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Games and Incomplete Information
A Survey
Part I

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Contents:

Games and Incomplete Information

A Survey

1. Introduction

This survey is based to a large extend on a lecture given by S. Sorin on the occasion of a five days course on "Games and Incomplete Information" carried out at Bielefeld University in 1985. It is intended to give a short introduction to the recent developments in this quickly developing field within game theory. Due to the ever growing amount of results herein it has necessarily to remain incomplete and certainly reflects to a large extend the authors' propensity towards some special questions.

The paper is divided into three parts; the first of it introduces into non – cooperative game theory by providing the prerequisites for the second and third part. Within those, results on the use and effect of information and the shortage thereof are given. Part two herein investigates one – shot explicit normal form games and provides existence results on equilibria subsequent to a section dedicated to the analysis of lack of information under the Bayesian assumption. Part three is reserved to the investigation of dynamical aspects of information in connection with the treatment of lack of information in multistage games.

Game theory comprehenses its task as to explain behavior, preferably human behavior, on the basis of the physical attributes of players, groups of players, and their surrounding. Two main streams of analysis developed, each giving answers to inquiries on different aspects of behavior. The first, called cooperative game theory, the one we shall not be occupied with, is confined to the investigation of fairness in connection with the distribution of (attractive) goods. Here the main point is that some good has to be distributed in harmony within some group. The other type of analysis, called non – cooperative game theory, is occupied with the distribution of goods where the bargainers, being present with their physical attributes, act on their own. This results from the assumption that transfers of a good from one acting subject to another are excluded and moreover no organization enables subjects to form contracts and guarantee keeping those agreements. The acting subjects, called players in the sequel, are present with their attitude towards

the goods and with an enumeration of their available ways of behavior. Further, they may be affected by their physical surrounding. Necessarily, the search for an explanation of behavior has to presume the existence of a subject – minded relation between the attitude and the manner of behavior. To that concern game – theory presupposes rationality of the players. This rationality assumption evidently is cogent since a non – rational way of behavior never can be subject to any kind of explanation. To formalize the attitude of the players towards the goods under consideration there is ascribed a value to them reducing a vector – valued description to a one – dimensional. We are thus allowed to speak of the payoff resulting from the behavior of the players. As a consequence we suppose the players to be interested in maximizing their payoff. It is assumed that the players act completely isolated, their opponents do only occur in their consideration as far as their potential behavior, not as their payoff is concerned. Now, what sort of behavior can be expected to be obeyed by rational players? According to the points above, a collective behavior can only be stable, if no one can improve his payoff by unilateral deviation. The remaining collective ways of behavior are known as equilibria and it is their existence under various assumptions which will be investigated.

For a given action, the decision whether it may be called rational or not, not only depends on the final payoff, but also on the information available at the time the action had to be performed. Therefore our analysis of human behavior has to face that information may not be complete. This understanding requires a modelling of information in mathematically treatable terms. In the history of mathematics there exist several attempts to define information and a measure thereof, one of them is due to the founder of game theory, J. von Neumann, at the end of the fourties.
His attempt, however, was not very successful in contrast to the approaches of N. Wiener and C.E. Shannon. The ideas of the latter ran into information theory. This approach shall be dealt with rather shortly, since within game theory a far simpler approach is widely used. This approach understands information on events as to spring off a partition of the set of events. Two events are either non – confoundable or completely indistinguishable, referring to be in two distinct or only in one information set. There is nothing "in between"; formalizing the notion of "in between" could be provided by: similarity according to some distance – function or, on the other hand, by statistical consideration on the ease of discrimination by best tests. Thus, the common game theoretical approach to deal with information is very rough. As a consequence, a quantitative statement concerning information may only be given by the coarseness of the partition; much information corre-

sponds to many classes with few elements. This does not allow for a rigorous esteem on the advantages to be derived from obtaining information, since the similarity of states of the environment (expressed by small variation of the payoff) and the information structure on the states of the environment, are not related. To put it the other way round, an efficient use can only be made of such a division of the states of the environment into different classes, which make states with nearly identical consequences indistinguishable and those with very much different consequences distinguishable by the receipt of information. The game theoretical approach towards handling lack of information certainly proves to be sufficient in many specific contexts, however, lacks for example for not allowing to describe a relation between the information available and the payoff obtainable thereof, more specifically, we are not allowed to view the payoff as a function of the amount of information available, ceteris paribus.

The lack of information in a game may be caused by different assumptions. As an example, the environment, in which a game takes place, may not be completely analyzable with respect to the consequences on the payoff, – here we assumed the payoff to depend on the state of the environment which is, after all, a most realistic assumption. On the other hand, lack of information may be due to the anonymity of players for instance arising in bidding situations. As an example the payoff function of the opponents or their strategical abilities may not be known. In contrast we shall not be concerned with lack of information due to "non–remembering", since it is difficult to motivate rationality on one hand and bounded mental capacities on the other.

The degree of ignorance may assume two different levels. First there may be uncertainty on a certain parameter defining – among others – the state of the environment. This may be caused by a non–observable random–mechanism and the most simple example arises in explicit normal form games. There the payoff to the players does not only depend on their actions, but also on a parameter to which there is ascribed value randomly according to a random–mechanism whose existence is common knowledge. On a higher stage those structures may be viewed at as if all parameters are known. Instead of the parameter to which there is a value ascribed randomly, the underlying probability distribution is viewed at and basically the expected payoff is subject to the strategical considerations of the players. Thereby those models are made tractable in the usual framework of game theory.

The above form of uncertainty is called imperfect information, which contrasts to the notion of incomplete information. The latter describes the ignorance of the players on a parameter for which also no probability distribution is given as common knowledge of the players.

From those models much more difficulties resulted, and in fact, they were treated in the early days of game theory by considering the worst case – a maxmin – approach. In two steps, each requiring 15 years of development of game theory, they were made at last manageable. Therein the Bayesian approach was used characterized by supposing everybody to ascribe a probability to all thinkable events.

Investigating the presence of information and its amount can only be relevant when the players can make use of it. Thus a word on some ways of using information is in order. One point seems to be intuitively clear, namely the possession of information augments the obtainable payoff. At a second view it is not clear as that, since in certain multistage games too much information can also reduce the achievable payoff. Also information may be used as to support the implementations of optional behavior by observing a sufficient amount of uncertainty. The obtained information may be disparate enough to enable the players to choose their actions deterministically on this information. As a last point it should be mentioned that common information makes correlation of strategies accessible. This was originally excluded by the rules of the game and now is introduced in a formalized manner. Since the information has not directly an impact on the payoff, this formalization attaches a smell of cooperativity into non – cooperative game theory, but due to the rigorous assumption on the information – processing system a lot of effects, which have to be faced in free communication situations, are ruled out.

2. Prerequisites

Games are given by an abstract set of values defining admissible behavior of the players and by assigning a payoff to them when a play of the game is over. Thus, a minimum set of ingredients needed for the definition of a game as a mathema-- tically treatable object consists of a set of players $\mathcal{N}$, a set of optional actions A_n for each player $n \in \mathcal{N}$ and a payoff – function, which combines the actions of the players to yield a payoff to each player. $\mathcal{N}$ will be assumed to be a finite set throughout. However, the above description is only a very coarse one and may not be judged at as being sufficient for all purposes. Games may be analyzed on various levels of accuracy reflecting the type of analysis an investigator wishes to use and the results he envisages. Remembering that only a few parlorgames are finished after each player performed a single action we may wish to pay closer attention to the games' dynamic sequential movement. In the first section we therefore shall recall some basic results on games in extensive form.

2.1. Games in Extensive Form

In this exposition the term extensive game will always refer to the finite case. In a preliminary version the definition of an extensive game will read:

An *extensive game* Γ consists firstly of a finite tree with a root denoted by 0. Secondly, the non –.terminal nodes $\mathcal{X}$– E of the tree are provided with a partition $\{ \mathfrak{M}_0, \mathfrak{M}_1, \dots , \mathfrak{M}_N \}$. To each of the terminal nodes of E there is attached an N – vector $u^N := u^N(e) := (u_1(e), \dots, u_N(e))$.

The nodes of the tree refer to the states a play of the game may reach, a state x belonging to $\mathfrak{M}_n$ indicates that it is player n's turn to take an action. Those ad- missible actions to be taken at state x are defined by the edges of the tree, star- ting from x. Assuming $\mathfrak{M}_n \neq \emptyset$ for all n we may call the game Γ an *N – person game*. An interpretation to $\mathfrak{M}_0$ will be given lateron.

The imposed structure on the set of nodes and the finiteness of its number ensure that any play of a game – corresponding to a path within the tree which is follo- wed up by the actions of the players – .finally reaches a terminal node. The vector attached to this node defines the payoff for the players.

Consider the following verbal description of a game.

Example: (Stone - Scissors - Paper)

There are two players, 1 and 2, each of them potentially using one of three actions

- doubling up his hand (Stone)
- showing two fingers (Scissors)
- showing his flat hand (Paper)

These actions are independently chosen by the player, stone wins against scissors, scissors wins against paper and paper wins against stone. The player showing the winning action gets one unit, the other loses one, in case of identical actions the payoff is zero to both players.

A tentative description of the game may be given as follows:
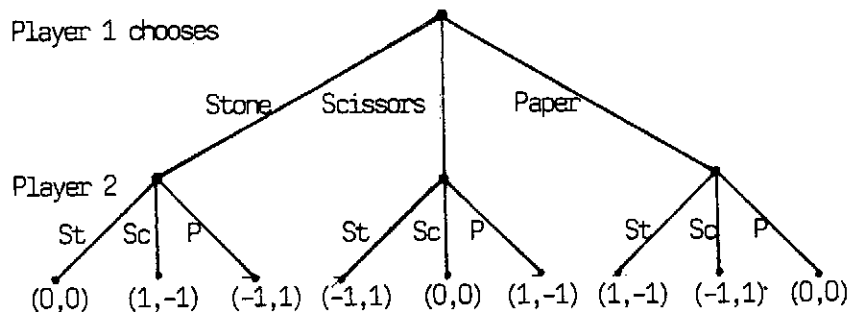
This representation is unsatisfying since the structure of the tree suggests the interpretation of different levels within the tree as to correspond to a difference in time. Obviously we may therefore only describe games which are played sequentially such as "Chess" and "Go". We observe that some mathematical ingredient must be missing to yield the equivalence of representations by game trees in which the roles of players 1 and 2 are reversed. We want to express simultaneousness of actions or, more precisely, since we do not want to refer explicitly to the physical flow of time, to express ignorance of the other player's decisions. Towards this aim H.W.KUHN [53] introduced the notion of "information sets".

On the set of non – terminal nodes $\mathcal{X}$ – E a refinement of the players' partition $\{\mathfrak{M}_{A}, \ldots , \mathfrak{M}_{N}\}$ is given. The elements A_{n} of the subpartition $\{\mathcal{O}_{A}, \ldots , \mathcal{O}_{N}\}$ are called *information – sets* of player n, $n \in \mathcal{N}$ For all A_{n} and all $x \in A_{n}$ the sets of successors of $x, \mathcal{F}_{x}$, have an identical cardinality $K(A_{n})$. Thus equivalence classes $\mathcal{C}_{n}(A_{n}) = (C_{A}(A_{n}, \ldots C_{K(A_{n})})$ on the set of successors of A_{n} may be defined, each class containing exactly one successor of each node $x \in A_{n}$. The classes are called *choices*.

We define $\mathcal{O}_{o} \supset \mathcal{P}(\mathfrak{M}_{o})$. Nodes x and $\widetilde{x}$ being contained in a fixed information – set are not discriminable by player n. We may thus represent the game stone – scissors – paper by
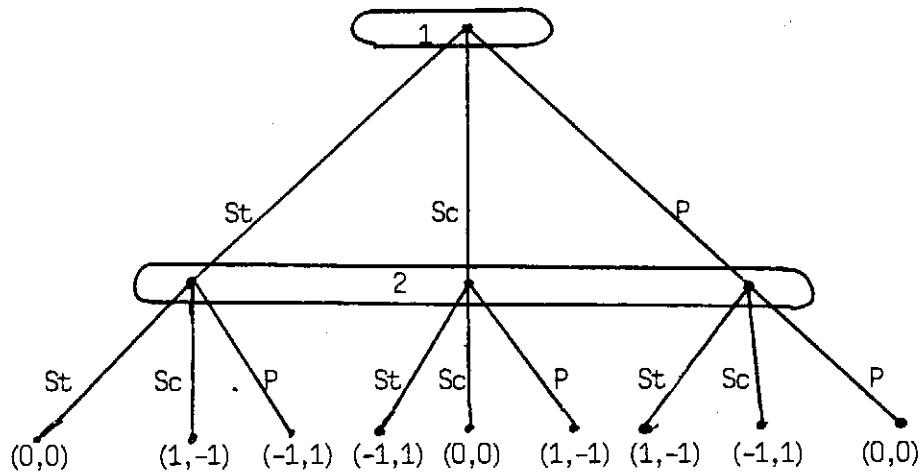
The – possibly void – element $\mathfrak{M}_{o}$ of the partition $\mathfrak{M}$ is defined to consist of nodes at which chance moves take place. If $x \in \mathfrak{M}_{o}$, then $\sigma_{o}(\cdot \mid x)$ is a conditional probability distribution on the set of successors $\mathcal{F}_{x}$ of x. Chance moves may occur in the beginning of a game, giving rise to the investigation of subgames, (frequently) of common structure.
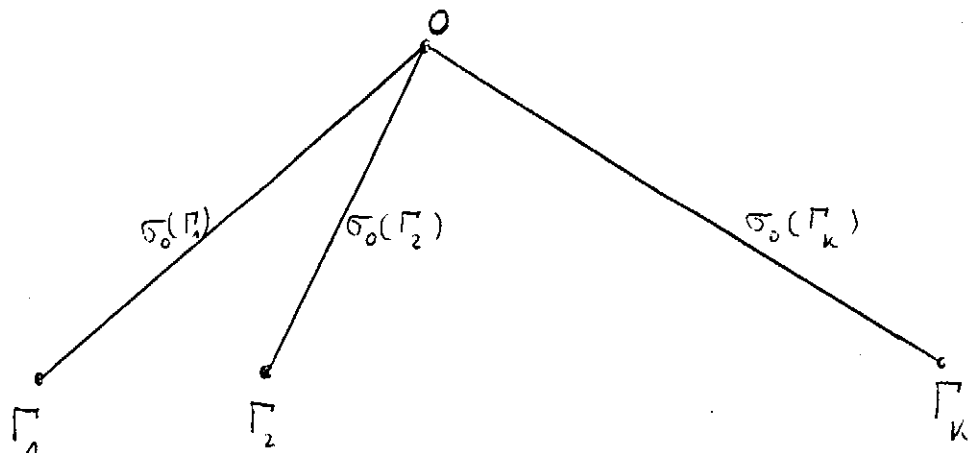
Models of this kind will be investigated in chapter 5. However, it is not excluded that random moves may take place at intermediate moves – although most (?) of our parlor games are not of this kind.

Summarizing, an extensive form game is characterized by giving

$$\Gamma = (\mathcal{G}, \{\mathfrak{m}_o, \ldots \mathfrak{m}_N\}, \{\mathcal{A}_1, \ldots, \mathcal{A}_N\}, u^N, \sigma_o)$$

In the sequel we shall assume that the players' mental capacities for reasoning and memorizing are unlimited. This allows for error–free remembering of all informations a player had during the play of a game and the knowledge of his own past actions. The notion of perfect recall as given by H.W. Kuhn [53] denotes the theoretical formalization of our intuition.

Observe first that the assumptions on the tree allows for forming the transitive closure of the successor–relation $\mathcal{J}_x$. The transitive closure of the set of successors of x is denoted by $\overline{\mathcal{J}}_x$.

