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Concave Utility and Individual Demand

by

Yakar Kannai

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University of Bielefeld

33501 Bielefeld, Germany

Introduction

It was discovered by de Finetti in the late forties that not every convex preference ordering may be represented by a concave utility function. The problem of characterizing those convex preferences which are representable by concave utility functions (the so–called concavifiable preference orderings), and the closely related problem of constructing those concave utility functions (in the concavifiable case) was the subject of detailed studies. The results of these studies are far from obvious (even though, as noted already by Fenchel, the problem appears at first to be trivial (in the original: "Diese Frage, die auf den ersten Blick trivial erscheinen mag, ...").

The subject is esthetically appealing and intellectually challenging. Is there more to it? Consider an individual consumer possessing a certain wealth and a well defined preference ordering, who is facing a price system. The individual will demand the commodity bundle(s) which maximizes his (her) preferences in the set of all affordable bundles. It stands to reason that demands generated by concavifiable preference orderings have properties not shared by all demands generated by convex preferences. The aim of the present lecture notes is to survey those properties – to study the implications for demand of the fact that the individual maximizes a preference representable by a concave utility function.

In Lecture 1 we present a simple result concerning the behavior of demand near singular points, as well as certain simple examples of non–concavifiable preference orderings. In Lecture 2 the method – due to Fenchel – of analyzing preferences given by twice differentiable utility functions is described. In order to simplify certain computations, we introduce and apply a special coordinate system. A similar coordinate system, designed specifically to facilitate calculations concerning demand, is presented at the end of Lecture 2 and in Lecture 3. This coordinate system is applied in Lecture 4 for the study of the variation of demand with income while prices are fixed (the behavior of the Engel curves) and for characterizing those preferences for which demand is monotone (decreasing) in price (for a fixed income). The fundamental concept of a least concave utility function is described in Lecture 5. The relevance of this concept for certain bargaining setups and for demand theory is explained.

No attempt was made to present results in their strongest, most general form. Quite the other way round. Keeping with the didactic nature of the Lectures, most results were stated and proved under simplifying conditions, such as extra differentiability assumptions, low dimension, non-zero Gaussian curvature, compactness, and so on. It is hoped that in this manner we managed to convey the spirit of the subject and to stress the essential points without getting entangled in technical intricacies. References were supplied, as a rule, to enable the reader to find more general statements and their proofs. (A reader familiar with convex analysis will not find it too hard, in most cases, to understand what happens in more general cases than those treated here.) Most of the material (except for certain steps in the proof of Theorem 4.1) is elementary and is accessible to students who have a working knowledge of advanced calculus, linear algebra, and elementary mathematical economics.

These notes are based upon a series of lectures delivered by the author at the Institute for Mathematical Economics of the University of Bielefeld. I wish to thank Prof. Joachim Rosenmüller for kindly inviting me to Bielefeld. Mrs. Karin Fairfield contributed a lot to the success of my visit, and over-saw the typing of the manuscript. Mrs. Anna Glow typed the notes. Ms. Dipl.Wirt.-Math. Doris Süssenbach and Mr. Dipl.-Wirt.-Math. Oliver Weigel have rendered an important service by taking notes of my lectures and proof-reading the manuscript. I express my deep gratitude to all of them.

Lecture I

We denote a commodity bundle by $x = (x_1,\dots,x_n) \in \mathbb{R}^n$. K is a convex subset of $\mathbb{R}^n$, and $\succsim$ is a complete continuous convex preference ordering defined on K. Let I denote the income of the individual, and w her initial endowment. Let p be a non-zero element of $\mathbb{R}^n$. Then the (possibly empty-valued) demand correspondence $f(p,w,I)$ is defined by $f(p,w,I) = \{x^0 \in K : x^0 \text{ maximizes } \succsim \text{ in the set } \{x : p(x-w) \leq I\}\}$.

The budget constraint $p(x - w) \leq I$ includes as special cases the budget constraints used in partial ($w = 0$) and general ($I = 0$) equilibrium theory.
Assume now that $\succsim$ is strictly convex. Then $f(p,w,I)$ is a point valued function. Let $\succsim$ be representable by a twice continuously differentiable utility function u. We wish to show that certain concavity properties of u have interesting consequences for the behavior of the demand at singular points.

Theorem 1.1:

Let $n = 2$, and assume that u is concave in K and is strictly monotone there, i.e., $u_i(x) > 0$[1], $i = 1,2$, for all $x \in K$. Let p^0 be a price vector for which the demand f is <u>not</u> differentiable with respect to p. Then

$$\lim_{p \to p^0} \frac{\partial f_i}{\partial p_i}(p) = -\infty \qquad i = 1,2.$$

This means, for example, that

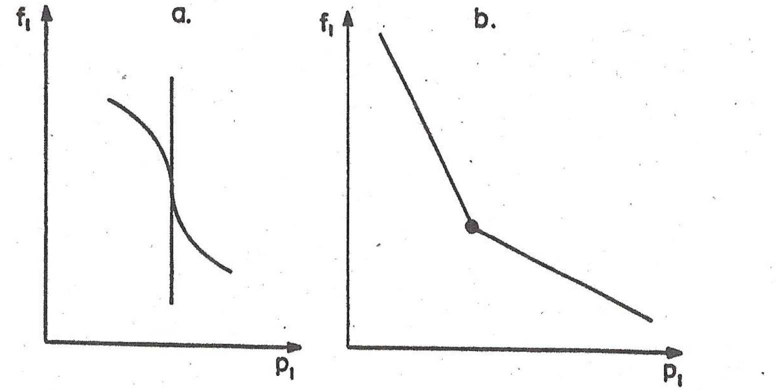

Fig. 1

[1] We denote derivatives by sub-indices. Thus, $u_i = \frac{\partial u}{\partial x_i}$.

(a) is possible, but (b) is not. An n-dimensional version of theorem 1.1, as well as a proof, may by found in [HJK]. To exhibit the elementary flavor of the argument, we prove here a somewhat weaker version of the theorem. Namely, we let p tend to p^0 in such a way that only one p_i varies. Thus, let e.g. p_1 be fixed at p_1^0, and set

$$t = \frac{p_2}{p_1}.$$

Then utility maximization implies that

$$\frac{u_2}{u_1} = \frac{p_2}{p_1} (= t)$$

and the budget constraint reads

$$p_1(x_1-w_1) + p_2(x_2-w_2) \leq I \qquad \text{or} \qquad x_1-w_2 + t(x_2-w_2) \leq \frac{I}{p_1}.$$

We denote differentiation with respect to t by $\dot{}$.
Differentiating the budget constraint yields (for the demanded x = f(p,w,I))

$$\dot{x}_1 + (x_2-w_2) + t\dot{x}_2 = 0$$

or

$$\dot{x}_1 + t\,\dot{x}_2 = w_2 - x_2. \tag{1.1}$$

Utility maximization leads to

$$tu_1(f_1(t), f_2(t)) - u_2(f_1(t),f_2(t)) = 0$$

(we have suppressed the dependence of f on w and I).

Using the chain rule, we obtain

$$u_1 + tu_{11}\dot{f}_1 + tu_{12}\dot{f}_2 - u_{21}\dot{f}_1 - u_{22}\dot{f}_2 = 0.$$

Rearranging, we get

$$(tu_{11} - u_{21})\dot{f}_1 + (tu_{12} - u_{22})\dot{f}_2 = -u_1 \tag{1.2}$$

(1.1) and (1.2) form a system of linear equations in the unknowns $\dot{f}_1$, $\dot{f}_2$, with determinant

$$\det = -t^2u_{11} + 2tu_{12} - u_{22} \tag{1.3}$$

If det $\neq 0$ at p^0, then by the implicit function theorem $\dot{f}_1$ and $\dot{f}_2$ exist and are continuous.

Non differentiability can occur only if det = 0.
The concavity of u is equivalent to the Hessian of u being negative semi-definite. Evaluate the Hessian on the vector (–t, 1):

$$\left[\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}\begin{pmatrix} -t \\ 1 \end{pmatrix}\right]\begin{pmatrix} -t \\ 1 \end{pmatrix} = t^2u_{11} - 2tu_{12} + u_{22} \leq 0$$

so det ≥ 0 for all real t. It follows that the discriminant of (1.3) satisfies

$$u_{12}^2 - u_{11}u_{22} \leq 0.$$

If det $(t^0) = 0$ $(t^0 = \frac{p_2^0}{p_1^0})$ then

$$t^0 = \frac{-2u_{12} \pm \sqrt{u_{12}^2 - u_{11}u_{22}}}{-2u_{11}}.$$

The square root, however, must vanish for t^0 real. Hence $t_0 = \frac{u_{12}}{u_{11}}$.
Solving (1.1), (1.2), for $\dot{f}_2$, we get

$$\dot{f}_2 = \frac{-u_1 - (w_2-x_2)(tu_{11}-u_{12})}{\det}. \tag{1.4}$$

The denominator is ≥ 0, the first term on the numerator is strictly negative, and the second term vanishes at $t = t^0$. Hence (1.4) implies that $\dot{f}_2 \to -\infty$ as t approaches t^0.

Example 1.2:
There exist preference relations $\succsim$ for which $\frac{\partial f_2}{\partial p_2}$ tends to $+\infty$ as p tends to p^0 – a point of non-differentiability. Those relations $\succsim$ are not concavifiable, i.e., there exist no concave u representing $\succsim$.
Let

$$u = \frac{x_2 - \alpha - x_1^4}{1 - x_1}$$

for $x_1 < 1$, $x_2 > \alpha > 0$. Then

$$u_1 = \frac{u - 4x_1^3}{1 - x_1}; \qquad u_2 = \frac{1}{1 - x_1}.$$

Choose $p_2^0 = 1$, $p_1^0 = x_2^0 - \alpha$, then $x^0 = (0,x_2^0)$. A simple computation (see [HJK]) shows that

$$\lim_{p \to p^0} \frac{\partial f_2}{\partial p_2} = +\infty.$$

This example is a slight modification of an example by Fenchel [F] of a strictly convex non–concavifiable preference ordering. Fenchel showed that the ordering induced by the utility function

$$u = \frac{x_2 - x_1^4}{1 - x_1} \qquad (1.5)$$

is non–concavifiable. The example (1.5) yields a strictly concave version of the most elementary example (also due to Fenchel) of a preference relation with straight indifference lines which are not parallel, such as given by $u = \frac{x_2}{1-x_1}$ (see figure).

