finite games guarantees that every game in standard form has at least one equilibrium point in mixed strategies (Nash 1951). Since we shall restrict our attention to games with perfect recall where Kuhn's theorem holds we shall be able to rely on the existence of equilibrium points in behavior strategies. (See the comment on payoff equivalence above.) In the framework of our theory equilibrium points in behavior strategies will be the most important ones. Therefore, the word equilibrium point without any qualifications will always refer to an equilibrium point in behavior strategies.

Local equilibrium points: A behavior strategy combination $b \in B$ is a local equilibrium point if b is a local best reply to itself. A local equilibrium point is not necessarily an equilibrium point in behavior strategies. However, our theory will be mainly concerned with standard forms where this is the case. The games to which our solution function is applied will have the property that a local best reply of a player is always a best reply.

Strong equilibrium points: An equilibrium point r in mixed strategies is called strong if for every player i his strategy r_i in r is a strong best reply to r.

Note that this use of the term "strong equilibrium point" is different from that introduced by Aumann (Aumann). We feel that in view of the connection to strong inequalities our use of the term is a very natural one. Moreover, we do not need a name for Aumann's cooperative concept which does not appear in our strictly non-cooperative theory.

A local equilibrium point $b \in B$ is called strong if for every agent $ij \in M$ the local strategy b_{ij} prescribed by b is a strong local best reply to b.

Obviously, strong equilibrium points and strong local equilibrium points must be pure strategy combinations.

We say that an equilibrium point in mixed strategies r is strong for player i if this player's strategy r_i in r is a strong best reply to r whereas the same condition is not necessarily satisfied for the other players. Similarily, a local equilibrium point $b \in B$ is strong for agent ij if ij's local strategy b_{ij} in b is a strong local best reply to b.

3. Standard forms with perfect recall

Games in standard from which can be derived from extensive games with perfect recall have important special properties. In order to find out what these special properties are we have to investigate the distinguishing features of extensive games with perfect recall. We shall do this in a somewhat informal way for we do not want to burden the analysis with the formalism of the extensive game. Those who are familiar with the relevant definitions will have no difficulty to see that our conclusions are correct.

An extensive game has perfect recall if every player at each of his information sets knows all his previous choices. This has the consequence that the agents ij of player i and their choices φ_{ij} can be thought of as nodes of a tree whose structure is closely connected to the tree structure of the extensive game. In order to prepare the description of this "tree of player i" we first introduce a convenient way of speaking on the relevant details of the extensive form.

In the following we shall look at a fixed extensive game with perfect recall and the standard form $G = (\Phi, H)$ derived from it.

Precedence: Let ij and ik be two agents of player i. We say that ik follows ij or equivalently that ij precedes ik by φ_{ij} if at ik's information set player i knows that he has taken choice φ_{ij} at ij's information set. If in addition to this at ik's information set player i knows that φ_{ij} was the last choice he has made up to now, we say that ik immediately follows ij or equivalently that ij immediately precedes ik by φ_{ij}.

The definition of perfect recall has the consequence that ik follows ij if and only if there is at least one play which intersects first ij's information set and then ik's information set. An agent ij who is not preceded by any other agent of player i is called a first agent. If ik is not a first agent then he immediately follows a uniquely determined agent ij by a uniquely determined choice φ_{ij}. We call this agent ij the <u>immediate predecessor</u> of ik and we speak of the choice φ_{ij} as the <u>choice</u> which <u>immediately precedes</u> ik and we say that ik <u>immediately follows</u> φ_{ij}.

A choice φ_{ij} which is not immediately followed by any agent of player i is called <u>terminal</u>. If φ_{ij} is not terminal one or several agents of player i may immediately follow φ_{ij}.

<u>The tree of a player</u>: We shall now look at a fixed player $i \in N$ and we shall construct a <u>tree</u> K_i <u>of player</u> i. This tree K_i has three kinds of <u>nodes</u>: 1) the origin o of the tree; 2) the agents $ij \in M_i$; 3) the choices $\varphi_{ij} \in \Phi_{ij}$ of the agents $ij \in M_i$. The <u>edges</u> of the tree are as follows: 1) For each first agent ij there is an edge (o,ij) which conncets the origin to this first agent. 2) For each agent $ij \in M_i$ and for each of his choices φ_{ij} there is an edge (ij,φ_{ij}) which connects ij with φ_{ij}. 3) For each choice φ_{ij} and each ik such that ik immediately follows φ_{ij} the tree contains an edge (φ_{ij},ik). (For an example see figures 2.2 and 2.3).

The terminal choices are also called <u>endpoints</u> of the tree K_i. A path from the origin o to an endpoint of K_i is called a <u>quasiplay</u>.

The extensive form does not specify the information of a player after the end of the game. Suppose that he receives only as much information as is implied by those information sets which are reached by the play. With this assumption in mind a quasiplay can be interpreted as a description of player i's information on the play after the end of the game.

<u>Quasiplay combinations</u>: Let C be the set of agents who are nodes on a fixed quasiplay and let φ_C be that subcombination for C which contains those choices of these agents which are nodes on the quasiplay. We call φ_C the <u>quasiplay combination</u> of the quasiplay. The set of all subcombinations φ_C which are quasiplay combinations for some quasiplay is

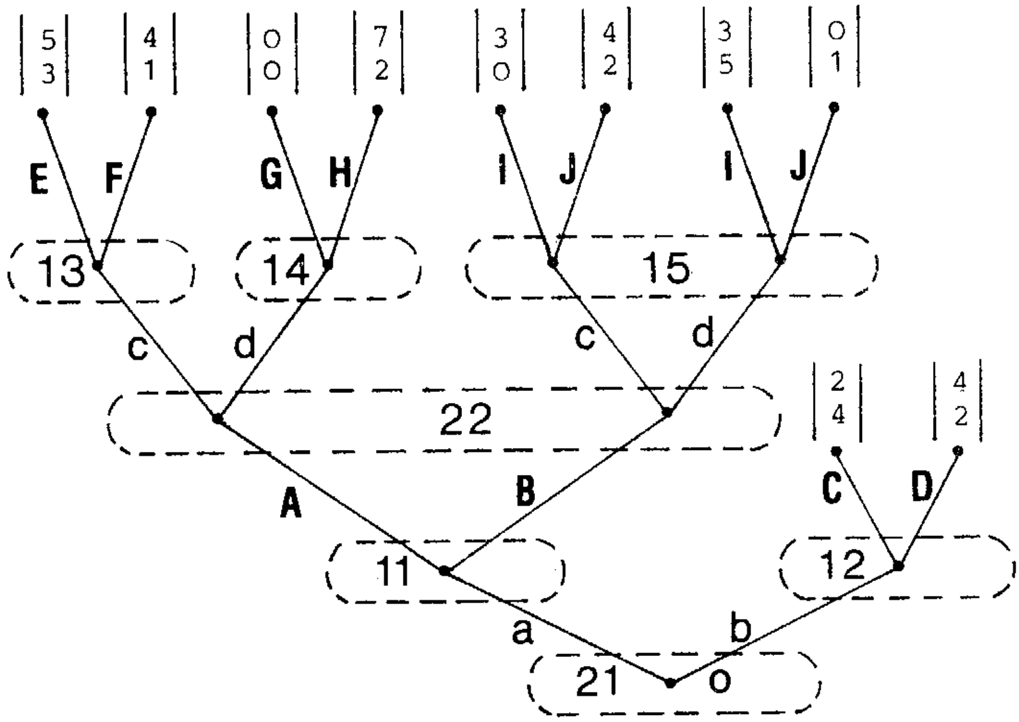


Figure 2.2: An extensive game with perfect recall. - Information sets are represented by dotted lines. Payoffs are given as column vectors with the entry for player 1 above. The letter on the left of an edge denotes the choice to which it belongs.

