INSTITUTE OF MATHEMATICAL ECONOMICS

WORKING PAPERS

No. 338

ZiF Research Group

on

PROCEDURAL APPROACHES TO CONFLICT RESOLUTION

October 2001 - August 2002

Fascicle 10

FUZZY JUDGMENT IN BARGAINING GAMES: Diverse Patterns of Price Determination and Transaction in Buyer-Seller Exchange

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Diverse Patterns of Price Determination and Transaction in Buyer-Seller Exchange

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Paper to be presented at the 13th World Congress of Economics, Lisbon, Portugal, September 9-13. The initial version of this paper was prepared and written while the authors were Research Fellows at the Center for Interdisciplinary Research (ZIF), University of Bielefeld, Germany, 2002. We are grateful to ZIF for this support.

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ABSTRACT

Fuzzy Judgment in Bargaining Games: Diverse Patterns of Price Determination and Transaction in Buyer-Seller Exchange.

Ewa Roszkowska and Tom R. Burns.

This paper draws on fuzzy judgment theory in the description and analysis of buyer-seller bargaining conditions and price determination processes, taking into account players' economic as well as non-economic values. Given the players' initial value (or utility) structures vis-à-vis one another, thirteen (13) distinct situations in their negotiation space can be identified and described formally (geometrically and algebraically), each situation defining a particular negotiation space and a settlement price range. Particular value structures derive from the players' social relationship and operate in two ways on the bargaining process: First, they orient players to, or focus them on, particular zone(s) of the settlement interval, namely those that most closely correspond to or fits the core value(s) of their social relationship. Second, they operate in adapting or transforming the players' goals or aspiration levels in the bargaining game in a manner consistent with their relationship.

1. INTRODUCTION

In the general theory of games (**GGT**) (Burns and Gomolinska, 1998, 2000a, 2000b, Burns et al, 2001; Gomolinska 1999), games are conceptualized in a uniform and general way in terms of the concepts of *rule*[3] and *rule complexes*.[4] A well-specified game at time t is a particular interaction situation where there are general rules for the players and they have well-defined roles (rule complexes) with respect to one another (however, not all games are necessarily well-defined with, for instance, clearly specified and consistent roles and role relationship(s)). A social role is a particular rule complex, serving as the basis of an incumbent's judgments and actions in relation to other players in their roles in the defined game[5].

Role and role relationships provide frames of appropriate rules including values and norms; these are particular ways in which actions are classified, judged, and given "internal" interpretations and meanings (Burns and Flam, 1987). "Non-cooperation" in,

[3] Rules are a type of knowledge (Burns and Flam, 1987; Burns, 1990). An abstract, formal conception of a **rule** may be expressed as follows (Burns and Gomolinska, 1998; Gomolinska, 2002): A rule r in a given language is a triple r=(X, Y, α) where X and Y are finite sets of formulas of the language called the set of premises or conditions and the set of justifications (default provisions or exception conditions), respectively, and where α is a formula called the conclusion of rule r. The latter either provides information, evaluation, or a directive or requirement for action (in this case of a directive, the actor is supposed to implement or perform it). All elements of X **should** hold and all elements of Y **may** hold (in the latter case, an actor presumes that justifications hold on the basis of lack of information to the contrary. When they do not hold, then an "exception" obtains, that is the rule cannot be applied). If the premises obtain and the justifications are not known to not apply, then the actor applies r and α is concluded. If the set of justifications is non-empty, then the rule is in fact a sort of default rule (Reiter's (1980) default logic). If the set of premises and justifications are both empty, then the rule is axiomatic – such rules may represent "facts" and unconditional directives. Axiomatic rules can be viewed as equivalent to their conclusions. Since all formulas can be rewritten in the form of axiomatic rules, the basic (but not atomic!) objects of our conceptual space are just rules. One can distinguish several types of rules, e.g., declarative, prescriptive, proscriptive, evaluative, decision rules, etc.

[4] The motivation behind the development of the concept of rule complex has been to consider repertoires of rules in all their complexity with complex interdependencies among the rules and, hence, to treat them not merely as sets of rules but as entities containing members relating to the relationships among members. The organization of rules in rule complexes provides us with a powerful tool to investigate and describe various sorts of rules with respect to their functions such as values, norms, judgment rules, prescriptive rules, and meta-rules as well as more complex objects consisting of rules such as roles, routines, algorithms, action modalities, models of reality as well as social relationships and games (see later). Informally speaking, a rule complex is a set consisting of rules and/or other rule complexes (see Gomolinska, 2002). More formally, a **rule complex** is a set obtained from rules according to the following formation rules: (1) Any set of rules is a rule complex; (2) If C is a family of rule complexes, then the union over the family C, $\bigcup C$, is a rule complex; (3) the power set P(C) of a rule complex C is a rule complex; (4) If $C \subseteq D$ and D is a rule complex, then C is a rule complex. In words, the class of rule complexes contains all sets of rules, is closed under the union and the power set, and preserves inclusion. Notice that for any rule complex C and a set X, C– X is a rule complex. Similarly, for any non-empty family of rule complexes C, the intersection of the family C, $\bigcap C$ is a rule complex.

[5] The notion of a *situation* is a primitive. S denotes situations with subscripts, if needed. We use the lower case t, possibly with subscripts, to denote points of time or context (or other reference). Thus, S_t denotes a situation at time or context t. Given a concrete situation S at t, a general *game structure* is represented as a particular rule complex **G(t)**. This complex includes roles as rule subcomplexes along with norms and other rules.

for instance, a prisoners' dilemma (PD for short) situation is not merely "defection" in the case that the players are friends or relatives in a solidary relationship, but viewed rather as a form of "disloyalty" or "betrayal" and subject to harsh social judgment and sanction (but the quality and extent of such sanctions may not be fully known beforehand or recognized *ex post* by those subject to them). In the case of enemies, "defection" in the PD game would be fully expected and considered "natural" -- neither shameful nor contemptible, but right and proper damage to the other, and, hence, not a matter of "defection" at all. Such considerations enable one to systematically identify and analyze the symbolic and moral aspects of games associated with particular social relationships and normative contexts.

An player's role is specified in **GGT** in terms of a few basic cognitive and normative components (formalized as mathematical objects in Burns and Gomolinska, 1998, 2000a, 2000b; Burns et al 1998, 2001; Gomolinska 1999). Each role consists of at least the following:

i) a *model* describing the players' "*situational view*" and providing the perspective on, and basis for understanding of, the reality of the interaction when the game is played. It consists of a complex of rules representing players' beliefs about themselves, their environment, interaction conditions and constraints;

ii) a complex of "*values*" consisting of the players' values, goals and commitments. In this complex there are rules assigning values to things and deeds, determining what is "good", "bad", "acceptable", "unacceptable";

iii) a complex of "*actions*" including acts, routines, programs, and strategies which can be used by the players in order to respond or to deal with problems and challenges in the context of the situation; in *open games* (Burns et al, 2001), the players construct and develop strategies as the game goes on, for instance, formulating proposals and counter-proposals.

iv) a "*modality*" complex defining a player's action mode for generating or determining actions. Among the important types of modality are instrumental rationality (the usual rational choice mode), normative orientation, habitual and ritualistic modes of action, and combinations of these.

In a generalized game or interaction situation, the players evaluate and regulate their actions, paying attention to systems of norms and values, and their relationships to one another (as well as to other players). Each player makes judgments about what is best or appropriate to do, or to avoid doing in the particular circumstances.[6]

This paper describes and analyzes buyer-seller exchange on the basis of fuzzy judgment theory, which is one of the core components of **GGT**. The necessary analytical tools are presented in section 2 and applied to buyer-seller bargaining in section 3. The paper specifies and analyzes the diverse patterns of exchange and price determination, arising as a function of the type of bargaining situation and the character of the players' social relationship.

[6] For a discussion of correspondences between these notions and some notions of classical game theory, see Burns et al (2001) and Burns and Roszkowska (2002).

2. CONCEPTUALIZING FUZZY BARGAINING GAMES

In open games, players are able to construct and elaborate strategies and outcomes in the course of their interaction, for instance in market exchange as a bargaining game (Burns et al, 2001). In such games there is a socially constructed "bargaining space" within which there are settlement possibilities varying as a function of the players' particular roles and social relationships in that context.

