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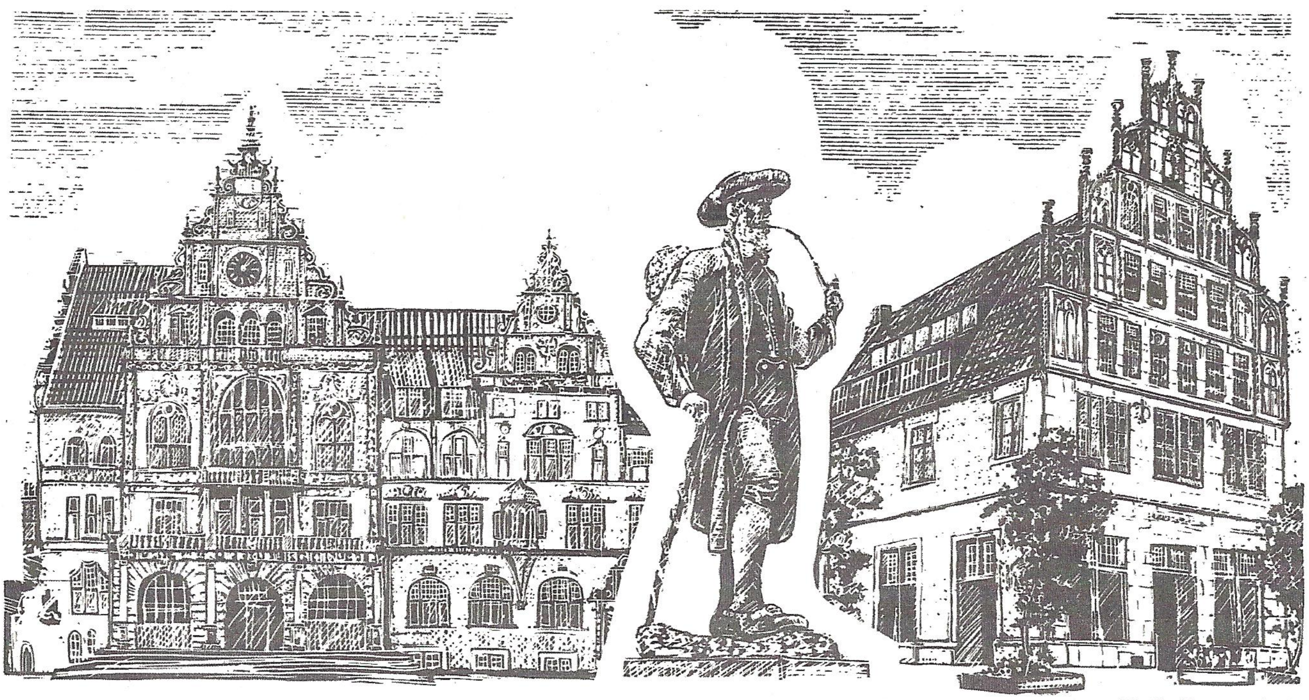
A GENERAL THEORY OF EQUILIBRIUM SELECTION IN GAMES

Chapter 3
Consequence of Desirable Properties

March 1982



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<u>Note</u>: This chapter belongs to a revised version of the earlier manuscript "A non-cooperative solution theory with cooperative applications" by John C. Harsanyi and Reinhard Selten. The new chapter 3 corresponds to the old chapter 2 (IMW Working Paper No. 92, March 1980).

Chapter 3

Chapter 3. Consequences of Desirable Properties

The nature of the problem of equilibrium point selection in non-cooperative games does not seem to permit a satisfactory solution concept which can be characterized by a set of simple axioms. Nevertheless, it is useful to look at desirable properties which one might want to require and to explore their consequences.

Even if full scale axiomazation cannot be achieved, important conclusions can be drawn from axiomatic considerations of limited scope. The simplest class of games where the equilibrium point selection problem occurs is that of all 2x2-games with two strong pure strategy equilibrium points. A central notion of our theory, namely that of risk dominance can be fully axiomatized for this admittedly very restricted class of games.

It is also important to see that certain properties which may seem to be desirable at first glance cannot be achieved. As we shall see, it is impossible to define a continuous solution function.

A reasonable solution concept should neither be influenced by positive linear payoff transformations nor by renamings of players, agents and choices. The notion of an isomorphism combines both kinds of operations. We look at invariance with respect to isomorphisms as an indispensable requirement.

One might wish to require that an increase of payoffs at a strong equilibrium point always enhances its chance to become the solution. However, examples of games with more than two players show that this kind of payoff monotonicity is not very convincing as a general requirement.

Structural features like subgames of extensive games cannot be neglected by a reasonable solution concept. In order to transfer this idea to the framework of the standard form we shall introduce special substructures called cells. This gives rise to powerful requirements called cell consistency and truncation consistency which reduce the task of finding a solution for general games to the simpler one of finding a solution for games without cells.

An impossibility result to be derived in this chapter concerns a way of subdividing one information set into two which we call "sequential agent splitting". An agent who has to choose between three choices α,β,γ is subdivided into two agents, one who first chooses between "α or β" and γ and another, who then, if necessary, decides between α and β. Unfortunately, it is not possible to require that this kind of agent splitting should not essentially change the limit solution of the game without violating other axioms like cell concistency and truncation consistency which we judge to be intuitively more compelling.

Further desirable requirements concern the elimination of superfluous strategic possibilities. An agent may have two choices α and β such that β is a local best reply wherever α has this property but not vice versa. In this case, α is called inferior to β. One might want to require that the removal of an inferior choice does not change the solution of the game. Unfortunately, we have to be satisfied with a much weaker partial invariance property with respect to inferior choices.

A similar requirement concerns classes of choices which are distinguished only by name. Such duplicate classes should be replacable by their centroids. An anologous requirement is considered for classes of semiduplicates which are indistinguishable in a weaker sense. Here, too, we have to be satisfied with partial invariance properties. It may matter in which order various superfluous strategic possibilities are eliminated.

Standard forms without cells, inferior choices, duplicate classes and semiduplicate classes are called irreducible. The three partial invariance properties mentioned above together with cell consistency, truncation consistency and invariance with respect to isomorphisms uniquely determine the extension of a solution function for irreducible games to general games. If these six requirements are satisfied the task of finding the solution of a general game can be transformed to the task of solving certain irreducible games. This can be done by a procedure of decomposition and reduction described by the flow chart of figure 3.29.

It will often be convenient to look at examples of games with normal form structure. Many important phenomena arise already there and can be more easily discussed in the simpler framework of such games where we need not distinguish between a player and his only agent. For 2-person games of this kind we shall employ the conventional bimatrix representation.

1. Continuity

Consider the class of all 2x1-games shown in figure 3.1. For $t \neq 0$ the game has only one equilibrium point, namely A for $t > 0$ and B for $t < 0$.

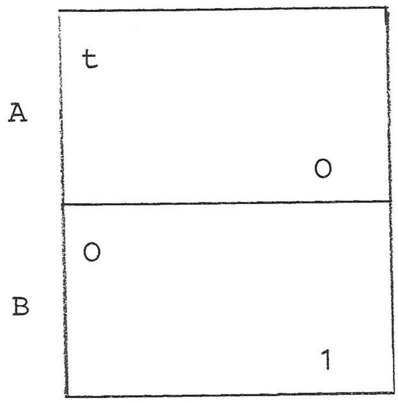


Figure 3.1: A class of 2x1-games. Player 1's payoff is given above and player 2's payoff is shown below.

For $t = 0$ every mixed strategy of player 1 is an equilibrium strategy. Clearly, no solution concept can assign a unique equilibrium point to every game in the class in a continuous way. Not only player 1's strategy but also player 2's payoff must behave discontinuously as a function of t at $t = 0$.

If a payoff parameter is varied continuously, some equilibrium points may suddenly disappear and others which have not been there before may suddenly appear. In order to show how this problem may arise in a less trivial way we add a further example. Consider the class of games given by figure 3.2. Here for $t < -1$ the strategy combination Aa is the only equilibrium point of game. For $-1 \leq t \leq +1$ both Aa and Bb are equilibrium points. Moreover, for $-1 \leq t \leq +1$ the game has a third equilibrium point in mixed strategies where player 1 uses A with probability $2/(3-t)$ and player 2 uses a with probability $(1+t)/(3+t)$. For $-1 < t < +1$ the game has no further equilibrium point. For $t = -1$ and $t = +1$

	a	b
A	2 ; 1-t	0 ; 0
B	0 ; 0	1+t ; 2

Figure 3.2: A class of 2x2-games.

there are infinitely many equilibrium points, but this does not matter as far as our argument is concerned. Any function which assigns a unique equilibrium point to every game in the class must behave discontinuously with respect to t at some point in the interval $-1 \leq t \leq +1$.

It is now clear that a certain amount of discontinuity cannot be avoided in a theory of equilibrium point selection. Continuity considerations seem to be of little relevance for the problem.

2. Positive linear payoff transformations

The payoffs of the players are von-Neumann-Morgenstern utilities. Interpersonal comparisons may be possible but they should not be considered as relevant for a non-cooperative solution theory where each player is assumed to be motivated by his own payoff exclusively.

Interpersonal utility comparisons are important for ethical theory but they have no room in a solution concept which is exclusively based on individualistic rationality assumption.

Since von Neumann-Morgenstern utilities are determined only up to positive linear transformations and since interpersonal comparisons are considered irrelevant, a game remains essentially unchanged if each player's payoff is subjected to a different positive linear transformation. This leads to the following definition of equivalence between games.

Equivalence: Two games in standard form $G = (\Phi,H)$ and $G'=(\Phi,H')$ with the same set Φ of pure strategy combinations are equivalent if constants $\alpha_i > 0$ and β_i can be found for every i in the player set N, such that

$$H'_i(\varphi) = \alpha_i H_i(\varphi) + \beta_i \tag{3.1}$$

holds for every $\varphi \in \Phi$ and every $i \in N$.

Invariance with respect to positive linear payoff transformations: A solution function L for a class $\mathcal{G}$ of games in standard form is called invariant with respect to positive linear payoff transformations, if for two equivalent games G and G' in $\mathcal{G}$ we always have $L(G) = L(G')$.

Invariance with respect to positive linear payoff transformations is a very important requirement. It is more than a desirable property. In our judgement it is indispensable.

3. Symmetry

A rational theory of equilibrium selection must determine a solution which is independent of strategically irrelevant features of the game. Names and numbers used to distinguish players, agents and choices should not matter. Games which do not differ in other ways must be considered as isomorphic and should not be treated differently.

Invariance with respect to renaming of players, agents and choices may be looked upon as a symmetry property since its most important implication can be seen in the fact that the solution must reflect the symmetries of the game.

Renamings: A renaming of players, agents and choices in a standard form $G = (\Phi,H)$ may be thought of as a system of mappings which relates G to another game $G' = (\Phi',H')$. The old names of players, agents and choices in G are replaced by new names in G'. We shall use the notation indicated in figure 3.3.

	old name	new name
player	i	$\sigma(i)$
agent	ij	$\sigma(i)\rho_i(j)$
choice	φ_{ij}	$f_{ij}(\varphi_{ij})$

Figure 3.3: The system of notation used for renaming.

Three kinds of mappings are involved: a mapping σ from the player set N of G onto the player set N' of G', for each player i a mapping ρ_i from his agent set M_i onto $\sigma(i)$'s agent set $M'_{\sigma(i)}$ and finally for every agent ij a mapping f_{ij} which maps his choice set Φ_{ij} onto agent $\sigma(i)\rho_i(j)$'s choice set in G'. All these mappings are one-to-one.

Actually, it is sufficient to describe the system $f = (f_{ij})_M$

of mappings from choice sets onto choice sets in order to specify a renaming. If one knows which choice in G is mapped on which choice in G' one also knows which player is mapped on which player and which agent is mapped on which agent. Therefore, it is natural to think of the system f as endowed with all the information on the mappings σ and ρ_i for $i \in N$. These auxiliary mappings need not be mentioned explicitely if we describe how G' results from G by a renaming.

We may look at f as a mapping from Φ to Φ'. This suggests the notation $f(\varphi)$ for that combination $\varphi' \in \Phi'$ whose components are related to those of φ as follows:

$$\varphi'_{kl} = f_{ij}(\varphi_{ij}) \qquad (3.2)$$
$$\text{with } k = \sigma(i) \text{ and } l = \rho_i(j)$$

for every $ij \in M$.

It is convenient to adopt a notion of isomorphism which permits us to say that equivalent games are isomorphic. Therefore, our definition of an isomorphism will involve a combination of a renaming with a system of positive linear payoff transformations.

<u>Isomorphism</u>: An <u>isomorphism</u> from $G = (\Phi,H)$ to $G' = (\Phi',H')$ is a system $f = f(\varphi_{ij})_M$ of one-to-one mappings f_{ij} of ij's choice set M_{ij} in G onto $\sigma(i)\rho_i(j)$'s choice set $M'_{\sigma(i)\varphi_i(j)}$ in G' such that the following conditions are satisfied:

(i) The mapping σ is a one-to-one mapping of the player set N of G onto the player set N' of G'.

(ii) For every $i \in N$ the mapping ρ_i is a one-to-one mapping from player i's agent set M_i in G onto player $\sigma(i)$'s agent set $M'_{\sigma(i)}$ in G'.

(iii) The payoff functions H and H' are related as follows:

(3.3) $$H'_{\sigma(i)}(f(\varphi)) = \alpha_i H_i(\varphi) + \beta_i$$
for every $i \in N$ and every $\varphi \in \Phi$ with constants $\alpha_i > 0$ and β_i.

An isomorphism is called a <u>renaming</u> if in (iii) we have $\alpha_i = 1$ and $\beta_i = 0$ for every $i \in N$.

Two games G and G' are called <u>isomorphic</u> if at least one isomorphism from G to G' exists.

<u>Simplifications for games with normal form structure</u>: In a game $G = (\Phi,H)$ with normal from structure where each player i has just one agent, it is convenient not to distinguish between a player and his only agent. For such games an <u>isomorphism</u> from $G = (\Phi,H)$ to $G' = (\Phi',H')$ can be described as a system of mappings $f = (f_i)_N$ where f_i maps i's pure strategy set Φ_i in G onto $\sigma(i)$'s pure strategy set $\Phi_{\sigma(i)}$ in G'. The notation $\varphi' = f(\varphi)$ is used in the sense that the components of φ' and φ are connected by $\varphi'_{\sigma(i)} = f_i(\varphi_i)$ for every $i \in N$. Of course, (i) and (iii) must hold as in the more general case.

<u>Extension of the mapping f</u>: Consider an isomorphism $f = (f_{ij})_M$ from a standard form $G = (\Phi,H)$ to a standard form $G' = (\Phi',H')$. Let b_{ij} be a local strategy of agent ij

in G. We write

(3.4) $$b'_{km} = f_{ij}(b_{ij})$$

if we have

(3.5) $$b'_{km}(f_{ij}(\varphi_{ij})) = b_{ij}(\varphi_{ij})$$
with $k = \sigma(i)$ and $m = \rho_i(j)$
for every $\varphi_{ij} \in \Phi_{ij}$.

In this way f_{ij} is extended from Φ_{ij} to B_{ij}. We write $b'=f(b)$ if the local strategies in the behavior strategy combination b' are related to those of b as in (3.5). Obviously, (3.3) and (3.5) imply

(3.6) $$H_{\sigma(i)}(f(b)) = \alpha_i H_i(b) + \beta_i$$
for every $b \in B$ and every $i \in N$.

It is clear that an isomorphism f looked upon as a mapping defined on B preserves best reply relationships and carries equilibrium points into equilibrium points.

Invariance with respect to isomorphisms: A solution function L for a class of standard form games $\mathcal{G}$ is invariant with respect to isomorphisms if for every isomorphism f from a game $G \in \mathcal{G}$ to a game $G' \in \mathcal{G}$ (which may or may not be different from G) we have

(3.7) $$L(G') = L(G)$$

Interpretation: Equation (3.7) is the formal expression of what is meant by saying that isomorphic games should not be treated differently. Invariance with respect to isomorphisms includes invariance with respect to positive linear utility transformations to which it adds an invariance with respect to renaming. A formal description of this latter invariance need not be given here. In our judgement invariance with respect to isomorphisms is an indispensable requirement for any rational theory of equilibrium point selection which is based on strategic considerations exclusively.

With the help of the notion of an isomorphism we can give a precise meaning to the idea that the solution should correctly reflect the symmetries of a game.

Symmetries: A symmetry of a game $G = (\Phi,H)$ is an isomorphism from G to itself.

Symmetry invariant equilibrium points: An equilibrium point r of $G=(\Phi,H)$ is called symmetry invariant if for every symmetry f of G we have $r = (f(r)$.

Theorem on symmetry invariance: Let $G=(\Phi,H)$ be an interior substructure of a game in standard form with perfect recall. Then G has a symmetry invariant equilibrium point in behavior strategies.

Proof: Nash has shown that every finite game in normal form has a symmetry invariant equilibrium point [Nash 1951]. In view of this result we can conclude that G has a local equilibrium point in behavior strategies. The local best reply property of interior substructures of standard forms with perfect recall (see chapter 2, section 6) has the consequence that a local equilibrium point of G is an equilibrium point.

A solution function L which is invariant with respect to isomorphisms must assign a symmetry invariant equilibrium point to every game in the class where it is defined.

An example of a game with a symmetry is given in figure 3.4. The game is a 2-person game with normal form structure. It has three equilibrium points, two in pure strategies, namely Aa and Bb and a mixed one $r=(r_1,r_2)$ with $r_1(A)=2/3$ and $r_2(a)=1/3$. The symmetry f carries Aa to Bb and vice versa. The mixed equilibriu point r is the only one which is symmetry invariant. Any solutio function L which is invariant with respect to isomorphism cannot assign anything else but L(G) = r to this game.

	a	b
A	2, 2	0, 0
B	0, 0	1, 4

→ renaming of players and strategies

$\sigma(1) = 2 \qquad \sigma(2) = 1$

$f_1(A) = b \qquad f_1(B) = a$

$f_2(a) = B \qquad f_2(b) = A$

	a	b
A	4, 1	1, 0
B	0, 0	2, 2

→ linear transformations of payoffs
player 1's payoff is divided by 2 and
player 2's payoff is multiplied by 2

	a	b
A	2, 2	0, 0
B	0, 0	1, 4

Figure 3.4: An example of a symmetry.

The payoff vector of r is $H(r) = (\frac{2}{3}, 1\frac{1}{3})$. Note that both players receive more at each of both pure strategy equilibrium points. Nevertheless, invariance with respect to isomorphism forces us to adopt r as the solution.

4. Best reply structure

In the last section we have argued that invariance with respect to positive linear payoff transformations has to be supplemented by invariance with respect to renamings of players, agents, and strategies. In this way, we obtained the stronger notion of invariance with respect to isomorphisms.