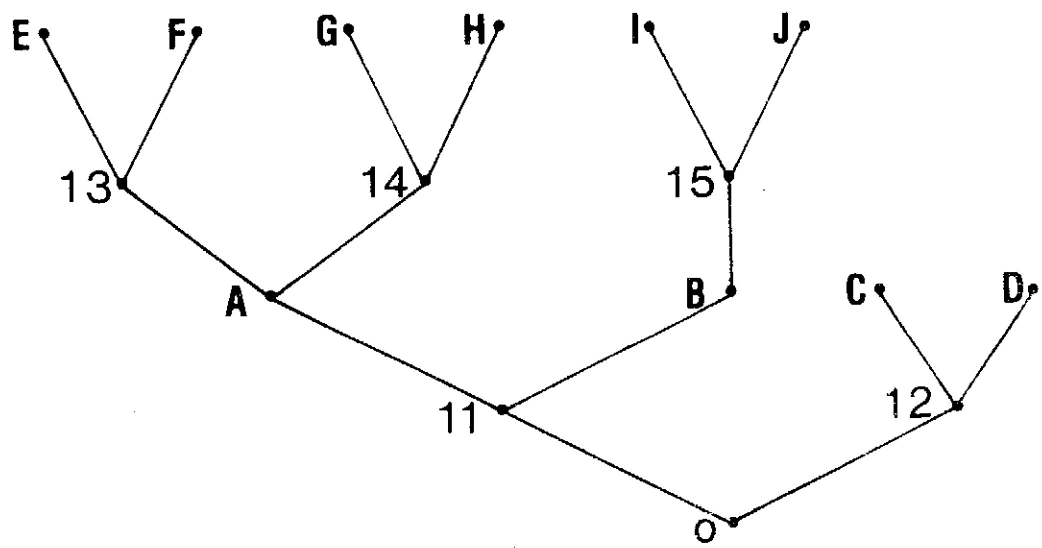


Figure 2.3: Player 1's tree K_1 for the game of figure 2.2.

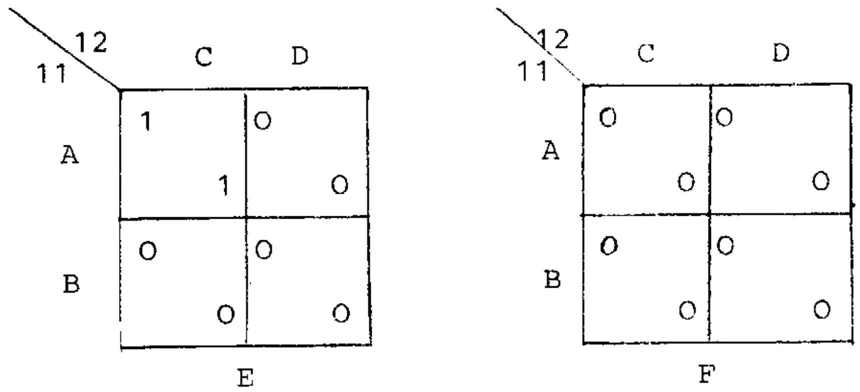
denoted by Δ_i. Obviously, there is a one-to-one correspondence between the quasiplays and the elements of Δ_i.

For every $\varphi_i \in \Phi_i$ let $\delta(\varphi_i)$ be the set of all quasiplay combinations $\varphi_C \in \Delta_i$ such that the choices in φ_C are prescribed by φ_i. Since several agents ik may immediately follow an agent ij by the same choice φ_{ij} the set $\delta(\varphi_i)$ will generally contain more than one element.

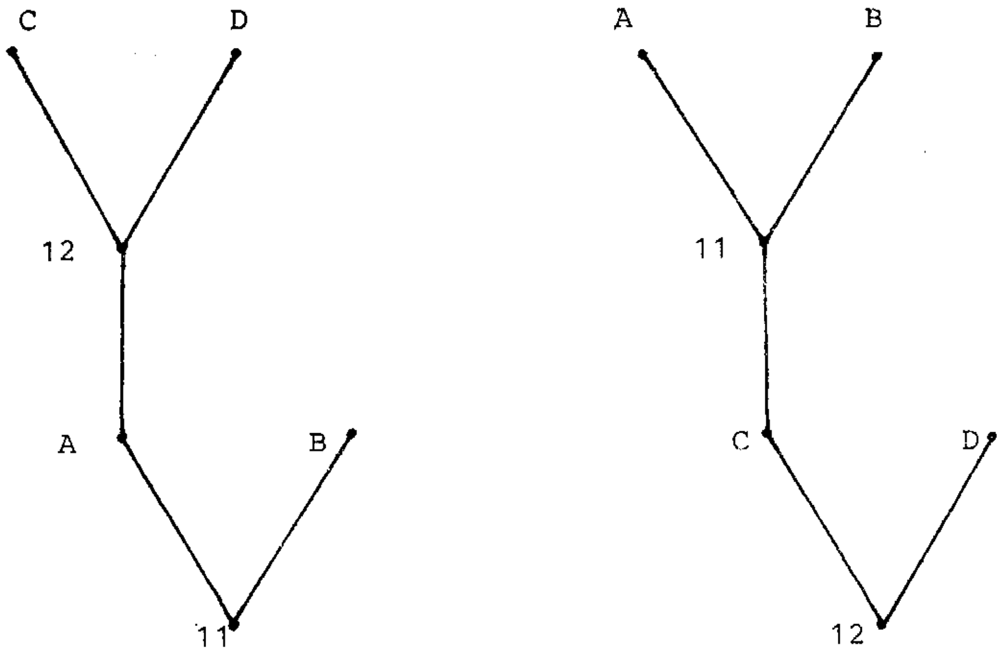
Succession probability:Let ik be an agent who immediately follows ij by the choice φ_{ij}. Let ξ_{ik} be the conditional probability that ik will be reached by a play of the game under the condition that ij has been reached and choice φ_{ij} has been taken. Obviously, this probability depends on the strategies of the other players but it does not depend on player i's strategy. In order to see this suppose that a pure combination $\varphi = \varphi_i\varphi_{-i}$ is being played. The condition that ij has been reached and that choice φ_{ij} has been taken implies that φ_i prescribes the choices on the path from o to φ_{ij}. All strategies φ_i with this property will have the same effect on the other players' information as long as no other agent of player i has been reached. We can conclude that ξ_{ik} is a function $\xi_{ik}(\varphi_{-i})$ of the strategies of the other players. We call $\xi_{ik}(\varphi_{-i})$ the succession probability of ik for φ_{-i}. A succession probability $\xi_{ij}(\varphi_{-i})$ is also defined for first agents ij as the probability that ij is reached if φ_{-i} is played. It may, of course, happen that ij cannot be reached if φ_{-i} is played. In this case, we define $\xi_{ij}(\varphi_{-i})$ as zero.

Payoff decomposition: Suppose that a pure combination $\varphi = \varphi_i\varphi_{-i}$ is being played. The set C of all agents which are reached by a play and their choices prescribed by φ_i form a quasiplay combination $\varphi_C \in \delta(\varphi_i)$. The probability that a specific $\varphi_C \in \delta(\varphi_i)$ will be generated in this way by a play if φ is played, is nothing else than the product of the succession probabilities of the agents in C. We denote this probability by $\xi(\varphi_{-i})$.

Consider the conditional expectation of the payoff vector under the condition that $\varphi_C \in \delta(\varphi_i)$ has been generated by the play in the way described above. This conditionally expected



<u>Figure 2.4</u>: Example of a standard form with perfect recall. Player 1 has two agents 11 and 12. Agent 11 selects rows, agent 12 selects columns, player 2 selects matrices. Payoffs of player 1 are shown in the upper left corner and payoffs of player 2 are shown in the lower right corner.



<u>Figure 2.5</u>: Two possibilities for the construction of player 1's tree K_1 in the example of figure 2.2.

payoff depends only on φ_{-i}. The reasons are the same as those for the independence of the succession probabilities on φ_i. The product of the conditionally expected payoff vector with $\xi(\varphi_{-i})$ is denoted by $h(\varphi_C\varphi_{-i})$. If φ_{-i} excludes the possibility that φ_C is generated the value of h is defined as the zero vector. We call $h(\varphi_C\varphi_{-i})$ the contribution of φ_C to $H(\varphi)$.It is clear that the following formula is true for every $\varphi = \varphi_i\varphi_{-i}\in\Phi$:

$$H(\ ) = \sum_{\varphi_C \in \delta(\varphi_i)} h(\varphi_C\varphi_{-i}) \tag{2.16}$$

The payoff vector function h is defined on $\Delta_i X \Phi_{-i}$. Equation (2.16) will be referred to as <u>player</u> i's <u>payoff</u> <u>decomposition</u> of $H(\varphi)$.

<u>Perfect recall in the standard form:</u> Consider a standard form $G = (\Phi, H)$ which may or may not have its origin in an extemsive form. We say that player $i\in N$ has <u>perfect</u> <u>recall</u> if it is possible to construct a tree K_i with the following properties:

(1) K_i has the following nodes: 1) The origin o; 2) The agents $ij\in M_i$; 3) the choices $\varphi_{ij}\in\Phi_{ij}$ of the agents $ij\in M_i$. The edges of K_i are as follows: 4) For every agent $ij\in M_i$ and every choice $\varphi_{ij}\in\phi_{ij}$ there is one edge (ij, φ_{ij}) which connects ij and φ_{ij}; 5) For every agent $ij\in M_{ij}$ there is either an edge (o,ij) which connects the origin o to ij or there is a choice φ_{ik} of another agent ik such that an edge (φ_{ik}, ij) connects φ_{ik} and ij.