Consider a buyer **B** and a seller **S** bargaining about the price p of a good or service X (Burns et al, 2001; Burns and Roszkowska, 2002). Seller **S** has a minimum or reserve price $p_S(min)$ and buyer **B** has a maximum price $p_B(max)$ where presumably $p_S(min)<p_B(max)$. We obtain the following spaces for a seller $P_S=[p_S(min),+\infty)$, a buyer $P_B=[0, p_B(max)]$, and the negotiation space for both, $\mathbf{NS}=P_S \cap P_B$. Each also has an operative goal, ambition level, or ideal conception of a "good deal" or possibly a "fair deal" in the particular situation: $p_S(ideal)$ and $p_B(ideal)$, where $p_S(ideal)\in P_S$ and $p_B(ideal)\in P_B$. Determination of these values is based, in part, on what they believe or guess about one another's limits (namely the reserve price of the seller and the value of the buyer) or the determination can be based on past experience (or on some theory, which may or may not be accurate). Typically, these are adjusted as the bargaining process goes on (Burns et al, 1998). The " anchoring points" of ideals and limits in players' value complexes make up a *fuzzy semantic space*, which is basic to the judgment processes that go on in the bargaining. By "fuzzy" we are referring to the vagueness, lack of precision, or roughness of concepts, judgments, and beliefs of players; in addressing these phenomena, we employ fuzzy set methods (Zadeh, 1965, 1973, 1996; Burns and Roszkowska, 2002, Gomolinska, 2002; Nurmi, 1978, 1981, 2000). Based on their fuzzy judgments, *the players propose prices and accept or reject one another's the proposals.* When one accepts the proposal of another, a deal is made. The proposal is the selling or final price.

In bargaining games, for instance in market exchange, the socially constructed "negotiation space" **NS** (with its settlement possibilities) varies as a function of the players' particular needs or limits defined in P_S and P_B as well as the particular social relationship between them in the context of which the bargaining interactions take place. In previous papers (Burns et al, 2001; Burns and Roszkowska, 2002), we assumed that $p_S(min)\leq p_B$ (max) obtained between **S**'s reserve or minimum price $p_S(min)$ and **B**'s maximum price or value $p_B(max)$. Here we relax this restriction and consider all possibilities which can be generated in such games.

GGT formulates models of the judgment process and, in particular, the judgment of similarity or dissimilarity and the ways in which players use approximate reasoning and deal with imprecise information in making decisions, interacting, and negotiating agreements. The conceptualization of fuzzy judgment entails a two process model:

(1) the judgment of similarity and dissimilarity (where threshold functions) are provided;
(2) the judgment of fit or degree of membership formulated as a fuzzy set M taking on values between [0,1].

As we discuss more fully below, the judgment expression J(i,t) for player i at time or context t along with its thresholds is transformed into a fuzzy function M[7]. Function M does two things. First, it normalizes judgment with a minimum (=0) for sufficiently dissimilar and a maximum (=1) for sufficiently similar. Second, when one has less than perfect similarity (or dissimilarity), it distinguishes the degree of fit or membership, which may be based on fuzzy verbal distinctions such as "moderately fitting", "borderline", and "not fitting very well." That is, it represents fuzzy judgments between the maximum and minimum with breakpoints or thresholds based on distinctions such as "moderately fitting" and "borderline", or other semantic distinctions. Rather than judgment being a matter of yes or no, it may express a degree of, for instance, preference, consensus, compliance, rule matching or equilibrium.[8]

In a market bargaining game, the negotiators' *fuzzy judgment functions* concerning price levels can be generated as follows. For seller S, the fuzzy evaluative judgment function, J(S,t), at time or context t is given by the membership function $M_{J(S,t)}$ which may be specified as follows: $M_{J(S,t)}: \Re \rightarrow [0,1]$

$$M_{J(S,t)}(x)=\begin{cases} 1 & \text{for} \quad x \geq p_S(\text{ideal}) \\ c & \text{for} \quad p_S(\text{min}) < x < p_S(\text{ideal}) \\ 0.5 & \text{for} \quad x = p_S(\text{min}) \\ 0 & \text{for} \quad x < p_S(\text{min}) \end{cases}$$

where x denotes an offer (or option) and $c \in (0.5, 1)$.

For player B, the fuzzy evaluative judgment function, J(B,t), at time or context t is given by its membership function $M_{J(B,t)}$ which may be represented as follows: $M_{J(B,t)}: \Re \rightarrow [0,1]$

$$M_{J(B,t)}(y)=\begin{cases} 1 & \text{for} \quad y \leq p_B(\text{ideal}) \\ d & \text{for} \quad p_B(\text{ideal}) < y < p_B(\text{max}) \\ 0.5 & \text{for} \quad y = p_B(\text{max}) \\ 0 & \text{for} \quad y > p_B(\text{max}) \end{cases}$$

where y denotes an offer (or option) and $d \in (0.5, 1)$.

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[7] Definition: A fuzzy set A in a universe of discourse U = {$u_1, u_2, \ldots, u_n$} will be represented by a set of pairs ($M_A(u)$,u) $\forall u \in U$, where M_A: U→ [0,1] is an actor's **judgment of the fit or degree of membership degree of u in A:** from full membership (=1) to full non-membership (=0) through all intermediate values. Consider that one of the objects in the universe U is a rule or standard r. We are interested in judgments about the degree of fit with, or membership degree of, some condition or action x with respect to a norm or value r that is, J(i,t)(x, r). In the case that norms and values as well as actions and outcomes are completely crisp, we have the classical case.

[8] In general, GGT is able to make use of key social concepts which are imprecise and ambiguous: definition of the situation, the game or type of game, role, norm, value, particular types of action such as "cooperation" and "non-cooperation", or "compliance" and "non-compliance".

$M_{J(S,t)}(x)$ (res. $M_{J(B,t)}(y)$) is interpreted as seller's (res. buyer's) "degree of satisfaction" with price for a good X. It can vary from unsatisfactory (when the price is beyond the player's limit, which defines acceptability) to fully satisfactory (in the latter case, the price equalling or exceeding the player's ideal or aspiration level). Each player i is motivated or driven to maximize $M_{J(i,t)}$, where $i \in \{B,S\}$, that is, the measure of the degree of fit or membership in her judgment function incorporating her underlying values and goals. Elsewhere (Burns et al, 2001a), we have shown that any number of positive settlement results may obtain within the bargaining space, defined by the bargainers' limits. Also, their beliefs (or guesses) about one another's limits are important factors in their bargaining behaviour. For instance, if the buyer believes or is led to believe that the seller has a higher reserve price than she actually has, she might be prepared to settle at a price between this estimated level and her own ideal level; this settlement price would be higher than in the case where a more accurate seller reserve price was known to the buyer. A simpler pattern holds, of course, for the seller. The bargaining process entails then players' communications which involve not only proposals and counter-proposals but adjustments of their estimates of one another's limits as well as adjustments of their ambition levels or ideals in the situation (Burns et al, 2001). These adjustments depend on their belief revision processes, the persuasiveness and even bluffs of the players, the time and resource constraints under which each operates (Burns and Gomolinska, 2001). The possibilities are several:

(1) Settlements are unambiguously reached (provided, of course, such an outcome exists) if: $p_S(\text{ideal}) \leq p^* \leq p_B(\text{ideal})$
(2) No settlement or deal is reached, because the offer is unambiguously unacceptable (given the particular limits of one or both players). That is, $M_{J(i,t)} = 0$ for either player: $p^* < p_S(\text{min})$ or $p^* > p_B(\text{max})$.
(3) A price agreement is attainable if the proposal p^* satisfies the following conditions for buyer as well as seller: $p_S(\text{min}) \leq p^* < p_S(\text{ideal})$ and $p_B(\text{ideal}) < p^* \leq p_B(\text{max})$. (Such a settlement may not be possible, but if it is, the agreement would entail a substantial degree of ambiguity. A variety of prices satisfy these conditions and are a function of various contingencies and conditions).
(4) Maximum ambiguity (maximum discontent agreement) obtains at the limits: for the seller, $p^* = p_S(\text{min})$ and for the buyer, $p^* = p_B(\text{max})$.

Remark 1: In such bargaining processes, established social relationships among the players involved guide adjustment processes, the construction of options and the patterns of interaction and outcomes (Burns and Gomolinska, 2001), as we discuss later.

Remark 2: Elsewhere, we have also shown that the particular social relationship – the particular social rules and expectations associated with the relationship – make for greater or lesser deception and communicative distortion, greater or lesser transaction costs, and likelihood of successful bargaining (Burns et al, 1998). The difficulties – and transaction costs – of reaching a settlement are greatest for pure rivals. They would be more likely to risk missing a settlement than pragmatic "egoists." This is because rivals tend to suppress the potential cooperative features of the game situation in favour of pursuing their rivalry. Pure "egoists" are more likely to effectively resolve some of the collective action dilemmas in the bargaining setting in order to achieve an optimal settlement. Friends may exclude bargaining altogether as a precaution against undermining their friendship relationship. Or, if they do choose to conduct business together, their predisposition to self-sacrifice for one another may also make for certain bargaining difficulties (but different from those of rivals) and increased transaction costs in reaching a settlement (Burns et al, 2001a).

Remark 3: Elsewhere (Burns and Gomolinska, 2001; Burns et al, 2001), we have shown that bargainers may try to manipulate what the other believes about their limits. For instance, the sellers convinces the buyer that $p_S(\text{min})$ is much higher than it is, approaching or equalling $p_S(\text{ideal})$. Similarly for the buyer. These processes of persuasion, fabrication, deception, etc. often prolong the bargaining. They may also result in an aborted process.

The anchoring points (the players ideals and limits) in players' value complexes making up the fuzzy judgment function are relatively stable. However, players may change their judgment function, for instance raising or lowering their operative "ideals" in the situation, as we illustrate later. Working out the effects of this is more or less straightforward.