As we have seen, isomorphisms preserve best reply relationships. One may take the point of view that these relationships contain the essence of a non-cooperative game since no other information is needed in order to determine the set of all equilibrium points. This suggests the idea that two games should be treated in the same way if they do not differ with respect to their best reply relationships. Unfortunately, invariance requirements of this kind turn out to be too strong if they are imposed on the solution function. As we shall see in a later section, one would have to accept counter-intuitive consequences.

Our solution concept is composed of a number of different parts which interact in a process of equilibrium point selection. One of the most important notions which enter the definition of the solution as a building block is that of risk dominance. The concept will be explained in later sections. In the limited context of 2x2-games it will be possible to axiomatize the notion of risk dominance. One of the axioms will be an invariance requirement based on best reply considerations. As far as risk dominance in 2x2-games is concerned the requirement is a very natural one, even if it is doubtful whether it should be extended to a wider context.

It will be necessary to introduce the notion of a best reply structure in order to obtain a formal description of the best reply relationships. However, we shall do this for games with normal form structure only, since we do not want to pursue the subject of invariance requirements based on best reply relationships beyond a very limited scope.

In the following all definitions will refer to a game $G = (\Phi,H)$ with normal form structure. No distinction is made between a player i and his only agent.

Best reply structure: The set of all pure best replies of player i to $q_{.i}$ is denoted by $A_i(q_{.i})$. The correspondence A_i which assigns the set $A_i(q_{.i})$ to $q_{.i} \in Q_{.i}$ is called player i's best reply correspondence. $A = (A_i)_N$ is the system of best reply correspondences.

The best reply structure $B = (\Phi, A)$ of $G = (\Phi, H)$ consists of the set of pure strategy combinations $\Phi = \underset{i \in N}{X} \Phi_i$ and the system $A = (A_i)_N$ of best reply correspondences.

It is clear that an isomorphism f from G to G' carries the best reply structure of G to that of G'.

Stability sets: The set of all $q_{.i} \in Q_{.i}$ such that a given pure strategy φ_i is a best reply to $q_{.i}$ is denoted by $S(\varphi_i)$. The set $S(\varphi_i)$ is called the stability set of φ_i. Obviously, $S(\varphi_i)$ is the set of all $q_{.i}$ with $\varphi_i \in A_i(q_{.i})$. One may look upon S as a correspondence from the union of all Φ_i to the union of all $Q_{.i}$. In a sense the correspondence S is the inverse of the system A of best reply correspondences. The pair (Φ, S) could also serve as a formal description of the best reply structure.

Graphical representation for 2x2-games: The best reply structure of 2x2-games can be vizualized with the help of a simple graphical representation. Consider the class of 2x2-games described by figure 3.5. These games have strong equilibrium points in the upper left and lower right corners. It is convenient to introduce the notation u_i and v_i for the losses faced by player i if he deviates from the equilibrium point $U = U_1U_2$ and $V = V_1V_2$, respectively, whereas the other player plays his equilibrium strategy (see figure 3.5).

	U_2	V_2
U_1	a_{11} b_{11}	a_{12} b_{12}
V_1	a_{21} b_{21}	a_{22} b_{22}

$$u_1 = a_{11} - a_{21} > 0$$
$$u_2 = b_{11} - b_{12} > 0$$
$$v_1 = a_{22} - a_{12} > 0$$
$$v_2 = b_{22} - b_{21} > 0$$

Figure 3.5: 2x2-games with strong equilibrium point in north-west and south-east corners.

A mixed strategy q_i in a 2x2-game is fully described by one of both probabilities. We shall use the notation

(3.8) $p_i = q_i(V_i)$

Player 1's strategy U_1 is a best reply for

(3.9) $a_{11}p_2 + a_{12}(1-p_2) \geq a_{21}p_2 + a_{22}(1-p_2)$

and V_1 is his best reply for

(3.10) $a_{11}p_2 + a_{12}(1-p_2) \leq a_{21}p_2 + a_{22}(1-p_2)$

This yields

(3.11) $U_1 \in A_1(q_2)$ for $0 \leq p_2 \leq \frac{u_1}{u_1+v_1}$

(3.12) $V_1 \in A_1(q_2)$ for $\frac{u_1}{u_1+v_1} \leq p_2 \leq 1$

Similarily we obtain

(3.13) $U_2 \in A_2(q_1)$ for $0 \leq p_1 \leq \frac{u_2}{u_2+v_2}$

$$(3.14)\quad V_2 \in A_2(q_1) \qquad \text{for} \quad \frac{u_2}{u_2+v_2} \le p_1 \le 1$$

We can draw a diagram which represents all mixed strategy combinations as points (p_1, p_2) in a rectangular coordinate system. This is done in figure 3.6 for a special case ($u_1 = 2$, $u_2 = 6$, $v_1 = 8$, $v_2 = 4$). The diagram will be called the <u>stability diagram</u> of the game.

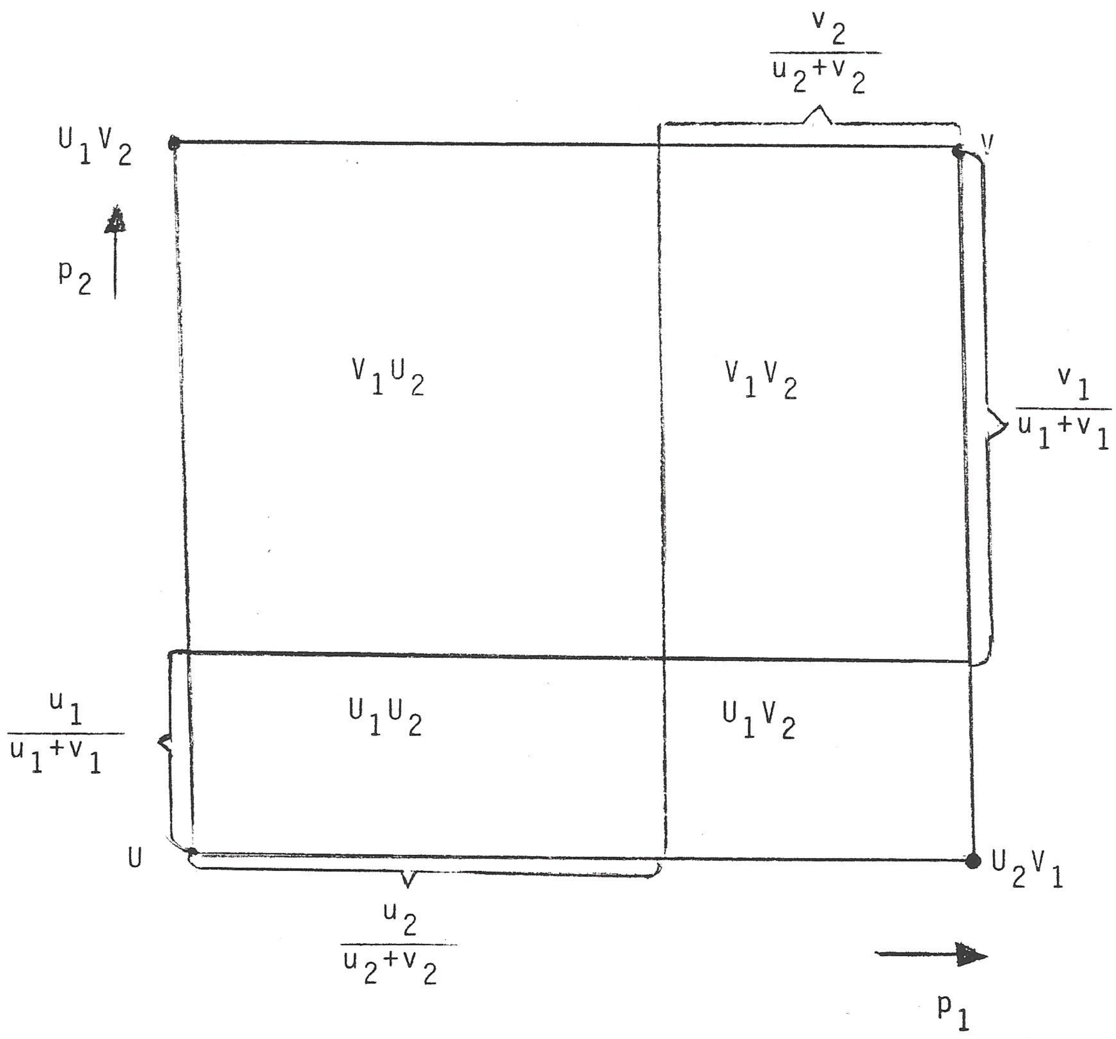


Figure 3.6: Stability diagram of the game of figure 3.5.

The regions where the four pure strategy combinations are best replies are indicated in figure 3.6. We call these regions the stability regions of the respective pure strategy combinations.

The stability regions are closed rectangles, all of which have one corner in common, the mixed equilibrium point with $p_1 = u_2/(u_2 + v_2)$ and $p_2 = u_1/(u_1 + v_1)$. The equilibrium points U and V belong to their stability region but the "cross combinations" U_1V_2 and V_1U_2 belong to the stability region of the opposite cross combination.

It is interesting to note that the best reply structure of a game in the class of figure 3.5 does not depend on anything else but the ratios u_1/v_1 and u_2/v_2 of the players' deviation losses at both strong equilibrium points. Absolute payoff levels do not matter. Only ratios of payoff differences are important.

<u>Payoff transformations which preserve the best reply structure</u> : Let $G = (\Phi, H)$ be a game with normal form structure and let ψ_{-j} be a fixed j-incomplete pure strategy combination for G. We construct a new game $G' = (\Phi, H')$ with the same set Φ of pure strategy combinations. For $i \neq j$ define

(3.15) $H_i'(\varphi) = H_i(\varphi)$ for every $\varphi \in \Phi$

Let λ be a constant. Player j's payoff is defined as follows:

(3.16) $H_j'(\varphi_j \psi_{-j}) = H_i(\varphi_j \psi_{-j}) + \lambda$

(3.17) $H_j'(\varphi_j \varphi_{-j}) = H_j(\varphi_j \varphi_{-j})$ for $\varphi_{-j} \neq \psi_{-j}$

We say that G' results from G by adding λ to player j's payoff at ψ_{-j}.

It is clear that the same amount $\lambda q_{.j}(\psi_{-j})$ is added to every payoff of the form $H_j(q_j q_{.j})$ in the transition from H_j to H_j'. Therefore, we obtain the following result: Adding λ to player j's payoff at ψ_{-j} does not change the best reply structure.

Consider the game of figure 3.5. We receive the game of figure 3.7 if we make the following changes one after the other:

1. We add $-a_{21}$ to player 1's payoffs at U_2
2. We add $-b_{12}$ to player 2's payoffs at U_1
3. We add $-a_{12}$ to player 1's payoffs at V_2
4. We add $-b_{21}$ to player 2's payoffs at V_1

This confirms once more what we already know from the investigation of the best reply structure of the games of figure 3.5: Every game in this class has the same best reply structure as the corresponding game of figure 3.7.

It may be worth-while to point out that not every payoff transformation which preserves the best reply structure can be obtained by a combination of positive linear payoff transformations with the repeated application of the operation of adding a constant to player j's payoffs at ψ_{-j}. 2x2-games are exceptional in this respect. Already in 2x3-games other best reply structure preserving payoff transformations are possible.

	U_2	V_2
U_1	u_1 ; u_2	0 ; 0
V_1	0 ; 0	v_1 ; v_2

Figure 3.7: Games received by best reply structure preserving transformations from those of figure 3.5.

An example is the class of games in figure 3.8. A positive linear transformation or adding a constant at player 2's payoffs at a or b cannot change the quotient

$$\text{(3.18)} \qquad \frac{H_2(bd) - H_2(bc)}{H_2(be) - H_2(bd)} = 3\,\frac{1+t}{1-t}$$

which clearly depends on t. Therefore, a combination of such transformations cannot yield the same result as a transition from one t to another.

	c	d	e
a	1 ; 2	0 ; 1-t	0 ; 0
b	0 ; 0	0 ; 1+t	1 ; $\frac{4}{3} + \frac{2}{3}t$

$0 < t < 1$

c is best reply for $0 \le q_1(b) \le \frac{1}{2}$

d is best reply for $\frac{1}{2} \le q_1(b) \le \frac{3}{4}$

e is best reply for $\frac{3}{4} \le q_1(b) \le 1$

Figure 3.8: A class of 2x3-games with the same best reply structure.

Invariance with respect to payoff transformations which preserves the best reply structure: A solution function L for a class $\mathcal{G}$ of games with normal form structure is called *invariant with respect to payoff transformations which preserve the best reply structure* or shortly *best reply invariant* if for any two games $G = (\Phi, H)$ and $G' = (\Phi, H')$ in $\mathcal{G}$ with the same best reply structure we have $L(G) = L(G')$.

Comment: As has been said before we do not insist on best reply invariance as a desirable property of a solution function. Nevertheless, it is an intuitively attractive requirement which should not be violated without a good reason. We want to keep as much of it as possible.

5. Payoff dominance

Consider the game of figure 3.9. the equilibrium point $U=U_1U_2$ yields higher payoffs for both players than the other pure strategy equilibrium point $V=V_1V_2$. The mixed equilibrium point with probabilities of .4 and .8 for U_1 and U_2, respectively, yields even worse payoffs, namely 7.2 for player 1 and 4 for player 2. Clearly, among the three equilibrium points of the game, U_1U_2 is the most attractive one for both players. This suggests that they should not have any trouble to coordinate their expectations at the commonly preferred equilibrium point U_1U_2. The solution of the game should be U_1U_2. The idea that equilibrium points with greater payoffs for all players should be given preference in problems of equilibrium point selection leads to the following definition.

Payoff dominance: Let r and s be two equilibrium points of $G = (\Phi,H)$ with $\Phi = \underset{i\in N}{X} \Phi_i$. We say that r payoff dominates s if we have

$$(3.19) \qquad H_i(r) > H_i(s) \qquad \text{for every } i\in N$$

In (3.19) we require strict inequality since we want to restrict considerations of payoff dominance to cases where the interest of all players unambigously points in the same direction.

The idea of payoff dominance must be handled with care. We cannot require that L(G) should never be payoff dominated by any other equilibrium point. As we have seen in section 3 invariances with respect to isomorphisms forces us to accept the mixed equilibrium point as the solutions of the game in figure 3.4 even if it is payoff dominated by both pure strategy equilibrium points.

	U_2	V_2
U_1	9, 7	0, 1
V_1	7, 2	8, 6

Figure 3.9: Example of a 2x2-game with payoff dominance.

	U_2	V_2
U_1	2, 6	0, 0
V_1	0, 0	8, 4

Figure 3.10: Game with the best reply structure of the game in figure 3.9.

The example of figure 3.4 shows that we should not pay attention to payoff dominance relationships where the dominating equilibrium point fails to be symmetry invariant. This leads to the following definitions.

Payoff efficiency: A symmetry invariant equilibrium point r of a game G = (Φ,H) is called payoff efficient if G has no other symmetry invariant equilibrium point s which payoff dominates r.

A solution function L for a class of games $\mathcal{G}$ is <u>payoff efficient</u> if L(G) is payoff efficient for every G ∈ $\mathcal{G}$. Unfortunately, payoff efficiency is a very strong requirement which cannot be easily satisfied by a solution concept such as ours. Moreover, there are reasons why it should not be satisfied in general. One of these reasons will be discussed in the section on cells.

Another reason is connected to the fact that a situation similar to that in figure 3.4 may arise without any lack of symmetry invariance. Two equilibrium points which both payoff dominate a third one but not each other may be equally strong in the sense that the theory does not yield a sufficient reason to select one rather than the other. In such situations it may be unavoidable to select an equilibrium point which fails to be payoff efficient.

In spite of the difficulties arising with this notion, payoff dominance is an important criterion of equilibrium point selection which cannot be completely ignored.

Payoff dominance relationships can easily be reversed by repeated additions of constants to a player j's payoff at some ψ_{-j}. Any strong equilibrium point ψ can be made the only payoff efficient one by performing the operations of adding a sufficiently great constant λ_j to the payoffs of every player j at his j-incomplete ψ_{-j} derived from ψ. This shows that best reply invariance and payoff efficiency are in conflict.

In the construction of our solution concept we have rejected full best reply invariance in favor of keeping the possibility of giving some room to considerations of payoff dominance without going as far as imposing the requirement of payoff efficiency.

6. The intuitive notion of risk dominance

Consider the game of figure 3.11. There is no payoff dominance relationship between both pure strategy equilibrium points $U = (U_1, U_2)$ and $V = (V_1, V_2)$. Player 1 has higher payoffs at U and player 2 has higher payoffs at V.

Suppose that the players are in a state of mind where they think that either U or V must be the solution of the game. What is the risk of deciding one way or the other? If Player 1 expects that player 2 will choose U_2 with a probability of more than .01 it is better for him to choose U_1. Only if player 2 chooses V_2 with a probability of at least .99 player 1's strategy V_1 will be the more profitable one. In this sense U_1 is much less risky than V_1.

	U_2	V_2
U_1	99 / 49	0 / 0
V_1	0 / 0	1 / 51

Figure 3.11: An extreme example of risk dominance.

Now let us look at the situation of player 2. His strategy V_2 is the better one if he expects player 1 to select V_1 with a probability of more than .49 and U_2 is preferable if he expects U_1 with a probability greater than .51. In terms of those numbers V_2 seems to be slightly less risky than U_2.

It is obvious that player 1's reason to select U_1 rather than V_1 is much stronger than player 2's reason to select V_2 rather than U_2. The players must take this into account when they try to form subjective probabilities on the other player's behavior. Presumably player 1 will select U_1 with high probability and since player 2 knows this,he is likely to think that it is better for him to choose U_2 rather than V_2. It is plausible to assume that at the end both players will come to the conclusion that both of them will play the equilibrium point U.

The same line of reasoning can be followed for less extreme situations. Consider a game of the form of figure 3.7 with $u_1 > v_1$ and $v_2 > u_2$. Player 1's risk situation is connected to the ratio u_1/v_1 and player 2's risk situation to the ratio v_2/u_2. Player 1 is more strongly attracted to U than player 2 to V if u_1/v_1 is greater than v_2/u_2. This is the case if and only if we have $u_1u_2 > v_1v_2$.

These considerations suggest the following notion of risk dominance for the games under consideration. U risk dominates V for $u_1u_2 > v_1v_2$ and V risk dominates U for $v_1v_2>u_1u_2$. The heuristic arguments which lead to this conclusion are fully in terms of the best reply structure. We have compared probabilities of the form $u_i/(u_i+u_i)$ and $v_i/(u_i+v_i)$. The probabilities which must be compared are the same in the more general situation of figure 3.5. These probabilities depend only on the best reply structure.

Since similar products appear in Nash's cooperative bargaining theory we call u_1u_2 and v_1v_2 the Nash-products of U and V, respectively.

It is interesting to note that the areas of the stability regions of U and V (see figure 3.6) are proportional to the Nash-products of U and V. This is a further argument for a notion of risk dominance based on the comparison of Nash-products.

