Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

Nr. 149

Games and Incomplete Information

A Survey

Part II

June 1986

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5. Multistage Games

We shall now return to the games in normal form introduced in section 2.2. Let us first of all recall the definition and the rules of an N-person normal form game. It is denoted by

$$\Gamma = ((\Omega, \mathfrak{A}, \mu), \mathfrak{A}_n, A_n, u_n) \quad \text{where}$$

$(\Omega, \mathfrak{A}, \mu)$ is a probability space

$\mathfrak{A}_n$ are sub-σ-algebras of $\mathfrak{A}$

A_n are arbitrary sets

$u_n : \Omega \times \prod_{n=1}^{N} A_n \to \mathbb{R}$ is player n's utility function

At stage O a random experiment represented by $(\Omega, \mathfrak{A}, \mu)$ is performed. Every player is informed about the outcome of the experiment according to his sub-σ-algebra $\mathfrak{A}_n$. (In the case of a finite probability space where every algebra $\mathfrak{A}_n$ is generated by a partition of Ω, this means that player n is told the element of his partition containing the selected ω.) Then the players choose their actions simultaneously.

As the existence theorems for equilibria are only formulated for games without random moves we prefer to remove $(\Omega, \mathfrak{A}, \mu)$ and $\mathfrak{A}_n$ from the explicit description of the game.

Let us define the strategy set Σ_n of player n as the set of all stochastic kernels from $(\Omega, \mathfrak{A}_n)$ to A_n,

$$\Sigma_n := \{\sigma_n : \sigma_n \mid (\Omega, \mathfrak{A}_n) \Rightarrow A_n\}$$

(We don't bother about a suitable σ-algebra on A_n because in future applications A_n will be finite.)

If $\omega \in \Omega$ was chosen by using the strategies $\sigma_1 .. \sigma_N$ the players generate the distribution $\bigotimes_n \sigma_n(\omega, \cdot)$ on $\prod_n A_n$. Thus we may define the payoff U_n by

$$U_n(\sigma_1 .. \sigma_N) := \int_\Omega \int_{\prod A_n} u_n \, d \bigotimes_n \sigma_n(\omega, \cdot) \, \mu(d\omega)$$

The game $((\Omega, \mathfrak{A}, \mu), \mathfrak{A}_n, A_n, u_n)$ is now represented by the game in standard normal form (without random moves) (Σ_n, U_n).

It is evident that this model is suitable for representing incomplete information arisng from the ignorance of the true payoff functions. But one can also describe uncertainty about the other player's information on the game. To demonstrate this we put the game of the first example of chapter 3 into the above model:

Example: Ω = {1,2} x {1,2}. The probability distribution μ is given by

		2nd component	
		1	2
1st component	1	1/4	1/2
	2	1/4	0

Player 1 is informed about the first and player 2 about the second component of the selected ω, i.e.
$\mathfrak{A}_1$ is generated by {{(1,1),(1,2)},{(2,1),(2,2)}} and $\mathfrak{A}_2$ is generated by {{(1,1),(2,1)},{(1,2),(2,2)}}. The payoff function is given by

$$u((1,1),\cdot) = u((1,2),\cdot) = A(\cdot)$$
$$u((2,1),\cdot) = B(\cdot)$$ ∎

In the above example the measure space $(\Omega,\mathfrak{A})$ and the sub-σ-algebras $\mathfrak{A}_n$ have a special shape that is considered very often in the literature on this subject: Ω is a product space, $\Omega = \prod_{n=1}^{N} T_n$, ($T_n$ finite) and the algebras $\mathfrak{A}_n$ are generated by T_n. That means player n only learns the n-th component of the selected $\omega = (t_1 ..., t_N)$. Player n is said to be given the type t_n. This terminology is a bit misleading because one could suspect that a player's type only influences his own payoffs which need not be the case. One reason for concentrating on product spaces Ω is the frequently assumed property that the player's private informations are independent. If $\Omega = \prod_{n=1}^{N} T_n$ this only means that μ is the product of its marginal distributions. Example 1 shows that this assumption is rather not typical, but it makes the game more tractable.

If we confine ourselves to finite probability spaces $(\Omega,\mathfrak{A},\mu)$ one can give a reason for restricting the attention to product spaces. Every algebra $\mathfrak{A}_n$ is generated by a partition of Ω which will be denoted by $T_n = \{t_n^1,...,t_n^{m_n}\}$. Define a mapping from Ω to $T = \prod_{n=1}^{N} T_n$ by asigning to every $\omega \in \Omega$ the n-tuple $(t_1 ... t_N) \in T$ that satisfies $\omega \in \bigcap_n t_n$. Let μ' be the image measure of μ under this mapping and $\mathfrak{B}_n$ the algebra generated by T_n on T.

If an N-tupel of types $(t_1 \dots t_N)$ determines precisely one ω define $u'(t_1 \dots t_N) = u(\omega, \cdot)$, otherwise (that means if none of the players can distinguish between them) take the expectation over all ω in the inverse image of $(t_1 \dots t_N)$. The games given by $(\Omega, \mathfrak{A}, \mu)$, $\mathfrak{A}_1 \dots \mathfrak{A}_N$ and u resp. (T, μ'), $\mathfrak{T}_1 \dots \mathfrak{T}_N$ and u' are isomorphic in a natural way.

5.1 The L.P.-Formulation of Finite Zero-Sum Games with Incomplete Information

The class of games with incomplete information considered here has the following additional properties:

- There are only two players involved
- The sets of types for both players are finite
- The distributions on the sets of types are independent, i.e. the knowledge of his own type doesn't enable a player to update his information on his opponent's type.
- We assume zero-sum property.

Maintaining the notation introduced above we have

$\Omega = R \times S$

R set of types for player 1

S set of types for player 2 — both finite

$\mathfrak{A} = \mathfrak{P}(\Omega)$ $\quad \mathfrak{A}_1 = \mathfrak{P}(R)$ $\quad \mathfrak{A}_2 = \mathfrak{P}(S)$

$\mu = p \otimes q$

p distribution on R

q distribution on S

I set of actions for player 1

J set of actions for player 2 — both finite

The strategies σ^1 and σ^2 are now denoted by $x = (x_i^r)_{i \in I, r \in R}$ resp. $y = (y_j^s)_{j \in J, s \in S}$ with

x_i^r = Prob (action i I r is announced)

$\Sigma_1 = X, \quad \Sigma_2 = Y$

The payoff function u reduces to a set of $I \times J$-matrices $A^{r,s}$, $r \in R, s \in S$. U is given by

$$U(x,y) = \sum_{r,s} p^r q^s \sum_{i,j} x_i^r y_j^s A^{r,s}(i,j).$$

According to the min-max-theorem the game G(p,q) = (X,Y,U) has a value V(p,q). We shall now investigate the properties of this value as a function of p and q.

(1) V is Lipschitz with respect to p and q.

(2) V is concave w.r.t. p and convex w.r.t. q.

Although a formal proof of this statement will follow from the L.P.-formulation of the game we shall now give an informal argument which does not only apply to this class of games. It is based on the intuitively appealing but mathematically meaningless conjecture that in every zero-sum game the value of information is positive:

Let p^1, p^2 be in $\Delta(R)$ and $\lambda \in [0,1]$ such that $\lambda p^1 + (1-\lambda) p^2 = p$. Consider the games $G'(\lambda, p^1, p^2, q)$ and $G''(\lambda, p^1, p^2, q)$. In $G'(\lambda, p^1, p^2, q)$ chance chooses r $\in \{1,2\}$ with probability $(\lambda, 1-\lambda)$ and player 2 is informed about the outcome. Then the game $G(p^r, q)$ is played. The value v' of $G'(\lambda, p^1, p^2, q)$ will be $v' = \lambda V(p^1, q) + (1-\lambda) V(p^2, q)$. In $G''(\lambda, p^1, p^2, q)$ player 2 does not learn the value of r. The value of v" of $G''(\lambda, p^1, p^2, q)$ will be v" = V(p,q) because player 2 can't to better than calculating with the expected distribution p. From the argument above it follows that

$$v' \leq v'' \qquad \text{resp.}$$

$$\lambda V(p^1, q) + (1-\lambda) V(p^2, q) \leq V(p,q) = V(\lambda p^1 + (1-\lambda) p^2, q)$$

which means concavity of V w.r.t. p.