In general, our analysis suggests a spectrum of settlement possibilities in negotiation games -- exactly how wide or narrow the particular space depends on the players' ambition levels and limits. Also, settlements depend in part on the players beliefs or estimates of one another's anchoring points and, in part, on their social relationships which orient and regulate their evaluations and judgments in the game process.

3. APPLICATIONS AND RESULTS

Buyer and sellers operate in a **negotiation space** or zone **NS** in which they make bids and offers respectively. Let us denote by p the *"settlement price of a transaction"*. We should assume that $p>0$. Observe that a transaction obtains when **S's** offer x matches **B**'s bid y, $x=y$. It follows that $p=x=y$. The prices acceptable to both sides of the negotiation are on the diagonal $y=x$. This line is referred to as the **settlement line.**

Our analysis entails consideration of the relationship of the players' value complexes (defining 4 points, that is, two points for each player) in relation to the line of potential settlement in the negotiation space. There is an interval of possible, acceptable solutions to the players (the set may, of course, be empty; see later).

The *"level of satisfaction"* for both negotiators is described by $M_{(S,B)}$ by the following:

$$M_{(S,B)}(p)=\left(M_{J(S,t)}(p), M_{J(B,t)}(p)\right),$$

where $p\in$ **NS**.

Given the players' value complexes, we obtain thirteen (13) cases of potential *"transaction prices"* on the settlement interval which can be described geometrically and by $M_{(S,B)}(p)$ (see representations in the figures of Appendix 1 and also Table 1). In this paper, given that each of the players of the bargaining game operate with an interval (defined by two points, an ideal or goal price and a limit price where the former is assumed greater than the latter), then the two intervals in relation to one another generate 13 possible situations. The GGT models of the situations specify the *potential* transaction prices on the settlement interval. In one of the situations (SIT 1), the players' value complexes are completely incompatible, and there is no settlement interval or point and, therefore, no transaction possibility. In two of the situations (SIT 12,13), there are potentially mutual and fully satisfactory situations in that their respective ideals can be realized in their transactions. In two other situations (SIT 2,3), there are potential settlements above the limits of both but below their goals or ideals and, therefore, not fully satisfying for either. Finally, most of the situations (SIT 4,5,6,7,8,9,10,11) are asymmetric providing a better result (in some cases according to a player's goal) for one of the players but not the other. Our analysis also shows how the social relationship of the players through the relationship's inherent values and meta-values – or a more general normative order applying in the situation S:

(1) orients them to or focuses them on particular zones of the settlement interval that most corresponds to or fits the core value(s) of their relationship;
(2) motivates them to adapt or transform their operative goals or aspiration levels in the bargaining game in a manner consistent with their relationship. This is a particularly important process in situations where the bargaining conditions are problematic or judgment dilemmas arise.

In our application of the theory, we consider two general types of game conditions:

I. Game situations where the players' value complexes are exogenously given and fixed but the players may orient differentially to the potential settlement interval as a function of their social relationship;

II. Situations where the players transform their value complexes, in particular their ideal or aspiration levels in a given game as a function of the particular relationship or normative order applying to the interaction situation, that is the shift or revision of value orientations endogenous to the game process.

Given either exogenously or endogenously determined value complexes, players generate potential "settlement intervals". The latter are a function of the players' value complexes vis-a-vis one another and their concrete social context (including non-market relationships). In the negotiation context where a social relationship is activated, each player i, operates with a value complex, VALUE(i,t)=(p_i(ideal), p_i(limit), v_i, meta-v_i), where $i \in \{B,S\}$; p_i(ideal) is as defined earlier; p_i(limit) is either p_S(min) or p_B(max); v_i and meta-v_i are characteristic of the players' particular social relationship; so, v_i is a value or value orientation defining right and proper actions of i and j vis-à-vis one another and also possibly right and proper outcomes. For instance, the core value orientation v_i of actor i in the case of a solidary relationship with j entails taking the other's desires and needs into account and preferring to share gains (and losses). In a domination relation, the core value v_i characterizing the relationship orients the players toward asymmetry in actions and outcomes. The v_i orientations for enemies are similar to one another in that the players are mutually oriented to causing distress and dis-benefit or harm to the other.

The meta-value, meta-v_i, in VALUE(i,t) orients the players in relation to their judgment situation and to making adjustments and transformations. It operates in two ways:

(1) It orients players to or focuses them on the zone(s) of the settlement interval that most correspond to or fits the core value(s) of the relationship, for example, as articulated in v_i. Thus, solidary negotiators are oriented to those potential prices on the settlement interval that correspond to or realize the value or norm of mutual benefit and gain (or sharing of losses), whereas participants in a relationship of domination are oriented to the asymmetric possibilities on the settlement interval. Such a mechanism operates even in the case of exogenously determined value complexes.

(2). It adapts or transforms value complexes in an interaction situation S_t in a manner consistent with the relationship. For instance, it prioritizes changes in p_i(ideal) according to the degree that the change is judged similar to or fitting v_i. Thus, solidary players adjust or transform p_i(ideal) so that in the concrete situation it expresses or realizes the core value(s) of the relationship, e.g. as articulated in v_i in terms of mutual benefit or justice. Thus, on a general level, the meta-value meta-v_i transforms the value complex, in particular it transforms p_i(ideal)

$$\text{meta-}v_i\text{: } p_i(\text{ideal},t) \rightarrow p_i(\text{ideal}, t+1)$$

This is a major mechanisms operating in the case of endogenously determined value complexes.

The following subsections defines and analyzes the relations between players' value complexes including their ideals or ambition levels, limits, and any meta-value deriving from their social relationship(s), the resultant negotiation situations, and likely bargaining results.

3.1. Exogenous determination of value complexes.

Exogenously determined value complexes are -- for our purposes here -- considered fixed. Observe that for all situations except number 1 (the empty settlement set) and number 13 (convergence on a point), there is an **interval of settlement prices**. The question is how do bargainers determine a final price or limit the region of satisfactory settlement prices. One major factor in limiting the settlement interval is the non-market social relation(s) of the players. A social relationship between players implies the activation of particular values and meta-values that orient their judgments and negotiations within the configuration of their value complexes. As indicated above, the meta-value meta-v_i(t) for player i at time or situation t is a part of an player's **operative value complex** in the situation. It orients the players to a particular region or regions of the settlement interval where the latter is defined by their exogenously determined ideals and limits in their respective value complexes. Through such mechanisms, considerations other than pure market oriented calculation may be activated and influence negotiation judgments and outcomes. For instance, the players see one another as members of a solidary group (family, friendship network, etc.) or as participants in a status or authority relationship; or as rivals or enemies. Evaluative judgments based on these diverse social relationships lead them to react in differentiated but predictable ways to the 13 situations, as we argue below.

For instance, a dominated player in a status or authority relationship would focus on an appropriate part of the interval – defined by the meta-value orienting the player to those potential prices which give priority to **asymmetric** levels of satisfaction or realization: in other words, the satisfaction or realization of the dominant player's ideal should be greater than that for the dominated player. The more or less shared value complex of solidary players would orient them to finding mutually satisfying settlements. Players who are indifferent to one another (for instance, self-interested rationalists) would act pragmatically, accepting a settlement or settlements as long as it (they) do not violate their limits.

There are four types of general patterns discernible in the thirteen (13) distinct situations (see appendix).

(1) The settlement zone is non-empty and is characterized by ideal results for both bargaining agents.

This type of occurrence is found in situations 12 and 13 where the negotiators' ideals converge. The convergence is a point in one case (situation 13), extending to a line in the other (situation 12). The meta-value(s) of the players having a solidary relationship would predispose them to focus on their mutually satisfactory zone of the settlement interval. The upper zone in figure 12 would be particularly satisfying to such players. On the other hand, rivals or enemies are predisposed to reject the opportunities for such satisfactory solutions, although their evaluations in fact were initially convergent. That is, they would be predisposed to **orient away from the region of mutual benefit and satisfaction.** In general, neither mutual gain nor asymmetrical outcomes are satisfactory to players who are rivals or in a hostile social relationship. **Their value orientations drive them rather to divergence than to convergence, thus drawing out the negotiations in time and increasing the likelihood of a breakdown, even causing them to miss or give up opportunities to make certain (mutual) gains with which, in other circumstances, they would be very pleased.** In the case of negotiators involved in a status or authority relationship, they would be predisposed to focus on the zone of the settlement characterized by the highest degree of asymmetry, favouring of course the satisfaction of the dominant player more than that of the dominated player but within the limit defined by the subordinate player's value complex. On the other hand, self-interested players lack by definition social passions, for instance, a deep concern for the other player characteristic of solidary relations, or the competitiveness of rivals or the animosity of enemies. Rational actors would of course bargain to obtain the best possible deal for self but would be prepared to accept any of the settlements near to or on the ideal settlement zone.

(2) The settlement interval contains ideal solutions but asymmetrically.

