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Bid and Ask Prices in Hierarchically Structured Economies with Two Commodities

by

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[1] This paper grew out of Chapter 8 of Spanjers (1992).

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Abstract

In this paper a model of a pure exchange economy with two commodities is analyzed in which the agents are organized as a hierarchical structure which consists of a set of agents and a set of asymmetric bilateral trade relationships between agents. We assume that the hierarchical structure can be represented by a tree and that there exists exactly one agent who is not dominated in any hierarchical relationship he is part of.

Each hierarchical relationship between two agents is interpreted as a trade relationship in which the dominating agent sets bid and ask prices for the two commodities in the economy and the dominated agent acts as a price taker with respect to this relationship.

The existence of an equilibrium in this economy is proven by use of backward induction. Two illustrative examples are provided.

Finally, we state a simple process according to which price formation and trade in the economy may take place. This exchange process relies on the possibility to write enforcable I-Owe-Yous in the economy.

1 Introduction

The general equilibrium model, as introduced by Walras (1874) and formulated by Arrow and Debreu (1954) and Debreu (1959), is one of the fundamental models in economics. It is a model in which decentralized selfish decision making at equilibrium prices leads to outcomes which are efficient for the economy as a whole, as is stated in the first theorem of welfare economics. Here, we are concerned with two weaknesses of the model. Firstly, all agents in the economy, with exception of an artificial auctioneer who strictly speaking is not an agent, are assumed to act as price takers, without any agent setting the prices. Secondly, it is assumed that each agent can trade with every other agent in the economy through "the market", an abstract institution where supply and demand of commodities meet and trade takes place. Thus, the problem of specifying and analyzing trade relationships and communication structures that may give rise to such a market is not addressed.

In dropping the assumption of price taking agents, one encounters the wider problem of integrating imperfect competition in general equilibrium models. In describing the behaviour of economic agents in general equilibrium models with imperfect competition, it seems appropriate to make explicit what an agent anticipates to be the consequences of (a change in) his actions. In Negishi (1961) it is assumed that some firms anticipate the prices for some of their inputs and outputs to be a linear function of the current state of the economy and the change in their demand for inputs and supply of outputs. In Hahn (1978) and Gale (1978), the notion of rational conjectures is introduced which is shown to be too strong to be of any use. In Vind (1983) the concept of conjectures is generalized through the model of equilibrium with coordination. Here agents may have veto power over changes in actions planned by others that would make them worse off. Once again, the existence of equilibrium is a problem. Interestingly, Vind introduces a formal concept of exchange institutions and provides a simple example in which the anticipations of the agents, called expectation functions, are derived from the set of exchange institutions in the economy.

Recently, a lot of attention is paid to models that specify the trade and communication structure of markets. An important line of research investigates dynamic models in which agents meet pairwise and then trade according to specified trade rules. The agents typically meet often, either in a prespecified order or at random. Eventually, some equilibrium allocation is approached. This line of research is initiated by Feldman (1973) and developed further by, amongst others, Rubinstein and Wollinski (1985). In the context of exchange economies with a continuum of agents we refer to Gale (1986) and, more recently, McLennan and Sonnenschein (1991). In a more restrictive setting, McAffee (1993) allows individual sellers to choose the transaction mechanism they want to use. Thus, what are called the transaction institutions in the market arise endogenously. A different approach to modelling communication in markets is sketched by Kirman (1983), who uses stochastic graph theory in analyzing communication structures.

Economic models with fixed restrictive communication structures have focussed on theories of spatial economics and theories on intermediaries. In the first part of Karmann (1981) a spatial general equilibrium model is given in which transportation technologies play an important role. These transportation technologies can be interpreted as describing the user's costs of exchange institutions that take the form of competitive markets. Models on intermediaries are mostly partial equilibrium models. We refer to Krelle (1976) for models of successive monopolies and vertical intergration. More recently, models of successive monopolies in the context of pure exchange economies have been analyzed in Spanjers (1992). Finally, in Grodal and Vind (1989) competitive markets are modelled as exchange institutions with prices.

The aim of our research is to introduce models in which exchange no longer takes place in markets are abstract aggregated institutions that veil an underlying organizational structures. We analyze models, with exogenously given structures of bilateral trade relationships, their specific trade rules, and compatible knowledge structures. The trade rules governing the bilateral trade relationships may change from relationship to relationship and are formalized by the institutional characteristic of the trade relationship under consideration. The institutional characteristics specify the messages agents send through the given trade relationship. Thus, our approach of specifying the messages sent over (bilateral) trade relationships is along the lines of modelling institutions as mechanisms as, e.g., discussed in Hurwicz (1989). A significant difference, however, is that we use "partial" mechanisms in specifying institutional characteristics of trade relationships. Thus, we "decentralize" the mechanism for the economy as a whole in several partial mechanisms which are independent with respect to their messages to be sent, as specified by the trade rules. Agents, however, may participate in a number of partial mechanisms. Therefore, the anticipations of such an agent concerning the reactions of the other agents participating in

these partial mechanisms on a change in his actions establish the interaction of the partial mechanisms. Thus, the anticipations play a crucial role in establishing the interaction of the partial mechanisms in an economy and their effect on the actions of the agents.

A disturbing problem in this kind of models is that of the existence of equilibrium. In Spanjers (1992, Chapter 6), a model is considered in which each agent has at most one leader and the structure of asymmetric trade relationships can be represented by a (weakly) connected directed graph, i.e., every agent can potentially exchange commodities with any other agent, either directly or through a chain of intermediaries. This kind of hierarchical structure is called a hierarchical tree. The institutional characteristic of the trade relationships is that of mono price setting, i.e., the dominating agent setting a vector of price at which the dominated agent can buy or sell any amount he wants. Thus, generalized models of successive monopolies are analyzed in the context of pure exchange economies with a finite number of commodities. Unfortunately, it in this kind of model equilibria may not exist, the problem being that the sole agent who gets no prices set by another agent may not be able to set a vector of prices that leaves him with a consumption bundle whitout negative components. The essence of the problem is that this agent does not have a "no-trade option". The problem is illustrated by a three agents, two commodity example.

One way to get results on the existence of equilibrium is by making different restrictive assumptions on the structure of trade relationships in the economy. In Spanjers (1994), this is achieved by assuming that the structure of trade relationships contains sufficiently many potential possibilities for arbitrage. We say arbitrage may take place when an agent has at least two leaders setting different vectors of prices for him. In buying commodities where they a relatively cheap, and selling them where they are relatively expensive, the agent in question can obtain any consumption bundle. If there are sufficiently many potential possibilities for arbitrage, this ensures, in equilibrium, that parts of the economy have uniform prices. Indeed, monopoly outcomes with and without price differentiation, as well as Walrasian equilibrium are obtained as special cases.

In the present paper we take a different approach to solving the problem of existence of equilibrium. We no longer focus on the institutional characteristic of mono pricing. Instead, we assume that the dominating agent in a trade relationship sets two prices for each commodity, one price for buying and one for selling. This institutional characteristic, which is called the institutional characteristic of bid and ask prices, enables the leader on a trade relationship to enforce zero trades by setting zero as the price at which he buys and setting positive prices at which he sells. As in Spanjers (1992, Chapter 6), we consider models in which the hierarchical structure is a hierarchical tree. Finally, we restrict ourselves to economies with only two commodities.

In Section 2 we present the model of a hierarchically structured economy with bid and ask prices and two commodities. In Section 3 the existence of equilibrium in such a model is proven. Some examples of hierarchically structured economies with bid and ask prices are given in Secion 4. One of the examples illustrates that the equilibrium allocation in a hierarchically structured economy with bid and ask prices is different from the equilibrium allocation in the corresponding hierarchically structured economy with mono price setting in which the leader of a trade relationship has the additional possibility to enforce zero trades. In Section 5 we specify a simple trade process for the model. This process shows how the aggregated information the agents are assumed to have about the part the economy that is downstream of them is disaggragated to lead to optimal individual decisions. Finally, some concluding remarks and suggestions for further research are made in Section 6.

