On leading terms of quaternionic Stickelberger elements over number fields

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Contents

Introduction

In this thesis we construct and study Stickelberger elements associated to automorphic representations of the multiplicative group of quaternion algebras over arbitrary algebraic number fields. We prove lower bounds on the order of vanishing of these Stickelberger elements and, moreover, interpolation and leading term formulae. As a special case the vanishing part of Mazur and Tate's refined "Birch and Swinnerton-Dyer type"-conjecture for elliptic curves of rank zero follows. Moreover, we use the Stickelberger elements to construct p-adic L-functions. The results for the Stickelberger elements can be transferred to p-adic L-functions to prove exceptional zero conjectures.

Historical Background

Let p be a prime number and A an elliptic curve over $\mathbb{Q}$ of conductor N. In the seminal article [MTT86] Mazur, Tate and Teitelbaum formulate a p-adic Birch and Swinnerton-Dyer conjecture for the p-adic L-function $L_p(A,s)$ associated to A. If A has split multiplicative reduction at p, the interpolation formula relating the p-adic and the complex L-functions ensures that the p-adic L-functions vanishes at $s=0$ even if the complex L-function $L(A,s)$ does not vanish at $s=1$. They conjectured that

$$\operatorname{ord}_{s=0} L_p(A,s) = \operatorname{ord}_{s=1} L(A,s) + 1$$

holds in general and further, if the analytic rank of $A(\mathbb{Q})$ is zero they stated the formula

$$L_p'(A,0) = \mathcal{L}(A)\ \frac{L(A,1)}{\Omega_A^+}. \tag{0.1}$$

Here $\mathcal{L}(A)$ is the (arithmetic) $\mathcal{L}$-invariant of A at p and Ω_A^+ is the real period attached to A. We give a definition of these invariants:

Since A has split multiplicative reduction there exists a rigid analytic uniformization

$$\mathbb{G}_m/q_{\mathrm{T}}^{\mathbb{Z}} \xrightarrow{\cong} A_{\mathbb{Q}_p},$$

where $q_{\mathrm{T}} \in \mathbb{Q}_p^*$ is called the Tate period of $A_{\mathbb{Q}_p}$. The $\mathcal{L}$-invariant is given by the quotient

$$\mathcal{L}(A) = \frac{\log_p(q_{\mathrm{T}})}{\operatorname{ord}_p(q_{\mathrm{T}})}.$$

The Néron lattice $\mathfrak{L}_A$ of A is obtained by integrating a Néron differential ω_A against all elements in $H_1(A(\mathbb{C}),\mathbb{Z})$. There exists a pair of positive real numbers $\Omega_A^+, \Omega_A^- \in \mathbb{R}_{>0}$, the real and imaginary periods attached to A, uniquely determined by the following property: If $A(\mathbb{R})$ has two connected components

$$\mathfrak{L}_A = \Omega_A^+\mathbb{Z} + i\Omega_A^-\mathbb{Z} \qquad \text{(the "rectangular case")}$$

holds and if $A(\mathbb{R})$ has only one connected component, we have

$$\mathfrak{L}_A \subseteq \Omega_A^+\mathbb{Z} + i\Omega_A^-\mathbb{Z} \qquad \text{(the "nonrectangular case")}$$

with index two. To be exact, in the nonrectangular case elements in $\mathfrak{L}_A$ are of the form $a\Omega_A^+ + ib\Omega_A^-$ with $a \equiv b \bmod 2$.

There are several generalizations of this conjectures, e.g. from Hida to totally real number fields (see [Hid09]) and Deppe (see [Dep16]) and Disegni (see [Dis16]) to arbitrary number fields. More important, the formula (0.1) was proven by Greenberg and Stevens (cf. [GS93]). Mok (see [Mok09]) and Spieß (see [Spi14]) prove Hida's conjecture for modular elliptic curves over totally real number fields in the case of analytic rank 0 (under some technical assumptions). Deppe generalizes Spieß' methods to prove the vanishing part of the exceptional zero conjecture for arbitrary number fields. Further, in [BD99] Bertolini and Darmon prove an analogue of (0.1) for the anticyclotomic p-adic L-function of the base change of A to an imaginary quadratic field in which p splits.

In [MT87] Mazur and Tate give an integral refinement of the p-adic Birch and Swinnerton-Dyer conjecture. For this, they define Stickelberger elements, which can be seen as an analogue of the p-adic L-function of A at a "finite level". They are defined in the following way:

Fixing an integer $M \geq 1$ we write μ_M for the group of M-th roots of unity, $L = \mathbb{Q}(\mu_M)^+$ for the maximal totally real subextension of $\mathbb{Q}(\mu_M)$ and put $\mathcal{G}_M = \mathrm{Gal}(L/\mathbb{Q})$. We have an isomorphism

$$(\mathbb{Z}/M\mathbb{Z})^*/\{\pm 1\} \xrightarrow{\cong} \mathcal{G}_M,$$

where the image of $a \in (\mathbb{Z}/M\mathbb{Z})^*$ is denoted by σ_a.

By the modularity theorem one can associate a normalized newform $f \in S_2(\Gamma_0(N))$ to A such that the corresponding L-functions coincide (cf. [Wil95], [TW95] and [BCDT01]). Let $\lambda_A \colon \mathbb{Q}/\mathbb{Z} \to \mathbb{C}$ be the modular symbol for A given by

$$\lambda_A(q) = 2\pi i \int_{i\infty}^{q} f(z)dz$$

as defined in [MTT86]. By a beautiful theorem of Manin and Drinfeld (cf. [Man72] and [Dri73]) there exists a proper subring $\mathcal{R} \subset \mathbb{Q}$ such that $\lambda_A(q) = [q]_A^+\Omega_A^+ + i[q]_A^-\Omega_A^-$ with functions $[\cdot]_A^\pm\colon \mathbb{Q}/\mathbb{Z} \to \mathcal{R}$. For example, if A is a strong Weil curve, we can take $\mathcal{R} = \mathbb{Z}[\frac{1}{\tau c_A}]$, where τ is the order of the finite group $A(\mathbb{Q})_{\mathrm{tors}}$ and c_A is the Manin constant of A. The Manin constant c_A is an integer, which is conjectured to be 1. This is known in many cases (cf. [Edi91]). Note that, in special cases, one has even better bounds for the denominators occurring in the ring $\mathcal{R}$ (cf. [Wut14]). The functions $[\cdot]_A^\pm$ are the so called "+" resp. "−" modular symbols. In the following we just treat the "+" modular symbol, so we write $[\cdot]_A$ for $[\cdot]_A^+$.

The Stickelberger element of modulus M associated to A is defined as

$$\Theta_{A,M}^{\mathrm{MT}} = \frac{1}{2} \sum_{a \in (\mathbb{Z}/M\mathbb{Z})^*} \left[\frac{a}{M}\right]_A \sigma_a \in \frac{1}{2}\mathcal{R}[\mathcal{G}_M],$$

where for an arbitrary (commutative and unital) ring R and an arbitrary group H the group algebra of H over R is denoted by $R[H]$. Since $[q]_A = [-q]_A$ for all $q \in \mathbb{Q}/\mathbb{Z}$ (cf. [MTT86]) we have

$$\Theta_{A,M}^{\mathrm{MT}} \in \mathcal{R}[\mathcal{G}_M]$$

as long as $M \geq 3$, which we will assume from now on.

Next, we state the vanishing part of Mazur and Tate's conjecture of the "Birch and Swinnerton-Dyer type". For general R and H as above, let $I_R(H) \subseteq R[H]$ be the kernel of the augmentation map $R[H] \to R$, $h \mapsto 1$.

Definition. The order of vanishing $\mathrm{ord}_R(\xi)$ of an element $\xi \in R[H]$ is defined as

$$\mathrm{ord}_R(\xi) = \begin{cases} r & \text{if } \xi \in I_R(H)^r \setminus I_R(H)^{r+1}, \\ \infty & \text{if } \xi \in I_R^j(H) \text{for all } j \geq 1. \end{cases}$$

Let S_M be the set of prime factors p of M such that A has split multiplicative reduction at p. We define

$$r_M = \operatorname{rank} A(\mathbb{Q}) + \#S_M.$$

Conjecture (Mazur-Tate)**.** *Let $\mathcal{R} \subset \mathbb{Q}$ be a subring, which contains $[q]_A$ for all $q \in \mathbb{Q}$. Then the inequality*

$$\mathrm{ord}_{\mathcal{R}}(\Theta_{A,M}^{\mathrm{MT}}) \geq r_M$$

holds, i.e. $\Theta_{A,M}^{\mathrm{MT}} \in I_{\mathcal{R}}(\mathcal{G}_M)^{r_M}$.

Outline of the thesis

One of our main objectives is to prove the following

Theorem. *For a subring $\mathcal{R} \subset \mathbb{Q}$ which contains $[q]_A$ for all $q \in \mathbb{Q}$, we have*

$$\operatorname{ord}_{\mathcal{R}}(\Theta^{\mathrm{MT}}_{A,M}) \geq \# S_M.$$

Thus the conjecture of Mazur and Tate is true if rank $A(\mathbb{Q}) = 0$. In fact, we prove a more general statement for Stickelberger elements associated to automorphic representations of the multiplicative group of quaternion algebras over algebraic number fields. Let us take a closer look on the structure of this thesis:

Let us fix an algebraic number field F. Further, let E over F be quadratic étale algebra, i.e. E is a field or isomorphic to $F \times F$. In Section 1 we generalize methods developed by Dasgupta and Spieß in [DS] from the multiplicative group F^* to an arbitrary one-dimensional torus (which F-rational points are given by E^*/F^*) to get vanishing results for certain homology classes c_χ, where χ is a character of the adelic points of the torus.

For the second section, let B be a quaternion algebra over F in which E can be embedded. We assume that at the Archimedean places of F the algebra B is split if an only if E is split. Moreover, we assume that E is isomorphic to $F \times F$ if and only if B is split (i.e. B is no division algebra). We define modular symbols κ in terms of the group cohomology of the algebraic group $G = B^*/\mathbb{G}_m$ over F with values in certain adelic function spaces (B^* denotes the F-algebraic group given by $B^*(M) = (B \otimes_F M)^*$). For every allowable modulus $\mathfrak{m}$ (see Definition 2.12) we can pull back the cohomology class κ via an embedding $T \longrightarrow G$ of conductor $\mathfrak{m}$ to get a distribution valued cohomology class $\Delta_{\mathfrak{m}}(\kappa)$.

The third section combines the results of Sections 1 and 2 to define Stickelberger elements. For this, let L over F be a finite Galois extension. We assume that L over F is E-anticyclotomic (see Definition 3.1) if B is non-split and abelian if B is split. Let $\mathcal{G}$ be the Galois group of L over E resp. F. Following Dasgupta and Spieß the cap product of the Artin reciprocity map for L over E resp. F with a fundamental class for the group of relative units of E over F gives a homology class c_L. By the assumption on the splitting behaviour at infinite places we can define the Stickelberger element as the cap product of c_L with $\Delta_{\mathfrak{m}}(\kappa)$. An analysis of the action of local points of the torus T on Bruhat-Tits trees (which are already carried out in Sections 2.1 and 2.2) gives functional equations for Stickelberger elements (Proposition 3.6). Moreover, using the results of Section 1 we bound their order of vanishing (Lemma 3.5).

In Section 4 we specialize to Stickelberger elements coming from automorphic representations. Let π_B be an automorphic representation of G that is cohomological with respect to the trivial coefficient system. Let R_π be the ring of integers of the field of definition of π_B. In Section 4.1 we choose a concrete modular symbol such that the corresponding Stickelberger element $\Theta_{\mathfrak{m}}(L/F, \pi_B)$ lies in $R_\pi[\mathcal{G}]$. We apply the results of Section 3 to $\Theta_{\mathfrak{m}}(L/F, \pi_B)$. To be more precise, we get

$$\mathrm{ord}_{R_\pi}(\Theta_{\mathfrak{m}}(L/F, \pi_B)) \geq \# S_{\mathfrak{m}},$$

where $S_{\mathfrak{m}}$ is the set of all primes $\mathfrak{p}$ of F dividing $\mathfrak{m}$ such that either $\pi_{B,\mathfrak{p}}$ is Steinberg or $\mathfrak{p}$ is inert in E and $\pi_{B,\mathfrak{p}}$ is the non-trivial unramified twist of the Steinberg representation. More generally, in Theorem 4.6 we show that the Stickelberger elements lie in a product of partial augmentation ideals.

In Section 4.2 we prove interpolation formulae. If B is non-split, we use results of File, Martin and Pitale (cf. [FMP]) on toric period integrals to show that our Stickelberger elements interpolate (square roots of) special values of the L-function of the base change of π_B with respect to E over F (Theorem 4.9), i.e. for every character $\chi\colon \mathcal{G} \to \mathbb{C}^*$ of conductor $\mathfrak{m}$ we have

$$|\chi(\Theta_{\mathfrak{m}}(L/F, \pi_B))|^2 \doteq L(1/2, \pi_{B,E} \otimes \chi),$$

where "$\doteq$" means equality up to explicit fudge factors.

If B is split, we evaluate the Stickelberger elements at primitive characters to relate them to special values of L-functions (Theorem 4.10), i.e.

$$\chi(\Theta_{\mathfrak{m}}(L/F, \pi_B)) \doteq L(1/2, \pi_B \otimes \chi)$$

This time around we get the result by concrete calculations.

Making use of the interpolation formulae, we prove in Section 4.3, Theorem 4.17, the following leading term formula: Suppose that all primes $\mathfrak{p}$ in $S_{\mathfrak{m}}$ are split in E and $\pi_{B,\mathfrak{p}}$ is Steinberg for all $\mathfrak{p} \in S_{\mathfrak{m}}$. In this situation we define "automorphic periods" $\mathfrak{q}_{\mathfrak{p}} \in F_{\mathfrak{p}}^* \otimes R_\pi$. Crucial in the definition of the automorphic periods are extension classes of the Steinberg representation, which were first studied by Breuil in [Bre04]. For non-split quaternion algebras we prove that

$$\prod_{\mathfrak{p} \in S_{\mathfrak{m}}} \mathrm{ord}_{\mathfrak{p}}(q_{\mathfrak{p}}) \cdot \Theta_{\mathfrak{m}}(L/F, \pi_B) \doteq \prod_{\mathfrak{p} \in S_{\mathfrak{m}}} (\mathrm{rec}_{\mathfrak{p}}(q_{\mathfrak{p}}) - 1) \cdot \sqrt{L(1/2, \pi_{B,E})} \bmod I_{R_\pi}(\mathcal{G})^{\# S_{\mathfrak{m}}+1}$$

holds. Here $\mathrm{rec}_{\mathfrak{p}}$ denotes the local reciprocity map at a prime of E lying above $\mathfrak{p}$. However, for a split-quaternion algebra we get

$$\prod_{\mathfrak{p}\in S_{\mathfrak{m}}} \mathrm{ord}_{\mathfrak{p}}(q_{\mathfrak{p}}) \cdot \Theta_{\mathfrak{m}}(L/F, \pi_B) \doteq \prod_{\mathfrak{p}\in S_{\mathfrak{m}}} (\mathrm{rec}_{\mathfrak{p}}(q_{\mathfrak{p}}) - 1) \cdot L(1/2, \pi_B) \bmod I_{R_\pi}(\mathcal{G})^{\# S_{\mathfrak{m}}+1}$$

with $\mathrm{rec}_{\mathfrak{p}}$ being the local reciprocity map over $F_{\mathfrak{p}}$.

Finally, in Section 4.4, we construct p-adic L-functions via Stickelberger elements. In particular, if B is split, they coincide with the multi-variable p-adic L-function constructed by Deppe (see [Dep16]). If B is non-split, our construction yields to anticyclotomic p-adic L-functions. Note that, if E is totally real, one expects that there exists no anticyclotomic $\mathbb{Z}_{\mathfrak{p}}$-extension but E-anticyclotomic Stickelberger elements at finite level can still be defined.

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Abgrenzung des eigenen Beitrags gemäß §10(2) der Promotionsordnung

Der Inhalt dieser Dissertation baut auf zwei Arbeiten auf, die der Autor zusammen mit Lennart Gehrmann in den Arbeiten [BG17] und [BG] veröffentlicht. Die Arbeit [BG17] wurde in der Zeitschrift *Proceedings of the London Mathematical Society* veröffentlicht und die Arbeit [BG] wurde von der Zeitschrift *Transactions of the American Mathematical Society* zur Veröffentlichung angenommen. In den Arbeiten zeigen L. Gehrmann und der Autor ähnliche Resultate einerseits für die algebraische Gruppe PGL_2 (in [BG17]) und andererseits für einen Quotienten der multiplikativen Gruppe einer nichtspaltenden Quaternionenalgebra (in [BG]), jeweils über einem total reellen Zahlkörper als Grundkörper. In dieser Dissertation hat der Autor beide Ansätze vereinheitlicht und von einem total reellen Grundkörper auf beliebige Zahlkörper als Grundkörper verallgemeinert. Die Idee der Verbesserung der Resultate von Spieß zu Erweiterungen von Steinberg-Darstellungen, siehe Lemma 4.13, stammt von L. Gehrmann. Dieses Resultat ist wichtig für den Beweis des 'Leading Term' Satzes, Theorem 4.17. Die Abschnitte 4.3 und 4.4 in [BG] enthalten einen Vergleich von automorphen und algebraischen L-Invarianten. Diese sind im wesentlichen von L. Gehrmann erarbeitet worden und daher nicht Teil der vorgelegten Dissertation.

Notations

We will use the following notations throughout this thesis. All rings are commutative and unital. The group of invertible elements of a ring R will be denoted by R^*. For a group H we will denote the group algebra of H over R by $R[H]$. We let $I_R(H) \subseteq R[H]$ be the kernel of the augmentation map $R[H] \to R$, $h \mapsto 1$. Let Θ be an element of $R[H]$. We write $\Theta^\vee$ for the image of Θ under the map induced by inversion on H. Given a group homomorphism $\chi \colon H \to R^*$ we let $R(\chi)$ be the representation of H whose underlying R-module is R itself and on which H acts via the character χ. If N is another $R[H]$-module, we put $N(\chi) = N \otimes_R R(\chi)$.

For a set X and a subset $A \subseteq X$ the characteristic function $\mathbb{1}_A \colon X \to \{0,1\}$ is defined by

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{else.} \end{cases}$$

Throughout the article we fix an algebraic number field F with ring of integers $\mathcal{O}_F$ and $r_1(F)$ real and $r_2(F)$ complex places. For a non-zero ideal $\mathfrak{a} \subseteq \mathcal{O}_F$ we set $N(\mathfrak{a}) = \#\mathcal{O}_F/\mathfrak{a}$. If v is a place of F, we denote by F_v the completion of F at v.

If $\mathfrak{p}$ is a finite place, we let $\mathcal{O}_{F_\mathfrak{p}}$ denote the valuation ring of $F_\mathfrak{p}$ and write $\operatorname{ord}_\mathfrak{p}$ for the normalized additive valuation, i.e. $\operatorname{ord}_\mathfrak{p}(\varpi_\mathfrak{p}) = 1$ for any local uniformizer $\varpi_\mathfrak{p} \in \mathcal{O}_{F_\mathfrak{p}}$. For an arbitrary place v let $|\cdot|_v$ be the normalized multiplicative norm. This means that $|x|_\mathfrak{p} = N(\mathfrak{p})^{-\operatorname{ord}_\mathfrak{p}(x)}$ if $\mathfrak{p}$ is a finite place, $|x|_v = |\sigma_v(x)|_\mathbb{R}$ if σ_v is the embedding $F \hookrightarrow \mathbb{R}$ corresponding to the real Archimedean place v (where $|\cdot|_\mathbb{R}$ is the usual absolute value on $\mathbb{R}$) and $|x|_v = |\sigma_v(x)|_\mathbb{C}$ if σ_v is the embedding $F \hookrightarrow \mathbb{C}$ corresponding to the complex Archimedean place v (where $|\cdot|_\mathbb{C}$ is the square of the usual absolute value on $\mathbb{C}$, i.e. $|z|_\mathbb{C} = z\overline{z}$).

For a finite (possibly empty) set S of places of F we define the "S-truncated adeles" $\mathbb{A}^S$ as the restricted product of the completions F_v over all places v which are not in S. We often write $\mathbb{A}^{S,\infty}$ instead of $\mathbb{A}^{S \cup S_\infty}$. Here S_∞ denotes the set of Archimedean places of F. We always drop the superscript $\emptyset$ if $S = \emptyset$.

If $\mathfrak{G}$ is an algebraic group over F and v is a place of F, we write $\mathfrak{G}_v = \mathfrak{G}(F_v)$ and put $\mathfrak{G}_S = \prod_{v \in S} \mathfrak{G}_v$ for a set of places S of F as above. Furthermore, if $K \subseteq \mathfrak{G}(\mathbb{A})$ is a subgroup, we define K^S as the image of K under the quotient map $\mathfrak{G}(\mathbb{A}) \to \mathfrak{G}(\mathbb{A}^S)$. If $\mathfrak{m} \subseteq \mathcal{O}_F$ is a non-zero ideal, we put

$$S_\mathfrak{m} = \{\mathfrak{p} \in S \text{ s.t. } \mathfrak{p} \mid \mathfrak{m}\}.$$

Generalities on functions and distributions

Given topological spaces X, Y we will write $C(X,Y)$ for the space of continuous functions from X to Y. If R is a topological ring, we define $C_c(X,R) \subseteq C(X,R)$ as the subspace of continuous functions with compact support. If we consider Y (resp. R) with the discrete topology, we will often write $C^0(X,Y)$ (resp. $C_c^0(X,R)$) instead.

For a ring R and an R-module N, we define the R-module of N-valued distributions on X as $\mathrm{Dist}(X,N) = \mathrm{Hom}_{\mathbb{Z}}(C_c^0(X,\mathbb{Z}), N)$. If X is discrete, we have the following pairing

$$C_c^0(X,\mathbb{Z}) \times C^0(X,N) \longrightarrow N,\ (\psi,\phi) \longmapsto \sum_{x\in X} (\psi\cdot\phi)(x),$$

which induces an isomorphism of R-modules

$$C^0(X,N) \longrightarrow \mathrm{Dist}(X,N). \tag{0.2}$$

We will always identify these two R-modules via the above isomorphism if X is discrete. In the case that X is a compact space, we denote the space of N-valued distributions of total volume 0 by $\mathrm{Dist}_0(X,N)$.

We say that an R-module N is prodiscrete if N is a topological group such that there exist open R-submodules

$$\ldots \subseteq N_2 \subseteq N_1 \subseteq N$$

with $\bigcap_i N_i = \{0\}$ and $N = \varprojlim_i N/N_i$. Let X be a totally disconnected compact space and N a prodiscrete R-module. We restrict the canonical pairing

$$\varprojlim_i C^0(X, N/N_i) \otimes \mathrm{Dist}(X,R) \longrightarrow \varprojlim_i N/N_i = N$$

to $C(X,N)$ via the embedding

$$C(X,N) \hookrightarrow \varprojlim_i C^0(X,N/N_i).$$

This yields an integration pairing

$$C(X,N) \otimes \mathrm{Dist}(X,R) \longrightarrow N. \tag{0.3}$$

1 Characters and homology classes

In Section 3 of [DS] Dasgupta and Spieß develop a machinery to bound the order of vanishing of Stickelberger elements coming from distributions on the split one-dimensional torus. In this section we indicate how to generalize their methods to non-split tori. At primes at which the torus splits essentially the same arguments as in [DS] apply. At a non-split prime $\mathfrak{p}$ the situation turns out to be even simpler: the local torus is compact and thus, the rank does not change if one passes from arithmetic subgroups to $\mathfrak{p}$-arithmetic subgroups of the torus.

Let us fix a quadratic étale algebra E of F, i.e. E is either isomorphic to $F \times F$ or a quadratic field extension of F. If E is a field, we write $\mathcal{O}$ for the ring of integers of E. If E is isomorphic to $F \times F$, we fix once and for all an isomorphism of E with $F \times F$ and let $\mathcal{O}$ be $\mathcal{O}_F \times \mathcal{O}_F$. In both cases, write τ for the generator of $\mathrm{Aut}_F(E)$.

We consider the algebraic torus $T = \mathrm{Res}_{E/F}\,\mathbb{G}_m/\mathbb{G}_m$ over F. Let us write d for the rank of $\mathcal{O}^*/\mathcal{O}_F^*$. If E is a field, d is equal to the number of Archimedean places of F that are split in E. In the other case the choice of the first F-coordinate of E yields an isomorphism $\mathbb{G}_m \cong T$ and so we get $d = r_1(F) + r_2(F) - 1$ by Dirichlet's unit theorem. For the rest of the paper we will identify $\mathbb{G}_m$ and T via the above isomorphism.

For a finite place $\mathfrak{p}$ of F all $\mathcal{O}_{F_\mathfrak{p}}$-orders in $E_\mathfrak{p}$ are of the form $\mathcal{O}_{F_\mathfrak{p}} + \mathfrak{p}^m\mathcal{O}_\mathfrak{p}$ for some $m \geq 0$. Here $\mathcal{O}_\mathfrak{p}$ denotes the maximal $\mathcal{O}_{F_\mathfrak{p}}$-order in $E_\mathfrak{p}$. Let us write $U_{T_\mathfrak{p}}^{(m)}$ for the image of $(\mathcal{O}_{F_\mathfrak{p}} + \mathfrak{p}^m\mathcal{O}_\mathfrak{p})^*$ in $T_\mathfrak{p}$. If v is an Archimedean place of F, we define U_{T_v} as the connected component of 1 in T_v. Further, we put

$$U_{T_\infty} = \prod_{v \in S_\infty} U_{T_v} \subset T_\infty.$$

Given a non-zero ideal $\mathfrak{m} \subseteq \mathcal{O}_F$ we define

$$U_T(\mathfrak{m}) = \prod_{\mathfrak{p} \notin S_\infty} U_{T_\mathfrak{p}}^{(\mathrm{ord}_\mathfrak{p}(\mathfrak{m}))} \times U_{T_\infty} \subseteq T(\mathbb{A}).$$

To ease the notation we write U_T instead of $U_T(\mathcal{O}_F)$.

