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A Walrasian Approach to Bargaining Games

by

Walter Trockel

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University of Bielefeld

33501 Bielefeld, Germany

Abstract

The paper presents and discusses an alternative approach to Bargaining Games. N-person Bargaining Games with complete information are shown to induce in a canonical way an Arrow-Debreu economy with production and private ownership. The unique Walras stable competitive equilibrium of this economy is shown to coincide with an asymmetric Nash-Bargaining solution of the underlying game with weights corresponding to the shares in production. In the case of an economy with equal shares in production the unique competitive equilibrium coincides with the symmetric Nash-Bargaining solution. As this in turn represents the unique Shapley NTU-value our paper solves a problem posed by Shubik, namely to find a model in which the Shapley NTU-value is a Walrasian equilibrium.

"...there remain several questions of interpretation, particularly the meaning of equitable..."
L. Shapley (1969)

"There has been some controversy about the interpretation of the λ-transfer-value...no consensus has yet emerged on the significance of these concerns, which had been addressed...in numerous explorations of the λ-transfer-value as a tool for analysing games and markets..."
Al Roth (1988)

0. Introduction

"There is a strong analogy though no formal equivalence that we know of between the comparison weights that we must introduce in order to obtain a feasable transfer value and the prices in a competitive market"
M. Shubik (1985)

The present paper solves for the particular class of (NTU–) Bargaining Games this problem posed by Shubik of characterizing Shapley's λ–transfer–value as an equilibrium price system in a specific competitive Arrow–Debreu economy with production and private ownership. As for this class of games the solution of the bargaining problem prescribed by the λ–transfer–value coincides with the Nash–solution where the equilibrium price system determines endogenously rates at which utilities of different players can be transferred at the solution point. Our result provides an alternative approach to the Nash–Bargaining solution.

Our starting point in section 1 is an abstract bargaining situation as modelled in Roth (1979). After recalling Shapley's notion of NTU–value in section 2 we shall associate in section 3 a family of bargaining economies to each abstract bargaining situation.

In section 4 the unique competitive equilibrium of each such economy is shown to characterize an asymmetric Nash–solution. Shapley's equity axiom is represented in that context by an "equal–shares–in–production"–assumption for the bargaining economy, whose unique stable equilibrium then turns out to coincide with the symmetric Nash–Bargaining solution.

The concluding section 5 discusses relations to the literature and ways of potential extensions.

1. The Nash-Solution

As a guiding principle for a value theory for games Shapley (1969) postulated the existence of a scaling vector $\lambda = (\lambda_1, \dots, \lambda_n)$ for the cardinal utility functions of the n players of a game, under which the outcome would be both <u>equitable</u> and <u>efficient</u>.

We shall illustrate these notions as conceived by Shapley in the special context of Bargaining Games.

In general cooperative games with non–transferable utility there is a set of feasable utility allocations for its members for each coalition of players. In bargaining games all these sets except the one for the grand coalition degenerate. Those sets for single coalitions together determine the <u>disagreement outcome.</u> Being in a world of cardinal utilities, being justified in the literature as von Neumann–Morgenstern utility functions, which allow to execute outcomes of the game via suitable underlying lotteries, we may assume without loss of generality (if necessary after a suitable positive affine transformation of utility scales) that the disagreement point is the origin in $\mathbb{R}^n$.

Accordingly we define an n–person Bargaining Game as a compact convex subset S of $\mathbb{R}^n$ with at least one element $s \gg 0$.

Denoting by B the set of all such bargaining games a <u>solution</u> is any function $f: B \to \mathbb{R}^n$ with $f(S) \in S$ for all $S \in B$.

The <u>Nash Bargaining solution</u>, which can be characterized by the four axioms of covariance under positive affine transformations, symmetry, Pareto optimality and Independence of Irrelevant Alternatives (cf. Roth (1979) p. 110) is alternatively described as that solution, which associates with any bargaining game that point, on which the so called <u>Nash product</u> $(x_1, \dots, x_n) \longmapsto \prod_{i=n}^{n} x_i$ becomes maximal.