2 The Model

In this section we define a hierarchically structured economy with bid and ask prices and two commodities. We describe such an economy by its hierarchical structure, by its agents and their individual characteristics, and by the institutional characteristics of the trade relationships. The hierarchical structure of the economy is described by a hierarchical tree, which describes between which agents bilateral asymmetric trade relationships exist and which agent dominates the other agent in such a trade relationship. Since we analyze a pure exchange economy, we describe each agent by his utility function and his initial endowments. We assume that every trade relationship has the institutional characteristic of bid and ask prices. This means that for every asymmetric trade relationship the dominating agent sets two prices for each commodity. One price, the bid price, is the price at which he buys the commodity. The other price, the ask price, is the price at which he sells the commodity. The

dominating agent has the obligation to buy and sell any amount the dominated agents want to trade at the corresponding prices. Finally, we assume the economy only has two commodities.

We start by introducing the concepts we use to describe the hierarchically structured economy. First, however, we introduce some terminology concerning graph theory.

A **(Simple) Directed Graph** is a pair (A, D) consisting of a finite non-empty set of vertices A and a set of arcs $D \subset \{(i,j) \in A \times A \mid i \neq j\}$. The **Indegree** of a point $a \in A$ in a (simple) directed graph (A, D) is the number of ingoing arrows of the point a, and is denoted as $\overleftarrow{\rho}\,(a) := \#\{(i,j) \in D \mid j = a\}$. A (simple) directed graph (A, D) is **(Weakly) Connected** if it cannot be expressed as the union of two (simple) directed graphs, i.e., there do not exist two (simple) directed graphs (A_1, D_1) and (A_2, D_2) such that $A = A_1 \cup A_2$ and $D = D_1 \cup D_2$. A (simple) directed graph (A, D) has a **Tree Structure** if it is (weakly) connected and $\#A - \#D = 1$.

Definition 2.1 *A* **Hierarchical Tree**, (A, W), *is a (weakly) connected, simple directed graph that has a tree structure and has exactly one $i \in A$ such that $\overleftarrow{\rho}\,(i) = 0$.*

Definition 2.2 *Let (A, W) be a hierarchical tree. Let $w := (i, j) \in W$. The* **Institutional Characteristic** *of w is the correspondence $T_w : \tilde{X}_w \rightrightarrows \tilde{Y}_w$.*

The institutional characteristic T_w for a relationship $w \in W$ is interpreted as specifying for each signal $s \in \tilde{X}_w$, chosen by agent i, the set $T_w(s) \subset \tilde{Y}_w$ of actions agent j can choose from with respect to the relationship w.

Definition 2.3 *A* **Hierarchically Structured Economy** *with 2 commodities is a tuple $\mathbf{E} = ((A, W), \{U_i, \omega_i\}_{i \in A}, \{T_w\}_{w \in W})$, where:*

1. *(A, W) is a hierarchical tree.*
2. *$U_i : \mathbb{R}^2_+ \to \mathbb{R}$ is the utility function of agent $i \in A$.*
3. *$\omega_i \in \mathbb{R}^2_+$ is the initial endowment of agent $i \in A$.*
4. *$T_w : \tilde{X}_w \rightrightarrows \mathbb{R}^2$ is the institutional characteristic of trade relationship $w \in W$.*

In this paper, we only consider economies with the institutional characteristics of mono pricing and of bid and ask prices.

The institutional characteristic of **Mono Pricing** is the correspondence $\mathcal{T}^{mon} : S^1 \rightrightarrows \mathbb{R}^l$ such that $\forall p \in S^1$:

$$\mathcal{T}^{mon}(p) := \{d \in \mathbb{R}^2 \mid p \cdot d \leq 0\},$$

where, as usual, $S^1 := \{p \in \mathbb{R}^2_+ \mid p_1 + p_2 = 1\}$ is the 1-dimensional unit simplex.

In order to introduce the institutional characteristic of bid and ask prices, let S^3 be the 3-dimensional unit simplex defined by

$$S^3 := \{q \in \mathbb{R}^4_+ \mid \sum_{a=1}^{4} q_a = 1\}.$$

Define

$$P := \{q := (\underline{q}, \overline{q}) := (\underline{q}_1, \underline{q}_2, \overline{q}_1, \overline{q}_2) \in S^3 \mid 0 \leq \underline{q} \leq \overline{q}\}.$$

The institutional characteristic of **Bid and Ask Prices** is the correspondence $\mathcal{T}^{bap} : P \rightrightarrows \mathbb{R}^l$ such that for each $p := (\underline{p}, \overline{p}) \in P$ we have

$$\mathcal{T}^{bap}(p) := \{d \in \mathbb{R}^2 \mid \sum_{c=1}^{2} \underline{p}_c \cdot \min\{0, d_c\} + \sum_{c=1}^{2} \overline{p}_c \cdot \max\{0, d_c\} \leq 0\}.$$

Now we are in a position to introduce the following definition.

Definition 2.4 *A* **Hierarchically Structured Economy with Bid and Ask Prices** *(with two commodities) is a hierarchically structured economy with two commodities, such that* $\forall w \in W$ *we have* $\mathcal{T}_w = \mathcal{T}^{bap}$*. Similarly, a* **Hierarchically Structured Economy with Mono Pricing** *(with two commodities) is a hierarchically structured economy with two commodities such that* $\forall w \in W$ *we have* $\mathcal{T}_w = \mathcal{T}^{mon}$.

For each agent $i \in A$ we use $L_i := \{h \in A \mid (h, i) \in W\}$ to denote the set of (direct) leaders of agent i, and we use $F_i := \{j \in A \mid i \in L_j\}$ as the set of (direct) followers of agent i. Since (A, W) is a hierarchical tree we have for one agent $k \in A$ that $L_k = \emptyset$. We refer to this agent as the **Top Agent** in the economy. We denote the set that has the top agent k as its only element by $S_1 := \{k\}$. For all other agents $i \in A \setminus S_1$ we have that $\#L_i = 1$. Finally, we use $L := \{1, 2\}$ to denote the set of commodities in $\mathbf{E}$.

We make the following assumption with respect to the utility functions and the initial endowments of the agents throughout the paper, with the exception of Section 4.

Assumption 2.5 *Let* $\mathbf{E}$ *be a hierarchically structured economy with bid and ask prices. For every agent* $i \in A$ *it holds that* $\omega_i \gg 0$ *and* U_i *represents a neoclassical preference relation.* [1]

The next step in completing the model is to describe the choice spaces of the agents in the economy. We can, without loss of generality, we restrict ourselves to a compact price space Q contained in P such that the vector of prices for selling is strictly larger than zero. Let $\hat{P} := \{(\underline{p}, \overline{p}) \in P \mid \overline{p} \gg 0\}$. Define the correspondence $\hat{B} : \hat{P} \times \mathbb{R}^2_+ \rightrightarrows \mathbb{R}^2_+$ such that $\forall (p, z) \in \hat{P} \times \mathbb{R}^2_+$, where p is a vector of bid and ask prices and z is a given commodity bundle, we have the set of attainable allocations:

$$\begin{aligned}\hat{B}(p, z) &:= \{y \in \mathbb{R}^2_+ \mid (y - z) \in \mathcal{T}^{bap}(p)\} \\ &= \{y \in \mathbb{R}^2_+ \mid \sum_{c \in L} \underline{p}_c \cdot \min\{0, y_c - z_c\} + \sum_{c \in L} \overline{p}_c \cdot \max\{0, y_c - z_c\} \leq 0\}.\end{aligned}$$

For each agent $i \in A$ the demand function $x^*_i : \hat{P} \times \mathbb{R}^2_{++} \to \mathbb{R}^2_+$ is defined such that for every $p = (\underline{p}, \overline{p}) \in \hat{P}$ and $\omega_i \in \mathbb{R}^2_{++}$:

$$x^*_i(p, \omega_i) \in \operatorname{argmax}_{x_i \in \hat{B}(p, \omega_i)} U_i(x_i).$$ [2]

Define for $\varepsilon > 0$:

$$Q(\varepsilon) := \{(\underline{q}, \overline{q}) \in P \mid \overline{q} \geq \varepsilon \cdot 1_2\}.$$

We choose some $\varepsilon^* > 0$ such that for each $p \in \hat{P} \setminus Q(\varepsilon^*)$ we have

$$\exists q \in Q(\varepsilon) : x^*_i(q, \omega_i) = x^*_i(p, \omega_i)$$

or

$$\forall i \in A \setminus S_1 : \exists c \in L : x^*_{ic}(p, \omega_i) > \omega_c,$$

[1] See also Aliprantis, Brown and Burkinshaw (1989, Def. 1.3.4). They say a preference relation $\succeq$ on some $\mathbb{R}^l_+$ is neoclassical whenever it is continuous and either:

1. $\succeq$ is stricly monotone and strictly convex.
2. $\succeq$ is stricly monotone and strictly convex on int $\mathbb{R}^l_+$, and everything in the interior is preferred to anything on the boundary.

