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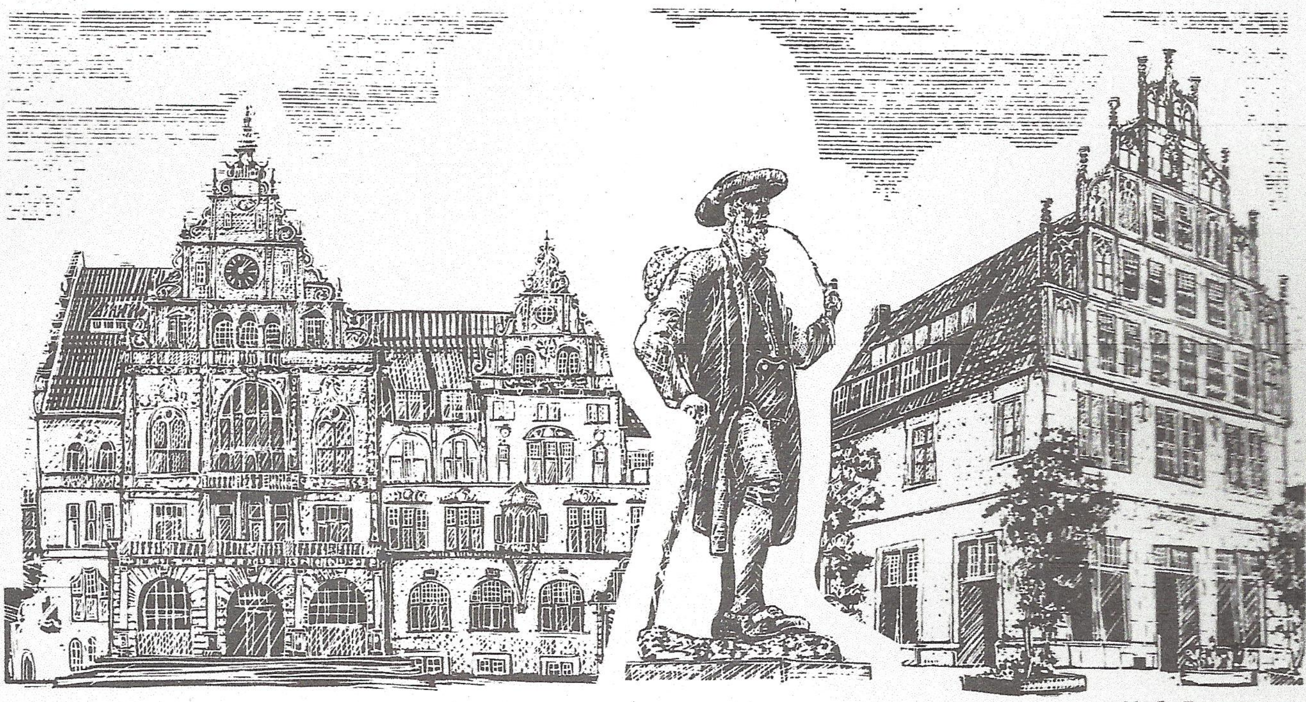

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A General Theory of Equilibrium Selection in Games

Chapter 7
A Bargaining Problem with Transaction Costs on one Side

It is the purpose of this chapter to investigate a two-person bargaining situation where one of the participants has transaction costs connected to making a proposal. One may think of an illegal deal where player 1, the seller, faces punishment,if he is caught bargaining, whether an agreement is reached or not. The transaction costs express the utility loss involved in this risk.

Bargaining is modelled in the same way as in the unanimity game. Both players make simultaneous decisions. Agreement is reached if both of them make the same proposal. Otherwise conflict results. Player 1 can choose to make no proposal in which case he is better off than in a conflict reached by disagreeing proposals.

In the bargaining situation considered here, the players can divide a fixed amount of money among themselves. Both of them are assumed to have utility functions which are linear in money.

A similar bargaining problem with only two possible agreements but with transaction costs on both sides has been explored in the literature (Selten and Leopold, 1983). Ulrike Leopold has investigated the much more difficult case of bargaining on the division of a fixed amount of money with transaction costs on both sides (Leopold-Wildburger 1982). Some remarks on the results obtained there will be made at the end of the chapter.

Loosely speaking, one may say that the model examined here is almost a special case of the much more complicated problem treated by Ulrike Leopold. The results presented in this chapter agree with those obtained by Ulrike Leopold for sufficiently small transaction costs of player 2. This is not surprising but, by no means trivial, since strictly speaking the simpler model is not a special case of the more complicated one.

Our theory has been conceived for finite games since this permits us to concentrate efforts on the basic problems of equilibrium selection without running into technical difficulties connected to infinite games. Therefore, it will be assumed that money is not infinitely divisible.

1. The model

The model has the form of a two-person game with normal form structure. As has been explained above, the model is similar to a unanimity game. In order to reach agreement each of both players must make the same proposal on the division of one money unit. However, player 1 has the option not to bargain at all.

The possible agreements can be characterized by the amount x assigned to player 1. The corresponding agreement payoff for player 2 is 1-x. It is assumed that there is a smallest piece of money worth 1/M where M is a positive even number. This piece of money cannot be further subdivided. Player 1's agreement payoff must be an integer multiple of 1/M with $0 < x < 1$. We exclude agreements which do not give positive amounts to both players. The set of all possible agreements is given by

$$(7.1) \qquad X = \{x \mid x = \frac{k}{M}, \; k = 1,\dots,M-1\}$$

The symbol W_1 denotes player 1's choice not to bargain at all. If player 1 selects W_1 he receives a positive payoff α and player 2 receives 0, independently of player 2's strategy. In order to exclude uninteresting cases which would require special attention we impose the following conditions on M and the transaction cost parameter α.

$$(7.2) \qquad M > 2$$

$$(7.3) \qquad \frac{2}{M-1} < \alpha < \frac{M-1}{M}$$

Since M is even (7.2) means that M can assume the values 4,6,... .
We are mainly interested in the behavior of the limit solution for $M \to \infty$.
In this respect (7.2) and (7.3) are no restriction of the generality of our analysis.

The bargaining situation is described by the following two-person game $G = (\Phi, H)$ with $\Phi = \Phi_1 \times \Phi_2$:

$$(7.4) \qquad \Phi_1 = X \cup \{W_1\}$$

$$(7.5) \qquad \Phi_2 = X$$

$$(7.6) \qquad H_1(\varphi) = \begin{cases} x & \text{for } \varphi_1 = \varphi_2 = x \\ \alpha & \text{for } \varphi_1 = W_1 \\ 0 & \text{for } \varphi_1 \neq \varphi_2 \text{ and } \varphi_1 \neq W_1 \end{cases}$$

$$(7.7) \qquad H_2(\varphi) = \begin{cases} 1-x & \text{for } \varphi_1 = \varphi_2 = x \\ 0 & \text{for } \varphi_1 \neq \varphi_2 \end{cases}$$

2. Properties of ε-perturbations

In order to apply our theory to the model we have to look at the ε-perturbations $G_\varepsilon = (\Phi_\varepsilon, H_\varepsilon)$ of the game G defined by (7.4) to (7.7). In the following we introduce some notational conventions.

In order to avoid confusion with algebraic expressions like xy the notation (x,y) is used for pure strategy pairs of G; as usual the first component is player 1's strategy and the second is player 2's strategy. The ε-extreme strategy corresponding to player i's pure strategy x is denoted by $[x]_{\varepsilon i}$ or by $x_{\varepsilon i}$ where the shorter notation does not lead to confusion. $W_{\varepsilon 1}$ is player 1's ε-extreme strategy corresponding to W_1. The symbol $X_{\varepsilon i}$ is used for the set of player i's ε-extreme strategies corresponding to proposals $x \in X$. The pure strategy sets $\Phi_{\varepsilon 1}$ and $\Phi_{\varepsilon 2}$ in G_ε are as follows:

$$(7.8) \qquad \Phi_{\varepsilon 1} = X_{\varepsilon 1} \cup \{W_{\varepsilon 1}\}$$

$$(7.9) \qquad \Phi_{\varepsilon 2} = X_{\varepsilon 2}$$

The payoff function H_ε agrees with H. As much as possible the analysis will be based on the short ε-perturbation $\hat{G}_\varepsilon = (\Phi_\varepsilon, \hat{H}_\varepsilon)$ rather than G_ε or the modified ε-perturbation $\bar{G}_\varepsilon = (\Phi_\varepsilon, \bar{H}_\varepsilon)$. (See chapter 5, section 5).

Two payoff inequalities: Let m be the smallest integer such that m/M is greater than α:

(7.10) $$m = \min \{k | k = 1,\dots,M-1 \text{ and } \frac{k}{M} > \alpha\}$$

It will be shown that the following two inequalities hold for sufficiently small ε:

(7.11) $$\hat{H}_{\varepsilon 1}(W_{\varepsilon 1} x_{\varepsilon 2}) > \hat{H}_{\varepsilon 1}(x_{\varepsilon 1} x_{\varepsilon 2}) \quad \text{for } x = \frac{k}{M} \text{ with } k = 1,\dots,m-1$$

(7.12) $$\hat{H}_{\varepsilon 1}(W_{\varepsilon 1} x_{\varepsilon 2}) < \hat{H}_{\varepsilon 1}(x_{\varepsilon 1} x_{\varepsilon 2}) \quad \text{for } x = \frac{k}{M} \text{ with } k = m,\dots,M-1$$

In order to show this we apply (5.51):

(7.13) $$\hat{H}_{\varepsilon 1}(W_{\varepsilon 1} x_{\varepsilon 2}) = \alpha + \varepsilon_1 (M-1)\alpha$$

(7.14) $$\hat{H}_{\varepsilon 1}(x_{\varepsilon 1} x_{\varepsilon 2}) = x + \varepsilon_1 x$$

Inequality (7.11) holds for $\alpha > x$. For $x = (m-1)/M$ we may have $x = \alpha$. In view of $M > 2$ inequality (7.11) holds in this case, too. For sufficiently small ε inequality (7.12) is valid in view of $x > \alpha$.