(2) K_i permits a payoff decomposition of the form &2.16), i.e. a payoff vector function h defined on $\Delta_i X \Phi_{-i}$ can be found such that equation (2.16) holds.

In this definition it is understood that Δ_i and $\delta(\varphi_i)$ have the same relationship to K_i as in the case of the tree of a player constructed from the extensive form. The definition of "precedes", "follows", "first agent", "quasiplay", etc. are transferred to the new context in the obvious way.

A standard form $G = (\Phi, H)$ has perfect recall if in G every player $i\in N$ has perfect recall.

The tree K_i of player i need not be uniquely determined by the structure of the standard form. A very simple example is given by figure 2.4. Two ways of constructing the tree of player 1 are shown in figure 2.5. Note that even if K_i is uniquely determined the payoff decomposition (2.16) need not be uniquely determined.

4. Properties of standard forms with perfect recall

In principle it is always possible to check whether any given standard form has perfect recall. There are only finitely many possibilities to arrange player i's agents and choices in a tree K_i with the properties required by condition (1) of the definition. In order to find out whether the payoff decomposition property holds one has to look at (2.16) as a linear equation system for the vectors $h(\varphi_C\varphi_{-i})$. If a function h with (2.16) exists for a given K_i one can find it by solving the system.

Luckily, applications of our theory do not make it necessary to engage in such tedious computations. Models of substantial interest are usually given as normal forms or as extensive games with perfect recall. In such cases it is clear from the beginning that the standard forms under consideration have perfect recall.

The significance of the definition of perfect recall in the framework of the standard form lies in the fact that it enables us to develop our theory without formal reference to the extensive form. Thereby, one can avoid unnecessary complications without loss of precision.

Construction of payoff equivalent behavior strategies: In the following we shall show how in a standard form with perfect recall a behavior strategy b_i can be constructed which is payoff equivalent to a given completely mixed strategy q_i. This amounts to a proof of a version of Kuhn's theorem in the framework of the standard form. The extension of the result to mixed strategies which are not completely mixed is easy but we shall not discuss this question here, since it has no significance for our theory.

Assume that $G = (\Phi,H)$ is a standard form with perfect recall and that q_i is a completely mixed strategy for player i. We can construct a tree K_i of player i with the properties required by the definition of perfect recall. Let K_i be a fixed tree of this kind and let h be a payoff vector function which satisfies (2.16) with respect to this tree.

It is convenient to introduce the notation $<ij)$ for the set of all agents who precede ij in K_i. Let $\gamma_{<ij)}$ be that subcombination for $<ij)$ which contains the choices on the path from the origin o to ij. We call $\gamma_{<ij)}$ the <u>precombination</u> of ij. In the case of a first agent ij the set $<ij)$ is empty. In this case $\gamma_{<ij)}$ is the <u>empty subcombination</u> containing no choice at all. For reasons of formal convenience we do not want to exclude the empty precombination.

The payoff decomposition property (2.16) has the consequence that agent ij's choice does not have any influence on the payoffs, unless φ_i prescribes the choices in the precombination to the agents in $<ij)$.

For every $\varphi_{ij} \in \Phi_{ij}$ and every $ij \in M_i$ define

$$(2.17) \qquad b_{ij}(\varphi_{ij}) = \frac{q_i(\gamma_{<ij)}\varphi_{ij})}{q_i(\gamma_{<ij)})}$$

where $\gamma_{<ij)}$ is the precombination of ij and where the denominator on the right hand side is interpreted as 1 in the case of $<ij) = \emptyset$. In (2.17) we use the notation introduced by (2.10). Realization probabilities of the form $q_i(\varphi_C)$ have been defined by (2.12).

Since q_i is completely mixed the denominator in (2.17) is always positive. Moreover, it is clear that the sum of all $b_{ij}(\varphi_{ij})$ with $\varphi_{ij} \in \phi_{ij}$ is equal to 1. Consequently, a local strategy b_{ij} is defined by (2.17). Let b_i be that behavior strategy of player i whose components are defined by (2.17). We shall show that b_i is payoff equivalent to q_i. In view of (2.16) it is sufficient to prove:

$$(2.18) \qquad b_i(\gamma_D) = q_i(\gamma_D) \quad \text{for every } \gamma_D \in \Delta_i$$

Equation (2.18) is a consequence of (2.17). We obtain $b_i(\gamma_D)$ by the multiplication of all $b_{ij}(\gamma_{ij})$ along the quasiplay corresponding to γ_D. In the product of the fractions taken from the right hand side of (2.17) numerators cancel against denumerators of the next term. The product is nothing else than the numerator of the last term. This is $q_i(\gamma_D)$. We have proved the following theorem:

Theorem (Kuhn's theorem): Let G = (ϕ,H) be a standard form with perfect recall and let q_i be a completely mixed strategy of a player i in G. The the behavior strategy b_i defined by (2.17) is payoff equivalent to q_i.

Further properties of standard forms with perfect recall: Our theory makes use of two further properties of standard forms with perfect recall. An important tool to be developed will be a payoff decomposition which focuses on the effects of a specific agent's choice. One may think of this relation as a rearrangement of the payoff decompotion (2.16) whose possibility is required by the definition of perfect recall in the standard form. This "agent payoff decomposition" will involve the definition of a local payoff for each agent. The second property to be investigated is a recursive relationship between the local payoff of the agents of one player.

Agent payoff decomposition: Let G = (Φ,H) be a standard form with perfect recall; let $i \in N$ be one of the players and let ij be one of his agents. According to the defintion of perfect recall we can construct a tree K_i of player i. Let K_i be a fixed tree of this kind and let h be a payoff vector function defined on $\Delta_i \times \Phi_{-i}$ which satisfies (2.16) with respect to this tree.

It is convenient to introduce the notation [ij> for the set of agents which consists of ij and all those agents of player i who follow ij in K_i. We call [ij> agent ij's forward set. The rectangular bracket indicates that ij is included in the set [ij> . Agent ij is not included in the set <ij) of agents who precede him. A subcombination for [ij> will be called a postcombination for agent ij.

For every postcombination $\varphi_{[ij>}$ we shall define a set of quasiplay combinations $\delta(\varphi_{[ij>})$. The set $\delta(\varphi_{[ij>})$ is the set of all $\varphi_C \in \Delta_i$ such that φ_C is prescribed by $\gamma_{<ij)}\varphi_{[ij>}$. The quasiplay combinations in $\delta(\varphi_{[ij>})$ correspond to those quasiplays in K_i which go through ij and after ij through choices in $\varphi_{[ij>}$ only. Define

$$(2.19) \qquad H_{ij}(\varphi_{[ij>}\varphi_{-i}) = \sum_{\varphi_C \in \delta(\varphi_{[ij>})} h_i(\varphi_C \varphi_{-i})$$

where h_i is the i-th component of the payoff vector function h. We call H_{ij} agent ij's <u>local payoff function</u>. Note that the definition of the local payoff function is relative to a given graph K_i and a given payoff vector function h. Local payoffs are not uniquely determined by the standard form.

One may think of H_{ij} as that part of player i's payoff which is influenced by ij's choices. ij has no influence on payoffs obtained by quasiplays which do not go through ij. (2.19) is generalized to hybrid combinations of the form $b_{[ij>}q_{\cdot i}$ by the following definition:

$$(2.20) \qquad H_{ij}(b_{[ij>}q_{\cdot i}) =$$

$$= \sum_{\varphi_{[ij>} \in \Phi_{[ij>}} \sum_{\varphi_{-i} \in \Phi_{-i}} b_{[ij>}(\varphi_{[ij>})q_{\cdot i}(\varphi_{-i})H_{ij}(\varphi_{[ij>}\varphi_{-i})$$

With the help of (2.16) it can be seen that player i's payoff $H_i(b_i q_{\cdot i})$ can be split into two parts, one involving H_{ij} and the other not depending on local strategies of agents in ij's forward set; the second part is represented by a function $\underline{h}_{ij}$ defined on $B_{M_i \setminus [ij>} \underline{X} Q_{\cdot i}$:

$$(2.21) \qquad H_i(b_i q_{\cdot i}) = b_{<ij)}(\gamma_{<ij)})H_{ij}(b_{[ij>}q_{\cdot i}) + \underline{h}_{ij}(b_{M_i \setminus [ij>}q_{\cdot i})$$

where the subcombinations $b_{<ij)}$, $b_{[ij>}$ and $b_{M_i \setminus [ij>}$ are those prescribed by b_i. We refer to (2.21) as the <u>agent payoff decomposition</u>. This relationship shows that the influence of ij's choice on player i's payoff works through his local payoff H_{ij}.