They call a preference relation $\succeq$ strictly monotone on a set X if $\forall x, y \in X : [x > y \Rightarrow x \succ y]$.

[2] Note that by the properties of Assumption 2.5 we have that for each $p \in \hat{P}$ and $\omega_i \in \mathbb{R}^2_{++}$, the set $\operatorname{argmax}_{x_i \in \hat{B}_i(p, \omega_i)} U_i(x_i)$ contains only one element.

where $\omega := \sum_{i \in A} \omega_i$. Such an ε^* exists because the initial endowments of the agents are strictly positive and the utility functions are strictly monotonic on $\mathbb{R}^2_{++}$. Clearly, the set $Q(\varepsilon^*)$ is compact.
Denote:

$$Q := Q(\varepsilon^*).$$

$$\tilde{\omega} := (\tilde{\omega}_c)_{c \in L}, \quad \text{where} \quad \tilde{\omega}_c := \sum_{i \in A} \max_{q \in Q} x^*_{ic}(q, \omega).$$

$$Y := \{d \in \mathbb{R}^2 \mid -\tilde{\omega} \le d \le \tilde{\omega}\}.$$

$$Y_+ := Y \cap \mathbb{R}^2_+.$$

$$X_i := Y^{L_i} \times Q^{F_i}.$$

$$X := \prod_{i \in A} X_i.$$

We give the following definition of a trade-price-allocation tuple in a given hierarchically structured economy with bid and ask prices.

Definition 2.6 *A* **Trade-Price-Allocation Tuple** *in the economy* $\mathbf{E}$ *is a tuple* $(d, p, x) \in Y^W \times Q^W \times Y_+^A$ *where:*

1. $d_{ji} \in Y$ *is the vector of net trade on the trade relationship* $(i,j) \in W$. *We denote* $d_i := (d_{ih})_{h \in L_i}$.

2. $p_{ij} = (\underline{p}_{ij}, \overline{p}_{ij}) \in Q$ *is the price vector denoting the bid prices* $\overline{p}_{ij}$ *and ask prices* $\underline{p}_{ij}$ *charged on the trade relationship* $(i,j) \in W$. *We denote* $p_i := (p_{ij})_{j \in F_i}$.

3. $x_i \in Y_+$ *is the consumption bundle for agent* $i \in A$.

Agents assume that their actions do not influence the prices their leader sets for them. On the other hand, we assume that each agent "correctly" anticipates the net trade with each of his followers as a function of the prices he sets. We assume he takes into account the consequences of possible changes in the prices his followers set for their followers etc. These anticipations may result from each agent having aggregated information about the part of the economy that is downstream from him and knowing nothing but the prices his leader sets for him about the rest of the

economy. We may assume that the downstream part of the economy for each agent is perfectly transparent for this agent, or at least that he has the information that summarizes what net trades to anticipate as a function of the prices he may set for his followers. Thus, we assume the **Downstream Information Structure** to hold. The anticipations of the agents are described by their anticipated net trade correspondences.

For any agents $i \in A$ and $j \in F_i$ the anticipated net trade correspondence $t_{ij} : Q \rightrightarrows Y$ is defined. This correspondence describes for each vector $q \in Q$ of bid and ask prices agent i may set on the trade relationship $(i,j) \in W$ the set of net trades $t_{ij}(q) \subset Y$ agent i anticipates to have with agent j. The anticipated net trade correspondences are defined recursively, using the following process. We start with the agents $j \in A_0 := \{j \in A \mid F_j = \emptyset\}$, who do not have any followers. Clearly, since those agents have no followers we do not need to define any anticipated net trade correspondence for them. Then, given the anticipated net trade correspondences for some non-empty set of agents A_n with $n \in \mathbb{N}$, with their followers, if they have any followers, we derive the anticipated net trade correspondences for the of agents $C_{n+1} := \{i \in A \setminus A_n \mid F_i \subset A_n\}$. Then we define $A_{n+1} := A_n \cup C_{n+1}$ etc. We stop this procedure when we reach an n^* such that $A_{n^*} = A$. Since (A, W) is an hierarchical tree, such an n^* exists. The anticipated net trade correspondences are derived using the following definition.

Definition 2.7 *Let* $i \in A$ *with* $F_i \neq \emptyset$. *Let for each agent* $j \in F_i$ *the tuple of anticipated net trade correspondences* $(t_{jm})_{m \in F_j}$ *be given.*[3] *The* **Anticipated Net Trade Correspondence** $t_{ij} : Q \rightrightarrows Y$ *of agent* i *with respect to agent* $j \in F_i$ *is given by:*

$$t_{ij}(q_{ij}) := \operatorname{argmax}_{e_{ji} \in T^{bap}(q_{ij}) \cap Y} \{ \max_{y_j \in \Psi_j(e_{ji})} U_j(y_j) \},$$

where $\Psi_j : Y \rightrightarrows \mathbb{R}^2_+$ *such that* $\forall e_{ij} \in Y$

$$\Psi_j(e_{ji}) := \{y \in \mathbb{R}^2_+ \mid y \le \omega_j + e_{ji} - \sum_{m \in F_j} e_{mj}$$

[3] As noticed above, the tuple may be the empty tuple if $F_j = \emptyset$. In this case the definition of the correspondence $\Psi_j : Y \rightrightarrows \mathbb{R}^2_+$ below reduces to:

$$\forall e_{ji} \in Y : \quad \Psi_j(e_{ji}) := \{y \in \mathbb{R}^2_+ \mid y \le \omega_j + e_{ji}\}.$$

$$\text{where } \forall m \in F_j : \exists q_{jm} \in Q : e_{mj} \in t_{jm}(q_{jm})\}.$$

In the above definition agent $j \in F_i$ is assumed by agent i to be "optimistic" with respect to the reactions of his followers $m \in F_j$, i.e., it is assumed by agent i that a follower $m \in F_j$ of agent j, when he is indifferent between two actions at prices as set by agent j, chooses those actions that are the best for agent j. This implies that certain coordination problems between agent j and his followers are solved to the benefit of agent j.[4]

Given the anticipated net trade correspondences we define the correspondences that describe the actions an individual agent anticipates to be feasible for him. We refer to these correspondences as the feasible actions correspondences of the agents in the economy.

Definition 2.8 *The* **Feasible Actions Correspondence** $B_i : Q \rightrightarrows X_i \times Y_+$ *of agent* $i \in A \setminus S_1$ *with* $L_i = \{h\}$ *is such that* $\forall p_{hi} \in Q$

$$\begin{aligned} B_i(p_{hi}) \ := \{(e_{ih}, q_i, y_i) \in X_i \times Y_+ \mid \ & e_{ih} \in T^{bap}(p_{hi}) \\ & y_i \leq \omega_i + e_{ih} - \sum_{j \in F_i} e_{ji} \\ & \text{where } \forall j \in F_i : e_{ji} \in t_{ij}(q_{ij})\}. \end{aligned}$$

The feasible actions set B_k for the top agent $k \in S_1$ is given by

$$\begin{aligned} B_k = \{(q_k, y_k) \in X_k \times Y_+ \mid \ & y_k \leq \omega_k - \sum_{j \in F_k} e_{jk} \\ & \text{where } e_{jk} \in t_{kj}(q_{kj})\ \}. \end{aligned}$$

The optimization problem of agent $i \in A \setminus S_1$ who has agent $h \in L_i$ as his direct leader is to maximize his utility over his feasible actions set. Here his feasible actions set depends on the prices p_{hi} his leader sets for him. Therefore agent i solves the following optimization problem

$$\max_{(e_{ih}, q_i, y_i) \in B_i(p_{hi})} U_i(y_i).$$

The optimization problem of the top agent $k \in S_1$ in the economy is given by

$$\max_{(q_k, y_k) \in B_k} U_k(y_k).$$

Now we have described the individual optimization problem for each agent $i \in A$ we define an equilibrium in the hierarchically structured economy.