Best replies to pure strategies: $A_{\varepsilon i}$ denotes player i's best reply correspondence which maps every strategy of the other player to the set of player i's pure best replies in G_ε. Inequalities (7.11) and (7.12) permit the following conclusion. For sufficiently small ε we have:

(7.15) $$A_{\varepsilon 1}(x_{\varepsilon 2}) = \begin{cases} \{W_{\varepsilon 1}\} & \text{for } x \leq \alpha \\ \{x_{\varepsilon 1}\} & \text{for } x > \alpha \end{cases}$$

Since $W_{\varepsilon 1}$ assigns the same probability ε to all $x \in X$ player 2's unique best reply to $W_{\varepsilon 1}$ in G_ε is $[1/M]_\varepsilon$. Obviously, for sufficiently small ε player 2's unique best reply to $x_{\varepsilon 1}$ in G_ε is $x_{\varepsilon 2}$. For sufficiently small ε we have:

(7.16) $A_{\varepsilon 2}(x_{\varepsilon 1}) = \{x_{\varepsilon 2}\}$

(7.17) $A_{\varepsilon 2}(W_{\varepsilon 1}) = \{[\frac{1}{M}]_{\varepsilon}\}$

Strong equilibrium points: It can be seen immediately that for sufficiently small ε the game G_{ε} has M-m strong equilibrium points of the form $x_{\varepsilon 1}x_{\varepsilon 2}$ with $x = k/M$ and $k = m,\dots,M-1$ and one additional strong equilibrium point $W_{\varepsilon 1}[1/M]_{\varepsilon}$. There are no further pure strategy equilibrium points. The symbol x_{ε} will be used as a short notation for $x_{\varepsilon 1}x_{\varepsilon 2}$.

3. Decomposition and reduction

The procedure of decomposition and reduction has to be applied to G_{ε}. In the following it will always be assumed that ε is sufficiently small in the sense that the results obtained in section 2 hold.

Neither player 1 nor player 2 forms a cell in G_{ε} since otherwise best replies could not depend on the other player's strategy. Since G_{ε} is a game with normal form structure we do not make any distinction between a player and his single agent. Inferior choices may also be called inferior pure strategies.

Inequality (7.11) permits the conclusion that player 1's pure strategies $x_{\varepsilon 1}$ with $x = k/M$ and $k = 1,\dots,m-1$ are inferior in G_{ε}. The other pure strategies of player 1 are not inferior since they are unique best replies somewhere. The same is true for all pure strategies of player 2. Let G'_{ε} be the game which results from G_{ε} by elimination of inferior choices (pure strategies).

We now have to ask the question whether G'_{ε} is decomposable (see figure 3.29). The argument used for G_{ε} also establishes the absence of cells in G'_{ε} .

It is now necessary to examine whether G'_{ε} has inferior pure strategies. As we shall see player 2's pure strategies $x_{\varepsilon 2}$ with $x = k/M$ and $k = 2,\dots,m-1$ are inferior in G'_{ε}. Let q_1 be a mixed strategy of player 1 in G'_{ε}. Equation (5.52) yields:

(7.18) $\hat{H}_{\varepsilon 2}(q_1[k/M]_{\varepsilon 2}) = \varepsilon_2 \frac{M-k}{M}$ for $k = 1,\dots,m-1$

This is due to the fact that player 1's pure strategies $[k/M]_{\varepsilon 1}$ with $k = 1,\dots,m-1$ have been eliminated already. q_1 assigns probability ε to each of the pure strategies 1/M,...,(m-1)/M. Obviously in G'_ε player 2's pure strategy $[1/M]_{\varepsilon 2}$ dominates his pure strategies $[k/M]_{\varepsilon 2}$ with k = 2,...,m-1. No other pure strategies are inferior in G'_ε.

Let $G''_\varepsilon = (\phi''_\varepsilon, H''_\varepsilon)$ be the game which results from G'_ε by elimination of inferior choices. Arguments very similar to those used above show that G''_ε has neither cells nor inferior strategies nor semiduplicates nor duplicates. G''_ε is irreducible.

Define

(7.19) $\quad X'' = \{x \mid x = \frac{k}{M} \text{ with } k = m,\ldots,M-1\}$

Let $X''_{\varepsilon i}$ be the set of all ε-extreme strategies of player i corresponding to proposals $x \in X''$. The pure strategy sets of G''_ε are as follows:

(7.20) $\quad \Phi''_{\varepsilon 1} = X''_{\varepsilon 1} \cup \{W_{1\varepsilon}\}$

(7.21) $\quad \Phi''_{\varepsilon 2} = X''_{\varepsilon 2} \cup \{[1/M]_{\varepsilon 2}\}$

Obviously, the best replies to pure strategies in G''_ε are the same as in G'_ε and both games have the same strong equilibrium points.

4. Initial candidates

The process of candidate elimination and substitution has to be followed in G''_ε. In order to find the first candidate set we have to determine the primitive formations of G''_ε. Each of the strong equilibrium points generates a primitive formation. Every pure strategy belongs to one of these strong equilibrium points. Consequently, the primitive formation of G''_ε are exactly those which are generated by the strong equilibrium points. It follows that the first candidate set Ω_1 is nothing else than the set of all strong equilibrium points.

The investigation of dominance relationships between pairs of initial candidates will lead to the conslusion that one of the candidates in Ω_1 is globally dominant in G''_ε. As has been pointed out in chapter 5, section 5, a globally dominant candidate is the solution.

Since the solution is found by global dominance there is no need to look at strategic distances. The payoff dominance relationships between pairs of initial candidates are easily discovered. In view of $x > \alpha$ for $x \varepsilon X''$ each of the candidates of the form $x_\varepsilon = x_{\varepsilon 1} x_{\varepsilon 2}$ payoff dominates $W_{\varepsilon 1}[1/M]_{\varepsilon 2}$ for sufficiently small ε. It is also clear that for sufficiently small ε there is no payoff dominance between two different candidates of the form $x_\varepsilon = x_{\varepsilon 1} x_{\varepsilon 2}$. The dominance relationship between two such candidates is determined by risk dominance.

It can now be seen that for sufficiently small ε a candidate of the form $x_\varepsilon = x_{\varepsilon 1} x_{\varepsilon 2}$ is globally dominant if it risk dominates all other strong equilibrium points of this form.

The investigation of risk dominance between pairs of strong equilibrium points of the form $x_\varepsilon = x_{\varepsilon 1} x_{\varepsilon 2}$ will lead to the conclusion that one of these strong equilibrium points risk dominates all others and therefore is the solution of the game.

5. Risk dominance

The risk dominance comparisons to be investigated do not require the use of the logarithmic tracing procedure. For a given prior the path of the linear tracing procedure depends only on the best reply structure. The bicentric prior is also determined by the best reply structure. Therefore, for the purpose of computing risk dominance relationships without the logarithmic tracing procedure a restricted game can be replaced by another game with the same pure strategy sets and the same best reply structure and a simpler payoff function.

In our case ε-perturbation payoffs will be replaced by payoffs of the short ε-perturbation. If in $G''_\varepsilon = (\phi''_\varepsilon, H''_\varepsilon)$ the payoff function H''_ε is replaced by the restriction of the payoff function for the short ε-perturbation to Φ''_ε we obtain the game $\hat{\tilde{G}}_\varepsilon = (\phi''_\varepsilon, \hat{\tilde{H}}_\varepsilon)$. Obviously, $\hat{\tilde{G}}_\varepsilon$ has the same best reply structure as G''_ε.

In our case no agents are fixed in the transition to the restricted game for a risk dominance comparison between two different strong equilibrium points $x_\varepsilon = x_{\varepsilon 1}x_{\varepsilon 2}$, $y_\varepsilon = y_{\varepsilon 1}y_{\varepsilon 2}$. Both players have different strategies in both equilibrium points. Therefore, the restricted game for the comparison is the formation spanned by both equilibrium points. The best reply structure of the restricted game is fully determined by the best reply structure of the whole game. (If agents are fixed in the transition to the restricted game this is not necessarily the case.)

It is now clear that for the purpose of computing risk dominance relationships without the logarithmic tracing procedure, we can replace G''_ε by $\hat{\tilde{G}}_\varepsilon$. This will be done in the following.

As long as the logarithmic tracing procedure is not used it does not matter whether the computations for the determination of risk dominance comparisons are based on the restricted game or on some larger formation. The bicentric prior and the result of the application of the linear tracing procedure are the same in both cases. We shall make use of this fact.

<u>A formation containing the restricted game</u>: Let $x_\varepsilon = x_{\varepsilon 1}x_{\varepsilon 2}$, $y_\varepsilon = y_{\varepsilon 1}y_{\varepsilon 2}$ be two different strong equilibrium points. We want to explore the risk dominance relationship between these equilibrium points in $\hat{\tilde{G}}_\varepsilon$. For this purpose we determine a formation F of $\hat{\tilde{G}}_\varepsilon$ which contains both equilibrium points. In this formation F player 1 has the pure strategies $x_{\varepsilon 1}, y_{\varepsilon 1}$ and $W_{\varepsilon 1}$ and player 2 has the pure strategies $x_{\varepsilon 2}, y_{\varepsilon 2}$ and $[1/M]_{\varepsilon 2}$. Figure 7.1 shows a bimatrix representation of F.

We have to show that for sufficiently small ε the substructure F is a formation of $\hat{\tilde{G}}_\varepsilon$. Consider a proposal $s \in X''$ different from x and y and let $s_{\varepsilon i}$ be the corresponding ε-extreme strategy of player i. Suppose that player 2 uses an arbitrary mixed strategy of F and player 1 plays $s_{\varepsilon 1}$. Then player 1's payoff in $\hat{\tilde{G}}_\varepsilon$ is $\varepsilon_1 s$. For sufficiently small ε this is smaller than α. Therefore in $\hat{\tilde{G}}_\varepsilon$ player 1's best reply to a mixed strategy available to player 2 in F cannot be $s_{\varepsilon 1}$.

Now suppose that player 1 uses an arbitrary mixed strategy available in F. Then player 2's payoff for $s_{\varepsilon 2}$ is $\varepsilon_2(1-s)$ which is smaller than his payoff for $[1/M]_{\varepsilon 2}$. Therefore in $\hat{\tilde{G}}_\varepsilon$ player 2's best reply to a mixed strategy

	$x_{\varepsilon 2}$	$y_{\varepsilon 2}$	$[\frac{1}{M}]_{\varepsilon 2}$
$x_{\varepsilon 1}$	$x+\varepsilon_1 x$ $1-x+\varepsilon_2(1-x)$	$\varepsilon_1 x$ $\varepsilon_2(1-y)$	$\varepsilon_1 x$ $\varepsilon_2(1-\frac{1}{M})$
$y_{\varepsilon 1}$	$\varepsilon_1 y$ $\varepsilon_2(1-x)$	$y+\varepsilon_1 y$ $1-y+\varepsilon_2(1-y)$	$\varepsilon_1 y$ $\varepsilon_2(1-\frac{1}{M})$
$W_{\varepsilon 1}$	$\alpha+\varepsilon_1(M-1)\alpha$ $\varepsilon_2(1-x)$	$\alpha+\varepsilon_1(M-1)\alpha$ $\varepsilon_2(1-y)$	$\alpha+\varepsilon_1(M-1)\alpha$ $\varepsilon_2(1-\frac{1}{M})$

$$\varepsilon_1 = \frac{\varepsilon}{1-(M-1)\varepsilon} \qquad \varepsilon_2 = \frac{\varepsilon}{1-M\varepsilon}$$

Figure 7.1 : The game F which determines the risk dominance relationship between $x_\varepsilon = x_{\varepsilon 1}x_{\varepsilon 2}$ and $y_\varepsilon = y_{\varepsilon 1}y_{\varepsilon 2}$.

of player 1 available in F cannot be $s_{\varepsilon 2}$.