In case E is a field we fix once and for all for every prime $\mathfrak{p}$ of F a prime $\mathfrak{P}$ of E lying above $\mathfrak{p}$ and a local uniformizer $\varpi_\mathfrak{P}$ at $\mathfrak{P}$. If $\mathfrak{p}$ is split in E, the choice of $\mathfrak{P}$ determines an isomorphism $T_\mathfrak{p} \cong F_\mathfrak{p}^*$. We will always identify these two groups via the above isomorphism. Likewise, for every split Archimedean place v of F we fix a place w of E above v and identify T_v with F_v^*.

To unify the notation, we call every place of F split in E if E is isomorphic to $F \times F$. We have $\mathcal{O}_\mathfrak{p} = \mathcal{O}_{F_\mathfrak{p}} \times \mathcal{O}_{F_\mathfrak{p}}$ for every finite place $\mathfrak{p}$ of F in this situation. The fixed isomorphism of T with $\mathbb{G}_m$ provides an isomorphism of $U_{T_\mathfrak{p}}^{(m)}$ with the m-th unit group $U_\mathfrak{p}^{(m)} = \{x \in \mathcal{O}_\mathfrak{p}^* \mid x \equiv 1 \bmod \mathfrak{p}^m\}$ of $F_\mathfrak{p}$. Further, we write $\mathfrak{P}$ for the prime ideal in $\mathcal{O}$ over $\mathfrak{p}$ which is $\mathfrak{p}$ in the first F-coordinate of E and $\mathcal{O}_F$ in the second. We fix a local uniformizer $\varpi_\mathfrak{p}$ of F and call $\varpi_\mathfrak{P} = (\varpi_\mathfrak{p}, 1)$ local uniformizer at $\mathfrak{P}$.

1.1 Fundamental classes

Suppose that there exists a real Archimedean place v of F which splits in E. The group $U_{T_v} \cong \mathbb{R}_{>0}^*$ is torsion-free. Therefore, for every subgroup $A \subseteq T(F)$ the group

$$A^+ = \ker\left(A \longrightarrow T_\infty / U_{T_\infty}\right)$$

is torsion-free. If there is no real Archimedean place that splits in E, we choose an auxiliary finite place $\mathfrak{q}$ of F and a maximal open torsion-free subgroup $U_{T_\mathfrak{q}}^+ \subseteq U_{T_\mathfrak{q}}$. If $A \subseteq T(F)$ is a subgroup such that the image of A under the embedding $T(F) \hookrightarrow T_\mathfrak{q}$ is contained in $U_{T_\mathfrak{q}}$, we define

$$A^+ = \ker\left(A \longrightarrow U_{T_\mathfrak{q}} / U_{T_\mathfrak{q}}^+\right).$$

Similarly, if $\widetilde{U} \subseteq U_T$ is any subgroup, we define $\widetilde{U}^+ \subseteq \widetilde{U}$ to be the subgroup of elements which $\mathfrak{q}$-component lies in $U_{T_\mathfrak{q}}^+$. To avoid distinguishing the two cases we simply put $\widetilde{U}^+ = \widetilde{U}$ if there is a real Archimedean place that splits in E.

Remark 1.1. Assume that there is no real Archimedean place that splits in E. For the rest of the article we use the following convention for this situation: Whenever we choose a set of finite primes S (resp. a non-zero ideal $\mathfrak{m}$) of F we will assume that the fixed prime $\mathfrak{q}$ is not contained in S (resp. co-prime to $\mathfrak{m}$).

Given a finite (possibly empty) set S of places of F, an open subgroup $\widetilde{U} \subseteq U_T^S$ and a ring R we define

$$\mathcal{C}_?(\widetilde{U}, R)^S = C_?^0(T(\mathbb{A}^S)/\widetilde{U}^+, R)$$

for $? \in \{\emptyset, c\}$. For a non-zero ideal $\mathfrak{m} \subseteq \mathcal{O}_F$ we set

$$\mathcal{C}_?(\mathfrak{m}, R)^S = \mathcal{C}_?(U_T(\mathfrak{m})^S, R)^S. \tag{1.1}$$

If S is the empty set, we drop it from the notation.

Further, if S is a finite set of finite places of F, we define

$$\mathcal{U}_S = \ker\left(T(F) \longrightarrow T(\mathbb{A}^S)/U_T^S\right).$$

By Dirichlet's unit theorem $\mathcal{U}_S^+$ is a free group of rank $d+r$, where r is the number of places in S which are split in E. Thus, the homology group $\mathrm{H}_{d+r}(\mathcal{U}_S^+, \mathbb{Z})$ is free of rank one. We fix a generator η^S of this group. Further, we fix a fundamental domain $\mathcal{F}^S$ for the action of $T(F)/\mathcal{U}_S^+$ on $T(\mathbb{A}^S)/U_T^{S,+}$. By Shapiro's lemma the identification

$$\mathcal{C}_c(\mathcal{O}, \mathbb{Z})^S = \text{c-ind}_{\mathcal{U}_S^+}^{T(F)} C(\mathcal{F}^S, \mathbb{Z})$$

induces an isomorphism

$$\mathrm{H}_{d+r}(\mathcal{U}_S^+, C(\mathcal{F}^S, \mathbb{Z})) \xrightarrow{\cong} \mathrm{H}_{d+r}(T(F), \mathcal{C}_c(\mathcal{O}, \mathbb{Z})^S).$$

Here $\text{c-ind}_{\mathcal{U}_S^+}^{T(F)} C(\mathcal{F}^S, \mathbb{Z})$ is the compact induction of $C(\mathcal{F}^S, \mathbb{Z})$ from $\mathcal{U}_S^+$ to $T(F)$, i.e. it is the space of locally constant functions $f\colon T(F) \to C(\mathcal{F}^S, \mathbb{Z})$ with compact support modulo $\mathcal{U}_S^+$ such that $f(hg) = hf(g)$ for all $h \in \mathcal{U}_S^+$.

The fundamental class ϑ^S is defined as the image of the cap product of η^S with the characteristic function $\mathbb{1}_{\mathcal{F}^S}$ under the above isomorphism. Similarly as before, we drop the superscript S if it is the empty set.

Remark 1.2. Let $S^+ \subseteq S_\infty$ be the set of all split Archimedean places of F. A generator η of $\mathrm{H}_d(\mathcal{U}^+, \mathbb{Z})$ can be identified with the fundamental class of the compact torus $U_{T_{S^+}}/\mathcal{U}^+$.

If E is a field and $\mathfrak{p} \in S$ is either inert or ramified in E, the group $\mathrm{H}^0(T_\mathfrak{p}, C(T_\mathfrak{p}/U_{T_\mathfrak{p}}, \mathbb{Z}))$ is free of rank one. Let $c_\mathfrak{p}$ be the normalized generator, i.e. the function that is constantly one.

If $\mathfrak{p} \in S$ is split in E, we have a sequence of $T_\mathfrak{p}$-modules

$$0 \longrightarrow C_c(T_\mathfrak{p}, \mathbb{Z}) \longrightarrow C_c(F_\mathfrak{p}, \mathbb{Z}) \xrightarrow{g \mapsto g(0)} \mathbb{Z} \longrightarrow 0, \tag{1.2}$$

where the first map is given by extension by zero making use of the identification $T_\mathfrak{p} \cong F_\mathfrak{p}^*$. Note that this isomorphism also provides the $T_\mathfrak{p}$-action on $C_c(F_\mathfrak{p}, \mathbb{Z})$. Taking $U_{T_\mathfrak{p}}$-invariants yields the exact sequence

$$0 \longrightarrow C_c(T_\mathfrak{p}/U_{T_\mathfrak{p}}, \mathbb{Z}) \longrightarrow C_c(F_\mathfrak{p}, \mathbb{Z})^{U_{T_\mathfrak{p}}} \xrightarrow{g \mapsto g(0)} \mathbb{Z} \longrightarrow 0. \tag{1.3}$$

We define $c_\mathfrak{p}$ as the image of $1 \in \mathbb{Z}$ under the connecting homomorphism $\mathbb{Z} \to \mathrm{H}^1(T_\mathfrak{p}, C_c(T_\mathfrak{p}/U_{T_\mathfrak{p}}, \mathbb{Z}))$.

Remark 1.3. If $\mathfrak{p}$ is split, the group $T_\mathfrak{p}/U_{T_\mathfrak{p}}$ is a free abelian group of rank 1. The exact sequence (1.3) is a projective resolution of the trivial $T_\mathfrak{p}/U_{T_\mathfrak{p}}$-module. Therefore, if $\eta_\mathfrak{p}$ is a generator of the free abelian group $\mathrm{H}_1(T_\mathfrak{p}/U_{T_\mathfrak{p}}, \mathbb{Z})$ of rank 1, we get

$$c_\mathfrak{p} \cap \eta_\mathfrak{p} = \pm 1 \in \mathrm{H}_0(T_\mathfrak{p}/U_{T_\mathfrak{p}}, C_c(T_\mathfrak{p}/U_{T_\mathfrak{p}}, \mathbb{Z})) \cong \mathbb{Z}.$$

The canonical pairing

$$\mathcal{C}_c(\mathcal{O}, \mathbb{Z})^S \times C_c(T_\mathfrak{p}/U_{T_\mathfrak{p}}, \mathbb{Z}) \longrightarrow \mathcal{C}_c(\mathcal{O}, \mathbb{Z})^{S-\{\mathfrak{p}\}}$$

induces a cap product pairing on (co)homology groups. The following lemma essentially follows from Remark 1.3.

Lemma 1.4. *For every $\mathfrak{p} \in S$ the equality $\vartheta^{S-\{\mathfrak{p}\}} = \pm c_\mathfrak{p} \cap \vartheta^S$ holds. The sign only depends on the choice of the generators η^S and $\eta^{S-\{\mathfrak{p}\}}$.*

1.2 Derivatives of local characters

In this section we fix a finite place $\mathfrak{p}$ of F. Let A be a group and $l_\mathfrak{p} \colon T_\mathfrak{p} \to A$ a locally constant homomorphism. We can view $l_\mathfrak{p}$ as an element of $C^0(T_\mathfrak{p}, A)$. Since $l_\mathfrak{p}$ is a group homomorphism the map $y \mapsto y.l_\mathfrak{p} - l_\mathfrak{p}$ is constant. Thus, the image of $l_\mathfrak{p}$ in $C^0(T_\mathfrak{p}, A)/A$ is fixed by the $T_\mathfrak{p}$-action.

If E is a field and $\mathfrak{p}$ is inert or ramified in E, we define

$$c_{l_\mathfrak{p}} \in \mathrm{H}^0(T_\mathfrak{p}, C^0(T_\mathfrak{p}, A)/A)$$

to be the image of $l_\mathfrak{p}$.

On the other hand, if $\mathfrak{p}$ is split in E, we define

$$c_{l_\mathfrak{p}} \in \mathrm{H}^1(T_\mathfrak{p}, C^0_c(F_\mathfrak{p}, A))$$

to be the class given by the cocycle

$$z_{l_\mathfrak{p}}(x)(y) = \mathbb{1}_{x\mathcal{O}_{F,\mathfrak{p}}}(y) \cdot l_\mathfrak{p}(x) + ((\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}} - \mathbb{1}_{x\mathcal{O}_{F,\mathfrak{p}}}) \cdot l_\mathfrak{p})(y) \tag{1.4}$$

for $x \in T_\mathfrak{p}$ and $y \in F_\mathfrak{p}$.

Remark 1.5. For a prime $\mathfrak{p}$, which is split in E, we consider the unique $T_\mathfrak{p}$-equivariant homomorphism

$$\alpha_\mathfrak{p} \colon C_c(T_\mathfrak{p}/U_{T_\mathfrak{p}}, \mathbb{Z}) \longrightarrow C_c(F_\mathfrak{p}, \mathbb{Z}) \tag{1.5}$$

that sends $\mathbb{1}_{U_\mathfrak{p}}$ to $\mathbb{1}_{\mathcal{O}_{F_\mathfrak{p}}}$. The class $c_{\mathrm{ord}_\mathfrak{p}}$ associated to the homomorphism

$$\mathrm{ord}_\mathfrak{p} \colon T_\mathfrak{p} \cong F_\mathfrak{p}^* \longrightarrow \mathbb{Z}$$

is equal to the image of the class $c_{\mathfrak{p}}$ under the homomorphism

$$\mathrm{H}^1(T_{\mathfrak{p}}, C_c(T_{\mathfrak{p}}/U_{T_{\mathfrak{p}}}, \mathbb{Z})) \xrightarrow{(\alpha_{\mathfrak{p}})_*} \mathrm{H}^1(T_{\mathfrak{p}}, C_c(F_{\mathfrak{p}}, \mathbb{Z})).$$

More generally, the class $c_{l_{\mathfrak{p}}}$ can be constructed as the image of $l_{\mathfrak{p}}$ under a boundary map

$$\delta\colon \mathrm{H}^0(T_{\mathfrak{p}}, C^0(T_{\mathfrak{p}}, A)/A) \longrightarrow \mathrm{H}^1(T_{\mathfrak{p}}, C_c^0(F_{\mathfrak{p}}, A)).$$

See Section 3.2 of [DS] for more details.

We are mostly interested in the following situation: We fix a ring R and an ideal $\mathfrak{a} \subseteq R$. We set $\overline{R} = R/\mathfrak{a}$ and similarly, write $\overline{N} = N \otimes_R \overline{R}$ for every R-module N. Let $\chi_{\mathfrak{p}}\colon T_{\mathfrak{p}} \to R^*$ be a character. Suppose we have given an ideal $\mathfrak{a}_{\mathfrak{p}} \subseteq \mathfrak{a}$ such that $\chi_{\mathfrak{p}} \equiv 1 \bmod \mathfrak{a}_{\mathfrak{p}}$. Then

$$\mathrm{d}\chi_{\mathfrak{p}}\colon T_{\mathfrak{p}} \longrightarrow \overline{\mathfrak{a}_{\mathfrak{p}}},\ x \longmapsto \chi_{\mathfrak{p}}(x) - 1 \bmod \mathfrak{a}\mathfrak{a}_{\mathfrak{p}}.$$

defines a group homomorphism, which yields a cohomology class $c_{\mathrm{d}\chi_{\mathfrak{p}}}$.

1.3 Derivatives of global characters

As above we fix a ring R and an ideal $\mathfrak{a} \subseteq R$. Let $\chi\colon T(\mathbb{A})/T(F) \to R^*$ be a locally constant character and write $\overline{\chi}\colon T(\mathbb{A})/T(F) \to \overline{R}$ for its reduction modulo $\mathfrak{a}$. For a place v of F we denote by χ_v the local component of χ at v, i.e. the composition

$$\chi_v\colon T_v \hookrightarrow T(\mathbb{A}) \xrightarrow{\chi} R^*.$$

Since the kernel of χ is open there exists a non-zero ideal $\mathfrak{m} \subseteq \mathcal{O}$ such that χ restricted to $U_T(\mathfrak{m})$ is trivial. The smallest such ideal is called the conductor of χ. Similarly, for a finite place $\mathfrak{p}$ of F we define the conductor of $\chi_{\mathfrak{p}}$ to be the $\mathfrak{p}$-component of the conductor of χ. In the following we will fix a non-zero ideal $\mathfrak{m}$ such that χ restricted to $U_T(\mathfrak{m})$ is trivial (not necessarily the conductor) and view χ as an element of $\mathrm{H}^0(T(F), \mathcal{C}(\mathfrak{m}, R))$.

Suppose we have given a finite set S of finite places of F and ideals $\mathfrak{a}_{\mathfrak{p}} \subseteq \mathfrak{a}$ such that $\chi_{\mathfrak{p}} \equiv 1 \bmod \mathfrak{a}_{\mathfrak{p}}$ holds for all $\mathfrak{p} \in S$. In this situation, we can regard the restriction $\overline{\chi}^S$ of $\overline{\chi}$ to $T(\mathbb{A}^S)$ as an element of $\mathrm{H}^0(T(F), \mathcal{C}(\mathfrak{m}, \overline{R})^S)$. Further, we want to take the Archimedean places into account. Let $\overline{\chi}^{S,\infty}$ be the restriction of $\overline{\chi}$ to $T(\mathbb{A}^{S,\infty})$. For every real Archimedean place v of F which is split in E we fix a character $\epsilon_v\colon T_v/U_{T_v} \to \{\pm 1\}$ and an ideal $\mathfrak{a}_v \subseteq \mathfrak{a}$ with

$$\chi_v(-1) \equiv -\epsilon_v(-1) \bmod \mathfrak{a}_v.$$

Thus, $\psi_v := 1 + (\chi_v\epsilon_v)(-1)$ is an element of $\mathfrak{a}_v$. If v is a complex or non-split real Archimedean place, we set $\psi_v = \epsilon_v = 1$, and $\mathfrak{a}_v = R$. Let us write $\epsilon = \prod_{v\in S_\infty} \epsilon_v \colon T_\infty \to \{\pm 1\}$. An easy calculation shows that $\widetilde{\chi}^S := \prod_{v\in S_\infty} \psi_v \cdot \overline{\chi}^{S,\infty}$ defines an element of $\mathrm{H}^0(T(F), \mathcal{C}(\mathfrak{m}, \overline{\prod_{v\in S_\infty} \mathfrak{a}_v})^{S,\infty}(\epsilon))$.

The ϵ-isotypical projection

$$C(T_\infty/U_{T_\infty}, R) \longrightarrow R(\epsilon),\ f \longmapsto \sum_{x\in T_\infty/U_{T_\infty}} \epsilon(x) f(x) \tag{1.6}$$

yields a $T(\mathbb{A})$-equivariant map

$$\mathcal{C}_c(\mathfrak{m}, R) \longrightarrow \mathcal{C}_c(\mathfrak{m}, R)^\infty(\epsilon). \tag{1.7}$$

As before, let r be the number of primes in S which are split in E. We define $\vartheta^{S,\infty}$ to be the image of ϑ^S under the map

$$\mathrm{H}_{d+r}(T(F), \mathcal{C}_c(\mathcal{O}, \mathbb{Z})^S) \longrightarrow \mathrm{H}_{d+r}(T(F), \mathcal{C}_c(\mathcal{O}, \mathbb{Z})^{S,\infty})$$

induced by (1.7) with $\epsilon = 1$. Furthermore, we define

$$\overline{c_\chi} = \overline{c_\chi}(\mathfrak{m}, S, \epsilon) \in \mathrm{H}_{d+r}(T(F), \mathcal{C}_c(\mathfrak{m}, \overline{\prod_{v\in S_\infty} \mathfrak{a}_v})^{S,\infty}(\epsilon))$$

as the cap product of $\widetilde{\chi}^S$ with $\vartheta^{S,\infty}$.

Next, we are going to attach a homology class c_χ to the character χ and compare it with the class $\overline{c_\chi}$ associated to its reduction. For this, we need to consider a slight generalization of (1.1). Let S' be another (possibly empty) finite set of finite places of F disjoint from S. For an open subgroup $\widetilde{U} \subseteq U_T^{S\cup S',\infty}$ we define

$$\mathcal{C}_c(\widetilde{U}, S, R)^{S',\infty} = \mathcal{C}_c(\widetilde{U}, R)^{S\cup S',\infty} \otimes \bigotimes_{\substack{\mathfrak{p}\in S \\ \mathfrak{p}\ \text{split}}} C_c^0(F_\mathfrak{p}, R) \otimes \bigotimes_{\substack{\mathfrak{p}\in S \\ \mathfrak{p}\ \text{non-split}}} C_c^0(T_\mathfrak{p}, R)/R.$$

As in (1.1), we put $\mathcal{C}_c(\mathfrak{m}, S, R)^{S',\infty} = \mathcal{C}_c(U_T(\mathfrak{m})^{S\cup S',\infty}, S, R)^{S',\infty}$ and drop S' from the notation if it is the empty set. Extension by zero at the split places in S together with the canonical projection at non-split places in S induces a map

$$\mathcal{C}_c(\mathfrak{m}, R)^\infty \longrightarrow \mathcal{C}_c(\mathfrak{m}, S, R)^\infty. \tag{1.8}$$

Let $c_\chi = c_\chi(\mathfrak{m}, S, \epsilon)$ denote the image of χ under the composition

$$\begin{aligned} H^0(T(F), \mathcal{C}(\mathfrak{m}, R)) &\xrightarrow{\cdot\cap\vartheta} \mathrm{H}_d(T(F), \mathcal{C}_c(\mathfrak{m}, R)) \\ &\xrightarrow{(1.7)} \mathrm{H}_d(T(F), \mathcal{C}_c(\mathfrak{m}, R)^\infty(\epsilon)) \\ &\xrightarrow{(1.8)} \mathrm{H}_d(T(F), \mathcal{C}_c(\mathfrak{m}, S, R)^\infty(\epsilon)). \end{aligned} \tag{1.9}$$

Now let us assume that $\mathfrak{a} \cdot \prod_{v\in S\cup S_\infty} \mathfrak{a}_v = 0$. Hence, multiplication in R induces a multilinear map

$$\mu\colon \overline{\mathfrak{a}_{\mathfrak{p}_1}} \times \ldots \times \overline{\mathfrak{a}_{\mathfrak{p}_s}} \times \overline{\prod_{v\in S_\infty} \mathfrak{a}_v} \longrightarrow \prod_{v\in S\cup S_\infty} \mathfrak{a}_v \hookrightarrow R,$$

where we write $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. The next proposition can be proved along the same lines as Proposition 3.8 of [DS].

Proposition 1.6. *The following equality of homology classes holds:*

$$c_\chi = \pm\, \mu_*((c_{\mathrm{d}\chi_{\mathfrak{p}_1}} \cup \ldots \cup c_{\mathrm{d}\chi_{\mathfrak{p}_s}}) \cap \overline{c_\chi})$$

In particular, $c_\chi = 0$ *if* $\prod_{v\in S\cup S_\infty} \mathfrak{a}_v = 0$.

2 Modular Symbols

We are going to introduce modular symbols for a quotient of the multiplicative group of a quaternion algebra over F in terms of group cohomology. The calculations needed for the constructions arise locally and are carried out in Sections 2.1 and 2.2.

Let us fix a quaternion algebra B over F such that

- E can be embedded into B, i.e. all places of F at which B is non-split are non-split in E as well,
- B is the split quaternion algebra (i.e. B is isomorphic to $\mathrm{Mat}_{2\times 2}$) if and only if E is isomorphic to $F \times F$ and
- B is non-split at all Archimedean places of F which are non-split in E.

The set of finite places of F at which B is ramified will be denoted by $\mathrm{ram}(B)$. So in particular, $\mathrm{ram}(B) = \emptyset$ if B is split.

We choose once and for all an embedding $\iota\colon E \hookrightarrow B$. By the Skolem-Noether-Theorem there exists a $J \in B^*$, unique up to multiplication by an element of E^*, such that

$$\iota(\tau(e)) = J\iota(e)J^{-1}$$

holds for all $e \in E$. Let us fix such an element $J \in B^*$.

By abuse of notation we write B^* for the F-algebraic group given by $B^*(M) = (B \otimes_F M)^*$ for any F-module M. We consider the reductive F-algebraic group $G = B^*/\mathbb{G}_m$ and view T as an algebraic subgroup of G via the embedding ι.

In addition, we fix a maximal order $\mathcal{R} \subseteq B$ such that $\iota(\mathcal{O}) \subseteq \mathcal{R}$. For all primes $\mathfrak{p}$ of F we write $\mathcal{R}_\mathfrak{p} \subseteq B_\mathfrak{p}$ for the induced maximal order and $K_\mathfrak{p}$ for the image of $\mathcal{R}_\mathfrak{p}^*$ in $G_\mathfrak{p}$.

Now let us assume that B is non-split and that $\mathfrak{p} \in \mathrm{ram}(B)$ is a prime which is inert in E. From the explicit description of the non-split quaternion algebra over a p-adic local field one gets that the element J is a $E_\mathfrak{p}^*$-multiple of a uniformizer of a ramified quadratic extension of $F_\mathfrak{p}$. Therefore, we have $J \notin K_\mathfrak{p}$ in this case.

2.1 Local norm relations

This section contains all local computations that we need to prove norm relations between Stickelberger elements of different moduli and functional equations for Stickelberger elements. Most local norm relations were already proven by Cornut and Vatsal in Section 6 of [CV07].

We fix a finite place $\mathfrak{p}$ of F at which B is split. In particular, the group $G_{\mathfrak{p}}$ is isomorphic to $\mathrm{PGL}_2(F_{\mathfrak{p}})$. Let $\mathcal{T}_{\mathfrak{p}} = (\mathcal{V}_{\mathfrak{p}}, \vec{\mathcal{E}}_{\mathfrak{p}})$ be the Bruhat-Tits tree of $G_{\mathfrak{p}}$, i.e.

- $\mathcal{V}_{\mathfrak{p}}$ is the set of maximal orders in $B_{\mathfrak{p}}$ and
- there exists an oriented edge $e = (v, v') \in \vec{\mathcal{E}}_{\mathfrak{p}}$ between two vertices $v, v' \in \mathcal{V}_{\mathfrak{p}}$ if and only if the intersection of the corresponding orders is an Eichler order of level $\mathfrak{p}$.