Definition 2.9 *Let* **E** *be a hierarchically structured economy with bid and ask prices. A tuple $(d^*, p^*, x^*) \in X \times Y_+$ is an* **Equilibrium** *in* **E** *if* $\forall i \in A \setminus S_1$ *such that* $\exists h \in A$ *with* $L_i = \{h\}$*:*

1. $(d_i^*, p_i^*, x_i^*) \in \text{argmax}_{(e_i, q_i, y_i) \in B_i(p_{hi}^*)} U_i(y_i)$
2. $x_i^* \leq \omega_i + d_{ih}^* - \sum_{j \in F_i} d_{ji}^*$

and for the top agent $k \in S_1$ we have that:

1. $(p_k^*, x_k^*) \in \text{argmax}_{(q_k, y_k) \in B_k} U_k(y_k)$
2. $x_k^* \leq \omega_k - \sum_{j \in F_k} d_{jk}^*$.

3 Existence of Equilibrium

In this section we prove the existence of an equilibrium in the model of an hierarchically structured economy with bid and ask prices and with two commodities. Unfortunately, we cannot use the line of proof of Spanjers (1992, Chapter 6). There the optimization problem of a middleman is separated in two problems. The first problem is setting the prices for the followers as to maximize the profits from his position as an intermediary, given the prices set by his leader. The second problem is choosing the optimal consumption bundle, given the income of the agent. The income (or wealth) of the agent consists of the value of the initial endowments of the middleman and the (maximal) profits from his position as a middleman, given the prices set by his leader.

[4]Below, we see that each agent is "optimistic" with respect to the reactions of his followers in the sense that any agent $i \in A$ correctly assumes that if (some of) his followers, $\hat{F}_i \subset F_i$, are indifferent between two actions at the prices set for them by agent i, then the agents in $\hat{F}_i$ chooses those (combinations of) such actions that are the best for agent i. Thus, we find that (implicitly) assuming agent j to be "optimistic" with respect to the reactions of the agents in F_j in Definition 2.7 is consistent with the kind of individual optimizing behaviour of the agents we assume below. In fact, in our model, we are looking for perfect equilibrium in "very nice" plays, with "very nice" plays defined as in Hellwig and Leininger (1987) in the corresponding multi-stage game.

In our present model with bid and ask prices we cannot separate the optimization problems of the agents in this way. The reason for this is that, depending of the prices some agent i quotes for his followers, he may have positive or negative net trade in some commodity with his leader. However, since this leader sets bid and ask prices, he may consequently face different relative prices for the commodity in question, depending on whether the net trade in the commodity with his leader is positive or negative. Such a difference in relative prices does not occur on a trade relationship with the institutional characteristic of mono pricing, which allows us to separate the optimization problems of the agents in the way described above in hierarchically structured economies with mono pricing.

For each agent $i \in A$ we define a correspondence of attainable allocations Δ_i which, in Lemma 3.3, we show to be continuous if for each $j \in F_i$ the correspondence t_{ij} has a compact graph. Using this result, we prove in Lemma 3.4, that for any $i \in A$ and $j \in F_i$ the anticipated net trade correspondence t_{ij} has a compact graph. This property is then used in the proof of Theorem 3.5 to construct an equilibrium in the hierarchically structured economy with bid and ask prices $\mathbb{E}$.

Before we define the correspondences of attainable allocations, we define for each agent $i \in A$ the set Γ_i of after downstream trade bundles. The set of after downstream trade bundles of agent $i \in A$ contains the bundles agent i anticipates to be attainable as a result of the trade with his followers, without trading with his leader.

Definition 3.1 *Let $\mathbb{E}$ be a hierarchically structured economy with bid and ask prices. Let $i \in A$. The set Γ_i is the set of* **After Downstream Trade Bundles** *of agent i where:*

$$\Gamma_i := \{x_i \in \mathbb{R}^2 \mid x_i = \omega_i + \sum_{j \in F_i} e_{ji} \quad \text{with } \forall j \in F_i : \exists p_{ij} \in Q : e_{ji} \in t_{ij}(p_{ij})\}.$$

The correspondence of attainable allocations of agent $i \in A$ with $L_i := \{h\}$ is the correspondence that for each price vector p_{hi} assigns the set of consumption bundles that are attainable for agent i.

Definition 3.2 *Let $\mathbb{E}$ be a hierarchically structured economy with bid and ask prices. Let $i \in A$ such that $L_i = \{h\}$. Then $\Delta_i : Q \rightrightarrows \mathbb{R}^2_+$ is the* **Correspondence of Attainable Allocations** *of agent i if for each $p_{hi} \in Q$:*

$$\Delta_i(p_{hi}) := \{y \in \mathbb{R}^2_+ \mid \exists x \in \Gamma_i : (y - x) \in \mathcal{T}^{bap}(p_{hi})\}.$$

Next we prove the following crucial lemma.

Lemma 3.3 *Let $\mathbb{E}$ be a hierarchically structured economy with bid and ask prices for which Assumption 2.5 holds. If for each $j \in F_i$ it holds that t_{ij} has a compact graph, then Δ_i is a continuous correspondence.*

Proof

(i) Δ_i is an upper hemi-continuous correspondence.

Since $\forall j \in F_i$ the correspondence t_{ij} has a compact graph, we have that Γ_i is a compact set. Therefore Γ_i is bounded and Δ_i has a bounded graph.

Next we prove that Δ_i has a closed graph and therefore is a upper hemi-continuous correspondence.

Suppose, for contradiction, that Δ_i does not have a closed graph, i.e.,

$$\exists p^q \to p^0, \; y^q \in \Delta_i(p^q), \; y^q \to y^0$$

such that

$$y^0 \notin \Delta_i(p^0).$$

For each $q \in \mathbb{N} \setminus \{0\}$, since $y^q \in \Delta_i(p^q)$, there exists some $x^q \in \Gamma_i$ such that

$$\sum_{c \in L} \underline{p}^q_c \cdot \min\{0, (y^q_c - x^q_c)\} + \sum_{c \in L} \overline{p}^q_c \cdot \max\{0, (y^q_c - x^q_c)\} \leq 0.$$

From the definition of Δ_i it follows that $y^0 \notin \Delta_i(p^0)$ implies $\nexists x^0 \in \Gamma_i$ such that

$$\sum_{c \in L} \underline{p}^0_c \cdot \min\{0, (y^0_c - x^0_c)\} + \sum_{c \in L} \overline{p}^0_c \cdot \max\{0, (y^0_c - x^0_c)\} \leq 0.$$

Since $p^q \to p^0 :=, (\underline{p}^0, \overline{p}^0)$, with $\overline{p}^0 \gg 0$ and $y^q \to y^0$ we have that $x^q \to x^0 \notin \Gamma_i$. This contradicts Γ_i being a closed set.

Thus it follows that Δ_i is a u.h.c. correspondence.

(ii) Δ_i is a lower hemi-continuous correspondence.

