Nonlocal operators on domains

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Contents

1. Introduction

In the thesis we consider variational solutions to equations that involve nonlocal operators of integro-differential type, such as

$$\mathcal{L}u(x) = \lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^d \setminus B_\varepsilon(x)} (u(x) - u(y))k(x,y)\,\mathrm{d}y. \tag{1.1}$$

We mainly address the following three problems: The well-posedness of nonlocal boundary value problems for a large class of admissible kernels k, the phase transition from nonlocal to local equations and the homogenization of nonlocal equations.

Preliminarily, let us fix the terms *local* and *nonlocal*. An operator acting on a function $u : \mathbb{R}^d \to \mathbb{R}$ is called *local*, if evaluating it at a point $x \in \mathbb{R}^d$ (if possible), it is sufficient to know the values of u in an arbitrary small neighborhood of x. Examples of local operators are the differential operator $\mathcal{L}u(x) = \nabla u(x)$ or the Laplace operator $\Delta u = \sum_{i=1}^d \partial_{ii} u$. The operator (1.1) is of different nature. To evaluate $\mathcal{L}u(x)$ we need to know the values $u(y)$ for all $y \in \operatorname{supp}(k(x,\cdot))$, which may be $\mathbb{R}^d$. Therefore we use the term *nonlocal*. Formally, an operator $\mathcal{L}$ is called local if $\operatorname{supp}(\mathcal{L}u) \subset \operatorname{supp}(u)$ and nonlocal otherwise.

As the title of the thesis indicates, we consider a domain Ω in $\mathbb{R}^d$ and a suitable function f defined on it. Our aim is to discuss variational solutions u of the nonlocal equation

$$\mathcal{L}u = f \quad \text{on } \Omega. \tag{1.2}$$

We emphasize that we do not assume (1.2) to hold pointwise. Further, we note that for constant u we have $\mathcal{L}u = 0$ and therefore, to expect uniqueness of a solution to (1.2), we need to prescribe boundary data. Due to the nonlocality of $\mathcal{L}$ the boundary data has to be defined on the complement of Ω. Thus when speaking about solutions to nonlocal equations, we say that u solves a nonlocal boundary value problem, with an nonlocal operator $\mathcal{L}$.

Nonlocal operators are closely related to stochastic processes. The celebrated example are here pure jump Lévy processes, whose infinitesimal generators are nonlocal operators. Moreover nonlocal operators play a crucial rule in models with long range interactions.

For example nonlocal equations are studied in physics, in particular in the theory of perodynamics. Peridynamics describes a nonlocal, in general vector valued, continuum model including deformations with discontinuities and is introduced by Silling in [Sil00]. The monograph [Sch03] deals with Lévy based models of financial markets. In particular, it is shown on the basis of historical data, that jump processes are more appropriate to model financial markets than models based only on Brownian motion. Another application in finance is given in [Lev04], where the American put option is analyzed with a stock return rate following regular Lévy process of exponential type.

In [GO08], Gilboa and Osher use nonlocal operators within the framework of image processing. They show advantages of the nonlocal approach, which allows interactions between any two points in the image domain, in handling textures and repetitive structures to classical PDE methods.

1.1. Existence and uniqueness of variational solutions

Given an open domain $\Omega \subset \mathbb{R}^d$ and functions $f : \Omega \to \mathbb{R}$, $g : \partial\Omega \to \mathbb{R}$, the classical Dirichlet problem is to find a function $u : \overline{\Omega} \to \mathbb{R}$ such that

$$-\Delta u = f \quad \text{in } \Omega, \tag{1.3a}$$
$$u = g \quad \text{on } \partial\Omega. \tag{1.3b}$$

More generally, one can replace the Laplace operator in (1.3a) by a second order differential operator of the form

$$\mathcal{L}u = -\sum_{i,j=1}^{d} \partial_j \big(a_{ij}(\cdot)\partial_i u(x) + b_i(\cdot)u\big), \tag{1.4}$$

where a_{ij}, b_i are coefficients defined in Ω. The operator (1.4) is called uniformly elliptic if there is a constant $\lambda > 0$, such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \lambda\,|\xi|^2$$

for all $\xi \in \mathbb{R}^d$ and for all $x \in \mathbb{R}^d$.

Consider the problem (1.3) with Δ replaced by $\mathcal{L}$ with coefficients a_{ij} assumed to be only measurable and bounded and a function f which is not necessarily smooth.

In this case, for the well-posedness it is convenient to use the concept of weak solutions, which require a choice of proper Hilbert spaces. Here the appropriate ones are the Sobolev spaces

$$H^1(\Omega) = \{u \in L^2(\Omega) \,|\, \nabla u \in L^2(\Omega)\}$$

and $H_0^1(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$, where $\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2$. Classical assumptions in this setting are $f \in H^{-1}(\Omega) = \big(H_0^1(\Omega)\big)^*$ and $g \in H^1(\Omega)$.

Let us go back to (1.3) with homogeneous boundary data $g = 0$. The Riesz representation theorem implies that there is a unique $u \in H_0^1(\Omega)$ such that

$$\langle \nabla u, \nabla \varphi \rangle_{L^2(\Omega)} = \langle f, \varphi \rangle_{L^2(\Omega)} \tag{1.5}$$

for every $\varphi \in H_0^1(\Omega)$. The equality (1.5) is called the weak formulation of the Dirichlet problem and $u \in H_0^1(\Omega)$ is called a weak solution. If we consider more general operators, the Riesz representation theorem cannot be applied. In this case existence and uniqueness results follow from an application of the Lax-Milgram Lemma, or if the bilinear form is not positive definite, from the application of the Fredholm alternative, see for example the monographs [LU68], [GT77] or [Eva10].

When considering nonzero boundary data $g : \partial\Omega \to \mathbb{R}$ in (1.3), it is necessary that g admits an extension $\widetilde{g} \in H^1(\Omega)$, to obtain the existence of a weak solution. Therefore it is natural to assume a priori $g \in H^1(\Omega)$. In this case u is called a weak solution of the boundary value problem (1.3) if u satisfies (1.5) and $u - g \in H^1_0(\Omega)$.

To postulate $u - g \in H^1_0(\Omega)$ is one natural way to interpret (1.3b). If the boundary $\partial\Omega$ is smooth, say $\partial\Omega$ is C^1, any function $\phi \in H^1(\Omega)$ admits a trace $\phi_{|\partial\Omega} \in H^{1/2}(\partial\Omega)$. Therefore another interpretation of (1.3b) is

$$u_{|\partial\Omega} = g_{|\partial\Omega} \tag{1.6}$$

in the sense of traces. This interpretation is of course equivalent to the first one, since the trace operator from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ is one-to-one.

One aim of the present work is to extent the classical Hilbert space techniques to nonlocal analogues of (1.3) for operators of the form

$$(\mathcal{L}u)(x) = \lim_{\varepsilon\to 0} \int\limits_{B^c_\varepsilon(x)} (u(x) - u(y))\, k(x,y)\,\mathrm{d}y. \tag{1.7}$$

Here $k : \mathbb{R}^d \times \mathbb{R}^d$ is a measurable kernel. The most common example is given by $k(x,y) = \mathcal{A}_{d,-\alpha}\,|x-y|^{-d-\alpha}$ for $\alpha \in (0,2)$. In this model case $\mathcal{L}$ becomes the fractional Laplace operator – the pseudo-differential operator with symbol $|\xi|^\alpha$ – and we denote $\mathcal{L} = (-\Delta)^{\alpha/2}$. Here $\mathcal{A}_{d,-\alpha}$ is a norming constant that can be defined explicitly in terms of the Euler Γ -function. When considering the limit cases $\alpha \to 0^+$ or $\alpha \to 2^-$ it is important to note, that $\mathcal{A}_{d,-\alpha} \asymp \alpha(2-\alpha)$, see Subsection 2.1.2.

Note that in (1.7) there is no lower order term, such as b_i in (1.4). Nonetheless, when considering possibly nonsymmetric kernels, as explained in Chapter 3, we assume

$$\sup_{x\in\mathbb{R}^d} \int\limits_{\{k_s(x,y)\neq 0\}} \frac{k_a^2(x,y)}{k_s(x,y)}\,\mathrm{d}y < \infty, \tag{1.8}$$

where for an arbitrary kernel k its symmetric and antisymmetric part are defined by

$$k_s(x,y) = \frac{1}{2}\left(k(x,y) + k(y,x)\right) \quad \text{and} \quad k_a(x,y) = \frac{1}{2}\left(k(x,y) - k(y,x)\right).$$

Therefore by (1.8) the antisymmetric part is of lower order and thus can be interpreted as an analogue to b_i in (1.4).

Formally, we call an operator of the form (1.7) uniformly elliptic of order $\alpha \in (0,2)$, if there is a constant $\lambda > 0$ such that for every $x \in \mathbb{R}^d$ $\nu_x(A) = \int_A k(x,y)\,\mathrm{d}y$ is an α-stable measure and

$$\int\limits_{B_1} |\langle \xi, h\rangle|^2\,\nu_x(\mathrm{d}h) \geq \lambda\,|\xi|^2 \tag{1.9}$$

for all $\xi \in \mathbb{R}^d$. A sufficient condition for ellipticity of order $\alpha \in (0,2)$ of a nonlocal operator (1.7) is

$$\lambda(2-\alpha)\,|x-y|^{-d-\alpha} \leq k(x,y) \leq \lambda^{-1}(2-\alpha)\,|x-y|^{-d-\alpha} \tag{1.10}$$

for some constant $\lambda > 0$. We also use the term elliptic of order $\alpha \in (0,2)$, if the kernel k is comparable to $(2-\alpha)\,|x-y|^{d-\alpha}$ in an integrated sense, see Section 4.1. Note that it is not clear, whether the integrated comparability on all scales implies (1.9).

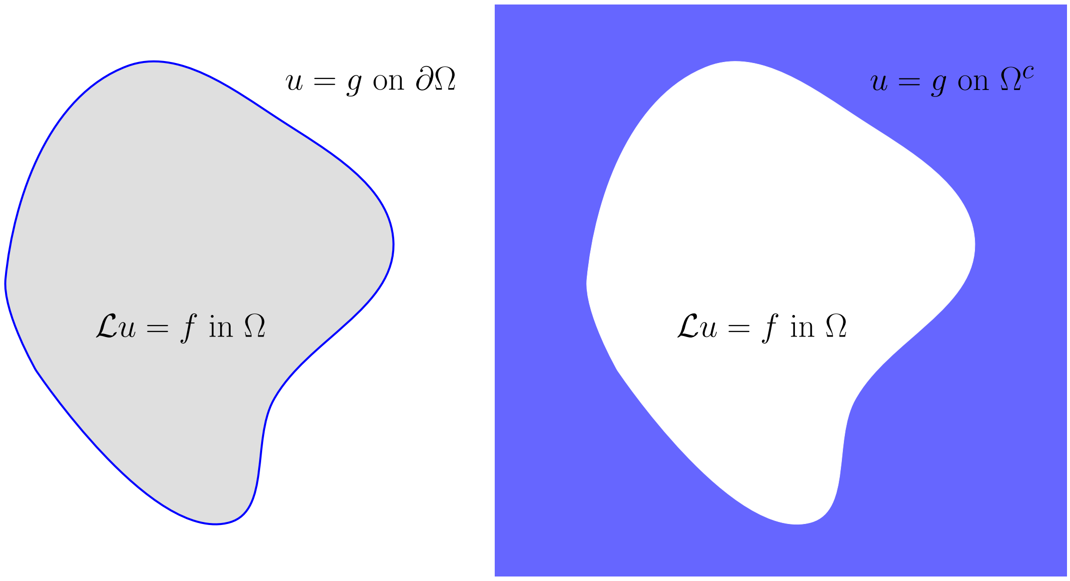

Figure 1.1.: Illustration of *local* vs. *nonlocal* Dirichlet boundary data

As already mentioned above, by the nonlocal character of $\mathcal{L}$, to evaluate the operator at any point $x \in \mathbb{R}^d$, one needs to know a function $u : \mathbb{R}^d \to \mathbb{R}$ in any point. On this account, when replacing the local operator in (1.3b) by a nonlocal operator of the form (1.7), we need to prescribe the boundary data on the complement of a given domain Ω. Nevertheless, to avoid the term *complement value problem*, we use the term *nonlocal boundary value problem*.

Now for given functions $f : \Omega \to \mathbb{R}$ and $g : \Omega^c \to \mathbb{R}$, the nonlocal Dirichlet problem is to find a function $u : \mathbb{R}^d \to \mathbb{R}$, such that

$$\mathcal{L}u = f \quad \text{in } \Omega, \tag{1.11a}$$

$$u = g \quad \text{on } \Omega^c. \tag{1.11b}$$

We develop a Hilbert space approach to solve (1.11), which is similar to the classical theory for second order PDE's. Let us look at the model case of the fractional Laplacian with homogeneous boundary condition $g = 0$ to explain the Hilbert space approach in the nonlocal setting. Under this assumptions a proper Hilbert space is given by the fractional Sobolev space of functions that vanish outside Ω:

$$H^{\alpha/2}_{\Omega}(\mathbb{R}^d) = \{u \in H^{\alpha/2}(\mathbb{R}^d) \,|\, u = 0 \text{ a.e. on } \Omega^c\},$$

where

$$H^{\alpha/2}(\mathbb{R}^d) = \Big\{u \in L^2(\mathbb{R}^d) \,|\, \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \,\mathrm{d}x\,\mathrm{d}y\Big\} < \infty$$

is the fractional Sobolev space of order $\alpha/2$ on $\mathbb{R}^d$. Note that functions in $H^{\alpha/2}_{\Omega}(\mathbb{R}^d)$ are defined on $\mathbb{R}^d$. If we define the associated bilinear form $\mathcal{E} : H^{\alpha/2}_{\Omega}(\mathbb{R}^d) \times H^{\alpha/2}_{\Omega}(\mathbb{R}^d) \to \mathbb{R}$ of $(-\Delta)^{\alpha/2}$ by

$$\mathcal{E}(u,v) = \frac{1}{2}\mathcal{A}_{d,-\alpha} \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}} \,\mathrm{d}x\,\mathrm{d}y,$$

the Riesz representation theorem implies the existence of a unique $u \in H^{\alpha/2}_{\Omega}(\mathbb{R}^d)$, such that

$$\mathcal{E}(u, \varphi) = \langle f, \varphi \rangle \,. \tag{1.12}$$

for $f \in \left(H^{\alpha/2}_{\Omega}(\mathbb{R}^d)\right)^*$. The equality (1.12) is called the weak formulation of the nonlocal Dirichlet problem and $u \in H^{\alpha/2}_{\Omega}(\mathbb{R}^d)$ is called a weak solution.

If we consider nonzero boundary data $g \in H^{\alpha/2}(\mathbb{R}^d)$, we can reduce the problem (1.11) to the case of zero boundary data by solving the equation

$$\mathcal{L}u = f - \mathcal{L}g \quad \text{in } \Omega, \tag{1.13a}$$

$$u = 0 \qquad\quad \text{on } \Omega^c. \tag{1.13b}$$

This is possible because $\mathcal{E}(g, \cdot)$ is a continuous linear functional on $H^{\alpha/2}_{\Omega}(\mathbb{R}^d)$. If $\widetilde{u}$ is a solution of (1.13), then $\widetilde{u} + g$ solves (1.11).

But the assumption $g \in H^{\alpha/2}(\mathbb{R}^d)$ implies already certain regularity of the function g everywhere in $\mathbb{R}^d$. It seems to be more natural to assume regularity only where the equation holds, namely on Ω. On this account we introduce function spaces $V^{\alpha/2}(\Omega|\mathbb{R}^d)$ that prescribe regularity of the functions in Ω and over the boundary $\partial\Omega$. To be precise

$$V^{\alpha/2}(\Omega|\mathbb{R}^d) = \Big\{ f \in L^2(\mathbb{R}^d) \,|\, \alpha(2-\alpha) \iint\limits_{(\Omega^c \times \Omega^c)^c} \frac{(f(x) - f(y))^2}{|x-y|^{d+\alpha}} \, \mathrm{d}x \, \mathrm{d}y < \infty \Big\}.$$

Note that $(\Omega^c \times \Omega^c)^c = (\Omega \times \Omega) \cup (\Omega \times \Omega^c) \cup (\Omega^c \times \Omega)$. Thus the singularity of the weight $|x-y|^{-d-\alpha}$ occurs only on $\Omega \times \Omega$ and on $\partial\Omega$. Therefore we do not assume regularity outside Ω, but $\mathcal{E}(g, \cdot)$ is still a linear continuous functional on $H^{\alpha/2}_{\Omega}(\mathbb{R}^d)$ for $g \in V^{\alpha/2}(\Omega|\mathbb{R}^d)$. For fixed α the norming constant $\alpha(2-\alpha)$ can be suppressed, anyhow it affects the asymptotics of $V^{\alpha/2}(\Omega|\mathbb{R}^d)$ as $\alpha \to 2^-$, see Subsection 2.2.4.

As in (1.6), one could suppose that it is sufficient to prescribe the boundary data on the (nonlocal) boundary of Ω, i.e. on Ω^c. Actually it turns out that there is a nonlocal analogue of the trace space $H^{1/2}(\partial\Omega)$ for sufficiently regular domains Ω. Consider a function $f : \Omega^c \to \mathbb{R}$. Then f can be extended to a function $\widetilde{f} \in V^{\alpha/2}(\Omega|\mathbb{R}^d)$, if

$$\iint\limits_{\Omega^c\,\Omega^c} \frac{(f(x) - f(y))^2}{(|x-y| + \delta_x + \delta_y)^{d+\alpha}} \, \mathrm{d}y \, \mathrm{d}x < \infty,$$

where $\delta_x = \mathrm{dist}(x, \Omega)$, see [KD16].

We examine the solvability of (1.11) under various assumptions on the measurable kernels $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$. Such as operators of the form (1.4) are natural generalization of $-\Delta$, operators of the form (1.7) constitute a natural generalization of $(-\Delta)^{\alpha/2}$, if the kernel k is comparable to $\mathcal{A}_{d,-\alpha} |x-y|^{-d-\alpha}$ in an integrated sense, i.e. if there is $\lambda > 0$, such that

$$\begin{aligned} \lambda \mathcal{A}_{d,-\alpha} \iint\limits_{(\Omega^c \times \Omega^c)^c} \frac{(u(x) - u(y))^2}{|x-y|^{d+\alpha}} \, \mathrm{d}y \, \mathrm{d}x &\leq \iint\limits_{(\Omega^c \times \Omega^c)^c} (u(x) - u(y))^2 k(x,y) \, \mathrm{d}y \, \mathrm{d}x \\ &\leq \lambda^{-1} \mathcal{A}_{d,-\alpha} \iint\limits_{(\Omega^c \times \Omega^c)^c} \frac{(u(x) - u(y))^2}{|x-y|^{d+\alpha}} \, \mathrm{d}y \, \mathrm{d}x \end{aligned}$$

for all $u \in L^2(\mathbb{R}^d)$.

Under these comparability assumptions, $H_\Omega^{\alpha/2}(\mathbb{R}^d)$ turns out to be an appropriate space to solve the nonlocal Dirichlet problem with homogeneous boundary data zero. In order to deal with a wider class of kernels, we use the given kernel k, or precisely its symmetric part, to define function spaces in which we solve the nonlocal Dirichlet problem.

For a general kernel k an appropriate Hilbert space to solve (1.13) is given by

$$H_0^k(\Omega|\mathbb{R}^d) = \Big\{ u \in L^2(\mathbb{R}^d) | u = 0 \text{ a.e. on } \Omega^c \text{ and } \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x) - u(y))^2 k_s(x,y) \,\mathrm{d}y \,\mathrm{d}x < \infty \Big\}$$

and an appropriate function space for the boundary data is given by

$$V^k(\Omega|\mathbb{R}^d) = \Big\{ f \in L^2(\Omega) \,|\, \iint\limits_{(\Omega^c\times\Omega^c)^c} (f(x) - f(y))^2 k(x,y) \,\mathrm{d}x \,\mathrm{d}y < \infty \Big\}.$$

Starting from the Dirichlet problem for second order elliptic operators, it is natural to consider kernels with a certain singularity on the diagonal and thus generating operators with a differential character. In addition the nonlocal character of the operator (1.7) allows us to consider integrable kernels k.

A kernel is called integrable if, for every $x \in \mathbb{R}^d$ the quantity $\int_{\mathbb{R}^d} k_s(x,y)\,\mathrm{d}y$ is finite and the mapping $x \mapsto \int_{\mathbb{R}^d} k_s(x,y)\,\mathrm{d}y$ is locally integrable whereas it is called non-integrable otherwise, see Definition 2.37.

If k is integrable the operator $\mathcal{L}$ is a well defined operator on L^2. Nevertheless also for integrable kernels the corresponding function spaces $H_0^k(\Omega|\mathbb{R}^d)$ may satisfy a Poincaré-Friedrichs inequality, which allows us to prove coercivity of the associated bilinear form. At a first glance, this seems to be surprising, since the operator (1.7) with an integrable kernel has no differential structure. A simple example of an integrable kernel is given by $k(x,y) = \mathbb{1}_{B_1}(x-y)$. and in this case $H(\Omega;k) = L^2(\Omega_1)$, where $\Omega_1 = \{x \in \mathbb{R}^d | \operatorname{dist}(x,\Omega) < 1\}$.

Let us comment on related results in the literature. Note that the results of Section 2.3 and Chapter 3, with exception of minor changes, e.g. Corollary 3.11, are published in [FKV14]. In contrast to [FKV14] we drop the assumption of boundedness of the domain Ω, where it is not needed in the proofs and simplify some assumptions on the comparability of the bilinear forms.

Advantages of our approach are, that we deal with operators with non constant coefficients, which are also allowed to be nonsymmetric. Further our approach allows us to deal with integrable and non-integrable kernels at the same time. We review related results on this part of the thesis only shortly and refer to the introduction of [FKV14] for a deeper embedding of the current results.

We should mention that variational solutions to nonlocal problems have been considered within the theory of peridynamics. Using variational techniques, the well-posedness of a perodynamic nonlocal diffusion model is proved in [DGLZ12]. [MD13], Mengesha and Du consider a scalar peridynamic model involving also sign changing kernels.

In our approach, functions are defined on the whole $\mathbb{R}^d$ and the equation holds on some domain $\Omega \subset \mathbb{R}^d$ and therefore the nonlocal boundary data are prescribed on Ω^c. On the contrary, in peridynamics, the nonlocal boundary data, which are called *volume constraints* in this context, are prescribed on a subset ω of a domain $D \subset \mathbb{R}^d$, where $|\omega| > 0$. Translating this to our setting would lead to $D = \Omega \setminus \omega$ and $\omega = \Omega^c$.

We would like to point out that we assume regularity of the boundary data only on the domain Ω and over its boundary $\partial\Omega$. In many works considering nonlocal boundary value problems, the authors assume regularity of the boundary data on the whole of $\mathbb{R}^d$, e.g. [HJ96], where the solvability of the Dirichlet problem with nonlocal boundary data is addressed or [DCKP14], where a Harnack-inequality for minimizers of integro-differential operators is proved. Also in the aforementioned papers from the theory of peridynamics regularity is assumed on the whole domain D which corresponds to regularity everywhere from our perspective.

1.2. Nonlocal to local phase transition

Consider a sequence of nonlocal uniformly elliptic operators $\mathcal{L}^\alpha$ of order $\alpha \in (0,2)$, indexed by the parameter α. In the sequel, we explain that a sequence of nonlocal operators can localize in the limit, i.e. converge in some sense to a local operator $\mathcal{L}$, as the order $\alpha \nearrow 2$. We call this *phase transition*[1].

We illustrate this with an example. Consider the model case $k(x,y) = \mathcal{A}_{d,-\alpha} |x-y|^{-d-\alpha}$ and let $u \in C_c^2(\mathbb{R}^d)$. Substituting $x - y = h$ we can rewrite the operator (1.7) in terms of second differences as

$$(-\Delta)^{\alpha/2} u(x) = -\frac{1}{2}\mathcal{A}_{d,-\alpha} \int\limits_{\mathbb{R}^d} \frac{(u(x-h) - 2u(x) + u(x+h))}{|h|^{d+\alpha}} \, \mathrm{d}h. \tag{1.14}$$

Note that in comparison with (1.1), we can omit the principal value. Now for a fixed $x \in \mathbb{R}^d$, the second order Taylor expansion of u in x is given by

$$(u(x-h) - 2u(x) + u(x+h)) = \sum_{i,j}^{d} \partial_i \partial_j u(x) h_i h_j + o(h^2). \tag{1.15}$$

Plugging (1.15) into (1.14) for small values of h and using the asymptotics of $\mathcal{A}_{d,-\alpha}$, we obtain

$$\lim_{\alpha \to 2^-} (-\Delta)^{\alpha/2} u(x) = -\Delta u(x).$$

Thus for fixed $u \in C_c^2(\mathbb{R}^d)$ the operators $\mathcal{L}^\alpha = (-\Delta)^{\alpha/2}$ converge in some sense to the operator $\mathcal{L} = -\Delta$ as $\alpha \nearrow 2$, or, in other words, there is a phase transition from a nonlocal to a local operator.

Considering this phase transition a natural question is the following: Consider a family of nonlocal Dirichlet problems

$$\mathcal{L}^{k^\alpha} u_\alpha = f \quad \text{in } \Omega, \tag{1.16a}$$
$$u_\alpha = g \quad \text{on } \Omega^c, \tag{1.16b}$$

[1]Within the theory of perodynamics this phenomenon is also denoted as vanishing nonlocality.

indexed by the parameter $\alpha \in (0,2)$. Note that, given $\alpha \in (0,2)$ fixed, u_α is the solution of the nonlocal Dirichlet problem (1.16) with an uniformly elliptic operator $\mathcal{L}^{k_\alpha}$ of order α.

Does the sequence (u_α) of solutions converge to the solution of a local Dirichlet problem

$$\sum_{i,j=1}^{d} \partial_i \left(a_{ij}(\cdot)\partial_j u(\cdot)\right) = f \quad \text{in } \Omega, \tag{1.17a}$$

$$u = g \quad \text{on } \Omega^c, \tag{1.17b}$$

in an appropriate norm?

We address this question in Chapter 4 for kernels k^α that generate uniformly elliptic operators of order $\alpha \in (0,2)$. It turns out that for appropriate boundary data g the sequence of solutions (u_α) converges to the solution of a second order boundary value problem in $L^2(\mathbb{R}^d)$ and also in the stronger norm of $V^{\alpha_0/2}(\Omega|\mathbb{R}^d)$ for any $\alpha_0 \in (0,2)$, cf Theorem 4.2. Moreover the limit equation can be characterized in terms of the given family of kernels k^α, namely the coefficients are given by

$$a_{ij}(x) = \lim_{\alpha\to 2^-} \int_0^1 \int_{S^{d-1}} t^{d+1}\sigma_i\sigma_j k^\alpha(x, x+t\sigma)\,\mathrm{d}\sigma\,\mathrm{d}t.$$

Instead of proving the convergence of solutions directly, we prove Γ-convergence of the associated energy functionals

$$F_g^\alpha(u) = \begin{cases} \frac{1}{4} \iint\limits_{(\Omega^c\times\Omega^c)^c} (u(x)-u(y))^2 k^\alpha(x,y)\,\mathrm{d}y\,\mathrm{d}x - \int\limits_\Omega u(x)f(x)\,\mathrm{d}x, & \text{if } u \in V_g^{\alpha/2}(\Omega|\mathbb{R}^d), \\ +\infty & \text{else,} \end{cases} \tag{1.18}$$

where $V_g^{\alpha/2}(\Omega|\mathbb{R}^d)$ is the subspace of functions from $V^{\alpha/2}(\Omega|\mathbb{R}^d)$, which are equal to g on Ω^c.

Note that minimizers of F_g^α solve (1.16) in a variational sense, if we assume the kernel k^α to be symmetric.

Studying the asymptotic behavior of solutions to (1.16) leads us to study the asymptotic behavior of the underlying function spaces $V^{\alpha/2}(\Omega|\mathbb{R}^d)$. In [BBM01] it is shown, that the norm of the fractional Sobolev spaces converges to the $W^{1,p}$-norm, i.e.

$$(2-\alpha) \iint\limits_{\Omega\,\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}y\,\mathrm{d}x \overset{\alpha\to 2^-}{\longrightarrow} K_d \int\limits_\Omega |\nabla u|^p\,\mathrm{d}x,$$

provided one uses the right norming constant $\alpha(2-\alpha) \asymp \mathcal{A}_{d,-\alpha}$. Using this fact, $H^1(\Omega)$ can be identified with the intersection of $H^{\alpha/2}(\Omega)$, $\alpha < 2$. The same questions turns out to be more involved in our setting, namely there is an interplay between the norming constant that behaves as $(2-\alpha)$ and the singularity $|x-y|^{-d-\alpha}$ over the boundary of Ω. It is worth mentioning that pointwise convergence of the functionals (1.18) to their local counterparts fails in general in our setting, cf. Remark 4.9.

We want to comment on results related to this part of the thesis in detail. The already mentioned paper [BBM01] can be seen as a starting point in this line of research. In [Pon04a] the same question is considered, replacing $|\cdot|^p$ by a continuous function and considering not necessarily radial (but still translation invariant) weights. Ponce also connects this to the $\Gamma-$convergence of the associated energies. Leoni and Spector [LS11], [LS14] prove similar characterizations of Sobolev spaces for quantities characterized by two exponents $1 < p < \infty$, $1 \leq q < \infty$, where the second parameter q is motivated by some application in image processing. In addition, they do not need any assumptions on the boundary $\partial\Omega$.

Aubert and Kornprobst [AK09] apply the $\Gamma-$convergence result proved by Ponce to approximate variational problems on $W^{1,p}(\Omega)$ by nonlocal quantities.

There are several articles in the field of peridynamics, concerning the phase transition from nonlocal to local. A nonlocal version of the gradient $\mathcal{G}$ and divergence operator $\mathcal{D}$ are defined in [DGLZ13] and convergence of this objects to the classical gradient and divergence operator is analyzed in [MS15].

The convergence of minimizers of nonlocal functionals to minimizers of local functionals is studied in [MD15], which is closely related to our results of Chapter 4. In contrast to our approach, the pointwise convergence of the nonlocal gradient $\mathcal{G}$ and nonlocal divergence $\mathcal{D}$ is used to obtain the Γ-convergence of the associated energies. Note that, as already mentioned before, also in [MD15] regularity of functions is required in the whole domain D, which corresponds to $\mathbb{R}^d$ in our setting.

1.3. Homogenization of nonlocal Dirichlet problem

Homogenization describes the phenomenon that the solutions to a sequence of equations with highly oscillating coefficients can be approximated by the solution of an effective equation. The problem naturally arises, when one asks for the macroscopic behavior of models with oscillations on a microscopic scale. Within this work, we consider the problem of homogenization in the setting of integro-differential operators of the form (1.7), where the kernels are assumed to be periodic.

For simplicity, let us start with a simple one-dimensional example for the homogenization of a second order differential equation. For $\Omega = (0,1) \subset \mathbb{R}$ and $f \in L^2(\Omega)$ consider the Dirichlet problem

$$\partial_x \left(a(\frac{\cdot}{\varepsilon}) \partial_x u(\cdot) \right) = f \quad \text{in } (0,1), \tag{1.19a}$$

$$u(0) = u(1) = 0, \tag{1.19b}$$

where $a : \mathbb{R} \to \mathbb{R}$ is one-periodic. For $\varepsilon \to 0$, the sequence (u_ε) of solutions to (1.19) converges to the solution of

$$\partial_x \left(a^* \partial_x u(\cdot) \right) = f \quad \text{in } (0,1), \tag{1.20a}$$

$$u(0) = u(1) = 0, \tag{1.20b}$$

where $a^* = \left(\int_0^1 \frac{1}{a(x)} \, \mathrm{d}x \right)^{-1}$ is the harmonic mean of a. This can be seen easily using the boundedness of the family $\xi_\varepsilon = a(\cdot)\partial_x u(\cdot)$ in $H_0^1((0,1))$. Also in higher dimension the limit equation can be completely characterized. In this case the homogenization formula involves a corrector equation.

We address this problem in the context of nonlocal operators of the form (1.7) for symmetric kernels k, which we assume to be periodic, i.e.

$$k(x + e_i, y) = k(x, y)$$

for all $i \in \{1, .., d\}$. Now we define

$$k_\varepsilon(x, y) = \varepsilon^{-d-\alpha} k\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right).$$

Let $\Omega \subset \mathbb{R}^d$, $f \in L^2(\Omega)$ and consider the equation

$$\mathcal{L}^{k_\varepsilon} u = f \quad \text{in } \Omega, \tag{1.21a}$$
$$u = 0 \quad \text{on } \Omega^c. \tag{1.21b}$$

The questions is, if the family (u_ε) converges to the solution of a homogenized equation

$$\mathcal{L}^{k_0} u = f \quad \text{in } \Omega, \tag{1.22a}$$
$$u = 0 \quad \text{on } \Omega^c, \tag{1.22b}$$

where the kernel k_0 satisfies (1.10) and, due to the ε-periodicity of k_ε, depends only on the differences $x - y$.

We use an approach based on $\Gamma-$convergence to face this problem and thus we consider the associated energy functionals

$$F_\varepsilon(u) = \iint\limits_{(\Omega^c \times \Omega^c)^c} (u(x) - u(y))^2 k_\varepsilon(x, y)\, \mathrm{d}y\, \mathrm{d}x = \mathcal{E}_\varepsilon(u, u).$$

Using the compactness property of $\Gamma-$convergence, we can assume that F_{ε_n} $\Gamma-$converges to an abstract functional F_0 for a sequence (ε_n) converging to zero. To obtain an integral representation of the limit functional we consider $F_\varepsilon(u) = \mathcal{E}_\varepsilon(u, u)$ as a bilinear form. This allows us to use the representation theory for Dirichlet forms due to Beurling and Deny and to rewrite the limit functional F_0 as a Dirichlet form

$$\mathcal{E}_0(u, u) = \iint\limits_{\mathbb{R}^d\, \mathbb{R}^d} (u(x) - u(y))^2 J(\mathrm{d}x, \mathrm{d}y),$$

for some Radon measure J on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\}$.

Unfortunately, we are not able to find a characterization of the measure J in terms of the periodic kernels k_ε, which would allow to prove $\Gamma-$convergence of F_ε to F_0 for any sequence $\varepsilon_n \to 0$. Therefore F_0 may still depend on the chosen sequence.

Let us comment on related results on this part of the thesis. A theory for the homogenization of second order equations was developed since the late 1960ies, mostly by the Italian, French and Russian school. Several techniques were developed to prove homogenization results, e.g. G-convergence and later Γ-convergence introduced by S. Spagnolo in [Spa68] and DeGiorgi in [DG75], H-convergence established by Murat and Tartar which is closely connected to the method of compensated compactness , see [MT97], or the method of two scale converges due to [All92].

There are only very few results concerning the homogenization of nonlocal equations. For fully nonlinear integro-differential operators Schwab obtained existence of a limit equation and convergence of a family of solutions in [Sch10] in the periodic setting and later in the stochastic setting in [Sch13].

Homogenization of nonlocal equations in periodically perforated domains is discussed in [Foc09] using $\Gamma-$convergence techniques. Recently, in [FRS16] Fernándes Bonder, Ritorto and Salort prove H-convergence of an arbitrary sequence of nonlocal functionals by a variant of Murats compensated compactness technique in the nonlocal setting. They prove a *div-curl-Lemma* for nonlocal divergence and nonlocal gradient. Moreover the equivalence to $\Gamma-$convergence of the associated energies is obtained, assuming H-convergence. However, even in the case of periodic homogenization no explicit formula for the homogenized limit is derived.

1.4. Connection of the three main parts

Finally let us comment on the connection of the three main parts of the underlying work. Consider a family of symmetric kernels (k_ε^α) indexed by two parameter ε and α and the Dirichlet problem

$$\mathcal{L}_\varepsilon^\alpha u = f \quad \text{in } \Omega, \tag{1.23a}$$
$$u = 0 \quad \text{on } \Omega^c, \tag{1.23b}$$

where $\mathcal{L}_\varepsilon^\alpha$ is an operator of the form (1.7) with k_ε^α instead of k.

Assume that for fixed ε k_ε^α generates nonlocal uniformly elliptic operator of order α with $\varepsilon-$periodic coefficients. By the results of Chapter 3 this problem is well-posed for all $\alpha \in (0,2)$ and $\varepsilon > 0$. We prove in Chapter 4 that the sequence (u_ε^α) of solutions converges to the solution of

$$\partial_i(a_{i,j}^\varepsilon(\cdot)\partial_j u) = f \quad \text{in } \Omega, \tag{1.24a}$$
$$u = 0 \quad \text{on } \Omega^c, \tag{1.24b}$$

where the coefficients inherit the periodicity of k_ε^α.

Given a sequence of translation invariant kernels, the corresponding solutions to the boundary value problem converge to the solution of a second order boundary value problem with constant coefficients a_{ij}^*.

Since we know that in the local setting homogenization takes place, a result for the nonlocal analogue considered in Chapter 5 would answer the question if the diagram

$$\begin{array}{ccc} k_\varepsilon^\alpha & \xrightarrow{\varepsilon\to 0} & k_0 \\ \big\downarrow{\scriptstyle \alpha\to 2} & & \big\downarrow{\scriptstyle \alpha\to 2} \\ a_{ij}^\varepsilon & \xrightarrow{\varepsilon\to 0} & a_{ij}^* \end{array}$$

commutates. Here the kernels and coefficient matrices are used as substitutes for the Dirichlet problem with the corresponding local or nonlocal operators.

1.5. Outline

Chapter 2 is devoted to function spaces tailor-made for the study of nonlocal operators on domains. Starting from classical function spaces on a domain Ω, we generalize these spaces to fit to the existence theory for nonlocal Dirichlet problems. In connection to the phase transition from nonlocal to local operators, we also study their asymptotic behavior.

In Chapter 3 the solvability of the nonlocal Dirichlet problem is analyzed under various assumptions on a given measurable kernel k. A variational formulation of the problem is deduced and basic properties of the associated bilinear form are discuses. From this well-posedness is proved using classical Hilbert space methods.

The phase transition from nonlocal to local equations is presented in Chapter 4. First, $\Gamma-$convergence of the associated energy functionals without boundary condition is proved and afterwards the compatibility of boundary data and Γ-limit is obtained. Subsequently the Γ-convergence is applied to obtain the convergence of solutions.

Chapter 5 deals with the homogenization of nonlocal equations. First, we review the Γ-convergence approach to prove homogenization of second order equations. Then we transfer some of the techniques to the nonlocal setting. Unfortunately we do not obtain an homogenization result.

To make the thesis self-contained, we collect basic properties of Γ-convergence and Dirichlet forms in the Appendix. Further, for the sake of completeness we give the definition of regular domains and collect the proofs of same technical lemmas.

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Abgrenzung des eigenen Beitrags gemäß §10(2) der Promotionsordnung

Den Inhalt des Abschnitts 2.3 sowie des Kapitels 3 hat der Autor dieser Dissertation zusammen mit Matthieu Felsinger und Moritz Kaßmann in der Arbeit [FKV14] veröffentlicht. Die Behandlung der in Abschnitt 2.2.3 betrachteten Funktionenräume stützt sich teilweise ebenfalls auf [FKV14]. Section 5 der gemeinsamen Arbeit [FKV14] ist im wesentlichen von M. Felsinger erarbeitet worden und nicht Teil der vorgelegten Dissertation. Lemma 3.14 dieser Arbeit (Lemma 4.3 in [FKV14]) ist von M. Kaßmann bewiesen worden. Dieses Resultat ist wichtig für den Beweis des schwachen Maximumprizips, Theorem 3.12.

2. Function spaces

In the first section of this chapter we give the definitions of Sobolev and fractional Sobolev or Sobolev-Slobodeckij spaces on a set $\Omega \subset \mathbb{R}^d$ and recall some results concerning the connection from the fractional Sobolev spaces to the first order Sobolev space. Subsequently we give an alternative definition of the above mentioned spaces in terms of second order differences.

From this on we introduce function spaces for functions defined on the whole $\mathbb{R}^d$ that have certain regularity properties on Ω and in addition some regularity over the boundary of Ω. These spaces are obtained as a generalization of the classical spaces by extending the area of integration. In this context we also examine the connection of this spaces to weighted L^2 spaces.