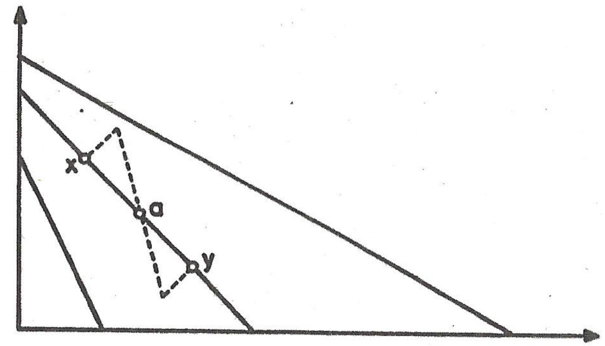

Fig. 2

There is an intuitive "mountaineering" argument for the non–concavifiability of $\frac{x_2}{1-x_1}$, due to Aumann [A] (himself a well–known mountain climber). In fact, if the order were representable by a differentiable concave utility function v, think of the figure as a contour map, the "height" being the utility v. The contour lines are closer at x than at y, which means the ground is steeper there. In particular, the ground rises faster along the dashed line perpendicular to the indifference curve at x than it falls along the parallel dashed line starting at y. So if one were to string a telegraph line along the dotted line and pull it taut, it would pass over the indifference curve containing x and y. This means that v cannot be concave.

Lecture 2

Here we consider one–point, 2nd order conditions for studying concavifiability of a preference ordering given by a C^2 utility function v. We follow the ideas of Fenchel [F], as simplified in [KN 77] by using a special coordinate system. This coordinate system is also used later on as in [KN 86], for investigating the problem when only the direction of the indifference surface is given at any point.

Thus let us assume, to begin with, that our preference ordering $\succsim$ is given by a C^2 function v defined on K, and we ask whether there exists a function $u \in C^2(K)$ such that u is concave and u represents $\succsim$, i.e., u is a strictly monotone increasing function of v. (Note that in many instances v might be easy to work with. Thus the ordering represented by the non–concave function $v(x_1, x_2) = x_1x_2$ may be also given by the utility function $u(x_1, x_2) = v^{1/2}$. A concavifiable function such as v is referred to as "indirectly concave function" by Birchenhall and Grout [BG].) Thus, we are looking for a C^2 real function $F: v(K) \to \mathbb{R}$ such that $F' > 0$ everywhere and $u = F(v)$ is concave.

Obviously, a concave function can have its gradient vanishing only at a maximal point. By the chain rule,

$$u_i(x) = F'(v(x))\, v_i(x), \quad 1 \le i \le n. \qquad (2.1)$$

Hence the condition

I) $(\nabla v)(x) = (v_1(x), \ldots, v_n(x)) \neq 0$ for all non–maximal $x \in K$.

Differentiating (2.1), we get

$$u_{ij} = F'(v)\, v_{ij} + F''(v)\, v_i v_j.$$

The 2nd order condition for concavity of u at x is that the Hessian of u at x (the matrix of second order derivatives at x) is negative semi–definite, i.e., that for all $\xi \in \mathbb{R}^n$,

$$\Sigma_{i,j=1}^{n} u_{ij}(x)\xi_i\xi_j = \Sigma_{i,j=1}^{n} F'(v)\, v_{ij}(x)\xi_i\xi_j + \Sigma_{i,j=1}^{n} F''(v)\, v_i(x)\, v_j(x)\xi_i\xi_j$$
$$= F'(v)\Big[\Sigma_{i,j=1}^{n} v_{ij}\xi_i\xi_j + \frac{F''}{F'}(v)\big(\Sigma_{i=1}^{n} v_i(x)\xi_i\big)^2\Big] \le 0. \qquad (2.2)$$

Choosing in particular a vector ξ such that $\Sigma_{i=1}^{n} v_i(x)\xi_i = 0$ we see that the second term in (2.2) vanishes and we are led to condition

II) $\Sigma_{i,j=1}^{n} v_{ij}(x)\xi_i\xi_j \leq 0$ if $\Sigma_{i=1}^{n} v_i(x)\xi_i = 0$.

Note that condition II) just states the convexity of the preference ordering $\succsim$ (or the quasi–concavity of the utility function v).

Considering now vectors ξ for which $\Sigma_{i=1}^{n} v_i(x)\xi_i \neq 0$, we may re–write (2.2) as

$$\frac{F''}{F'}(v(x)) \leq -\frac{\Sigma_{i,j=1}^{n} v_{ij}(x)\xi_i\xi_j}{(\Sigma_{i=1}^{n} v_i(x)\xi_i)^2}. \qquad (2.3)$$

Setting

$$a(x) = \sup_{\{\xi:\ \Sigma_{i=1}^{n} v_i(x)\xi_i \neq 0\}} \frac{\Sigma_{i,j=1}^{n} v_{ij}(x)\xi_i\xi_j}{(\Sigma_{i=1}^{n} v_i(x)\xi_i)^2} \qquad (2.4)$$

we see that the finiteness of the expression on the left of (2.3) implies condition

III) $a(x) < \infty$ for all $x \in K$.

Consider, for example, the quasi–concave utility function $v = \ln x_2 - \ln x_1$, representing (a rotation of) the ordering shown in Fig. 2. The Hessian of F(v) is easily computed to be

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} \frac{F'+F''}{x_1^2} & \frac{-F''}{x_1x_2} \\ \frac{-F''}{x_1x_2} & \frac{-F'+F''}{x_2^2} \end{bmatrix}$$

having a negative determinant $\frac{-F'^2}{x_1^2x_2^2}$, and thus cannot be concave. Indeed, condition III) is not satisfied.

The left hand side of (2.3) depends only on v(x) (only on the indifferent surface). Hence

IV) $G(t) = \inf_{\{x:\ v(x) = t\}} [-a(x)] > -\infty$.

Rewriting (2.3) as $\frac{F''}{F'}(t) \leq G(t)$, we obtain

V) The function G(t) majorizes the logarithmic derivative of a function H(t), where $H(t) > 0$ for $t \in \text{int}\{v(K)\}$ and $H \in C^1(v(K))$. Thus, we have proved

Fenchel's theorem: The conditions I) through V) are necessary and sufficient for the existence of a C^2 concave utility function representing $\succsim$.

We can give this theorem a more precise form. In order to avoid complications, we state here, without proving, a somewhat special case.

Theorem 2.1: If K is compact, then $\succsim$ is concavifiable in K iff G(t) is Lebesgue integrable on v(K). A concave utility function is given by $u(x) = F(v(x))$, where F is defined in v(K) by means of

$$F(v) = \int_{v(p_0)}^{v} \exp\left[\int_{v(p_0)}^{t} G(s)ds\right] dt \qquad (2.5)$$

when $p_0 \in K$ is arbitrary.

(The importance of this F will become clearer in Lecture 5.)

We wish to evaluate $a(\bar{x})$ for $\bar{x}$ non–maximal. We accomplish this by choosing a special coordinate system, at which the point under consideration becomes the origin, and the tangent hyperplane to the indifference surface through $\bar{x}$ – $\{y: y \sim \bar{x}\} = \{y: v(y) = v(\bar{x})\}$ – equals the hyperplane $x_n = 0$. Orient x_n so that $x_n > 0$ and small is preferred to $x_n = 0$. The fact that $\nabla v \perp$ tangent hyperplane implies that $(\nabla v)(\bar{x}) = (0,\ldots,0,\lambda)$, where $\lambda(\bar{x}) = |\nabla v(\bar{x})| = (\Sigma_{i=1}^{n} v_i(\bar{x})^2)^{1/2}$, or $v_i(\bar{x}) = 0$, $1 \leq i \leq n-1$, $v_n(\bar{x}) = \lambda(\bar{x})$. We also choose coordinates $x_1,\ldots, x_{n-1}$ in such a way that the matrix $(v_{ij}(\bar{x}))_{i,j=1}^{n-1}$ is diagonal. Denote the rank of this matrix by $r - 1$ $(= r(\bar{x}) - 1)$. (Thus $r(x) \leq n$.) Without loss of generality let the first $r - 1$ eigenvalues be non zero (by II) they are then negative). Then in the special coordinate system the Hessian of v looks like:

$$\begin{bmatrix} v_{11} & & & & v_{1,n} \\ & \ddots & & & \\ & & v_{r-1,r-1} & & \\ & & & 0 & v_{r,n} \\ & & & & \ddots \\ v_{n,1} & \cdots & v_{n,r} & \cdots & v_{nn} \end{bmatrix}$$

We claim that III) is equivalent to: $v_{j,n}(\bar{x}) = 0$ whenever $v_{j,j}(\bar{x}) = 0$. In fact, conside-

ring the vector $\xi = (0,\dots,\eta, 0,\dots,1)$, we see that
$\uparrow_j$

$$\frac{\sum_{i,j=1}^{n} v_{ij}\xi_i\xi_j}{(\Sigma v_i \xi_i)^2} = \frac{v_{jj}\eta^2 + 2v_{jn}\eta + v_{nn}}{\lambda^2} = (v_{nn} + 2v_{jn}\eta)/\lambda^2$$

which cannot be bounded for $\eta \in \mathbb{R}$, unless $v_{j,n} = 0$. Conversely, if $v_{j,n}(\bar{x}) = 0$ for all $r(\bar{x}) \leq j \leq n-1$, we may evaluate a(x) explicitly. Observe that $\sum_{i=1}^{n} v_i(\bar{x})\xi_i = \lambda\xi_n$. Hence by (2.4), we obtain, setting $\eta_i = \frac{\xi_i}{\xi_n}$, $1 \leq i \leq n-1$, that

$$a(x) = \sup_{\xi_n \neq 0} \frac{\sum_{i,j=1}^{n} v_{ij}\xi_i\xi_j}{\lambda^2 \xi_n^2}$$

$$= \sup_{\eta \in \mathbb{R}^{n-1}} \frac{1}{\lambda^2} \left[\sum_{i=1}^{r-1} v_{ii}\eta_i^2 + 2\sum_{i=1}^{r-1} v_{in}\eta_i + v_{nn} \right]$$

$$= \sup_{\eta \in \mathbb{R}^{n-1}} \frac{1}{\lambda^2} \left[v_{nn} + \sum_{i=1}^{r-1} v_{ii}\left(\eta_i + \frac{v_{in}}{v_{ii}}\right)^2 - \sum_{i=1}^{r-1} \frac{v_{in}^2}{v_{ii}} \right].$$

But $v_{ii} \leq 0$. Hence the supremum is attained when

$$\eta_i + \frac{v_{in}}{v_{ii}} = 0, \quad 1 \leq i \leq r-1, \quad \text{and}$$

$$a(x) = \frac{1}{\lambda^2(x)} \left[v_{nn}(x) - \sum_{i=1}^{r-1} \frac{v_{in}^2(x)}{v_{ii}(x)} \right]. \tag{2.6}$$

We can reformulate III) as

III') $\quad v_{jn} = 0 \quad$ for $\quad r \leq j \leq n-1$.

