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A General Theory of Equilibrium Selection in Games, Chapter 5: The Solution Concept

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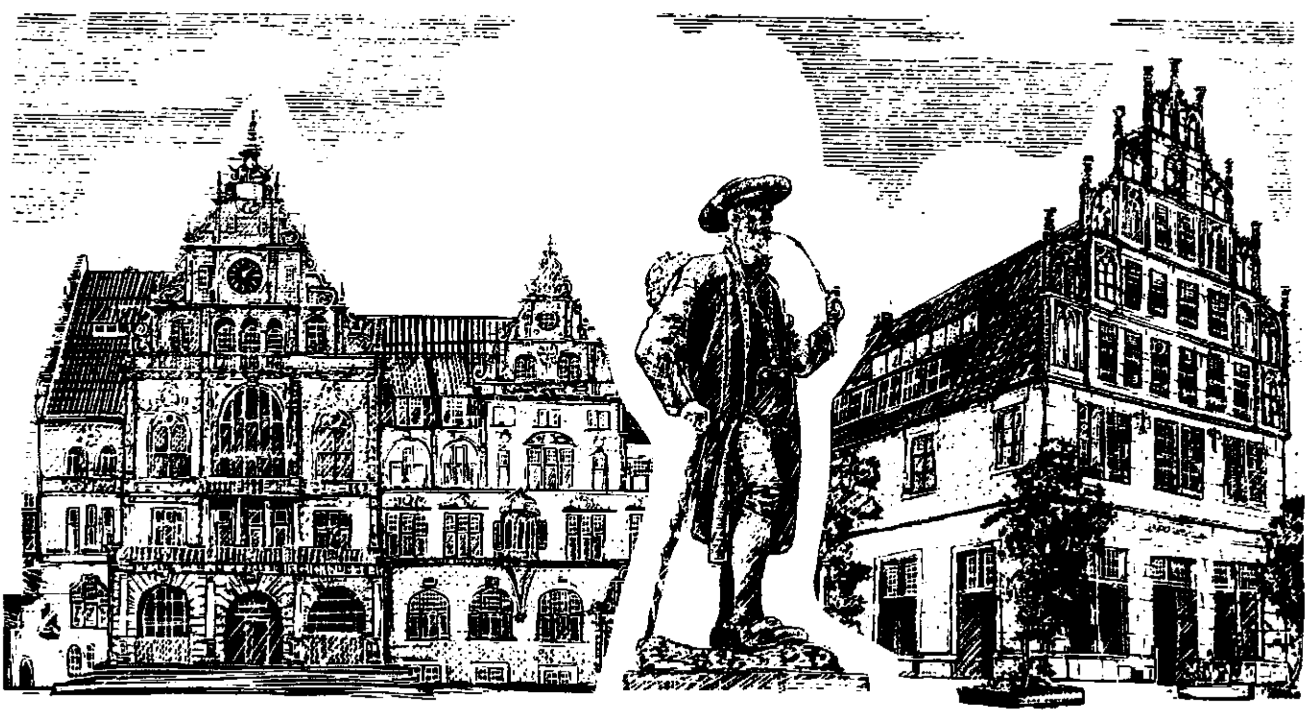

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CHAPTER 5

Chapter 5. The Solution Concept

Some of the ideas, which are used in our solution concept, have been introduced already in earlier chapters. Our preliminary discussion has addressed the fundamental problems to be dealt with. Chapter 2 has shown how our theory is based on a solution function for interior substructures of standard forms with perfect recall. Our approach to the perfectness problem has been explained there. Chapter 3 has described how our theory extends a solution function for irreducible games to reducible games. The last chapter has introduced the tracing procedure which is of central importance for our solution concept.

In this chapter we shall explain how our theory solves irreducible games. For this purpose, we employ a "<u>process of candidate elimination and substitution</u>. The process starts with a set of natural solution candidates and then generates a finite sequence of sets of equilibrium points, called <u>candidate sets</u>. The last candidate set contains only one equilibrium point, the <u>solution</u> of the game. As an introduction to this chapter we shall give an informal overview over the new ideas to be introduced. Detailed and precise definitions can be found in sections 1, 2 and 4.

Our theory makes use of special substructures of games, called <u>formations</u>. Roughly speaking, a formation is a proper substructure obtained by eliminating some of the agents' choices which is closed with repsect to local best replies.Those formations, which are smallest in the sense that they contain no other formation,are called primitive. The primitive formations are of special significance for our theory. In irreducible games with formations we shall look at the solutions of all the primitive formations as the natural candidates for a solution of the game. We shall try to find the solution among these natural candidates.

Irreducible games without formations are called <u>basic</u>. In order to solve a basic game the tracing procedure is applied to the centroid of the game, i.e. to that strategy combination where each agent uses all his choices with the same probability. The result of doing this is the solution of the game.

In irreducible games with formations we shall make use of payoff dominance and risk dominance for pairwise comparisons of solution candidates. Our definition of risk dominance between two equilibrium points will be based on the tracing procedure applied to a special prior. If this yields one of both equilibrium points as the result of the tracing procedure then this equilibrium point risk dominates the other. The special prior distribution depends on the two equilibrium points and therefore is called the bicentric prior.

A risk dominance comparison is not performed in the original game, but in a restricted game, a substructure of the smallest formation which contains both equilibrium points. In this formation those players who have the same strategy in both equilibrium points are fixed at these equilibrium strategies. The result is the restricted game.

The bicentric prior is looked upon as a preliminary theory to be improved by the tracing procedure. The construction of the bicentric prior starts from a picture of a player's initial beliefs: All other players play their strategies in the same equilibrium point. With subjective probabilities adding up to one this will be either one of both equilibrium points to be compared. - The player plays his central local best reply to the joint mixture corresponding to his beliefs. His bicentric prior strategy is the result of averaging over all possible subjective probability distributions over both equilibrium points. The bicentric prior is the mixed strategy combination for the restricted game which contains the bicentric prior strategies as components.

Risk dominance and payoff dominance are combined to a dominance relationship which gives precedence to payoff dominance. Risk dominance determines dominance, if neither of the two equilibrium points payoff dominates the other.

Not all dominance relationships are regarded as equally important. It is reasonable to emphasize comparisons between equilibrium points which are similar to each other. For this purpose a measure of strategic distance between two equilibrium points is introduced. The definition of strategic distance is closely related to that of the bicentric prior. If the subjective probability of a player for one of both equilibrium points is increased from zero to one, this probability will pass a finite number of critical points where the central local best reply of the player is changed. The number of critical points for all players in the restricted game is the strategic distance.

The measure of strategic distance is further refined to a measure of strategic net distance within a candidate set. For each of both equilibrium points the strategic distance to the next neighbor in the candidate set is subtracted from the strategic distance between both equilibrium points; the sum of both surpluses plus 1 is the strategic net distance. 1 is added in order to make sure that the strategic net distance is positive. Strategic net distance is small if both equilibrium points are relatively near to each other compared with the strategic distances to next neighbors in the candidate set.

The stability of an equilibrium point within a candidate set is measured by its stability index. The stability index is the greatest strategic net distance assumed in the set, such that within this net distance the candidate is not dominated by another candidate in the set. A candidate is maximally stable in a candidate set if its stability index is maximal in the set.

The process of candidate elimination and substitution starts with the set of all solutions of primitive formations, the first candidate set. Whenever possible an elimination step is performed in the transition from one candidate set to the next. An elimination step eliminates all candidates which are not maximally stable.

If a candidate set with more than one element cannot be narrowed down by an elimination step, a substitution step has to be performed. For this purpose, a substitute of the candidate set is determined. The substitute is obtained by tracing the centroid of the candidate set. In the centroid of the candidate set each player uses the unweighted average of his strategies in the candidates of the set.

Generally, the substitute is not yet the solution. Among the candidates which have been eliminated before there may be preferable ones. Therefore, in a substitution step a new candidate set is formed which contains the substitute together with all candidates which have been eliminated before, but not yet substituted. Once a candidate has been substituted it is finally removed from consideration, but elimination is cancelled if substitution becomes necessary. The flow chart of the process of candidate elimination and substitution is shown in figure 5.3.

Our solution concept is based on a recursive definition. We need to know the solutions of the primitive formations in order to start the process of candidate elimination and substitution. The primitive formations are smaller than the game under consideration; therefore, we can assume that their solutions are known.

A primitive formation of an irreducible game may not be irreducible. In this case, we have to apply the procedure of decomposition and reduction in order to find the solution (chapter 3, section 12).

As has been explained at the end of chapter 2, the solution function specified by our theory is meant to be applied to uniformly perturbed games. The limit solution is obtained by letting the perturbance parameter go to zero.

The solution concept combines a number of separate ideas such as cell decomposition, reduction, primitive formations, the tracing procedure, payoff dominance, risk dominance, strategic net distance, candidate elimination and substitution. The coherence of the composite structure cannot be made clear without a thorough discussion of the building blocks. Therefore, we shall not follow the shortest path to the completion of mathematical definition of the solution. Auxiliary concepts will be motivated and examined in the light of examples where they are introduced.

1. Initial Candidates

Games which arise in the context of economic theory often have many strong equilibrium points. Obviously, in such cases it is more natural to select a strong equilibrium point rather than a weak one. Of course, strong equilibrium points are not always available, and if they are available they may be ineligible in view of lack of symmetry invariance.

Even if strongness is unsuitable as a selection criterion, it is still possible to look for a principle which helps us to avoid those weak equilibrium points which are especially unstable. An example of a very unstable equilibrium point is the completely mixed one in a 2x2-game with two strong ones. In figure 3.6 this unstable equilibrium point corresponds to the point where the stability regions of U and V meet. Both U and V are best replies to the unstable equilibrium point. Mixed equilibrium points at the border of stability regions of strong ones can be found in more complicated games, too. H is clearly desirable to restrict the selection of such equilibrium points to exceptional cases which cannot be avoided for good reasons such as symmetry considerations.

The way in which we shall approach this problem makes use of certain substructures of games, called formations. With the help of such substructures we shall be able to identify a class of equilibrium points without unnecessary instability properties. Among these we shall find natural solution candidates which will be called initial candidates.

Formations: Let $G = (\Phi,H)$ with $\Phi = \underset{i\in N}{X} \Phi_i$ and $\Phi_i = \underset{ij\in M_i}{X} \Phi_{ij}$ be a game in standard form.

A subset Ψ of Φ is called cartesian, if Ψ is a non-empty proper subset of Φ of the form $\Psi = \underset{i\in N}{X} \Psi_i$ with $\Psi_i = \underset{ij\in M_i}{X} \Psi_{ij}$.

Consider a cartesian set Ψ. Let $G' = (\Psi,H')$ be the game which results from G by narrowing the choice sets in Φ to Ψ. (See chapter 2, section 5). Obviously, the Ψ_{ij} are admissable new choice sets. G' is a substructure of G.

For every agent ij let $B'_{i\setminus ij}$ be the set of all ij-incomplete behavior strategies in G'. For every player i let $Q'_{.i}$ be the set of all i-incomplete joint mixtures in G'. The sets $B'_{i\setminus ij}$ and $Q'_{.i}$ are subsets of the corresponding sets $B_{i\setminus ij}$ and $Q_{.i}$ for G (see chapter 2, section 5).

$G' = (\Psi,H')$ is a formation of $G = (\Phi,H)$ if the following condition is satisfied for every agent ij and for every hybrid combination $b'_{i\setminus ij}\, q'_{.i}$ with $b'_{i\setminus ij} \in B'_{i\setminus ij}$ and $q'_{.i} \in Q'_{.i}$:

(5.1) $A_{ij}(b'_{i\setminus ij}\, q'_{.i}) \subseteq \Psi_{ij}$

Here A_{ij} is the local best reply correspondence of agent ij for G, which has been introduced in section 6 of chapter 2. It is important to notice that on the left hand side of (5.1) we find the set of all pure local best replies to $b'_{i\setminus ij}\, q'_{.i}$ in G. If G' is a formation then the local best replies in G' are also local best replies in G. This is a consequence of (5.1) and the lemma on local best replies in section 2 of chapter 2.

Condition (5.1) can be expressed by saying that Ψ is closed with respect to local best replies in G. If this is the case we call $G' = (\Psi,H')$ the formation of G generated by Ψ.

Let Λ and Ψ both be cartesian subsets of Φ which are closed with respect to local best replies in G and let Λ be a proper subset of Ψ. Then we say that the formation generated by Λ is a <u>subformation</u> of the formation generated by Ψ.

Remarks: A behavior strategy b_i cannot be a best reply to a joint mixture $q_{.i}$ unless it is a local best reply. Therefore (5.1) has the following consequence. For every player i and every i-incomplete joint mixture $q'_{.i} \in Q'_{.i}$ we have:

$$(5.2) \qquad A_i(q'_{.i}) \subseteq \Psi_i$$

where A_i is player i's best reply correspondence which has been introduced in section 4 of chapter 3.

It follows by (5.2) that an equilibrium point of a formation G' of G is also an equilibrium point of G. Since a formation is a game, it follows by Nash's theorem that every formation has at least one equilibrium point.

A formation is defined by the local condition (5.1) rather than the global condition (5.2). This has important consequences for the case that G is an interior substructure of a standard form with perfect recall. For such games the notion of a central local best reply $a_i(q_{.i})$ to a joint mixture has been introduced in section 6 of chapter 2. It follows by (5.1) together with the definition of the central local best reply that it does not matter whether $a_i(q'_{.i})$ is computed in G or G' if $q'_{.i}$ is a joint mixture for a formation G' of G.