Situations 4,5,6,7,8,9,10 and 11 are characterized by asymmetry, where 4 and 7 provide ideal settlements only for the buyer, 5 and 6 only for the seller, and where 8,9,10, and 11 provide ideal settlements for both but not in common. In such situations, players with a status or authority relationship are predisposed to focus on the substantial zone of the settlement interval characterized by asymmetry, favouring the satisfaction of the dominant player more than that of the dominated player. Solidary players would try unsuccessfully to identify a zone of prices or price which represents a "fair deal" or as much gain for both as possible, thus minimizing asymmetric possibilities. Rational, self-interested players would bargain to obtain the best possible deal for self but would settle even for an asymmetric result, provided it does not violate their limits. A hostile player or rival would accept inferior settlement prices that also disadvantage the other, for instance any of the asymmetric outcomes disfavoring the other. In general, rivals would each reject asymmetric settlements and, thus, abort the game. Of course, if the player who gains more satisfaction can conceal this from the other, then a settlement might be reached. But rivals as well as enemies typically gain satisfaction from pointing out an asymmetric result which disadvantages the other.

(3) The settlement interval is not empty but it contains no ideal solutions for either of the players.

This is characterized by situations 2 and 3. Neither player can fully realize her ideal but the interval of possible solutions is within their respective limits (maximum price for the buyer and minimum price for the seller). Solidary players might either try to sacrifice for one another, or to find a "fair division," according to some norm or principle of distributive justice appropriate for their relationship. In the case of players in a status or authority relationship, Situation 2 offers no satisfactory pattern, either in terms of the dominant player's ideal settlement or in terms of realizing or satisfying the meta-value of asymmetry which defines in part their relationship. Situation 3 offers partial satisfaction to the dominant player.

(4) The settlement interval or zone is empty.

The maximum of the buyer is less than the minimum of the seller (SIT 1). This is characterized by situation 1 where the rectangle of lines defined by their respective value complexes do not intersect with the settlement line. Given exogenously determined, fixed value complexes, situation 1 offer no openings for settlement, regardless of the social relationship between the players (an exception to this rule is discussed below). On the one hand, the no settlement situation is fully expected or "natural" for a relationship of animosity, where both players are oriented to mutual non-cooperation (–C–C). On the other hand, this situation clashes with the value orientations which characterize a solidary relation (CC) or those which characterize players in a status or authority relationship (C–C or –CC). In these cases, the players would be predisposed to adjust their goals or ideal levels, for instance, the subordinate player would sacrifice by lowering her goal and accepting a worse settlement vis-a-vis the higher status person or authority. This type of adjustment is analyzed in section 3.2 dealing with endogenous formation of value complexes.

This sub-section has considered the role of established relationships, external to the market relation, in limiting or determining the price on a given interval. There are several other possible factors or mechanisms in determining a final price or a more narrow range of settlement prices, for instance:

- Players choose the final price in the middle between two acceptable prices.
- They choose the final price in the optimal interval proportional to the "level of satisfaction" of both players in the negotiation.
- They ask a mediator (or arbiter) to assist in determining a fair procedure and/or fair price (range), or they themselves apply an agreed upon fair division procedure.
- The fuzzy judgment process could be repeated on another level, meaning that both sides agree on the interval prices, and then they construct their fuzzy judgment function once again but in such a way that (p_S(min), p_S(ideal)), (p_B(ideal), p_B(max)) defines an "interval of optimal prices", and then they repeat the procedure.
- They apply other procedures on which both parties agree before initiating negotiation.

Consideration of these extensions of the theory would take us beyond the scope of this paper.

3.2 Endogenous Determination of Value Complexes

In the case of endogenous processes, players adjust their **operative ideal or aspiration level** as a function of their social relationship or the normative context in which they interact. Normally, they maintain stability in their value complexes, because these play a key role in orienting and guiding them in their judgments and actions and in giving them identity. Established value complexes enable them to be predictable and trustworthy. But if, in a given situation, no satisfactory deal obtains in the settlement interval, they are motivated to reconsider and possibly restructure their value complexes - at least for operative purposes. This is done as a function of the meta- value meta-v_i,(t) defined by their social relationship (where $i \in (B,S)$ and t is time or context). That is, the value sub-complex defined by their particular social relationship and activated in the situation operates on the players' value complexes:

$$\text{meta-}v_i\text{: VALUE}(i,t) \rightarrow \text{VALUE}(i,t+1)$$

where $i \in \{B,S\}$ and VALUE(i,t)=(p_i(ideal, t), p_i(limit, t), v_i(t), meta-v_i(t)).

In sum, value sub-complexes derived from non-market social relationships may be activated and influence players' operative value complexes and, therefore, their judgments and, ultimately, market negotiation outcomes. In this subsection, we are interested in the transformation of value complexes, in particular players' ideals or aspiration levels in the given negotiation situation. The following analyses consider different types of social relationship and their impact on players' adjustments and revisions of their operative ideals or goals in the situation.

(1) Solidary relationships and mutual adjustment to construct common ideal levels. Solidary players are predisposed to adjust their "ideal levels" to take one another into account. Genuine mutual cooperativeness (CC) is valued in the relationship and is likely to be generated in a wide range of situations (although there are limits, defining the scope or field of the relationship (Burns et al, 2001). In a bargaining process, they are disposed to adjust their ideal levels closer to one another's limits, whether minimum or maximum levels, as the case may be. Thus, they tend to generate overlapping areas such as in situations 12 and 13. The players are thus likely to find a price or prices which satisfy both of them. One interpretation of this is that each would like to obtain a result that no one of them would experience as a decrease in "level of satisfaction" at the same time the other gains in "level of satisfaction" (similar to Pareto optimal solution).

In general, given a bargaining situation in the context of a solidary relation, the players would be predisposed to mutually adjust their aspiration levels so that they converge or overlap. Such convergence already obtains in situations 12 and 13. In these situations, their ambition levels are fully compatible:

$$p_S(\text{ideal}) \leq p_B(\text{ideal}).$$

Both realize their ideals in situation 12: $p \in (p_S(\text{ideal}), p_B(\text{ideal}))$

and in situation 13, $p = p_S(\text{ideal}) = p_B(\text{ideal})$.

The situations 2,3,4,5,6,7,8,9,10,11 entail partial incompatibility or contradiction. In these cases, solidary players are motivated to adjust their operative ideals, reducing them so as to bring about convergence between the ideal levels (that is, departing substantially from their initial buyer and seller positions). Thus, buyer moves her ideal toward $p_B(\text{max})$ and seller moves hers toward $p_S(\text{min})$, thereby approaching the case of situation 13. They may even sacrifice vis-à-vis one another to such an extent that $p_S(\text{ideal}) < p_B(\text{ideal})$, generating the pattern of situation 12.

Formally, they may transform their individual ideals into a **mutual or collective ideal**:

$$p_S(\text{ideal}), p_B(\text{ideal}) \rightarrow p_{BS}(\text{ideal}).$$

They apply this collective ideal in their deliberations on and determinations of the settlement price. And they find partial realizations, that is "solutions," with respect to their shared or collective ideal, $p_{BS}(\text{ideal})$. This is accomplished within their respective limits, $p_S(\text{min})$ and $p_B(\text{max})$, for instance in the determination of a price that is judged to be "a fair deal for both."

In situation 1, there is an obvious contradiction. If their respective limits are fixed, then there will be no settlement, even given their solidary relationship. Again, they would be predisposed to revise their limits in order to find or create a common zone. Further considerations of such matters would take us beyond the analyses presented in this paper.

(2) **Domination relationships and asymmetrical adjustment of value complexes (ideals).**

Players having a particular status or authority relationship are predisposed in the bargaining game to adjust their ideal levels as follows:

Dominant player: Either she makes no adjustment or she raises her operative ideal in the situation.

Dominated player: She adjusts her ideal level downward to fit the dominant player's expectations or demands (at least up to her own limit). This is a –CC (or, alternatively a C–C) situation, that is, one player "cooperates" by making a sacrifice and accepting a less satisfactory settlement. The dominant player does not cooperate in this way but maintains or increases her ideal (exceptions arise in contexts where norms of *nobless oblige* apply to the dominant player).

In the case that the players have no established status or authority relationship, but have **unequal power** (because of differential knowledge or capabilities, or because one has alternative possibilities and the other does not), they are inclined to **asymmetrically** adjust their operative aspiration levels (and possibly their limits). Thus, a dominated buyer adjusts her operative goal or aspiration level in the situation to accommodate the demand or expectation of the dominant seller:

$$\text{meta-v}_B : p_B(\text{ideal},t) \rightarrow p_B(\text{ideal}, t+1) \approx p_S(\text{ideal}).$$

The settlement price would satisfy the seller more (relative to her initial ideal or aspiration price) that that of the buyer. The same mechanism would operate in the case of a powerful, assertive buyer vis-a-vis a weak seller who is compelled to accommodate. The situation can be seen as involving a parallel mechanism to that described above in the case of a status or authority relationship.