It follows that for sufficiently small ε the substructure F is a formation of $\tilde{G}_{\varepsilon}$. Since this formation contains x_{ε} and y_{ε} it also contains the formation spanned by both equilibrium points or in other words the restricted game for the comparison of both equilibrium points. In order to determine risk dominance between x_{ε} and y_{ε} we can concentrate our attention on F.

Player 1's bicentric prior: Player 1's bicentric prior p_1 for the comparison between x_{ε} and y_{ε} can be determined as indicated in figure 7.2. The vertical axis shows the z-line. Player 1's payoffs for $x_{\varepsilon 1}, y_{\varepsilon 1}$ and $W_{\varepsilon 1}$ against strategies of the form $zx_{\varepsilon 2} + (1-z)y_{\varepsilon 2}$ are represented by straight lines marked $x_{\varepsilon 1}, y_{\varepsilon 1}$ and $W_{\varepsilon 1}$, respectively.

In the case shown in the diagram the intersection of the lines for $x_{\varepsilon 1}$ and $y_{\varepsilon 1}$ is below the line for $W_{\varepsilon 1}$. It may also happen that the intersection is not below this line. We have to distinguish both cases. For this purpose we determine z_o.

(7.22) $$z_o = \frac{y}{x+y} + \varepsilon_1 \frac{y-x}{x+y}$$

The intersection is below the line for $W_{\varepsilon 1}$ if we have :

(7.23) $$\frac{xy}{x+y}(1+2\varepsilon_1) < \alpha(1+(M-1)\varepsilon_1)$$

In view of $M-1 \geq 3$ inequality (7.23) holds for

(7.24) $$\frac{xy}{x+y} \leq \alpha$$

In the opposite case

(7.25) $$\frac{xy}{x+y} > \alpha$$

the intersection is above the line for $W_{\varepsilon 1}$ if ε is sufficiently small.

With the help of elementary computations guided by figure 7.2 we can now determine player 1's prior. For sufficiently small ε we obtain the following result:

(7.26) $$p_1(x_{\varepsilon 1}) = 1 - \frac{\alpha}{x} - \varepsilon_1((M-1)\frac{\alpha}{x} - 1) \quad \text{for } \frac{xy}{x+y} \leq \alpha$$

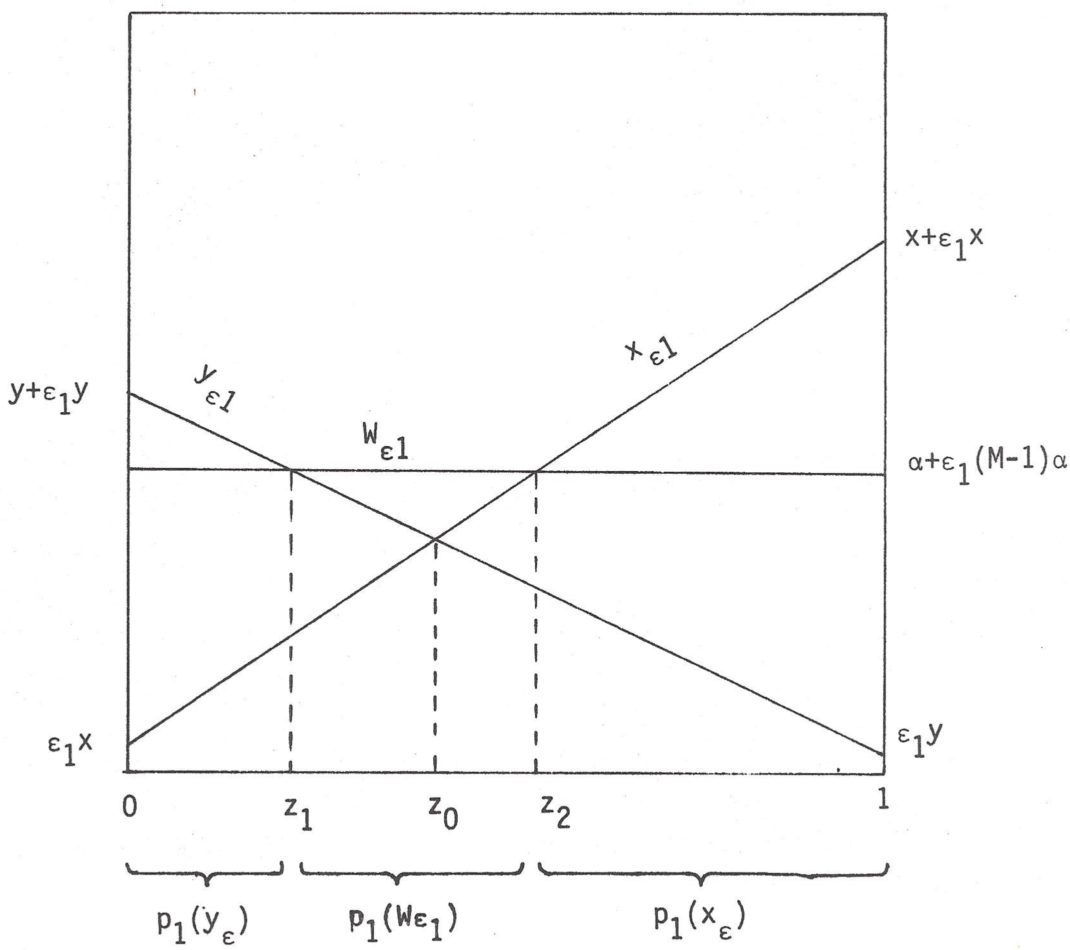


Figure 7.2: Determination of player 1's bicentric prior for the comparison of $x_\varepsilon = x_{\varepsilon 1}x_{\varepsilon 2}$ and $y_\varepsilon = y_{\varepsilon 1}y_{\varepsilon 2}$.

(7.27) $p_1(y_{\varepsilon 1}) = 1 - \frac{\alpha}{y} - \varepsilon_1((M-1)\frac{\alpha}{y} - 1)$ for $\frac{xy}{x+y} \leq \alpha$

(7.28) $p_1(W_{\varepsilon 1}) = \alpha \frac{x+y}{xy} - 1 + \varepsilon_1((M-1)\alpha \frac{x+y}{xy} - 2)$ for $\frac{xy}{x+y} \leq \alpha$

(7.29) $p_1(x_{\varepsilon 1}) = \frac{x}{x+y} + \varepsilon_1 \frac{x-y}{x+y}$ for $\frac{xy}{x+y} > \alpha$

(7.30) $p_1(y_{\varepsilon 1}) = \frac{y}{x+y} + \varepsilon_1 \frac{y-x}{x+y}$ for $\frac{xy}{x+y} > \alpha$

(7.31) $p_1(W_{\varepsilon 1}) = 0$ for $\frac{xy}{x+y} > \alpha$

Player 2's bicentric prior: Player 2's bicentric prior can be computed in essentially the same way as player 1's bicentric prior. One obtains:

(7.32) $p_2(x_{\varepsilon 2}) = \frac{1-x}{2-x-y} - \varepsilon_2 \frac{x-y}{2-x-y}$

(7.33) $p_2(y_{\varepsilon 2}) = \frac{1-y}{2-x-y} - \varepsilon_2 \frac{y-x}{2-x-y}$

(7.34) $p_2([\frac{1}{M}]_{\varepsilon 2}) = 0$

Payoffs obtained against bicentric prior: In order to determine the risk dominance relationship between x_ε and y_ε we must apply the tracing procedure to the bicentric prior p. The first step is the determination of the payoffs obtained for the pure strategies in F against the bicentric prior strategy of the other player.

(7.35) $H_{\varepsilon 1}(x_{\varepsilon 1}p_2) = \frac{x(1-x)}{2-x-y} + \varepsilon_2 \frac{x(y-x)}{2-x-y} + \varepsilon_1 x$

(7.36) $H_{\varepsilon 1}(y_{\varepsilon 1}p_2) = \frac{y(1-y)}{2-x-y} + \varepsilon_2 \frac{y(x-y)}{2-x-y} + \varepsilon_1 y$

(7.37) $H_{\varepsilon 1}(W_{\varepsilon 1}p_2) = \alpha + \varepsilon_1(M-1)\alpha$

(7.38) $H_{\varepsilon 2}(p_1 x_{\varepsilon 2}) = (1-x)[p_1(x_{\varepsilon 1}) + \varepsilon_2]$

(7.39) $\hat{H}_{\varepsilon 2}(p_1 y_{\varepsilon 2}) = (1-y)[p_1(y_{\varepsilon 1}) + \varepsilon_2]$

(7.40) $\hat{H}_{\varepsilon 2}(p_1 [\frac{1}{M}]_{\varepsilon 2}) = \varepsilon_2 \frac{M-1}{M}$

Conditions for $W_{\varepsilon 1}$ being the best reply to p_2: It is necessary to determine the best reply to the bicentric prior. We first ask the question under which circumstances $W_{\varepsilon 1}$ is player 1's unique best reply to player 2's prior strategy p_2. For this purpose,we must compare the payoff in (7.37) with the payoffs in (7.35) and (7.36). We shall show that for sufficiently small ε player 1's best reply set $A_{\varepsilon 1}(p_2)$ has the following property:

(7.41) $A_{\varepsilon 1}(p_2) = \{W_{\varepsilon 1}\}$ if and only if $\alpha \geq \frac{x(1-x)}{2-x-y}$ and $\alpha \geq \frac{y(1-y)}{2-x-y}$

It is clear that for sufficiently small ε strategy $W_{\varepsilon 1}$ is the only best reply to p_2 if the inequalities for α hold with $>$ instead of $\geq$. Moreover, for sufficiently small ε the strategy $W_{\varepsilon 1}$ cannot be a best reply to p_2 if one of both inequalities for α does not hold. In order to show that assertion (7.41) holds for border cases, too, we shall show that for sufficiently small ε the ε-term in (7.37) outweighs the ε-terms in (7.35) and (7.36):

(7.42) $\varepsilon_1(M-1)\alpha > \varepsilon_2 \frac{x(y-x)}{2-x-y} + \varepsilon_1 x$

