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in Economic Situations
by

Bezalel Peleg and Peter Sudhölter

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University of Bielefeld
33501 Bielefeld, Germany

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Bezalel Peleg**) and Peter Sudhölter

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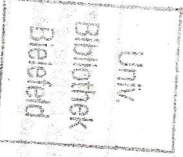
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Abstract

We consider the class of all abstract economies with convex and compact strategy sets, continuous and quasi-concave payoff functions, and continuous and convex-valued feasibility correspondences. We prove that the Nash correspondence is the unique solution on the foregoing class of abstract economies that satisfies non-emptiness, rationality in one-person economies, and consistency.

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§1 Introduction

Abstract economies were introduced by Debreu (1952) in order to help in the proof of existence of Walras equilibrium for competitive economies (Arrow and Debreu (1954)). They have since been used extensively in general equilibrium theory (see, e.g., Shafer and Sonnenschein (1975), Ichishi (1983), and Border (1985)). Abstract economies are, essentially, strategic games in which the feasible set of strategies of a player may depend on the strategies chosen by the other players. Thus, an abstract economy is a strategic game combined with a finite sequence of feasibility correspondences, one for each player. The main solution concept for abstract economies is the social equilibrium (see Debreu (1952)), which is a generalization of the Nash equilibrium. We provide an axiomatization of the Nash correspondence of abstract economies.

We now briefly describe our result. Consider the class of all abstract economies with the following properties: (i) The strategy sets are convex and compact. (ii) The payoff functions are continuous and quasi-concave. (iii) The feasibility correspondences are continuous and have non-empty convex values. Then, the Nash correspondence is the unique solution on the foregoing class of abstract economies that satisfies non-emptiness, rationality in one-person games, and consistency.

The Nash correspondence has been axiomatized recently for various classes of games (see Peleg and Tjis (1993), Peleg et al. (1994), and Norde et al. (1993)). In particular, Norde et al. (1993) contains a complete characterization for the class of mixed extensions of finite games and for the class of games with continuous concave payoff functions. This paper is devoted to the characterization of the Nash correspondence of generalized games and quasi-concave payoff functions. Although we use some results of Peleg et al. (1994) and Norde et al. (1993), our result does not directly follow from the foregoing papers. The main difficulty is that the class of quasi-concave functions, unlike the class of concave functions, is not closed under addition (i.e., the sum of two quasi-concave functions may not be quasi-concave).

§2 The Model

Let Ω be an infinite set (the set of "potential players"). An abstract economy is a list $E = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N}, (F_i)_{i \in N} \rangle$ where $N = N(E) \subset \Omega$ is a finite non-empty set (the set of players), A_i is the (non-empty) set of strategies of player $i \in N$; $u_i : A(E) \rightarrow E_i$ is the payoff function for $i \in N$ (here $A(E) = \prod_{i \in N} A_i$ and E_i is the real line); and $F_i, i \in N$, is a correspondence from $A_{-i} = \prod_{j \neq i} A_j$ to A_i . F_i is the feasibility correspondence of $i \in N$. Thus, if $x \in A(E)$ then $F_i(x_{-i}) \subset A_i$ is the set of feasible strategies of i (when the rest of the players choose $x_{-i} = (x_j)_{j \in N \setminus \{i\}}$). $\bar{x} \in A(E)$ is a Nash equilibrium (NE), or a social equilibrium (SE), if: (i) $\bar{x}_i \in F_i(\bar{x}_{-i})$ for all $i \in N$, and (ii) $u_i(\bar{x}) \geq u_i(x_i, \bar{x}_{-i})$ for all $x_i \in F_i(\bar{x}_{-i})$ and $i \in N$.

Abstract economies were first studied in Debreu (1952) in connection with the Arrow-Debreu existence theorem of Walras equilibrium. They also later continued to play an important rôle in general equilibrium theory (see, e.g. Shafer and Sonnenschein (1975) and Border (1985, Chapters 19 and 20)).

We shall consider the class ξ_q of abstract economies that have the following properties.