(Note that III') is really the idea behind the proof of theorem 2.1.)
Using (2.6) we may re-write IV) as

IV') $$\inf_{\{x:v(x)=t\}} \frac{1}{\lambda^2}\left(-v_{nn}(x) + \sum_{i=1}^{r-1} \frac{v_{in}^2(x)}{v_{ii}(x)}\right) > -\infty.$$

What is the geometric meaning of the eigenvalues $v_{jj}(x)$? Up to a second order, we have for small y

$$v(x+y) = v(x) + (\nabla v(x), y) + \tfrac{1}{2}\sum_{i,j=1}^{n} v_{ij}(x)\, y_i y_j \; [+ 0(|y|^2)]$$

or in the special coordinates

$$v(x+y) = v(x) + \lambda y_n + \tfrac{1}{2}\sum_{i=1}^{r-1} v_{ii} y_i^2 + \sum_{i=1}^{r-1} v_{in} y_i y_n + \tfrac{1}{2} v_{nn} y_n^2 + \dots$$

The indifference surface through x satisfies $v(x+y) = v(x)$ or, up to second order,

$$\tfrac{1}{2} v_{nn} y_n^2 + \left(\lambda + \sum_{i=1}^{r-1} v_{in} y_i\right) y_n + \tfrac{1}{2}\sum_{i=1}^{r-1} v_{ii} y_i^2 = 0.$$

For small $|y|$ we get

$$y_n \simeq \frac{-\sum_{i=1}^{r-1} v_{ii} y_i^2}{2\lambda}.$$

The indifference surface is approximately a paraboloid, and the v_{ii} are proportional to the principal curvatures. In particular, if the Gaussian curvature is non zero, then all the numbers $v_{ii} \neq 0$ and $r(x) = n$. The condition III') is autimatically satisfied. If, in addition, K is compact, then IV') and V) are also satisfied and we have:

Proposition 2.2: If $\succsim$ is representable by a C^2 utility function satisfying I) and II), K is compact and the Gaussian curvature of the indifference surfaces never vanishes, then $\succsim$ is concavifiable, and there exists a C^2 concave utility function representing $\succsim$.

In order to deal more directly with questions of demand, we follow the approach of Debreu [D 72], and assume that our primary data consist of a unit vector function g(x) defined on K and representing the unit normal to the indifference surface through x, directed towards increasing utility. (Note that g(x) is proportional to the price system p at which x is demanded). We fix a point $\bar{x} \in K$, and introduce (as above) a special coordinate system adapted for analyzing the behavior of g near $\bar{x}$. We do not move the origin. But we choose the coordinates so as to make the hyperplane $x_n = \bar{x}_n$ tangent to the indifference hypersurface $\{y: y \sim \bar{x}\}$ at $\bar{x}$.

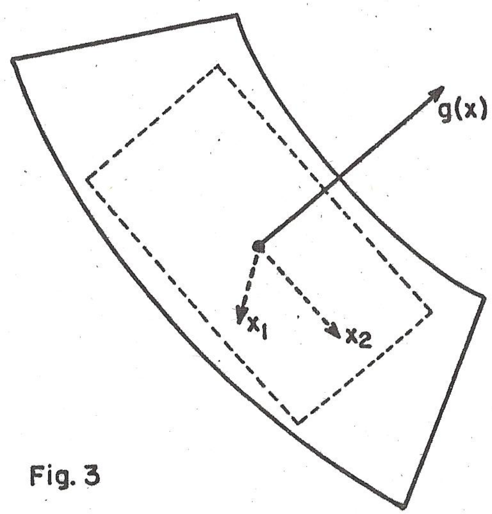

Fig. 3

In this system

$$g_i(\bar{x}) = 0 \qquad 1 \le i \le n-1 \qquad g_n(\bar{x}) = 1. \tag{2.7}$$

Differenciation of the identity

$$\Sigma_{i=1}^{n} g_i^2(x) - 1 = 0$$

yields

$$\Sigma_{i=1}^{n} g_i(x) \frac{\partial g_i}{\partial x_k} = 0, \qquad 1 \le k \le n,$$

which, together with (2.7), implies

$$\frac{\partial g_n}{\partial x_k}(\bar{x}) = 0, \qquad 1 \le k \le n. \tag{2.8}$$

We wish to investigate the matrix $(\frac{\partial g_i}{\partial x_j}(\bar{x}))^{n-1}_{i,j=1}$.
The famous integrability conditions in [D 72] just state that this matrix is symmetric.

Proposition 2.3: Assume that there exists a C^2 utility function u representing $\succsim$ near $\bar{x}$. Then the matrix $(\frac{\partial g_i}{\partial x_j}(\bar{x}))^{n-1}_{i,j=1}$ is symmetric.

Proof: The gradient of u at x, $\nabla u(x)$, is orthogonal (at x) to the indifference surface through x, hence is proportional to the unit vector g(x), and

$$\nabla u(x) = \lambda(x) g(x) \qquad \text{where} \quad \lambda(x) = |\nabla u(x)|, \qquad \text{or}$$

$$\frac{\partial u}{\partial x_i}(x) = \lambda(x)\, g_i(x) \qquad 1 \le i \le n. \tag{2.9}$$

Differentiating (2.9) with respect to x_j we get

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \frac{\partial \lambda}{\partial x_j}(x)\, g_i(x) + \lambda(x) \frac{\partial g_i}{\partial x_j}(x). \tag{2.10}$$

Evaluating (2.10) at $x = \bar{x}$ and using (2.7), we get

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(\bar{x}) = \lambda(\bar{x}) \frac{\partial g_i}{\partial x_j}(\bar{x}), \qquad 1 \le i,j \le n-1. \tag{2.11}$$

The equality of the mixed derivatives states that $\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}$.

Hence $\frac{\partial g_i}{\partial x_j}(\bar{x}) = \frac{\partial g_j}{\partial x_i}(\bar{x})$ for $1 \le i,j \le n-1$.

We choose now coordinates $x_1, \ldots, x_{n-1}$ (in the directions of the tangent hyperplane to the indifference surface through $\bar{x}$) so that the symmetric (n–1)x(n–1) matrix $(\frac{\partial g_i}{\partial x_j}(\bar{x}))^{n-1}_{i,j=1}$ is diagonal. Denote the eigenvalues by $\alpha_1, \ldots, \alpha_{n-1}$. They are proportional to the principal curvatures of the indifference hypersurface $\{x: x \sim \bar{x}\}$, and are non–positive by convexity of $\succsim$.

Evaluating (2.10) for i = n, substituting $x = \bar{x}$ and using (2.8), we obtain

$$\frac{\partial^2 u}{\partial x_n \partial x_j}(\bar{x}) = \frac{\partial \lambda}{\partial x_j}(\bar{x}) \qquad (\text{for } 1 \le j \le n). \tag{2.12}$$

On the other hand, evaluating (2.10) for j = n, substituting $x = \bar{x}$, and using (2.7), we obtain

$$\frac{\partial^2 u}{\partial x_i \partial x_n}(\bar{x}) = \lambda(\bar{x}) \frac{\partial g_i}{\partial x_n}(\bar{x}) \qquad (\text{for } 1 \le i \le n-1). \tag{2.13}$$

By the equality of mixed derivatives $\frac{\partial^2 u}{\partial x_n \partial x_i} = \frac{\partial^2 u}{\partial x_i \partial x_n}$, we get from (2.12) and (2.13), setting

$$\frac{\partial g_i}{\partial x_n}(\bar{x}) = \beta_i(\bar{x}) \qquad 1 \le i \le n-1 \tag{2.14}$$

that

$$\frac{\partial \lambda}{\partial x_i}(\bar{x}) = \lambda(\bar{x})\, \beta_i(\bar{x}) \qquad 1 \le i \le n-1. \tag{2.15}$$

The relations (2.15) mean that the value of $\lambda(x)$ in any point of the indifference surface through $\bar{x}$ is uniquely determined, once $\lambda(\bar{x})$ is given. For the equations (2.15) determine the derivatives of λ in all directions parallel to the indifference hypersurface. This is clear from a geometric point of view.

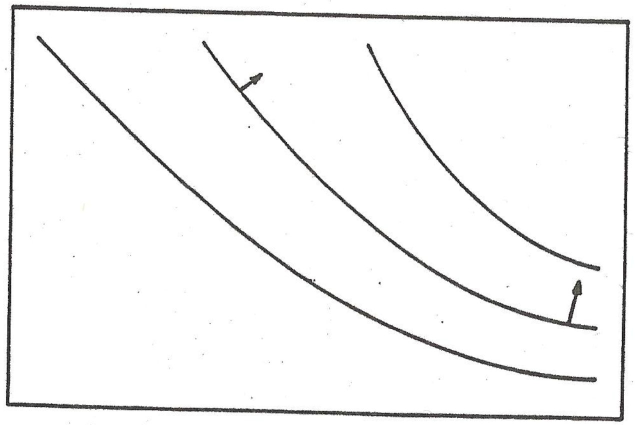

Fig. 4

For $\lambda(x)$, or the size of the gradient of u at x, determines how close the indifference surfaces come to each other – how dense they are at the point x. There is no absolute measure for this. But the relative densities of indifference surfaces along the same indifference surface are well defined. – Compare also the discussion in [KN 77, pp. 18–20], in particular the derivation of formula (3.7) there, where a similar argument, which covers the case of non–smooth indifference sets, is given.
Using (2.11) (in the diagonal form) for $1 \le i,j \le n-1$, (2.12), (2.13) and (2.15), we see that the Hessian matrix of u at $\bar{x}$ is given by

$$\begin{bmatrix} \lambda\alpha_1 & & & \lambda\beta_1 \\ & \ddots & & \vdots \\ & & \lambda\alpha_{n-1} & \lambda\beta_{n-1} \\ \lambda\beta_1 & \cdots & \lambda\beta_{n-1} & \frac{\partial\lambda}{\partial x_n} \end{bmatrix} \qquad \text{for } i \le j \le n.$$

(That $u_{nn} = \frac{\partial\lambda}{\partial x_n}$ follows from (2.10) and (2.8).)

Using condition III') or the argument leading to it, we see that a necessary condition for concavity of u is that $\beta_i = 0$ whenever $\alpha_i = 0$ $(\lambda \neq 0)$. We assume that $\alpha_1(\bar{x}),\dots,\alpha_{r-1}(\bar{x}) < 0$, but $\alpha_r(\bar{x}),\dots,\alpha_{n-1}(\bar{x}) = 0$. Consider the quadratic form

$$q(\xi) = (D^2u(\bar{x})\xi,\xi) = \lambda\Sigma_{i=1}^{r-1}\alpha_i\xi_i^2 + 2\lambda\Sigma_{i=1}^{r-1}\beta_i\xi_i\xi_n + \frac{\partial\lambda}{\partial x_n}\xi_n^2.$$

Then $q(\xi) \le 0$ if $\xi_n = 0$. Set $\eta_i = \frac{\xi_i}{\xi_n}$, $1 \le i \le n-1$, for $\xi_n \neq 0$. Then

$$q(\xi) = \xi_n^2\,[\lambda\Sigma_{i=1}^{r-1}\alpha_i\eta_i^2 + 2\lambda\Sigma_{i=1}^{r-1}\beta_i\eta_i + \frac{\partial\lambda}{\partial x_n}]$$

$$= \xi_n^2\,[\lambda\Sigma_{i=1}^{r-1}\alpha_i(\eta_i + \frac{\beta_i}{\alpha_i})^2 + \frac{\partial\lambda}{\partial x_n} - \lambda\Sigma_{i=1}^{r-1}\frac{\beta_i^2}{\alpha_i})] \le \xi_n^2(\frac{\partial\lambda}{\partial x_n} - \lambda\Sigma_{i=1}^{r-1}\frac{\beta_i^2}{\alpha_i}). \qquad (2.16)$$

(Notice the similarity between (2.16) and (2.6).)