Convexity w.r.t. q follows by duality.

(3) V(p,q) is piecewise bilinear on $\Delta(R) \times \Delta(S)$.

This result can be obtained from a linear programming formulation of the game. It is well known that a pair of optimal strategies and the value of a zero-sum matrix game are the solution of a suitable linear program. An analogue to this program in the case of incomplete information is given by

Lemma: V(p,q) is the value of the linear program

$$\max \quad v \mapsto \sum_s q^s v^s$$

subject to

$$\bigwedge_{j,s} \quad \sum_{r,i} A^{r,s}(i,j)\, p^r x_i^r \geq v^s$$

$$\bigwedge_{r} \quad \sum_{i} x_i^r = 1$$

$$\bigwedge_{i,r} \quad x_i^r \geq 0$$

Proof:

$$V(p,q) = \max_x \min_Y \sum_{r,s,i,j} p^r q^s A^{r,s}(i,j)\, x_i^r y_j^s$$

$$= \max_x \sum_s q^s \underbrace{\min_j \sum_{r,i} p^r A^{r,s}(i,j)\, x_i^r}$$

replaced by v^s in the objective function.

The change of variables $\alpha_i^r = p^r \cdot x_i^r$ gives the following program:

Lemma: V(p,q) is the value of the linear program

$$\max \quad v \mapsto \sum_s q^s v^s$$

subject to

$$\bigwedge_{j,s} \quad \sum_{i,r} A^{r,s}(i,j)\, \alpha_i^r \geq v^s$$

$$\bigwedge_{r} \quad \sum_{i} \alpha_i^r = p^r$$

$$\bigwedge_{i,r} \quad \alpha_i^r \geq 0$$

This program enables us to give a formal proof of the cav – vex property of V(p,q):

If (v^0, α^0) (resp. (v^1, α^1)) are feasible for (p^0, q) (resp. (p^1, q)) then for all $\lambda \in [0,1]$

$$(v^\lambda, \alpha^\lambda) = (\lambda v^1 + (1-\lambda) v^0, \lambda \alpha^1 + (1-\lambda)\alpha^0) \text{ is feasible}$$

for $(p^\lambda, q) = (\lambda p^1 + (1-\lambda) p^0, q)$ implying

$$V(p^\lambda, q) \geq \lambda\, V(p^1, q] + (1-\lambda)\, V(p^0, q).$$

The program is also used to prove

Proposition: $V(p,q)$ is piecewise bilinear on $\Delta(R) \times \Delta(S)$. That means there exist finite partitions $\{\Delta(R)_k\}_k$ of $\Delta(R)$ and $\{\Delta(S)_\ell\}_\ell$ of $\Delta(S)$ into convex polyedra such that the restriction of V to each product $\Delta(R) \times \Delta(S)$ is bilinear.

Observe that in the second program p is on the right side of the constraints so that the matrix of the linear program is independent of p and q. This feature is essential for the proof of the proposition. The requirement that the probabilities on the sets of types are independent appears implicitly in the formulation of the proposition. A very similar program to determine the value and the equilibirum strategies of such a game can also be put up for the case of correlated types. For details of the proof see Ponssard and Sorin [80].

From the above proposition follows that in order to define V on $\Delta(R) \times \Delta(S)$ it suffices to compute a finite number of values $V(p_k, q_\ell)$, p_k and q_ℓ defined by the partitions of $\Delta(R)$ and $\Delta(S)$.

Example: $|I| = |J| = |R| = |S| = 2$

$$A^{11} = A^{12} = A^{22} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

$$A^{21} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

It is obvious that $\bigwedge_p V(p,0) = 0$
$\bigwedge_q V(1,q) = 0$

because in both cases player 2 will choose his second parameter.

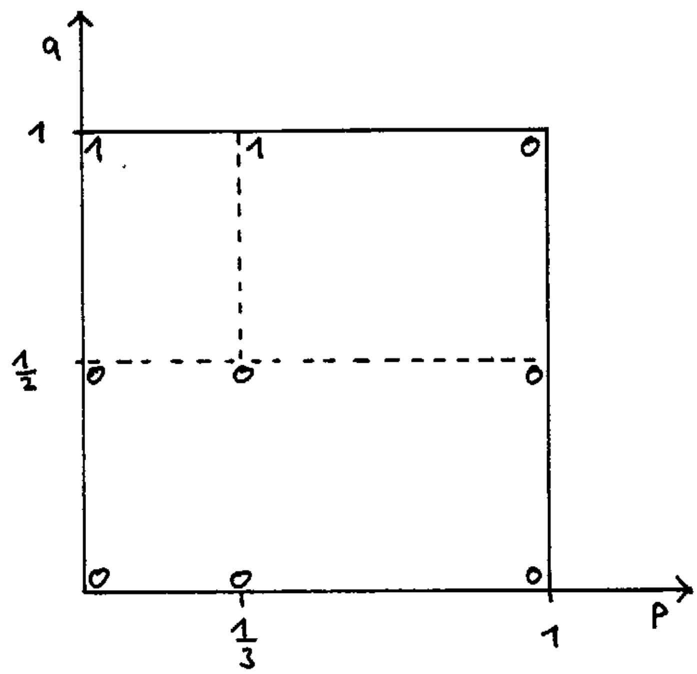
$$V(0,q) = \begin{cases} 2q - 1 & \text{for } q \geq 1/2 \\ 0 & \phantom{\text{for }} q \leq 1/2 \end{cases}$$

(For $q \geq 1/2$ player 1 chooses the first row, otherwise the second row)

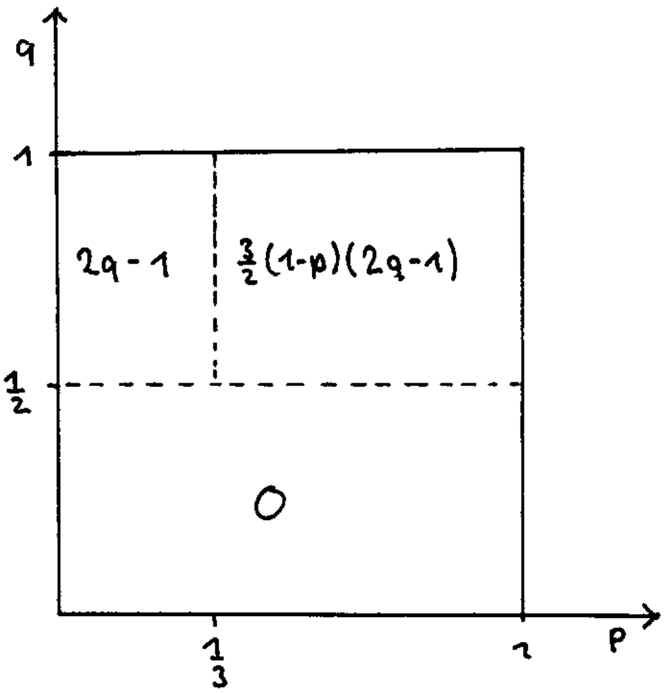
$$V(p,1) = \begin{cases} 1 & \text{for } p \leq 1/3 \\ 2/3(1-p) & \phantom{\text{for }} p > 1/3 \end{cases}$$

(In this case player 2 chooses first and second column with probability 1/2)

We have now computed the value V(p,q) on the boundary of the unit square.



Using piecewise bilinearity and the cav – vex property it can easily be checked that this suffices to give a complete description of V(p,q):



The foregoing results especially apply to the following class of games:

5.2 Games with Almost Perfect Information

At the beginning of such a game a chance more determines the player's type, i.e. $r \in R$ and $s \in S$ are chosen independently according to probability distribution p and q and both players are told their own type.

Then player 1 decides upon a move $i_1 \in I_1$ which is revealed to player 2. Knowing his opponent's previous move player 2 selects a move $j_1 \in J_1$ which is told to player 1 who chooses $i_2 \in I_2$ and so on. Finally player 1 receives $A^{r,s}(i_1, j_1, i_2, j_2, \ldots, i_T, j_T)$ from player 2. (The sets of actions $I_1, \ldots I_T$ and $J_1, \ldots, J_T$ are again finite).