- (2.1) $E = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N}, (F_i)_{i \in N} \rangle \in \xi_q$ if
 - A_i is a non-empty, compact, and convex subset of some (finite-dimensional) Euclidean space E_i for every $i \in N$.
 - u_i is continuous on the graph of F_i , $gr F_i$, for every $i \in N$. (We recall that $gr F_i = \{x \in A(E) \mid x_i \in F_i(x_{-i})\}$.)
- (2.2) For every $i \in N$ and $x_{-i} \in A_{-i}$, $u_i(\cdot, x_{-i})$ is a quasi-concave function on $F_i(x_{-i})$.
- (2.3) For every $i \in N$, F_i is continuous (i.e., both upper hemi-continuous and lower hemi-continuous) on A_{-i} .
- (2.4) For every $i \in N$ and $x_{-i} \in A_{-i}$, $F_i(x_{-i})$ is non-empty, closed, and convex.

A solution on ξ_q is a function φ that assigns to each $E \in \xi_q$ a subset $\varphi(E)$ of $A(E)$. For example, the function SE which assigns to each $E \in \xi_q$ its set of social equilibria $SE(E)$, is a solution. A solution φ on ξ_q satisfies non-emptiness (NEM), if $\varphi(E) \neq \emptyset$ for every $E \in \xi_q$. (We remark that, by Theorem 4.3.1 in Ichihashi (1983), $SE(\cdot)$ satisfies NEM.) φ satisfies one-person rationality (OPR), if for every one-person abstract economy $E = \langle \{i\}, A_i, u_i, F_i \rangle$ in ξ_q ,

$$\varphi(E) \subset \{x_i \in F_i \mid u_i(x_i) \geq u_i(y_i) \text{ for all } y_i \in F_i\}.$$

(F_i is constant for one-person abstract economies.) Obviously $SE(\cdot)$ satisfies OPR.

Let $E \in \xi_q$, $x \in A(E)$, and $S \subset N(E)$, $S \neq \emptyset$. The reduced abstract economy of E with respect to S and x , E^S, x is given by

$$E^S, x = \langle S, (A_i)_{i \in S}, (u_i)_{i \in S}, (F_i)_{i \in S} \rangle$$

where $u_i^S(a_i) = u_i(a_i, x_{N \setminus S})$ and $F_i^S(a_i) = F_i(a_i, x_{N \setminus S})$ for every $a_i \in A_i, i \in S$ and $i \in S$. (Here the notation $x_T = (x_j)_{j \in T}$ is used whenever $x \in A(E)$ and $T \subset N$.) Under the foregoing assumptions $E^S, x \in \xi_q$, that is, ξ_q is closed in the sense of Peleg and Tijs (1993). A solution φ on ξ_q is consistent (CONS) if for every $E \in \xi_q$, $S \subset N(E)$, $S \neq \emptyset$ and $x \in \varphi(E)$, $x_S \in \varphi(E^S, x)$. The interpretation of the consistency property for abstract economies is actually the same as the interpretation for strategic games (see Peleg and Tijs (1993)). Indeed, φ is consistent if for every $E \in \xi_q$ the following condition is satisfied: If x is an "equilibrium", that is, $x \in \varphi(E)$, $S \subset N(E)$, $S \neq \emptyset$, and all the members of $N \setminus S$ announce their strategies $x_i, i \in N \setminus S$, and leave the economy E , then the members of S do not have to revise their strategies $x_j, j \in S$. As the reader may easily verify $SE(\cdot)$ satisfies CONS on ξ_q .

§3 A Characterization of the Nash Correspondence on ξ_q

Theorem 3.1: If a solution φ on ξ_q satisfies NEM, OPR, and CONS, then $\varphi(E) = SE(E)$ for every $E \in \xi_q$.

The following simple result is needed for the proof of Theorem 3.1.

Lemma 3.2: Let $E \in \xi_q$ and $x \in A(E)$. If $x_i \in SE(E^{\{i\}}, x)$ for every $i \in N(E)$, then $x \in SE(E)$.

Proof:

Let $i \in N(E)$. Because $x_i \in SE(E^{i\setminus i}, x)$ it follows that

$$(3.1) \quad x_i \in F_i^* \text{ and } u_i^*(x_i) \geq u_i^*(y_i) \text{ for all } y_i \in F_i^*.$$

By the definition of reduced games, (3.1) is equivalent to

$$x_i \in F_i(x_{-i}) \text{ and } u_i(x) \geq u_i(y_i, x_{-i}). \text{ Thus, } x \in SE(E). \quad \text{q.e.d.}$$

Now we shall prove that $SE(\cdot)$ is the maximum solution on ξ_q which satisfies OPR and CONS.

Lemma 3.3: If a solution φ on ξ_q satisfies OPR and CONS, then $\varphi(E) \subset SE(E)$

for every $E \in \xi_q$.