In the third part of the chapter we define function spaces with general, e.g. singular or nonsingular weights, which are tailor-made for the Hilbert space based existence theory for nonlocal Dirichlet problems established in Chapter 3. Thereunto we give conditions for a nonlocal version of a Poincaré-Friedrichs inequality to hold on these spaces, in terms of a given kernel k.

2.1. Classical function spaces

2.1.1. Sobolev and fractional Sobolev spaces

First we give the definition of Sobolev spaces of order one and Sobolev-Slobodeckij spaces of fractional order $s \in (0,1)$ on an arbitrary domain Ω. There are severals ways to define these spaces on $\mathbb{R}^d$, i.e. Fourier transform or interpolation theory. From this it is possible to define spaces on domains by restriction. To ensure the equivalence of the so defined spaces, one needs to assume some regularity of the boundary $\partial\Omega$. We use intrinsic definitions that do not need regularity assumptions on the boundary $\partial\Omega$. Since we do not use Sobolev spaces $W^{k,p}(\Omega)$ for $p \neq 2$ and $k > 1$, we restrict ourself to the Hilbert space case $p = 2$.

Definition 2.1. Let $\Omega \subset \mathbb{R}^d$ be open.

1. Then
$$H^1(\Omega) = \{f \in L^2(\Omega) \mid \int_\Omega |\nabla f|^2 \, \mathrm{d}x < \infty\} \tag{2.1}$$
and a norm on $H^1(\Omega)$ is defined by
$$\|f\|^2_{H^1(\Omega)} = \|f\|^2_{L^2(\Omega)} + \int_\Omega |\nabla f(x)|^2 \, \mathrm{d}x.$$
2. We denote by $H^1_0(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$.

3. We denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$. If $x \in H^{-1}(\Omega)$, we define the norm
$$\|x\|_{H^{-1}(\Omega)} = \sup\left\{\langle x, f\rangle \mid f \in H_0^1(\Omega), \|f\|_{H^1(\Omega)} = 1\right\}.$$

Next we define Sobelev spaces of fractional order, the so called Sobolev-Slobodeckij spaces.

Definition 2.2. Let $\Omega \subset \mathbb{R}^d$ be open.

1. For $0 < s < 1$ we define
$$[f]^2_{H^s(\Omega)} = \mathcal{A}_{d,-2s} \iint\limits_{\Omega\,\Omega} \frac{(f(x) - f(y))^2}{|x-y|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}x.$$
The linear space $H^s(\Omega)$ is defined as
$$H^s(\Omega) = \{f \in L^2(\Omega) \mid [f]_{H^s(\Omega)} < \infty\}$$
and a norm on $H^s(\Omega)$ is given by
$$\|f\|_{H^s(\Omega)} = \left(\|f\|^2_{L^2(\Omega)} + [f]^2_{H^s(\Omega)}\right)^{1/2}.$$
2. We denote by $H_0^s(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^s(\Omega)$.
3. We define
$$H^s_\Omega(\mathbb{R}^d) = \{u \in H^s(\mathbb{R}^d) \mid u = 0 \text{ a.e. on } \Omega^c\}.$$
4. We denote by $H^{-s}(\Omega)$ the dual space of $H_0^s(\Omega)$. If $x \in H^{-s}(\Omega)$, we define the norm
$$\|x\|_{H^{-s}(\Omega)} = \sup\left\{\langle x, f\rangle \mid f \in H_0^s(\Omega), \|f\|_{H^s(\Omega)} = 1\right\}.$$

The seminorm $[f]_{H^s(\Omega)}$ is called Gagliardo-seminorm. Here $\mathcal{A}_{d,-2s}$ is a norming constant that ensures that the Fourier symbol of the fractional Laplacian equals $|\xi|^{2s}$. In our context this constant is important when considering the asymptotic behavior of the spaces $H^s(\Omega)$ for $s \to 1^-$.

Remark 2.3. 1. We would like to point out, that in the definitions above the case $\Omega = \mathbb{R}^d$ is included.

2. For $f \in C_c^\infty(\Omega)$ $\|f\|_{H^1(\Omega)} = \|f\|_{H^1(\mathbb{R}^d)}$. Therefore we can equivalently define $H_0^1(\Omega)$ as the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^1(\mathbb{R}^d)}$. Doing so, functions in $H_0^1(\Omega)$ are defined on $\mathbb{R}^d$ automatically.

   In general this is wrong for the fractional Sobolev spaces, but for Lipschitz domains Ω we have for $s \neq \frac{1}{2}$
$$\text{completion of } C_c^\infty(\Omega) \text{ w.r.t. } \|\cdot\|_{H^s(\mathbb{R}^d)} = \text{completion of } C_c^\infty(\Omega) \text{ w.r.t. } \|\cdot\|_{H^s(\Omega)}.$$
3. $H^s_\Omega(\mathbb{R}^d)$ is the subspace of functions from $H^s(\mathbb{R}^d)$ that vanish outside Ω. One can prove that
$$H^s_\Omega(\mathbb{R}^d) = \text{completion of } C_c^\infty(\Omega) \text{ w.r.t. } \|\cdot\|_{H^s(\mathbb{R}^d)}.$$
   A proof of the last two statements can be found in [McL00, Thm. 3.33].

We want to collect some basic properties of the Sobolev spaces $H^s(\Omega)$, $s \in (0,1]$. We omit the proof and refer to [Wlo87, Thm. 3.1] and [McL00, P. 87].

Theorem 2.4. *$H^1(\Omega)$ and $H^s(\Omega)$ are separable Hilbert spaces. If Ω is a C^1-domain, then $C^\infty(\overline{\Omega})$ is dense in both spaces.*

2.1.2. Asymtotics as $s \nearrow 1$

The explicit value of the constant $\mathcal{A}_{d,-2s}$ can be given in terms of the Euler Γ-function, namely

$$\mathcal{A}_{d,-2s} = \frac{2^{2s-1}}{\pi^{d/2}} \frac{\Gamma(\frac{d+2s}{2})}{\Gamma(-s)}, \quad s \in (0,1), d \in \mathbb{N}.$$

The constant $\mathcal{A}_{d,-2s}$ ensures that

$$\lim_{s\to 1^-} \|f\|_{H^s(\mathbb{R}^d)} = \|f\|_{H^1(\mathbb{R}^d)}, \text{ and } \lim_{s\to 0^+} \|f\|_{H^s(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}.$$

We discuss this property in detail below. Within the scope of this work the explicit value of $\mathcal{A}_{d,-2s}$ plays a minor role. More important is its asymptotic behavior, that is described by

$$\lim_{s\to 1^-} \frac{\mathcal{A}_{d,-2s}}{(1-s)} = \frac{d}{|S^{d-1}|}, \qquad \lim_{s\to 0^+} \frac{\mathcal{A}_{d,-2s}}{s} = \frac{1}{|S^{d-1}|}.$$

We refer to [Fel13, Prop. 2.24] or [DPV11, Prop. 4.1] for more details on the constant $\mathcal{A}_{d,-2s}$.

We want to emphasize that even for smooth functions f the quantity

$$\iint_{\Omega\,\Omega} \frac{(f(x)-f(y))^2}{|x-y|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}x$$

may blow up when $s \to 1^-$. Adding the norming constant $(1-s)$, Bourgain, Brezis and Mirunescu prove in [BBM01] that for smooth domains the above quantity converges to the $H^1(\Omega)$ norm of f, up to a constant. To be precise, let $\Omega \subset \mathbb{R}^d$ be open and $\partial\Omega$ smooth, $f \in L^2(\Omega)$, then

$$\lim_{s\to 1^-} (1-s) \iint_{\Omega\,\Omega} \frac{(f(x)-f(y))^2}{|x-y|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}x = K_\Omega \int_\Omega |\nabla u|^2 \,\mathrm{d}x, \tag{2.2}$$

where both sides are infinite if $f \notin H^1(\Omega)$ and K_Ω is a positive constant depending only on Ω. In this sense, the left-hand side of (2.2) gives an alternative definition of the Sobolev space $H^1(\Omega)$. The same holds for general domains Ω under an additional assumption, see [LS11] [1]. Maz'ya and Shaposhnikova prove in [MS02a] that for $f \in \bigcap_{0<s<1} H^s(\mathbb{R}^d)$

$$\lim_{s\to 0^+} s \iint_{\mathbb{R}^d\,\mathbb{R}^d} \frac{(f(x)-f(y))^2}{|x-y|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}x = |S^{d-1}| \, \|f\|^2_{L^2(\mathbb{R}^d)}.$$

[1] [BBM01] and [LS11] prove these results in the more general case $p \neq 2$. Nevertheless we concentrate on the Hilbert space case $p = 2$ in this work

The following lemma extends the characterization of $H^1(\Omega)$ given by the left-hand side of (2.2) to the case of a sequence $(f_n) \in L^2(\Omega)$. The proposition is a direct consequence of [BBM01, Thm. 4], but since we use this particular result in the following we would like to fix it, for the sake of completeness.

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^d$ be a C^1-domain. Let $f \in L^2(\Omega)$, $(f_n) \in L^2(\Omega)$, $\|f_n - f\|_{L^2(\Omega)} \overset{n\to\infty}{\longrightarrow} 0$ and (s_n) a sequence in $(0,1)$ converging to 1. Assume that there is $A > 0$, such that*

$$\lim_{n\to\infty} (1 - s_n) \iint\limits_{\Omega\,\Omega} \frac{(f_n(x) - f_n(y))^2}{|x-y|^{d+2s_n}} \,\mathrm{d}y\,\mathrm{d}x \le A.$$

Then $f \in H^1(\Omega)$.

For an easier presentation we write

$$(1 - s_n)\,|x-y|^{2-d-2s_n} = \rho_n(x-y) + r_n(x-y),$$

where

$$\rho_n(x-y) = (1-s_n)\,|x-y|^{2-d-2s_n}\,\mathbb{1}_{\{|x-y|<1\}},$$
$$r_n(x-y) = (1-s_n)\,|x-y|^{2-d-2s_n}\,\mathbb{1}_{\{|x-y|\ge 1\}}.$$

Note that $\rho_n \in L^1(\mathbb{R}^d)$ and

$$\int\limits_{\mathbb{R}^d} \rho_n(h)\,\mathrm{d}h = 1 \text{ for all } n \in \mathbb{N}. \tag{2.3}$$

For the proof we need the following technical Lemma, taken from [BBM01, Lem. 1].

Lemma 2.6. *Assume $g \in L^1(\mathbb{R}^d)$, $\phi \in C_0^\infty(\mathbb{R}^d)$ and $\rho \in L^1(\mathbb{R}^d)$. Then*

$$\left| \int\limits_{\mathbb{R}^d} g(x) \int\limits_{(y-x)\cdot e_i \ge 0} \frac{\phi(y)-\phi(x)}{|x-y|} \rho(x-y)\,\mathrm{d}y\,\mathrm{d}x \right| \le \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{|g(x)-g(y)|}{|x-y|}\,|\phi(y)|\,\rho(x-y)\,\mathrm{d}y\,\mathrm{d}x.$$

We omit the proof and refer to [BBM01, Lem. 1]. As already mentioned the proof of the proposition extents the proof of of [BBM01, Thm. 2] to the case of a sequence (f_n).

Proof. The inequality $(a-b)^2 \le 2(a^2+b^2)$ allows us to deduce

$$\iint\limits_{\Omega\,\Omega} \frac{(f_n(x)-f_n(y))^2}{|x-y|^2} r_n(x-y)\,\mathrm{d}y\,\mathrm{d}x \le C(1-s_n)\,\|f_n\|^2_{L^2(\Omega)} \overset{n\to\infty}{\longrightarrow} 0.$$

Let $\phi \in C_0^\infty(\Omega)$. Using Taylor's formula it is easy to see, that

$$\int\limits_{(y-x)\cdot e_i} \frac{\phi(y)-\phi(x)}{|y-x|} \rho_n(x-y)\,\mathrm{d}y \overset{n\to\infty}{\longrightarrow} K\nabla\phi(x)\cdot e_i,$$

where the constant K can be computed explicitly.

We extend f_n and ϕ to the whole $\mathbb{R}^d$ by zero outside Ω and apply Lemma 2.6 with $g = f_n$ and $\rho = \rho_n$ to obtain

$$\begin{aligned}\left|\int\limits_{\Omega} f_n(x) \int\limits_{(y-x)\cdot e_i \geq 0} \frac{\phi(y)-\phi(x)}{|x-y|}\rho_n(x-y)\,\mathrm{d}y\,\mathrm{d}x\right| &\leq \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{|f_n(x)-f_n(y)|}{|x-y|}\,|\phi(y)|\,\rho_n(x-y)\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \iint\limits_{\Omega\,\Omega} \frac{|f_n(x)-f_n(y)|}{|x-y|}\,|\phi(y)|\,\rho_n(x-y)\,\mathrm{d}y\,\mathrm{d}x \\ &+ \int\limits_{\Omega^c} \int\limits_{\mathrm{supp}(\phi)} |f_n(y)|\,|\phi(y)|\,\frac{\rho_n(x-y)}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x \\ &= I_n + II_n. \end{aligned} \tag{2.4}$$

Hölders inequality, (2.3) and a change of variables yields

$$\begin{aligned} I_n &\leq \left(\iint\limits_{\Omega\,\Omega} \frac{(f_n(x)-f_n(y))^2}{|x-y|^2}\rho_n(x-y)\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2} \left(\iint\limits_{\mathbb{R}^d\,\Omega} \phi^2(y)\rho_n(h)\,\mathrm{d}y\,\mathrm{d}h\right)^{1/2} \\ &\leq \left(\iint\limits_{\Omega\,\Omega} \frac{(f_n(x)-f_n(y))^2}{|x-y|^2}\rho_n(x-y)\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2} \|\phi\|_{L^2(\Omega)}\,. \end{aligned}$$

Set $\delta = \mathrm{dist}(\Omega^c, \mathrm{supp}(\phi))$. Then

$$\int\limits_{|h|>\delta} \rho_n(h)\,\mathrm{d}h = (1-s_n) \int\limits_{|h|>\delta} |h|^{2-d-2s_n}\,\mathrm{d}h \leq (1-s_n)C_\delta. \tag{2.5}$$

By Hölders inequality and (2.5)

$$II_n \leq \frac{1}{\delta} C_\delta (1-s_n)\,\|f_n\|_{L^2(\Omega)}\,\|\phi\|_{L^2(\Omega)} \longrightarrow 0$$

for $n \to \infty$. Letting $n \to \infty$ in (2.4), dominated convergence implies

$$K \left|\int\limits_{\Omega} f(x)(\nabla\phi(x)\cdot e_i)\,\mathrm{d}x\right| \leq A\,\|\phi\|_{L^2(\Omega)}\,.$$

This holds for e_i, $i = 1,..,d$ and thus

$$\left|\int\limits_{\Omega} f\frac{\partial_i \phi}{\partial x_i}\,\mathrm{d}x\right| \leq \frac{A}{K}\,\|\phi\|_{L^2(\Omega)}\,.$$

This proves the assertion. □

2.1.3. Further characterizations of Sobolev spaces

Now we give an alternative definition of the above defined spaces. To this end we use an intrinsic characterization of Besov spaces $B^s_{pq}(\Omega)$, $s \in \mathbb{R}$, $1 \le p, q \le \infty$ taken from [Tri06], that is available for bounded Lipschitz domains Ω. In the special case $p = q = 2$ these spaces coincide with the Sobolev spaces $H^s(\Omega)$. This can be seen easily from the Fourier definition of both spaces.

Define first and second order differences of a function f by

$$\begin{aligned}\Delta_h f(x) &= f(x) - f(x+h),\\ \Delta_h^2 f(x) &= 2f(x+h) - f(x) - f(x+2h).\end{aligned}$$

We begin with the following

Proposition 2.7. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and $0 < s \le 1$. For $x \in \Omega$ define the set*

$$V(x,t) = \{h \in \mathbb{R}^d : |h| < t \text{ and } x + \tau h \in \Omega \text{ for } 0 \le \tau \le 2\}.$$

Define the ball means of second differences of a function f in Ω by

$$d_t^\Omega f(x) = \left(t^{-d} \int\limits_{V(x,t)} \left|\left(\Delta_h^2 f\right)(x)\right|^2 \,\mathrm{d}h \right)^{1/2}, \quad x \in \Omega, t > 0. \tag{2.6}$$

Then $H^s(\Omega)$ is the collection of all $f \in L^2(\Omega)$ such that

$$\|f\|_{L^2(\Omega)} + \left(\int\limits_0^1 t^{-2s} \left\|d_t^\Omega f\right\|^2_{L^2(\Omega)} \frac{\mathrm{d}t}{t} \right)^{1/2} < \infty \tag{2.7}$$

in the sense of equivalent norms.

Proof. The proposition is a direct consequence of [Tri06, Thm. 1.118 (ii)] with $p = q = M = u = 2$. (2.7) characterizes the Besov space $B^s_{22}(\Omega)$. For $p = q = 2$ the Besov space $B^s_{pq}(\Omega)$ coincides with the (fractional) Sobolev space $H^s(\Omega)$. ☐

In the following we replace the ball means by an appropriate integration over a suitable subset of Ω. For this we define the following set:

$$V_\Omega^x = \{y \in \mathbb{R}^d : y \in \Omega \text{ and } \frac{x+y}{2} \in \Omega\}.$$

This is the set of all points $y \in \Omega$ such that the middle point $\frac{x+y}{2}$ is also in Ω. Of course, for any convex set Ω, $V_\Omega^x = \Omega$. We end up with a definition of classical and fractional Sobolev spaces in terms of second differences.

Corollary 2.8. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. $H^s(\Omega)$, $0 < s \leq 1$, is the collection of all $f \in L^2(\Omega)$ such that*

$$\|f\|_{L^2(\Omega)} + \left(\iint\limits_{\Omega\, V_\Omega^x} \frac{\left(f(x) - 2f(\frac{x+y}{2}) + f(y)\right)^2}{|x-y|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}x \right)^{1/2} < \infty$$

in the sense of equivalent norms.

Proof. Define for $x \in \Omega$

$$\widetilde{V}(x,t) = \{y \in \Omega : |x-y| < 2t \text{ and } \frac{x+y}{2} \in \Omega\}.$$

Substituting $y = x + 2h$ and using the definition of V_Ω^x together with Fubini's theorem yields

$$\begin{aligned}
\int\limits_\Omega \int\limits_0^1 \int\limits_{V(x,t)} \left|\Delta_h^2 f(x)\right|^2 \mathrm{d}h t^{-d-2s} \frac{\mathrm{d}t}{t} \,\mathrm{d}x &= \int\limits_\Omega \int\limits_0^1 \int\limits_{V(x,t)} |f(x) - 2f(x+h) + f(x+2h)|^2 \,\mathrm{d}h t^{-d-2s} \frac{\mathrm{d}t}{t} \,\mathrm{d}x \\
&\leq \int\limits_\Omega \int\limits_0^1 \int\limits_{\widetilde{V}(x,t)} \left|f(x) - 2f(\frac{x+y}{2}) + f(y)\right|^2 \mathrm{d}y t^{-d-2s} \frac{\mathrm{d}t}{t} \,\mathrm{d}x \\
&= \int\limits_\Omega \int\limits_0^1 \int\limits_{V_\Omega^x} \left|f(x) - 2f(\frac{x+y}{2}) + f(y)\right|^2 \mathbb{1}_{\{|x-y|<2t\}} \,\mathrm{d}y t^{-d-2s} \frac{\mathrm{d}t}{t} \,\mathrm{d}x \\
&= \int\limits_\Omega \int\limits_{V_\Omega^x} \left|f(x) - 2f(\frac{x+y}{2}) + f(y)\right|^2 \int\limits_{\frac{|x-y|}{2}}^1 t^{-d-2s} \frac{\mathrm{d}t}{t} \,\mathrm{d}y \,\mathrm{d}x \\
&= \frac{2^{d+2s}}{d+2s} \int\limits_\Omega \int\limits_{V_\Omega^x} \frac{\left|f(x) - 2f(\frac{x+y}{2}) + f(y)\right|^2}{|x-y|^{d+2s}} \,\mathrm{d}y \,\mathrm{d}x \\
&\quad - \frac{1}{d+2s} \int\limits_\Omega \int\limits_{V_\Omega^x} \left|f(x) - 2f(\frac{x+y}{2}) + f(y)\right|^2 \mathrm{d}y \,\mathrm{d}x.
\end{aligned}$$

Since Ω is bounded, the second term on the right-hand side is absolutely bounded by $\|f\|_{L^2(\Omega)}^2$. This proves one direction in the asserted equivalence of norms. The proof of the inverse inequality can be obtained in an analogous way using the regularity of $\partial\Omega$ and by adding an L^2-term to get the inverse inequality in the second line of the above computation. □

Remark 2.9. 1. For $s = 1$ this is a gradient free definition of $H^1(\Omega)$.

2. For $0 < s < 1$ we get an alternative definition of the fractional Sobolev spaces in terms of second differences. Note that there is no norming constant $(1-s)$ in this definition. Nevertheless it is possible to obtain

$$\lim_{s \to 1^-} \|f\|_{H^s(\Omega)} = C \|f\|_{H^1(\Omega)}.$$

To conclude this one has to check that the constants in the equivalence of norms in Proposition 2.7 does not depend on s.

In the next section we use this characterization of $H^1(\Omega)$ to generalize this spaces by replacing V_Ω^x with $\mathbb{R}^d$.

2.2. Function spaces with regularity over the boundary

In this section we define function spaces for functions defined on the whole $\mathbb{R}^d$ whose restriction to a suitable set $\Omega \subset \mathbb{R}^d$ has the regularity properties of $H^1(\Omega)$, $H^s(\Omega)$ respectively. In addition they also have some regularity over the boundary of Ω.

2.2.1. Nonlocal generalization of $H^1(\Omega)$

The following definition is motivated by the characterization of Corollary 2.8.

Definition 2.10. Let Ω be an arbitrary subset of $\mathbb{R}^d$. We define the following linear space:

$$V(\Omega|\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) | \, \|f\|_{V(\Omega|\mathbb{R}^d)} < \infty\},$$

where

$$\|f\|^2_{V(\Omega|\mathbb{R}^d)} = \|f\|^2_{L^2(\mathbb{R}^d)} + \iint\limits_{\Omega\,\mathbb{R}^d} \frac{(f(x) - 2f(\frac{x+y}{2}) + f(y))^2}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x.$$

We set

$$[f]^2_{V(\Omega|\mathbb{R}^d)} = \iint\limits_{\Omega\,\mathbb{R}^d} \frac{(f(x) - 2f(\frac{x+y}{2}) + f(y))^2}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x.$$

Proposition 2.11. *$V(\Omega|\mathbb{R}^d)$ endowed with the norm $\|\cdot\|_{V(\Omega|\mathbb{R}^d)}$ is a Hilbert space.*

The proof follows analogously to the proof of Lemma 2.19, below. Since we do not use the Hilbert property of $V(\Omega|\mathbb{R}^d)$ we omit the proof. For Lipschitz domains Ω, it follows from Corollary 2.8 that $\nabla f \in L^2(\Omega)$ if $f \in V(\Omega|\mathbb{R}^d)$. In addition, the integration over differences where at least one node is in Ω^c gives some regularity of the function over the boundary of Ω. Below we will give a second definition of $V(\Omega|\mathbb{R}^d)$ containing the $H^1(\Omega)$-norm. Our next result shows that we can approximate functions in $V(\Omega|\mathbb{R}^d)$ by smooth functions, at least for sufficiently smooth domains $\Omega \subset \mathbb{R}^d$.

Lemma 2.12. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded and $\partial\Omega$ be C^1. Let $u \in V(\Omega|\mathbb{R}^d)$. Then there exist functions $u_n \in C_c^\infty(\mathbb{R}^d)$ such that*

$$u_n \overset{n\to\infty}{\longrightarrow} u \quad \text{in } V(\Omega|\mathbb{R}^d).$$

Our proof uses the standard technique used to prove that $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$, cf [Eva10, 5.3.3. Thm. 3] and extents it to the nonlocal setting.

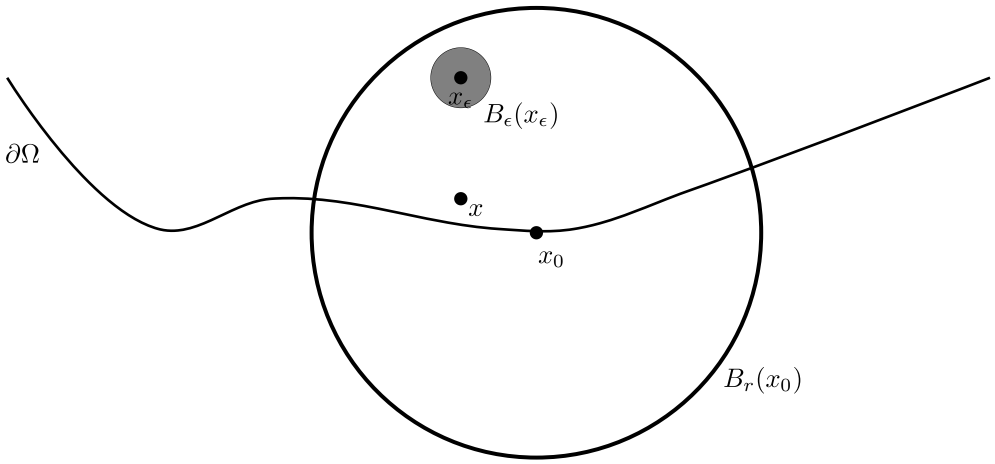

Figure 2.1.: The shifted point x_ε and the area of convolution $B_\varepsilon(x_\varepsilon)$

Proof. We prove that there is a sequence (u_n) of functions in $C_c^\infty(\mathbb{R}^d)$ such that the seminorms $[\cdot]_{V(\Omega|\mathbb{R}^d)}$ of u_n converge to u as $n \to \infty$. The convergence of the L^2-norms of u_n follows by standard arguments, since the sequence (u_n) is constructed by translation and convolution of the function u with a mollifier.

In order to simplify the notation, we define

$$\Delta u(x;y) = u(x) - 2u(\frac{x+y}{2}) + u(y). \tag{2.8}$$

Step 1:
Let $x_0 \in \partial\Omega$. Since $\partial\Omega$ is C^1, there exists $r > 0$ and a function $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$, $\gamma \in C^1$, such that (upon relabeling the coordinates)

$$\Omega \cap B_r(x_0) = \{x \in B_r(x_0) | x_d > \gamma(x_1, ..., x_{d-1})\}.$$

Set $x = (x_1, ..., x_{d-1}, x_d) = (x', x_d)$. For $x \in B_{r/2}(x_0)$ we define the shifted point

$$x_\varepsilon = x + 2\varepsilon e_n.$$

For small ε, $B_\varepsilon(x_\varepsilon) \Subset \Omega$ for $x \in \Omega$. We define $u_\varepsilon(x) = u(x_\varepsilon)$ and

$$v_\varepsilon = \eta_\varepsilon * u_\varepsilon$$

where η_ε is a smooth mollifier having support in $B_\varepsilon(0)$.

Step 2:
We prove that for $\operatorname{supp} u \Subset B_{r/2}(x_0)$

$$[v_\varepsilon - u]_{V(\Omega|\mathbb{R}^d)} \overset{\varepsilon\to 0}{\longrightarrow} 0.$$

We have

$$[v_\varepsilon - u]_{V(\Omega|\mathbb{R}^d)} \leq [v_\varepsilon - u_\varepsilon]_{V(\Omega|\mathbb{R}^d)} + [u_\varepsilon - u]_{V(\Omega|\mathbb{R}^d)} = I + II.$$

For the term II we estimate:

$$\begin{aligned}
[u_\varepsilon]^2_{V(\Omega|\mathbb{R}^d)} &= \iint\limits_{\Omega\,\mathbb{R}^d} \frac{(\Delta u_\varepsilon(x;y))^2}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x \\
&= \iint\limits_{\Omega\,\mathbb{R}^d} \mathbb{1}_{\{x_n - 2\varepsilon > \gamma(x')\}}(x) \frac{(\Delta u(x;y))^2}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x \\
&\leq [u]^2_{V(\Omega|\mathbb{R}^d)} .
\end{aligned}$$

Vitali's convergence theorem yields

$$[u_\varepsilon - u]_{V(\Omega|\mathbb{R}^d)} \to 0.$$

Now we look at the first term I:

$$\begin{aligned}
[v_\varepsilon - u_\varepsilon]^2_{V(\Omega|\mathbb{R}^d)} &\leq \iint\limits_{\Omega\,\mathbb{R}^d} \frac{(\Delta v_\varepsilon(x;y) - \Delta u_\varepsilon(x;y))^2}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x \\
&\leq \iint\limits_{\Omega\,\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} (\Delta u_\varepsilon(x-z, y-z)) \eta_\varepsilon(z)\,\mathrm{d}z - \Delta u_\varepsilon(x;y) \right)^2 |x-y|^{-d-2} \,\mathrm{d}y\,\mathrm{d}x \\
&\leq \iint\limits_{\Omega\,\mathbb{R}^d} \left(\int\limits_{B_1(0)} (\Delta u_\varepsilon(x-\varepsilon z; y-\varepsilon z) - \Delta u_\varepsilon(x;y)) \eta(z)\,\mathrm{d}z \right)^2 |x-y|^{-d-2} \,\mathrm{d}y\,\mathrm{d}x \\
&\leq \iiint\limits_{\Omega\times\mathbb{R}^d\times B_1(0)} (\Delta u_\varepsilon(x-\varepsilon z; y-\varepsilon z) - \Delta u_\varepsilon(x;y))^2 |x-y|^{-d-2} \eta(z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \iiint\limits_{B_1(0)\times\Omega\times\mathbb{R}^d} (\Delta u_\varepsilon(x-\varepsilon z; y-\varepsilon z) - \Delta u_\varepsilon(x;y))^2 |x-y|^{-d-2} \,\mathrm{d}y\,\mathrm{d}x\,\eta(z)\,\mathrm{d}z\,.
\end{aligned}$$

Using the dominated convergence theorem twice for $u_\varepsilon(\cdot)$ and $u_\varepsilon(\cdot - \varepsilon z)$ with majorant u gives for fixed $z \in B_1(0)$

$$\iint\limits_{\Omega\,\mathbb{R}^d} (\Delta u_\varepsilon(x-\varepsilon z; y-\varepsilon z) - \Delta u_\varepsilon(x;y))^2 |x-y|^{-d-2} \,\mathrm{d}y\,\mathrm{d}x \longrightarrow 0.$$

Further, the function

$$z \mapsto \left| \eta(z) \iint\limits_{\Omega\,\mathbb{R}^d} (\Delta u_\varepsilon(x-\varepsilon z; y-\varepsilon z) - \Delta u_\varepsilon(x;y))^2 |x-y|^{-d-2} \,\mathrm{d}y\,\mathrm{d}x \right|$$

is bounded by $4[u]_{V(\Omega|\mathbb{R}^d)}$ for all $\varepsilon > 0$ and a.e. $z \in B_1(0)$. Thus, again by Lebesgues dominated convergence theorem

$$\iiint\limits_{B_1(0)\times\Omega\times\mathbb{R}^d} (\Delta u_\varepsilon(x-\varepsilon z; y-\varepsilon z) - \Delta u_\varepsilon(x;y))^2 |x-y|^{-d-2} \,\mathrm{d}y\,\mathrm{d}x\,\eta(z)\,\mathrm{d}z \longrightarrow 0$$

for $\varepsilon \to 0$.

Step 3:
Let $u \in V(\Omega|\mathbb{R}^d)$ be arbitrary. Let $R > 0$ such that $\Omega \subset B_R(0)$. Let $f_R \in C_c^\infty(B_{3R}(0))$ with $f_R \le 1$ and $f_R(x) = 1$ for all $x \in B_{2R}(0)$. Define $u_R = f_R u$. Then $\operatorname{supp}(u_R) \subset B_{3R}(0)$ and $[u - u_R]_{V(\Omega|\mathbb{R}^d)} \overset{R\to\infty}{\longrightarrow} 0$.
Step 4:
Let $x_i \in \partial\Omega$, $r_i > 0$, $i = 1, .., N$, such that

$$\partial\Omega \subset \bigcup_{i=1}^{N} B_{r_i/2}(x_i),$$

where the r_i are chosen small enough, such that (again opon relabeling the coordinates) we can assume

$$\Omega \cap B_{2r_i}(x_i) = \{x \in B_{r_i}(x_i) | x_d > \gamma(x')\}$$

for some smooth $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ as in Step 1. Let $\Omega^* = \{x \in \mathbb{R}^d | \operatorname{dist}(x, \Omega) > \frac{1}{2} \min_{i=\{1,..,N\}} r_i\}$ and $\Omega_0 = \{x \in \Omega | \operatorname{dist}(x, \Omega^c) > \frac{1}{2} \min_{i=\{1,..,N\}} r_i\}$. Then

$$\bigcup_{i=1}^{N} B_{r_i}(x_i) \cup \Omega^* \cup \Omega_0 = \mathbb{R}^d.$$

Let $\{\xi_i\}_{i=0}^{N+1}$ be a smooth partition of unity subordinated to the above constructed sets.
We define

$$u_i = \xi_i \cdot u_R \text{ for all } i \in \{0, .., N+1\},$$

and thus

$$\begin{aligned}
&\operatorname{supp} u_i \subset B_{r_i}(x_i) \text{ for } i \in \{1, ..N\},\\
&\operatorname{supp} u_0 \subset \Omega_0,\\
&\operatorname{supp} u_{N+1} \subset \Omega^*.
\end{aligned}$$

Step 5:
Let $\delta > 0$. Let $i \in \{1, .., N\}$. By Step 2 there exists a sequence $v_\varepsilon^i \in C_c^\infty(B_{r_i}(x_i))$ such that

$$[u_i - v_\varepsilon^i]_{V(\Omega|\mathbb{R}^d)} \longrightarrow 0$$

for $\varepsilon \to 0$. Thus we can choose $\varepsilon_0 > 0$ such that $[u_i - v_\varepsilon^i]_{V(\Omega|\mathbb{R}^d)} < \frac{\delta}{N+2}$ for all $i \in \{1, .., N\}$.
For $i = N+1$ define $v_\varepsilon^{N+1} = \eta_\varepsilon * u_{N+1}$ and set $r = \frac{1}{4} \min_{i\in\{1,..,N\}} r_i$. Choosing $\varepsilon < r$ and since $\operatorname{supp} u_{N+1} \subset \Omega^*$ for all $x \in \Omega$, $y \in \mathbb{R}^d$ and $z \in B_\varepsilon(0)$

$$\Delta u_{N+1}(x; y) = \Delta v_\varepsilon^{N+1}(x - z; y - z) = 0 \quad \text{or} \quad |x - y| > r.$$

Thus

$$
\begin{aligned}
[v_\varepsilon^{N+1} - u_{N+1}]^2_{V(\Omega|\mathbb{R}^d)} &= \iint_{\Omega\,\mathbb{R}^d} (\Delta v_\varepsilon^{N+1}(x;y) - \Delta u^{N+1}(x;y))^2 \, |x-y|^{-d-2} \, \mathrm{d}y \, \mathrm{d}x \\
&= \iint_{\Omega\mathbb{R}^d} \left(\int_{B_\varepsilon(0)} \Delta u_{N+1}(x-z; y-z) - \Delta u_{N+1}(x;y)\eta_\varepsilon(z) \, \mathrm{d}z \right)^2 |x-y|^{-d-2} \, \mathrm{d}x \, \mathrm{d}y \\
&\le C_r \iiint_{B_1(0)\times\Omega\times\mathbb{R}^d} (\Delta u_{N+1}(x-\varepsilon z; y-\varepsilon z) - \Delta u_{N+1}(x;y))^2 \, \mathrm{d}y \, \mathrm{d}x \eta(z) \, \mathrm{d}z.
\end{aligned}
$$

By the continuity of the shift in $L^2(\mathbb{R}^d)$

$$
\iint_{\Omega\,\mathbb{R}^d} (\Delta u_{N+1}(x-\varepsilon z; y-\varepsilon z) - \Delta u_{N+1}(x;y))^2 \, \mathrm{d}y \, \mathrm{d}x \longrightarrow 0.
$$

Further, for any $z \in B_1(0)$, the map

$$
z \mapsto \left| \eta(z) \int_{\Omega\,\mathbb{R}^d} (\Delta u_{N+1}(x-\varepsilon z; y-\varepsilon z) - \Delta u_{N+1}(x;y))^2 \, \mathrm{d}y \, \mathrm{d}x \right|
$$

is bounded. Thus $[v_\varepsilon^{N+1} - u_{N+1}]_{V(\Omega|\mathbb{R}^d)} \to 0$ by dominated convergence and we find $\varepsilon_0 > 0$, such that $[v_\varepsilon^{N+1} - u_{N+1}]_{V(\Omega|\mathbb{R}^d)} < \frac{\delta}{N+2}$ for all $\varepsilon < \varepsilon_0$. We define $v_\varepsilon^0 = \eta_\varepsilon * u_0$. Thus for $\varepsilon < r$

$$
\operatorname{supp} v_\varepsilon^0 \Subset \Omega.
$$

The convergence $v_\varepsilon^0 \to u_0$ follows by the same arguments as above and we find $\varepsilon_0 > 0$ such that $[v_\varepsilon^0 - u_0]_{V(\Omega|\mathbb{R}^d)} < \frac{\delta}{N+2}$ for all $\varepsilon, \varepsilon_0$.

Step 6:
Define $v_\varepsilon = \sum_{i=0}^{N+1} \xi_i * v_\varepsilon^i \in C_c^\infty(\mathbb{R}^d)$. Since $u_R(x) = \sum_{i=0}^{N+1} u_i(x)$, we have

$$
\begin{aligned}
[u_R - v_\varepsilon]_{V(\Omega|\mathbb{R}^d)} &\le \left[\sum_{i=0}^{N+1} \left(\xi_i v_\varepsilon^i - \xi_i u \right) \right]_{V(\Omega|\mathbb{R}^d)} \\
&\le \sum_{i=0}^{N+1} [\left(\xi_i v_\varepsilon^i - \xi_i u \right)]_{V(\Omega|\mathbb{R}^d)} \\
&\le (N+2) \frac{\delta}{N+2}.
\end{aligned}
$$

Choosing $R = \frac{1}{\varepsilon}$ in Step 3, concludes

$$
[u - v_\varepsilon]_{V(\Omega|\mathbb{R}^d)} \le [u - u_R]_{V(\Omega|\mathbb{R}^d)} + [u_R - v_\varepsilon|_{V(\Omega|\mathbb{R}^d)} \overset{\varepsilon\to 0}{\longrightarrow} 0.
$$

The convergence in $L^2(\mathbb{R}^d)$ follows from the continuity of the shift in $L^2(\mathbb{R}^d)$. □

In the following we introduce an equivalent norm on the space $V(\Omega|\mathbb{R}^d)$. This second norm allows us to examine the relation between $V(\Omega|\mathbb{R}^d)$ and the space $V^s(\Omega|\mathbb{R}^d)$, cf, Subsection 2.2.3. In Subsection 2.2.2 we use this norm together with the density property of Lemma 2.12 to obtain the existence of a regular Dirichlet form on $L^2(\mathbb{R}^d)$.

Theorem 2.13. *Let Ω be a C^1-domain in $\mathbb{R}^d$. Then an equivalent norm on $V(\Omega|\mathbb{R}^d)$ is given by*

$$\|f\|^{\clubsuit}_{V(\Omega|\mathbb{R}^d)} = \left(\|f\|^2_{L^2(\mathbb{R}^d)} + \|\nabla f\|^2_{L^2(\Omega)} + \iint\limits_{\Omega\,\Omega^c} \frac{(f(x)-f(y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x \right)^{1/2}.$$

The idea of the proof is following. By Corollary 2.8 we can estimate $\nabla u \in L^2(\Omega)$. To estimate the double integral we proceed as follows:
1. Step: Assume that $\Omega = \mathbb{R}^d_+$ and that $\operatorname{supp} f \subset B_r(0)$.
2. Step: To estimate the double-integral, we use the identity

$$2\Delta_h f(x) = \Delta_{2h} f(x) - \Delta_h^2 f(x+h),$$

where $x, h \in \mathbb{R}^d$. This allows us to rewrite the first differences as second differences plus 'large' first ones. Since we integrate over all differences, we can compensate the 'large' first differences on the right-hand side by the left-hand-side.
3. Step: For a general smooth domain Ω we cover Ω and $\partial\Omega$ by a finite number of balls. Using a coordinate transformation and the above steps on any of the sets yields the assertion.

