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Cost Sharing in a Joint Project

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Cost sharing in a joint project*

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Abstract

The focus of this paper is on cooperation in compound joint projects. A group of agents aims to work together in a joint project which can have different forms. Each feasible form corresponds to a subset of a given set of basic units. The cost of the chosen project is the sum of the costs of the basic units involved in the project. The benefit of each of the agents is dependent on the form of the chosen project. A related cooperative game may be helpful in solving the question of how to share the costs. Under certain conditions this game turns out to be a convex game. For structured joint projects also methods using simple cost sharing rules from the taxation literature are indicated. Many well-known cases in the cost sharing literature fit in our model and some earlier results are special cases of the results which we obtain in this paper.

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1 Introduction

Cooperation is an essential part of human interaction. Especially environmental problems call for cooperation. Game theory can contribute in smoothening cooperation by developing attractive and transparant rules for the allocation of costs or rewards among the participants in joint projects. There is a huge literature dealing with cost sharing problems using game theory. For surveys see Tijs and Driessen (1986) and Young (1994).

In this paper we consider situations where agents plan to cooperate in a complex project. The agents have to decide about the form of the project and about the associated cost sharing. Both facets depend on the involved costs and the budgets, which we identify with the rewards, of the agents for the different forms which the project may finally have. In our model fit e.g. cooperation in irrigation systems (cf. Aadland and Kolpin (1998), Kolpin and Aadland (2001)), airport landing networks (cf. Brânzei et al. (2002), Littlechild and Thompson (1977), Potters and Sudhölter (1999), Koster et al. (2001)), railway networks with facilities (Fragnelli et al. (2000)) and Norde et al. (2002)) and also car pooling, sharing a club house and sharing play fields by different clubs etc.

In an irrigation system the wishes of the participants differ and are determined by the position of the pieces of land owned by the participants. In a railway system intercity trains will have wishes different from local trains etc. In an airport landing network the wishes of the participants depend on the size of their planes and their offered flights.

To make our life not too difficult in this paper we suppose from now on that there is a collection of basic units (components) such that each feasible project consists of a subset of these components, and such that the cost of such a feasible project is equal to the sum of the costs of the involved components. Further we suppose that the benefits increase if the set of involved components increases. In railway projects the basic units are tracks between two neighbouring railway stations and available facilities at the railway stations. In irrigation systems and in airport landing systems the basic units are ditch pieces and landing strip pieces, respectively.

The outline of this paper is as follows. In Section 2 we introduce the formal model of a joint project situation and a related cooperative game. Sufficient conditions are given which guarantee that the game is a convex game. In Section 3 for structured joint projects transparant solutions for related cost sharing problems are introduced, which are based on cost sharing rules from the taxation literature.

2 Joint project situations and joint enterprise games

A *joint project situation* is a tuple $< N, A, c, F, (R_i)_{i\in N} >$ where N is the set of agents involved in the cooperation, A is the set of basic units, $c: A \to \mathbb{R}_+$ the cost function, $F \subset 2^A$ the set of feasible projects, and $R_i : F \to \mathbb{R}_+$ the reward function of agent $i \in N$. In the following we suppose (J.1) and (J.2), with

(J.1) $\phi \in F$ and $R_i(\phi) = 0$ for each $i \in N$.

(J.2) If $\pi_1, \pi_2 \in F$ and $\pi_1 \subset \pi_2$, then $R_i(\pi_1) \leq R_i(\pi_2)$ for each $i \in N$ *(Monotonicity)*.

We will say that F is a *lattice* if (J.3) holds, with

(J.3) If $\pi_1, \pi_2 \in F$, then $\pi_1 \cap \pi_2 \in F$, $\pi_1 \cup \pi_2 \in F$ (*Lattice property*).

We will say that a joint project situation $< N, A, c, F, (R_i)_{i\in N} >$ is based on the tree $< V, A >$, with root $v_0 \in V$, if the basic units are the arcs of the tree, and if each feasible project consits of the arcs of a subtree of $< V, A >$ with root v_0.

Note that F is a lattice for tree–based joint project situations.