In Situation 1, if the dominated player's limit is fixed, then there will be no resolution, she would refuse to transact. But, of course, under some conditions, she might be prepared to adjust her limit (in self-sacrifice) in order to make possible a transaction, satisfying the demands or expectations of the dominant agent. Such considerations would take us beyond the analyses of this paper.

In other situations, the dominated player negotiates within her initial limit and provides self-sacrificing solutions which realize or approach realizing the ideal of the dominant player. For instance, in the case of a dominant seller, the buyer accedes to her,

$$p \approx p_B(\max).$$

Or, alternatively for a dominant buyer, the seller accedes,

$$p \approx p_S(\min).$$

In the case that the dominated player also adjusts her operative ideal to be closer to her limit (and closer to the ideal of the dominant player), then in the limit:[9]
Dominant seller: $p \approx p_B(max) \approx p_B(ideal) \leq p_S(ideal)$.
Dominant buyer: $p \approx p_S(min) \approx p_S(ideal) \leq p_B(ideal)$.

(3) Relationships of rivalry or hostility and disruptive adjustment of value complexes.
If the players in the situation are hostile to one another (–C–C) – or are rivals – they would be mutually predisposed to increase their ideal or aspiration levels vis-a-vis one another. Each would be oriented to cause distress in the other. The seller would set her $p_S(ideal)$ as high as possible, possibly above $p_B(max)$, and the buyer would set $p_B(ideal)$ as low as possible, even below $p_S(min)$. This might result in Situation 1, or at least in Situations 2 or 3 with a much **narrowed** settlement interval than otherwise would be the case, for instance if they were purely rational actors.

Hostile players would also be predisposed to set their limits close to their ideal or aspiration levels. That is, the seller would tend to increase $p_S(min)$, and the buyer to decrease $p_B(max)$, thus, reducing the potential settlement interval. This can be interpreted as a judgment to refuse to make any sacrifice or suffer any burden whatsoever for the sake of the other or for reaching a common settlement. Under such conditions of antagonism, negotiation would tend to break down entirely. There are limits to these tendencies, having to do with, among other things, the degree of relative importance for the players of making a transaction with one another (that is, the opportunity costs of aborted transactions).

(4) Indifference without adjustment.
If the players in the situation are purely self-interested – without concern or passion – that is, neither has responsibility for or claims on the other (a type of anomie), then they tend to act pragmatically. They are prepared to accept a settlement which is above their minimum (or alternatively maximum) level. In this sense, they **cooperate** in the adjustment process.

Thus, given an anomic relationship among self-interested players, they are predisposed to try to get the best for self but also are prepared to compromise in order to obtain a settlement, at least within the space defined by their limits and their ideals. They would settle in the manner analyzed earlier in terms of exogenously fixed value complexes (with given ideals and limits). **This pattern arises because the players have no compelling social relationship and related value structures inducing or obligating them to alter their value complexes.**

Table 1 summarises the relationships between type of situation, character of the social relationship, and transaction patterns.

[9] Such adjustments reduce the experience of dissonance between the dominated player's aspiration level and her actual exchange conditions (Burns and Gomolinska, 2000).

4. CONCLUSIONS AND FURTHER DEVELOPMENTS

(1) As indicated earlier, one future development would be to consider the judgment processes and procedures which players use in determining a transaction price **within** a settlement interval or zone.

(2) A further consideration is to define and analyze the transformation of value complexes, in particular their limits (that is, maxima or minima).

(3) Processes of persuasion and deception can be modelled and analyzed in that the players operate with models of the situation which are constructions with incomplete and imperfect information (possibly, false information) (Burns and Gomolinska, 2001).

(4) The bargaining process can be analysed further in terms of players' styles of negotiation (for instance, cooperative or competitive negotiation styles). Such styles of negotiations do not always correspond fully to their social relationships. The use of a particular style may depends on the specific issue of negotiation, the context of the situation, personality factors, and so on. Thus, solidary players may find themselves in a negotiation situation where non-cooperative styles are expected or appropriate. Or, conversely, rivals or enemies may find themselves dealing with an issue or interacting in a situation where more cooperative styles are expected by key outside agents or by general norms and laws applying to the situation. In a bargaining situation where both negotiators use competitive styles and aim to outdo the other, the likelihood of agreement would be small except in situation 2 (price $p=p_B(max)=p_S(min)$) and situation 13 price $p=p_B(ideal)=p_S(ideal)$, where equality of outcomes obtain. Similarly, they may also negotiate and arrive at a settlement in situation 12 (where "equality" in terms of each realizing her ideal obtains in the settlement interval). In the other situations, no agreement would be acceptable to both, since the outcomes are asymmetric.

When one of the negotiators consistently uses the competitive style, and the other a cooperative one, agreement would tend to be at the limits, namely price $p=p_B(max)$ or $p=p_S(min)$. When both negotiators follow cooperative styles, the interval of acceptable prices arises – as characterized in this paper in a number of our cases -- and the question remains of determining the final settlement price.

(5) Finally, the model of fuzzy membership function has a very simple form in this paper -- only four levels of satisfaction: full, partially, minimum, and unsatisfactory. One may readily extend the analysis to consider functions with more levels of satisfaction, on one side, or with more complicated analytical forms, on the other side. Thus, in a certain sense, the model presented here is a starting point to analyze bargaining situations, where membership functions have many levels of satisfaction or take on more complex forms.

Table 1. Relationships between Type of Situation, Character of the Social Relationship, and Transaction patterns.

	CC (Solidary Relation)	**-CC/C-C (Domination Relationship)**	**-C-C (Relation of Animosity or Rivalry**	**OO (Indifferent or Anomic)**
Potentially mutually satisfactory situations SIT 12, 13	Price determination and transaction take place within the configuration of the players' value complexes. No change expected in value complexes	The dominated player adjusts her value complex, generating one of the asymmetric situations, SIT 8, 9, or 10. Price determination and transaction takes place within these resultant configurations.	Players transform value complexes, generating SITs 1,2, or 3 with a narrowed (or in SIT 1 no) settlement interval. Any price negotiation and transactions takes place within the configuration of value complexes defining situations 2 or 3.	Price determination and transaction. No change expected in value complex
Asymmetric situations providing a much better result for one of the players than the other: SIT 4, 5, 6, 7, 8, 9, 10, 11	The players adjust their value complexes to construct a collective or common ideal. And they determine a price which is as close to equally satisfying for both as possible.	Price determination and transaction within the configuration of value complexes. The dominated player operates in the region of the settlement interval that generates appropriate asymmetric outcome, consistent with the relationship but within her limit. There is no tendency on the players' part to transform their value complexes[10]	Low likelihood of a settlement. Transaction possible if the player who gains more satisfaction manages to conceal this from the other. But rivals typically gain satisfaction from revealing the accomplishment of a better result.	Transaction, after bargaining, although differences occur, of course, in the levels of satisfaction.
Situations without fully satisfactory patterns for either player: SIT 2, 3	Transaction. The players adjust their value complexes in order to construct a collective or common ideal. And then determine a price for both which is equally satisfying.	The dominated player operates with the region of the settlement interval that generates appropriate asymmetric outcome, that is consistent with the relationship. But SIT 2 is highly problematic since there is no asymmetry. This is alien to players in a genuine relation of domination. And SIT 3 is problematic for either a dominant buyer or a dominant seller.	Low likelihood of a settlement. But transaction possible within the limits. But rivals typically gain satisfaction from revealing the accomplishment of a better result.	Transaction after bargaining
Incompatible value complexes, SIT 1	No transaction, in the preliminary assessment. However, the players are predisposed to adjust limits in their value complexes, so as to construct situations such as SITs 2,3 6, 7, or 11	No transaction. However, the dominated player is predisposed (and expected) to adjust her limit resulting in situations such as SITs 4,5,6,7	No transaction and no tendency to adjust limits of their value complexes. SIT 1 fits – is a "natural" expression of – their relationship	No transaction and no tendency to adjust limits, other things being equal.

[10] SIT,4,5 6,7 are problematic for one or the other player (as SITs 2 and 3 are not fully satisfactory). For instance SIT 4 and 6 are problematic for a dominant seller, and SIT 5 and SIT 7 are problematic for a dominant buyer. (And situations 2 and 3 are problematic for either a dominant buyer or dominant seller).