We begin with the following

Lemma 2.14. *There is $C \geq 1$, such that for $f \in L^2(\mathbb{R}^d)$*

$$\int\limits_{\{0<x_d<1\}} \int\limits_{\{-x_d>y_d>-1\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x \leq C \int\limits_{\{0<x_d<3\}} \int\limits_{\{0>z_d>-x_d\}} \frac{(f(x)-f(z))^2}{|x-z|^{d+2}}\,\mathrm{d}z\,\mathrm{d}x. \tag{2.9}$$

The proof of the lemma in the one dimensional case was given by Luis Silvestre. The key idea is to integrate inequality (2.10), below. This idea is extended to the higher dimensional case by an appropriate parametrization of the involved integrals. [2]

Proof. Without loss of generality assume $d = 2$. Let $x \in \{0 < x_2 < 1\}, y \in \{-x_2 > y_2 > 1\}$ and $z, v \in \mathbb{R}^2$ to be chosen later. By the triangle inequality

$$|f(x) - f(y)| \leq |f(v) - f(y)| + |f(v) - f(z)| + |f(x) - f(z)|$$

and thus by Young's inequality

$$\frac{1}{3}\frac{|f(x)-f(y)|^2}{|x-y|^4} \leq \frac{|f(v)-f(y)|^2}{|x-y|^4} + \frac{|f(v)-f(z)|^2}{|x-y|^4} + \frac{|f(x)-f(z)|^2}{|x-y|^4}. \tag{2.10}$$

[2]There was a mistake in an earlier version of Lemma 2.14, which is corrected in the published version of the thesis.

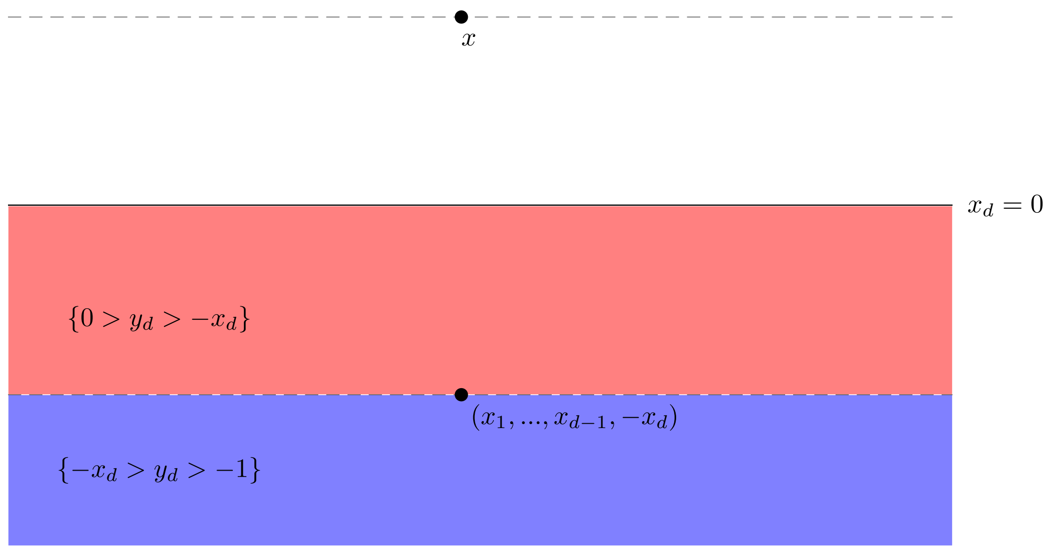

Figure 2.2.: Area of integration in Lemma 2.14

Let $x \in \mathbb{R} \times (0,1)$. Then for any $y \in \mathbb{R} \times (-1, -x_2)$ we find $\tau_1, \tau_2 \in \mathbb{R}$ and $t_1, t_2 \in (0,1)$, such that

$$y = \gamma(\tau_1, \tau_2, t_1) \quad \text{and} \quad x = \gamma(\tau_1, \tau_2, t_2)$$

where

$$\gamma(\tau_1, \tau_2, t) = ((1-t)\tau_2 + t\tau_1, -1 + 2t), \quad t \in [0,1].$$

Given this parametrization, we choose $v = 2x - y$ and $z = \gamma(\tau_1, \tau_2, t_3) \in \overline{(\gamma(\tau_1, \tau_2, 1-t_2), \gamma(\tau_1, \tau_2, \frac{3-2t_2}{4}))}$, which means that $t_3 \in (1-t_2, \frac{3-2t_2}{4})$. For all these point we have

$$|x - y| = \frac{1}{2} |v - y|, \tag{2.11}$$

$$|x - y| \geq \frac{1}{2} |v - z|, \tag{2.12}$$

$$|x - y| \geq |x - z|. \tag{2.13}$$

Integrating the inequality (2.10) in t_3 from $1 - t_2$ to $\frac{3-2t_2}{4}$ yields

$$\frac{1}{3} \frac{|f(x) - f(y)|^2}{|x-y|^{d+2}} \leq \frac{|f(v) - f(y)|^2}{|x-y|^{d+2}} \tag{2.14}$$

$$+ \frac{1}{\left|\gamma(\tau_1, \tau_2, 1-t_2) - \gamma(\tau_1, \tau_2, \frac{3-2t_2}{4})\right|} \int_{1-t_2}^{\frac{3-2t_2}{4}} \left(\frac{|f(v) - f(\gamma(\tau_1, \tau_2, t_3))|^2}{|x-y|^{d+2}} + \frac{|f(x) - f(\gamma(\tau_1, \tau_2, t_3))|^2}{|x-y|^{d+2}} \right) \mathrm{d}t,$$

Recall that

$$\begin{aligned} x &= \gamma(\tau_1, \tau_2, t_2), \\ y &= \gamma(\tau_1, \tau_2, t_1), \\ v &= \gamma(\tau_1, \tau_2, 2t_2 - t_1). \end{aligned}$$

In order to prove the lemma, we integrate both sides of the above inequality and estimate the three terms on the right-hand side. Note that instead of integrating w.r.t $x = (x_1, x_2)$ and $y = (y_1, y_2)$ we can also integrate over τ_1, τ_2, t_1 and t_2. Then the left hand side of (2.9) becomes

$$\int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \int\limits_{\frac{1}{2}}^{1} \int\limits_{0}^{1-t_2} \frac{(f(\gamma(\tau_1, \tau_2, t_2)) - f(\gamma(\tau_1, \tau_2, t_1)))^2}{|\gamma(\tau_1, \tau_2, t_2) - \gamma(\tau_1, \tau_2, t_1)|^4} \left|\gamma'(\tau_1, \tau_2, t_2)\right| \left|\gamma'(\tau_1, \tau_2, t_1)\right| \mathrm{d}t_1 \, \mathrm{d}t_2 \, \mathrm{d}\tau_2 \, \mathrm{d}\tau_1.$$

We use this notation to estimate the second and third term.
1st term: Since $v = 2x - y$ we have $\mathrm{d}x = 2^{-2}\,\mathrm{d}v$ and by (2.11)

$$\begin{aligned} &\int\limits_{\{0<x_d<1\}} \int\limits_{\{-x_d>y_d>-1\}} \frac{|f(2x-y) - f(y)|^2}{|x-y|^{d+2}} \mathrm{d}y \, \mathrm{d}x \\ =&2^{d+2} 2^{-d} \int\limits_{\{0<v_d<3\}} \int\limits_{\{-v_d/3>y_d>\max\{-1,-v_d\}\}} \frac{|f(v) - f(y)|^2}{|v-y|^{d+2}} \mathrm{d}y \, \mathrm{d}v \\ \leq& 4 \int\limits_{\{0<v_d<3\}} \int\limits_{\{0>y_d>-v_d\}} \frac{|f(v) - f(y)|^2}{|v-y|^{d+2}} \mathrm{d}y \, \mathrm{d}v \, . \end{aligned}$$

To estimate the second and third term we first note that

$$|\partial_t \gamma(\tau_1, \tau_2, t)| = \left|\gamma'(\tau_1, \tau_2, t)\right| = \sqrt{(\tau_1 - \tau_2)^2 + 4}$$

for all $t \in (0,1)$ and for abbreviation we write $|\gamma'(\tau_1, \tau_2|)$. Next we examine the behavior of $|\gamma(\tau_1, \tau_2, s_1) - \gamma(\tau_1, \tau_2, s_2)|$
for $0 < s_1, s_2 < 1$. It is easy to check that

$$|\gamma(\tau_1, \tau_2, s_1) - \gamma(\tau_1, \tau_2, s_2)| = |s_1 - s_2| \sqrt{(\tau_1 - \tau_2)^2 + 4} = |s_1 - s_2| \left|\gamma'(\tau_1, \tau_2)\right| \tag{2.15}$$

2nd term: After integrating in τ_1, τ_2, t_1 and t_2, we obtain

$$\int\limits_{\mathbb{R}^2} \int\limits_{\frac{1}{2}}^{1} \int\limits_{0}^{1-t_2} \frac{1}{\left|\frac{1}{2}(\frac{1}{2} - t_2)\right| |\gamma'(\tau_1, \tau_2)|} \int\limits_{1-t_2}^{\frac{3-2t_2}{4}} \frac{(f(\gamma(\tau_1, \tau_2, 2t_2 - t_1)) - f(\gamma(\tau_1, \tau_2, t_3)))^2}{|\gamma(\tau_1, \tau_2, t_2) - \gamma(\tau_1, \tau_2, t_1)|^4} \left|\gamma'(\tau_1, \tau_2)\right|^3 \mathrm{d}t_3 \, \mathrm{d}t_1 \, \mathrm{d}t_2 \, \mathrm{d}\tau_2 \, \mathrm{d}\tau_1.$$

The change of variables $s_1 = 2t_2 - t_1$ and changing the order of integration yields

$$\leq \int_{\mathbb{R}^2} \int_{\frac{1}{2}}^{1} \int_{2t_2}^{3t_2-1} \frac{1}{\left|\frac{1}{2}(\frac{1}{2}-t_2)\right|} \int_{1-t_2}^{\frac{3-2t_2}{4}} \frac{(f(\gamma(\tau_1,\tau_2,s_1)) - f(\gamma(\tau_1,\tau_2,t_3)))^2}{|\gamma(\tau_1,\tau_2,t_3) - \gamma(\tau_1,\tau_2,s_1)|^4} \left|\gamma'(\tau_1,\tau_2)\right|^2 \,\mathrm{d}t_3 \,\mathrm{d}s_1 \,\mathrm{d}t_2 \,\mathrm{d}\tau_2 \,\mathrm{d}\tau_1$$

$$\overset{(2.12)}{\leq} \int_{\mathbb{R}^2} \int_{\frac{1}{2}}^{3} \int_{1-s_1}^{\frac{1}{2}} \frac{(f(\gamma(\tau_1,\tau_2,s_1)) - f(\gamma(\tau_1,\tau_2,t_3)))^2}{|\gamma(\tau_1,\tau_2,t_2) - \gamma(\tau_1,\tau_2,t_1)|^4} \left|\gamma'(\tau_1,\tau_2)\right|^2 \int_{\max(\frac{s_1+1}{3},1-t_3)}^{\min(\frac{s_1}{2},\frac{3-4t_3}{2})} \frac{1}{\left|\frac{1}{2}(\frac{1}{2}-t_2)\right|} \,\mathrm{d}t_2 \,\mathrm{d}t_3 \,\mathrm{d}s_1 \,\mathrm{d}\tau_2 \,\mathrm{d}\tau_1$$

$$\leq 2 \int_{\mathbb{R}^2} \int_{\frac{1}{2}}^{3} \int_{1-s_1}^{\frac{1}{2}} \frac{(f(\gamma(\tau_1,\tau_2,s_1)) - f(\gamma(\tau_1,\tau_2,t_3)))^2}{|\gamma(\tau_1,\tau_2,t_2) - \gamma(\tau_1,\tau_2,t_1)|^4} \left|\gamma'(\tau_1,\tau_2)\right|^2 \int_{1-t_3}^{\frac{3-4t_3}{2}} \frac{1}{\left|(\frac{1}{2}-t_2)\right|} \,\mathrm{d}t_2 \,\mathrm{d}t_3 \,\mathrm{d}s_1 \,\mathrm{d}\tau_2 \,\mathrm{d}\tau_1$$

$$\leq \log(2)2 \int_{\mathbb{R}^2} \int_{\frac{1}{2}}^{3} \int_{1-s_1}^{\frac{1}{2}} \frac{(f(\gamma(\tau_1,\tau_2,s_1)) - f(\gamma(\tau_1,\tau_2,t_3)))^2}{|\gamma(\tau_1,\tau_2,t_2) - \gamma(\tau_1,\tau_2,t_1)|^4} \left|\gamma'(\tau_1,\tau_2)\right|^2 \,\mathrm{d}t_3 \,\mathrm{d}s_1 \,\mathrm{d}\tau_2 \,\mathrm{d}\tau_1$$

$$= 2\log(2) \int_{\{0<v_2<3\}} \int_{\{0>z_2>-v_2\}} \frac{(f(v) - f(z))^2}{|v-z|^4} \,\mathrm{d}z \,\mathrm{d}v.$$

Next we estimate the third term on the right hand side of (2.14).

3rd term: After changing to the parametrized integrals, the third term a reads

$$\int_{\mathbb{R}^2} \int_{\frac{1}{2}}^{1} \int_{0}^{1-t_2} \frac{1}{\left|\frac{1}{2}(\frac{1}{2}-t_2)\right| \left|\gamma'(\tau_1,\tau_2)\right|} \int_{1-t_2}^{\frac{3-2t_2}{4}} \frac{(f(\gamma(\tau_1,\tau_2,t_2)) - f(\gamma(\tau_1,\tau_2,t_3)))^2}{|\gamma(\tau_1,\tau_2,t_2) - \gamma(\tau_1,\tau_2,t_1)|^4} \left|\gamma'(\tau_1,\tau_2)\right|^3 \,\mathrm{d}t_3 \,\mathrm{d}t_1 \,\mathrm{d}t_2 \,\mathrm{d}\tau_2 \,\mathrm{d}\tau_1$$

Here we just need to interchange the order of integration between t_3 and t_1. Using (2.15) this yields

$$\int_{\mathbb{R}^2} \int_{\frac{1}{2}}^{1} \int_{1-t_2}^{\frac{3-2t_2}{4}} \frac{(f(\gamma(\tau_1,\tau_2,t_2)) - f(\gamma(\tau_1,\tau_2,t_3)))^2}{\left|\frac{1}{2}(\frac{1}{2}-t_2)\right|} \int_{0}^{1-t_2} \frac{1}{|t_2-t_1|^4} \,\mathrm{d}t_1 \left|\gamma'(\tau_1,\tau_2)\right|^{-2} \,\mathrm{d}t_3 \,\mathrm{d}t_2 \,\mathrm{d}\tau_2 \,\mathrm{d}\tau_1$$

$$\leq 8 \int_{\mathbb{R}^2} \int_{\frac{1}{2}}^{1} \int_{1-t_2}^{\frac{3-2t_2}{4}} \frac{(f(\gamma(\tau_1,\tau_2,t_2)) - f(\gamma(\tau_1,\tau_2,t_3)))^2}{\left|(\frac{1}{2}-t_2)\right|} \frac{1}{(2t_2-1)^3} \left|\gamma'(\tau_1,\tau_2)\right|^{-2} \,\mathrm{d}t_3 \,\mathrm{d}t_2 \,\mathrm{d}\tau_2 \,\mathrm{d}\tau_1$$

According to (2.15) we need to show that

$$(t_2 - t_3)^4 \asymp \frac{1}{2}(2(t_2 - \frac{1}{2}))^4 \quad \text{for } t_2 \in (\frac{1}{2}, 1), t_3 \in (1-t_2, \frac{3-2t_2}{4}),$$

which is easy to check. Thus we obtain

$$\leq C \int_{\mathbb{R}^2} \int_{\frac{1}{2}}^{1} \int_{1-t_2}^{\frac{3-2t_2}{4}} \frac{(f(\gamma(\tau_1,\tau_2,t_2)) - f(\gamma(\tau_1,\tau_2,t_3)))^2}{|(t_3-t_2)|^4 \, |\gamma'(\tau_1,\tau_2)|^4} \left|\gamma'(\tau_1,\tau_2)\right|^2 \,\mathrm{d}t_3 \,\mathrm{d}t_2 \,\mathrm{d}\tau_2 \,\mathrm{d}\tau_1$$
$$= C \int_{\{0<x_2<1\}} \int_{\{0>z_2>-x_2\}} \frac{(f(x)-f(z))^2}{|x-z|^4} \,\mathrm{d}z \,\mathrm{d}x.$$

The conclusion follows by adding these three inequalities. □

Note that the constant is far from being optimal. With the help of the above lemma, we are able to estimate first order differences on the half space by second order ones. Recall that $\Delta f(x;y) = f(x) - 2f(\frac{x+y}{2}) + f(y)$.

Lemma 2.15. *There is $C \geq 1$ such that for all $f \in L^2(\mathbb{R}^d)$*

$$\iint_{\mathbb{R}^d_+ \, \mathbb{R}^d_-} \frac{(f(x)-f(y))^2}{|x-y|^{d+2}} \,\mathrm{d}y \,\mathrm{d}x \leq C \left(\iint_{\mathbb{R}^d_+ \, \mathbb{R}^d_-} \frac{(\Delta f(x;y))^2}{|x-y|^{d+2}} + \|f\|^2_{L^2(\mathbb{R}^d)} \right). \tag{2.16}$$

Proof. Let us assume that the right-hand side of (2.16) is finite. Otherwise there is nothing to prove. First note that for $\delta > 0$

$$\iint_{\{|x-y|>\delta\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+2}} \,\mathrm{d}y \,\mathrm{d}x \leq \frac{4}{\delta} \|f\|^2_{L^2(\mathbb{R}^d)}. \tag{2.17}$$

To switch from first to second differences, consider the identity

$$2\Delta_h f(x) = \Delta_{2h} f(x) - \Delta_h^2 f(x+h).$$

Combining this with Youngs inequality yields

$$\frac{(\Delta_h f(x))^2}{|h|^2} \leq (1+\varepsilon)\frac{(\Delta_{2h} f(x))^2}{|2h|^2} + (1+\frac{4}{\varepsilon})\frac{\Delta_h^2 f(x+h))^2}{|h|^2}, \tag{2.18}$$

where $\varepsilon > 0$ will be chosen later. Substitute $y = x+h$ and using (2.17) with $\delta = 1$ gives

$$\iint_{\mathbb{R}^d_+ \, \mathbb{R}^d_-} \frac{(f(x)-f(y))^2}{|x-y|^{d+2}} \,\mathrm{d}y \,\mathrm{d}x \leq \int_{\mathbb{R}^d_+} \int_{\{1>-h_d>x_d\}} \frac{(\Delta_h f(x))^2}{|h|^{d+2}} \,\mathrm{d}h \,\mathrm{d}x + 4\|f\|^2_{L^2(\mathbb{R}^d)}. \tag{2.19}$$

By (2.18) the first term on the right-hand side can be estimated by

$$\begin{aligned}\int_{\mathbb{R}^d_+}\int_{\{1>-h_d>x_d\}} \frac{(\Delta_h f(x))^2}{|h|^{d+2}}\,\mathrm{d}h\,\mathrm{d}x &\le (1+\varepsilon)\int_{\mathbb{R}^d_+}\int_{\{1>-h_d>x_d\}} \frac{(\Delta_{2h} f(x))^2}{|2h|^2}\frac{\mathrm{d}h}{|h|^d}\,\mathrm{d}x\\ &\quad + (1+\frac{4}{\varepsilon})\int_{\mathbb{R}^d_+}\int_{\{1>-h_d>x_d\}} \frac{\Delta_h^2 f(x+h))^2}{|h|^2}\frac{\mathrm{d}h}{|h|^d}\,\mathrm{d}x\\ &= (1+\varepsilon)\int_{\mathbb{R}^d_+}\int_{\{2>-h_d>2x_d\}} \frac{(\Delta_h f(x))^2}{|h|^2}\frac{\mathrm{d}h}{|h|^d}\,\mathrm{d}x\\ &\quad + (1+\frac{4}{\varepsilon})\int_{\mathbb{R}^d_+}\int_{\{1>-h_d>x_d\}} \frac{(\Delta_h^2 f(x+h))^2}{|h|^2}\frac{\mathrm{d}h}{|h|^d}\,\mathrm{d}x.\end{aligned}$$

The second differences on the right-hand side have one node in $\mathbb{R}^d_+$ and two nodes in $\mathbb{R}^d_-$. Substituting $h = y - x$ yields

$$\int_{\mathbb{R}^d_+}\int_{\{1>-h_d>x_d\}} \frac{(\Delta_h^2 f(x+h))^2}{|h|^2}\frac{\mathrm{d}h}{|h|^d}\,\mathrm{d}x \le \iint_{\mathbb{R}^d_+\,\mathbb{R}^d_-} \frac{(\Delta f(x;y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x.$$

Subtracting the first differences with $-h_d > x_d$ from both sides we have

$$\begin{aligned}\int_{\mathbb{R}^d_+}\int_{\{0>y_d>-x_d\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x &\le \varepsilon\int_{\mathbb{R}^d_+}\int_{\{-x_d>y_d>-2\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x\\ &\quad + (1+\frac{4}{\varepsilon})\iint_{\mathbb{R}^d_+\,\mathbb{R}^d_-} \frac{(\Delta f(x;y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x.\end{aligned} \tag{2.20}$$

By Lemma 2.14 and (2.17) there is $C > 1$ such that

$$\int_{\mathbb{R}^d_+}\int_{\{-x_d>y_d>-2\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x \le C\left(\int_{\mathbb{R}^d_+}\int_{\{0>y_d>-1\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x + \|f\|^2_{L^2(\mathbb{R}^d)}\right). \tag{2.21}$$

Consequently, choosing $\varepsilon < \frac{1}{2}C$ we conclude from (2.20)

$$\frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\{0>y_d>-x_d\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x \le (1+\frac{4}{\varepsilon})\int_{\mathbb{R}^d_+}\int_{\{1>-h_d>x_d\}} \frac{(\Delta_h^2 f(x+h))^2}{|h|^2}\frac{\mathrm{d}h}{|h|^d}\,\mathrm{d}x + C\,\|f\|^2_{L^2(\mathbb{R}^d)}.$$

Using again (2.21) and (2.20), we obtain

$$\int_{\mathbb{R}^d_+}\int_{\{0>y_d>-1\}} \frac{(f(x)-f(y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x \le C_1\left(\iint_{\mathbb{R}^d_+\,\mathbb{R}^d_-} \frac{(\Delta f(x;y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x + \|f\|^2_{L^2(\mathbb{R}^d)}\right)$$

Combining this with (2.17) completes the proof of (2.16). □

We are now in the position to give the proof of Theorem 2.13.

Proof of Theorem 2.13. First we prove that there is $C_1 > 0$ such that

$$\|f\|_{V(\Omega|\mathbb{R}^d)} \leq C_1 \|f\|^*_{V(\Omega|\mathbb{R}^d)}. \tag{2.22}$$

Recall the two identities

$$f(x) - 2f(\frac{x+y}{2}) + f(y) = (f(y) - f(x)) - 2(f(\frac{x+y}{2}) - f(x))$$

and

$$f(x) - 2f(\frac{x+y}{2}) + f(y) = (f(x) - f(y)) - 2(f(\frac{x+y}{2}) - f(y)).$$

If $\frac{x+y}{2} \in \Omega^c$, we use the first identity, if $\frac{x+y}{2} \in \Omega$ we use the second one. This yields

$$\begin{aligned}
\iint\limits_{\Omega\,\Omega^c} \frac{(f(x) - 2f(\frac{x+y}{2}) + f(y))^2}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x &\leq 2 \iint\limits_{\Omega\,\Omega^c} \frac{(f(x) - f(y))^2}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x \\
&\quad + 4 \iint\limits_{\substack{\Omega\times\Omega^c\cap \\ \{\frac{x+y}{2}\in\Omega\}}} \frac{(f(y) - f(\frac{x+y}{2}))^2}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x \\
&\quad + 4 \iint\limits_{\substack{\Omega\times\Omega^c\cap \\ \{\frac{x+y}{2}\in\Omega^c\}}} \frac{(f(x) - f(\frac{x+y}{2}))^2}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x \\
&\leq (2^{d+4} + 2) \iint\limits_{\Omega\,\Omega^c} \frac{(f(x) - f(y))^2}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

If $x, y \in \Omega$ but $\frac{x+y}{2} \in \Omega^c$ we estimate by the triangle inequality

$$\iint\limits_{\substack{\Omega\times\Omega\cap \\ \{\frac{x+y}{2}\in\Omega^c\}}} \frac{(f(x) - 2f(\frac{x+y}{2}) + f(y))^2}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x \leq 2^{d+4} \iint\limits_{\Omega\,\Omega^c} \frac{(f(x) - f(y))^2}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x.$$

Further by Corollary 2.8 there is $C > 0$ such that

$$\|\nabla f\|_{L^2(\Omega)} \leq C \iint\limits_{\Omega\,V^x_\Omega} \frac{(\Delta f(x;y))^2}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x.$$

Altogether we find $C_1 > 0$, such that

$$\|f\|_{V(\Omega|\mathbb{R}^d)} \leq C_1 \|f\|^{\clubsuit}_{V(\Omega|\mathbb{R}^d)}.$$

Now we prove the inverse inequality. We want to apply Lemma 2.15 thus we need to change coordinates near to a point $x_0 \in \partial\Omega$. For $\delta > 0$ we denote by Ω_δ the set

$$\Omega_\delta = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, \partial\Omega) < \delta\}.$$

Since Ω is a C^1-domain, there is a covering of the boundary by finitely many balls K_i centered in $x_i \in \partial\Omega$ and C^1-diffeomorphisms $\psi_i : K_i \to \mathbb{R}^d$, $i = 1, .., J$, with the following properties

$$\begin{aligned} \psi_i(\Omega \cap K_i) &\subset \mathbb{R}^d_+ \\ \psi_i(\Omega^c \cap K_i) &\subset \mathbb{R}^d_-. \end{aligned}$$

For the explicit construction of ψ_i we refer to Appendix C. Let Ω_0 be the set

$$\Omega_0 = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \delta/2\}.$$

We choose $\delta > 0$ such that

$$\Omega \subset \Omega_0 \cup \left(\bigcup_{i=1}^{J} K_i \right).$$

Without loss of generality we can assume that all balls K_i have the same radius $r > 0$ by possibly increasing the number of balls. Furthermore we can assume (by possibly decreasing δ and increasing the number of balls) that we can divide Ω_δ in J subsets A_i, such that $A_i \Subset K_i$ and also the set

$$\psi_i^{-1}(\{y \in \mathbb{R}^d \mid \forall z \in \psi_i(A_i)\ \|z - y\|_\infty < 3 \operatorname{diam}(\psi_i(A_i))\}) \subset K_i.$$

Now we have

$$\begin{aligned} \iint\limits_{\Omega\,\Omega^c} \frac{(f(x) - f(y))^2}{|x - y|^{d+2}} \, \mathrm{d}y \, \mathrm{d}x \leq & \iint\limits_{\Omega_0\,\Omega^c} \frac{(f(x) - f(y))^2}{|x - y|^{d+2}} \, \mathrm{d}y \, \mathrm{d}x \\ & + \sum_{i=1}^{J} \int\limits_{\Omega \cap A_i} \int\limits_{\Omega^c} \frac{(f(x) - f(y))^2}{|x - y|^{d+2}} \, \mathrm{d}y \, \mathrm{d}x. \end{aligned}$$

By (2.17) the first term on the right-hand side can easily be estimated

$$\iint\limits_{\Omega_0\,\Omega^c} \frac{(f(x) - f(y))^2}{|x - y|^{d+2}} \, \mathrm{d}y \, \mathrm{d}x \leq \frac{4}{\delta} \|f\|^2_{L^2(\mathbb{R}^d)}.$$

Let us regard $i \in \{1, .., J\}$ as fixed and drop the index in the following. Then

$$\begin{aligned} \int\limits_{\Omega \cap A} \int\limits_{\Omega^c} \frac{(f(x) - f(y))^2}{|x - y|^{d+2}} \, \mathrm{d}y \, \mathrm{d}x \leq & \int\limits_{\Omega \cap A} \int\limits_{\Omega^c \cap K} \frac{(f(x) - f(y))^2}{|x - y|^{d+2}} \, \mathrm{d}y \, \mathrm{d}x \\ & + \int\limits_{\Omega \cap A} \int\limits_{\Omega^c \cap K^c} \frac{(f(x) - f(y))^2}{|x - y|^{d+2}} \, \mathrm{d}y \, \mathrm{d}x \\ \leq & \int\limits_{\Omega \cap A} \int\limits_{\Omega^c \cap K} \frac{(f(x) - f(y))^2}{|x - y|^{d+2}} \, \mathrm{d}y \, \mathrm{d}x + \frac{4}{r} \|f\|^2_{L^2(\mathbb{R}^d)} \end{aligned}$$

where $r = \text{dist}(A, K^c) > 0$. Define $g = f \circ \psi$, according to Lemma 2.15 we obtain

$$\begin{aligned}\int\limits_{\Omega\cap A}\int\limits_{\Omega^c\cap K}\frac{(f(x)-f(y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x &\leq C_1\int\limits_{\mathbb{R}^d_+\cap\psi(A)}\int\limits_{\mathbb{R}^d_-\cap\psi(K)}\frac{(g(x)-g(y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x\\ &\leq C_2\left(\int\limits_{\mathbb{R}^d_+\cap\psi(K)}\int\limits_{\mathbb{R}^d_-\cap\psi(K)}\frac{(\Delta g(x;y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x + \|g\|^2_{L^2(\psi(K))}\right)\\ &\leq C_3\left(\int\limits_{\Omega\cap K}\int\limits_{\Omega^c\cap K}\frac{(\Delta f(x;y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x + \|f\|^2_{L^2(K)}\right).\end{aligned}$$

Proceed analogously for all $i \in \{1, .., J\}$, we have

$$\iint\limits_{\Omega\,\Omega^c}\frac{(f(x)-f(y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x \leq C_4\left(\iint\limits_{\Omega\,\Omega^c}\frac{(\Delta f(x;y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x + \|f\|^2_{L^2(\mathbb{R}^d)}\right).$$

Finally by Corollary 2.8 we conclude

$$\|f\|^{\clubsuit}_{V(\Omega|\mathbb{R}^d)} \leq C_5\left(\iint\limits_{\Omega\,\mathbb{R}^d}\frac{(\Delta f(x;y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x + \|f\|^2_{L^2(\mathbb{R}^d)}\right)$$

which completes the proof. □

2.2.2. Dirichlet forms associated to $V(\Omega|\mathbb{R}^d)$

As a corollary of Theorem 2.13 we obtain the existence of regular Dirichlet form on $V(\Omega|\mathbb{R}^d)$. To the best of the authors knowledge, a form of this type is not studied in the literature.

Consider a bilinear form $\mathcal{E} : V(\Omega|\mathbb{R}^d) \times V(\Omega|\mathbb{R}^d) \to \mathbb{R}$,

$$\mathcal{E}(u,v) = \int\limits_{\Omega}\sum_{i,j=1}^{d} a_{ij}(x)\partial_i u(x)\partial_j v(x)\,\mathrm{d}x + \iint\limits_{\Omega\,\Omega^c}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2}}a(x,y)\,\mathrm{d}y\,\mathrm{d}x \quad (2.23)$$

where the matrix a_{ij} is uniformly elliptic and $a \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is positive. Using Theorem 2.13 and Lemma 2.12 $(\mathcal{E}, V(\Omega|\mathbb{R}^d))$ becomes a regular symmetric Dirichlet form on $L^2(\mathbb{R}^d)$.

Corollary 2.16. *Let $\Omega \subset \mathbb{R}^d$ be a C^1-domain. Then $(\mathcal{E}, V(\Omega))$ is a regular Dirichlet form on $L^2(\mathbb{R}^d)$.*

Proof. It is clear from the gradient structure of the first integral and the first order difference structure of the second integral, that $\mathcal{E}$ satisfies the contraction property, Definition B.1(4), see [FU12, Ex. 1.2.1].

Further by Lemma 2.12 $C_0^\infty(\mathbb{R}^d)$ is dense in $V(\Omega|\mathbb{R}^d)$. From this we obtain that $(\mathcal{E}, V(\Omega))$ is regular. □

2.2.3. Generalization of fractional Sobolev spaces

We want to generalize the space $H^s(\Omega)$ in the same sense as $H^1(\Omega)$. Thus we integrate also over the set $\Omega \times \Omega^c$. This leads to the following definition:

Definition 2.17. Let $\Omega \subset \mathbb{R}^d$ be open. For $0 < s < 1$ we define the linear space $V^s(\Omega|\mathbb{R}^d)$ as

$$V^s(\Omega|\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) \mid \frac{f(x) - f(y)}{|x-y|^{d/2+s}} \in L^2((\Omega^c \times \Omega^c)^c)\}$$

and a norm on $V^s(\Omega|\mathbb{R}^d)$ by

$$\|f\|_{V^s(\Omega|\mathbb{R}^d)} = \left(\|f\|^2_{L^2(\mathbb{R}^d)} + s(1-s) \iint\limits_{(\Omega^c \times \Omega^c)^c} \frac{(f(x) - f(y))^2}{|x-y|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}x \right)^{1/2} .$$

Remark 2.18. 1. Note that

$$(\Omega^c \times \Omega^c)^c = (\Omega \times \Omega) \cup (\Omega^c \times \Omega) \cup (\Omega \times \Omega^c)$$

and, although we integrate over a set in the product space $\mathbb{R}^d \times \mathbb{R}^d$, we use the notation of double integrals on $\mathbb{R}^d$ instead of one single integral on the product space.

2. Note that for $f \in V^s(\Omega|\mathbb{R}^d)$ an equivalent norm is given by

$$\|f\|^2_{L^2(\mathbb{R}^d)} + s(1-s) \iint\limits_{\Omega\,\mathbb{R}^d} \frac{(f(x) - f(y))^2}{|x-y|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}x.$$

The two norms are comparable with constant two. The first definition is the natural definition starting in a variational setup. The second definition is more appropriate in computations.

The following example illustrates that finiteness of the seminorm on $V^s(\Omega|\mathbb{R}^d)$ requires some regularity of the function across $\partial\Omega$:

Example 1. Let $\Omega = B_1(0)$, $s \in (0,1)$. Define $g : \mathbb{R}^d \to \mathbb{R}$ by

$$g(x) = \begin{cases} (|x| - 1)^\beta & \text{if } 1 \leq |x| \leq 2\,, \\ 0, & \text{else.} \end{cases}$$

We show that $g \in V^s(\Omega|\mathbb{R}^d)$ if and only if $\beta > \frac{2s-1}{2}$. Since s is fixed we omit the factor $s(1-s)$. Then

$$\begin{aligned} [g]_{V^s(\Omega|\mathbb{R}^d)} &= 2 \int\limits_{B_1} \int\limits_{B_2 \setminus B_1} (|x| - 1)^{2\beta} |x-y|^{-d-2s} \,\mathrm{d}x\,\mathrm{d}y \\ &= 2 \int\limits_{B_2 \setminus B_1} (|x| - 1)^{2\beta} \int\limits_{B_1} |x-y|^{-d-2s} \,\mathrm{d}y\,\mathrm{d}x\,. \end{aligned}$$

For $1 < |x| < 2$ choose $\xi = \left(\frac{3-|x|}{2|x|}\right) x$. Then we may estimate

$$\int_{B_1} |x-y|^{-d-2s} \,\mathrm{d}y \geq \int_{B_{(|x|-1)/2}(\xi)} |x-y|^{-d-2s} \,\mathrm{d}y \geq \int_{B_{(|x|-1)/2}(\xi)} \left(2\left(|x|-1\right)\right)^{-d-2s} \,\mathrm{d}y$$
$$\geq |B_1| \left(\frac{|x|-1}{2}\right)^d \left(2\left(|x|-1\right)\right)^{-d-2s} .$$

Thus $[g]_{V^s(\Omega|\mathbb{R}^d)} \geq C \int_{B_2\setminus B_1} \left(|x|-1\right)^{2\beta-2s} \,\mathrm{d}x$ for some constant $C = C(d) > 0$. This integral is finite if $2\beta - 2s > -1$. On the other hand, for $x \in B_2 \setminus B_1$, we have

$$\int_{B_1} |x-y|^{-d-2s} \,\mathrm{d}y \leq \int_{B_2(x)\setminus B_\delta(x)} |x-y|^{-d-2s} \,\mathrm{d}y \leq C' \operatorname{dist}(x, \partial B_1)^{-2s} .$$

Therefore, $[g]_{V^s(\Omega|\mathbb{R}^d)} \leq C' \int_{B_2\setminus B_1} \left(|x|-1\right)^{2\beta-2s} \,\mathrm{d}x$, which shows that $g \in V^s(\Omega|\mathbb{R}^d)$ if and only if $\beta > \frac{2s-1}{2}$.

In the second order case ($s = 1$) the function g, interpreted as a function on $B_2 \setminus B_1$ has a trace on $\partial\Omega$ if and only if $\beta > \frac{1}{2}$. We note that $g \notin H^s(\mathbb{R}^d)$ for $s > \frac{1}{2}$ because of the discontinuity at $|x| = 2$. An analogues computation shows that $g \in V(\Omega|\mathbb{R}^d)$ if and only if $\beta > \frac{1}{2}$.

Lemma 2.19. *Let $\Omega \subset \mathbb{R}^d$ be open and $s \in (0,1)$. The linear space $V^s(\Omega|\mathbb{R}^d)$ endowed with the norm*

$$\|f\|_{V^s(\Omega|\mathbb{R}^d)} = \left(\|f\|^2_{L^2(\mathbb{R}^d)} + [f,f]_{V^s(\Omega|\mathbb{R}^d)}\right)^{1/2} ,$$

where

$$[f,g]_{V^s(\Omega|\mathbb{R}^d)} = s(1-s) \iint_{(\Omega^c\times\Omega^c)^c} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}x$$

is a separable Hilbert space.

Proof. The norm on $V^s(\Omega|\mathbb{R}^d)$ is obviously induced by a scalar product

$$\langle f,g\rangle_{V^s(\Omega|\mathbb{R}^d)} = (f,g)_{L^2(\mathbb{R}^d)} + [f,g]_{V^s(\Omega|\mathbb{R}^d)} .$$

The proof of the completeness and separability follows the argumentation in [Wlo87, Thm. 3.1].

First we show the completeness of $V^s(\Omega|\mathbb{R}^d)$. Let (f_n) be a Cauchy sequence with respect to the norm $\|\cdot\|_{V^s(\Omega|\mathbb{R}^d)}$. Set

$$v_n(x,y) = \frac{(f_n(x)-f_n(y))}{|x-y|^{d/2+s}} .$$

Then, by definition of $\|\cdot\|_{V^s(\Omega|\mathbb{R}^d)}$ and the completeness of $L^2(\mathbb{R}^d)$, (f_n) converges to some f in the norm of $L^2(\mathbb{R}^d)$. We may chose a subsequence f_{n_k} that converges a.e. to f. Then v_{n_k} converges a.e. on $\mathbb{R}^d \times \mathbb{R}^d$ to the function

$$v(x,y) = \frac{(f(x)-f(y))}{|x-y|^{d/2+s}} .$$

By Fatou's Lemma,

$$\iint\limits_{(\Omega^c\times\Omega^c)^c} \frac{(f(x)-f(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y \le \liminf_{k\to\infty} \iint\limits_{(\Omega^c\times\Omega^c)^c} (v_{n_k}(x,y))^2\,\mathrm{d}x\,\mathrm{d}y \le \sup_{k\in\mathbb{N}} \|v_{n_k}\|^2_{L^2((\Omega^c\times\Omega^c)^c)}.$$

Since (v_n) is a Cauchy sequence (and hence bounded) in $L^2((\Omega^c\times\Omega^c)^c)$, this shows that $\left[f,f\right]_{V^s(\Omega|\mathbb{R}^d)} < \infty$, i.e. $f \in V^s(\Omega|\mathbb{R}^d)$. Another application of Fatou's Lemma shows that

$$\begin{aligned}\left[f_{n_k}-f, f_{n_k}-f\right]_{V^s(\Omega|\mathbb{R}^d)} &= \iint\limits_{(\Omega^c\times\Omega^c)^c} |v_{n_k}(x,y)-v(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\le \liminf_{l\to\infty} \iint\limits_{(\Omega^c\times\Omega^c)^c} |v_{n_k}(x,y)-v_{n_l}(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \xrightarrow{k\to\infty} 0.\end{aligned}$$

This shows that $\|f_{n_k}-f\|_{V^s(\Omega|\mathbb{R}^d)} \to 0$ for $k\to\infty$ for the subsequence (n_k) chosen above and thus $\|f_n-f\|_{V^s(\Omega|\mathbb{R}^d)} \to 0$ as $n\to\infty$, since (f_n) was assumed to be a Cauchy sequence. The completeness of $V^s(\Omega|\mathbb{R}^d)$ is proved.