Note that $(v, v') \in \vec{\mathcal{E}}_{\mathfrak{p}}$ if and only if $(v', v) \in \vec{\mathcal{E}}_{\mathfrak{p}}$. In this situation we say that v and v' are neighbours and write $v \sim v'$. Each vertex has $N(\mathfrak{p}) + 1$ neighbours.

For an integer $n \geq 0$ we define $\vec{\mathcal{E}}_{\mathfrak{p},n}$ as the set of non-backtracking paths in $\mathcal{T}_{\mathfrak{p}}$ of length n, i.e.

$$\vec{\mathcal{E}}_{\mathfrak{p},n} = \left\{ (v_0, \ldots, v_n) \in \mathcal{V}_{\mathfrak{p}}^{n+1} \mid (v_i, v_{i+1}) \in \vec{\mathcal{E}}_{\mathfrak{p}} \text{ and } v_i \neq v_{i+2} \text{ for all } i \right\}.$$

In particular, we have $\vec{\mathcal{E}}_{\mathfrak{p},0} = \mathcal{V}_{\mathfrak{p}}$ and $\vec{\mathcal{E}}_{\mathfrak{p},1} = \vec{\mathcal{E}}_{\mathfrak{p}}$. The group $G_{\mathfrak{p}}$ acts on $\vec{\mathcal{E}}_{\mathfrak{p},n}$ via conjugation in each component.

Let R be a ring and N an R-module. In the following we consider $\vec{\mathcal{E}}_{\mathfrak{p},n}$ as a discrete topological space. The Atkin-Lehner involution $W_{\mathfrak{p}^n}$ on $C^0(\vec{\mathcal{E}}_{\mathfrak{p},n}, N)$ is given by interchanging the orientation, i.e.

$$W_{\mathfrak{p}^n}(\phi)(v_0, \ldots, v_n) = \phi(v_n, \ldots, v_0).$$

The Hecke operator

$$\mathbb{T}_{\mathfrak{p}} \colon C^0(\vec{\mathcal{E}}_{\mathfrak{p},n}, N) \longrightarrow C^0(\vec{\mathcal{E}}_{\mathfrak{p},n}, N) \tag{2.1}$$

is defined by

$$\phi \longmapsto \left((v_0, \ldots, v_n) \longmapsto \sum_{v_{n-1} \neq v \sim v_n} \phi(v_1, \ldots, v_n, v) \right).$$

Note that, if $n = 0$, the condition $v_{n-1} \neq v$ is empty. For $(v_0, \ldots, v_n) \in \vec{\mathcal{E}}_{\mathfrak{p},n}$ we define

$$\partial_{(v_0,\ldots,v_n)} \colon C^0(\vec{\mathcal{E}}_{\mathfrak{p},n}, N) \longrightarrow \mathrm{Dist}(T_{\mathfrak{p}}/\operatorname{Stab}_{T_{\mathfrak{p}}}(v_0, \ldots, v_n), N) \tag{2.2}$$

to be the $T_{\mathfrak{p}}$-equivariant map given by

$$\phi \longmapsto (t \mapsto \phi(t(v_0, \ldots, v_n)))\,.$$

Here we used the identification (0.2) of distribution and function spaces on the discrete space $T_{\mathfrak{p}}/\operatorname{Stab}_{T_{\mathfrak{p}}}(v_0, \ldots, v_n)$. For $v \in \mathcal{V}_{\mathfrak{p}}$ let $l_{\mathfrak{p}}(v)$ be the uniquely determined integer given by

$$U_{T_{\mathfrak{p}}}^{(l_{\mathfrak{p}}(v))} = \operatorname{Stab}_{T_{\mathfrak{p}}}(v).$$

Remark 2.1. Let $\mathcal{R}_{\mathfrak{p}}(\mathfrak{p}^n) \subseteq B_{\mathfrak{p}}$ be an Eichler order of level $\mathfrak{p}^n$ contained in the fixed maximal order $\mathcal{R}_{\mathfrak{p}}$. We write $K_{\mathfrak{p}}(\mathfrak{p}^n)$ for the image of $\mathcal{R}_{\mathfrak{p}}(\mathfrak{p}^n)^*$ in $G_{\mathfrak{p}}$. There exists a unique vertex in $\mathcal{V}_{\mathfrak{p}}$ fixed by $K_{\mathfrak{p}}$ and thus, we get a canonical isomorphism

$$C^0(G_{\mathfrak{p}}/K_{\mathfrak{p}}, N) \xrightarrow{\cong} C^0(\mathcal{V}_{\mathfrak{p}}, N).$$

In the case $n \geq 1$ there is an up to orientation unique element in $\vec{\mathcal{E}}_{\mathfrak{p},n}$ fixed by $K_{\mathfrak{p}}(\mathfrak{p}^n)$. Therefore, there are two natural isomorphisms

$$C^0(G_{\mathfrak{p}}/K_{\mathfrak{p}}(\mathfrak{p}^n), N) \xrightarrow{\cong} C^0(\vec{\mathcal{E}}_{\mathfrak{p},n}, N),$$

which are interchanged by the Atkin-Lehner involution.

The following lemma is essentially Lemma 6.5 of [CV07].

Lemma 2.2. *Let $v \in \mathcal{V}_{\mathfrak{p}}$ be a vertex of $\mathcal{T}_{\mathfrak{p}}$.*

(i) Let $l_{\mathfrak{p}}(v) = 0$.

- *If $\mathfrak{p}$ is split in E, there are exactly two neighbours v' of v such that $l_{\mathfrak{p}}(v') = 0$. They are given by $\varpi_{\mathfrak{P}} v$ and $\varpi_{\mathfrak{P}}^{\tau} v$.*
- *If $\mathfrak{p}$ is ramified in E, there is exactly one neighbour v' of v such that $l_{\mathfrak{p}}(v') = 0$. It is given by $\varpi_{\mathfrak{P}} v$.*
- *If $\mathfrak{p}$ is inert in E, there is no such neighbour.*

(ii) Let $l_{\mathfrak{p}}(v) \geq 1$. Then there exists a unique neighbour v' of v with

$$l_{\mathfrak{p}}(v') = l_{\mathfrak{p}}(v) - 1.$$

(iii) In both cases, (i) and (ii), the remaining neighbours v' of v satisfy

$$l_{\mathfrak{p}}(v') = l_{\mathfrak{p}}(v) + 1.$$

They are permuted faithfully and transitively by $U_{T_{\mathfrak{p}}}^{(l_{\mathfrak{p}}(v))}/U_{T_{\mathfrak{p}}}^{(l_{\mathfrak{p}}(v)+1)}$.

We will construct a sequence of vertices of $\mathcal{T}_\mathfrak{p}$ which are compatible in the sense of the above Lemma as follows: Let $\vec{\mathcal{E}}_{\mathfrak{p},\infty} = \varprojlim_n \vec{\mathcal{E}}_{\mathfrak{p},n}$ be the set of infinite, non-backtracking sequences of adjacent vertices. Let w_0 be the vertex corresponding to $\mathcal{R}_\mathfrak{p}$ or, equivalently, the unique vertex fixed by the action of $K_\mathfrak{p}$. By our assumptions we have $l_\mathfrak{p}(w_0) = 0$. Using Lemma 2.2 (iii) we consecutively choose vertices w_i such that $w_i \sim w_{i-1}$ and $l_\mathfrak{p}(w_i) = i$ for all $i \geq 1$. We set $w_\infty = (w_0, w_1, w_2, \dots) \in \vec{\mathcal{E}}_{\mathfrak{p},\infty}$. Further, we define $w_{-1} = \varpi_\mathfrak{P} w_0$ if $\mathfrak{p}$ is ramified in E. If $\mathfrak{p}$ splits in E, we set $w_{-j} = \varpi_\mathfrak{P}^{-j} w_0$ for every integer $j > 0$.

It will be convenient to introduce the following notation:

$$\eta_\mathfrak{p} = \begin{cases} 0 & \text{if } \mathfrak{p} \text{ is inert in } E, \\ -1 & \text{if } \mathfrak{p} \text{ is ramified in } E, \\ -\infty & \text{if } \mathfrak{p} \text{ is split in } E. \end{cases}$$

By definition we have

$$\operatorname{Stab}_{T_\mathfrak{p}}(w_{m-n}, \dots, w_m) = U_{T_\mathfrak{p}}^{(m)} \tag{2.3}$$

for all integers $m, n \geq 0$ such that $m - n \geq \eta_\mathfrak{p}$.

Definition (2.2) yields

$$\partial_m := \partial_{(w_{m-n},\dots,w_m)} \colon C^0(\vec{\mathcal{E}}_{\mathfrak{p},n}, N) \to \operatorname{Dist}(T_\mathfrak{p}/U_{T_\mathfrak{p}}^{(m)}, N)$$

for all integers m, n as above.

For $m \geq 0$, the projection $\pi_m \colon T_\mathfrak{p}/U_{T_\mathfrak{p}}^{(m+1)} \to T_\mathfrak{p}/U_{T_\mathfrak{p}}^{(m)}$ yields maps

$$\begin{aligned} (\pi_m)^* \colon \operatorname{Dist}(T_\mathfrak{p}/U_{T_\mathfrak{p}}^{(m)}, N) &\longrightarrow \operatorname{Dist}(T_\mathfrak{p}/U_{T_\mathfrak{p}}^{(m+1)}, N)) \\ f &\longmapsto f \circ \pi_m \end{aligned}$$

and

$$\begin{aligned} (\pi_m)_* \colon \operatorname{Dist}(T_\mathfrak{p}/U_{T_\mathfrak{p}}^{(m+1)}, N) &\longrightarrow \operatorname{Dist}(T_\mathfrak{p}/U_{T_\mathfrak{p}}^{(m)}, N) \\ f &\longmapsto \sum_{t \in U_{T_\mathfrak{p}}^{(m)}/U_{T_\mathfrak{p}}^{(m+1)}} t.f. \end{aligned}$$

Lemma 2.3. *Let $n \geq 0$ be an integer. The following formulas hold for all $\phi \in C^0(\vec{\mathcal{E}}_{\mathfrak{p},n}, N)$:*

(i) For $m \geq \max\{1, n + \eta_\mathfrak{p} + 1\}$ the equality

$$(\partial_m \circ \mathbb{T}_\mathfrak{p})(\phi) = ((\pi_m)_* \circ \partial_{m+1})(\phi) + \mathbb{1}_\mathfrak{p}(\mathfrak{p}^n)((\pi_{m-1})^* \circ \partial_{m-1})(\phi)$$

holds with

$$\mathbb{1}_\mathfrak{p}(\mathfrak{p}^n) = \begin{cases} 1 & \textit{if } n = 0, \\ 0 & \textit{else.} \end{cases}$$

(ii) If $n + \eta_{\mathfrak{p}} \leq 0$, the following equality holds:

$$(\partial_0 \circ \mathbb{T}_{\mathfrak{p}})(\phi) = ((\pi_0)_* \circ \partial_1)(\phi) + (*),$$

where

$$(*) = \begin{cases} 0 & \text{if } \mathfrak{p} \text{ is inert in } E, \\ \mathbb{1}_{\mathfrak{p}}(\mathfrak{p}^n)\varpi_{\mathfrak{P}}\partial_0(\phi) & \text{if } \mathfrak{p} \text{ is ramified in } E, \\ \mathbb{1}_{\mathfrak{p}}(\mathfrak{p}^n)\varpi_{\mathfrak{P}}\partial_0(\phi) + (\varpi_{\mathfrak{P}})^{-1}\partial_0(\phi) & \text{if } \mathfrak{p} \text{ is split in } E. \end{cases}$$

(iii) If $\mathfrak{p}$ is inert in E and $n = 1$, then

$$(\partial_1 \circ \mathbb{T}_{\mathfrak{p}} \circ W_{\mathfrak{p}})(\phi) + \partial_1(\phi) = ((\pi_0)_* \circ \partial_1)(\phi)$$

holds.

Proof. We will give a proof of part (iii). The other parts are proven similarly. See e.g. [CV07], Section 6, where most of the cases are already dealt with.

Let $\phi \in C^0(\vec{\mathcal{E}}_{\mathfrak{p},n}, N)$ and $t \in T_{\mathfrak{p}}/U_{T_{\mathfrak{p}}}$. Then we have

$$\begin{aligned} ((\pi_0)_* \circ \partial_1)(\phi)(t) &= (\pi_0)_*(\phi(t(w_0, w_1))) \\ &= \sum_{\widetilde{t} \in U_{T_{\mathfrak{p}}}/U_{T_{\mathfrak{p}}}^{(1)}} \widetilde{t}\phi(t(w_0, w_1)) \\ &= \sum_{v \sim w_0} \phi(t(w_0, v)), \end{aligned}$$

where the last equality follows from Lemma 2.2, (iii). On the other hand, we have

$$\begin{aligned} (\partial_1 \circ \mathbb{T}_{\mathfrak{p}} \circ W_{\mathfrak{p}})(\phi)(t) &= (\mathbb{T}_{\mathfrak{p}} \circ W_{\mathfrak{p}})(\phi)(t(w_0, w_1)) \\ &= \mathbb{T}_{\mathfrak{p}}(\phi(t(w_1, w_0))) \\ &= \sum_{\substack{v \sim w_0 \\ v \neq w_1}} \phi(t(w_0, v)). \end{aligned}$$

Together with

$$\partial_1(\phi)(t) = \phi(t(w_0, w_1))$$

the equality follows. □

Remark 2.4. Let $m \geq 1$ and $n \geq 0$ be integers such that $m - n \geq \eta_p$. The only cases where we do not have a formula involving $(\pi_{m-1})_* \circ \partial_m$ are the following: $n \geq 2$ and either $\mathfrak{p}$ is inert in E and $m = n$ or $\mathfrak{p}$ is ramified and $m = n - 1$.

Let us denote by

$$\mathrm{inv}\colon \mathrm{Dist}(T_{\mathfrak{p}}/U_{T_{\mathfrak{p}}^{(m)}}, N) \longrightarrow \mathrm{Dist}(T_{\mathfrak{p}}/U_{T_{\mathfrak{p}}^{(m)}}, N)$$

the map induced by inversion. The following lemma is the main ingredient for proving a functional equation for Stickelberger elements.

Lemma 2.5. *Let $n \geq 0$ be an integer.*

(i) Assume $n \leq -\eta_{\mathfrak{p}}$. Then for all $\phi \in C^0(\vec{\mathcal{E}}_{\mathfrak{p},n}, N)$ the equality

$$(\partial_0 \circ W_{\mathfrak{p}^n})(\phi) = (\mathrm{inv} \circ \partial_0)(J\phi)$$

holds up to multiplication by an element of $T_{\mathfrak{p}}$.

(ii) Assume $m \geq n$. Then for all $\phi \in C^0(\vec{\mathcal{E}}_{\mathfrak{p},n}, N)$ the equality

$$\partial_m(\phi) = (\mathrm{inv} \circ \partial_m)(J\phi)$$

holds up to multiplication by an element of $T_{\mathfrak{p}}$.

Proof. To prove (i), note that for $t \in T_{\mathfrak{p}}/U_{T_{\mathfrak{p}}}$ we have

$$\begin{aligned}\mathrm{inv}(\partial_0(J\phi))(t) &= (J\phi)(t^{-1}(w_{-n}, \ldots, w_1, w_0)) \\ &= \phi(J^{-1}t^{-1}(w_{-n}, \ldots, w_1, w_0)) \\ &= \phi(tJ^{-1}(w_{-n}, \ldots, w_1, w_0)) \\ &= \phi(t(\varpi_{\mathfrak{P}}^n J^{-1}w_0, \ldots, \varpi_{\mathfrak{P}} J^{-1}w_0, J^{-1}w_0)).\end{aligned}$$

Since $t' J^{-1} w_0 = J^{-1} t'^{-1} w_0 = J^{-1} w_0$ holds for all $t' \in U_{T_{\mathfrak{p}}}$ it follows from Lemma 2.2 that $J^{-1} w_0 = \varpi_{\mathfrak{P}}^k w_0$ for some $k \in \mathbb{Z}$. This leads to

$$\begin{aligned}\mathrm{inv}(\partial_0(J\phi))(t) &= \phi(t(\varpi_{\mathfrak{P}}^n J^{-1}w_0, \ldots, \varpi_{\mathfrak{P}} J^{-1}w_0, J^{-1}w_0)) \\ &= \phi(t(\varpi_{\mathfrak{P}}^{k+n} w_0, \ldots, \varpi_{\mathfrak{P}}^{k-1} w_0, \varpi_{\mathfrak{P}}^k w_0))\end{aligned}$$

and we get

$$\begin{aligned}\varpi_{\mathfrak{P}}^{-k-n}\, \mathrm{inv}(\partial_0(J\phi))(t) &= \phi(t(w_0, \ldots, \varpi_{\mathfrak{P}}^{-n+1} w_0, \varpi_{\mathfrak{P}}^{-n} w_0)) \\ &= (W_{\mathfrak{p}}^n \phi)(t(w_{-n}, \ldots, w_1, w_0)).\end{aligned}$$

Claim (ii) follows by a similar calculation as in the first part using that, by Lemma 2.2, there exists an element $x \in T_{\mathfrak{p}}$ such that

$$x(J^{-1}w_{m-n}, \ldots, J^{-1}w_m) = (w_{m-n}, \ldots, w_m)$$

holds. □

Remark 2.6. If B is the split quaternion algebra, we can give explicit versions of the above definitions and calculations in terms of matrices by using the isomorphisms of Remark 2.1. This has been carried out in [BG17]. We only give an overview:

Let us identify B with $\mathrm{Mat}_{2\times 2}$ and choose ι to be the canonical embedding

$$\iota\colon E \longrightarrow B,\ (x,y) \longmapsto \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

In this way we can see T as the torus of diagonal matrices in $G = \mathrm{PGL}_2$. Further, the fixed isomorphism of $\mathbb{G}_m$ with T is given by

$$x \longmapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$

We choose J as the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathcal{R} = \mathrm{Mat}_{2\times 2}(\mathcal{O}_F)$.

The elements $w_m \in \mathcal{V}_\mathfrak{p}$ for $m \geq 0$ defined after Lemma 2.2 are given as follows: We take w_0 corresponding to $\mathcal{R}_\mathfrak{p} = \mathrm{Mat}_{2\times 2}(\mathcal{O}_{F_\mathfrak{p}})$ and for $m \geq 1$ we define $w_m = h_m.w_0$ with $h_m = \begin{pmatrix} \varpi_\mathfrak{p}^m & 1 \\ 0 & 1 \end{pmatrix}$, where $\varpi_\mathfrak{p} \in \mathcal{O}_{F_p}$ is a local uniformizer at $\mathfrak{p}$. Obviously, we have $\mathrm{Stab}_{G_\mathfrak{p}}(w_0) = K_\mathfrak{p}$ (which induces the isomorphism $G_\mathfrak{p}/K_\mathfrak{p} \xrightarrow{\cong} \mathcal{V}_\mathfrak{p}$ of Remark 2.1) and thereby $\mathrm{Stab}_{T_\mathfrak{p}}(w_0) = U_\mathfrak{p}$. For $m \geq 1$ it follows that $\mathrm{Stab}_{G_\mathfrak{p}}(w_m) = h_m K_\mathfrak{p} h_m^{-1}$. An easy calculation shows $\mathrm{Stab}_{T_\mathfrak{p}}(w_m) = U_\mathfrak{p}^{(m)}$, as desired.

In particular, we get a refined version of Lemma 2.5 with equalities not only up to multiplication by an element in $T_\mathfrak{p}$ (see Lemma 1.5 of [BG17]), i.e. we have

$$(\partial_0 \circ W_{\mathfrak{p}^n})(\phi) = \varpi_\mathfrak{p}^n (\mathrm{inv} \circ \partial_0)(J\phi) \tag{2.4}$$

and

$$\partial_m(\phi) = (\mathrm{inv} \circ \partial_m)(J\phi). \tag{2.5}$$

2.2 Ends and the Steinberg representation

We will give a quick review of the theory of ends of the Bruhat-Tits tree. As before, let $\mathfrak{p}$ be a finite place of F which is split in B. By realizing the Steinberg representation as a space of functions on the set of ends, we construct a map $\delta_\mathfrak{p}^*$ from the dual of the Steinberg representation to the space of distributions on the local torus, which is compatible with the maps ∂_m for $m \geq 1$.

We say that two elements $(v_i)_{i\geq 0}$ and $(v'_i)_{i\geq 0}$ in $\vec{\mathcal{E}}_{\mathfrak{p},\infty}$ are equivalent if there exist integers $N, N' \geq 0$ such that $v_{N+i} = v'_{N'+i}$ for all $i \geq 0$. An end in $\mathcal{T}_{\mathfrak{p}}$ is defined as an equivalence class of elements in $\vec{\mathcal{E}}_{\mathfrak{p},\infty}$. The set of ends is denoted by $\mathrm{Ends}_{\mathfrak{p}}$. To an edge $e \in \vec{\mathcal{E}}_{\mathfrak{p}}$ we assign the set $V(e)$ of ends that have a representative containing e. The sets $V(e)$ form a basis of a topology on $\mathrm{Ends}_{\mathfrak{p}}$, which turns $\mathrm{Ends}_{\mathfrak{p}}$ into a compact space. The natural action of $G_{\mathfrak{p}}$ on $\vec{\mathcal{E}}_{\mathfrak{p},n}$ extends to an action on $\mathrm{Ends}_{\mathfrak{p}}$.

Let $\mathfrak{F} \subseteq \mathrm{Ends}_{\mathfrak{p}}$ be the set of fix points under the action of $T_{\mathfrak{p}}$. As a consequence of Lemma 2.2 we see that $T_{\mathfrak{p}}$ acts simply transitively on the complement of $\mathfrak{F}$. Hence, choosing a base point $[v_\infty]$ in the complement yields a homeomorphism $\kappa_{[v_\infty]}\colon T_{\mathfrak{p}} \to \mathrm{Ends}_{\mathfrak{p}} - \mathfrak{F}$ via $t \mapsto t[v_\infty]$. In the following we will choose the class of w_∞ as our base point and write $\kappa = \kappa_{[w_\infty]}$.

Remark 2.7. The set $\mathfrak{F}$ is non-zero only if $\mathfrak{p}$ is split in E. In this case $\mathfrak{F}$ consists of two elements given as follows: Clearly, the equivalence classes of the elements

$$o_{\mathfrak{P}} = (w_0, \varpi_{\mathfrak{P}} w_0, \varpi_{\mathfrak{P}}^2 w_0, \dots) \text{ and } o_{\mathfrak{P}^\tau} = (w_0, \varpi_{\mathfrak{P}}^\tau w_0, (\varpi_{\mathfrak{P}}^\tau)^2 w_0, \dots)$$

are fixed by $T_{\mathfrak{p}}$. Using Lemma 2.2 one can show that $\mathfrak{F} = \{[o_{\mathfrak{P}}], [o_{\mathfrak{P}^\tau}]\}$ holds. In particular, the choice of the prime $\mathfrak{P}$ lying above $\mathfrak{p}$ if E is a field resp. the choice of one F-coordinate of E if B is split (and hence, in both cases, the choice of the vertices w_i for $i \leq -1$) is equivalent to the choice of the element $[o_{\mathfrak{P}}]$ of $\mathfrak{F}$. In particular, there exists a unique $G_{\mathfrak{p}}$-equivariant homeomorphism of $\mathrm{Ends}_{\mathfrak{p}}$ with $\mathbb{P}^1(F_{\mathfrak{p}})$ which maps $o_{\mathfrak{P}}$ to 0, $o_{\mathfrak{P}^\tau}$ to ∞ and w_∞ to 1 (where we write $\infty, 0, 1$ for the points $[1:0], [0:1], [1:1]$ of $\mathbb{P}^1(F_{\mathfrak{p}})$).