An extensive game Γ is called *extensive game with perfect recall* if the following conditions hold for any player:

(1) available information never gets lost during the play of a game, i.e.

$$\bigwedge_{\mathcal{A}} \bigwedge_{\tilde{\mathcal{A}}} \tilde{\mathcal{A}} \cap \bigcup_{y \in \mathcal{A}} \overline{\mathcal{J}}_y \in \{\emptyset, \tilde{\mathcal{A}}\}$$

(2) a player remembers all his choices performed in the course of a play, i.e.

$$\bigwedge_{\mathcal{C}(\mathcal{A})} \bigwedge_{\tilde{\mathcal{A}}} \tilde{\mathcal{A}} \cap \bigcup_{y \in \mathcal{C}(\mathcal{A})} \overline{\mathcal{J}}_y \in \{\emptyset, \tilde{\mathcal{A}}\}$$

(3) a play of a game never reaches an information set twice, i.e.

$$\bigwedge_{\mathcal{A}} \bigwedge_{x \in \mathcal{A}} \overline{\mathcal{J}}_x \cap \mathcal{A} = \emptyset$$

Those assumptions exclude information structures of the following kind:
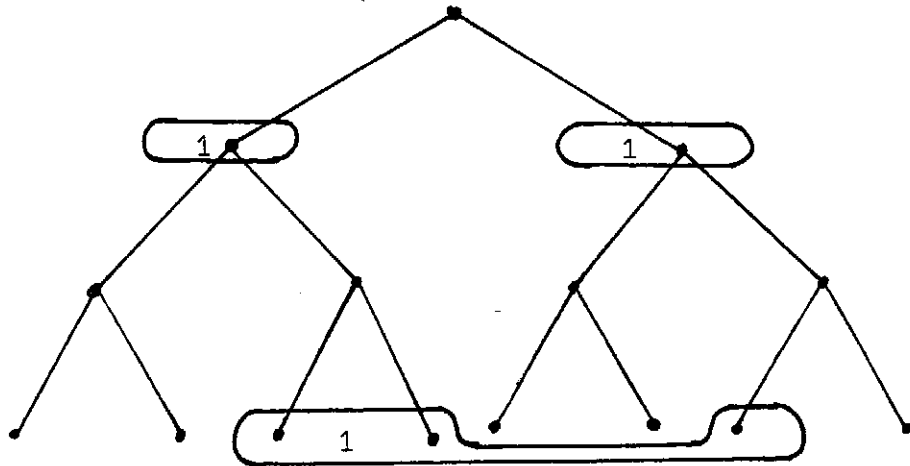
This should be excluded since player   should remember his information in his previous move.

Secondly, the figure
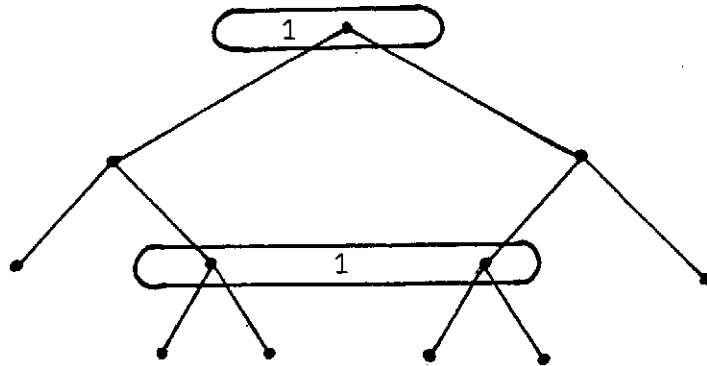
is excluded since player 1 remembers his previous choice and thirdly

is removed from the set of games to be considered to avoid interpretatorial difficulties. For a discussion of this point see J. McKINSEY [52], page 114 ff.

As a consequence of the assumption of perfect recall we obtain a more perspicious form of the strategies which also allow for an easier implementation of them. This advantage will emerge from a theorem of H.W.KUHN [53] telling us that a player may "postpone crossing his bridges until he gets to them" (R. AUMANN [6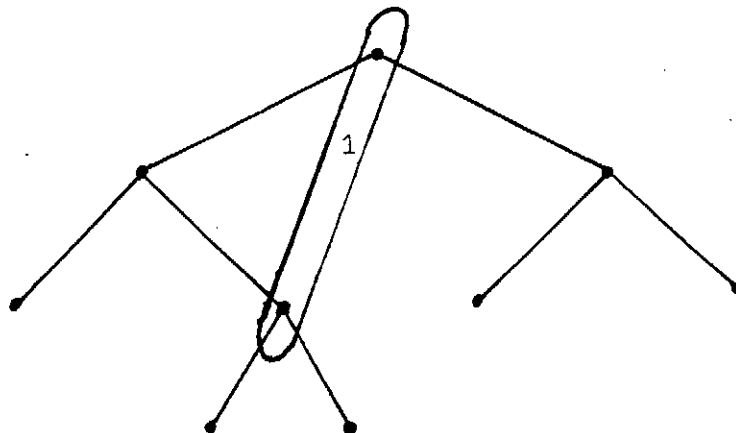4], p.684); somewhat less poetical it means that the players do not choose an action for all elements of   in advance but - sequentially - only for those nodes in the tree which are really reached in the course of a play.

We shall now provide the theoretical formulation for the behavior of the players.

1. A *pure strategy* for player $n \in \mathcal{N}$ is an $\mathcal{A}_n$-measurable function

$$\sigma_n : \mathcal{M}_n \longrightarrow \mathcal{C}$$

such that for $x \in \mathcal{A}$: $\sigma_n(x) \in \mathcal{C}(\mathcal{A})$

2. A *behavior strategy* for player $n \in \mathcal{N}$ is an $\mathcal{A}_n$-measurable function

such that for $x \in \mathcal{A}$: $\text{supp}(\sigma_n(x)) \subset \mathcal{C}(\mathcal{A})$

3. A *mixed strategy* for player $n \in \mathcal{N}$ is a probability distribution on the set of all pure strategies.

Obviously, pure strategies are special cases of behavior- and mixed strategies. Also it is easily seen that the set of mixed strategies covers the set of behavior strategies. The remarkable fact concerning behavior and mixed strategies was indicated above:

Using the notion: strategies σ_n and $\tilde{\sigma}_n$ are called equivalent if for any terminal node its marginal probability remains fixed (given any combination of strategies of the players $N - \{n\}$), then we may formulate:

Theorem: In a game of perfect recall, every mixed strategy admits the existence of an equivalent behavior strategy.

Warning: It should be observed that the latter definition really makes use of the finiteness assumption on the game. See R. AUMANN [64] for a thorough discussion.

At this point we have to reflect the basis and stimulus for the decision of the players.

As any mathematical theory describing the behavior of mankind game theory assumes rationality of the acting subjects. Since in any play of a game any player is endowed with some payoff we therefore assume the players to behave in order to maximize their payoff. Now maximizing the payoff in N-person games for N $\neq$ 1 is not an easy thing to do for player n since the actions of all the other players may influence his payoff. From this observation the central question of game theory arises:

What is the solution of a game?

This means asking for the "most likely" result being obtained from playing a game.

In analyzing properties of some given model any (mathematical) theory has to confine itself to the paramters, rules,.... as given in this model. For non-cooperative game theory which is the field to be worked in as far as this survey is concerned, this means that there exists no exogenous commitment power to enter into binding contracts with other players. Neither credible promises nor threats may be made and moreover, there is no communication possible between the players. In later models we shall allow communication along some well defined rules but they will be inserted into the description of the game. However, we shall not allow preplay or intraplay communication going beyond communication going beyond the specified rules. Any free communication between the players immediately gives rise to involving other players utility into one's own consideration by threatening or introducing correlation as a consequence of self-binding as far as fucture actions are concerned. "Self binding power", as introduced by T.C. SCHELLING [60] is nowadays considered as demarcating cooperative and non-cooperative game theory, the former going beyond the scope of this survey (and definitely asking for a proper one).

Our assumptions on the structure of the underlying tree yield that any pure strategy (= path within the tree) specifies a payoff to all players. The existence of a final payoff is now extended to all mixed- and behavior strategies.

For a strategy vector $\sigma^N = (\sigma_1, \ldots \sigma_N)$ we define the outcome of the game to be the expected payoff-vector $(U_n(\sigma_0, \sigma^N))_{n \in \mathcal{N}}$

$$U_n(\sigma_0, \sigma^N)$$

$$= E_{\sigma_0, \sigma^N} \left[u_n \right]$$

$$= \sum_{e \in E} \Pr\{e\} \cdot u_n(e)$$

Denoting the unique path from 0 to the terminal node e by x^L, $\Pr\{e\}$ is given by

$$\Pr\{e\} = \sum_{l=1}^{L} \sum_{n=0}^{N} 1_{\mathcal{G}_n}(x_{l-1}) \cdot \sigma_n(x_l \mid x_{l-1}) \qquad {}^{*)}$$

From the non-cooperative point of view player n analyzes his situation as follows. Expecting the other players to choose the strategies $\sigma_1, \ldots \sigma_{n-1}, \sigma_{n+1}, \ldots \sigma_N$ independently he can be reckoned upon selection σ_n^* as to maximize his own payoff. The maximizing σ_n^* given his anticipation of the other player's strategies $(\sigma_m)_{m \neq n}$ is called best reply to $(\sigma_m)_{m \neq n}$. However, if the same degree of rationality is ascribed to all players this usually leads to an expectational circle of the kind "if they know that I know that they know...". This circle only terminates — even in the first stage — if there exists a strategy vector $\sigma_1^* \ldots \sigma_N^*$ such that for all n σ_n^* is the best reply to $(\sigma_m^*)_{m \neq n}$. This observation gives rise to the definition of an equilibrium introduced by J. NASH [50]:

A strategy vector $\sigma^{*N} = (\sigma_1^*, \ldots \sigma_N^*)$ is called *(Nash-)equilibrium* if

$$\bigwedge_{n \in \mathcal{N}} \bigwedge_{\sigma_n} E_{\sigma_0, \sigma^{*N}} \left[u_n \right] \geqslant E_{\sigma_0, \sigma_1^* \ldots \sigma_n \ldots \sigma_N^*} \left[u_n \right]$$

We shall use the following convention:

Given an equilibrium vector $\sigma^{*N} = (\sigma_1^*, \ldots \sigma_N^*)$ the strategy σ_n^* will be referred to as an equilibrium strategy of player n. No confusion should arise.

We observe first that a corollary to the previous theorem is easily at hand.

${}^{*)}$ Here it should be observed that any mixed strategy induces in a canonical way a probability concerning the continuation of the path — a behavior strategy.

Corollary (KUHN [53])

> In an extensive game with perfect recall there exists to any equili-
> brium in mixed strategies a payoff – equivalent equilibrium in beha-
> vior strategies.

Leaving aside the problem of ensuring the existence of equilibria for some large class of games we observe that equilibrium p oints usually are not unique as the following example shows
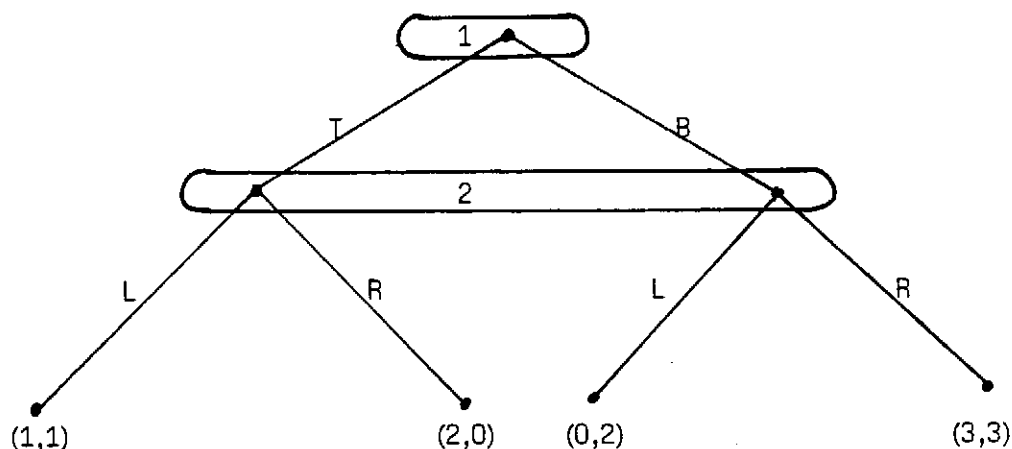
Moreover, as is well known, some equuilibrium points are unsatisfactory as far as their properties are concerned. Interpreting non – cooperative game theory as a theory suggesting ways of rational behavior to players (postulate of rational recommendation), the theory has to select one equilibrium. This problem however, is not conclusively solved, only some material for discussion is provided by J.C. HARSANYI and R. SELTEN in [80]. The huge amount of literature on the "prisoners dilemma" indicates efforts of game theorists even in a specific case to deal with it. We refer the reader to [74], part 1.

A second game theoretic approach to explain the observable behavior of players is concerned with enlargening the set of points being viewed at as potential outcomes in a non – cooperative game. This point of view reflects the weakening of the assumptions on the notion of an equilibrium. R. SELTEN [65], [73] found it unsatisfactory that the equilibrium property of a strategy vector can be destroyed by disequilibria of unreached parts of the game. He therefore defined the concept of perfect and subgame – perfect equili bria in [75].

Persuing a path – the play of a game – in the graph of an extensive form game, some of its stages x induce the structure of an extensive game on $\overline{J_x}$, the set of successors of x. The necessary and sufficient condition for this is called regularity of the subgraph together with its imposed information structure. It means that any information set containing one node of $\overline{}_x$ does not contain any vertices outside of $\overline{J_x}$. In this case the subgame Γ is defined by the restriction of the structure on G to the graph–structure of $\overline{J_x}$. Moreover, the strategies of Γ are to be restricted canonically to Γ_x . We may then define a *subgame–perfect equilibrium* to be an equilibrium which induces equilibria on every subgame of Γ.

An interesting feature in conjunction with subgame perfectness is the following concerning the existence of equilibria within pure strategies. Generally, restriction on pure strategies admits no equilibrium strategy vector.

An extensive game is refered to be of *perfect information*,
if for all $n \in \mathcal{N}$: $\quad \alpha_n = \mathcal{P}(\mathfrak{M}_n)$

This definition expresses all players' knowledge on the state of a play when they have to take an action. These condition is met e.g. in some of our parlor games such as Chess, Go, Kalahari....

For those the following result of KUHN [53] applies, assuming the games are provided with some stopping rule which ensures their termination in finitely many steps.

Theorem: Every extensive game with perfect information has a subgame perfect equilibrium within the set of pure strategies.

It may be seen that the notion of subgame–perfectness rules out a lot of difficulties arising with unreached subgames. However, the concept is not strong enough to exclude them all (see R. SELTEN [75], Sec. 6).

Introducing perturbations of games as consequences of break–downs of rationality with some small probability there is no longer any unreached subgame. Thus using an adequate equilibrium concept for perturbed game it may be shown that sequences of equilibrium points belonging to perturbations becoming smaller and smaller, converge to an equilibrium in the original game. The equilibrium points which may be approximated in this way are called *perfect equilibrium* points. They

fall within the class of subgame – perfect equilibrium points. The discussion concerning perfect equilibria would be meaningless if their existence could be ensured only for a small class of games. However, Selten proved:

Theorem: Every extensive game Γ with perfect recall has at least one perfect equilibrium point.

It should be remarked that all the existence results cited stem from an existence theorem for equilibria of games in normal form which will be tackled in the next section. We have decided to present those results on extensive form games mainly for three reasons. First the existence result of Nash is by no means a constructive one since it uses some fixed point theorem for its proof. Thus the computation of equilibria is generally unsolved. However, using the time dependent structure of extensive form games sometimes equilibria may be calculated. Secondly, a wide field of investigation of equilibria for classes of games deals with multistage games. Here it is assumed that parts of the structure of the multistage game are obtained as repetitions of one – shot games, the repeated parts of the structure define the parameters of the game whereof the solution of the multistage game is found to be the value of some function (to be derived). Thirdly, as Selten carries out, subgame perfectness cannot be detected by analysis of the normal form of a game. Thus, the normal form is an inadequate representation of the extensive form as far as recommendation of equilibria is concerned.

2.2 Games in Normal Form

From the previous section we learned that in an extensive form game the payoff to the players is completely determined by the strategies of the players and the random moves. In fact, the strategies were combined with the random moves to yield some unique path within the graph which defines the payoff to the players through its terminal node. Therefore, the – sequential – structure of the strategies is of no relevance. In order to look at an extensive form game as a normal form game we define

$$A_n = \prod_{A \in \mathcal{A}_n} C(A)$$

to be the set of actions available to player n.