Proof:

Let $E \in \xi_q$ and $x \in \varphi(E)$. If $|N(E)| = 1$ then $\varphi(E) \subset SE(E)$ by OPR. (If S is a finite set, then $|S|$ is the cardinality of S). Assume now $|N(E)| \geq 2$. By CONS, $x_i \in \varphi(E^{i\setminus i}, x)$ for all $i \in N(E)$. Hence, by the first part of the proof, $x_i \in SE(E^{i\setminus i}, x)$ for all $i \in N(E)$. Therefore, by Lemma 3.2, $x \in SE(E)$. q.e.d.

In order to prove the converse inclusion we shall show that $SE(\cdot)$ has the ancestors property (AP) on ξ_q (see Peleg et al. (1994) and Proposition 3 of Norde et al. (1993)).

Lemma 3.4 (AP): If $E \in \xi_q$ and $\hat{x} \in SE(E)$, then there exists $H \in \xi_q$ such that the

following conditions hold:

$$(3.2) \quad N(H) \supset N(E);$$

$$(3.3) \quad H \text{ has exactly one social equilibrium } y;$$

$$(3.4) \quad y_{N(E)} = \hat{x};$$

$$(3.5) \quad H^{N(E), y} = E.$$

We postpone the proof of Lemma 3.4 and shall now prove Theorem 3.1.

Proof of Theorem 3.1:

Let $E \in \xi_q$ and $\hat{x} \in SE(E)$. By Lemma 3.4 there exists $H \in \xi_q$ such that (3.2)-(3.5) are satisfied. By NEM of φ and Lemma 3.3, $\varphi(H) = \{y\}$. By CONS of φ , (3.4), and (3.5) we obtain

$$\hat{x} = y_{N(E)} \in \varphi(H^{N(E), y}) = \varphi(E).$$

Thus, $\varphi(E) \supset SE(E)$. Because of Lemma 3.3, $\varphi(E) = SE(E)$. q.e.d.



Proof of Lemma 3.4:

Let $E = \langle N(E), (A_i)_{i \in N(E)}, (u_i)_{i \in N(E)}, (F_i)_{i \in N(E)} \rangle$ be a member of ξ_q and $\hat{x} \in SE(E)$. We define $H = \langle N(H), (A_i^*)_{i \in N(H)}, (u_i^*)_{i \in N(H)}, (F_i^*)_{i \in N(H)} \rangle$ in the following way.

$$(3.6) \quad N(H) = N \cup N^+ \text{ where } N \cup N^+ = \emptyset, |N| = |N^+|, \text{ and } N = N(E).$$

The existence of $N(H) \subset \Omega$ is guaranteed by the fact that Ω is infinite.

We denote by t a bijection of N onto N^+ and we choose.

$$(3.7) \quad A_i^* = A_{t(i)}^* = A_i \text{ for all } i \in N.$$

In order to define u_i^* and F_i^* , $i \in N(H)$, we need the following concepts and notations. A

continuous function $\pi_i : E_i \rightarrow A_i$, $i \in N(H)$ (see (2.1)), is a retraction if $\pi_i(x_i) = x_i$ for all $x_i \in A_i$. Because A_i is a convex and compact subset of E_i there exists a retraction π_i of E_i on A_i . Thus we may define:

$$(3.8) \quad F_i^*(x_{N \setminus \{i\}}, x_{N^+}) = F_i(\pi_j(x_j + \hat{x}_j - x_{t(j)}))_{j \in N \setminus \{i\}}$$

$$(3.9) \quad F_{t(i)}^*(x_{-t(i)}) = A_i$$

for every $i \in N$ and every $x_{-t(i)} \in A_{-t(i)}^*$ where $A_{-t(i)}^* = \prod_{j \in N} A_j \times \prod_{j \in N^+} A_{t(j)}$. As the reader may verify, F_i^* satisfies (2.4) and (2.5) for all $i \in N(H)$.