Interpretation: The stability condition (5.2) can be interpreted as follows: Suppose that player i is convinced that the other players will use strategies $\varphi_j \in \Psi_j$ in the formation. If player i's expectations are compatible with this assumption he will never have a pure best reply outside Ψ_i. Whatever his subjective probability distribution over Ψ_{-i} may be, his best replies will be strategies within the formation. A pure strategy outside Ψ_i will always yield less expected payoff than some pure strategy inside Ψ_i. In this respect,the stability properties of a formation are similar to those of a strong equilibrium point.

Formations are defined in terms of best replies to joint mixtures rather than to i-incomplete combinations of mixed strategies. As has been explained in section 2 of chapter 2 player i may hold subjective beliefs on his opponents which cannot be expressed by an i-imcomplete combination of mixed strategies. In our theory such beliefs occur as preliminary expectations which are gradually revised by the tracing procedure. Of course, after the solution has been found the beliefs of the players are nothing else than the i-incomplete mixed strategy combinations generated by the solution. Preliminary beliefs are disequilibrium beliefs and, therefore, need not have the same properties as the final solution.

A local best reply condition rather than a global one is used for the definition of a formation. This seems to be appropriate in the framework of the standard form. A definition in terms of global best replies would not immediately lead to a standard form, but to a normal form. The pure strategy combinations in this normal form do not necessarily form a cartesian set. Any definition which leads to a standard form on the basis of a global best reply condition would have to be more complicated and probably somewhat artificial.

<u>Intersections of formations</u>: Let $G' = (\Psi,H')$ and $G'' = (\Lambda,H'')$ be two formations of $G = (\Phi,H)$ such that the intersection $\Delta = \Lambda \cap \Psi$ is non-empty. The game $\bar{G} = (\Delta,\bar{H})$ which results from G by narrowing the choice sets in Φ to Δ is called the <u>intersection</u> of the formations G' and G''.

<u>Intersection Lemma</u>: If $\bar{G}$ is the intersection of two formations G' and G'' of G, then $\bar{G}$ is a formation of G.

<u>Proof</u>: Condition (5.1) applied to $\bar{G}$ is an immediate consequence of (5.1) for G' and G''.

<u>Primitive formations</u>: A formation $G' = (\Psi,H')$ of G is called <u>primitive</u>, if G' has no subformation. A <u>primitive game</u> is a game without formations.

Remarks: It follows by the intersection lemma that two primitive formations of a game do not intersect. Obviously, a game which is not primitive must have at least one primitive formation.

Let ψ be a strong equilibrium point of G. Then $\{\psi\}$ generates a primitive formation of G.

Comment: Primitive formations are the smallest substructures with similar stability properties as strong equilibrium points. An equilibrium point in a primitive formation may be weak as far as the strategies in the formation are concerned, but it is strong with respect to outside strategies in the sense that an outside strategy incurs a positive deviation loss. Our solution concept favors the selection of such equilibrium points in order to obtain as much of the desirable stability properties of strong equilibrium points as possible.

Suppose that r is an equilibrium point of a primitive formation G' of G. Then it cannot happen that a strong equilibrium point ψ of G is a best reply for r. The reason for this is that $\{\psi\}$ generates a primitive formation which would have to belong to G'. This is impossible since G' is primitive. The stability property which has been described is certainly a desirable feature of equilibrium points of primitive formations.

It would not be reasonable to prefer strong equilibrium points to weak ones under all circumstances. This can be seen with the help of the game with normal form structure shown in figure 5.1. There $\Psi = \Psi_1 \times \Psi_2$ with $\Psi_1 = \{U_1, V_1\}$ and $\Psi_2 = \{U_2, V_2\}$ generates a primitive formation G' and $\{W\}$ with $W = (W_1, W_2)$ generates another primitive formation G". The formation G' is equivalent to a matching pennies game and has a unique equilibrium point $r = (r_1, r_2)$ where both players use both strategies with probability 1/2. The equilibrium payoffs $H(r) = (3,3)$ are much better for both players than the equilibrium payoffs in $H(W) = (1,1)$. In view of the fact that in G' all payoffs are greater than 1, it is of little importance that W is strong and r is weak. Clearly, in the case of figure 5.1 it is more reasonable to select r as the solution rather than W.

	U_2	V_2	W_2
U_1	4, 2	2, 4	0, 0
V_1	2, 4	4, 2	0, 0
W_1	0, 0	0, 0	1, 1

Figure 5.1: A 3x3-game with two primitive formations.

The game has also a third equilibrium point q which is not in a primitive formation $q_i(U_i) = 1/8$, $q_i(V_i) = 1/8$ and $q_i(W_i) = 3/4$. Both r and W are best replies to q.

We shall take the point of view that the solutions of primitive formations are natural candidates for the solution of the whole game. Therefore, we must now turn our attention to the question of how to define the solution of primitive games. It may happen that a primitive game has cells, inferior choices, duplicates or semiduplicates. In such cases one has to apply our procedure of reduction and decomposition in order to find the solution. A direct definition of the solution without reference to substructures is required only for irreducible primitive games.

Solution of basic games: A game is called basic if it is primitive and irreducible or, in other words, if it has neither cells nor inferior choices nor duplicate classes nor semiduplicate classes nor formations. Let $G = (\Phi,H)$ be a basic game. The centroid $c(\Phi)$ of G is that behavior strategy combination where every agent assigns equal probabilities to all his choices. Let $\mathcal{G}_1$ be the set of all basic games in the class $J(\mathcal{R})$ of interior substructures of standard forms with perfect recall (see chapter 2, section 12). We define a solution function L_1 for $\mathcal{G}_1$ as follows: For every $G = (\Phi,H) \in \mathcal{G}_1$ the solution $L_1(G)$ is the result $T(G,c(\Phi))$ of the logarithmic two speed tracing procedure applied to G with $c(\Phi)$ as prior distribution. This solution function L_1 is called the basic solution function.

Initial candidates: We say that a game $G = (\Phi,H)$ has size K if K is the number of all choices in G, i.e. if K is the sum of all K_{ij} with $ij \in M$ where K_{ij} is the number of agent ij's choices. Let L be a solution function for the class of all games in $J(\mathcal{R})$ whose size is smaller than K and let $G = (\Phi,H)$ be an irreducible game of size K. An initial candidate for G with respect to L is defined as follows: If G is basic, then the basic solution $L_1(G)$ is the only initial candidate for G. If G is not basic, then the solutions $L(G')$ of the primitive formations G' of G are the initial candidates of G. This definition is a meaningful one since in the case of a non-basic game the primitive formations must be of smaller size. The set of all initial candidates is denoted by Ω_1. We call Ω_1 the first candidate set.

Comment: Since the notion of an initial candidate is a part of the recursive definition of the solution function proposed in this book, it has to be relative to a solution function for games whose primitive formations are all basic. Our proposed solution function agrees with the basic solution function for basic games.

It may happen that the first candidate set of a non-basic game has only one initial candidate, but generally such games will have many initial candidates. In this case, Ω_1 is only the first in a sequence of candidate sets generated by a process of candidate elimination and substitution which has been loosely described in the introduction of the chapter. Our notion of risk dominance is used as an important criterion of candidate elimination. It will be the task of the next section to introduce the formal definition of risk dominance and to discuss the underlying conceptual ideas.

2. Risk dominance

In sections 6 to 8 of chapter 3 the notion of risk dominance has been discussed in the framework of 2x2-games. The axiomization in section 8 of chapter 3 shows that for this narrow class of games the comparison of Nash-products is a natural criterion of risk dominance. Obviously, a general definition should agree with that of chapter 3 for 2x2-games with two strong equilibrium points. Of couse, this is not the only desirable property which one might want to achieve.

Since we cannot really motivate our general definition by desirable properties we shall emphasize the plausibility of its direct interpretation. In the long run, one might want to extend the axiomatic approach to risk dominance beyond the narrow range of 2x2-games but no attempt in this direction will be made here. In section 4 we shall discuss some of the desirable properties which distinguish our present notion of risk dominance from other definitions which we considered in earlier stages of the development of our theory.

The nature of risk dominance comparisons: Risk dominance is concerned with pairwise comparisons between equilibrium points. Consider two equilibrium points $U = (V_i)_N$ and $V = (V_i)_N$, not necessarily in pure strategies, of a game $G = (\Phi,H)$. Imagine a hypothetical situation where it is common knowledge that all players think that either U or V must be the solution without knowing which of both equilibrium points is the solution. Risk dominance tries to capture the idea that in this state of confusion the players enter a process of expectation formation which may lead to the conclusion that in some sense one of both equilibrium points is less risky than the other.

Bayesian rationality requires that a decision maker must have a subjective probability distribution over the states of the world which determine the consequences of his possible actions. We take the point of view that in game theory the subjective probabilities of players should not be arbitrary. A rational player should have a rational way of deriving his subjective probabilities from the structure of the game situation.

Imagine a rational outside observer who shares the common knowledge of the players and tries to form expectations on the game. We assume that there is just one rational way in which the outside observer can form his expectations on the behavior of the players. Obviously, rational players must form their expectations on other players in the same way as the outside observer as far as the behavior of other players is concerned. This means that we do not have to consider different processes of expectation formation for different players but at just one way of forming expectations, namely that of a rational outside observer.

Our theory looks at the rational formation of expectations as a process which proceeds in two stages. The first stage yields a preliminary theory on the players' behavior. This theory takes the form of a mixed strategy combination, the bicentric prior, which already has been mentioned in the introduction of the chapter. With the help of the tracing procedure the expectations of the preliminary theory are then gradually transformed to final expectations.

The preliminary theory will look at the players as Bayesian decision makers and, therefore, must involve expectations on the players' expectations. Therefore, the preliminary theory is focussed on players rather than agents. The players are the centers of expectation formation. All agents of the same player must have the same expectations. Consequently, expectations on expectations must concern players rather than agents. This is the main reason why our theory had to be developed in terms of the standard form rather than the agent normal form (see chapter 2, section 2).

Some of the players may have the same strategy in U and V. Consider a player i with $U_i = V_i$. On the basis of the assumption that either U or V is the solution, it is natural to expect that player i will play his equilibrium strategy $U_i = V_i$; he has no need to know whether U or V is the solution in order to play his solution strategy.

Therefore, our theory describes a process of forming expectations where expectations on players i with $U_i = V_i$ are always fixed at these equilibrium strategies. Accordingly, risk dominance comparisons will be performed in a restricted game where such players are fixed at their equilibrium strategies.

The restricted game has an additional feature which serves to secure a desirable property called formation consistency.It should not matter whether risk dominance is determined in the game as a whole or in one of its formations which contains both equilibrium points.

Restricted game: Let U and V be two different equilibrium points, not necessarily in pure strategies, of a game $G = (\Phi,H)$ in standard form. In view of the fact that the intersection of two formations is a formation (intersection lemma) a smallest formation F exists such that U and V belong to F. We call this formation F, the formation spanned by U and V. Let D be the set of players i who have the same strategy $U_i = V_i$ in both equlibrium points and let C be the set of all agents of players in D. The restricted game $G' = (\Phi',H')$ for the comparison between U and V is the game which results from the formation F spanned by U and V by fixing every agent $ij \in C$ at the local strategy prescribed by the common equilibrium strategy $U_i = V_i$ in both equilibrium points.

Let N' be the set of all $i \in N$ with $U_i \neq V_i$ or in other words the player set of the restricted game. Clearly, the combinations $U' = (U_i)_{N'}$ and $V' = (V_i)_{N'}$ are equilibrium points of the restricted game G'. We say that U' and V' correspond to U and V, respectively, in G'.

Comment: In exceptional cases the logarithmic tracing procedure has to be used in order to determine risk dominance. If this happens the weights α_i^* of the logarithmic terms (see chapter 4, section) may have an influence on the result and it may matter whether the logarithmic tracing procedure is performed in the game as a whole or in one of its formations. The definition of the restricted game as a substructure of the formation spanned by U and V has the consequences that risk dominance in the game as a whole is not different from risk dominance in one of its formations. This is the desirable property of formation consistency which has been mentioned above.

The restricted game may have cells, inferior strategies, duplicate or semiduplicate classes. We do not apply our procedure of decomposition and reduction in such cases. We look at the restricted game as a constraint on the process of forming expectations in the original game. Attention is focussed on the formation spanned by U and V and expectations on players with $U_i = V_i$ are fixed on these strategies from the beginning to the end of the process. In view of this interpretation of the restricted game we do not attach any significance to structural features which are not present in the game as a whole.

A theory of preliminary expectations: Let U and V be two equilibrium points of G = (Φ,H) such that for every player i the strategies U_i and V_i in U and V, respectively, are different. (G may be the original game under consideration or it may be the relevant restricted game.) We continue to look at the hypothetical situation where it is common knowledge that all players believe that either U or V is the solution.

It will be convenient to look at the problem of forming preliminary expectations from the point of view of an outside observer. He may approach the problem by asking the following question: What could a player do if he had to make his decision in an initial state of uncertainty where he does not yet know whether U or V is the solution? Of course, finally he will know, but the problem at hand is not yet the derivation of final expectations but the derivation of preliminary expectations.