Risk dominance and payoff dominance may point in different directions. An example is the game of figure 3.9 where U payoff dominates V but V has the greater Nash-product (the Nash-products are the same as in figure 3.10).

The notion of risk dominance between strong equilibrium points which has been obtained heuristically can be characterized by a set of simple axioms. This will be done in a later section.

7. Payoff monotonicity

In this section we shall discuss the requirement of payoff monotonicity which has been mentioned in the introduction of the chapter. Since we shall argue that this property is a less reasonable one than one might think, we shall restrict our attention to games with normal form structure. The phenomenon which we want to exhibit occurs already there.

Consider a game $G = (\Phi, H)$ with normal form structure and let ψ be a pure strategy equilibrium point of G. We construct a new game $G' = (\Phi, H')$ with the same set Φ of pure strategy combinations. Let λ_i with $i \in N$ be non-negative constants at least one of which is positive. Define

(3.20) $H'(\varphi) = H(\varphi)$ for $\varphi \neq \psi$

(3.21) $H_i'(\psi) = H_i(\psi) + \lambda_i$ for every $i \in N$

If G and G' are related in this way we say that G' <u>results from G by strengthening</u> ψ. The only difference between G and G' consists in the fact that some players receive more at ψ.

<u>Payoff monotonicity</u>: A solution function L for a class of games with normal form structure is called payoff monotonous if the following is true: If the solution L(G) of a game $G \in \mathcal{G}$ is a pure strategy equilibrium point and if G' results from G by strengthening L(G) then we have L(G') = L(G).

<u>Interpretation</u>: The requirement of payoff monotonicity is a very appealing one. Why should an equilibrium point become less attractive if some of its payoffs are increased? Nevertheless, an objection can be raised which makes it doubtful whether one should insist on payoff monotonicity as a general property.

In order to explain the nature of the counter-argument we look at the example of the three-person games of figure 2.12 and of figure 2.13. The game of figure 2.13 results from that of figure 2.12 by strengthening $U = U_1U_2U_3$. In the second game player 3 receives 1 unit more at U than in the first one. Otherwise both games agree in all payoffs.

It is reasonable to start a crude analysis of the risk situation in both games with the assumption that player 3 is more likely to choose U_3 in the second game. But does this strengthen U more than $V = V_1V_2V_3$? Suppose that each of the players 1 and 2 expects the other to behave in the same way in both games. Then an increase of their subjective probability for U_3 will increase their incentive to use their strategies V_1 and V_2. The numbers are chosen in such a way that it is not unreasonable to expect that the change from the first game to the second one enhances the stability of V more than that of U.

U_3:

	U_2	V_2
U_1	7, 7, 1	0, 0, 0
V_1	0, 0, 0	15, 15, 0

V_3:

	U_2	V_2
U_1	3, 3, 0	0, 0, 0
V_1	0, 0, 0	4, 4, 3

Figure 3.12: A three-person game. Player 3 chooses between the left and the right matrix.

U_3:

	U_2	V_2
U_1	7, 7, 2	0, 0, 0
V_1	0, 0, 0	15, 15, 0

V_3:

	U_2	V_2
U_1	3, 3, 0	0, 0, 0
V_1	0, 0, 0	4, 4, 3

Figure 3.13: A game which results from that of figure 3.12 by strengthening U.

The solution concept which we shall propose here actually assigns the solution U to the first game and the solution V to the second. It does not have the payoff monotonicity property.

In spite of the fact that we reject payoff monotonicity as a general property we think that it is a very reasonable requirement for 2x2-games. There we cannot find any reason to suppose that one of two strong equilibrium points can be made more attractive by strengthening the other. The nature of the example seems to indicate that at least three players are needed in order to produce an example where payoff monotonicity fails to be convincing.

8. Axiomatic characterization of risk dominance between strong equilibrium points in 2x2-games

Let $\mathcal{R}$ be the class of all 2x2-games with 2 strong equilibrium points. We shall axiomatise a <u>risk dominance relationship</u> which is defined between the two strong equilibrium points of any game in $\mathcal{R}$. The notation $U \succ V$ is used in order to indicate that U risk dominates V. We also permit that neither U risk dominates V nor V risk dominates U and we write $U|V$ if this is the case. For any game $G \in \mathcal{R}$ with strong equilibrium points U and V exactly one of the following statements must hold:

1.	$U \succ V$	U risk dominates V in G
2.	$V \succ U$	V risk dominates U in G
3.	$U\|V$	There is no risk dominance between U and V in G.

This is part of the definition of the concept of a risk dominance relationship and not yet a requirement to be imposed on it.

The axioms are stated below. It will always be understood that U and V are the strong equilibrium points of a game $G = (\Phi, H) \in \mathcal{R}$.

<u>(I). Invariance with respect to isomorphisms:</u> Let f be an isomorphism from G to G'. Then we have $f(U) \succ f(V)$ in G' if and only if we have $U \succ V$ in G.

<u>(II). Best reply invariance:</u> Let $G' = (\Phi, H')$ be a game which has the same best reply structure as $G = (\Phi, H)$. Then $U \succ V$ holds in G' if and only if it holds in G.

<u>(III). Payoff monotonicity:</u> Let $G' = (\Phi, H')$ be a game which results from $G = (\Phi, H)$ by strengthening U. If $U \succ V$ or $U|V$ holds in G then $U \succ V$ holds in G'.

<u>Interpretation</u>: It is clear that we must require invariance with respect to isomorphisms. The reasons are the same as those discussed in section 3. As we have seen in section 6 the intuitive arguments which we have used in order to compare risks attached to different equilibrium points run in terms of the best reply structure. Imposing axiom (II) means that we look for a concept of this kind without specifying a precise way in which risk comparisons should be made.

Payoff monotonicity has been discussed in section 7. As far as 2x2-games are concerned it seems to be a very desirable property even if for more complicated games the situation is less clear.

<u>Theorem</u>: There is one and only one risk dominance relationship for $\mathcal{R}$ which satisfies (I), (II) and (III). As in figure 3.5 let u_i and v_i with $i = 1,2$ be the deviation losses of player i at the strong equilibrium points U and V of a

game $G \in \mathcal{R}$. Then we have

(3.22) $U \succ V$ for $u_1u_2 > v_1v_2$

(3.23) $V \succ U$ for $v_1v_2 > u_1u_2$

(3.24) $U | V$ for $u_1u_2 = v_1v_2$

Proof: Up to renamings of the strategies every game $G \in \mathcal{R}$ is in the class of games of figure 3.5. Any such game has the same best reply structure as the corresponding game of figure 3.7 (see section 4). Multiplication of player 1's payoff by $1/v_1$ and player 2's payoff by $1/u_2$ transforms a game of figure 3.7 into a game of figure 3.14.

	V_1	V_2
U_1	u ; 1	0 ; 0
U_2	0 ; 0	1 ; v

$$u = \frac{u_1}{v_1} \qquad v = \frac{v_2}{u_2}$$

Figure 3.14: Games equivalent to those of figure 3.7.

For $u = v$ the game of figure 3.14 has a symmetry which carries U to V (renaming of strategies and exchanging the players). Therefore, in view of (I) for $u = v$ we must have $U|V$.

A game of figure 3.14 with $u > v$ results from a game with $u = v$ from strengthening U. Therefore, in view of (III) we must have $U \succ V$ for every game of figure 3.14 with $u > v$ and similarily $V \succ U$ for every game of figure 3.14 with $v > u$.

Since the best reply structure of a game of figure 3.5 is the same as that of the corresponding game of figure 3.14

we must have $U \succ V$ for $u > v$ there, too. We have $u > v$ if and only if $u_1 u_2 > v_1 v_2$. Analogously, we have $V \succ U$ if and only if $v_1 v_2 > u_1 u_2$. This proves the theorem.

Comment: The theorem gives a firm basis to our intuitive considerations on risk dominance between strong equilibrium points in 2x2-games. The only notion of risk dominance which agrees with the axioms can be described as a comparison of Nash-products of deviation losses.

It is interesting that our result supports Nash's bargaining theory under fixed threats without relying on anything similar to the axiom of irrelevant alternatives which plays a crucial role in his axiomization.

On the basis of the risk dominance relationship characterized by the theorem one can define a solution function which will be called pure risk dominance solution function since it completely ignores the aspect of payoff dominance.

The pure risk dominance solution: The pure risk dominance solution function $\bar{L}$ on $\mathcal{R}$ is defined as follows: Let U and V be the strong equilibrium points of $G = (\Phi, H)$ and let u_i and v_i for $i = 1.2$ be the deviation losses at U and V (as in figure 3.5). Let $r = (r_1, r_2)$ with

$$(3.25) \qquad r_1(U_1) = \frac{v_2}{u_2+v_2} \quad , \quad r_2(U_2) = \frac{v_1}{u_1+v_1}$$

be the third equilibrium point of G. Then we have:

$$(3.26) \qquad \bar{L}(G) = \begin{cases} U & \text{for } u_1 u_2 > v_1 v_2 \\ V & \text{for } v_1 v_2 > u_1 u_2 \\ r & \text{for } u_1 u_2 = v_1 v_2 \end{cases}$$

<u>Conflict between risk dominance and payoff dominance:</u> We have already pointed out in section 6 that a risk dominance relationship in one direction is compatible with a payoff dominance relationship in the other direction. It is maybe useful to look at the extreme example of figure 3.15.

Here U payoff dominates V but V strongly risk dominates U. It is reasonable to expect that most players would prefer to play V_i rather than U if the game is played for a considerable amount of money (say $ 1000,- per unit) without preplay communication. On the other hand, with preplay communication they may very well come to the conclusion that they can trust each other to choose $U = (U_1, U_2)$. An agreement to do so is selfstabilizing and does not need any commitment power.

	U_2	V_1
U_1	9, 9	0, 8
V_2	8, 0	8, 8

<u>Figure 3.15:</u> Example of payoff dominance and risk dominance in opposite directions.

If it is common knowledge of both players that both are fully rational then there should not be any need to enter preplay communication before the beginning of this game since the outcome can be predicted easily anyhow. Therefore, even under conditions which do not permit preplay communication they should trust each other to play U.

The pure risk dominance solution involves a certain lack of rationality. Nevertheless, under certain circumstance distrust may be justified. Suppose for example that in the game under consideration preplay communication has taken place and for some mysterious reason the players could not agree on U. Then, after the breakdown of communication, it is certainly justified not to look at payoff dominance and to rely on risk dominance only.

For a long time the authors took the point of view that everything which goes beyond pure risk dominance should be captured by formal models of preplay communication which explicitly describe how trust is developed rationally under the threat of conflict. In a theory of this type the pure risk dominance solution would serve as a threat point of preplay bargaining. Preplay bargaining itself would be described as a game where an equilibrium point has to be selected. Hopefully in this bargaining game the conflict between risk dominance and payoff dominance may not occur. Otherwise, one would meet the difficulty that bargaining on bargaining is required before the beginning of the barbaining game. In spite of the difficulties involved in this approach it may still be worth trying.

It is our impression that a theory which gives room to both payoff dominance and risk dominance is more in agreement with the usual image of what constitutes rational behavior. Moreover, it avoids some of the difficulties of the approach outlined above even if models of preplay communication may still be necessary for some purposes.

<u>The proposed solution function for 2x2-games with two strong equilibrium points</u>: The solution function L for $\hat{\mathcal{R}}$ which results from the application of our general concept to this class gives absolute priority to payoff dominance. It can be described as follows. Let U and V be the strong equilibrium points of G = (Φ,H). Then we have:

$$L(G) = \begin{cases} U & \text{if } H_i(U) > H_i(V) \text{ for } i = 1,2 \\ V & \text{if } H_i(U) < H_i(V) \text{ for } i = 1,2 \\ \bar{L}(G) & \text{else} \end{cases} \tag{3.27}$$

where $\bar{L}(G)$ is the pure risk dominance solution function introduced in section 8. We call this solution function L the <u>proposed solution function</u> for $\hat{\mathcal{R}}$.

One may ask how the solution function L should be extended to the class of all 2x2-games. Obviously, those games which have only one equilibrium point raise no difficulties. Some degenerate cases with an infinity of equilibrium points like the example of figure 3.16 cannot be fully discussed before the introduction of further basic concepts. An important definition, namely that of a cell will be introduced in the next section in order to prepare the discussion of further desirable properties. The notion of a cell permits us to decompose some games into smaller games.

	U_2	V_2
U_1	1, 1	0, 1
V_i	1, 0	0, 0

<u>Figure 3.16</u>: A degenerate 2x2-game.

The game of figure 3.16 turns out to be decomposable in this sense. We shall come back to this example in section 10.

9. Cells

It is natural to require that a solution function for extensive games is subgame consistent in the sense that the behavior prescribed on a subgame is nothing else than the solution of the subgame. After all, once the subgame has been reached all other parts of the game are strategically irrelevant.

It is not immediately clear how subgame consistency can be achieved in the framework of the standard form. The definition of a subgame depends on the tree structure of the extensive form. The standard form abstracts from the information on the sequential order in which choices are made.

A further complication is added by the fact that we do not apply our solution function directly to the standard form of an extensive game, but to its ε-perturbations. We must do this in a way which achieves subgame consistency of the limit solution for the original game.

The essential features of a subgame are not lost in the transition from the extensive form to the ε-perturbed form. In order to capture these essential features we shall define substructures of the standard form which will be called cells. As we shall see a subgame always corresponds to a cell of an ε-perturbation, but it is also possible that an ε-perturbation has a cell which does not arise from a subgame of the underlying extensive form.

It may seem to be somewhat confusing that a subgame of an extensive game generally does not correspond to a cell of its unperturbed standard form. In the unperturbed standard form a subgame corresponds to a slightly different kind of substructure which will be called a semicell. A semicell is very similar to a cell but its distinguishing properties are less stringent.

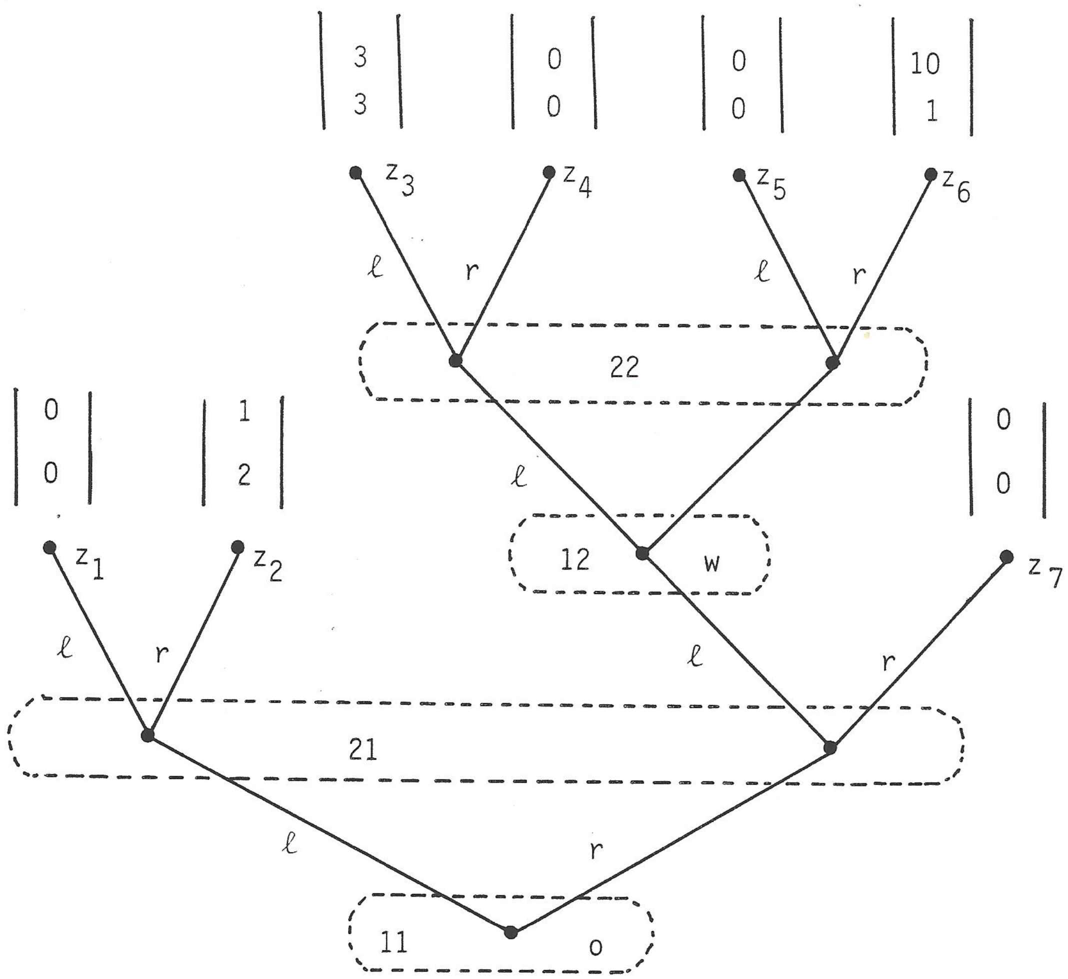
Since subgames are related to problems of perfectness, it is not too surprising that the substructures generated by subgames in ε-perturbed standard forms have better properties than those generated in the unperturbed standard form.

Before we go on to define semicells and cells we shall discuss the problem with the help of a specific example. In this way, it will be easier to understand the intuitive ideas underlying our definitions.

<u>The notion of a subgame</u>: Since we want to avoid the formalism of extensive games we cannot give a precise definition of a subgame. However, the following description will be suffcient for our purposes: Consider a node x of the tree of an extensive game Γ. Let K_x be the subtree containing x and all nodes after x. The subtree K_x is the tree of a subgame Γ_x if the following condition is satisfied: Every information set with nodes in K_x does not contain any nodes outside K_x. The rules of Γ_x are those

specified by Γ after x has been reached.

<u>The example of figure 3.17</u>: Let Γ be the game of figure 3.17. This game has a subgame Γ_w at w. The information sets of agents 12 and 22 do not contain any nodes outside the subtree K_w at w.



<u>Figure 3.17</u>: An extensive game with a subgame Γ_w at w. The names of the agents of both players are given within the dotted lines representing their information sets. Player 1's payoff is above and player 2's payoff is shown below. Choices are indicated by the letters ℓ and r standing for left and right.