The random moves are grasped using the definitions

$$\Omega = \prod_{A \in \mathcal{O}_0} C(A) \, ,$$

$\mathcal{O} = \mathcal{P}(\Omega)$ and, interpreting ω as a vector $(\omega_A)_{A \in \mathcal{O}_0}$ representing the choices of chance at nodes from $\mathcal{M}_0$, setting

$$\sigma_0(\omega) = \prod_{A \in \mathcal{O}_0} \sigma_0(\omega_A | A).$$

Identifying (ω, a^N) and the induced terminal node e achieved by the path (ω, a^N) we may put $u_n(e) := u_n(\omega, a^N)$ and denote a normal form game Γ by

$$\Gamma = ((\Omega, \mathcal{O}, \mu), \mathcal{O}_n, A_n, u_n)$$

In this vector a complete enumeration of the parameters of an explicit normal form game is given, their interaction is defined by the rules as follows: In the 0-th stage according to μ some $\omega \in \Omega$ is chosen. Player n now bases the selection of an action a_n on an observation of the "least" set $A_n \in \mathcal{O}_n$ containing ω. All actions now are chosen in ignorance of the other players behavior. The rules may be expressed by the following diagram:
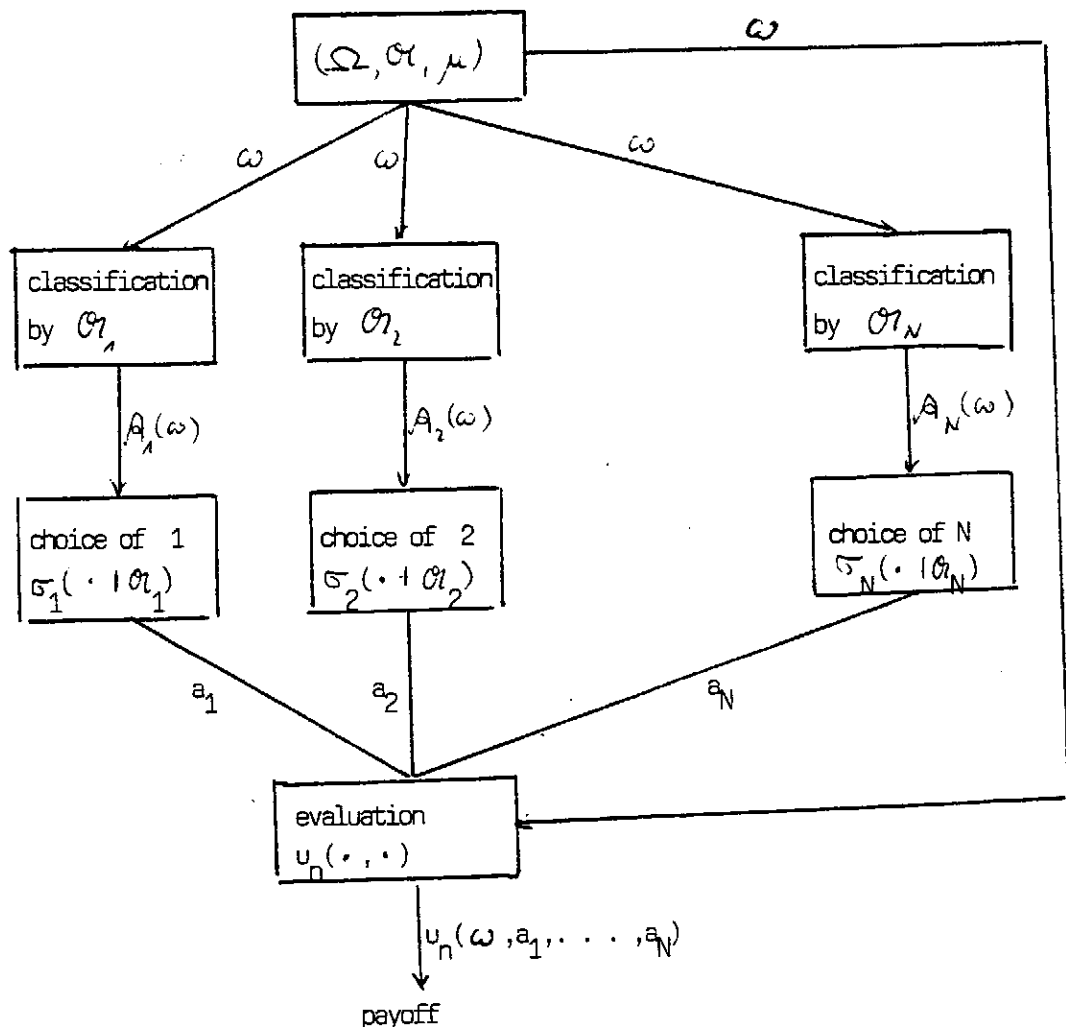
In case of finite sets Ω and A_n, $n \in \mathcal{N}$, pure strategies may be defined in the natural way as $\mathcal{O}_n$-measurable functions $\sigma_n : \Omega \longrightarrow A_n$. The notion of a mixed strategy then is easily extended to yield conditional probabilities $\sigma_n \mid \Omega \Longrightarrow A_n$.

A vector σ^N of strategies defines an expected payoff to player n by

$$U_n(\sigma^N) = \sum_{\omega} \sum_{a^N} \mu(\omega) \; \sigma^N(a^N \mid \omega) \; u_n(\omega, a^N)$$

The finiteness assumption on Ω may be replaced by the requirement of isomorphy to $\mathcal{R}$, in that case strategies are defined as $\mathcal{O}_n$-measurable conditional probabilities $\sigma_n \mid \Omega \Longrightarrow A_n$ *); if for some $\varphi : \Omega \longrightarrow A_n$ $\sigma_n(\cdot \mid x) = \mathcal{E}_{\varphi(x)}(\cdot)$ μ-almost everywhere then σ_n is called pure strategy.

The definition of mixed strategies in the general case (infinite A_n and Ω) is burdened with some difficulties. As R. AUMANN showed in the series of papers [61], [63], and [64], caused by measurability problems, mixed strategies may not be defined as probability distributions on the set of pure strategies in general. Observing the intuitive meaning of a mixed strategy to consist of a method for choosing pure strategies by means of some random device, he proposes another approach. Selecting the pure strategies according to some random variable defined on an appropriate probability space comes out to be suited to define mixed strategies. In contrast to the former idea, it is not the distribution on the set of pure strategies, but it is the random variable itself which has to be worked with. Defining mixed strategies as measurable functions

$$\sigma_n : [0,1] \times \Omega \longrightarrow A_n$$

proves to be appropriate.

P. MILGROM and R. WEBER [80] chose another approach to overcome the difficulties being bound up with the definition of mixed strategies. Their method is well-suited to ensure the existence of equilibrium strategies in incomplete information games since using their "distributional strategies" it can be shown that the merely unavoidable co mpactness condition on the set of strategies is satisfied. For the case $\Omega = \Omega_0 \times \dots \times \Omega_N$ they defined *distributional strategies* to consist of probability distributions on the Borel-subsets of $\Omega_n \times A_n$ such that their marginal distribution on Ω_n is equal to the marginal distribution on Ω_n induced by μ, i.e. for all measurable $B \subset \Omega_n$ the identity

$$\mu(\Omega_0 \times \Omega_1 \times \dots \times B \times \dots \Omega_N) = \delta_n(B \times A_n) , \quad n \in \mathcal{N}$$

---

*) a conditional probability $W \mid X \Longrightarrow \mathcal{Y}$ is said to be $\mathcal{O}$-measurable, iff $W : X \longrightarrow \Delta(\mathcal{Y})$ is $\mathcal{O}$-measurable.

holds. The payoff induced by a distributional strategy vector δ^N is given by

$$\tilde{U}_n(\delta^N) = \int_{A^N \times \Omega} u_n(\omega^{N+1}, a^N) \bigotimes_{n \in \mathcal{N}} \delta^N(da^N, \omega^{N+1}))$$

$$= \int_{A^N \times \Omega} u_n(\omega^{N+1}, a^N)\, \delta_1(da_1 \mid \omega_1) \ldots \delta_N(da_N \mid \omega_N)\, \mu(d\omega)$$

where $\delta_n(d a_n \mid \omega_n)$ denotes the conditional probability on A_n induced by the distributional strategy δ_n. To ensure $\tilde{U}_n(\delta^N)$ to be well-defined we assume the following properties of the formal elements contained in the model:

- For each n Ω_n is a complete, separable metric space

- the action spaces A_n are complete, separable metric spaces,

- the set of states of nature Ω_0 is a complete, separable and metric space

and

- the payoff functions

$$u_n : \Omega \times \overline{\prod_{m \in \mathcal{N}}} A_m \longrightarrow \mathbb{R}$$

are bounded and measurable.

Remembering the definition of the expected payoff from common mixed strategies $\sigma_n : [0,1] \times \Omega \to A_n$ as given by

$$U_n(\sigma^N)$$

$$= \int_{[0,1]^N \times \Omega} u_n(\omega [\sigma_1(s_1, \omega_1), \ldots \sigma_N(s_N, \omega_N)]) \, \lambda(ds_1) \otimes \ldots \otimes \lambda(ds_N) \otimes \mu(d\omega)$$

Milgrom and Weber state an equivalence theorem relating mixed and distributional strategies. It shows that the set of distributional strategies may be embedded into the set of mixed strategies and to every mixed strategy there exists a payoff-equivalent distributional strategy. Thus the players do not lose strength when being restricted to use distributional strategies, formally we quote:

Let M_n be player n's set of mixed strategies, and D_n his set of distributional strategies.

Theorem: There exists a collection $\{m_n, d_n\}_{n \in \mathcal{N}}$ of functions $m_n : D_n \longrightarrow M_n$ and $d_n : M_n \longrightarrow D_n$ such that

(i) $\quad d_n \circ m_n = id_{D_n}$

and

(ii) for all mixed strategies $\sigma_1, \dots \sigma_N$ and all n :

$$U_n(\sigma_1, \dots \sigma_N) = U_n(d_1(\sigma_1) \dots d_N(\sigma_N))$$

In the remainder of this section we shall be solely concerned with the common mixed strategies.

The equilibrium conditions for normal form games are just rewritten versions of those for extensive form games.

A strategy – vector σ^{*N} is called *(Nash –) equilibrium* if

$$\bigwedge_n \bigwedge_{\sigma_n} U_n(\sigma^{*N}) \geqslant U_n(\sigma_n | \sigma^{*-n})$$

Obviously equilibria of extensive form games are equilibria of the corresponding normal form games and vice versa.

Assuming $|\Omega| = 1$ (no random moves) J. NASH [50] proved an existence theorem for equilibria.

We omit $(\Omega, \mathcal{O}, \mu)$ and $\mathcal{O}_n$ in the description of the game which may graphically represented as
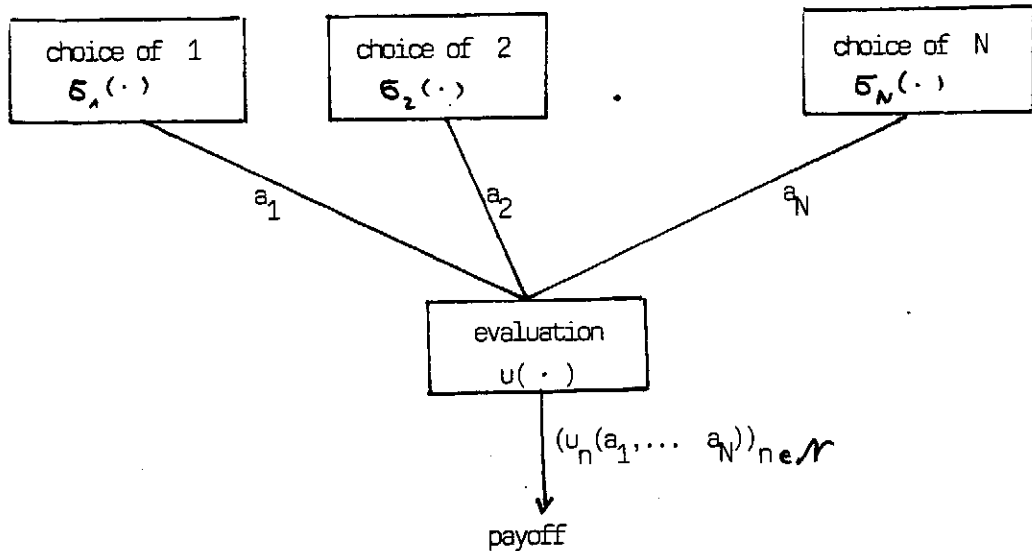
Theorem: Let $\Gamma = (A_n, u_n)$ be a normal form game. Then there exists an equilibrium of Γ.

Sometimes it is more convenient to insert mixed strategies directly into the description of a game. A game $\Gamma = (A_n, u_n)^*$ then may preferably denoted as $\Gamma = (\Sigma_n, U_n)$ where $\Sigma_n = \Delta(A_n)$ and $U_n = E_\sigma(u_n)$. Forgetting about the underlying finite set of available actions and simply regarding Σ_n, U_n as to be given, Nash's result may be extended. Basically convexity of the set of strategies Σ_n and concavity of the payoff functions U_n as depending on the strategies of player n is needed, see e.g. NIKAIDO – ISODA [55].

Besides the specific game theoretical assumption that the payoff to player n is affected by the actions of the other players even in the case that the set of admissible actions of player n itself may be restricted by the other players the existence of an equilibrium could be shown. This result of K.J.ARROW and G. DEBREU [53] has a wide field of applications in theoretical economics.

It should be noted that Nash's result itself generalized the famous maxmin – theorem of J.v. NEUMANN [28]. The latter considers a two – player situation in which one player gains what the other loses. This zero – sum assumption yields that the interests of the players are diametrically opposed. The theorem shows that the minimal payoff that player 1 (the maximizer) may ensure for himself is exactly the maximal payoff player 2 (the minimizer) cannot avoid to pay.

Theorem: Let $\Gamma = ((A_1, A_2), u_1, u_2)$ where A_1 and A_2 are finite sets and $u_2(a_1, a_2) = - u_1(a_1, a_2)$. $^{**})$ Then

$$\max_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2} \left\{ E_{\sigma_1 \sigma_2}(u_1) \right\} = \min_{\sigma_2 \in \Sigma_2} \max_{\sigma_1 \in \Sigma_1} \left\{ E_{\sigma_1 \sigma_2}(u_1) \right\}$$

---

*) Warning: Several (most?) authors understand the form

$\Gamma = (A_n, u_n)$ to restrict the set of available strategies

to pure ones in our terminology. Our notation may be viewed

at as being more adequate in chapter 3.

**) Recall that the strategies of the players are elements of $\Sigma_n = \Delta(A_n)$, n = 1,2.

It is not too surprising that generalizations of the minmax – theorem with respect to a weakening on the conditions on the sets of strategies Σ_1, Σ_2, and on the properties of u, $-u$ find analogous sufficient conditions to the existence of a value to (coincidence of "maxmin" and "minmax") as those used in the generalizations of Nash's theorem in case of more than two players and non zero – sum payoff.

In fact M. SION's theorem [58] stated below shows the coincidence of infsup and supinf for a quasiconcave / quasiconvex payoff function and convex sets of strategies, more precisely

Theorem: Let Σ_1, Σ_2 denote convex topological spaces, one of them being compact. For $U : \Sigma_1 \times \Sigma_2 \longrightarrow \mathbb{R} \cup \{\infty\}$ assume the sets

$$\{\sigma \in \Sigma_2 / U(\sigma_1, \sigma) \leq c\} \quad \text{and} \quad \{\sigma \in \Sigma_1 / U(\sigma, \sigma_2) \geq c\}$$

to be closed and convex for every $(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2$ and $c \in \mathbb{R}$. Then

$$\sup_{\sigma_1} \inf_{\sigma_2} \{U(\sigma_1, \sigma_2)\} = \inf_{\sigma_2} \sup_{\sigma_1} \{U(\sigma_1, \sigma_2)\}$$

Moreover, Σ_n being compact, for some $n = 1,2$, then the corresponding operator may be replaced by max resp. min.

Also in view of later applications, an analog of the minmax – theorem deduced by D. BLACKWELL [54] will be given. In some classes of multi-stage two – person games one of the players is not aware of the payoff – function of his opponent. Knowing his opponent to be one of a finite number of types he therefore is interested in controlling the vector consisting of the payoffs "to all of his opponents". In particular we are interested whether in a multistage game given by the repetitive play of a randomly and unobserved chosen game the vector – payoff may be shown to approach some subsets of $\mathbb{R}^M$ where M denotes the number of types.

D. Blackwell investigated the following model.

Let $\mathcal{X}$ be a finite set of vectors of $\mathbb{R}^N$ defined as the entries of a matrix

$$\begin{pmatrix} x(a_1^1, a_2^1) \dots & x(a_1^1, a_2^{K_2}) \\ \vdots & \vdots \\ x(a^{K_1}, a) \dots & x(a_1^{K_1}, a_2^{K_2}) \end{pmatrix}$$

This vector – payoff matrix (player 1 selecting rows and player 2 selecting columns) induces a multistage game by the following rules. Some pair $(a_1, a_2) \in A_1 \times A_2$ is selected (randomly) independently by (σ_1, σ_2). The resulting payoff – vector $x(a_1, a_2)$ is told to both players. In our earlier terminology this means that their information algebra is induced by the partition (first stage)

$$\mathcal{O}_1 = \left\{ \left\{ \text{proj}_2 (a_1, a_2) / x(a_1, a_2) = x \right\} \times \{a_1\} / a_1 \in A_1, x \in \mathcal{X} \right\}$$
$\mathcal{O}_2$ analogously.

In the next step they choose another pair of actions depending on their information, according to the $\mathcal{O}_1$ – measurable conditional probabilities (σ_n), $\sigma_n^1 | A_1 \times A_2 \Longrightarrow A_1 \times A_2$, they get to know the resulting payoff, and so on. The construction of the sequence of information – algebras is obvious. Generally $\sigma_n^\ell | (A_1 \times A_2)^{\ell-1} \Longrightarrow A_1 \times A_2$.

