INSTITUTE OF MATHEMATICAL ECONOMICS

WORKING PAPERS

No. 347

A Combinatorial Topological Theorem and Its Application

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Abstract. We prove the following combinatorial topological theorem: Let m and n be any positive integers with $m \leq n$ and let $T(n;1,2) = \{x \in \mathbb{R}^n_+ \mid \sum_{h=1}^n x_h = t, \text{ for } 1 \leq t \leq 2\}$ be a subset of the n-dimensional Euclidean space $\mathbb{R}^n$. For every $i = 1, \cdots, m$, there is a class $\{C_i^j \mid j = 1, \cdots, n\}$ of subsets C_i^j of $T(n;1,2)$. If C_i^j is closed for all i, j; and if for every $i = 1, \cdots, m$, and every subset I of the set $\{1, 2, \cdots, n\}$, the set $F(I) = \{x \in T(n;1,2) \mid \sum_{h \in I} x_h = \sum_{h=1}^n x_h\}$ is a subset of the union of the sets C_i^j, $j \in I$, then there exists a connected subset $\mathcal{C}$ of the set $T(n;1,2)$ such that there exist $x, y \in \mathcal{C}$ with $\sum_{h=1}^n x_h = 1$ and $\sum_{h=1}^n y_h = 2$, and for every $x \in \mathcal{C}$, there exists a partition $\Pi = (\Pi(1), \Pi(2), \cdots, \Pi(m))$ of the set $\{1, 2, \cdots, n\}$ so that $\Pi(i) \neq \emptyset$ for every i and

$$x \in \cap_{i=1}^m \cap_{j \in \Pi(i)} C_i^j.$$

We prove this theorem based upon a generalization of Birkhoff-von Neumann theorem and a theorem of continuum of zero points of Herings, Talman and Yang. This new result gives a substantial generalization of the well-known lemma of Knaster, Kuratowski, and Mazurkiewicz (KKM Lemma) in combinatorial topology. In addition, we give an economic application of this new result which solves a multi-person collective combinatorial optimization problem.

Key Words. KKM lemma, combinatorial topological theorem, multi-person decision, combinatorial optimization.

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[1]This research is partially supported by the Alexander von Humboldt Foundation.

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1 Introduction

Let n be any positive integer. Given a positive real number t, let $S(n;t) = \{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i = t\}$ be a subset of the non-negative orthant $\mathbb{R}^n_+$ of the n-dimensional Euclidean space $\mathbb{R}^n$. For any subset I of the set $\{1, \cdots, n\}$, let $F(I) = \{x \in S(n;1) \mid \sum_{i \in I} x_i = 1\}$. The well-known lemma of Knaster, Kuratowski, and Mazurkiewicz [11] states that if C^1, C^2, $\cdots$, C^n are closed subsets of $S(n;1)$, and if, for each subset I of the set $\{1, 2, \cdots, n\}$, $F(I)$ is a subset of the union of the sets C^i, $i \in I$, then these n sets have a nonempty intersection. The same conclusion holds when for every point x in $S(n;1)$, $x_i = 0$ implies $x \in C^i$. The latter result is due to Scarf [16]. Brouwer's fixed point theorem can be immediately derived by using KKM Lemma. Let f be a continuous mapping from $S(n;1)$ into itself. For each i, let $C^i = \{x \in S(n;1) \mid f_i(x) \leq x_i\}$. Clearly, all the conditions of KKM Lemma are satisfied. Thus, there exists a point $x^* \in S(n;1)$ such that

$$f_i(x^*) \leq x_i^*, \forall i.$$

Obviously, $f(x^*) = x^*$.

The KKM Lemma is probably one of the most elegant results in combinatorial topology and along with Sperner's lemma [19] has also widely been known in the fields of mathematical programming and economic theory due to the successful computation of economic equilibria and fixed points; see Scarf [16]. Many results have evolved out of the KKM Lemma and Sperner's lemma and found successful applications in the fields of equilibrium theory, game theory, and optimization theory; see Tucker [21], Fan [2], Scarf [16], Shapley [18], van der Laan and Talman [12], Gale [8], Freund [4, 5], van der Laan, Talman, and Van der Heyden [13], Yamamoto [22], Bapat [1], Zhou [25], van der Laan, Talman, and Yang [14, 15], Herings and Talman [9], Yang [23, 24] among others.