The description of the initial state of uncertainty between U and V must be made more precise. What does player i think about the other players in this state of uncertainty? He must think that finally they will find out whether U or V is the solution; they will all follow the same rational reasoning process and therefore all of them will come to the same final conclusion and will act accordingly. Player i being a Bayesian must have subjective probabilities z_i and $1-z_i$ for both of these possibilities.

A player in the initial state of uncertainty must expect that finally not only the other players but also he himself will know which of both equilibrium points is the solution. However, if he had to make his decision in his initial state of uncertainty he could do nothing else than to choose a best reply against the i-imcomplete mixture with probabilities z_i for U_{-i} and $1-z_i$ for V_{-i}.

What should the outside observer think about the parameters z_i? As a Bayesian he must form a prior distribution. Obviously, the distributions of the z_i should be independent of each other since the players form their expectations independently of each other. Moreover, it is natural to form a flat prior on z_i, i.e. a uniform distribution over [0,1].

It may happen that among the behavior strategies of player i there is more than one best reply to the joint mixture with probabilities z_i for U_{-i} and $1-z_i$ for V_{-i}. In this case, it is natural for the outside observer to assume that player i will choose his central local best reply to the joint mixture (see chapter 2, section 6).

A plausible chain of reasoning has led us to a complete description of a preliminary theory an outside observer should have on the player's behavior in the hypothetical situation.

It is convenient to introduce the symbolic expression $z_iU_{-i} + (1-z_i)V_{-i}$ for the i-imcomplete joint mixture with probabilities z_i and $1-z_i$ for U_{-i} and V_{-i}, respectively. The preliminary theory can be summarized as follows:

1. Each player i believes that either all other players behave according to U_{-i} or all other players behave according to V_{-i}.

2. Each player i has a subjective probability z_i for U_{-i} and subjective probability $1-z_i$ for V_{-i}.

3. Each player i plays his central local best reply $a_i(z_iU_{-i}+(1-z_i)V_{-i})$ to the i-incomplete joint mixture $z_iU_{-i} + (1-z_i)V_{-i}$.

4. The z_i are independently distributed (subjective) random variables; each of them has an even distribution over the interval [0,1].

The expectations specified by the preliminary theory take the form of a mixed strategy combination which will be called the <u>bicentric prior</u>, since it is a special prior distribution concerning a hypothetical comparison between two equilibrium points.

<u>Bicentric prior</u>: Let $G = (\Phi,H)$ be an interior substructure of a standard form with perfect recall, i.e. a game in $J(\mathcal{R})$. (See chapter 2, section 12). Let U and V be two equilibrium points of G and let i be a player such that his strategies U_i and V_i in U and V, respectively, are different. For every z with $0 \leq z \leq 1$ define:

(5.3) $\quad r_i^z = a_i(zU_{-i}+(1-z)V_{-i})$

where a_i denotes the central local best reply (chapter 2, section 6). The bicentric prior strategy of player i for the comparison of U and V is defined as follows:

(5.4) $$p_i(\varphi_i) = \int_0^1 r_i^z(\varphi_i)dz$$

for every $\varphi_i \in \Phi_i$

In the next section we shall prove a lemma which shows that no difficulty arises with respect to the integrability of $r_i^z(\varphi_i)$. As we shall see, there the interval [0,1] can be subdivided into a finite number of subintervals where r_i^z is constant. The bicentric prior for the comparison between U and V is that strategy combination p which contains the bicentric prior strategies as components.

As before let $U = (U_i)_N$ and $V = (V_i)_N$ be two different equilibrium points of a game $G = (\Phi,H)$, but such that we have $U_k = V_k$ for some players k. Let $G' = (\Phi',H')$ be the restricted game for the comparison of U and V and let $U' = (U_i)_{N'}$ and $V' = (V_i)_{N'}$ be the equilibrium points of G' which correspond to U and V, respectively. Then the bicentric prior strategy of a player i with $U_i \neq V_i$ for the comparison of U and V is the bicentric prior strategy p_i of this player for the comparison of U' and V' in the restricted game G'; the bicentric prior for the comparison between U and V is the bicentric prior p' for the comparison between U' and V' in G'.

Risk dominance: Let $U = (U_i)_N$ and $V = (V_i)_N$ be two different equilibrium points of a game $G \in J(\mathcal{R})$ and let p' be the bicentric prior for the comparison between U and V. Let G' be the restricted game for the comparison between U and V. We say that U risk dominates V if we have:

(5.5) $\quad T(G',p') = U' = (U_i)_{N'}$

Analogously, V risk dominates U if we have:

(5.6) $\quad T(G',p') = V' = (V_i)_{N'}$

where N' is the player set of the restricted game G'.

Interpretation: The definition of risk dominance is based on a hypothetical process of forming expectations starting from a state of uncertainty between U and V. A preliminary view of the risks involved in the uncertainty between U and V is embodied in the bicentric prior. If the gradual adaption of expectations with the help of the tracing procedure converges to one of both equilibrium points, then the risks arising from the initial state of uncertainty favor this equilibtrium point. The word "risk dominant" can be understood as "dominant in the players' expectation after due consideration of the risks involved in the initial state of uncertainty". This justifies our language use.

As in section 8 of chapter 3 we permit the possibility that none of both equilibrium points risk dominates the other. T(G',p') may be different both from U' and V'. If this happens none of both equilibrium points is clearly favored by the risks involved in the uncertainty between U and V.

3. Properties of risk dominance

It will be the first task of this section to show that the integral in (5.4) is always well defined. Since r_i^z is a central local best reply which is obtained as the result of an iterative process, this is not obvious (see section 6 of chapter 2).

We shall prove two lemmas on desirable properties of our notion of risk dominance. The property of formation consistency which has been mentioned already in section 3 is expressed by the first one of these two lemmas. The second one concerns another desirable property, namely invariance with respect to isomorphisms.

A special class of games, called unanimity games will be examined in detail. In non-degenerate cases risk dominance in such games can be characterized in a simple way which is reminiscent of Nash's cooperative bargaining theory with fixed threats. Therefore, the name Nash-property will be used in this connection.

A lemma on 2x2-games will show that our general concept of risk dominance agrees with the special one axiomatized in chapter 3, section 8 for 2x2-games with two strong equilibrium points. We shall also reconsider the payoff monotonicity counterexample from chapter 3, section 7, in order to verify that our notion of risk dominance does not have this property. The discussion will show that nevertheless the result is not unreasonable.

Stability regions with respect to central local best replies: Let $G=(\Phi,H)$ be the interior substructure of a standard form with perfect recall or, in other words, a game in the class $J(\mathcal{R})$. In order to show that our definition (5.4) of the bicentric prior strategy in fact describes a well defined mixed strategy we shall prove a more general result on central local best replies. Some auxiliary definitions and notations will now be introduced in order to prepare this result.

Let L_i be the set of all behavior strategies r_i of player i, such that for every agent ij of player i the local strategy r_{ij} prescribed by r_i to agent ij is the centroid of some subset of agent ij's choice set Φ_{ij}. Obviously, a strategy r_i which is a central local best reply $a_i(q_{.i})$ to some i-incomplete joint mixture must be an element of R_i. Therefore, we call L_i the set of potential central local best replies of player i. It is clear that R_i is a finite set.

For every $r_i \in L_i$ let $R(r_i)$ be the set of all $q_{.i} \in Q_{.i}$ such that r_i is the central local best reply to $q_{.i}$. We call $R(r_i)$ the central local best reply stability region of r_i or shortly the stability region of r_i where there is no danger of confusion with the stability region $S(\varphi_i)$ introduced in section 4 of chapter 3. One may think of the correspondence R as the inverse of the central local best reply function a_i.

The following lemma will assert that $R(r_i)$ is convex; this means that for $q_{.i} \in R(r_i)$ and $r_{.i} \in R(r_i)$ and $0 < \alpha < 1$ the joint mixture $s_{.i}$ with

$$(5.7) \qquad s_{.i}(\varphi_{-i}) = \alpha q_{.i}(\varphi_{-i}) + (1-\alpha) r_{.i}(\varphi_{-i})$$

for every $\varphi_{-i} \in \Phi_{-i}$ is also in $R(r_i)$.

Convexity lemma: Let $G = (\Phi,H)$ be an interior substructure of a standard form with perfect recall and let $r_i \in L_i$ be a potential central local best reply of player i in G. Then the central local best reply stability region $R(r_i)$ of r_i in G is convex.

Proof: Let b_i^0, b_i^1, ... be a best reply sequence for a joint mixture $q_{.i}$. According to the theorem on coordination in chapter 2 the sequence $b_i^0, b_i^1, \ldots$

converges after a finite number of steps and it has been pointed out in the remark after the proof that $|M_i|$ is an upper bound of this number.

In the proof of the theorem on coordination a partition M_i^1, M_i^2 of M_i has been introduced. The construction was based on a fixed tree K_i of player i (a tree of player i in the game with perfect recall whose substructure G is assumed to be.) It is an important property of the construction that for $ij \in M_i^{k+1}$ the forward set $[ij\rangle$ relative to K_i belongs to $M_i^1 \cup \ldots \cup M_i^k$. (For the definition of $[ij\rangle$ see section 4 of chapter 2). In a best reply sequence $b_i^o, b_i^1, \ldots$ for $q_{.i}$ the local strategies of agents in M_i^k do not change any more after b_i^k. The local best replies of agents $ij \in M_i^{k+1}$ to $b_i^k q_{.i}$ maximize his local payoff $H_{ij}(b_{[ij\rangle} q_{.i})$. It is important to note that for fixed $b_{[ij\rangle}$ this local payoff is a linear function of the probabilities $q_{.i}(\varphi_{-i})$.

Let c_i be that behavior strategy of player i which for every agent ij assigns equal probabilities to all choices in Φ_{ij}. A best reply sequence $b_i^o, b_i^1, \ldots$ to a joint mixture $q_{.i}$ will be called <u>normal</u>, if it starts from $b_i^o = c_i$. Obviously, exactly one normal best reply structure belongs to every joint mixture $q_{.i}$.

Consider two joint mixtures $r_{.i}$ and $q_{.i}$. Let $b_i^o, b_i^1, \ldots$ and $g_i^o, g_i^1, \ldots$ be the normal best reply sequences for $r_{.i}$ and $q_{.i}$, respectively. Suppose that we have $b_i^k \neq g_i^k$ for some k. Since the local strategies of agents in M_i^k do not change after b_i^k and g_i^k, respectively, in the two normal best reply sequences, we must have $a_i(r_{.i}) \neq a_i(q_{.i})$. Therefore, the same normal best reply sequence $b_i^o, b_i^1, \ldots$ belongs to every $r_{.i} \in R(r_i)$.

For every sequence $b_i^o, \ldots, b_i^k$ with $b_i^o = c_i$ and $b_i^m \in L_i$ for $m = 1, \ldots, k$ let $N(b_i^o, \ldots, b_i^k)$ be the set of all joint mixtures $q_{.i}$ such that the first k+1 members of the normal best reply sequence for $q_{.i}$ are the strategies $b_i^o, \ldots, b_i^k$. We shall prove by induction on k that $N(b_i^o, \ldots, b_i^k)$ is convex. In view of the fact that the same normal best reply sequence belongs to every $r_i \in R(r_i)$ this is sufficient for the convexity of $R(r_i)$.

Obviously, the assertion that $N(b_i^o, \ldots, b_i^k)$ is convex holds for k = o. In order to see that the assertion holds for k + 1 if it holds for k consider a sequence $b_i^o, \ldots, b_i^{k+1}$ with $b_i^o = c_i$ and $b_i^m \in L_i$ for $m=1, \ldots, k+1$.

We can assume that $N(b_i^o,\ldots,b_i^{k+1})$ is non-empty since the empty set is convex anyhow. Consider three joint mixtures $q_{.i}$, $r_{.i}$ and $s_{.i}$ related as in (5.7) and with $q_{.i}$ and $r_{.i}$ in $N(b_i^o,\ldots,b_i^{k+1})$. In view of the definition of b_i^{k+1} an agent $ij \in M_i^{k+1}$ has the same local best replies to $b_i^k\, q_{.i}$ and to $b_i^k\, r_{.i}$. In view of the convexity of $N(b_i^o,\ldots,b_i^k)$ and the linearity of $H_{ij}(b_{[ij>}q_{.i})$ with respect to $q_{.i}$ we can conclude that the local best replies of an agent $ij \in M_i^{k+1}$ to $b_i^k\, s_{.i}$ are the local best replies of this agent to $b_i^k\, q_{.i}$ and $b_i^k\, r_{.i}$. This shows that $N(b_i^o,\ldots,b_i^{k+1})$ is convex.

Consequences for the bicentric prior: The convexity of the central local best reply stability regions has the consequence that under the assumptions underlying the definition of the bicentric prior the interval $0 \leq z \leq 1$ is partitioned into finitely many subintervals where r_i^z is constant. Each of these subintervals corresponds to the intersection of a stability region $R(r_i)$ with the set of all joint mixtures of the form $zU_{-i}+(1-z)V_{-i}$. The values of z for mixtures in the intersection form the subinterval which will be denoted by $Z(r_i)$. We call $Z(r_i)$ the z-line subinterval for r_i. Of course, in many cases $Z(r_i)$ may be empty and in others it may consist of a single point. For every $r_i \in L_i$ let $|Z(r_i)|$ be the length of the subinterval $Z(r_i)$. Obviously, instead of (5.4) we can also write:

$$(5.8) \qquad p_i(\varphi_i) = \sum_{r_i \in L_i} |Z(r_i)|\, r_i(\varphi_i)$$

for every $\varphi_i \in \Phi_i$

We say that $Z(r_i)$ is an essential subinterval if $|Z(r_i)|$ is positive. A strategy $r_i \in L_i$ is called essential for p_i if $Z(r_i)$ is an essential subinterval. Obviously, only the essential $r_i \in L_i$ contribute anything to the sum (5.8).