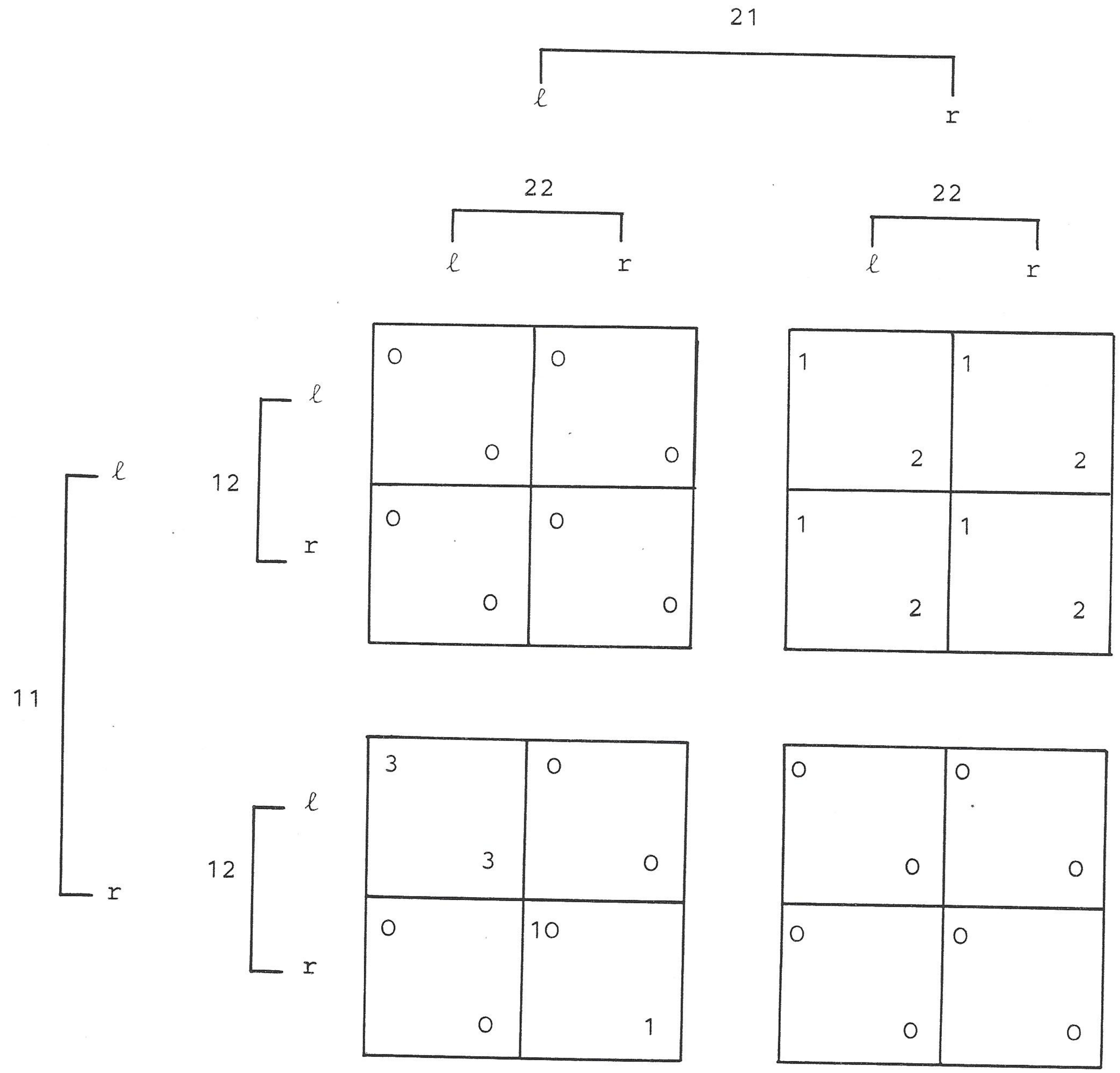


Figure 3.18: The standard form of the extensive game of figure 3.17. The choices controlled by agent ij are indicated by connecting lines marked by ij. Agent 11 selects a row of bimatrices; 21 selects a column of bimatrices; 12 selects a row within the bimatrix and 22 selects a column within the bimatrix. Player 1's payoffs are shown above and player 2's payoffs are shown below.

The standard form of Γ is shown in figure 3.18. It is represented as an array of 4 bimatrices. The subgame Γ_w is reached if agent 11 chooses r and agent 21 chooses ℓ. Therefore, the strategic situation of the subgame Γ_w is that of the 2x2-game represented by the bimatrix in the lower left corner of figure 3.18. We shall refer to this bimatrix as the <u>bimatrix of the subgame</u> Γ_w.

The bimatrix of the subgame Γ_w can be obtained as a substructure of the standard form by fixing agent 11 at r and agent 21 at ℓ. What are the special features which distinguish this substructure from other substructures obtainable in a similar way ? In order to answer this question we shall look more closely at the payoff functions H_1 and H_2 of both players in the standard form of figure 3.18. Let x_{ij} be agent ij's probability of choosing ℓ and let y_{ij} be agent ij's probability of choosing r. A behavior strategy combination can be represented by a vector

(3.28) $$x = (x_{11}, x_{12}, x_{21}, x_{22})$$

Accordingly, we write $H_i(x)$ for i's expected payoff if the behavior strategy combination corresponding to x is played. The payoffs are as follows:

(3.29) $$H_1(x) = x_{11}x_{21} + y_{11}x_{21}[3x_{12}x_{22} + 10y_{12}y_{22}]$$

(3.30) $$H_2(x) = 2x_{11}x_{21} + y_{11}x_{21}[3x_{12}x_{22} + y_{12}y_{22}]$$

The expression in the rectangular brackets have an obvious interpretation. Let $H_{w1}(x_{12}x_{22})$ and $H_{w2}(x_{12}x_{22})$ be player 1's and player 2's payoffs in the 2x2-game represented by the bimatrix of Γ_w. We have

(3.31) $$H_{w1}(x_{12}x_{22}) = 3x_{12}x_{22} + 10y_{12}y_{22}$$

(3.32) $$H_{w2}(x_{12}x_{22}) = 3x_{12}x_{22} + y_{12}y_{22}$$

Obviously, agents 12 and 22 need not be concerned with anything else but H_{w1} and H_{w2}. The local strategies of agent 11 and 12 do not really matter for them. This is due to the fact that H_1 is always a non-negative linear transformation of H_{x1} and H_2 is always a non-negative linear transformation of H_{x2}. The coefficients of these transformations are determined by the local strategies of agents 11 and 21, but these coefficients do not have any essential influence on the strategic situation of agents 12 and 22. It does not matter what 12 and 22 do if we have $y_{11}x_{21} = 0$, but for $y_{11}x_{21} > 0$ it always matters in the same way. As far as the subgame agents 12 and 22 are concerned, a transition from one pair of completely mixed local strategies for the outside agents 11 and 21 to another amounts to a positive linear payoff transformation and therefore is analogous to the transition to an equivalent game.

The fact that H_1 and H_2 are non-negative linear transformations of expressions involving the local strategies of 12 and 22 with coefficients determined by the local strategies of 11 and 21 can be checked in the standard form without any reference to the underlying extensive form. It can be seen immediately that a subgame will always lead to an analogous situation in the standard form.

Total payoffs can be written as subgame payoffs multiplied by the probability of reaching the subgame supplemented by an additional term which reflects the payoffs at end-points outside the subgame.

If we want to obtain a substructure which corresponds to the subgame, it does not really matter where we fix the agents outside the subgame as long as we avoid local strategies which produce zero probability of reaching the subgame. It will be convenient to define semicells and cells in such a way that all outside agents kj are fixed at the centroids c_{ij} of their choice sets ϕ_{ij}. (For the definition of the centroid, see chapter 2, section 6).

It is useful to look at the situation in uniformly perturbed games before we begin to state formal definitions. Consider an ε-perturbation of the standard form shown in figure 3.18. Obviously, there H_1 and H_2 will always be positive linear transformations of H_{w1} and H_{w2}, since no choice can be taken with a probability less than ε. It is clear that a subgame will always lead to an analogous situation in the ε-perturbation of the standard form.

The definition of a semicell will involve non-negative linear payoff transformations and the definition of a cell will involve positive linear transformations. In this way we solve the task of recognizing subgame like substructures in the standard form. However, it must be pointed out that semicells of standard forms and cells of ε-perturbed standard forms may not always correspond to subgames in an underlying extensive game. An example for this will be given later.

<u>Semicells and cells</u>: Let $G = (\Phi, H)$ with

$$(3.33) \qquad \Phi = \underset{i \in N}{X} \Phi_i = \underset{ij \in M}{X} \Phi_{ij}$$

be a game in standard form and let C be a non-empty proper subset of M. Let $G^C = (\Phi^C, H^C)$ be the game which results from G by fixing the agents $ij \in M \setminus C$ at the centroids c_{ij} of their choice sets Φ_{ij}. We shall use the symbol $-C$ as an abbreviation for $M \setminus C$.

The game G^C is a <u>semicell</u> if for every $\psi_{-C} \in \Phi_{-C}$ and for every i with $M_i \cap C \neq \emptyset$ a number $\alpha_i(\psi_{-C}) \geq 0$ and a number $\beta_i(\psi_{-C})$ can be found such that we have:

$$(3.34) \qquad H_i(\psi_{-C}\varphi_C) = \alpha_i(\Psi_{-C}) H_i^C(\varphi_C) + \beta_i(\psi_{-C})$$

G^C is a <u>cell</u>, if for every $\psi_{-C} \in \Phi_{-C}$ and for every i with $M_i \cap C \neq \emptyset$ a number $\alpha_i(\psi_{-C}) > 0$ and a number $\beta_i(\psi_{-C})$ can be found such that (3.34) holds.

We say that C <u>forms a semicell</u> or a <u>cell</u> if G^C is a semicell or cell, respectively. If this is the case then G^C is called the semicell or cell <u>formed</u> by C.

<u>Remark</u>: It is an immediate consequence of (3.34) that numbers $\alpha_i(b_{-C})$ and $\beta_i(b_{-C})$ can be found for every subcombination $b_{-C} \in B_{-C}$, such that (3.34) holds with b_{-C} in place of ψ_{-C} with $\alpha_i(b_{-C}) \geq 0$ in the case of a semicell and with $\alpha_i(b_{-C}) > 0$ in the case of a cell. Obviously,

$$(3.35) \qquad \alpha_i(b_{-C}) = \sum_{\varphi_{-C} \in \Phi_{-C}} b_{-C}(\varphi_{-C}) \alpha_i(\varphi_{-C})$$

and

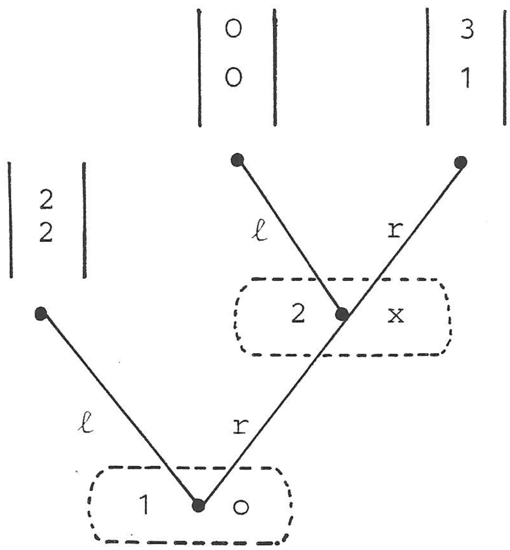
$$(3.36) \qquad \beta_i(b_{-C}) = \sum_{\varphi_{-C} \in \Phi_{-C}} b_{-C}(\varphi_{-C})\beta_i(\varphi_{-C})$$

satisfy these requirements.

Interpretation: A semicell or a cell may have fewer players than the original game, since $M_i \cap C$ may be empty for some players. If a semicell or a cell arises from a subgame of an underlying extensive form, then (3.34) can be satisfied for these outside players, too. Moreover, if $\alpha_i(\psi_{-C})$ is the probability of reaching the subgame by ψ_{-C}, then the multiplicative constant in (3.34) does not depend on i.

This shows that it would have been possible to define subgame-like substructures in a much more restrictive way. However, our definitions are based on the idea that any subset C of agents which is strategically independent should be treated in the same way.

The standard form does not permit us to reconstruct subgames of the extensive form. This is shown by the example of figure 3.19. The upper game has a subgame and the lower has none and both have the same standard form. Should these games be treated differently? We think that the fact that in the lower game player 2's information set does not tell him whether x or y has been reached is without relevance. His decision is important for him only if x has been reached.



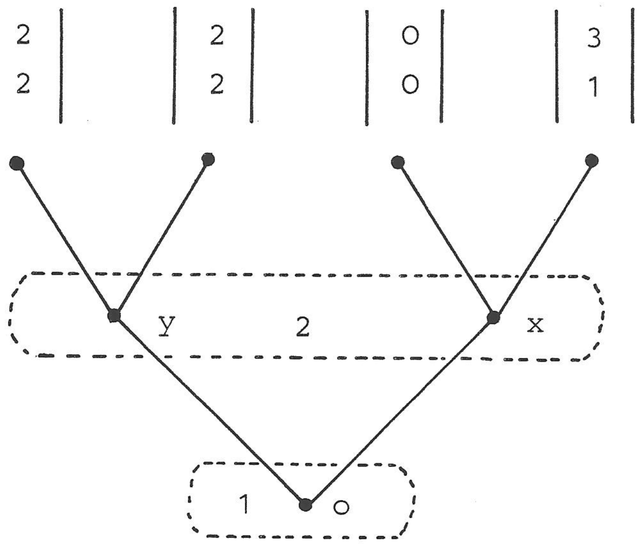


Figure 3.19: Two extensive games with the same standard form. The upper one has a subgame at x, the lower one has no subgame. For the conventions of graphical representation see figure 3.17.

The next example throws further light on the cell definition.

Example of a cell not arising from a subgame: Suppose that two players 1 and 2 with linear utilities for money are involved in a bimatrix game whose entries are in terms of unknown currencies. Before they make their choice a third player secretly selects between two alternative possibilities:

(a) player 1 receives U.S. dollars and player 2 receives Israeli shekels.

(b) player 1 receives French francs and player 2 receives German marks.

It is reasonable to cover the strategic situation of players 1 and 2 in this game by the definition of a cell. For every fixed strategy choice of player 3 their situation is the same. Therefore, they do not even have to think about player 3's strategy choice. They only interact with each other and as far as they are concerned the game is a two person game.

Remark: If G is the standard form of an extensive game Γ and C is the set of all agents in a subgame of Γ, then C forms a semicell in G. Moreover, C forms a cell in every ε-perturbation G_ε of G. This is clear from the discussion preceding the definition of semicells and cells.

It is worthwhile to show that a semicell of a standard form always corresponds to a cell of its ε-perturbation, regardless of whether it arises from a subgame of an underlying extensive form or not. This is a consequence of the following lemma.

<u>Lemma on semicells</u>: Let $G = (\Phi, H)$ be a standard form and let C be a subset of agents in G which forms a semicell of G. Then C forms a cell of every interior substructure $G' = (\phi', H')$ of G such that the agent set M' of G' contains C.

<u>Proof</u>: With the help of (3.35) and (3.36) we can determine numbers $\alpha_i(\varphi'_{M' \setminus C})$ and $\beta_i(\varphi'_{M' \setminus C})$ which satisfy the requirements for a semicell in G'. Assume that at least one the $\alpha_i(\varphi_{-C})$ is positive. Then all $\alpha_i(\varphi'_{M \setminus C})$ are positive, since the choices in G' are completely mixed local strategies in G. In this case C forms a cell in G'. Assume, on the contrary, that $\alpha_i(\varphi_{-C}) = 0$ holds for every $\varphi_{-C} \in \Phi_{-C}$. Then by (3.34) we have

$$\text{(3.35)} \qquad H_i(\psi_{-C}\varphi_C) = \beta_i(\Psi_{-C})$$

for players with agents in C. Moreover, we also have

$$\text{(3.36)} \qquad H_i^C(\varphi_C) = \beta_i(\psi_{-C})$$

by the definition of the semicell payoff function. Define $\hat{\alpha}_i(\varphi_{-C}) = 1$ and $\hat{\beta}_i(\varphi_{-C}) = 0$ for every $\varphi_{-C} \in \Phi_{-C}$. With these coefficients instead of $\alpha_i(\varphi_{-C})$ and $\beta_i(\varphi_{-C})$, respectively, (3.34) holds, too, and the conditions required by the definition of a cell are satisfied. C forms a cell already in G and as we have seen above, in G' too.

Comment: The solution function of our theory is not directly applied to standard forms but to their ε-perturbations. The fact that semicells of a standard form correspond to cells of its ε-perturbations permits us to concentrate attention on cells. Some further results on cells will be derived in the following

Lemma on cells: Let C and C' with $C \cap C' \neq \emptyset$ be two proper subsets of M which both form cells of G = (Φ,H) with

$$(3.37) \qquad \Phi = \underset{i \in N}{X} \Phi_i = \underset{ij \in M}{X} \Phi_{ij}$$

Then $D = C \cap C'$ forms a cell of G, too.

Proof: Any change of the subcombination b_{-D} for $M \setminus D$ can be achieved by two successive changes, such that first only the local strategies of agents in $M \setminus C$ are changed and then those of the agents in $C \setminus D$. Both changes are connected with the positive linear payoff transformations for the players i with $M_i \cap D \neq \emptyset$, in the first case since C forms a cell and in the second case since C' forms a cell. Two successive positive linear transformations performed one after another are equivalent to one positive linear transformation. In this way we receive the positive linear transformations whose existence are required by the definition of a cell as applied to G.

Counterexample: One may think that the union of two subsets C and C' forms a cell if C and C' form a cell. The example of figure 3.20 shows that this is not necessarily true. The game of figure 3.20 is a game with normal form structure. We need not distinguish between a player and his only agent. Both {1} and {2} form cells, since for fixed strategies of the other players the difference between the payoffs for U_i and V_i is always 1 for i = 1,2. Nevertheless, {1,2} is not a cell and not even a semicell since a shift from U_3 to V_3 reverses the payoff differences between U_1U_2 and V_1V_2. No non-negative linear payoff transformation can produce this result.

U_3:

	U_2	V_2
U_1	3, 3, 1	0, 4, 0
V_1	4, 0, 0	1, 1, 0

V_3:

	U_2	V_2
U_1	0, 0, 0	2, 1, 0
V_1	1, 2, 0	3, 3, 1

Figure 3.20: A counterexample. {1} and {2} form cells but {1,2} does not form a cell. Player 1, 2 and 3 choose rows, columns and matrices, respectively.

Elementary cells: Let G = (Φ,H) be a standard form and let C be a non-empty proper subset of its agent set M such that C forms a cell G^G of G. The cell G^G is called elementary if no proper subset of C forms a cell of G. It follows by the lemma on cells that subsets which form elementary cells do not intersect.

Comment: The fact that elementary cells do not intersect is an important one since it enables us to define a solution function which is based on the idea that a game with cells should be solved by first solving the elementary cells and then solving the game which results by fixing the agents of the elementary cells at the local strategies prescribed by the solution of these cells.

10. Cell consistency and truncation consistency

In this section we shall look at two desirable properties of solution functions. Roughly speaking, cell consistency requires that the solution of the whole game agrees with that of its cells as far as the cell agents are concerned. Truncation concerns the "truncated" game which results if the agents in a cell are fixed at the solution of the cell. The requirement postulates that the solution of the truncated game should agree with the solution of the whole game as far as its agents are concerned.