Our main concern in this paper is it to give an alternative characterization of this Nash Bargaining solution.

2. Shapley's NTU-Value

Consider a bargaining game S. We illustrate in figure 1 the standard case where S in addition to being compact and convex is comprehensive (with respect to $\mathbb{R}^n_+$).

Figure 1 (see appendix)

While any point on the boundary of S is efficient it is not so obvious, what equity means. In a strategically simple context like the present one of bargaining games it very roughly means "equal shares".

We shall discuss briefly Shapley's basic idea without being dependent on a rigorous definition in the sequel because we shall approach the problem from quite a different point of view.

For an arbitrary efficient point x on the boundary of S the transformation rate for utilities of different players is typically different from the rates of their total utilities at that point. The difference of these rates reflects the difference between efficiency and equity. Requiring both properties means coincidence of the (absolute values) of slopes of the boundary of S at x and the vector x itself. For x in Figure 1 to maximize the sum of players' utilities the utility payoff of player 2 has to be tripled. For the players to share equally at x the payoff of player 1 has to be doubled. Accordingly the efficiency and equity weights are 1/3 and 2/1 respectively.

Shapley's NTU–value requires both sets of weights to coincide in a solution.

The problem of finding a solution for the λ–translated bargaining set amounts to finding the λ–transfer–value or Shapley NTU–value.

So suppose we have problem

I: Find $x^* \in S$ such that

1) $x_1^* = \ldots = x_n^*$ equitability

2) $\sum_{i=1}^{n} x_i^* = \max_{x \in S} \sum_{i=1}^{n} x_i$ efficiency

If I has no solution one is lead to the following problem:

II: Find $\lambda \in \mathbb{R}_+^n /\{0\}$ such that I has a solution if S is replaced by

$\lambda * S := \{(\lambda_1 x_1, \ldots \lambda_n x_n) \mid x \in S\}$.

Shapley (1969) has proven the (generally non unique) existence of a λ–transfer–value for NTU–games by a fixed point argument. In the case of bargaining games the unique solution coincides with the Nash bargaining solution.

An alternative way of looking at this is to support the bargaining game by a hyperplane at the Nash solution. This generates a TU–game, whose Shapley–value is just the Nash–solution of both the underlying NTU as well as the derived TU bargaining game.

The situation, which is generated by extending the set of alternatives in an NTU game via a supporting hyperplane to those in a TU–game is comparable to that where a single agent, who is an isolated consumer and producer, gets the additional opportunity to sell and buy on a market at given market prices.

If in addition we interpret the set S of utility allocations as a production possibility set for the n players it is natural to think of the bargaining game as of a particular Arrow–Debreu economy.

3. The Bargaining Economy

Given an n–person bargaining game $S \subset \mathbb{R}_+^n$ as defined above. We define an associated bargaining economy E_θ^S as follows:

We have n–commodities, which are the n–players former cardinal utility levels. So our commodity space is $\mathbb{R}^n$. All n consumers have the same consumption set $\mathbb{R}_+^n$. The endowments are $e_i = 0 \in \mathbb{R}_+^n$ for all $i \in \{1..., n\}$. The preference of consumer i is represented by the utility function u_i with $u_i(x_1, \ldots, x_n) = x_i$. There is one firm with production possibility set $Y :\equiv S$. Ownership of this firm is distributed among the n consumers, who have shares $\theta_i > 0$, $i = 1..., n$ with $\sum_{i=1}^{n} \theta_i = 1$.

In this specific economy, which is illustrated in Figure 2, every consumer is only interested in "his" commodity i.e. his cardinal utility in the underlying bargaining game. Any potential altruistic considerations would have been aptly represented already in those cardinal utility functions of the bargaining game.