It is now clear that no problems arise with respect to the integrability of r_i^z in the definition of the bicentric prior. Moreover, (5.8) indicates how the bicentric prior can be computed in applications to specific examples.

We shall now turn our attention to the properties of formation consistency and invariance with respect to isomorphisms.

Formation consistency lemma: Let G be an interior substructure of a standard form with perfect recall and let U and V be two different equilibrium points of G, not necessarily in pure strategies. Moreover, let $\bar{G}$ be a formation which contains both U and V. Then U risk dominates V in $\bar{G}$,if and only if U risk dominates V in G.

Proof: The formation F spanned by U and V in $\bar{G}$ is also the formation spanned by U and V in G. In both cases we receive the same restricted game and the same bicentric prior.

Lemma on invariance with respect to isomorphisms: Let f be an isomorphism from a game $G = (\Phi,H) \in J(\mathcal{R})$ to a game $\bar{G} = (\bar{\Phi},\bar{H}) \in J(\mathcal{R})$ and let U and V be two different equilibrium points of G. Then U risk dominates V in G, if and only if f(U) risk dominates f(V) in $\bar{G}$.

Proof: Let $G' = (\Phi',H')$ and $\bar{G}' = (\bar{\Phi}',\bar{H}')$ be the restricted games of G and $\bar{G}$, respectively. Obviously, the restriction f' of f to Φ' is an isomorphism from G' to $\bar{G}'$.

The linear payoff transformations connected to the isomorphism f' also carry the logarithmic payoffs of the auxiliary games in the logarithmic tracing procedure for G' to the corresponding payoffs for $\bar{G}'$. This can be seen from the definition of the logarithmic tracing procedure. The weights α_i^* of the logarithmic terms are defined as maximal payoff differences and therefore change in the appropriate way (chapter 4, section).

The isomorphism f' maps the bicentric prior for the comparison between U and V to the bicentric prior for the comparison between f(U) and f(V) since isomorphisms preserve the best reply structure. It follows that the assertion of the lemma holds.

Unanimity games: In the following we shall investigate risk dominance in a special class of games.

A unanimity game $G = (\Phi,H)$ with $\Phi = \Phi_1 \times \ldots \times \Phi_n$ is a game with normal form structure whose strategy sets and payoffs are as follows: For i=1,...,n the strategy set Φ_i contains m pure strategies $U_i^1,\ldots,U_i^m$. We use the notation $U^j = (U_1^j, \ldots, U_n^j)$. Payoffs are defined as follows:

$$(5.9)\qquad H_i(\varphi) = \begin{cases} u_i^j & \text{for } \varphi = U^j, \quad j = 1, \ldots, m \\ 0 & \text{else} \end{cases}$$

where all u_i^j are positive numbers ($i=1, \ldots, n$; $j=1, \ldots, n$). We use the notation $u^j = (u_1^j, \ldots, u_n^j)$.

The vectors u^j can be interpreted as payoff vectors attached to possible agreements. A pure strategy consists in voting for one of these agreements. An agreement is reached if and only if the players unanimously vote for it.

The set of all u^j with $j = 1, \ldots, m$ is denoted by X. In view of the interpretation given above, X is called the <u>agreement set</u>. Sometimes we shall distinguish the elements of X by different letters u, v, ... rather than by upper indices; in such cases indexed capital letter U_i, V_i, ... will be used for the corresponding pure strategies; thereby we avoid double indices.

The pure strategy combinations U^j are strong equilibrium points of $G=(\Phi,H)$. The game may have additional pure strategy equilibrium points like (U_1, V_2, W_3) in a 3-person unanimity game with $X = \{u,v,w\}$ but these equilibrium points are weak since no player loses anything by deviation.

The <u>Nash product</u> of a strong equilibrium point U is the product $u_1 \cdot u_2 \cdot \ldots \cdot u_n$ of all components of its payoff vector $u = (u_1, \ldots, u_n)$. For the special case of 2-person unanimity games this definition coincides with that of chapter 3, section 6. Nash's cooperative bargaining theory with fixed threats selects that agreement which has the highest Nash product. We shall show that our concept of risk dominance is in harmony with Nash's theory. In order to do this we shall compute risk dominance between strong equilibrium points in unanimity games.

The definition of risk dominance has been given for games in the class $J(\mathcal{P})$ but it can be applied to other games as well as long as no difficulties arise. We do not want to discuss the question whether unanimity games belong to $J(\mathcal{P})$. In order to compute the limit solution of unanimity games we would have to look at their ε-perturbations. As we shall argue later the theorem stated below remains true for ε-perturbations with sufficiently small ε.

Nash-product theorem: Let U and V be two strong equilibrium points of a unanimity game $G = (\Phi, H)$. The equilibrium point U risk dominates V, if the Nash-product of U is greater than that of V.

Proof: Since we have $U_i \neq V_i$ for $i = 1, \ldots, n$ no player i is fixed in the restricted game. In the 2-person case $\Psi = \Psi_1 \times \Psi_2$ with $\Psi_i = \{U_i, V_i\}$ is the set of pure strategy combinations of the formation spanned by U and V. For $n > 2$ the whole game is spanned by U and V since every pure strategy is a best reply to an i-incomplete pure strategy combination where two players j and k use U_j and V_k. Therefore, for $n > 2$ the restricted game agrees with the whole game.

In order to compute the bicentric prior strategies p_i we look at the following payoffs.

$$(5.10) \qquad H_i(U_i[zU_{-i} + (1-z)V_{-i}]) = zu_i$$

$$(5.11) \qquad H_i(V_i[zU_{-i} + (1-z)V_{-i}]) = (1-z)v_i$$

The comparison of (5.10) and (5.11) shows that the following is true for the best reply r_i^z to $zU_{-i} + (1-z)V_{-i}$:

$$(5.12) \qquad r_i^z = \begin{cases} U_i & \text{for } 1 > z > \dfrac{v_i}{u_i+v_i} \\ \\ V_i & \text{for } 0 < z < \dfrac{v_i}{u_i+v_i} \end{cases}$$

Wherever there is only one best reply, this is also the central local best reply. Figure 5.2 graphically represents the result. The joint mixtures $zU_{-i}+(1-z)V_{-i}$ are shown as points on the line segment $0 \leq z \leq 1$. The essential subintervals $Z(U_i)$ and $Z_1(V_i)$ meet at the critical point $v_i/(u_i + v_i)$. The graphical representation in figure 5.2 will be referred to as player i's z-line.

According to (5.8) the probabilities assigned to U_i and V_i by player i's bicentric strategy p_i are determined by the length of the subintervals for U_i and V_i:

$$(5.13) \qquad p_i(U_i) = \frac{u_i}{u_i+v_i}$$

$$(5.14) \qquad p_i(V_i) = \frac{v_i}{u_i+v_i}$$

for $i = 1, \ldots, n$. We now look at player i's payoff obtained at p_{-i}.

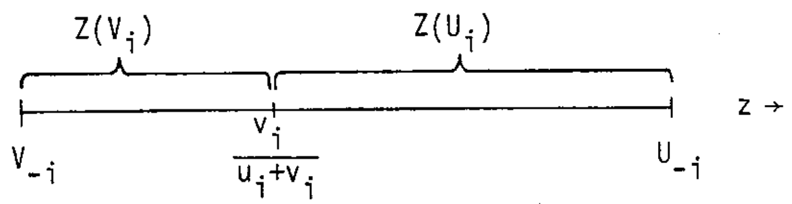

Figure 5.2: Player i's z-line.

Let N_i be the set of all players except i.

$$(5.15) \quad H_i(U_i p_{-i}) = u_i \prod_{k \in N_i} \frac{u_k}{u_k+v_k}$$

$$(5.16) \quad H_i(V_i p_{-i}) = v_i \prod_{k \in N_i} \frac{v_k}{u_k+v_k}$$

This shows that we have:

$$(5.17) \quad H_i(U_i p_{-i}) > H_i(V_i p_{-i})$$

for $i = 1,\ldots,n$ if the Nash product of U is greater than that of V. If this is the case then U is the only best reply to the bicentric prior $p = (p_1,\ldots,p_n)$ and the tracing procedure yields $T(G,p) = U$. This shows that the assertion is true.

Comment: In order to have a short name the property of our risk dominance definition exhibited by the Nash-product theorem will be referred to as the Nash-property.

In their attempts to define risk dominance in a satisfactory way the authors have been guided by the idea that it is desirable to reproduce the result of Nash's cooperative bargaining theory with fixed threats. The Nash-property is not an unintended byproduct of our theory.

One may object that other axiomatic bargaining theories like that of Kalai and Smorodinsky seem to be equally plausible (Kalai-Smorodinsky 1975). We think that we can reject this point of view since our axiomatic characterization of risk dominance between strong equilibrium points in 2x2-games supports the Nash-product as a selection criterion.

Note that the risk dominating equilibrium point with the higher Nash-product is obtained as the strong best reply to the bicentric prior and not in a more substantial application of the tracing procedure. We think that this is a desirable feature of our theory. Unanimity games have a very simple structure. Therefore, a reasonable equilibrium selection theory should be expected to solve them in a simple way.

For the class $\hat{\mathcal{G}}$ of 2x2-games with two strong equilibrium points a risk dominance relation has been characterized by three axioms in chapter 3, section 8. The following lemma shows that the definition introduced in this chapter yields the same risk dominance relation on $\hat{\mathcal{G}}$.

Lemma on 2x2-games: Let U and V be two different strong equilibrium points of a 2x2-game and for i = 1,2 let u_i and v_i be the deviation losses of U and V, respectively, (see figure 3.5). The U risk dominates V if and only if we have :

$$(5.18) \qquad u_1 u_2 > v_1 v_2$$

Proof: As has been shown in chapter 3, section 4, the 2x2-game can be transformed into the unanimity game of figure 3.7 without changing the best reply structure. This game is equivalent to the game in figure 3.14 (chapter 3, section 8). Figure 3.7 permits the conclusion that the bicentric prior strategy p_i of player i is given by (5.13) and (5.14) and that U is the only best reply to $p = (p_1, p_2)$ if (5.18) holds. Therefore, (5.16) is sufficient for risk dominance of U over V. Moreover, V risk dominates U for $v_1v_2 > u_1u_2$. In order to see that for $u_1u_2 = v_1v_2$ none of both equilibrium points risk dominates the other we remember that in this case the game of figure 3.14 has a symmetry f which carries U to V and vice versa (see chapter 3,setion 8). In view of the lemma on invariance with respect to isomorphisms this excludes risk dominance of one of both equilibrium points over the other. It follows that U risk dominates V if and only if (5.18) holds.

Intransitivities: A unanimity game is called non-degenerate if any two strong equilibrium points have different Nash-products. The Nash-product theorem shows that in non-degenerate unanimity games risk dominance between strong equilibrium points is a transitive relationship. Unfortunately, we cannot expect this kind of transitivity in general.

Consider the game G in figure 3.25. This game has three strong equilibrium points (α,α), (β,β) and (γ,γ). We are going to show that the following is true:

(i) (α,α) risk dominates (β,β)
(ii) (β,β) risk dominates (γ,γ)
(iii) (γ,γ) risk dominates (α,α)

In order to see this we need not apply our definition in detail. Any formation consistent risk dominance relation which agrees with ours on 2x2-games with two strong equilibrium points must satisfy (i), (ii) and (iii).

A formation is obtained if γ is removed from the strategy sets of both players. The same is true with respect to the other two pure strategies. These formations are 2x2-games. In order to determine risk dominance it is sufficient to compare Nash-products. The Nash-products for the comparison between (α,α) and (β,β) are 21 and 18, those for (β,β) versus (γ,γ) are 18 and 16 and those for (γ,γ) versus (α,α) are 32 and 21.

Comment: Our notion of risk dominance is based on the idea of a hypothetical situation where it is generally believed that one of two equilibrium points U and V is the solution. As long as the players follow

Bayesian reasoning processes the uncertainty between U and V cannot move their behavioral inclinations out of the formation spanned by U and V. Therefore, the property of formation consistency seems to be unavoidable. This together with our axiomatic characterization of risk dominance for 2x2-games with two strong equilibrium points leads to the conclusion that we should not expect transitivity.

Obviously, the intransitivities in the game of figure 3.25 are connected with the impossibility theorem of chapter 3, section 11. The cells which are produced by sequential agent splitting are formations of the game. The cell structure obtained by sequential agent splitting has the consequence that one of the three risk dominance comparisons becomes irrelevant. This fact is exploited by the proof of the theorem.

The payoff monotonicity counterexample: In chapter 3, section 7, we have discussed the numerical example of figures 3.12 and 3.13 which throws doubt on payoff monotonicity as a desirable property of a risk dominance relation. The numbers have been chosen in such a way that payoff monotonicity does not hold for the example with the definition introduced in this chapter. We shall show that U risk dominates V in figure 3.12 and that V risk dominates U in figure 3.13.