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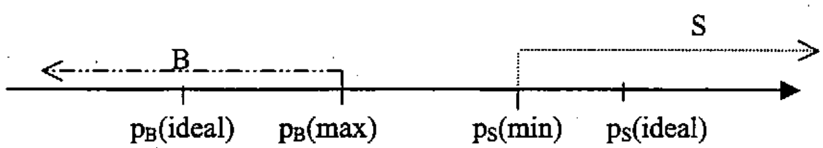
APPENDIX 1:

Symbols:

B - Buyer
S - Seller
p_B(ideal) - ideal price for the Buyer
p_B(max) - maximal price for the Buyer
p_S(min) - minimal price for the Seller
p_S(ideal) - ideal price for the Seller
NS – bargaining space
$M_{(S,B)}$ -level of satisfaction for both negotiators

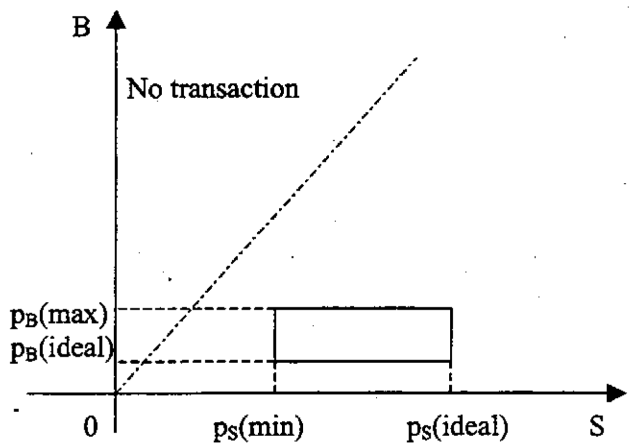
Situation 1.

$p_B(ideal)<p_B(max)<p_S(min)<p_S(ideal)$



NS=∅.No se ttlement interval or point. No transaction.

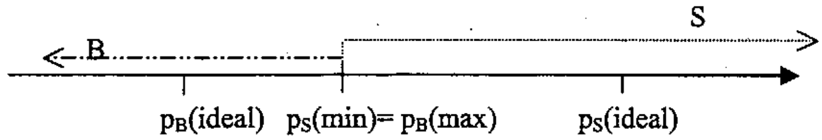
Figure 1.



No transaction. $M_{(S,B)}(p)$ cannot be generated.

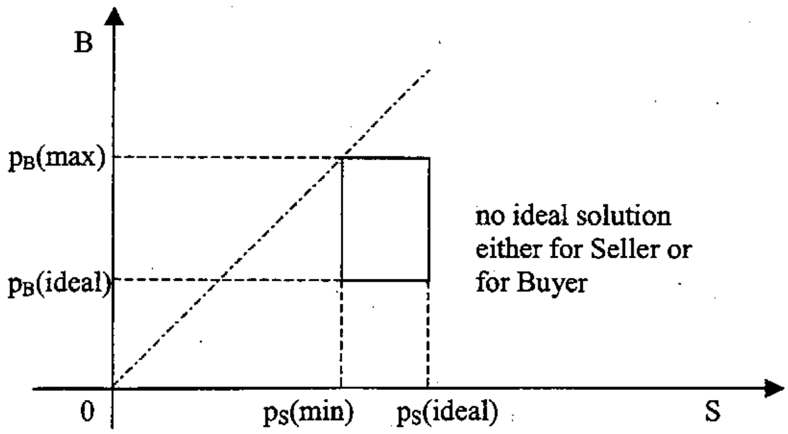
Situation 2.

$p_S(ideal) < p_S(min) = p_B(max) < p_S(ideal)$



NS= $\{p = p_S(min) = p_B(max)\}$. This point is the only possible transaction.

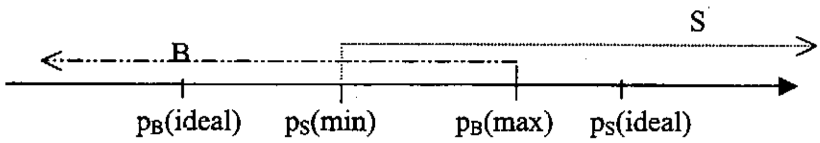
Figure 2.



$M_{(S,B)}(p) = (0.5;\ 0.5) \quad \text{if} \quad p = p_S(min) = p_B(max)$

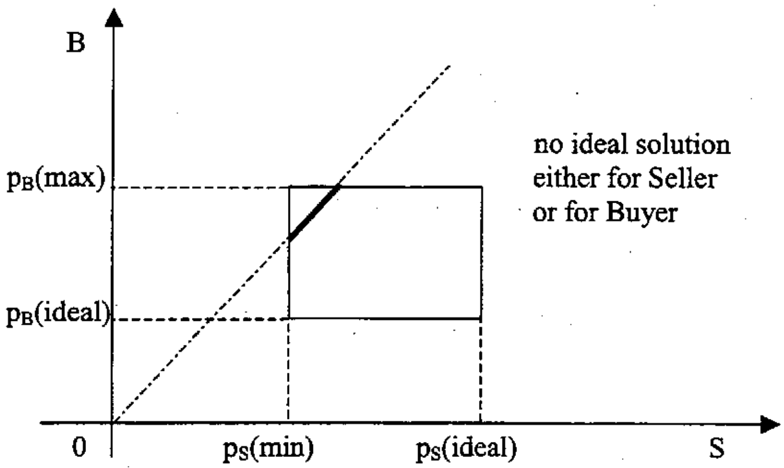
Situation 3.

$p_B(\text{ideal}) < p_S(\text{min}) < p_B(\text{max}) < p_S(\text{ideal})$



NS=$[p_S(\text{min}), p_B(\text{max})]$

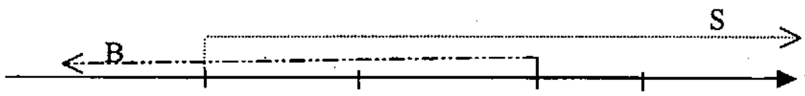
Figure 3.



$$M_{(S,B)}(p)=\begin{cases}(c; 0.5) & \text{if } p = p_B(\text{max}) \\ (c; d) & \text{if } p \in (p_S(\text{min}), p_B(\text{max})) \\ (0.5; d) & \text{if } p = p_S(\text{min})\end{cases}$$

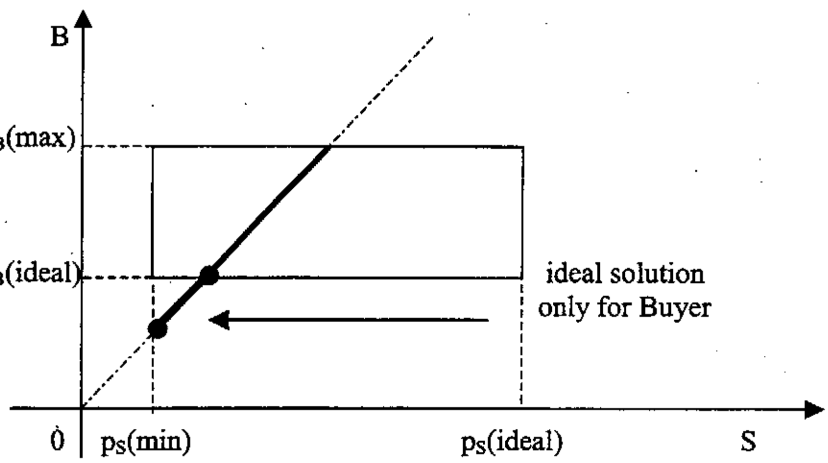
Situation 4.

$p_S(min)<p_B(ideal)<p_B(max)< p_S(ideal)$



NS=[$p_S(min)$, $p_B(max)$]

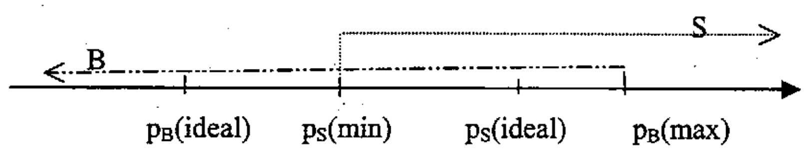
Figure 4.



$$M_{(S,B)}(p)=\begin{cases} (c;0.5) & \text{if} \quad p=p_B(max) \\ (c;d) & \text{if} \quad p\in(p_B(ideal),p_B(max)) \\ (c;1) & \text{if} \quad p\in(p_S(min),p_B(ideal)] \\ (0.5;1) & \text{if} \quad p=p_S(min) \end{cases}$$

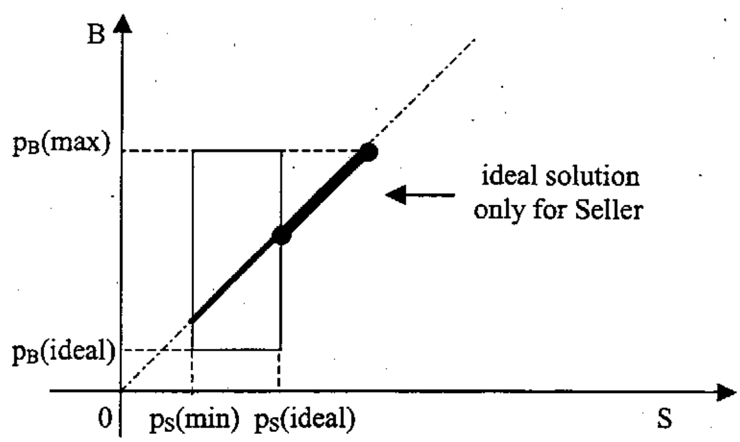
Situation 5.