In order to define u_i^* , $i \in N(E)$, we first introduce auxiliary functions

$$(3.10) \quad v_i^*(x) = u_i(x_i, (\pi_j(x_j + \hat{x}_j - x_{t(j)}))_{j \in N \setminus \{i\}}) \text{ for all } x \in A(H) \text{ and } i \in N.$$

$v_i^*(\cdot, x_{-i})$ is quasi-concave on $F_i^*(x_{-i}) = F_i((\pi_j(x_j + \hat{x}_j - x_{t(j)}))_{j \in N \setminus \{i\}})$ for every $x_{-i} \in A_{-i}^*$, and continuous on $gr F_i^*$. Using (3.10) we further define $\beta_i : \prod_{j \in N} A_j \times \prod_{j \in N^+} A_{t(j)} \rightarrow E_i$ by

$$\beta_i(x_{N \setminus \{i\}}, x_{N^+}) = \beta_i(x_{-i}) = \max \{v_i^*(x_i, x_{-i}) \mid x_i \in F_i^*(x_{-i})\} \text{ for } i \in N.$$

By the Maximum Theorem (see Section 2.3 of Ichishi (1983)), $\beta_i(\cdot)$ is continuous. Furthermore

$$(3.11) \quad \beta_i((x_i)_{j \in N \setminus \{i\}}, (\hat{x}_h)_{h \in N^+}) = \max \{u_i(x_i, x_{N \setminus \{i\}}) \mid x_i \in F_i(x_{N \setminus \{i\}})\}.$$

(Here $\hat{x} \in A(H)$ is defined by \hat{x}_i for $i \in N$.) Now we introduce our second family of auxiliary functions. For every $i \in N$ let $w_i^* : A(H) \rightarrow E_i$ be given by

$$(3.12) \quad w_i^*(x) = w_i^*(x_i, x_{-i}) = \beta_i(x_{-i}) - \|x_{t(i)} - \hat{x}_i\| \|x_i - \hat{x}_i\|.$$

(Here $\|\cdot\|$ denotes the Euclidean norm on E_i .)

It is obvious that w_i^* satisfies (2.2) and (2.3). Hence

$$u_i^*(x) = \min \{v_i^*(x), w_i^*(x)\}$$

is an admissible payoff function for $i \in N(E)$. Finally, we define

$$(3.13) \quad u_{i(i)}^*(x) = -\|x_i - x_{i(i)}\| \text{ for all } i \in N(E).$$

This completes the definition of H . Clearly, $H \in \xi_q$. Now we claim:

$$(3.14) \quad \text{If } y \in SE(H) \text{ then } y_i = y_{i(i)} = \hat{x}_i \text{ for all } i \in N(E).$$

Indeed, if $y \in SE(H)$ then $y_i = y_{i(i)}$ for all $i \in N(E)$ by (3.13). Hence $v_i^*(y) = u_i(y_i, \hat{x}_{N \setminus \{i}\})$ and $w_i^*(y) = u_i(\hat{x}_N) - \|y_i - \hat{x}_i\|^2$. Thus, for $i \in N(E)$, y_i is a best response to y_{-i} iff $y_i = \hat{x}_i$. Therefore, our claim is proved. So far we have proved (3.2)-(3.4). In order to prove (3.5) we observe that for all $i \in N(E)$:

$$\begin{aligned} & \min \{v_i^*(x_i), \beta_i(x_i, (\hat{x}_h)_{h \in N^*}), w_i^*(x_i, (\hat{x}_h)_{h \in N^*})\} = \\ & \min \{u_i(x_N), \beta_i(x_i, (\hat{x}_h)_{h \in N^*})\} = \\ & \min \{u_i(x_N), \max \{u_i(z_i, x_{N \setminus \{i}\}) \mid z_i \in F_i(x_{N \setminus \{i}\})\}\} = \\ & u_i(x_N) \text{ (see (3.11)).} \end{aligned}$$

Also, $F_i^*(x_i, (\hat{x}_h)_{h \in N^*}) = F_i(x_i, (\hat{x}_h)_{h \in N^*})$ for each $i \in N(E)$. q.e.d.

§4 Concluding Remarks

We considered the set of abstract economies with convex and compact strategy sets, continuous and quasi-concave payoff functions, and continuous and convex-valued feasibility correspondences. The Nash correspondence is completely characterized on the foregoing class of economies by the following three axioms: non-emptiness, rationality for one-person games, and consistency.

Let $E = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N}, (F_i)_{i \in N} \rangle$ be an abstract economy. E is a game if for each $i \in N$ and $x_{-i} \in A_{-i}$, $F_i(x_{-i}) = A_i$. Thus, E is a game if there are no feasibility constraints. If E is a game, then we shall also write $E = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$.

Denote by Γ_q the set of all games in ξ_q . By modifying the proof of Theorem 3.1 we can show the following result.