Set

$$M(x) = \lim_{y \to x}\sup\ [-\Sigma_{i=1}^{r-1}\frac{\beta_i^2}{\alpha_i}(y)]. \qquad (2.17)$$

We have essentially proved:

Proposition 2.4: The ordering $\succsim$ is concavifiable near $\bar{x}$ with a C^2 concave utility function if

(i) $\alpha_i \le 0$, $\qquad 1 \le i \le n-1$

and

(ii) there exists a C^1 function λ near $\bar{x}$ such that $\lambda(\bar{x}) \neq 0$, the equations (2.15) are satisfied, and

$$\frac{\partial\lambda}{\partial x_n}(x) + M(x) \le 0 \qquad (2.18)$$

for all x near $\bar{x}$.

Note 2.5: It is well–known (see e.g. [BT]) that the set of points for which all α_i are non–zero is dense in every strictly convex hypersurface, in particular in every indifference surface. Hence we may modify (2.17) and set

$$M(x) = \lim_{\{y:\, y \sim x,\ \Pi_{i=1}^{n-1}\alpha_i(y) \neq 0\}}\sup\ [-\Sigma_{i=1}^{n-1}\frac{\beta_i^2}{\alpha_i}(y)]. \qquad (2.17')$$

We supply here a proof, based upon Sard's theorem, of the fact that the set of points for which the curvature of a strictly convex curve does not vanish is dense (the restriction to the one–dimensional case is done only to simplify the exposition). The problem being local, consider a strictly convex curve defined by $y = \varphi(x)$ for $x \in (a,b) \subset \mathbb{R}^1$, where $\varphi \in C^2((a,b))$ and φ is strictly concave in (a,b). Consider the map $\psi: (a,b) \to \mathbb{R}^1$ given by

$$\psi(x) = \varphi'(x). \qquad (2.19)$$

Then x is a critical point for ψ if and only if $\psi'(x) = \varphi''(x) = 0$.
The strict concavity of φ implies that ψ is strictly monotone, hence invertible with a continuous inverse. By Sard's theorem, the set $\mathbb{E} = \{t \in \mathbb{R}^1$: There exists x such that $\psi(x) = t$ and $\psi'(x) = 0\}$ has Lebesgue measure zero. We can write

$$\mathbb{E} = \{t \in \mathbb{R}^1 : \psi'(\psi^{-1}(t)) = 0\}.$$

Let now $\bar{x} \in (a,b)$ be such that the curvature of the curve $y = \varphi(x)$ vanishes at $(\bar{x},\varphi(\bar{x}))$. Then $\varphi''(\bar{x}) = 0$. Set $\varphi'(\bar{x}) = \bar{t}$, so that $\bar{t} \in \mathbb{E}$. There exists a sequence $t_n \to \bar{t}$ with $t_n \notin \mathbb{E}$ – otherwise $\mathbb{E}$ would contain an open interval and have positive Lebesgue measure. Let $x_n = \psi^{-1}(t_n)$. By continuity, $x_n \to \bar{x}$, and $\varphi''(x_n) = \psi'(x_n) \neq 0$.

Note also that the set of points where none of the α_i vanishes is open (by continuity of the functions α_i, $1 \le i \le n$).

Lecture 3

We apply the special coordinate system(s) constructed earlier for studying the derivatives of an individual demand function with respect to prices and income. We consider for simplicity the case $w = 0$, the formulas requiring obvious modifications in the more general case. Let thus $\bar{p}$, $\bar{I}$ be fixed price vector and income and let $\bar{x} = f(\bar{p},\bar{I})$ be the bundle demanded at $\bar{p}$, $\bar{I}$. We consider f near $(\bar{p},\bar{I})$. The budget constraint equation is

$$\Sigma_{j=1}^{n} p_j x_j = px = I \tag{3.1}$$

Utility maximization implies that price p is the normal to the indifference surface through x. A unit normal is given by $g(x)$. Hence

$$g(x) = \frac{p}{|p|} \tag{3.2}$$

(here and in the sequel $|p|$ denotes the Euclidean norm of p).
Differentiating formally (3.1) and (3.2) near $\bar{p}$, $\bar{I}$ with respect to p and I, we obtain from the chain rule

$$\Sigma_{j=1}^{n} p_j \frac{\partial f_j}{\partial p_k} = -x_k, \qquad 1 \le k \le n \tag{3.3}$$

$$\Sigma_{j=1}^{n} \frac{\partial g_i}{\partial x_j} \frac{\partial f_j}{\partial p_k} = \frac{\delta_{ik}}{|p|} - \frac{p_i p_k}{|p|^3}, \qquad 1 \le i,k \le n \tag{3.4}$$

$$\Sigma_{j=1}^{n} p_j \frac{\partial f_j}{\partial I} = 1, \tag{3.5}$$

$$\Sigma_{j=1}^{n} \frac{\partial g_i}{\partial x_j} \frac{\partial f_j}{\partial I} = 0, \qquad 1 \le i \le n \tag{3.6}$$

Let us normalize prices (and income) so that $|\bar{p}| = 1$, and use the special coordinate system introduced earlier. Then

$$\bar{p}_i = g_i(\bar{x}) = 0, \qquad 1 \le i \le n-1,$$

$$\bar{p}_n = g_n(\bar{x}) = 1 \tag{3.7}$$

Substituting (3.7) in (3.3) and (3.5), we get

$$\frac{\partial f_n}{\partial p_k}(\bar{p},\bar{I}) = -\bar{x}_k \qquad 1 \le k \le n \tag{3.3'}$$

and

$$\frac{\partial f_n}{\partial I}(\bar{p},\bar{I}) = 1 \tag{3.5'}$$

We also see from (2.8) that the equations (3.4) and (3.6) are vacuous for $i = n$ (all

coefficients on the left side vanish identically).
For $1 \le i < n$, we use the diagonal form of the matrix $\left(\frac{\partial g_i}{\partial x_j}(\bar{x})\right)_{i,j=1}^{n-1}$, as well as the formulas (2.14), to rewrite the left hand sides of (3.4) and (3.6) at $\bar{x}$, $\bar{p}$, $\bar{I}$ as follows:

$$\Sigma_{j=1}^{n} \frac{\partial g_i}{\partial x_j} \frac{\partial f_j}{\partial p_k} = \Sigma_{j=1}^{n-1} \frac{\partial g_i}{\partial x_j} \frac{\partial f_j}{\partial p_k} + \frac{\partial g_i}{\partial x_n} \frac{\partial f_n}{\partial p_k} = \alpha_i \frac{\partial f_i}{\partial p_k} + \beta_i \frac{\partial f_n}{\partial p_k}$$

and

$$\alpha_i \frac{\partial f_i}{\partial I} + \beta_i \frac{\partial f_n}{\partial I}, \qquad 1 \le i \le n-1, 1 \le k \le n.$$

Using now (3.7) for the right hand sides of (3.4) and applying (3.3') and (3.5'), we obtain

$$\alpha_i(\bar{x}) \frac{\partial f_i}{\partial p_k}(\bar{p},\bar{I}) - \beta_i(\bar{x})\,\bar{x}_k = \frac{\delta_{ik}}{|\bar{p}|}$$

or

$$\frac{\partial f_i}{\partial p_k}(\bar{p},\bar{I}) = \frac{\delta_{ik} + \beta_i(\bar{x})\,\bar{x}_k}{\alpha_i}, \qquad 1 \le i \le n-1, \quad 1 \le k \le n \tag{3.4'}$$

and

$$\frac{\partial f_i}{\partial I}(\bar{p},\bar{I}) = -\frac{\beta_i(\bar{x})}{\alpha_i(\bar{x})}, \qquad 1 \le i \le n-1. \tag{3.6'}$$

We may now make a rigorous argument, using the implicit function theorem, and conclude that if $\alpha_i(\bar{x}) \neq 0$ for all $1 \le i \le n-1$ (i.e. the Gaussian curvature of the indifference hypersurface through $\bar{x}$ does not vanish at $\bar{x}$) then the demand f is continuously differentiable with respect to prices (and income), the derivatives being given by the formulas (3.3') through (3.6') (these formulas are just the Slutsky formulas). Conversely, if the demand is price differentiable at $(\bar{p},\bar{I})$, and $\alpha_i(\bar{x}) = 0$ for some $1 \le i \le n-1$, then by (3.4') the finiteness of $\frac{\partial f_i}{\partial p_n}$ implies that $\beta_i(\bar{x})$ vanishes as well $(\bar{x}_n = (\bar{p},\bar{x}) = \bar{I} > 0)$. But then $\frac{\partial f_i}{\partial p_i}$ cannot be finite. Hence $\alpha_i(\bar{x}) \neq 0$ for all $1 \le i \le n-1$ and we have proved

Theorem 3.1 (Debreu [D 72]): The demand function $f(\bar{p}, \bar{I})$ is price differentiable at $(\bar{p},\bar{I})$ if, and only if, the Gaussian curvature of the indifference hypersurface through $\bar{x} = f(\bar{p},\bar{I})$ is non zero at $\bar{x}$, i.e., all the $\alpha_i(\bar{x})$, $1 \le i \le n-1$, are non zero.

Note that by (3.5') and (3.6') (or by homogeneity of $f(\bar{p}, \bar{I})$) the demand is income differentiable whenever the demand is price differentiable. The converse is not true.

Example 3.2: Consider the preference ordering represented near $\bar{x} = (1, 1)$ (in $\mathbb{R}^2$) by the utility function

$$u(x_1, x_2) = x_1 + x_2 - (x_1 - x_2)^4.$$

Then

$$\nabla u = (1 - 4(x_1 - x_2)^3,\ 1 + 4(x_1 - x_2)^3)$$

and $\quad u_i(x) > 0, \quad i = 1, 2, \quad$ for x near $\bar{x}$.