(7.43) $\varepsilon_1(M-1)\alpha > \varepsilon_2 \frac{y(x-y)}{2-x-y} + \varepsilon_1 y$

In view of (5.48) and (5.49) we have:

(7.44) $\varepsilon_1 = \frac{\varepsilon}{1-(M-1)\varepsilon}$

(7.45) $\varepsilon_2 = \frac{\varepsilon}{1-M\varepsilon}$

In order to obtain a relationship between ε_1 and ε_2 we divide ε_2 by ε_1:

(7.46) $\frac{\varepsilon_1}{\varepsilon_2} = \frac{1-M\varepsilon}{1-(M-1)\varepsilon} = 1-\varepsilon_1$

This yields

(7.47) $\varepsilon_2 = \frac{\varepsilon_1}{1-\varepsilon_1}$

Consequently (7.42) and (7.43) are equivalent to the following inequalities (7.48) and (7.49), respectively:

$$(7.48) \qquad (M-1)\alpha > \frac{1}{1-\varepsilon_1} \cdot \frac{x(y-x)}{2-x-y} + x$$

$$(7.49) \qquad (M-1)\alpha > \frac{1}{1-\varepsilon_1} \cdot \frac{y(x-y)}{2-x-y} + y$$

We can make use of two simple algebraic identities:

$$(7.50) \qquad \frac{x(y-x)}{2-x-y} + x = \frac{2x(1-x)}{2-x-y}$$

$$(7.51) \qquad \frac{y(x-y)}{2-x-y} + y = \frac{2y(1-y)}{2-x-y}$$

Since 2-x-y is the sum of 1-x and 1-y in both cases the right hand side is smaller than 2. Therefore, for sufficiently small ε the right hand sides of (7.48) and (7.49) are smaller than 2. On the other hand, assumption (7.3) has the consequence that the left hand side is greater than 2. Therefore (7.41) holds.

<u>Conditions for $x_{\varepsilon 1}$ and $y_{\varepsilon 1}$ being best replies to</u> p_2: If one of the conditions on α in (7.41) is not satisfied then $x_{\varepsilon 1}$ or $y_{\varepsilon 1}$ is a best reply of player 1 to p_2 for sufficiently small ε. In order to find out where $x_{\varepsilon 1}$ or $y_{\varepsilon 1}$ is player 1's unique best reply to p_2 we form the difference of his payoffs in (7.35) and (7.36).

$$(7.52) \qquad \mathcal{H}_{\varepsilon 1}(x_{\varepsilon 1}p_2) - \mathcal{H}_{\varepsilon 1}(y_{\varepsilon 1}p_2) = \frac{(x-y)(1-x-y)}{2-x-y} + \varepsilon_2 \frac{y^2-x^2}{2-x-y} + \varepsilon_1(x-y)$$

It will be convenient to concentrate attention on the case $x > y$. After having derived the results for this case the results for the opposite case $y > x$ can simply be obtained by exchanging the roles of x and y. With the help of (7.52) we shall show that for sufficiently small ε player 1's best reply set $A_{\varepsilon 1}(p_2)$ has the following properties:

$$(7.53) \qquad A_{\varepsilon 1}(p_2) = \{x_{\varepsilon 1}\} \quad \text{for } x > y \quad \text{if and only if } x+y < 1 \text{ and } \frac{x(1-x)}{2-x-y} > \alpha$$

(7.54) $\quad A_{\varepsilon 1}(p_2) = \{y_{\varepsilon 1}\}$ for $x > y$ if and only if $x+y \geq 1$ and $\frac{y(1-y)}{2-x-y} > \alpha$

Under the condition on α in (7.53) strategy $W_{\varepsilon 1}$ cannot be a best reply to p_2 and the right hand side of (7.52) is positive for sufficiently small ε. Therefore (7.53) holds. Under the condition on α in (7.54) strategy W_{ε_1} cannot be a best reply to p_2 either. For $x+y>1$ the right hand side of (7.52) is negative. Clearly, in this case y_ε is player 1's unique best reply to p_2 . Now consider the case $x+y=1$. In this case we have:

(7.55) $\quad \tilde{H}_{\varepsilon 1}(x_{\varepsilon 1}p_2) - \tilde{H}_{\varepsilon 1}(y_{\varepsilon 1}p_2) = (x-y)(\varepsilon_1-\varepsilon_2)$

Equation (7.47) yields

(7.56) $\quad \varepsilon_2 > \varepsilon_1$

This shows that $y_{\varepsilon 1}$ is player 1's unique best reply if the conditions of (7.54) are satisfied with $x+y=1$. It is easy to see that the conditions of (7.53) and (7.54) exhaust the set of all possible pairs with $x>y$ where $W_{\varepsilon 1}$ is not the unique best reply to p_2. Therefore, the conditions in (7.53) and (7.54) are not only sufficient but also necessary for the assertions.

The results show that for sufficiently small ε player 1's best reply to p_2 is uniquely determined. It will be shown that the analogous statement holds for player 2, too. The uniqueness of the vector best reply to the prior is important for the applicability of the linear tracing procedure in the determination of the risk dominance relationship between x_ε and y_ε.

Player 2's best replies to the prior: In order to determine player 2's best reply set $A_{\varepsilon 2}(p_1)$ we first notice that for sufficiently small ε player 2's strategy $[1/M]_{\varepsilon 2}$ cannot be a best reply to p_1. This is an immediate consequence of (7.38), (7.39) and (7.40) together with the fact that the main terms of $p_1(x_{\varepsilon 1})$ and $p_1(y_{\varepsilon 1})$ are always positive. In order to compare the payoffs in (7.38) and (7.39) it is necessary to distinguish the cases (7.24) and (7.25). For the sake of notational shortness we introduce the following definition:

(7.57) $\quad \Delta = \frac{1}{x-y}\,[\tilde{H}_{\varepsilon 2}(p_1x_{\varepsilon 2}) - \tilde{H}_{\varepsilon 2}(p_1y_{\varepsilon 2})]$

After some computations (7.38) and (7.39) together with (7.26) to (7.31) yield the following result:

$$(7.58) \qquad \Delta = \frac{\alpha}{xy} - 1 + \varepsilon_1 \left(\frac{(M-1)\alpha}{xy} - 1\right) - \varepsilon_2 \quad \text{for } \frac{xy}{x+y} \leq \alpha$$

$$(7.59) \qquad \Delta = \frac{1-x-y}{x+y} + \varepsilon_1 \frac{2-x-y}{x+y} - \varepsilon_2 \quad \text{for } \frac{xy}{x+y} > \alpha$$

We continue to concentrate our attention on the case $x > y$. With the help of (7.58) and (7.59) it will be shown that the following is true:

$$(7.60) \qquad A_{\varepsilon 2}(p_1) = \begin{cases} \{x_{\varepsilon 2}\} & \text{for } x + y < 1 \text{ or } xy \leq \alpha \\ \{y_{\varepsilon 2}\} & \text{for } x + y \geq 1 \text{ and } xy > \alpha \end{cases}$$

The condition on α in (7.58) can be rewritten as follows:

$$(7.61) \qquad \frac{\alpha}{xy} \geq \frac{1}{x+y}$$

In view of $M-1 \geq 2$ this together with (7.58) yields

$$(7.62) \qquad \Delta \geq \frac{1-x-y}{x+y} + \varepsilon_1 \frac{2-x-y}{x+y} - \varepsilon_2$$

Inequality (7.62) holds regardless of the value of α. It follows that for sufficiently small ε the assertion (7.60) holds in the subcase $x+y<1$.

Now consider the subcase $x+y \geq 1$ and $xy \leq \alpha$. In this case the condition on α in (7.58) holds. Consequently Δ is positive for $xy<\alpha$ if ε is sufficiently small. Moreover, we have:

$$(7.63) \qquad \Delta = \varepsilon_1(M-2) - \varepsilon_2 \quad \text{for } x+y \geq 1 \text{ and } \alpha = xy$$

In view of (7.47) and $M \geq 4$ the right hand side of (7.63) is positive for sufficiently small ε. We can conclude that the assertion in the first line of (7.60) holds for sufficiently small ε.

Now consider the subcase $x+y \geq 1$ and $xy > \alpha$. If the condition on α in (7.58) holds, then Δ is negative for sufficiently small ε. If the condition on α in (7.59) holds, then for sufficiently small ε the right hand side of (7.59) is negative for $x+y > 1$. Moreover, we have:

$$(7.64) \qquad \Delta = \varepsilon_1 - \varepsilon_2 \quad \text{for } x+y = 1 \text{ and } xy > \alpha$$

In view of $\varepsilon_1 < \varepsilon_2$ the right hand side of (7.64) is negative for $x > y$. Therefore, the assertion of the second line of (7.60) holds for sufficiently small ε.

Our results show that for sufficiently small ε player 2's best reply to the bicentric prior is always uniquely determined.

Exclusion of $x_{\varepsilon 1}y_{\varepsilon 2}$ and $y_{\varepsilon 1}x_{\varepsilon 2}$ as best replies to the bicentric prior: In the following it will be shown that for sufficiently small ε neither $x_{\varepsilon 1}y_{\varepsilon 2}$ nor $y_{\varepsilon 1}x_{\varepsilon 2}$ can be vector best replies to the bicentric prior. Without loss of generality we can restrict our attention to the case $x > y$.

Suppose that $x_{\varepsilon 1}y_{\varepsilon 2}$ is the vector best reply to the bicentric prior. (7.53) requires $x + y < 1$ and (7.60) requires $x + y \geq 1$. Obviously, this is impossible.

Now assume that $y_{\varepsilon 1}x_{\varepsilon 2}$ is the vector best reply to the bicentric prior. (7.54) requires $x + y \geq 1$. This condition permits the following conclusion:

$$(7.65) \qquad xy \geq \frac{y(1-y)}{2-x-y} \qquad \text{for } x+y \geq 1$$

Therefore (7.54) requires $xy > \alpha$. Contrary to the assumption, it follows by (7.60) that $y_{\varepsilon 2}$ is player 2's only best reply to the bicentric prior.

Risk dominance relationships: It is clear that x_ε risk dominates y_ε if x_ε is the vector best reply to the bicentric prior. Analogously, y_ε risk dominates x_ε if y_ε is the vector best reply to the bicentric prior. The only other possibilities for the vector best reply to p are $W_{\varepsilon 1}x_{\varepsilon 2}$ and $W_{\varepsilon 1}y_{\varepsilon 2}$. For these two cases we apply the linear tracing procedure in order to determine the risk dominance relationship between x_ε and y_ε. As we shall see, x_ε risk dominates y_ε if the best reply to p is $W_{\varepsilon 1}x_{\varepsilon 2}$ and y_ε risk dominates x_ε if the best reply to p is $W_{\varepsilon 1}y_{\varepsilon 2}$.