Both games are symmetric with respect to players 1 and 2. Therefore, it is sufficient to compute the bicentric prior strategies for player 1 and player 3. A player's bicentric prior depends only on his own payoffs. Therefore, the bicentric prior strategy p_1 of player 1 is the same in both games. Player 3's bicentric prior strategy in the game of figure 3.12 will be denoted by p_3 and that for figure 3.13 will be denoted by p_3'. We obtain the following results:

(5.19) $\quad p_1(U_1) = \frac{7}{11} \qquad p_1(V_1) = \frac{4}{11}$

(5.20) $\quad p_3(U_3) = \frac{1}{4} \qquad p_3(V_3) = \frac{3}{4}$

(5.21) $\quad p_3'(U_3) = \frac{2}{5} \qquad p_3'(V_3) = \frac{3}{5}$

We can now compute player 1's payoff for U_1 and V_1 if the others use

$p_{-1} = p_2p_3$ or $p'_{-1} = p_2p'_3$.

(5.22) $$H_1(U_1p_{-1}) = \frac{7}{44} \cdot 7 + \frac{21}{44} \cdot 3 = \frac{112}{44}$$

(5.23) $$H_1(V_1p_{-1}) = \frac{4}{44} \cdot 15 + \frac{12}{44} \cdot 4 = \frac{108}{44}$$

(5.24) $$H'_1(U_1p'_{-1}) = \frac{14}{55} \cdot 7 + \frac{21}{55} \cdot 3 = \frac{161}{55}$$

(5.25) $$H'_1(V_1p'_{-1}) = \frac{8}{55} \cdot 15 + \frac{12}{55} \cdot 4 = \frac{168}{55}$$

The payoff functions for figures 3.12 and 3.13 are denoted by H and H', respectively. Player 3 is faced with $p_{-3} = p_1p_2$ in both games.

(5.26) $$H_3(U_3p_{-3}) = \frac{49}{121} \cdot 1 = \frac{49}{121}$$

(5.27) $$H_3(V_3p_{-3}) = \frac{16}{121} \cdot 3 = \frac{48}{121}$$

(5.28) $$H'_3(U_3p_{-3}) = \frac{49}{121} \cdot 2 = \frac{98}{121}$$

(5.29) $$H'_3(V_3p_{-3}) = \frac{16}{121} \cdot 3 = \frac{48}{121}$$

These computations show that U is the only best reply to the bicentric prior in the game of figure 3.12. Consequently, in this game U risk dominates V.

In the game of figure 3.13 the only best reply to the bicentric prior is (V_1,V_2,V_3). At this strategy combination players 1 and 2 have an incentive to play V_1 and V_2. Therefore, in the games G^t which arise in the application of the tracing procedure their best replies to (V_1,V_2,V_3) will always be V_1 and V_2. At some critical value of t player 3 will switch over to V_3 since his best reply to (V_1,V_2,V_3) is V_3. The final result is V. This shows that in the game of figure 3.13, where player 3 has a higher payoff at U, the equilibrium point V risk dominates U.

Comment: The result obtained for the payoff monotonicity counterexample does not look unreasonable. In figure 3.13 player 3's incentive to use U_3 is stronger than in figure 3.12. Therefore, we have $p_3'(U_3) > p_3(U_3)$. Since V_1 and V_2 are very advantageous for players 1 and 2 if player 3 plays U_3 their best replies to the bicentric prior shift from U_1 and U_2 to V_1 and V_2 in the transition from figure 3.12 to 3.13.

In many cases the deviation of one player from an equilibrium point decreases the other players' incentive to stick to it. At the equilibrium point V of figure 3.12 and 3.13 the situation is reversed as far as deviations of player 3 are concerned. His deviation to U_3 has a stabilizing effect in the sense that it increases the other players' incentive to stick to V. Therefore, it works in favor of V that player 3 is more strongly attracted to U_3 in figure 3.13.

Risk dominance in ε-perturbations: In the application of our theory to special classes of games of substantial interest, like bargaining games or oligopoly games, risk dominance comparisons often have to be computed for pairs of strong equilibrium only. With the exception of degenerate border cases, it rarely makes a difference whether risk dominance between two strong equilibrium points is determined in an ε-perturbation with sufficiently small ε or in the unperturbed game. In the following we shall look at the important case where one of both equilibrium points is the only best reply to the bicentric prior. Under a mild regularity condition on the z-line "conspicuous risk dominance" will be used in order to describe this situation.

Conspicuous risk dominance: Let $G = (\Phi,H)$ be a standard form, not necessarily in J ($\mathcal{K}$) and let U and V be two strong equilibrium points of G. We say that player i has a regular z-line if in the interval $0 \leq z \leq 1$ with the exception of finitely many points there is only one pure strategy φ_i for every z such that φ_i is a best reply to $zU_{-i} + (1-z)V_{-i}$. Obviously, φ_i is the central local best reply r_i^z if this is the case. Moreover, in view of the linearity of $H_i(\varphi_i[zU_{-i} + (1-z)V_{-i}])$ as a function of z, the set $Z(\varphi_i)$ of all z such that φ_i is a best reply to $zU_{-i}+(1-z)V_{-i}$ is a subinterval of $0 \leq z \leq 1$. Therefore, a bicentric prior strategy p_i can be computed according to (5.4) if player is has a regular z-line. We say that U conspicuously risk dominates V in G if every player i in the player set N' of the restricted game G' for the comparison between U and V has a regular z-line and if in G' the equilibrium point U' corresponding to U is the only best reply to the bicentric prior $p' = (P_i)_{N'}$.

Conspicuous risk dominance lemma: Let U and V be two strong equilibrium points of a standard from $G = (\Phi,H)$ with perfect recall such that U conspicuously risk dominates V in G. For every ε-perturbation $G_\varepsilon = (\Phi_\varepsilon,H_\varepsilon)$ of G let U_ε and V_ε be the strategy combinations whose local strategies $U_{\varepsilon ij}$ and $V_{\varepsilon ij}$ are the ε-extreme local strategies corresponding to the local strategies U_{ij} and V_{ij} in U and V, respectively. Then an $\bar{\varepsilon} > 0$ can be found such that for every $0 < \varepsilon \leq \bar{\varepsilon}$ the strategy combination U_ε and V_ε are strong equilibrium points of G_ε such that U_ε risk dominates V_ε.

Proof: As can be seen by (2.42) the payoff vectors $H(U_\varepsilon)$ and $H(V_\varepsilon)$ are continuous functions of ε. Consequently, for sufficiently small ε the strategy combinations U_ε and V_ε are strong equilibrium points of G_ε. We shall assume that all players i in G have different equilibrium strategies in U and V. If this is not the case the argument can be applied to the game which results from G by fixing the players with the same strategy in U and V at their equilibrium strategies in G_ε.

Let $\varphi_{\varepsilon i}$ be the ε-extreme strategy corresponding to a pure strategy φ_i; this means that the local strategies $\varphi_{\varepsilon ij}$ prescribed by $\varphi_{\varepsilon i}$ are the ε-extreme strategies corresponding to the choices φ_{ij} prescribed by φ_i. The payoff

$$H_{\varepsilon i}(\varphi_{\varepsilon i}[zU_{\varepsilon -i} + (1-z)V_{\varepsilon -i}])$$

is a continuous function of ε. Consequently, for sufficiently small ε the ε-extreme strategy $\varphi_{\varepsilon i}$ is the only best reply to $zU_{\varepsilon -i} + (1-z)V_{\varepsilon -i}$ if φ_i is the only best reply to $zU_{-i} + (1-z)V_{-i}$. This shows that for $\varepsilon \to 0$ the bicentric prior p_ε for the comparison between U_ε and V_ε in G_ε converges to the bicentric prior p for the comparison between U and V in G. We can conclude that for sufficiently small ε the equilibrium point U_ε is the only best reply to p_ε in G_ε. Therefore, the assertion of the lemma holds.

Comment: The conspicuous risk dominance lemma can be applied to the special case of unanimity games. In this way we receive the following analogy to the Nash product theorem.

Nash-product theorem for perturbations: Let U and V be two strong equilibrium points of a unanimity game $G = (\Phi,H)$, such that the Nash-product of U is greater than the Nash-product of V. For every ε-perturbation G_ε of G let U_ε and V_ε be defined as in the conspicuous risk dominance lemma. Then an $\bar{\varepsilon} > 0$ can be found with the property that for $0 < \varepsilon \leq \bar{\varepsilon}$ the strategy combinations U_ε and V_ε are strong equilibrium points of G_ε, such that U_ε risk dominates V_ε in G_ε.

Proof: The proof of the Nash-product theorem has shown that U conspicuosly risk dominates V. Unanimity games have normal form structure and therefore are standard forms with perfect recall. The assertion is an immediate consequence of the conspicuous risk dominance lemma.

4. Candidate elimination and substitution

The last missing piece in the definition of our solution concept is the process of candidate elimination and substitution which will be described in this section. In the introduction of the chapter we have already mentioned the auxiliary notions which are used in the process of candidate elimination and substitution. We shall first define dominance, then strategic distance, stretegic net distance and maximal stability and, finally, the substitute of a candidate set. The process is described by the flow chart in figure 5.3. In the same way as in earlier chapters we shall try to motivate our definitions where they are introduced.

Dominance: Let $U = (U_i)_N$ and $V = (V_i)_N$ be two different equilibrium points, not necessarily in pure strategies, of a game $G = (\Phi,H)$ in the class $J(\mathcal{R})$. We say that U dominates V if one of the following two statements (i) and (ii) holds:

(i) $H_i(U) > H_i(V)$ for every $i \in N$ with $U_i \neq V_i$

(ii) U risk dominates V and $H_i(U) \geq H_i(V)$ for at least one $i \in N$ with $U_i \neq V_i$.

We write $U \succ V$ if U dominates V and $U \| V$ if none of both equilibrium points dominates the other.

Interpretation: Dominance is a combination of payoff dominance and risk dominance. As in the interpretation of risk dominance we look at a hypothetical situation where it is generally believed that either U or V is the solution. Players with $U_i = V_i$ can be expected to play this strategy. Therefore, payoff dominance as well as risk dominance concerns only players with $U_i \neq V_i$ or, in other words, the players who belong to the restricted game. Risk dominance of U over V does not matter if V payoff dominates U in the restricted game. Therefore, in (ii) we require $H_i(U) \geq H_i(V)$ for at least one $i \in N$ with $U_i \neq V_i$.

In the definition of dominance, payoff dominance has priority over risk dominance. We take the point of view that there is no risk involved in a situation where expectations can be coordinated by common payoff interests of the relevant players (see chapter 3, section 8).

The idea of strategic distance: Before we formally define the measure of strategic distance we want to indicate our reasons for doing this. Not all risk dominance comparisons can be regarded as equally important. We think that it is reasonable to give more weight to comparisons between equilibrium points which are near to each other in a strategically relevant sense. In order to give a precise meaning to this intuitive idea one needs a measure of strategic distance.

Consider two equilibrium points U and V of a game G. We think of the strategic distance between U and V as connected to the differences between the strategies used in U and V. Therefore, we take the point of view that those players whose strategies in U and V agree do not contribute anything to the strategic distance between U and V.

Consider a player i whose strategies in U and V are different from each other. How can one measure the difference between his equilibrium strategies U_i and V_i? In order to answer this question we imagine that player i's beliefs are described by a joint mixture $zU_{-i} + (1-z)V_{-i}$. This idea has been used already for the definition of the bicentric prior. Suppose that player i first firmly believes in V_{-i} which corresponds to $z = 0$ and then gradually increases his confidence into U_{-i} until $z = 1$ is reached. Each z is connected to a central local best reply r_i^z to $zU_{-i}+(1-z)V_{-i}$. (See (5.3) in section 2 of this chapter). As z is increased from 0 to 1 the strategy r_i^z changes at finitely many critical points and remains constant in the open subinterval between two neighboring critical points of this kind. Our measure of strategic distance will be the number of all critical points summed up over all players with $U_i \neq V_i$.

The greater the number of critical points is the more difficult is the comparison between U and V. One may think of our definition of strategic distance as a measure of the intensity of initial confusion arising in the hypothetical situation where all players believe that either U or V is the solution. The initial confusion is measured before the bicentric prior has been formed in the first stage of the emergence of Bayesian expectations. This seems to be reasonable since the bicentric prior can be looked upon as a preliminary resolution of initial confusion.

Strategic distance: Let $U = (U_i)_N$ and $V = (V_i)_N$ be two different equilibrium points, not necessarily in pure strategies of a game $G = (\Phi,H)$ in the class $J(\mathcal{R})$ of interior substructures of standard forms with perfect recall. let N' be the player set of the restricted game G' for the comparison of U and V. Consider a player $i \in N'$ and the essential subintervals $Z(r_i)$ on his z-line (see consequences for the bicentric prior, section 3). There are finitely many points $z_1,\ldots,z_s$ where two adjacent essential subintervals meet; these points $z_1,\ldots,z_s$ are called critical points of player i; the number of player i's critical points is denoted by $e_i(U,V)$. The strategic distance $e(U,V)$ between U and V is defined as follows:

$$e(U,V) = \sum_{i \in N'} e_i(U,V) \tag{5.30}$$

Strategic distance in unanimity games: Consider two strong equilibrium points U and V of a unanimity game G. As we have seen in section 3 (see figure 5.2) in this case every player has exactly one critical point, namely $v_i/(u_i + v_i)$. The strategic distance $e(U,V)$ is nothing else than the number of players.