If we prove that Δ_i is lower hemi-continuous it follows by the definition of continuity

of correspondences that Δ_i is a continuous correspondence.
The correspondence Δ_i is lower hemi-continuous if and only if $\forall q \in \mathbb{N},\ p^q \in Q$:

$$p^q \to p^0 \ \text{ and } \ y^0 \in \Delta_i(p^0)$$

implies that

$$\exists \{y^q\}_{q=1}^{\infty} \ \text{ with } \ \forall q \in \mathbb{N} \setminus \{0\} : \ y^q \in \Delta_i(p^q) \ \text{ and } \ y^q \to y^0.$$

Since $y^0 \in \Delta_i(p^0)$: $\exists x^0 \in \Gamma_i$ such that

$$\sum_{c \in L} \underline{p}_c^0 \cdot \min\{0, (y_c^0 - x_c^0)\} + \sum_{c \in L} \overline{p}_c^0 \cdot \max\{0, (y_c^0 - x_c^0)\} \leq 0. \quad (1)$$

We consider three cases.
(iia) $y^0 = 0$.
Since $\forall p \in Q$: $0 \in \Delta_i(p)$, take $\forall q \in \mathbb{N} \setminus \{0\}$: $y^q := 0$.
(iib) $y^0 \in \operatorname{int} \mathbb{R}_+^2$.
Define $B_i^q := \{z \in \Delta_i(p) \mid (z - x^0) \in T^{bap}(p^q)\}$. Since $y^0 \in \mathbb{R}_+^2$ we have, for q sufficiently large, that $B_i^q \neq \emptyset$.
For each q such sufficiently large, take $y^q \in \operatorname{argmin}_{z \in B_i^q} \| z - y^0 \|$.
The sequence $\{y^q\}$ is in a compact set, since it is in the closed ball around y^0 with radius $\max_{q \in \mathbb{N}\setminus\{0\}} \| y^q - y^0 \|$, so the sequence converges.
By the continuity of multiplication, addition and the operators min and max, we have that $y^q \to y^0$.
(iic) $y_0 \in \partial \mathbb{R}_+^2 \setminus \{0\}$.
We consider three cases.
(1) Suppose $y^0 \leq x^0$.
Now, $\forall p \in Q$: $(y^0 - x^0) \in T^{bap}(p)$. Taking $y^q := y^0$ for each $q \in \mathbb{N} \setminus \{0\}$ gives us the required sequence.
(2) Suppose $\exists c \in L$: $[y_c^0 > x_c^0 \ \wedge \ y_c^0 > 0]$.
By the definition of Q we have $\overline{p}_c^0 > 0$. As before, for q sufficiently large, we have

$$B_i^q := \{z \in \Delta_i(p^q) \mid (z - x^0) \in T^{bap}(p^q)\} \neq \emptyset,$$

and construct $\{y^q\}$ as in *(iib)*.
(3) Suppose $y^0 \not\leq x^0$ *and* $\forall c \in L$: $[y_c^0 > x_c^0 \Rightarrow y_c^0 = 0]$. (2)
Clearly, $\exists \tilde{c} \in L$: $y_{\tilde{c}}^0 > x_{\tilde{c}}^0$ and, since $p^0 \in Q$, we have $\overline{p}_{\tilde{c}} > 0$.

This implies, by (1), that $\exists \overline{c} \in L : [y_{\overline{c}}^0 < x_{\overline{c}}^0 \ \wedge \ \underline{p}_{\overline{c}}^0 > 0]$. (3)
Consider the case that $\exists \overline{c} \in L$: $[y_{\overline{c}}^0 < x_{\overline{c}}^0 \ \wedge \ y_{\overline{c}}^0 > 0]$. Now the same procedure as in (iib) can be applied.
So we are only left to check the case that $\forall \overline{c} \in L$:
$[(y_{\overline{c}}^0 < x_{\overline{c}}^0 \ \wedge \ \underline{p}_{\overline{c}}^0 > 0) \Rightarrow y_{\overline{c}}^0 = 0]$ (4)
can not occur. We show that because the number of commodities does not exceed 2 this implies $y^0 = 0$, which contradicts $y \in \partial \mathbb{R}_+^l \setminus \{0\}$.
Without loss of generality, take $y_1^0 > x_1^0$ and $y_1^0 = 0$ to satisfy (2). By (3) we must have $y_2^0 > x_2^0$ and $\underline{p}_2^0 > 0$, which, by (4), implies $y_2^0 = 0$.

Q.E.D.

Lemma 3.4 *Let* $\mathbb{E}$ *be a hierarchically structured economy with bid and ask prices for which Assumption 2.5 holds. For each* $i \in A \setminus S_1$, $h \in L_i$, *the correspondence* t_{hi} *has a compact graph.*

Proof
We prove this lemma by induction, following the recursive procedure we used to construct the anticipated net trade correspondences in Section 2.
(i) Starting Condition.
For each $j \in A$: $F_j = \emptyset$, $i \in L_j$ the correspondence $t_{ij} : Q \rightrightarrows Y$ can be represented by a continuous function on Q, because, by definition, we have $\forall p \in Q$ that $\overline{p} \geq \varepsilon^* \cdot 1_l$.
Thus, t_{ij} has a compact graph.
(ii) Induction Hypothesis.
Let $i \in A \setminus S_1$, $j \in F_i$, $h \in L_i$. Suppose for each $j \in F_i$ the correspondence t_{ij} has a compact graph. Then t_{hi} has a compact graph.
Proof
Since $\forall j \in F_i$ we have that t_{ij} has a compact graph it follows by Lemma 3.3 that Δ_i is a continuous correspondence.
Define $X_i : Q \rightrightarrows Y_+$ such that:

$$X_i(p_{hi}) := \operatorname{argmax}_{y_i \in \Delta_i(p_{hi})} U_i(y_i).$$

By the Maximum Theorem it follows that X_i has a compact graph.

To prove that t_{hi} has a compact graph we must show that if $p_{hi}^q \in Q$ and $e_{ih}^q \in Y$ for each $q \in \mathbb{N}$, such that $\{p_{hi}^q\}_{q=1}^{\infty} \to p_{hi}^0$, $\{e_{ih}^q\}_{q=1}^{\infty} \to e_{ih}^0$ and $\forall q \in \mathbb{N}\setminus\{0\}$: $e_{ih}^q \in t_{hi}(p_{hi}^q)$,

then $e^0_{ih} \in t_{hi}(p^0_{hi})$.
We have that $e_{ih} \in t_{hi}(p_{hi})$ if and only if for each $j \in F_i$ there exist p_{ij} and e_{ji} such that $e_{ji} \in t_{ij}(p_{ij})$ and

$$e_{hi} - \sum_{j \in F_i} e_{ji} + \omega_i \in X_i(p_{ih}).$$

Since X_i is a correspondence with a compact graph, it follows from $\{p^q_{hi}\}^\infty_{q=1} \to p^0_{hi}$ that $\{X_i(p^q_{hi})\}^\infty_{q=1} \to X_i(p^0_{hi})$. Furthermore $\{e^q_{ih}\}^\infty_{q=1} \to e^0_{ih}$, so it follows that $e^0_{ih} \in t_{hi}(p^0_{hi})$ if and only if $\{(\sum_{j\in F_i} e^q_{ji})\}^\infty_{q=1} \to (\sum_{j \in F_i} e^0_{ji})$, where for each $q \in \mathbb{N} \setminus \{0\}$ and $j \in F_i$ we have $e^q_{ji} \in t_{ij}(p^q_{ij})$ for some $p^q_{ij} \in Q$.
Since $\{X_i(p^q_{hi})\}^\infty_{q=1} \to X_i(p^0_{hi})$ it follows from the definitions of Δ_i and Γ_i and from t_{ij} having a compact graph for each $j \in F_i$, that such a sequence $\{(e^q_{ji}, p^q_{ij})_{j\in F_i}\}^\infty_{q=1}$ exists.
This proves the induction hypothesis.

Q.E.D.

The next theorem states that an equilibrium in a hierarchically structured economy with bid and ask prices $\mathbb{E}$ exists. To prove this theorem we construct an equilibrium, which proves its existence.

Theorem 3.5 [Existence Theorem]
In each hierarchically structured economy with bid and ask prices $\mathbb{E}$ for which Assumption 2.5 holds, an equilibrium exists.