The term "game with almost perfect information" can be explained from the fact that both players have perfect information on their opponent's moves but imperfect information on their opponent's type.

The game described above is denoted by G(p,q). – Let V(p,q) be its value.

Theorem: $V(p,q) = \underset{p}{\mathrm{cav}} \; \max_{i_1 \in I_1} \; \underset{q}{\mathrm{vex}} \; \min_{j_1 \in J_1} \; \ldots$

$$\underset{p}{\mathrm{cav}} \; \max_{i_T \in I_T} \; \underset{q}{\mathrm{vex}} \min_{j_T \in J_T} \left(\sum_{r,s} p^r q^s A^{r,s} (i_1 j_1 \ldots i_T j_T) \right)$$

Proof (by Induction):

Denote by $V^i(p,q)$ the value of the game obtained by restricting I_1 to the unique element i. It is sufficient to show that

$$V(p,q) = \underset{p}{\mathrm{cav}} \max_{i \in I_1} V^i(p,q)$$

Obviously player 1 can guarantee $\max_{i \in I_1} V^i(p,q)$ without relating to his type. Since the value is concave in p he can also get $\underset{p}{\mathrm{cav}} \max_{i \in I_1} V^i(p,q)$. But he is not able to obtain more than that. According to the min – max theorem the value V(p,q) exists, thus we may assume that player 1's strategy is known to player 2. In this case he is able to compute posterior probabilities on the set R after observing his opponent's first move. Assume that the signal $i \in I_1$ occurs with a total probability of λ_i and denote player 2's posteriors after observing i by p_i. Then he can prevent player 1 from getting more than

$$\sum_{i \in I_1} \lambda_i V^i(p_i, q)$$

$$\leq \sum_{i \in I_1} \lambda_i \max_{j \in I_1} V^j(p_i, q)$$

$$\leq \underset{p}{\mathrm{cav}} \max_{i} V^i(p,q)$$

as p_i satisfies $\sum_i \lambda_i p_i = p$.

Once we have found the value of the game with the above formula we can determine optimal strategies. In contrast to the calculation of the value this can be carried out by forward computation, i.e. starting with an optimal strategy for player 1 at stage 1 and ending with an optimal strategy for player 2 at stage T without having to construct optimal strategies for all possible past histories that could occur but only for the history that develops in the course of the game.

Theorem: An optimal strategy for player 1 in $G(p_\circ, q_\circ)$ at stage 1 is defined as follows:

(1) Choose $h \in \mathbb{R}^S$ satisfying

$$V(p_\circ, q_\circ) = hq_\circ, \quad \bigwedge_{q \in \Delta(S)} V(p_\circ, q) \geq hq$$

Then choose for all $i \in I_1$

$$\lambda_i \in [0,1], \quad p_i \in \Delta(R), \quad h_i \in \mathbb{R}^S$$

such that

$$- \sum_{i \in I_1} \lambda_i = 1, \quad \sum_{i \in I_1} \lambda_i p_i = p_\circ$$

$$- V(p_i, q_\circ) = h_i q_\circ, \quad \bigwedge_{q \in \Delta(S)} V(p_i, q) \geq h_i q$$

$$- \sum_{i \in I_1} \lambda_i h_i = h$$

(2) Denoting by x_i^r the probability of playing i given type r, player 1's initial move is given by

$$x_i^r = \lambda_i \cdot \frac{p_i^r}{p_\circ^r}$$

The first thing player 1 has to do is to select a supporting hyperplane of the convex function $V(p_\circ, \cdot)$ at point $q_\circ$ represented by the vector h. h can be interpreted as a vector payoff. Using the strategy we are going to construct h^s is the minimal amount player 1 will obtain from player 2 if his type is s. The limitation of the vector payoff is done without recourse to the value of $q_\circ$. From this it is clear that the vector h must represent a supporting hyperplane because otherwise for some $q_\circ$ player 1 could do better then the value permits. From the existence of the value we know that player 1 must be able to realize at least one vector payoff h satisfying the supporting property. It requires some additional argument, which won't be carried out here, that, if the supporting hyperplane is not unique, he may choose any of them. Then he has to fix the following parameters:

λ_i : total probability with which he is going to select parameter i

p_i : a posteriori probability on R player 2 can compute after observing parameter i (knowing p and player 1's strategy)

h_i : vector payoff he is going to realize after playing i at stage 1

The interpretation explains the required properties of λ_i, p_i, h_i. The computation of the transition probabilities x_i^r is obvious. We already pointed out that player 1 does not use the distribution q_o on player 2's types to compute his optimal strategy. His relevant state parameters are the vector payoff he grants player 2 and the information he reveals himself. Of course player 2 using an analogue equilibrium strategy never bothers about these revelations, but equilibrium conditions force him to take into consideration the case that player 2 is able to calculate the correct posteriors (i.e. knows his strategy) and exploits this information.

If (i,j) were selected at stage 1, player 1 has to perform similar considerations at stage 2 using $V^{ij}(p_i, q_j)$ instead of $V(p_o, q_o)$, $V^{ij}(\cdot)$ being the value of the game obtained by restricting the payoff function $A^{r,s}(\cdot\ )$ to $A^{r,s}(i,j,...)$. The only difference is that he does not have to choose a supporting hyperplane in the first place, it is already given by h_i.

5.3 Repeated Games with Incomplete Information

5.3.1 Lack of Information on One Side

A finitely repeated two-person zero-sum game with lack of information on one side is based on the following data:

- finite sets I and J
  (sets of actions for players 1 and 2)
- a finite set R
  (set of types for player 1)
- for every $r \in R$ an $|I| \times |J|$ -matrix A
  (payoff matrix)
- a probability distribution p on R
- a natural number T
  (number of stages)

The game runs as follows:

- At stage 0 chance chooses $r \in R$ according to p.
  Both players know p but only player 1 is
  informed about the outcome of the lottery.
- At each stage t = 1,...,T both players choose
  independently parameters $i_t \in I$ resp. $j_t \in J$.
- Then (i_t, j_t) is announced to both players
  but they are not told the corresponding value
  $A^r(i_t, j_t)$ of the selected payoff matrix so
  that player 2 can't draw any conclusions
  concerning his opponent's type.
- Both players have perfect recall, i.e. they
  can use all information they get in the course
  of the above procedure up to stage t for their
  decision at stage t + 1.
- After stage T player 1 recieves from player 2
  the amount $\frac{1}{T} \sum_{t=1}^{T} A^r(i_t, j_t)$

Dividing the payoffs by T has the advantage that the average payoffs can be compared for different numbers of stages and that the payoffs are bounded for all T uniformly.

According to the description above player 1 can make his choice of parameters at stage t + 1 depend on his type r and the history up to stage t $(i_1,j_1),\dots,(i_t,j_t)$. We define $H := I \times J$ so that the set of histories up to stage t is denoted by H^t. Then a strategy of player 1 consists of a sequence $\sigma = (\sigma_1 \dots \sigma_T)$ of mappings $\sigma_t : R \times H^{t-1} \to \Delta(I)$ (resp. stochastic kernels $\sigma_t \mid R \times H^{t-1} \Rightarrow I$)
A strategy for player 2 is given by a sequence $\tau = (\tau_1 \dots \tau_T)$ of mappings $\tau_t : H^{t-1} \to \Delta(J)$ (resp. stochastic kernels $\tau_t \mid H^{t-1} \Rightarrow J$)
The sets of strategies are denoted by Σ_T resp. Π_T.