Moreover equating $\frac{p_2}{p_1}$ to $\frac{u_2}{u_1}$ we see that at the demanded point x

$$\frac{1 + 4(x_1 - x_2)^3}{1 - 4(x_1 - x_2)^3} = \frac{p_2}{p_1}$$

from which we easily compute the demand function

$$x_1 = f_1 = \frac{I}{p_1 + p_2} + \frac{p_2(p_2 - p_1)^{1/3}}{4^{1/3}(p_1 + p_2)^{4/3}},$$

$$x_2 = f_2 = \frac{I}{p_1 + p_2} - \frac{p_1(p_2 - p_1)^{1/3}}{4^{1/3}(p_1 + p_2)^{4/3}}. \tag{3.7}$$

Setting $\bar{p} = (1, 1)$, $\bar{I} = 2$, we see from (3.7) that the demand is differentiable in I (actually linear in I) without being differentiable with respect to p_1 and p_2 (due to the vanishing of the fourth-order indifference curve at (1, 1)).

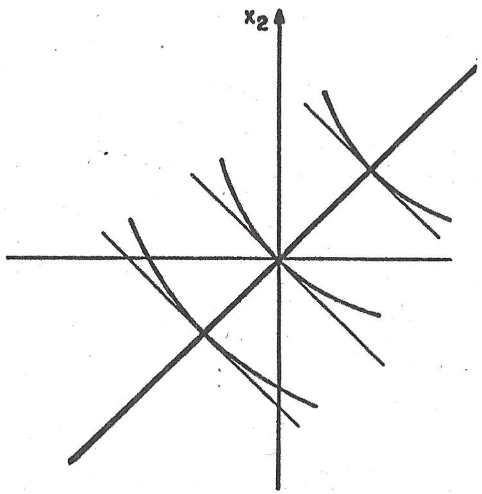

Fig. 5

If follows from the discussion in Lecture 2 and Theorem 3.1 that the only difference between local behavior of demand functions generated by concavifiable preference orderings and those generated by non-concavifiable ones may be perceived near points at which demand is not price differentiable, i.e., points where some α_i vanish. Thus, Theorem 1.1 is no mere coincidence.

Note 3.3: The formula (3.5') is valid whether or not demand is price differentiable. In fact, the stronger relation

$$f_n(p, I_2) - f_n(p, I_1) = |p|(I_2 - I_1) \tag{3.8}$$

follows at once from the budget equations

$$f_n(p, I_j) = (f(p, I_j), \frac{p}{|p|}) = \frac{I_j}{|p|}$$

valid for $j = 1, 2$.

The proportionality factor $\lambda(x)$, appearing in (2.9) (and equal to $|\nabla u(x)|$) may be interpreted as the marginal utility of income [SA]. For $u(f(p, I))$ is the maximal utility attainable by the individual possessing wealth I and facing prices p (the indirect utility function).

However

$$\frac{\partial u(f(p,I))}{\partial I} = \Sigma_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial f_i}{\partial I} = \lambda\, \Sigma_{i=1}^{n} g_i \frac{\partial f_i}{\partial I}. \tag{3.9}$$

For normalized prices ($|p| = 1$) we obtain from (3.9) that at $\bar{p}$, $\bar{I}$,

$$\frac{\partial u}{\partial I} = \lambda(\bar{x}) = \lambda(f(\bar{p}, \bar{I})) \tag{3.10}$$

Differentiating (3.10) once more with respect to income we get

$$\frac{\partial^2 u}{\partial I^2} = \frac{\partial \lambda}{\partial I}(f(\bar{p}, \bar{I})) = \Sigma_{i=1}^{n} \frac{\partial \lambda}{\partial x_i}\frac{\partial f_i}{\partial I}. \tag{3.11}$$

Substituting (2.15), (3.5') and (3.6') in (3.11), we conclude that

$$\frac{\partial^2 u}{\partial I^2} = \frac{\partial \lambda}{\partial I}(f(\bar{p}, \bar{I})) = -\lambda(\bar{x})\ \Sigma_{i=1}^{n-1} \frac{\beta_i^2(\bar{x})}{\alpha_i(\bar{x})} + \frac{\partial \lambda}{\partial x_n}(\bar{x}). \tag{3.12}$$

From (2.16) we get

Proposition 3.4: The utility function u is concave if and only if the marginal utility of income is a non-increasing function of income along the Engel curve, or, equivalently, if and only if the indirect utility is a concave function of income.

Note also the similarity between the right hand side of (3.12) and the expression (2.18).

Lecture 4

The formulas developed in lecture 3 may be used in order to study many properties of income and price derivatives of demand, particularly (but not limited to) near points of non–differentiability. Theorem 1.1 might be formulated so as to hold in $n > 2$ dimensions, see [HJK]. We will consider in this lecture the behavior of income derivatives near points where the Gaussian curvature vanishes. We will also characterize price–monotone demand functions.

Recall that an Engel curve is the graph of the map $f(p, I)$ mapping income I into $\mathbb{R}^n$ while prices p are held fixed. Thus the Engel curve is the inverse image of g parameterized by I (= g).

Recall that a smooth curve is said to intersect a smooth hypersurface **transversally** at a point x if the tangent to the curve at x is not contained in the tangent space of the hypersurface. The intersection of a family of curves with a family of hypersurfaces is **uniformly transversal** if the angles between the tangents of the curves and the tangent spaces of the hypersurfaces are bounded away from zero. We wish to generalize these concepts for Engel curves which might not be smooth. Accordingly, we say that the curve C intersects the smooth hypersurface transversally at x if there exists a neighborhood U of x and a (double sheeted) circular (proper) cone K whose vertex is at x and whose axis is normal to the hypersurface at x, such that $U \cap C \subset K$. We say that the intersection of a family of curves with a family of hypersurfaces is uniformly transversal if the cones K can be taken to be isometric to the same proper (pointed) cone. (It is obvious that the definitions coincide with the earlier ones in the smooth case.)

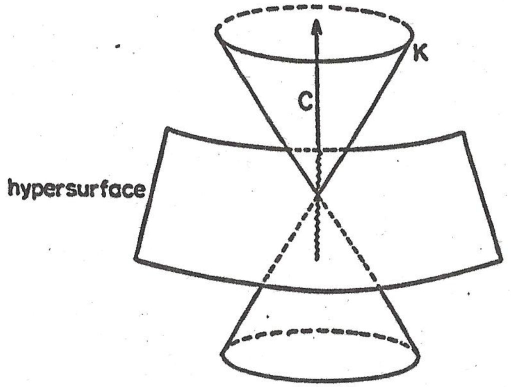

Fig. 6

We state now our main result about the behavior of Engel curves.

Theorem 4.1: Let $\bar{x} = \bar{x}(\bar{p}, \bar{I})$. The following statements are equivalent:

1) The demand function $f(p, I)$ is uniformly Lipschitz in income for p near $\bar{p}$ and I near $\bar{I}$.
2) The Engel curves intersect the indifference hypersurfaces uniformly transversally near $\bar{x}$.
3) There exists a constant M and a neighborhood V of $\bar{x}$ such that for all $x \in V$ and all $1 \leq j \leq n-1$ for which $\alpha_j(x) \neq 0$, we have

$$\left|\frac{\beta_i(x)}{\alpha_j(x)}\right| \leq M \qquad (4.1)$$

Supplement 4.2: If statement 3) from Theorem 4.1 holds, then

1) Almost all Engel curves are continuously differentiable in a neighborhood of $\bar{x}$.
2) The Engel curves are differentiable almost everywhere in I (near $\bar{I}$) for all p near $\bar{p}$.
3) The preference relation $\succsim$ is concavifiable.

Remark 4.3: Uzawa's theorem [U], that demand is income Lipschitzian if there are no inferior goods, is a special case of Theorem 4.1. In fact, one has only to take a (double sheeted) circular cone containing the positive and the negative orthants and having the line through the price vector p as axis. Then the cone K can be taken to be a translate of that cone.

In order to avoid technical difficulties which arise from the possibility that $\alpha_j(x)$ might be different from zero for one, but not all $1 \leq j \leq n-1$, we will prove Theorem 4.1 and Supplement 4.2 for the case $n = 2$. The proof for the general case may be found in [KN 86]. Note that the equivalence of statements 1) and 2) (of Theorem 4.1) is not more difficult for general n than for $n = 2$.

Proof:

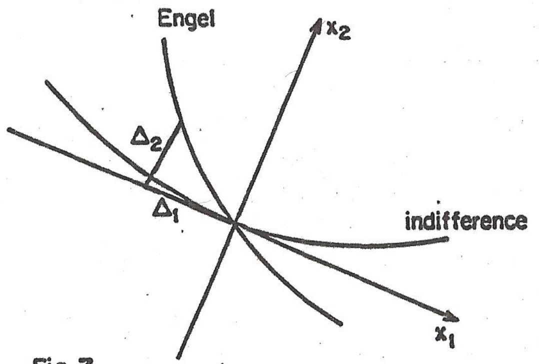

Fig. 7

We show first the equivalence of statements 1) and 2) of Theorem 4.1. In fact, let I_1, I_2 be two incomes. Then (see Fig.7; the coordinate system is taken with respect to $f(p, I_1)$)

$$\frac{\Delta_1}{\Delta_2} = \frac{|f_1(p, I_2) - f_1(p, I_1)|}{|f_2(p, I_2) - f_2(p, I_1)|}. \qquad (4.2)$$

By (3.8) the component f_2 of the demand satisfies always a Lipschitz condition, namely

$$\frac{|f_2(p, I_2) - f_2(p, I_1)|}{|I_2 - I_1|} = |p|. \qquad (4.3)$$

Hence the demand satisfies a uniform Lipschitz condition if and only if there exists a constant L such that for all I_1, I_2 (near $\bar{I}$) we have

$$\frac{|f_1(p, I_2) - f_1(p, I_1)|}{|I_2 - I_1|} \leq L. \qquad (4.4)$$

Hence the Engel curves intersect the indifference surface uniformly transversally, i.e., the ratio $\frac{\Delta_1}{\Delta_2}$ is uniformly bounded, if and only if the left hand side of (4.4) (which is just $\frac{\Delta_1}{\Delta_2}/|p|$ by (4.3)) is uniformly bounded.

If statement 1) of Theorem 4.1 holds and $\alpha_1(\bar{x}) \neq 0$, then by (3.6')

$$-\frac{\beta_1(\bar{x})}{\alpha_1(\bar{x})} = \frac{\partial f_1}{\partial I}(\bar{p}, \bar{I}) = \lim_{h \to 0} \frac{f_1(\bar{p}, \bar{I} + h) - f_1(\bar{p}, \bar{I})}{h}. \qquad (4.5)$$

The Lipschitz bound (4.4) implies that the right hand side of (4.5) is bounded by L, hence (4.1) holds (with M = L).