In this article we will establish the following combinatorial topological theorem: Let m and n be any positive integers with $m \leq n$ and let $T(n;1,2) = \{x \in \mathbb{R}^n_+ \mid \sum_{h=1}^n x_h = t, \text{ for } 1 \leq t \leq 2\}$. For every $i = 1, \cdots, m$, there is a class $\{C_i^j \mid j = 1, \cdots, n\}$ of subsets C_i^j of $T(n;1,2)$. If C_i^j is closed for all i, j; and if for every $i = 1, \cdots, m$, and every subset I of the set $\{1, 2, \cdots, n\}$, the set $F(I) = \{x \in T(n;1,2) \mid \sum_{h \in I} x_h = \sum_{h=1}^n x_h\}$ is a subset of the union of the sets C_i^j, $j \in I$, then there exists a connected subset $\mathcal{C}$ of the set $T(n;1,2)$ such that $S(n;1) \cap \mathcal{C} \neq \emptyset$, $S(n;2) \cap \mathcal{C} \neq \emptyset$, and for every $x \in \mathcal{C}$, there exists a partition $\Pi = (\Pi(1), \Pi(2), \cdots, \Pi(m))$ of the set $\{1, 2, \cdots, n\}$ so that $\Pi(i) \neq \emptyset$ for every i and

$$x \in \cap_{i=1}^m \cap_{j \in \Pi(i)} C_i^j.$$

Notice that the partition Π depends on the point x. We prove this theorem based upon a generalization of Birkhoff-von Neumann theorem and a theorem of continuum of zero points of Herings, Talman and Yang [10]. This new result gives a substantial generalization of the KKM Lemma. In addition, we give an economic application of this new result which solves a multi-person collective combinatorial optimization problem.

This article is organized as follows. In Section 2 we present our main results. Finally, we apply these results to solve a multi-person collective combinatorial optimization problem in Section 3.

2 Main Results

We first introduce some notation. Let I_k be the set of the first k positive integers and $\mathbb{R}^k$ the k-dimensional Euclidean space. Given a finite set K, $|K|$ denotes the number of elements in K. The vectors 0^k and 1^k represent the vectors of $0's$ and $1's$ in $\mathbb{R}^k$, respectively. For any $x, y \in \mathbb{R}^k$, $< x, y >$ stands for the inner product of x and y. For any two subsets A and B in $\mathbb{R}^k$, $A \subset B$ means that A is a proper subset of B, and $A \subseteq B$ means that either $A \subset B$ or $A = B$. For each $i \in I_k$, $e^k(i)$ denotes the i-th unit vector in $\mathbb{R}^k$. For two positive numbers t_1 and t_2 with $t_1 \leq t_2$ and a positive integer k, define

$$T(k; t_1, t_2) = \{x \in \mathbb{R}^k_+ \mid \sum_{h \in I_k} x_h = t, \quad \text{for} \quad t_1 \leq t \leq t_2\}.$$

We recall two results. The first lemma is recently introduced by Yang [24] and might be seen as a generalization of Birkhoff-von Neumann theorem; see Schrijver [17].

Lemma 2.1 *For two positive integers m and n with $m \leq n$, if an $m \times n$ matrix $U = [u(i,j)]$ satisfies the following system of linear equations*

$$\begin{aligned} &\textstyle\sum_{j=1}^{n} u(i,j) = \frac{1}{m}, \ \forall i \in I_m; \\ &\textstyle\sum_{i=1}^{m} u(i,j) = \frac{1}{n}, \ \forall j \in I_n; \\ &u(i,j) \geq 0, \ \forall i \in I_m, \forall j \in I_n, \end{aligned}$$

then there exists a partition $(\Pi(1), \cdots, \Pi(m))$ of the elements of I_n such that for every $i \in I_m$, $\Pi(i) \neq \emptyset$ and

$$u(i,j) > 0, \ \forall j \in \Pi(i).$$

The following theorem of continuum of zero points is due to Herings, Talman and Yang [10]. To state their result, consider an arbitrary nonempty simple polytope P in $\mathbb{R}^n$ that has the following representation and has no redundant constraint:

$$P = \{x \in \mathbb{R}^n \mid < a^i, x > \leq b_i, \forall i \in K; \ < c^j, x > = d_j, \forall j \in L\},$$

where $|K| \geq n+1$ and $|L| \leq n$. For each subset I of K, define

$$F(I) = \{x \in P \mid < a^i, x > = b_i, \forall i \in I\}.$$

Then $F(I)$ is called a *face* of P unless it is empty. Note that $F(\emptyset) = P$. Let

$$\mathcal{I} = \{I \subseteq I_m \mid F(I) \text{ is a face of } P\}.$$

The polytope P is said to be *simple* if the dimension of any face $F(I)$ of P is equal to $n - |I| - |L|$.