Next neighbors: Let Ω be a set of equilibrium points for a game $G \in J(\mathcal{R})$. (We may think of a candidate set, e.g. the first candidate set Ω_1, defined in section 1). We assume that Ω has at least two elements. Consider two different equilibrium points U and V in Ω. We say that V is a next neighbor of U in Ω if we have :

(5.31) $$e(U,V) = \min_{W \in \Omega \setminus \{U\}} e(U,W)$$

We use the notation

(5.32) $$e(U,\Omega) = \min_{W \in \Omega \setminus \{U\}} e(U,W)$$

for the distance of U to a next neighbor in Ω.

The idea of strategic net distance: Our theory gives more weight to dominance comparisons between equilibrium points which are relatively near to each other. The most important dominance comparisons within a candidate set Ω are those between two equilibrium points which are next neighbors of each other in Ω. Note that U is not necessarily a next neighbor of V in Ω if V is a next neighbor of U in Ω. In order to judge the importance of a dominance comparison of U and V relative to a candiate set Ω our theory looks at the question how close U and V come to being next neighbors in Ω. Closeness to the condition of being next neighbors to each other in Ω can be measured by the sum of the surpluses $e(U,V) - e(U,\Omega)$ and $e(U,V) - e(V,\Omega)$ of the distance between U and V over the distances of U and V to their next neighbors. In order to avoid distances of zero between two different candidates we add 1 to this sum of surpluses. In this way, we obtain our measure of strategic net distance.

One may ask why we do not take strategic distance rather than strategic net distance as a measure of importance of dominance comparisons within a candidate set. In fact, an earlier version of our theory was based on strategic distance rather than strategic net distance. Examples have led us to the conclusion that an equilibrium point should not be judged as extraordinarily stable simply because it is far away from its next neighbors in the candidate set whereas all other candidates are near to each other. Strategic net distance as a measure of importance of dominance comparisons does not give any advantage to candidates which are far off from other candidates. The stability of each candidate is judged in terms of comparisons with other candidates who are relatively near to it.

Strategic net distance: Let Ω be a set of equilibrium points for a game $G \in J(\mathcal{P})$ and let U and V be two different equilibrium points in

Ω. The *strategic net distance* $e(U,V,\Omega)$ *of* U *and* V *in* Ω is defined as follows:

$$(5.33) \qquad e(U,V,\Omega) = 2e(U,V) - e(U,\Omega) - e(V,\Omega) + 1$$

It can be seen immediately that $e(U,V,\Omega)$ is equal to $e(V,U,\Omega)$. Moreover, it follows by (5.32) that $e(U,V,\Omega)$ is always positive.

The use of the same symbol e for both strategic distance and strategic net distance does not lead to confusion since distance is a function of two arguments and net distance is a function of three arguments. The *maximal net distance within* Ω is defined as follows:

$$(5.34) \qquad e(\Omega) = \max_{U,V\in\Omega} e(U,V,\Omega)$$

Stability: As before let Ω be a set of at least two equilibrium points for a game $G \in J(\mathcal{R})$. We say that $U \in \Omega$ is *undominated* in Ω if no $V \in \Omega$ with $V \neq U$ dominates U; otherwise we says that U is *dominated* in Ω. For every $U \in \Omega$ we define a *stability index* $\sigma(U,\Omega)$ of U in Ω:

$$(5.35) \qquad \sigma(U,\Omega) = e(\Omega)$$
if U is undominated in Ω

$$(5.36) \qquad \sigma(U,\Omega) = \min[e(U,V,\Omega) \mid V\in\Omega \text{ and } V \succ U]-1$$
if U is dominated in Ω

The right hand side of (5.36) is the smallest number k such that no $V\in\Omega$ with $V \neq U$ and $e(U,V,\Omega) \leq k$ dominates U. Equation (5.36) has no meaning if U is undominated in Ω. If U is dominated in Ω then the right hand side of (5.36) is at most $e(\Omega)-1$. This means that (5.35) assigns the highest possible stability index to undominated equilibrium points in Ω if there are any. Define:

$$(5.37) \qquad \sigma(\Omega) = \max_{U\in\Omega} \sigma(U,\Omega)$$

$\sigma(\Omega)$ is the *maximal stability index in* Ω. We say that U is *maximally stable in* Ω if we have:

$$(5.38) \qquad \sigma(U,\Omega) = \sigma(\Omega)$$

The set of all maximally stable elements of Ω is denoted by $\mathfrak{M}(\Omega)$.

Remarks: If Ω contains equilibrium points which are undominated in Ω then the set $\mathfrak{M}(\Omega)$ of maximally stable elements of Ω is the set of all $U \in \Omega$ which are undominated in Ω. This is an immediate consequence of (5.35) and (5.36).

The stability index $\sigma(U,\Omega)$ is one of the integers $0, \ldots, e(\Omega)$. It may happen that all the $U \in \Omega$ have the same stability index $\sigma(U,\Omega)$. If this is the case we have $\mathfrak{M}(\Omega) = \Omega$. There are two ways in which $\mathfrak{M}(\Omega)$ may fail to be smaller than Ω. It may happen that all elements of Ω are undominated in Ω; then we have $\sigma(U,\Omega) = e(\Omega)$ for all $U \in \Omega$. Since risk dominance may be cyclical it is also possible that all elements of Ω are dominated by other elements of Ω and that all of them have the same stability index $\sigma(U,\Omega)$.

Comments: On the basis of strategic net distance as a measure for the importance of dominance comparisons it is reasonable to use the stability index in order to determine those elements of a candidate set which are considered to be maximally stable relative to this set. Whenever $\mathfrak{M}(\Omega)$ is smaller than Ω our process of candidate elimination and substitution will perform an elimination step which eliminates all candidates not in $\mathfrak{M}(\Omega)$. Generally, stability indices with respect to $\Omega' = \mathfrak{M}(\Omega)$ are different from stability indices with respect to Ω and it may be possible to continue the elimination by the application of the elimination step to the new candidate set. Sometimes the first candidate set Ω can be narrowed down to a single element by repeated application of the elimination step. However, this is not always possible.

If after a number of elimination steps we obtain a candidate set Ω with $\mathfrak{M}(\Omega) = \Omega$ all equilibrium points in Ω must be considered equally good or rather equally bad since no selection can be made among them on the basis of their stability within Ω. In this situation the process of elimination and substitution performs a substitution step. An imprecise description of the substitution step has been given already in the introduction of the chapter.

In order to prepare the definition of the substitute of a candidate set we shall first define the centroid of a candidate set. The centroid of a candidate set is a mixed strategy combination. Like the bicentric prior it has the interpretation of a special prior distribution for the tracing procedure. Each player is expected to use his equilibrium strategy in each of the candidates with the same probability. This means that in the computation of a player's centroid strategy each equilibrium strategy is counted as many times as it occurs in candidates of the set. The equilibrium strategies are behavior strategies but the resulting mixture generally is no behavior strategy but a mixed strategy. The substitute of a candidate set is the result of tracing its centroid.

<u>Substitute of a candidate set</u>: Let Ω be a set of equilibrium points of a game $G = (\Phi,H)$ in $J(\mathcal{R})$. Let $b^1,\dots,b^m$ be the elements of Ω. For every player i in G let b_i^k be player i's behavior strategy in b^k. The <u>centroid</u> $c(\Omega)$ of Ω is the mixed strategy combination c whose elements c_i are defined as follows:

$$c_i(\varphi_i) = \frac{1}{m}\sum_{k=1}^{m} b_i^k(\varphi_i) \qquad (5.39)$$

for every $\varphi_i \in \Phi_i$. The <u>substitute of</u> Ω is the result $T(G,c(\Omega))$ of tracing the centroid of Ω.

<u>Comment</u>: One may ask why we do not define the substitute in a way which generalizes the bicentric prior to something which could be called the "multilateral prior". One could define i-incomplete joint mixtures of the form

$$q_{.i} = \sum_{k=1}^{m} z_i b_{-i}^k \qquad (5.40)$$

For every vector $z = (z_1,\dots,z_m)$ with $z_i \geq 0$ and

$$\sum_{i=1}^{m} z_i = 1 \qquad (5.41)$$

one could form the central local best reply r_i^z to the corresponding $q_{.i}$. Integration over the simplex of the vectors z would then yield a "multilateral prior strategy", an obvious generalization of the bicentric prior strategy.

We admit that one of the reasons why we did not take this approach is the complexity of the computations which have to be performed if Ω contains many candidates. However, this is not the only reason. We feel that the circumstances which require the computations of a substitute are not exactly analogous to the hypothetical situations where one of two equilibrium points is generally believed to be the solution. If all candidates in a candidate set are maximally stable then it is quite likely that none of them is the solution. In fact, this happens in cases where all the elements of Ω fail to be symmetry invariant and cannot be selected for this reason. In this case the application of the tracing procedure to the centroid of Ω leads to an equilibrium point which is in some sense "between" the equilibrium points of Ω . Of course, the same would be true for the tracing procedure when applied to the multilateral prior instead of the centroid. However, there is no strong reason to prefer the "multilateral prior" to the centroid.

We look at the impasse faced in a situation where all candidates in a candidate set are maximally stable as a "dominance failure" in the sense that considerations of risk dominance and payoff dominance have reached a dead end and therefore must be supplemented by a different principle.This new principle is the coordination of expectations by the application of the tracing procedure to the centroid of the candidate set. Unlike the bicentric prior the centroid does not even superficially take the risk situation into account which have been considered already in the determination of dominance. Since dominance considerations have failed the prior is now formed in the most simple way by taking averages over the candidates.

<u>The process of candidate elimination and substitution</u>: In order to distinguish different candidate sets which appear in the process of candidate elimination and substitution we use lower indices: $\Omega_1, \Omega_2, \ldots, \Omega_m$ is the sequence of candidate sets in the order in which it is generated by the process. However, in the description of the process by the flow chart of figure 5.3 it is more convenient to use a dynamic notation. Ω stands for the last candidate set which has been generated. At the beginning Ω_1 is the first candidate set but later it may become the candidate set generated by the last substitution step.

As figure 5.3 shows the process begins with the determination of the first candidate set in <u>rectangle 1</u> . The process then moves to <u>rectangle 2</u>

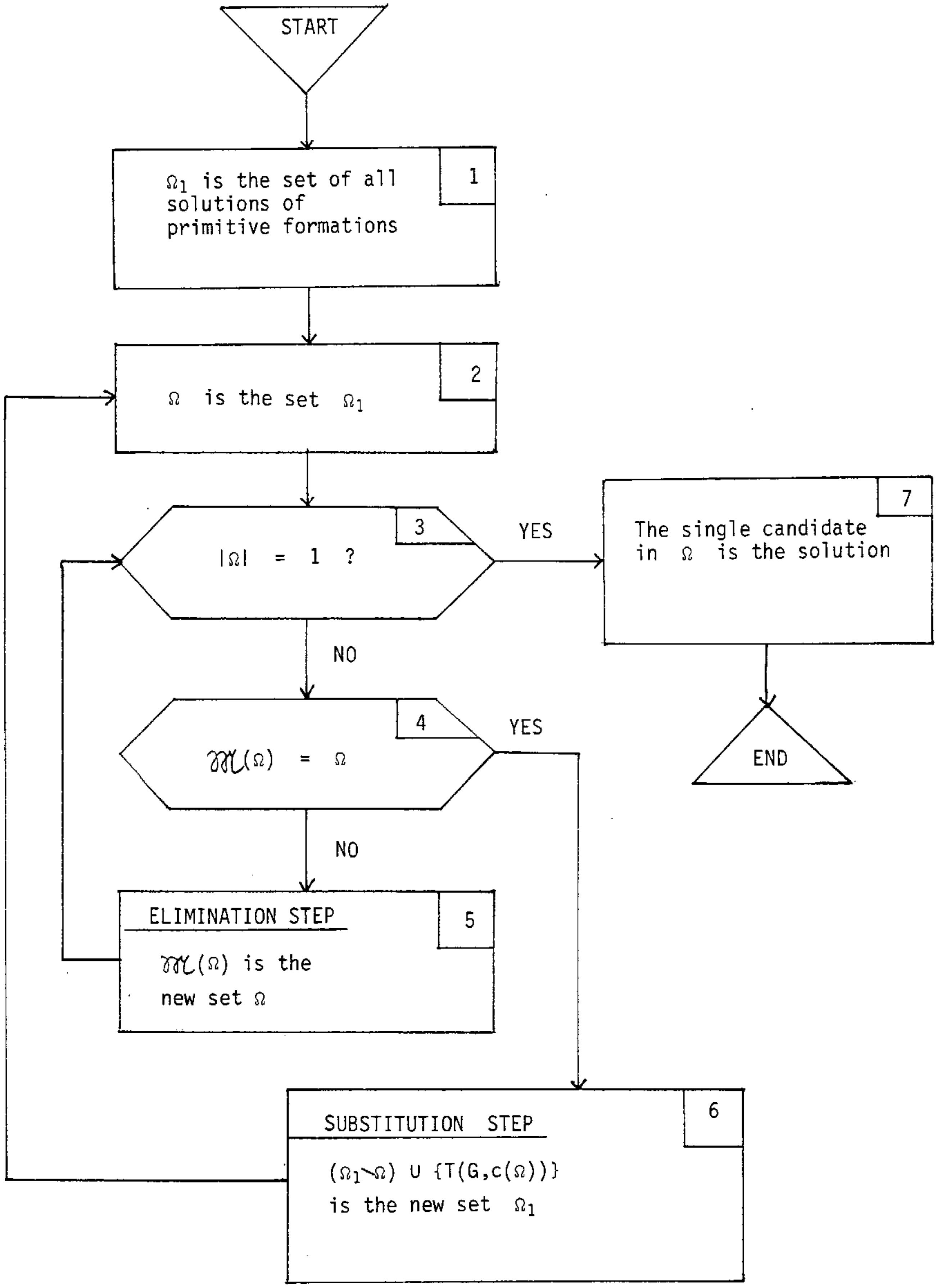

Figure 5.3: Flow chart for the process of candidate elimination and substitution. - $\mathfrak{M}(\Omega)$ is the set of maximally stable candidates in Ω and $c(\Omega)$ is the centroid of Ω.

where Ω receives the meaning of Ω_1. The process goes on to rhomboid 3. Rhomboids contain questions whose answers determine the next step. Rectangles contain operations including the change of names. In rhomboid 3 the question is asked whether the number $|\Omega|$ of elements in Ω is 1. The process stops after rectangle 1 if the answer is yes. In this case the single element of Ω is the solution.