We suppose that the agents choose a feasible optimal project π_1 where $\pi_1 \in \arg\max_{\pi\in F}\left(\sum_{i\in N} R_i(\pi) - c(\pi)\right)$ and $c(\pi) = \sum_{a\in\pi} c(a)$. To solve the cost sharing problem, or equivalently, to solve the problem of dividing the total benefit $\sum_{i\in N} R_i(\pi_1) - c(\pi_1)$, the related cooperative game $< N, v >$, which we call the *joint enterprise game*, may be helpful, where for the coalition $S \in 2^N$, the worth $v(S)$ is equal to

$$\max_{\pi\in F}\left(\sum_{i\in S} R_i(\pi) - c(\pi)\right).$$

Then one can use for this joint enterprise game standard solutions as the Shapley value (Shapley (1953)), the nucleolus (Schmeidler (1969)) or the τ-value (Tijs (1981)) to solve the benefit allocation problem. Especially in case the game is convex the Shapley value is appealing, because in this case the core is large and the Shapley value is the barycenter of the core. Recall (cf. Shapley (1971)) that a game $< N, v >$ is a *convex game* if for all $S, T \in 2^N : v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$. In Theorem 2.1 sufficient conditions on a joint project situation are given to guarantee that the corresponding joint enterprise game is convex. A role plays here the supermodularity property of R_i for each $i \in N$, if F is a lattice. Recall that $R_i : F \to \mathbb{R}_+$ is a *supermodular function* if

$$R_i(\pi_1 \cup \pi_2) + R_i(\pi_1 \cap \pi_2) \geq R_i(\pi_1) + R_i(\pi_2).$$

In general, a joint enterprise game is not necessarily convex. Even the core may be empty (cf. Feltkamp et al. (1996)).

Example 2.1. (A connection problem) Consider the graph $< V, A >$ with vertex set $\{v_0, v_1, v_2, v_3\}$ and arc set $A = \{a_1, a_2, a_3, a_4 | a_1 = \{v_0, v_1\},\ a_2 = \{v_1, v_2\},\ a_3 = \{v_2, v_3\}$ and $a_4 = \{v_0, v_3\}\}$. Suppose agent i wants to connect v_0 with v_i, via a path, where $i \in N = \{1, 2, 3\}$ and the cost of using an arc a equals $c(a) = 10$. Suppose that $F = 2^A$ and that a right connection corresponds to a benefit 12 for the involved agent. Then this situation corresponds to the joint project situation $< N, A, c, 2^A, (R_i)_{i \in N} >$, where $R_i(\pi) = 12$ if π contains a path connecting v_i with v_0 and $R_i(\pi) = 0$ otherwise. The corresponding joint enterprise game $< N, v >$ is given by $v(\{1\}) = v(\{3\}) = 2$, $v(\{2\}) = 0$, $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 4$, and $v(N) = 6$. The Shapley value of this game equals $\left(2\frac{1}{3}, 1\frac{1}{3}, 2\frac{1}{3}\right)$ and is unequal to the unique core element $(2, 2, 2)$. The game is not convex because $v(\{1,2\}) + v(\{2,3\}) > v(\{1,2,3\}) + v(\{2\})$. Note that $F = 2^A$ is a lattice but $R_2 : F \to \mathbb{R}_+$ is not supermodular: $R_2(\pi_1) + R_2(\pi_2) = 24 > 12 = R_2(\pi_1 \cap \pi_2) + R_2(\pi_1 \cup \pi_2)$, with $\pi_1 = \{a_1, a_2\}$ and $\pi_2 = \{a_3, a_4\}$.

Now we arrive at our main result.

Theorem 2.1. *Let $< N, v >$ be the cooperative enterprise game corresponding to the joint project situation $< N, A, c, F, (R_i)_{i \in N} >$. Suppose that F is a lattice and that $R_i : F \to R_+$ is supermodular for each $i \in N$. Then $< N, v >$ is a convex game.*

Proof. Take $S, T \in 2^N$. Let $\alpha \in F$ and $\beta \in F$ be such that

(i) $\sum_{i \in S} R_i(\alpha) - c(\alpha) = v(S)$, $\sum_{i \in T} R_i(\beta) - c(\beta) = v(T)$.

Note that from (J.2) it follows

(ii) $R_i(\alpha) \leq R_i(\alpha \vee \beta)$ for $i \in S \setminus T$

(iii) $R_i(\beta) \leq R_i(\alpha \vee \beta)$ for $i \in T \setminus S$.

and from the supermodularity of R_i

(iv) $R_i(\alpha) + R_i(\beta) \leq R_i(\alpha \vee \beta) + R_i(\alpha \wedge \beta)$ for $i \in S \cap T$.

Adding the inequalities in (ii), (iii) and (iv) we obtain

(v) $\sum_{i \in S} R_i(\alpha) + \sum_{i \in T} R_i(\beta) \leq \sum_{i \in S \cup T} R_i(\alpha \vee \beta) + \sum_{i \in S \cap T} R_i(\alpha \wedge \beta)$.