Let c be an arbitrary nonzero vector in $\mathbb{R}^n$. Then F^+ will denote the face of P such that for each $x^+ \in F^+$ it holds that $< c, x^+ >= \max_{x \in P} < c, x >$, and F^- will denote the face of P such that for each $x^- \in F^-$ it holds that $< c, x^- >= \min_{x \in P} < c, x >$. For each $I \in \mathcal{I}$, we define

$$\begin{aligned} A(I) = \{y \in \mathbb{R}^n \mid & \; y = \textstyle\sum_{i \in I} \mu_i a^i + \sum_{j \in L} \beta_j c^j + \gamma c, \\ & \; \mu_i \geq 0, \forall i \in I, \; \beta_j \in \mathbb{R}, \forall j \in L, \text{ and } \gamma \in \mathbb{R}\}, \\ A^*(I) = \{x \in \mathbb{R}^n \mid & < x, y > \leq 0, \; \forall y \in A(I)\}. \end{aligned}$$

Theorem 2.2 *Let $\varphi : P \Longrightarrow \mathbb{R}^n$ be any nonempty valued, compact valued, convex valued and upper semicontinuous correspondence. If for any $I \in \mathcal{I}$ and any $x \in F(I)$, it holds that*

$$\varphi(x) \cap A^*(I) \neq \emptyset,$$

then there exists a connected set C of zero points of φ such that $C \cap F^- \neq \emptyset$ and $C \cap F^+ \neq \emptyset$.

Based upon the above theorems, we will be able to prove the following existence theorem.

Theorem 2.3 *Let m and n be any positive integers with $m \leq n$. For every $i \in I_m$, there is a class $\{C_i^j \mid j \in I_n\}$ of subsets C_i^j of $T(n; 1, 2)$. If C_i^j is closed for all i, j, and if for every $i \in I_m$, and every $I \subseteq I_n$, the set $F(I) = \{x \in T(n; 1, 2) \mid \sum_{h \in I} x_h = \sum_{h \in I_n} x_h\}$ is a subset of the union of the sets C_i^j, $j \in I$, then there exists a connected subset C of the set $T(n; 1, 2)$ such that $S(n; 1) \cap C \neq \emptyset$, $S(n; 2) \cap C \neq \emptyset$, and for every $x \in C$, there exists a partition $\Pi = (\Pi(1), \Pi(2), \cdots, \Pi(m))$ of the set I_n so that $\Pi(i) \neq \emptyset$ for every i and*

$$x \in \cap_{i=1}^m \cap_{j \in \Pi(i)} C_i^j.$$

Proof: Let $S(m; 1) = \{x \in \mathbb{R}_+^m \mid \sum_{h=1}^m x_h = 1\}$. Define $C^i = \{x \in S(m; 1) \mid x_i \geq 1/m\}$ for each $i \in I_m$. Let V denote the set $S(m; 1) \times T(n; 1, 2)$. For each $(i, j) \in I_m \times I_n$, define

$$C^{(i,j)} = C^i \times C_i^j.$$

Clearly, $C^{(i,j)}$ is a closed and nonempty subset of $S(m; 1) \times T(n; 1, 2)$. Moreover, the collection of sets $\{ C^{(i,j)} \mid (i, j) \in I_m \times I_n \}$ has *the covering property* that for every $(x_1, x_2) \in V$ it holds $(x_1, x_2) \in C^{(i,j)}$ for some (i, j) with $x_{1,i} > 0$ and $x_{2,j} > 0$.

For each $i \in I_m$, let p^i denote the vector $1^m/m - e^m(i)$ in $\mathbb{R}^m$ and for each $j \in I_n$, let q^j denote the vector $1^n/n - e^n(j)$ in $\mathbb{R}^n$. Note that for any i and any j, $\sum_{h\in I_m} p^i_h = 0$ and $\sum_{h\in I_n} q^j_h = 0$. For each $(i,j) \in I_m \times I_n$, define the vector $c^{(i,j)} \in \mathbb{R}^m \times \mathbb{R}^n$ by

$$c^{(i,j)} = (p^i, q^j).$$

Now define the point-to-set mapping φ from V to the collection of subsets of $\mathbb{R}^m \times \mathbb{R}^n$ as

$$\varphi(x_1, x_2) = Conv(\{ c^{(i,j)} \mid (x_1, x_2) \in C^{(i,j)}\}),$$

where $Conv(D)$ denotes the convex hull of a set D. It is easy to check that φ is upper semi-continuous. Furthermore, for every $(x_1, x_2) \in V$, the set $\varphi(x_1, x_2)$ is nonempty, convex and compact.