If type r was selected using the strategies $\sigma = (\sigma_1 \dots \sigma_T)$ and $\tau = (\tau_1 \dots \tau_T)$ the players generate the probability distribution $P^r_{(\sigma,\tau)}$ on the set of histories H^T defined by

$$P^r_{(\sigma,\tau)}(h_1 \dots h_T) = \prod_{t=1}^{T} \sigma_t(r, h_1 \dots h_{t-1}; h^1_t) \cdot \tau_t(h_1 \dots h_{t-1}; h^2_t)$$

We can extend $P^r_{(\sigma,\tau)}$ to a distribution on $R \times H^T$ by

$$P^p_{(\sigma,\tau)}(r, h_1 \dots h_T) = p^r \cdot P^r_{(\sigma,\tau)}(h_1 \dots h_T)$$

The payoff function of the game is naturally defined as the expectation of the average payoff with respect to this distribution, explicitly

$$\mathcal{L}^p_T(\sigma,\tau) = \sum_{r, h_1 \dots h_T} P^p_{(\sigma,\tau)}(r, h_1 \dots h_T)\ \frac{1}{T}\sum_{t=1}^{T} A^r(h_t)$$

Hence we have a noncooperative two-person zero-sum game $\Gamma_T(p) = (\Sigma_T, \Pi_T, \mathcal{L}^p_T)$. The sets of strategies Σ_T and Π_T are compact and $\mathcal{L}^p_T$ depends linearly on σ resp. τ. Thus the min-max theorem guarantees the existence of a value $v_T(p)$. For further considerations we may assume that player 2 knows player 1's strategy. In this case any move by player 1 that depends on his type enables player 2 to compute new posteriors on R. Using the private information means revealing it. The following three examples show how different the consequences of revealing information can be:

Example: $|I| = |J| = |R| = 2 \qquad p = (\frac{1}{2}, \frac{1}{2})$

(1) $$A^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Knowing his own type, player 1 always has a dominant strategy in the one shot game, namely choosing the first row if he is of type 1 and choosing the second row if he has type 2. But if he acts in this way at the first stage player 2 knows his type and he will react by selecting only the second or only the first column respectively from stage two on. The resulting payoff converges to zero as T tends to infinity. On the other hand if player 1 decides not to use his information at all, player 2 faces a game with payoff matrix

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

and player 1 will get a payoff of 1/4.

(2) $$A^1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

In this case the situation is reversed. Playing a dominant strategy also means complete revelation of player 1's private information, but he has nothing to lose acting this way. His payoff will be zero, the maximum he can get.

In the preceeding examples an extreme treatment of private information is favorable: In the first case player 1 conceals it completely, in the second case he reveals it completely. There are more interesting examples situated somewhat in between, where player 1 makes his moves dependent on his type, but not in a deterministic way, so that player 2 can compute new posteriors on R but never gains absolute certainty about his opponent's true type.

(3) $$|I| = |R| = 2\,, \quad |J| = 3\,, \quad p = (\tfrac{1}{2}, \tfrac{1}{2})$$

$$A^1 = \begin{pmatrix} 4 & 0 & 2 \\ 4 & 0 & -2 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & 4 & -2 \\ 0 & 4 & 2 \end{pmatrix}$$

If player 1 plays completely revealing his limiting payoff is zero since the values of both matrices are zero. If he plays non-revealing, he will get the value of

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{pmatrix}$$

which is zero as well.

Let us suppose for a moment that the distribution on player 1's type is not $\left(\frac{1}{2},\frac{1}{2}\right)$ but e.g. $\left(\frac{3}{4},\frac{1}{4}\right)$ or $\left(\frac{1}{4},\frac{3}{4}\right)$. In the first case the weighted payoff matrix is

$$\begin{pmatrix} 3 & 1 & 1 \\ 3 & 1 & -1 \end{pmatrix} \text{ in the second case } \begin{pmatrix} 1 & 3 & -1 \\ 1 & 3 & 1 \end{pmatrix}$$

Playing without relation to his private information he could always guarantee himself a payoff of one by choosing the first resp. the second row. So he must bring about that from player 2's point of view either $\left(\frac{3}{4},\frac{1}{4}\right)$ or $\left(\frac{1}{4},\frac{3}{4}\right)$ are the correct posteriors on R and play non-revealing from then on. The way he can give rise to this effect is quite simple: If he is of type 1, he uses at the first stage the distribution $\left(\frac{3}{4},\frac{1}{4}\right)$ on his actions. If he is of type 2 he performs a similar lottery with the probabilities interchanged. After observing i = 1 player 2 calculates

$$\mathrm{Prob}(r=1 \mid i=1) = \frac{\mathrm{Prob}(r=1, i=1)}{\mathrm{Prob}(i=1)} = \frac{\frac{1}{2}\cdot\frac{3}{4}}{\frac{1}{2}\cdot\frac{3}{4}+\frac{1}{2}\cdot\frac{1}{4}} = \frac{3}{4}$$

and if he observes i = 2

$\mathrm{Prob}(r=1 \mid i=2) = \frac{1}{4}$

Thus player 1 can obtain a limiting payoff of one.

Definition: A strategy $\sigma=(\sigma_1 \ldots \sigma_T)$ of player 1 is called non-revealing if the kernels σ_t, t = 1...T are independent of r.

Given a game $\Gamma_T(p)$ with payoff matrices A^r, $r\in R$, the NR-game is defined as the (one-shot) matrix game $A(p) = \sum_r p^r A^r$.

If player 1 decides to play non-revealing (i.e. to use only non-revealing strategies) the maximum payoff he can get is the value u(p) of the NR-game A(p).

Let us now have a closer look at the strategy spaces:

$$\begin{aligned}\Sigma_T &= \Delta(I)^R \times \Delta(I)^{R\times H} \times \ldots \times \Delta(I)^{R\times H^{T-1}} \\ &= \Delta(I)^R \times \Delta(I)^{H\times(R \cup R\times H \cup \ldots \cup R\times H^{T-2})} \\ &= \Delta(I)^R \times \Sigma_{T-1}^H\end{aligned}$$

Analogously for player 2:

$$\begin{aligned}\Pi_T &= \Delta(J) \times \Delta(J)^H \times \ldots \times \Delta(J)^{H^{T-1}} \\ &= \Delta(J) \times \Pi_{T-1}^H\end{aligned}$$

A strategy in the T-stage game can be viewed as a pair consisting of a strategy for stage 1 and a strategy for a (T-1)-stage game dependent on what happened at stage 1.

$$\min_{\tau\in\mathcal{T}_T}\ \max_{\sigma\in\Sigma_T}\ \mathcal{L}^p_T(\sigma,\tau)$$

$$=\min_{\tau_1}\ \min_{(\tau_2\ldots\tau_T)}\ \max_{\sigma_1}\ \max_{(\sigma_2\ldots\sigma_T)}\ \mathcal{L}^p_T(\sigma,\tau)$$

$$\geq \min_{\tau_1}\ \max_{\sigma_1}\ \min_{(\tau_2\ldots\tau_T)}\ \max_{(\sigma_2\ldots\sigma_T)}\ \mathcal{L}^p_T(\sigma,\tau)$$

$$\geq \max_{\sigma}\ \min_{\tau}\ \mathcal{L}^p_T(\sigma,\tau)$$

From the min-max theorem it follows that both inequalities can be replaced by equalities. For the following calculation we abbreviate $\mathbb{P}^p_{(\sigma,\tau)}$ by $\mathbb{P}$ and introduce

$$\bar{\sigma}_1(i)=\sum_r p^r\,\sigma_1(r;i)$$

(total probability of parameter i at stage 1)

$$p(r\mid i)=\frac{p^r\,\sigma_1(r;i)}{\bar{\sigma}_1(i)}$$

(conditional probability of type r given parameter i)

$$\min_{\tau}\ \max_{\sigma}\ \mathcal{L}^p_T(\sigma,\tau)$$

$$=\min_{\tau_1}\ \max_{\sigma_1}\ \min_{(\tau_2\ldots\tau_T)}\ \max_{(\sigma_2\ldots\sigma_T)}\ \mathcal{L}^p_T(\sigma,\tau)$$

$$=\min_{\tau_1}\ \max_{\sigma_1}\ \min_{(\tau_2\ldots\tau_T)}\ \max_{(\sigma_2\ldots\sigma_T)}\ \frac{1}{T}\sum_{r,\ell_1\ldots\ell_T}\mathbb{P}(r,\ell_1\ldots\ell_T)\sum_{t=1}^{T}A^r(\ell_t)$$

$$=\min_{\tau_1}\ \max_{\sigma_1}\ \min_{(\tau_2\ldots\tau_T)}\ \max_{(\sigma_2\ldots\sigma_T)}\ \frac{1}{T}\Big(\sum_{r,\ell_1}\mathbb{P}(r,\ell_1)\,A^r(\ell_1)+$$