Theorem 4.1: If a solution φ on Γ_q satisfies NEM, OPR, and CONS, then $\varphi(G) = NE(G)$ for every game $G \in \Gamma_q$ (for $G \in \Gamma_q$ we denote $NE(G) = SE(G)$).

The axiomatization of the Nash correspondence on Γ_q is not covered by the results of Norde et al. (1993). Also, our results may be generalized to infinite-dimensional strategy spaces. Indeed, one may replace (2.1) by a weaker assumption. First, we recall that a normed linear space \bar{X} (with norm $\|\cdot\|$), is strictly convex if for every pair of linearly independent vectors $x, y \in \bar{X}$ it holds that $\|x+y\| < \|x\| + \|y\|$. The following result is useful.

Lemma 4.2: If A is a non-empty, convex, and compact subset of a convex compact subset B of a normed space \bar{X} , then there is a retraction of B on A .

Proof: It is well-known that a compact set is separable, hence the affine hull of B (the set $\text{aff } B = \{\sum_{i=1}^r \gamma_i b_i \mid r \in \mathbb{N}, \sum_{i=1}^r \gamma_i = 1, \gamma_i \in \mathbb{R}, b_i \in B \text{ for all } i \text{ with } 1 \leq i \leq r\}$) is separable.

Indeed, if $\{b_i\}_{i \in \mathbb{N}}$ is a countable dense subset of B , then $\{\sum_{i=1}^r \gamma_i b_i \mid r \in \mathbb{N}, \sum_{i=1}^r \gamma_i = 1, \gamma_i \in \mathbb{Q}\}$ is a countable dense subset of $\text{aff } B$.
If there is a retraction of $B - \{b_0\}$ on $A - \{b_0\}$ for some $b_0 \in B$, then there is a retraction of B on A , because translations are continuous. Therefore we assume w.l.o.g. that 0 is a member of B , hence $\text{aff } B$ is a linear subspace of \bar{X} . Moreover, it can be assumed that $\text{aff } B$ is a Banach space (otherwise take a "smallest" completion of $\text{aff } B$ which does not effect separability, compactness, or convexity). Thus $\text{aff } B$ has a strictly convex isomorphic norm (see p. 60 of Day (1973)). The proof of existence of a retraction of a strictly convex normed space on a nonvoid convex compact subset is the same as the proof when the normed space is Euclidean. q.e.d.

Now, using the generalizations of existence of social equilibria and the Maximum Theorem for infinite-dimensional spaces (see Section 2.3 and Theorem 4.7.2 of Ichishi (1983)), we obtain

Corollary 4.3: Theorem 3.1 remains true if (2.1) is replaced by:
(4.1) For every $i \in N$, A_i is a non-empty, compact, and convex subset of a normed linear space.

Indeed, in view of (3.10) it is sufficient to verify the existence of a retraction of $A_{i+1} - A_i$ to A_i for $i \in N(E)$. This existence is guaranteed by Lemma 4.2 applied to $B = A_{i+1} - A_i$ and $A = A_i$.

Finally, it should be remarked that obvious examples show the logical independence of NEM, OPR, and CONS in the results (Theorems 3.1, 4.1, and Corollary 4.3).

References:

- [1] Arrow, K.J. and Debreu, G. (1954): Existence of an equilibrium for a competitive economy. *Econometrica* 22, 265-290
- [2] Border, K.C. (1985): Fixed point theorems with applications to economics and game theory. Cambridge University Press, Cambridge, UK
- [3] Day, M.M. (1973): Normed linear spaces. Springer-Verlag, Berlin
- [4] Debreu, G. (1952): A social equilibrium existence theorem. Proceedings of the National Academy of Sciences 38, 886-893
- [5] Ichishi, T. (1983): Game theory for economic analysis. Academic Press, New York
- [6] Norde, H., Potkers, J., Reijnders, H., and Vermeulen, D. (1983): Equilibrium selection and consistency. Report 9341, Department of Mathematics, University of Nijmegen, NL
- [7] Peleg, B., Potkers, J., and Tjis, S. (1994): Minimality of consistent solutions for strategic games, in particular for potential games. Center for Economic Research, Tilburg University, mimeo
- [8] Peleg, B., and Tjis, S. (1993): The consistency principle for games in strategic form. Center for Economic Research, Tilburg University, mimeo
- [9] Shafer, W.J., and Sonnenschein, H. (1975): Equilibrium in abstract economies without ordered preferences. *Journal of Mathematical Economics* 2, 345-348