Recursive local payoff relationship: We continue to work under the assumptions of the previous subsection. Let θ_{ij} be the set of all terminal choices of agent ij. Agent ij's local payoff can be split into two parts, one which is obtained by his terminal choices $\varphi_{ij}\in\theta_{ij}$ and another which is due to his non-terminal choices $\varphi_{ij}\in\Phi_{ij}\setminus\theta_{ij}$.
Define

$$(2.22)\quad h_{ij}(b_{ij}q_{\cdot i}) = \sum_{\varphi_{ij}\in\theta_{ij}} \sum_{\varphi_{-i}\in\phi_{-i}} b_{ij}(\varphi_{ij})q_{\cdot i}(\varphi_{-i})h_i(\gamma_{<ij)}\varphi_{ij}\varphi_{-i})$$

where h_i stands for the i-th component of h. Equation (2.22) describes that part of agent ij's local payoff which is due to his terminal choices. That part of his choices which is due to his non-terminal choices can be expressed with the help of the local payoffs of his immediate followers. We use the symbol $F(\varphi_{ij})$ for the set of all agents of player i who immediately follow ij by φ_{ij}. With this notation we obtain the following relationship.

$$(2.23)\quad H_{ij}(b_{[ij>}q_{\cdot i}) = h_{ij}(b_{ij}q_{\cdot i}) + \sum_{\varphi_{ij}\in\Phi_{ij}\setminus\theta_{ij}} \sum_{ik\in F(\varphi_{ij})} b_{ij}(\varphi_{ij})H_{ik}(b_{[ik>}q_{\cdot i})$$

Equation (2.23) shows that agent ij's local payoff can be expressed as a function of his own local strategy b_{ij}, the i-incomplete mexture $q_{\cdot i}$ and the local payoffs of his immediate followers. We refer to (2.23) as the recursive local payoff relationship.

5. Substructures

In our theory it will often be necessary to look at substructures of standard forms with perfect recall. The solution concept is not applied directly to standard forms with perfect recall but to perturbed games derived from them. These perturbed games can be interpreted as special substructures. Moreover, the solution concept is recursive in the sense that in many cases it is necessary to solve substructures of a perturbed game in order to find its solution. Substructures of standard forms are standard forms, too, but the property of perfect recall is not necessarily inherited from the superstructure. The substructures appearing in our theory generally do not have perfect recall. Nevertheless, they have important special properties which in a sense are even stronger than those of games with perfect recall. It is necessary to investigate these properties.

We shall give a general definition of a substructure which is sufficiently wide to cover a variety of quite different special cases which will be important for our theory. All substructures can be obtained by the application of two operations. The first one is fixing an agent at a local strategy; intuitively this has the interpretation that in the new game the agent must use this local strategy and thereby becomes a dummy who can be left out in the description of the new game. The new game does not contain this agent anymore. The second operation is narrowing the choice set of an agent; this means that in the new game his choice set consists of a finite number of his local strategies in the old game. This can be described as the restriction of his strategy choice to the convex hull of these local strategies.

In the most general case, the transition from a standard form to a substructure will involve a set C of agents who are fixed at local strategies and a set D of agents whose choice sets are narrowed down.

It will be convenient to identify local strategies in the new game with local strategies in the old game. This can be done in a natural way. However, in order to avoid confusion one must make sure that two different local strategies in the new game are also different in the old game.

Therefore, a linear independence restriction has to be imposed on a set of local strategies which is admissable as a new choice set.

The identification of local strategies of the substructure with local strategies of the original game avoids unnecessary notational complications. Moreover, in this way one achieves the useful effect that a substructure of a substructure is also a substructure of the original game.

The substructures which are important for our theory will be interior in the sense that the agents, including those who are fixed, are restricted to completely mixed local strategies of the old game. Interior substructures of standard forms with perfect recall have special features which will be called decentralization properties. The content and the significance of these decentralization properties will be discussed in section 6.

<u>Fixing agents</u>: Let $G = (\Phi,H)$ be a game in standard form and let $r_C = (r_{ij})_C$ be a subcombination for a non-empty set C of agents with $C \neq M$. Let M' be the set $M \setminus C$ and let N' be the set of all players $i \in N$ with at least one agent in $M'_i = M_i \setminus C$. For every $i \in N'$ define

$$(2.24) \qquad H'_i(\varphi_{M'}) = H_i(\varphi_M r_C)$$

Obviously, $\Phi' = \Phi_{M'}$ together with $H' = (H_i)_{M'}$ forms a game $G' = (\Phi',H')$ with the player set N' and the agent set M'. This game G' is called the <u>game which results from</u> G <u>by fixing the agents in</u> C at r_C.

<u>Narrowing choice sets</u>: Let $G = (\Phi,H)$ be a game in standard form. Let D be a non-empty subset of the agent set M. For every $ij \in D$ let R_{ij} be a finite subset of ij's local strategy set B_{ij}. Define

$$(2.25) \qquad R_D = \underset{ij \in D}{\times} R_{ij}$$

$$(2.26) \qquad \Phi' = \Phi_{M \setminus D} \times R_D$$

We construct a game $G' = (\Phi',H')$ whose payoffs are defined as follws:

(2.27) $\quad H'(\varphi') = H(\varphi')$ for every $\varphi' \in \Phi'$

We say that $G' = (\Phi',H')$ <u>results from</u> $G = (\Phi,H)$ <u>by narrowing the choice sets in</u> Φ_D to R_D.

Not every game G' which can be obtained in this way will be considered to be a substructure of G. We shall impose a restriction on the new choice sets R_{ij} which enables us to identify local strategies in G' with local strategies in G without running into the difficulty that two different local strategies in G' must be identified with the same local strategy in G. Consider a local strategy b'_{ij} of an agent $ij \in D$ in G'. Suppose that in G agent ij uses each of his local strategies $r_{ij} \in R_{ij}$ with probability $b'_{ij}(r_{ij})$. If he does this he will actually play a local strategy b_{ij} in G. Obviously, the probabilities assigned to the choices $\varphi_{ij} \in \Phi_{ij}$ by this local strategy are as follows:

(2.28)
$$b_{ij}(\varphi_{ij}) = \sum_{r_{ij} \in R_{ij}} b'_{ij}(r_{ij})\, r_{ij}(\varphi_{ij})$$

Let $|\Phi_{ij}|$ be the number of elements in Φ_{ij}. The local strategies $b_{ij} \in B_{ij}$ can be interpreted as $|\Phi_{ij}|$-dimensional vectors whose components are the probabilities $b_{ij}(\varphi_{ij})$ arranged in some fixed order. We say that R_{ij} is a <u>set of independent local strategies</u> if the vectors corresponding to the elements of R_{ij} are linearily independent. Obviously, the order in which the $b_{ij}(\varphi_{ij})$ are arranged does not matter for this definition. If R_{ij} is a set of independent local strategies then there can be at most one $b'_{ij} \in B'_{ij}$ for every $b_{ij} \in B_{ij}$ such that both local strategies are related as in (2.28). This becomes obvious if one looks at (2.28) as an equation system for the probabilities $b'_{ij}(\varphi_{ij})$.

We want to use a system of notation which enables us to denote the right hand side of (2.28) by $b'_{ij}(\varphi_{ij})$. If this can be done we need not use different symbols for a local strategy in G' and the corresponding local strategy in G. It is not quite sufficient to require that R_{ij} is a set of independent local strategies in order to achieve this goal. We must impose the following additional condition:

<u>Notational unambiguity condition</u>: If $\varphi_{ij} \in \Phi_{ij}$ belongs to R_{ij} then $r_{ij}(\varphi_{ij}) = 0$ holds for every $r_{ij} \neq \varphi_{ij}$ in R_{ij}.

It is clear that this notational unambiguity condition excludes the possibility that different probabilities are assigned to φ_{ij} by b'_{ij} and the corresponding local strategy b_{ij} in (2.28).

We say that R_{ij} is an <u>admissable new choice set</u> if R_{ij} is a set of independent local strategies and if the notational unambiguity condition is satisfied for R_{ij}. If R_{ij} is an admissable new choice set then we shall make no distinction between a local strategy b'_{ij} for G' and that local strategy b_{ij} for G which corresponds to it by (2.28).
We say that the game $G' = (\Phi',H')$ which results from $G = (\Phi,H)$ by narrowing down the choice sets in Φ_D to R_D is <u>imbedded in</u> G if all the R_{ij} in R_D are admissible new choice sets. If G' is imbedded in G then equation (2.27) can be immediately generalized to behavior strategy combinations b' for G':

$$(2.29) \qquad H'(b') = H(b')$$

The identification of behavior strategy combinations for G' with behavior strategy combinations for G is a consequence of the identification of local strategies. In view of (2.26) the payoff function H' can be described as the restriction of H to B'.