The mapping $\mathcal{I}$

$$\begin{aligned}\mathcal{I}\colon V^s(\Omega|\mathbb{R}^d) &\to L^2(\Omega)\times L^2((\Omega^c\times\Omega^c)^c), \\ \mathcal{I}(f) &= \left(f, \frac{(f(x)-f(y))}{|x-y|^{d/2+s}}\right),\end{aligned} \tag{2.24}$$

is isometric due to the definition of the norm in $V^s(\Omega|\mathbb{R}^d)$. Having shown the completeness of $V^s(\Omega|\mathbb{R}^d)$ we obtain that $\mathcal{I}(V^s(\Omega|\mathbb{R}^d))$ is a closed subspace of the Cartesian product on the right-hand side of (2.24). This product is separable, which implies (cf. [Wlo87, Lem. 3.1]) the separability of $V^s(\Omega|\mathbb{R}^d)$.

Hence, $V^s(\Omega|\mathbb{R}^d)$ is separable. □

Lemma 2.20. *Let $\Omega\subset\mathbb{R}^d$ be open and $\partial\Omega$ be C^1. Let $u\in V^s(\Omega|\mathbb{R}^d)$. Then there exist functions $u_n\in C_c^\infty(\mathbb{R}^d)$ such that*

$$u_n \xrightarrow{n\to\infty} u \quad \text{in } V^s(\Omega|\mathbb{R}^d).$$

Proof. The proof is analogue to the proof of Lemma 2.12. □

We define the following affine subspaces of $L^2(\mathbb{R}^d)$ and $V^s(\Omega|\mathbb{R}^d)$.

Definition 2.21. Let $\Omega\subset\mathbb{R}^d$ be open. For $g\in L^2(\mathbb{R}^d)$, $g\in V^s(\Omega|\mathbb{R}^d)$ respectively, we define the following linear spaces:

(i) $L^2_0(\Omega|\mathbb{R}^d) = \{u\in L^2(\mathbb{R}^d)\colon u=0 \text{ a.e. on } \Omega^c\}$,

(ii) $L^2_g(\Omega|\mathbb{R}^d) := \{u\in L^2(\mathbb{R}^d) | u=g \text{ a.e. on } \Omega^c\}$,

(iii) $V^s_g(\Omega|\mathbb{R}^d) := \{u\in V^s(\Omega|\mathbb{R}^d) | u=g \text{ a.e. on } \Omega^c\}$.

Remark 2.22. Let $g\in L^2(\mathbb{R}^d)$, $g\in V^s(\Omega|\mathbb{R}^d)$ respectively. Then $L^2_g(\Omega|\mathbb{R}^d)$ and $V^s_g(\Omega|\mathbb{R}^d)$ are closed subsets of $L^2(\mathbb{R}^d)$ and $V^s(\Omega|\mathbb{R}^d)$. This follows directly from the completeness of $L^2(\mathbb{R}^d)$ and $V^s(\Omega|\mathbb{R}^d)$.

This definition is useful to encode Dirichlet boundary data in the function space and is used in the variational formulation of the nonlocal Dirichlet problem.

From the classical compact embedding of the Sobolev-Slobodeckij spaces we obtain the analogous result for the spaces $V_g^s(\Omega|\mathbb{R}^d)$. Let us recall a classical result about Sobolev-Slobodeckij spaces from [Wlo87, Thm. 7.8]

Theorem 2.23. *Let $\Omega \subset \mathbb{R}^d$ open and bounded and let $l_2 < l_1$, $l_1, l_2 \in \mathbb{R}_+$. Then the embedding $H_0^{l_1}(\Omega) \hookrightarrow H_0^{l_2}(\Omega)$ is compact.*

As a corollary of this we have

Corollary 2.24. *Let $\Omega \subset \mathbb{R}^d$ open and bounded, let $1/2 < s_1 < s_2 < 1$ and let $g \in V^{s_2}(\Omega)$. Then the embedding $V_g^{s_1}(\Omega|\mathbb{R}^d) \hookrightarrow V_g^{s_2}(\Omega|\mathbb{R}^d)$ is compact.*

Proof. One can write

$$V_g^s(\Omega|\mathbb{R}^d) = g + H_\Omega^s(\mathbb{R}^d).$$

By [McL00, Thm. 3.33] $H_\Omega^s(\mathbb{R}^d) = H_0^s(\Omega)$ for $s \neq \frac{1}{2}$, thus the proof follows directly from Theorem 2.23. □

2.2.4. Asymptotics for $s \nearrow 1$ in the generalized setting

In this section we analyze the connection between the spaces $V^s(\Omega|\mathbb{R}^d)$, $H^1(\Omega)$ and $V(\Omega|\mathbb{R}^d)$. We have seen in Subsection 2.1.2 that the constant $\mathcal{A}_{d,-2s}$ guaranties that $\|f\|_{H^s(\Omega)} \to K\|f\|_{H^1(\Omega)}$, where $K > 0$. This implies that

$$\bigcap_{s<1} H^s(\Omega) = H^1(\Omega).$$

If we consider instead of $H^s(\Omega)$ the spaces $V^s(\Omega|\mathbb{R}^d)$, we need to examine the behavior of

$$(1-s) \iint\limits_{\Omega\,\Omega^c} \frac{(f(x)-f(y))^2}{|x-y|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}x.$$

The interaction of the norming constant $(1-s)$ and the singularity at $x = y$ takes place only on $\partial\Omega$ and thus one could assume

$$(1-s) \iint\limits_{\Omega\,\Omega^c} \frac{(f(x)-f(y))^2}{|x-y|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}x \overset{s\to 1^-}{\longrightarrow} 0.$$

This is right for smooth functions – at least $C_0^1(\mathbb{R}^d)$ functions, see the proof of Theorem 4.6 – but fails to be true in general, see the example below.

Since $V^s(\Omega|\mathbb{R}^d)$ and $V(\Omega|\mathbb{R}^d)$ generalize $H^s(\Omega)$ and $H^1(\Omega)$ in the same way, another possible conjecture is

$$\bigcap_{s<1} V^s(\Omega|\mathbb{R}^d) = V(\Omega|\mathbb{R}^d).$$

The following example shows that both conjectures are false in general. We define a function $f : \mathbb{R} \to \mathbb{R}$ that belongs to $V^s(\Omega|\mathbb{R}^d)$ for any $s \in (0,1)$, but $f \notin V(\Omega|\mathbb{R}^d)$. Further the $V^s(\Omega|\mathbb{R}^d)$-seminorm does not converge to the $H^1(\Omega)$-seminorm.

Example 2. Let $\Omega = (-1, 1)$ and define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} (|x| - 1)^{1/2} & \text{if } 1 < |x| < 2, \\ 0 & \text{else.} \end{cases}$$

Of course $f \in H^1(\Omega)$ and $\|f\|_{H^1(\Omega)} = 0$. Let us calculate the $V^s(\Omega|\mathbb{R}^d)$-seminorm of f.

$$\begin{aligned} [f]^2_{V^s(\Omega|\mathbb{R}^d)} &= (1-s) \iint\limits_{(\Omega^c\times\Omega^c)^c} \frac{(f(x) - f(y))^2}{|x-y|^{1+2s}} \,\mathrm{d}y\,\mathrm{d}x \\ &= 2(1-s) \int_{-1}^{1} \int_{1}^{2} \frac{(y-1)}{|x-y|^{1+2s}} \,\mathrm{d}y\,\mathrm{d}x \\ &= -2(1-s) \int_{1}^{2} (y-1) \int_{y-1}^{y+1} |x|^{-1-2s} \,\mathrm{d}x\,\mathrm{d}y \\ &= 2(1-s) \int_{1}^{2} (y-1) \left[(y-1)^{-2s} - (y+1)^{-}2s\right] \,\mathrm{d}y \\ &= -\frac{(1-s)}{s} \int_{0}^{1} y^{1-2s} - y(y+2)^{-2s} \,\mathrm{d}y \\ &= \frac{1}{2s} 1^{2-2s} + \frac{(1-s)}{s} \frac{3^{2s+1}(2s-1) + 2^{2s+2}}{(2s+1)(2s+2)} . \end{aligned}$$

The second term in the last line goes to zero when $s \to 1^-$, while the first term goes to $\frac{1}{2}$. Thus

$$\lim_{s\to 1^-} [f]_{V^s(\Omega|\mathbb{R}^d)} = \frac{1}{2}.$$

Note that $f \notin V(\Omega|\mathbb{R}^d)$, since $\|f\|_{V(\Omega|\mathbb{R}^d)} = \infty$.

Let us define the following function space.

Definition 2.25. Let $\Omega \subset \mathbb{R}^d$ be open and bounded. We define the linear space $\widetilde{V}(\Omega|\mathbb{R}^d)$ as

$$\widetilde{V}(\Omega|\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) \mid \sup_{s<1} \|f\|_{V^s(\Omega|\mathbb{R}^d)} < \infty \right\}.$$

If $f \in \widetilde{V}(\Omega|\mathbb{R}^d)$, we define the norm

$$\|f\|_{\widetilde{V}(\Omega|\mathbb{R}^d)} = \sup_{s<1} \|f\|_{V^s(\Omega|\mathbb{R}^d)} .$$

Remark 2.26.

1. We can interpret $\widetilde{V}(\Omega|\mathbb{R}^d)$ as the intersection of all $V^s(\Omega|\mathbb{R}^d)$, i.e

$$\widetilde{V}(\Omega|\mathbb{R}^d) = \bigcap_{s<1} V^s(\Omega|\mathbb{R}^d).$$

2. Since the norm of $V^s(\Omega|\mathbb{R}^d)$ dominates the norm of $H^s(\Omega)$,

$$\lim_{s\to 1^-} \|f\|_{V^s(\Omega|\mathbb{R}^d)} < \infty$$

implies that $f \in H^1(\Omega)$, see Subsection 2.1.2.

The above example leads to the following observation. The intersection of $V^s(\Omega|\mathbb{R}^d)$ is not equal to $V(\Omega|\mathbb{R}^d)$. As a consequence of the next proposition we obtain $V(\Omega|\mathbb{R}^d) \subset \widetilde{V}(\Omega|\mathbb{R}^d)$.

Proposition 2.27. *Let Ω be a C^1-domain in $\mathbb{R}^d$. For any $s \in (0,1)$ the space $V(\Omega|\mathbb{R}^d)$ is continuously embedded in $V^s(\Omega|\mathbb{R}^d)$, i.e there is a constant $C > 0$ such that*

$$\|u\|_{V^s(\Omega|\mathbb{R}^d)} \leq C \|u\|_{V(\Omega|\mathbb{R}^d)}$$

for any $u \in V(\Omega|\mathbb{R}^d)$. Furthermore the constant C does not depend on s.

Proof. Let $u \in V(\Omega|\mathbb{R}^d)$. Note that the function $s \mapsto s(1-s)$ is bounded by $\frac{1}{4}$ on $(0,1)$. Let $|x-y| > 1$, then for all $s \in (0,1)$

$$s(1-s) \int\limits_{|h|>1} |h|^{-d-2s} \,\mathrm{d}h = s(1-s) \int\limits_1^\infty r^{-1-2s} \,\mathrm{d}r = -(1-s) r^{-2s}\Big|_1^\infty \leq 1 \tag{2.25}$$

Thus

$$\begin{aligned} s(1-s) \iint\limits_{\Omega\,\mathbb{R}^d} \frac{(u(x)-u(y))^2}{|x-y|^{d+2s}} \mathbb{1}_{\{|x-y|>1\}} \,\mathrm{d}y \,\mathrm{d}x &\leq 2s(1-s) \iint\limits_{\Omega\,\mathbb{R}^d} \frac{u^2(x)+u^2(y)}{|x-y|^{d+2s}} \mathbb{1}_{\{|x-y|>1\}} \,\mathrm{d}y \,\mathrm{d}x \\ &\leq \|u\|^2_{L^2(\mathbb{R}^d)} . \end{aligned}$$

Now for $|x-y| < 1$

$$\begin{aligned} s(1-s) \int\limits_\Omega \int\limits_{\Omega^c \cap B_1(x)} \frac{(u(x)-u(y))^2}{|x-y|^{d+2s}} \,\mathrm{d}y \,\mathrm{d}x &\leq \frac{1}{4} \int\limits_\Omega \int\limits_{\Omega^c \cap B_1(x)} \frac{(u(x)-u(y))^2}{|x-y|^{d+2}} \,\mathrm{d}y \,\mathrm{d}x \\ &\leq \frac{1}{4} \iint\limits_{\Omega\,\Omega^c} \frac{(u(x)-u(y))^2}{|x-y|^{d+2}} \,\mathrm{d}y \,\mathrm{d}x. \end{aligned}$$

Next we estimate the integral on $\Omega \times \Omega$. By Theorem 2.13 $u \in V(\Omega|\mathbb{R}^d)$ implies $\nabla u \in L^2(\Omega)$. Since Ω is smooth, there is an extension $\widetilde{u}$ of $u_{|\Omega}$ to $\mathbb{R}^d$ such that $\widetilde{u} \in H^1(\mathbb{R}^d)$ and

$$\|\widetilde{u}\|_{H^1(\mathbb{R}^d)} \leq C \|u\|_{H^1(\Omega)} ,$$

see [AF03, Thm. 5.24]. First, let us assume that $\widetilde{u} \in C_c^\infty(\mathbb{R}^d)$. By the mean value theorem

$$\begin{aligned} \int\limits_{\mathbb{R}^d} (\widetilde{u}(x) - \widetilde{u}(x+h))^2 \,\mathrm{d}x &= \int\limits_{\mathbb{R}^d} |h|^2 \int\limits_0^1 (\nabla \widetilde{u}(x+th))^2 \,\mathrm{d}t \,\mathrm{d}x \\ &= \int\limits_{\mathbb{R}^d} |h|^2 (\nabla \widetilde{u}(\widetilde{x}))^2 \,\mathrm{d}\widetilde{x} \int\limits_0^1 \mathrm{d}t \\ &= |h|^2 \int\limits_{\mathbb{R}^d} (\nabla \widetilde{u}(x))^2 \,\mathrm{d}x. \end{aligned}$$

Now this implies

$$
\begin{aligned}
s(1-s) \iint\limits_{\Omega\,\Omega} \frac{(u(x)-u(y))^2}{|x-y|^{d+2s}} \mathbb{1}_{\{|x-y|<1\}} \,\mathrm{d}y\,\mathrm{d}x &\le s(1-s) \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{(\widetilde{u}(x)-\widetilde{u}(y))^2}{|x-y|^{d+2s}} \mathbb{1}_{\{|x-y|<1\}} \,\mathrm{d}y\,\mathrm{d}x \\
&\le s(1-s) \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{(\widetilde{u}(x)-\widetilde{u}(x+h))^2}{|h|^{d+2s}} \mathbb{1}_{\{|h|<1\}} \,\mathrm{d}h\,\mathrm{d}x \\
&\le 2 \left\|\nabla \widetilde{u}\right\|^2_{H^1(\mathbb{R}^d)} \\
&\le C \left\|\nabla u\right\|^2_{H^1(\Omega)}
\end{aligned}
$$

For $\widetilde{u} \notin C_c^\infty(\mathbb{R}^d)$ the argument follows from the density of $C_c^\infty(\mathbb{R}^d)$ in $H^1(\mathbb{R}^d)$. Altogether we obtain

$$
\|u\|_{V^s(\Omega|\mathbb{R}^d)} \le C \|u\|_{V(\Omega|\mathbb{R}^d)} .
$$

Note that there is no dependence of the constants arising in the proof of $s \in (0,1)$. □

We want to fix the relation between $H_0^1(\Omega)$ and $V(\Omega|\mathbb{R}^d)$, where we consider $H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}^{H^1(\mathbb{R}^d)}$, thus a function $f \in H_0^1(\Omega)$ is defined on $\mathbb{R}^d$.

Proposition 2.28. *Let $\Omega \subset \mathbb{R}^d$ be a bounded C^1-domain. Then $H_0^1(\Omega)$ is continuously embedded in $V(\Omega|\mathbb{R}^d)$.*

Proof. Let $u \in H_0^1(\Omega)$. By Theorem 2.13 it is sufficient to prove

$$
\iint\limits_{\Omega\,\Omega^c} \frac{u^2(x)}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x \le C \|u\|_{H^1(\Omega)} .
$$

This is a consequence of the classical Hardy inequality for domains. For $x \in \Omega$ set $\delta_x = \operatorname{dist}(x, \partial\Omega)$, then by the Hardy inequality

$$
\int\limits_{\Omega} \frac{u^2(x)}{\delta_x^2} \,\mathrm{d}x \le C_{d,\Omega} \|\nabla u\|^2_{L^2(\Omega)}
$$

for all $u \in H_0^1(\Omega)$, see for instance [KM97] for a concise proof of the Hardy inequality. Now

$$
\begin{aligned}
\iint\limits_{\Omega\,\Omega^c} \frac{u^2(x)}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x &\le \int\limits_{\Omega} u^2(x) \int\limits_{\delta_x}^{\infty} r^{-3} \,\mathrm{d}r\,\mathrm{d}x \\
&\le \frac{1}{2} \int\limits_{\Omega} \frac{u^2(x)}{\delta_x^2} \,\mathrm{d}x \\
&\le C \|\nabla u\|^2_{L^2(\Omega)}
\end{aligned}
$$

□

2.2.5. Changing the asymptotics

Let us examine the relation between $V^s(\Omega|\mathbb{R}^d)$ and $V(\Omega|\mathbb{R}^d)$ in more details. As mentioned above the seminorm $[\cdot]_{H^s(\Omega)}$ blows up without the norming constant $(1-s)$. This is caused by the strong singularity at $x=y$ in the double integral.

We now look at the integral

$$\iint\limits_{\Omega\,\Omega^c} \frac{(u(x)-u(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}y\,\mathrm{d}x\,.$$

This integral has a singularity only on the boundary $\partial\Omega$ and thus one do not need the norming constant $(1-s)$ on $\Omega\times\Omega^c$ to avoid a blow up of this quantity. Motivated by this observation we introduce a second norm $\|\cdot\|^{\dagger}_{V^s(\Omega|\mathbb{R}^d)}$ on $V^s(\Omega|\mathbb{R}^d)$. This norm is equivalent to the standard norm for fixed $s\in(0,1)$ but has a different asymptotic behavior for $s\to 1$.

Definition 2.29. Let $\Omega\subset\mathbb{R}^d$ be open and bounded. Let $f\in V^s(\Omega|\mathbb{R}^d)$. We define a norm on $V^s(\Omega|\mathbb{R}^d)$ by

$$\|f\|^{\dagger}_{V^s(\Omega|\mathbb{R}^d)} = \left(\iint\limits_{(\Omega^c\times\Omega^c)^c} (f(x)-f(y))^2 k_\Omega(x,y)\,\mathrm{d}y\,\mathrm{d}x + \|f\|^2_{L^2(\mathbb{R}^d)} \right)^{1/2},$$

where $k_\Omega : (\Omega^c\times\Omega^c)^c \to [0,\infty]$ defined by

$$k_\Omega(x,y) = \begin{cases} (1-s)\,|x-y|^{-d-2s} & \text{if } (x,y)\in\Omega\times\Omega,\\ |x-y|^{-d-2s} & \text{else.} \end{cases}$$

Remark 2.30. Note that for fixed $s\in(0,1)$ this norm is comparable to the standard norm with comparability constant $(1-s)$, i.e.

$$(1-s)\,\|f\|^{\dagger}_{V^s(\Omega|\mathbb{R}^d)} \le \|f\|_{V^s(\Omega|\mathbb{R}^d)} \le \frac{1}{(1-s)}\,\|f\|^{\dagger}_{V^s(\Omega|\mathbb{R}^d)}\,.$$

By a slight change in the proof Proposition 2.27 we obtain the following continuous embedding.

Corollary 2.31. *Let Ω be a C^1-domain in $\mathbb{R}^d$. For any $s\in(0,1)$ the space $V(\Omega|\mathbb{R}^d)$ is continuously embedded in $V^s(\Omega|\mathbb{R}^d)$ with respect to the norm $\|\cdot\|^{\dagger}_{V^s(\Omega|\mathbb{R}^d)}$, i.e there is a constant $C>0$ such that*

$$\|u\|^{\dagger}_{V^s(\Omega|\mathbb{R}^d)} \le C\,\|u\|_{V(\Omega|\mathbb{R}^d)}$$

for any $u\in V(\Omega|\mathbb{R}^d)$. Furthermore the constant C does not depend on s.

Proof. Since

$$\int\limits_{\Omega}\int\limits_{\Omega^c\cap B_1(x)} \frac{(u(x)-u(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}y\,\mathrm{d}x \le \int\limits_{\Omega}\int\limits_{\Omega^c\cap B_1(x)} \frac{(u(x)-u(y))^2}{|x-y|^{d+2}}\,\mathrm{d}y\,\mathrm{d}x\,.$$

The assertion follows analogues to the proof of Proposition 2.27. □

The proof gives even more: We can identify the intersection of all $V^s(\Omega|\mathbb{R}^d)$ with respect to the norms $\|\cdot\|^\dagger_{V^s(\Omega|\mathbb{R}^d)}$ with $V(\Omega|\mathbb{R}^d)$.

Proposition 2.32. *Let $\Omega \subset \mathbb{R}^d$ be a C^1-domain. Then*

$$V(\Omega|\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) \mid \sup_{s<1} \|f\|^\dagger_{V^s(\Omega|\mathbb{R}^d)} < \infty \right\}.$$

Proof. Let $f \in V(\Omega|\mathbb{R}^d)$. Then by Theorem 2.13 $\nabla f \in L^2(\Omega)$. Following the arguments of the proof of Proposition 2.27, we obtain

$$s(1-s) \iint\limits_{\Omega\,\Omega} \frac{(f(x)-f(y))^2}{|x-y|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}x \leq C \|\nabla f\|^2_{L^2(\Omega)}$$

for any $s \in (0,1)$. Further

$$\iint\limits_{\Omega\,\Omega^c} \frac{(f(x)-f(y))^2}{|x-y|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}x \leq \iint\limits_{\Omega\,\Omega^c} \frac{(f(x)-f(y))^2}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x + C \|f\|^2_{L^2(\mathbb{R}^d)}.$$

Now let $g \in \left\{ f \in L^2(\mathbb{R}^d) \mid \sup_{s<1} \|f\|^\dagger_{V^s(\Omega|\mathbb{R}^d)} < \infty \right\}$. The inequality

$$\sup_{s<1} \|g\|_{V^s(\Omega|\mathbb{R}^d)} \geq \lim_{s\to 1} \|g\|_{V^s(\Omega|\mathbb{R}^d)}$$

and [BBM01, Thm. 1] imply

$$\lim_{s\to 1} \|g\|_{V^s(\Omega|\mathbb{R}^d)} = C(\Omega) \|\nabla g\|_{L^2(\Omega)}.$$

Further it is easily seen that

$$\lim_{s\to 1} \|g\|_{V^s(\Omega|\mathbb{R}^d)} \geq \iint\limits_{\Omega\,\Omega^c} \frac{(g(x)-g(y))^2}{|x-y|^{d+2}} \,\mathrm{d}y\,\mathrm{d}x.$$

This proves the assertion. □

2.2.6. Weighted L^2-spaces

If one considers the seminorm of $V^s(\Omega|\mathbb{R}^d)$ and $V(\Omega|\mathbb{R}^d)$, one may notice that the requirement that a function f belonging to this spaces needs to be in $L^2(\mathbb{R}^d)$ is a too strong or even artificial assumption. In this section we point out how this assumption can be weaken in a way that preserves the Hilbert space property of the afore mentioned spaces.

We introduce weighted L^2-spaces as follows.

Definition 2.33. Let $\gamma \in \mathbb{R}_+$ and $w^\gamma(x) = ((1+|x|)^{-d-\gamma}$. Set $w^\gamma(x)\,\mathrm{d}x = \nu^\gamma(\mathrm{d}x)$. We define the w-weighted L^2-space as

$$L^2(\mathbb{R}^d, \nu^\gamma) = \left\{ f : \mathbb{R}^d \to \mathbb{R} \mid \|f\|_{L^2(\mathbb{R}^d,\nu^\gamma)} < \infty \right\},$$

where the norm is defined by

$$\|f\|^2_{L^2(\mathbb{R}^d,\nu^\gamma)} = \int\limits_{\mathbb{R}^d} |f(x)|^2 \, w^\gamma(x)\,\mathrm{d}x.$$

The next lemma connects the weighted L^2-spaces to the finiteness of the seminorms of $V^s(\Omega|\mathbb{R}^d)$ and $V(\Omega|\mathbb{R}^d)$. As a consequence a function f does not necessarily go to zero at infinity to ensure the finiteness of the seminorms. The idea of the proof is taken from [KD16, Prop. 9].

Lemma 2.34. *Let $\Omega \subset \mathbb{R}^d$ be open and let $f : \mathbb{R}^d \to \mathbb{R}$ be measurable. Assume*

$$\iint_{\Omega\,\mathbb{R}^d} \frac{(f(x)-f(y))^2}{|x-y|^{d+\gamma}}\,\mathrm{d}y\,\mathrm{d}x < \infty.$$

Then $f \in L^2(\mathbb{R}^d, \nu^\gamma)$.

Proof. First we prove that following algebraic inequality: Let $0 < a \leq n < 2n \leq b$. Then

$$b^2 \leq 4(a-b)^2. \tag{2.26}$$

This can be easily checked, since $b^2 = (b-a+a)^2 \leq 2[(b-a)^2 + a^2] \leq 2(b-a)^2 + \frac{b^2}{2}$.
Set for $n \in \mathbb{N}$

$$E_n = \{x \in \mathbb{R}^d |\, |f(x)| \leq n\}$$

and choose $R > 1$ such that $B_R \cap \Omega \neq \emptyset$. Set $F_n = E_n \cap B_R$ and choose $n \in \mathbb{N}$, such that $|F_n| > 0$. Now (2.26) and the fact that $|w-v| \leq R(1+|w|)$ for $w \in \mathbb{R}^d, v \in B_r$ yield

$$\begin{aligned}
2\iint_{\Omega\,\mathbb{R}^d} \frac{(f(x)-f(y))^2}{|x-y|^{d+\gamma}}\,\mathrm{d}y\,\mathrm{d}x &\geq \iint_{F_n\,\mathbb{R}^d\setminus E_{2n}} \frac{(f(x)-f(y))^2}{|x-y|^{d+\gamma}}\,\mathrm{d}y\,\mathrm{d}x + \iint_{\Omega\setminus E_{2n}\,F_n} \frac{(f(x)-f(y))^2}{|x-y|^{d+\gamma}}\,\mathrm{d}y\,\mathrm{d}x \\
&\geq \frac{1}{4R^{d+\gamma}} \left(\iint_{F_n\,\mathbb{R}^d\setminus E_{2n}} \frac{f^2(y)}{(1+|y|)^{d+\gamma}}\,\mathrm{d}y\,\mathrm{d}x + \iint_{\Omega\setminus E_{2n}\,F_n} \frac{f^2(x)}{(1+|x|)^{d+\gamma}}\,\mathrm{d}y\,\mathrm{d}x \right) \\
&\leq \frac{|F_n|}{4R^{d+\gamma}} \int_{\mathbb{R}^d\setminus E_{2n}} \frac{f^2(x)}{(1+|x|)^{d+\gamma}}\,\mathrm{d}x
\end{aligned}$$

Since $|f| \leq 2n$ on E_n this finishes the proof. $\square$

The next lemma proves that also the inversion of the lemma holds true, if we cut out the diagonal $|x-y| < 1$ and assume that Ω is bounded

Lemma 2.35. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $f \in L^2(\mathbb{R}^d, \nu^\gamma)$. Then*

$$\iint_{\substack{\Omega\,\mathbb{R}^d\cap\\ \{|x-y|>1\}}} \frac{(f(x)-f(y))^2}{|x-y|^{d+\gamma}}\,\mathrm{d}y\,\mathrm{d}x < \infty.$$

Proof. Since Ω is bounded, we find $R > 0$, such that $\Omega \subset B_R(0)$. For $x \in B^c_{2R}(0)$ and $y \in \Omega$

$$|x-y| \geq |x| - R \geq \frac{1}{2}\,|x|\ .$$

Now we can estimate

$$\begin{aligned}\iint\limits_{\substack{(\Omega^c\times\Omega^c)^c\\ \cap\{|x-y|>1\}}} \frac{(f(x)-f(y))^2}{|x-y|^{d+\gamma}}\,\mathrm{d}x\,\mathrm{d}y &\leq 4 \iint\limits_{\substack{\Omega\times\mathbb{R}^d\\ \cap\{|x-y|>1\}}} \frac{f^2(x)}{|x-y|^{d+\gamma}}\,\mathrm{d}x\,\mathrm{d}y\\ &\leq |\Omega| \int\limits_{B_{2R}(0)} f^2(x)\,\mathrm{d}x\\ &+ |\Omega| \int\limits_{B_{2R}^c(0)} \frac{f^2(x)}{|x^{d+\gamma}|}\,\mathrm{d}x\\ &\leq C(\Omega)\,\|f\|^2_{L^2(\mathbb{R}^d,\nu^\gamma)}\,.\end{aligned}$$

□

Example 3. Consider the function $f:\mathbb{R}^d\to\mathbb{R}$ defined by

$$f(x)=|x|^\beta\,.$$

for some $\beta>0$. Then $f\notin L^2(\mathbb{R}^d)$, but $f\in L^2(\mathbb{R}^d,\nu^\gamma)$, if $\gamma>2\beta$:

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{R}^d,\nu^\gamma)} &= \int\limits_{\mathbb{R}^d} \frac{|x|^{2\beta}}{(1+|x|)^{d+\gamma}}\,\mathrm{d}x\\ &\leq C\int\limits_0^1 r^{d-1}r^{2\beta}\,\mathrm{d}r + C\int\limits_1^\infty r^{d-1}r^{2\beta}r^{-d-\gamma}\,\mathrm{d}r\\ &= C\frac{1}{d+2\beta} + C\int\limits_1^\infty r^{-1-\gamma+2\beta}\,\mathrm{d}r.\end{aligned}$$

Now the second integral is finite if and only if $\gamma>2\beta$ and it is easy to check the reverse inequality.

Now we can enlarge the spaces $V^s(\Omega|\mathbb{R}^d)$ and $V(\Omega|\mathbb{R}^d)$ as follows. Replacing the L^2-norm in Definition 2.17 and Definition 2.10 by a weighted L^2-norm with $\gamma=2s$ for $V^s(\Omega|\mathbb{R}^d)$ and $\gamma=2$ for $V(\Omega|\mathbb{R}^d)$ allows a more general behavior at infinity, but does not change the properties of the before mentioned spaces (at least for fixed $s\in(0,1)$ in the case $V^s(\Omega|\mathbb{R}^d)$).

Remark 2.36. For fixed $0<s_1<s_2<1$ it can be easily seen that

$$V^{s_2}(\Omega|\mathbb{R}^d)\hookrightarrow V^{s_1}(\Omega|\mathbb{R}^d).$$

This is a natural property since the seminorm measures the regularity of a function in Ω and over the boundary of Ω. Replacing the L^2-norm by the weighted variant infringes this property, since a stronger weight allows more grow at infinity.

For this reason we do not change the original definitions, but we point out where we can replace the L^2-norm. In Section 3.3 we will give an alternative approach to allow even more general behavior at infinity. In contrast to the above approach, this approach is not consistent with the Hilbert space property of the solution space.

2.3. Function spaces with a general kernel as weight

2.3.1. Definition and basic properties

In this section we define function spaces tailor-made to deal with the nonlocal Dirichlet problem in a very general framework. Starting from the spaces $V^s(\Omega|\mathbb{R}^d)$ defined in Section 2.2 we replace the weight $|x-y|^{-d-2s}$ by a general symmetric kernel $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$. Adapting the function spaces to the kernel k makes it possible to deal with integro-differential and integral operators at the same time. For this purpose we introduce two classes of kernels

Definition 2.37. Let $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ be measurable and symmetric. We call k integrable, if for every $x \in \mathbb{R}^d$ the quantity $\int_{\mathbb{R}^d} k(x,y)\,\mathrm{d}y$ is finite and the mapping

$$x \mapsto \int\limits_{\mathbb{R}^d} k(x,y)\,\mathrm{d}y \in L^1_{loc}(\mathbb{R}^d).$$

If k is not integrable in the above sense, we call k non-integrable.

A simple integrable example is given by $k(x,y) = \mathbb{1}_{B_1}(x-y)$. The standard non-integrable example is given by $k(x,y) = |x-y|^{-d-\alpha}$ for some $\alpha \in (0,2)$. Another non-integrable kernel is given by $k(x,y) = -\frac{\ln(|x-y|)}{|x-y|^d}\mathbb{1}_{B_1}(x-y)$.

We start with the definition of the following linear spaces:

Definition 2.38. Let $\Omega \subset \mathbb{R}^d$ be open and $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ be measurable and symmetric. We define the following linear spaces:

(i) Define

$$V^k(\Omega|\mathbb{R}^d) = \Big\{ v : \mathbb{R}^d \to \mathbb{R} \colon v|_\Omega \in L^2(\Omega), (v(x)-v(y))\, k^{1/2}(x,y) \in L^2((\Omega^c \times \Omega^c)^c) \Big\},$$
$$[u,v]_{V^k(\Omega|\mathbb{R}^d)} = \iint\limits_{(\Omega^c \times \Omega^c)^c} [u(x)-u(y)]\,[v(x)-v(y)]\, k(x,y)\,\mathrm{d}y\,\mathrm{d}x\,.$$

A seminorm on $V^k(\Omega|\mathbb{R}^d)$ is given by $[v,v]_{V^k(\Omega|\mathbb{R}^d)}$.

(ii) In the case $\Omega = \mathbb{R}^d$ we write $V^k(\mathbb{R}^d|\mathbb{R}^d) = H^k(\mathbb{R}^d)$ and a norm on this space is defined by

$$\|v\|^2_{H^k(\mathbb{R}^d)} = \|v\|^2_{L^2(\mathbb{R}^d)} + \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (v(x)-v(y))^2 k(x,y)\,\mathrm{d}y\,\mathrm{d}x\,.$$

(iii) $H^k_0(\Omega|\mathbb{R}^d) = \big\{ u \in H^k(\mathbb{R}^d) \colon u = 0 \text{ a.e. on } \Omega^c \big\}$ endowed with the norm $\|\cdot\|_{H^k(\mathbb{R}^d)}$.

(iv) We denote by $H^k_0(\Omega|\mathbb{R}^d)^*$ the dual space of $H^k_0(\Omega|\mathbb{R}^d)$. If $x \in H^k_0(\Omega|\mathbb{R}^d)^*$, we define the norm

$$\|x\|_{H^k_0(\Omega|\mathbb{R}^d)^*} = \sup \Big\{ \langle x, v\rangle \mid v \in H^k_0(\Omega|\mathbb{R}^d), \|v\|_{H^k(\mathbb{R}^d)} = 1 \Big\}.$$

Remark 2.39.

a) Note that the properties of a function belonging to $V^k(\Omega|\mathbb{R}^d)$ depend heavily on the kernel k.

b) It is clear from the definition that for any $\Omega \subset \mathbb{R}^d$ open
$$\left(H_0^k(\Omega|\mathbb{R}^d), \|\cdot\|_{H^k(\mathbb{R}^d)}\right) \hookrightarrow \left(H^k(\mathbb{R}^d), \|\cdot\|_{H^k(\mathbb{R}^d)}\right) \tag{2.27}$$
and $H^k(\mathbb{R}^d) \subset V^k(\Omega|\mathbb{R}^d)$. Moreover, if $g \in V^k(\Omega|\mathbb{R}^d)$ and $g = 0$ a.e. on Ω^c, then $g \in H_0^k(\Omega|\mathbb{R}^d)$.

We want to emphasize that symmetry of k is not needed in Definition 2.38, since the integrand in the seminorm is symmetric with respect to x and y. Thus, for a general, possibly nonsymmetric kernel k, its symmetrization $k_s = \frac{1}{2}(k(x,y) + k(y,x)$ defines the same function space.

The function spaces defined in this section are use to obtain existence and uniqueness of weak solutions to nonlocal Dirichlet problems, see Chapter 3 below. Due to the linearity of the operator one can transform the Dirichlet problem with boundary data g to a problem with zero boundary data. On this account, we introduce the space $H_0^k(\Omega|\mathbb{R}^d)$ that encodes the homogeneous boundary data.

Note that a function $v \in V^k(\Omega|\mathbb{R}^d)$ belongs only to $L^2(\Omega)$. Because of this for general k the space $V^k(\Omega|\mathbb{R}^d)$ is not a Hilbert space due to the lack of completeness. If $k > 0$ almost everywhere on $(\Omega^c \times \Omega^c)^c$, we can prove the completeness of $V^k(\Omega|\mathbb{R}^d)$ following the proof of [DRV14, Prop. 3.1].

We want to mention two special cases. First we consider a nonsingular example, let $k(x,y) = \mathbb{1}_{B_1}(x-y)$. Then $V^k(\Omega|\mathbb{R}^d)$ equals $L^2(\Omega_1)$, where Ω_1 is the thickening of Ω,
$$\Omega_1 = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x,\Omega) < 1\}.$$
Another important example is given by $k(x,y) = \mathcal{A}_{d,-2s}\,|x-y|^{-d-2s}$. In this case the seminorm of $V^k(\Omega|\mathbb{R}^d)$ equals the seminorm of $V^s(\Omega|\mathbb{R}^d)$. Because of the L^2-part of the norms, the norm of $V^k(\Omega|\mathbb{R}^d)$ is dominated by the norm of $V^s(\Omega|\mathbb{R}^d)$.

Let us fix the Hilbert space property of the spaces $H_0^k(\Omega|\mathbb{R}^d)$ and $H(\Omega;k)$ for a general kernel k.

Lemma 2.40. *Let $\Omega \subset \mathbb{R}^d$ be an open set and assume that $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ is measurable and symmetric. Then the spaces $H_0^k(\Omega|\mathbb{R}^d)$ and $H^k(\mathbb{R}^d)$ are separable Hilbert spaces.*

Proof. The proof is analogous to the proof of Lemma 2.19 when we replace the weight $|x-y|^{-d-2s}$ by $k(x,y)$. □

If Ω is a C^1-domain and the kernel depends only on the differences $x-y$, the density of smooth functions in $V^k(\Omega|\mathbb{R}^d)$ can be proven by a slight change of the proof of Lemma 2.12.