Proof
By Lemma 3.3 and Lemma 3.4 it follows that for each $i \in A \setminus S_1$ the correspondence Δ_i is continuous and has non-empty values.
From Lemma 3.4 it follows that for each $i \in A \setminus S_1$ the correspondences B_i has a compact graph. Similarly it follows that B_k, $k \in S_1$, is a compact set.
Finally, we have that for each $i \in A$, $j \in F_i$ it holds that $t_{ij}(0, \overline{p}_{ij}) = 0$, so that each agent $i \in A$ always has the no trade option. Therefore $B_k \neq \emptyset$ for $k \in S_1$ and for each $i \in A \setminus S_1$ the correspondence B_i has non empty-values only.

We recursively construct an equilibrium tuple (d^*, p^*, x^*).

Define $D_0 := \{k \in A \mid L_k = \emptyset\}$. Since B_k is a compact non-empty set there exists a tuple $(p^*_k, x^*_k) \in X_k \times Y_+$ such that

$$(p^*_k, x^*_k) \in \text{argmax}_{(q_k, y_k) \in B_k} U_k(y_k).$$

By the definition of B_k there exists a tuple $(d^*_{jk})_{j\in F_k} \in Y^{F_k}$ such that for each $j \in F_k$ we have that $d^*_{jk} \in t_{kj}(p^*_{kj})$ and $x^*_k \leq \omega_k - \sum_{j\in F_k} d^*_{jk}$.
Let for $t \in \mathbb{N}$ the set D_t be obtained through earlier steps of the procedure, such that $A \setminus D_t \neq \emptyset$, where D_t is the set of agents on which the procedure has already been applied.
Choose $h \in D_t$ Such that $F_h \setminus D_t \neq \emptyset$ and choose $i \in F_h \setminus D_t$. The vector $d^*_{ih} \in t_{hi}(p^*_{hi})$ is already constructed in a previous step of the procedure. By the definition of t_{hi} there exists a tuple $(d^*_i, p^*_i, x^*_i) \in X_i \times Y_+$ with $d^*_i := d^*_{hi}$ such that

$$(d^*_i, p^*_i, x^*_i) \in \text{argmax}_{(q_i, e_i, y_i) \in B_i(p^*_{hi})} U_i(y_i).$$

By the definition of B_i there exists a tuple $(d^*_{ji})_{j\in F_i} \in Y^{F_i}$ such that for each $j \in F_i$ we have $d^*_{ji} \in t_{ij}(p^*_{ij})$ and $x^*_i \leq \omega_i + d^*_{ih} - \sum_{j\in F_i} d^*_{ji}$.
Now define $D_{t+1} := D_t \cup \{i\}$.

Repeating this procedure until for some t^* we have $D_{t^*} = A$ yields an equilibrium tuple.

Q.E.D.

4 Two Examples

In this section we consider two examples of (parts of) hierarchically structured economies with bid and ask prices. In the first example we consider three cases in which the equilibrium allocations in hierarchically structured economies with bid and ask prices coincide with those in the corresponding economies with mono pricing. In Table 1 the equilibria in the economies with mono pricing are given. For more details and some results on hierarchically structured economies with mono pricing we refer to Spanjers (1992, Chapter 6), from which Table 1 originates. In the second example we illustrate why this coinciding of equilibrium allocations need not be the case in general if an equilibrium in the hierarchically structured economy with mono pricing exists. It is illustrated that even if the top agent trades voluntarily in a hierarchically structured economy with mono pricing, it need not be the case that the equilibrium allocations in this economy and the corresponding economy with bid and ask prices coincide.

Example 4.1
Consider the economy with the set of agents $A := \{a, b, c\}$ and two commodities, where $L := \{1, 2\}$ is the set of commodities. The individual characteristics of the agents are:

$$\begin{array}{lll} a: & U_a(x_a) := \sqrt{x_{a1}} + \sqrt{x_{a2}} & \omega_a := (0,0) \\ b: & U_b(x_b) := \sqrt{x_{b1}} + \sqrt{x_{b2}} & \omega_b := (1,0) \\ c: & U_c(x_c) := \sqrt{x_{c1}} + \sqrt{x_{c2}} & \omega_c := (0,1). \end{array}$$

Consider the economy that follows if the hierarchical tree $\mathcal{T}_1 := (A, \{(a,b),(a,c)\})$ organizes the above defined set of agents. The optimization problem of agent b at given prices $p_{ab} := (\underline{p}_{ab}, \overline{p}_{ab})$ is:

$$\max_{(d_{ba}, x_b) \in X_b \times Y_+} \sqrt{x_{b1}} + \sqrt{x_{b2}}$$

such that

$$\underline{p}_{ab1} \cdot \min\{0, d_{ba1}\} + \underline{p}_{ab2} \cdot \min\{0, d_{ba2}\} +$$

$$\overline{p}_{ab1} \cdot \max\{0, d_{ba1}\} + \overline{p}_{ab2} \cdot \max\{0, d_{ba2}\} \leq 0$$

$$x_b \leq \omega_b + d_{ba}.$$

Since $\omega_b := (1, 0)$ we have the peculiarity that

$$d_{ba1} = \min\{0, d_{ba1}\}$$

$$d_{ba2} = \max\{0, d_{ba2}\}.$$

So this optimization problem boils down to:

$$\max_{(d_{ba}, x_b) \in X_b \times Y_+} \sqrt{x_{b1}} + \sqrt{x_{b2}}$$

such that

$$\underline{p}_{ab1} \cdot d_{ba1} + \overline{p}_{ab2} \cdot d_{ba2} \leq 0$$

TABLE 1: Equilibrium Values

Variable	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$
	$q^*_{ba1} \approx 0.7071$ $q^*_{ba2} \approx 0.2929$	$q^*_{ab1} \approx 0.5232$ $q_{ab2} \approx 0.4768$	$q^*_{ab1} \approx 0.4042$ $q^*_{ab2} \approx 0.5958$
	$q^*_{ac1} \approx 0.2929$ $q^*_{ac2} \approx 0.7071$	$q^*_{bc1} \approx 0.7233$ $q^*_{bc2} \approx 0.2767$	$q^*_{ac1} \approx 0.6358$ $q^*_{ac2} \approx 0.3642$
x^*_{a1}	0.1716	0.0698	0
x^*_{a2}	0.1716	0.0840	0
x^*_{b1}	0.7071	0.8244	0.7913
x^*_{b2}	0.1213	0.1927	0.3642
x^*_{c1}	0.1213	0.1058	0.2087
x^*_{c2}	0.7071	0.7233	0.6358
$U_a(x^*_a)$	0.6864	0.3069	0
$U_b(x^*_b)$	1.4142	1.8141	2.2290
$U_c(x^*_c)$	1.4142	1.3823	1.5730

$x_b \le \omega_b + d_{ba}.$

After rescaling the prices $\overline{p}_{ab1}, \underline{p}_{ab2}$ to prices $q_{ab1} := \frac{\overline{p}_{ab1}}{\overline{p}_{ab1}+\underline{p}_{ab2}}$ and $q_{ab1} := \frac{\underline{p}_{ab2}}{\overline{p}_{ab1}+\underline{p}_{ab2}}$, this is the problem for the corresponding economy with mono pricing as in Table 1. The anticipated net trade correspondence t_{ab} therefore corresponds to the anticipated net trade correspondence in the case of mono pricing. The same line of reasoning holds with respect to agent c, and therefore the optimization problem of agent a also corresponds to that in the corresponding economy with mono pricing. As a consequence we have that, although in our example we have a continuum of equilibrium price vectors p_{ac} and p_{ab}, the equilibrium allocation in the economy with bid and ask prices is the equilibrium allocation in the corresponding hierarchically structured economy with mono pricing.