<u>Substructures of standard forms</u>: We also say that $G' = (\Phi',H')$ is <u>imbedded in</u> $G = (\Phi,H)$ if G' results from G by fixing agents at local strategies. In this case we face no problem of identification of local strategies since local strategies in G' are local strategies in G anyhow.

Consider a sequence of games $G^1,\dots,G^m$ such that for $k = 1,\dots,m-1$ the game G^{k+1} results from G^k either by fixing agents or by narrowing choice sets in such a way that G^{k+1}

is imbedded in G^k. A sequence of this kind will be called a chain of substructures from G^1 to G^m. A game $G'=(\Phi',H')$ will also be called imbedded in G if there is a chain of substructures from G to G'. In a chain of substructures $G^1,\dots,G^m$ the local strategies in G^{k+1} are local strategies in G^k. Therefore, the local strategies in G^m are local strategies in G^1. This justifies the extended use of the word "imbedded".

A game G' which is imbedded in G will also be called a substructure of G. It is clear that every substructure G' of G can actually be obtained in two steps by first fixing those agents of G which do not belong to G' and then narrowing the choice sets of those agents who have different choice sets in G'. We can think of both operations being performed simultaneously since they do not interfere with each other. In this sense we speak of the game $G' = (\Phi',H')$ which results from $G = (\Phi,H)$ by fixing the agents in G at r_C and by narrowing the choice sets in Φ_D to R_D. Of course, this manner of speaking presupposes that C and D are non-intersecting and that the other conditions are satisfied which are required for both operations separately. If all choice sets in R_D are admissible then a game G' which results from G in this way is imbedded in G. All substructures of G can be obtained as games G' of this kind. We state the result in the form of a lemma.

Lemma on substructures: If $G' = (\Phi',H')$ is a substructure of a standard form $G = (\Phi,H)$ then for some C,D,r_C and R_C such that the R_{ij} in R_C are admissible new choice sets, the game $G' = (\Phi',H')$ results from $G = (\Phi,H)$ by fixing the agents in C at r_C and by narrowing the choice sets in Φ_D to R_D.

Interior substructures: Let $G = (\Phi,H)$ be a standard form and let $G' = (\Phi',H')$ be a substructure of G. Assume that G' results from G by fixing the agents in C at r_C and by narrowing the choice sets in Φ_D to R_D. We say G' is an interior substructure of G if for every $ij\in C$ the local strategy r_{ij} in r_C is completely mixed and for every $ij\in D$ all choices r_{ij} in the new choice sets R_{ij} are completely mixed local strategies in G.

6. Decentralization properties of interior substructures of standard forms with perfect recall

Interior substructures of standard forms with perfect recall have special features for which the interpretation suggests itself that they permit a player to delegate certain aspects of his strategy choice to his agents. We refer to these special features as decentralization properties. As we have seen, games with perfect recall have an important property of this kind which is expressed by Kuhn's theorem. Randomization need not be performed centrally by the player; one can rely on behavior strategies where randomization is decentralized. It is not quite obvious that Kuhn's theorem also holds for interior substructures of standard forms with perfect recall. We have to show that this is the case.

Another important decentralization property consists in the fact that a behavior strategy of a player is a best reply to a joint mixture for the other players if and only if it is a local best reply to this joint mixture. We call this the "local best reply property". It permits the player to delegate the task of checking the best reply properties of a behavior strategy to his agents. It also has the consequence that local equilibrium points are equilibrium points. Generally, games with perfect recall do not have the local best reply property but their interior substructures do have this property.

The availability of a decentralized way of checking whether a given behavior strategy is a best reply to a joint mixture does not yet mean that a player can delegate the task of finding a best reply to his agents. Generally, an agent needs to know the local strategies of other agents of the same player in order to determine a local best reply of his own. This poses a coordination problem. As we shall see later interior substructures of standard forms with perfect recall permit a decentralized iterative procedure which achieves coordination at a best reply for the player. We call this the coordination property.

<u>Construction of payoff equivalent behavior strategies</u>: Let $G = (\Phi,H)$ be a standard form with perfect recall and let $G' = (R,H')$ be an interior substructure of G which results from G by narrowing the choice sets in Φ to R. We shall show that for every mixed strategy q_i' for G' we can find a payoff equivalent behavior strategy b_i'. Obviously, we do not have to look at a more general case since a subsequent fixing of agents at local strategies in G' will not destroy the possibility of constructing payoff equivalent behavior strategies.

Let R_i be the set of pure strategies of player i in G'. Every $r_i \in R_i$ can be interpreted as a completely mixed strategy of player i. Suppose that in G player i uses each strategy $r_i \in R_i$ with its probability $q_i'(r_i)$. If he does this he will actually play a mixed strategy where each $\varphi_i \in \Phi_i$ is used with the following probability

$$(2.30) \qquad q_i'(\varphi_i) = \sum_{r_i \in R_i} q_i'(r_i) r_i(\varphi_i)$$

There is no risk of ambiguity in this notation since the strategies φ_i do not belong to R_i. Equation (2.30) permits us to interprete q_i' as a mixed strategy for G. Consider a specific agent ij of player i and one of his choices $s_{ij} \in R_{ij}$ in G'. Let S_i^j be the set of all pure strategies $r_i \in R_i$ in G' which contain s_{ij} as agent ij's component. Let K_i be a fixed tree of player i in G and let $\gamma_{<ij)}$ be agent ij's precombination in this tree. Define

$$(2.31) \qquad b_{ij}'(s_{ij}) = \frac{\sum_{r_i \in S_i^j} q_i'(r_i) r_i(\gamma_{<ij)})}{\sum_{r_i \in R_i} q_i'(r_i) r_i(\gamma_{<ij)})}$$

for every $s_{ij} \in R_{ij}$. Equation (2.31) defines a local strategy for ij in G'. As we shall see (2.31) yields the same result as the construction (2.17) applied to q_i' as a mixed strategy for G. Obviously, the denominator of (2.31) is nothing else than the realization probability $q_i'(\gamma_{<ij)})$ of ij's precombi-

nation $\gamma_{<ij)}$. The nominator can be interpreted as the probability that first the precombination is realized and then s_{ij} is played by q_i'. The probability of choosing $\varphi_{ij} \in \Phi_{ij}$ by b_{ij}' is as follows:

$$(2.32) \quad b_{ij}'(\varphi_{ij}) = \sum_{s_{ij} \in R_{ij}} b_{ij}'(s_{ij}) s_{ij}(\varphi_{ij})$$

This shows that the constructions (2.31) and (2.17) yield the same local strategy. For every agent ij of player i let b_{ij}' be the local strategy obtained in this way. Moreover, let b_i' be the behavior strategy which contains the b_{ij}' as components. Our version of Kuhn's theorem in section 3 shows that q_i' and b_i' are payoff equivalents. We state the result as a theorem.

Theorem on substructures (Kuhn's theorem): Let $G' = (\Phi', H')$ be an interior substructure of a standard form $G = (\Phi, H)$ with perfect recall and let q_i' be a mixed strategy of a player i in G'. Then the behavior strategy b_i' defined by (2.31) is payoff equivalent to q_i'.

Local best reply property: A game $G = (\Phi, H)$ in standard form has the local best reply property if the following is true for every player $i \in N$ and for every i-incomplete joint mixture $q_{\cdot i} \in Q_{\cdot i}$ in G: If $b_i \in B_i$ is a local best reply of player i to $q_{\cdot i}$ then b_i is a best reply to $q_{\cdot i}$.

Theorem on local best replies: Let $G' = (\Phi', H')$ be an interior substructure of a standard form $G = (\Phi, H)$ with perfect recall. Then G' has the local best reply property.

Proof: As in the construction of payoff equivalent behavior strategies we can restrict our attention to the case that $G' = (\Phi', H')$ results from G by narrowing the choice sets in Φ to Φ'. Obviously, a subsequent fixing of agents cannot destroy the local best reply property.

For the purpose of this proof it is convenient to think of local strategies, behavior strategies and i-incomplete joint mixtures in G' as special objects of this kind of G. Since G' is an interior substructure every choice $\varphi_{ij} \in \Phi_{ij}$ must be taken with positive probability. Let K_i be a fixed tree of player i in G. The realization probability of the pre-combination $\gamma_{<ij)}$ will always be positive in G'. It follows by the agent payoff decomposition (2.21) that a local best reply of ij in G' must maximize ij's local payoff over the region B'_{ij} of local strategies of ij in G'.