Lemma 2.41. *Let Ω be a C^1-domain and $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ be measurable. Assume there is a measurable function $\widetilde{k} : \mathbb{R}^d \to [0,\infty]$, such that $k(x,y) = \widetilde{k}(x-y)$. Then for every $u \in H_0^k(\Omega|\mathbb{R}^d)$ there exist functions $u_n \in C_c^\infty(\Omega)$ such that*
$$u_n \overset{n\to\infty}{\longrightarrow} u \quad \text{in } H_0^k(\Omega|\mathbb{R}^d).$$

Proof. Since $\widetilde{k}$ is translation invariant, we can apply the proof of Lemma 2.12 with the obvious changes, namely replace $\Delta u(x;y)$ by $u(x) - u(y)$ and the weight $|x-y|^{-d-2s}$ by $\widetilde{k}(x-y)$. Note that since $u = 0$ a.e. on Ω^c, the constructed sequence can be chosen from $C_c^\infty(\Omega)$ instead of $C_c^\infty(\mathbb{R}^d)$. □

2.3.2. Poincaré-Friedrichs inequality

Let us formulate a nonlocal version of the Poincaré-Friedrichs inequality in our set-up: There exists a constant $C_P > 0$ such that for all $u \in L^2_0(\Omega|\mathbb{R}^d)$

$$\|u\|^2_{L^2(\Omega)} \leq C_P \iint_{\mathbb{R}^d\,\mathbb{R}^d} (u(x) - u(y))^2 k(x, y)\,\mathrm{d}x\,\mathrm{d}y\,. \tag{P}$$

This inequality appears as an assumption, explicitly or implicitly, in most of the existence results of the nonlocal Dirichlet problem in Chapter 3 below. In this section we provide sufficient conditions of a kernel k for (P) to hold. Here k may be nonsymmetric, nevertheless we can replace k by its symmetrization in (P) since the integrand is symmetric. Since the classical Poincaré-Friedrichs inequality deals with derivatives on the right-hand side it seems to be surprising that there is an analogue for integrable kernel.

Note that throughout this section for the sake of convenience we replace the integration over $(\Omega^c \times \Omega^c)^c$ by the integration over $\mathbb{R}^d \times \mathbb{R}^d$, since we only deal with functions u that vanish almost everywhere outside Ω.

The following result generalizes the Poincaré-Friedrichs inequalities from [AM10] and [AP09, Prop. 1], respectively, to a larger class of integrable and non-integrable kernels. (In these references, the Poincaré-Friedrichs inequality is stated for functions with values in $\mathbb{R}^d$.)

Lemma 2.42. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ be measurable. Assume that there is a symmetric, a.e. nonnegative function $L \in L^1(\mathbb{R}^d)$ satisfying the following properties: $|\{L > 0\}| > 0$ and there is $c_0 > 0$ such that for all $u \in L^2(\Omega)$*

$$\iint_{\mathbb{R}^d\,\mathbb{R}^d} (u(x) - u(y))^2\, k(x, y)\,\mathrm{d}y\,\mathrm{d}x \geq c_0 \iint_{\mathbb{R}^d\,\mathbb{R}^d} (u(x) - u(y))^2\, L(x - y)\,\mathrm{d}y\,\mathrm{d}x\,. \tag{2.28}$$

Then the following Poincaré-Friedrichs inequality holds: There is $C_P = C_P(\Omega, c_0, L) > 0$ such that for all $u \in H^k_0(\Omega|\mathbb{R}^d)$

$$\|u\|^2_{L^2(\mathbb{R}^d)} \leq C_P \iint_{\mathbb{R}^d\,\mathbb{R}^d} (u(x) - u(y))^2 k(x, y)\,\mathrm{d}y\,\mathrm{d}x\,. \tag{2.29}$$

The example in [AVMRTM10, Rem. 6.20] shows that Lemma 2.42 fails to hold if one replaces the domain of integration $\mathbb{R}^d \times \mathbb{R}^d$ by $\Omega \times \Omega$ in (2.28) and (2.29).

For the proof of the Poincaré-Friedrichs inequality, we need the following technical Lemma taken from [DK11, Lem. 10].

Lemma 2.43. *Let $q \in L^1(\mathbb{R}^d)$ be nonnegative almost everywhere and let $\operatorname{supp} q \subset B_\rho(0)$ for some $\rho > 0$. Then for all $R > 0$ and all functions u:*

$$\iint_{B_R\,B_R} (u(x) - u(y))^2\, (q * q)\,(x - y)\,\mathrm{d}y\,\mathrm{d}x \leq 4\|q\|_{L^1(\mathbb{R}^d)} \iint_{B_{R+\rho}\,B_{R+\rho}} (u(x) - u(y))^2\, q(x - y)\,\mathrm{d}y\,\mathrm{d}x\,.$$

Repeated application of Lemma 2.43 and the fact that convolution of a function with itself enlarges its support allows us to prove the Poincaré inequality.

Proof of Lemma 2.42. Let L satisfy the assumptions of the lemma. Without loss of generality $0 \in \Omega$ (otherwise shift Ω). Furthermore, we may assume that there is $\rho > 0$ such that $\operatorname{supp} L \subset B_\rho(0)$ (otherwise replace L by $L\,\mathbb{1}_{B_\rho(0)}$). Fix $R > 0$ such that $\Omega \Subset B_R(0)$. For $\nu \in \mathbb{N}$ define

$$L_\nu = \underbrace{L * L * \ldots * L}_{2^\nu \text{ times}} .$$

By the properties of L we have

$$(L * L)(0) = L_1(0) = \int\limits_{\mathbb{R}^d} L(z)L(-z)\,\mathrm{d}z = \int\limits_{\mathbb{R}^d} L^2(z)\,\mathrm{d}z > 0$$

and $L_1 = L * L \in C_b(\mathbb{R}^d)$, which implies that we may find $\delta > 0$ (depending on L) such that $L_1 > 0$ on $B_\delta(0)$. By the property of the convolution there is $m \in \mathbb{N}$ depending on L and Ω such that $L_m > 0$ on $B_R(0)$. Let $u \in H^k_0(\Omega|\mathbb{R}^d)$. Then we may estimate

$$\begin{aligned} E^{L_m}_{B_R}(u,u) &:= \iint\limits_{B_R\,B_R} (u(x) - u(y))^2\, L_m(x-y)\,\mathrm{d}y\,\mathrm{d}x \\ &\geq \int_\Omega u^2(x) \int_{\Omega^c \cap B_R} L_m(x-y)\,\mathrm{d}y\,\mathrm{d}x \geq C(L,\Omega)\,\|u\|^2_{L^2(\mathbb{R}^d)} . \end{aligned} \tag{2.30}$$

Iterated application of Lemma 2.43 (with $\rho' = 2^m\rho$ and $q = L_j$, $j = m-1, \ldots, 0$) yields

$$E^{L_m}_{B_R}(u,u) \leq 4\|L_{m-1}\|_{L^1(\mathbb{R}^d)} E^{L_{m-1}}_{B_{R+\rho'}}(u,u) \leq \ldots \leq 4^m E^{L}_{B_{R+m\rho'}}(u,u) \prod_{j=0}^{m-1} \|L_i\|_{L^1(\mathbb{R}^d)} . \tag{2.31}$$

(2.30), (2.31) and the assumption (2.28) imply

$$\begin{aligned} \|u\|^2_{L^2(\mathbb{R}^d)} &\leq \frac{1}{C(L,\Omega)} E^{L_m}_{B_R}(u,u) \leq \frac{4^m}{C(L,\Omega)} \prod_{j=0}^{m-1} \|L_i\|_{L^1(\mathbb{R}^d)} \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))^2 L(x-y)\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \frac{4^m}{c_0\, C(L,\Omega)} \prod_{j=0}^{m-1} \|L_i\|_{L^1(\mathbb{R}^d)} \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))^2 k(x,y)\,\mathrm{d}y\,\mathrm{d}x . \end{aligned}$$

This finishes the proof of Lemma 2.42. □

For non-integrable kernels k we have the following Poincaré-Friedrichs inequality:

Lemma 2.44. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be measurable and nonnegative almost everywhere. We assume that for some $\alpha \in (0,2)$, some $\lambda > 0$ and all $u \in L^2(\mathbb{R}^d)$ the kernel k satisfies* (E_α) *(see p. 67). Then there is $C_P > 0$ such that for all $u \in L^2_\Omega(\mathbb{R}^d)$*

$$\|u\|_{L^2(\mathbb{R}^d)} \leq C_P \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))^2 k(x,y)\,\mathrm{d}y\,\mathrm{d}x . \tag{2.32}$$

Given $\alpha_0 \in (0,2)$ and $\alpha \in [\alpha_0, 2)$, the constant C_P can be chosen independently of α.

The proof of the main assertion is simple. The statement about the independence of C_P on α is proved in [MS02a, Thm. 1].

3. Existence and uniqueness of solutions for nonlocal boundary value problems

Given an open domain $\Omega \subset \mathbb{R}^d$ and functions $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$, the nonlocal Dirichlet problem is to find a function $u : \mathbb{R}^d \to \mathbb{R}$ such that

$$\mathcal{L}u = f \qquad \text{in } \Omega\,, \tag{3.1a}$$
$$u = g \qquad \text{on } \Omega^c\,. \tag{3.1b}$$

In this chapter we establish a Hilbert space approach to solve the Dirichlet problem (3.1) associated with nonlocal operators of the form

$$(\mathcal{L}u)\,(x) = \lim_{\varepsilon \to 0+} \int\limits_{y \in \mathbb{R}^d \setminus B_\varepsilon(x)} (u(x) - u(y))k(x, y)\,\mathrm{d}y\,, \tag{3.2}$$

where $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ is assumed to be measurable. For this purpose we use the function spaces defined in Section 2.3, which are tailor-made for a given kernel k.

We focus on two different aspects. On the one side we consider kernels k with the following three properties: k is not necessarily symmetric, k might be singular on the diagonal and k is allowed to be discontinuous. On the other hand we focus on boundary data, that are not assumed to be regular outside the given domain Ω.

Let us start with an easy example in the simple setting where Ω equals the unit ball $B_1 \subset \mathbb{R}^d$.

Example 4. Assume $0 < \beta < \frac{\alpha}{2} < 1$. Let I_1, I_2 be arbitrary nonempty open subsets of S^{d-1} with $I_1 = -I_1$. Set $\mathcal{C}_j = \{h \in \mathbb{R}^d |\ \frac{h}{|h|} \in I_j\}$ for $j \in \{1, 2\}$ and

$$k(x, y) = |x - y|^{-d-\alpha} \mathbb{1}_{\mathcal{C}_1}(x - y) + |x - y|^{-d-\beta} \mathbb{1}_{\mathcal{C}_2}(x - y) \mathbb{1}_{B_1}(x - y)\,.$$

The part involving $|x - y|^{-d-\beta}$ can be seen as a lower order perturbation of the main part of the kernel resp. integro-differential operator produced by $|x - y|^{-d-\alpha} \mathbb{1}_{\mathcal{C}_1}(x - y)$.

Define boundary data $g : \mathbb{R}^d \to \mathbb{R}$ by

$$g(x) = \begin{cases} (|x| - 1)^\gamma, & \text{if } 1 \le |x| \le 2\,, \\ 0, & \text{else.} \end{cases}$$

where γ is an arbitrary real number satisfying $\gamma > \frac{\alpha - 1}{2}$. Note that g may be unbounded if $\alpha < 1$. Let $f \in L^2(\Omega)$ be arbitrary. For this choice of a kernel k and such data g and f we obtain well-posedness of the Dirichlet problem (5.9) for $\Omega = B_1 \subset \mathbb{R}^d$.

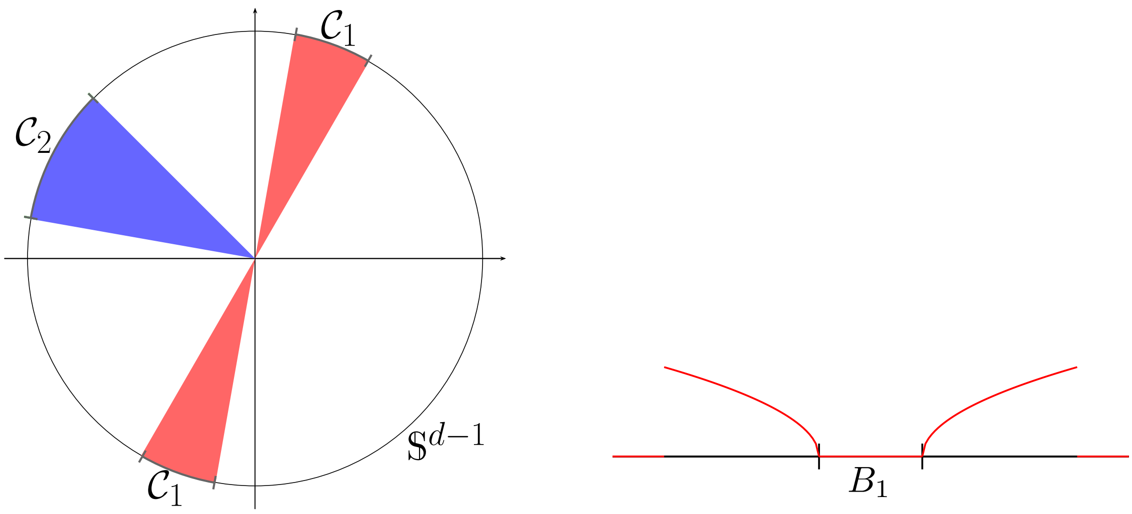

Figure 3.1.: Support of the kernel k in the product space and profile of the boundary data g

Note that all results of this chapter rely on the article [FKV14], unless it is explicitly mentioned.

In the first section of this chapter we fix some assumption on admissible kernels. Afterwards we derive a variational formulation of (3.1) introducing the bilinear form $\mathcal{E}$ associated to $\mathcal{L}$. In Section 3.3 we prove a Gårding inequality and apply the Lax-Milgram Lemma to the nonlocal Dirichlet problem for a class of kernels, for which the bilinear form is positive definite. Section 3.4 is devoted to the weak maximum principle for integro-differential operators in bounded domains. This tool is applied when using the Fredholm alternative in Subsection 3.4.2, which allows us to consider also kernels for which the bilinear form is no longer positive definite. Finally, in Section 3.5 we provide many detailed examples of kernels k and discuss their properties.

3.1. Setting

In most of our existence results we deal with integrable and non-integrable kernels – see Definition 2.37 – at the same time. Simple examples of integrable and non-integrable kernels are given by $k(x,y) = \mathbb{1}_{B_1}(x-y)$ and $k(x,y) = |x-y|^{-d-1}$.

For non-integrable kernels the operator $\mathcal{L}$ acts as an integro-differential operator, while for integrable kernels $\mathcal{L}$ has no differential structure and is a well-defined operator on L^2.

For a given kernel $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ its symmetric and anti-symmetric parts are defined by

$$k_s(x,y) = \frac{1}{2}\left(k(x,y) + k(y,x)\right) \quad \text{and} \quad k_a(x,y) = \frac{1}{2}\left(k(x,y) - k(y,x)\right).$$

By the positivity of k it follows that $|k_a(x,y)| \le k_s(x,y)$ for almost all $x, y \in \mathbb{R}^d$. Throughout this chapter we assume that the symmetric part of the kernel satisfies the following integrability condition[1]:

$$x \mapsto \int_{\mathbb{R}^d} \left(1 \wedge |x-y|^2\right) k_s(x,y)\,\mathrm{d}y \in L^1_{loc}(\mathbb{R}^d). \tag{L}$$

[1]Condition (L) corresponds to the integrability condition of the Levy measure in the Levy-Khinchin formula

In addition, in order to prove well-posedness of the bilinear form associated to the operator $\mathcal{L}$, we need to impose a condition on how the symmetric part of k dominates the anti-symmetric part of k. We assume that there exist a symmetric kernel $\widetilde{k} : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ with $|\{y \in \mathbb{R}^d | \, \widetilde{k}(x,y) = 0, k_a(x,y) \neq 0\}| = 0$ for all x, and constants $A_1 \geq 1, A_2 \geq 1$ such that

$$\iint\limits_{(\Omega^c \times \Omega^c)^c} (u(x) - u(y))^2 \widetilde{k}(x,y) \, \mathrm{d}x \, \mathrm{d}y \leq A_1 \iint\limits_{(\Omega^c \times \Omega^c)^c} (u(x) - u(y))^2 k_s(x,y) \, \mathrm{d}x \, \mathrm{d}y \qquad (\widetilde{K}_1)$$

for all $u \in V^k(\Omega | \mathbb{R}^d)$, and at the same time

$$\sup_{x \in \mathbb{R}^d} \int \frac{k_a^2(x,y)}{\widetilde{k}(x,y)} \, \mathrm{d}y \leq A_2. \qquad (\widetilde{K}_2)$$

A natural choice is $\widetilde{k} = k_s$ because in this case $(\widetilde{K}_1)$ trivially holds and $\widetilde{k}(x,y) = 0$ implies $k_a(x,y) = 0$. Assumption $(\widetilde{K})$ would then reduce to the condition

$$\sup_{x \in \mathbb{R}^d} \int\limits_{\{k_s(x,y) \neq 0\}} \frac{k_a^2(x,y)}{k_s(x,y)} \, \mathrm{d}y \leq A \,. \qquad \text{(K)}$$

Condition (K) appears in [SW11, (1.1)] and is sufficient for that $(\mathcal{E}, C_c^{0,1}(\mathbb{R}^d))$ extends to a regular lower bounded semi-Dirichlet form. Note that our assumption $(\widetilde{K})$ is weaker and thus we can extend [SW11, Thm 1.1], see Lemma 3.3. The kernel in Example 4 illustrates the difference between $(\widetilde{K})$ and (K). While (K) does not hold, choosing $\widetilde{k} = |x-y|^{-d-\alpha}$ $(\widetilde{K})$ is satisfied. We discuss this in detail in Section 3.5.

Remark 3.1. We would like to point out, that in contrast to [FKV14] in $(\widetilde{K}_1)$ the area of integration is $(\Omega^c \times \Omega^c)^c$ instead of $\mathbb{R}^d \times \mathbb{R}^d$. This allows us to drop a second comparability assumption in our existence theorem, cf [FKV14, Thm. 3.5] and Theorem 3.10 below. Note that in the case $u = 0$ a.e. on Ω^c both assumptions are equivalent. Further one easily obtain the comparability condition on sets of the form $\Omega \times \mathbb{R}^d$ from $(\widetilde{K}_1)$.

3.2. Variational formulation of the Dirichlet problem

Consider a bilinear form defined by

$$\mathcal{E}^k(u,v) = \iint\limits_{(\Omega^c \times \Omega^c)^c} (u(x) - u(y)) v(x) k(x,y) \, \mathrm{d}y \, \mathrm{d}x \,. \qquad (3.4)$$

First we prove well-posedness of this expression and that the bilinear form is associated to $\mathcal{L}$. For this purpose we use $(\widetilde{K})$ on how the symmetric part of k dominates the anti-symmetric part of k.

Remark 3.2. Note that in the definition of $\mathcal{E}^k$ we integrate only over $(\Omega^c \times \Omega^c)^c$. This definition is equivalent to the one given in [FKV14], when the function v vanishes outside Ω. Nevertheless, when considering symmetric kernels and nonzero boundary data, the Dirichlet energy of the solution is given by

$$\mathcal{E}^k(u,u) = \frac{1}{2} \iint\limits_{(\Omega^c \times \Omega^c)^c} (u(x)-u(y))^2 k(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$

while the integral over $\mathbb{R}^d \times \mathbb{R}^d$ may be infinite.

Let us show that the bilinear form defined in (3.4) is associated to $\mathcal{L}$ and that the integrand in (3.4) is – in contrast to the integrand in (3.2) – integrable in the Lebesgue sense.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^d$ be open and assume that k satisfies* (L) *and* $(\widetilde{K})$. *Define for $n \in \mathbb{N}$ the set $D_n = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \colon |x-y| > 1/n\}$ and*

$$\begin{aligned} \mathcal{L}_n u(x) &= \int_{B^c_{1/n}(x)} (u(x)-u(y))\, k(x,y)\,\mathrm{d}y, \\ \mathcal{E}^k_n(u,v) &= \iint\limits_{D_n} (u(x)-u(y))\, v(x) k(x,y)\,\mathrm{d}y\,\mathrm{d}x\,. \end{aligned}$$

Then we have $(\mathcal{L}_n u, v)_{L^2(\mathbb{R}^d)} = \mathcal{E}^k_n(u,v)$ and $\lim_{n\to\infty} \mathcal{E}^k_n(u,v) = \mathcal{E}^k(u,v)$ for all $u, v \in C^\infty_c(\Omega)$. Moreover, $\mathcal{E}^k \colon H^k(\mathbb{R}^d) \times H^k(\mathbb{R}^d) \to \mathbb{R}$ is continuous. By (2.27)*, $\mathcal{E}^k$ is also continuous on $H^k_0(\Omega|\mathbb{R}^d)$.*

As mentioned above, our proof is an extension of the proof of [SW11, Thm. 1.1].

Proof. Assume $u, v \in C^\infty_c(\mathbb{R}^d)$. Splitting k in its symmetric and antisymmetric part yields

$$\begin{aligned} (\mathcal{L}_n u, v)_{L^2(\mathbb{R}^d)} &= \int\limits_{\mathbb{R}^d} \int\limits_{B^c_{1/n}(x)} (u(x)-u(y)) k(x,y)\,\mathrm{d}y\; v(x)\,\mathrm{d}x \\ &= \frac{1}{2} \iint\limits_{D_n} (u(x)-u(y))(v(x)-v(y)) k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\quad + \iint\limits_{D_n} (u(x)-u(y)) v(x) k_a(x,y)\,\mathrm{d}y\,\mathrm{d}x\,. \end{aligned}$$

The first integral is finite due to (L). In order to show the integrability of the second integrand we use $(\widetilde{K})$ with $A = \max(A_1, A_2)$ and the Cauchy-Schwarz inequality:

$$\begin{aligned} &\iint\limits_{D_n} |u(x)-u(y)|\,|v(x)|\,|k_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x \\ &\quad = \iint\limits_{D_n} |u(x)-u(y)|\,|v(x)|\,\widetilde{k}^{1/2}(x,y)\,|k_a(x,y)|\,\widetilde{k}^{-1/2}(x,y)\,\mathrm{d}y\,\mathrm{d}x \end{aligned}$$

$$\le \left(\iint\limits_{D_n} (u(x)-u(y))^2 \widetilde{k}(x,y)\,\mathrm{d}x\,\mathrm{d}y \right)^{1/2} \left(\int\limits_{\mathbb{R}^d} v(x)^2 \int\limits_{B^c_{1/n}(x)} \frac{k_a^2(x,y)}{\widetilde{k}(x,y)}\,\mathrm{d}y\,\mathrm{d}x \right)^{1/2}$$

$$\le A \left(\iint\limits_{D_n} (u(x)-u(y))^2 k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y \right)^{1/2} \|v\|_{L^2(\mathbb{R}^d)} .$$

This shows $\langle \mathcal{L}_n u, v\rangle_{L^2(\mathbb{R}^d)} = \mathcal{E}^k_n(u,v)$ and that all expressions in this equality are well-defined. In particular, by dominated convergence $\lim\limits_{n\to\infty} \mathcal{E}^k_n(u,v) = \mathcal{E}^k(u,v)$. Moreover, $\mathcal{E}^k(u,v) < \infty$ for $u, v \in H^k(\mathbb{R}^d)$.

Now let us prove the continuity of $\mathcal{E}\colon H^k(\mathbb{R}^d) \times H^k(\mathbb{R}^d) \to \mathbb{R}$. Let $u, v \in H^k(\mathbb{R}^d)$. Again by the symmetry of k_s and by $(\widetilde{K})$ we obtain

$$\begin{aligned}
\left|\mathcal{E}^k(u,v)\right| &= \left| \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))v(x)k(x,y)\,\mathrm{d}y\,\mathrm{d}x \right| \\
&\le \left| \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))v(x)k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \right| \\
&\quad + \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|\, \widetilde{k}(x,y)^{1/2}\, |v(x)|\, |k_a(x,y)|\, \widetilde{k}^{-1/2}\,\mathrm{d}y\,\mathrm{d}x \\
&\le \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|\, |v(x)-v(y)|\, k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad + \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|\, \widetilde{k}(x,y)^{1/2}\, |v(x)|\, |k_a(x,y)|\, \widetilde{k}^{-1/2}(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\le \left(\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \right)^{1/2} \left(\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (v(x)-v(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \right)^{1/2} \\
&\quad + A \left(\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \right)^{1/2} \|v\|_{L^2(\mathbb{R}^d)} \\
&\le C \|u\|_{H^k(\mathbb{R}^d)} \|v\|_{H^k(\mathbb{R}^d)} .
\end{aligned}$$

This shows that $\mathcal{E}^k$ is a continuous bilinear form on $H^k(\mathbb{R}^d)$ and on $H^k_0(\Omega|\mathbb{R}^d)$. □

Now we are able to provide a variational formulation of the Dirichlet problem (5.9) with the help of the bilinear form $\mathcal{E}^k$.

Definition 3.4. Assume (L) and $(\widetilde{K})$. Let Ω be open and $f \in H^*_\Omega(\mathbb{R}^d; k)$.

(i) $u \in H_0^k(\Omega|\mathbb{R}^d)$ is called a solution of

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = 0 & \text{on } \Omega^c\,, \end{cases} \tag{D$_0$}$$

if

$$\mathcal{E}^k(u,\varphi) = \langle f, \varphi \rangle \qquad \text{for all } \varphi \in H_0^k(\Omega|\mathbb{R}^d)\,. \tag{3.5}$$

(ii) Let $g \in V^k(\Omega|\mathbb{R}^d)$. A function $u \in V^k(\Omega|\mathbb{R}^d)$ is called a solution of

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = g & \text{on } \Omega^c\,, \end{cases} \tag{D}$$

if $u - g \in H_0^k(\Omega|\mathbb{R}^d)$ and (3.5) holds.

In the subsequent sections we will show how to solve this problem using classical Hilbert space techniques. As long as the bilinear form is positive definite, we obtain existence and uniqueness via the Lax-Milgram Lemma. If we want to consider kernels k for which the form is not positive definite anymore, see (C) and Remark 3.8 below, we have to use Fredholms alternative. Since Fredholms alternative needs compactness, we have to restrict ourself to non-integrable kernels in this case.

Remark 3.5. If $C_c^\infty(\Omega)$ is dense in $H_0^k(\Omega|\mathbb{R}^d)$ then $H_\Omega^*(\mathbb{R}^d;k)$ is a space of distributions on Ω. In this case solutions in the sense of Definition 3.4 are weak solutions to (D$_0$) and (D), respectively. By Lemma 2.41 this is true for example if $k(x,y)$ depends only on the difference $x-y$.

In the following we weaken the concept of a solution to allow more general behavior of the boundary data at infinity. As already mentioned before we consider only linear problems and thus the boundary value problem with inhomogeneous boundary data can be rewritten as a boundary value problem with homogeneous boundary data but another inhomogeneity

$$\widetilde{f} = f - \mathcal{L}g.$$

We need to prove that $\widetilde{f}$ is a continuous linear functional in $H_\Omega^*(\mathbb{R}^d;k)$. In [FKV14] and Definition 3.4 it is assumed that the boundary data itself are in $V^k(\Omega|\mathbb{R}^d)$. On the one hand this is a simple way to guarantee $\widetilde{f} \in H_\Omega^*(\mathbb{R}^d;k)$. On the other hand this naturally implies some L^2-condition on the boundary data g at infinity, see Subsection 2.2.6. To avoid this, we give a second weaker definition of a solution.

Definition 3.6. Assume (L) and $(\widetilde{K})$. Let Ω be open, $f \in H_\Omega^*(\mathbb{R}^d;k)$. Let $g : \mathbb{R}^d \to \mathbb{R}$. A function $u : \mathbb{R}^d \to \mathbb{R}$ is called a generalized solution of

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = g & \text{on } \Omega^c\,, \end{cases} \tag{$\widetilde{D}$}$$

if $u - g \in H_0^k(\Omega|\mathbb{R}^d)$ and (3.5) holds.

3.3. Gårding inequality and Lax-Milgram Lemma

In this section we discuss basic properties of the bilinear form $\mathcal{E}^k$ which can be used in order to prove solvability of the Dirichlet problem. First, we establish a Gårding inequality under the conditions (L) and $(\widetilde{K})$. In a second subsection we show that if, in addition, the Poincaré-Friedrichs inequality and a certain cancellation property hold, the bilinear form $\mathcal{E}^k$ is positive definite and coercive. This allows to establish a first existence result with the help of the well-known Lax-Milgram Lemma, see Theorem 3.10.

3.3.1. Gårding inequality

Lemma 3.7 (Gårding inequality). *Let k satisfy* (L) *and* $(\widetilde{K})$. *Then there is $\gamma = \gamma(A_1, A_2) > 0$ such that*

$$\mathcal{E}^k(u,u) \geq \frac{1}{4}\|u\|^2_{H^k(\mathbb{R}^d)} - \gamma\|u\|^2_{L^2(\mathbb{R}^d)} \qquad \textit{for all } u \in H^k(\mathbb{R}^d)\,. \tag{3.6}$$

Proof. Let $u \in H^k(\mathbb{R}^d)$ and let $A = \max\{A_1, A_2\}$. By $(\widetilde{K})$ we obtain

$$\begin{aligned}
\mathcal{E}^k(u,u) &\geq \frac{1}{2}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x - \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |(u(x)-u(y))u(x)k_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{1}{2}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad - \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \left|(u(x)-u(y))\widetilde{k}^{1/2}(x,y)u(x)k_a(x,y)\widetilde{k}^{-1/2}(x,y)\right|\,\mathrm{d}y\,\mathrm{d}x \\
&\geq \frac{1}{2}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad - \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \left[\varepsilon\,|u(x)-u(y)|^2\,\widetilde{k}(x,y) + \frac{1}{4\varepsilon}u^2(x)k_a^2(x,y)\widetilde{k}^{-1}(x,y)\right]\mathrm{d}y\,\mathrm{d}x \\
&\geq \frac{1}{4}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x - \frac{1}{4\varepsilon}A\|u\|^2_{L^2(\mathbb{R}^d)} \\
&\geq \tfrac{1}{4}\|u\|^2_{H^k(\mathbb{R}^d)} - \gamma\|u\|^2_{L^2(\mathbb{R}^d)}\,,
\end{aligned}$$

if we choose ε sufficiently small such that $A\varepsilon < \frac{1}{4}$ and then $\gamma = \gamma(A)$ sufficiently large. □

3.3.2. Application of the Lax-Milgram Lemma

To verify that the bilinear form $\mathcal{E}^k$ is positive definite, we assume the following cancellation condition:

$$\inf_{x\in\mathbb{R}^d} \liminf_{\varepsilon\to 0+} \int\limits_{B^c_\varepsilon(x)} k_a(x,y)\,\mathrm{d}y \geq 0. \tag{C}$$

Remark 3.8. As the proof below shows, assumption (C) can be relaxed. It is sufficient to assume

$$\inf_{x\in\mathbb{R}^d} \liminf_{\varepsilon\to 0+} \int\limits_{B_\varepsilon^c(x)} k_a(x,y)\,\mathrm{d}y > -\frac{1}{2C_P}\,,$$

with C_P as in (P). The bilinear form $\mathcal{E}^k$ would still be coercive.

There are many cases for which condition (C) holds. If $k_a(x,y)$ depends only on $x-y$, then for every $x\in\mathbb{R}^d$ and every $\varepsilon>0$ one obtains $\int_{B_\varepsilon^c(x)} k_a(x,y)\,\mathrm{d}y = 0$ which trivially implies (C). But there are also many interesting cases for which condition (C) is not satisfied, see Section 3.5.

Proposition 3.9. *Let $\Omega\subset\mathbb{R}^d$ be open. Let $f\in H_\Omega^*(\mathbb{R}^d;k)$ and let k satisfy* (L), (P), $(\widetilde{K})$ *and* (C). *Then there is a unique solution $u\in H_0^k(\Omega|\mathbb{R}^d)$ to* (D_0).

Proof. In Lemma 3.3 it was shown that $\mathcal{E}^k$ is a continuous bilinear form on $H_0^k(\Omega|\mathbb{R}^d)$. First, we show that (C) implies that $\mathcal{E}^k$ is positive definite. Let $u\in H_0^k(\Omega|\mathbb{R}^d)$. Observe that $k=k_s+k_a$ and for every $\varepsilon>0$

$$\begin{aligned}
\iint\limits_{\{|x-y|>\varepsilon\}} (u(x)-u(y))u(y)k_a(x,y)\,\mathrm{d}y\,\mathrm{d}x &= \frac{1}{2}\iint\limits_{\{|x-y|>\varepsilon\}} (u(x)-u(y))(u(x)+u(y))k_a(x,y)\,\mathrm{d}y\,\mathrm{d}x\\
&= \frac{1}{2}\iint\limits_{\{|x-y|>\varepsilon\}} (u^2(x)-u^2(y))k_a(x,y)\,\mathrm{d}y\,\mathrm{d}x\\
&= \frac{1}{2}\Bigg(\iint\limits_{\{|x-y|>\varepsilon\}} u^2(x)k_a(x,y)\,\mathrm{d}y\,\mathrm{d}x - \iint\limits_{\{|x-y|>\varepsilon\}} u^2(y)k_a(x,y)\,\mathrm{d}y\,\mathrm{d}x\Bigg)\\
&= \int\limits_{\mathbb{R}^d} u^2(x)\int\limits_{B_\varepsilon^c(x)} k_a(x,y)\,\mathrm{d}y\,\mathrm{d}x\,.
\end{aligned}$$

From (C) we obtain

$$\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))u(y)k_a(x,y)\,\mathrm{d}y\,\mathrm{d}x = \lim_{\varepsilon\to 0}\iint\limits_{\{|x-y|>\varepsilon\}} (u(x)-u(y))u(y)k_a(x,y)\,\mathrm{d}y\,\mathrm{d}x \geq 0\,.$$

Hence,

$$\mathcal{E}^k(u,u) = \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))u(x)k(x,y)\,\mathrm{d}y\,\mathrm{d}x \geq \frac{1}{2}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x\,, \tag{3.7}$$

i.e. $\mathcal{E}^k(u,u)\geq 0$ for all $u\in H_0^k(\Omega|\mathbb{R}^d)$. By (P) and (3.7)

$$\mathcal{E}^k(u,u) \geq \frac{1}{4C_P}\|u\|^2_{L^2(\Omega)} + \frac{1}{4}\big[u,u\big]_{H^k(\mathbb{R}^d)} \geq \frac{1}{4C_P}\,\|u\|^2_{H^k(\mathbb{R}^d)}\,,$$

which shows that $\mathcal{E}(u,u)$ is coercive.

By the Lax-Milgram Lemma, there is a unique u in $H_0^k(\Omega|\mathbb{R}^d)$, such that

$$\mathcal{E}^k(u,\varphi) = \langle f,\varphi\rangle \qquad \text{for all } \varphi\in H_0^k(\Omega|\mathbb{R}^d)\,.$$ □

Next, we show how the Dirichlet problem with nonzero boundary data can be transformed in a problem with homogeneous boundary data and thus that the Dirichlet problem with suitable complement data $g \in V^k(\Omega|\mathbb{R}^d)$ has also a unique solution.

Theorem 3.10. *Let $\Omega \subset \mathbb{R}^d$ be open and let k satisfy* (L), (P), $(\widetilde{K})$ *and* (C). *Then* (D) *has a unique solution $u \in V^k(\Omega|\mathbb{R}^d)$. Moreover,*

$$[u,u]_{V^k(\Omega|\mathbb{R}^d)} \le C\left(\|f\|^2_{H^*_\Omega(\mathbb{R}^d;k)} + [g,g]_{V^k(\Omega|\mathbb{R}^d)}\right), \tag{3.8}$$

where $C = C(C_P, A_1, A_2)$ is a positive constant.

Proof. To prove the theorem we show that under the above assumptions on g the problem (D) can be transformed into a problem of the form (D$_0$). If $\widetilde{u} \in H^k_0(\Omega|\mathbb{R}^d)$ is a solution to

$$\begin{cases} \mathcal{L}\widetilde{u} = f - \mathcal{L}g & \text{in } \Omega \\ \quad \widetilde{u} = 0 & \text{on } \Omega^c \end{cases} \tag{3.9}$$

then $u = \widetilde{u} + g$ belongs to $V^k(\Omega|\mathbb{R}^d)$ and solves (D). In order to apply Proposition 3.9 to (3.9) it remains to show that $\mathcal{L}g = \mathcal{E}^k(g,\cdot) \in H^*_\Omega(\mathbb{R}^d;k)$. We have

$$\begin{aligned}
|\mathcal{E}^k(g,\varphi)| &= \left| \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (g(x)-g(y))\varphi(x)k(x,y)\,\mathrm{d}y\,\mathrm{d}x \right| \\
&\le \frac{1}{2}\left| \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (g(x)-g(y))(\varphi(x)-\varphi(y))k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y \right| + \left| \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (g(x)-g(y))\varphi(x)k_a(x,y)\,\mathrm{d}x\,\mathrm{d}y \right| \\
&=: I_1 + I_2\,.
\end{aligned}$$

Since $\varphi = 0$ a.e. on Ω^c an application of the Cauchy-Schwarz inequality yields

$$\begin{aligned}
I_1 &\le \Bigg(\iint\limits_{\Omega\,\mathbb{R}^d} (g(x)-g(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \Bigg)^{1/2} \Bigg(\iint\limits_{\Omega\,\mathbb{R}^d} (\varphi(x)-\varphi(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \Bigg)^{1/2} \\
&= [g,g]^{1/2}_{V^k(\Omega|\mathbb{R}^d)} [\varphi,\varphi]^{1/2}_{H^k(\mathbb{R}^d)}\,.
\end{aligned}$$

The term I_2 can be estimated as follows: By $(\widetilde{K}_1)$ and $(\widetilde{K}_2)$, (see Remark 3.1)

$$\begin{aligned}
I_2 &\le \iint\limits_{\Omega\,\mathbb{R}^d} |(g(x)-g(y))|\,\widetilde{k}^{1/2}(x,y)\,|\varphi(x)|\,|k_a(x,y)|\,\widetilde{k}^{-1/2}(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\le \Bigg(\iint\limits_{\Omega\,\mathbb{R}^d} (g(x)-g(y))^2 \widetilde{k}(x,y)\,\mathrm{d}x\,\mathrm{d}y \Bigg)^{1/2} \Bigg(\iint\limits_{\Omega\,\mathbb{R}^d} \varphi^2(x) \frac{k_a^2(x,y)}{\widetilde{k}(x,y)}\,\mathrm{d}y\,\mathrm{d}x \Bigg)^{1/2} \\
&\le A_1^{1/2} A_2^{1/2} [g,g]^{1/2}_{V^k(\Omega|\mathbb{R}^d)} \|\varphi\|_{L^2(\Omega)}\,.
\end{aligned}$$

This shows the continuity of $\mathcal{E}(g,\cdot)\colon H_0^k(\Omega|\mathbb{R}^d) \to \mathbb{R}$ and hence (3.9) has a unique solution $\widetilde{u} \in H_0^k(\Omega|\mathbb{R}^d)$.