We define the Steinberg representation $\mathrm{St}_{\mathfrak{p}}$ to be the space of locally constant $\mathbb{Z}$-valued functions on $\mathrm{Ends}_{\mathfrak{p}}$ modulo constant functions, i.e. $\mathrm{St}_{\mathfrak{p}} = C_c^0(\mathrm{Ends}_{\mathfrak{p}}, \mathbb{Z})/\mathbb{Z}$. The $G_{\mathfrak{p}}$-action on $\mathrm{Ends}_{\mathfrak{p}}$ extends to an action on $\mathrm{St}_{\mathfrak{p}}$ via $(\gamma.\varphi)([v_\infty]) = \varphi(\gamma^{-1}[v_\infty])$ for $\gamma \in G_{\mathfrak{p}}$, $\varphi \in \mathrm{St}_{\mathfrak{p}}$ and $[v_\infty] \in \mathrm{Ends}_{\mathfrak{p}}$. The open embedding $\kappa\colon T_{\mathfrak{p}} \hookrightarrow \mathrm{Ends}_{\mathfrak{p}}$ induces a $T_{\mathfrak{p}}$-equivariant map

$$\delta_{\mathfrak{p}}\colon C_c^0(T_{\mathfrak{p}}, \mathbb{Z}) \longrightarrow \mathrm{St}_{\mathfrak{p}}$$

via extension by zero and thus, by dualizing we get a map

$$\delta_{\mathfrak{p}}^*\colon \mathrm{Hom}(\mathrm{St}_{\mathfrak{p}}, N) \longrightarrow \mathrm{Dist}(T_{\mathfrak{p}}, N).$$

If $\mathfrak{p}$ is split in E, we can extend κ to a map from $F_{\mathfrak{p}}$ to $\mathrm{Ends}_{\mathfrak{p}}$ by mapping 0 to $o_{\mathfrak{P}}$. Thus, we can extend $\delta_{\mathfrak{p}}$ to a map

$$\delta_{\mathfrak{p}}\colon C_c^0(F_{\mathfrak{p}}, \mathbb{Z}) \longrightarrow \mathrm{St}_{\mathfrak{p}}, \tag{2.6}$$

which in turn induces a $T_{\mathfrak{p}}$-equivariant map

$$\delta_{\mathfrak{p}}^*\colon \operatorname{Hom}(\mathrm{St}_{\mathfrak{p}}, N) \longrightarrow \operatorname{Dist}(F_{\mathfrak{p}}, N).$$

If $\mathfrak{p}$ is non-split, the image of κ is equal to $\mathrm{Ends}_{\mathfrak{p}}$. Therefore, $\delta_{\mathfrak{p}}$ descends to a map

$$\delta_{\mathfrak{p}}\colon C_c^0(F_{\mathfrak{p}}, \mathbb{Z})/\mathbb{Z} \longrightarrow \mathrm{St}_{\mathfrak{p}} \tag{2.7}$$

and thus, we have

$$\delta_{\mathfrak{p}}^*\colon \operatorname{Hom}(\mathrm{St}_{\mathfrak{p}}, N) \longrightarrow \operatorname{Dist}_0(T_{\mathfrak{p}}, N) \subseteq \operatorname{Dist}(T_{\mathfrak{p}}, N).$$

Dualizing the canonical map $\vec{\mathcal{E}}_{\mathfrak{p}} \to \mathrm{St}_{\mathfrak{p}}$ given by $e \mapsto \mathbb{1}_{V(e)}$ yields the $G_{\mathfrak{p}}$-equivariant evaluation map

$$\mathrm{ev}_{\mathfrak{p}}\colon \operatorname{Hom}(\mathrm{St}_{\mathfrak{p}}, N) \longrightarrow C^0(\vec{\mathcal{E}}_{\mathfrak{p}}, N). \tag{2.8}$$

Further, there is the natural map

$$j_m\colon \operatorname{Dist}(T_{\mathfrak{p}}, N) \longrightarrow \operatorname{Dist}(T_{\mathfrak{p}}/U_{T_{\mathfrak{p}}}^{(m)}, N)$$

induced by the projection $T_{\mathfrak{p}} \to T_{\mathfrak{p}}/U_{T_{\mathfrak{p}}}^{(m)}$. By definition we have

$$V(w_{m-1}, w_m) = \kappa(U_{T_{\mathfrak{p}}}^{(m)}) \tag{2.9}$$

for all $m \geq 1$ and, if $\mathfrak{p}$ is split in E, we also have

$$V(w_{-1}, w_0) = \kappa(\mathcal{O}_{F_{\mathfrak{p}}}).$$

From this, one easily gets

Lemma 2.8. *(i) Let $m \geq 1$ be an integer. The following diagram is commutative:*

$$\begin{array}{ccc}
\operatorname{Hom}(\mathrm{St}_{\mathfrak{p}}, N) & \xrightarrow{\ \mathrm{ev}_{\mathfrak{p}}\ } & C^0(\vec{\mathcal{E}}_{\mathfrak{p}}, N) \\
\Big\downarrow{\scriptstyle \delta_{\mathfrak{p}}^*} & & \Big\downarrow{\scriptstyle \partial_m} \\
\operatorname{Dist}(T_{\mathfrak{p}}, N) & \xrightarrow{\ j_m\ } & \operatorname{Dist}(T_{\mathfrak{p}}/U_{T_{\mathfrak{p}}}^{(m)}, N)
\end{array}$$

(ii) Suppose that $\mathfrak{p}$ is split in E. Let

$$\alpha_{\mathfrak{p}}^{*}\colon \operatorname{Dist}(F_{\mathfrak{p}}, N) \longrightarrow \operatorname{Dist}(T_{\mathfrak{p}}/U_{T_{\mathfrak{p}}}, N)$$

be the dual of the map (1.5)*. Then the following diagram is commutative:*

$$\begin{array}{ccc}
\operatorname{Hom}(\operatorname{St}_{\mathfrak{p}}, N) & \xrightarrow{\quad \operatorname{ev}_{\mathfrak{p}} \quad} & C^{0}(\vec{\mathcal{E}}_{\mathfrak{p}}, N) \\
\delta_{\mathfrak{p}}^{*} \Big\downarrow & & \Big\downarrow \partial_{0} \\
\operatorname{Dist}(F_{\mathfrak{p}}, N) & \xrightarrow{\quad \alpha_{\mathfrak{p}}^{*} \quad} & \operatorname{Dist}(T_{\mathfrak{p}}/U_{T_{\mathfrak{p}}}, N)
\end{array}$$

Proof. Let us give a proof of part (i). For this, let $\xi \in \operatorname{Hom}(\operatorname{St}_{\mathfrak{p}}, N)$ and $t \in T_{\mathfrak{p}}/U_{T_{\mathfrak{p}}}^{(m)}$. Note that the set of functions $\mathbb{1}_{tU_{T_{\mathfrak{p}}}^{(m)}}$, where t ranges over representatives of $T_{\mathfrak{p}}$ modulo $U_{T_{\mathfrak{p}}}^{(m)}$, is a $\mathbb{Z}$-basis for $C^{0}(\vec{\mathcal{E}}_{\mathfrak{p}}, N)$. Using (2.9) we get

$$\begin{aligned}
\partial_{m}(\operatorname{ev}_{\mathfrak{p}}(\xi))(\mathbb{1}_{tU_{T_{\mathfrak{p}}}^{(m)}}) &= \operatorname{ev}_{\mathfrak{p}}(\xi)(t(w_{m-1}, w_{m})) \\
&= \xi(\mathbb{1}_{V(t.V(w_{m-1}, w_{m}))}) \\
&= \xi(\mathbb{1}_{\kappa(tU_{T_{\mathfrak{p}}}^{(m)})}) \\
&= j(\delta_{\mathfrak{p}}^{*}(\xi))(\mathbb{1}_{tU_{T_{\mathfrak{p}}}^{(m)}}),
\end{aligned}$$

where the last equality holds by definition. □

There is also a twisted version of the above constructions if $\mathfrak{p}$ is inert in E. Let $\operatorname{nr}\colon B_{\mathfrak{p}}^{*} \to F_{\mathfrak{p}}^{*}$ denote the reduced norm. The character

$$\chi_{-1}\colon B_{\mathfrak{p}}^{*} \longrightarrow \{\pm 1\}, \; g \longmapsto (-1)^{\operatorname{ord}_{\mathfrak{p}}(\operatorname{nr}(g))}$$

is trivial on the center and thus, descents to a character on $G_{\mathfrak{p}}$. The twisted Steinberg representation is defined by

$$\operatorname{St}_{\mathfrak{p}}^{\mathrm{tw}} = \operatorname{St}_{\mathfrak{p}}(\chi_{-1}).$$

Since $\mathfrak{p}$ is inert in E we have $\operatorname{ord}_{\mathfrak{p}}(\operatorname{nr}(t)) \equiv 0 \bmod 2$ for all $t \in T_{\mathfrak{p}}$. Therefore, the map

$$\delta_{\mathfrak{p}}^{\mathrm{tw}}\colon C^{0}(T_{\mathfrak{p}}, \mathbb{Z})/\mathbb{Z} \longrightarrow \operatorname{St}_{\mathfrak{p}}^{\mathrm{tw}}, \; f \longmapsto \delta_{\mathfrak{p}}(f) \otimes 1 \tag{2.10}$$

is $T_{\mathfrak{p}}$-equivariant.

There is also a ($G_{\mathfrak{p}}$-equivariant) twisted evaluation map

$$\mathrm{ev}_{\mathfrak{p}}^{\mathrm{tw}} \colon \operatorname{Hom}(\mathrm{St}_{\mathfrak{p}}^{\mathrm{tw}}, N) \longrightarrow C^0(\vec{\mathcal{E}}_{\mathfrak{p}}, N). \tag{2.11}$$

It is given by dualizing the map

$$\vec{\mathcal{E}}_{\mathfrak{p}} \longrightarrow \mathrm{St}_{\mathfrak{p}}^{\mathrm{tw}},\ e \longmapsto \chi_{-1}(g_e) \cdot \mathbb{1}_{V(e)} \otimes 1,$$

where $g_e \in G_{\mathfrak{p}}$ is any element such that $g_e.(w_0, w_1) = e$. Again, using (2.9) and following the same lines as in the proof of Lemma 2.8 we get

Lemma 2.9. *Let $m \geq 1$ be an integer and let $\mathfrak{p}$ be a prime, which is inert in E. The following diagram is commutative:*

$$\begin{array}{ccc} \operatorname{Hom}(\mathrm{St}_{\mathfrak{p}}^{\mathrm{tw}}, N) & \xrightarrow{\ \mathrm{ev}_{\mathfrak{p}}^{\mathrm{tw}}\ } & C^0(\vec{\mathcal{E}}_{\mathfrak{p}}, N) \\ \Big\downarrow{\scriptstyle (\delta_{\mathfrak{p}}^{\mathrm{tw}})^*} & & \Big\downarrow{\scriptstyle (-1)^{m+1}\partial_m} \\ \mathrm{Dist}_0(T_{\mathfrak{p}}, N) & \longrightarrow & \mathrm{Dist}(T_{\mathfrak{p}}/U_{T_{\mathfrak{p}}}^{(m)}, N) \end{array}$$

2.3 Global cohomology classes and pullback to the torus

In this section we globalize the constructions of the previous sections. Unfortunately, we have to take different approaches depending on B being split or non-split since the dualizing module of arithmetic subgroups of G is trivial (resp. non-trivial) if B is non-split (resp. split).

Let $\mathrm{Div}(\mathbb{P}^1(F))$ be the free abelian group over $\mathbb{P}^1(F)$ and let $\mathrm{Div}_0(\mathbb{P}^1(F))$ be the kernel of the map

$$\mathrm{Div}(\mathbb{P}^1(F)) \to \mathbb{Z},\ \sum_P m_P P \mapsto \sum_P m_P.$$

Note that we have a $G(F)$-action on $\mathrm{Div}_0(\mathbb{P}^1(F))$ induced by the $G(F)$-action on $\mathbb{P}^1(F)$. We write

$$D_G = \begin{cases} \mathbb{Z} & \text{if } B \text{ is non-split and} \\ \mathrm{Div}_0(\mathbb{P}^1(F)) & \text{if } B \text{ is split.} \end{cases}$$

We fix pairwise disjoint finite sets S_{St}, S_{tw} and S' of finite places of F disjoint from $\mathrm{ram}(B)$ and put $S = S_{\mathrm{St}} \cup S_{\mathrm{tw}}$. From now on we assume that every $\mathfrak{p}$ in S_{tw} is inert in E. For an R-module N and a compact open subgroup $K \subseteq G(\mathbb{A}^{S,\infty})$ we consider

$$\mathcal{A}(K, S_{\mathrm{St}}, S_{\mathrm{tw}}; N)^{S'} = C\Big(G(\mathbb{A}^{S \cup S',\infty})/K, \qquad\qquad (2.12)$$
$$\mathrm{Hom}\Big(\bigotimes_{\mathfrak{p}\in S_{\mathrm{St}}} \mathrm{St}_{\mathfrak{p}} \otimes \bigotimes_{\mathfrak{p}\in S_{\mathrm{tw}}} \mathrm{St}_{\mathfrak{p}}^{\mathrm{tw}}, \mathrm{Hom}(D_G, N)\Big)\Big)$$

with its natural $G(F)$-action, i.e. for every $\mathfrak{p} \in S_{\mathrm{St}}$ (resp. $\mathfrak{p} \in S_{\mathrm{tw}}$) we view $\mathrm{St}_{\mathfrak{p}}$ (resp. $\mathrm{St}_{\mathfrak{p}}^{\mathrm{tw}}$) as a $G(F)$-module via the embedding $G(F) \hookrightarrow G_{\mathfrak{p}}$ and put

$$(g.\Phi)(x)(f_{\mathrm{St}} \otimes f_{\mathrm{tw}}) = \Phi(g^{-1}x)((g^{-1}f_{\mathrm{St}}) \otimes (g^{-1}f_{\mathrm{tw}}))$$

for $g \in G(F)$, $\Phi \in \mathcal{A}(K, S_{\mathrm{St}}, S_{\mathrm{tw}}; N)^{S'}$, $x \in G(\mathbb{A}^{S\cup S',\infty})/K$, $f_{\mathrm{St}} \in \bigotimes_{\mathfrak{p}\in S_{\mathrm{St}}} \mathrm{St}_{\mathfrak{p}}$ and $f_{\mathrm{tw}} \in \bigotimes_{\mathfrak{p}\in S_{\mathrm{tw}}} \mathrm{St}_{\mathfrak{p}}^{\mathrm{tw}}$. Further, we fix a locally constant character

$$\epsilon\colon T_\infty \longrightarrow \{\pm 1\}\,.$$

We will often view ϵ as a character on $T(F)$ via the embedding $T(F) \hookrightarrow T_\infty$. There exists a unique extension $\epsilon\colon G_\infty \to \{\pm 1\}$ such that the diagram

$$\begin{array}{ccc} T_\infty & \xrightarrow{\ \epsilon\ } & \{\pm 1\} \\ {\scriptstyle\iota}\big\downarrow & \nearrow_{\epsilon} & \\ G_\infty & & \end{array}$$

is commutative. Again, we view ϵ also as a character on $G(F)$ via the embedding $G(F) \hookrightarrow G_\infty$. We are interested in the cohomology of the $G(F)$-modules $\mathcal{A}(K, S_{\mathrm{St}}, S_{\mathrm{tw}}; N)(\epsilon)$.

Proposition 2.10. *Let $S_{\mathrm{St}}, S_{\mathrm{tw}}$ be disjoint finite sets of finite places of F disjoint from* $\mathrm{ram}(B)$ *and $K \subseteq G(\mathbb{A}^{S,\infty})$ a compact open subgroup.*

(i) Let N be a flat R-module equipped with the trivial $G(F)$-action. Then the canonical map

$$H^q(G(F), \mathcal{A}(K, S_{\mathrm{St}}, S_{\mathrm{tw}}; R)(\epsilon)) \otimes_R N \to H^q(G(F), \mathcal{A}(K, S_{\mathrm{St}}, S_{\mathrm{tw}}; N)(\epsilon))$$

is an isomorphism for all $q \geq 0$.

(ii) If R is Noetherian, then the groups $H^q(G(F), \mathcal{A}(K, S_{\mathrm{St}}, S_{\mathrm{tw}}; R)(\epsilon))$ are finitely generated R-modules for all $q \geq 0$.

Proof. This is almost verbatim Proposition 4.6. of [Spi14]. □

Definition 2.11. The space of N-valued, $(S_{\mathrm{St}}, S_{\mathrm{tw}})$-special modular symbols on G of level K and sign ϵ is defined to be

$$\mathcal{M}(K, S_{\mathrm{St}}, S_{\mathrm{tw}}; N)^{\epsilon} = \mathrm{H}^d(G(F), \mathcal{A}(K, S_{\mathrm{St}}, S_{\mathrm{tw}}; N)(\epsilon)).$$

Let $\mathfrak{n} \subseteq \mathcal{O}_F$ be a non-zero ideal coprime to $\mathrm{ram}(B)$. We fix an Eichler order $\mathcal{R}(\mathfrak{n}) \subseteq \mathcal{R}$ of level $\mathfrak{n}$ contained in the fixed maximal order $\mathcal{R}$. As in the local case, we write $K_{\mathfrak{p}}$ (resp. $K_{\mathfrak{p}}(\mathfrak{n})$) for the image of $\mathcal{R}_{\mathfrak{p}}^*$ (resp. $\mathcal{R}(\mathfrak{n})_{\mathfrak{p}}^*$) in $G_{\mathfrak{p}}$ and set

$$K = \prod_{\mathfrak{p} \notin S_\infty} K_{\mathfrak{p}} \quad \left(\text{resp. } K(\mathfrak{n}) = \prod_{\mathfrak{p} \notin S_\infty} K_{\mathfrak{p}}(\mathfrak{n}) \right).$$

Following Remark 2.6 we can identify $\mathcal{R}(\mathfrak{n})$ with

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathcal{O}_F) \text{ s.t. } c \equiv 0 \bmod \mathfrak{n} \right\}$$

in case B is split.

We put

$$\mathcal{M}(\mathfrak{n}, S_{\mathrm{St}}, S_{\mathrm{tw}}; N) = \mathcal{M}(K(\mathfrak{n})^S, S_{\mathrm{St}}, S_{\mathrm{tw}}; N)$$

and

$$\mathcal{M}(\mathfrak{n}; N) = \mathcal{M}(\mathfrak{n}, \emptyset, \emptyset; N).$$

Without loss of generality we will always assume that every $\mathfrak{p} \in S$ divides $\mathfrak{n}$ exactly once.

For an open subgroup $\widetilde{U} \subseteq U_T^{S\cup S',\infty}$ we define

$$\mathcal{D}(\widetilde{U}, S; N)^{S',\infty} = \mathrm{Hom}_R(\mathcal{C}_c(\widetilde{U}, S, R)^{S',\infty}, N).$$

In case $\widetilde{U} = U_T(\mathfrak{m})$ with $\mathfrak{m} \subseteq \mathcal{O}_F$ a non-zero ideal we write $\mathcal{D}(\mathfrak{m}, S; N)^{S',\infty}$ for the corresponding distribution space.

Since we assume that every prime $\mathfrak{p}$ in S_{tw} is inert in E the local maps (2.6) and (2.7) (resp. (2.10)) induce the semi-local maps

$$\delta_{S_{\mathrm{St}}} = \otimes_{\mathfrak{p}\in S_{\mathrm{St}}} \delta_{\mathfrak{p}} \colon \bigotimes_{\substack{\mathfrak{p}\in S_{\mathrm{St}},\\ \mathfrak{p} \text{ split}}} C_c(F_{\mathfrak{p}}, \mathbb{Z}) \otimes \bigotimes_{\substack{\mathfrak{p}\in S_{\mathrm{St}},\\ \mathfrak{p} \text{ non-split}}} C_c(T_{\mathfrak{p}}, \mathbb{Z})/\mathbb{Z} \longrightarrow \bigotimes_{\mathfrak{p}\in S_{\mathrm{St}}} \mathrm{St}_{\mathfrak{p}}$$

respectively

$$\delta_{S_{\mathrm{tw}}}^{\mathrm{tw}} = \otimes_{\mathfrak{p}\in S_{\mathrm{tw}}} \delta_{\mathfrak{p}}^{\mathrm{tw}} \colon \bigotimes_{\mathfrak{p}\in S_{\mathrm{tw}}} C_c(T_{\mathfrak{p}}, \mathbb{Z})/\mathbb{Z} \longrightarrow \bigotimes_{\mathfrak{p}\in S_{\mathrm{tw}}} \mathrm{St}_{\mathfrak{p}}^{\mathrm{tw}}.$$

For every compact open subgroup $K \subseteq G(\mathbb{A}^{S\cup S',\infty})$ and every element $g \in G(\mathbb{A}^{S\cup S',\infty})$ we get a $T(F)$-equivariant homomorphism

$$\Delta^{S'}_{g,S_{\mathrm{St}},S_{\mathrm{tw}}} \colon \mathcal{A}(K, S_{\mathrm{St}}, S_{\mathrm{tw}}; N)^{S'} \longrightarrow \mathcal{D}(\iota^{-1}(gKg^{-1}), S; N)^{S'}$$

given by

$$\begin{aligned} &\Delta^{S'}_{g,S_{\mathrm{St}},S_{\mathrm{tw}}}(\Phi)(x)(f_{S_{\mathrm{St}}} \otimes f_{S_{\mathrm{tw}}}) \\ &\quad = \begin{cases} \Phi(\iota(x)g)(\delta_{S_{\mathrm{St}}}(f_{S_{\mathrm{St}}}) \otimes \delta^{\mathrm{tw}}_{S_{\mathrm{tw}}}(f_{S_{\mathrm{tw}}})) \text{ if } B \text{ is non-split and} \\ \Phi(\iota(x)g)(\delta_{S_{\mathrm{St}}}(f_{S_{\mathrm{St}}}) \otimes \delta^{\mathrm{tw}}_{S_{\mathrm{tw}}}(f_{S_{\mathrm{tw}}}))(0-\infty) \text{ if } B \text{ is split} \end{cases} \end{aligned} \tag{2.13}$$

for $\Phi \in \mathcal{A}(K, S_{\mathrm{St}}, S_{\mathrm{tw}}; N)^{S'}$, $x \in T(\mathbb{A}^{S\cup S',\infty})/\iota^{-1}(gKg^{-1})$ and $f_{S_{\mathrm{St}}}$, as well as $f_{S_{\mathrm{tw}}}$, in the appropriate semi-local function spaces.

Composing $\Delta_{g,S_{\mathrm{St}},S_{\mathrm{tw}}}$ with the restriction map

$$\mathcal{M}(K, S_{\mathrm{St}}, S_{\mathrm{tw}}; N)^{\epsilon} \longrightarrow \mathrm{H}^d(T(F), \mathcal{A}(K, S_{\mathrm{St}}, S_{\mathrm{tw}}; N)(\epsilon))$$

on cohomology yields a map

$$\mathcal{M}(K, S_{\mathrm{St}}, S_{\mathrm{tw}}; N)^{\epsilon} \longrightarrow \mathrm{H}^d(T(F), \mathcal{D}(\iota^{-1}(gKg^{-1}), S; N)^{\infty}(\epsilon)),$$

which we will also denote by $\Delta_{g,S_{\mathrm{St}},S_{\mathrm{tw}}}$.

Keep in mind that by Remark 2.1 there is an up to orientation unique $G(\mathbb{A}^{\mathrm{ram}(B)\cup S,\infty})$-equivariant isomorphism

$$G(\mathbb{A}^{\mathrm{ram}(B)\cup S,\infty})/K(\mathfrak{n})^{\mathrm{ram}(B)\cup S} \cong {\prod}'_{\mathfrak{p}\notin \mathrm{ram}(B)\cup S\cup S_\infty} \vec{\mathcal{E}}_{\mathfrak{p},\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})}. \tag{2.14}$$

Definition 2.12. A non-zero ideal $\mathfrak{m} \subseteq \mathcal{O}_F$ is called $\mathfrak{n}$-allowable if $\mathfrak{m}$ is coprime to $\mathrm{ram}(B)$ and $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{m}) - \mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}) > \eta_{\mathfrak{p}}$ for all $\mathfrak{p} \notin \mathrm{ram}(B)$.

Let us fix an $\mathfrak{n}$-allowable ideal $\mathfrak{m}$. For a finite place $\mathfrak{p}$ of F that is not in $S \cup \mathrm{ram}(B)$ we define $e_{\mathfrak{p}} = (w_{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{m})-\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})}, \ldots, w_{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{m})})$, where the w_i are the vertices chosen in Section 2.1. Let $g_{\mathfrak{m}} = (g_{\mathfrak{p}})_{\mathfrak{p}} \in G(\mathbb{A}^{S,\infty})/K(\mathfrak{n})^S$ be the element that is equal to one at places in $\mathrm{ram}(B)$ and corresponds to $(e_{\mathfrak{p}})_{\mathfrak{p}}$ under the above isomorphism for all places $\mathfrak{p} \notin S \cup \mathrm{ram}(B)$. In this case, the equality

$$U_T(\mathfrak{m}) = \iota^{-1}(g_{\mathfrak{m}} K(\mathfrak{n}) g_{\mathfrak{m}}^{-1})$$

holds and hence, we have a map

$$\Delta_{\mathfrak{m},S_{\mathrm{St}},S_{\mathrm{tw}}} = \Delta_{g_{\mathfrak{m}},S_{\mathrm{St}},S_{\mathrm{tw}}} \colon \mathcal{M}(\mathfrak{n}, S_{\mathrm{St}}, S_{\mathrm{tw}}; N)^{\epsilon} \longrightarrow \mathrm{H}^d(T(F), \mathcal{D}(\mathfrak{m}, S; N)^{\infty}(\epsilon)). \tag{2.15}$$

As always, we drop S_{St} and S_{tw} from the notation if they are empty.

For every $\mathfrak{p} \notin \mathrm{ram}(B)$ the Hecke operator $\mathbb{T}_{\mathfrak{p}}$ as defined in (2.1) acts on $\mathcal{M}(\mathfrak{n}; N)^{\epsilon}$ via the isomorphism (2.14). Similarly, for $\mathfrak{n}' \mid \mathfrak{n}$ the global Atkin-Lehner involution $W_{\mathfrak{n}'}$ is given by applying the local Atkin-Lehner involutions $W_{\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}')}}$ at the places $\mathfrak{p} \mid \mathfrak{n}'$. Also, for every $\mathfrak{p} \in \mathrm{ram}(B)$ the local Atkin-Lehner involution $W_{\mathfrak{p}}$ is given by interchanging the two elements in the set $G_{\mathfrak{p}}/K_{\mathfrak{p}}$.

Remark 2.13. Once again, as in Remark 2.6 we can give an explicit description of the elements $g_{\mathfrak{m}}$ if B is a split quaternion algebra in terms of matrices by identifying B with $\mathrm{Mat}_{2\times 2}$. We choose $g_{\mathfrak{m}}$ as the projection of the matrix $(h_{\mathfrak{p}})_{\mathfrak{p}} \in \mathrm{PGL}_2(\mathbb{A}^{\infty})$ given by

$$h_{\mathfrak{p}} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \mathrm{ord}_{\mathfrak{p}}(\mathfrak{m}) = 0, \\ \begin{pmatrix} \varpi_{\mathfrak{p}}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{m})} & 1 \\ 0 & 1 \end{pmatrix} & \text{else,} \end{cases} \tag{2.16}$$

where $\varpi_{\mathfrak{p}} \in \mathcal{O}_{F_{\mathfrak{p}}}$ is a local uniformizer at $\mathfrak{p}$.