$$\sum_{\ell_1}\mathbb{P}(\ell_1)\sum_r\mathbb{P}(r\mid\ell_1)\cdot\sum_{\ell_2\ldots\ell_T}\mathbb{P}(\ell_2\ldots\ell_T\mid r,\ell_1)\sum_{t=2}^{T}A^r(\ell_t)\Big)$$

$$=\min_{\tau_1}\ \max_{\sigma_1}\ \frac{1}{T}\Big(\sum_r p^r\sum_{i,j}\sigma_1(r;i)\,\tau_1(j)\,A^r(i,j)+$$

$$\sum_{\ell_1}\mathbb{P}(\ell_1)\cdot\min_{(\tau_2\ldots\tau_T)}\ \max_{(\sigma_2\ldots\sigma_T)}\ \sum_r\mathbb{P}(r\mid\ell_1)$$

$$\sum_{\ell_2\ldots\ell_T}\mathbb{P}(\ell_2\ldots\ell_T\mid r,\ell_1)\sum_{t=2}^{T}A^r(\ell_t)\Big)$$

$$=\min_{\tau_1}\ \max_{\sigma_1}\ \frac{1}{T}\Big(\sum_r p^r\sum_{i,j}\sigma_1(r;i)\,\tau_1(j)\,A^r(i,j)+\sum_i\bar{\sigma}_1(i)\cdot$$

$$\min_{(\tau_2 .. \tau_T)} \max_{(\sigma_2 .. \sigma_T)} \sum_r p(r|i)$$
$$\sum_{l_2 .. l_T} P(l_2 .. l_T | r, i) \sum_{t=2}^{T} A^r(l_t) \Big)$$
$$= \min_{\tau_1} \max_{\sigma_1} \frac{1}{T} \Big(\sum_r p^r \sum_{i,j} \sigma_1(r,i)\, \tau_1(j) \cdot$$
$$\Big(A^r(i,j) + \min_{(\tau_2 - \tau_T)} \max_{(\sigma_2 .. \sigma_T)} \sum_r p(r|i)$$
$$\sum_{l_2 .. l_T} P(l_2 .. l_T | r, i) \sum_{t=2}^{T} A^r(l_t) \Big)\Big)$$

The second min max term already resembles very much the value of a $(T-1)$-stage game with probability distribution $p(\cdot|i)$, only the strategies $(\sigma_2 \dots \sigma_T)$ and $(\tau_2 .. \tau_T)$ still belong to the T-stage game. But the first stage history only influences the future payoffs via $p(\cdot|i)$, thus for every pair h' = (i',j') inserted in the corresponding component of $(\sigma_2 .. \sigma_T)$ and $(\tau_2 \dots \tau_T)$ the value of

$$\min_{(\tau_2 .. \tau_T)} \max_{(\sigma_2 .. \sigma_T)} \sum_r p(r|i) \sum_{l_2 .. l_T} \quad \dots$$

remains the same. So we can omit the first stage history and continue by

$$\dots = \min_{\tau_1} \max_{\sigma_1} \frac{1}{T} \Big(\sum_r p^r \sum_{i,j} \sigma_1(r; i)\, \tau_1(j)$$
$$\Big(A^r(i,j) + \min_{(\tau_1 .. \tau_{T-1})} \max_{(\sigma_1 .. \sigma_{T-1})} \sum_r p(r|i)$$
$$\sum_{l_1 .. l_{T-1}} P^r_{(\tau_1 .. \tau_{T-1}, \sigma_1 .. \sigma_{T-1})}(l_1 .. l_{T-1}) \sum_{t=1}^{T-1} A^r(l_t) \Big)\Big)$$

We have just proved the following

Theorem: $$v_T(p) = \frac{1}{T} \min_{\tau_1} \max_{\sigma_1} \Big(\sum_r p^r \sum_{i,j} \sigma_1(r; i)\, \tau_1(j) \Big(A^r(i,j) + (T-1)\, v_{T-1}(p(\cdot|i)) \Big)\Big)$$

The above proof is a formulation of Armbruster [83] for the case of lack of information on one side. But it should be pointed out that the result also holds for lack of information on both sides and the dependent case (cf. sections 5.3.2 and 5.3.3). The proof remains virtually unchanged.

The theorem implies that the value of $\Gamma_T(p)$ can be computed recursively, but in order to do this for one special p we have to know the value $v_{T-1}(\cdot)$ for every distribution on R.

The formula also shows what is known as the recursive structure of the game $\Gamma_T(p)$. Imagine a game where at first $\Gamma_1(p)$ is played. Then player 1 tells his strategy to a referee who announces the posterior probability $p(\cdot|i)$ to player 2. Afterwards $\Gamma_{T-1}(p(\cdot|i))$ is played. The total payoff is $\frac{1}{T}$(payoff in $\Gamma_1(p)$ + $(T-1)$ payoff in $\Gamma_{T-1}(p(\cdot|i))$). The above theorem indicates that for a proper formalization of this game its value coincides with $v_T(p)$. By induction the result can be extended to games where the posteriors are announced after each stage.

Intuitively it could be suspected that the payoff player 1 can guarantee himself decreases with the number of stages because the amount of information player 2 collects can only increase with the number of stages. The recursive formula can be used to prove this.

Proposition: For all $p \in \Delta(R)$ the sequence $v_T(p)$ is decreasing. Consequently $v_T(\cdot)$ has a limit function.

Theorem: $\lim_{T \to \infty} v_T(p) = \operatorname{cav}_p u(p)$

To prove this it suffices to show that $\bigwedge_T v_T(p) \geq \operatorname{cav} u(p)$ and $\lim_{T \to \infty} v_T(p) \leq \operatorname{cav} u(p)$. The first inequality is easy to verify. Player 1 can always play non-revealing, thus $v_T(p) \geq u(p)$. By paragraph 5.1 we know that $v_T(\cdot)$ is concave, therefore $v_T(p) \geq \operatorname{cav}_p u(p)$.

The proof of the second inequality is much more involved. We will have to estimate the payoff at a certain stage t conditionally to the history up to that stage. Let player 1 use an arbitrary strategy σ which is known to player 2. Let us fix any history $h_1 \ldots h_{t-1}$ and denote player 2's posterior probability on R conditionally to $h_1 \ldots h_{t-1}$ by p_t. $\gamma_t(\sigma_t, \tau_t)$ is the payoff at stage t conditionally to the history $h_1 \ldots h_{t-1}$ provided the players use the strategies σ_t and τ_t at stage t.

First of all we estimate the difference in payoffs that arises if player 1 uses a non-revealing strategy that results from σ_t by weighting the types according to p_t instead of σ_t itself. Define

$$\bar{\sigma}_t(\cdot) = \sum_r p_t^r \, \sigma_t(r, \cdot)$$

Then we find

$$|\gamma_t(\sigma_t, \tau_t) - \gamma_t(\bar{\sigma}_t, \tau_t)| \leq C \sum_{r,i} \bar{\sigma}_t(i) \, | p^r_{t+1}(i) - p^r_t |$$

with

$$C = 2 \cdot \max_{r,i,j} |A^r(i,j)|$$

especially

$$\gamma_t(\sigma_t, \tau_t) \leq \gamma_t(\bar{\sigma}_t, \tau_t) + C \sum_{r,i} \bar{\sigma}_t(i) \, | p^r_{t+1}(i) - p^r_t |$$

Since $\bar{\sigma}_t$ is nonrevealing, the maximum payoff player 2 can obtain at stage t is $u(p_t)$. Consequently

$$\gamma_t(\sigma_t, \tau_t) \leq u(p_t) + C \sum_{r,i} \bar{\sigma}_t(i) \, | p^r_{t+1}(i) - p^r_t |$$

Taking the expectation over all histories $h_1 \ldots h_{t-1}$ gives

$$E(\gamma_t(\sigma_t, \tau_t)) \leq \operatorname{cav} u(p) + C \sum_r E \, | p^r_{t+1} - p^r_t |$$

since $E(u(p_t)) \leq E(\operatorname{cav}_p u(p_t)) \leq \operatorname{cav}_p u(E(p_t)) = \operatorname{cav}_p u(p)$ by Jensen's inequality.