Now define a set S to be *approachable* by player 1 if there exists $(\sigma_1^\ell)_\ell$ such that

$$\bigwedge_{\varepsilon > 0} \bigwedge_{K_0} \bigwedge_{(\sigma_2^\ell)_\ell} \mathcal{P}_{r(\sigma_1^\ell)(\sigma_2^\ell)} \left\{ d_k \geq \varepsilon \text{ for some } k \geq K_0 \right\} < \varepsilon$$

where d_k denotes the distance of the arithmetic mean $n^{-1} \sum X_i$ from S, and $X_1, X_2, \ldots$ are the random variables with distribution induced by (σ_1^ℓ) and (σ_2^ℓ).

Analogously we define S to be *excludable* by player 2 if there exists (σ_2^ℓ) such that

$$\bigvee_{d > 0} \bigwedge_{\varepsilon > 0} \bigvee_{K_0} \bigwedge_{k \geq K_0} \bigwedge_{(\sigma_1^\ell)} \mathcal{P}_{r(\sigma_1^\ell),(\sigma_2^\ell)} \left\{ d_k \geq d \text{ for all } k \geq K_0 \right\} > 1 - \varepsilon$$

Defining $\text{Hull}(q) = \text{conv} \left(\left\{ \sum_{a_2} q(a_2) x(a_1, a_2) / a_1 \in A_1 \right\} \right)$, D. Blackwell proved:

Theorem: (i)   A closed convex set S is approachable if and only if

$$\bigwedge_q \quad S \cap \text{Hull}(q) \neq \emptyset .$$

             (ii)   If for some q_0 $S \cap \text{Hull}(q) = \emptyset$,then q_0 may be used by player 2 to exclude S.

3. Incomplete Information versus Imperfect Information

The study of games with incomplete information began with the pioneering papers [67a], [67b], and [68] of J.C.HARSANYI in 1967. In contrast to imperfect information which means ignorance of the players of some previous moves performed within the the game, incomplete information is interpreted as lack of full information about the normal or extensive form of the game. Such incomplete information may arise from several cases among them ignorance on

- the payoff functions
- the moves available

or

- the information which the opponents have on the game.

Whereas the first two forms of ignorance are obvious we give an example concerning the latter, thereby following S. SORIN and S. ZAMIR [85]:

EXAMPLE: (on lack of information on 1 1/2 sides)

Given two payoff-matrices A and B accordingly to a random move some of them is chosen. The structure of the random mechanism and the information situation for the players is depicted in the following figure
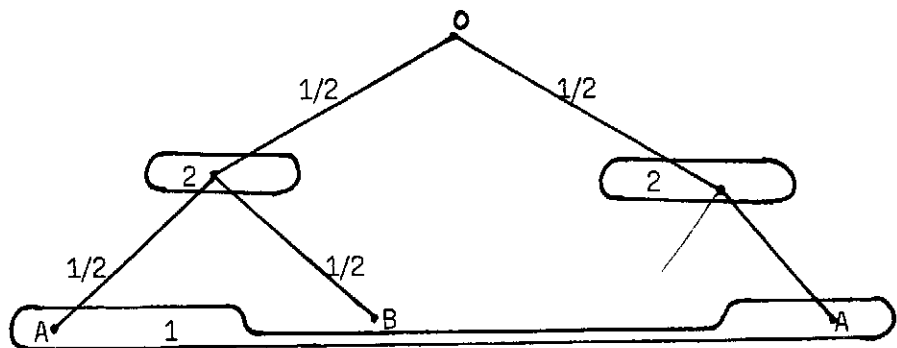
Player 1 knows the true game ("state of nature") but does not know what player 2 knows.

We shall now try to shed a light on the situation concerning incomplete information before the papers of Harsanyi. At that time a majority of game theorists considered uncertainty with respect to the state of nature to differ from uncertainty with respect to the moves of a rational player. Therefore, earlier attempts to handle the problem of incomplete information involved a pessimistic valuation of the situation and therefore were treated by a maxmin approach, e.g. J.W. MILNOR [51] and D.R. LUCE and H. RAIFFA [57]. Thus in contrast to the investigation of imperfect information games which were already treated by J. von NEUMANN and O. MORGENSTERN [44] there was little progress as far as incomplete information was concerned. Following Harsanyi this was mainly due to the fact that even in the simple case of one player's uncertainty on the other player's payoff there results a potentially infinite hierarchy of beliefs of one player on the other players' behavior. Under this kind of uncertainty player 1's action will depend on his belief on player 2's payoff—function as an important determinant of player 2's behavior. This expectation is called player 1's first order expectation. On the other hand his action also hinges upon player 2's first order expectation on player 1's payoff—function. This belief on player 2's action may be called second order expectation. This hierarchy of beliefs must be viewed at as being infinite (just as the sequence of reasoning was in conjunction with the introduction of equilibria). Assuming rationality as basis of a theoretical treatment of analyzing human behavior L. SAVAGE [54] deducted decision—makers to act as if they used some subjective probability distribution on all parameters of the world. This gives a convincing basis to the Bayesian point of view which enabled Harsanyi to give a heuristic answer to the treatment of incomplete information. Harsanyi gave plausible arguments that a game with incomplete information should be analyzed by its "Bayes—equivalent game", a game with imperfect information. This "equivalent" game was constructed by introducing a chance move in advance in which "types" of players are chosen to act as players in subgames with imperfect information thereafter. The argument of Harsanyi was made mathematically rigorous only recently by J.—F. MERTENS and S. ZAMIR [85].

Assume to be given a set of states of nature Ω_o or, far more general, a set of states of the world Ω in which Ω_o is contained. Further let A_n denote a set of actions to player n assumed to be independent of the state of the world. (This condition is not very restrictive). Further let $u_n : \Omega \times \prod_m A_m \longrightarrow \mathbb{R}$ denote the payoff function of player n.

Harsanyi's idea was to summarize all parameters and beliefs as expressed by some player via subjective probabilities in what he calls an attribute vector. It shall be first characterized what an attribute vector has to comprize. Suppose Ω_o, the space of states defining the set of potentially played games to be a compact set.

A *beliefs – hierarchy of level K* is a sequence $(C_0, C_1, ..., C_K)$ such that

1.   (i) $C_0 \subset \Omega_o$ is compact

      (ii) $C_k \subset C_{k-1} \times \left[\Delta(C_{k-1}) \right]^N$ is compact

      and

      (iii) $proj_0 (C_k) = C_{k-1}$ and

2.   (i) $P^n_{C_k}(\cdot , \Delta(C_{k-2}), ... \Delta(C_{k-2}), (\Delta(C_{k-1}))^N)$

         $= P^n_{C_{k-1}}(\cdot , \Delta(C_{k-2}), ... \Delta(C_{k-2}))$

      and

      (ii) $P^n_{C_k}(C_{k-1}, \Delta(C_{k-1}), ... , \cdot , ... \Delta(C_{k-1}))$

         $= \varepsilon_{P^n_{C_{k-1}}} (\cdot)$

The assumption 1 (ii) means that a belief up to level k consists of a belief up to level $(k-1)$ and a vector of assumptions on the beliefs of the players $n \in \mathcal{N}$ up to level $(k-1)$. Due to the Bayesian hypothesis the elements of this vector are probability distributions. Condition 1. (iii) shows that nothing gets lost climbing up the hierarchy of beliefs. The second group of conditions describe the mental abilities of the players. 2 (ii) is the familiar assumption that player n recalls his previous order beliefs ("perfect recall"). Condition 2 (i) says that player n's k-level beliefs coincide with his $(k-1)$ – level beliefs as far as hierarchies up to level $(k-1)$ are concerned.

The central problem to realize Harsanyi's idea is to define properly a set Ω of states of the world including any sequence of hierarchies of beliefs such that a point of Ω fully describes the state of nature and the attribute vector for the players. In the following definition the properties of the space of states of the world are expressed.

An *abstract beliefs – space* $(C, \Omega_0, f, (P^n)_{n \in \mathcal{N}})$ is defined

by a compact set C of attribute vectors, a continuous

function $f : C \longrightarrow \Omega_0$ which describes the state of nature

$f(\ ) \in \Omega_0$ as belonging to a given state of the world

and continuous mappings $P^n : C \longrightarrow \Delta(C)$ (with respect to

the weak* – topology) satisfying

$$\bigwedge_{\tilde{\omega} \in supp(P^n_\omega)} P^n_{\tilde{\omega}} = P^n_\omega$$

for all $n \in \mathcal{N}$.

The last condition is due to the definition of the attribute vector of player n as consisting of his beliefs. This suggests an extended version of the coherence condition 2 (i) as stating that in every state of the world as being considered possible by player n his type P^n is remains the same.

Mertens und Zamir show that Harsanyi's idea of identifying the limits of coherent beliefs with points of an appropriate space of states of the world can be mathematically formalized. In fact, they derived the following result.

Theorem: There exists a set Ω containing Ω_0 such that a point $\omega \in \Omega$ gives a full description of the state of nature and every player's beliefs P^n_ω ; i.e., $(\Omega, \Omega_0, proj_{\Omega_0} (id_\Omega), (P^n_\omega)_{n \in \mathcal{N}})$ is an Ω_0 - based beliefs space.

Ω is obtained as the projective limit of compact spaces Y_k (with respect to the natural projections $proj_k : Y_{k+1} \longrightarrow Y_k$) such that for all $k \in \mathcal{N}$ ($Y_0, Y_1, \dots Y_k$) denotes a beliefs – hierarchy of level K. The construction of the beliefs space Ω does not exclude the existence of different $\omega, \tilde{\omega}$ which are considered as equivalent states of the world by all players. Looking for a non – redundant description those Ω_0 – based abstract beliefs – spaces Ω_{NR} are shown to be embeddable

into Ω as compact subsets. The universal Ω_o-based abstract beliefs-space Ω may be equally characterized as

$$\Omega = \Omega_o \times T^N$$
$$T = \Delta(\Omega_o \times T^{N-1})$$

up to an appropriate concept of structural similiarity called BL-homeomorphy.

The above theorem shows that incomplete information may be viewed at as lack of information concerning the value of an attribute vector. As a consequence of this theorem, in all game theoretical models any player should be endowed with the knowledge of his own type P_ω^n . This stems from P_ω^n reflecting player n's beliefs on the world he is in. In a large number of models involving incomplete information the states of the world, which are assumed to be modelled by states of nature, are assumed to form a finite set. In view of the generality of those models the following theorem of Mertens and Zamir is important.

Theorem: The finite beliefs-subspaces of Ω are dense in the set of all beliefs-subspaces of Ω with respect to the Hausdorff topology on closed subsets on Ω.

This means that we do not lose too much by assuming finite parameter spaces in game-theoretical models involving incomplete information. Thus we shall restrict ourselves subsequently on considering only the finite case.

Assumption: For the remaining part of this section let Ω be a finite set.

Given some player we have shown so far that any kind of lack of information structure within the game may be reduced to lack of information concerning the value of a parameter defining the types of the other players, provided we assume the Bayesian point of view. We have not yet shown the modified game with parameter space Ω to be already a game with imperfect information. To that concern we are still missing a probability distribution on Ω which is common knowledge of the players. If its existence could also be deduced this provided a justification of the notion "incomplete information game" to a large number of imperfect information games studied in the literature. However, it turns out that not for all elements $\omega \in \Omega$ the beliefs of the players are compatible and common

knowledge, thereby providing a basis for the existence of a commonly known probability distribution on the states of the world. To clarify this statement let us consider the following example.

Assume N = 2 and let each player be one of the types from {A,B} and {a,b} respectively. Thus Ω = { Aa Ab, Ba,Bb }. Player 1, knowing his type, has a probability distribution on player 2's type. In case of being A he expects player 2 to be of type a with probability 1/3 and of type b with probability 2/3. The complete enumeration may be provided by the matrices

player 1's beliefs

$$
\begin{array}{c}
 & a & b \\
A & \begin{pmatrix} 1/3 & & 2/3 \\ B \\ 3/4 & & 1/4 \end{pmatrix} & \text{player 1's type}
\end{array}
$$

and

player 2's type

$$
\begin{array}{cc}
& A & \begin{pmatrix} 3/8 & & 5/8 \\ & & \\ B & 5/8 & & 3/8 \end{pmatrix}
\end{array}
$$

player 2's beliefs

The extensive form – not depicting the subjective probabilities – is given by
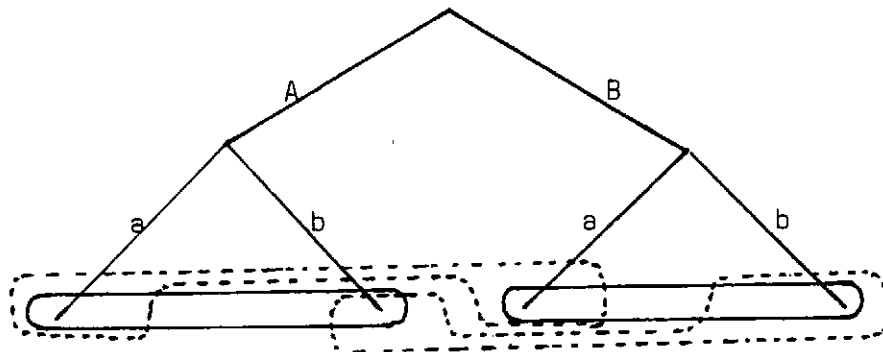
where the solid line denotes an information set of player 1 and the dashed line an information set of player 2. It is easy to see that there does not exist a probability distribution on Ω which is compatible with the beliefs of both players. The example suggests that the consistent case is rather the exception than the rule. In the sequel we shall analyze incomplete information games in order to find an equivalent imperfect information game. The notion of equivalence of an incomplete

and an imperfect information game means that to each equilibrium payoff in one of the games there also is one in the other game. In the (more general) non-consistent case the equilibrium payoffs may be derived via the agent-normal form in a somewhat unsatisfactory way. We shall be concerned with this concept of Harsanyi and Selten after having tackled the consistent case.

In the remainder of this section we shall give the complete proofs of the results provided by Mertens and Zamir since the proof of their proposition 4.4 is not correct. The subsequent propositions and particularly their theorem 4.8 heavily depends on the claim of proposition 4.4 and partly on its proof.

Definition: A probability distribution $\mu \in \Delta(\Omega)$ is called *consistent* with the beliefs $(\omega \longmapsto P_\omega^\flat)$ if

$$\bigwedge_n \bigwedge_{B \subset \Omega} \mu(B) = \sum_\omega \mu(\omega) \, P_\omega^n(B)$$

Let $\mathcal{O}_n$ denote the information algebra in Ω for player n generated by the projections

$$\text{proj}_n : \begin{array}{ccc} \Omega & \longrightarrow & \mathcal{T} \\ (\omega_o, P_\omega^1, \ldots P_\omega^N) & \longmapsto & P_\omega^n \end{array}$$

(Remember $\Omega = \Omega_o \times \mathcal{T}^N$).

The first result shows the subjective probability of player n to be the conditional probability derived from the consistent distribution μ, given his information.

Lemma: Let $\mu \in \Delta(\Omega)$ be consistent, then

$$\bigwedge_n \bigwedge_{B \subset \Omega} P_\omega^n(B) = \mu(B \mid \mathcal{O}_n)(\omega)$$

$$= \mu(B \mid \mathcal{F}^n(\omega))$$

where $\mathcal{F}^n(\omega)$ denotes the least member of $\mathcal{O}_n$ containing ω,

$$\mathcal{F}^n(\omega) = \{ \tilde{\omega} \ / \ \text{proj}_n(\tilde{\omega}) = \text{proj}_n(\omega) \}$$

$$= \{ \tilde{\omega} \ / \ P_\omega^n = P_{\tilde{\omega}}^n \}$$

Proof: Since Ω is a Ω_o-based abstract beliefs space, the property

$$\bigwedge_{\tilde\omega \in supp(P_\omega^h)} P_\omega^h = P_{\tilde\omega}^h$$

holds, yielding

$$supp(P_\omega^n) \subset \mathcal{F}^n(\omega).$$

Using the consistency assumption we get for all n and B

$$\mu(B \cap \mathcal{F}^n(\omega)) = \sum_{\tilde\omega} P_{\tilde\omega}^n (B \cap \mathcal{F}^n(\omega)) \, \mu(\tilde\omega)$$

$$= \sum_{\substack{\tilde\omega: \\ \mathcal{F}^n(\omega) \cap supp(P_{\tilde\omega}^n) \neq \emptyset}} P_{\tilde\omega}^n (B \cap \mathcal{F}^n(\omega)) \, \mu(\tilde\omega)$$

$$= \sum_{\tilde\omega \in \mathcal{F}(\omega)} P_\omega^n (B \cap \mathcal{F}^n(\omega)) \, \mu(\tilde\omega)$$

$$= \sum_{\tilde\omega \in \mathcal{F}^n(\omega)} P_\omega^n (B) \, \mu(\tilde\omega)$$

$$= P_\omega^n (B) \, \mu(\mathcal{F}^n(\omega)). \qquad \#$$

Remark that the existence of a consistent distribution establishes some weak equivalence between the original incomplete information game and the imperfect information game defined by initially choosing a state ω according to μ before starting to play. The information concerning ω is expressed by the information algebras $\mathcal{O}_{\mathbf{u}}$. Anticipating the one to one relation of equilibrium payoffs we may say that Harsanyi had this weak equivalence in mind in his series of papers from 1967. It is a weak equivalence in as much the consistent distribution yet only may be communicated to the players by some external being. However, Mertens and Zamir show that even a stronger notion of equivalence is available, showing that μ may be derived as common knowledge.