3 Stickelberger elements for modular symbols

In this section we study Stickelberger elements coming from the modular symbols defined in Section 2. They are constructed by taking the cap product of the pullback of the modular symbols via the map (2.15) with the homology class defined in Section 1.3 associated to the Artin reciprocity map. As an immediate consequence of the results of Section 1 we can bound their order of vanishing from below and, furthermore, prove a functional equation.

Throughout this section we fix a ring R, an R-module N, a non-zero ideal $\mathfrak{n} \subseteq \mathcal{O}$, which is coprime to $\mathrm{ram}(B)$, and a character ϵ as before. In addition, we fix a modular symbol $\kappa \in \mathcal{M}(\mathfrak{n}; N)^{\epsilon}$.

Definition 3.1. Assume that E is a field. Let L be a finite Galois extension of F. We say L is E-anticyclotomic if it contains E, is abelian over E and if $\tau\sigma\tau^{-1} = \sigma^{-1}$ holds for all $\sigma \in \mathrm{Gal}(L/E)$.

Let us fix a finite Galois extension L of F which is E-anticyclotomic (resp. abelian) if B is non-split (resp. split). Write $\mathcal{G}$ for the Galois group of L over E (resp. F). The Artin reciprocity map of L over E (resp. F) induces a group homomorphism

$$\mathrm{r}_L \colon T(\mathbb{A})/T(F) \longrightarrow \mathcal{G}.$$

In addition, we fix an $\mathfrak{n}$-allowable ideal $\mathfrak{m}$ of $\mathcal{O}_F$ that bounds the ramification of L over E (resp. L over F), i.e. $U_T(\mathfrak{m})$ is contained in the kernel of r_L. Let

$$c_L = c_{\mathrm{r}_L} \in \mathrm{H}_d(T(F), \mathcal{C}(\mathfrak{m}, \mathbb{Z}[\mathcal{G}])^{\infty}(\epsilon))$$

be the image of r_L under (1.9) with $S = \emptyset$. We adopt similar notations if S is not the empty set, e.g. we set $c_L(\mathfrak{m}, S, \epsilon) = c_{\mathrm{r}_L}(\mathfrak{m}, S, \epsilon)$.

The natural pairing

$$\mathcal{C}_c(\mathfrak{m}, \mathbb{Z}[\mathcal{G}])^{\infty} \times \mathcal{D}(\mathfrak{m}; N)^{\infty} \longrightarrow \mathbb{Z}[\mathcal{G}] \otimes N$$

induces a cap-product pairing

$$\mathrm{H}_d(T(F), \mathcal{C}_c(\mathfrak{m}, \mathbb{Z}[\mathcal{G}])^{\infty}(\epsilon)) \times \mathrm{H}^d(T(F), \mathcal{D}(\mathfrak{m}; N)^{\infty}(\epsilon)) \longrightarrow \mathbb{Z}[\mathcal{G}] \otimes N.$$

Definition 3.2. The Stickelberger element of modulus $\mathfrak{m}$ associated with κ and L/F is defined as the cap-product

$$\Theta_{\mathfrak{m}}(L/F, \kappa) = \Delta_{\mathfrak{m}}(\kappa) \cap c_L \in \mathbb{Z}[\mathcal{G}] \otimes N.$$

As a direct consequence of functoriality of the Artin reciprocity map we get the following compatibility property:

Proposition 3.3. *Let L' be an intermediate extension of L over F containing E if E is a field. We will denote by $\mathcal{G}'$ the Galois group of L over E (resp. F) in case B is non-split (resp. split). Then we have*

$$\pi_{L/L'}(\Theta_{\mathfrak{m}}(L/F,\kappa)) = \Theta_{\mathfrak{m}}(L'/F,\kappa),$$

where

$$\pi_{L/L'}\colon \mathbb{Z}[\mathcal{G}]\otimes N \longrightarrow \mathbb{Z}[\mathcal{G}']\otimes N$$

is the canonical projection.

Let k be an R-algebra and $\chi\colon \mathcal{G}\to k^*$ a character. The character also induces an R-linear map $\chi\colon \mathbb{Z}[\mathcal{G}]\otimes N \longrightarrow k\otimes_R N$. Via the Artin reciprocity map we can view χ as a character of $T(\mathbb{A})$. We write $\chi_\infty\colon T(F_\infty)\to\mu_2(k)$ for the component at infinity of χ.

Proposition 3.4. *Let k be an R-algebra which is a field and let $\chi\colon \mathcal{G}\to k^*$ be a character. If $\chi_\infty\neq\epsilon$, we have*

$$\chi(\Theta_{\mathfrak{m}}(L/F,\kappa)) = 0.$$

Proof. We have

$$\begin{aligned}\chi(\Theta_{\mathfrak{m}}(L/F,\kappa)) &= \chi(c_L)\cap\Delta_{\mathfrak{m}}(\kappa)\\ &= c_\chi(\mathfrak{m},\emptyset,\epsilon)\cap\Delta_{\mathfrak{m}}(\kappa).\end{aligned}$$

Making the construction of $c_\chi(\mathfrak{m},\emptyset,\epsilon)$ (in particular (1.6)) present, one sees that orthogonality of characters yields the vanishing of $c_\chi(\mathfrak{m},\emptyset,\epsilon)$ for $\chi_\infty\neq\epsilon$. □

Let S_{St} and S_{tw} be finite disjoint sets of finite places of F with

- $\mathfrak{p}$ divides $\mathfrak{n}$ exactly once for all $\mathfrak{p}\in S = S_{\mathrm{St}}\cup S_{\mathrm{tw}}$,
- S is disjoint from $\mathrm{ram}(B)$ and
- every prime in S_{tw} is inert in E.

The local evaluation maps (2.8) and (2.11) induce a map

$$\mathrm{Ev}_{S_{\mathrm{St}},S_{\mathrm{tw}}}\colon \mathcal{M}(\mathfrak{n}, S_{\mathrm{St}}, S_{\mathrm{tw}}; N)^{\epsilon} \longrightarrow \mathcal{M}(\mathfrak{n}; N)^{\epsilon}.$$

For a place v of F we let $\mathcal{G}_v \subseteq \mathcal{G}$ be the decomposition group at v. Note that even if E is a field and v is split in E this is well defined since L is E-anticyclotomic. If $\mathfrak{p} \in S$, we define $I_{\mathfrak{p}} \subseteq \mathbb{Z}[\mathcal{G}]$ as the kernel of the projection $\mathbb{Z}[\mathcal{G}] \twoheadrightarrow \mathbb{Z}[\mathcal{G}/\mathcal{G}_{\mathfrak{p}}]$. If $v \in S_\infty$ is real and split in E, we let σ_v be a generator of $\mathcal{G}_v$ and define $I_v^{\pm 1} \subseteq \mathbb{Z}[\mathcal{G}]$ as the ideal generated by $\sigma_v \mp 1$. For complex and non-split real Archimedean places we define $I_v^{\pm} = \mathbb{Z}[\mathcal{G}]$.

Lemma 3.5. *Assume that N is $\mathbb{Z}$-flat and that there exists an $(S_{\mathrm{St}}, S_{\mathrm{tw}})$-special modular symbol $\kappa' \in \mathcal{M}(\mathfrak{n}, S_{\mathrm{St}}, S_{\mathrm{tw}}, ; N)^{\epsilon}$ lifting κ, i.e. $\mathrm{Ev}_{S_{\mathrm{St}},S_{\mathrm{tw}}}(\kappa_S) = \kappa$ holds. Then we have*

$$\Theta_{\mathfrak{m}}(L/F, \kappa) \in \left(\prod_{v \in S_\infty} I_v^{-\epsilon_v(-1)} \cdot \prod_{\mathfrak{p} \in S_{\mathfrak{m}}} I_{\mathfrak{p}} \right) \otimes N.$$

In particular, if $N = R$ is a $\mathbb{Z}$-flat ring and ϵ is trivial, we have

$$2^{-\widetilde{d}}\Theta_{\mathfrak{m}}(L/F, \kappa) \in R[\mathcal{G}]$$

and

$$\mathrm{ord}_R(2^{-\widetilde{d}}\Theta_{\mathfrak{m}}(L/F, \kappa)) \geq \# S_{\mathfrak{m}},$$

where $\widetilde{d}$ is the number of real Archimedean places of F which are split in E.

Proof. By Lemma 2.8 (i) and Lemma 2.9 we have

$$\begin{aligned}\Theta_{\mathfrak{m}}(L/F, \kappa) &= \Delta_{\mathfrak{m}}(\kappa) \cap c_L(\mathfrak{m}, \emptyset, \epsilon) \\ &= \pm\, \Delta_{\mathfrak{m}, S_{\mathrm{St}}, S_{\mathrm{tw}}}(\kappa') \cap c_L(\mathfrak{m}, S, \epsilon).\end{aligned}$$

We set $I = \prod_{v \in S_\infty} I_v^{-\epsilon_v(-1)} \cdot \prod_{\mathfrak{p} \in S_{\mathfrak{m}}} I_{\mathfrak{p}}$ and consider the ring $A = \mathbb{Z}[\mathcal{G}]/I$ together with the projection maps $\pi\colon \mathbb{Z}[\mathcal{G}] \to A$ and $\pi_N\colon \mathbb{Z}[\mathcal{G}] \otimes N \to A \otimes N$. We have

$$\pi_N(\Theta_{\mathfrak{m}}(L/F, \kappa)) = \pm\, \Delta_{\mathfrak{m}, S_{\mathrm{St}}, S_{\mathrm{tw}}}(\kappa') \cap \pi_*(c_L(\mathfrak{m}, S, \epsilon)) = 0$$

since the homology class $\pi_*(c_L(\mathfrak{m}, S, \epsilon)) = c_{\pi \circ r_L}(\mathfrak{m}, S, \epsilon)$ vanishes by applying Proposition 1.6 with $\mathfrak{a} = A$. □

Lemma 3.6. *Suppose that every $\mathfrak{p} \in \mathrm{ram}(B)$ is inert in E and that we can decompose $\mathfrak{n} = \mathfrak{n}_1\mathfrak{n}_2$ such that $\mathfrak{n}_1$ is coprime to $\mathfrak{m}$ and $\mathfrak{n}_2 \mid \mathfrak{m}$. Write $\mathfrak{n}_1 = \prod_{i=1}^{r} \mathfrak{p}_i^{n_i}$, with $n_i \geq 1$ for $1 \leq i \leq r$. Let κ be an eigenvector of $W_{\mathfrak{p}_i^{n_i}}$ with eigenvalue $\varepsilon_i \in \{\pm 1\}$ for $1 \leq i \leq r$ and of $W_\mathfrak{p}$ with eigenvalue $\varepsilon_\mathfrak{p} \in \{\pm 1\}$ for every $\mathfrak{p} \in \mathrm{ram}(B)$. Further, write $\varepsilon_{\mathfrak{n}_1} = \prod_{i=1}^{r} \varepsilon_i$ for the eigenvalue of $W_{\mathfrak{n}_1}$. Then*

$$\Theta_\mathfrak{m}(L/F, \kappa)^\vee = (-1)^{d+\delta} \cdot \epsilon(-1) \cdot \varepsilon_{\mathfrak{n}_1} \prod_{\mathfrak{p} \in \mathrm{ram}(B)} \varepsilon_\mathfrak{p} \cdot \Theta_\mathfrak{m}(L/F, \kappa)$$

holds up to multiplication with an element in $\mathcal{G}$ with

$$\delta = \begin{cases} 0 & \text{if } B \text{ is non-split and} \\ 1 & \text{if } B \text{ is split.} \end{cases}$$

Proof. Let Φ be an element in $\mathcal{A}(K(\mathfrak{n}); N)(\epsilon)$. By Lemma 2.5 we have

$$\Delta_\mathfrak{m}(W_{\mathfrak{n}_1} \cdot \prod_{\mathfrak{p} \in \mathrm{ram}(B)} W_\mathfrak{p} \cdot \Phi) = (-1)^\delta \epsilon(-1)(\mathrm{inv} \circ \Delta_\mathfrak{m})(J\Phi))$$

up to multiplication with an element in $T(F)$. For the places $\mathfrak{p} \in \mathrm{ram}(B)$ we use that we have $J \notin K_\mathfrak{p}$ (see the beginning of Section 2). The factor $\epsilon(-1)$ is the contribution of the Archimedean places and the factor $(-1)^\delta$ is coming from (2.13), i.e. it follows from

$$J(0 - \infty) = -(0 - \infty)$$

if B is split.

Hence, by passing to cohomology we get

$$\begin{aligned}
\varepsilon_{\mathfrak{n}_1} \cdot \prod_{\mathfrak{p} \in \mathrm{ram}(B)} \varepsilon_\mathfrak{p} \cdot \Theta_\mathfrak{m}(L/F, \kappa) &= \Theta_\mathfrak{m}(L/F, W_{\mathfrak{n}_1} \cdot \prod_{\mathfrak{p} \in \mathrm{ram}(B)} W_\mathfrak{p} \cdot \kappa) \\
&= (-1)^\delta \epsilon(-1) c_L \cap \mathrm{inv}(\Delta_\mathfrak{m}(\kappa)) \\
&= (-1)^\delta \epsilon(-1)\, \mathrm{inv}(c_L) \cap \Delta_\mathfrak{m}(\kappa) \\
&= (-1)^{d+\delta} \epsilon(-1)(c_L)^\vee \cap \Delta_\mathfrak{m}(\kappa),
\end{aligned}$$

up to multiplication with an element in $\mathcal{G}$. The last equality holds for the following reason: The $T(F)$-action on $\mathcal{C}(\mathfrak{m}, \mathbb{Z}[\mathcal{G}])^\infty(\epsilon)$ is inverted by applying inv and inverting the $\mathcal{U}_S^+$-action induces multiplication by $(-1)^d$ on $H_d(\mathcal{U}_S^+, \mathbb{Z})$. Thus we get

$$\mathrm{inv}(c_L) = (-1)^d (c_L)^\vee$$

and the claim follows. □

Remark 3.7. As a consequence of the choices made in Remark 2.6 or rather by the equations (2.4) and (2.5), we have a more precise statement of Lemma 3.6 in case B is split:

$$\Theta_{\mathfrak{m}}(L/F,\kappa)^{\vee} = (-1)^{d+1} \cdot \epsilon(-1) \cdot \varepsilon_{\mathfrak{n}_1} \cdot \sigma_{\mathfrak{n}_1}^{-1} \cdot \Theta_{\mathfrak{m}}(L/F,\kappa)$$

Here $\sigma_{\mathfrak{n}_1} = \prod_{\mathfrak{p}|\mathfrak{n}_1} \sigma_{\mathfrak{p}}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}_1)}$ and $\sigma_{\mathfrak{p}} \in \mathcal{G}$ is the (arithmetic) Frobenius at $\mathfrak{p}$ (note that in particular the $\mathfrak{p} \mid \mathfrak{n}_1$ are unramified by the definition of $\mathfrak{n}_1$). See [BG17], Proposition 1.15, for more details.

As a consequence, we can compute the parity of the order of vanishing of Stickelberger elements. With the same hypothesis as in the previous proposition we get:

Corollary 3.8. *Suppose that $N = R$, $r = \mathrm{ord}_R(\Theta_{\mathfrak{m}}(L/F,\kappa)) < \infty$ and that 2 acts invertibly on $I_R(\mathcal{G})^r/I_R(\mathcal{G})^{r+1}$. Then we have*

$$(-1)^r = (-1)^{d+\delta} \cdot \epsilon(-1) \cdot \varepsilon_{\mathfrak{n}_1}.$$

Proof. The involution $(\cdot)^{\vee}$ induces multiplication by (-1) on $I_R(\mathcal{G})/I_R(\mathcal{G})^2$ and thus, it induces multiplication by $(-1)^r$ on $I_R(\mathcal{G})^r/I_R(\mathcal{G})^{r+1}$. Since the equality in Lemma 3.6 holds up to multiplication with an element in $\mathcal{G}$, we have

$$(-1)^r\Theta_{\mathfrak{m}}(L/F,\kappa) = \Theta_{\mathfrak{m}}(L/F,\kappa)^{\vee} = (-1)^d\epsilon(-1) \cdot \epsilon_{\mathfrak{n}_1} \cdot \Theta_{\mathfrak{m}}(L/F,\kappa)$$

in $I_R(\mathcal{G})^r/I_R(\mathcal{G})^{r+1}$. Assume that $(-1)^r \neq (-1)^d\epsilon(-1) \cdot \epsilon_{\mathfrak{n}_1}$. Then the above equation would imply

$$2\Theta_{\mathfrak{m}}(L/F,\kappa) \equiv 0 \bmod I_R(\mathcal{G})^{r+1}$$

and thus we get

$$\Theta_{\mathfrak{m}}(L/F,\kappa) \equiv 0 \bmod I_R(\mathcal{G})^{r+1}$$

since by assumption 2 acts invertibly on $I_R(\mathcal{G})^r/I_R(\mathcal{G})^{r+1}$. But this contradicts our assumption that the order of vanishing of $\Theta_{\mathfrak{m}}(L/F,\kappa)$ is exactly r. □

4 Automorphic Stickelberger elements

We will apply the results of the previous sections to cohomology classes coming from automorphic forms. After constructing Stickelberger elements associated to automorphic representations and proving norm relations for them (Section 4.1) we get lower bounds for the order of vanishing. Moreover, we prove interpolation formulae (Section 4.2) and a leading term formula (Section 4.3). We close our discussion by giving a positive answer to the conjecture of Mazur and Tate in the analytic rank zero situation formulated in the introduction and by giving a construction of p-adic L-functions in Section 4.4.

4.1 Stickelberger elements associated to automorphic representations

Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ with the following properties:

- π_v is a discrete series representation of weight 2 for all real Archimedean places v of F,
- π_v is isomorphic to the principal series representation $\pi(\mu_1, \mu_2)$ with $\mu_1(z) = z^{\frac{1}{2}}\overline{z}^{-\frac{1}{2}}$ and $\mu_2(z) = z^{-\frac{1}{2}}\overline{z}^{\frac{1}{2}}$ for all complex Archimedean places v of F and
- $\pi_\mathfrak{p}$ is special, i.e. a twist of the Steinberg representation, for all $\mathfrak{p} \in \mathrm{ram}(B)$.

A result of Clozel (cf. [Clo90]) tells us that there exists a smallest subfield $\mathbb{Q}_\pi \subseteq \mathbb{C}$, which is a finite extension of $\mathbb{Q}$ such that $\pi^\infty = \prod_{\mathfrak{p} \notin S_\infty} \pi_\mathfrak{p}$ can be defined over $\mathbb{Q}_\pi$. $\mathbb{Q}_\pi$ is called the field of definition of π. We write $\Gamma_0(\mathfrak{n}) \subseteq \mathrm{PGL}_2(\mathbb{A})$ for the usual adelic congruence subgroup of level $\mathfrak{n}$. By the automorphic formulation of Atkin-Lehner theory due to Casselman (see [Cas73]) there exists a unique non-zero ideal $\mathfrak{f}(\pi) \subseteq \mathcal{O}_F$ such that $(\pi^\infty)^{\Gamma_0(\mathfrak{f}(\pi))}$ is one-dimensional. Thus, the standard Hecke operator $\mathbb{T}_\mathfrak{p}$ (resp. the Atkin-Lehner involutions $\mathrm{W}_\mathfrak{p}$) acts on $(\pi^\infty)^{\Gamma_0(\mathfrak{f}(\pi))}$ via multiplication by a scalar which we denote by $\lambda_\mathfrak{p}$ (resp. $\omega_\mathfrak{p}$). More precisely, the Hecke eigenvalues $\lambda_\mathfrak{p}$ are elements of the ring of integers R_π of $\mathbb{Q}_\pi$.

If B is a non-split quaternion algebra, our assumptions on π provide a transfer of π to B, proven by Jacquet and Langlands in [JL70], i.e. there exists an automorphic representation π_B of $G(\mathbb{A})$ such that

- $\pi_{B,v} \cong \pi_v$ for all places v at which B is split,

- $\pi_{B,v}$ is the trivial one-dimensional representation for all $v \in S_\infty$ at which B is non-split and

- $\pi_{B,\mathfrak{p}}$ is the trivial (resp. non-trivial) smooth one-dimensional representation of $G_\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{ram}(B)$ for which $\pi_\mathfrak{p}$ is the (twisted) Steinberg representation. In particular, the eigenvalue of $W_\mathfrak{p}$ acting on $\pi_{B,\mathfrak{p}}$ is the negative of the root number of $\pi_\mathfrak{p}$.

As before, we identify G and PGL_2 if the fixed quaternion algebra B is split. In this situation we will write $\pi_B = \pi$ etc. to unify the notation.

In either case, let $\mathfrak{f}(\pi_B)$ be the maximal divisor of $\mathfrak{f}(\pi)$ which is coprime to $\mathrm{ram}(B)$. We define

$$\mathcal{M}(\mathfrak{f}(\pi_B); \mathbb{Q}_\pi)^{\epsilon,\pi} \subseteq \mathcal{M}(\mathfrak{f}(\pi_B); \mathbb{Q}_\pi)^{\epsilon}$$

to be the common eigenspace of the operators $\mathbb{T}_\mathfrak{p}$ for $\mathfrak{p} \notin \mathrm{ram}(B)$ with eigenvalues $\lambda_\mathfrak{p}$. The formalism of $(\mathfrak{g}, K)$-cohomology together with the strong multiplicity one theorem implies that $\mathcal{M}(\mathfrak{f}(\pi_B); \mathbb{Q}_\pi)^{\epsilon,\pi}$ is one-dimensional for every sign character ϵ. It follows by Proposition 2.10 (i) that

$$\mathcal{M}(\mathfrak{f}(\pi_B); R_\pi) \otimes_{R_\pi} \mathbb{Q}_\pi \longrightarrow \mathcal{M}(\mathfrak{f}(\pi_B); \mathbb{Q}_\pi)$$

is an isomorphism. Therefore, the intersection of $\mathcal{M}(\mathfrak{f}(\pi_B); \mathbb{Q}_\pi)^{\epsilon,\pi}$ with the image of $\mathcal{M}(\mathfrak{f}(\pi_B); R_\pi)$ in $\mathcal{M}(\mathfrak{f}(\pi_B); \mathbb{Q}_\pi)$ is a locally free R_π-module of rank one. We choose a maximal element $\kappa^{\pi_B,\epsilon}$ of this module.

Remark 4.1. (i) If R_π is a PID, the generator $\kappa^{\pi_B,\epsilon}$ is unique up to multiplication by an element in R_π^*. In particular, if the automorphic representation π corresponds to a modular elliptic curve over F, then $\mathbb{Q}_\pi$ is equal to $\mathbb{Q}$ and thus, $\kappa^{\pi_B,\epsilon}$ is unique up to sign.

(ii) We could weaken the assumptions on $\pi_\mathfrak{p}$ for $\mathfrak{p} \in \mathrm{ram}(B)$. It is enough to assume that $\pi_\mathfrak{p}$ is either special or supercuspidal. But in the supercuspidal case there is no canonical local new vector for $\pi_{B,\mathfrak{p}}$. To ease the exposition, we stick to the special case.

(iii) In [BG17] we have chosen a different approach for the construction of a modular symbol. Starting with an automorphic form Φ of parallel weight 2 we construct an Eichler-Shimura homomorphism to get an element $\kappa_\Phi^\epsilon \in \mathcal{M}(f(\pi_B); \mathbb{C})^\epsilon$ (see Section 2.1 of [BG17]). Using this approach, one has to choose Φ properly such that $\kappa_\Phi^\epsilon \in \mathcal{M}(f(\pi_B); R_\pi)^\epsilon$.

As in Section 3, we fix a finite Galois extension L of F which is E-anticyclotomic (resp. abelian) if B is non-split (resp. split) and write $\mathcal{G}$ for the Galois group of L over E (resp. F). Further, let $\mathfrak{m}$ be an $f(\pi_B)$-allowable ideal of $\mathcal{O}_F$ that bounds the ramification of L over E (resp. F).

Definition 4.2. The Stickelberger element of modulus $\mathfrak{m}$ and sign ϵ associated to π_B and L/F is defined by

$$\Theta_{\mathfrak{m}}(L/F, \pi_B)^{\epsilon} = \Theta_{\mathfrak{m}}(L/F, \kappa^{\pi_B, \epsilon}) \in R_{\pi}[\mathcal{G}].$$

Remark 4.3. The element $\Theta_{\mathfrak{m}}(L/F, \pi_B)^{\epsilon}$ depends on the choice of an $U_{T_{\mathfrak{p}}}$-stable vertex and an end of the Bruhat-Tits tree for every prime $\mathfrak{p} \notin \mathrm{ram}(B)$. If we take different choices, $\Theta_{\mathfrak{m}}(L/F, \pi_B)^{\epsilon}$ is multiplied by an element of $\mathcal{G}$. Therefore, the element

$$\mathfrak{L}_{\mathfrak{m}}(L/F, \pi_B)^{\epsilon} = \Theta_{\mathfrak{m}}(L/F, \pi_B)^{\epsilon} \cdot (\Theta_{\mathfrak{m}}(L/F, \pi_B)^{\epsilon})^{\vee} \in R_{\pi}[\mathcal{G}]$$

is independent of these choices.