If Ω has more than one element the answer to the question in rhomboid 3 is no and the process moves to rhomboid 4. If the set $\mathfrak{M}(\Omega)$ of maximally stable candidates in Ω is smaller than Ω then an elimination step is performed in rectangle 5. The set $\mathfrak{M}(\Omega)$ becomes the new set Ω and the process returns to rhomboid 3.

If all elements of Ω are maximally stable in Ω the answer to the question in rhomboid 4 is yes and a substitution step has to be performed in rectangle 6. In Ω_1 the elements of Ω are removed and replaced by the substitute $T(G,c(\Omega))$ of Ω. In this way, we receive a new set Ω_1. Then the process returns to rectangle 2.

Remark: It is clear that the process of candidate elimination and substitution stops after a finite number of candidate sets has been generated. The first candidate set is finite. An elimination step reduces the size of the candidate set. A substitution step temporarily may increase the number of candidates but each further substitution step results in a number of candidates smaller than after the previous substitution step. Finally, a candidate set with only one element must be reached.

Solution function: The description of the process of candidate elimination and substitution completes the definition of the solution function L specified by our theory. This solution function L is defined for the class $J(\mathcal{P})$ of all interior substructures of standard forms with perfect recall.

It is often possible to extend the definition of our solution function L beyond the class $J(\mathcal{P})$. However, it must be kept in mind that there the six properties mentioned in the extension theorem of chapter 3, section 12, do not necessarily hold.

Our solution concept for games in $\mathcal{P}$ is the limit solution function $\underset{\rightarrow}{L}$ for our solution function L. (See chapter 2, section 9). From now on L will always refer to the solution function specified by our theory and $\underset{\rightarrow}{L}$ to the limit solution function for L.

5. Solutions of special games

In this section we shall apply our solution concept to some special classes of games and to numerical examples which illustrate certain aspects of our definitions. We shall first show that in non-degenerate unanimity games our solution concept selects the equilibrium point with the highest Nash-product. Then we shall prove that the proposed solution function for 2x2-games with two strong equilibrium points defined in section 8 of chapter 3 agrees with our solution function L. Finally, we shall turn our attention to two numerical examples of special interest.

Non-degenerate unanimity games: A unanimity game $G = (\Phi,H)$ is called non-degenerate if G has one strong equilibrium point U whose Nash-product is greater than that of every other strong equilibrium point.

We do not want to examine the question which non-degenerate unanimity games belong to the class $J(\mathcal{P})$. In any case, no difficulties arise in the extension of L to such games.

Theorem on non-degenerate unanimity games: Let $G = (\Phi,H)$ be a non-degenerate unanimity game and let U be that strong equilibrium point of G which has the greatest Nash-product. Then we have:

$$(5.42) \qquad L(G) = \underset{\rightarrow}{L}(G) = U$$

Proof: Let $U^1,\dots,U^m$ be the strong equilibrium points of G. We first show that neither G nor ε-perturbations G_ε of G with sufficiently small ε have cells. In order to see this we shall construct an i-incomplete mixed combination q_{-i} of completely mixed strategies with the property that an arbitrarily small deviation of a player k with $k \neq i$ from q_{-i} changes the best replies of player i. This excludes the possibility of a cell which contains i but not k. Since the construction can be based on any pair of players i and k it can be shown in this way that there is no cell. It will be convenient to use the following notation:

$$(5.43) \qquad \gamma_h = \frac{1}{\sqrt[n-1]{u_i^h}}$$

and

$$(5.44) \qquad \gamma = \sum_{h=1}^{m} \gamma_h$$

Here u_1^h is player 1's payoff in U^h. The components q_j of q_{-i} are as follows:

$$(5.45) \qquad q_j(U_j^h) = \frac{\gamma_h}{\gamma}$$

for $h = 1,\ldots,m$ and for every q_j in q_{-i}. For this q_{-i} player i's payoffs $H_i(U_i^h q_{-i})$ are the same for every $h = 1,\ldots,m$. Therefore, player i's best replies change if a player $k \neq i$ deviates from q_k in the direction of one of the strategies U_k^h . For sufficiently small ε the strategies q_j are completely mixed not only in G but also in G_ε. The same is true for sufficiently small deviations from q_j. We can conclude that neither G nor G_ε for sufficiently small ε have cells.

For $i = 1,\ldots,n$ and $h = 1,\ldots,m$ let $U_{\varepsilon i}^h$ be the extreme strategy corresponding to U_i^h in G_ε. It is clear that for sufficiently small ε the strategy combination $U_\varepsilon^h = (U_{\varepsilon i}^h)_N$ is a strong equilibrium point of G_ε. This shows that in G_ε as well as in G every pure strategy is the only best reply to some i-incomplete strategy combination. Therefore neither G nor G_ε for sufficiently small ε has inferior strategies, duplicates or semiduplicates; these games are irreducible. It is not necessary to apply the procedure of decomposition and reduction.

In G and G_ε for sufficiently small ε every pure strategy belongs to a strong equilibrium point. Therefore, the primitive formations are exactly those generated by the strong equilibrium points. The first candidate set Ω_1 is the set of all strong equilibrium points.

U cannot be payoff dominated by another strong equlibrium point of G or of G_ε for sufficiently small ε; the Nash-product theorem of section 3 permits the conclusion that U dominates all other equilibrium points in the first candidate set Ω_1 of G. Therefore $L(G) = U$ holds. On the basis of the Nash-product theorem for perturbations of section 3 the same argument applied to G_ε with sufficiently small ε yields $L(G_\varepsilon) = U_\varepsilon$ where

U_ε is the equilibrium point of G_ε corresponding to U. This shows that we have $\underset{\to}{L}(G) = U$.

Global dominance: Let Ω_1 be the first candidate set for an irreducible game G. We say that an equilibrium point U of G is called globally dominant in G if it belongs to the first candidate set Ω_1 of G and, in addition to this, dominates every other equilibrium point in Ω_1.

Remarks: If U is globally dominant in G then U is the solution L(G) of G. This follows by $\Omega_2 = \{U\}$. The proof of the theorem on non-degenerate unanimity games has shown that in such games the equilibrium point U with the highest Nash-product is globally dominant and that for sufficiently small ε the equilibrium point U_ε corresponding to U is globally dominant in the ε-perturbation G_ε.

Comment: The Nash-property of our notion of risk dominance leads to a "Nash-property" for our solution concept which is expressed by the theorem on non-degenerate unanimity games. We feel that it is a desirable feature of our theory that non-degenerate unanimity games are solved in an especially simple way, namely by global dominance. Moreover, all dominance comparisons are decided by payoff dominance or conspicuous risk dominance (see section 3).

The investigation of 2x2-games with two strong equilibrium points will make it necessary to look more closely at the ε-perturbations of such games. Not only in this context, but in general for 2-person-games with normal form structure it will be useful to replace an ε-perturbation by an equivalent game with a simpler payoff function. For this purpose, we introduce the notion of a modified ε-perturbation.

Modified ε-perturbations: Let $G = (\Phi, H)$ with $\Phi = \Phi_1 x \Phi_2$ be a 2-person-game with normal form structure. Consider an ε-perturbation $G_\varepsilon = (\Phi_\varepsilon, H_\varepsilon)$ of G, For every $\varphi_i \in \Phi_i$ let $\varphi_{\varepsilon i}$ be the corresponding ε-extreme strategy. The connection between H_ε and H has been explored in section 7 of chapter 2. Analogously to (2.39) define:

(5.46) $$\eta_i = 1 - |\phi_i|\varepsilon \qquad \text{for } i = 1,2$$

Consider a pure strategy combination $\psi = (\psi_1, \psi_2)$ and the corresponding

ε-extreme combination $\psi_\varepsilon = (\psi_{\varepsilon 1}, \psi_{\varepsilon 2})$. In view of (2.42) the payoff vector for ψ_ε can be written as follows:

$$(5.47)\quad H_\varepsilon(\psi_\varepsilon) = \eta_1\eta_2 H(\psi) + \varepsilon\eta_1 \sum_{\varphi_2\in\Phi_2} H(\psi_1\varphi_2)$$

$$+ \varepsilon\eta_2 \sum_{\varphi_1\in\Phi_1} H(\varphi_1\psi_2)$$

$$+ \varepsilon^2 \sum_{\varphi\in\Phi} H(\varphi)$$

The <u>modified</u> <u>ε-perturbation</u> $\bar{G}_\varepsilon = (\Phi_\varepsilon, \bar{H}_\varepsilon)$ differs from G_ε only with respect to payoffs. In order to describe $\bar{H}_\varepsilon$ in a convenient way we introduce the following notation:

$$(5.48)\quad \varepsilon_1 = \frac{\varepsilon}{\eta_2} = \frac{\varepsilon}{1-|\Phi_2|\varepsilon}$$

$$(5.49)\quad \varepsilon_2 = \frac{\varepsilon}{\eta_1} = \frac{\varepsilon}{1-|\Phi_1|\varepsilon}$$

On the right hand side of (5.47) we neglect the last term and divide by $\eta_1\eta_2$. Obviously, this amounts to positive linear payoff transformations. in this way, we obtain $\bar{H}_\varepsilon$:

$$(5.50)\quad \bar{H}_\varepsilon(\psi_\varepsilon) = H(\psi) + \varepsilon_1 \sum_{\varphi_2\in\Phi_2} H(\psi_1\varphi_2)$$

$$+ \varepsilon_2 \sum_{\varphi_1\in\Phi_1} H(\varphi_1\psi_2)$$

By construction $\bar{G}_\varepsilon$ and G_ε are equivalent (see chapter 2, section 2). Both games have the same solution. For some purposes it will be useful to consider a game $\hat{G}_\varepsilon = (\Phi_\varepsilon, \hat{H}_\varepsilon)$ with an even shorter payoff. We call this game $\hat{G}_\varepsilon$ the <u>short</u> <u>ε-perturbation</u> of G. Player i's best replies remain unchanged if the term which does not depend on ψ_i in (5.50) is dropped. The payoffs of the short ε-perturbation are as follows:

$$(5.51) \qquad \hat{H}_{\varepsilon 1}(\psi_\varepsilon) = H_1(\psi) + \varepsilon_1 \sum_{\varphi_2 \in \Phi_2} H_1(\psi_1 \varphi_2)$$

$$(5.52) \qquad \hat{H}_{\varepsilon 2}(\psi_\varepsilon) = H_2(\psi) + \varepsilon_2 \sum_{\varphi_1 \in \Phi_1} H_2(\varphi_1 \psi_2)$$

Generally, the short ε-perturbation is not equivalent to the ε-perturbation. Payoff dominance relationships may differ in both games. However, the best reply structure is the same and the weights of the logarithmic terms in the logarithmic tracing procedure are the same in $\hat{G}_\varepsilon$ and $\bar{G}_\varepsilon$ since they are determined by payoff differences where they matter (see chapter 4, section). This permits us to draw the following conclusion.

<u>Lemma on ε-perturbations</u>: Let G_ε be an ε-perturbation of a 2-person game G with normal form structure. Let $\bar{G}_\varepsilon$ be the modified ε-perturbation of G and let $\hat{G}_\varepsilon$ be the short ε-perturbation of G. Payoff dominance relationships between equilibrium points are the same in G_ε and $\bar{G}_\varepsilon$. Risk dominance relationships between equilibrium points are the same in G_ε and $\hat{G}_\varepsilon$. The substitute of a candidate set Ω is the same in G_ε and in $\hat{G}_\varepsilon$.

<u>Proof</u>: The proof has been given above.

<u>Theorem on 2x2-games</u>: On the class $\hat{\mathcal{R}}$ of all 2x2-games with two strong equilibrium points the solution function L specified by our theory agrees with the proposed solution function defined by (3.27) in chapter 3, section 8. Moreover, on the chlass $\hat{\mathcal{R}}$ the limit solution function $\underset{\to}{L}$ of L agrees with L.

<u>Proof</u>: Let U and V be two strong equilibrium points of a game $G \in \hat{\mathcal{R}}$. Let G_ε be the ε-perturbation, $\bar{G}_\varepsilon$ the modified ε-perturbation and $\hat{G}_\varepsilon$ the short ε-perturbation of G. Moreover, let U_ε and V_ε be the strategy combinations corresponding to U and V in G_ε. For sufficiently small ε both U_ε and V_ε are strong equilibrium points of G_ε. It is clear that the first candidate set is {U,V} in the case of G and $\{U_\varepsilon, V_\varepsilon\}$ in the case of G_ε with sufficiently small ε.