Suppose that b'_i is a local best reply of player i in G' to $q'_{\cdot i}$. Assume that b'_i is not a best reply of player i to $q'_{\cdot i}$ in G'. Let r'_i be a best reply of player i in G' to $q'_{\cdot i}$. There must be some agents ij for whom the following inequality holds:

$$(2.33) \quad H_{ij}(r'_{[ij>}q'_{\cdot i}) \neq H_{ij}(b'_{[ij>}q'_{\cdot i})$$

Otherwise r'_i and b'_i would yield equal payoffs. This follows by the payoff decomposition (2.16) and by application of (2.19) to first agents. Among the agents for whom (2.33) holds, there must be at least one such that both local payoffs in (2.33) are equal for all agents who follow him. Let ij be an agent of this kind. Consider his local strategies b'_{ij} and r'_{ij} prescribed by b'_i and r'_i, respectively. Since for all later agents the local payoffs on both sides of (2.33) are equal it follows by the recursive local payoff relationship (2.23) that b'_{ij} is a local best reply both to $b'_i q'_{\cdot i}$ and to $r'_i q'_i$. This together with (2.33) has the consequence that r'_{ij} cannot be a local best reply to $r'_i q'_{\cdot i}$. However, a behavior strategy cannot be a best reply unless it is a local best reply. This is a contradiction to the assumption that r'_i is a best reply.

<u>Remarks</u>: Since a behavior strategy cannot be a best reply unless it is a local best reply the local best reply property has the consequence that $b'_i \in B'_i$ is a best reply to $q'_{\cdot i}$ if and only if b'_i is a local best reply to $q'_{\cdot i}$.

In order to prepare the statement of the coordination property it is necessary to introduce some additional definitions and notations which will refer to a game $G = (\Phi, P)$ in standard form.

<u>Centroid</u>: Let $\Psi_{ij} \subseteq \Phi_{ij}$ be a non-empty set of choices of agent ij. The <u>centroid</u> $c(\Psi_{ij})$ of Ψ_{ij} is the following local strategy b_{ij} of ij:

$$(2.34) \qquad b_{ij}(\varphi_{ij}) = \begin{cases} \frac{1}{|\Psi_{ij}|} & \text{for } \varphi_{ij} \in \Psi_{ij} \\ 0 & \text{for } \varphi_{ij} \notin \Psi_{ij} \end{cases}$$

where $|\Psi_{ij}|$ is the number of elements in Ψ_{ij}. The centroid of ij's choice set Φ_{ij} is denoted by c_{ij}. We call c_{ij} the <u>centroid</u> of agent ij. The centroid c_i of player i is that one of his behavior strategies which prescribes the centroid $c_{ij} = c(\Phi_{ij})$ to every agent $ij \in M_i$.

<u>Local best reply set</u>: For every hybrid combination of the form $b_{i\setminus ij}q_{\cdot i}$ we define a <u>local best reply set</u> $A_{ij}(b_{i\setminus ij}q_{\cdot i})$. The set $A_{ij}(b_{i\setminus ij}q_{\cdot i})$ is the set of all choices $\varphi_{ij} \in \Phi_{ij}$ which are local best replies to $b_{i\setminus ij}q_{\cdot i}$. In this way <u>local best reply correspondences</u> A_{ij} from $B_{i\setminus ij} \times Q_{\cdot i}$ to Φ_{ij} are defined for every $ij \in M$.

<u>Central local best reply</u>: The centroid $c(A_{ij}(b_{i\setminus ij}q_{\cdot i}))$ of the local best reply set $A_{ij}(b_{i\setminus ij}q_{\cdot i})$ is denoted by $a_{ij}(b_{i\setminus ij}q_{\cdot i})$. This local strategy of agent ij is called ij's <u>central local best reply to</u> $b_{i\setminus ij}q_{\cdot i}$.

<u>Comment on the coordination problem</u>: Suppose that $b_{i\setminus ij}q_{\cdot i}$ describes the expectations of agent ij. Then, from his point of view, only the choices in $A_{ij}(b_{i\setminus ij}q_{\cdot i})$ are reasonable ones and all of them are equally good. Under these circumstances it is very natural to assume that he will use all these choices with equal probabilities. This is the idea behind the definition of the central local best reply.

Assume that $q_{\cdot i}$ represents player i's expectations on the other players before he has decided on his own strategy. Suppose that the local best reply property holds for the game $G = (\Phi, H)$ under consideration. However, this alone does not permit the player to delegate the task of choosing a best reply to his agents. Since the local best reply set

$A_{ij}(b_{i\setminus ij}q_{.i})$ depends on $b_{i\setminus ij}$ the agents of player i have to form expectations on the other agents of the same player in order to be able to determine local best replies. Moreover, the local best reply property does not guarantee a global best reply unless all agents form correct expectations on each other.

Actually for interior substructures of standard forms with perfect recall the agents' problem of forming coordinated expectations is less severe than it might seem at first glance. The recursive local payoff relationship permits a recursive determination of local best replies for the agents of player i. One starts with the agents not followed by others, one continues with the immediately preceding ones, etc.until a local best reply for the whole player is obtained.

However, this way of coordinating the expectations of player i's agents on each other is not completely satisfactory. It is based on the tree of player i which fails to be uniquely determined. For the purpose of our theory it is necessary to select a unique best reply of player i. Moreover, it seems to be desirable to avoid explicit use of the tree of player i if one wants to obtain a theory which is as simple as possible.

Our iterative procedure arrives at a uniquely determined local best reply of player i without explicit reference to his tree. Actually, the construction is not essentially different from the procedure based on the tree which has been outlined above. This will become apparent in the proof of its effectiveness for interior substructures of standard forms with perfect recall.

The iterative procedure can be thought of as a decentralized interaction process involving the player and his agents. First the player sends to his agents a message containing an initial hybrid combination $b_i^0q_{.i}$. Then the agents determine central local best replies to $b_i^0q_{.i}$ and inform the player. He puts these central best replies together and thereby forms a new hybrid combination $b_i^1q_{.i}$.

The agents again determine central local best replies, etc. For interior substructures of standard forms with perfect recall the procedure converges after a finite number of steps. Moreover, the end result does not depend on the initial strategy b_i^o.

It is justified to speak of a decentralized procedure since the player performs a passive role as a clearing house for messages.

<u>Central local best reply of a player</u>: Let $G = (\Phi,H)$ be a game in standard form and let $q_{.i}$ an i-incomplete joint mixture for G. Moreover, let b_i^o be a behavior strategy for player i. We construct a sequence $b_i^o, b_i^1, \ldots$ of behabior strategies for player i. The local strategies b_{ij}^k prescribed by b_i^k are defined as follows:

$$(2.35) \qquad b_{ij}^k = a_{ij}(b_{i\setminus ij}^{k-1} q_{.i})$$

for every $ij \in M_i$ and for $k = 1,2, \ldots$ where $b_{i\setminus ij}^{k-1}$ is the ij-incomplete behavior strategy prescribed by b_i^{k-1}. We call the sequence $b_i^o, b_i^1, \ldots$ the <u>reply sequence for $q_{.i}$ starting from b_i^o</u>.

Suppose that all reply sequences for $q_{.i}$ regardless of the initial strategy b_i^o converge to the same behavior strategy b_i. If this is the case the common limit b_i of all reply sequences for $q_{.i}$ is called <u>player i's central local best reply to $q_{.i}$</u>. Player i's central local best reply to $q_{.i}$ is denoted by $a_i(q_{.i})$.

<u>Coordination property</u>: A game $G = (\Phi,H)$ in standard form has the <u>coordination property</u> if for every $i \in N$ and for every i-incomplete joint mixture $q_{.i} \in Q_{.i}$ all reply sequences $b_i^o, b_i^1, \ldots$ defined by (2.35) converge to the same limit b_i after a finite number of steps.

<u>Theorem on coordination</u>: Let $G' = (\Phi',H')$ be an interior substructure of a standard form $G = (\Phi,H)$ with perfect recall. Then $G' = (\Phi',H')$ has the coordination property.

Proof: As far as the assertion to be proved is concerned there is no essential difference between a substructure where the agents in C are fixed at r_C and another one where the choice sets of these agents are narrowed down to sets $\{r_{ij}\}$ containing the local strategy in r_C as the only element. Therefore, we shall assume that $G' = (\Phi',H')$ results from $G = (\Phi,H)$ by narrowing the choice sets in Φ to Φ'.

Let K_i be a fixed tree of player i in G. Let M_i^1 be the set of all agents who in K_i are not followed by other agents of player i. For $k = 1,2,...$ let M_i^{k+1} be the set of all agents of player i such that all the agents who follow an agent of this kind are in $M_i^1 \cup ... \cup M_i^k$. It is clear that there can be only finitely many non-empty sets M_i^k. The number $|M_i|$ of agents of player i is an upper bound for the number of non-empty sets M_i^k.