In order to prove estimate (3.8) we apply $\widetilde{u} \in H_0^k(\Omega|\mathbb{R}^d)$ as test function and obtain

$$
\begin{aligned}
\langle f, \widetilde{u}\rangle_{H_\Omega^*(\mathbb{R}^d;k)} + \mathcal{E}^k(g,\widetilde{u}) = \mathcal{E}^k(\widetilde{u},\widetilde{u}) &= \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (\widetilde{u}(x)-\widetilde{u}(y))\widetilde{u}(x)k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (\widetilde{u}(x)-\widetilde{u}(y))\widetilde{u}(x)k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (\widetilde{u}(x)-\widetilde{u}(y))\,\widetilde{u}(x)k_a(x,y)\,\mathrm{d}x\,\mathrm{d}y\,,
\end{aligned}
$$

where the second term on the right-hand side is non-negative due to (C). Hence,

$$
\begin{aligned}
\frac{1}{2}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (\widetilde{u}(x)-\widetilde{u}(y))^2 k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y &\le \langle f,\widetilde{u}\rangle_{H_\Omega^*(\mathbb{R}^d;k)} + \mathcal{E}^k(g,\widetilde{u}) \\
&\le \|f\|_{H_\Omega^*(\mathbb{R}^d;k)}\,\|\widetilde{u}\|_{H^k(\mathbb{R}^d)} + \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (g(x)-g(y))\widetilde{u}(x)k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (g(x)-g(y))\widetilde{u}(x)k_a(x,y)\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}
$$

The Young inequality and the fact that $v = 0$ a.e. on Ω^c imply for $\varepsilon > 0$, to be specified later,

$$
\begin{aligned}
\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (g(x)-g(y))\widetilde{u}(x)k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y &\le \frac{1}{2}\iint\limits_{(\Omega^c\times\Omega^c)^c} |g(x)-g(y)|\,|\widetilde{u}(x)-\widetilde{u}(y)|\,k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\le \frac{1}{8\varepsilon}\iint\limits_{(\Omega^c\times\Omega^c)^c} |g(x)-g(y)|^2\,k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x + \varepsilon \iint\limits_{(\Omega^c\times\Omega^c)^c} |\widetilde{u}(x)-\widetilde{u}(y)|^2\,k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}
$$

Similarly, using $(\widetilde{K}_1)$ and $(\widetilde{K}_2)$ we obtain

$$
\begin{aligned}
\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (g(x)-g(y))\widetilde{u}(x)k_a(x,y)\,\mathrm{d}x\,\mathrm{d}y &\le \iint\limits_{\Omega\,\mathbb{R}^d} |g(x)-g(y)|\,|\widetilde{u}(x)|\,|k_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x \\
&\le \frac{A}{4\varepsilon}\iint\limits_{\Omega\,\mathbb{R}^d} |g(x)-g(y)|^2\,\widetilde{k}(x,y)\,\mathrm{d}y\,\mathrm{d}x + \frac{\varepsilon}{A}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \widetilde{u}^2(x)\frac{k_a^2(x,y)}{\widetilde{k}(x,y)}\,\mathrm{d}y\,\mathrm{d}x \\
&\le \frac{A^2}{2\varepsilon}\iint\limits_{(\Omega^c\times\Omega^c)^c} |g(x)-g(y)|^2\,k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x + \varepsilon\|\widetilde{u}\|_{L^2(\Omega)}^2\,.
\end{aligned}
$$

Altogether we obtain

$$\begin{aligned}
\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (\widetilde{u}(x)-\widetilde{u}(y))^2 k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y &\le 2\,\|f\|_{H^*_\Omega(\mathbb{R}^d;k)}\,\|\widetilde{u}\|_{H^k(\mathbb{R}^d)} \\
&\quad + \frac{1}{4\varepsilon}\iint\limits_{(\Omega^c\times\Omega^c)^c} |g(x)-g(y)|^2\,k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x + 2\varepsilon \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |\widetilde{u}(x)-\widetilde{u}(y)|^2\,k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{A^2}{\varepsilon}\iint\limits_{(\Omega^c\times\Omega^c)^c} |g(x)-g(y)|^2\,k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x + 2\varepsilon\|\widetilde{u}\|^2_{L^2(\Omega)} \\
&\le \frac{1}{2\varepsilon}\,\|f\|^2_{H^*_\Omega(\mathbb{R}^d;k)} + \left(\frac{4A^2+1}{4\varepsilon}\right)\big[g,g\big]_{V^k(\Omega|\mathbb{R}^d)} + 4\varepsilon\|\widetilde{u}\|^2_{H^k(\mathbb{R}^d)}\,.
\end{aligned}$$

Applying the Poincaré-Friedrichs inequality (P), and choosing $\varepsilon = \frac{1}{16C_P}$ we deduce

$$\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (\widetilde{u}(x)-\widetilde{u}(y))^2 k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y \le c_1\,\|f\|^2_{H^*_\Omega(\mathbb{R}^d;k)} + c_2\big[g,g\big]_{V^k(\Omega|\mathbb{R}^d)}\,,$$

where $c_1 \ge 1$ depends on C_P and $c_2 \ge 1$ depends on A and C_P. Since $H^k_0(\Omega|\mathbb{R}^d) \subset V^k(\Omega|\mathbb{R}^d)$ and $u = \widetilde{u} + g$ the assertion (3.8) follows. □

Next, we want to weaken the assumptions on the boundary data g. Doing so, we do not obtain a solution in the sense of Definition 3.4, since the solution itself is not in $V^k(\Omega|\mathbb{R}^d)$. Nevertheless, we still obtain existence of a generalized solution in the sense of Definition 3.6. To avoid a comparability assumption of the type $(\widetilde{K}_1)$ on a thickening of Ω, for simplicity, we assume that k satisfies the stronger assumption (K). Define for $\delta > 0$

$$\Omega_\delta = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x,\Omega) < \delta\}.$$

The following corollary is not contained in [FKV14]. In short, it gives a simple approach to weaken the assumptions on the growth of the boundary data g.

Corollary 3.11. *Let $\Omega \subset \mathbb{R}^d$ be open and let k satisfy* (L), (K), (P) *and* (C). *Assume further, that the boundary data g satisfies the following integrability conditions for some $\delta > 0$:*

$$g \in L^2(\Omega), \tag{3.10}$$

$$\iint\limits_{\Omega_\delta\,\Omega_\delta} (g(x)-g(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x < \infty, \tag{3.11}$$

$$G(x) = \int\limits_{\Omega^c_\delta} g(y) k_s(x,y)\,\mathrm{d}y \in L^2(\Omega). \tag{3.12}$$

Then $(\widetilde{D})$ has a unique solution u.

Proof. According to the proof of Theorem 3.10 we only need to check that $\mathcal{E}^k(g,\cdot) \in H^*_\Omega(\mathbb{R}^d;k)$. Let $\varphi \in H^k_0(\Omega|\mathbb{R}^d)$. Then

$$
\begin{aligned}
|\mathcal{E}^k(g,\varphi)| &= \left| \iint\limits_{\Omega\,\mathbb{R}^d} (g(x)-g(y))\varphi(x)k(x,y)\,\mathrm{d}y\,\mathrm{d}x \right| \\
&\leq \left| \iint\limits_{\Omega\,\Omega_\delta} (g(x)-g(y))\varphi(x)k(x,y)\,\mathrm{d}y\,\mathrm{d}x \right| + \left| \iint\limits_{\Omega\,\Omega_\delta^c} (g(x)-g(y))\varphi(x)k(x,y)\,\mathrm{d}y\,\mathrm{d}x \right| \\
&\leq \frac{1}{2}\left| \iint\limits_{\Omega_\delta\,\Omega_\delta} (g(x)-g(y))(\varphi(x)-\varphi(y))k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y \right| + \left| \iint\limits_{\Omega\,\Omega_\delta} (g(x)-g(y))\varphi(x)k_a(x,y)\,\mathrm{d}x\,\mathrm{d}y \right| \\
&\quad + \int\limits_{\Omega} |g(x)|\,|\varphi(x)| \int\limits_{\Omega_\delta^c} k(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int\limits_{\Omega} |\varphi(x)| \left| \int_{\Omega_\delta^c} g(y)k(x,y)\,\mathrm{d}y \right| \mathrm{d}x \\
&=: I_1 + I_2 + II_1 + II_2\,.
\end{aligned}
$$

By the Cauchy-Schwarz Inequality and (3.11) we obtain

$$
\begin{aligned}
I_1 &\leq \left(\iint\limits_{\Omega_\delta\,\Omega_\delta} (g(x)-g(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \right)^{1/2} \left(\iint\limits_{\Omega_\delta\,\Omega_\delta} (\varphi(x)-\varphi(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \right)^{1/2} \\
&\leq C_g \left[\varphi,\varphi\right]^{1/2}_{H^k(\mathbb{R}^d)}.
\end{aligned}
$$

I_2 can be estimated as above using (K):

$$
I_2 \leq A \left(\iint\limits_{\Omega_\delta\,\Omega_\delta} (g(x)-g(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \right)^{1/2} \|\varphi\|_{L^2(\Omega)}\,.
$$

Further by (L), (3.10) and (3.12)

$$
II_1 + II_2 \leq C_g \|\varphi\|_{L^2(\Omega)}\,.
$$

Altogether this proves the continuity of $\mathcal{E}^k(g,\cdot) : H^*_\Omega(\mathbb{R}^d;k) \to \mathbb{R}$. □

3.4. Weak maximum principle and Fredholm alternative

To prove existence and uniqueness of solutions when the bilinear form $\mathcal{E}^k$ is no longer positive definite, we apply Fredholm's alternative. To this end we need to establish a weak maximum principle implying that the homogeneous equation has only the trivial solution.

3.4.1. Weak maximum principle

This subsection deals with a weak maximum principle for subsolutions $u \in H_0^k(\Omega|\mathbb{R}^d)$ of the homogeneous equation

$$\mathcal{E}^k(u,\varphi) = 0 \quad \text{for all } \varphi \in H_0^k(\Omega|\mathbb{R}^d).$$

Since the proof of the weak maximum principle uses classical techniques from second order equations, we need to assume that the the kernels k exhibit a non-integrable singularity at the diagonal.

We assume that for some $\alpha \in (0,2)$, some $\lambda > 0$ and all $u \in L^2(\mathbb{R}^d)$ the estimate

$$\iint\limits_{(\Omega^c\times\Omega^c)^c} (u(x)-u(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \geq \lambda\, \alpha(2-\alpha) \iint\limits_{(\Omega^c\times\Omega^c)^c} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x \tag{E$_\alpha$}$$

holds true. Condition (E$_\alpha$) requires some minimal singularity of k_s at the diagonal and allows us to use the classical Sobolev embedding theorems. Note that (E$_\alpha$) implies that

$$V^{\alpha/2}(\Omega|\mathbb{R}^d) \hookrightarrow V^k(\Omega|\mathbb{R}^d)$$

if we replace the L^2-norm in the definition of $V^{\alpha/2}(\Omega|\mathbb{R}^d)$ by a weighted variant with weight $\frac{1}{(1+|x|)^{d+\alpha}}$, see Subsection 2.2.6. An energy estimate in the sense of (E$_\alpha$) would imply the reverse embedding.

Further, we assume that the symmetric part dominates the antisymmetric part of the kernel strictly. To be precise, we assume that there is $D > 1$ such that for almost every $x, y \in \mathbb{R}^d$

$$|k_a(x,y)| \leq D^{-1} k_s(x,y) \tag{3.13}$$

Note that, by the positivity of k, the inequality $|k_a(x,y)| \leq k_s(x,y)$ holds for almost every $x, y \in \mathbb{R}^d$. Condition (3.13) is satisfied by several examples, e.g for

$$k(x,y) = |x-y|^{-d-\alpha} + g(x,y)\mathbb{1}_{B_1}(x-y)|x-y|^{-\mathrm{d}-\beta}\,,$$

if $0 < \beta < \alpha/2$ and $\|g\|_\infty \leq \frac{1}{2}$, cp. Example (11) in Section 3.5. But there are also examples which violate the condition, e.g. $k(x,y) = |x-y|^{-d-\alpha}\mathbb{1}_{\mathbb{R}^d_+}(x-y)$, cf. Example (10).

Under the above conditions we can prove the following weak maximum principle:

Theorem 3.12. *Let k satisfy* (L), (K), (E$_\alpha$), (3.13). *Let $u \in H_0^k(\Omega|\mathbb{R}^d)$ satisfy*

$$\mathcal{E}^k(u,\varphi) \leq 0 \quad \textit{for all } \varphi \in H_0^k(\Omega|\mathbb{R}^d)\,. \tag{3.14}$$

Then $\sup_\Omega u \leq 0$.

Remark 3.13. As the proof reveals, it is possible to weaken assumption (3.13) significantly because the estimate under consideration is needed only in an integrated sense. However, it seems challenging to provide a simple appropriate alternative to (3.13).

For the proof we need the following algebraic lemma:

Lemma 3.14. *Assume $\theta > 1$ and $a, b \in [0, 1)$. Then*

$$\frac{1-a}{1-b} \leq \theta^2 \frac{(b-a)^2}{(1-b)(1-a)} + \frac{\theta}{\theta - 1}, \tag{3.15}$$

$$\frac{1-b}{1-a} + \frac{1-a}{1-b} \leq 2\theta^2 \frac{(b-a)^2}{(1-b)(1-a)} + \frac{2\theta}{\theta - 1}. \tag{3.16}$$

Proof. It is sufficient to establish assertion (3.15) since it implies (3.16). For the proof of (3.15) it is sufficient to assume $a \leq b$. Assume $\theta > 1$ and $0 \leq a \leq b < 1$. Then, for $t = \frac{b}{a}$

$$0 \leq a < 1 \leq t < \frac{1}{a}$$

and inequality (3.15) reads

$$\frac{1-a}{1-ta} \leq \theta^2 \frac{a^2(t-1)^2}{(1-ta)(1-a)} + \frac{\theta}{\theta - 1}. \tag{3.17}$$

Case 1: $\frac{1}{a} - 1 \leq \theta(t-1)$. In this case

$$\left(\tfrac{1}{a} - 1\right)^2 \leq \theta^2 (t-1)^2 \quad \Rightarrow \quad (1-a) \leq \theta^2 \frac{a^2(t-1)^2}{(1-a)} \quad \Rightarrow \quad \frac{1-a}{1-ta} \leq \theta^2 \frac{a^2(t-1)^2}{(1-ta)(1-a)},$$

which proves (3.17).

Case 2: $\frac{1}{a} - 1 > \theta(t-1) \;\Leftrightarrow\; (t-1) < \frac{1}{\theta}\left(\frac{1}{a} - 1\right) \;\Leftrightarrow\; t < \frac{1}{\theta}\left(\frac{1}{a} - 1\right) + 1$. Therefore

$$\frac{1-a}{1-ta} = \frac{a\left(\frac{1}{a} - 1\right)}{a\left(\frac{1}{a} - t\right)} \leq \frac{\frac{1}{a} - 1}{\frac{1}{a} - 1 - \frac{1}{\theta}\left(\frac{1}{a} - 1\right)} = \frac{\frac{1}{a} - 1}{(\frac{1}{a} - 1)(1 - \frac{1}{\theta})} = \frac{\theta}{\theta - 1},$$

which again proves (3.17). □

Proof of Theorem 3.12. We apply a strategy which is often used in the proof of the weak maximum principle for second order differential operators (e.g. [GT77]). We first show that u attains its supremum on a set of positive measure. In a second step we show that this leads to a contradiction if the supremum is positive.

We choose as test function $v = (u - k)^+$, where $0 \leq k < \sup_\Omega u$. Then $v \in H_0^k(\Omega|\mathbb{R}^d)$ and

$$\mathcal{E}^k(u, v) = \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x) - u(y)) v(x) k(x, y) \,\mathrm{d}y \,\mathrm{d}x \leq 0 \tag{3.18}$$

Since $(u-k)^-(x)v(x) = 0$, we have $(u(x)-u(y))v(x) = \left[(u-k)^+(x) + (u-k)^+(y) + (u-k)^-(y)\right] v(x)$ and deduce

$$\frac{1}{2} \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (v(x)-v(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x + \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(y)-k)^-(u(x)-k)^+ k(x,y)\,\mathrm{d}y\,\mathrm{d}x$$
$$\leq - \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (v(x)-v(y))v(x)k_a(x,y)\,\mathrm{d}y\,\mathrm{d}x\,,$$

and since the second term on the left-hand side is positive

$$\frac{1}{2} \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (v(x)-v(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \leq - \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (v(x)-v(y))v(x)k_a(x,y)\,\mathrm{d}y\,\mathrm{d}x\,.$$

Now by Cauchy-Schwarz and the assumptions on k_a

$$\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (v(x)-v(y))v(x)k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x$$
$$\leq 2 \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |v(x)-v(y)|\, k_s^{1/2}(x,y)\, |v(x)|\, k_s^{-1/2}(x,y)\, |k_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x$$
$$\leq 2A^{1/2}\|v\|_{L^2(\mathbb{R}^d)} \Bigg(\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (v(x)-v(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \Bigg)^{1/2}\,,$$

or equivalently

$$\Bigg(\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (v(x)-v(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \Bigg)^{1/2} \leq 2A^{1/2}\|v\|_{L^2(\mathbb{R}^d)}.$$

By (E_α) and since $v = 0$ on Ω^c, the Sobolev and the Hölder inequality imply that there is a constant $C = C(d) > 0$ such that

$$\|v\|_{L^{2d/(d-\alpha)}(\mathbb{R}^d)} \leq C\|v\|_{L^2(\Omega)} \leq C|\operatorname{supp} v|^{\alpha/2d}\|v\|_{L^{2d/(d-\alpha)}(\Omega)}.$$

(If $d \leq 2$ the critical exponent may be replaced by any number greater than 2.) Thus

$$|\operatorname{supp} v| \geq C^{-2d/\alpha}.$$

This inequality is independent of k and therefore it holds for $k \nearrow \sup_\Omega u$. Therefore u must attain its supremum on a set of positive measure. This completes the first step of the proof.

We now derive a contradiction. Without loss of generality we may assume $\sup_\Omega u = 1$. Set $v = u^+$. We define a new function $\overline{v}$ by

$$\overline{v} = \frac{v}{1-v} = \frac{1}{1-v} - 1.$$

We want to use $\overline{v}$ as a test function in (3.14) but it is not clear whether $\overline{v}$ belongs to $H_0^k(\Omega|\mathbb{R}^d)$. Thus we look at approximations and define for small $\varepsilon > 0$

$$v_\varepsilon = (1-\varepsilon)v \quad \text{and} \quad \overline{v}_\varepsilon = \frac{v_\varepsilon}{1-v_\varepsilon}\,.$$

The function $\overline{v}_\varepsilon$ is an admissible test function. However, in order to simplify the presentation, we use $\overline{v}$ instead of $\overline{v}_\varepsilon$ and postpone this issue until the end of the proof. Plugging $\overline{v}$ into (3.18), we obtain

$$\begin{aligned}\frac{1}{2}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} &(v(x)-v(y))\left(\frac{1}{1-v(x)}-\frac{1}{1-v(y)}\right)k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &\le -\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d}(v(x)-v(y))\frac{v(x)}{1-v(x)}k_a(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &= -\frac{1}{2}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d}(v(x)-v(y))\left(\frac{v(x)}{1-v(x)}+\frac{v(y)}{1-v(y)}\right)k_a(x,y)\,\mathrm{d}y\,\mathrm{d}x\,.\end{aligned}$$

This is equivalent to

$$\begin{aligned}&\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d}\frac{(v(x)-v(y))^2}{(1-v(x))(1-v(y))}k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &\le -\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d}\frac{(v(x)-v(y))}{(1-v(x))^{1/2}(1-v(y))^{1/2}}\left(\frac{v(x)(1-v(y))+v(y)(1-v(x))}{(1-v(x))^{1/2}(1-v(y))^{1/2}}\right)k_a(x,y)\,\mathrm{d}y\,\mathrm{d}x\,.\end{aligned}$$

An application of the Young inequality leads to

$$\begin{aligned}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d}\frac{(v(x)-v(y))^2}{(1-v(x))(1-v(y))}k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x &\le \frac{1}{2}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d}\frac{(v(x)-v(y))^2}{(1-v(x))(1-v(y))}k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &+\frac{1}{2}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d}\frac{[v(x)(1-v(y))+v(y)(1-v(x))]^2}{(1-v(x))(1-v(y))}\frac{k_a^2(x,y)}{k_s(x,y)}\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

and hence

$$\begin{aligned}\frac{1}{2}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d}&\frac{(v(x)-v(y))^2}{(1-v(x))(1-v(y))}k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &\le\frac{1}{2}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d}\frac{[v(x)(1-v(y))+v(y)(1-v(x)))^2]}{(1-v(x))(1-v(y))}\frac{k_a^2(x,y)}{k_s(x,y)}\,\mathrm{d}y\,\mathrm{d}x\,.\end{aligned}\tag{3.19}$$

Using $v=0$ on Ω^c, $v\le 1$ and that $\frac{k_a^2(x,y)}{k_s(x,y)}$ is symmetric, the right-hand side can be estimated from above as follows:

$$\begin{aligned}
&\frac{1}{2}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{[v(x)(1-v(y))+v(y)(1-v(x)))^2]}{(1-v(x))(1-v(y))}\frac{k_a^2(x,y)}{k_s(x,y)}\,\mathrm{d}y\,\mathrm{d}x\\
&\le \iint\limits_{\Omega\,\mathbb{R}^d} \left(\frac{v^2(x)(1-v(y))}{1-v(x)}+v(x)v(y)\right)\frac{k_a^2(x,y)}{k_s(x,y)}\,\mathrm{d}y\,\mathrm{d}x\\
&\le \theta^2 \iint\limits_{\Omega\,\mathbb{R}^d} \frac{(v(x)-v(y))^2}{(1-v(x))(1-v(y))}\frac{k_a^2(x,y)}{k_s(x,y)}\,\mathrm{d}y\,\mathrm{d}x+\left(\tfrac{\theta}{\theta-1}+1\right)\iint\limits_{\Omega\,\mathbb{R}^d}\frac{k_a^2(x,y)}{k_s(x,y)}\,\mathrm{d}y\,\mathrm{d}x\\
&\le \frac{\theta^2}{D}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{(v(x)-v(y))^2}{(1-v(x))(1-v(y))}k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x+\left(\tfrac{\theta}{\theta-1}+1\right)A|\Omega|
\end{aligned}$$

where we have applied Lemma 3.14 and (3.13). Now, we choose $\theta=\sqrt{\frac{D+1}{2}}$ such that $\frac{\theta^2}{D}<1$. Combining the above estimate and (3.19) leads to

$$\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{(v(x)-v(y))^2}{(1-v(x))(1-v(y))}k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x\le c_1A|\Omega|\,,$$

for some positive constant $c_1=c_1(D)$. Next, we want to estimate the left-hand side from below. We apply the inequality

$$\frac{(a-b)}{ab}=(a-b)(b^{-1}-a^{-1})\ge(\log a-\log b)^2\,,$$

which holds for positive reals a,b, to $a=1-v(y)$ and $b=1-v(x)$. Thus we obtain

$$\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d}\Big(\log(1-v(x))-\log(1-v(y))\Big)^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x\le c_1A|\Omega|\,.$$

Due to condition (E_α) we can apply the Sobolev inequality and obtain

$$\|w\|_{L^{2d/(d-\alpha)}}\le c_2A|\Omega|\,,$$

where $c_2\ge 1$ and $w=\log(1-v)$. Recall that, in fact, we have proved $\|w_\varepsilon\|_{L^{2d/(d-\alpha)}}\le c_2A|\Omega|$ for $w_\varepsilon=\log(1-v_\varepsilon)$ and every $\varepsilon\in(0,\frac{1}{2})$, where c_2 is independent of ε.

By Fatou's lemma, this contradicts the fact that $v=u^+$ attains is supremum 1 on a set of positive measure. The proof is complete. □

3.4.2. Fredholm alternative

The aim of this subsection is to prove existence and uniqueness of solutions to (D) without assuming positive definiteness of the bilinear form $\mathcal{E}^k$, i.e. without assuming the cancellation assumption (C). Since we use the weak maximum principle obtained in the previous section, we need to assume that the kernel k satisfies (E_α) and (3.13). Note that (E_α) implies (P) by Lemma 2.44.

We prove the following well-posedness result:

Theorem 3.15. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $f \in H^*_\Omega(\mathbb{R}^d;k)$ and let k satisfy* (L), (K), (3.13) *and* (E_α). *Then the Dirichlet problem* (D) *has a unique solution $u \in V^k(\Omega|\mathbb{R}^d)$. Moreover, there is a constant $C = C(C_P, A, D) > 0$ such that*

$$\big[u,u\big]_{V^k(\Omega|\mathbb{R}^d)} \leq C\left(\|f\|^2_{H^*_\Omega(\mathbb{R}^d;k)} + \|g\|^2_{L^2(\Omega)} + \big[g,g\big]_{V^k(\Omega|\mathbb{R}^d)} + \|u\|^2_{L^2(\Omega)}\right). \tag{3.20}$$

Proof. We use the Fredholm alternative (see e.g. [Eva10]).

Step 1: We will use (E_α) to show that the embedding $H^k_0(\Omega|\mathbb{R}^d) \hookrightarrow L^2_\Omega(\mathbb{R}^d)$ is compact. Since the embedding $L^2(\Omega) \hookrightarrow H^*_\Omega(\mathbb{R}^d;k)$ is continuous we obtain then the compactness of the embedding $H^k_0(\Omega|\mathbb{R}^d) \hookrightarrow H^*_\Omega(\mathbb{R}^d;k)$.
Let $\mathcal{A} \subset H^k_0(\Omega|\mathbb{R}^d)$ with $\|u\|_{H^k(\mathbb{R}^d)} \leq C$ for all $u \in \mathcal{A}$ and some $C < \infty$. Let $B \subset \mathbb{R}^d$ be an open ball with $\Omega \subset B$. Let us recall that the embedding $H^{\alpha/2}(B) \hookrightarrow L^2(B)$ is compact. Then, for $u \in \mathcal{A}$

$$\begin{aligned}\|u\|^2_{H^{\alpha/2}(B)} &= \iint_{B\,B} (u(x)-u(y))^2\,|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \iint_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))^2\,|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \lambda^{-1} \iint_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))^2 k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \leq \lambda^{-1}C^2,\end{aligned}$$

where we used (E_α). Therefore $\mathcal{A}$ is bounded in $H^{\alpha/2}(B)$ and thus precompact in $L^2(B)$. By the definition of $H^k_0(\Omega|\mathbb{R}^d)$ we know $u=0$ on Ω^c and thus the set $\mathcal{A}$ is also precompact in $L^2_\Omega(\mathbb{R}^d)$ and in $H^*_\Omega(\mathbb{R}^d;k)$.

Step 2: Existence and uniqueness of (D_0). By Lemma 3.7 the bilinear form

$$(u,v) \mapsto \mathcal{E}^k(u,v) + \gamma(u,v)_{L^2(\Omega)}$$

is coercive for some $\gamma = \gamma(A) > 0$ and therefore there is a unique solution $u \in H^k_0(\Omega|\mathbb{R}^d)$ to the problem

$$\begin{cases} \mathcal{E}^k(u,v) + \gamma(u,v)_{L^2(\Omega)} = \langle f,v\rangle & \text{for all } v \in H^k_0(\Omega|\mathbb{R}^d), \\ \qquad\qquad u = 0 & \text{on } \Omega^c. \end{cases} \tag{3.21}$$

Moreover, due to Lemma 3.7 the solution u satisfies

$$\|u\|^2_{H^k_0(\Omega|\mathbb{R}^d)} \leq 4\mathcal{E}^k(u,u) + 4\gamma\,\|u\|^2_{L^2(\Omega)} = 4\,\langle f,v\rangle \leq 4\,\|f\|_{H^*_\Omega(\mathbb{R}^d;k)}\,\|u\|_{H^k_0(\Omega|\mathbb{R}^d)}.$$

This estimate together with Step 1 shows that the operator $K\colon H^*_\Omega(\mathbb{R}^d;k) \to H^*_\Omega(\mathbb{R}^d;k)$, which maps the inhomogeneity f to the solution $u \in H^k_0(\Omega|\mathbb{R}^d) \Subset H^*_\Omega(\mathbb{R}^d;k)$ of (3.21), is a compact operator. Fredholm's theorem in combination with the weak maximum principle Theorem 3.12 shows that (D_0) has a unique solution $u \in H^k_0(\Omega|\mathbb{R}^d)$.

Step 3: The well-posedness of (D) follows in the same way as in the proof of Theorem 3.10. It remains to prove the estimate (3.20). Let u be the solution of (D). We apply $v = u - g \in$

$H_0^k(\Omega|\mathbb{R}^d)$ as test function:

$$\begin{aligned}
&\langle f, v\rangle_{H_\Omega^*(\mathbb{R}^d;k)} - \mathcal{E}^k(g,v) = \mathcal{E}^k(v,v) \\
&= \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (v(x)-v(y))v(x)k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (v(x)-v(y))v(x)k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (v(x)-v(y))\,v(x)k_a(x,y)\,\mathrm{d}x\,\mathrm{d}y\,,
\end{aligned}$$

As in the proof of Theorem 3.10 we may estimate

$$\begin{aligned}
\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (v(x)-v(y))^2 k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y &\le \frac{1}{2\varepsilon}\,\|f\|^2_{H_\Omega^*(\mathbb{R}^d;k)} + 2\varepsilon\,\|v\|^2_{H^k(\mathbb{R}^d)} \\
&+ \frac{1}{2\varepsilon}\iint\limits_{\Omega\,\mathbb{R}^d} |g(x)-g(y)|^2\,k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x + 2\varepsilon \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |v(x)-v(y)|^2\,k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&+ \frac{A^2}{2\varepsilon}\iint\limits_{\Omega\,\mathbb{R}^d} |g(x)-g(y)|^2\,k_s(x,y)\,\mathrm{d}y\,\mathrm{d}x + 2\varepsilon\|v\|^2_{L^2(\Omega)} \\
&+ 2\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |v(x)-v(y)|\,|v(x)|\,|k_a(x,y)|\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}$$

Due to $v=0$ on Ω^c, the last term can be estimated as follows:

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |v(x)-v(y)|\,|v(x)|\,|k_a(x,y)|\,\mathrm{d}x\,\mathrm{d}y \\
&\le \varepsilon \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (v(x)-v(y))^2 k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y + \frac{A}{4\varepsilon}\|v\|^2_{L^2(\Omega)}\,.
\end{aligned}$$

Hence, after choosing ε appropriately,

$$\iint\limits_{\Omega\,\mathbb{R}^d} (v(x)-v(y))^2 k_s(x,y)\,\mathrm{d}x\,\mathrm{d}y \le \|f\|^2_{H_\Omega^*(\mathbb{R}^d;k)} + c_1\big[g,g\big]_{V^k(\Omega|\mathbb{R}^d)} + c_2\|v\|^2_{L^2(\Omega)}\,,$$

where $c_1, c_2 > 0$ depend on A. This implies (3.20). □

Remark 3.16. Using the above proof with the obvious changes, it is possible to relax the assumptions on the boundary data g as in Corollary 3.11 and thus to obtain existence of generalized solutions in the sense of Definition 3.6 under the assumptions of Theorem 3.15 for k.

3.5. Examples of kernels

In this section we provide several examples of kernels $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ to which the theory above can be applied directly. We also give examples of kernels which are not covered by the above results, but which lead to a better understanding of our main assumptions on the admissible kernels. Further the examples below can be used as building blocks for more specific examples.

Recall that all kernels studied in this chapter satisfy assumption (L). Further, the kernels are distinguished in two cases, **integrable** and **non-integrable** kernels, see Definition 2.37. At the end of this section we list all examples together with their corresponding properties.

3.5.1. Integrable kernels

Let us start with a simple observation. Every kernel with the property that the antisymmetric part is of the form $k_a(x,y) = g(x-y)$ for some function g satisfies the assumption (C). This follows from the fact that for $x \in \mathbb{R}^d$

$$\int\limits_{B_\varepsilon^c(x)} k_a(x,y)\,\mathrm{d}y = \int\limits_{B_\varepsilon^c(x)} g(x-y)\,\mathrm{d}y = \int\limits_{B_\varepsilon^c(0)} g(z)dz = 0\,.$$

1. $k(x,y) := \mathbb{1}_{B_1}(x-y)$. The kernel is obviously symmetric. Thus it satisfies (C). It also satisfies the Poincaré-Friedrichs inequality (P) as shown in Subsection 2.3.2.
2. $k(x,y) := \mathbb{1}_{B_R \setminus B_r}(x-y)$ for some numbers $0 < r < R$. Again, (C) and the Poincaré-Friedrichs inequality (P) hold.
3. $k(x,y) := \mathbb{1}_{B_1 \cap \mathbb{R}^d_+}(x-y)$. Symmetrization leads to

$$\begin{aligned} k_s(x,y) &= \tfrac{1}{2}\mathbb{1}_{B_1 \cap \mathbb{R}^d_+}(x-y) + \tfrac{1}{2}\mathbb{1}_{B_1 \cap \mathbb{R}^d_+}(y-x) = \tfrac{1}{2}\mathbb{1}_{B_1}(x-y) \\ k_a(x,y) &= \tfrac{1}{2}\mathbb{1}_{B_1 \cap \mathbb{R}^d_+}(x-y) - \tfrac{1}{2}\mathbb{1}_{B_1 \cap \mathbb{R}^d_-}(x-y)\,. \end{aligned}$$

   Since k depends only on $x-y$, condition (C) holds. Concerning the Poincaré-Friedrichs inequality (P), k is not different from example (1).
4. This example is more general than Example 3. Set $k(x,y) := \mathbb{1}_{B_1}(x-y)\mathbb{1}_{\mathcal{C}}(x-y)$ where the set $\mathcal{C}$ is defined by $\mathcal{C} = \{h \in \mathbb{R}^d | \frac{h}{|h|} \in I\}$ and I is an arbitrary nonempty open subset of S^{d-1}. If I is of the form $I = B_r(\xi) \cap S^{d-1}$ for some $\xi \in S^{d-1}$ and some $r > 0$, then $\mathcal{C}$ is a cone. In any case, we obtain

$$\begin{aligned} k_s(x,y) &= \tfrac{1}{2}\mathbb{1}_{B_1 \cap (\mathcal{C} \cup -\mathcal{C})}(x-y)\,, \\ k_a(x,y) &= \tfrac{1}{2}\mathbb{1}_{B_1 \cap \mathcal{C}}(x-y) - \tfrac{1}{2}\mathbb{1}_{B_1 \cap -\mathcal{C}}(x-y)\,. \end{aligned}$$

In the examples above, $k(x,y)$ depends only on $x-y$. As a result, one can choose $L(z) = k(0, y-x)$ in the condition (2.28). Let us look at examples where this is not possible.

5. $k(x,y) := g(x,y)\mathbb{1}_{B_1}(x-y)$, where g is any measurable bounded function satisfying $g \geq c$ almost everywhere for some constant $c > 0$. Note that g does not need to be symmetric. Then

$$\begin{aligned} k_s(x,y) &= \tfrac{1}{2}(g(x,y) + g(y,x))\mathbb{1}_{B_1}(x-y) \\ k_a(x,y) &= \tfrac{1}{2}(g(x,y) - g(y,x))\mathbb{1}_{B_1}(x-y)\,. \end{aligned}$$

   Condition (C) does not hold in general but (K) holds because $k_s(x,y) \geq c\mathbb{1}_{B_1}(x-y)$ which allows us to apply the Poincaré-Friedrichs inequality (P) choosing $L(z) = c\mathbb{1}_{B_1}(z)$ in (2.28).

6. Here, we set $d = 1$ and define a kernel $k : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ as follows. Define
$$D = [-1, 0] \times [0, 1] \cup \{(x, y) \in \mathbb{R}^2 | \, (x \leq y \leq x + 1)\} .$$
Set $k(x, y) := 2 \cdot \mathbb{1}_D(x, y)$. Then the antisymmetric part of k is given by
$$k_a(x, y) = \mathbb{1}_D(x, y) - \mathbb{1}_{(-D)}(x, y) .$$
Due to the construction of D we obtain for $|x| > 1$ $\lim\limits_{\varepsilon \to 0+} \int_{B^c_\varepsilon(x)} k_a(x, y) \, \mathrm{d}y = 0$ whereas for $x \in (-1, 1)$ we obtain $\lim\limits_{\varepsilon \to 0+} \int_{B^c_\varepsilon(x)} k_a(x, y) \, \mathrm{d}y = -x$, which implies that k does not satisfy condition (C). Though, conditions (K) and (2.28) hold true because of $k_s(x, y) \geq \mathbb{1}_{B_1}(x - y)$.
7. Again, set $d = 1$. We define $k(x, y)$ by $k(x, y) = 2 \cdot \mathbb{1}_{(-4,4)}(x - y) + k_a(x, y)$ where
$$k_a(x, y) = \begin{cases} g(x - 1, y - 3) & \text{if } x < y \\ -g(y - 1, x - 3) & \text{else} , \end{cases}$$
and
$$g(x, y) = \operatorname{sgn}(xy) \mathbb{1}_{(-1,1)\times(-1,1)}(x, y) .$$
By construction k_a is antisymmetric and satisfies condition (C). Thus k is not a function of $x - y$ but still satisfies (C). Conditions (K) and (2.28) hold true, too.

3.5.2. Non-integrable kernels

Here are several examples of kernels $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ with a singularity at the diagonal. See above for our definition of when we call a kernel non-integrable. Recall that we want all examples to satisfy (L). Throughout this section (with one exception) $\alpha \in (0, 2)$ is an arbitrary fixed number.

8. $k(x, y) := |x - y|^{-d-\alpha}$. Obviously, k is symmetric and satisfies (L). Conditions (C) and (K) hold due to the symmetry. Lemma 2.44 can be directly applied. This kernel k is very special because the space $H^k(\mathbb{R}^d)$ is isomorphic to the fractional Sobolev space $H^{\alpha/2}(\mathbb{R}^d)$ (cf. Remark 2.39b). There is a constant $C \geq 1$, independent of α, such that for all $v \in C_c^\infty(\mathbb{R}^d)$
$$C^{-1} \|v\|_{H^{\alpha/2}(\mathbb{R}^d)} \leq \alpha(2 - \alpha) \|v\|_{H^k(\mathbb{R}^d)} \leq C \|v\|_{H^{\alpha/2}(\mathbb{R}^d)} ,$$
where $\|v\|^2_{H^{\alpha/2}(\mathbb{R}^d)} = \int (1 + |\xi|^2)^{\alpha/2} |\widehat{v}(\xi)|^2 \, \mathrm{d}\xi$. Thus, for fixed $v \in C_c^\infty(\mathbb{R}^d)$,
$$\alpha(2 - \alpha) \, [v, v]_{H^k(\mathbb{R}^d)} \longrightarrow [v, v]_{H^1(\mathbb{R}^d)} \qquad \text{for } \alpha \to 2^-$$
$$\alpha(2 - \alpha) \|v\|_{H^k(\mathbb{R}^d)} \longrightarrow \|v\|_{L^2(\mathbb{R}^d)} \qquad \text{for } \alpha \to 0^+.$$
Similar results hold true for $\mathbb{R}^d$ replaced by a bounded domain [BBM02, MS02b].
9. Let I be an arbitrary nonempty open subset of S^{d-1} with the property $I = -I$. Set $\mathcal{C} = \{h \in \mathbb{R}^d | \, \frac{h}{|h|} \in I\}$ and $k(x, y) := |x - y|^{-d-\alpha} \mathbb{1}_{\mathcal{C}}(x - y)$. Again, k is symmetric and satisfies (L). It turns out that k is comparable to example (8) in the sense of Lemma 2.44. The only difference is that the constant λ depends on I.

10. $k(x,y) := |x-y|^{-d-\alpha}\mathbb{1}_{\mathbb{R}^d_+}(x-y)$. This example is different from example (9) because k is not symmetric anymore. The symmetric and antisymmetric parts are given by

$$\begin{aligned} k_s(x,y) &= \tfrac{1}{2}|x-y|^{-d-\alpha} \\ k_a(x,y) &= |x-y|^{-d-\alpha}\big(\tfrac{1}{2}\mathbb{1}_{\mathbb{R}^d_+}(x-y) - \tfrac{1}{2}\mathbb{1}_{\mathbb{R}^d_-}(x-y)\big)\,. \end{aligned}$$

Lemma 2.44 can still be applied but conditions (K) and $(\widetilde{K})$ do not hold. Condition (C) does hold, though.

11. Assume $0 < \beta < \frac{\alpha}{2}$ and $g : \mathbb{R}^d \times \mathbb{R}^d \to [-K, L]$ measurable for some $K, L > 0$. Define

$$k(x,y) := |x-y|^{-d-\alpha} + g(x,y)\mathbb{1}_{B_1}(x-y)|x-y|^{-d-\beta}$$

Additionally, we assume that k is nonnegative. This property does not follow in general under the assumptions above. However, for every choice of K there are many admissible cases with $\inf g = -K$. We obtain $k_s(x,y) \geq \frac{1}{2}|x-y|^{-d-\alpha}$ for $|x-y| \leq (2K)^{\frac{-1}{\alpha-\beta}}$. Since k_s is nonnegative, we can apply Lemma 2.44 and (P) holds. Further $(\widetilde{K})$ is satisfied with $\widetilde{k}(x,y) = |x-y|^{-d-\alpha}\mathbb{1}_{B_1}(x-y)$ and $A_1 = 1$. Conditions (C) and (K) hold for some but not for all choices of g.

The following example is an extension and, at the same time, a special case of Example (11). Example (12) shows that our condition $(\widetilde{K})$ is indeed a relaxation of (K) or [SW11, (1.1)].