The set V can be rewritten as

$$\begin{aligned} V = \{ (x_1, x_2) \in \mathbb{R}^m \times \mathbb{R}^n \quad | \quad & < x_1, -e^m(i) > \leq 0, \ \forall i \in I_m \\ & < x_1, 1^m > = 1 \\ & < x_2, -e^n(j) > \leq 0, \ \forall j \in I_n \\ & < x_2, -1^n > \leq -1 \\ & < x_2, 1^n > \leq 2 \ \} \end{aligned}$$

We call $< x_2, -1^n > \leq -1$ the $(n+1)$th inequality w.r.t. x_2 and $< x_2, 1^n > \leq 2$ the $(n+2)$th inequality w.r.t. x_2. For each $I \subseteq I_m$ and each $J \subseteq I_{n+2}$, let

$$\begin{aligned} F(I,J) = \{(x_1, x_2) \in V \quad | \quad & < x_1, -e^m(i) > \leq 0, \ \text{if } i \in I \\ & < x_2, -e^n(j) > \leq 0, \ \text{if } i \in J \cap I_n \\ & < x_2, -1^n > \leq -1, \ \text{if } \ n+1 \in J \\ & < x_2, 1^n > \leq 2, \ \text{if } \ n+2 \in J\} \end{aligned}$$

If $F(I,J)$ is not empty, then it is a face of V. Obviously, $F(\emptyset, \emptyset) = V$. Let $\mathcal{I} = \{(I,J) \mid F(I,J) \neq \emptyset\}$. Notice that if $(I,J) \in \mathcal{I}$, then I_m cannot be a subset of I, neither I_n can be a subset of J, nor $\{n+1, n+2\}$ can be a subset of J.

Let $c = (0^m, 1^n)$. Then we have

$$F^+ = \{ (x_1, x_2) \in V \mid \sum_{j\in I_n} x_{2,j} = 2 \} \text{ and } \quad F^- = \{ (x_1, x_2) \in V \mid \sum_{j\in I_n} x_{2,j} = 1 \}.$$

For each $(I,J) \in \mathcal{I}$, we have

$$\begin{aligned} A(I,J) = \{z \in \mathbb{R}^m \times \mathbb{R}^n \quad | \quad & z = \textstyle\sum_{i\in I} \alpha_i(-e^m(i), 0^n) \\ & + \textstyle\sum_{j\in J\cap I_m} \beta(j)(0^m, -e^n(j)) \\ & + \textstyle\sum_{n+1\in J} \beta(n+1)(0^m, -1^n) \\ & + \textstyle\sum_{n+2\in J} \beta(n+2)(0^m, 1^n) \\ & + \gamma_1(1^m, 0^n) + \gamma_2(0^m, 1^n) \\ & \alpha(i) \geq 0, \ \forall i \in I, \ \beta(j) \geq 0, \ \forall j \in J, \\ & \gamma_1, \gamma_2 \in \mathbb{R}\} \quad . \end{aligned}$$

and $A^*(I,J) = \{x \in \mathbb{R}^m \times \mathbb{R}^n \mid < x, z > \leq 0, \ \ \forall z \in A(I,J)\}$.

We will show that for any $(I,J) \in \mathcal{I}$ and any $(x_1, x_2) \in F(I,J)$, it holds

$$\varphi(x_1, x_2) \cap A^*(I,J) \neq \emptyset.$$

For any point (x_1, x_2) in the relative interior of V, i.e., $(x_1, x_2) \in F(I,J)$ with $(I,J) = \emptyset$, we have $\varphi(x_1, x_2) \subseteq A^*(\emptyset)$ since $< \gamma_1(0^m, 1^n) + \gamma_2(1^m, 0^n), z >= 0$ for any $z \in \phi(x_1, x_2)$ and any numbers γ_1, γ_2.

Now we consider the points on the boundary of the set V. Take any $(I,J) \in \mathcal{I}$ so that $F(I,J)$ is a proper face of the set V. Then at least one of the sets I and J is not empty. Take any $(x_1, x_2) \in F(I,J)$. Then we have $x_{1,i} = 0$ for every $i \in I$ and $x_{2,j} = 0$ for every $j \in J \cap I_n$. Then the covering property implies that there exists some $(i^*, j^*) \in I_m \times I_n$ such that $(x_1, x_2) \in C^{(i^*, j^*)}$ with $x_{1,i^*} > 0$ and $x_{2,j^*} > 0$. Clearly, $(p^{i^*}, q^{j^*}) \in \varphi(x_1, x_2)$. Notice that $i^* \notin I$ and $j^* \notin J$. It is easy to check that