The converse (3) implies 1)) is deeper. We prove first statement 1) of Supplement 4.2. Thus, assume that (4.1) holds, and let x^0 be a point where α_1 vanishes. By [BT] (see Note (2.5)) there exists a sequence $x_n \to x^0$ such that $\alpha_1(x_n) \neq 0$. The normal direction at x and the direction parallel to the indifference curve at x depends smoothly on x (here the assumption that the dimension n = 2 is used in an essential way). Hence $\alpha_1(x_n) \to \alpha_1(x^0)$ and $\beta_1(x_n) \to \beta_1(x^0)$.
By (4.1), $|\beta_1(x_n)| \leq M\,|\alpha_1(x_n)|$. Hence $\beta_1(x^0) = 0$. Consider now the map $g\colon V \to S^1$. Then we may write the differential of g at x (using our coordinate system attached to x) in the form

$$Dg = \begin{bmatrix} \alpha_1(x), & \beta_1(x) \\ 0 & 0 \end{bmatrix} \qquad (4.6)$$

By the "hard" Sard theorem [ST, p.47] the set of critical values of g has measure zero. In our case the maximal rank of Dg (x) is one. If x is not a critical point of g, then $\alpha_1(x) \neq 0$. For if $\alpha_1(x) = 0$ we just proved that $\beta_1(x) = 0$ and Dg (x) = 0. Hence for almost all p (near $\bar{p}$), $x \in g^{-1}(p)$ is non–critical and this in turn implies that $\alpha_1(x) \neq 0$. Thus for all x on the Engel curve $x = f(p,\cdot)$ we have $\alpha_1(x) \neq 0$, so that f is continuously differentiable, and statement 1) of supplement 4.2 follows. Let now p be a non–critical value of g, so that $f_1(p, I)$ is continuously differentiable and (3.6') holds. By the mean value theorem of the differential Calculus,

$$\frac{f_1(p, I_2) - f_1(p, I_1)}{I_1 - I_2} = \frac{\partial f_1}{\partial I}(p, I_3) = -\frac{\beta_1(f(p, I_3))}{\alpha_1(f(p, I_3))} \qquad (4.7)$$

for a certain $I_3 \in (I_1, I_2)$. It follows from (4.1) that the left hand side of (4.7) is bounded (in absolute value) by M. Let now p be arbitrary (near $\bar{p}$). There exists a sequence $p_n \to p$ with p_n non–critical values of g. By uniform continuity, (4.1) and (4.7),

$$|f_1(p, I_2) - f_1(p, I_1)| = \lim_{n \to \infty} |f_1(p_n, I_2) - f_1(p_n, I_1)| \leq M|I_2 - I_1|.$$

Hence (4.4) holds and the demand is uniformly Lipschitz continuous.

Statement 2) of supplement 4.2 follows from the income Lipschitz continuity of the demand and the well–known Rademacher Theorem on the differentiability almost everywhere of Lipschitz functions.

Statement 3) of the supplement follows from Proposition 2.4 and Note 2.5. For if (4.1) holds for almost all x in V then the continuity of the function $\beta_j(x)$ implies that M(x) as given by (2.17') is bounded. Then a non zero λ satisfying (2.18) may be found.

Example 4.4: The utility function u is given near the origin by

$$u = x_2 - x_1^4 - x_2^2 - x_1 x_2^2 - 2 x_1^2 x_2^2 \tag{4.8}$$

(A simple rotation and translation turn this into a monotone ordering with positive income.) Then

$$u_1 = -4 x_1^3 - x_2^2 - 4 x_1 x_2^2, \qquad u_2 = 1 - 2 x_2 - 2 x_1 x_2 - 4 x_1^2 x_2 \tag{4.9}$$

and a simple computation shows that near the origin $u_{11} \leq 0$, $u_{22} < 0$ and that

$$u_{11} u_{22} - u_{12}^2 = 24 x_1^2 + 4 x_2^2 + 0(x_1^3) + 0(x_1 x_2^2) \tag{4.10}$$

for $|x| \to 0$. Hence the ordering represented by u is strictly convex near the origin, u is concave near the origin, the demand is differentiable except at those (p, I) for which f(p, I) = 0, i.e., I = 0, $p_1 = 0$. The Engel curve passing through the origin is given implicitly by $u_1 = 0$, or, according to (4.9), by $x_2^2(1 + 4x_1) = -4x_1^3$. It follows that for small x_1, $x_2 \approx \pm 2(-x_1)^{3/2}$. Hence the Engel curve has a cusp at the origin, the branches being tangent to the indifference curve (see Figure 8). Clearly, the non–smooth Engel curve is not transversal to the indifference surface.

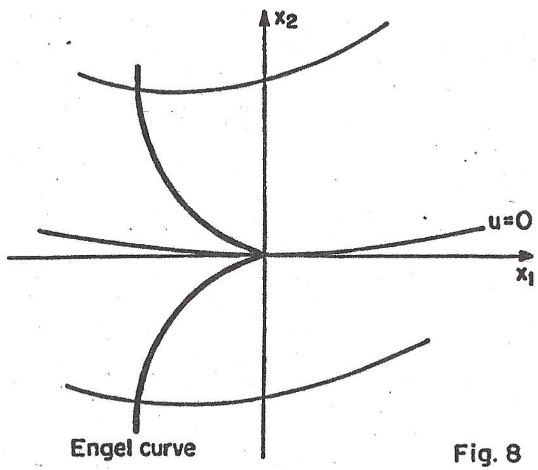

Fig. 8

Note 4.5: The Engel curve may be tangent to the indifference surface and be nevertheless smooth (even though the parametric representation of the curve as a function of I is non–differentiable), see Remark 1 on page 313–314 in [KN 86] and Figure 3 there (the equation of the Engel curve is given approximately by $x_2 = x_1^3$).

We turn now our attention to the study of the price dependence of the demand. We recall

Definition 4.6: The demand is (price) monotone at $\bar{p}$, $\bar{I}$ if for all $\xi \in \mathbb{R}^n$,

$$\Sigma_{i,j=1}^{n} \frac{\partial f_i}{\partial p_j}(\bar{p}, \bar{I}) \xi_i \xi_j \leq 0 . \tag{4.11}$$

Observe that (i) the matrix $\frac{\partial f_i}{\partial p_j}$ is not symmetric in general and (ii) by choosing ξ to be a unit vector e_i we see that (4.11) implies that $\frac{\partial f_i}{\partial p_i} \leq 0$, i.e., there are no Giffen goods.

It was proved by Mitjuschin and Polterovitch [MP] that if there exists a C^2 concave utility function u near $\bar{x} = f(\bar{p}, \bar{I})$ such that

$$\Sigma_{i,j=1}^{n} u_{ij}(\bar{x}) \bar{x}_i \bar{x}_j + 4 \Sigma_{j=1}^{n} u_j \bar{x}_j \geq 0 \tag{4.12}$$

then the demand is monotone at $\bar{p}$, $\bar{I}$. This result is not quite satisfactory, for it gives a sufficient condition only and the function u is not intrinsically given as part of the data. While these difficulties could be overcome, we prefer to give a more geometric characterization of monotonicity. We use the coordinate systems and notations introduced in Lectures 2 and 3.

Theorem 4.7: Let $\alpha_i(\bar{x}) \neq 0$ for $1 \leq i \leq n-1$. Then demand is monotone at $\bar{p}$, $\bar{I}$, if and only if

$$4 \bar{x}_n + \Sigma_{i=1}^{n-1} \alpha_i (\bar{x}_n \frac{\beta_i}{\alpha_i} + \bar{x}_i)^2 \geq 0 . \tag{4.13}$$

We prove the theorem for the case n = 2, the general case presenting no conceptual difficulties, but requiring a little more computation (see Kannai [KN 89]).

Proof: Substituting (3.3') and (3.4') in (4.11), we see that demand is monotone at $\bar{p}$, $\bar{I}$ if and only if

$$\frac{1 + \beta_1(\bar{x})\bar{x}_1}{\alpha_1(\bar{x})} \xi_1^2 + \frac{\beta_1(\bar{x})\bar{x}_2}{\alpha_1(\bar{x})} \xi_1 \xi_2 - \bar{x}_1 \xi_1 \xi_2 - \bar{x}_2 \xi_2^2 \leq 0 \tag{4.14}$$

for all $(\xi_1, \xi_2) \in \mathbb{R}^2$. The inequality (4.14) is always satisfied if $\xi_1 = 0$, for

$\bar{x}_2 = (p\,\bar{x})/\,|p| = \bar{I}/\,|p|$. We may therefore assume that $\xi_1 \neq 0$. Set $\mu = \frac{\xi_2}{\xi_1}$. Then (4.14) is equivalent to

$$-\bar{x}_2\,\mu^2 + \left(\frac{\beta_1\bar{x}_2}{\alpha_1} - \bar{x}_1\right)\mu + \frac{1 + \beta_1\bar{x}_1}{\alpha_1} \leq 0 . \tag{4.15}$$

The left hand side of (4.15) attains its maximum at

$$\bar{\mu} = \frac{\left(\frac{\beta_1\bar{x}_2}{\alpha_1} - \bar{x}_1\right)}{2\bar{x}_2} = \frac{\beta_1\bar{x}_2 - \alpha_1\bar{x}_1}{2\alpha_1\bar{x}_2},$$

and the maximum is equal to

$$\frac{-\bar{x}_2(\beta_1\bar{x}_2 - \alpha_1\bar{x}_1)^2}{4\alpha_1^2\,\bar{x}_2^2} + \frac{(\beta_1\bar{x}_2 - \alpha_1\bar{x}_1)}{\alpha_1} \cdot \frac{(\beta_1\bar{x}_2 - \alpha_1\bar{x}_1)}{2\alpha_1\bar{x}_2} + \frac{1 + \beta_1\bar{x}_1}{\alpha_1} =$$

$$= \frac{-(\beta_1\bar{x}_2 - \alpha_1\bar{x}_1)^2 + 2(\beta_1\bar{x}_2 - \alpha_1\bar{x}_1)^2 + 4\alpha_1\bar{x}_2(1 + \beta_1\bar{x}_1)}{4\alpha_1^2\bar{x}_2} =$$

$$= \frac{(\beta_1\bar{x}_2 - \alpha_1\bar{x}_1)^2 + 4\alpha_1\bar{x}_2 + 4\alpha_1\beta_1\bar{x}_1\bar{x}_2}{4\alpha_1^2\bar{x}_2} = \frac{(\beta_1\bar{x}_2 + \alpha_1\bar{x}_1)^2 + 4\alpha_1\bar{x}_2}{4\alpha_1^2\bar{x}_2}.$$

The denominator $4\,\alpha_1^2\,\bar{x}_2$ is always positive. Hence (4.15) holds if and only if

$$(\beta_1\bar{x}_2 + \alpha_1\bar{x}_2)^2 + 4\alpha_1\bar{x}_2 \leq 0$$

or, after dividing by α_1, if and only if

$$\alpha_1\left(\frac{\beta_1\bar{x}_2}{\alpha_1} + \bar{x}_1\right)^2 + 4\bar{x}_2 \geq 0 ,$$

which is (4.13).

Note 4.8: For an extension of Theorem 4.7 to the case where some $\alpha_i(\bar{x})$ may vanish see [KN 89, p. 93].