Towards this aim we define for $\omega \in \Omega$ and $n \in \mathcal{N}$

$$C_{\omega,1}^n = supp(P_\omega^n)$$

and, inductively for k = 1,2,....

$$C_{\omega,k+1}^n = C_{\omega,k}^n \cup \bigcup_{\tilde\omega \in C_{\omega,k}^n} \bigcup_{m \in \mathcal{N}} supp(P_{\tilde\omega}^m) .$$

Since $C_{\omega,1}^n \subset C_{\omega,2}^n \subseteq ...$ and because of the finiteness of Ω a limiting set C_ω^n will be reached.

This is, according to the belief of player n, the minimal set containing the real state of the world. Of course, the real state must not be contained in it since $\omega \in C_\omega^h$ is not generally true.

Lemma: Let $\mu \in \Delta(\Omega)$ be consistent, then

(i) $$\bigwedge_{\omega \in \text{supp}(\mu)} \bigwedge_{n \in \mathcal{J}} \omega \in \text{supp}(P_\omega^n)$$

(ii) $$\bigwedge_{n \in \mathcal{N}} \text{supp}(\mu) = \bigcup_\omega \text{supp}(P_\omega^n)$$

Proof: According to the previous lemma

$$P_\omega^n(\omega) = \mu(\omega \mid \mathcal{F}^n(\omega))$$

$$= \frac{\mu(\{\omega\} \cap \mathcal{F}^n(\omega))}{\mu(\mathcal{F}^n(\omega))}$$

$$= \frac{\mu(\omega)}{\mu(\omega) + \sum_{\substack{\tilde{\omega} \in \mathcal{F}(\omega) \\ \tilde{\omega} \neq \omega}} \mu(\tilde{\omega})}$$

$$> 0$$

thereby showing (i) and the inclusion "$\subset$" of (ii). To see "$\supset$" we use the previous lemma likewise;

$\tilde{\omega} \in \bigcup_\omega \text{supp}(P_\omega^h)$ yields

$$0 < P_\omega^n(\tilde{\omega}) = \mu(\tilde{\omega} \mid \mathcal{F}^n(\omega)) = \frac{\mu(\{\tilde{\omega}\} \cap \mathcal{F}^n(\omega))}{\mu(\mathcal{F}^n(\omega))}$$

whence the claim follows.   #

As a consequence we observe $C_\omega^h \subset \text{supp}(\mu)$ for all $\omega \in \Omega$. From the lemma, (i) we also derive a simplification of the inductive formula for the players' esteems on the real state of the world. For $\omega \in \text{supp}(\mu)$ and any $n \in \mathcal{N}$ it reduces to $C_{\omega,1}^n = \text{supp}(P_\omega^h)$ and $C_{\omega,k+1}^n = \bigcup_{\tilde{\omega} \in C_{\omega,k}^n} \bigcup_{m \in \mathcal{N}} \text{supp}(P_{\tilde{\omega}}^m)$ for $k = 1,2,\dots$.

The next lemma shows that all players have the same esteem on the minimal set containing the real state of the world in the consistent case.

Lemma:

$$\bigwedge_{\omega \in \text{supp}(\mu)} \bigwedge_{n,m \in \mathcal{N}} C_\omega^n = C_\omega^m$$

Proof: First observe that due to the stationarity of $(C_{\omega,k}^n)_{k \geqslant K_n}$ it is sufficient to show the inclusion

$$\bigwedge_{\omega \in \text{supp}(\mu)} \bigwedge_{n,m \in \mathcal{N}} C_{\omega,k+1}^n \supset C_{\omega,k}^m .$$

We shall prove this claim by induction. Using the reduced formula for $C_{\omega,k+1}^n$ we have as induction basis for all $\omega \in \text{supp}(\mu)$ and $m,n \in \mathcal{N}$:

$$\begin{aligned}
C_{\omega,1}^m &= \text{supp}(P_\omega^m) \\
&\subset \bigcup_{\widetilde{m}} \text{supp}(P_\omega^{\widetilde{m}}) \\
&\subset \bigcup_{\widetilde{\omega} \in \text{supp}(P_\omega^n)} \bigcup_{\widetilde{m}} \text{supp}(P_\omega^{\widetilde{m}}) \\
&= C_{\omega,2}^n .
\end{aligned}$$

The induction step is performed as follows:

Suppose

$$\bigwedge_{\omega \in \text{supp}(\mu)} \bigwedge_{m,n} C_{\omega,k}^n \supset C_{\omega,k-1}^m$$

and let ω^* be an arbitrary element of $C_{\omega,k}^m$. Then

$$\omega^* \in \bigcup_{\widetilde{\omega} \in C_{\omega,k-1}^m} \bigcup_{\widetilde{m}} \text{supp}(P_{\widetilde{\omega}}^{\widetilde{m}}),$$

say $\omega^* \in \bigcup_{\widetilde{m}} \text{supp}(P_{\widetilde{\omega}^*}^{\widetilde{m}})$.

According to the induction hypothesis $\omega^* \in C_{\omega,k}^n$. Then

$$\bigcup_{\widetilde{m}} \text{supp}(P_{\widetilde{\omega}^*}^{\widetilde{m}}) \subset \bigcup_{\widetilde{\omega} \in C_{\omega,k}^n} \bigcup_{\widetilde{m}} \text{supp}(P_{\widetilde{\omega}}^{\widetilde{m}})$$

$$= C_{\omega,k+1}^n$$

whence $\omega^* \in C_{\omega,k+1}^n$ which was to be shown. #

Note that the esteem of the players concerning the potential states of the world coincide provided the existence of a consistent distribution containing the real state of the world is assumed. $C_\omega = C_\omega^h$ is common knowledge! In fact, a stronger result shall be proved now. All players may compute a conditional probability distribution on the states of the world given their common knowledge C_ω and we shall show that those conditional probability distributions also coincide and may be viewed at as common knowledge.

Lemma: The conditional probability distribution $\mu(\cdot | C_\omega^h)$ derived from consistent probability distributions $\mu(\cdot)$ coincide for all C_ω^h.

Proof: Let $\mu(\cdot)$ be consistent. Since

$$C_\omega^n \subset \bigcup_m \bigcup_\omega \text{supp}(\cdot P_\omega^n) = \text{supp}(\mu),$$

we infer $\mu(\omega_0) > 0$ for arbitrary $\omega_0 \in \text{supp}(P_\omega^h) \subset C_\omega^h$. Now observe that $\omega_0 \in \text{supp}(P_\omega^h)$ yields $P_{\omega_0}^h = P_\omega^h$ and consequently $C_{\omega_0,1}^h = C_{\omega,1}^h$. Reminding the definition of C^h we infer $C_{\omega_0}^h = C_\omega^h$. It shall now be proved that $\mu(\cdot)$ is uniquely defined on $C_{\omega_0}^h$ given some specific value for $\mu(\omega_0)$. Let us be given ω_0 such that $\mu(\omega_0) > 0$ and assume $\tilde\omega \in \text{supp}(P_{\omega_0}^m) \subset \mathcal{F}^1(\omega_0)$. Then

$$\frac{\mu(\tilde\omega)}{\mu(\omega_0)} = \frac{\dfrac{\mu(\{\tilde\omega\} \cap \mathcal{F}^m(\omega_0))}{\mu(\mathcal{F}^m(\omega_0))}}{\dfrac{\mu(\{\omega_0\} \cap \mathcal{F}^m(\omega_0))}{\mu(\mathcal{F}^m(\omega_0))}}$$

$$= \frac{\mu(\tilde\omega \mid \mathcal{F}^m(\omega_0))}{\mu(\omega_0 \mid \mathcal{F}^m(\omega_0))} = \frac{P_{\omega_0}^m(\tilde\omega)}{P_{\omega_0}^m(\omega_0)}$$

thereby yielding

$$\mu(\tilde\omega) = \frac{P_{\omega_0}^m(\tilde\omega)}{P_{\omega_0}^m(\omega_0)} \cdot \mu(\omega_0) > 0$$

Thus, starting from $C_{\omega,1}^m = \text{supp}(P_\omega^m)$ we found $\mu(\cdot)$ to be uniquely defined on $C_{\omega,k}^h$, inductively, depending on the initial value $\mu(\omega_0)$. At least we find $\mu(\cdot)$ to be uniquely defined on C_ω, likewise depending on the initial value $\mu(\omega_0)$. In defining the conditional distributions $\mu(\cdot | C_\omega^h)$ the different factors $\mu(\omega_0)$ cancel out, thereby proving the assertion.   #

We define a state of the world ω to be *consistent* if there exists a consistent $\mu \in \Delta(\Omega)$ such that $\omega \in \text{supp}(\mu)$.

Using this notion Mertens and Zamir now proved

Theorem: (i) There exists a test on consistency of a state yielding this property of the given state $\omega \in \Omega$ of the world as common knowledge.

(ii) In case of consistency of ω the sets C_ω^h containing it are common knowledge. Likewise the conditional distribution $\mu(\cdot \mid C_\omega^h)$ on the set of potential states of nature is common knowledge.

As far as statement (i) is concerned we refer to the original article of Mertens and Zamir. The claim (ii) is proved by the preceding lemmata.

It will now be deduced that any incomplete information game may be viewed at as an imperfect information game as far as their Nash-equilibria are concerned - provided the beliefs of the players are assumed to be consistent.

Summarizing, we get as a consequence of the existence theorem on Ω_o-based beliefs spaces that any incomplete information game is - under the Bayesian assumption - nothing more then a point in the universal beliefs-space Ω. For all practical purpose further we may assume Ω to be finite. At this time now we shall embed the incomplete information situation given by $\omega \in \Omega$ into an imperfect information game. Recall that a player set $\mathcal{N} = \{1,...,N\}$, action sets $A_n, n \in \mathcal{N}$ and payoff-functions $u_n : \Omega \times \prod_m A_m \longrightarrow \mathbb{R}$ were given. A vector-payoff game is now given by defining player n's strategies to be

$$\Sigma_n = \{\sigma_n / \sigma_n \mid \Omega \Longrightarrow A_n\}$$

and the payoff to player n to consist of the vector

$$\bar{U}_n(\sigma^N) = (\bar{U}_{F_n}(\sigma^N))_{F_n \in \mathcal{F}_n}$$

where
$$\bar{U}_{F_n}(\sigma^N) = \sum_\omega \sum_{a^N} u_n(\omega, a^N)\sigma^N(a^N \mid \omega) \, P_{F_n}(\omega)$$

and $\mathcal{F}_n$ is a partition of Ω, each member $F_n \in \mathcal{F}_n$ containing all ω having some fixed type $P^h \in \Delta(\Omega)$.

For vector – payoff games the notion of a (Nash –) equilibrium is easily at hand, namely $\sigma^{*N} = (\sigma_1^*, \ldots \sigma_N^*)$ is a *(Nash –)Equilibrium* if

$$\bigwedge_n \bigwedge_{F_n} \bigwedge_{\sigma_n} \overline{U}_{F_n}(\sigma^{*N}) \geqslant \overline{U}_{F_n}(\sigma_n | \sigma^{*-n})$$

Endowing the parameters of the vector payoff game with a different set of rules, a different interpretation, a game in "agent normal – form" arises. In the specific context it is known as Selten game G^{**} (see HARSANYI [67a], p. 179). There are $\prod |\mathcal{F}_n|$ "agents" acting in the game, selecting a strategy and then choosing their $N-1$ partners, one from each $\widetilde{\mathcal{F}}_n$, according to their subjective probabilities. This game may be transposed into the common game theoretical context and is called by R. SELTEN [80], p.48, "game with subjective random moves". In those games partnership is not necessarily a symmetric relationship, which in particular implies the payoffs to be considered as being fictitious. The existence of equilibrium points may be established in games with subjective random moves analogously to that in games with (objective) random moves, see e.g. SELTEN [75]. Further, even Kuhn's theorem holds such that the restriction on behavioral strategies is available.

The different interpretation of the parameters of the game does not affect the equilibrium points which are easily seen to coincide since the different components of the payoff vector to be optimized in the first game correspond to numerical payoffs for independent agents in the second. Of course we have to observe additionally that the information of player n in the vector payoff game defined by ω coincides with the information of agent $F_n = \widetilde{\mathcal{F}}_n(\omega)$ in the "Selten – game". In the general – inconsistent – case of games with incomplete information the somewhat less satisfying situation arises that the obtained information did not outsprang the observation of a hidden random mechanism. The players even know that there does not exist some objectificating mechanism and are concious of the incompatability of their beliefs. Despite that they have to base their actions in a rational way on their beliefs to achieve an equilibrium payoff. Its existence is guaranteed according to the foregoing.

In the consistent case the subjective beliefs were transformed into objective and commonly known parameters, a probability distribution μ on a consistent subset C of Ω resulted. In this case for the given (constant) incomplete information situation an explicit normal form game is defined reducing incomplete to imperfect information. Following these lines the equilibria of the vector payoff game defined above are retained in the newly defined game. The subsequent theorem condenses the intuition of Harsanyi.

Theorem: Let $\omega \in \Omega$ be a consistent state. Then the incomplete information situation defined by ω corresponds to the vector payoff game given above and its equilibrium points may alternatively computed by finding the equilibrium points of the explicit normal form (Bayesequivalent) game $\Gamma = ((\Omega, \mathcal{P}(\Omega), \mu_\omega), \mathcal{O}_{\mu}, u_{\mu})$, where μ_ω is the consistent distribution defined by ω, and $\mathcal{O}_{\mu}$ is defined by the partition $\tilde{\mathcal{F}}_{\mu}$.

4. Use of Information

In the present chapter it will be observed that there are different forms of using information going far beyond the most popular aspect investigated in the previous chapter by considering explicit normal form games. In those games the amount of information available to the players was defined by the coarseness of the information σ - algebras of the players. Switching from one σ-algebra to a finer one for exactly one player we may easily observe that he may now ensure himself a higher payoff than that which was achievable previously. Whereas the existence of equilibria could be derived, under finiteness assumptions, the calculation of the equilibrium payoffs, preferably as a function of some measure of the coarseness of the information σ - algebras can not be performed easily. It should be remarked that in those one - shot games the information of the players is a statical notion. This assumption will be maintained in this chapter but be given up in a later chapter where we shall be occupied with the dynamical growth of information of the players on a firstly unknown realization of a random variable. More precisely, we shall assume a set of payoff - matrices being given in advance and in the 0 - th step one of them is selected by an only partially observable random mechanism. This payoff - matrix defines the payoff in a repeated (multistage) game and the information provided to the players consists of the (partially) observed actions of the opponent. The asymptotical value and the value in the infinite - horizon case will be investigated and given computably - at least in principle - in terms of the one - stage parameters of the game.

The stationary counterpart of using information to derive higher payoffs is investigated in the last section of this chapter. In the two - person zero - sum case we shall provide the existence of equilibria when the information available to the players is not considered as being fixed but subject to the decision of the players. Of course we shall not assume the players to be free to choose any system which may provide information but we shall assume the maximum amount of derivable information being fixed such that the players have to keep well defined restrictions on the choice of those systems. The tools used in this section and the measure of information will be provided by information theory.

A second form of using information has meta - theoretical aspects. Remembering that in normal form games equilibria are usually not to be found in pure strategies, the problem of implementing equilibrium strategies arises. If one of the tasks of game theory is to give rational and complete recommendations how to play in a

given situation, then equilibria derived from a mixed strategy proposal are unsatisfying as long as the construction of a random mechanism with given probabilities is not easily at hand. This may be viewed as one reason that random behavior is seldom observed in practice. Conversely, this sheds a light on the application of game theory as a theory describing human behavior. For those reasons it is important that − provided some assumptions on an explicit normal form game are satisfied − for any (equilibrium −) strategy there exists a pure strategy yielding an (approximately) identical payoff. The conditions needed concern the shape of the opponents observations of one's private information on the states of nature and may be viewed at as a form of independency of observations. The matter is refered to section 4.2.

A third form of using information emerges when playing cooperatively in a non − cooperative nonzero − sum game yields a higher payoff to all players than that coming out from using (non − cooperative) Nash − equilibrium strategies. Cooperation of the players is made available by assuming a commonly observable event − usually different precision of the observation as performed by the players is assumed. The model may be embedded into the explicit normal form games by supposing the payoff − functions to be independent of the states of nature. The idea to consider an external random mechanism sprang off the investigation of the theory of bargaining with incomplete information and ran into the theory of correlated equilibria. In this field the set of equilibrium payoffs derived from varying the probability spaces of states of nature and the information σ − algebras of players is analyzed whereas the payoff − functions are assumed to be fixed. It should be observed, however, that the theory is not occupied with the strategical problem defined by the negotiation on the selection of some space of states of nature together with the information σ − algebras.