We shall restrict our attention to parameter pairs (x,y) with $x > y$ since for $x < y$ the same arguments can be applied with the roles of x and y interchanged. Let us first consider the case where for sufficiently small ε the best reply to p is $W_{\varepsilon 1}x_{\varepsilon 2}$. It follows by (7.41) and (7.60) that we must have:

(7.66) $\alpha \geq \frac{x(1-x)}{2-x-y}$

and

(7.67) $x+y < 1$ or $xy \leq \alpha$

It can be seen that under these conditions the difference (7.52) between player 1's payoffs for $x_{\varepsilon 1}$ and $y_{\varepsilon 1}$ against player 2's prior strategy is positive. Therefore, we can exclude the possibility that along the path of the tracing procedure applied to p in F player 1 shifts to $y_{\varepsilon 1}$. In view of the fact that $\hat{H}_{\varepsilon 1}(x_\varepsilon)$ is greater than $\hat{H}_{\varepsilon 1}(W_{\varepsilon 1}x_{\varepsilon 1})$ there will be a reversal point where $x_{\varepsilon 1}$ becomes player 1's best reply. This reversal point can be determined as follows:

(7.68) $$t_{\varepsilon 1} = \frac{\tilde{H}_{\varepsilon 1}(W_{\varepsilon 1}p_2) - \hat{H}_{\varepsilon 1}(x_{\varepsilon 1}p_2)}{\hat{H}_{\varepsilon 1}(W_{\varepsilon 1}p_2) - \hat{H}_{\varepsilon 1}(x_{\varepsilon 1}p_2) + \hat{H}_{\varepsilon 1}(x_\varepsilon) - \hat{H}(W_{\varepsilon 1}x_{\varepsilon 2})}$$

With the help of (7.35),(7.37) and figure 7.1 we can compute the limit t_1 of $t_{\varepsilon 1}$ for $\varepsilon \to 0$:

(7.69) $$t_1 = \lim_{\varepsilon \to 0} t_{\varepsilon 1} = \frac{\alpha - \frac{x(1-x)}{2-x-y}}{\alpha - \frac{x(1-x)}{2-x-y} + x-\alpha}$$

In view of $x > \alpha$ we have:

(7.70) $0 \leq t_1 < 1$

Figure 7.1 shows that $\hat{H}_{\varepsilon 2}(W_{\varepsilon 1}[1/M]_{\varepsilon 2})$ is greater than $\hat{H}_{\varepsilon 2}(W_{\varepsilon 1}y_{\varepsilon 2})$. Therefore, player 2 cannot be destabilized to $y_{\varepsilon 2}$. However, there will be a reversal point $t_{\varepsilon 2}$ with $0 < t_{\varepsilon 2} < 1$ where player 2 shifts to $[1/M]_{\varepsilon 2}$. This reversal point can be determined as follows:

(7.71) $$t_{\varepsilon 2} = \frac{\hat{H}_{\varepsilon 2}(p_1 x_{\varepsilon 2}) - \hat{H}_{\varepsilon 2}(p_1[1/M]_{\varepsilon 2})}{\hat{H}_{\varepsilon 2}(p_1 x_{\varepsilon 2}) - \hat{H}_{\varepsilon 2}(p_1[1/M]_{\varepsilon 2}) + \hat{H}_{\varepsilon 2}(W_{\varepsilon 1}[1/M]_{\varepsilon 2}) - \hat{H}_{\varepsilon 2}(W_{\varepsilon 1}x_{\varepsilon 2})}$$

Figure 7.1 together with (7.38), (7.39) and (7.40) shows that for $\varepsilon \to 0$ all payoffs in (7.71) with the exception of $\hat{H}_{\varepsilon 2}(p_1 x_{\varepsilon 3})$ vanish. This yields the following conclusion:

$$(7.72) \qquad t_2 = \lim_{\varepsilon \to 0} t_{\varepsilon 2} = 1$$

The comparison between (7.71) and (7.72) shows that for sufficiently small ε we have:

$$(7.73) \qquad t_{\varepsilon 1} < t_{\varepsilon 2}$$

Therefore player 1 is the first to shift. He shifts to $x_{\varepsilon 1}$. Since x_ε is a strong equilibrium point of F player 2's strategy $x_{\varepsilon 2}$ is his unique best reply on the whole jump sequent. x_ε risk dominates y_ε.

The case where $W_{\varepsilon 1} y_{\varepsilon 1}$ is the best reply to the bicentric prior can be treated in a very similar way. We shall not repeat essentially the same arguments in detail. The trace remains at $W_{\varepsilon 1} y_{\varepsilon 2}$ until player 1 shifts to $y_{\varepsilon 1}$ at a point $t'_{\varepsilon 1}$. If in (7.69) the roles of x and y are interchanged one receives the limit t'_1 of $t'_{\varepsilon 1}$ for $\varepsilon \to 0$. The strong equilibrium point y_ε is the result of the tracing procedure which shows that y_ε risk dominates x_ε.

As we have seen x_ε risk dominates y_ε for sufficiently small ε if and only if the best reply to the bicentric prior is either x_ε or $W_{\varepsilon 1} x_{\varepsilon 2}$. In other words, x_ε risk dominates y_ε for sufficiently small ε if player 2's best reply to p_1 is $x_{\varepsilon 2}$. Analogously, y_ε risk dominates x_ε if player 2's best reply to p_1 is $y_{\varepsilon 2}$. It is interesting to note that the direction of risk dominance depends only on player 2's best reply to player 1's prior strategy. Our results are summarized by the following theorem.

<u>Theorem on risk dominance in G''_ε :</u> Let $x_\varepsilon = x_{\varepsilon 1} x_{\varepsilon 2}$ and $y_\varepsilon = y_{\varepsilon 1} y_{\varepsilon 2}$ be two different strong equilibrium points of G''_ε. Then for sufficiently small ε the risk dominance relationships between x_ε and y_ε in G''_ε are as follows:

(7.74) $\quad x_\varepsilon$ risk dominates y_ε for $x > y$ if $x + y < 1$ or $xy \leq \alpha$

(7.75) $\quad y_\varepsilon$ risk dominates x_ε for $x > y$ if $x + y \geq 1$ and $xy > \alpha$

<u>Remarks:</u> The risk dominance relationships for $x < y$ can be obtained by interchanging the roles of x and y in (7.74) and (7.75). - Since G''_ε has

only finitely many strong equilibrium points we can find a number ε_0 such that for every ε with $\varepsilon \leq \varepsilon_0$ the risk dominance relationships between pairs of equilibrium points x_ε and y_ε in G''_ε are correctly described by (7.74) and (7.75).

6. The limit solution

In the following we shall always assume that ε is sufficiently small in the sense that risk dominance in G''_ε is correctly described by (7.74) and (7.75). The theorem on risk dominance in G''_ε will be used in order to determine the limit solution of the bargaining model. For this purpose, we shall introduce a useful graphical tool, the risk dominance diagram.

The risk dominance diagram: Let R be the set of all pairs (x,y) of real numbers with the following properties.

(7.76) $\quad \alpha < x < 1$

(7.77) $\quad \alpha < y < 1$

(7.78) $\quad x \neq y$

Each risk dominance comparison between two different strong equilibrium points x_ε and y_ε corresponds to a pair (x,y) in R. The risk dominance diagram is a graphical representation of R which indicates the regions where one of both equilibrium points risk dominates the other. x_ε risk dominates y_ε if we have:

(7.79) $\quad x + y < 1 \text{ or } xy \leq \alpha \quad \text{for } x > y$

and

(7.80) $\quad x + y \geq 1 \text{ and } xy > \alpha \quad \text{for } x < y$

The first condition is taken from (7.74) in the risk dominance theorem. The second condition is obtained by interchanging the roles of x and y. Let R_x be the set of all pairs $(x,y) \in R$ with (7.79) or (7.80). Analogously, we define R_y as the set of all pairs (x,y) satisfying:

(7.81) $x + y \geq 1$ and $xy > \alpha$ for $x > y$

and

(7.82) $x + y < 1$ or $xy \leq \alpha$ for $x < y$

It is clear that y_ε risk dominates x_ε if (x,y) is in R_y. We call R_x and R_y the risk dominance regions for x and y, respectively. The risk dominance diagram is a graphical representation of the risk dominance regions.

Figure 7.3 and 7.4 show the risk dominance diagrams for $\alpha = .2$ and $\alpha = .4$. The diagram in figure 7.3 is typical for values of α with $\alpha < .25$ and figure 7.4 is typical for $\alpha \geq .25$. This is due to the fact that the intersection point of $xy = \alpha$ with the 45^0-degree line is at $(\sqrt{\alpha},\sqrt{\alpha})$. For $\alpha < .25$ this intersection point is below the line $x+y = 1$. Therefore, in these cases the line $x+y = 1$ determines part of the border between both risk dominance regions. For $\alpha = .25$ the line $x+y = 1$ is a tangent of the curve $xy = \alpha$ and for $\alpha > .25$ the line is completely below the curve.

The limit solution for $\alpha < .25$: Consider the case $\alpha < .25$. Since M is even G''_ε has a strong equilibrium point $\tilde{x}_\varepsilon = [.5]_{\varepsilon 1}[.5]_{\varepsilon 2}$. In the risk dominance diagram all risk dominance comparisons of this $\tilde{x}_\varepsilon$ with other strong equilibrium points y_ε correspond to pairs (x,y) on the vertical line through $(.5,.5)$. As can be seen in figure 7.3 the intersection of this vertical line with R is completely in R_x. (The 45^0-line does not belong to R.) Therefore $\tilde{x}_\varepsilon$ risk donimates all other strong equilibrium points of the form $y_\varepsilon = y_{\varepsilon 1}y_{\varepsilon 2}$. It follows that $\tilde{x}_\varepsilon$ is globally dominant. Therefore $\tilde{x}_\varepsilon$ is the solution of G_ε. Consequently, $(.5,.5)$ is the limit solution of G. We have obtained the following result.

Result: For $\alpha < .25$ the strong equilibrium point $(.5,.5)$ is the limit solution of G.

The case $\alpha \geq .25$: In the following we shall assume $\alpha \geq .25$. The intersection point of $xy = \alpha$ and the 45^0-line in figure 7.4 is at $(\sqrt{\alpha},\sqrt{\alpha})$. Suppose that $\sqrt{\alpha}$ is an integer multiple k/M of the smallest money unit. In this exceptional case $[\sqrt{\alpha}]_{\varepsilon 1}[\sqrt{\alpha}]_{\varepsilon 2}$ is the solution of G_ε and $(\sqrt{\alpha},\sqrt{\alpha})$ is the limit solution of G since all the point of R on the vertical line through the intersection point belong to R_x.