Completeness: A class $\mathcal{G}$ of games in standard form is called *complete* if a substructure G' of a game $G \in \mathcal{G}$ always belongs to $\mathcal{G}$. Obviously, $\mathcal{G}$ is complete if $\mathcal{G}$ is the interior substructure class $\mathcal{J}(\mathcal{G}')$ of another class $\mathcal{G}'$ of standard forms. In our theory we only consider solution functions for complete subclasses of the class $\mathcal{J}(\mathcal{R})$ of all interior substructures of standard forms with perfect recall.

Truncations: Let L be a solution function for a complete class $\mathcal{G} \subseteq \mathcal{J}(\mathcal{R})$. Let $G \in \mathcal{G}$ be a game with a cell G^C. The *truncation of* G *with respect to* G^C *and* L is the Game $\bar{G}$ which results from G by fixing the agents G^C at their local strategies in the solution $L(G^C)$ of G^C.

Remark: The completeness of $\mathcal{G}$ is important since it guarantees that both G^C and the truncation $\bar{G}$ of G with respect to G^C and L are in $\mathcal{G}$ if G^C is a cell of G. Both G^C and $\bar{G}$ result from G by fixing some of the agents but not all of them.

Cell consistency: A solution function L for a complete class $\mathcal{G} \subseteq \mathcal{J}(\mathcal{R})$ is called cell consistent if for a cell G^C of a game $G \in \mathcal{G}$ the solution $L(G^C)$ and $L(G)$ of G^C and G always prescribe the same local strategies to all agents in G^C.

Truncation consistency: A solution function L for a complete class $\mathcal{G} \subseteq \mathcal{J}(\mathcal{R})$ is called truncation consistent, if for a truncation $\bar{G}$ of a game $G \in \mathcal{G}$ with respect to a cell G^C and L the solutions $L(\bar{G})$ and $L(G)$ of $\bar{G}$ and G always prescribe the same local strategies for all agents of $\bar{G}$.

Interpretation: As far as their strategic situation is concerned, the agents in a cell do not depend on outside agents. This has been discussed in the last section. Obviously, cell consistency is a very natural requirement.

If cell consistency is accepted then truncation consistency becomes an almost unavoidable additional requirement. If it is rational to expect that the cell agents will play their local strategies of the cell solution, it should be possible to replace the analysis of the whole game by the analysis of the cell and the truncated game.

As we shall see, cell consistency and truncation consistency have the consequence that it is sufficient to know the solutions of games without cells in order to compute the solution for all games in the complete class $\mathcal{G} \subseteq \mathcal{J}(\mathcal{R})$ where the solution function is defined.

Decomposibility: A game G is called decomposable if it has at least one cell. Games without cells are called indecomposable. We say that G is fully decomposable if every agent belongs to an elementary cell. Decomposable games which are not fully decomposable are called partially decomposable.

Main truncation: Let L be a solution function for a complete class $\mathcal{G} \subseteq \mathcal{J}(\mathcal{R})$. For every partially decomposable game $G \in \mathcal{G}$ we construct a game $\hat{G}$ which is called the main truncation of G with respect to L. Let $G^1,\ldots,G^k$ be the elementary cells of G. The game $\hat{G}$ results from G by fixing the agents of the elementary cells at their local strategies in the solutions $L(G^1),\ldots,L(G^k)$ of the elementary cells.

Composition: Let L be a solution function for a complete class $\mathcal{G} \subseteq \mathcal{J}(\mathcal{R})$ and let $G \in \mathcal{G}$ be a fully decomposable game. Let r be that behavior strategy combination for G which contains for every agent ij his local strategy prescribed by the solution $L(G^j)$ of the elementary cell to which he belongs. This behavior strategy combination r is called the composition of the elementary cell solutions. Now consider a partially decomposable game $G \in \mathcal{G}$. Let r be that behavior strategy combination for G which (a) for every agent in an elementary cell G^j of G contains his local strategy in $L(G^j)$ and (b) for every agent in the main truncation $\hat{G}$ of G with respect to L contains his local strategy in $L(\hat{G})$. This behavior strategy combination r is the <u>composition of the main truncation and elementary cell solutions</u>.

<u>Composition lemma</u>: Let L be a solution function for a complete class $\mathcal{G} \subseteq \mathcal{J}(\mathcal{R})$. Then for every fully decomposable $G \in \mathcal{G}$ the composition of the main truncation and elementary cell solutions is an equilibrium point of G.

<u>Proof</u>: L assigns equilibrium points in behavior strategies to games $G \in \mathcal{G}$. Since these games are interior substructures of standard forms with perfect recall they have the local best reply property. (See the theorem on local best replies in chapter 2, section 6). The local best replies of the cell agents do not depend on local strategies of outside agents. The construction of the main truncation embodies the expectation that cell agents use their local strategies in the cell solution. Therefore, in both cases (full and partial decomposibility) the composition is a local equilibrium of G and, consequently, an equilibrium point of G.

<u>Extension</u>: Let $\mathcal{G}_o$ be the subclass of all indecomposable games in a class $\mathcal{G} \subseteq \mathcal{J}(\mathcal{R})$ and let L_o be a solution function for $\mathcal{G}_o$. On the basis of L_o we shall construct a solution function L for $\mathcal{G}$ which will be called the <u>extension of L_o to $\mathcal{G}$</u>. The extension L is recursively defined by the following properties (A), (B) and (C):

(A) For $G \in \mathcal{G}_o$ we have $L(G) = L_o(G)$

(B) If $G \in \mathcal{G}$ is fully decomposable, then $L(G)$ is the composition of the elementary cell solutions.

(C) If $G \in \mathcal{G}$ is partially decomposable, then $L(G)$ is the composition of the main truncation and the cell solutions.

<u>Extension lemma</u>: Let L_o be a solution function for the subclass $\mathcal{G}_o$ of all indecomposable games in a class $\mathcal{G} \subseteq \mathcal{J}(\mathcal{R})$. There is one and only one solution function L for $\mathcal{G}$ with the properties (A), (B) and (C).

<u>Proof</u>: It is clear that a unique behavior strategy combination $L(G)$ is defined by (A), (B) and (C) for every $G \in \mathcal{G}$. Property (C) may have to be applied several times, first to the game itself, then to its main truncation, etc. But finally a main truncation will arise which is either indecomposable or fully decomposable. An easy induction argument based on the composition lemma shows that $L(G)$ is an equilibrium point in behavior strategies of G.

<u>Extension theorem</u>: Let L_o be a solution function for the subclass $\mathcal{G}_o$ of all indecomposable games in a class $\mathcal{G} \subseteq \mathcal{J}(\mathcal{R}_o)$; let L_o be invariant with respect to positive linear payoff transformations. There is one and only one cell consistent and truncation consistent solution function for $\mathcal{G}$ which agrees with L_o on $\mathcal{G}_o$, namely the extension L of $\mathcal{G}_o$ to $\mathcal{G}$.

In order to prove this theorem we need the following "decomposition lemma".

Decomposition lemma: Let L be a solution function for a complete class $\mathcal{G} \subseteq \mathfrak{Z}(\mathcal{R})$. Let G^C be a cell of a game $G \in \mathcal{G}$ and let $\bar{G}$ be the truncation of G with respect to G^C and L. Then every agent subset D with $C \cap D = \emptyset$ which forms a cell in G, forms a cell in $\bar{G}$, too. Moreover, the cells formed by D in G and $\bar{G}$ are equivalent games.

Proof: In the same way as in the proof of the lemma on semicells in the last section one can use (3.35) and (3.36) in order to find the linear transformations which show that D forms a cell in $\bar{G}$ and that the cells formed by D in G and $\bar{G}$ are equivalent.

Proof of the extension theorem: We first show that a cell consistent and truncation consistent solution function L which agrees with L_o on $\mathcal{G}_o$ must be the extention of L_o to $\mathcal{G}$.

Suppose that the agent subsets $C_1,\ldots,C_k$ form the elementary cells $G^1,\ldots,G^k$ of the standard form $G \in \mathcal{G}$. Let $\tilde{G}$ be the truncation with respect to one of the elementary cells, say G^1, and to L. The decomposition lemma permits us to conclude that $C_2,\ldots,C_k$ form cells $\tilde{G}^2,\ldots,\tilde{G}^k$. Moreover, $\tilde{G}^j$ and G^j are equivalent for $j = 2,\ldots,k$. Since L_o is invariant with respect to positive linear transformations we have $L(G^j) = L(\tilde{G}^j)$ for $j = 2,\ldots,k$. It is now clear that we can obtain properties (B) and (C) of the extension by repeated application of cell consistency and truncation consistency. It does not matter in which order the elementary cells are removed one after the other.

It remains to show that the extension of L of L_o has the properties of cell consistency and truncation consistency. This will be done by induction on the number m of agents in G.

Both properties trivially hold for games in $\mathcal{G}$ with only one agent. Assume that they hold for games with at most m-1 agents. Consider a decomposable game $G \in \mathcal{G}$ with m agents. Let C be a subset of agents which forms a cell G^C of G. Let $\bar{G}$ be the truncation of G with respect to G^C and L.

The elementary cells of G^C are elementary cells of G, too. Suppose that G^C is fully decomposable or indecomposable. In this case the agreement of L(G) with $L(G^C)$ and $L(\bar{G})$ is an immediate consequence of (B) and (C).

Now suppose that G^C is partially decomposable. Then G is partially decomposable, too. Let $\hat{G}$ be the main truncation of G and let G^D be the main truncation of G^C where D stands for the set of agents in G^D. Obviously, the agents in D are agents of $\hat{G}$, too.

It follows by (B) and (C) that L(G) agrees with $L(G^C)$ and $L(\hat{G})$ as far as the agents in the elementary cells of G are concerned. Therefore G^D is a cell of $\hat{G}$.

Since C has less than m agents, truncation consistency can be applied to G^C. Therefore $L(G^C)$ agrees with $L(G^D)$ for the agents in D. Consequently, the truncation of $\hat{G}$ with respect to G^D is $\bar{G}$. Since $\hat{G}$ has fewer than m agents we can rely on cell consistency and truncation consistency in order to conclude that $L(\hat{G})$ is the composition of $L(\bar{G})$ and $L(G^D)$. Since by definition $L(G)$ agrees with $L(\hat{G})$ for the agents in $\hat{G}$, it follows that $L(G)$ is the composition of $L(G^C)$ and $L(\bar{G})$.

<u>Comment</u>: The extension theorem shows that cell consistency and truncation consistency are powerful properties which reduce the task of defining a solution concept to the task of defining a solution concept for indecomposable games.

Cell consistency and truncation consistency require that all considerations which may influence the selection of equilibrium points are applied strictly locally, i.e. only to those indecomposable games which appear in the process of computing the solution with the help of (A), (B) and (C) on the basis of a solution concept for indecomposable games. These indecomposable games shall be called the <u>bricks</u> of the original game.

<u>Local and global payoff efficiency</u>: Payoff efficiency is an example of a selection criterion which cannot be applied to the game as a whole but only locally to its bricks.

Figure 3.21 shows an extensive game Γ which may serve as an example. The game has two subgames Γ_ℓ after agent 11's choice ℓ and Γ_r after agent 11's choice r. Let G_ε be an ε-perturbed standard form of Γ and let G_ε^ℓ be the cells G_ε which correspond to Γ_ℓ and Γ_r, respectively. For every agent let ℓ_ε and r_ε be his ε-extreme local strategy corresponding to ℓ and r, respectively. Note that player 1 does not belong to G_ε^ℓ and G_ε^r. Each of both cells has only one payoff efficient equilibrium point. In both cases the pay-

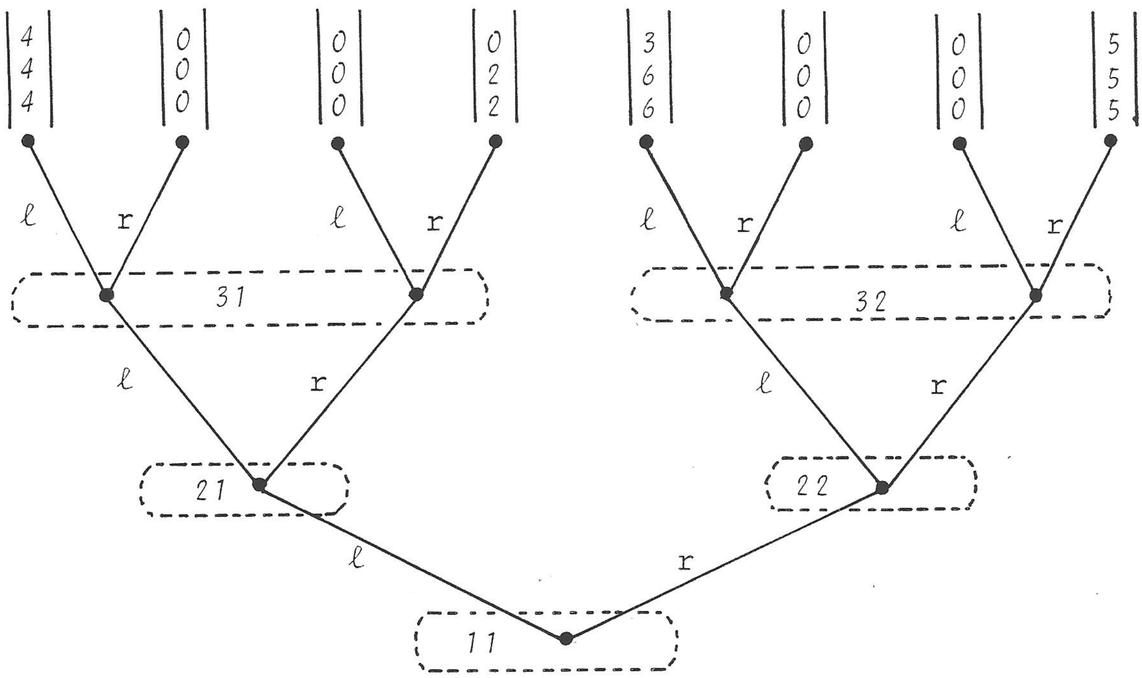


Figure 3.21: Example of a conflict between local and global payoff efficiency. For the conventions of graphical representation see figure 3.17.

off efficient equilibrium point prescribes ℓ_ε to both agents. This shows that local payoff efficiency, i.e. the application of the principle to bricks of the game rather than to the whole game together with cell consistency and truncation consistency result in the selection of that equilibrium point of G_ε which prescribes ℓ_ε to each of the agents. However, this equilibrium point is not globally payoff efficient in G_ε. The equilibrium point which prescribes r_ε to all agents yields better payoffs for all players.

It is in the interest of all players to play r_ε everywhere rather than ℓ_ε everywhere. Unfortunately, this is true only in the beginning of the game. In the subgame Γ_r the interests of player 1 do not count any more. Agents 22 and 32 must be expected to choose ℓ_ε rather than r_ε.

The degenerate 2x2-game of figure 3.16: The game of figure 3.16 has only one payoff efficient equilibrium point, namely $U = (U_1,U_2)$. Nevertheless, our theory does not select this equilibrium point as the limit solution. This is a consequence of cell consistency together with invariance with respect to isomorphisms. Obviously {1} and {2} form cells in all ε-perturbations of this game. These cells are one-person games with only one agent who gets the same payoff for both of his choices. In view of invariance with respect to isomorphisms the solution assigns probability 1/2 to each of both choices. Cell consistency leads to the result that the limit solution of the game of figure 3.16 is that equilibrium point where both players choose each of their pure strategies with probability 1/2. In spite of the fact that this equilibrium point fails to be payoff efficient, the result is not unreasonable. Each player is interested in his own payoff only. He does not care for the other player's payoff and has no reason to prefer one of his pure strategies over the other for any fixed expectation on the other player's behavior.

Decomposition properties of the limit solution: One may ask the question whether cell consistency and truncation consistency induce similar properties on the limit solution. After all, the most important reason for the introduction of the cell notion was the idea that the limit solution should be *subgame consistent* in the sense that as far as the agents of a subgame are concerned, the solution of the subgame agrees with the solution of the whole game. Of course, if we speak of the limit solution of an extensive game we really mean the limit solution of its standard form. The above definition of subgame consistency should be understood in this way. We shall use the symbol $\underset{\rightarrow}{L}(\Gamma)$ for the limit solution of an extensive game.

Cell consistency of a solution function L implies subgame consistency of the limit solution function $\underset{\rightarrow}{L}$ of L. This follows by the lemma on semicells.

One might expect that an analogous result can be obtained with respect to truncation consistency. However, this is not the case. In order to show this we must first define the *truncation* $\bar{\Gamma}$ of an extensive game Γ with respect to a subgame Γ' of Γ and $\underset{\rightarrow}{L}$. One obtains this extensive game $\bar{\Gamma}$ if in Γ the subgame Γ' is replaced by its solution payoff $H'(\underset{\rightarrow}{L}(\Gamma')$. This means that the starting point of Γ' becomes an endpoint with payoffs according to $H'(\underset{\rightarrow}{L}(\Gamma'))$ and that nothing is changed outside the subgame Γ'.

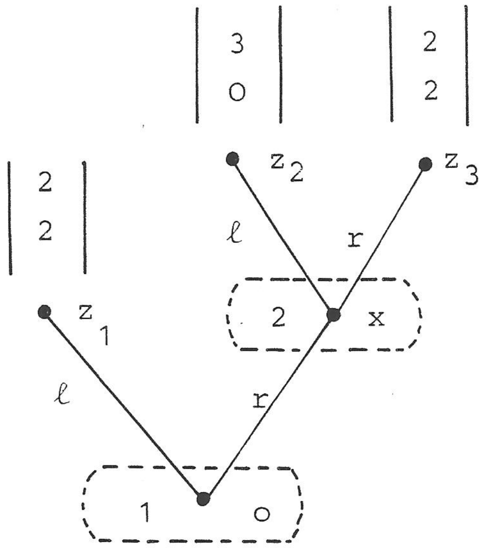
We say that $\underset{\rightarrow}{L}$ is *subgame truncation consistent* if the following condition is satisfied: Let Γ be an extensive form and Γ' be a subgame of Γ such that $\underset{\rightarrow}{L}$ is defined for Γ, Γ' and the truncation $\bar{\Gamma}$ of Γ with respect to Γ' and $\underset{\rightarrow}{L}$. Then $\underset{\rightarrow}{L}(\Gamma)$ and $\underset{\rightarrow}{L}(\bar{\Gamma})$ prescribe the same local strategies to the agents in $\bar{\Gamma}$.

Before we go on to show why a limit solution function cannot be expected to have the property of subgame truncation consisteny, we want to clarify a small point concerning the standard form of an extensive game. Our definition of the standard form does not permit players without agents. However, such players are possible in extensive games. In the transition to the standard form we simply remove such players and their payoffs. This has the consequence that the standard form of a subgame or a truncation with respect to a subgame may have fewer players than the standard form of the whole game. This must be kept in mind in the interpretation of the properties of subgame consistency and subgame truncation consistency.