4.1. Information and Cooperation

Cooperative game theory basically is occupied with the inquiry of desirable properties of the payoff − distribution among the players. The fairness problem for the distribution of achievable payoffs is not dealt with in non − cooperative game theory due to the absence of institutions guaranteeing the adherence of contracts. Thus the payoff is determined by the physical abilities of the players which yields as most important the notion of an equilibrium payoff. Nevertheless, since generally equilibrium points are not uniquely defined, one of them has to be re-

commended – thereby inducing the necessity of comparison of their properties. This was already noted in an earlier section and led to the investigation of perfect equilibria. In fact, apart from the class of zero – sum games a whole lot of games in normal form are such that equilibria miss to be pareto – optimal by far or are unsatisfying because of their asymmetric payoffs in a game with symmetric players.

Sometimes correlating strategies would be a remedy to overcome those problems. A famous example is known as the "battle of sexes" defined by the payoff – bimatrix

$$\begin{pmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{pmatrix}$$

with asymmetric pure – strategy equilibrium payoffs (2,1) and (1,2) and the non – pareto – optimal mixed strategy equilibrium yielding (2/3, 2/3). Uniting forces, which means correlating the strategies requires the existence of some communication systems between the players or at least some signalling mechanism whose outcome is to a certain extend observable by the players. The most simple model has already been introduced into normal form games as a special case of explicit normal form games. The given game $\Gamma = (A_n, u_n)$ is extended by introducing additionally a probability space $(\Omega, \mathcal{O}l, \mu)$ and information σ – algebras $\mathcal{O}l_n$ to describe the information given to the players on the outcome of a random experiment performed according to μ in order to correlate their strategies.

We emphasize that the payoff received by the players is not affected by the outcomes of the experiments, in contrast to the common assumption concerning explicit normal form games. Since the states of nature do not enter the payoff – functions, they have no direct impact on the strategies of the players and therefore cannot be used a priori to correlate the strategies. Therefore it is assumed that preplay communication of the players is available to enable them using a well – defined interpretation of the obtained observations in the course of a play. The mechanism allowing preplay communication however, will not be specified and thus does not enter the formal description of a game. This is unfamiliar as far as non – cooperative game theory is concerned but is usually assumed for cooperative game theory.

Given any normal form game, correlation may be performed according to an arbitrary information device. As an example observe that by preplay communication the players of the "battle of sexes" may agree upon flipping a coin to decide on playing the pairs of actions (top, left) and (bottom, right). Obviously, the resulting expected payoff is pareto - optimal and yields the symmetric allocation (3/2, 3/2). The above arrangement is also an equilibrium since no player can gain from unilateral deviation. Formally we define:

The strategy - vector $\sigma^{*N} = (\sigma_1^*,...,\sigma_N^*)$ is
called *correlated equilibrium* for the game $\Gamma = (A_k, u_k)$
if there exists $(\Omega, \mathcal{Q}, \mu)$ and information σ-algebras $\mathcal{O}_k$,
$n \in \mathcal{N}$ such that σ^{*N} is a (Nash -) equilibrium for
$\Gamma_e = ((\Omega, \mathcal{Q}, \mu), \mathcal{O}_k, A_k, u_k)$.

J. HARSANYI and R. SELTEN [72] noticed that for all normal form games any point within the convex hull of Nash - equilibrium payoffs may be obtained as a correlated equilibrium payoff. Only little later in his pioneering article on correlated equilibria R. AUMANN [74] provided an example showing the existence of correlated equilibria outside the convex hull of Nash - equilibria. A most illustrating example concerning this point was given by L.A. GERARD - VARET and H. MOULIN [78]:

Example: Define Γ by means of the payoff - bimatrix

$$\begin{pmatrix} (5,4) & (4,5) & (0,0) \\ (0,0) & (5,4) & (4,5) \\ (4,5) & (0,0) & (5,4) \end{pmatrix}$$

No pair of pure strategies is an equilibrium since they all give rise to the same cycle. The only Nash - equilibrium is obtained as the uniform distribution on the set of strategies of both players. Defining
$\Omega = \{1,2,3\} \times \{1,2,3\}$, $\mathcal{Q} = \mathcal{P}(\Omega)$, $\mathcal{O}_1 = \mathcal{P}(\{1,2,3\}) \times \{\emptyset \, \{1,2,3\}\}$, $\mathcal{O}_2$ analogously
$\mu(i,j) = 1/6$ for $j = (i + 1) \bmod 3$ or $i = j$, $i = 1,2,3$, a correlated
equilibrium arises with payoff (4.5, 4.5). In fact, μ chooses a pair of actions, its n - th component being told to player n, n = 1,2.

Choosing the action as suggested is immediately seen to be an equilibrium strategy. The resulting payoff (4.5, 4.5) is not contained in the convex hull of Nash – equilibria, – the latter contains only the point (3,3).

The set of correlated equilibria achievable by variation of $(\Omega, \mathcal{A}, \mu)$ and $(\mathcal{A}_n)$ may be shown to be already obtainable by canonical representations such as the one described above, more precisely we may choose $\Omega = \Pi \, A_n$, $\mathcal{A} = \Pi \, \mathcal{P}(A_n)$ and $\mathcal{A}_n = \Pi \, \{\emptyset, A_n\} \times \mathcal{P}(A_n)$. This result of R. AUMANN [84] has a strong impact on the computation of the set of correlated equilibria which could not be performed in the original case.

As a consequence of the existence of a canonical representation for all correlated equilibria they all may be viewed at as to differ only by the underlying probability measure μ. Therefore a condition on μ to yield a correlated equilibrium may be formulated as

μ induces a correlated equilibrium in conjunction with $(\Omega, \mathcal{A})$ and $(\mathcal{A}_n)$ as defined above if and only if

$$\bigwedge_{n \in \mathcal{N}} \bigwedge_{a_n, \tilde{a}_n \in A_n} \sum_{a^{-n}} \mu(a^{-n}, a_n) \, u_n(a^{-n}, a_n) \geqslant \sum_{a^{-n}} \mu(a^{-n}, a_n) \, u_n(a^{-n}, \tilde{a}_n)$$

In the context of extensive form (multi-stage) games the richer structure allows for introducing information providing – correlating – mechanisms at all stages. Those games have been investigated by R. MYERSON [84] and, within the context of "repeated games" to be tackled in a later chapter by F..Forges. An example of Myerson shows that there may exist correlated sequential equilibrium payoffs which are not obtainable as correlated equilibrium payoffs.

Example: The game is visualized as follows:
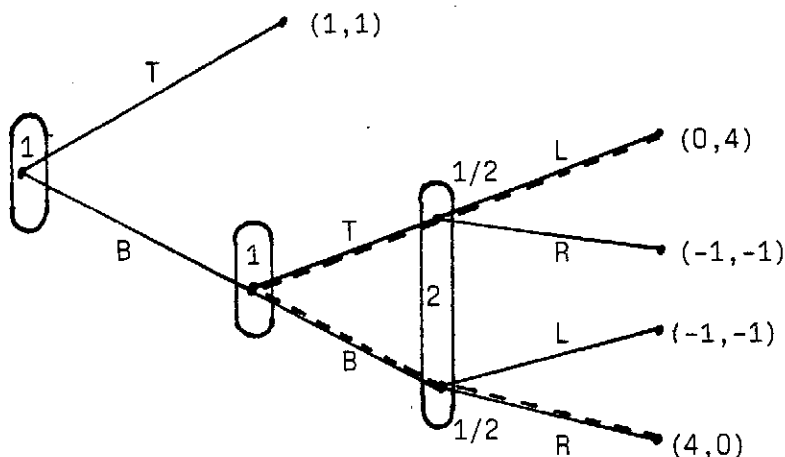
At stage 1 player 1 has two possible actions T and B, provided he plays T the play is over with the payoff vector (1,1); in case he plays B they simultaneously choose a further motion. If the second – stages actions are correlated by observing the outcome of flipping a coin deciding whether (T,L) or (B,R) is proposed, then the correlated sequential equilibrium (2,2) is obtained. However, regarding the normal form of the game, obviously a random choice of the sequence of actions leading to (0,4) would not be followed by player 1 since the payoff is dominated by (1,1). Thus player 2's payoff never exceeds 1 showing (2,2) to be not obtainable as correlated equilibrium payoff.

An even further extension of the correlating mechanisms was provided by F. FORGES [84]. She introduced the notion of communication equilibria. The communication systems defined as signalling matrices provide the receiver with some randomly disturbed variant of the symbol used by the sender. Thereby they allow intraplay communication.

Given a maximal length T of the play of a multistage game Γ with perfect recall, a *communication device* d^T is defined to be a stochastic system $d^T = (d_t)_{t=1,\ldots,T}$

$$d^T : \prod_{n \in N} \prod_{\tau=1}^{t-1} \mathcal{Y}_{d\tau}^n \times \mathfrak{Z}_\tau^n \Longrightarrow \prod_n \mathfrak{Z}_t^n$$

with finite sets $\mathcal{Y}_{d\tau}^n$, and measurable spaces $\mathfrak{Z}_\tau^n$, $\tau = \{1,\ldots,T\}$. $\mathcal{Y}_{d\tau}^n$ denotes the set of symbols which can be used to provide messages to the other players at time τ, whereas a symbol z_t^n is received by

player n at time t, being (stochastically) dependent of all previously sent symbols $(y_1^h, \ldots, y_{t-1}^h)_{h \in \mathcal{N}}$ and the previously received symbols $(z_1^h, \ldots, z_{t-1}^h)_{h \in \mathcal{N}}$.

It should be noted that correlating and sequential devices form subclasses of communication devices by assuming d^T to be independent of the inputs and, additionally for the correlating device, assuming all outputs preceeding the first stage. Now the definition of a communication equilibrium and a sequential equilibrium may be given analogously to that of a correlation equilibrium. We omit this formulation. Due to the above remark and the previously given example the following theorem becomes obvious.

Theorem: conv (Nash – equilibrium payoffs for Γ)

$\subsetneq$ {correlated equilibrium payoffs for Γ}

$\subsetneq$ {sequential equilibrium payoffs for Γ}

$\subset$ {communication – equilibrium payoffs for Γ}

At this point we only mention that a result on the computability on the sets of sequential – and communication – equilibrium payoffs is available (F. FORGES [85]) just as in case of correlated equilibria.

By some game – theorists preplay communication is felt to be consistent with the rules of a normal form game. Thus correlated equilibria are inherent in the game and we may ask whether also the more general equilibria can be viewed at as implicitely attached to the description of a game. This point of view would be justified if for all games, or less ambitious, for a well – shaped class of games the coincidence of correlated equilibria with sequential – or communication – equilibria could be shown. Such type of result was provided by F. FORGES in a series of papers [82], [83], and [84], which additionally show (in case of the coincidence of correlated – and communication – equilibrium payoffs) that they all are payoff – equivalent to the payoff resulting from an incentive compatible mechanism (R. MYERSON [82]) or a "noisy – channel" *) (F .Forges)

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*) The notion of a "noisy channel" used by F. Forges is not related to the same notion used in information theory!

We propose another point of view, meaning that introducing communication devices or, more specifically sequential or correlating mechanisms each go beyond the scope of normal form games. Whereas in explicit normal form games the criterion to choose some strategy is explicitly contained in the description of the game by the definition of the payoff-functions as depending on the states of nature, the extension of a normal form game by a communication device provides a priori no possibility of rational considerations concerning the other players strategies. They are only available given some probability space and a common interpretation of the payoffs and then ran into the availability of equilibria. Both, the interpretation of observed results and the definition of the communication device require some complicated communication system for preplay communication and are not formalized and put into the description of the games yet. Moreover, since different communication devices generally yield different equilibrium payoffs, there is a game theoretical problem behind the selection of a certain device requiring an analysis of the game on a higher level. An approach towards this field is provided in section 4.3. Admitting free preplay communication also allows for the following feature. Observe that even if the players do not agree on the form or use of a communication device single players may use self-binding rules to enlarge their payoff — if acting as announced seems to be credible to the other players. An easy example to this is provided by the above example of Myerson. The normal form Nash-equilibrium yielding the payoff-vector (4,0) is destroyed when player 2 announces to behave depending on the outcome of publicly flipping a fair coin. But, as we mentioned earlier, we thereby reach the field of cooperative game theory.

4.2. Information and Implementation of Strategies

One reason counteracting the application of game theory is that equilibria usually only exist withing the set of mixed strategies, forcing the players to construct random devices. Consequently, an objection raised against the analysis of (human) behavior by game theoretical models is that people apparently do not base their decisions on random mechanisms such as the roll of a die or the toss of a coin. Some kinds of responses are given to that criticism by several authors.

As a first answer one may try to identify classes of games giving rise to the existence of equilibria within the set of pure strategies. One result of this type was given in chapter 2, section 1, stating the existence of pure – strategy equilibria in games with perfect information.

A second kind of answering the criticism was at first heuristically provided by the common belief among game theorists that once there is sufficient randomness in the environment of the players, then randomization on actions is not needed. In the real world information is sufficiently disparate among the players such that for every player the distribution on the observable events is sufficiently diffuse. The players may therefore base their decisions deterministically on the observed events and despite of that the opponents are not able to predict the decisions. Thus the problem of using random devices in order to achieve equilibria becomes insignificant and the need for using mixed strategies is not as compelling as sometimes suspected.

Thirdly, the criticism can be answered by pointing out the fact that for some classes of games there is no observable difference between selecting actions deterministically by a function of the observable environment or stochastically, likewise depending on the observations of nature. More precise, we claim the existence of models in which for any mixed strategy, or at least for any mixed equilibrium strategy, there is some pure strategy, yielding for all combinations of the other players strategies the same payoff as the former strategy does.

The pioneering work within this area was performed by R. BELLMAN and D. BLACKWELL [49] and A. DVORETZKY, A. WALD and J. WOLFOWITZ [51]. Both treated two – person zero – sum games defined by finite action sets A_n and pay-off – functions u_n. The games are additionally endowed with some random mechanism creating uncertainty for one of the players on the true state of nature.

Whereas the Bellman – Blackwell – model is given in explicit normal form (just as all the games to be investigated later), Dvoretzky et al assign non – symmetric roles to the players. This model provides an illustrating example as far as the second answer to critics (the existence of pure equilibrium strategies) is concerned. The game – tree of the sequential extensive form game is depicted in the following figure:
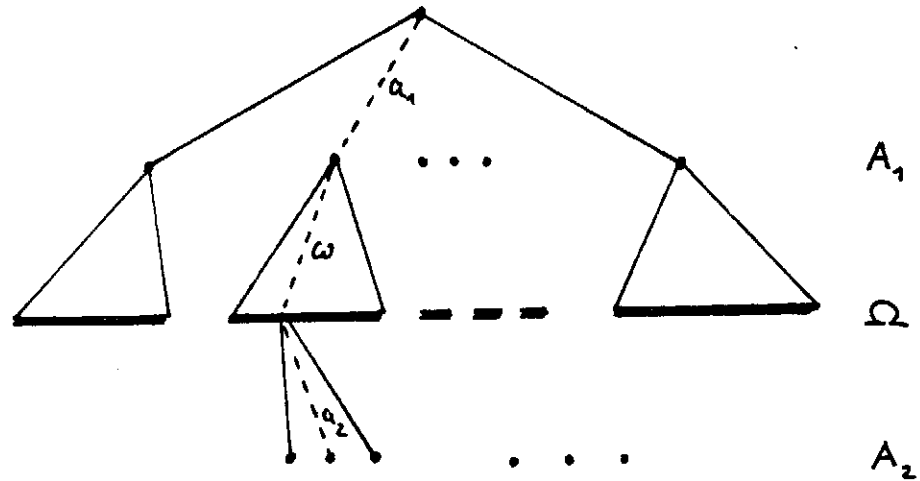
A play of the game consists of performing three actions in a series. First player 1 chooses an action $a_1 \in A_1$. According to some exogeneously given $\sigma_0(\cdot \mid \ )$ some $\omega \in \Omega$ is chosen by the random move on the second stage. Player 2 then observes ω and chooses some action $a_2 \in A_2$ such that after all a payoff $u(\omega, a_1, a_2)$ is obtained by player 1. This verbal description of the rules corresponds to the formal definitions: $\Sigma_1 = \{ \sigma_1 \mid \sigma_1 \in \Delta(A_1) \}$ as the set of strategies of player 1 and $\Sigma_2 = \{ \sigma_2 \mid \sigma_2 \mid \Omega \Longrightarrow A_2, \ \mathcal{O}_2\text{-measurable} \}$ for the abilities of player 2. $\sigma_2 \in \Sigma_2$ is called *pure strategy* of player 2 if $\sigma_2(\cdot \mid \omega) \in \{ \varepsilon_{a_2}(\cdot) \mid a_2 \in A_2 \}$ $\sigma_0(\cdot \mid a_1)$ – almost everywhere, $a_1 \in A_1$. Thus, basically, pure strategies refer to some partition of Ω. Using an extension of Lyapunov's result, namely the existence of some partition $\{ M_{a_2} \mid a_2 \in A_2 \}$ of Ω such that the equality

$$ \int_{\Omega} \sigma_2(a_2 \mid \omega) \, \sigma_0(d\omega \mid a_1) = \sigma_0(M_{a_2} \mid a_1) $$

holds for any $(a_1, a_2) \in A_1 \times A_2$, the authors prove the existence of a payoff – equivalent pure strategy to any given strategy.