Next we consider the economy with the set of agents A as before and the hierarchical tree $\mathcal{T}_2 := (A, \{(b,a),(a,c)\})$. By the same type of reasoning as before we have that the optimization problem of agent c, and therefore the anticipated net trade correspondence t_{ac}, corresponds to the optimization problem in the corresponding economy with mono pricing. Furthermore we know that, because $\omega_a = (0,0)$ and $\omega_c = (0,1)$, for the trade between agent a and agent b it holds that

$$d_{ba1} = \max\{0, d_{ba1}\}$$

$$d_{ba2} = \min\{0, d_{ba2}\}.$$

This once again implies that the optimization problem of agent b corresponds to his optimization problem in the model with mono pricing. As a consequence we find that, although we have a continuum of equilibrium tuples of price vectors p_{ab} and p_{ac}, the equilibrium allocation in the hierarchically structured economy with bid and ask prices is the equilibrium allocation of the corresponding hierarchically structured economy with mono pricing as in Table 1.

Finally, consider the economy with the set of agents as before and with the hierarchical tree $\mathcal{T}_3 := (A, \{(a,b),(b,c)\})$. The optimization problem of agent c and the anticipated net trade correspondence t_{bc} correspond to those in the situation with

mono pricing.

The optimization problem of agent b now becomes, at given prices $p_{ab} := (\underline{p}_{ab}, \overline{p}_{ab})$:

$$\max_{(d_{ba}, q_{bc}, x_b) \in X_b \times Y_+} \sqrt{x_{b1}} + \sqrt{x_{b2}}$$

such that

$$\underline{p}_{ab1} \cdot \min\{0, d_{cb1}\} + \underline{p}_{ab2} \cdot \min\{0, d_{cb2}\} +$$

$$\overline{p}_{ab1} \cdot \max\{0, d_{cb1}\} + \overline{p}_{ab2} \cdot \max\{0, d_{cb2}\} \le 0$$

$$x_b \le \omega_b + d_{ba} - e_{cb}$$

where $e_{cb} \in t_{bc}(q_{bc})$.

Since $\omega_a = (0,0)$ we must have $d_{ba} = 0$ in equilibrium. Once again this is attained at prices that correspond to those in the corresponding model with mono pricing. So we find a continuum of equilibrium tuples of price vectors p_{ab} and p_{bc}, all of which yield in the same equilibrium allocation in the hierarchically structured economy with bid and aks prices under consideration, as that of the corresponding hierarchically structured economy with mono pricing. Once again, the equilibrium allocations are given in Table 1.

Example 4.2

Suppose we have an economy with bid and ask prices with two commodities, $L := \{1,2\}$ and a set of agents $A \supset \{a,b,c\}$ in which consumer b is such that $F_b = \{c\}$, $L_b = \{a\}$, and the anticipated net trade correspondence t_{bc} with respect to commodity 1 as depicted in Figure 1, in which the prices are standardized as to have $\frac{\overline{p}_{ab1}}{\underline{p}_{ab2}} = 1$. This leads to a set of after downstream trade allocations Γ_b as in Figure 2.

The set Γ_b in Figure 2 denotes (a part of) the set of commodity bundles agent b can obtain through downstream trade. Since these commodity bundles are obtained through downstream trade only, the set Γ_b does not depend on the prices p_{ab} set by the leader of agent b, agent a. In general the set Γ_b is not restricted to $\mathbb{R}^2_+$. The set of allocations attainable for agent b, Δ_b, depends both on Γ_b and on the prices p_{ab}.

In Figure 2 the point on the outer frontier of $\Delta_b(p_{ab})$ to the left of the point B equal to (x_1^B, x_2^B) can be obtained by agent b by generating trades with agent c as to end up with the after downstream trade bundle (x_1^B, x_2^B) and selling commodity 1 for

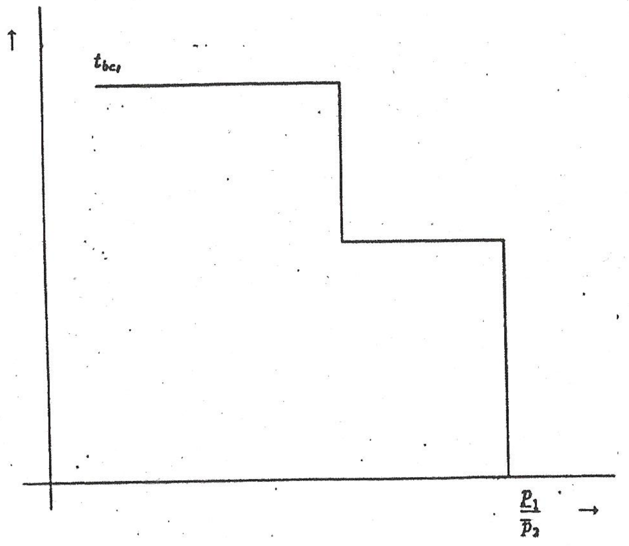

FIGURE 1: The Anticipated Net Trade Correspondence t_{bc}.

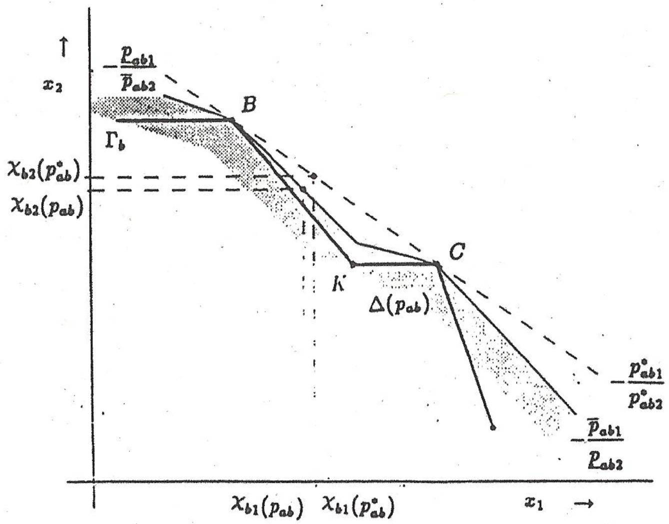

FIGURE 2: The After Downstream Trade Set Γ_b.

commodity 2 at the prices set by agent a. The points on the outer frontier of $\Delta_b(p_{ab})$ to the right of the point B but to the left of the kink K, are obtained by starting from the point B and buying commodity 1 for commodity 2. The difference in the slope to the right and to the left of B is caused by the difference in the bid and ask prices set by agent a. The point K of the kink in the outer frontier of $\Delta_b(p_{ab})$ can be obtained in two different ways. Firstly by starting from B and buying commodity 1 for commodity 2. Secondly by starting from point C and selling commodity 1 for commodity 2. The points still further to the right of this kink are obtained by starting from point C and choosing suitable trades with agent a.

The set $\{y \in \mathbb{R}^2_+ \mid \exists x \in \Gamma_b \text{ such that } y \leq x\}$ in Figure 2 is not convex. Contrary to the situation with mono pricing, agent a has the possibility to set prices p_{ab} which are such that, at least to some extend, this non convexity carries over to the set of attainable allocations $\Delta_b(p_{ab})$. Now suppose the utility function of agent b satisfies the conditions of Assumption 2.5 and the utility of agent b of the allocations (x^B_{b1}, x^B_{b2}) and (x^C_{b1}, x^C_{b2}) in Figure 2 is the same. In this case the non convexity in $\Delta_b(p_{ab})$ may lead to an optimal consumption bundle $X_b(p_{ab})$ which is strictly smaller than the corresponding consumption bundle $X_b(p^*_{ba})$ at the optimal price vector p^*_{ab} without differences in bid and ask prices. Assuming that both p_{ab} and p^*_{ab} lead to the same optimal trades between the agents b and c, the strictly smaller consumption bundle of agent b leads to a strictly larger after downstream trade bundle for agent a which, all other things remaining equal, leads to a preferred consumption bundle for agent a. This implies that, because of the non convexity of the set $\{y \in \mathbb{R}^2_+ \mid \exists x \in \Gamma_b \text{ such that } y \leq x\}$, in this stylized example the equilibrium allocations in the models with bid and ask prices and with mono pricing differ.