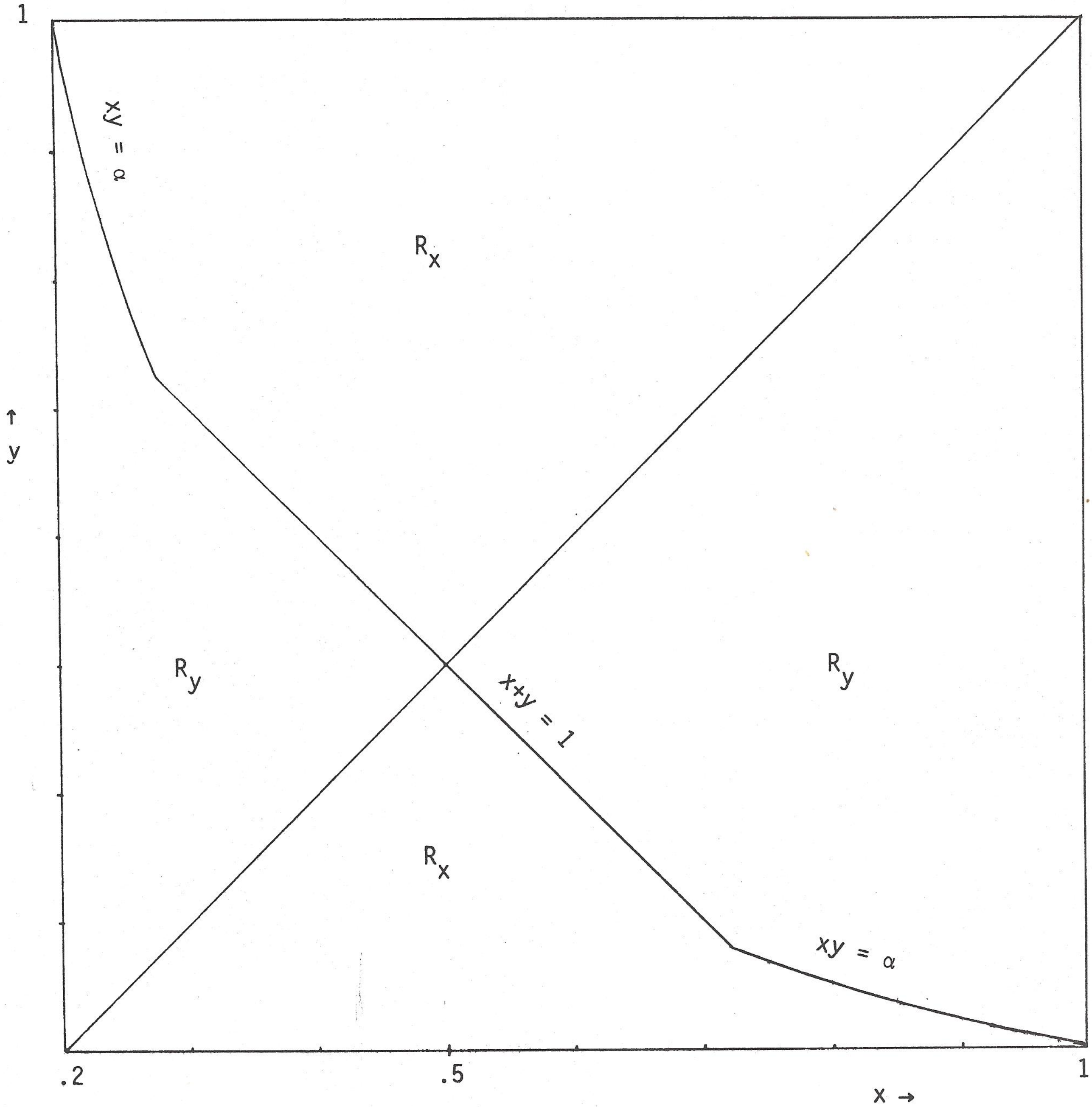


Figure 7.3: Risk dominance diagram for $\alpha = .2$.
Border points with $xy = \alpha$ belong to the lower dominance region. Border points with $x+y = 1$ and $xy > \alpha$ belong to the upper dominance region.

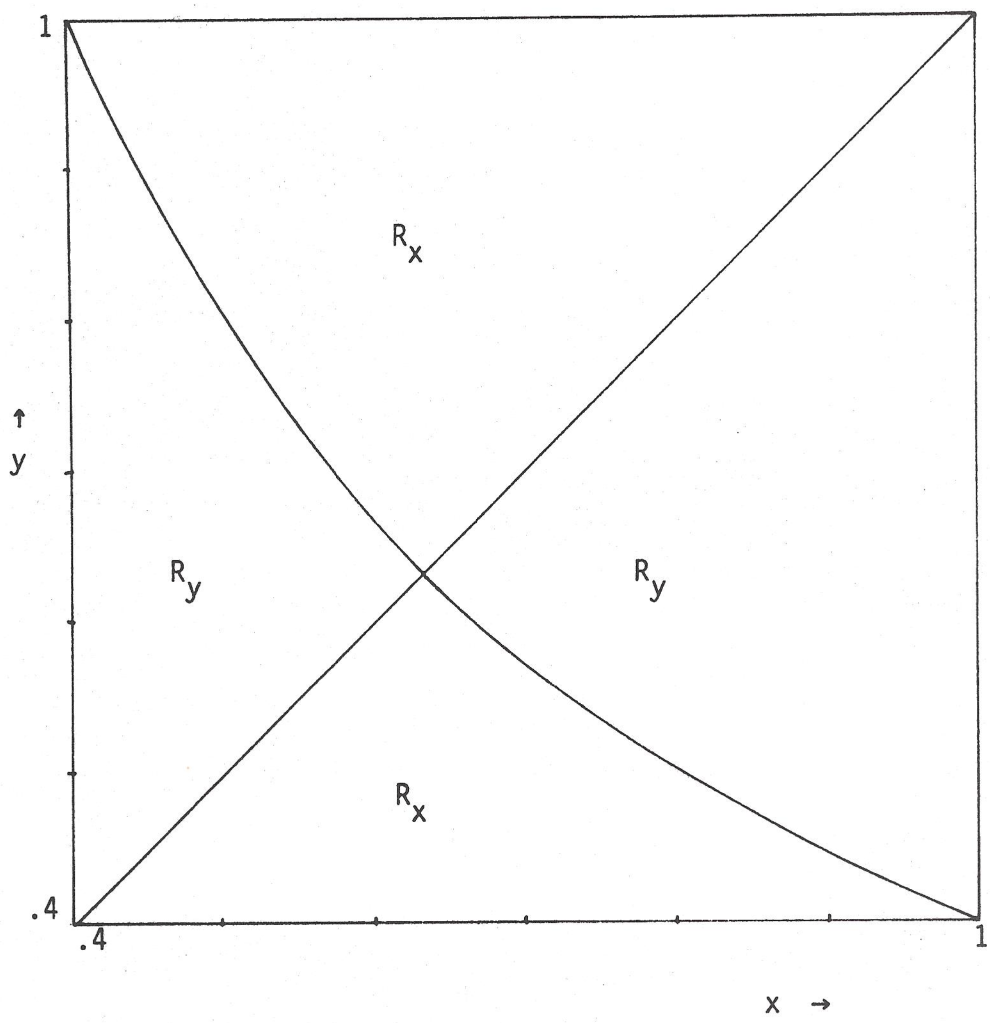


Figure 7.4: Risk dominance diagram for α = .4. Border points with $xy = \alpha$ belong to the lower risk dominance region.

It is not surprising that the limit solution can be found near $(\sqrt{\alpha},\sqrt{\alpha})$ if $\sqrt{\alpha}$ is not an integer multiple of the smallest money unit. Let g be that integer which satisfies the following inequality:

$$\frac{g}{M} \leq \sqrt{\alpha} < \frac{g+1}{M} \tag{7.83}$$

Define

$$\underline{x} = \frac{g}{M} \tag{7.84}$$

and

$$\bar{x} = \frac{g+1}{M} \tag{7.85}$$

We shall show that either $\underline{x}_\varepsilon = \underline{x}_{\varepsilon 1}\underline{x}_{\varepsilon 2}$ or $\bar{x}_\varepsilon = \bar{x}_{\varepsilon 1}\bar{x}_{\varepsilon 2}$ is the solution of G_ε.

Not all pairs $(x,y) \in R$ correspond to risk dominance comparisons but only those which are <u>grid points</u> in the sense that both x and y are multiples of 1/M. In order to find the solution of G_ε one has to look at the grid points in the vicinity of $(\sqrt{\alpha},\sqrt{\alpha})$. Figures 7.5 and 7.6 show two situations which can arise. In the case of figure 7.5 all grid points $(\underline{x},y)$ of R belong to R_x. Therefore, in this case $\underline{x}_\varepsilon$ is globally dominant. Similarly in figure 7.6 the grid points $(\bar{x},y)$ belong to R_y and $\bar{x}_\varepsilon$ is globally dominant.

If the grid points $(\underline{x},\bar{x})$ and $(\bar{x},\underline{x})$ are above the curve $xy = \alpha$ then $\underline{x}_\varepsilon$ is the solution of G_ε. This is the case if the following condition is satisfied:

$$\underline{x}\bar{x} > \alpha \tag{7.86}$$

If we have:

$$\underline{x}\bar{x} \leq \alpha \tag{7.87}$$

then $\bar{x}_\varepsilon$ is the solution of G_ε. The special case $\underline{x}\bar{x} = \alpha$ leads to $\bar{x}_\varepsilon$ as the solution of G_ε since the border points with $xy = \alpha$ belong to the lower risk dominance region. (See figure 7.4).

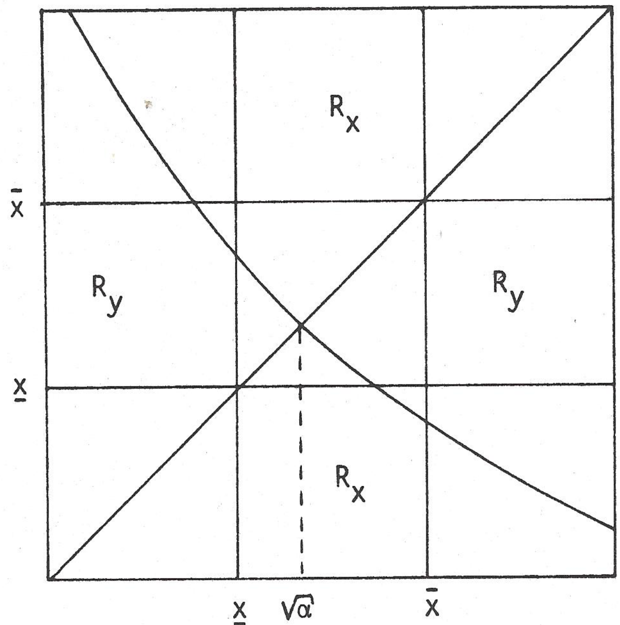


Figure 7.5: Vicinity of $(\sqrt{\alpha},\sqrt{\alpha})$ in the risk dominance diagram with $.25 \leq \alpha \leq 1$. A case where $\underline{x}_\varepsilon$ is the solution of G_ε.