$p_B(\text{ideal}) < p_S(\text{min}) < p_S(\text{ideal}) < p_B(\text{max})$



NS=$[p_S(\text{min}), p_B(\text{max})]$

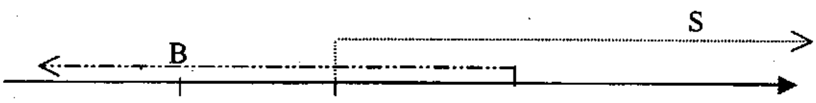
Figure 5.



$$M_{(S,B)}(p)=\begin{cases}(1;0.5) & \text{if } p=p_B(\text{max})\\ (1;d) & \text{if } p\in[p_S(\text{ideal}),p_B(\text{max}))\\ (c;d) & \text{if } p\in(p_S(\text{min}),p_S(\text{ideal}))\\ (0.5;d) & \text{if } p=p_S(\text{min})\end{cases}$$

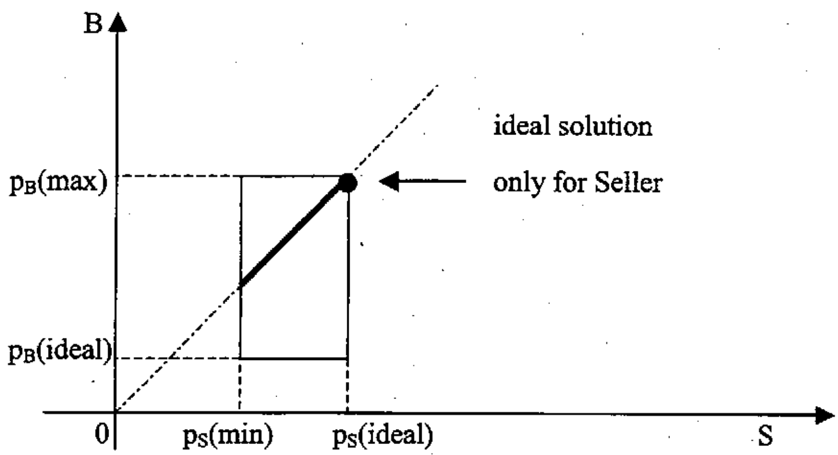
Situation 6.

$p_B(ideal) < p_S(min) < p_S(ideal) = p_B(max)$



NS= $[p_S(min), p_B(max)] = [p_S(min), p_S(ideal)]$

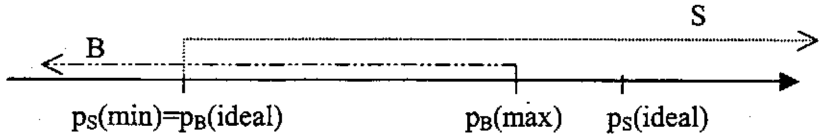
Figure 6.



$$M_{(S,B)}(p) = \begin{cases} (1; 0.5) & \text{if } p = p_B(max) = p_S(ideal) \\ (c; d) & \text{if } p \in (p_S(min), p_S(ideal)) \\ (0.5; d) & \text{if } p = p_S(min) \end{cases}$$

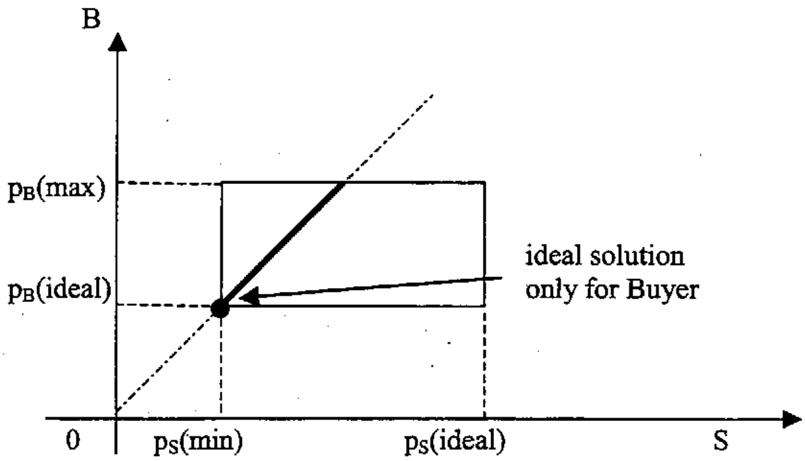
Situation 7.

$p_S(min)=p_B(ideal)<p_B(max)<p_S(ideal)$



NS=$[p_S(min), p_B(max)]=[p_B(ideal), p_B(max)]$

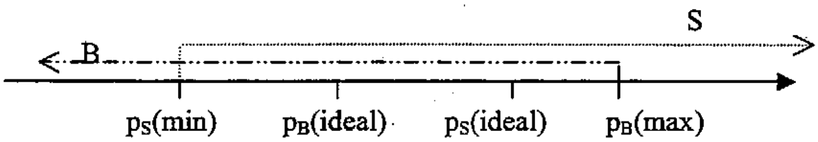
Figure 7.



$$M_{(S,B)}(p)=\begin{cases}(c;0.5) & \text{if } p=p_B(max)\\ (c;d) & \text{if } p\in(p_S(min),p_B(max))=(p_B(ideal),p_B(max))\\ (0.5;1) & \text{if } p=p_S(min)=p_B(ideal)\end{cases}$$

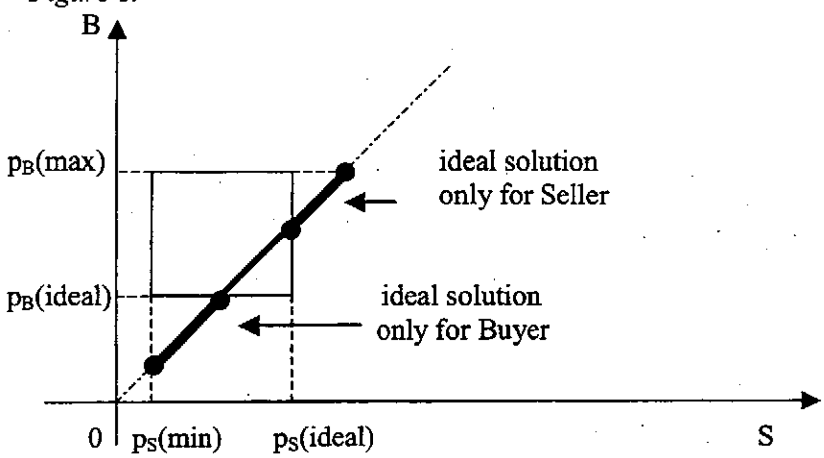
Situation 8.

$p_S(min)<p_B(ideal)< p_S(ideal)<p_B(max)$



NS=[$p_S(min)$, $p_B(max)$]

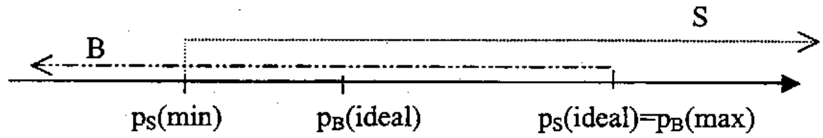
Figure 8.



$$M_{(S,B)}(p)=\begin{cases}(1;0.5) & \text{if } p=p_B(max)\\ (1;d) & \text{if } p\in[p_S(ideal),p_B(max))\\ (c;d) & \text{if } p\in(p_B(ideal),p_S(ideal))\\ (c;1) & \text{if } p\in(p_S(min),p_B(ideal)]\\ (0.5;1) & \text{if } p=p_S(min)\end{cases}$$

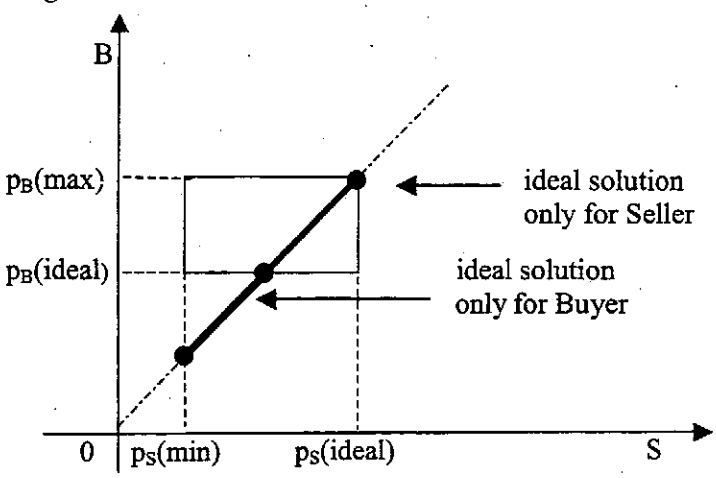
Situation 9.