12. Assume $0 < \beta < \frac{\alpha}{2}$. Let I_1, I_2 be arbitrary nonempty disjoint open subsets of S^{d-1} with $I_1 = -I_1$ and $|-I_2 \setminus I_2| > 0$. Set $\mathcal{C}_j = \{h \in \mathbb{R}^d | \frac{h}{|h|} \in I_j\}$ for $j \in \{1,2\}$. Set

$$k(x,y) = |x-y|^{-d-\alpha}\mathbb{1}_{\mathcal{C}_1}(x-y) + |x-y|^{-d-\beta}\mathbb{1}_{\mathcal{C}_2}(x-y)\mathbb{1}_{B_1}(x-y)\,.$$

The symmetric and antisymmetric parts of k are given by

$$\begin{aligned} k_s(x,y) &= |x-y|^{-d-\alpha}\mathbb{1}_{\mathcal{C}_1}(x-y) + \tfrac{1}{2}|x-y|^{-d-\beta}\mathbb{1}_{\mathcal{C}_2\cup(-\mathcal{C}_2)}(x-y)\mathbb{1}_{B_1}(x-y)\,, \\ k_a(x,y) &= \tfrac{1}{2}|x-y|^{-d-\beta}\mathbb{1}_{\mathcal{C}_2\cap B_1}(x-y) - \tfrac{1}{2}|x-y|^{-d-\beta}\mathbb{1}_{(-\mathcal{C}_2)\cap B_1}(x-y)\,. \end{aligned}$$

Let us show that condition (K) does not hold, i.e. $(\widetilde{K}_2)$ is not satisfied for $\widetilde{k} = k_s$. Let $h, h_a, h_s, \widetilde{h} : \mathbb{R}^d \to [0,\infty]$ be defined by $h(x-y) = k(x,y)$ and $h_a, h_s, \widetilde{h}$ accordingly. Note that $|h_a| = h_s$ on $\mathcal{C}_2 \cup -\mathcal{C}_2$. Then

$$\begin{aligned} \sup_{x\in\mathbb{R}^d} \int\limits_{\{k_s(x,y)\neq 0\}} \frac{k_a(x,y)^2}{k_s(x,y)}\,\mathrm{d}y &= \int\limits_{\mathbb{R}^d} \frac{h_a^2(z)}{h_s(z)}\mathbb{1}_{B_1}(z)dz = \int\limits_{\{\mathcal{C}_2\cup(-\mathcal{C}_2)\}} \frac{h_a^2(z)}{h_s(z)}\mathbb{1}_{B_1}(z)dz \\ &= \int\limits_{\{\mathcal{C}_2\cup(-\mathcal{C}_2)\}} h_s(z)\mathbb{1}_{B_1}(z)dz = \int\limits_{\{\mathcal{C}_2\cup(-\mathcal{C}_2)\}} \frac{1}{2}|z|^{-d-\beta}\mathbb{1}_{B_1}(z)dz = +\infty \end{aligned}$$

Let us explain why $(\widetilde{K}_1)$ and $(\widetilde{K}_2)$ hold for $\widetilde{k}(x,y) = |x-y|^{-d-\alpha}$. $(\widetilde{K}_1)$ follows easily from $k_s(x,y) \geq |x-y|^{-d-\alpha}\mathbb{1}_{\mathcal{C}_1}(x-y)$, the constant A_1 needs to be chosen in dependence of $\mathcal{C}_1$

resp. I_1. Let us check $(\widetilde{K}_2)$:

$$\sup_{x\in\mathbb{R}^d}\int \frac{k_a^2(x,y)}{\widetilde{k}(x,y)}\,\mathrm{d}y = \int_{\mathbb{R}^d} \frac{h_a^2(z)}{\widetilde{h}(z)}dz = \int_{\{\mathcal{C}_2\cup-(\mathcal{C}_2)\}} \frac{h_a^2(z)}{\widetilde{h}(z)}\mathbb{1}_{B_1}(z)dz$$
$$= \int_{\{\mathcal{C}_2\cup(-\mathcal{C}_2)\}} \frac{1}{4}|z|^{-d-2\beta+\alpha}\mathbb{1}_{B_1}(z)dz \leq A_2\,,$$

where A_2 depends on I_2 and $\alpha/2-\beta$. **Note:** If we modify the example by choosing $I_1 = S^{d-1}$, i.e. $\mathcal{C}_1 = \mathbb{R}^d$, then condition (K) does hold.

13. The following example appears in [DK11, Ex. 12]. Assume $0 < b < 1$ and $0 < \alpha' < 1 + \frac{1}{b}$. Define $\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 |\, |x_1| \geq |x_2|^b \text{ or } |x_2| \geq |x_1|^b\}$ and set

$$k(x,y) = \mathbb{1}_{\Gamma\cap B_1}(x-y)\,|x-y|^{-d-\alpha'}\,.$$

Note that the kernel k depends only on $x-y$ and is symmetric but condition (L) is not obvious. Using integration in polar coordinates one can show that there is $C \geq 1$ such that for $\alpha = \alpha' - (1/b-1)$, $h(z) = k(x, x+z)$ and every $r \in (0,1)$

$$r^2\int_{B_r} |z|^2 h(z)\,\mathrm{d}z + \int_{\mathbb{R}^d\setminus B_r} h(z)\,\mathrm{d}z \leq Cr^{-\alpha}\,.$$

Thus α is the effective order of differentiability of the corresponding integro-differential operator. In [DK11] the comparability of the quadratic forms needed for Lemma 2.44 is established. From the point of view of this article the kernel k is very similar to the kernel $|x-y|^{-d-\alpha}\mathbb{1}_{B_1}(x-y)$. Of course, one can now produce related nonsymmetric examples.

14. The following example is taken from [FU12], [SW11]. It provides a nonsymmetric kernel k with a singularity on the diagonal which is non-constant. Assume $0 < \alpha_1 \leq \alpha_2 < 2$ and let $\alpha : \mathbb{R}^d \to [\alpha_1, \alpha_2]$ be a measurable function. We assume that α is continuous and that the modulus of continuity ω of the function α satisfies

$$\int_0^1 \frac{(\omega(r)|\log r|)^2}{r^{1+\alpha_2}}dr < \infty\,.$$

Note that, as a result, there are $\beta \in (0,1)$ and $C_H > 0$ such that $[\alpha]_{C^{0,\beta}(\mathbb{R}^d)} \leq C_H$. Let $b : \mathbb{R}^d \to \mathbb{R}$ be another measurable function which is bounded between two positive constants and satisfies $|b(x) - b(y)| \leq c|\alpha(x) - \alpha(y)|$ as long as $|x-y| \leq 1$ for some constant $c > 0$. Finally, set

$$k(x,y) = b(x)|x-y|^{-d-\alpha(x)}\,.$$

In [SW11] it is proved, that k satisfies (L) and (K). Since α is bounded from below by α_1, the Poincaré-Friedrichs inequality (P) holds. Condition (3.13) does not hold for this example since $\lim\limits_{|x-y|\to\infty} \dfrac{|k_a(x,y)|}{k_s(x,y)} = 1$.

Let us slightly modify the example and look at $k'(x,y) = \mathbb{1}_{B_R}(x-y)k(x,y)$ for some $R \gg 1$. Then conditions (L), (K) and (P) still hold true for k'.

Lemma 3.17. *The kernel k' satisfies* (3.13).

Proof. We have to show that $\frac{|k'_a(x,y)|}{k'_s(x,y)} \leq \Theta < 1$ for all $x, y \in \{|x-y| < R\}$. By assumption there are $c_1, c_2 > 0$ such that

$$c_1 \leq b(x) \leq c_2 \quad \text{for all } x \in \mathbb{R}^d .$$

We can assume that $\alpha(x) \leq \alpha(y)$ due to the symmetry of $\frac{|k'_a(x,y)|}{k'_s(x,y)}$.
Case 1a: $|x-y| \leq 1$ and $k'_a(x,y) > 0$. Then

$$\frac{|k'_a(x,y)|}{k'_s(x,y)} = \frac{b(x)|x-y|^{-d-\alpha(x)} - b(y)|x-y|^{-d-\alpha(y)}}{b(x)|x-y|^{-d-\alpha(x)} + b(y)|x-y|^{-d-\alpha(y)}}$$
$$= \frac{b(x) - b(y)|x-y|^{\alpha(x)-\alpha(y)}}{b(x) + b(y)|x-y|^{\alpha(x)-\alpha(y)}} \leq \frac{1-\frac{c_1}{c_2}}{1+\frac{c_1}{c_2}} =: \Theta_1$$

Case 1b: $|x-y| \leq 1$ and $k'_a(x,y) < 0$. Then

$$\frac{|k'_a(x,y)|}{k'_s(x,y)} = \frac{b(y)|x-y|^{-d-\alpha(y)} - b(x)|x-y|^{-d-\alpha(x)}}{b(x)|x-y|^{-d-\alpha(x)} + b(y)|x-y|^{-d-\alpha(y)}} = \frac{b(y) - b(x)|x-y|^{\alpha(y)-\alpha(x)}}{b(y) + b(x)|x-y|^{\alpha(y)-\alpha(x)}}$$

Since $|\alpha(y) - \alpha(x)| \leq C_H \, |x-y|^{\beta}$, we obtain

$$|x-y|^{\alpha(y)-\alpha(x)} \geq |x-y|^{C_H |x-y|^{\beta}} \geq \delta(C_H, \beta) > 0.$$

Thus

$$\frac{|k'_a(x,y)|}{k'_s(x,y)} \leq \frac{1-\frac{c_1}{c_2}\delta}{1+\frac{c_1}{c_2}\delta} =: \Theta_2$$

Case 2a: $1 < |x-y| < R$ and $k'_a(x,y) < 0$. Then

$$\frac{|k'_a(x,y)|}{k'_s(x,y)} = \frac{b(y)|x-y|^{-d-\alpha(y)} - b(x)|x-y|^{-d-\alpha(x)}}{b(x)|x-y|^{-d-\alpha(x)} + b(y)|x-y|^{-d-\alpha(y)}}$$
$$= \frac{b(y) - b(x)|x-y|^{\alpha(y)-\alpha(x)}}{b(y) + b(x)|x-y|^{\alpha(y)-\alpha(x)}} \leq \frac{1-\frac{c_1}{c_2}}{1+\frac{c_1}{c_2}} = \Theta_1$$

Case 2b: $1 < |x-y| < R$ and $k'_a(x,y) > 0$. Then

$$\frac{|k'_a(x,y)|}{k'_s(x,y)} = \frac{b(x)|x-y|^{-d-\alpha(x)} - b(y)|x-y|^{-d-\alpha(y)}}{b(x)|x-y|^{-d-\alpha(x)} + b(y)|x-y|^{-d-\alpha(y)}}$$
$$= \frac{b(x) - b(y)|x-y|^{\alpha(x)-\alpha(y)}}{b(x) + b(y)|x-y|^{\alpha(x)-\alpha(y)}} \leq \frac{1-\frac{c_1}{c_2}R^{\alpha_1-\alpha_2}}{1+\frac{c_1}{c_2}R^{\alpha_1-\alpha_2}} =: \Theta_3$$

□

We have shown than k' satisfies all conditions needed in order to apply Theorem 3.15.

All examples of non-integrable kernels from above relate, in one way or another, to the standard kernel $|x-y|^{-d-\alpha}$ for some $\alpha \in (0,2)$ and the Sobolev-Slobodeckij space $H^{\alpha/2}(\mathbb{R}^d)$.

15. $k(x,y) = -|x-y|^{-d}\ln(|x-y|)\mathbb{1}_{B_1}(x-y)$. In this case k is non-integrable but the operator generated by k is of differentiability order less than any $\alpha > 0$. This example is symmetric. Anyway one can create nonsymmetric variants of this kernel. Lemma 2.42 can be applied and therefore k satisfies the Poinaré-Friedrichs inequality.
16. $k(x,y) = |x-y|^{-d}\, l(|x-y|)$, where $l : [0,\infty) \to [0,\infty)$ is a slowly varying function at zero. Example 15 is a special case of this example with $l(r) = \ln(r)\mathbb{1}_{B_1}(r)$. k is again symmetric, but nonsymmetric variants can easily be generated.

We could also study examples with kernels which relate to a generic standard kernel $|x-y|^{-d}\phi(|x-y|^2)^{-1}$ where ϕ itself can be chosen from a rather general class of functions, e.g. the class of complete Bernstein functions.

Finally, let us summarize the examples from above in a table with focus on the assumptions on the kernels k in the existence and uniqueness results. Recall that all examples satisfy (L). In the tabular below, the symbol **?** indicates that the answer depends on the concrete specification of the example.

Examples:	(1)	(3)	(2)	(4)	(5)	(6)	(7)	⋮	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
(P)	✓	✓	✓	✓	✓	✓	✓	⋮	✓	✓	✓	✓	✓	✓	✓	✓	✓
(C)	✓	✓	✓	✓	**?**	–	✓	⋮	✓	✓	✓	**?**	✓	✓	–	✓	✓
$(\widetilde{K})$	✓	✓	✓	✓	✓	✓	✓	⋮	✓	✓	–	✓	✓	✓	✓	✓	✓
(K)	✓	✓	✓	✓	✓	✓	✓	⋮	✓	✓	–	**?**	–	✓	✓	✓	✓
symmetry	✓	✓	–	–	**?**	–	–	⋮	✓	✓	–	**?**	–	✓	–	✓	✓
	integrable kernels								**non-integrable kernels**								

4. Nonlocal to local phase transition

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain. For given $f : \Omega \to \mathbb{R}$ and $g : \Omega^c \to \mathbb{R}$ we consider a family of nonlocal Dirichlet problems

$$\mathcal{L}^\alpha u_\alpha = f \qquad \text{in } \Omega, \tag{4.1a}$$

$$u_\alpha = g \qquad \text{on } \Omega^c, \tag{4.1b}$$

indexed by a parameter $\alpha \in (0, 2)$, where $\mathcal{L}^\alpha$ is a uniformly elliptic integro-differential operator of order α of the form

$$\mathcal{L}^\alpha u(x) = P.V. \int_{\mathbb{R}^d} (u(x) - u(y))\, k^\alpha(x, y)\, \mathrm{d}y. \tag{4.2}$$

Assume that $k^\alpha(x, y)$ is comparable to $(2 - \alpha)\, |x - y|^{-d-\alpha}$ in the following sense: There is a constant $\lambda > 0$ such that

$$\lambda \mathcal{A}_{d,-\alpha} \iint_{(\Omega^c \times \Omega^c)^c} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} \,\mathrm{d}y \,\mathrm{d}x \leq \iint_{(\Omega^c \times \Omega^c)^c} (u(x) - u(y))^2 k^\alpha(x, y) \,\mathrm{d}y \,\mathrm{d}x$$
$$\leq \lambda^{-1} \mathcal{A}_{d,-\alpha} \iint_{(\Omega^c \times \Omega^c)^c} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} \,\mathrm{d}y \,\mathrm{d}x.$$

Under this assumption, for fixed $\alpha \in (0, 2)$, the Dirichlet problem is well-posed, if $f \in H^*_\Omega(\mathbb{R}^d; k)$, $g \in V^{\alpha/2}(\Omega|\mathbb{R}^d)$, cf. Theorem 3.10.

In this chapter we prove that the solutions u_α of (4.1) converge to the solution of a local Dirichlet problem of second order

$$\mathcal{L}u = f \qquad \text{in } \Omega, \tag{4.3a}$$

$$u = g \qquad \text{on } \partial\Omega, \tag{4.3b}$$

when the order α of $\mathcal{L}^\alpha$ goes to 2^-. Here $\mathcal{L}$ is a uniformly elliptic second order differential operator of the form

$$\mathcal{L}u(x) = -\operatorname{div}\left(A(\cdot)\nabla u\right). \tag{4.4}$$

The coefficients a_{ij} of the matrix A can be computed from a the given family of kernels k^α, namely

$$a_{ij}(x) = \lim_{\alpha \to 2^-} \int_0^1 \int_{S^{d-1}} t^{d+1} \sigma_i \sigma_j k^\alpha(x, x + t\sigma) \,\mathrm{d}\sigma \,\mathrm{d}t.$$

To prove the convergence of solutions, we prove that the associated nonlocal energy functionals $\Gamma-$converge to the energy functional of the local Dirichlet problem.

4.1. Setting and main result

Throughout this chapter we assume that $\Omega \subset \mathbb{R}^d$ is a bounded C^1-domain. Since we prove the convergence of solutions via the $\Gamma-$convergence of the associated energies, we have to limit ourself to the case of symmetric kernels k^α, i.e.

$$k^\alpha(x,y) = k^\alpha(y,x). \tag{4.5}$$

Besides we make the following assumptions on the kernels k^α:

1. There is a constant $\lambda > 0$, such that for a.e. $x, y \in \mathbb{R}^d$

$$k^\alpha(x,y) \leq (2-\alpha)\lambda \left|x-y\right|^{-d-\alpha}. \tag{4.6}$$

2. There is a constant $\lambda > 0$ such that for all $u \in L^2(\mathbb{R}^d)$

$$\iint\limits_{(\Omega^c\times\Omega^c)^c} (u(x)-u(y))^2 k^\alpha(x,y)\,\mathrm{d}x\,\mathrm{d}y \geq \lambda^{-1}\alpha(2-\alpha) \iint\limits_{(\Omega^c\times\Omega^c)^c} (u(x)-u(y))^2 \left|x-y\right|^{-d-\alpha}\,\mathrm{d}x\,\mathrm{d}y \tag{4.7}$$

Inequality (4.6) implies the following integrability condition on k^α:

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \left(1 \wedge |x-y|^2\right) k^\alpha(x,y)\,\mathrm{d}y < C_L \tag{4.8}$$

for some $C_L > 0$, see (L) in Chapter 3.

In this section we call a nonlocal operator uniformly elliptic of order α, if the kernel satisfies the comparability assumptions (4.6) and (4.7). Note that this is not the formal definition of uniform ellipticity given in the introduction of this thesis, cf. (1.9). To the best of the author knowledge, it is not clear, whether the integrated lower bound implies the formal definition of uniformly ellipticity. Nevertheless, if a family (k^α) of kernels satisfies (4.7) with λ independent of α, the local operator $\mathcal{L}$ generated by the family (k^α) for $\alpha \to 2$ satisfies an integrated ellipticity condition, i.e. there is $\lambda' > 0$, such that

$$\sum_{i,j=1}^d \int_\Omega a_{ij}(x)\partial_i u(x)\partial_j u(x)\,\mathrm{d}x \geq \lambda' \sum_{i,j=1}^d \int \partial_i u(x)\partial_j u(x)\,\mathrm{d}x \tag{4.9}$$

for all $u \in L^2(\Omega)$. For $x \in \Omega$, $\xi \in \mathbb{R}^d$ and $r > 0$, define $u_\xi^x(y) = y\xi \mathbb{1}_{B_r(x)}(y)$. Then (4.9) yields

$$\sum_{i,j=1}^d \xi_i\xi_j \fint_{B_r(x)} a_{ij}(y)\,\mathrm{d}y \geq \lambda' \left|\xi\right|^2.$$

Thus the pointwise estimate can be obtained almost everywhere using the local character of $\mathcal{L}$.

Remark 4.1. Consider a family of kernels $(k^\alpha)_{\alpha\in(0,2)}$ and the corresponding operators $\mathcal{L}^{k^\alpha}$. To obtain the convergence of these operators to a local operator $\mathcal{L}$, it is sufficient to assume that the family k^α localizes, i.e. for all $\delta > 0$

$$\sup_{x\in\mathbb{R}^d} \lim_{\alpha\to 2} \int_{|x-y|>\delta} |x-y|\, k^\alpha(x,y)\,\mathrm{d}y = 0\,.$$

Assumption (4.6) is stronger, but we need the stronger assumption to prove the compatibility of the boundary data with the Γ-limit, see Theorem 4.8.

Due to (4.6),(4.7) it is convenient to call $\mathcal{L}^\alpha$ an operator of 'order' α. The main result of this chapter is the following

Theorem 4.2. *Let $g \in V(\Omega|\mathbb{R}^d)$ and let $(k^\alpha)_{(\alpha\in(0,2))}$ be a family of symmetric kernels satisfying* (4.6) *and* (4.7) *with λ independent of α. Let u_α be the solution to the nonlocal Dirichlet problem* (4.1).

1. *Then the local Dirichlet problem* (4.3) *with coefficients $a_{i,j}$ given by*

$$a_{ij}(x) = \lim_{\alpha\to 2^-} \int_0^1 \int_{\mathbb{S}^{d-1}} t^{d+1}\sigma_i\sigma_j k^\alpha(x, x+t\sigma)\,\mathrm{d}\sigma\,\mathrm{d}t.$$

   has a solution $u \in H^1(\Omega)$. Further the sequence of solutions (u_α) converges to u in $L^2(\mathbb{R}^d)$ as $\alpha \to 2^-$.

2. *Let $\alpha_0 \in (0,2)$ be fixed. Then the sequence of solutions (u_α) also converges to u in $V^{\alpha_0/2}(\Omega|\mathbb{R}^d)$ as $\alpha \to 2^-$.*

Remark 4.3. The assumptions (4.7) and (4.6) on a given kernel k^α are sufficient to obtain a variational solution to the nonlocal Dirichlet problem, cf. Theorem 3.10.

As mentioned before, we use a variational approach to prove Theorem 4.2. To this aim, we introduce the corresponding energy functionals of (4.1) and (4.3) $\widetilde{F}_g^\alpha : L^2(\mathbb{R}^d) \to [-\infty,\infty]$, $\widetilde{F}_g : L^2(\mathbb{R}^d) \to [-\infty,\infty]$ given by

$$\widetilde{F}_g^\alpha(u) = \begin{cases} \frac{1}{4} \iint\limits_{(\Omega^c\times\Omega^c)^c} (u(x)-u(y))^2 k^\alpha(x,y)\,\mathrm{d}y\,\mathrm{d}x - \int\limits_\Omega u(x)f(x)\,\mathrm{d}x, & \text{if } u \in V_g^{\alpha/2}(\Omega|\mathbb{R}^d),\\ +\infty, & \text{else,}\end{cases}$$

$$\widetilde{F}_g(u) = \begin{cases} \frac{1}{2} \int\limits_\Omega \langle A(x)\nabla u(x), u(x)\rangle\,\mathrm{d}x - \int\limits_\Omega u(x)f(x)\,\mathrm{d}x & \text{if } u-g \in H_0^1(\Omega),\\ +\infty, & \text{else.}\end{cases}$$

Note that for both functionals the functions are defined on the whole $\mathbb{R}^d$, in particular $H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^1(\mathbb{R}^d)}}$.

For the sake of completeness let us prove that $\widetilde{F}_g^\alpha$ is associated to (4.1).

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. If u is a minimizer of the functional $\widetilde{F}_g^\alpha$ in $L^2(\mathbb{R}^d)$, then u solves the Dirichlet problem* (4.1).

Proof. Let $u \in V_g^{\alpha/2}(\Omega|\mathbb{R}^d)$, otherwise $\widetilde{F}_g^\alpha = +\infty$. Let $v \in V_0^{\alpha/2}(\Omega|\mathbb{R}^d)$. We formally compute $\frac{d}{dt} F(u+tv)\big|_{t=0}$.

$$\begin{aligned}\frac{d}{dt}\widetilde{F}_g^\alpha(u+tv)\big|_{t=0} &= \frac{d}{dt}\Big(\frac{1}{4}\iint\limits_{(\Omega^c\times\Omega^c)^c} (u(x)+tv(x)-u(y)-tv(y))^2 k^\alpha(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\quad - \int\limits_\Omega (u(x)+tv(x))f(x)\,\mathrm{d}x\Big)\Big|_{t=0} \\ &= \frac{1}{2}\iint\limits_{(\Omega^c\times\Omega^c)^c} (u(x)-u(y))(v(x)-v(y))k^\alpha(x,y)\,\mathrm{d}y\,\mathrm{d}x - \int\limits_\Omega v(x)f(x)\,\mathrm{d}x.\end{aligned}$$

This is the weak formulation of (4.1). □

An analogous computation proves that $\widetilde{F}_g$ is associated to (4.3).

4.2. Gamma-Convergence of the energies

In this section we prove Γ-convergence of the above defined energy functionals. For a short survey on Γ−convergence we refer to Appendix A, where we collect some basic properties of Γ−convergence.

First we compute the Γ−limit of the free energy, i.e. without any boundary condition. Since Γ−convergence is stable under continuous perturbations, we can neglect the forcing term $\langle u, f\rangle_{L^2(\Omega)}$ in the computation of the Γ−limit. Therefore we consider the functionals $F^\alpha : L^2(\mathbb{R}^d) \to [0,\infty]$ and $F : L^2(\mathbb{R}^d) \to [0,\infty]$ defined by

$$F^\alpha(u) = \begin{cases} \frac{1}{4}\iint\limits_{(\Omega^c\times\Omega^c)^c} (u(x)-u(y))^2 k^\alpha(x,y)\,\mathrm{d}y\,\mathrm{d}x, & \text{if } u \in V^{\alpha/2}(\Omega|\mathbb{R}^d), \\ +\infty, & \text{else,} \end{cases} \tag{4.10}$$

$$F(u) = \begin{cases} \frac{1}{2}\int\limits_\Omega \langle A(x)\nabla u(x), u(x)\rangle\,\mathrm{d}x, & \text{if } u \in H^1(\Omega), \\ +\infty, & \text{else.} \end{cases} \tag{4.11}$$

Recall that the entries $a_{ij}(x)$ of the matrix $A(x)$ are given by

$$a_{ij}(x) = \lim_{\alpha\to 2^-} \int\limits_0^1 \int\limits_{S^{d-1}} t^{d+1}\sigma_i\sigma_j k^\alpha(x, x+t\sigma)\,\mathrm{d}\sigma\,\mathrm{d}t. \tag{4.12}$$

Remark 4.5. 1. If k^α satisfies the followings assumption we can rewrite (4.12) explicitly, i.e. without a limit $\alpha \to 2$.

For all $\alpha \in (0,2)$, there exists $\delta > 0$ and a function $K : \mathbb{R}^d \times B_\delta(0) \to \mathbb{R}$ such that

$$K(x,h) = K(x, rh) \text{ for all } r \in (0,\delta), x \in \mathbb{R}^d, h \in B_1(0)$$

and $k^\alpha(x,y) = (2-\alpha)\,|x-y|^{-d-\alpha}\,K(x, y-x)$ for all $x \in \mathbb{R}^d$ and $y \in B_\delta(x)$.

Then we obtain

$$a_{ij}(x) = \int\limits_{\mathbb{S}^{d-1}} \sigma_i \sigma_j K(x,\sigma)\,\mathrm{d}\sigma.$$

We can interpret this assumption as follows: For every $x \in \mathbb{R}^d$, in some small ball around x, the value of the kernel depends only on the direction of $x - y$.

2. The functional F^α and F are associated to the nonlocal and local Neumann problem with homogeneous boundary data. The nonlocal Neumann problem is to find a function $u \in V^{\alpha/2}(\Omega|\mathbb{R}^d)$, such that

$$\begin{aligned} \mathcal{L}u &= 0 && \text{in } \Omega \\ \mathcal{N}u &= 0 && \text{on } \Omega^c, \end{aligned}$$

where $\mathcal{N}$ is a nonlocal integro-differential operator given by

$$\mathcal{N}u(y) = -\int\limits_{\Omega} (u(x) - u(y))k(x,y)\,\mathrm{d}x.$$

It can be shown that this problem converges to the classical Neumann problem as $\alpha \to 2^-$ using Theorem 4.6, below. For details on the nonlocal Neumann problem we refer to [DRV14].

We are now in the position to formulate a Γ-convergence result for the above defined functionals.

Theorem 4.6. *Let $\Omega \subset \mathbb{R}^d$ be a C^1-domain. Let $\alpha_0 \in (0,2)$ and let the family $(k^\alpha)_{\alpha\in(0,2)}$ satisfy* (4.6) *with λ independent of α. Then*

$$\Gamma\!-\lim_{\alpha\to 2} F^\alpha = F \tag{4.13}$$

where the Γ-limit is taken with respect to the topology of $V^{\alpha_0/2}(\Omega|\mathbb{R}^d)$ and the entries of $A(x)$ are given by (4.12).

Remark 4.7. Due to the symmetry of the double integral with respect to x and y in the definition of F^α, the kernels k^α need not to be symmetric. If k is not symmetric, the symmetrization $k_s(x,y) = \frac{1}{2}(k(x,y) + k(y,x))$ defines the same functional. Note that we use the symmetry in the computation of the Euler-Lagrange equation, cf. Lemma 4.4.

The proof of the Γ-convergence consists of two steps, the lim sup– and the lim inf-inequality. The proof of the lim inf-inequality uses the idea to regularize the functions by a smooth mollifier, coming from [Pon04b, Lem. 8] and also [LS14, Thm. 1.2]. One difference from our to the above mentioned results is the area of integration in the definition of the functionals F^α. Furthermore, we prove $\Gamma-$convergence in with respect to the stronger topology of $V^{\alpha_0/2}(\Omega|\mathbb{R}^d)$.

Proof. **L**imsup-Inequality:
Let (α_n) be a sequence in $(0,2)$ with $\alpha_n \to 2$. To avoid double indices, we write n instead of α_n, if there is no risk of confusion. Further we set $\|\cdot\|_{V^{\alpha_0/2}(\Omega|\mathbb{R}^d)} = \|\cdot\|_{\alpha_0}$. We have to construct a sequence $u_n \in V^{\alpha_0/2}(\Omega|\mathbb{R}^d)$, such that $\|u_n - u\|_{\alpha_0} \overset{n\to\infty}{\longrightarrow} 0$ and

$$F(u) \geq \limsup_{n\to\infty} F^{\alpha_n}(u_n). \tag{4.14}$$

If $F(u) = \infty$ we can take the constant sequence $u_n = u$ for all $n \in \mathbb{N}$. Thus we can assume that $u \in H^1(\Omega)$. Consider the metric defined on $H^1(\Omega) \cap V^{\alpha_0/2}(\Omega|\mathbb{R}^d)$ by $d(u,v)^2 = \|u-v\|_{\alpha_0}^2 + \|u-v\|_{H^1(\Omega)}^2$. Then $d(u,v) \geq \|u-v\|_{\alpha_0}$ and $F(u)$ is continuous w.r.t. d on $H^1(\Omega) \cap V^{\alpha_0/2}(\Omega|\mathbb{R}^d)$. By Lemma 2.20 $C_c^\infty(\mathbb{R}^d)$ is dense in $V^{\alpha_0/2}(\Omega|\mathbb{R}^d)$ and also in $H^1(\Omega)$.

Thus it is sufficient to prove the limsup-inequality for all $u \in C_c^\infty(\mathbb{R}^d)$, cf. Remark A.6. Let $u \in C_c^\infty(\mathbb{R}^d)$ and set $u_n = u$ for all $n \in \mathbb{N}$. Since $u \in C_c^\infty(\mathbb{R}^d)$, by Taylor's formula

$$(u(x) - u(y))^2 = \sum_{i,j=1}^{d} \partial_i u(x) \partial_j u(x)(x-y)_i (x-y)_j + r(x,y)\,|x-y|^3, \tag{4.15}$$

where $r(x,y)$ is bounded for all $x, y \in \mathbb{R}^d$, say $|r(x,y)| \leq C_r$. Further there is $C > 0$ such that

$$|u(x) - u(y)| < C\,|x-y| \quad \text{for } x, y \in \mathbb{R}^d$$

and there is $R > 0$, such that $\operatorname{supp}(u) \subset B_R(0)$. First we prove

$$\lim_{n\to\infty} \iint_{\Omega\,\Omega^c} (u(x) - u(y))^2 k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x = 0.$$

Fix $x \in \Omega$ and choose $R > 0$, such that $\operatorname{supp}(u) \subset B_R(0)$. Set $\delta = \operatorname{dist}(x, \Omega^c) > 0$. Now by (4.6)

$$\begin{aligned}
\int_{\Omega^c} (u(x) - u(y))^2 k^n(x,y)\,\mathrm{d}y &\leq \int_{|x-y|>\delta} (u(x) - u(y))^2 k^n(x,y)\,\mathrm{d}y \\
&\leq C \int_{|x-y|>\delta} |x-y|^2\, k^n(x,y)\,\mathrm{d}y \\
&\leq (2-\alpha_n) C_1 \int_{R>|x-y|>\delta} |x-y|^{2-d-\alpha}\,\mathrm{d}y \\
&\leq C_2(2-\alpha_n) \int_\delta^R r^{1-\alpha}\,\mathrm{d}r \\
&\leq C_3(R^{2-\alpha_n} - \delta^{2-\alpha_n}) \overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Next we prove

$$\limsup_{n\to\infty} \iint_{\Omega\,\Omega} (u(x) - u(y))^2 k^n(x,y) \leq F(u).$$

Again fix $x \in \Omega$.

$$\begin{aligned}\int_{\Omega} (u(x)-u(y))^2 k^n(x,y)\,\mathrm{d}y &\leq \int_{|x-y|<1} (u(x)-u(y))^2 k^n(x,y)\,\mathrm{d}y \\ &+ \int_{|x-y|\geq 1} (u(x)-u(y))^2 k^n(x,y)\,\mathrm{d}y \\ &= I + II\end{aligned}$$

By the same arguments as above, the second term goes to zero as $n \to \infty$. Using (4.15) we can estimate I:

$$\begin{aligned}I = \sum_{i,j=1}^{d} \partial_i u(x) \partial_j u(x) &\int_{|x-y|<1} (x-y)_i (x-y)_j k^n(x,y)\,\mathrm{d}y \\ + \int_{|x-y|<1} & r(x,y)\,|x-y|^3\, k^n(x,y)\,\mathrm{d}y.\end{aligned}$$

The second term can be estimated using (4.6):

$$\begin{aligned}\int_{|x-y|<1} r(x,y)\,|x-y|^3\, k^n(x,y)\,\mathrm{d}y &\leq (2-\alpha_n)\lambda C_r \int_{B_1(0)} |y|^{3-d-\alpha_n}\,\mathrm{d}y \\ &\leq (2-\alpha_n)\lambda C_r \int_0^1 \tau^{2-\alpha_n}\,\mathrm{d}\tau \\ &\leq C_r \frac{2-\alpha_n}{3-\alpha_n}\lambda \to 0\end{aligned}$$

as $n \to \infty$. Now we look at the integral in the first term. Let $i, j \in \{1, ..., d\}$. Then

$$\begin{aligned}\int_{|x-y|<1} (x-y)_i (x-y)_j k^n(x,y)\,\mathrm{d}y &= \int_{|h|<1} \frac{h_i h_j}{|h|^2} k^n(x, x+h)\,\mathrm{d}h \\ &= \int_{S^{d-1}} \int_0^1 t^{d+1} \sigma_i \sigma_j k^n(x, x+t\sigma)\,\mathrm{d}t\,\mathrm{d}\sigma \\ &\overset{n\to\infty}{\longrightarrow} a_{ij}(x)\end{aligned}$$

by definition of (a_{ij}).

Altogether we obtain that for fixed $x \in \Omega$ and $u \in C_c^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} (u(x)-u(y))^2 k^n(x,y)\,\mathrm{d}y \overset{n\to\infty}{\longrightarrow} \sum_{i,j=1}^{d} \partial_i u(x) \partial_j u(x) a_{ij}(x).$$

Since

$$\int_{\mathbb{R}^d} (u(x)-u(y))^2 k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x \leq C_u\,,$$

by the Dominated Convergence Theorem

$$\iint\limits_{(\Omega^c\times\Omega^c)^c} (u(x)-u(y))^2 k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x \overset{n\to\infty}{\longrightarrow} \int\limits_\Omega \sum_{i,j=1}^d \partial_i u(x)\partial_j u(x) a_{ij}(x)\,\mathrm{d}x$$

and since the sequence (α_n) was arbitrary

$$\limsup_{\alpha\to 2^-} F^\alpha(u) \le F(u).$$

Liminf-Inequality:
For simplicity, we prove the liminf-inequality for all sequences (u_n) that converge to u in $L^2(\mathbb{R}^d)$. Of course this implies the liminf-inequality with respect to the stronger norm $\|\cdot\|_{\alpha_0}$.

Let (α_n) be a sequence in $(0,2)$ with $\alpha_n\to 2$. We have to prove that for all $u,(u_n)_{n\in\mathbb{N}}\in L^2(\mathbb{R}^d)$ with $\|u_n-u\|_{L^2(\mathbb{R}^d)} \overset{n\to\infty}{\longrightarrow} 0$

$$\liminf_{\to\infty} F^{\alpha_n}(u_n) \ge F(u). \tag{4.16}$$

First we notice, that if $\liminf_{n\to\infty} F_{\alpha_n}(u_n)=\infty$ there is nothing to prove. Thus we can assume that the left-hand side of (4.16) is finite.

Let $u,(u_n)_{n\in\mathbb{N}}\in L^2(\mathbb{R}^d)$ with $\|u_n-u\|_{L^2(\mathbb{R}^d)} \overset{n\to\infty}{\longrightarrow} 0$. We define

$$f^\varepsilon(x) = f*\eta_\varepsilon(x),$$

where η_ε is a smooth mollifier having support in $B_\varepsilon(0)$. Then $u^\varepsilon, u_n^\varepsilon \in C^\infty(\mathbb{R}^d)$ and by Lemma C.4

$$u_n^\varepsilon \overset{n\to\infty}{\longrightarrow} u^\varepsilon \text{ in } C^k(\overline{\Omega}) \text{ for any } k\in\mathbb{N}. \tag{4.17}$$

Define

$$\Omega_\delta = \{x\in\Omega \,|\, \operatorname{dist}(x,\partial\Omega)>\delta\}.$$

Using Jensen's inequality , Fubini and a change of variables we obtain for all $n\in\mathbb{N}$ and $\varepsilon\in(0,\delta)$

$$\begin{aligned}
\iint\limits_{\Omega_\delta\,\Omega_\delta} (u_n^\varepsilon(x)-u_n^\varepsilon(y))^2 k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x &= \iint\limits_{\Omega_\delta\,\Omega_\delta} \left(\int\limits_{B_\varepsilon} (u_n(x-z)-u_n(y-z))\eta_\varepsilon(z)\,\mathrm{d}z\right)^2 k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\le \int\limits_{B_\varepsilon}\iint\limits_{\Omega_\delta\,\Omega_\delta} (u_n(x-z)-u_n(y-z))^2 k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x\eta_\varepsilon(z)\,\mathrm{d}z \\
&\le \int\limits_{B_\varepsilon}\iint\limits_{\Omega\,\Omega} (u_n(x')-u_n(y'))^2 k^n(x',y')\,\mathrm{d}y'\,\mathrm{d}x'\eta_\varepsilon(z)\,\mathrm{d}z \\
&\le F^{\alpha_n}(u_n).
\end{aligned}$$

In the last step we use the positivity of the integrands in the definition of F^n and the fact that $\int_{\mathbb{R}^d}\eta_\varepsilon = 1$ for any $\varepsilon>0$. By Taylor's formula we have

$$|u_n^\varepsilon(x)-u_n^\varepsilon(y)| \le \left|\sum_{i=1}^d \partial_i u_n^\varepsilon(x)(x-y)_i\right| + C_n^\varepsilon |x-y|^2$$

$$|u^\varepsilon(x)-u^\varepsilon(y)| \le \left|\sum_{i=1}^d \partial_i u^\varepsilon(x)(x-y)_i\right| + C^\varepsilon |x-y|^2$$

and by (4.17) $C_n^\delta \overset{n\to\infty}{\longrightarrow} C_\delta$. Thus for some $r>0$ sufficiently small

$$
\begin{aligned}
\liminf_{n\to\infty} F_{\alpha_n}(u_n) &\geq \liminf_{n\to\infty} \iint\limits_{\Omega_\delta\,\Omega_\delta} (u_n^\varepsilon(x)-u_n^\varepsilon(y))^2 k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\geq \liminf_{n\to\infty} \int\limits_{\Omega_\delta} \int\limits_{|x-y|\leq r\cap\Omega_\delta} (u_n^\varepsilon(x)-u_n^\varepsilon(y))^2 k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\geq \int\limits_{\Omega_\delta} \liminf_{n\to\infty} \sum_{i,j=1}^d \partial_i u_n^\varepsilon(x)\partial_j u_n^\varepsilon(x) \int\limits_{|x-y|\leq r\cap\Omega_\delta} (x-y)_i(x-y)_j k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\overset{n\to\infty}{\longrightarrow} \int\limits_{\Omega_\delta} \sum_{i,j=1}^d a_{ij}(x)\partial_i u(x)\partial_j u(x)\,\mathrm{d}x\,,
\end{aligned}
$$

where we used Fatou's Lemma and (4.17). Set $A=(a_{ij})$, the above inequality is equivalent to

$$
\liminf_{n\to\infty} F_{\alpha_n}(u_n) \geq \int\limits_{\Omega_\delta} \langle A(x)\nabla u^\varepsilon(x), \nabla u^\varepsilon(x)\rangle\ \mathrm{d}x.
$$

Since

$$
\liminf_{n\to\infty} \iint\limits_{(\Omega^c\times\Omega^c)^c} (u_n(x)-u_n(y))^2 k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x < \infty\,,
$$

Proposition 2.5 implies that $u\in H^1(\Omega)$. Thus $\nabla u^\varepsilon \to \nabla u$ in $L^2(\Omega_\delta)$, $\varepsilon\in(0,\delta)$. Using the continuity of the scalar product and let $\varepsilon\to 0$ yields

$$
\liminf_{n\to\infty} F_{\alpha_n}(u_n) \geq \int\limits_{\Omega_\delta} \langle A(x)\nabla u(x), \nabla u(x)\rangle\ \mathrm{d}x.
$$

Since $u\in H^1(\Omega)$, by absolute continuity of the Lebesgues integral, there is $\delta>0$ such that for any $\sigma>0$

$$
\int\limits_{\Omega\setminus\Omega_\delta} \langle A(x)\nabla u(x), \nabla u(x)\rangle\ \mathrm{d}x < \sigma.
$$

Altogether,

$$
\liminf_{n\to\infty} F_{\alpha_n}(u_n) \geq F(u)-\sigma.
$$

Since σ is arbitrary small, the assertion follows. □

The following theorem proves the compatibility of the Dirichlet boundary condition and $\Gamma-$convergence. We set again $A=(a_{ij})$ and define $F_g^\alpha : L^2(\mathbb{R}^d)\to[0,\infty]$ and $F_g : L^2(\mathbb{R}^d)\to[0,\infty]$ by

$$
F_g^\alpha(v) = \begin{cases} \frac{1}{4} \iint\limits_{(\Omega^c\times\Omega^c)^c} (v(x)-v(y))^2 k^\alpha(x,y)\,\mathrm{d}y\,\mathrm{d}x, & \text{if } v\in V_g^{\alpha/2}(\Omega|\mathbb{R}^d), \\ +\infty, & \text{else,} \end{cases} \tag{4.18}
$$

$$
F_g(v) = \begin{cases} \frac{1}{2}\int\limits_\Omega \langle A(x)\nabla v(x), \nabla v(x)\rangle\ \mathrm{d}x, & \text{if } v-g\in H_0^1(\Omega), \\ +\infty, & \text{else.} \end{cases} \tag{4.19}
$$

Recall that the matrix can be computed in terms of the family (k^α) by the formula

$$a_{ij}(x) = \lim_{\alpha\to 2^-} \int_0^1 \int_{S^{d-1}} t^{d+1}\sigma_i\sigma_j k^\alpha(x, x+t\sigma)\,\mathrm{d}\sigma\,\mathrm{d}t.$$

Theorem 4.8. *Let $\Omega \subset \mathbb{R}^d$ be a bounded C^1-domain. Let $g \in V(\Omega|\mathbb{R}^d)$ and $\alpha_0 \in (0,2)$. Further assume that the family $(k^\alpha)_{\alpha\in(0,2)}$ satisfies (4.6) independent of α. Then*

$$\Gamma - \lim_{\alpha\to 2} F_g^\alpha = F_g\,, \tag{4.20}$$

where the Γ-limit is taken with respect to the topology of $V^{\alpha_0/2}(\Omega|\mathbb{R}^d)$.