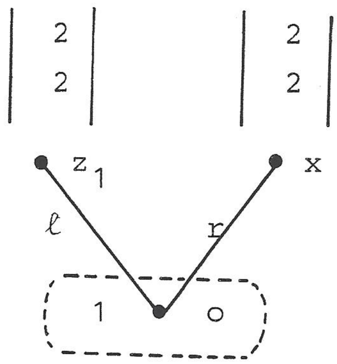
Consider the extensive game Γ of figure 3.22. This game has a subgame Γ_x at node x. Since Γ_x has only one equilibrium point, namely player 2's choice of r, this equilibrium point is the limit solution $\underset{\rightarrow}{L}(\Gamma_x)$ of the subgame. Figure 3.23 shows the truncation $\bar{\Gamma}$ of Γ with respect to Γ_x and $\underset{\rightarrow}{L}$. The standard form of $\bar{\Gamma}$ has a symmetry which maps one choice to the other. The same is true for all its ε-perturbations. If L satisfies the requirement of invariance with respect to isomorphisms then $\underset{\rightarrow}{L}(\bar{\Gamma})$ must assign equal probabilities to ℓ and r in figure 3.23.

Every ε-perturbation of the standard form of the game Γ of figure 3.22 has only one equilibrium point which prescribes the ε-extreme choices r_ε corresponding to r to both agents. Therefore the limit solution $\underset{\rightarrow}{L}(\Gamma)$ assigns r to both agents. We have obtained the following result.

<u>Result:</u> Let L be a solution function for the interior substructure class $\mathfrak{J}(\mathcal{G})$ of a class of standard forms $\mathcal{G}$, such



<u>Figure 3.22</u>: An extensive 2-person game.



<u>Figure 3.23</u>: "Truncation" of the game of figure 3.21.

that $\mathcal{G}$ contains the standard forms of the extensive games of figure 3.22 and 3.23. If L is invariant with respect to isomorphisms then the limit solution $\underset{\rightarrow}{L}$ of L is not subgame truncation consistent.

<u>Comment</u>: Since invariance with respect to isomorphism is an indispensible requirement, we cannot expect subgame

truncation consistency of the limit solution function. The nature of the example shows why we need not be disturbed by this lack of subgame truncation consistency. In the only equilibrium point of the ε-perturbed standard form of figure 3.22 player 1 receives a payoff of $2+2\varepsilon-\varepsilon^2$. A deviation to his ε-extreme strategy ℓ_ε corresponding to ℓ would yield only $2+\varepsilon^2$. The loss of $2\varepsilon(1-\varepsilon)$ approaches the limit 0 for $\varepsilon \to 0$. The fact that the choice of r_ε by both agents is a strong equilibrium point is lost in the transition to the limit. One can say that important information on the game structure may be suppressed if the truncation with respect to a subgame and $\underset{\to}{L}$ is formed. Truncations formed in ε-perturbations with respect to the corresponding cells preserve this information. Clearly, this is an advantage rather than a disadvantage.

The result obtained above remains correct if invariance with respect to isomorphisms is required of $\underset{\to}{L}$ rather than L. This shows that any solution concept addressed directly to unperturbed standard forms is bound to run into difficulties. Our roundabout approach via the ε-perturbations may seem to be cumbersome, but it recommends itself by more than one reason.

11. Sequential agent splitting

Figure 3.24 shows what sequential agent splitting means in the extensive form. An agent hk of a player h who has to choose between α, β and γ is split into two agents hm and hk such that hm has to select either γ or -γ and then in case of -γ agent hk has to choose between α and β. The symbol -γ is used as an abbreviation for "a or b". One has to imagine the graphical representation of an extensive form where the upper part of figure 3.24 is taken out and is replaced by the lower part of figure 3.24. Of course, we have to assume that originally there was no agent hm in the game.

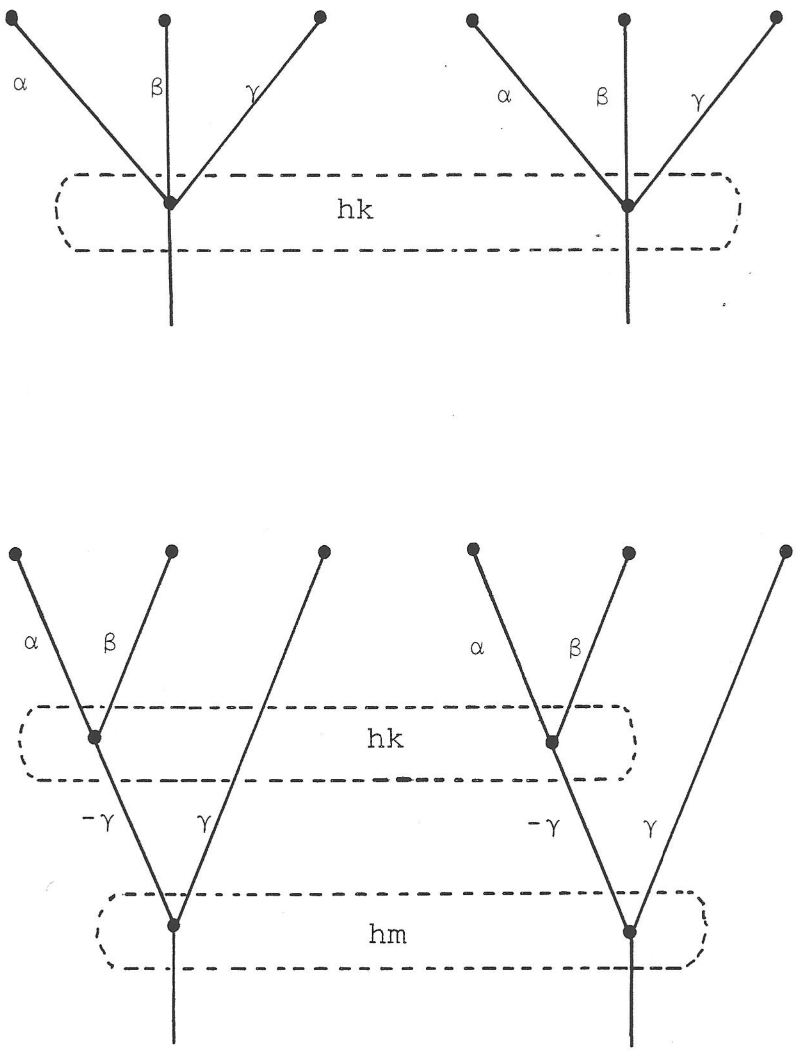


Figure 3.24: An example of sequential agent splitting in the extensive form.

At least at first glance it is hard to imagine why sequential agent splitting should in any way change the strategic situation. Nevertheless, as we shall see, one cannot avoid the conclusion that sequential agent splitting does have a considerable influence on risk comparisons between equilibrium points in some games.

In order to connect sequential agent splitting to our theory of equilibrium selection we have to introduce a formal definition in the framework of the standard form.

<u>Sequential agent splitting in the standard form</u>: Let G' = (Φ, H) with

(3.38) $$\Phi = \underset{i\in N}{X} \Phi_i = \underset{ij\in M}{X} \Phi_{ij}$$

be a game in standard form. Let $hk \in M_h$ be one of the agents of a player $h\in N$ and let $\gamma \in \Phi_{hk}$ be one of agent hk's choices. Moreover, let m be a positive integer with $hm \notin M_h$. We construct a standard form G' = (Φ', H') with

(3.39) $$\Phi' = \underset{i\in N}{X} \Phi'_i = \underset{ij\in M'}{X} \Phi'_{ij}$$

(3.40) $$M' = M \cup \{hm\}$$

(3.41) $$\Phi'_{ij} = \Phi_{ij} \quad \text{for } ij \in M \setminus \{hk\}$$

(3.42) $$\Phi'_{hk} = \Phi_{hk} \setminus \{\gamma\}$$

(3.43) $$\Phi'_{hm} = \{\gamma, -\gamma\}$$

(In (3.43) agent hm's alternative of not choosing γ is symbolized by $-\gamma$). We say that the behavior strategy combination $b' = (b'_{ij})_{M'}$ corresponds to $b = (b_{ij})_M$ and write $b' \to b$ if we have

(3.44) $b'_{ij} = b_{ij}$ for $ij \in M \setminus \{hk\}$

(3.45) $b'_{hm}(\gamma) = b_{hk}(\gamma)$

(3.46) $b'_{hm}(-\gamma) b'_{hk}(\varphi_{hk}) = b_{hk}(\varphi_{hk})$ for $\varphi_{hk} \in \Phi'_{hk}$

The payoffs of G' are defined as follows:

(3.47) $H'(\varphi') = H(\varphi)$ with $\varphi' \to \varphi$

The game $G' = (\Phi, H')$ is called the game which _results from_ $G = (\Phi, H)$ _by splitting off an agent_ hm _for_ hk's _choice_ γ.

Remark: The following equation is an immediate consequence of (3.44) to (3.47):

(3.48) $H'(b') = H(b)$ for $b' \to b$

Interpretation: Even if the formal definition may seem to be somewhat complicated it can be seen easily that is the correct translation of the idea of sequential agent splitting into the language of the standard form.

It is important to point that the operations of ε-perturbation and sequential agent splitting are not interchangeable. It matters what is done first. If in the example of figure 3.24 we first form an ε-perturbation and then split off agent km then the task of the new agents hm and hk can be described as the choice between the ε-extreme local strategies α_ε, β_ε and γ_ε corresponding to α, β and γ . One may think of the new agents as decision makers who do not make their own mistakes but simply administrate the mistakes of the old agent hk. Each of the choices α, β and γ will be taken with minimum probability ε.

The picture is different if agent hm is split off first and then the ε-perturbation is formed. If the intentional choices are $-\gamma$ and α then β will be chosen with probability $\varepsilon(1-\varepsilon)$ and γ with probability ε; the situation is analogous if the intentional choices are $-\gamma$ and β. These probabilities are almost the same as in the reversed case considered above. However, if agent hm intends to choose γ then the probabilities of α and β depend on the intentions of agent hk; if he intends to choose β then α is chosen with probability ε^2 and β with probability $\varepsilon(1-\varepsilon)$. The situation is analogous if he intends to choose β.

Obviously, one has to expect that for specific values of ε sequential agent splitting before and after ε-perturbation leads to essentially different solutions. This suggests that one should not be concerned about the solutions of ε-perturbations as such but only about limit solutions. At least at first glance, it seems to be reasonable to require that for unperturbed standard forms with perfect recall sequential agent splitting does not produce an essential change of the limit solution.

Invariance with respect to sequential agent splitting: Let $\mathcal{G}$ be a class of games with perfect recall and let L be a solution function for the interior substructure class $\mathcal{J}(\mathcal{G})$ of $\mathcal{G}$. We say that L is invariant with respect to sequential agent splitting if the following condition is always satisfied: Let G and G' be two games in $\mathcal{G}$ such that G' results from G by splitting off an agent hm for a choice γ of an agent hk. If $\underset{\rightarrow}{L}(G)$ and $\underset{\rightarrow}{L}(G')$ exist, then we have $\underset{\rightarrow}{L}(G') \rightarrow \underset{\rightarrow}{L}(G)$.

Interpretation: The requirement postulates that the limit solution of G' corresponds to the limit solution of G provided that both limit solutions exist. In the case that $\underset{\rightarrow}{L}(G)$ prescribes γ with probability 1, agent hk's behavior in the limit solution of G' is not restricted by the requirement since it does not matter what he does if agent hm selects γ.

We shall prove an impossibility theorem which forces us to reject the requirement of invariance with respect to sequential agent splitting. The proof will make use of the specific 3x3-game shown in figure 3.25. Therefore, one needs the technical presupposition that this game has a limit solution. In a reasonable solution theory non-existence of a limit solution should not be a problem for games as simple as that of figure 3.25.

Impossibility theorem: Let $\mathcal{G}$ be a class of standard form games which contains all 2-person games with at most two agents for each player and at most three choices for each agent. Let L be a solution function for the interior substructure class $\mathcal{J}(\mathcal{G})$ of $\mathcal{G}$ which satisfies the requirement of cell consistency and truncation consistency and which for 2x2-games with two strong equilibrium points either agrees with the proposed solution function or with the pure risk dominance solution function (see section 8).

	α	β	γ
α	7 5	0 2	0 2
β	0 0	2 9	0 0
γ	0 0	0 2	8 4

Figure 3.25: A special 3x3-game.

Moreover, let L be such that the limit solution $\underset{\rightarrow}{L}$ of L exists for the game of figure 3.25. Then L does not satisfy the requirement of invariance with respect to sequential agent splitting.

Proof: It will be shown that two different ways of sequential agent splitting in the game of figure 3.25 lead to a contradiction. Figure 3.26 summarizes the argument of the proof . At the top of figure 3.26 we find the game G of figure 3.25. Two arrows point to two standard forms G^1 and G^2 with two agents for each player. Both games are obtainable from G by repeated sequential agent splitting.

G has only two agents 11 and 21. The game G^1 results from G by first splitting of an agent 12 for agent 11's choice β and then splitting off an agent 22 for agent 21's choice β.

In G^1 the agents 11 and 21 form a semicell $\hat{G}^1$. The choice between α and γ does not matter unless the other agents 12 and 22 both choose -β. In every ε-perturbation G^1_ε of G^1 this semicell $\hat{G}^1$ corresponds to a cell $\hat{G}^1_\varepsilon$.

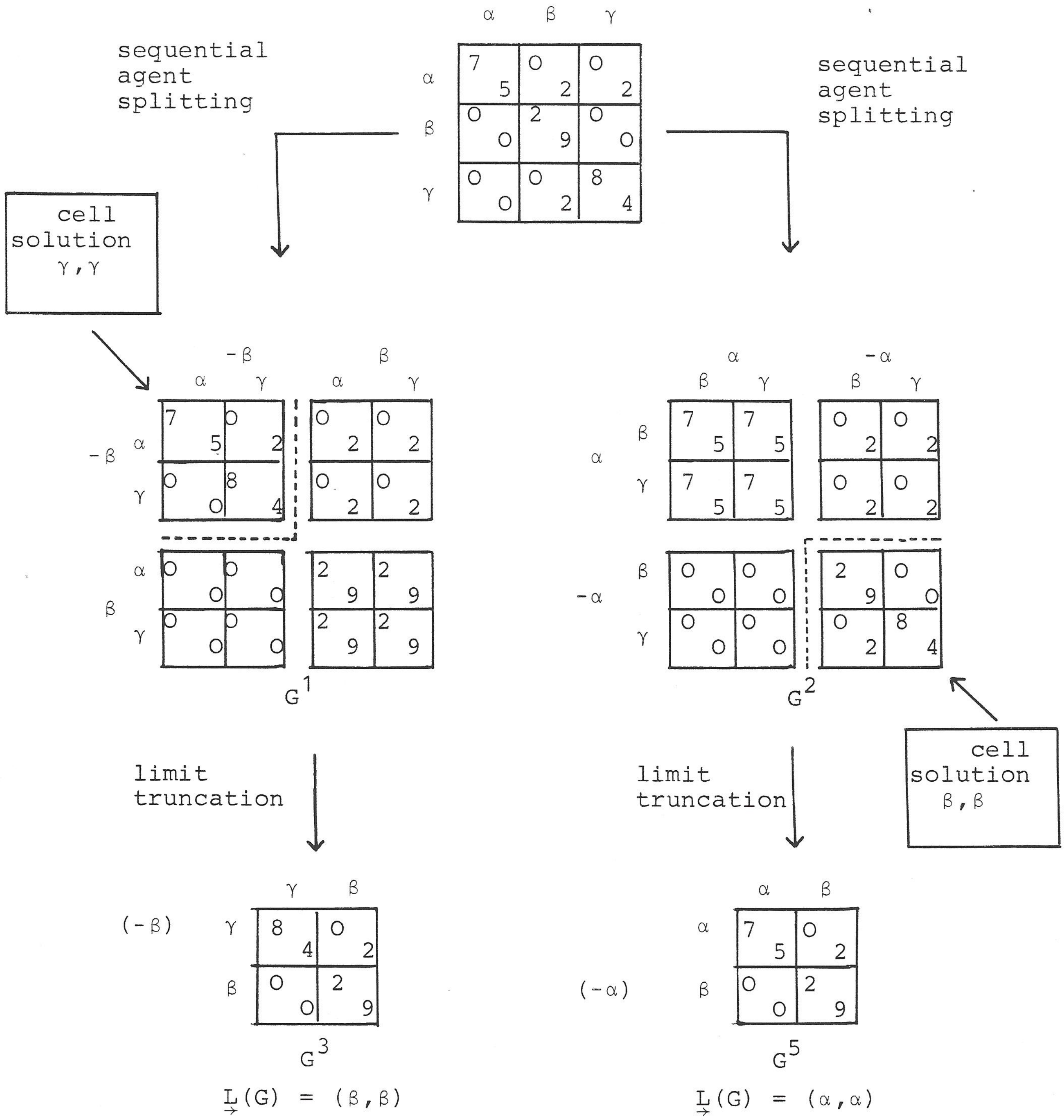


Figure 3.26: Proof of the impossibility theorem. Player 1's payoffs are shown above and player 2's payoffs are shown below. Agents 11 and 21 choose between α and γ in G^1 and between β and γ in G^2. Agents 12 and 22 choose between β and −β in G' and between α and −α in G^2.

For sufficiently small ε the solution $L(\hat{G}^1_\varepsilon)$ of the cell is (γ,γ). This follows by a comparison of the Nash-products for (α,α) and (γ,γ) which for small ε are near to 21 and 32, respectively. Let $\bar{G}^1_2$ be the truncation of G^1_ε with respect to G^1_ε and L. For $\varepsilon \to 0$ the payoffs of $\bar{G}^1_\varepsilon$ approach those of the game G^3 shown below G^1 in figure 3.26. In this sense we may say that G^3 is the "limit truncation" of G^1 with respect to $\hat{G}^1_\varepsilon$ and L. A comparison of Nash-products in G^3 shows that for sufficiently small ε we must have $L(\bar{G}^1_\varepsilon)=(\beta,\beta)$.

Cell consistency and truncation consistenvy of L lead to the conclusion that $L(G^1_\varepsilon)$ prescribes the choices $\beta,\gamma,\beta,\gamma$ to the agents 11, 12, 21, 22 in that order. Obviously, the same is true for the limit solution $\underset{\rightarrow}{L}(G^1)$. Invariance with respect to sequential agent splitting requires that the limit solution of G^1 corresponds to that of G. Therefore, we must have $\underset{\rightarrow}{L}(G) = (\beta,\beta)$.

A similar argument is indicated by the right hand side of figure 3.26. The game G^2 results from G by splitting off agents for the choice α. The cell formed by 12 and 22 in the ε-perturbation G^1_ε of G^1 has the solution (β,β). The "limit truncation" G^5 shows that we must have $\underset{\rightarrow}{L}(G) = (\alpha,\alpha)$.