$$< p^{i^*}, -e^m(i) >= -\frac{1}{m} < 0, \ \forall i \in I;$$

$$< p^{i^*}, 1^m >= 0;$$

$$< q^{j^*}, -e^n(j) >= -\frac{1}{n} < 0, \ \forall j \in J \cap I_n;$$

$$< q^{j^*}, -1^n >= 0, \ \text{if} \ n+1 \in J;$$

$$< q^{j^*}, 1^n >= 0, \ \text{if} \ \ n+2 \in J;$$

$$< q^{j^*}, 1^n >= 0.$$

Take any $z \in A(I,J)$. By using the above six inequalities, we have $< z, (p^{i^*}, q^{j^*}) > \leq 0$. This implies that $(p^{i^*}, q^{j^*}) \in A^*(I,J)$ and so $\varphi(x_1, x_2) \cap A^*(I,J) \neq \emptyset$.

So, we have shown that all the conditions of Theorem 2.2 are satisfied. Then there exists a connected set H of zero points of φ such that $H \cap F^- \neq \emptyset$ and $H \cap F^+ \neq \emptyset$. Let $C = \{x_2 \mid (x_1, x_2) \in H\}$. Then we know that C is a connected subset of $T(n; 1, 2)$ with $C \cap S(n; 1) \neq \emptyset$ and $C \cap S(n; 2) \neq \emptyset$.

For each $(x_1^*, x_2^*) \in H$, we have that

$$(0^m, 0^n) \in \varphi(x_1^*, x_2^*).$$

Let the collection $\mathcal{L}$ of elements of $I_m \times I_n$ be defined by

$$\mathcal{L} = \{L = (i,j) \in I_m \times I_n \mid (x_1^*, x_2^*) \in C^L\}.$$

Suppose that $\mathcal{L} = \{L^1, \cdots, L^l\}$, where $L^k = (i^k, j^k)$ for $k = 1, \cdots, l$. Then there exist nonnegative numbers μ_k, $k \in I_l$ so that $\sum_{k \in I_l} \mu_k = 1$ and

$$\sum_{k=1}^{l} \mu_k c^{L^k} = (0^m, 0^n). \tag{2.1}$$

It follows from (2.1) that

$$\sum_{(i,j)\in\mathcal{L}} \mu_{(i,j)}(p^i, q^j) = (0^m, 0^n)$$

and that

$$\sum_{(i,j)\in\mathcal{L}} \mu_{(i,j)} = 1$$

for certain $\mu_{(i,j)} \geq 0$ for $(i,j) \in \mathcal{L}$. If $(i,j) \notin \mathcal{L}$, let $\mu_{(i,j)} = 0$. It follows that the system

$$\begin{array}{l} \sum_{(i,j)\in I_m\times I_n} \mu_{(i,j)}(p^i, q^j) = \mathbf{0} \\ \sum_{(i,j)\in I_m\times I_n} \mu_{(i,j)} = 1 \\ \mu_{i,j} \geq 0, \ \forall (i,j) \in I_m \times I_n \end{array}$$

has solutions. This system implies that for each $i \in I_m$,

$$\sum_j \mu_{(i,j)} = 1/m$$

and for each $j \in I_n$,

$$\sum_i \mu_{(i,j)} = 1/n.$$

From this property it follows that the $m \times n$ matrix U with entries $\mu_{(i,j)}$ satisfies the conditions of Lemma 2.1. So, there exists a partition $\Pi = (\Pi(1), \Pi(2), \cdots, \Pi(m))$ of the set I_n such that $\Pi(i) \neq \emptyset$ for every $i \in I_m$ and

$$\mu_{(i,j)} > 0, \forall j \in \Pi(j).$$

This implies that for every $i \in I_m$,

$$(i,j) \in \mathcal{L}, \forall j \in \Pi(i).$$

Since $(x_1^*, x_2^*) \in \cap_{h=1}^{l} C^{L^h}$, we have

$$(x_1^*, x_2^*) \in \cap_{i\in I_m} \cap_{j\in\Pi(i)} C^i \times C_i^j.$$

This implies that

$$x_2^* \in \cap_{i=1}^{m} \cap_{j\in\Pi(i)} C_i^j.$$

This completes the proof. $\square$

It is worth pointing out that Theorem 2.2 was actually proved in a constructive way. This implies that the connected set of solutions in the above theorem can be also computed. From a practical point of view, this is quite useful in real life applications; see the next section. The above theorem implies the following three special cases. For $m = 1$, we have:

Theorem 2.4 *Let n be any positive integer. If C^1, C^2, $\cdots$, C^n are closed subsets of $T(n;1,2)$, and if for every subset I of the set I_n, the set $F(I) = \{x \in T(n;1,2) \mid \sum_{h\in I} x_h = \sum_{h\in I_n} x_h\}$ is a subset of the union of the sets C^j, $j \in I$, then there exists a connected subset $\mathcal{C}$ of the set $T(n;1,2)$ such that $S(n;1)\cap \mathcal{C} \neq \emptyset$, $S(n;2)\cap \mathcal{C} \neq \emptyset$, and for every $x \in \mathcal{C}$, it holds that*

$$x \in \cap_{j=1}^{n} C^j.$$

For $m = n$, we have:

Theorem 2.5 *Let n be any positive integer. For every $i \in I_n$, there is a class $\{C_i^j \mid j \in I_n\}$ of subsets C_i^j of $T(n;1,2)$. If C_i^j is closed for all i, j, and if for every $i \in I_n$, and every subset I of I_n, the set $F(I) = \{x \in T(n;1,2) \mid \sum_{h\in I} x_h = \sum_{h\in I_n} x_h\}$ is a subset of the union of the sets C_i^j, $j \in I$, then there exists a connected subset $\mathcal{C}$ of the set $T(n;1,2)$ such that $S(n;1)\cap \mathcal{C} \neq \emptyset$, $S(n;2)\cap \mathcal{C} \neq \emptyset$, and for every $x \in \mathcal{C}$, there exists a permutation $\pi = (\pi(1), \pi(2), \cdots, \pi(n))$ of $(1, 2, \cdots, n)$ so that*

$$x \in \cap_{i=1}^{n} C_i^{\pi(i)}.$$

For a fixed t, say $t = 1$, we have:

Theorem 2.6 *Let m and n be any positive integers with $m \leq n$. For each $i \in I_m$, there is a class $\{C_i^j \mid j \in I_n\}$ of subsets C_i^j of $S(n;1)$. If C_i^j is closed for all i, j, and if for every $i \in I_m$, and every $I \subseteq I_n$, the set $F(I) = \{x \in S(n;1) \mid \sum_{h\in I} x_h = 1\}$ is a subset of the union of the sets C_i^j, $j \in I$, then there exists a partition $\Pi = (\Pi(1), \Pi(2), \cdots, \Pi(m))$ of the set I_n so that $\Pi(i) \neq \emptyset$ for every i and*

$$\cap_{i=1}^{m} \cap_{j\in\Pi(i)} C_i^j \neq \emptyset.$$

For $m = 1$, the above theorem implies the KKM Lemma, and for $m = n$, the above theorem implies the intersection results of Gale [8] and van der Laan et al. [14]. We can strengthen the boundary condition of Theorem 2.3 so that the intersection points always lie in the interior of $S(n;t)$ for $1 \leq t \leq 2$.

Theorem 2.7 *Let m and n be any positive integers with $m \leq n$. For every $i \in I_m$, there is a class $\{C_i^j \mid j \in I_n\}$ of subsets C_i^j of $T(n;1,2)$. If C_i^j is closed for all i,j; and if for every $i \in I_m$, we have $\cup_{j\in I_n} C_i^j = T(n;1,2)$ so that for every $x \in T(n;1,2)$, $x_j = 0$ implies $x \notin C_i^j$, then there exists a connected subset $\mathcal{C}$ of the set $T(n;1,2)$ such that $S(n;1) \cap \mathcal{C} \neq \emptyset$, $S(n;2) \cap \mathcal{C} \neq \emptyset$, and for every $x \in \mathcal{C}$, there exists a partition $\Pi = (\Pi(1), \Pi(2), \cdots, \Pi(m))$ of the set I_n so that $\Pi(i) \neq \emptyset$ for every i and*

$$x \in \cap_{i=1}^{n} \cap_{j\in\Pi(i)} C_i^j.$$

Now we will derive a generalization of Brouwer's fixed point theorem with a continuum of partition based well-behaved inequalities.

Theorem 2.8 *Let m and n be any two positive integers with $m \leq n$. For every $i \in I_m$, let $f^i : T(n;1,2) \mapsto \mathbf{R}^n$ be a continuous function such that for every $1 \leq t \leq 2$, $x \in S(n;t)$ implies $f^i(x) \in S(n;t)$. Then there exists a connected subset $\mathcal{C}$ of the set $T(n;1,2)$ such that $S(n;1) \cap \mathcal{C} \neq \emptyset$, $S(n;2) \cap \mathcal{C} \neq \emptyset$, and for every $x \in \mathcal{C}$, there exists a partition $\Pi = (\Pi(1), \Pi(2), \cdots, \Pi(m))$ of the set I_n so that for every $i \in I_m$, $\Pi(i) \neq \emptyset$ and*