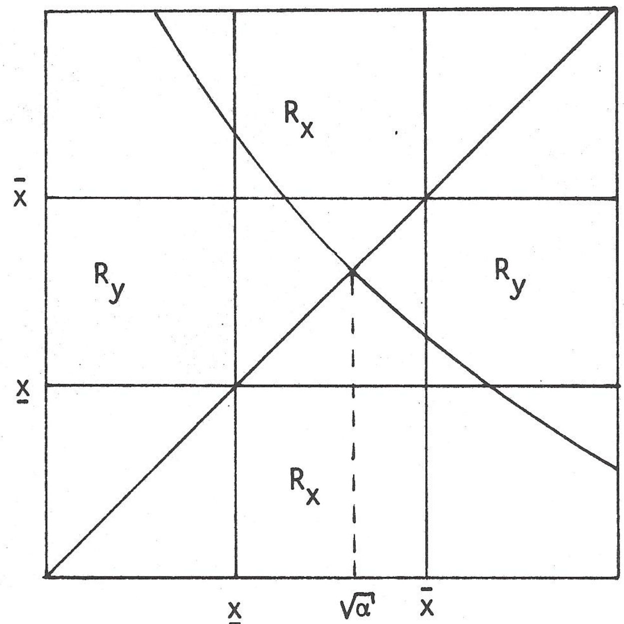


Figure 7.6: Vicinity of $(\sqrt{\alpha},\sqrt{\alpha})$ in the risk dominance diagram with $.25 \leq \alpha \leq 1$. A case where $\bar{x}_\varepsilon$ is the solution of G_ε.

The limit solution of G is $(\underline{x},\underline{x})$ if $\underline{x}_\varepsilon$ is the solution of G_ε and $(\bar{x},\bar{x})$ if $\bar{x}_\varepsilon$ is the solution of G_ε. This yields the following theorem.

<u>Theorem on the limit solution</u>: Let $\underline{x}$ be the greatest integer multiple of 1/M with $\underline{x} \leq \sqrt{\alpha}$ and let $\bar{x}$ be the smallest integer multiple of 1/M with $\bar{x} > \sqrt{\alpha}$. The game G described in section 1 has the following limit solution :

$$(7.88) \qquad \underset{\to}{L}(G) = \begin{cases} (\frac{1}{2}, \frac{1}{2}) & \text{for } \alpha < .25 \\ (\underline{x}, \underline{x}) & \text{for } \alpha \geq .25 \text{ and } \bar{x}\underline{x} > \alpha \\ (\bar{x}, \bar{x}) & \text{for } \alpha \geq .25 \text{ and } \bar{x}\underline{x} \leq \alpha \end{cases}$$

7. The asymptotic solution

A smallest money unit 1/M has been introduced as a feature of the bargaining model considered here in order to obtain a finite game. It is natural to think of 1/M as very small. Therefore, we are interested in the behavior of the limit solution for large M. As M goes to infinity the limit solution approaches (.5,.5) for $\alpha < .25$ and $(\sqrt{\alpha},\sqrt{\alpha})$ for $\alpha \geq .25$. Define

$$(7.89) \qquad (\tilde{x},\tilde{x}) \qquad \begin{cases} (.5,.5) & \text{for } 0 < \alpha < .25 \\ (\sqrt{\alpha},\sqrt{\alpha}) & \text{for } .25 \leq \alpha < 1 \end{cases}$$

We call $(\tilde{x},\tilde{x})$ the <u>asymptotic solution</u> for large M. A graph of $\tilde{x}$ as a function of α is shown in figure 7.7.

<u>Interpretation</u>: Consider the game which results from our model if W_1 is removed from player 1's pure strategy set. This game is a non-degenerate unanimity game whose limit solution is (.5,.5). (See the theorem on non-degenerate unanimity in chapter 5, section 5). For $\alpha \leq .25$ the availability of W_1 does not change this limit solution. We may say that small transaction costs do not improve the bargaining position of player 1.

For $\alpha > .25$ player 1 receives more than 1/2 in the asymptotic solution. Moreover,in this range player 1's asymptotic solution payoff $\sqrt{\alpha}$ is an increasing concave function of α; an increase of α strengthens player 1's bargaining position but the incremental effect becomes weaker for higher α.

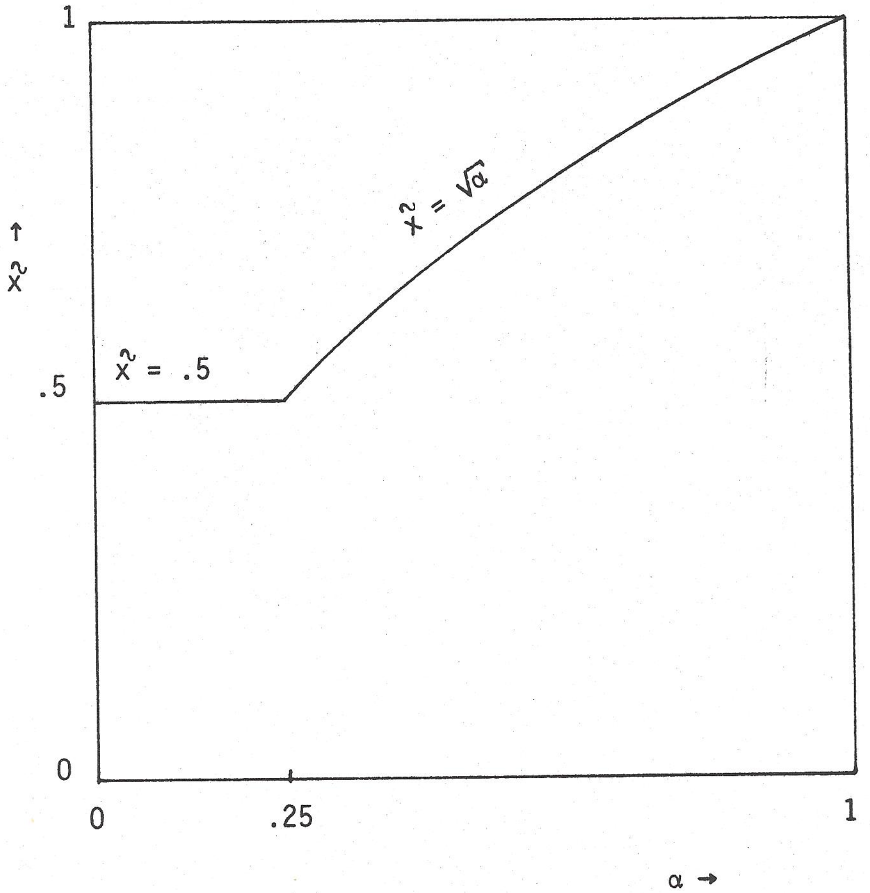


Figure 7.7: Player 1's payoff in the asymptotic solution for large M as a function of the transaction cost parameter α.

It is interesting to compare the asymptotic solution with a naive approach to the same bargaining situation. One might base a naive theory on the levels which the players can guarantee for themselves, namely α for player 1 and 0 for player 2. If the players split the difference above these levels player 1 receives the following agreement payoff:

$$(7.90) \qquad \hat{x} = \frac{1+\alpha}{2}$$

In the diagram of figure 7.7 equation (7.90) could be represented by a straight line connecting the points (0,.5) and (1,1). Obviously, $\hat{x}^2$ is greater than α. Therefore we have:

$$(7.91) \qquad \tilde{x} < \hat{x} \quad \text{for } 0 < \alpha < 1$$

This shows that the transaction cost parameter α does not improve player 1's bargaining position as much as the naive argument suggests. In fact, this is very reasonable since there is an important difference between both players with respect to the way in which their security levels α and 0 can be guaranteed. Player 1 must risk to get 0 if he tries to get more than α, whereas player 2 receives at least 0, no matter what he does. Inequality (7.91) shows that our equilibrium selection theory is sensitive to this difference.

Asymptotic solutions for other models: It is possible to give a more general definition of an asymptotic solution. In the following we shall indicate how this can be done without going into formal detail.

Consider a situation which could be modelled as a game in standard form where some of the agents or all agents have choice sets which convex and compact subsets of some euclidian space. For this purpose of applying our theory this game is replaced by a sequence of finite games depending on a parameter M such that for sufficiently large M the distance between a choice in the infinite game and the nearest choice of the same agent in the finite game becomes arbitrarily small. The asymptotic solution can be defined as the limit approached by the limit solution as M goes to infinity.

Of course, an aymptotic solution need not exist. Moreover, the asymptotic solution may depend on the way in which the infinite game is replaced by a sequence of finite substructures.

Difficulties with the convergence to an asymptotic solution do not pose a serious problem for our theory of equilibrium selection. We take the point of view that infinite games are useful as convenient idealizations of finite games with a large number of pure strategies. Infinite games cannot really be found in a finite world. Therefore, difficulties posed by infinite games should be looked upon as caused by overidealization. In view of Nash's existence theorem for finite games one should not be worried by the non-existence of equilibrium points in infinite games. Similarly, one may suspect that the infinite game does not represent important features of the underlying finite situation if difficulties with the convergence to an asymptotic solution arise.

8. Other kinds of transaction costs

Transaction costs enter the bargaining model considered here in a specific way. They are offer-related in the sense that player 1 has to bear costs of α whether an agreement is reached or not. Moreover, player 1's decision situation is simultaneous rather than sequential. He does not have to commit himself to making a proposal before he selects a specific proposal.

In the following we shall look at two variants of the model considered here. In both cases we shall sketch the process of finding the limit solution without going into formal detail. The first variant will deal with agreement-related transaction costs incurred only if agreement is reached. In the second variant player 1's decision is sequential in the sense that he has to commit himself to making a proposal before he selects a specific proposal.

Agreement-related transaction costs: Assume that transaction costs are connected to reaching an agreement. We may think of an illegal trade where bargaining in itself is not punishable but the seller player 1 can be punished if an agreement has actually been reached. This means that the transaction costs can be deducted from player 1's agreement payoff x in order to obtain his payoff for the strategy combination (x,x).