Theorem: Assuming $\sigma(\cdot \mid a_1)$ to be a σ-finite and atomless measure on the measurable space $(\Omega, \mathcal{Q})$ for any $a_1 \in A_1$, then for any (mixed) strategy σ_2 of player 2 there exists a pure strategy $\hat{\sigma}_2$ of player 2 such that

$$U(\sigma_1, \sigma_2) = U(\sigma_1, \hat{\sigma}_2)$$

where

$$U(\sigma_1, \sigma_2) = \sum_{a_1 \in A_1} \sigma(a_1) \int_\Omega \sum_{a_2} \sigma_2(a_2 \mid \omega) \cdot \sigma_0(d\omega \mid a_1)$$

The model of Bellman and Blackwell applies to the second kind of answer towards the criticism on the applicability of game theory to explain human behavior. In their model mixed strategies can be approximated by pure ones. A further difference to the class of games investigated by Dvoretzky et al concerns the symmetric roles of the players as far as the flow of time is concerned.

They assumed to be given an explicit normal form game $\Gamma = ((\Omega, \mathcal{O}_1, \mu), \mathcal{O}_1, \mathcal{O}_2, A_1, A_2, u_1, u_2)$ with a finite set of actions A_1 for player 1. The space of states of nature is defined to be $[0,1]$ endowed with the Borel σ-algebra and the Lebesgue-measure λ, say. Assume $\mathcal{O}_1$ containing $\{\omega\}, \omega \in \Omega$ as atoms where $\mathcal{O}_2$ denotes the trivial algebra $\{\emptyset, \Omega\}$. This means that player 1 is fully informed on the state of nature whereas player 2 gets no information whatsoever. Further, assume $u_1: [0,1] \times A_1 \times A_2 \longrightarrow \mathbb{R}$ to be bounded and piecewise continuous such that the modulus of continuity of the pieces is uniform over (a_1, a_2). Under these conditions mixed strategies may be approximated by pure ones (as far as the induced payoff is concerned). The approximation idea is as follows: First Ω is cut into pieces $\mathcal{M}^1, \dots, \mathcal{M}^L$ such that the variation of the payoff-function u on each set $\mathcal{M}^\ell$ is small. Then, in a second step the partition $\mathcal{M}$ is refined by partitioning every $\mathcal{M}^\ell$. The refinement $\mathfrak{m}^\ell = \{M_{a_1} / a_1 \in A_1\}$ results. The sets $M_{a_1}^\ell$ are chosen as to satisfy the condition

$$\mu(M_{a_1}^1) = \int_{M_{a_1}^1} \sigma_1(a_1 \mid \omega) \, \mu(d\omega)$$

Of course the quality of the approximation depends on the partition $\mathcal{M}$; generally there is no payoff-equivalent pure strategy to a given mixed strategy. The above argument shows

Theorem: For all mixed strategies $\sigma_1 \mid \Omega \Longrightarrow A_1$ and any $\varepsilon > 0$ there exists a pure strategy $\hat{\sigma}_1 : \Omega \longrightarrow A_1$ such that for any $a_2 \in A_2$:

$$\int_0^1 \sum_{a_1} \sigma_1(a_1 \mid \omega) \, u(\omega, a_1, a_2) \, \lambda(d\omega)$$

$$\leq \int_0^1 u(\omega, \hat{\sigma}_1(\omega), a_2) \, \lambda(d\omega) + \varepsilon$$

Using the definition

> Given some $\varepsilon > 0$ the strategies $\sigma_h, \tilde{\sigma}_h$ are called ε-*equivalent* if for all strategies σ^{-h} of the players from $\mathcal{N} - \{h\}$
>
> $$\bigwedge_{m \in \mathcal{N}} \; | \; U_m(\sigma_n, \sigma^{-n}) \; - \; U_m(\tilde{\sigma}_n, \sigma^{-n}) \; | \; < \; \varepsilon$$

The above result may be paraphrased into:

> To all mixed strategies of player 1 there exists an ε-equivalent pure strategy.

Assuming u to be independent of ω we may easily infer from the theorem the existence of pure-strategy correlated equilibria being induced by state spaces $\Omega = \Omega_\Lambda \times \Omega_2$ and information σ-algebras $\mathcal{O}_\Lambda, \mathcal{O}_2$ providing independent information on the realizations of an atomlessness probability distribution μ on Ω. This result may also be extended to the N-person, non-zero-sum case as R. AUMANN remarked in [74].

A more ambitious approach to derive the existence of pure strategy equilibria was set forth by R. RADNER and R. ROSENTHAL [82]. They investigated explicit normal form games and found conditions which ensure the existence of 0-equivalent pure-strategy equilibria to all mixed strategy equilibria. They showed that non-atomicity of the distribution on states of nature and an appropriate form of independency of the pooled information of the players $\mathcal{N} - \{n\}$ on one hand and of player n on the other concerning the payoff relevant states of nature admits purification of all mixed strategy equilibria. We give their result only informally since it is included in the results of Aumann et al to be given subsequently – apart from their existence result on pure-strategy equilibria.

As far as the existence of pure-strategy equilibria is concerned Radner and Rosenthal show by an example that mere atomlessness is not sufficient and some kind of independency of the players' observation must be required.

Example: There are given two players, 1 and 2, each of them disposing of two possible actions, $A_1 = \{a_1', a_1''\}$ and $A_2 = \{a_2', a_2''\}$. The payoff-functions $u_1, u_2 : A_1 \times A_2 \rightarrow \mathbb{R}$ satisfy $u_1(a_1', a_2) \neq u_1(a_1'', a_2)$ for all $a_2 \in A_2$ and $u_2(a_1, a_2') \neq u_2(a_1, a_2'')$ for all $a_1 \in A_1$. The set of states of nature $\Omega = \Omega_\Lambda \times \Omega_2$ is equal to $[0,1]$ endowed with the σ-algebra of Borel-sets $\mathcal{B}$. For $B \in \mathcal{B}$ let $\mu(B)$ be defined as $\mu(B) = 2 \cdot \lambda^2(B \cap \{(\omega_\Lambda, \omega_2) / \omega_2 > \omega_\Lambda\})$ where λ^2 denotes the

Lebesgue – measure on $\mathbb{R}^2$. A pair $\omega = (\omega_1, \omega_2)$ being chosen according to μ, player 1 gets to know its first component whereas player 2 is informed on the second one.

For any $\omega_2 \in \Omega_2$ the conditional distribution on Ω_1, $\mu(\cdot \mid \omega_2)$ is equal to $\omega_2^{-1} \lambda^1(\cdot)$. Purifying the mixed equilibrium strategy $\sigma_1^*(a_1' \mid \omega_1) = 2^{-1} = \sigma_1^*(a_1^4 \mid \omega_1)$ means to find a partition B_1, B_2 of Ω corresponding to the choice of actions a_1' and a_1'' such that in particular for all moves a_2 of player 2 the equality

$$U_1(\sigma_1^*, a_2) = \int_{\Omega_1 \times \Omega_2} 2^{-1}(u_1(a_1', a_2) + u_1(a_1'', a_2))\mu(d\omega_1 \mid \omega_2)\mu(d\omega_2)$$

$$= \int_{\Omega_1 \times \Omega_2} \sum_i 1_{B_i}(\omega_1) \, u_1(a_1^{(i)}, a_2) \, \mu(d\omega_1 \mid \omega_2) \, \mu(d\omega_2)$$

holds. This yields, using the assumption on the shape of u_1 that equality has to hold independently of player 2's information ω_2, i.e.

$$\int_{\Omega_1} 2^{-1}(u_1(a_1', a_2) + u_1(a_1'', a_2)) \, \mu(d\omega_1 \mid \omega_2)$$

$$= \int_{\Omega_1} \sum_i 1_{B_i}(\omega_1) \, u_1(a_1^{(i)}, a_2) \, \mu(d\omega_1 \mid \omega_2)$$

Thus purifying means to find a set $B_1 \in \mathcal{X}$ such that for almost all ω_2 : $\mu(B_1 \mid \omega_2) = 1/2$ or, equivalently,

$$\omega_2 \cdot \lambda^1(B_1 \cap [\omega_2, 1]) = 1/2$$

As is well known, such a set B_1 does not exist, thereby excluding the existence of a pure – strategy equilibrium.

Preceding their investigation concerning purifications and ε – purifications of equilibrium strategies P. MILGROM and R. WEBER [80] are concerned with an analysis of the sensitivity of equilibria to the modelling assumptions of a game. Before stating their theorem showing upper – hemicontinuity of the set of equilibria as depending on the payoff – functions and information structure their example (War of Attrition) shall be given.

Example: We consider a two player game in explicit normal form. The set of states of nature is given by the probability space $(\mathbb{R}_+^2, \mathcal{Y}^2, \mu)$ where $\mathcal{Y}^2$ denotes the Borel σ-algebra restricted on $\mathbb{R}_+^2$. μ is assumed to be a product measure $\mu = \nu \otimes \nu$ with ν having a continuous density with respect to the Lebesgue-measure. The realization of μ gives a pair of "incentive values" (ω_1, ω_2) to the players since the shape of the payoff-function is given as

$$u_n : \Omega \times \prod A_m \longrightarrow \mathbb{R}$$

$$(\omega_1, \omega_2, a_1, a_2) \longmapsto \begin{cases} \omega_n - a_m & \text{if } a_n > a_m \\ -a_n & \text{otherwise,} \quad n = 1, 2, m \neq n \end{cases}$$

The information σ-algebras of the players are defined by $\mathcal{A}_1 = \mathcal{A}_1' \otimes \{\emptyset, \mathbb{R}_+\}$, where $\mathcal{A}_1'$ contains the one-elementary sets as atoms, and $\mathcal{A}_2 = \{\emptyset, \mathbb{R}_+\} \otimes \mathcal{A}_2'$ satisfies an analogous condition. Thereby each player gets to know his own incentive value and is completely uninformed on his opponent's.

The unique symmetric pure-strategy equilibrium of this game is known to be $(\bar{\sigma}, \bar{\sigma})$ such that $\bar{\sigma}(\cdot \mid \omega_1)$ is the point measure $\varepsilon_{T(\omega_n)}(\cdot)$ on $(\mathbb{R}_+, \mathcal{Y})$,

$$T(\omega_0) = \int_0^{\omega_0} \frac{w \, f(w)}{1 - \nu(w)} \lambda(dw)$$

Assuming the extreme situation that the incentive-values are chosen deterministically, i.e. according to some point measure ε_w, such that both players obtain the same incentive, then there is no pure-strategy equilibrium, moreover there is only one symmetric mixed strategy equilibrium given as

$$\sigma^*(A \mid w) = \int_A 1 - \exp\{-a/w\} \, \lambda(da) \quad , A \in \mathcal{Y}$$

It yields as the expectational stopping time (action) just the incentive value w. This case may be viewed at as the limiting case for states of nature being chosen according to the product of some probability distribution with support being contained in the interval $[w-h, w+h]$. For the cumulative distribution function $F^h(\cdot)$ resulting from the equilibrium strategies (obtained from consideration of the continuous case) we get the bounds

$$1 - \exp\{-a/w+h\} \leqslant F^h(a) \leqslant 1 - \exp\{-a/w-h\}$$

Thus, for $h \to 0$ we get pointwise convergence of $F^h(\cdot)$ to the cumulative distribution function $1 - \exp\{-a/w\}$ derived from the pure equilibrium strategies of the deterministic case.

The "convergence theorem" of Milgrom and Weber shows that the possibility of "approximating" a mixed strategy by a sequence of pure strategies in "near–by" games is not a singular apparition. The general assumptions of Milgrom and Weber on the class of explicit normal form games to be considered were given in 2., section 2. We just list the formulas:

(i) $\quad \Omega = \Omega_0 \times \Omega_1 \times \ldots \Omega_N$

where Ω_n $n = 0,\ldots,N$ are complete, separable metric spaces

(ii) the action spaces $A_n, n \in \mathcal{N}$ are complete, separable and metric spaces.

(iii) μ is a probability distribution on the Borel – subsets of Ω

and

(iv) the payoff – functions

$$u_n : \Omega \times \prod A_m \longrightarrow \mathbb{R}$$

are measurable and bounded.

We remind a distributional strategy to be a probability distribution δ_n on $\Omega_n \times A_n$ such that for measurable subsets B of Ω_n

$$\delta_n (B \times A_n) = \mu_n(B)$$

where μ_n denotes the marginal distribution of μ on its $(n + 1)$ – th component, i.e.

$$\mu_n(B) = \mu(\Omega_0 \times \Omega_1 \times \ldots \times B \times \ldots \times \Omega_N), \quad n \in \mathcal{N}$$

Assuming the atomlessness of the marginal distributions μ_n there is enough exogeneous randomness such that for any distributional strategy an ε – equivalent pure strategy may be found. This denseness theorem shows that pure strategies and mixed strategies are empirically indistinguishable which observation provides an answer to the third of the criticisms on the application of game theory as mentioned above. However, the very weak assumption of atomlessness is not sufficient to provide the existence of equilibria and thereby the pure – strategy approximation to them. In order to ensure this, appropriate continuity conditions on

the payoff – functions and the information structure have to be satisfied. They are quoted in the following

Theorem: Given any $\varepsilon > 0$, suppose equicontinuity of the payoff – functions $u_n(\omega_0, \omega^N, \cdot)$ for all $(\omega_0, \omega^N) \epsilon \Omega$ to be given. Further, assume $\mu(\cdot)$ to be absolute continuous with respect to the product of its atomlessness marginals, $\otimes \mu_n(\cdot)$ and the action spaces A_n to be compact. Then there exists a pure strategy ε–equilibrium.

This theorem is a direct consequence of the existence result on equilibria in distributional strategies and the denseness theorem provided by Milgrom and Weber. The existence of pure – strategy equilibria require stronger assumptions. In fact, the examples of DVORETZKY et al [51] and the one of Radner and Rosenthal given above suggest that the existence theorem for pure strategy equilibria as will be given below will not hold under much further weakening of the assumptions. They show that on one hand some independency condition of the observations received by the players is unavoidable and also an assumption concerning the continuity of the payoffs is cogent.

The most general existence result on pure – strategy equilibria (as far as compact action spaces are concerned) is up to now given by Milgrom and Weber.

Theorem: Suppose Ω_0 to be finite, the marginal distribution μ_n of μ to be atomless and the marginals
$$\mu_n(\cdot \mid \omega_0) = \mu_n(\Omega_1 \times \ldots \times \times \ldots \Omega_N \mid \omega_0)$$ to be independent. Let the payoff – functions $u_n(\cdot)$ be continuous and the action spaces be compact. Then to any equilibrium strategy there exists a payoff – equivalent pure equilibrium strategy. The latter set is non – void.

This concludes the application of information as far as the implementation of players' behavior within a class of games is concerned.

From Milgrom and Weber's theorem we learned that the conditions needed to establish the existence of pure – strategy ε – equilibria are significantly weaker then those needed for pure – strategy equilibria. To approximate a mixed strategy equilibrium only some form of continuity of the underlying probability distribution has to be required whereas to ensure the existence of pure – strategy equilibria moreover some independency of the information obtained is to be presupposed.

An interesting feature concerning approximation was derived by AUMANN et al [83]. They gave sufficient conditions for approximation of all mixed strategies by pure one's on one hand and weaker conditions for approximations of equilibrium strategies on the other and showed by an example that the weaker conditions are not sufficient for the approximation of all strategies. Their assumptions on the parameters of the explicit normal form game are as follows:

The underlying probability space $(\Omega, \mathcal{O}, \mu)$ is obtained by addition of some probability–measure to the product of measurable spaces, i.e. $(\Omega, \mathcal{O})$ is defined by $\Omega = \Pi \Omega_n$ and $\mathcal{O} = \otimes \mathcal{O}_n$, where $(\Omega_n, \mathcal{O}_n)$ are measurable spaces (assumed to be isomorphic to $(\mathbb{R}, \mathcal{B})$) in order to avoid technical difficulties in defining conditional probabilities). This ensures a regular version of the conditional distribution of μ on Ω_n to exist. It will be denoted as $\mu_n(\cdot \mid \omega^{-n})$

Further the action sets A_n are assumed to be finite.

Defining the measure μ to be *conditionally atomless* for player n if $\mu_n(\cdot \mid \omega^{-n})$ is atomless μ^{-n} – almost everywhere, we may formulate their approximation result on all strategies.