$$f_j^i(x) \geq x_j, \ \ \forall j \in \Pi(i).$$

Proof: For each $(i,j) \in I_m \times I_n$, define

$$C_i^j = \{x \in T(n;1,2) \mid f_j^i(x) \geq x_j\}.$$

Clearly, the conditions of Theorem 2.3 are satisfied. The conclusion follows. □

3 A Fair Voluntary Job Assignment Problem

Consider a factory or a company in which there are a number of workers and jobs. These jobs have to be done by workers. All workers are able to do the jobs but they differ in the extent to which they like or dislike the jobs and how much compensation is associated with each job. The manager has to consider who should do which job at what level of compensation in such a way that every worker has incentive to do his job and also feels he is being treated equally among his colleagues, and furthermore the whole arrangement should be efficient. Such problems arise naturally in many situations. Related problems have been studied before; see for example, Foley [3], Svensson [20], Fujishige, Katoh, and Ichimori [7], Fujishige [6], and Yang [24].

Formally, the economic model can be described as follows. There are n workers and n jobs. The set of workers will be denoted by I_n and the set of jobs denoted also by I_n. Each worker $i \in I_n$ has preferences over jobs and compensations. This preference relation can be represented by a utility function $u_i : N \times \mathbb{R} \mapsto \mathbb{R}$ where $N = I_n \cup \{0\}$ and 0 represents a dummy job. It is natural to assume that $u_i(j, m)$ is a continuous and nondecreasing function in money (m) for each given $j \in N$. Here money will be treated as a perfectly divisible good. A function $x : I_n \mapsto \mathbb{R}_+$ will be called *a compensation scheme.* Any permutation π of jobs is *an assignment of jobs. A feasible assignment* consists of an assignment π of jobs and a compensation scheme x. At the feasible assignment (π, x), worker i is assigned to job $\pi(i)$ with compensation $x_{\pi(i)}$. A feasible assignment (π, x) satisfies *the participation incentive constraint* if $u_i(\pi(i), x_{\pi(i)}) > u_i(0,0)$ for every $i \in I_n$. This condition says that it is better for every worker to take his job than not to participate at all. In other words, every worker has incentive to do his job because if he does not do any job, there will be no income (i.e., compensation) for him. This also means every worker is voluntarily taking his job. A feasible assignment (π, x) is *envy-free* if for every worker $i \in I_n$,

$$u_i(\pi(i), x_{\pi(i)}) \geq u_i(\pi(j), x_{\pi(j)}), \ \forall j \in I_n.$$

This condition says that every worker is being treated equally and does not envy any other worker, and that every worker gets what he likes best. A feasible assignment (π, x) is *efficient* if there exists no other feasible assignment (ρ, y) such that $u_i(\rho(i), y_{\rho(i)}) \geq u_i(\pi(i), x_{\pi(i)})$ for all $i \in I_n$ and $u_j(\rho(j), y_{\rho(j)}) > u_j(\pi(j), x_{\rho(j)})$ for some $j \in I_n$ and $\sum_{i \in I_n} x_i = \sum_{i \in I_n} y_i$.

A feasible assignment (π, x) is called *a fair voluntary job assignment* if it is efficient, envy-free and satisfies the participation incentive constraint. Clearly, the fair voluntary job assignment problem is a multi-person collective (mixed) combinatorial optimization problem.

Let M_1 and M_2 be positive numbers with $M_1 < M_2$ and let $T(n; M_1, M_2) = \{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i = t, \text{ for } M_1 \leq t \leq M_2 \}$. For the existence of fair voluntary job assignment we impose the following conditions.

Assumption 3.1 *For any $(i, j) \in I_n \times I_n$ and any $x \in T(n; M_1, M_2)$, it holds that*

$$u_i(j, x_j) \leq \max_{k \in I_n} u_i(k, x_k) \quad \text{if } x_j = 0.$$

This assumption simply states that no worker would like to do any job if there is no compensation.

Assumption 3.2 *For every worker i, there exists a positive number $\bar{m}_i$ such that*

$$u_i(j, \bar{m}_i) > u_i(0,0) \quad \text{for all } j \in I_n.$$

This assumption states that for every worker i, if the compensation exceeds or is equal to the minimum value $\bar{m}_i$, it is better for him to take a job than not to do any job.

Assumption 3.3 *The amount M_1 of money is so big that for every worker $i \in I_n$, it holds that $M_1 \geq n \times \bar{m}_i$ and $u_i(j, \frac{M_1}{n}) > u_i(k, \bar{m}_i)$ for all $j, k \in I_n$, where the numbers $\bar{m}_i$, $i \in I_n$, are stated in the previous assumption.*

This assumption states that for any worker, his minimum compensation requirement is no greater than the average compensation and his welfare of taking any job with the average compensation is no worse than that of taking any job with his minimum compensation requirement.