The situation is most naturally modelled by a game G^a where both players have the same pure strategy set X. In G^a the payoff vector for strategy combination (x,x) is $(x-\alpha,1-x)$. Strategy combinations (x,y) with $x \neq y$ yield zero for both players.

It would make no difference for the analysis if W_1 were included in player 1's pure strategy set with zero payoffs for both players whenever W_1 is used.

The application of the procedure of decomposition and reduction to G_ε^a first removes player 1's ε-extreme strategies $x_{\varepsilon 1}$ corresponding to $x \in X$ with $x \leq \alpha$. In a second step player 2's ε-extreme strategies $x_{\varepsilon 2}$ with $x \leq \alpha$ are eliminated. For sufficiently small ε and sufficiently large M the resulting game $\hat{G}_\varepsilon^a$ is irreducible.

The game $\hat{G}_\varepsilon^a$ is very similar to the ε-perturbed game of a unanimity game even if the perturbances are different. Suppose that X" contains exactly one element x_o where the Nash-product $(x-\alpha)(1-x)$ assumes its maximum. Similar arguments as in the proof of the theorem on non-degenerate unanimity games in chapter 5, section 5 can be used in order to show that (x_o,x_o) is the limit solution of G^a.

Obviously, for almost all values of α and M, the value where $(x-\alpha)(1-x)$ assumes its maximum, is uniquely determined. For large M this value is near to $\hat{x}$ in (7.90).

Inequality (7.91) shows that agreement-related transaction costs are more favorable for player 1's bargaining position than offer-related transaction costs. This result may be interpreted as due to the fact that under agreement-related transaction costs player 1 avoids transaction costs in the conflict case where the proposals of both players are different from each other.

Sunk transaction costs: Assume that player 1 first has to decide whether he wants to bargain or not; if he chooses to bargain he has then to make a second decision where he selects his proposal. As before, both players make their decision without any information on previous or simultaneous decisions of the other player. Once player 1 has made the decision to bargain, he has to bear the transaction costs α. In this sense the transaction costs are sunk when he makes his second decision.

The situation is described by a game $G^s = (\Phi^s,H^s)$ in standard form, where player 1 has two agents 11 and 12. Agent 11 has two choices W_1 and X and the choice set of agent 12 is X. We need not distinguish between player 2 and his single agent. Player 2's choice set is X.

If agent 11 chooses W_1 then player 1 receives α and player 2 receives 0, regardless of what agent 12 and player 2 do. If agent 11 chooses X, agent 12 selects x and player 2 plays y, then the players receive their payoffs for (x,y) in G.

It can be seen easily that the ε-perturbation G^S_ε is decomposable. (G^S is indecomposable). Agent 12 and player 2 form a cell. This cell is equivalent to the ε-perturbation of a unanimity game. In the solution of the cell both players use their ε-extreme strategies corresponding to the proposal .5.

The main truncation of G^S_ε is a one-person game where agent 11 chooses between his ε-extreme strategies corresponding to W_1 and X. For sufficiently small ε the solution of this main truncation X for $0 < \alpha < .5$ and W_1 for $.5 < \alpha < 1$. We do not want to look at the border case $\alpha = .5$ since this would force us to investigate ε-terms. We can conclude that for sufficiently small ε the choices of agents 11, 12 and player 2 prescribed by the limit solution of G_ε are ε-extreme strategies corresponding to X, .5,.5, respectively, for $0 < \alpha < .5$ and to W_1, .5, .5, respectively, for $.5 < \alpha < 1$.

Player 1's limit solution payoff is the maximum of α and .5. For $\alpha > .25$ this is below his asymptotic solution payoff $\sqrt{\alpha}$ in G.

If player 1 has to sink his transaction costs before he can select a proposal his bargaining position is not improved. For $.5 < \alpha < 1$ no agreement is reached. Player 1 knows that his sunk transaction costs do not have any influence on the bargaining outcome and, therefore, cannot afford to bargain if his transaction costs are greater than .5.

The example shows that it is important to find the right way of modelling the internal sequential structure of a player's decision situation. The modelling choice between G and G^S depends on the question whether player 1 has to commit himself to bargain before he can select a specific proposal or whether the choice to bargain can be delayed until it finally has to be made simultaneously with the selection of a proposal. In the first case G and in the second case G^S is the adequate model.

The game G^S may be described as the game which results from G by splitting off an agent 11 for player 1's choice W_1. The difference between the limit solutions of G and G^S illustrates the lack of invariance with respect to sequential agent splitting discussed in section 11 of chapter 3. As we have seen there, this lack of invariance is unavoidable if one does not want to sacrifice even more compelling requirements for a theory of equilibrium selection. Our theory takes the point of view that the players face different risk situations in G and G^S. In G the choice W_1 is still available to player 1 when he has the opportunity to select a proposal. In G^S player 1 cannot choose W_1 anymore when he selects a proposal. Therefore, player 1's choice W_1 influences risk comparisons between different agreement possibilities in G but not in G^S. Upon reflection this is not as unreasonable as it may appear to be at first glance.

9. Transaction costs on both sides

It is interesting to look at a bargaining situation where not only player 1 but also player 2 has transaction costs. Ulrike Leopold has explored this problem (Leopold-Wildburger 1982). In the bargaining game G^b with transaction costs on both sides, player 2 has an additional pure strategy W_2. The payoffs for (x,W_2) are 0 for player 1 and β for player 2, where β with $0 < \beta < 1$ is player 2's transaction cost parameter. The payoffs for (W_1,W_2) are α for player 1 and β for player 2. Otherwise, the game G^b agrees with the game G.

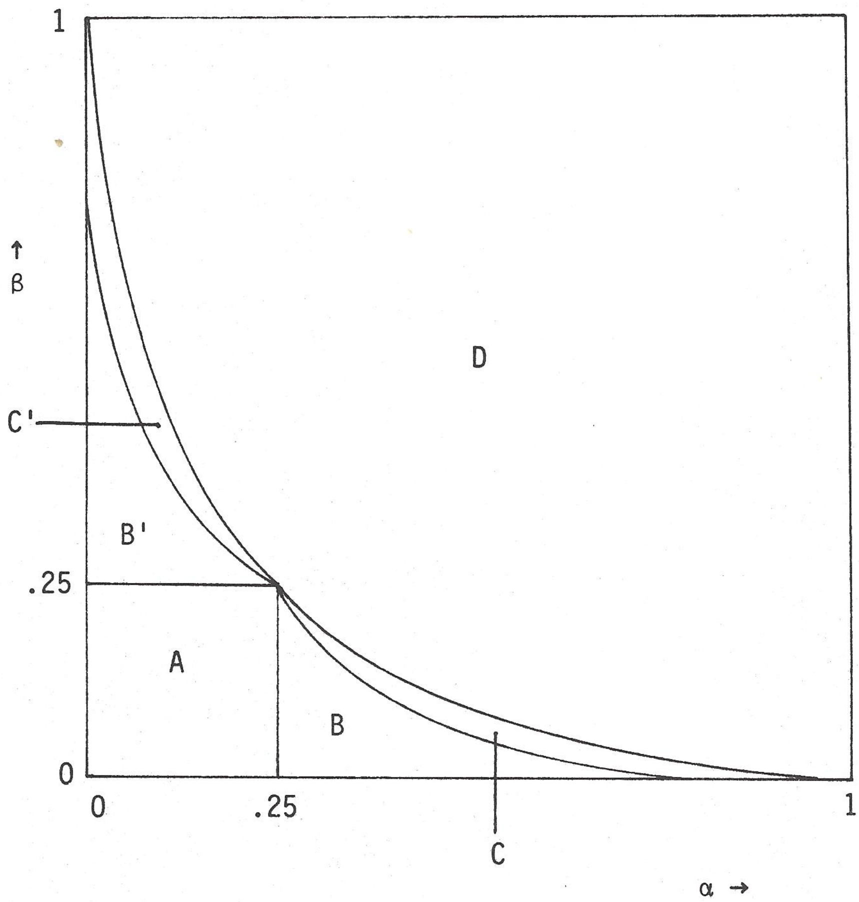
Ulrike Leopld has shown that an asymptotic solution exists for every parameter combination (α,β). Figure 7.8 summarizes her results. The upper part shows a parameter diagram in the (α,β)-plane which indicates the regions where different types of asymptotic solutions are obtained. The table below the diagram indicates the asymptotic solutions obtained in these regions.

Player 1's payoff $\tilde{x}$ in the asymptotic solution for region C is obtained as a root of the following cubic equation:

$$(7.92) \qquad \tilde{x}^3 - (1+2\alpha - \frac{\beta}{2})\tilde{x}^2 + (2\alpha + \alpha^2)\tilde{x} - \alpha^2 = 0$$

An interchange of the role of both players yields the asymptotic solution for region G'.

The asymtotic solutions for regions A and B are exactly those which have been obtained for $\alpha < .25$ and $\alpha \geq .25$ in the case of transaction costs



Region	Asymptotic solution
A	$(.5,.5)$
B	$(\sqrt{\alpha},\sqrt{\alpha})$
B'	$(\sqrt{\beta},\sqrt{\beta})$
C	$(\tilde{x},\tilde{x})$ *)
C'	$(\tilde{x}',\tilde{x}')$ *)
D	(W_1,W_2)

Figure 7.8: Asymptotic solutions of the bargaining problem with transaction costs on both sides.

*) Explained in the text.

on one side. This is not surprising, but also not trivial since the model with transaction costs on one side is not really a special case of the model with transaction costs on both sides. In the first case, both players have a different number of pure strategies and in the second case, both have the same number of pure strategies. The difference is small, but it cannot be disregarded in the application of our equilibrium selection theory.

An interesting feature of the model with transaction costs on both sides can be seen in the fact that the asymptotic solution (W_1,W_2) is obtained for parameter combinations with $\alpha + \beta < 1$ where α or β are relatively high. In these cases, our theory does not select a payoff efficient equilibrium point; in spite of the availability of agreements with payoffs greater than the transaction costs, the asymptotic solution recommends the option not to bargain at all.

The failure to reach a profitable agreement in the presence of relatively high transaction costs is not an unreasonable result. One may say that in such cases the strategic uncertainty underlying the definition of the bicentric prior involves a high risk for making a proposal and, therefore, points to the selection of the "safe" equilibrium point (W_1,W_2).

Preliminary references

Leopold-Wildburger, Ulrike, Gleichgewichtsauswahl in einem Verhandlungsspiel mit Opportunitätskosten, Pfeffersche Buchhandlung, Bielefeld 1982.

Selten, Reinhard and Ulrike Leopold, Equilibrium point selection in a bargaining situation with opportunity costs, Economie appliquée, tome XXXVI(1983)pp.611-648.