Summing up over all stages t = 1...T and dividing by T we arrive at

$$\alpha^p_T(\sigma, \tau) \leq \operatorname{cav} u(p) + C \frac{1}{T} \sum_r E\left(\sum_{t=1}^{T} | p^r_{t+1} - p^r_t | \right)$$

Finally, using the martingale property of p_t, Jensen's inequality and Cauchy-Schwarz inequality one can show that

$$\frac{1}{T} \sum_r E\left(\sum_{t=1}^{T} | p^r_{t+1} - p^r_t | \right) \leq \frac{1}{\sqrt{T}} \sum_r \sqrt{p^r(1-p^r)}$$

Thus the proof of the theorem is completed by the following result:

Theorem: $v_T(p) \leq \operatorname{cav} u(p) + O\left(\frac{1}{\sqrt{T}}\right)$

There are examples showing that $O\left(\frac{1}{\sqrt{T}}\right)$ is the best bound. (See Zamir [71]).

Generally a strategy of player 1 that is optimal in a game of length T is no longer optimal in a game with a different number of stages. He often faces the problem that striving for a high payoff at the beginning of the game means giving away a lot of his private information what mostly results in a lower payoff at the following stages. If the number of stages is small such a behaviour may be favourable, if T becomes large it will be disadvantageous. As player 1's optimal strategy is different for different lengths of the game player 2's best response will also change. It is a natural question whether there is a pair of strategies that are "nearly" best responses to each other for any T large enough. One is induced to consider infinitely repeated games.

The description of $\Gamma_T(p)$ exactly applies to the infinitely repeated game $\Gamma_\infty(p)$ except for the fact that the strategies σ, τ of player 1 resp. 2 consist of infinite sequences $(\sigma_1, \sigma_2, \ldots)$ resp. $(\tau_1, \tau_2, \ldots)$. We are still interested in an average payoff, but for arbitratry strategies σ, τ it may of course happen that the limiting payoff $\lim_{T\to\infty} \mathcal{L}^p_T(\sigma, \tau)$ does not exist. We avoid the definition of a payoff for every pair of strategies and specify only equilibrium payoffs:

Definition: Player 1 can guarantee f(p) in $\Gamma_\infty(p)$ if for all $p \in \Delta(R)$, all $\varepsilon > 0$ there exist a strategy σ_ε of player 1 and a natural number T_ε such that for all $t > T_\varepsilon$ and all strategies τ of player 2

$$\mathcal{L}^p_t(\sigma_\varepsilon, \tau) \geq f(p) - \varepsilon$$

is satisfied.

$\Gamma_\infty(p)$ has a value $v_\infty(p)$ if both players can guarantee $v_\infty(p)$. Explicitly:

$$\bigwedge_{p\in\Delta(R)} \ \bigwedge_{\varepsilon>0} \ \bigvee_{(\sigma_\varepsilon, \tau_\varepsilon)} \ \bigvee_{T_\varepsilon} \ \bigwedge_{t>T_\varepsilon}$$

$$\Big(\bigwedge_{\tau} \mathcal{L}^p_t(\sigma_\varepsilon, \tau) \geq v_\infty(p) - \varepsilon \ , \ \bigwedge_{\sigma} \mathcal{L}^p_t(\sigma, \tau_\varepsilon) \leq v_\infty(p) + \varepsilon \Big)$$

The following lemma is crucial for all theory of incomplete information:

Lemma: If player 1 can guarantee f(p), he can also guarantee cav f(p).

Proof: From the Caratheodory theorem it follows that for all $p \in \Delta(R)$ there exist $\lambda_r \in [0,1]$, $p_r \in \Delta(R)$ such that

$$\sum_r \lambda_r = 1 \ , \ \sum_r \lambda_r p_r = p \text{ and cav } f(p) = \sum_r \lambda_r f(p_r) \ .$$

Let σ_r be a strategy which guarantees $f(p_r)$ up to ε in $\Gamma_\infty(p_r)$, $r \in R$.
Define a strategy σ_0 as follows:
If $r \in R$ is chosen, play σ_s with probability $\lambda_s \cdot \frac{p^r_s}{p^r}$ for all r. Player 1 uses a lottery dependent on his type and plays nonrevealing from then on.
The total probability of playing σ_s is

$$\sum_r p^r \operatorname{Prob}(\sigma_s \mid r) = \sum_r p^r \Big(\lambda_s \frac{p^r_s}{p^r}\Big) = \sum_r \lambda_s p^r_s = \lambda_s$$

and the probability on R conditionally to σ_s is

$$\operatorname{Prob}(r \mid \sigma_s) = \frac{p^r \cdot \lambda_s \frac{p_s^r}{p^r}}{\lambda_s} = p_s^r$$

Since σ_s guarantees $f(p_s)$ up to ε there is a natural number T_ε such that $\bigwedge_{t > T_\varepsilon}$

$$\mathcal{L}_t^p(\sigma_0, \tau) \geq \sum_{s \in R} \lambda_s (f(p_s) - \varepsilon) = \operatorname{cav} f(p) - \varepsilon \qquad \blacksquare$$

Theorem: For all $p \in \Delta(R)$ player 1 has a strategy σ_0 such that for all t and all τ

$$\mathcal{L}_t^p(\sigma_0, \tau) \geq \operatorname{cav} u(p) \quad .$$

Proof: Since player 1 can guarantee u(p) at every stage by playing an optimal strategy among his nonrevealing strategies, he can also guarantee cav u(p). Actually the result is stronger than in the above lemma, because player 1 cannot only guarantee $u(p_r)$ up to ε but achieve a payoff grater than $u(p_r)$.

Our next aim is to find a strategy that enables player 2 to reduce player 1's payoff to at most cav u(p). From the first the existence of such a strategy is not clear. We can't apply the min – max theorem directly, in the foreground because the payoff is not defined for every pair of strategies. But even if the payoff function is extended to all pairs of strategies (e.g. by means of a Banach limit) it is no longer continuous. In fact minmax enters the scene in the shape of Blackwell's theorem (cf. 2.2).

Theorem: Player 2 can guarantee cav u(p) in $\Gamma_\infty(p)$, i.e. $v_\infty(p)$ = cav u(p).

Proof: Let H be a supporting hyperplane to cav u at point p defined by $h \in \mathbb{R}^R$. (As in section 5.2 h can be interpreted as a vector payoff, i.e. the first component of h gives the payoff that would arise if r = 1 were chosen etc.)

h satisfies

cav u(p) = h·p and $\bigwedge_{q \in \Delta(R)} u(q) \leq hq$.

It suffices to show that the set

$$C = \{ \ell \in \mathbb{R}^R : \bigwedge_{r \in R} \ell^r \leq h^r \}$$

is approachable in the sense of Blackwell. According to Blackwell's theorem the following condition is sufficient:

For all $\xi \notin C$ there exists a probability y on J such that if $\eta \in C$ is a closest point to ξ, the hyperplane perpendicular to the line $\xi - \eta$ through η separates ξ from

$$Co\left\{ \sum_{j \in J} y_j A(i,j) : i \in I \right\}$$

Let ξ and η be as above, $p' \in \Delta(R)$ parallel to $\xi - \eta$ i.e.

$H' = \{ g \in \mathbb{R}^R : p'g = p' \cdot \eta \}$ is a hyperplane through η perpendicular to $\xi - \eta$.

Let y be an optimal strategy for player 2 in the one-shot game

$A(p') = \sum_r p'^r A^r$. For all $x \in \Delta(I)$ we find

$$\begin{aligned}
& \sum_r p'^r x A^r y && (\ni Co\{\textstyle\sum_{j \in J} y_j A(i,j) : i \in I\}) \\
\leq \; & u(p') && \text{(by optimality of y)} \\
\leq \; & h \cdot p' && \text{(supporting property of h resp.H)} \\
= \; & \eta \cdot p' && (\text{if } p'^r > 0 \text{ then } \eta^r = h^r) \\
< \; & \xi \cdot p' && (\xi > \eta \in C,\ \xi \notin C)
\end{aligned}$$

■

The approaching strategy for player 2 according to Blackwell runs as follows:

- At stage 1 or if the average vector payoff up to stage t ξ_t is located in C play anything
- If $\xi_t \notin C$ determine the separating hyperplane between ξ_t and C given by p' and play optimally in A(p').