Next, we study the behaviour of Stickelberger elements under change of modulus.

Theorem 4.4 (Norm relations)**.** *(i) Let $\mathfrak{p}$ be a finite place of F that does not divide $\mathfrak{m}$. Write $\sigma_{\mathfrak{P}}$ for the the image of the uniformizer $\varpi_{\mathfrak{P}}$ under the Artin reciprocity map r_L of L over E (resp. F). Then the equality*

$$\Theta_{\mathfrak{mp}}(L/F, \pi_B)^{\epsilon} = (\lambda_{\mathfrak{p}} - (*))\Theta_{\mathfrak{m}}(L/F, \pi_B)^{\epsilon}$$

holds with

$$(*) = \begin{cases} 0 & \textit{if } \mathfrak{p} \textit{ is inert in } E, \\ \mathbb{1}_{\mathfrak{p}}(\mathfrak{f}(\pi_B))\sigma_{\mathfrak{P}} & \textit{if } \mathfrak{p} \textit{ is ramified in } E, \\ \sigma_{\mathfrak{P}}^{-1} + \mathbb{1}_{\mathfrak{p}}(\mathfrak{f}(\pi_B))\sigma_{\mathfrak{P}} & \textit{if } \mathfrak{p} \textit{ is split in } E, \end{cases}$$

where

$$\mathbb{1}_{\mathfrak{p}}(\mathfrak{f}(\pi_B)) = \begin{cases} 1 & \textit{if } \mathrm{ord}_{\mathfrak{p}}(\mathfrak{f}(\pi_B)) = 0 \textit{ and} \\ 0 & \textit{else.} \end{cases}$$

(ii) Let $\mathfrak{p}$ be a finite place of F that does divide $\mathfrak{m}$ and write $m = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{m})$. Then we have a decomposition

$$\Theta_{\mathfrak{mp}}(L/F, \pi_B)^{\epsilon} = \lambda_{\mathfrak{p}}\Theta_{\mathfrak{m}}(L/F, \pi_B)^{\epsilon} + \mathbb{1}_{\mathfrak{p}}(\mathfrak{f}(\pi_B))v_{\mathfrak{m}}(\Theta_{\mathfrak{mp}^{-1}}(L/F, \pi_B)^{\epsilon}),$$

where the elements $v_{\mathfrak{m}}(\Theta_{\mathfrak{mp}^{-1}}(L/F, \pi_B)^{\epsilon})$ can be characterized by the following properties:

- *$\pi_{L/L'}(v_{\mathfrak{m}}(\Theta_{\mathfrak{mp}^{-1}}(L/F, \pi_B)^{\epsilon}) = v_{\mathfrak{m}}(\Theta_{\mathfrak{mp}^{-1}}(L'/F, \pi_B)^{\epsilon})$ for all intermediate extensions L' of L over F (which contain E if E is a field)*

- $v_{\mathfrak{m}}(\Theta_{\mathfrak{m}\mathfrak{p}^{-1}}(L/F,\pi_B)^\epsilon) = [U_{T_\mathfrak{p}}^{(m-1)} : U_{T_\mathfrak{p}}^{(m)}](\Theta_{\mathfrak{m}\mathfrak{p}^{-1}}(L/F,\pi_B)^\epsilon)$ *in case the Artin reciprocity map for L over E (resp. F) is trivial on $U_T(\mathfrak{m}\mathfrak{p}^{-1})$*
- *Let k be a field which is an R_π-algebra and $\chi\colon \mathcal{G} \to k^*$ a character such that $\chi_\mathfrak{p}$ has conductor $\mathfrak{p}^m$. Then we have*

$$\chi(v_{\mathfrak{m}}(\Theta_{\mathfrak{m}\mathfrak{p}^{-1}}(L/F,\pi_B)^\epsilon)) = 0.$$

(iii) Suppose that $\mathfrak{p}$ is inert and divides $\mathfrak{m}$ as well as $\mathfrak{f}(\pi_B)$ exactly once. Let k be an R_π-algebra and $\chi\colon \mathcal{G} \to k^*$ a character which is unramified at $\mathfrak{p}$. Then we have

$$\chi(\Theta_{\mathfrak{m}}(L/F,\pi_B)^\epsilon) = 0.$$

Proof. This is a direct consequences of the local norm relations of Lemma 2.3. For part (iii), note that the local representation at $\pi_\mathfrak{p}$ is a (twisted) Steinberg representation and thus, the eigenvalue of $\mathbb{T}_\mathfrak{p} \circ W_\mathfrak{p}$ on a local new vector is -1. □

In the following we use the same notation as in the discussion before Lemma 3.5. Let S_{St} (resp. S_{tw}) be the set of finite places $\mathfrak{p}$ of F which are disjoint from $\mathrm{ram}(B)$ (and inert in E) such that the local component $\pi_\mathfrak{p}$ is the (twisted) Steinberg representation. As always, we set $S = S_{\mathrm{St}} \cup S_{\mathrm{tw}}$. Fix subsets $\mathfrak{S}_{\mathrm{St}} \subseteq S_{\mathrm{St}}$ and $\mathfrak{S}_{\mathrm{tw}} \subseteq S_{\mathrm{tw}}$ and put $\mathfrak{S} = \mathfrak{S}_{\mathrm{St}} \cup \mathfrak{S}_{\mathrm{tw}}$.

Let $t_{\mathfrak{S}} \in \mathbb{Z}$ be the product of the exponent of the 2-torsion subgroup of $\mathcal{M}(f(\pi_B); R_\pi)^\epsilon$ and the exponent of the torsion subgroup of

$$\bigoplus_{\mathfrak{p}\in\mathfrak{S}} \mathcal{M}(f(\pi_B)\mathfrak{p}^{-1}; R_\pi)^\epsilon.$$

If $d > 0$, we define

$$c_{\mathfrak{S}} = \gcd\left\{\prod_{\mathfrak{p}\in\mathfrak{S}'}(N(\mathfrak{p})+1) \mid \mathfrak{S}' \subset \mathfrak{S} \text{ with } \#\mathfrak{S} = \#\mathfrak{S}' + 1\right\}.$$

In the case $d = 0$ the above cohomology groups are torsion-free and hence, $t_{\mathfrak{S}} = 1$. Further, we put $c_{\mathfrak{S}} = 1$.

Finally, we define $n_{\mathfrak{S}} = c_{\mathfrak{S}} \cdot\ t_{\mathfrak{S}}$.

Lemma 4.5. *Let $\mathfrak{S} = \mathfrak{S}_{\mathrm{St}} \cup \mathfrak{S}_{\mathrm{tw}}$ as above. Then the cokernel of the map*

$$\mathrm{Ev}_{\mathfrak{S}_{\mathrm{St}},\mathfrak{S}_{\mathrm{tw}}}\colon \mathcal{M}(f(\pi_B),\mathfrak{S}_{\mathrm{St}},\mathfrak{S}_{\mathrm{tw}}; R_\pi)^\epsilon \to \mathcal{M}(f(\pi_B); R_\pi)^\epsilon$$

is annihilated by $n_{\mathfrak{S}}$.

Proof. Let κ be a preimage of $\kappa^{\pi_B,\epsilon}$ in $\mathcal{M}(f(\pi_B);R_\pi)$. Since π is new at $\mathfrak{p}$ for all $\mathfrak{p}\in S$ we know that the image of κ under the trace maps

$$\mathrm{Tr}_\mathfrak{p}\colon \mathcal{M}(f(\pi_B);R_\pi)^\epsilon \longrightarrow \mathcal{M}(f(\pi_B)\mathfrak{p}^{-1};R_\pi)^\epsilon$$

is torsion for all $\mathfrak{p}\in S$. Thus, $n_\mathfrak{S}\cdot\kappa$ lies in the kernel of all trace maps $\mathrm{Tr}_\mathfrak{p}$ for $\mathfrak{p}\in\mathfrak{S}$ and is also an eigenvector under the Atkin-Lehner operator $W_\mathfrak{p}$ with eigenvalue -1 (resp. 1) if $\mathfrak{p}\in\mathfrak{S}_{\mathrm{St}}$ (resp. $\mathfrak{S}_{\mathrm{tw}}$).

We are going to prove the claim by induction on $\#\mathfrak{S}$. The well-known short exact sequences

$$0\longrightarrow \text{c-ind}_{K_\mathfrak{p}(\mathfrak{p})}^{G(F_\mathfrak{p})}\mathbb{Z}\longrightarrow \left(\text{c-ind}_{K_\mathfrak{p}}^{G(F_\mathfrak{p})}\mathbb{Z}\right)^{W_\mathfrak{p}=-1}\longrightarrow \mathrm{St}_\mathfrak{p}\longrightarrow 0$$

for $\mathfrak{p}\in S_{\mathrm{St}}$ and

$$0\longrightarrow \text{c-ind}_{K_\mathfrak{p}(\mathfrak{p})}^{G(F_\mathfrak{p})}\mathbb{Z}\longrightarrow \left(\text{c-ind}_{K_\mathfrak{p}}^{G(F_\mathfrak{p})}\mathbb{Z}\right)^{W_\mathfrak{p}=1}\longrightarrow \mathrm{St}_\mathfrak{p}^{\mathrm{tw}}\longrightarrow 0$$

for $\mathfrak{p}\in S_{\mathrm{tw}}$ (see for example Section 2.4 of [Spi14]) induce an exact sequence

$$\mathcal{M}(f(\pi_B),\{\mathfrak{p}\};R_\pi)^\epsilon \xrightarrow{\mathrm{Ev}_{\{\mathfrak{p}\}}} (\mathcal{M}(f(\pi_B);R_\pi)^\epsilon)^{W_\mathfrak{p}=\pm1} \xrightarrow{\mathrm{Tr}_\mathfrak{p}} \mathcal{M}(f(\pi_B)\mathfrak{p}^{-1};R_\pi)^\epsilon$$

in cohomology for all $\mathfrak{p}\in S$. Hence, we can lift the class $n_\mathfrak{S}\cdot\kappa$ if $\mathfrak{S}=\{\mathfrak{p}\}$.

Now, let $\#\mathfrak{S}\geq 2$. We pick an element $\mathfrak{p}\in\mathfrak{S}$ and consider the following commutative diagram with exact columns (where, by abuse of notation, we write $\mathcal{M}(f(\pi_B),\mathfrak{S};R_\pi)^\epsilon$ for $\mathcal{M}(f(\pi_B),\mathfrak{S}_{\mathrm{St}},\mathfrak{S}_{\mathrm{tw}};R_\pi)^\epsilon$):

$$\begin{array}{ccc}
\mathcal{M}(f(\pi_B),\mathfrak{S};R_\pi)^\epsilon & \xrightarrow{\mathrm{Ev}_{\mathfrak{S}-\{\mathfrak{p}\}}} & \mathcal{M}(f(\pi_B),\{\mathfrak{p}\};R_\pi)^\epsilon \\
\downarrow{\scriptstyle \mathrm{Ev}_{\{\mathfrak{p}\}}} & & \downarrow{\scriptstyle \mathrm{Ev}_{\{\mathfrak{p}\}}} \\
(\mathcal{M}(f(\pi_B),\mathfrak{S}-\{\mathfrak{p}\};R_\pi)^\epsilon)^{W_\mathfrak{p}=\pm1} & \xrightarrow{\mathrm{Ev}_{\mathfrak{S}-\{\mathfrak{p}\}}} & (\mathcal{M}(f(\pi_B);R_\pi)^\epsilon)^{W_\mathfrak{p}=\pm1} \\
\downarrow{\scriptstyle \mathrm{Tr}_\mathfrak{p}} & & \downarrow{\scriptstyle \mathrm{Tr}_\mathfrak{p}} \\
\mathcal{M}(f(\pi_B)\mathfrak{p}^{-1},\mathfrak{S}-\{\mathfrak{p}\};R_\pi)^\epsilon & \xrightarrow{\mathrm{Ev}_{\mathfrak{S}-\{\mathfrak{p}\}}} & \mathcal{M}(f(\pi_B)\mathfrak{p}^{-1};R_\pi)^\epsilon
\end{array}$$

By the induction hypothesis we can lift $n_{\mathfrak{S}-\{\mathfrak{p}\}}\cdot\kappa$ to a class

$$\widetilde{\kappa}\in(\mathcal{M}(f(\pi_B),\mathfrak{S}-\{\mathfrak{p}\};R_\pi)^\epsilon)^{W_\mathfrak{p}=\pm1}.$$

If $d=0$, the map

$$\mathrm{Ev}_{\mathfrak{S}-\{\mathfrak{p}\}}\colon \mathcal{M}(f(\pi_B)\mathfrak{p}^{-1},\mathfrak{S}-\{\mathfrak{p}\};R_\pi)^\epsilon\longrightarrow\mathcal{M}(f(\pi_B)\mathfrak{p}^{-1};R_\pi)^\epsilon$$

is injective. Therefore, the claim follows from $\mathrm{Tr}_{\mathfrak{p}}(n_{\mathfrak{S}-\{\mathfrak{p}\}} \cdot \kappa) = 0$.

If $d > 0$, we consider the canonical map

$$\iota \colon \mathcal{M}(f(\pi_B)\mathfrak{p}^{-1}, \mathfrak{S} - \{\mathfrak{p}\}; R_\pi)^\epsilon \longrightarrow \mathcal{M}(f(\pi_B), \mathfrak{S} - \{\mathfrak{p}\}; R_\pi)^\epsilon.$$

Since $\mathrm{Tr}_{\mathfrak{p}} \circ \iota$ is equal to multiplication by $N(\mathfrak{p}) + 1$ we see that the element $(N(\mathfrak{p}) + 1) \cdot \widetilde{\kappa} - \iota(\mathrm{Tr}_{\mathfrak{p}}(\widetilde{\kappa}))$ can be lifted to a class in $\mathcal{M}(f(\pi_B), \mathfrak{S}; R_\pi)^\epsilon$. The commutativity of the diagram above together with the fact that $\mathrm{Tr}_{\mathfrak{p}}(n_{\mathfrak{S}-\{\mathfrak{p}\}} \cdot \kappa)$ vanishes implies that $(N(\mathfrak{p})+1) \cdot \widetilde{\kappa} - \iota(\mathrm{Tr}_{\mathfrak{p}}(\widetilde{\kappa}))$ is also a lift of $(N(\mathfrak{p})+1) n_{\mathfrak{S}-\{\mathfrak{p}\}} \cdot \kappa$. □

Let $I_v \subseteq \mathbb{Z}[\mathcal{G}]$ for $v \in S \cup S_\infty$ be the ideals defined in the discussion before Lemma 3.5. Then we have

Theorem 4.6 (Order of vanishing)**.** *For every $\mathfrak{f}(\pi_B)$-allowable modulus $\mathfrak{m}$ that bounds the ramification of L over E (resp. F) we have*

$$n_{S_\mathfrak{m}} \Theta_\mathfrak{m}(L/F, \pi_B)^\epsilon \in \left(\prod_{v \in S_\infty} I_v^{-\epsilon_v(-1)} \cdot \prod_{\mathfrak{p} \in S_\mathfrak{m}} I_\mathfrak{p} \right) \otimes R_\pi.$$

Proof. It is easy to see that the map

$$\mathrm{Ev}_{S_{\mathrm{St}}, S_{\mathrm{tw}}} \colon \mathcal{M}(\mathfrak{f}(\pi_B), S_{\mathrm{St}}, S_{\mathrm{tw}}; \mathbb{Q}_\pi)^{\epsilon,\pi} \longrightarrow \mathcal{M}(\mathfrak{f}(\pi_B); \mathbb{Q}_\pi)^{\epsilon,\pi}$$

is an isomorphism of one-dimensional $\mathbb{Q}_\pi$-vector spaces (cf. [Dep16], Proposition 4.10 for a proof in the GL_2-case). Therefore, the map

$$\mathcal{M}(\mathfrak{f}(\pi_B), S_{\mathrm{St}}, S_{\mathrm{tw}}; R_\pi)^{\epsilon,\pi} \longrightarrow \mathcal{M}(\mathfrak{f}(\pi_B); R_\pi)^{\epsilon,\pi}$$

has finite cokernel. Lemma 4.5 tells us that $n_{S_\mathfrak{m}}$ annihilates this cokernel. Therefore, the claim is a direct consequence of Lemma 3.5. □

As a consequence of Lemma 3.6 and Corollary 3.8 we get the following results:

Proposition 4.7 (Functional equation)**.** *Suppose that every $\mathfrak{p} \in \mathrm{ram}(B)$ is inert in E and that we can decompose $\mathfrak{f}(\pi_B) = \mathfrak{n}_1\mathfrak{n}_2$ with $\mathfrak{n}_1$ coprime to $\mathfrak{m}$ and $\mathfrak{n}_2$ dividing $\mathfrak{m}$. Let ε be the root number of π and $\varepsilon_{\mathfrak{n}_2}$ the product of the local root numbers of primes dividing $\mathfrak{n}_2$. Then the equality*

$$(\Theta_\mathfrak{m}(L/F, \pi_B)^\epsilon)^\vee = \epsilon(-1) \cdot \varepsilon \cdot \varepsilon_{\mathfrak{n}_2} \cdot \Theta_\mathfrak{m}(L/F, \pi_B)^\epsilon$$

holds up to multiplication with an element in $\mathcal{G}$.

Proof. We have to convince ourself that this is a consequence of Lemma 3.6.

Let us decompose ε into the local ε-factors. The part corresponding to the finite places is given by $\varepsilon_{\mathfrak{n}_1} \cdot \varepsilon_{\mathfrak{n}_2} \cdot \prod_{\mathfrak{p}\in \mathrm{ram}(B)} \varepsilon_{\mathfrak{p}}$. In case that B is non-split, the passage from π to π_B yields $\varepsilon_{\mathfrak{p}} = -\varepsilon_{\mathfrak{p}}^B$, where $\varepsilon_{\mathfrak{p}}^B$ is the Atkin-Lehner eigenvalue corresponding to π_B (see the definition of π_B).

For a infinite place v of F, the ε-factor ε_v is equal to -1. If v is real, this is well known (e.g. see [Gel75], Theorem 6.16). For a complex place we state a proof: The local representation is given by $\pi_v = \pi(\mu_1, \mu_2)$ with $\mu_1(z) = z^{\frac{1}{2}}\overline{z}^{-\frac{1}{2}}$ and $\mu_2(z) = z^{-\frac{1}{2}}\overline{z}^{\frac{1}{2}}$. By Definition, ε_v is given as a product $\varepsilon(\mu_1)\cdot\varepsilon(\mu_2)$ (see [JL70], p. 118). Using a shifting formula (see [Kud03]), (3.28)) we see that we can replace μ_1 by $\widetilde{\mu}_1 = z^0\overline{z}^{-1}$ and μ_2 by $\widetilde{\mu}_2 = z^{-1}\overline{z}^0$ without changing the ε-factors. In this situation Proposition 3.8 (iv) of [Kud03] states that $\varepsilon(\widetilde{\mu_1}) = \varepsilon(\widetilde{\mu_2}) = i$, so we get $\varepsilon_v = -1$.

Altogether, for B a non-split quaternion algebra we get

$$\begin{aligned}\varepsilon\cdot\varepsilon_{\mathfrak{n}_2} &= \varepsilon_{\mathfrak{n}_1}\cdot \prod_{\mathfrak{p}\in\mathrm{ram}(B)} \varepsilon_{\mathfrak{p}}^B \cdot (-1)^{\#\,\mathrm{ram}(B)}\cdot(-1)^{\# S_\infty}\\ &= \varepsilon_{\mathfrak{n}_1}\cdot \prod_{\mathfrak{p}\in\mathrm{ram}(B)} \varepsilon_{\mathfrak{p}}^B \cdot (-1)^{\#\{v\in S_\infty \text{ s.t. } B_v \text{ is split}\}}\\ &= \varepsilon_{\mathfrak{n}_1}\cdot \prod_{\mathfrak{p}\in\mathrm{ram}(B)} \varepsilon_{\mathfrak{p}}^B \cdot (-1)^{d}\end{aligned}$$

using the fact that the total number of ramified places in B is even and the conditions on the splitting behaviour of B. On the other hand, if B is split, we have $\#S_\infty = d + 1$. The claim follows. □

Corollary 4.8 (Parity)**.** *Suppose that every $\mathfrak{p} \in \mathrm{ram}(B)$ is unramified in E and that there is a decomposition $\mathfrak{f}(\pi_B) = \mathfrak{n}_1\mathfrak{n}_2$ with $\mathfrak{n}_1$ coprime to $\mathfrak{m}$ and $\mathfrak{n}_2$ dividing $\mathfrak{m}$. Moreover, we assume that $r = \mathrm{ord}_{R_\pi}(\Theta_{\mathfrak{m}}(L/F, \pi_B)^\epsilon) < \infty$ holds and that 2 acts invertibly on $I_{R_\pi}(\mathcal{G})^r/I_{R_\pi}(\mathcal{G})^{r+1}$. Then we have*

$$(-1)^r = \epsilon(-1)\cdot\varepsilon\cdot\varepsilon_{\mathfrak{n}_2}.$$

4.2 Interpolation formulae

We relate Stickelberger elements to special values of L-functions. For this, we will use different approaches for non-split and split quaternion algebras. To be more precise, in case that B is non-split we get a formula for $\mathfrak{L}_{\mathfrak{m}}(L/F, \pi_B)^\epsilon$ (see Theorem 4.9) using a computation of toric period integrals by File, Martin and Pitale (cf. [FMP]) as a crucial input whereas we get a formula for $\Theta_{\mathfrak{m}}(L/F, \pi_B)^\epsilon$ in the split case using concrete calculations which we will carry out (see Theorem 4.10). We keep the notations from the previous section.

Given any automorphic representation $\widetilde{\pi}$ of a reductive algebraic group over F and a finite set S of places of F we write $L^S(s,\widetilde{\pi})$ for the L-function without the Euler factors at places in S and $L_S(s,\widetilde{\pi})$ for the product of the Euler factors of places in S. In particular, if $S = S_{\mathfrak{m}}$ for a non-zero ideal $\mathfrak{m} \subseteq \mathcal{O}_F$, we write $L^{(\mathfrak{m})}(s,\widetilde{\pi})$ for $L^{S_{\mathfrak{m}}}(s,\widetilde{\pi})$.

Let us start with Stickelberger elements associated to π_B with B a non-split quaternion algebra. Let $\chi_{E/F}\colon \mathrm{Gal}(E/F) \to \mathbb{C}^*$ be the non-trivial character. Given a character $\chi\colon \mathcal{G} \to \mathbb{C}^*$ and a finite place $\mathfrak{p}$ of F we denote by $\varepsilon(1/2,\pi_{E,\mathfrak{p}}\otimes\chi_{\mathfrak{p}})$ the local epsilon factor of the base change of π to $\mathrm{PGL}_2(E)$ twisted by χ. Here we view characters χ as characters on $T(\mathbb{A})$ via the Artin reciprocity map. We say that χ fulfills the Saito-Tunnell condition with respect to B if for all finite places $\mathfrak{p}$ of F the following equality holds:

$$\varepsilon(1/2,\pi_{E,\mathfrak{p}}\otimes\chi_{\mathfrak{p}}) = \chi_{E/F,\mathfrak{p}}(-1)\,\mathrm{inv}(B_{\mathfrak{p}})$$

Here $\mathrm{inv}(B_{\mathfrak{p}}) \in \{\pm 1\}$ denotes the local invariant of B at $\mathfrak{p}$. By our assumptions on the splitting behaviour of B there is no condition at the Archimedean places.

Let $S(\pi)$ be the set of finite places at which π is ramified. For a character χ as above we set $S(\chi)$ to be the set of finite places at which χ is ramified. Finally, let $\Sigma(\pi,\chi)$ be the set of all finite places $\mathfrak{p}$ such that either the local conductor of π at $\mathfrak{p}$ is greater than one or the local conductor of π at $\mathfrak{p}$ is exactly one, E over F is ramified at $\mathfrak{p}$ and $\chi_{\mathfrak{p}}$ is unramified. The ramification index of E over F at a prime $\mathfrak{p}$ will be denoted by $e_{\mathfrak{p}}(E_{\mathfrak{p}}/F_{\mathfrak{p}})$.

Theorem 4.9. *There exists a constant $C \in \mathbb{C}^*$ such that for all $\mathfrak{f}(\pi_B)$-allowable moduli $\mathfrak{m}$ and all characters $\chi\colon \mathcal{G} \to \mathbb{C}^*$ of exact conductor $\mathfrak{m}$ with $\chi_\infty = \epsilon$ we have*

$$\begin{aligned}\chi(\mathfrak{L}_{\mathfrak{m}}(L/F,\pi_B)^{\epsilon}) = C\frac{[U_T : U_T(\mathfrak{m})]^2}{N(\mathfrak{m})} L_{S(\chi)}(1,\eta) L_{S(\pi)\cup S(\chi)}(1,\eta)\\ \times L_{S(\pi)\cap S(\chi)}(1,1_F) \prod_{\mathfrak{p}\in S(\pi)\cap S(\chi)^c} e_{\mathfrak{p}}(E_{\mathfrak{p}}/F_{\mathfrak{p}}) \cdot \frac{L^{\Sigma(\pi,\chi)}(1/2,\pi_E\otimes\chi)}{L^{\Sigma(\pi,\chi)}(1,\pi,\mathrm{Ad})},\end{aligned}$$

if χ fulfills the Saito-Tunnell condition and

$$\chi(\mathfrak{L}_{\mathfrak{m}}(L/F,\pi_B)^{\epsilon}) = 0$$

if χ does not fulfill the Saito-Tunnell condition.