Figure 2 (see appendix)

Note that the associated bargaining economy works on a purely ordinal basis. Whatever the game whose formulation is based on players' cardinal utilities means to the players, the economist is only confronted with joint production plans and (ordinal) preferences. It is only the substitution rates in equilibrium he is interested in. So any monotonic transformation of the utility functions of the bargaining economy would be allowed without changing anything.

The incomes by which consumption bundles of the consumers can be financed result from the partial ownership in the firm and the according profit shares. Therefore all consumers are interested in profit maximizing behaviour of the firm. On the other hand each consumer wants his "own" commodity to be as cheap as possible to allow large consumption.

Consumer i's budget restriction given the production plan y and the price system p is
$p \cdot x_i \leq \theta_i p \cdot y$

In equilibrium production y^* is profit maximizing at the price system p^*. The consumption optima are corner solutions x_i^* for every consumer i at "his" commodity axis satisfying $x_i^* = (0,...0, \theta_i \frac{p^* \cdot y^*}{p_i^*}, 0,...0) \in \mathbb{R}^n_+$ with the non–zero entry in the i–th coordinate. Accordingly the aggregate demand vector is

$$\left(\theta_1 \frac{p^* \cdot y^*}{p_1^*}, ..., \theta_n \frac{p^* \cdot y^*}{p_n^*}\right).$$

This can be seen as the aggregate demand of a representative consumer, whose preference is represented by the Cobb–Douglas utility function

$$(x_1, ...x_n) \longmapsto x_1^{\theta_1} \cdot ... \cdot x_n^{\theta_n}.$$

In equilibrium this demand vector coincides with the production vector y^*.

Existence of an equilibrium is straightforward because it is the result of the representative consumers' utility maximization problem on Y resp. on the budget set
$\{y \in Y \mid p^* \cdot y \leq p^* \cdot y^*\}$.

Moreover the resulting equilibrium is unique and stable with respect to the standard Walrasian tatonnement process. This follows from the fact, which may easily be verified that the excess demand function Z of our economy satisfies the Weak Axiom of Revealed Preferences, i.e. $\forall p \neq p^*: p^* \cdot Z(p) > 0$.

If (x^*, p^*) is the equilibrium, which coincides with the Nash solution and the NTU–value the λ–transfer has to be such that (x^*, p^*) is transferred to $((t,...,t), (1,...,1))$ by equity and efficiency.
For the canonical basis vectors $e_i = (0,...,1,...,0)$, $i = 1,...,n$ with 1 in the i–th coordinate the inner product $(1,...,1) \cdot (te_i - te_j)$ for any $i, j = 1,...,n$ is zero.
Therefore $\lambda_1,...,\lambda_n) \cdot (\frac{1}{\lambda_i} te_i - \frac{1}{\lambda_j} te_j) = 0$, $i,j = 1,...,n$.

In the non transferred world we have $p^* \cdot (\frac{1}{\lambda_i} te_i - \frac{1}{\lambda_j} te_j) = 0$ because $\frac{1}{\lambda_i} te_i, \frac{1}{\lambda_j} te_j$ are on the budget hyperplane. Accordingly p^* equals λ up to normalization.

4. Results

We collect our observations in the following

<u>Proposition 1:</u> Given an n–person bargaining game S. For any $\theta \in \mathbb{R}^n_{++}$ with $\sum_{i=1}^{n} \theta_i = 1$ the bargaining economy E^S_θ has a unique stable Walrasian equilibrium, whose consumption vector coincides with the asymmetric Nash–Bargaining solution with weights θ_i, $i = 1,...,n$.

An immediate consequence of this observation is of particular interest and will be stated as

<u>Proposition 2:</u> The (symmetric) Nash–solution of an n–person bargaining game S coincides with the equilibrium consumption vector of the associated equal shares bargaining economy $E^S_{(\frac{1}{n}, ..., \frac{1}{n})}$.