5 A Dynamic Interpretation of the Exchange Process

The model of hierarchically structured economies with bid and ask prices, as introduced in this paper, allows for a dynamic interpretation of the exchange process in the economy. This exchange process depends on the downstream information structure and on the possibility of leaders to write enforceable I-Owe-Yous for their direct followers.

The exchange process consists of three main parts, each consisting of a finite

number of stages. In the first part of the exchange process, the aggregated information about the equilibrium prices and net trades is disseminated through the economy. In the second part of the process the bundles of commodities that are to be sold by the individual agents to their direct leaders, given the equilibrium prices and net trades, are transferred to those leaders in exchange for enforceable I-Owe-Yous. These I-Owe-Yous promise to deliver the bundle of commodities the agent plans to buy at given equilibrium prices and net trade bundles. In the third and last part of the exchange process, the commodity bundles, as promised in the I-Owe-Yous are delivered.

Before we describe the separate parts of the exchange process, we recursively construct an ordered partition of the set of agent A, which we refer to as an echelon partition of the set of agents starting from the hierarchical tree (A, W). We define

$$S_1 = \{a \in A \mid L_a = \emptyset\}.$$

Let $n \in \mathbb{N}$ and $(S_i)_{i=1}^n$ be such that $\bigcup_{i=1}^n S_i \neq A$. Then we define

$$S_{n+1} = \{a \in A \setminus (\bigcup_{i=1}^{n} S_i) \mid L_a \cap S_n \neq \emptyset\}.$$

We stop this procedure when we find some $m \in \mathbb{N}$ such that $\bigcup_{i=1}^m S_i = A$. The **Echelon Partition** of the set of agents A that is compatible with the hierarchical tree (A, W) is the tuple $\xi := (S_1, \ldots, S_m)$ as construcuted above We find that each of the three parts of the exchange process consists of $m - 1$ stages.

The first part of the exchange process is such that the aggregated information the top agent has about the equilibria in the economy is disaggregated to the individual agents in the economy. This disaggregation of information takes place as described in the proof of Theorem 3.5, where an equilibrium for the economy is constructed. As a consequence, this disaggregation results in equilibrium prices and net trade, from which equilibrium consumption bundles result. In the first stage agent $k \in S_1$ sets prices $(p_{kj})_{j \in F_k} \in Q^{F_k}$ for his followers and proposes for each agent $j \in F_k$ the net trades $d_{jk} \in t_{kj}(p_{kj})$ which he likes to result as a consequence of these prices. For any $i \in A$ define the correspondence $t_i : Q^{F_i} \rightrightarrows Y^{F_i}$ with $\forall q_i \in Q^{F_i} : t_i(q_i) := (t_{ij}(q_{ij}))_{j \in F_i}$. We use Υ_i to denote the graph of the correspondence t_i, i.e.,

$$\Upsilon_i := \{(q_i, (e_{ji})_{j \in F_i}) \in Q^{F_i} \times Y^{F_i} \mid \forall j \in F_i : e_{ji} \in t_{ij}(q_{ij})\}.$$

The prices p_k set and net trades $(d_{jk})_{j\in F_k}$ proposed by agent k to his (direct) followers are such that

$$(p_k, (d_{jk})_{j\in F_k}) \in \text{argmax}_{(q_k,(e_{jk})_{j\in F_k})\in T_k}\{\max_{x_j\in\Phi_k((e_{jk})_{j\in F_k})} U_k(x_k)\},$$

where $\Phi_k: \ Y^{F_k} \rightrightarrows Y_+$ such that for any $(e_{jk})_{j\in F_k} \in Y^{F_k}$:

$$\Phi_k((e_{jk})_{j\in F_k}) \ := \{y_k \in Y_+ \mid \ y_k \leq \omega_k - \sum_{j\in F_k} e_{jk}\}.$$

Since, by the definition of t_{kj}, the followers can not do better by deviating, they oblige. In stage $n \in \{2,\ldots,m-1\}$ any agent $i \in S_n$ with $L_i = \{h\}$ gets prices $p_{hi} \in Q$ set for him by agent h, and net-trades $d_{ih} \in t_{hi}(p_{hi})$ proposed by this leader. Agent i chooses $(p_i, (d_{ji})_{j\in F_i}) \in Q^{F_i} \times Y^{F_i}$ such that

$$(p_i, (d_{ji})_{j\in F_i}) \in \text{argmax}_{(q_i,(e_{ji})_{j\in F_i})\in T_i}\{\max_{x_i\in\Phi_i(d_{hi},(e_{ji})_{j\in F_i})} U_i(x_i)\}$$

where $\Phi_i: \ Y \times Y^{F_i} \rightrightarrows Y_+$ such that for any $(d_{hi}, (e_{ji})_{j\in F_i}) \in Y \times Y^{F_i}$:

$$\Phi_i(d_{hi}, (e_{ji})_{j\in F_i}) \ := \{y_i \in Y_+ \mid \ y_i \leq \omega_i + d_{hi} - \sum_{j\in F_i} e_{ji}\}.$$

After $m-1$ stages this part of the exchange process, the disaggregation of information is finished. The prices set and the net trades proposed in the first part of the exchange process are equilibrium prices and corresponding equilibrium net trades. Therefore the net trades are feasible for every agent.

In the second part of the exchange process, the commodities that are to be sold by the followers to their leaders are transferred to those leaders. In return for these commodities bundles, the leaders write enforceable I-Owe-Yous that promise to deliver, in the third part of the exchange process, the commodities the followers are to buy from their leaders. In stage $n \in \{1,\ldots,m-1\}$ in this part of the exchange process, any agent $i \in S_{m+1-n}$ with $L_i = \{h\}$ transfers the bundle $d^+_{ih} := (\max\{0, d_{ihc}\})_{c\in L}$ to agent h in exchange for an enforceable I-Owe-You of agent h that promises to deliver the bundle $d^-_{ih} := (-\min\{0, d_{ihc}\})_{c\in L}$ to agent i in the third part of the exchange process.

In the third and last part of the exchange process the commodity bundles promised on the I-Owe-Yous of part two are actually delivered. In stage $n \in \{1,\ldots,m-1\}$ agent

$i \in S_n$ delivers to each of his followers $j \in F_i$ the commodity bundle d^-_{ih} in exchange for the corresponding I-Owe-You from the second stage of the exchange process. Since the I-Owe-Yous are enforceable, the corresponding commodity bundles are delivered, and every agent ends up with his (corresponding) equilibrium consumption bundle.

6 Concluding Remarks

In this paper a special kind of hierarchically structured economy is introduced and analyzed. Although the model is quite restrictive, (it only allows for two commodities, it is restricited to hierarchical structures that are hierarchical trees, and the institutional characteristics of the trade relationships are restricted to that of bid and ask prices) it allows for a positive answer to the crucial question of the existence of equilibrium. Clearly, the existence of equilibrium is a necessity if one wants to builds models in which the hierarchical structure or, in different models as the ones analyzed in this paper, the institutional characteristics of the trade relationships are to be endogenized.

In further research, the framework of the model allows us to endogenize the hierarchical structure of the economy in a way similar to the one outlined is Spanjers (1992, Chapter 4). In this sketch, the hierarchical tree of an economy is endogenously obtained starting from a structure of communication links and a particular kind of partition of the set of agents in hierarchical levels, which describes the information structure of the economy. The agents choose which of their communication links with an agent of a higher hierarchical level is transformed into a trade relationship. Thus, under certain assumptions, the choices of the agents lead to a hierarchical tree, and may lead to a model of a hierarchcally structured economy with bid and ask prices.

A different avenue to follow would be in adapting the model in a way that allows a quite general theorem on the existence of equilibrium in hierarchically structured economies. The models of hierarchically structured economies can be adapted in order to allow for the existence of equilibrium in a broad class of institutional characteristics and a broad class of hierarchical structures. Sketching this kind adaptions would leave the framework of these concluding remarks. This kind of models would not only allow for endogenizing the hierarchical structure of the economy, but they might also allow for endogenizing the institutional characteristics of the trade relationships. Last, but not least, the adaptions may allow the models to become more realistic.

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