Proof. By the strong approximation theorem there exist finitely many elements $g_1, \ldots, g_n \in G(\mathbb{A}^\infty)$ such that

$$G(A^\infty) = \bigcup_{i=1}^{n} G(F) g_i K(f(\pi_B)).$$

Note that $\mathcal{A}(f(\pi_B); \mathbb{C})(\epsilon) = \mathrm{Coind}_{K(f(\pi_B))}^{G(\mathbb{A}^\infty)} \mathbb{C}(\epsilon)$ holds, where $\mathrm{Coind}_K^{G(\mathbb{A}^\infty)} \mathbb{C}(\epsilon)$ is the coinduction of $\mathbb{C}(\epsilon)$ from $K(f(\pi_B))$ to $G(\mathbb{A}^\infty)$, i.e. it is the space of $K(f(\pi_B))$-invariant functions from $G(\mathbb{A}^\infty)$ to $\mathbb{C}(\epsilon)$.

Let us write $\Gamma_i = G(F) \cap g_i K(f(\pi_B)) g_i^{-1}$. Then Shapiro's Lemma yields

$$\mathcal{M}(f(\pi_B); \mathbb{C})^\epsilon \cong \bigoplus_{i=1}^{n} H^d(\Gamma_i, \mathbb{C}(\epsilon)).$$

The group cohomology of a discrete group is naturally isomorphic to the singular cohomology of its associated classifying space. We choose a torsion-free normal subgroup $\Gamma_i' \subseteq \Gamma_i$ of finite index. Then the classifying space $K(\Gamma_i', 1)$ is isomorphic to $\Gamma_i' \backslash X_G$. Here X_G is given by $G(F_\infty)/K_\infty$, where K_∞ is a maximal compact subgroup of $G(F_\infty)$. We can choose $K_\infty = \prod_{v \in S_\infty} K_{F_v}$ with $K_{F_v} = \mathrm{SO}(2)$ if v is real and split, $K_{F_v} = G_v$ if v is real and non-split and $K_{F_v} = \mathrm{SU}(2)$ if v is complex. This gives us

$$X_G \cong \prod_{\substack{v \in S_\infty \\ v \text{ split in } G}} \mathbb{H}_{F_v}$$

with $\mathbb{H}_{F_v}$ being the (usual) complex upper half-plane if v is real and $\mathbb{H}_{F_v} = \mathbb{C} \times \mathbb{R}_+^*$ the upper half-space of dimension 3 if v is complex.

The space X_G is a differentiable manifold and so the singular cohomology of $\Gamma_i' \backslash X_G$ with complex coefficients is isomorphic to its de Rham cohomology, i.e. we have

$$\mathrm{H}^d(\Gamma_i', \mathbb{C}(\epsilon)) \xrightarrow{\cong} \mathrm{H}^d_{\mathrm{dR}}(\Gamma_i' \backslash X_G, \mathbb{C}(\epsilon)).$$

Since Γ_i / Γ_i' is invertible on $\mathbb{C}(\epsilon)$ we also get this isomorphism with Γ_i' replaced by Γ_i. In total, we have an isomorphism

$$\begin{aligned}
\mathrm{ES}\colon \mathcal{M}(f(\pi_B); \mathbb{C})^\epsilon &\xrightarrow{\cong} \bigoplus_{i=1}^{n} \mathrm{H}^d_{\mathrm{dR}}(\Gamma_i \backslash X_G, \mathbb{C}(\epsilon)) \\
&\xrightarrow{\cong} \mathrm{H}^d_{\mathrm{dR}}(\bigcup_{i=1}^{n} \Gamma_i \backslash X_G, \mathbb{C}(\epsilon)) \\
&\xrightarrow{\cong} \mathrm{H}^d_{\mathrm{dR}}(G(F) \backslash (G(\mathbb{A}^\infty)/K(f(\pi_B)) \times X_G), \mathbb{C}(\epsilon)).
\end{aligned}$$

By Matsushima's formula the latter space is generated by cohomological automorphic forms. In particular, the image of $\kappa^{\pi_B,\epsilon}$ under ES is the differential form associated with a global (cohomological) new vector Φ of π_B. The fact that the above identifications of cohomology groups behave well under pullback and cup products together with Remark 1.2 implies that

$$\chi(\Theta_{\mathfrak{m}}(L/F,\pi_B)^{\epsilon}) = [U_T : U_T(\mathfrak{m})] P_B(g_{\mathfrak{m}}.\Phi,\chi) \tag{4.1}$$

holds up to multiplication by a non-zero constant, which is independent of χ and $\mathfrak{m}$. Here $g_{\mathfrak{m}} \in G(\mathbb{A}^\infty)$ is (a lift of) the element chosen at the end of Section 2.3 and

$$P_B(\phi,\chi) = \int_{T(F)\backslash T(\mathbb{A})} \phi(t)\chi(t)\; dt$$

denotes the global toric period integral of $\phi \in \pi$. Therefore, we also get the formula

$$\chi(\mathfrak{L}_{\mathfrak{m}}(L/F,\pi_B)^{\epsilon}) = [U_T : U_T(\mathfrak{m})]^2\,|P_B(g_{\mathfrak{m}}.\Phi,\chi)|^2$$

up to multiplication with a non-zero constant.

The second assertion follows from the vanishing criterion of toric periods integrals by Saito and Tunnell (see [Sai93] and [Tun83]). Since $g_{\mathfrak{m}}.\Phi$ is a test vector in the sense of [FMP], § 7.1, the first assertion follows from the main theorem of *loc.cit.* □

Now let us assume that B is split. As before, we will identify G with PGL_2 in this situation. In particular, we use the identifications introduced in Remark 2.6 and 2.13. In the following the Haar measure $dx = \prod_v dx_v$ on $T(\mathbb{A}) = \mathbb{A}^*$ is normalized such that $\mathrm{vol}(U_{\mathfrak{p}}, dx_{\mathfrak{p}}) = 1$ for all finite places $\mathfrak{p}$ of F. Let us fix a non-zero character $\psi\colon F\backslash\mathbb{A} \to \mathbb{C}^*$.

Theorem 4.10. *There exists a constant $C \in \mathbb{C}^*$ such that for all moduli $\mathfrak{m}$ and all characters $\chi\colon \mathbb{A}^* \to \mathcal{G} \to \mathbb{C}^*$ of conductor $\mathfrak{m}$ with $\chi_\infty = \epsilon$ we have*

$$\chi(\Theta_{\mathfrak{m}}(L/F,\pi_B)^{\epsilon}) = C\; \tau(\chi^{-1}) L^{(\mathfrak{m})}(1/2, \pi_B \otimes \chi).$$

Here $\tau(\chi^{-1}) = \tau(\chi^{-1},\psi,dx)$ is the Gauss sum of χ with respect to our choice of a Haar measure dx on $\mathbb{A}^*$ and the additive character ψ.

Proof. Following the same lines as in the proof of Theorem 4.9 we see that (4.1) holds in this situation as well, i.e. there exists $c \in \mathbb{C}^*$ such that we have

$$\begin{aligned}\chi(\Theta_{\mathfrak{m}}(L/F,\pi_B)^{\epsilon}) &= c\cdot [U_T : U_T(\mathfrak{m})] P_B(g_{\mathfrak{m}}.\Phi,\chi)\\ &= c\cdot [U : U(\mathfrak{m})] \int_{F^*\backslash\mathbb{A}^*} \Phi\left(\begin{pmatrix} x & 0\\ 0 & 1\end{pmatrix} g_{\mathfrak{m}}\right)\chi(x)dx\end{aligned}$$

for a global (cohomological) new vector Φ of $\pi = \pi_B$. For $s \in \mathbb{C}$ the integral

$$\int_{F^*\backslash\mathbb{A}^*} \Phi\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g_{\mathfrak{m}}\right) \chi(x)|x|^s dx$$

defines a holomorphic function. Let W denote the ψ-Whittaker function of $\mathcal{R}(g_{\mathfrak{m}})\Phi$, the vector obtained by right multiplication by $g_{\mathfrak{m}}$. Since $\mathcal{R}(g_{\mathfrak{m}})\Phi \in \pi$ is a pure tensor we can factor W as a product of local Whittaker functions W_v. For $\Re(s)$ large we can unfold the above integral to get

$$\begin{aligned}\int_{F^*\backslash\mathbb{A}^*} \Phi\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g_{\mathfrak{m}}\right) \chi(x)|x|^s dx &= \int_{\mathbb{A}^*} W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) \chi(x) dx \\ &= \prod_v \int_{F_v^*} W_v\left(\begin{pmatrix} x_v & 0 \\ 0 & 1 \end{pmatrix}\right) \chi_v(x_v) dx_v.\end{aligned}$$

Therefore, we are reduced to a computation of local integrals, which we will carry out in the rest of this section. □

Let $\mathfrak{p}$ be a finite place of F and $\pi_\mathfrak{p}$ an infinite dimensional, irreducible, smooth representations of $G(F_\mathfrak{p})$ of conductor $\mathfrak{p}^n$, i.e. we have

$$\dim_{\mathbb{C}} \pi_\mathfrak{p}^{K_\mathfrak{p}(\mathfrak{p}^n)} = 1.$$

The non-zero elements of $\pi_\mathfrak{p}^{K_\mathfrak{p}(\mathfrak{p}^n)}$ are called local newforms. Let Λ be a $\psi_\mathfrak{p}$-Whittaker functional of $\pi_\mathfrak{p}$. By definition Λ is a non-zero linear functional on $\pi_\mathfrak{p}$ such that

$$\Lambda\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \varphi\right) = \psi_\mathfrak{p}(x)\Lambda(\varphi)$$

for all $\varphi \in \pi_\mathfrak{p}$ and all $x \in F_\mathfrak{p}$.

Lemma 4.11. *Let $\varphi \in \pi_\mathfrak{p}$ be a local newform. For every character $\chi_\mathfrak{p}\colon F_\mathfrak{p}^* \to \mathbb{C}^*$ of conductor $\mathfrak{p}^m$ the following integral converges for $\Re(s)$ large and we have an equality*

$$\begin{aligned}&[U_\mathfrak{p} : U_\mathfrak{p}^{(m)}] \int_{F_\mathfrak{p}^*} \Lambda\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varpi_\mathfrak{p}^m & 1 \\ 0 & 1 \end{pmatrix} \varphi\right) \chi_\mathfrak{p}(x)|x|_\mathfrak{p}^s dx \\ =&c \cdot \tau(\chi_\mathfrak{p}^{-1}, \psi_\mathfrak{p}) N(\mathfrak{p})^{(t+m)s} L^{(\mathfrak{p}^m)}(s + 1/2, \pi_\mathfrak{p} \otimes \chi_\mathfrak{p}),\end{aligned}$$

where $\varpi_\mathfrak{p}$ is a local uniformizer at $\mathfrak{p}$, $c \in \mathbb{C}$ and $t \in \mathbb{Z}$ are constants independent of $\chi_\mathfrak{p}$ and m and

$$L^{(\mathfrak{p}^m)}(s, \pi_\mathfrak{p} \otimes \chi_\mathfrak{p}) = \begin{cases} L(s, \pi_\mathfrak{p} \otimes \chi_\mathfrak{p}) & \textit{if } m = 0, \\ 1 & \textit{if } m > 0. \end{cases}$$

Proof. Let $\mathfrak{p}^{-t}$ be the conductor of $\psi_{\mathfrak{p}}$. A straightforward calculation shows that

$$\Lambda'(\varphi) = \Lambda\left(\begin{pmatrix} \varpi_{\mathfrak{p}}^{-t} & 0 \\ 0 & 1 \end{pmatrix}\varphi\right)$$

defines a Whittaker functional with respect to an additive character $\psi'_{\mathfrak{p}}$ of conductor $\mathcal{O}_{\mathfrak{p}}$. It is well known that

$$\Lambda'\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\varphi\right) = 0$$

if $\mathrm{ord}_{\mathfrak{p}}(x) < 0$ and equal to a non-zero complex number $c \in \mathbb{C}^*$ for $\mathrm{ord}_{\mathfrak{p}}(x) = 0$ (see for example [Miy14]). Without loss of generality we may assume that $c = 1$. Hence, for $\Re(s)$ large we have the following equality

$$\begin{aligned}
&\int_{F_{\mathfrak{p}}^*} \Lambda\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \varpi_{\mathfrak{p}}^{m} & 1 \\ 0 & 1 \end{pmatrix}\varphi\right)\chi_{\mathfrak{p}}(x)|x|_{\mathfrak{p}}^{s}dx \\
=&\int_{F_{\mathfrak{p}}^*} \Lambda'\left(\begin{pmatrix} x\varpi_{\mathfrak{p}}^{t} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \varpi_{\mathfrak{p}}^{m} & 1 \\ 0 & 1 \end{pmatrix}\varphi\right)\chi_{\mathfrak{p}}(x)|x|_{\mathfrak{p}}^{s}dx \\
=&\int_{F_{\mathfrak{p}}^*} \Lambda'\left(\begin{pmatrix} 1 & x\varpi_{\mathfrak{p}}^{t} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x\varpi_{\mathfrak{p}}^{m+t} & 0 \\ 0 & 1 \end{pmatrix}\varphi\right)\chi_{\mathfrak{p}}(x)|x|_{\mathfrak{p}}^{s}dx \\
=&\int_{F_{\mathfrak{p}}^*} \Lambda'\left(\begin{pmatrix} x\varpi_{\mathfrak{p}}^{m+t} & 0 \\ 0 & 1 \end{pmatrix}\varphi\right)\psi'_{\mathfrak{p}}(x\varpi_{\mathfrak{p}}^{t})\chi_{\mathfrak{p}}(x)|x|_{\mathfrak{p}}^{s}dx \\
=&\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}}^{-t})|\varpi_{\mathfrak{p}}^{-t}|_{p}^{s}\sum_{k=0}^{\infty}\Lambda'\left(\begin{pmatrix} \varpi_{\mathfrak{p}}^{k} & 0 \\ 0 & 1 \end{pmatrix}\varphi\right)\int_{\varpi_{\mathfrak{p}}^{k-m}U_{\mathfrak{p}}}\psi'_{\mathfrak{p}}(x)\chi_{\mathfrak{p}}(x)|x|_{\mathfrak{p}}^{s}dx.
\end{aligned}$$

By classical formulas for the Whittaker functional of a newform (see for example [Miy14]) we have

$$\Lambda'\left(\begin{pmatrix} \varpi_{\mathfrak{p}}^{k} & 0 \\ 0 & 1 \end{pmatrix}\varphi\right) = |\varpi_{\mathfrak{p}}^{k}|_{\mathfrak{p}}^{1/2}\sum_{\substack{r+s=k \\ r,s\geq 0}}\alpha_1^{r}\alpha_2^{s},$$

where $\alpha_i \in \mathbb{C}$, $1 \leq i \leq 2$, are the complex numbers such that

$$L(s,\pi_{\mathfrak{p}}) = \prod_{i=1}^{2}(1-\alpha_i|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{s})^{-1}.$$

Therefore, if $m=0$, we obtain

$$\begin{aligned}
&\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}}^{-t})|\varpi_{\mathfrak{p}}^{-t}|_{\mathfrak{p}}^{s}\sum_{k=0}^{\infty}\Lambda'\left(\begin{pmatrix}\varpi_{\mathfrak{p}}^{k} & 0\\ 0 & 1\end{pmatrix}\varphi\right)\int_{\varpi_{\mathfrak{p}}^{k}U_{\mathfrak{p}}}\psi'_{\mathfrak{p}}(x)\chi_{\mathfrak{p}}(x)|x|_{\mathfrak{p}}^{s}dx\\
=&\tau(\chi_{\mathfrak{p}}^{-1},\psi_{\mathfrak{p}})N(\mathfrak{p})^{ts}\sum_{k=0}^{\infty}\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}}^{k})|\varpi_{\mathfrak{p}}^{k}|_{\mathfrak{p}}^{s+1/2}\left(\sum_{\substack{r+s=k\\ r,s\geq 0}}\alpha_1^{r}\alpha_2^{s}\right)\\
=&\tau(\chi_{\mathfrak{p}}^{-1},\psi_{\mathfrak{p}})N(\mathfrak{p})^{ts}\prod_{i=1}^{2}(1-\alpha_i\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}})|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{s+1/2})^{-1}\\
=&\tau(\chi_{\mathfrak{p}}^{-1},\psi_{\mathfrak{p}})N(\mathfrak{p})^{ts}L(s+1/2,\pi_{\mathfrak{p}}\otimes\chi_{\mathfrak{p}}).
\end{aligned}$$

In the case $m\geq 1$ we can use Lemma 2.2 of [Spi14] to get

$$\begin{aligned}
&\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}}^{-t})|\varpi_{\mathfrak{p}}^{-t}|_{p}^{s}\sum_{k=0}^{\infty}\Lambda'\left(\begin{pmatrix}\varpi_{\mathfrak{p}}^{k} & 0\\ 0 & 1\end{pmatrix}\varphi\right)\int_{\varpi_{\mathfrak{p}}^{k-m}U_{\mathfrak{p}}}\psi'_{\mathfrak{p}}(x)\chi_{\mathfrak{p}}(x)|x|_{\mathfrak{p}}^{s}dx\\
=&\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}}^{-t})|\varpi_{\mathfrak{p}}^{-t}|_{p}^{s}\int_{\varpi_{\mathfrak{p}}^{-m}U_{\mathfrak{p}}}\psi'_{\mathfrak{p}}(x)\chi_{\mathfrak{p}}(x)|x|_{\mathfrak{p}}^{s}dx\\
=&[U_{\mathfrak{p}}:U_{\mathfrak{p}}^{(m)}]^{-1}\tau(\chi_{\mathfrak{p}}^{-1},\psi_{\mathfrak{p}})N(\mathfrak{p})^{(t+m)s}
\end{aligned}$$

and thus the claim follows. □

Remark 4.12. If $\pi_{\mathfrak{p}}$ is an unramified principal series, an unramified twist of the Steinberg representation or a supercuspidal representation and $\chi_{\mathfrak{p}}\colon F_{\mathfrak{p}}^{*}\to\mathbb{C}^{*}$ is a character of conductor $\mathfrak{p}^{m}$ with $m\geq 1$, then the local Euler factor $L(s,\pi_{\mathfrak{p}}\otimes\chi_{\mathfrak{p}})$ is equal to 1.

4.3 Leading terms

In Section 3.7 of [Spi14], Spieß constructs extensions of the Steinberg representation associated to characters of the multiplicative group of a p-adic field. Such extensions were already constructed by Breuil in [Bre04] in case the character under consideration is a branch of the p-adic logarithm. After introducing a slightly improved version of Spieß' construction we will use it to give formulas for the leading term of Stickelberger elements in the analytic rank zero situation.

Let us fix a finite place $\mathfrak{p}$ of F which is split in E. Further, let R be a ring and N a prodiscrete R-module. We define the N-valued Steinberg representation by

$$\mathrm{St}_{\mathfrak{p}}(N)=C(\mathrm{Ends}_{\mathfrak{p}},N)/N.$$

A continuous homomorphism $f\colon N \to N'$ between prodiscrete R-modules induces a homomorphism

$$f_*\colon \mathrm{St}_\mathfrak{p}(N) \longrightarrow \mathrm{St}_\mathfrak{p}(N').$$

The canonical map

$$\mathrm{St}_\mathfrak{p} \otimes N \longrightarrow \mathrm{St}_\mathfrak{p}(N)$$

is an isomorphism if N is discrete. In this case, the map (2.6) induces a $T_\mathfrak{p}$-equivariant isomorphism

$$\delta_{\mathfrak{p},N}\colon C_c^0(F_\mathfrak{p}, N) \longrightarrow \mathrm{St}_\mathfrak{p}(N).$$

Let $U_\mathfrak{p}$ be the unipotent radical of $\mathrm{Stab}_{B_\mathfrak{p}^*}(o_{\mathfrak{P}^\tau})$, i.e. we have $\mathrm{Stab}_{B_\mathfrak{p}^*}(o_{\mathfrak{P}^\tau}) = E_\mathfrak{p}^*\, U_\mathfrak{p}$. As before, if E is a field, the choice of the prime $\mathfrak{P}$ lying above $\mathfrak{p}$ gives rise to an identification $E_\mathfrak{p}^* \cong F_\mathfrak{p}^* \times F_\mathfrak{p}^*$. Otherwise the isomorphism comes naturally. For a continuous homomorphism $l_\mathfrak{p}\colon F_\mathfrak{p}^* \to N$ we define $\widetilde{\mathcal{E}}(l_\mathfrak{p})$ as the set of pairs $(\varphi, y) \in C(B_\mathfrak{p}^*, N) \times R$ with

$$\varphi\left(gu(t_1, t_2)\right) = \varphi(g) + y \cdot l_\mathfrak{p}(t_1)$$

for all $g \in B_\mathfrak{p}^*$, $u \in U_\mathfrak{p}$ and $(t_1, t_2) \in F_\mathfrak{p}^* \times F_\mathfrak{p}^* \cong E_\mathfrak{p}^*$. The group $B_\mathfrak{p}^*$ acts on $\widetilde{\mathcal{E}}(l_\mathfrak{p})$ via

$$g.(\varphi(h), y) = (\varphi(g^{-1}h), y).$$

The subspace $\widetilde{\mathcal{E}}(l_\mathfrak{p})_0$ of tuples of the type $(\varphi, 0)$ with constant φ is $B_\mathfrak{p}^*$-invariant. Hence, we get an induced action of $G_\mathfrak{p}$ on the quotient $\mathcal{E}(l_\mathfrak{p}) = \widetilde{\mathcal{E}}(l_\mathfrak{p})/\widetilde{\mathcal{E}}(l_\mathfrak{p})_0$.

Lemma 4.13. *(i) Let $\pi\colon G_\mathfrak{p} \to \mathrm{Ends}_\mathfrak{p}$ be the projection given by $g \mapsto g[o_{\mathfrak{P}^\tau}]$. The following sequence of $R[G_\mathfrak{p}]$-modules is exact:*

$$0 \longrightarrow \mathrm{St}_\mathfrak{p}(N) \xrightarrow{(\pi^*,0)} \mathcal{E}(l_\mathfrak{p}) \xrightarrow{(0,\mathrm{id}_R)} R \longrightarrow 0$$

We define $b_{l_\mathfrak{p}}$ to be the associated cohomology class in $\mathrm{H}^1(G_\mathfrak{p}, \mathrm{St}_\mathfrak{p}(N))$.

(ii) For every continuous homomorphism $f\colon N \to N'$ between prodiscrete R-modules the equality

$$b_{f\circ l_\mathfrak{p}} = f_*(b_{l_\mathfrak{p}})$$

holds.

(iii) Suppose that N is discrete. Then, for the cohomology class $c_{l_\mathfrak{p}}$ defined in (1.4) we have

$$\delta_{\mathfrak{p},N}^*(b_{l_\mathfrak{p}}) = c_{l_\mathfrak{p}}.$$

Proof. Parts (i) and (iii) are essentially proven in Lemma 3.11 of [Spi14]. For the proof of (ii) let $f_*(\mathcal{E}(l_\mathfrak{p}))$ be the pushout of the following diagram:

$$\begin{array}{ccc} \mathrm{St}_\mathfrak{p}(N) & \xrightarrow{(\pi^*,0)} & \mathcal{E}(l_\mathfrak{p}) \\ {\scriptstyle f_*}\big\downarrow & & \\ \mathrm{St}_\mathfrak{p}(N') & & \end{array}$$

The homomorphism

$$\widetilde{\mathcal{E}}(l_\mathfrak{p}) \longrightarrow \widetilde{\mathcal{E}}(f \circ l_\mathfrak{p}),\ (\varphi, y) \longmapsto (f \circ \varphi, y)$$

induces a map from $f_*(\mathcal{E}(l_\mathfrak{p}))$ to $\mathcal{E}(f \circ l_\mathfrak{p})$. Hence, they yield isomorphic extensions. □

Remark 4.14. Note that we get rid of the factor 2 showing up in Lemma 3.11 of [Spi14], i.e. the extension class constructed above is "one half" of the extension class constructed in *loc.cit.*

Let π be an automorphic representation as in Section 4.1 and let $S_{\mathrm{St}}, S_{\mathrm{tw}}$ be disjoint finite sets of finite places of F disjoint from $\mathrm{ram}(B)$ such that every $\mathfrak{p}$ in S_{tw} is inert in E. Then for every prodiscrete R-module N and every $\mathfrak{p} \in S_{\mathrm{St}}$ the integration pairing

$$\mathrm{Hom}(\mathrm{St}_\mathfrak{p}, R) \otimes \mathrm{St}_\mathfrak{p}(N) \longrightarrow N,$$

which was defined in (0.3), induces a cup product pairing

$$\begin{aligned} &\mathcal{M}(f(\pi_B), S_{\mathrm{St}}, S_{\mathrm{tw}}; R)^\epsilon \otimes \mathrm{H}^1(G(F), \mathrm{St}_\mathfrak{p}(N)) \\ &\xrightarrow{\cup} \mathrm{H}^{d+1}(G(F), \mathcal{A}(f(\pi_B), S_{\mathrm{St}} - \{\mathfrak{p}\}, S_{\mathrm{tw}}; N)^{\{\mathfrak{p}\}}(\epsilon)). \end{aligned} \tag{4.2}$$

As a direct consequence of Lemma 4.13 (iii) we get

Corollary 4.15. *Let R be a ring, N an R-module and $l\colon T_\mathfrak{p} \longrightarrow N$ a locally constant character. For $\mathfrak{p} \in S_{\mathrm{St}}$ the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{M}(f(\pi_B), S_{\mathrm{St}}, S_{\mathrm{tw}}; R)^\epsilon & \xrightarrow{\cup b_{l_\mathfrak{p}}} & \mathrm{H}^{d+1}(G(F), \mathcal{A}(f(\pi_B), S_{\mathrm{St}} - \{\mathfrak{p}\}, S_{\mathrm{tw}}; N)^{\{\mathfrak{p}\}}(\epsilon)) \\ {\scriptstyle \Delta_{\mathfrak{m}, S_{\mathrm{St}}, S_{\mathrm{tw}}}}\big\downarrow & & \big\downarrow {\scriptstyle \Delta^{\{\mathfrak{p}\}}_{\mathfrak{m}, S_{\mathrm{St}}-\{\mathfrak{p}\}, S_{\mathrm{tw}}}} \\ \mathrm{H}^d(T(F), \mathcal{D}(\mathfrak{m}, S; R)^\infty(\epsilon)) & \xrightarrow{\cup c_{l_\mathfrak{p}}} & \mathrm{H}^{d+1}(T(F), \mathcal{D}(\mathfrak{m}, S - \{\mathfrak{p}\}; N)^{\{\mathfrak{p}\},\infty}(\epsilon)) \end{array}$$

From now on, let S be the set of finite places $\mathfrak{p}$ of F which are disjoint from $\mathrm{ram}(B)$, split in E and such that the local component $\pi_\mathfrak{p}$ is the Steinberg representation. To ease the notation, we are going to write $\mathcal{M}(f(\pi_B), S; N)$ instead of $\mathcal{M}(f(\pi_B), S, \emptyset; N)$ etc. For simplicity we assume that R_π is a principal ideal domain.