These propositions do not only solve the problem posed by Shubik for the special case of bargaining games. They also provide a new way of looking at Shapley's equity requirement. In the associated bargaining economy equity is reflected by the assumption of equal shares of consumers.

5. Concluding Remarks

The characterization of the Nash–Bargaining solution and, in this context, the Shapley–NTU value by an equilibrium price system of an associated economy can be seen as a kind of a second welfare theorem. Although mathematically trivial in our case it shets some light on the meaning of Shapley's equity requirement and of the λ–transfer–value. Equity corresponds to equal shares, the λ–transfer–vector is the equilibrium price vector.

As the given characterization is true for any, even small, n it might seem confusing that a, say, 2–person–bargaining outcome should be aptly described by competitive prices. It is, however, easy to consider the 2 players as intervals of types of 2 players with uniform distribution and to reformulate our bargaining economy as a coalition production economy as treated in Hildenbrand (1974).

The competitive interpretation is also supported by papers of Gale (1986) and of Rubinstein and Wolinsky (1984).

The puzzling fact of the bargaining economy that production is possible without input allows an easy remedy. One can supplement the model by a third good, an input (the dollar, which has to be split), which does not enter the agents' utilities but represents a a fixed input necessary for production.

There are certainly relations of our work to that of Moulin, Roemer and others on fair division on a joint ownership or fair allocation of goods (cf. Moulin (1989 a,b), Roemer (1986)) but there seems to be no formal equivalence.

There are some problems, which arise in a natural way from our results.

One can easily associate with general cooperative NTU–games economies where coalitions utility possibility sets are looked at as production possibility sets owned by the members of the coalitions. Again one should check the relation between the competitive equilibrium price system and the λ–transfer–value.

A second possible extension is to bargaining with incomplete information as modelled in Harsanyi and Selten (1972). An analogous approach via economies appears possible.

The economies here, however, have to deal with contingent commodities and different agents', different probabilistic beliefs on the states of the word representing players' types.

Finally there seems to be a relation in spirit, which should be clarified between subjective utility evaluations of games as treated in the work of Roth (1977a,b), (1978) and objective price evaluations as proposed in the present paper. One might think even of introducing markets for specific assets guaranteeing players' roles in specific games and leading to market evaluations of games.

Appendix

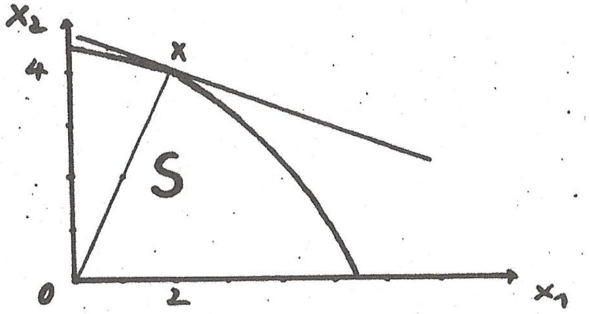

Figure 1

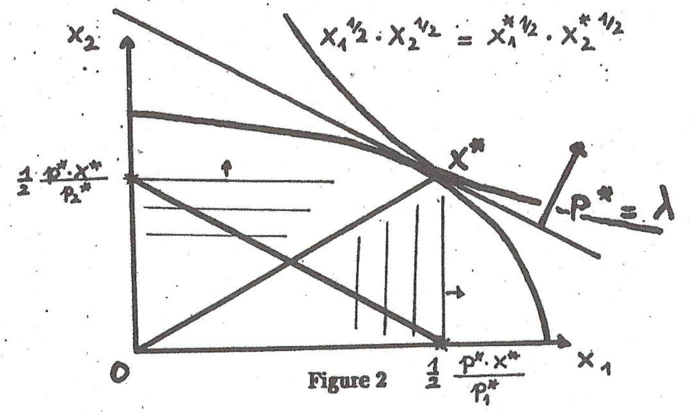

Figure 2

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