Remark: It is interesting to ask the question whether the result could be avoided by a more restrictive definition of a cell which would narrow down the applicability of the cell and truncation consistency requirement. In any case, a more restrictive definition of a cell would have to cover those substructures which correspond to subgames in the ε-perturbed standard form. Therefore it is worth pointing out that G^1 and G^2 can be interpreted as the standard forms of two extensive forms Γ^1 and Γ^2 both of whom have subgames corresponding to the respective semicells formed by agents 11 and 12. These extensive forms are shown in figures 3.27 and 3.28. Consequently, the impossibility result cannot be avoided by a more restrictive definition of cells.

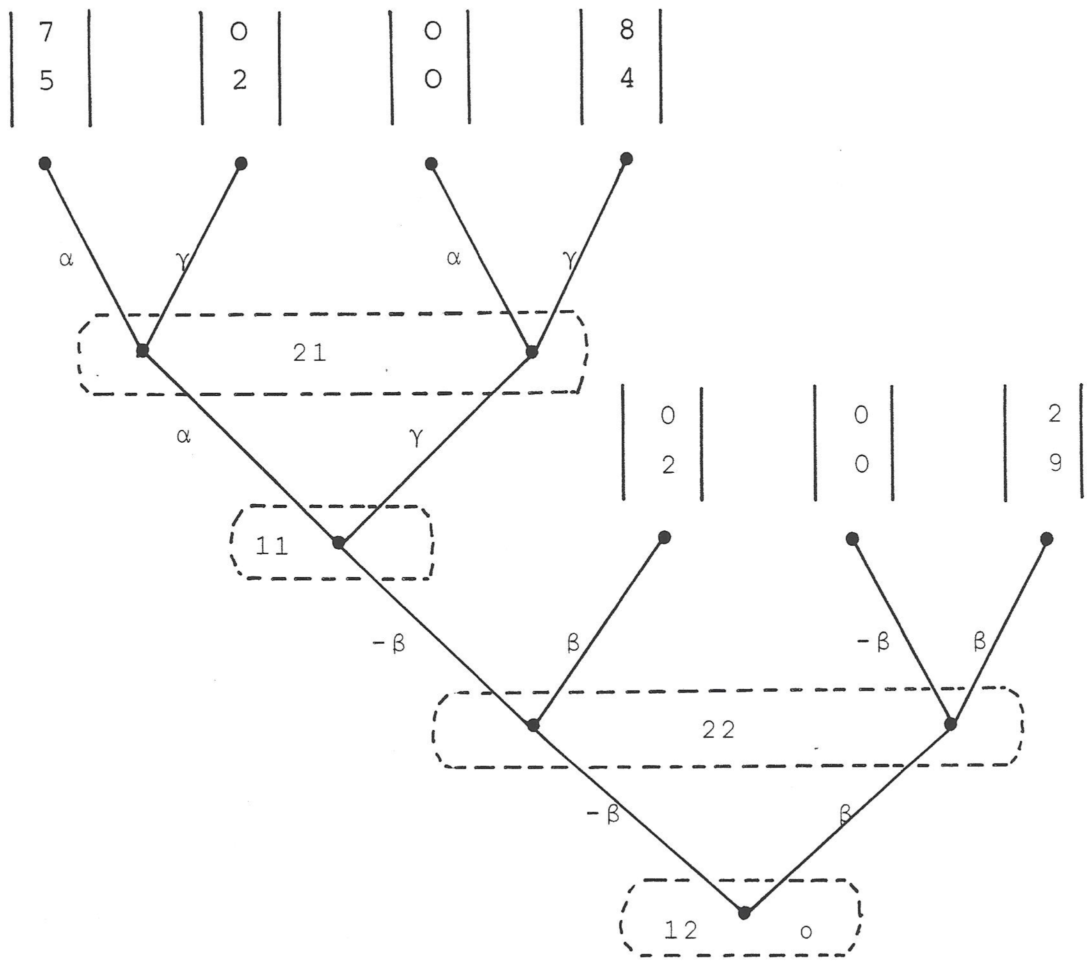


Figure 3.27: The extensive game Γ^1 whose standard form agrees with G^1 in figure 3.26.

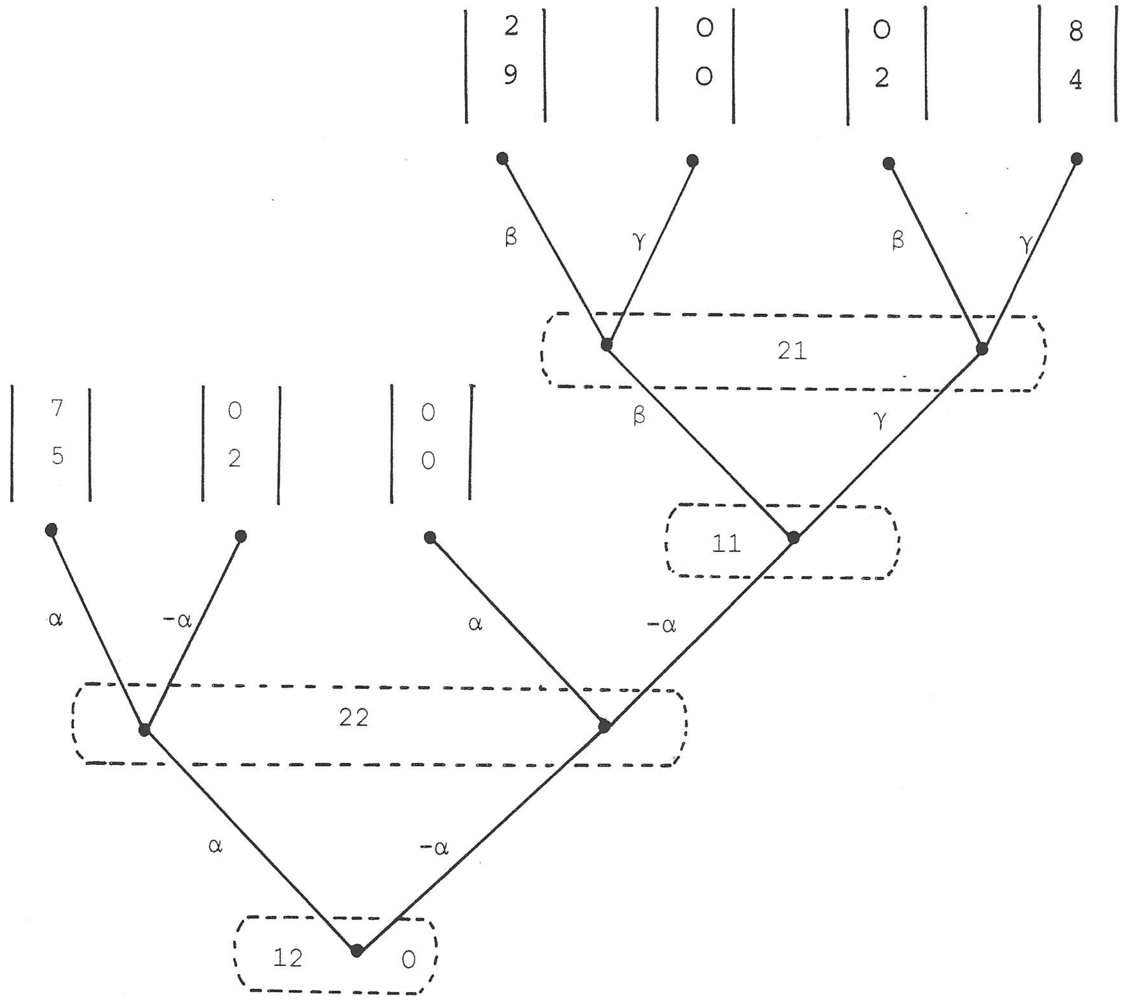


Figure 3.28: The extensive game Γ^2 whose standard form agrees with G^2 in figure 3.26.

Interpretation: We must draw the conclusion that it is by no means irrelevant whether a choice between α, β and γ has a sequential structure or not. Games where a simultaneous choice has to be made can be different from others where the decision is split into two steps involving choices between "α or β" and γ in the first step and between α and β in the second step. If we do not want to give up the idea of a solution function altogether we must abolish one of the properties which lead to the impossibility result. Among those properties invariance with respect to sequential agent splitting seems to be the least compelling one. Upon reflection it does not appear to be an unreasonable idea that risk comparisons between three alternatives may be changed by the imposition of a sequential structure.

After all, one must think of the fact that after a decision between "α or β" and γ has been made in favor of " α or β", alternative γ has become irrelevant and the risk comparisons may look quite different from those which would arise in a simultaneous choice situation. Different sequential orders may require different ways of looking at the situation. Even if it is not easy to understand why this should be so, it is reasonable to suppose that the basic reason for the impossibility result must be searched in this direction.

The proof of the impossibility result makes use of the fact that both ways of sequential agent splitting in figure 3.26 reduce the risk dominance comparisons to comparisons in 2x2-games which result from G by removing either α, β or γ from the pure strategy sets of both players. The three comparisons which can be made in this way result in an intransitive pattern: (α,α) dominates (β,β) and (β,β) dominates (γ,γ), but (γ,γ) dominates (α,α). Moreover, each of both ways of sequential agent splitting removes one of the three comparisons, namely that between (α,α) and (β,β) in the case of G^1 and that between (γ,γ) and (α,α) in the case of G^2. In this way we can see already here that the impossibility result is connected to intransitivities of risk dominance. We shall return to the phenomenon in chapter 5, section 3, after the introduction of our general definition of risk dominance.

12. Decomposition and reduction

As we have shown in section 10 cell consistency and truncation consistency reduce the task of finding solutions for decomposable games to the simpler one of finding solutions for indecomposable games. An indecomposable game may be further simplified by the elimination of superfluous strategic possibilities. We shall look at the elimination of inferior choices, duplicate classes and semiduplicate classes. These concepts have already been mentioned in the introduction of the chapter.
The partial invariance properties with respect to the elimination of such superfluous strategic possibilities have been mentioned there, too, as well as the difficulties confronting stronger requirements of a similar nature.

Our solution concepts permit the simultaneous elimination of all inferior choices in indecomposable games, the simultaneous elimination of all duplicate classes in indecomposable games without inferior choices and the simultaneous elimination of all semiduplicate classes in indecomposable games without inferior choices and duplicate classes. The solution is not changed by these operations. This is expressed by the three partial invariance properties.

In section 10 we have defined extensions of solution functions for indecomposable games to decomposable games. In a similar fashion we shall define extensions of solution functions for irreducible games to general games. Irreducible games are indecomposable games without inferior choices, duplicate classes and semiduplicate classes.

Unfortunately, we cannot subdivide the extension of a solution function for irreducible games to general games into two steps, one from irreducible games to decomposable games and another from decomposable games to general games. This is due to the fact that the application of one of the three elimination operations to an indecomposable game may produce a decomposable game.

If we speak of reduction as opposed to decomposition we mean the application of the three elimination operations. Decomposition into a cell and the truncation with respect to this cell substitutes the analysis of two games for the analysis of one game. Reduction simply reduces the size of the game to be analysed. The procedure of decomposition and reduction described by the flow chart of figure 3.29 shows how both types of operations interact in the determination of the solution of a general game.

<u>Stability sets</u>: Let $G = (\Phi, H)$ with

(3.49) $$\Phi = \underset{i \in N}{X} \Phi_i = \underset{ij \in M}{X} \Phi_{ij}$$

be a game in standard form und let $\varphi_{ij} \in \Phi_{ij}$ be a choice of an agent $ij \in M$. The set of all hybrid combinations $b_{i \setminus ij}q_{.i}$ such that ψ_{ij} is a local best reply to $b_{i \setminus ij}q_{.i}$ is denoted by $S_{ij}(b_{i \setminus ij}q_{.i})$. This set is called the <u>stability set</u> of ψ_{ij}. The stability set of ψ_{ij} may be described as the set of all hybrid combinations $b_{i \setminus ij}q_{.i}$ such that ψ_{ij} is in the local best reply set $A_{ij}(b_{i \setminus ij}q_{.i})$. (See chapter 2, section 6).

<u>Elimination of inferior choices</u>: As above let $G = (\Phi,H)$ be a game in standard form with (3.49) and let $ij \in M$ be an agent in G. A choice $\varphi_{ij} \in \Phi_{ij}$ is called <u>inferior to a choice</u> $\varphi_{ij} \in \Phi_{ij}$ if we have

$$(3.50) \qquad S_{ij}(\varphi_{ij}) \subsetneq S_{ij}(\psi_{ij})$$

A choice $\varphi_{ij} \in \Phi_{ij}$ is called <u>inferior</u> if it is inferior to at least one other choice $\psi_{ij} \in \Phi_{ij}$. Choices which are not inferior to any other choice are called <u>non-inferior</u>.

For every $ij \in M$ let Ψ_{ij} be the set of all non-inferior choices of agent ij. Obviously ψ_{ij} cannot be empty. Define

$$(3.51) \qquad \Psi = \underset{i \in N}{X} \Psi_i = \underset{ij \in M}{X} \Psi_{ij}$$

The game $G' = (\Psi,H')$ which results from $G = (\Phi,H)$ by narrowing the choice sets in Φ to Ψ (see chapter 2, section 5) is called the game which <u>results from</u> G <u>by elimination of inferior choices</u>.

<u>Comment</u>: Consider a choice φ_{ij} which is <u>weakly dominated</u>

by a choice ψ_{ij} in the sense that for every $\varphi_{-ij}\in\Phi_{-ij}$ player i's payoff $H_i(\varphi_{ij}\varphi_{-ij})$ is not greater than $H_i(\psi_{ij}\varphi_{-ij})$ and smaller than $H_i(\psi_{ij}\varphi_{-ij})$ for at least one $\varphi_{-ij}\in\Phi_{-ij}$. A weakly dominated choice is always inferior, but an inferior choice need not be weakly dominated.

We feel that in the framework of our theory the notion of an inferior choice is more relevant than that of a weakly dominated choice. A Bayesian player will always play a best reply to his expectations on the other players, i.e. to a joint mixture $q_{\cdot i}$. The games to which our solution function is applied are interior substructures of standard forms with perfect recall. Such games have the local best reply property (see chapter 2, section 6). This has the consequence that a behavior strategy b_i which is a best reply to $q_{\cdot i}$ assigns positive probabilities only to such choices which are local best replies to $b_i q_{\cdot i}$. Therefore, a choice is superfluous if it is less suitable as a local best reply than another one. This idea is expressed by condition (3.50) in the definition of an inferior choice.

<u>Duplicates and Semiduplicates</u>: Two choices φ_{ij} and ψ_{ij} of an agent ij in a standard from G = (Φ,H) are called <u>duplicates</u> if the condition

(3.52) $$H(\varphi_{ij}\varphi_{-ij}) = H(\psi_{ij}\varphi_{-ij})$$

is satisfied for every $\varphi_{-ij}\in\Phi_{-ij}$. They are called <u>semiduplicates</u> if instead of (3.52) the weaker condition

(3.53) $$H_i(\varphi_{ij}\varphi_{-ij}) = H_i(\psi_{ij}\varphi_{-ij})$$

is satisfied for every $\varphi_{-ij} \in \Phi_{-ij}$. Obviously, both the duplicate and semiduplicate relations are equivalence relations. Those equivalence classes with respect to these relations which contain more than one element are called <u>duplicate classes</u> and <u>semiduplicate classes</u>, respectively.

<u>Remarks</u>: If φ_{-ij} is replaced by an arbitrary hydrid combination $b_{i \smallsetminus ij} q_{.i}$. The same is true for (3.53) if φ_{ij} and ψ_{ij} are semiduplicates.

<u>Elimination of duplicate classes and semiduplicate classes</u>.
Let R_{ij} be that subset of agent ij's local strategy set B_{ij} which contains all choices without duplicates and the centroids $c(\Lambda_{ij})$ all of all duplicate classes Λ_{ij} in Φ_{ij}. It is clear that R_{ij} is an admissable new choice set of agent ij (see chapter 2, section 5).
Define

$$(3.54) \quad R = \underset{i \in N}{X} R_i = \underset{ij \in N}{X} R_{ij}$$

The game G' = (R,H') which results from G = (Φ,H) by narrowing the choice sets in Φ to R is called the game which <u>results from</u> G <u>by elimiation of duplicate classes</u>.

The game which <u>results from</u> G <u>by elimination of semiduplicate classes</u> is defined analogously. Instead of the duplicate classes the semiduplicate classes are removed from Φ_{ij} and replaced by their centroids.

<u>Comment</u>: Unlike the definition of inferior choices which is based on local best reply properties, the definition

of duplicates and semiduplicates is in terms of payoffs. Two choices φ_{ij} and ψ_{ij} with the same stability set need not be semiduplicate, since payoffs may differ where both choices fail to be local best replies. One may ask why we approach the conceptual problems behind inferior choices on the one hand and duplicates and semiduplicates on the other hand in a different spirit. We feel that this is justified in view of important differences between the elimination of inferior choices on the one hand and the elimination of duplicate and semiduplicate classes on the other hand. The arguments in favor of the removal of inferior choices are considerations of comparative strategic suitability. The elimination operations for duplicate and semiduplicate classes do not really remove strategic alternatives but simply stipulate that choices which are in some sense indistinguishable are used with equal probabilities. Structural indistinguishability is similar to symmetry. Payoff relationships underly the definition of isomorphisms. This suggests payoff oriented notions of duplicates and semiduplicates.

<u>Partial invariance properties</u>:At least at first glance it seems to be reasonable to require that the solution of a game is not changed neither by the elimination of inferior choices nor by the elimination of duplicate classes nor by the elimination of semiduplicate classes. Unfortunately, such requirements cannot be imposed in full generality.

In order in which the three operations of elimination are applied successively does sometimes influence the final result. After the elimination of inferior choices some semiduplicates may become duplicates and after the elimination of semiduplicate classes, choices may be inferior which were not inferior before. Moreover, a choice which is inferior in a decomposable game may be a duplicate or a semiduplicate in a truncation of this game.

In order to avoid conflicts with cell consistency and truncation consistency we adopt the point of view that properties related to the three operations should be strictly local in the sense that they do not apply directly to a decomposable game but rather to the indecomposable games which arise in the computation of its solution. This means that cell decomposition is given priority over the three elimination operations.

Among the three elimination operations, elimination of inferior choices is clearly the most important one. We shall require that elimination of inferior choices does not change the solution of any indecomposable games. Among the remaining two operations it seems to be reasonable to give precedence to the elimination of duplicate classes since duplicates are more closely related to each other than semiduplicates. These considerations lead us to the following partial invariance requirements.

<u>Partial invariance with respect to inferior choices</u>: A solution function L for a complete class $\mathcal{G} \subseteq \mathcal{J}(\mathcal{R})$ is called <u>partially invariant with respect to inferior choices</u> if L(G) = L(G') holds for every indecomposable game G $\in \mathcal{G}$ and the game G' which results from G by elimination of inferior choices.