We first show that the payoff dominance relationship between U_ε and V_ε in G_ε is the same as that between U and V in G. In view of the lemma on ε-perturbations this question can be examined in $\bar{G}_\varepsilon$. Let $\bar{a}_{ij}$ be player 1's payoff in $\bar{G}_\varepsilon$ which corresponds to a_{ij} in figure 3.5 (chapter 3, section 4). Equation (5.47) yields:

$$\bar{a}_{11} = a_{11} + \varepsilon_1(a_{11} + a_{12}) + \varepsilon_2(a_{11} + a_{21}) \quad (5.53)$$

$$\bar{a}_{22} = a_{22} + \varepsilon_1(a_{22} + a_{21}) + \varepsilon_2(a_{22} + a_{12}) \quad (5.54)$$

In view of

$$\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{1-2\varepsilon} = \bar{\varepsilon} \quad (5.55)$$

we obtain

$$\bar{a}_{11} - \bar{a}_{22} = \frac{1}{1-2\varepsilon}(a_{11} - a_{22}) \quad (5.56)$$

An analogous equation can be derived for player 2. This shows that payoff dominance does not differ in G and G_ε.

Risk dominance in G_ε can be investigated in the short ε-perturbation $\hat{G}_\varepsilon$. This game is shown in figure 5.4. We shall show that risk dominance in G_ε

	$U_{\varepsilon 2}$	$V_{\varepsilon 2}$
$U_{\varepsilon 2}$	$a_{11}+\bar{\varepsilon}(a_{11}+a_{12})$ $b_{11}+\bar{\varepsilon}(b_{11}+b_{21})$	$a_{12}+\bar{\varepsilon}(a_{11}+a_{12})$ $b_{12}+\bar{\varepsilon}(b_{12}+b_{22})$
$V_{\varepsilon 1}$	$a_{21}+\bar{\varepsilon}(a_{21}+a_{22})$ $b_{21}+\bar{\varepsilon}(b_{11}+b_{21})$	$a_{22}+\bar{\varepsilon}(a_{21}+a_{22})$ $b_{22}+\bar{\varepsilon}(b_{12}+b_{22})$

Figure 5.4: The short ε-perturbation $\hat{G}_\varepsilon$ of the game G of figure 3.5

agrees with risk dominance in G. For this purpose we compute the deviation losses $\hat{u}_{\varepsilon i}$ and $\hat{v}_{\varepsilon i}$ in $\hat{G}_\varepsilon$ and connect them to the deviation losses u_i and v_i in G. We obtain:

(5.57) $\hat{u}_i = u_i (1+\bar{\varepsilon})-\bar{\varepsilon}v_i$ for $i = 1,2$

(5.58) $\hat{v}_i = v_i (1+\bar{\varepsilon})-\bar{\varepsilon}u_i$ for $i = 1,2$

This yields:

(5.59) $\hat{u}_1\hat{u}_2 - \hat{v}_1\hat{v}_2 = (u_1u_2 - v_1v_2)(1+2\bar{\varepsilon})$

In view of the lemma on 2x2-games in section 3 equation (5.59) permits the conclusion that the risk dominance relationship between U_ε and V_ε in G_ε is the same as between U and V in G. Moreover, the lemma has shown that our general notion of risk dominance agrees with the special one of chapter 3.

Our results on payoff dominance and risk dominance show that the dominance relationship between U_ε and V_ε in G_ε is the same as that between U and V, respectively, in G. Suppose that U dominates V. Then the second candidate set contains only U in the case of G and only U_ε in the case of G_ε. Consequently U is selected both by L and $\underset{\rightarrow}{L}$. It is also clear that U is selected by the proposed solution function of chapter 3 if U dominates V.

It is now clear that the theorem holds in all cases where one of both strong equilibrium points dominates the other. Assume that neither U dominates V nor V dominates U in G. Then the same is true with respect to U_ε and V_ε in the case of G_ε. In order to determine the solution we have to compute the substitute of {U,V} in the case of G and of $\{U_\varepsilon,V_\varepsilon\}$ in the case of G_ε. In both cases Ω_3 contains only this substitute which therefore is the solution. We have to show that the substitute is none of both strong equilibrium points but the mixed equilibrium point which is also selected by the proposed solution function of chapter 3.

The transformations which have been applied in chapter 3 in order to obtain the form of figure 3.14 amount to positive linear transformations of payoff differences and, therefore, do not influence the path of the logarithmic tracing procedure. In the case of G and $\hat{G}_\varepsilon$ we obtain u = v in the transformed game of figure 3.14. For $\hat{G}_\varepsilon$ this follows by (5.59).

Obviously, the transformed game has only one symmetry invariant equilibrium point, namely the mixed one, which therefore must be the result of the logarithmic tracing procedure. This follows by the fact that the centroid of the second candidate set and the definition of the logarithmic tracing procedure are invariant with respect to isomorphisms.

Remark: The proof of the theorem has shown that for games $G \in \mathfrak{R}$ the dominance relationship between the two strong equilibrium points U and V is the same as the dominance relationship between U_ε and V_ε in an ε-perturbation with sufficiently small ε. Here we have to add the words "for sufficiently small ε" merely because V_ε may not be an equilibrium point of G_ε if ε is not small enough.

Payoff dominance in G and G_ε: We shall now look at the numerical example of figure 5.5 in order to illustrate the point that sometimes $\underset{\to}{L}(G)$ may be different from L(G) since the ε-perturbations G_ε of G show a payoff dominance relationship which is not present in G.

The modified ε-pertubation $\bar{G}_\varepsilon$ of the game G in figure 5.5 is shown in figure 5.6. In order to determine the solution $L(G_\varepsilon)$ of G_ε we first apply the procedure of decomposition and reduction to $\bar{G}_\varepsilon$. Since G_ε and $\bar{G}_\varepsilon$ are equivalent we can solve $\bar{G}_\varepsilon$ instead of G_ε. It can be seen easily that $\bar{G}_\varepsilon$ has no cells. However, the strategies W_1 and W_2 are inferior (both are dominated). After the elimination of W_1 and W_2 we obtain an irreducible game $\bar{G}^1_\varepsilon$ with two strong equilibrium points. Obviously, $V_\varepsilon=(V_{\varepsilon 1},V_{\varepsilon 2})$ payoff dominates $U_\varepsilon = (U_{\varepsilon 1},U_{\varepsilon 2})$. The first candidate set is $\{U_\varepsilon,V_\varepsilon\}$ and the second candidate set contains only V_ε. We have $L(\bar{G}_\varepsilon) = V_\varepsilon$. This yields $\underset{\to}{L}(G) = V$.

In the direct application of L to G the procedure of decomposition and reduction also removes W_1 and W_2. According to the theorem on 2x2-games the mixed equilibrium point is the solution of the game G^1 which results in this way. Consequently, $L(G) = (q_1,q_2)$ with $q_i(U_i) = q_i(V_i) = 1/2$ for $i = 1,2$. The solution L(G) is different from the limit solution $\underset{\to}{L}(G)$.

A degenerate unanimity game: In order to illustrate some aspects of candidate elimination and substitution we determine the solution L(G) of the degenerate unanimity game shown in figure 5.7. Since no additional insight could be gained in this way the analysis of the ε-perturbations and

	U_2	V_2	W_2
U_1	1, 1	0, 0	0, 0
V_1	0, 0	1, 1	1, 0
W_1	0, 0	0, 1	0, 0

Figure 5.5: A numerical example

	$U_{\varepsilon 2}$	$V_{\varepsilon 2}$	$W_{\varepsilon 2}$
$U_{\varepsilon 1}$	$1+2\bar{\varepsilon}$, $1+2\bar{\varepsilon}$	$2\bar{\varepsilon}$, $3\bar{\varepsilon}$	$2\bar{\varepsilon}$, $\bar{\varepsilon}$
$V_{\varepsilon 1}$	$3\bar{\varepsilon}$, $2\bar{\varepsilon}$	$1+3\bar{\varepsilon}$, $1+3\bar{\varepsilon}$	$1+3\bar{\varepsilon}$, $\bar{\varepsilon}$
$W_{\varepsilon 1}$	$\bar{\varepsilon}$, $2\bar{\varepsilon}$	$\bar{\varepsilon}$, $1+3\bar{\varepsilon}$	$\bar{\varepsilon}$, $\bar{\varepsilon}$

$$\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{1-3\varepsilon} = \bar{\varepsilon}$$

Figure 5.6: Modified ε-perturbation of the game of figure 5.5

the limit solution is omitted here.

The three strong equilibrium points $U = (U_1,U_2)$, $V = (V_1V_2)$ and $W = (W_1,W_2)$ are the elements of the first candidate set. U and V do not dominate each other (their Nash-products are equal) and both of them payoff dominate W. Therefore, U and V are the undominated elements of $\Omega_1 = \{U,V,W\}$. Consequently, the second candidate set is $\Omega_2 = \{U,V\}$.

	U_2	V_2	W_2
U_1	6, 4	0, 0	0, 0
V_1	0, 0	4, 6	0, 0
W_1	0, 0	0, 0	3, 3

Figure 5.7: A degenerate unanimity game

We have to compute the substitute $T(G,c(\Omega_2))$. In $c(\Omega_2)$ player i uses his strategies U_i and V_i with probabilities 1/2 and his strategy W_i with probability zero (i=1,2). The linear tracing procedure with the prior $c(\Omega_2)$ fails to be well defined. For $0 \leq t < .2$ the games G^t arising in the linear tracing procedure have exactly one equilibrium, namely (U_1,V_2). At t = .2 the graph of equilibrium points splits into three paths leading to U, V and the mixed equilibrium point $q = (q_1,q_2)$ whose components are as follows:

(5.60) $q_1(U_1) = .6 \qquad q_1(V_1) = .4 \qquad q_1(W_1) = 0$

(5.61) $q_2(U_2) = .4 \qquad q_2(V_2) = .6 \qquad q_2(W_2) = 0$

The symmetry which carries U to V excludes the possibility that either U or V is the result of the logarithmic tracing procedure. Hence

(5.62) $T(G,c(\Omega_2)) = q$

The substitution step yields the following third candidate set $\Omega_3 = \{W,q\}$. The payoffs attached to g are 2.4 for each of both players. This shows that q is payoff dominated by W. Therefore $\Omega_4 = \{W\}$ is the fourth candidate set. $L(G) = W$ is the solution.

Comment: At least at first glance it might seem to be natural to look at the substitute as the solution where a substitution step has to be performed. We do not want to define the solution in this way, since the substitute may actually be much less stable than one of the candidates which have been eliminated before. This is illustrated by the game of figure 5.7. In the transition from Ω_1 to Ω_2 the payoff dominated equilibrium point W is eliminated. The substitute of Ω_2 is the mixed strategy equilibrium point q. It would be undesirable to obtain this rather unstable equilibrium point as the solution. The substitution step requires a comparison of q with W and thereby gives the more stable equilibrium point W the chance to emerge as the solution.

6. Summary of procedures

In this section we shall try to give an overview over the structure of our theory. We shall outline the steps to be taken in the application to specific examples.

Even if our solution concept does not really specify an algorithm, it is convenient to summarize it in a way which looks like the description of a computer program. Each step specifies certain tasks which are broken down into subtasks if necessary. Once a task like finding the solution of an auxiliary game has been completed, one has to go back to the last unfinished task which may involve the solution of further auxiliary games.

It is assumed that the game to be solved is given as an unperturbed extensive form with perfect recall or as a game in normal form. The final aim is the determination of its limit solution.

Step 1: If the game is given in extensive form, construct its standard form (see chapter 2). The standard form is the game to which the theory is applied. A game which is given in normal form, is looked upon as a standard form with normal form structure. Continue with step 2.

Step 2: Select a sufficiently small ε_0 and form the ε-perturbed games with $0 < \varepsilon < \varepsilon_0$ (see chapter 2, section 7). Find the solutions for all these games. For each game to be solved, the required procedures begin with step 3. (The procedures must be followed parametrically, but it is convenient to describe them as they apply to single ε-perturbed games.) After the solutions of all ε-perturbed games have been found continue with step 6.

Step 3: Start with the procedure of decomposition and reduction described by figure 3.29 on page 89 of chapter 3. During this procedure solutions of irreducible games have to be computed (rectangle 10 in figure 3.29). The determination of the solution of each of the irreducible games begins with step 4.

Step 4: Find out whether the game is basic (it is basic if it has no formation). If the game is basic find its solution by tracing its centroid. If the game is non-basic continue with step 5.

Step 5: Find the primitive formations. For each primitive formation determine its solution, beginning with step 3. Form the first candidate set (the set of all solutions of primitive formations). Determine the payoff dominance relationships, strategic distances, strategic net distances and risk dominance relationships for all pairs of primitive formation solutions. Follow the process of candidate elimination and substitution described in figure 5.3 on page 40 of chapter 5. In the course of this process substitutes may come in as new candidates; if this happens payoff dominance relationships, strategic distances, strategic net distances and risk dominance relationships have to be determined between the substitute and the other equilibrium points in the candidate set produced by the substitution step. The end result of the process of candidate elimination and substitution is the solution.

Step 6: Determine the limit solution (see chapter 2, section 9).

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