Note 4.9: If the individual comes to the market with an initial endowment vector w (rather than with scalar wealth I) so that the budget constraint now reads $p\,x \leq p\,w$, then (3.3') and (3.4') still hold, with $\bar{x}_i$ replaced by $(\bar{x}_i - w_i)$. The crucial difference between this case and the former one is that now $\bar{x}_2 - w_2 = 0$. Then the quadratic form (4.14) cannot be bounded (and (4.13) cannot hold) unless $\bar{x} = w$. Thus our results are compatible with the well-known fact [MC, p. 216], that the excess demand function $f(p)$ is not monotone whenever $f(p) \neq 0$.

Lecture 5

Here we introduce least concave utility functions, survey some of their properties, and discuss some bargaining and economic applications. For the definition we follow Debreu [D 76]. Denote by V the set of continuous, concave functions v on K representing (the fixed) preference ordering $\succsim$.

Definition 5.1: Let $v_1, v_2 \in V$. If there exists a real valued concave function h on $v_2(K)$ such that $v_1(x) = h(v_2(x))$, then v_1 is more concave than v_2.

Note that the set V is preordered by the relation "v_1 is more concave than v_2". Note that h is a strictly increasing and continuous map of the interval $v_2(K)$ onto the interval $v_1(K)$ (i.e., $h(t) = v_1(v_2^{-1}(t))$), see Fig. 9.

$$\begin{array}{rl} v_2: & K \rightarrow R \\ & \quad\;\; \downarrow h \\ v_1: & K \rightarrow R \end{array}$$

Fig. 9

It is not immediate that least elements (i.e., utility functions u such that every utility v is representable as a concave function of u) do exist. However, Debreu [D 76] stated and proved clearly the following (implicitly stated but not proved by de Finetti [dF]):

Theorem 5.2: If $\succsim$ is concavifiable on K, then there exists a least element in V.

Note 5.3: The least elements in V are unique up to an increasing linear transformation from $\mathbb{R}$ to $\mathbb{R}$ of the form $t = as + b$ with $a > 0$. It is clear that if $u: K \rightarrow \mathbb{R}$ is least concave, so is $au(x) + b$ if $a > 0$. Moreover, if v is least concave, then both u is a concave function of v and v is a concave function of u. Hence u is a linear function of v and this linear function is increasing. It follows that a least concave utility function representing $\succsim$ is a cardinal utility. (Note also that a least concave utility function is uniquely determined by its values on two non–indifferent points.)

A least concave utility function possesses several interesting extremal properties (among all elements of V). Recall the concept of directional derivative (see also [KN 81]). Let v be any real function defined on K. Let p∈K and assume that $\{p + \lambda y : 0 \leq \lambda \leq \epsilon\} \subset K$ for a sufficiently small ϵ. If the limit

$$\lim_{\lambda \to 0_+} (v(p + \lambda y) - v(p)) / \lambda$$

exists ($+\infty$ and $-\infty$ are allowed as limits) we denote it by $v'(p,y)$, and call it the one-sided directional derivative of v at p with respect to y. (Clearly, $v'(p, \alpha y) = \alpha v'(p,y)$ if $\alpha > 0$.)

Note that if v is actually differentiable at p then $v'(p,y) = < \nabla v(p), y >$ where $\nabla v(p)$ is the gradient of v at p and we denote the inner product in R^m by < , >. But $v'(p,y)$ exists in many cases in which $\nabla v(p)$ does not exist. As a simple example, consider the concave function $u(x) = -|x|$ defined on $\mathbb{R}^1$. Then $u'(p,1) = -1$ for $p \geq 0$ and $u'(p,1) = 1$ for $p \leq 0$, the directional derivatives existing even at $p = 0$ where u is not differentiable. Concave functions do possess directional derivatives, as summarized in the following theorem. The proofs can be found in Rockafellar [R]. We use the following notation: For $p \in K$, $y \in R^n$, set $I(p, y) = \{\lambda \in R^1 : p + \lambda y \in K\}$.

Theorem 5.4: Let u be a concave function defined in a convex subset K of R^n. If $p \in \text{int } K$ then $u'(p,y)$ exists and is finite for all $y \in R^n$. If $p \in \partial K$ and $y \in K - \{p\}$, then $u'(p, y)$ exists (but might be infinite). The (extended) real function $u(\lambda;p,y)$, defined in the real interval $\lambda \in I(p,y)$ by the equation

$$u(\lambda;p,y) = u'(p + \lambda y, y) \qquad (5.1)$$

is monotone non–increasing in $\lambda \in I(p,y)$.

In order to avoid technical complications, we restrict ourselves to the case of compact K.

Theorem 5.5: Let $\succsim$ be a concavifiable preference ordering defined on the compact convex set K. The following statements are equivalent:

1) The function u is a least concave utility function representing $\succsim$ on K.

2) If v is a concave utility function representing $\succsim$ on K such that $v(p) = u(p)$ for p maximal and for p minimal with respect to $\succsim$, then $v(x) \geq u(x)$ for all $x \in K$.

3) If $p, q \in K$, $p \prec q$, v is concave on the layer $\{x: p \precsim x \precsim q\}$ and represents $\succsim$ on that layer, and $v(p) = u(p)$, $v(q) = u(q)$, then $v(x) \geq u(x)$ for all x in the layer.

4) If $p,q \in K$, $p \prec q$, the preference ordering $\succsim$ is monotone on the interval [p,q], and v is a concave utility function representing $\succsim$ on K, then

$$\frac{u'(p, q-p)}{u'(q, q-p)} \leq \frac{v'(p, q-p)}{v'(q, q-p)}. \qquad (5.2)$$

For the proof, note that statement 2) is a special case of statement 3). It is obvious that statement 1) implies both statement 2) and statement 4). It is not difficult to see that statement 3) implies statement 1), and an integration argument is needed to pass from statement 4) to statement 2). That statement 2) implies statement 3) (or that statement 2) implies statement 4)) needs some work, see e.g. [D 76] or [KN 77, p.12]. (While Theorem 5.5 has not been stated explicitly in this form earlier, all the elements of its proof may be found in [D 76], [KN 77], and [KN 81]).

Because of properties such as 2) in Theorem 5.5, least concave utility functions are also referred to as "minimally concave" ([KN 77] and [KN 81]) or "funzione minimamente convessa" by de Finetti [dF]. It is interesting to observe that de Finetti just states, without any comment or proof, that a function minimally concave in a layer (i.e., satisfying property 3) of Theorem 5.5) is obviously minimally concave in any sub–layer.

Note 5.6: It is possible to prove existence of least concave utility functions (at least for compact K), using Theorem 5.5. In fact, let $\succsim$ be concavifiable, so that V is non–empty. Set $u(x) = \inf \{v(x)\}$, where the infimum is taken over the set of all $v \in V$ for which $v(K) = [0,1]$. Then u(x) is a least concave utility representing $\succsim$ on K (for details, see [D 76]).

In [KN 77] and [KN 81] several methods are exhibited for constructing least concave utility functions. Actually, those methods make it possible to determine whether or not $\succsim$ is concavifiable, and if $\succsim$ is concavifiable, a least concave utility is constructed. We state here a sample

Theorem 5.7: Let K be a compact convex subset of $\mathbb{R}^n$, and let $\succsim$ be representable by a C^2 utility function v satisfying Fenchel's conditions I and II. Then the utility function F(v) given by the formula (2.5) is least concave.

Thus, Theorem 2.1 states that $\succsim$ is concavifiable iff G(s) is integrable, i.e., iff the formula (2.5) makes sense. Theorem 5.7 adds that if the formula (2.5) does make sense, then this formula yields a least concave utility function representing $\succsim$ on K. The two constants of integration appearing in (2.5) correspond to the fact (Note 5.3) that least concave utility functions are determined up to an increasing linear transformation.

Although the concept of a least concave utility function has obvious theoretical importance, its economic and game–theoretic applications might be of even greater significance. We illustrate the manner in which least concave utility representations may be interpreted and applied in four distinct contexts.

Example 5.8: Following Debreu [D 76], we consider a risk–bearing agent who preorders the set of probabilities on K. We identify each point $x \in K$ with the unit mass distribution supported on {x}, i.e., the probability of a (Borel) subset A of K is equal to one if and only if $x \in A$. Suppose that the agent has a bounded von Neumann–Morgenstern utility whose restriction v to K is a concave, real–valued continuous function representing the restriction $\succsim$ to K of his preferences on the set of probabilities on K. By Theorem 5.2 there exists a least concave utility u on K representing $\succsim$, and one has $v(x) = f(u(x))$ where f is concave. Thus in this context, one can separate the preferences ($\succsim$) of the agent for the commodity vectors in K, represented by u, from his attitude towards risk, described by the strictly increasing continuous concave function f. (The function u contains the "bare minimum" amount of concavity compatible with $\succsim$; every agent having the same ordering on K must have at least as much concavity as u in her utility function.)

Example 5.9: According to Proposition 3.4, the utility function u is concave if and only if the marginal utility of income is a non–increasing function of income, i.e., if and only if $\frac{\partial\lambda}{\partial I} = \frac{\partial^2 u}{\partial I^2} \leq 0$. It is too much to expect that u is least concave if and only if the marginal utility of income is constant along the Engel curves (i.e., $\frac{\partial\lambda}{\partial I} = \frac{\partial^2 u}{\partial I^2} = 0$). For $\frac{\partial\lambda}{\partial I}(f(p_1,I_1)) = 0$ might force $\frac{\partial\lambda}{\partial I}$ to be positive for some p_2, I_2. The most one could reasonably expect is that on each indifference surface one could find a point at which $\frac{\partial\lambda}{\partial I}$ vanishes. This is indeed the case, under certain assumptions. Again, a more general result may be found in [KN 86, Theorem 2]. We restrict ourselves here to a relatively simple case.

Theorem 5.10: Let u be a C^2 concave utility function representing $\succsim$ in a compact convex set K. Let the Gaussian curvature of all indifference hypersurfaces be positive in K. Then u is least concave in K if and only if on every indifference hypersurface there exists a point x such that $\frac{\partial\lambda}{\partial I}(x) = \frac{\partial^2 u}{\partial I^2}(x) = 0$.