Since $c(\alpha)+c(\beta)=c(\alpha\vee\beta)+c(\alpha\wedge\beta)$, and $\sum_{i\in S\cup T} R_i(\alpha\vee\beta)-c(\alpha\vee\beta)\le v(S\cup T)$, $\sum_{i\in S\cap T} R_i(\alpha\wedge\beta)-c(\alpha\wedge\beta)\le v(S\cap T)$, we obtain from (i) and (v):

(vi) $v(S)+v(T)\le v(S\cup T)+v(S\cap T)$.

Hence, $<N,v>$ is a convex game.

Let us call a function $R_i : F\rightarrow \mathbb{R}$ a *one-step reward function* if there is a $b_i>0$ and a $\pi_i\in F$ such that $R_i(\pi)=b_i$ if $\pi_i\subset\pi$ and $R_i(\pi)=0$ otherwise.

If F is a lattice, then a one-step reward function is supermodular. From Theorem 2.1 we obtain then

Corollary 2.2. *Let $<N,v>$ be the joint enterprise game corresponding to the tree-based joint project situation $<N,A,c,F,(R_i)_{i\in N}>$ and suppose that the reward functions R_i are one-step reward functions. Then $<N,v>$ is a convex game.*

Corollary 2.3. *Let $<N,v>$ be the joint enterprise game corresponding to the joint project situation $<N,A,c,2^A,(R_i)_{i\in N}>$ and suppose that the reward functions R_i are one-step reward functions. Then $<N,v>$ is a convex game.*

A special case of Corollary 2.2, where the underlying tree is a rooted line graph was proved in Brânzei et al. (2002). Corollary 2.3 was also proved in Koster et al. (2002).

3 Structured projects and simple cost sharing rules

In this section we want to describe how well-known cost sharing rules for simple cost sharing problems can be helpful for solving in an appealing and transparant manner, the reward sharing problem related to many complex joint project situations. Here *a simple cost sharing problem* is a tuple $<N,c,b>$, where N is the set of agents, $c\in\mathbb{R}_+$ is the cost to be paid by the agents and $b\in\mathbb{R}^n_+$, the maximal contribution vector, where b_i is the maximum contribution to c, which agent $i\in N$ is willing to pay. Further one assumes that $c\le\sum_{i\in N} b_i$. A cost sharing rule T assigns to problems of the form $<N,c,b>$ a vector $T(c,b)\in\mathbb{R}^N$, where $0\le T_i(c,b)\le b_i$ for each $i\in N$ and $\sum_{i\in N} T_i(c,b)=c$. Well-known from the taxation literature (H. Young (1987)) and the bankruptcy literature (Aumann and Maschler (1985)) are the cost sharing rules PROP (the proportional rule) and CEC (the constrained equal contribution rule). For each

$i \in N$, $\text{PROP}_i(c,b) = (\Sigma b_i)^{-1} b_i c$, and $\text{CEC}_i(c,b) = \min(b_i, \alpha)$, where $\alpha \in \mathbb{R}_+$ is the unique real number such that $\sum_{i \in N} \text{CEC}_i(c,b) = c$.

So, according to the proportional rule, the cost c is divided among the players proportionally to their individual maximal contribution b_i to c, while the constrained equal contribution rule assigns to the players with $b_i \geq \alpha$ a cost contribution share of α and for the other players, with $b_i < \alpha$, the cost share is equal to their individual maximal contribution b_i to c.

To use these simple cost sharing rules, for a joint project situation $< N, A, c, F, (R_i)_{i \in N} >$ we consider a sequence $\sigma =< \pi_1, \pi_2, ..., \pi_m >$ of feasible plans, where

(i) π_1 is an optimal plan consisting of m elements of A

(ii) $\pi_1 \supset \pi_2 \supset \pi_r \supset \cdots \supset \pi_m$

(iii) $|\pi_r \setminus \pi_{r+1}| = 1$ for each $r \in \{1, 2, ..., m-1\}$.