5.3.2 Lack of Information on Both Sides

A finitely repeated two-person zero-sum game with incomplete information on both sides is based on the following data (very similar to the one sided case):

- finite sets I and J (sets of actions for player 1 and 2)
- finite sets R and S (sets of types for player 1 and 2)

- for every (r,s) RxS an $|I| \times |J|$ – matrix $A^{r,s}$ (payoff matrix)
- probability distributions p and q on R resp. S
- a natural number T

The game runs as follows:

- At stage 0 chance chooses $r \in R$ according to p and $s \in S$ according to q (independently!). Both players know both probabilities but player 1 is only informed about the outcome of the first lottery and player 2 about the outcome of the second one.
- The further description is exactly as in the one sided case except for player 2's strategies. He can, of course, let his decisions depend on s.

As in the one – sided case a recursive formula for the value of the finitely repeated game is valid:

Theorem: $$v_T(p,q) = \frac{1}{T} \max_{\sigma_1} \min_{\tau_1} \Big(\sum_{r,s} p^r q^s \sum_{i,j} \sigma_1(r;i)\, \tau_1(s;j) \big(A^{r,s}(i,j) + (T-1)\, v_{T-1}(p(\cdot|i), q(\cdot|j))\big)\Big)$$

with obvious notation.

It is also convenient to define the non – revealing game A(p,q) as the one – shot matrix game $\sum_{r,s} p^r q^s A^{r,s}$ and denote its value by u(p,q).

Theorem: $\lim_{T \to \infty} v_T(p,q)$ exists for all $p \in \Delta(R)$, $q \in \Delta(S)$ and is the only simultaneous solution of the functional equations

$$v(p,q) = \operatorname{vex}_q \max \{u(p,q), v(p,q)\}$$

$$v(p,q) = \operatorname{cav}_p \min \{u(p,q), v(p,q)\}$$

The proof is beyond the scope of this survey.
See Mertens and Zamir [71].

We now turn to the infinitely repeated game $\Gamma_\infty(p,q)$. Again we shall not attempt to define payoffs for every pair of strategies. Instead we are going to define min max and max min for the infinitely repeated game.

Definition: f(p,q) is the minmax of $\Gamma_\infty(p,q)$ if

$$-\ \bigwedge_{\tau}\ \bigwedge_{\varepsilon>0}\ \bigvee_{\sigma}\ \bigvee_{T}\ \bigwedge_{t>T}\ L_t^{p,q}(\sigma,\tau) > f(p,q)-\varepsilon$$

$$-\ \bigwedge_{\varepsilon>0}\ \bigvee_{T_\varepsilon}\ \bigvee_{\tau_\varepsilon}\ \bigwedge_{\sigma}\ \bigwedge_{t>T_\varepsilon}\ L_t^{p,q}(\sigma,\tau_\varepsilon) < f(p,q)+\varepsilon$$

If such an f(p,q) exists it is the lowest payoff player 2 can guarantee in $\Gamma_\infty(p,q)$. The max min is defined analogously and we say that $\Gamma_\infty(p,q)$ has a value if the min max equals the max min.

Theorem: The min max of $\Gamma_\infty(p,q)$ equals $\operatorname{vex}_q \operatorname{cav}_p u(p,q)$.

The max min of $\Gamma_\infty(p,q)$ equals $\operatorname{cav}_p \operatorname{vex}_q u(p,q)$.

Outline of a proof: For duality reasons it is sufficient to prove the assertion for the min max.

If player 2 decides to play non – revealing, $\Gamma_\infty(p,q)$ reduces to a game with lack of information on one side defined by the payoff matrices $\sum_s q^s A^{r,s}$ and the distribution p on R. The one sided theory says that in this game player 2 can guarantee the payoff $\operatorname{cav}_p u(p,q)$. Using his information he can of course hold player 1 down to at most $\operatorname{vex}_q \operatorname{cav}_p u(p,q)$, which gives the second inequality of the above definition.

The proof of the fact that player 2 cannot guarantee more can be summerized as follows: Knowing τ player 1 can compute posteriors q_t on S after each stage. He will play non – revealing during a large number of stages T until the expected variation of the posteriors after stage T becomes negligible. After player 1 has exhausted the maximal amount of information we can assume player 2 to play non – revealing. Player 1 can thus obtain $u(p,q_T)$, consequently he can also get cav $u(p,q_T)$. His expected payoff will be

$$E(\operatorname{cav}_p u(p,q_T)) \geq \operatorname{vex}_q \operatorname{cav}_p u(p,q)$$

(due to Jensen's inequality). ■

It follows that the infinitely repeated game only has a value if $\operatorname{cav}_p \operatorname{vex}_q u(p,q) = \operatorname{vex}_q \operatorname{cav}_p u(p,q)$.

This is generally not the case:

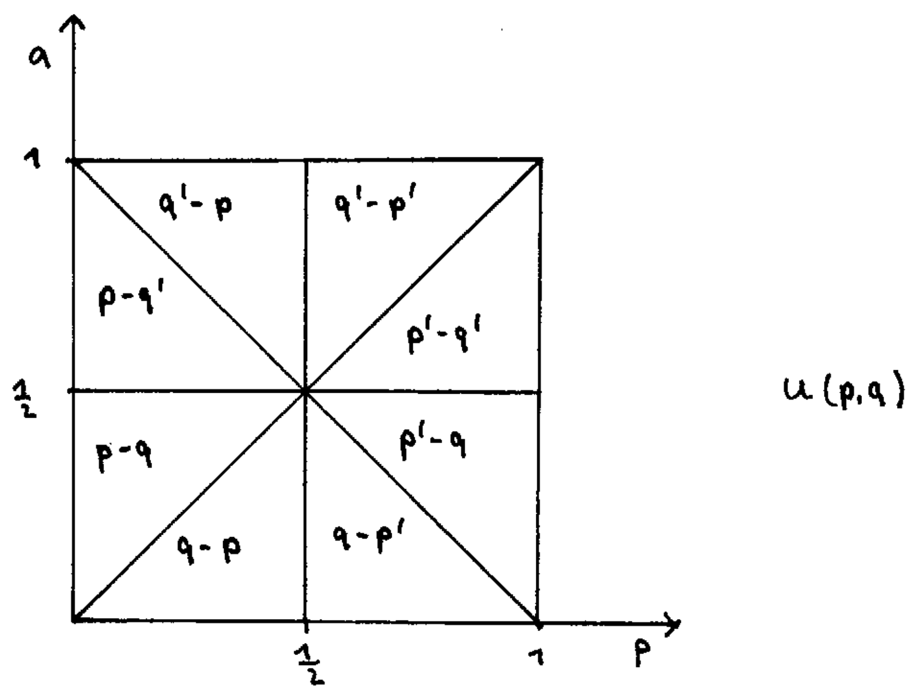
Example: $A^{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \end{pmatrix}$ $\quad A^{12} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

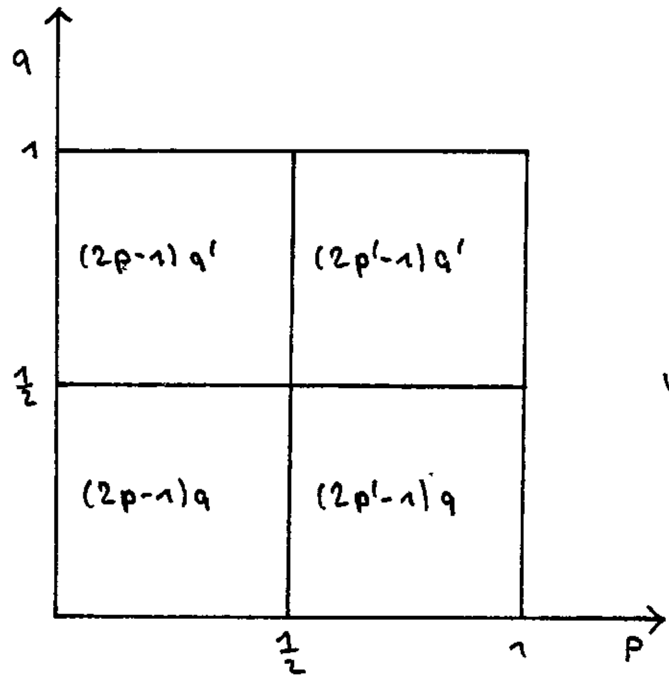
$$A^{21} = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A^{22} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$u(p,q)$ is the value of