Since G' is an internal substructure a local best reply in G' can be described as a local strategy which maximizes local payoffs in G'. This follows by the agent payoff decomposition (2.21).

The recursive local payoff relationship (2.23) has the consequence that in G' the central local best replies of agents in M_i^1 to hybrid combinations of the form $b_i' q_{\cdot i}'$ depend only on $q_{\cdot i}'$. Similarily, the central local best replies of agents in M_i^{k+1} depend only on $q_{\cdot i}'$ and the local strategies b_{ij}' of agents ij in $M_i^1 \cup ... \cup M_i^k$. This has the consequence that in the reply sequence $b_i'^{o}, b_i'^{1}, ...$ the local strategies of agents in M_i^1 do not change after $b_i'^{1}$, and those of agents in M_i^k do not change after $b_i'^{k}$. The sequence converges after at most $|M_i|$ steps.

The final result does not depend on the initial strategy b_o'. This is clear for the agents in M_i^1 and immediately follows by induction for the agents in every one of the sets M_i^k.

Remark: The proof has shown that the number $|M_i|$ of agents of player i is an upper bound for the number of steps needed until convergence of the reply sequence is reached.

7. Uniformly perturbed games

Our solution theory is not directly applied to games in standard form. As we have explained in the introduction of this chapter we first determine solutions of uniformly perturbed games and then find the limit solution by letting the perturbance go to zero.

Uniformly perturbed games of a standard form or ε-perturbations as we shall call them by a shorter name differ from the original game only by the fact that every choice must be taken with at least probability ε. Formally, an ε-perturbation will be defined as an interior substructure whose new choice sets consist of "extreme" local strategies which select a choice with maximal admissable probability.

<u>ε-perturbations</u>: Consider a standard form $G = (\Phi,H)$ and a positive number ε which is sufficiently small in the sense that the following condition is satisfied:

(2.36) $$\varepsilon < \frac{1}{|\Phi_{ij}|} \qquad \text{for every } ij \in M_i$$

where $|\Phi_{ij}|$ is the number of choices of agent ij. For every $\varepsilon>0$ with (2.36) we shall define the ε-perturbation $G_\varepsilon = (\Phi_\varepsilon,H_\varepsilon)$ of G. This game will be an interior substructure of G obtained by narrowing the choice sets in Φ to Φ_ε. In order to define the new choice sets $\Phi_{\varepsilon ij}$ we introduce the notion of an ε-extreme local strategy.

The <u>ε-extreme</u> local strategy $\varphi_{\varepsilon ij}$ for an agent ij in G <u>corresponding</u> to his choice $\varphi_{ij} \in \Phi_{ij}$ is defined as follows:

(2.37) $$\varphi_{\varepsilon ij}(\phi_{ij}) = \begin{cases} 1 - (|\Phi_{ij}|-1)\varepsilon & \text{for } \phi_{ij} = \varphi_{ij} \\ \varepsilon & \text{for } \phi_{ij} \neq \varphi_{ij} \end{cases}$$

For every $ij \in M$, the set of all ε-extreme local strategies of ij is agent ij's new choice set $\Phi_{\varepsilon ij}$ in $G_\varepsilon = (\Phi_\varepsilon,H_\varepsilon)$.

In order to show that G_ε is a substructure of G we have to prove that the $\Phi_{\varepsilon ij}$ are admissable new choice sets. The notational unambiguity condition is trivially satisfied since no φ_{ij} belongs to $\Phi_{\varepsilon ij}$. In view of inequality (2.36) together with (2.37) it is clear that none of the ε-extreme local strategies can be obtained as a linear combination of the others. $\Phi_{\varepsilon ij}$ is a set of independent local strategies. Consequently, $\Phi_{\varepsilon ij}$ is an admissable new choice set.

Since all ε-extreme local strategies are completely mixed the ε-perturbations of G are interior substructures of G.

Our notational conventions for the standard form are also used for ε-perturbations, with the only difference that the lower index ε is added everywhere in front of other lower indices if there are any. Where several different standard forms and their ε-perturbations appear, upper indices are used in order to make the necessary distinctions.

A behavior strategy combination b_ε for G_ε is also a behavior strategy combination for G. In view of the definition of G_ε as the game which results from G by narrowing the choice sets in Φ to Φ_ε we have:

(2.38) $$H_\varepsilon(b_\varepsilon) = H(b_\varepsilon)$$

In many cases we can shorten formulas by using H instead of H_ε.

Interpretation: The interpretation of uniformly perturbed games is based on the following idea. Agent ij cannot avoid to select any of his choices by mistake. The probability of selecting any given choice φ_{ij} by mistake is ε. There will be a probability $|\Phi_{ij}|\varepsilon$ of making a mistake. The probability of making no mistake is $1-|\Phi_{ij}|\varepsilon$. Whenever a mistake is made all choices are equally likely including that one which should have been chosen intentionally. The probabilities $b_{\varepsilon ij}(\varphi_{\varepsilon ij})$ describe the agent's intentions whereas the probabilities $b_{\varepsilon ij}(\varphi_{ij})$ describe his actual behavior. We shall now look at the connection between the probabilities of corresponding choices in G and G_ε assigned by a local strategy in G_ε.

Probabilities of choices in G: Define

$$\eta_{ij} = 1-|\Phi_{ij}|\varepsilon \tag{2.39}$$

for every $ij\in M$. According to the interpretation given above this is agent ij's probability of making no mistake. The following relationship is a consequence of (2.37):

$$b_{\varepsilon ij}(\varphi_{ij}) = \varepsilon + \eta_{ij} b_{\varepsilon ij}(\varphi_{\varepsilon ij}) \tag{2.40}$$

where $\varphi_{\varepsilon ij}$ corresponds to φ_{ij}. The choice φ_{ij} is selected with probability ε by mistake and with probability $\eta_{ij} b_{\varepsilon ij}(\varphi_{\varepsilon ij})$ intentionally. η_{ij} can be interpreted as realization probability of ij's intentions.

Payoffs in terms of pure strategy combinations for G: For some purposes it will be important to express the payoffs for a combination $\psi_\varepsilon \in \Phi_\varepsilon$ in terms of payoffs for combinations $\varphi \in \Phi$. In order to obtain a simpler formula we introduce the following notation:

$$\eta_C = \prod_{ij\in C} \eta_{ij} \tag{2.41}$$

for every $C \subseteq M$ with $\eta_C = 1$ for $C \neq \emptyset$. If we look at ψ_ε as a behavior strategy combination for G we see that η_C can be interpreted as the probability that the agents in C make no mistake. The payoff vector for φ_ε is easily obtained if, in addition to this, one takes into account that $\varepsilon^{|M|-|C|}$ is the probability for the agents in $M\setminus C$ of jointly selecting a specific φ_{-C} by mistake:

$$H(\psi_\varepsilon) = \sum_{C \subseteq M} \sum_{\varphi_{-C}\in\Phi_{-C}} \varepsilon^{|M|-|C|} \eta_C H(\psi_C \varphi_{-C}) \tag{2.42}$$

where ψ_C corresponds to the subcombination $\psi_{\varepsilon C}$ for C prescribed by ψ_ε.

Absence of perfect recall: Consider a standard form $G = (\Phi,H)$ with perfect recall and an ε-perturbation $G_\varepsilon = (\Phi_\varepsilon,H_\varepsilon)$ of G. It is interesting to point out that generally G_ε does not have perfect recall. The reasons are as follows.

Let $\psi_\varepsilon \in \Phi_\varepsilon$ and $\psi \in \Phi$ be pure strategy combinations in G_ε and G which correspond to each other and let ψ_i be player i's strategy in ψ. Let K_i be a tree of player i in G. One might think that with this tree the defintion of perfect recall is also satisfied for G_ε. However, with the help of the payoff decomposition (2.16) and the relationship (2.42) it can be seen immediately that this is not the case. (2.42) shows that $H(\psi_\varepsilon)$ does not only depend on the quasiplay combinations in $\delta(\psi_i)$ but on all quasiplay combinations in Δ_i and, therefore, on all choices prescribed by ψ_i.

Generally, it will also not be possible to find another tree, say $K_{\varepsilon i}$, such that the requirements of the definition of perfect recall are satisfied in G_ε. With the exceptions of special cases $H(\psi_\varepsilon)$ will depend on all choices ψ_{ij} in ψ_i. Therefore, all these choices would have to appear in each quasiplay combination in $\delta(\psi_{\varepsilon i})$. This is impossible unless the tree $K_{\varepsilon i}$ has only one quasiplay which would mean that $H(\psi_\varepsilon)$ does not depend on $\psi_{\varepsilon i}$.