Let L be a finite Galois extension of F which is E-anticyclotomic (resp. abelian) if B is a non-split (resp. split) quaternion algebra. Write $\mathcal{G}$ for the Galois group of L over E (resp. F). We denote by $I_\mathcal{G}$ the augmentation ideal of $R_\pi[\mathcal{G}]$, i.e. the kernel of the projection $R_\pi[\mathcal{G}] \twoheadrightarrow R_\pi$. For $\mathfrak{p} \in S$ we denote the local reciprocity map by $\mathrm{rec}_\mathfrak{p}$, i.e.

$$\mathrm{rec}_\mathfrak{p} \colon T_\mathfrak{p} \hookrightarrow T(\mathbb{A}) \xrightarrow{\mathrm{r}_L} \mathcal{G}.$$

We also consider the homomorphism

$$\mathrm{univ}_\mathfrak{p} \colon T_\mathfrak{p} \to F_\mathfrak{p}^* \otimes R_\pi \tag{4.3}$$

given by composing the isomorphism $T_\mathfrak{p} \cong F_\mathfrak{p}^*$ with the inclusion of $F_\mathfrak{p}^*$ into $F_\mathfrak{p}^* \otimes R_\pi$. Given an R_π-module N and a subset $\mathfrak{S} \subseteq S$ we let

$$\mathcal{M}(\mathfrak{f}(\pi_B), \mathfrak{S}; N)^{\epsilon,\pi} \subseteq \mathcal{M}(\mathfrak{f}(\pi_B), \mathfrak{S}; N)^{\epsilon}$$

be the the submodule on which $\mathbb{T}_\mathfrak{p}$ acts via $\lambda_\mathfrak{p}$ for all $\mathfrak{p} \notin \mathfrak{S} \cup \mathrm{ram}(B)$. Exactly as in Lemma 6.2 of [Spi14], one can prove that for every $\mathfrak{p} \in \mathfrak{S}$ the map

$$\cup b_{\mathrm{ord}_p} \colon \mathcal{M}(\mathfrak{f}(\pi_B), \mathfrak{S}; R_\pi)^{\epsilon,\pi} \longrightarrow \mathrm{H}^{d+1}(G(F), \mathcal{A}(\mathfrak{f}(\pi_B), \mathfrak{S} - \{\mathfrak{p}\}; R_\pi)^{\{\mathfrak{p}\}}(\epsilon))^\pi$$

has finite cokernel and that both modules are free of rank one modulo torsion.

By a theorem of Borel and Serre (cf. [BS76]) $S_\mathfrak{p}$-arithmetic groups are of type (VFL). It follows that $H^*(\Gamma, N)$ is finitely generated if N is a finitely generated R_π-module and that the functor $N \to H^*(\Gamma, N)$ commutes with direct limits. It follows that the canonical map

$$\begin{aligned}\omega_{\mathfrak{S},\mathfrak{p}} \colon \mathrm{H}^{d+1}(G(F), \mathcal{A}(\mathfrak{f}(\pi_B), \mathfrak{S} - \{\mathfrak{p}\}; R_\pi)^{\{\mathfrak{p}\}}(\epsilon))^\pi \otimes_{R_\pi} (F_\mathfrak{p}^* \otimes R_\pi) \\ \longrightarrow \mathrm{H}^{d+1}(G(F), \mathcal{A}(\mathfrak{f}(\pi_B), \mathfrak{S} - \{\mathfrak{p}\}; F_\mathfrak{p}^* \otimes R_\pi)^{\{\mathfrak{p}\}}(\epsilon))^\pi\end{aligned}$$

has finite kernel and cokernel. Let $\kappa_\mathfrak{S}$ be a generator of the maximal torsion-free quotient of $\mathcal{M}(\mathfrak{f}(\pi_B), \mathfrak{S}; R_\pi)^{\epsilon,\pi}$. For $\mathfrak{p} \in \mathfrak{S}$ we define $n_{\mathfrak{S},\mathfrak{p}}$ to be the lowest common multiple of the exponents of

- the cokernel of $\omega_{\mathfrak{S},\mathfrak{p}}$ and
- the torsion submodule of $\mathrm{H}^{d+1}(G(F), \mathcal{A}(\mathfrak{f}(\pi_B), \mathfrak{S} - \{\mathfrak{p}\}; R_\pi)^{\{\mathfrak{p}\}}(\epsilon))^\pi$.

Let $\gamma_{\mathfrak{S},\mathfrak{p}}$ be the order of the cokernel of the homomorphism

$$n_{\mathfrak{S},\mathfrak{p}}\mathcal{M}(\mathfrak{f}(\pi_B),\mathfrak{S};R_\pi)^{\epsilon,\pi} \xrightarrow{\cup b_{\mathrm{ord}_\mathfrak{p}}} n_{\mathfrak{S},\mathfrak{p}}\mathrm{H}^{d+1}(G(F),\mathcal{A}(\mathfrak{f}(\pi_B),\mathfrak{S}-\{\mathfrak{p}\};R_\pi)^{\{\mathfrak{p}\}}(\epsilon))^\pi.$$

Definition 4.16. An element $q_{\mathfrak{S},\mathfrak{p}} \in F_\mathfrak{p}^* \otimes R_\pi$ fulfilling

$$n_{\mathfrak{S},\mathfrak{p}}((\kappa_\mathfrak{S} \cup b_{\mathrm{ord}_\mathfrak{p}}) \otimes q_{\mathfrak{S},\mathfrak{p}}) = n_{\mathfrak{S},\mathfrak{p}}(\gamma_{\mathfrak{S},\mathfrak{p}} \cdot \kappa_\mathfrak{S} \cup b_{\mathrm{univ}_\mathfrak{p}}) \tag{4.4}$$

is called automorphic period of π at $\mathfrak{p}$ (with respect to $\mathfrak{S}$). If $\mathfrak{S} = \{\mathfrak{p}\}$, we simply write $q_\mathfrak{p} = q_{\{\mathfrak{p}\},\mathfrak{p}}$.

From the discussion above it follows that automorphic periods exist and are at least unique up to torsion. It is easy to see that the $\mathbb{Q}_\pi$-vector subspace generated by $q_{\mathfrak{S},\mathfrak{p}}$ in $F_\mathfrak{p}^* \otimes \mathbb{Q}_\pi$ is independent of $\mathfrak{S}$.

Theorem 4.17 (Leading term)**.** *For every extension L of F as above and every $\mathfrak{f}(\pi_B)$-allowable modulus $\mathfrak{m}$ that bounds the ramification of L over E (resp. F) the following equality holds in $I_\mathcal{G}^{\#S_\mathfrak{m}}/I_\mathcal{G}^{\#S_\mathfrak{m}+1}$ (up to sign):*

$$\begin{aligned} &n_{S_\mathfrak{m}}\left(\prod_{\mathfrak{p}\in S_\mathfrak{m}} n_{S_\mathfrak{m},\mathfrak{p}}\,\mathrm{ord}_\mathfrak{p}(q_{S_\mathfrak{m},\mathfrak{p}})\right)\Theta_\mathfrak{m}(L/F,\pi_B)^\epsilon \\ =&n_{S_\mathfrak{m}}\left(\prod_{\mathfrak{p}\in S_\mathfrak{m}} n_{S_\mathfrak{m},\mathfrak{p}}\,(\mathrm{rec}_\mathfrak{p}(q_{S_\mathfrak{m},\mathfrak{p}})-1)\right)\Theta_{\mathfrak{m}^S}(\diamondsuit/F,\pi_B)^\epsilon \end{aligned}$$

Here

$$\diamondsuit = \begin{cases} E & \text{if } B \text{ is non-split and} \\ F & \text{if } B \text{ is split,} \end{cases}$$

$\mathfrak{m}^S$ denotes the maximal divisor of $\mathfrak{m}$, which is coprime to S and $n_{S_\mathfrak{m}}$ is the non-zero integer defined in the discussion before Theorem 4.6.

Proof. As in the proof of Lemma 3.5 there exists $\kappa' \in \mathcal{M}(\mathfrak{f}(\pi_B),S_\mathfrak{m};R_\pi)^\epsilon$ with

$$n_{S_\mathfrak{m}}\Theta_\mathfrak{m}(L/F,\pi_B)^\epsilon = \Delta_{\mathfrak{m},S_\mathfrak{m}}(\kappa') \cap c_L(\mathfrak{m},S_\mathfrak{m},\epsilon).$$

If we apply Proposition 1.6 with $\mathfrak{a}_\mathfrak{p} = \mathfrak{a} = I_\mathcal{G}$, we get

$$n_{S_\mathfrak{m}}\Theta_\mathfrak{m}(L/F,\pi_B)^\epsilon = \Delta_{\mathfrak{m},S_\mathfrak{m}}(\kappa') \cap ((c_{\mathrm{drec}_{\mathfrak{p}_1}} \cup \dots \cup c_{\mathrm{drec}_{\mathfrak{p}_s}}) \cap \overline{c_L}(\mathfrak{m},S_\mathfrak{m},\epsilon)),$$

where $S_\mathfrak{m} = \{\mathfrak{p}_1,\dots,\mathfrak{p}_s\}$. By Corollary 4.15 we obtain

$$\begin{aligned} n_{S_\mathfrak{m}}\Theta_\mathfrak{m}(L/F,\pi_B)^\epsilon = {}&\Delta^{\{\mathfrak{p}_i\}}_{\mathfrak{m},S_\mathfrak{m}-\{\mathfrak{p}_i\}}(\kappa' \cup b_{\mathrm{drec}_{\mathfrak{p}_i}}) \\ &\cap ((c_{\mathrm{drec}_{\mathfrak{p}_1}} \cup \dots \cup \widehat{c_{\mathrm{drec}_{\mathfrak{p}_i}}} \cup \dots \cup c_{\mathrm{drec}_{\mathfrak{p}_s}}) \cap \overline{c_L}(\mathfrak{m},S_\mathfrak{m},\epsilon)) \end{aligned}$$

for every $i \in \{1,\dots,s\}$.

By Lemma 4.13 (ii) the following diagram is commutative for every $\mathfrak{p} \in S_\mathfrak{m}$:

$$
\begin{array}{ccc}
 & & \mathrm{H}^{d+1}(G(F), \mathcal{A}(\mathfrak{f}(\pi_B), S_{\mathfrak{m}} - \{\mathfrak{p}\}; R_\pi)^{\{\mathfrak{p}\}}(\epsilon)) \\
 & \nearrow^{\cup b_{\mathrm{ord}_{\mathfrak{p}}}} & \uparrow (\mathrm{ord}_{\mathfrak{p}})_* \\
\mathcal{M}(\mathfrak{f}(\pi_B), S_{\mathfrak{m}}; R_\pi)^\epsilon & \xrightarrow{\cup b_{\mathrm{univ}_{\mathfrak{p}}}} & \mathrm{H}^{d+1}(G(F), \mathcal{A}(\mathfrak{f}(\pi_B), S_{\mathfrak{m}} - \{\mathfrak{p}\}; F_{\mathfrak{p}}^* \otimes R_\pi)^{\{\mathfrak{p}\}}(\epsilon)) \\
 & \searrow_{\cup b_{\mathrm{drec}_{\mathfrak{p}}}} & \downarrow (\mathrm{drec}_{\mathfrak{p}})_* \\
 & & \mathrm{H}^{d+1}(G(F), \mathcal{A}(\mathfrak{f}(\pi_B), S_{\mathfrak{m}} - \{\mathfrak{p}\}; I_{\mathfrak{p}}/I_{\mathfrak{p}}^2)^{\{\mathfrak{p}\}}(\epsilon))
\end{array}
$$

Applying $\mathrm{drec}_{\mathfrak{p}}$ to (4.4) and using the commutativity of the lower triangle of the diagram we get

$$n_{S_{\mathfrak{m}},\mathfrak{p}}(\mathrm{rec}_{\mathfrak{p}}(q_{S_{\mathfrak{m}},\mathfrak{p}}) - 1)\kappa' \cup b_{\mathrm{ord}_{\mathfrak{p}}} = n_{S_{\mathfrak{m}},\mathfrak{p}}\gamma_{S_{\mathfrak{m}},\mathfrak{p}}\kappa' \cup b_{\mathrm{drec}_{\mathfrak{p}}}.$$

By the commutativity of the upper triangle of the diagram we see that

$$\gamma_{S_{\mathfrak{m}},\mathfrak{p}} = \mathrm{ord}_{\mathfrak{p}}(q_{S_{\mathfrak{m}},\mathfrak{p}}).$$

Hence, it is enough to show the following lemma. □

Lemma 4.18. *The equality*

$$(\Delta_{\mathfrak{m},S_{\mathfrak{m}}}(\kappa') \cup c_{\mathrm{ord}_{\mathfrak{p}_1}} \cup \cdots \cup c_{\mathrm{ord}_{\mathfrak{p}_s}}) \cap \overline{c_{\diamondsuit}}(\mathfrak{m}, S_{\mathfrak{m}}, \epsilon) = \pm n_{S_{\mathfrak{m}}} \ \Theta_{\mathfrak{m}^S}(\diamondsuit/F, \pi_B)^\epsilon$$

holds in R_π.

Proof. By Remark 1.5 we have $c_{\mathrm{ord}_{\mathfrak{p}_i}} = (\alpha_{\mathfrak{p}_i})_*(c_{\mathfrak{p}_i})$ for all $1 \leq i \leq s$. Thus, we get

$$
\begin{aligned}
&(c_{\mathrm{ord}_{\mathfrak{p}_1}} \cup \cdots \cup c_{\mathrm{ord}_{\mathfrak{p}_s}}) \cap \overline{c_{\diamondsuit}}(\mathfrak{m}, S_{\mathfrak{m}}, \epsilon) \\
=&(\alpha_{\mathfrak{p}_1} \otimes \ldots \otimes \alpha_{\mathfrak{p}_s})_*((c_{\mathfrak{p}_1} \cup \cdots \cup c_{\mathfrak{p}_s}) \cap \overline{c_{\diamondsuit}}(\mathfrak{m}, S_{\mathfrak{m}}, \epsilon)) \\
=& \pm (\alpha_{\mathfrak{p}_1} \otimes \ldots \otimes \alpha_{\mathfrak{p}_s})_*(c_{\diamondsuit}(\mathfrak{m}^S, \emptyset, \epsilon)).
\end{aligned}
$$

The second equality holds by Lemma 1.4. By Lemma 2.8 (ii) we have

$$
\begin{aligned}
&\Delta_{\mathfrak{m},S}(\kappa') \cap (\alpha_{\mathfrak{p}_1} \otimes \ldots \otimes \alpha_{\mathfrak{p}_s})_* \cap c_{\diamondsuit}(\mathfrak{m}^S, \emptyset, \epsilon) \\
=&(\alpha_{\mathfrak{p}_1} \otimes \ldots \otimes \alpha_{\mathfrak{p}_s})^*(\Delta_{\mathfrak{m},S}(\kappa')) \cap c_{\diamondsuit}(\mathfrak{m}^S, \emptyset, \epsilon) \\
=&\Delta_{\mathfrak{m}^S}(n_{S_{\mathfrak{m}}}\kappa) \cap c_{\diamondsuit}(\mathfrak{m}^S, \emptyset, \epsilon) \\
=&n_{S_{\mathfrak{m}}}\Theta_{\mathfrak{m}^S}(\diamondsuit/F, \pi_B)^\epsilon
\end{aligned}
$$

and thus, the claim follows. □

Remark 4.19. (i) Suppose $d = 0$ and that $S_{\mathfrak{m}} = \{\mathfrak{p}\}$. Then the module

$$\mathrm{H}^1(G(F), \mathcal{A}(\mathfrak{f}(\pi_B), \emptyset; R_\pi)^{\{\mathfrak{p}\}}(\epsilon))^\pi$$

is torsion-free. Thus, $n_{S_{\mathfrak{m}},\mathfrak{p}}$ is just the exponent of the cokernel of $\omega_{S_{\mathfrak{m}},\mathfrak{p}}$.

(ii) Using the norm relations and the interpolation formulae one can determine $\Theta_{\mathfrak{m}^S}(E/F, \pi_B)^\epsilon$ (resp. $\Theta_{\mathfrak{m}^S}(F/F, \pi_B)^\epsilon$) explicitly in terms of the special value at $1/2$ of the untwisted L-function $L(s, \pi_E)$ (resp. $L(s, \pi)$).

(iii) From the proof of Theorem 4.17 we see that $\mathrm{ord}_{\mathfrak{p}}(q_{\mathfrak{p}})$ is non-zero for every automorphic period $q_{\mathfrak{p}}$.

(iv) Assume that B is non-split, F is totally real and E is totally imaginary (so in particular B is totally definite). In this situation, Gehrmann and the author have given a comparison of the automorphic periods with Tate periods of the abelian variety associated to π_B (see [BG17], Sections 4.3 and 4.4).

4.4 Final remarks

We start by constructing p-adic L-functions via Stickelberger elements. For this, we have to take slightly different approaches for the non-split and split quaternion algebras.

We start with the construction in case that B is split. Let p be a rational prime and R the valuation ring of a p-adic field. Choose a non-zero ideal $\mathfrak{n} \subseteq \mathcal{O}_F$ such that every place $\mathfrak{p}$ lying over p divides $\mathfrak{n}$ exactly once. Further, let $\kappa \in \mathcal{M}(\mathfrak{n}, R)^\epsilon$ be an eigenvector of $T_{\mathfrak{p}}$ for all places $\mathfrak{p}$ of F dividing p with eigenvalue $\lambda_{\mathfrak{p}} \in R^*$. For every modulus $\mathfrak{m} \subseteq \mathcal{O}_F$ of the form $\mathfrak{m} = \prod_{\mathfrak{p}|p} \mathfrak{p}^{m_{\mathfrak{p}}}$ with $m_{\mathfrak{p}} \geq 1$ for all $\mathfrak{p}$ over p we define

$$\widetilde{\Theta}_{\mathfrak{m}}(\cdot/F, \kappa) = \left(\prod_{\mathfrak{p}|p} \lambda_{\mathfrak{p}}^{-m_{\mathfrak{p}}} \right) \Theta_{\mathfrak{m}}(\cdot/F, \kappa)$$

Then by an analogue of Theorem 4.4, (ii), this is a norm-compatible family and hence defines an element $\widetilde{\Theta}(F, \kappa)$ in the completed group ring $R[\![\mathcal{G}_p]\!]$, where $\mathcal{G}_p$ is the Galois group of the maximal abelian extension of F unramified outside p and ∞.

Let Z be a $\mathbb{Z}_p$-extension of F. The Galois group $\mathrm{Gal}(Z/F) \cong p^{\beta_p}\mathbb{Z}_p$, with $\beta_p = 2$ for $p = 2$ and $\beta_p = 1$ else, is a quotient of $\mathcal{G}_p$ (β_p is chosen such that $p^{\beta_p}\mathbb{Z}_p$ is the space of definition of the p-adic exponential map $\exp_p$). Write $l\colon \mathcal{G}_p \to p^{\beta_p}\mathbb{Z}_p$ for the induced surjective homomorphism. The $\mathbb{Z}_p$-rank of $\mathcal{G}_p$ is equal to t, where $r_2(F)+1 \leq t \leq [F:\mathbb{Q}]$ (with $t = r_2(F)+1$ if the Leopoldt conjecture holds). Let $L_1, \ldots, L_t$ be pairwise linear disjoint $\mathbb{Z}_p$-extensions of F and $l_i\colon \mathcal{G}_p \to p^{\beta_p}\mathbb{Z}_p$ the corresponding homomorphisms. For $s \in \mathbb{Z}_p$ and $\gamma \in \mathcal{G}_p$ we put $\langle\gamma\rangle_i^s = \exp_p(s\; l_i(\gamma))$. The p-adic L-function of κ is the t-variabled function

$$L_p(\underline{s},\kappa) = \prod_{i=1}^{t} \langle\cdot\rangle_i^{s_i}\left(\widetilde{\Theta}(F,\kappa)\right)$$

with $\underline{s} = (s_1, \ldots, s_t) \in \mathbb{Z}_p^t$. This coincides with the p-adic L-function constructed by Deppe (see [Dep16], Definition 3.5).

Let π be an automorphic representation over F as in the beginning of Section 4.1. If π is p-ordinary, we can replace the modular symbol κ^π associated to π (with trivial character) by its ordinary p-stabilization κ_p^π, The construction above yields the p-adic L-function of π:

$$L_p(\underline{s},\pi) = L_p(\underline{s},\kappa_p^\pi)$$

Observing that the element n_{S_p} for $S_p = \{\mathfrak{p} \mid p\}$ is independent of the modulus, we can deduce that the order of vanishing of $L_p(\underline{s},\pi)$ at $\underline{s} = (0,\ldots,0)$ is at least the number of primes lying above p at which $\pi_\mathfrak{p}$ is Steinberg representation. This was first proven by Spieß in [Spi14] if F is totally real and by Deppe in [Dep16] for arbitrary number fields. Moreover, the leading term theorem (Theorem 4.17) gives a positive answer to the exceptional zero conjecture 4.15 of [Dep16] with automorphic L-invariants replacing arithmetic L-invariants. Note that the main result of [Geh17] gives a comparison of automorphic and arithmetic L-invariants in some cases.

For non-split quaternion algebras we get similar statements by taking a slightly different approach: We have to consider the Galois group $\mathcal{G}_p^{\mathrm{anti}} = \mathrm{Gal}(\mathcal{Z}/E)$ of the maximal E-anticyclotomic extension $\mathcal{Z}$ of F which is unramified outside p and ∞ over E. Denoting by t the number of pairwise linearly disjoint $\mathbb{Z}_p$-extensions of E in $\mathcal{Z}$ and assuming $t > 0$ we follow the same lines as in the above construction to get a p-adic L-function $L_p^{\mathrm{anti}}(\underline{s},\pi_B)$. This time around, the order of vanishing of $L_p^{\mathrm{anti}}(\underline{s},\pi_B)$ at $\underline{s} = (0,\ldots,0)$ is at least the number of primes $\mathfrak{p}$ lying above p at which $\pi_{B,\mathfrak{p}}$ is Steinberg or which are inert with $\pi_{B,\mathfrak{p}}$ the twisted Steinberg. Theorem 4.17 generalizes Molina's work on exceptional zeros of anticyclotomic p-adic L-functions in the CM-case (see [Mol15]).

Finally, we want to outline how the theorem of the introduction can be obtained as a special case of Theorem 4.6. Note that Ota in [Ota] has announced a proof for special cases (to be exact only for square-free level) of this result using divisibility of certain derivatives of Kato's Euler system.

Let $M > 2$ be a natural number and A an elliptic curve over $\mathbb{Q}$ with corresponding normalized newform $f \in S_2(\Gamma_0(N))$ (and corresponding automorphic representation π). Let $+\colon \mathbb{R}^* \to \{\pm 1\}$ be the trivial character. Comparing Theorem 4.4 and Theorem 4.10 with the corresponding norm relations and interpolation formula for the Stickelberger elements of Mazur and Tate (see [MT87], Sections (1.3) and (1.4)) shows that there exists a constant $c \in \mathbb{C}^*$ such that

$$(\Theta^{\mathrm{MT}}_{A,M})^\vee = c \cdot \Theta_M \left(\mathbb{Q}(\mu_M)^+/\mathbb{Q}, \kappa^+_{\frac{2\pi i}{\Omega^+_A} f} \right)^+ .$$

A direct calculation shows that $c = 2$. See Remark 4.1, (iii), for the definition of $\kappa^+_{\frac{2\pi i}{\Omega^+_A} f}$. The modular symbol $\kappa^+_{\frac{2\pi i}{\Omega^+_A} f}$ is contained in $\mathcal{M}(N, \mathcal{R})^+$ whenever $[q]_A \in \mathcal{R}$ for all $q \in \mathbb{Q}/\mathbb{Z}$. For example we can choose $\mathcal{R} = \mathbb{Z}[\frac{1}{\tau c_A}]$ (see the introduction for the definition of τ and c_A). Thus, our main theorem follows from Theorem 4.6 together with the fact that A has split multiplicative reduction at p if and only if π_p is Steinberg.

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