A project with such a sequence will be called *structured.*

Such a sequence will not exist in general. For the case when $F = 2^A$ and also in tree-based cases we have a structured project. Given such a sequence σ, let a_r be the unique element of $\pi_r \setminus \pi_{r+1}$ for $r < m$ and a_m the unique element of π_m. Let T be the cost sharing rule which will be used. The idea then is to consider m simple cost sharing problems for N, where in the r-th problem the cost $c(a_r)$ has to be shared, and where the maximum contributions of the agents depend on the contributions of the agents in the costs of the first $r-1$ problems. To be more formal, given a sequence σ as above and a cost sharing rule T we denote the final reward vector by $B(T, \sigma)$. Then the final reward vector $B(T, \sigma)$ will have the form $B(T, \sigma) = R(\pi_1) - \sum_{r=1}^{m} T(c(a_r), b^r)$, where we have to explain what in each of the m simple cost sharing problems $< N, c(a_r), b^r >$ the maximal contribution vector b^r is. We introduce them in a recursive way. First $b^1 = R(\pi_1) - R(\pi_2)$, $h^1 = T(c(a_1), b^1)$. Then $b^2 = R(\pi_1) - R_1(\pi_3) - h^1$, $h^2 = T(c(a_2), b^2)$. If for $k \in \{3, ..., m\}$, $b^1, b^2, ..., b^{k-1}, h^1, h^2, ..., h^{k-1}$ are determined, then $b^k = R(\pi_1) - R(\pi_{k+1}) - \sum_{r=1}^{k-1} h^r$ with $\pi_{m+1} = \emptyset$, and $h^k = T(c(a_k), b^k)$. Then $B(T, \sigma) = R(\pi_1) - \sum_{r=1}^{m} h^r$.

It follows straightforwardly from

$$\sum_{i \in N} h_i^r = c(a_r), \; 0 \leq h_i^r \leq b_i^r$$

that

$$\sum_{i \in N} B_i(T,\sigma) = \sum_{i \in N} R_i(\pi_1) - \sum_{r=1}^{m} c(a_r) = \sum_{i \in N} R_i(\pi_1) - c(\pi_1),$$

so $B(T,\sigma)$ is a reward distribution of the maximal reward of N; further $0 \leq B_i(T,\sigma) \leq R_i(\pi_1)$.

Concluding, if the players agree about the optimal plan π_1, the simple cost sharing rule F, and the order $a_1, a_2, ..., a_m$ in which the cost shares $c(a_1), c(a_2), ..., c(a_m)$ are determined, then $B(T,\sigma)$ is the resulting reward distribution in the joint project π_1.

To illustrate the procedure we give an example.

Example 3.1. Consider the joint project $< N, A, c, F, (R_i)_{i \in N} >$ where $N = \{1,2\}$, $A = \{a_1, a_2, a_3\}$, $c(a_1) = c(a_2) = c(a_3) = 10$, $F = \{\emptyset, \{a_3\}, \{a_3, a_1\}, \{a_3, a_2\}, \{a_1, a_2, a_3\}\}$ and $R_1(\emptyset) = R_2(\emptyset) = R_1(\{a_3\}) = R_2(\{a_3\}) = 0$, $R_1(\{a_1, a_3\}) = 27$; $R_2(\{a_2, a_3\}) = 16$, $R_2(\{a_1, a_3\}) = R_1(\{a_2, a_3\}) = 0$, $R_1(\{a_1, a_2, a_3\}) = 28$, $R_2(\{a_1, a_2, a_3\}) = 18$. Then this is a project based on the tree $< V, A >$ with $V = \{v_0, v_1, v_2, v_3\}$, and with arcs $a_3 = (v_0, v_1)$, $a_1 = (v_1, v_2)$ and $a_2 = (v_1, v_3)$. Take CEC as rule for handling the simple cost sharing problems and a_1, a_2, a_3 as order of treatment of costs, so $\pi_1 = \{a_1, a_2, a_3\}$, $\pi_2 = \{a_2, a_3\}$ and $\pi_3 = \{a_3\}$. Then $b^1 = R(\pi_1) - R(\pi_2) = (28, 18) - (0, 16) = (28, 2)$. So $h^1 = \mathrm{CEC}(c(a_1), b^1) = \mathrm{CEC}(10, (28, 2)) = (8, 2)$. Then $b^2 = R(\pi_1) - R(\pi_3) - h^1 = (28, 18) - (0, 0) - (8, 2) = (20, 16)$. So $h^2 = \mathrm{CEC}(10, (20, 16)) = (5, 5)$. Then $b^3 = R(\pi_1) - R(\pi_4) - h^1 - h^2 = (15, 11)$, and $h^3 = \mathrm{CEC}(c(a_3), b^3) = (5, 5)$. So $B(\mathrm{CEC}), \sigma) = (28, 18) - (8, 2) - (5, 5) - (5, 5) = (10, 6)$.

Remark. In a tree-based problem first the cost sharing of a leaf of the original tree is solved, then the cost sharing of a leaf of that tree which we obtain from the original tree by removing the treated leaf. In general the resulting reward distribution depends on the order of treating the simple cost sharing problems.

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