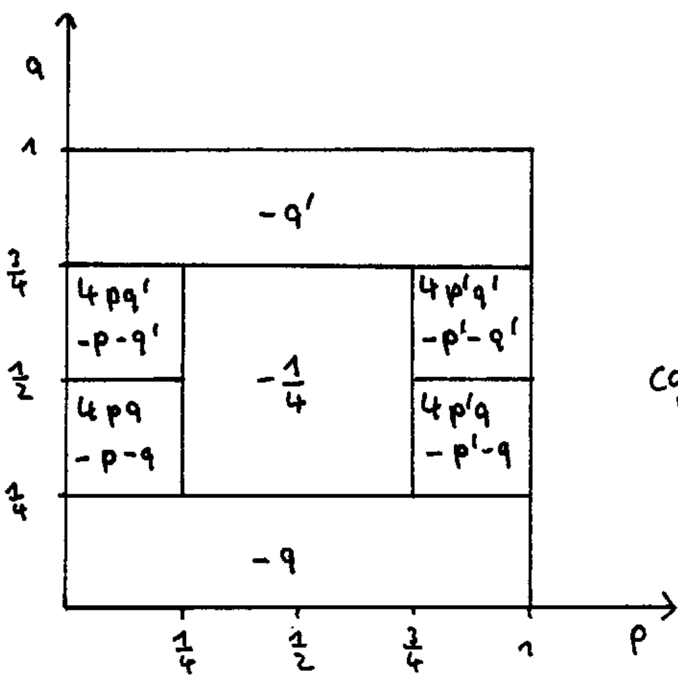
$$\begin{pmatrix} p-q & q-p & p-q & q-p \\ q'-p & p-q' & p-q' & q'-p \end{pmatrix}$$

where $p = p^1$, $p' = p^2$, $q = q^1$, $q' = q^2$.





$\operatorname{vex}_q \operatorname{cav}_p u(p,q)$



$\operatorname{cav}_p \operatorname{vex}_q u(p,q)$

5.3.3 The Dependent Case

Here the probability on the product space of types R x S is optional, the marginals p and q on R resp. S need not be independent. The main difference between the dependent and the independent case arises from the fact that the posteriors after each stage cannot be decomposed into p_t function of p, σ, h_t and q_t function of q, τ, h_t. If we don't confine ourselves to product spaces of types but consider an arbitrary random experiment with two information algebras a modified concept of concavity resp. convexity becomes necessary. Proofs are more involved, but the main ideas and results are the same.

5.3.4 Information Matrices

The general model is given by the following data:

- finite sets I and J (again sets of actions)
- a finite set K (states of nature)
- two partitions $K^{I} = \{K^{I}_1 \ldots K^{I}_A\}$ and $K^{II} = \{K^{II}_1 \ldots K^{II}_B\}$ (representing the player's private information)
- payoff matrices A^k, $k \in K$
- two families of $|I| \times |J|$-matrices H^k_{I} and H^k_{II}, $k \in K$ (information matrices)
- a probability distribution p on K

The game runs as follows:

- At stage 0 chance chooses $k \in K$ according to p. Then a is announced to player 1 and b to player 2 with $k \in K^{I}_a \wedge K^{II}_b$
- At each stage t both players choose independently parameters $i_t \in I$ resp. $j_t \in J$.
- Then $H^k_{I}(i_t, j_t)$ is announced to player 1 and $H^k_{II}(i_t, j_t)$ to player 2.

From this point on the description of repeated games with lack of information on both sides applies, except for the fact that there is no longer a common history $H^T = (I \times J)^T$ but each player has his private history consisting of a sequence of elements of his information matrices and the sequence of his own previous moves.

Note that the information matrices may reveal information on both the opponent's move and chance's move. The information matrices may even reveal more information about the state of nature than both players together have. That is the reason why in this case we would indeed loose generality considering only product spaces of types R x S. Transforming an arbitrary random experiment with information partitions into one of product type we can no longer replace states of nature, which none of the player can distinguish, by their expectation. The case studied before is obtained if $H^k_I(i,j) = H^k_{II}(i,j) = (i,j)$. It is called the standard information case. The main change in comparison with the standard information case is related to the non-revealing strategies. In the model without information matrices using information means revealing it. This is no longer true.

Example: $|I| = |J| = 2$. We have lack of information on one side, $|R| = 2$ with probability $(p, 1-p)$.

$$H^1_{II} = \begin{pmatrix} a & a' \\ b & b' \end{pmatrix} \qquad H^2_{II} = \begin{pmatrix} b & b' \\ a & a' \end{pmatrix}$$

Let player 1 use his private information in the following way: If he is of type 1 he chooses parameter 1 with probability x (and paramter 2 with probability $1-x$). If he is of type 2 he uses the distribution $(y, 1-y)$. Player 2 can compute the following posterior:

$$\mathrm{Prob}(r=1 \mid a) = \frac{\mathrm{Prob}(r=1, a)}{\mathrm{Prob}(r=1, a) + \mathrm{Prob}(r=2, a)} = \frac{px}{px + (1-p)(1-y)}$$

Player 1 does not give away any private information, if $\mathrm{Prob}(r = 1 | a) = p$ is satisfied. This is the case if $x = 1-y$. That means except for the case $x = y = 1/2$ player 1 has to make use of his information in order to conceal it.

So we have to call a player's strategy non-revealing if his opponent is not able to compute other posteriors on his type than the initial probabilities. Of course it can happen that there are no non-revealing strategies at all, e.g. if the information matrices for different types have no elements in common.

There are no general results for this model. Only special cases have been covered so far.

(1) Lack of information on one side

Here we have a generalization of theorems concerning the limiting value of the finitely repeated game and the value of the infinitely repeated game of section 5.3.1. Let us denote by u(p) the game where (the informed) player 1 is restricted to non-revealing strategies (with u(p) = $-\infty$ if there are no non-revealing strategies. But note that if p is an extreme point of $\Delta(R)$ every strategy is non-revealing)

Theorem: $\lim_{T\to\infty} v_T(p)$ and $v_\infty(p)$ exist and

$$\lim_{T\to\infty} v_T(p) = v_\infty(p) = \text{cav } u(p).$$

(2) Games with symmetric information

These games correspond to the following case:

- $K^{I} = K^{II} = K$ (there is lack of information on both sides and no private information)
- $H^k_I = H^k_{II}$ for all $k \in K$
- $H^k(i,j) \neq H^{k'}(i',j')$ whenever $i \neq i'$ or $j \neq j'$.

  (Both players know the opponent's previous moves)

Theorem: Under the above hypothesis $\lim_{T\to\infty} v_T(p)$ and $v_\infty(p)$ both exist and are equal.

The theorem is proved by transforming such a game into a repeated game with absorbing payoffs where it is known that, provided each player knows his opponent's past actions, v and lim v exist and are equal.

Example: $|K| = |I| = |J| = 2$ A^1, A^2 not specified, (p^1, p^2)

$$H^1 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad H^2 = \begin{pmatrix} \alpha' & \beta \\ \gamma & \delta \end{pmatrix}$$

This game is equivalent to the following repeated game:

$$\bar{A} = \begin{pmatrix} (p^1 \text{val} A^1 + p^2 \text{val} A^2)^* & p^1 A^1(1,2) + p^2 A^2(1,2) \\ p^1 A^1(2,1) + p^2 A^2(2,1) & p^1 A^1(2,2) + p^2 A^2(2,2) \end{pmatrix}$$

The asterisk indicates that if once (top, left) was played, the payoff is $(p^1 \text{val}A^1 + p^2 \text{val}A^2)$ for all following stages regardless of the player's actions.

Looking at the information matrices H^1, H^2 we find that unless (top, left) is played the player's only learn their opponents moves and they can expect the payoff listed in $\tilde{A}$. But if (top, left) occurs both players know the true payoff matrix and from then on they can both guarantee its value.

(3) Games where the information matrices are independent of k

The theorems concerning the limiting value of the finitely repeated game and the min max resp. max min of the infinitely repeated game of section 5.3.2 (lack of information on both sides) can be extended here but the proofs are much more difficult.

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