This condition expresses that even when pooling their information the players $\mathcal{N} - \{n\}$ may not ascribe positive probability to any particular observation of player n.

Theorem: Let μ be conditionally atomless for player n. Then for $\varepsilon > 0$ and every mixed strategy of player n there exists an ε–equivalent pure strategy.

Non–atomicity of the conditional distribution on the events observable by player n as being inferred by the "coalition" $\mathcal{N} - \{n\}$ may not be weakened to non–atomicity of the conditional distribution as being calculable by each player separately in order to get the above approximation result. Aumann et al provide an example to this phenomenon. However, a weaker assumption is sufficient to show that at least equilibrium strategies may be approximated.

For m $\neq$ n let μ_{mn} denote the marginal distribution on $\Omega_m \times \Omega_n$ derived from μ.

μ is called *weakly conditional atomless* for player n if for all m $\neq$ n the probability distribution μ_{mn} is conditional atomless for player n.

Theorem: Let μ be weakly conditional atomless for player n. Then for $\varepsilon > 0$ and every equilibrium strategy for player n there exists an. ε-equivalent pure strategy.

4.3. Quantifying the Utility of Information.

In this section we shall be occupied with the derivation of a functional relationship of the amount of information available to the players and the payoff attainable thereof. Here the information providing device will not be fixed but considered as a variable being subject to the decisions of the players. Therefore our results will describe the payoff coming from a most profitable management of information.

At first a companion – piece to the model of Dvoretzky, Wald and Wolfowitz will be considered. Recall that the latter was given by sequential acting of the two players with a random move in between. The choice of the first player defined the random – mechanism to be used at the intermediate stage, the event resulting from it was supposed to be observable by the second player, who chooses an action answering that of the first player thereupon.

The random – mechanism $\sigma_0(\cdot \mid a_1), a_1 \in A_1$ provides some information on the actions of player 1 for player 2 before the latter has to choose his action. Obviously, the utility of the information, expressed by the resulting equilibrium payoff, is affected by the similarity of the distributions $\sigma_0(\cdot \mid a_1)$ for varying actions a_1 of the first player. In particular, $\sigma_0(\cdot \mid a_1) = \sigma_0(\cdot)$ provides no information and the equilibrium payoff is just the value of $\Gamma = (A_1, A_2, u)$. By an appropriate specification of $(\sigma_0(\cdot \mid a_1))_{a_1 \in A_1}$ evidently all payoffs between

$$\max_{a_1 \in A_1} \min_{a_2 \in A_2} \{ u(a_1, a_2) \} \quad \text{and} \quad \max_{\sigma_1 \in \Delta(A_1)} \min_{a_2} \{ \sum_{a_1 \in A_1} \sigma(a_1) \, u(a_1, a_2) \}$$

are obtainable as equilibrium payoffs and by any given $(\sigma_0(\cdot \mid a_1))_{a_1 \in A_1}$ an equilibrium payoff is well defined. Thus a game on a higher stage may be defined

by considering the information providing mechanism as decision – variable of player 2, of course it has to be subject to some constraint. This problem was considered by H. – M. WALLMEIER [83].

Given $T \in \mathbb{N}$ he defined $A_1 = I^T$, $A_2 = J^T$ for some finite sets I and J, respectively and assumed the constraints on the information providing mechanism to be due to the transmission of the received information via some system with bounded efficiency. Assuming the system to be given by a "discrete, memoryless channel" $W \mid \mathcal{Y} \Longrightarrow \mathcal{Z}$ a stochastic system transforming T – sequences of inputs y^T to T – sequences of outputs z^T according to

$$W^T(z^T \mid y^T) = \prod_{t=1}^{T} W(z_t \mid y_t)$$

the strategies of the post – playing subject are given as pairs of encoding and decoding rules $P_E \mid I^T \Longrightarrow \mathcal{Y}^T$ and $P_D \mid \mathcal{Z}^T \Longrightarrow J^T$. The model can be depicted as follows:
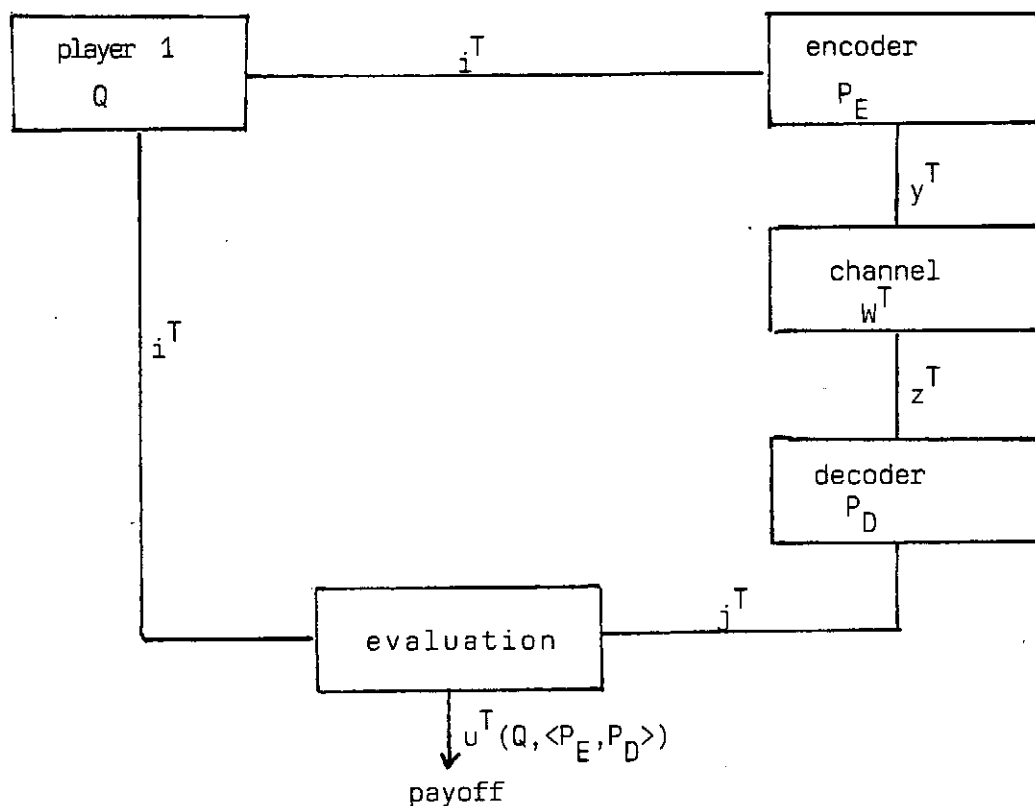
The main result of the paper provides a computable formula for the asymptotics to the values for the games with fixed blocklength T. Formally

Theorem: Given $u : I \times J \longrightarrow \mathbb{R}$ and a discrete, memoryless channel $W \mid \mathfrak{Y} \Longrightarrow \mathfrak{Z}$ with positive capacity $C(W)$, then

$$\lim_{T \to \infty} \max_{Q^T} \min_{\substack{P_E \mid I^T \Longrightarrow \mathfrak{Y}^T \\ P_D \mid \mathfrak{Z}^T \Longrightarrow J^T}} \left\{ n^{-1} \sum_{i,j^T,y^T,z^T} Q^T(i^T) P_E(y^T \mid i^T) W(z^T \mid y^T) P_D(j^T \mid z^T) u(i^T, j^T) \right\}$$

$$= \lim_{T \to \infty} \min_{\substack{P_E \mid I^T \Longrightarrow \mathfrak{Y}^T \\ P_D \mid \mathfrak{Z}^T \Longrightarrow J^T}} \max_{i^T} \left\{ n^{-1} \sum_{j^T,y^T,z^T} P_E(y^T \mid i^T) W(z^T \mid y^T) P_D(j^T \mid z^T) u(i^T, j^T) \right\}$$

$$= \max_{Q \in \Delta(J)} \left\{ D_{Q,u}(C(W)) \right\} ,$$

where the distortion – rate function $D_{Q,u}(\cdot)$ is defined by

$$D_{Q,u}(R) = \min_{\substack{V \mid I \Longrightarrow J \\ I(Q;V) \leqslant R}} \left\{ \sum_i Q(i) \sum_j V(j \mid i) u(i,j) \right\} .$$

The mutual information is given as

$$I(Q;V) = \sum_{i,j} Q(i) \, V(j \mid i) \, \log \frac{V(j \mid i)}{\sum_{i'} Q(i') \, V(j \mid i')}$$

and $C(W)$, the capacity of the channel W, is given as the maximum of mutual informations, $C(W) = \max \left\{ I(P;W) \right\}$. All these functions are well – known in information theory.

In a second model the players are found in symmetric roles as far as the flow of time is concerned. Here the amount of information on the states of nature, available to two players, is set into conjunction with the payoff resulting from it. The model of H. – M. WALLMEIER [84] could be classed as a higher stage mating part to explicit normal form games. Whereas usually information algebras describe the information attainable on the states of nature, here the information providing mechanisms to a certain extend are decision – parameters of the players. Given a finite set of states of nature endowed with some probability measure μ and finite set of actions, the information on the present state of nature has to be processed via two independent channels $W \mid \mathfrak{Y}_n \Longrightarrow \mathfrak{Z}_n$, $n = 1,2$. On the basis of the observed outcomes of these channels the actions are to be chosen. Again the equi-

librium in a game resulting from the consideration of blocks of states, x^T and actions i^T, j^T from I^T and J^T respectively, is investigated. As before, the asymptotical payoff is characterized. Graphically the model is shown as
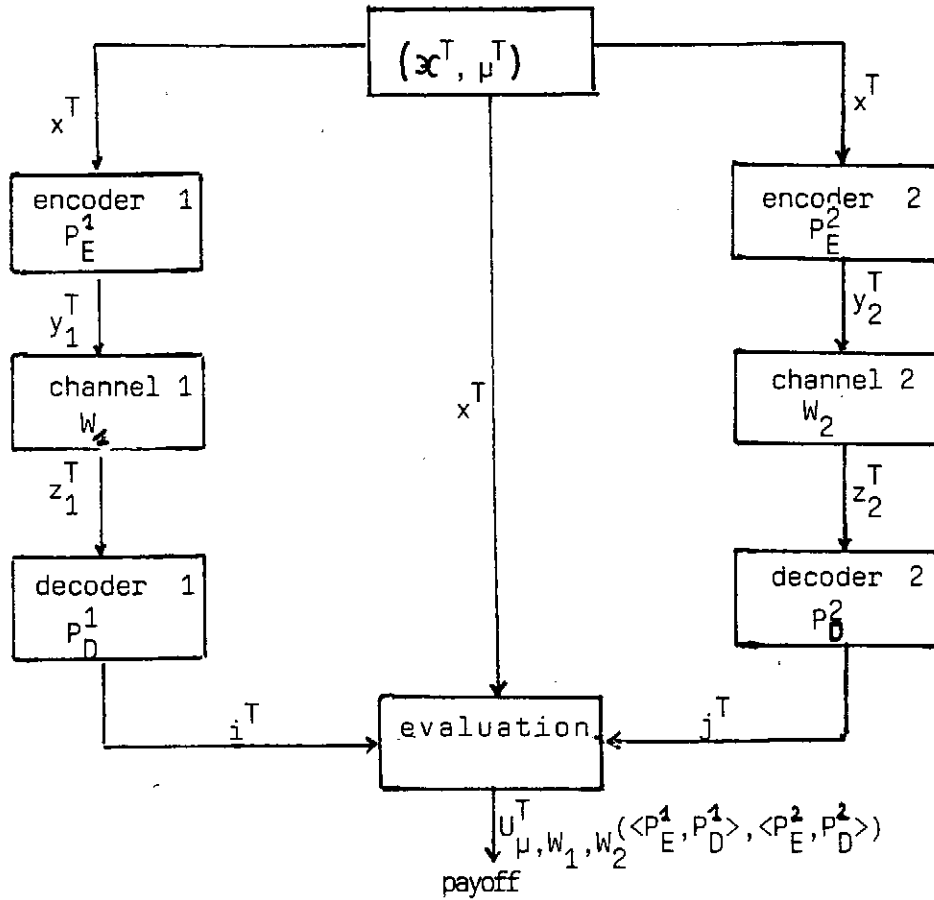
The states of nature have to be "encoded" which means: to be represented (possibly by a random – mechanism) as an input of the channel, the outputs have to be decoded to yield actions i^T and j^T respectively. The combined mechanisms $\langle P_E^n$, $P_D^n \rangle$ of player n are viewed at as the strategies of player n. From the distribution on the set of states of nature, the channel transition probabilities and the strategies of the players 1 and 2 an expected payoff results.

$$U^T_{\mu,W_1,W_2}(\langle P_E^1, P_D^1 \rangle, \langle P_E^2, P_D^2 \rangle)$$

$$= T^{-1} \sum_T \mu^T(x^T) \Big[\sum_{y_1^T z_1^T i^T} P_E^1(y_1^T|x^T) W_1(z_1^T|y_1^T) P_D^1(i^T|z_1^T)$$

$$\sum_{y_2^T z_2^T j^T} P_E^2(y_2^T|x^T) W_2(z_2^T|y_2^T) P_D^2(j^T|z_2^T) \Big] u^T(x^T, i^T, j^T)$$

(observe the zero – sum assumption).

The main theorem gives the equilibrium payoff in a computable manner and the proof moreover suggests how the equilibrium payoff may be ensured by the players to themselves.

Theorem: Given $u: I \times J \longrightarrow \mathbb{R}$ and discrete, memoryless channels $W_1 \mid \mathcal{U}_1 {=\!=} \mathcal{Z}_1$, $W_2 \mid \mathcal{U}_2 {=\!=} \mathcal{Z}_2$ with positive capacities, then

$$
\lim_{T \to \infty} \max_{\substack{P_E^1 \mid \mathcal{X}^T \Longrightarrow \mathcal{U}_{d_1}^T \\ P_D^1 \mid \mathcal{Z}_1^T \Longrightarrow I^T}} \min_{\substack{P_E^2 \mid \mathcal{X}^T \Longrightarrow \mathcal{U}_{d_2}^T \\ P_D^2 \mid \mathcal{Z}_2^T \Longrightarrow J^T}} \left\{ U(\langle P_E^1, P_D^1 \rangle, \langle P_E^2, P_D^2 \rangle) \right\}
$$

$$
= \lim_{T \to \infty} \min_{\substack{P_E^2 \mid \mathcal{X}^T \Longrightarrow \mathcal{U}_{d_2}^T \\ P_D^2 \mid \mathcal{Z}_2^T \Longrightarrow J^T}} \max_{\substack{P_E^1 \mid \mathcal{X}^T \Longrightarrow \mathcal{U}_{d_1}^T \\ P_D^1 \mid \mathcal{Z}_1^T \Longrightarrow I^T}} \left\{ U(\langle P_E^1, P_D^1 \rangle, \langle P_E^2, P_D^2 \rangle) \right\}
$$

$$
= \max_{\substack{V_1 \\ I(\mu;V_1) \leqslant C(W_1)}} \min_{\substack{V_2 \\ I(\mu;V_2) \leqslant C(W_2)}} \sum_{x} \mu(x) \sum_{i,j} V_1(i|x) V_2(j|X) u(x,i,j)
$$

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Index of Symbols

$\prod \mathcal{X}_n$ cartesian products of sets $\mathcal{X}_n$.

$\otimes \mathcal{O}_n$ σ – algebra induced by the cartesian product of σ– algebras $\mathcal{O}_n$ on the cartesian product space $\prod \mathcal{X}_n$

$\otimes \mu_n$ measure induced by measures μ_n on the cartesian product space $\prod \mathcal{X}_n$

$f : \mathcal{X} \rightarrow \mathcal{Y}$ function from $\mathcal{X}$ to $\mathcal{Y}$

$\Delta(\mathcal{X})$ set of probability distributions on $\mathcal{X}$

$supp(\mu)$ = support of the probability measure μ

ε_x measure on $\mathcal{X}$ with supp (ε_x) = $\{x\}$

$W | \mathcal{X} \Rightarrow \mathcal{Y}$ conditional probability on $\mathcal{Y}$ for any given $x \in \mathcal{X}$ (synonymous to $W : \mathcal{X} \longrightarrow \Delta(\mathcal{Y})$)

$(\Omega, \mathcal{O})$ measurable space

$(\Omega, \mathcal{O}, \mu)$ probability space

$\mathcal{P}(\mathcal{X})$ set of all subsets of $\mathcal{X}$

$conv(\mathcal{X})$ convex hull of $\mathcal{X}$

$\mathcal{X}^{-n}$ $= \prod_{m \neq n} \mathcal{X}_m$ cartesian product where $\mathcal{X}_n$ is deleted

μ^{-n} marginal distribution of μ on $\mathcal{X}^{-n}$

$(\tilde{\sigma}_n | \sigma^{-n})$ $= (\sigma_1, \ldots \sigma_{n-1}, \tilde{\sigma}_n, \sigma_{n+1}, \ldots \sigma_N)$

\#   denotes the end of a proof