$p_S(min) < p_B(ideal) < p_S(ideal) = p_B(max)$



NS=$[p_S(min), p_B(max)] = [p_S(min), p_S(ideal)]$

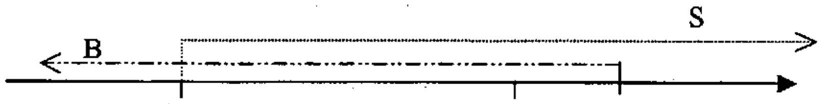
Figure 9.



$$M_{(S,B)}(p) = \begin{cases} (1;0.5) & \text{if } p = p_B(max) = p_S(ideal) \\ (c;d) & \text{if } p \in (p_B(ideal),\ p_S(ideal)) \\ (c;1) & \text{if } p \in (p_S(min), p_B(ideal)] \\ (0.5;1) & \text{if } p = p_S(min) \end{cases}$$

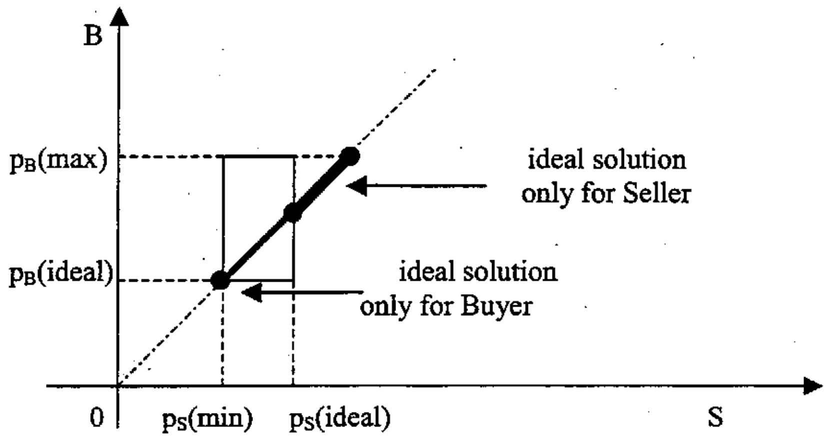
Situation 10.

$p_S(min)=p_B(ideal)< p_S(ideal) < p_B(max)$



NS=$[p_S(min), p_B(max)]=[p_B(ideal), p_B(max)]$

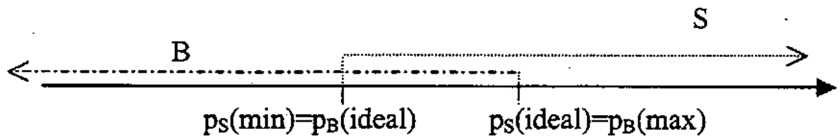
Figure 10.



$$M_{(S,B)}(p)=\begin{cases}(1;0.5) & \text{if } p=p_B(max)\\ (1;d) & \text{if } p\in[p_S(ideal),p_B(max))\\ (c;d) & \text{if } p\in(p_S(min),p_S(ideal))=(p_B(ideal),p_S(ideal))\\ (0.5;1) & \text{if } p=p_S(min)=p_B(ideal)\end{cases}.$$

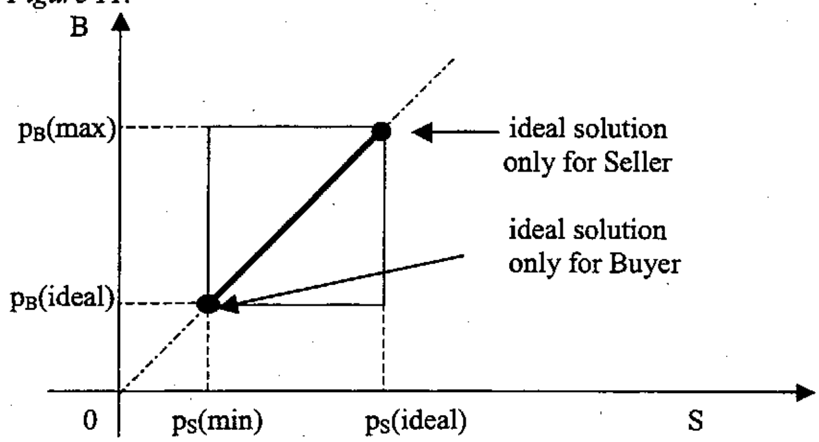
Situation 11.

$p_S(min)=p_B(ideal)<p_S(ideal)=p_B(max)$



NS=$[p_S(min), p_S(ideal)]=[p_B(ideal), p_B(max)]$

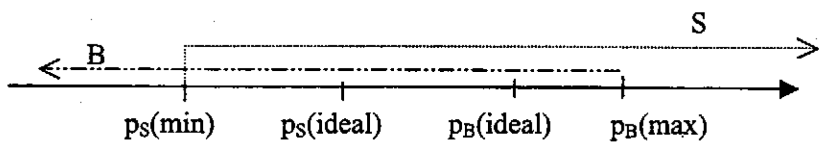
Figure 11.



$$M_{(S,B)}(p)=\begin{cases}(1;0.5) & \text{if } p=p_S(ideal)=p_B(max)\\(c;d) & \text{if } p\in(p_S(min),\ p_S(ideal))=(p_B(ideal),p_B(max))\\(0.5;1) & \text{if } p=p_S(min)=p_B(ideal)\end{cases}$$

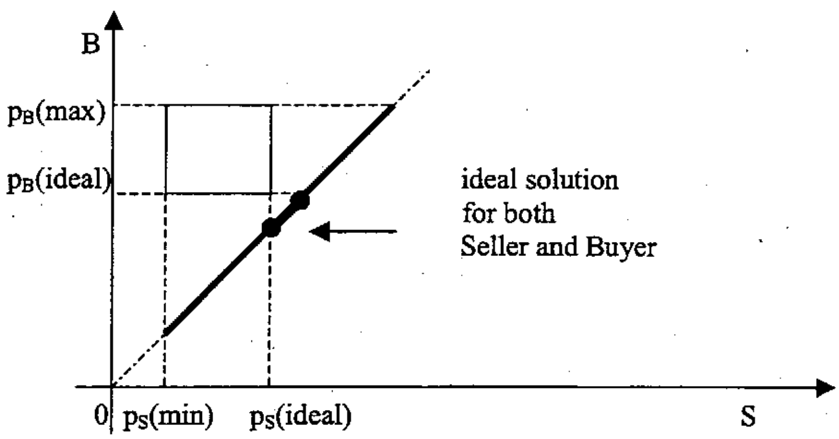
Situation 12.

$p_S(min) < p_S(ideal) < p_B(ideal) < p_B(max)$



NS=$[p_S(min), p_B(max)]$

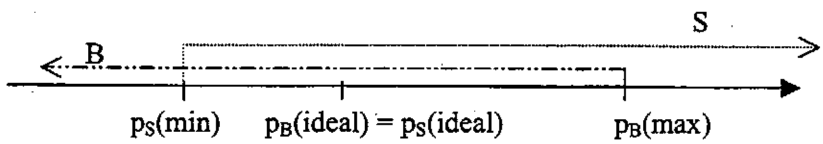
Figure 12.



$$M_{(S,B)}(p)=\begin{cases}(1;\ 0.5) & \text{if } p = p_B(\max)\\ (1;\ d) & \text{if } p\in(p_B(\text{ideal}),p_B(\max))\\ (1;\ 1) & \text{if } p\in[p_S(\text{ideal}),p_B(\text{ideal})]\\ (c;\ 1) & \text{if } p\in(p_S(\min),p_S(\text{ideal}))\\ (0.5;\ 1) & \text{if } p = p_S(\min)\end{cases}$$

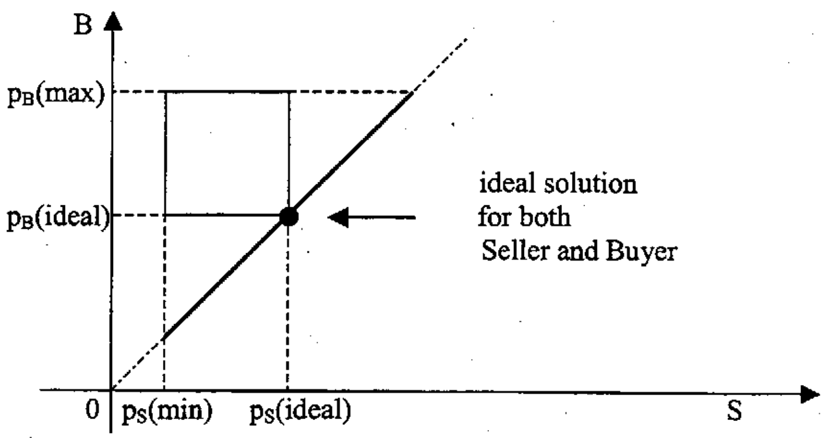
Situation 13.

$p_S(min) < p_B(ideal) = p_S(ideal) < p_B(max)$



NS=[p_S(min), p_B(max)]

Figure 13.



$$M_{(S,B)}(p)=\begin{cases}(1;0.5) & \text{if } p=p_B(\max)\\ (1;d) & \text{if } p\in(p_S(\text{ideal}),p_B(\max))=(p_B(\text{ideal}),p_B(\max))\\ (1;1) & \text{if } p=p_S(\text{ideal})=p_B(\text{ideal})\\ (c;1) & \text{if } p\in(p_S(\min),p_S(\text{ideal}))=(p_S(\min),p_B(\text{ideal}))\\ (0.5;1) & \text{if } p=p_S(\min)\end{cases}$$