By applying Theorem 2.5, we will be able to prove the following existence theorem. This theorem states the existence of a continuum of fair voluntary job assignments and says that if the total amount t of compensation money is growing continuously, the manager can continuously and gradually adjust the compensation with each job so that there exists a feasible assignment associated with the adjusted compensations which will constitute a fair voluntary job assignment.

Theorem 3.4 *If Assumptions 3.1, 3.2, and 3.3 are satisfied, then there exists a connected subset $\mathcal{C}$ of the set $T(n; M_1, M_2)$ such that $S(n; M_1) \cap \mathcal{C} \neq \emptyset$, $S(n; M_2) \cap \mathcal{C} \neq \emptyset$, and for every $x \in \mathcal{C}$, there exists a permutation π of jobs I_n so that (π, x) is a fair voluntary job assignment.*

Proof: We proceed in three steps. First we prove there exists a connected set of envy-free assignments and then we prove every envy-free assignment satisfies the participation incentive constraint, and finally show that every envy-free assignment is efficient. To prove the first part we only need to verify that Theorem 2.5 in Section 2 can be applied here. For each $(i, j) \in I_n \times I_n$, define $C_i^j = \{x \in T(n; M_1, M_2) \mid u_i(j, x_j) \geq u_i(k, x_k), \forall k \in I_n\}$. Clearly, all the assumptions in Theorem 2.5 are satisfied by Assumption 3.1 made here.

It follows from Theorem 2.5 that there exists a connected subset $\mathcal{C}$ of the set $T(n; M_1, M_2)$ such that $S(n; M_1) \cap \mathcal{C} \neq \emptyset$, $S(n; M_2) \cap \mathcal{C} \neq \emptyset$, and for each $x \in \mathcal{C}$, there exists a permutation π of the set I_n so that

$$x \in \cap_{i \in I_n} C_i^{\pi(i)}.$$

So, for every $i \in I_n$, it holds

$$u_i(\pi(i), x_{\pi(i)}) \geq u_i(\pi(j), x_{\pi(j)}), \ \forall\, j \in I_n \tag{3.2}$$

It is readily seen that (π, x) is an envy-free assignment. Suppose that (π, x) does not satisfy the participation incentive constraint. There would exist some worker i with $x_{\pi(i)} < \bar{m}_i$. Since

$$\sum_{h \in I_n} x_h \geq M_1 \geq n \times \bar{m}_i,$$

there exists some $j \in I_n$ such that $x_j > \frac{M_1}{n}$. The weak monotonicity and Assumption 3.3 imply that

$$u_i(j, x_j) \geq u_i(j, \frac{M_1}{n}) > u_i(\pi(i), \bar{m}_i) \geq u_i(\pi(i), x_{\pi(i)}).$$

This contradicts the fact that (π, x) is envy-free. So we have $x_{\pi(i)} \geq \bar{m}_i$ for every $i \in I_n$. Assumption 3.2 immediately implies that (π, x) satisfies the participation incentive constraint.

It remains to show that (π, x) is efficient. Now suppose to the contrary that (π, x) is not efficient. Then there would exist another feasible allocation (ρ, y) with $\sum_{i \in I_n} x_i = \sum_{i \in I_n} y_i$ such that for every $i \in I_n$, it holds

$$u_i(\rho(i), y_{\rho(i)}) \geq u_i(\pi(i), x_{\pi(i)}); \tag{3.3}$$

and there is some $j \in I_n$ satisfying

$$u_j(\rho(j), y_{\rho(j)}) > u_j(\pi(j), x_{\pi(j)}). \tag{3.4}$$

The definition of C_i^j together with (3.2), (3.3) and (3.4) implies that for all $i \in I_n$,

$$u_i(\rho(i), y_{\rho(i)}) \geq u_i(\rho(i), x_{\rho(i)})$$

and

$$u_j(\rho(j), y_{\rho(j)}) > u_j(\rho(j), x_{\rho(j)}).$$

Since $u_i(j, m)$, $i \in I_n$, are continuous and nondecreasing in money (m) for each given j, we have that for all $i \in I_n$, $y_{\rho(i)} \geq x_{\rho(i)}$, and $y_{\rho(j)} > x_{\rho(j)}$. This implies that $\sum_{j=1}^n y_j > \sum_{j=1}^n x_j$, yielding a contradiction. Therefore, (π, x) must be efficient. In conclusion, (π, x) is a fair voluntary job assignment. □

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