Proof: Using the coordinate systems introduced in Lecture 2 and the formulas for the Hessian matrix developed there, we see that

$$-u_{nn} + \Sigma_{i=1}^{n-1}\frac{u_{in}^2}{u_{ii}} = -\frac{\partial\lambda}{\partial x_n} + \Sigma_{i=1}^{n-1}\frac{\lambda^2\beta_i^2}{\lambda\alpha_i} = -\frac{\partial\lambda}{\partial x_n} + \lambda\,\Sigma_{i=1}^{n-1}\frac{\beta_i^2}{\alpha_i} \tag{5.3}$$

(Compare also (2.18) and (2.17) and note that by assumption, $\alpha_i(x) \neq 0$ for all $x \in K$, $1 \leq i \leq n-1$.). Choosing v = u and substituting (5.3) in (2.6), we obtain the formula

$$a(x) = \frac{1}{\lambda^2(x)}\left[-\frac{\partial\lambda}{\partial x_n} + \lambda\,\Sigma_{i=1}^{n-1}\frac{\beta_i^2(x)}{\alpha_i(x)}\right]. \tag{5.4}$$

It follows from (3.12) that

$$a(x) = -\frac{1}{\lambda^2(x)}\frac{\partial^2 u}{\partial I^2}(x). \tag{5.5}$$

Hence

$$G(t) = \min_{\{x:\ ux)=t\}}[-a(x)] = \min_{\{x:\ ux)=t\}}\left[\frac{\partial^2 u}{\partial I^2}(x)\cdot\frac{1}{\lambda^2(x)}\right] \tag{5.6}$$

for all $t \in u(K)$. (Note that the infimum, appearing in the definition of G(t) in IV) of Lecture 2, may be replaced by minimum due to compactness.) By Theorem 5.7, the function F(u) given by formula (2.5) is least concave. By Note 5.3, u is least concave if and only if F(t) is an increasing linear function of t. Differentiating (2.5) we find that

$F'(t) = \exp\left[\int_{u(p_0)}^{t} G(s)ds\right]$. Hence F is linear (i.e., F'(t) is constant) if and only $G(s) = 0$ for almost all $s \in u(K)$. Under the assumptions of Theorem 5.10 G(s) is continuous. Hence u is least concave if and only if $G(s) = 0$ for all $s \in u(K)$. By (5.6), this means that for every $s \in u(K)$ there exists $x \in K$ with $u(x) = s$ and $\frac{\partial^2 u}{\partial I^2}(x) = 0$.

Note 5.11: Theorem 5.10 asserts that under certain conditions, on each indifference hypersurface there is a point at which the income derivative of the marginal utility of income vanishes. Thus the marginal utility of income is "almost" constant (precisely:

constant to a first order accuracy) on the Engel curve through that point. This yields a new interpretation of the Marshallian concept of constancy of the marginal utility of income, enabling one to rescue (to some extent) the concept from the discredit it has received by Samuelson [SA]. This interpretation suffers however from the fact that the points where $\frac{\partial \lambda}{\partial I} = 0$ may be very few and far from each other (even though there is one on every indifference hypersurface).

Example 5.12: The concepts of complementary and substitute goods were defined in the 19th century using (cardinal) utility functions. These definitions were criticized in the 20th century (see [KN 80] for a very short historical review). Thus, consider for simplicity the case of two commodities. Let h_1, h_2 be small positive numbers, and compare the increase in utility caused by adding both the amount h_1 to the quantity x_1 consumed of first good and the amount h_2 to the quantity x_2 consumed of the second good, i.e., $u(x_1 + h_1, x_2 + h_2) - u(x_1, x_2)$, to the sum of the increases in utility caused by adding h_1 to x_1 and adding h_2 to x_2 separately, i.e.,

$$[u(x_1 + h_1, x_2) - u(x_1,x_2)] + [u(x_1, x_2 + h_2) - u(x_1, x_2)].$$

In other words, consider the difference

$$[u(x_1 + h_1, x_2 + h_2) - u(x_1,x_2)] - \{[u(x_1 + h_1, x_2) - u(x_1,x_2)] + [u(x_1, x_2 + h_2) - u(x_1,x_2)]\}$$
$$= u(x_1 + h_1, x_2 + h_2) + u(x_1,x_2) - u(x_1 + h_1, x_2) - u(x_1, x_2 + h_2). \qquad (5.7)$$

(The right hand side of (5.7) lends itself to other interpretations as well, such as the difference between $u(x_1 + h_1, x_2 + h_2) - u(x_1 + h_1, x_2)$ and $u(x_1, x_2 + h_2) - u(x_1,x_2)$.) If the goods are complementary (an increase in the consumption of one of them is usually accompanied by an increase in the consumption of the other, e.g., bread and butter) one expects the expression (5.7) to be positive. If the goods are **substitutes** (an increase in the consumption of one of them would be associated with a decrease in the consumption of the other, e.g., tea and coffee) then the expression (5.7) is expected to be negative. Deviding by h_1h_2 and letting h_1, h_2 tend to zero, and assuming that $u \in C^2$, we are led to the classical

Definition 5.13: The commodities 1 and 2 are said to be complementary at x_1, x_2 if $\frac{\partial^2 u}{\partial x_1 \partial x_2}(x_1, x_2) > 0$, and are said to be substitutes if $\frac{\partial^2 u}{\partial x_1 \partial x_2}(x_1, x_2) < 0$.

Definition 5.13 fell into disgrace (see e.g. [KN 80] and [SA]) because the sign of the

mixed derivative $u_{12}(x)$ is not invariant even under simple transformations of the utility scale. As computed in Lecture 2, the mixed derivative of $F(u)$ is given by

$$[F(u)]_{12} = F'(u)[u_{12}(x) + \frac{F''(u)}{F'(u)} u_1(x)\, u_2(x)] \qquad (5.8)$$

and can have an arbitrary sign, independent of the sign of $u_{12}(x)$. Suppose, however, that u is a least concave utility function representing our consumer's preferences, and that we are interested only in concave utility representations. If $u_{12}(x) < 0$ then $[F(u)]_{12} < 0$ (if one assumes monotonicity as well). Hence if $v_{12}(x) > 0$ for any concave utility $v = F(u)$, then $u_{12}(x) > 0$. These facts suggest that the definition is (if the least concave utility functions are twice differentiable):

Definition 5.13 bis: The commodities 1 and 2 are said to be complementary at x_1, x_2 if $\frac{\partial^2 u}{\partial x_1 \partial x_2}(x_1, x_2) > 0$ for a least concave utility function u representing $\succsim$, and are said to be substitutes at x_1, x_2 if $\frac{\partial^2 u}{\partial x_1 \partial x_2}(x_1, x_2) < 0$ for such a u.

Observe that the sign of $u_{12}(x_1, x_2)$ is the same for all least concave utility functions representing the same preference ordering.

The following might illustrate the ideas presented here.
Let $K = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$ and let $(x_1, x_2) \succsim (y_1, y_2)$ if and only if $x_1x_2 \geq y_1y_2$. Commodities 1 and 2 appear to be independent (everywhere) if one chooses the separable (concave) utility function $\ln x_1 + \ln x_2$, and appear to be substitutes everywhere if the concave utility function $-1/(x_1x_2)$ is chosen instead. Considering the least concave utility function $(x_1x_2)^{1/2}$, we see that commodities 1 and 2 are indeed complementary everywhere, according to definition 5.13 bis.

Example 5.14: (The "Nash²" game). Least concave utility functions are advantageous, in some bargaining situations, over concave utility functions which are not least concave. Let $a = (a_1,\ldots, a_n)$ be a vector in $\mathbb{R}^n$ with positive components. Two agents have to agree on sharing a. We assume that the ith agent has a preference ordering $\succsim_i$ on $K = \{x \in R^n : 0 \leq x_j \leq a_j\}$ with $\succsim_i$ continuous, monotone, and concavifiable on K, $i = 1,2$. Let U_i denote the set of concave utility functions u of $\succsim_i$, normalized by $u(0) = 0$, $u(a) = 1$, $i = 1,2$. Let the bargaining between the agents proceed according to the Nash bargaining model, i.e., each agent announces a cardinal utility $v_i \in U_i$, and we look for a vector $y \in K$ such that the Nash social utility $v_1(y)v_2(a-y)$ satisfies the condition

$$v_1(y)v_2(a-y) \geq v_1(z)v_2(a-z), \qquad (5.9)$$

for all $z \in K$. The disposable hull of the utility possibility set of K, i.e., the set $\{(v_1, v_2); v_1 \geq 0, v_2 \geq 0$, and there exists $x \in K$ such that $v_1 \leq v_1(x), v_2 \leq v_2(a-x)\}$ is convex. Hence if $z \neq y$ then we have a strict inequality in (5.9) and thus $v_1(y)$, $v_2(a-y)$ are unique even if y is not. We can consider therefore the following non-zero sum, two players game: The (pure) strategies spaces for the players are U_1 and U_2, and the vector-valued payoff functions are given by

$$M_1(v_1, v_2) = y, \qquad M_2(v_1, v_2) = a - y,$$

where y satisfies (5.9). The least concave utility functions form a Nash equilibrium pair of this game (formed from a Nash bargaining problem – hence the nick-name Nash²) as the following shows:

Proposition 5.15: Let u_i denote the least concave utility function for $\succsim_i$ satisfying $u_i \in U_i$, $i = 1,2$. Then for all $v_1 \in U_1$, $v_2 \in U_2$,

$$M_1(u_1, u_2) \succsim_1 M_1(v_1, u_2), \qquad M_2(u_1, u_2) \succsim_2 M_2(u_1, v_2). \qquad (5.10)$$

For a proof, see [KN 77, p.55]. As an illustration consider the one-dimensional case and let $\succsim_1 = \succsim_2$ be the usual order $\geq$. Then we may assume without loss of generality that $a = 1$ and $u_1 = u_2 = x$. Hence $y = a - y = \frac{1}{2}$. If participant 1 announces a concave utility function $v_1 \in C^1$ such that v_1 is not linear on $[0, \frac{1}{2}]$, then by concavity $\frac{1}{2} v_1'(\frac{1}{2}) < v_1(\frac{1}{2})$ implying that the derivative $v_1'(x)(1-x) - v_1(x)$ of the Nash social utility function is negative at $x = \frac{1}{2}$, while being positive at $x = 0$. Hence the maximum y is achieved at $y < \frac{1}{2}$, or $M_1(u_1, u_2) > M_1(v_1, u_2)$.

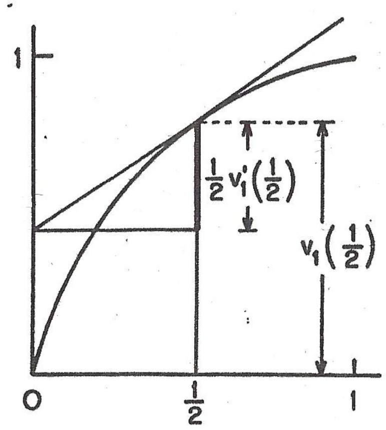

Fig. 10

It should be noted that in the preceding discussion the preferences are fixed ("revealed") and the players cannot conceal them or lie about them. They are free to choose a normalized utility as their cardinal utility. Proposition 5.15 suggests that it is best for you to pretend that your marginal utility decreases by not more than is absolutely necessary (Theorem 5.5). Note also that the Nash game over $U_1 x U_2$ possesses many similarities with zero sum games, for the interests of the bargainers are opposed as they move along the Pareto boundary of the utility possibility set.

Note 5.16: Least concave utility functions are advantageous also for many other solutions of the bargaining problems (e.g., the Kalai–Smorodinsky solution). For further details and references see [TP].

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