The absence of perfect recall in ε-perturbations is in complete agreement with the interpretation of perfect recall in the extensive form. Let ij be an agent who is preceded by an agent ik. In the ε-pertubation agent ij does not know which of his choices $\varphi_{\varepsilon ik}$ agent ik has selected. Agent ij knows φ_{ik} but not $\varphi_{\varepsilon ik}$ or, in other words, he does not know whether φ_{ik} has been selected intentionally or by mistake.

8. Uniform Perfectness

Perfect equilibrium points can be losely described as equilibrium points which can be approximated with any degree of precision by equilibrium points of perturbed games. In our theory we are interested only in uniformly perfect equilibrium points which can be approximated by equilibrium points of ε-perturbations of the game under consideration. We shall not give a formal definition of perfectness in general but only of uniform perfectness.

Limit equilibrium points: Let $G = (\Phi,H)$ be a standard form with perfect recall. Consider a monotonically decreasing sequence $\varepsilon_1, \varepsilon_2, \ldots$ of positive numbers converging to zero where ε_i is sufficiently small to satisfy condition (2.36) which has been imposed on perturbance parameters in the previous section. Let G^i be the ε_i-perturbation of G. A sequence $G^1, G^2, \ldots$ of ε-perturbations of G, which arises from a sequence $\varepsilon_1, \varepsilon_2, \ldots$ of this kind, is called a test sequence for G.

A behavior strategy combination r for G is called a limit equilibrium point of the test sequence $G^1, G^2, \ldots$ if for $k = 1,2, \ldots$ an equilibrium point r^k in behavior strategies of G^k can be found such that for $k \to \infty$ the sequence of the r^k converges to r. In this definition the r^k are interpreted as behavior strategies for G. Convergence is to be understood in this way. A behavior strategy combination r for G is called a limit equilibrium point of G if it is a limit equilibrium point of at least one test sequence for G.

The fact that a limit equilibrium point of G is an equilibrium point of G needs to be pointed out formally since it is not an immediate consequence of the definition. The proof will be omitted here since essentially the same result has been obtained elsewhere (Selten 1975, lemma 3). The argument used there can be easily transferred to the framework of the standard form.

Lemma on limit equilibrium points: A limit equilibrium point of a standard form $G = (\Phi,H)$ with perfect recall is an equilibrium point of G.

Uniformly perfect equilibrium points: A behavior strategy combination r for a standard form $G = (\Phi,H)$ with perfect recall is a uniformly perfect equilibrium point of G if it is a limit equilibrium point of G.

A theorem on the existence of uniformly perfect equilibrium points will be stated without proof since the result can be obtained in essentially the same way as a similar result which has been proved elsewhere (Selten 1975, theorem 5). In addition to the argument given there one has to make use of the fact that Kuhn's theorem holds for interior substructures of games with perfect recall and, therefore, expecially for ε-perturbations of games with perfect recall. This implies the existence of equilibrium points in behavior strategies for ε-perturbations of games with perfect recall.

Theorem on uniformly perfect equilibrium points: Let $G=(\Phi,H)$ be a standard form with perfect recall. Then G has at least one uniformly perfect equilibrium point.

9. Solution functions and limit solution functions

The equilibrium selection theory proposed in this book specifies a solution function which selects a unique equilibrium point for every interior substructure of a standard form with perfect recall. In particular a solution is determined for every ε-perturbation of a standard form with perfect recall. A limit solution for the unperturbed game is obtained by letting the perturbance parameter go to zero. The limit solution is a uniformly perfect equilibrium point. This is the way in which our theory deals with the perfectness problem.

Solution function: A solution function L for a class $\mathcal{G}$ of games in standard form is a function which assigns an equilibrium point $r = L(G)$ of G to every standard form $G \in \mathcal{G}$.

The class of all standard forms with perfect recall is denoted by $\mathcal{R}$. We call $\mathcal{R}$ the perfect recall class. If $\mathcal{G}$ is a class of games then $\mathcal{J}(\mathcal{G})$ denotes the class of all interior substructures of games in $\mathcal{G}$. We call $\mathcal{J}(\mathcal{G})$ the interior substructure class of $\mathcal{G}$. The solution concept in this book is based on a solution function for the interior substructure class $\mathcal{J}(\mathcal{R})$ of the perfect recall class $\mathcal{R}$.

For the purpose of studying desirable properties of solution functions it will often be useful to look at solution functions defined on more limited classes of games.

Limit solution functions: Let $\mathcal{G}$ be a class of games with perfect recall or, in other words, a subclass of $\mathcal{R}$. Let L be a solution function for the interior substructure class

$\mathfrak{J}(\mathcal{G})$ of $\mathcal{G}$. Let G be a game in $\mathcal{G}$. For the case that the limit exists define

$$(2.43) \quad \underset{\rightarrow}{L}(G) = \lim_{\varepsilon \to 0} L(G_\varepsilon)$$

where G_ε is the ε-perturbation of G. Obviously, $\mathfrak{J}(\mathcal{G})$ contains all ε-perturbations G_ε of G. We call $\underset{\rightarrow}{L}(G)$ the <u>limit solution of G with respect to L</u>. Suppose that the limit solution $\underset{\rightarrow}{L}$ (G) exists for all games $G \in \mathcal{G}$. Then a solution function $\underset{\rightarrow}{L}$ on $\mathcal{G}$ is defined by (2.43). We call this solution function $\underset{\rightarrow}{L}$ the <u>limit solution function of</u> L.

<u>The existence problem</u>: The answer to the question whether a limit solution function $\underset{\rightarrow}{L}$ for given solution function L on $\mathfrak{J}(\mathcal{G})$ exists depends on the mathematical structure of L. In spite of the fact that we deal with finite games only the mathematical structure of the solution function for $\mathfrak{J}(\mathfrak{R})$ specified by the equilibrium selection theory proposed in this book is not a very easy one. We shall come back to the existence problem for the limit solution function after we shall have given the full definition of the solution function in chapter 5. We shall not attempt to give a detailed and mathematically strict existence proof which is bound to be lengthy and very technical. Instead of this, we shall indicate strong reasons for the existence of the limit solution which seem to be capable of being worked out in detail.

<u>Uniform perfectness of the limit solution</u>: Suppose that $\mathcal{G}$ is a class of standard forms with perfect recall and that L is a solution function for $\mathfrak{J}(\mathcal{G})$. If the limit solution $\underset{\rightarrow}{L}(G)$ of a game $G \in \mathcal{G}$ with respect to L exists then $\underset{\rightarrow}{L}(G)$ is a uniformly perfect equilibrium point of G. This is an immediate consequence of (2.41) and the definition of uniform perfectness.

" WIRTSCHAFTSTHEORETISCHE ENTSCHEIDUNGSFORSCHUNG"

A series of books published by the Institute of Mathematical Economics, University of Bielefeld.

Wolfgang Rohde
Ein spieltheoretisches Modell eines Terminmarktes (A Game Theoretical Model of a Futures Market)
The model takes the form of a multistage game with imperfect information and strategic price formation by a specialist. The analysis throws light on theoretically difficult empirical phenomena.

Vol. 1 176 pages price: DM 24,80

Klaus Binder
Oligopolistische Preisbildung und Markteintritte (Oligopolistic Pricing and Market Entry)
The book investigates special subgame perfect equilibrium points of a three-stage game model of oligopoly with decisions on entry, on expenditures for market potential and on prices.

Vol. 2 132 pages price: DM 22,80

Karin Wagner
Ein Modell der Preisbildung in der Zementindustrie (A Model of Pricing in the Cement Industry)
A location theory model is applied in order to explain observed prices and quantities in the cement industry of the Federal Republic of Germany.

Vol. 3 170 pages price: DM 24,80

Rolf Stoecker
Experimentelle Untersuchung des Entscheidungsverhaltens im Bertrand-Oligopol (Experimental Investigation of Decision-Behavior in Bertrand-Oligopoly Games)
The book contains laboratory experiments on repeated supergames with two, three and five bargainers. Special emphasis is put on the end-effect behavior of experimental subjects and the influence of altruism on cooperation.

Vol. 4 197 pages price: DM 28,80

Angela Klopstech
Eingeschränkt rationale Marktprozesse (Market processes with Bounded Rationality)
The book investigates two stochastic market models with bounded rationality, one model describes an evolutionary competitive market and the other an adaptive oligopoly market with Markovian interaction.

Vol. 5 104 pages price: DM 29,80

Orders should be sent to:

Pfeffersche Buchhandlung, Alter Markt 7, 4800 Bielefeld 1, West Germany.