Partial invariance with respect to duplicates: A solution function L for a complete class $\mathcal{G} \subseteq \mathcal{S}(\mathcal{R})$ is called *partially invariant with respect to duplicates* if L(G) = L(G') holds for every indecomposable game G $\in \mathcal{G}$ without inferior choices and for the game G' which results from G by elimination of duplicate classes.

Partial invariance with respect to semiduplicates: A solution function L for a complete class $\mathcal{G} \subseteq \mathcal{S}(\mathcal{R})$ is called *partially invariant with respect to semiduplicates* if L(G) = L(G') holds for every indecomposable game G $\in \mathcal{G}$ without inferior choices and with duplicate classes and for the game G' which results from G by elimination of semiduplicate classes.

Irreducible games: A game G in standard form is called *irreducible* if it is indecomposable and neither has any inferior choices nor any duplicate classes nor any semiduplicate classes. Other games are called *reducible*.

Extension: Let $\mathcal{G}'$ be the subclass of all irreducible games in a complete class $\mathcal{G} \subseteq \mathcal{S}(\mathcal{R})$. Morevoer, let L' be a solution function for $\mathcal{G}'$. On the basis of L' we shall construct a solution function L for $\mathcal{G}$ which will be called the *extension of* L' *to* $\mathcal{G}$. The extension is defined recursively by the following properties (A) to (F):

(A) For G $\in \mathcal{G}'$ we have L(G) = L'(G).

(B) If G $\in \mathcal{G}$ is fully decomposable then L(G) is the composition of the elementary cell solutions.

(C) If G $\in \mathcal{G}$ is partially decomposable then L(G) is the composition of the main truncation and the elementary cell solutions.

(D) If $G \in \mathcal{G}$ is an indecomposable game with inferior choices then we have $L(G) = L(G')$ where G' is the game which results from G by elimination of inferior choices.

(E) If $G \in \mathcal{G}$ is an indecomposable game without inferior choices and with duplicate classes then we have $L(G) = L(G')$ where G' is the game which results from G by elimination of duplicate classes.

(F) If $G \in \mathcal{G}$ is an indecomposable game without inferior choices and without duplicate classes but with semi-duplicate classes then we have $L(G) = L(G')$ where G' is the game which results from G by elimination of semiduplicate classes.

It has to be shown that properties (A) to (F) determine a unique solution function. This is done by the following lemma.

Extension lemma: Let $\mathcal{G}'$ be the subclass of all irreducible games in a complete class $\mathcal{G} \subseteq \mathcal{J}(\mathcal{P})$. Moreover, let L' be a solution function for $\mathcal{G}'$. Then there is one and only one solution function L for $\mathcal{G}$ with properties (A) to (F).

Proof: We have to show (a) that $L(G)$ is uniquely determined by (A) to (F) and (b) that L is in fact a solution function, i.e. a function which assigns an equilibrium point to every $G \in \mathcal{G}$.

Both assertions are proved by induction on the total number Z of choices of all agents in G. The assertions are trivially true for $Z = 1$. Suppose that they hold for total numbers of choices up to Z-1. Let G be a game whose total number of choices is Z. Exactly one of the properties (A) to (F) is applicable to G. If G is irreducible then L(G) is a uniquely determined equilibrium point. If G is decomposable the compositions in (B) and (C) are uniquely determined equilibrium points (see the composition lemma in section 10). If one of the properties (D), (E) and (F) applies, it can be seen immediately that L(G) is a uniquely determined equilibrium point of G. All properties relate the solution of G to solutions of games with a smaller total number of choices.

<u>Comment</u>: The extension defined above will be used as a part of our solution concept. It permits us to define a solution function for the class $\mathcal{J}(\mathcal{R})$ of all interior substructures of standard forms with perfect recall starting from a solution function for the subclass of irreducible games in $\mathcal{J}(\mathcal{R})$. In this way, we want to secure cell consistency, truncation consistency and the three partial invariance properties as well as invariance with respect to isomorphisms. We have to show that we really reach this aim. This will be done by an "extension theorem". The proof will not immediately follow the theorem. It will be based on several results to be derived first.

<u>Extension theorem</u>: Let $\mathcal{G}'$ be the subclass of all irreducible games in a complete class $\mathcal{G} \subseteq \mathcal{J}(\mathcal{R})$. Let L' be a solution function for $\mathcal{G}'$ which is invariant with respect to isomorphisms. Then the extension L of L' to $\mathcal{G}$ is cell consistent, truncation consistent, partially in-

variant with respect to inferior choices, duplicates and semiduplicates and invariant with respect to isomorphisms. Moreover, there is no other solution function for $\mathcal{G}$ which agrees with L' on $\mathcal{G}'$ and satisfies these six requirements.

<u>Lemma on partial invariance properties</u>: Under the assumptions of the extension theorem the extension L is partially invariant with respect to inferior choices, duplicates and semiduplicates.

<u>Proof</u>: The assertion is an immediate consequence of (D), (E) and (F) in the definition of the extension.

<u>Lemma on consistency properties</u>: Under the assumptions of the extension theorem the extension L is cell consistent and truncation consistent.

<u>Proof</u>: In order to prove the theorem it is sufficient to show by induction on the number m of agents that no violation of cell consistency and truncation consistency can occur for decomposable games. The same argument as in the second part of the proof of the extension theorem in section 10 can be used here. We need not repeat this argument.

<u>Lemma on invariance with respect to isomorphisms</u>: Under the assumptions of the extension theorem the extension L is invariant with respect to isomorphisms.

<u>Proof</u>: Let G and G' be two isomorphic games in $\mathcal{G}$ and let f be an isomorphism from G to G'. We have to show L(G') = f(L(G)) is trivially true in the case Z = 1 where both games have only one agent with only one choice. Assume

that the assertion holds for total numbers of choices up to Z-1 where both games have only one agent with only one choice. Assume that the assertion hold for total numbers of choices up to Z-1 and let G and G' be games with Z choices. We distinguish three cases:

(1) G and G' are irreducible

(2) G and G' are indecomposable and reducible

(3) G and G' are decomposable.

Since an isomorphism carries inferior choices,duplicate classes, semiduplicate classes and cells to inferior choices, duplicate classes, semiduplicate classes and cells, respectively, the three cases are mutually exclusive and exhaustive.

Nothing needs to be proved in case (1). Consider case (2). One of the properties (D), (E) and (F) in the definition of the extension is applicable. Let $\hat{G}$ and $\hat{G}'$ be the games which result from G and G', respectively, by the application of the relevant elimination operation. Obviously, an isomorphism $\hat{f}$ from $\hat{G}$ to $\hat{G}'$ is induced by f. Since $\hat{G}$ and $\hat{G}'$ have fewer than Z choices we have $L(\hat{G}') = \hat{f}(L(\hat{G}))$. It follows that $L(G') = f(L(G))$ holds.

Now consider case (3). The isomorphism f carries an elementary cell of G to an elementary cell of G'. An elementary cell will always have fewer than Z choices. Therefore f carries the solution of an elementary cell of G to the solution of its counterpart in G'. It follows that the main truncation of G (if it exists) is carried to the main truncation of G'. Since the main truncation has fewer than Z choices the solution of the main truncation is carried to the solution of the main truncation of G'. It follows that we have L(G') = f(L(G)).

Proof of the extension theorem: The results obtained up to now show that the extension L satisfies the six requirements. It remains to show that there is no other solution function $\bar{L}$ for $\mathcal{G}$ which agrees with L' on $\mathcal{G}'$ and satisfies the six requirements. Let Z be the smallest number such that a game G ∈ $\mathcal{G}$ with a total number of Z choices and with $\bar{L}(G) \neq L(G)$ can be found. Let G be a game of this kind. Obviously, G must be reducible since L and $\bar{L}$ agree with L' for reducible games.

Suppose that G is indecomposable. The three partial invariance properties applied to $\bar{L}$ together with the properties (D), (E) and (F) of L lead to the conclusion that we must have $\bar{L}(G) = L(G)$, since the relevant elimination operation yields a game with less than Z choices. This shows that G cannot be indecomposable.

Now assume that G is decomposable. In the same way as in the proof of the extension theorem in section 10 it can be seen that properties (B) and (C) of L can be obtained as a consequence of cell consistency, truncation consistency and invariance with respect to positive linear payoff transformations. The result of applying (B) or (C) is achieved if truncations are formed for one elementary cell after the other. The order of doing this does not matter (see the first part of the proof of the extension theorem in section 10). This shows that (B) and

(C) hold for $\bar{L}$, too. Since the elementary cells and the main truncation of G (if it exists) have fewer than Z choices we must have $\bar{L}(G) = L(G)$. This shows that G cannot be decomposable either. Consequently, L is the only solution function for $\mathcal{G}$ which agrees with L' and $\mathcal{G}'$ and satisfies the six requirements.

<u>Comment</u>: The extension theorem shows that the way in which we reduce the task of solving games to the task of solving irreducible games is not an arbitrary one. If we want to obtain the properties mentioned in the theorem we have no essentially different choice.

It must be admitted, however, that the partial invariance properties with respect to inferior choices, duplicates and semiduplicates are very weak. Our definition severely restricts the applicability of the three elimination operations to special classes of games. Moreover, the elimination always removes all inferior choices, duplicate classes or semiduplicate classes at the same time. Nothing is said about what happens if only some inferior choices, duplicate classes or semiduplicate classes are removed. It is doubtful whether significantly stronger forms of the partial invariance properties can be satisfied together with cell consistency and truncation consistency by any solution function for the class of all interior substructures of standard forms with perfect recall.

<u>The procedure of decomposition and reduction</u>: As before let L' be a solution function for the subclass $\mathcal{G}'$ of

of irreducible games in a complete class $\mathcal{G}' \subseteq \mathcal{J}(\mathcal{P})$ such that L' is invariant with respect to isomorphisms, amd let L be the extension of L' to $\mathcal{G}$. On the basis of the knowledge of L' we can compute L(G) for every $G \in \mathcal{G}$. In principle the recursive definition of the extension enables us to do this. Therefore, it is not really necessary to say more about the computation of L on the basis of L'. Nevertheless, it is not without interest to specify a more detailed procedure, where the task of finding the solution is broken down into a succession of elementary steps. Additional insight may be gained in this way. Moreover, the application of the solution concept becomes easier by a more detailed description of what should be done in which order to which games. These questions will be answered by the <u>procedure of decomposition and reduction</u> described by the flow chart of figure 3.29. In the following, we shall give further explanations and exhibit the reasons why under the assumptions made above the procedure succeeds in computing the solution L(G) of the game G under consideration.

<u>Dynamic notation</u>: A dynamic notational convention is used in the flow chart. At any step of the procedure there will be a list of games $G^1,\dots,G^m$. The length of this list and the meaning of the symbol G^k varies during the procedure.

<u>Solution agreement</u>: At any point during the procedure every game G^k on the list will have the property that its solution $L(G^k)$ agrees with the solution L(G) of the game G to be solved, as far as the agents in G^k are

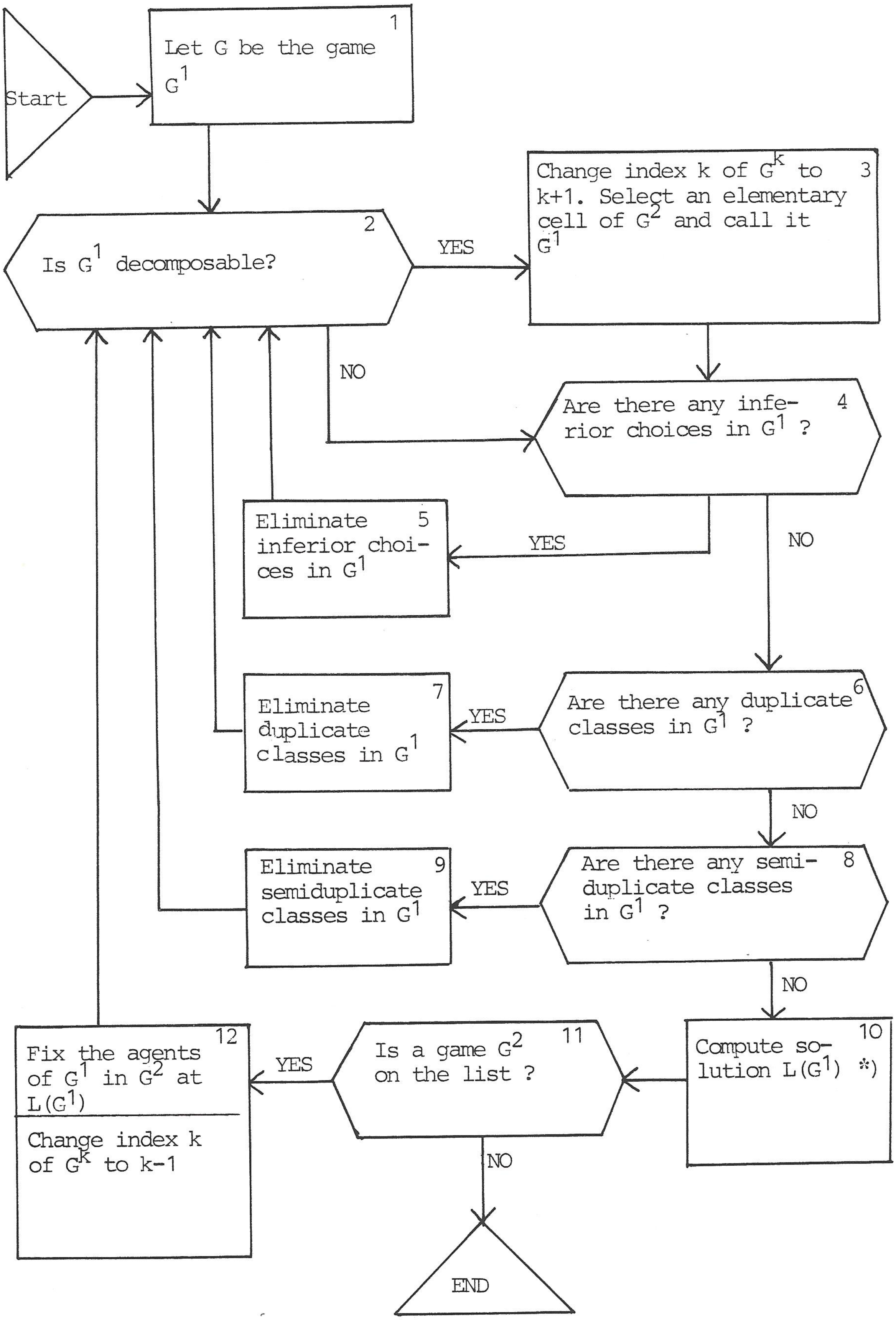


Figure 3.29: Flow chart for the procedure of decomposition and reduction.

*) By solution agreement $L(G^1)$ agrees with $L(G)$ for the agents in G^1.

concerned. We shall refer to this property of G^k as _solution agreement_. Wherever a new game is introduced, one of the six requirements satisfied by L in view of the extention theorem will guarantee solution agreement for the newly introduced game.

Course of the procedure: The procedure starts at _rectangle 1_ where the game G to be solved becomes the game G^1. The the procedure moves to _rhomboid 2._ Rhomboids contain questions whose answers determine the next step. Rectangles contain operations to be performed including the change of names.

If G^1 is decomposable the procedure moves from rhomboid 2 to _retangle 3_. Whenever retangle 3 is reached, the indices of $G^1,\ldots,G^m$ are moved up by 1. Thereby, these games receive the new names $G^2,\ldots,G^{m+1}$. An arbitrarily selected elementary cell of G^2 (formerly G^1) becomes the new game G^1. Cell consistency guarantees solution agreement for the new game G^1.

Rhomboid 4 can be reached from rectangle 3 or directly from rhomboid 2. In both cases G^1 is indecomposable at rhomboid 4. Elimination of inferior choices at _rectangle 5_ creates a new game G^1 whose solution agreement is guaranteed by partial invariance with respect to inferior choices.

If at rhomboid 4 the game G^1 has no inferior choices, then _rhomboid 6_ is reached. Solution agreement for the new game G^1 created at _rectangle 7_ follows by partial invariance with respect to duplicates.

If rhomboid 8 is reached then G' has neither inferior choices nor duplicate classes. Solution agreement of the new game G^1 created at rectangle 9 follows by partial invariance with respect to semiduplicates.

The procedure always turns back to rhomboid 2 after rectangles 5,7 and 9. Therefore at rectangle 10 the game G^1 will be irreducible. The computation of $L(G^1) = L'(G^1)$ determines the components of L(G) for the agents in G^1.

If at rhomboid 11 the game G^1 is not the only one on the list the procedure must have passed rectangle 3 at least once in the past. Consider the last time when this has happened. At that time G^1 was an elementary cell, say G' of the game now called G^2. In view of cell consistency and solution agreement we have $L(G^1) = L(G')$. This shows that at rectangle 12 the truncation of G^2 with respect to G' and L is formed by fixing the agents on G^1 at $L(G^1)$. Temporarily this truncation will be called G^2 in rectangle 12, but then the indices are moved down by 1 and the truncation becomes the new game G^1 to be considered at rhomboid 2. Solution agreement for this game follows by truncation consistency.

Note that as long as there are several games $G^2,\dots,G^m$ on the list, G^k is derived by reduction at rectangles 5,7 and 9 and/or by truncation at rectangle 12 from an elementary cell of G^{k+1}, $(k=1,\dots,m-1)$. In the same way the game G^m is derived from G. This is still true if at rhomboid 11 we have m = 1. Consequently, the local solution strategies have been computed for all agents as soon as at rhomboid 11 there is no game G^2 on the list.

<u>Comment</u>: Compared with the recursive definition of the extension by properties (A) to (F) the procedure of decomposition and reduction is nearer to an algorithm. In order to write a computer program on the basis of the flow chart of figure 3.29 one would have to supply subprograms for the rhomboids and rectangles. As an example consider rhomboid 2. For every non-empty proper subset G of agents we would have to check whether a linear equation system derived from (3.34) can be solved with positive $\alpha_i(\phi_{-C})$.

If one looks at the procedure in the light of computational considerations of this kind one may get the impression that we cannot hope to apply it unless the number of agents and choices is quite small. However, this impression is misleading since game models arising in economics and other fields often have a special structure which permits to exclude many possible complications beforehand. It may, for example, be clear from the structure of the model that the question in rhomboid 2 is always answered with NO.

Our theory will be based on a solution function for the class $\mathcal{J}(\mathcal{P})$ of all interior substructures of standard forms with perfect recall. It will be the extension of a solution function for the subclass of all irreducible games in $\mathcal{J}(\mathcal{P})$. In this way, we obtain a solution function with the six properties mentioned in the extention theorem. Moreover, the extension theorem axiomatizes the way in which we connect solutions of reducible games to solutions of irreducible games.

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