The technique we use goes back to the paper [DG75] and is often called the DeGiorgi method for matching boundary values. We extent this technique to the nonlocal setting.

Proof. The $\liminf -$inequality is a direct consequence of the one without boundary condition, since $L^2_g(\Omega|\mathbb{R}^d)$ is a closed subspace of $L^2(\Omega)$. Therefore it is sufficient to prove the $\limsup -$inequality for $u \in V^{\alpha_0/2}(\Omega|\mathbb{R}^d)$. We can assume, that $F_g(u)$ is finite, otherwise there is nothing to prove. Again, we write n instead of α_n when there is no risk of confusion. Note that $F_g(u) < \infty$ implies $u - g \in H^1_0(\Omega)$ and we can assume that $F^n_g(u)$ is bounded. We write

$$F^n_g(u) = F^n_g(u - g + g) \leq 2F^n_g(u-g) + 2F^n_g(g)$$

Since $g \in V(\Omega|\mathbb{R}^d)$, Proposition 2.28 and Proposition 2.27 imply that the right-hand-side is bounded.

Let (α_n) be a sequence in $(0,2)$ with $\alpha_n \overset{n\to\infty}{\longrightarrow} 2^-$. By Theorem 4.6

$$\Gamma - \lim_{\alpha\to 2^-} F^\alpha(u) = F(u).$$

Thus there is a sequence $(u_n) \in V^{\alpha_0/2}(\Omega|\mathbb{R}^d)$, such that $u_n \to u$ in $V^{\alpha_0/2}(\Omega|\mathbb{R}^d)$ and

$$\limsup_{n\to\infty} F^n(u_n) \leq F(u).$$

We want to modify (u_n) such that the modified sequence (v_n) is in $V_g^{\alpha_0/2}(\Omega|\mathbb{R}^d)$, but still satisfies

$$\limsup_{n\to\infty} F^n(v_n) \leq F(u).$$

Let $K \Subset \Omega$ compact. Set $\delta = \operatorname{dist}(\partial\Omega, K)$. We define for $\nu \in \mathbb{N}$:

$$D_0 = K, \qquad D_i = \{x \in \Omega : \operatorname{dist}(x,K) < \frac{i\delta}{\nu}\}, \quad i = 1, ..., \nu.$$

Further let $\phi_i, ..., \phi_\nu$ be cut-off functions defined by:

$$\begin{cases} \phi_i \in C^1_0(D_i), & 0 \leq \phi(x) \leq 1 \forall x \in D_i, \\ \phi_i(x) = 1, & \forall x \in D_{i-1}, \\ |D\phi_i| \leq \frac{\nu+1}{\delta}. \end{cases} \tag{4.21}$$

Set for all $n \in \mathbb{N}$ and $i = 1, ..., \nu$

$$v_{i_n} = (1 - \phi_i)u + \phi_i u_n = u - \phi_i(u - u_n).$$

Note that the sets D_i and thereby the sequence (v_{i_n}) depend implicitly on ν. We suppress this dependence. For $A \subset \mathbb{R}^d$ we define

$$F_n(u, A) = \iint\limits_{(A^c \times A^c)^c} (u(x) - u(y))^2 k^n(x, y) \, \mathrm{d}y \, \mathrm{d}x$$

and

$$G_n(u, A) = \left(\iint\limits_{(A^c \times A^c)^c} (u(x) - u(y))^2 k^n(x, y) \, \mathrm{d}y \, \mathrm{d}x \right)^{1/2} = \sqrt{F_n(u, A)}.$$

To shorten the notation, we introduce

$$\Gamma_n(u, v) = \int\limits_{\mathbb{R}^d} (u(x) - u(y))(v(x) - v(y)) k^n(x, y) \, \mathrm{d}y$$

and set $\Gamma_n(u, u) = \Gamma_n(u)$. See Section C.2 for some calculus rules for Γ^n we use in the sequel. Note that if $A = A_1 \dot{\cup} A_2$, then $(A^c \times A^c)^c \subset (A_1^c \times A_1^c)^c \cup (A_2^c \times A_2^c)^c$, cf. Lemma C.3. Using this and the positivity of $F^n(u, \cdot)$, we obtain for all $i \in \{1, ..., \nu\}$:

$$F_n(v_{i,n}, \Omega) - F_n(u_n, \Omega) \leq F_n(v_{i,n}, \Omega \setminus D_i) + F_n(v_{i,n}, D_i) - F_n(u_n, D_i). \tag{4.22}$$

The goal is to prove that the left-hand-side can be made arbitrary small independent of $n \in \mathbb{N}$. The first term on the right-hand-side of (4.22) can be estimated as follows. Set $\mathcal{B}_i = \Omega \setminus D_i$, then

$$\begin{aligned} F_n(v_{i,n}, \mathcal{B}_i) &\leq 2 \int\limits_{\mathcal{B}_i} \Gamma_n(u + (u_n - u)\phi_i) \, \mathrm{d}x \\ &\leq 4 \left(\int\limits_{\mathcal{B}_i} \Gamma_n(u) \, \mathrm{d}x + \int\limits_{\mathcal{B}_i} \Gamma_n((u_n - u)\phi_i) \, \mathrm{d}x \right). \end{aligned}$$

The first term can be estimated independent of i and ν using (4.6):

$$2 \int\limits_{\mathcal{B}_i} \Gamma_n(u) \, \mathrm{d}x \leq \lambda(2 - \alpha_n) \int\limits_{\Omega \setminus K} \int\limits_{\mathbb{R}^d} (u(x) - u(y))^2 \, |x - y|^{-d-\alpha_n} \, \mathrm{d}y \, \mathrm{d}x.$$

Observe that for $x \in \mathcal{B}_i$ $(u_n - u)\phi_i(x) = 0$. Using again (4.6), we obtain for the second term

$$\begin{aligned}
\int\limits_{\mathcal{B}_i} \Gamma_n((u_n-u)\phi_i)\,\mathrm{d}x &= \int\limits_{\mathcal{B}_i}\int\limits_{\mathbb{R}^d} \left[((u_n-u)\phi_i)(y)\right]^2 k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int\limits_{\mathcal{B}_i}\int\limits_{\mathbb{R}^d} (u_n(y)-u(y))^2(\phi_i(y) - \underbrace{\phi_i(x)}_{=0})^2 k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\le \int\limits_{\mathbb{R}^d} (u_n(y)-u(y))^2 \int\limits_{\Omega\setminus K} (\phi_i(y)-\phi_i(x))^2 k^n(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\le \lambda(2-\alpha_n)\left(\frac{\nu+1}{\delta}\right)^2 \int\limits_{\mathbb{R}^d} (u_n(y)-u(y))^2 \int\limits_{\Omega\setminus K} |x-y|^{-d-\alpha_n+2}\,\mathrm{d}x\,\mathrm{d}y \\
&\le C(\lambda,\Omega)\left(\frac{\nu+1}{\delta}\right)^2 \|u_n-u\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Next we estimate the second term on the right-hand-side of (4.22): We use the inequality

$$\int f^2 - g^2 \le \left(\int (f-g)^2\right)^{1/2}\left(\int (f+g)^2\right)^{1/2},$$

to obtain

$$F_n(v_{i,n}, Di) - F_n(u_n, D_i) \le G_n(v_{i,n} - u_n, D_i) G_n(v_{i,n} + u_n, D_i). \tag{4.23}$$

The first term on the right-hand-side of (4.23) can be estimated as follows:

$$\begin{aligned}
G_n(v_{i,n}-u_n, D_i) &\le \sqrt{2}\left(\int\limits_{D_i} \Gamma_n(v_{i,n}-u_n)\,\mathrm{d}x\right)^{1/2} = \sqrt{2}\left(\int\limits_{D_i} \Gamma_n((1-\phi_i)(u-u_n))\,\mathrm{d}x\right)^{1/2} \\
&\le \sqrt{2}\left(\int\limits_{D_i\setminus D_{i-1}} \Gamma_n((1-\phi_i)(u-u_n))\,\mathrm{d}x\right)^{1/2} + \sqrt{2}\left(\int\limits_{D_{i-1}} \Gamma_n((1-\phi_i)(u-u_n))\,\mathrm{d}x\right)^{1/2} \\
&= I + II
\end{aligned}$$

Set $D_i \setminus D_{i-1} = \mathcal{D}_i$. Choose $R > 0$ such that $\Omega \subset B_{R-1}(0)$. Now by (4.6):

$$\begin{aligned}
I &\le 2\left(\iint\limits_{\mathcal{D}_i\,\mathbb{R}^d} \left[(1-\phi_i(x))(u_n(x)-u(x)) - (1-\phi_i(y))(u_n(y)-u(y))\right]^2 k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2} \\
&\le \left(\iint\limits_{\mathcal{D}_i\,\mathbb{R}^d} \left[\phi_i(y)-\phi_i(x)\right]^2\left[(u_n(x)-u(x)) + (u_n(y)-u(y))\right]^2 k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2} \\
&\quad + \left(\iint\limits_{\mathcal{D}_i\,\mathbb{R}^d} \left[(u_n(x)-u(x)) - (u_n(y)-u(y))\right]^2\left[(1-\phi_i(x)) + (1-\phi_i(y))\right]^2 k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\le C\left(\frac{\nu+1}{\delta}\right)\lambda(2-\alpha_n)\left(\int\limits_{\mathbb{R}^d}\big[u_n(x)-u(x)\big]^2\int\limits_{B_R}|x-y|^{-d-\alpha_n+2}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\\
&\quad+C\lambda(2-\alpha_n)\left(\int\limits_{\mathcal{D}_i}\big[(u_n(x)-u(x)\big]^2\int\limits_{B_R^c}|x-y|^{-d-\alpha_n}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\\
&\quad+C\lambda(2-\alpha_n)\left(\frac{\nu+1}{\delta}\right)\left(\int\limits_{\mathbb{R}^d}\big[u_n(y)-u(y)\big]^2\int\limits_{\mathcal{D}_i}|x-y|^{-d-\alpha_n+2}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}\\
&\quad+C\lambda(2-\alpha_n)\left(\iint\limits_{\mathcal{D}_i\,\mathbb{R}^d}\big[(u_n(x)-u(x))-(u_n(y)-u(y))\big]^2|x-y|^{-d-\alpha_n}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\\
&\le C(\lambda,R)\left(\frac{\nu+1}{\delta}+1\right)\|u_n-u\|_{L^2(\mathbb{R}^d)}\\
&\quad+C\lambda(2-\alpha_n)\left(\iint\limits_{\mathcal{D}_i\,\mathbb{R}^d}\big[(u_n(x)-u(x))-(u_n(y)-u(y))\big]^2|x-y|^{-d-\alpha_n}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}.
\end{aligned}
$$

Now we estimate II. Note that $(1-\phi_i(x))=0$ for $x\in\mathcal{D}_{i-1}$.

$$
\begin{aligned}
II&\le 2\left(\iint\limits_{D_{i-1}\,D_{i-1}^c}\big[(1-\phi_i(y))(u_n(y)-u(y))\big]^2k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac12}\\
&\le 2\left(\int\limits_{D_{i-1}^c}(u_n(y)-u(y))^2\int\limits_{D_{i-1}}(\underbrace{1}_{=\phi_i(x)}-\phi_i(y))^2k^n(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac12}\\
&\le 2\left(\int\limits_{D_{i-1}^c}(u_n(y)-u(y))^2\int\limits_{D_{i-1}}(\phi_i(x)-\phi_i(y))^2k^n(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac12}\\
&\le C\lambda(2-\alpha_n)\left(\frac{\nu+1}{\delta}\right)\left(\int\limits_{D_{i-1}^c}(u_n(y)-u(y))^2\int\limits_{D_{i-1}}|x-y|^{-d-\alpha_n+2}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac12}\\
&\le C(\lambda,\Omega)\left(\frac{\nu+1}{\delta}\right)\|u_n-u\|_{L^2(D_{i-1})}\\
&\le C(\lambda,\Omega)\left(\frac{\nu+1}{\delta}\right)\|u_n-u\|_{L^2(\mathbb{R}^d)}
\end{aligned}
$$

Next we look at the second term on the right-hand side of (4.23). Recall that $v_{i_n} = u+\phi_i(u_n-u)$. Using the definition of G_n, we obtain

$$\begin{aligned} G_n(v_{i,n}+u_n, D_i) &\leq G_n(u, D_i) + G_n(u_n, D_i) + G_n(\phi_i(u_n-u), D_i) \\ &= a+b+c \end{aligned}$$

Since

$$\lim_{n\to\infty} F^{\alpha_n}(u_n) = F(u) \leq A$$

for a constant $A>0$ depending only on u, b is bounded for all $n \geq N_0 \in \mathbb{N}$. As explained at the beginning of the proof, also the term a is bounded and we can assume $a, b \leq A$. Further, using $\phi_i \leq 1$, we have

$$\begin{aligned} c &\leq 2\left(\int\limits_{D_i} \Gamma_n(\phi_i(u_n-u))\,\mathrm{d}x\right)^{1/2} \\ &\leq C\left(\int\limits_{D_i} \Gamma_n(u_n-u)\,\mathrm{d}x\right)^{1/2} \\ &\quad + \sqrt{2}\left(\iint\limits_{D_i\,\mathbb{R}^d} ((u_n-u)(x)+(u_n-u)(y))^2(\phi(x)-\phi(y))^2 k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2} \\ &\leq C(A) + \left(\int\limits_{D_i}(u_n-u)(x)^2\int\limits_{\mathbb{R}^d}(\phi(x)-\phi(y))^2k^n(x,y)\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2} \\ &\quad + \left(\int\limits_{\mathbb{R}^d}(u_n-u)(y)^2\int\limits_{D_i}(\phi(x)-\phi(y))^2k^n(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} \\ &\leq C(A) + C(\lambda,\Omega)\left(\frac{\nu+1}{\delta}\right)\|u_n-u\|_{L^2(\mathbb{R}^d)}\,. \end{aligned}$$

Altogether we can estimate the right-hand-side of (4.23) by

$$\begin{aligned} &G_n(v_{i,n}-u_n, D_i)\cdot G_n(v_{i,n}+u_n, D_i) \\ \leq &C\left(\left(\frac{\nu+1}{\delta}\right)^2\left(\|u_n-u\|^2_{L^2(\mathbb{R}^d)}+\|u_n-u\|_{L^2(\mathbb{R}^d)}\right)+(2-\alpha_n)\Big(\int\limits_{\mathcal{D}_i}\Gamma_n(u_n-u)\,\mathrm{d}x\Big)^{1/2}\right), \end{aligned}$$

where the constant $C>1$ depends only on λ, Ω and A. Here we use that

$$\left(\int_{\mathcal{D}_i}\Gamma_n(u_n-u)\,\mathrm{d}x\right)^{1/2} \leq 2A$$

and assumed without loss of generality that $\frac{\nu+1}{\delta}>1$. Adding all this inequalities, we obtain for all $i=1,..\nu$:

$$F_n(v_{i,n},\Omega) - F_n(u_n,\Omega) \leq \lambda(2-\alpha_n)\int\limits_{\Omega\setminus K}\int\limits_{\mathbb{R}^d}(u(x)-u(y))^2\,|x-y|^{-d-\alpha_n}\,\mathrm{d}y\,\mathrm{d}x$$

$$+ C\left(\frac{\nu+1}{\delta}\right)^2 \left(\|u_n - u\|^2_{L^2(\mathbb{R}^d)} + \|u_n - u\|_{L^2(\mathbb{R}^d)}\right)$$
$$+ C(2-\alpha_n)\left(\iint_{\mathcal{D}_i\,\mathbb{R}^d} ((u_n(x)-u(x)) - (u_n(y)-u(y)))^2 \,|x-y|^{-d-\alpha_n} \,\mathrm{d}y\,\mathrm{d}x\right)^{1/2},$$

where the constant C depends on λ, Ω, α_0 and A.

Summing up this inequality for all $i = 1, ..\nu$, dividing by ν and $\sum_{i=1}^n \sqrt{a_i} \le \sqrt{n\sum_{i=1}^n a_i}$ yields

$$\sum_{i=1}^{\nu} F_n(v_{i,n},\Omega) - \nu F_n(u_n,\Omega) < \nu\lambda(2-\alpha_n) \int_{\Omega\setminus K}\int_{\mathbb{R}^d} (u(x)-u(y))^2 \,|x-y|^{-d-\alpha_n} \,\mathrm{d}y\,\mathrm{d}x$$
$$+ \nu C\left(\frac{\nu+1}{\delta}\right)^2 \left(\|u_n - u\|^2_{L^2(\mathbb{R}^d)} + \|u_n - u\|_{L^2(\mathbb{R}^d)}\right)$$
$$+ \sqrt{\nu} C(2-\alpha_n)\left(\iint_{\Omega\,\mathbb{R}^d} ((u_n(x)-u(x)) - (u_n(y)-u(y)))^2 \,|x-y|^{-d-\alpha_n} \,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}$$

Hence, we find $v_n = v_{i,n}$, such that

$$F_n(v_n,\Omega) - F_n(u_n,\Omega) \le \lambda(2-\alpha_n) \int_{\Omega\setminus K}\int_{\mathbb{R}^d} (u(x)-u(y))^2 \,|x-y|^{-d-\alpha_n} \,\mathrm{d}y\,\mathrm{d}x$$
$$+ C\left(\frac{\nu+1}{\delta}\right)^2 \left(\|u_n - u\|^2_{L^2(\mathbb{R}^d)} + \|u_n - u\|_{L^2(\mathbb{R}^d)}\right)$$
$$+ \frac{1}{\sqrt{\nu}} C(2-\alpha_n)\left(\iint_{\Omega\,\mathbb{R}^d} ((u_n(x)-u(x)) - (u_n(y)-u(y)))^2 \,|x-y|^{-d-\alpha_n} \,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}$$
$$= (\clubsuit) + (\spadesuit) + (\bigstar)$$

The right-hand side can be made arbitrary small. Fix $\varepsilon > 0$. Choosing $K \Subset \Omega$, such that $(\clubsuit) < \frac{\varepsilon}{3}$, then ν such that $(\bigstar) < \frac{\varepsilon}{3}$ and then n, such that $(\spadesuit) < \frac{\varepsilon}{3}$.

Define a sequence (v_n), $v_n = \frac{1}{\nu}\sum_{i=1}^{\nu} v_{i,n}$. Then $v_n \in L^2_{g,\Omega}(\mathbb{R}^d)$, $v_n \to u$ in $L^2(\mathbb{R}^d)$ and

$$\limsup_{\alpha_n \to 2^-} F^{\alpha_n}(v_n) \le \limsup_{\alpha_n \to 2^-} F^{\alpha_n}(u_n) \le F(u).$$

This finishes the proof. □

Remark 4.9. Note that the functionals F^α defined by (4.10) does not converge pointwise to the functional F given by (4.11). We refer to Chapter 2 Example 2 for a counter example. We like to point out that this is caused by the area of integration $(\Omega^c \times \Omega^c)^c$ in the definition of F^α, which represents the regularity properties of the function under consideration.

4.3. Application of $\Gamma-$Convergence

In this section we apply the Γ-convergence results obtained in the previous section to proof the main result of this chapter, the phase transition form nonlocal to local Dirichlet problems and the convergence of the associated solutions. In the sequel, we construct a family of kernels k^α from a given positive definite matrix A. Unfortunately we are not able to approximate an arbitrary given second order operator, since there arises an extra term in the computation of the limit matrix.

Proof of Theorem 4.2. Without loss of generality we assume $\alpha_0 > 1$. For fixed α, the nonlocal Dirichlet problem has a unique variational solution $u_\alpha \in V^{k_\alpha}(\Omega|\mathbb{R}^d)$, see Proposition 3.9. Using the comparability assumptions (4.6) and (4.7) we obtain $u_\alpha \in V_g^{\alpha/2}(\Omega|\mathbb{R}^d)$.

Now let u_n be the solution corresponding to some sequence α_n converging to 2. Then there is an $N \in \mathbb{N}$ such that $\alpha_n > \alpha_0$ for all $n \geq N$. Since $V_g^{\alpha/2}(\Omega|\mathbb{R}^d) \hookrightarrow V_g^{\alpha_0/2}(\Omega|\mathbb{R}^d)$ compact, the sequence $(u_n)_{n\geq N}$ is precompact in in $V_g^{\alpha_0/2}(\Omega|\mathbb{R}^d)$, see Corollary 2.24.

Since $A(x)$ is uniformly elliptic the Dirichlet Problem (4.3) has a unique solution. Now Theorem A.8 implies that $u_n \to u$ in $V^{\alpha_0/2}(\Omega|\mathbb{R}^d)$, where u is a minimizer of the functional F_g. Since (4.3) is the Euler equation of F_g, u solves the local Dirichlet problem, see Lemma 4.4. □

We would like to use Theorem 4.2 to approximate a given a local second order boundary value problem

$$\begin{aligned} \partial_j(a_{ij}(\cdot)\partial_i u) &= f && \text{in } \Omega \\ u &= g && \text{on } \partial\Omega, \end{aligned}$$

by a family of nonlocal ones as follows: Since the boundary data $g \in H^1(\Omega)$, we can extend g to the whole space, such that $g \in H^1(\mathbb{R}^d)$. This implies $g \in V(\Omega|\mathbb{R}^d)$. Next we have to chose an appropriate family k^α of kernels depending on the given matrix (a_{ij}), such that

$$\lim_{\alpha\to 2^-} \int_0^1 \int_{\mathbb{S}^{d-1}} t^{d+1}\sigma_i\sigma_j k^\alpha(x, x+t\sigma)\,\mathrm{d}\sigma\,\mathrm{d}t = a_{ij}(x)\,. \tag{4.24}$$

One natural approach is to use the coefficients a_{ij} to define the behavior of $k^\alpha(x,y)$ in a specific direction and symmetrize in the obvious way: For given $\alpha \in (0,2)$ and $x, y \in \Omega$ we define

$$k^\alpha(x,y) = \mathcal{K}_d\alpha(2-\alpha)\,|x-y|^{-d-\alpha} \sum_{i,j=1}^{d} \frac{a_{ij}(x)-a_{ij}(y)}{2} \frac{(x-y)_i(x-y)_j}{|x-y|^2}\,,$$

where $\mathcal{K}_d$ is a constant depending on the dimension.

Unfortunately, plugging this choice of k^α in (4.24) leads to a coefficient matrix

$$\widetilde{A} = (2A + \operatorname{tr} A\, I_d)\,,$$

with $A = (a_{ij})$, i.e. there appears an additional term $\operatorname{tr} A\, I_d$. More generally we have the following

Proposition 4.10. *Let $n \in \mathbb{N}$ be odd. For $\alpha \in (0,2)$ and $x, y \in (\Omega^c \times \Omega^c)^c$ set*

$$k^\alpha(x,y) = \alpha(2-\alpha)\,|x-y|^{-d-\alpha} \sum_{i,j=1}^{d} \frac{a_{ij}(x) - a_{ij}(y)}{2} \frac{(x-y)_i^n (x-y)_j^n}{|x-y|^{2n}} .$$

Then the family $\mathcal{L}^{k^\alpha}$ of operators generated by the family k^α converges to the second order differential operator $\mathcal{L}$ with a coefficients matrix B given by

$$b_{ij} = \begin{cases} 2\mathcal{K}_d(n+1,n+1)a_{ij} & \text{if } i \neq j \\ \mathcal{K}_d(2n+2,0)a_{ii} + \mathcal{K}_d(n+1,n+1)\sum_{k=1,k\neq i}^{d} a_{kk} & \text{if } i = j\,, \end{cases}$$

where

$$\mathcal{K}_d(p,q) = \frac{2\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)\pi^{d-2/2}}{\Gamma\left(\frac{p+q+d}{2}\right)} .$$

Note that $\mathcal{K}_d(n+1,n+1) >> \mathcal{K}_d(2n+2,0)$ for large n. This allows us to manipulate k^α such that (4.24) holds and k^α is still positive at least when (a_{ij}) is a diagonal matrix. The Idea to use powers of $(x-y)_i(x-y)_j$ goes back to an idea of Bartek Dyda. For the proof we use the following result from [Bak97, section 3] about integration on spheres.

Corollary 4.11. *Let $\gamma = (\gamma_1, ..., \gamma_d) \in \mathbb{N}_0^d$. Define $p_\gamma : \mathbb{R}^d \to \mathbb{R}$ by*

$$p_\gamma(x) = x_1^{\gamma_1} \cdots x_d^{\gamma_d}.$$

Then

$$\int\limits_{\mathbb{S}^{d-1}} p_\gamma(\sigma)\,\mathrm{d}\sigma = \begin{cases} \left(\sum_{i=1}^d \gamma_i + d\right) \frac{\Gamma\left(\frac{\gamma_1+1}{2}\right)\cdots\Gamma\left(\frac{\gamma_d+1}{2}\right)}{\Gamma\left(\frac{\gamma_1+\cdots+\gamma_n+2+d}{2}\right)}, & \text{if } \gamma_i \equiv 0 \mod 2 \text{ for all } i \in \{1,..,d\} \\ 0, & \text{else.} \end{cases}$$

Proof of Proposition 4.10. For all $x, y \in \mathbb{R}^d$ we define

$$K(x,h) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{h_i h_j}{|h|^2}.$$

Note that $k^\alpha(x,y) = (2-\alpha)\,|x-y|^{-d-\alpha}\,(K(x,y-x) + K(y,x-y))$ and that it is sufficient to consider $K(x,h)$, see Remark 4.7. For this choice of K, we have $K(x,h) = K(x,rh)$ for all $x \in \mathbb{R}^d$ and $r \in \mathbb{R}$ and thus

$$\int_{\mathbb{S}^{d-1}} \sigma_k \sigma_l K(x,\sigma)\,\mathrm{d}\sigma = \sum_{i,j=1}^{d} b_{ij}(x) \int_{\mathbb{S}^{d-1}} \sigma_k \sigma_l \sigma_i^n \sigma_j^n\,\mathrm{d}\sigma. \tag{4.25}$$

The assertion follows directly from Corollary 4.11. □

Example 5. Let $n = 1$. By Corollary 4.11 the terms in (4.25) are nonzero if $k = l = i = j$, if $k = i$ and $l = j$ or $k = j$ and $l = i$ or if $i = j$ and $k = l$.
Case 1: $k = i$ and $l = j$ We have that

$$\int_{\mathbb{S}^{d-1}} \sigma_k^2 \sigma_l^2 \, \mathrm{d}\sigma = (4+d) \frac{\Gamma^2(\frac{3}{2})\Gamma^{d-2}(\frac{1}{2})}{\Gamma(3+\frac{d}{2})} = \frac{\frac{1}{4}\Gamma^d(\frac{1}{2})}{\frac{1}{2}\Gamma(2+\frac{d}{2})} = \frac{\pi^d}{(d+2)\Gamma(1+\frac{d}{2})} = \frac{\omega_d}{(d+2)}.$$

Case 2: Let $k = l = i = j$. Then

$$\int_{\mathbb{S}^{d-1}} \sigma_k^4 \, \mathrm{d}\sigma = (4+d) \frac{\Gamma(\frac{5}{2})\Gamma^{d-1}(\frac{1}{2})}{\Gamma(3+\frac{d}{2})} = \frac{\frac{3}{4}\Gamma^d(\frac{1}{2})}{\frac{1}{4}(d+2)\Gamma(1+\frac{d}{2})} = \frac{3\omega_d}{(2+d)}.$$

Case 3: Let $k = l$, $i = j$. As in Case 1 we obtain

$$\int_{\mathbb{S}^{d-1}} \sigma_k^2 \sigma_l^2 \, \mathrm{d}\sigma = \frac{\omega_d}{(d+2)}.$$

Thus

$$\sum_{i,j=1}^{d} \partial_i u(x) \partial_j u(x) \int\limits_{\mathbb{S}^{d-1}} \sigma_i \sigma_j K(x,\sigma) \, \mathrm{d}\sigma = \frac{\omega_d}{d+2} \left\langle (\operatorname{tr} A(x) I_d + 2A(x)) \nabla u(x), \nabla u(x) \right\rangle . \quad (4.26)$$

A formula of the type (4.26) appears in a different context in [Men12, Cor. 2.7], where nonlocal characterizations of Sobolev vector fields are studied. The idea to use Bakers result on the integration over spheres is taken from this work.

5. Homogenization of nonlocal Dirichlet Problem

The aim of this chapter is to develop a setup to obtain a homogenization result for nonlocal integro-differential operators of the form

$$\mathcal{L}u(x) = P.V. \int_{\mathbb{R}^d} (u(x) - u(y))k(x, y)\,\mathrm{d}y\,. \tag{5.1}$$

First we summarize the main ideas of the $\Gamma-$convergence approach to homogenization in the setting of second order differential equations in divergence form.

Afterwards we apply the Γ-convergence approach to the nonlocal case. To deal with problems that arise due to the nonlocality of the operator we use the theory of Dirichlet forms. Nevertheless we do not obtain a homogenization formula in the nonlocal case.

5.1. Homogenization of second order elliptic equations

In this section we briefly recall the Γ-convergence approach of homogenization in the setting of second order elliptic equations. This section should review the main steps to deduce a homogenization formula for second order differential equations, which we want to transfer to the nonlocal setting in Section 5.2. We do not give full proves, since we are only interested in the technical ideas. For the sake of completeness we give references to the literature.

Suppose that Ω is an open smooth subset of $\mathbb{R}^d$ and $Y = (0,1)^d$ is the unit cell in $\mathbb{R}^d$. A function $f : \mathbb{R}^d \to \mathbb{R}$ is called 1-periodic, if $f(x + e_i) = f(x)$ for all $i = 1, .., d$. Let (a_{ij}) be a symmetric $d \times d$ matrix of functions $a_{ij} : \mathbb{R}^d \to \mathbb{R}$, such that

$$\lambda\,|\xi|^2 \le \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \le \Lambda\,|\xi|^2 \tag{5.2}$$

for some constants $\lambda, \Lambda > 0$ and all $x \in \mathbb{R}^d$. Further let $a_{ij} : \mathbb{R}^d \to \mathbb{R}$ be 1-periodic for all $i, j \in \{1, ..., d\}$. We define

$$a^{\varepsilon}_{ij}(x) = a_{ij}\left(\frac{x}{\varepsilon}\right)$$

and consider for $f \in L^2(\Omega)$ the Dirichlet problem

$$\partial_i(a^{\varepsilon}_{ij}(\cdot)\partial_j u_\varepsilon) = f \quad \text{in } \Omega, \tag{5.3a}$$

$$u_\varepsilon = 0 \quad \text{on } \partial\Omega\,. \tag{5.3b}$$

For $\varepsilon \to 0$ the family of solutions (u_ε) converges to the solution u_0 of the so-called homogenized equation

$$\partial_i(a^*_{ij}\partial_j u_0) = f \quad \text{in } \Omega, \tag{5.4a}$$
$$u_0 = 0 \quad \text{on } \partial\Omega, \tag{5.4b}$$

where (a^*_{ij}) are the effective or homogenized coefficients. One says that u_0 solves the homogenized equation or that homogenization takes place. This can be proved by several techniques, e.g. asymptotic expansion, two scale convergence, the energy method or by use of Γ-convergence.

We will give the ideas of the approach using Γ-convergence and follow the presentation in [DM93], [BD98]. Usually, for this kind of problems the Γ-limit is taken with respect to the strong topology of $L^2(\Omega)$, since this implies precompactness of minimizers of the corresponding Dirichlet energy. Instead of looking at the equation (5.3), we consider the corresponding energy functionals

$$\widetilde{F}_\varepsilon(u) = \begin{cases} \int\limits_\Omega f_\varepsilon(x, \nabla u(x))\,\mathrm{d}x - \int\limits_\Omega u(x)f(x)\,\mathrm{d}x, & \text{if } u \in H^1_0(\Omega), \\ +\infty, & \text{else.} \end{cases}$$

where

$$f_\varepsilon(x, \xi) = \sum_{i,j=1}^{d} a^\varepsilon_{ij}(x)\xi_i\xi_j.$$

Note that a minimizer of F_ε in $H^1_0(\Omega)$ solves (5.3). As usual we neglect the forcing term in the computation of the Γ-limit and consider instead the functionals

$$F_\varepsilon(u) = \begin{cases} \int\limits_\Omega f_\varepsilon(x, \nabla u)\,\mathrm{d}x, & \text{if } u \in H^1_0(\Omega), \\ +\infty, & \text{else.} \end{cases}$$

Henceforward we will use the extension of a functional to $+\infty$ outside $H^1_0(\Omega)$, without writing it. By the compactness of Γ-convergence, c.f. Proposition A.9, there is a sequence (ε_k) and a functional F_0, such that

$$F_0(u) = \Gamma - \lim_{k\to\infty} F_{\varepsilon_k}(u).$$

for all $u \in L^2(\Omega)$.

Step 1: The first step is to prove an integral representation of the limit functional $F_0(u)$, i.e. there is a function $f_0 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that

$$F_0(u) = \int\limits_\Omega f_0(x, \nabla u)\,\mathrm{d}x.$$

To prove this one uses the localization method. Let $\mathcal{A}(\Omega)$ be the family of all open subsets of Ω. Now we consider

$$F_\varepsilon(u, A) = \int\limits_A f_\varepsilon(x, \nabla u)\,\mathrm{d}x$$

as a functional on $L^2(\Omega)$ and as a set function $F_\varepsilon(u, \cdot)$ for $A \in \mathcal{A}(\Omega)$. By a compactness argument there is a dense countable family $(A_i) \in \mathcal{A}(\Omega)$ and a sequence (ε_n) converging to zero, such that

$$F_0(u, A_i) = \Gamma - \lim_{n\to\infty} F_{\varepsilon_n}(u, A_i)$$

exists for all $i \in \mathbb{N}$. By the *DeGiorgi-Letta measure criterion* introduced in [DGL77] one concludes that for fixed u the set function $F_0(u,\cdot)$ is the restriction of a Borel measure to $\mathcal{A}(\Omega)$ and thus obtain an integral representation. By approximation with piecewise affine functions one proves that the integral representation is independent of u and thus

$$F_0(u,A) = \int_A f_0(x,\nabla u)\,\mathrm{d}x$$

for all $u \in L^2(\Omega)$. Furthermore $F_0(u,\cdot)$ is inner regular, i.e. $F_0(u,A) = \sup\{F_0(u,A')|A' \Subset A\}$ and we recover

$$F_0(u) = F_0(u,\Omega) = \int_\Omega f(x,\nabla u)\,\mathrm{d}x$$

for all $u \in L^2(\Omega)$, where $F_0(u) = +\infty$ if $u \notin H_0^1(\Omega)$. The function f_0 satisfies the same growth conditions as f_ε. For details on the localization method we refer to [DM93, Chap. 20] or [BD98, Chap.9 et seq.]. The integral representation is proved in [BD98, Thm. 12.5].

Step 2: In this step we prove that the function f_0 in the limit functional F_0 can be chosen independent of of the first variable. To do so, one first proves that the localized functionals $F_\varepsilon(\cdot,A)$ are translation invariant, i.e. for every $y \in \mathbb{R}^d$

$$F_0(u,A) = F_0(\tau_y u, \tau_y A),$$

where $\tau_y u(x) = u(x-y)$ and $\tau_y A = \{x \in \mathbb{R}^d : x - y \in A\}$. This follows from the periodicity of f_ε ([DM93, Thm. 24.1] / [BD98, Prop. 14.3]). The translation invariance of F and the integral representation obtained in Step 1 now yield

$$f_0(y,\nabla u) = \lim_{r\to 0} \frac{F_0(u,B_r(y))}{|B_r|} = \lim_{r\to 0} \frac{F_0(\tau_\rho u, B_r(y+\rho))}{|B_r|} = f(y+\rho,\nabla u)$$

and thus f_0 is independent of the first variable. Moreover f_0 can be expressed as the minimizer of the energy of $f = f_1$ over one periodicity cell. To be precise

$$f_0(\xi) = \min_{\phi \in W(\xi,Y)} \frac{1}{|Y|} \int_Y f(x, \xi + \nabla\phi(x))\,\mathrm{d}x, \tag{5.5}$$

where $W(\xi,Y)$ is the space of all $\phi \in H^1(\mathbb{R}^d)$ such that $\nabla\phi$ is Y-periodic and $\int_Y \nabla\phi\,\mathrm{d}x = \xi$. The minimum in (5.5) is achieved due to the growth estimates (5.2) on f. This expression for f_0 does not depend on the Γ-converging subsequence and thus by the Urysohn property ([DM93, Prop. 8.3]) for every sequence (ε_k) converging to zero F_{ε_k} Γ-converges to F_0. This is the crucial step to get from the level of Γ-converging subsequences to the Γ-convergence of the whole family F_ε.

Step 3: The minimizer of (5.5) can be given in terms of an Euler condition, which leads to a corrector result for the homogenized equation (5.4). The following is equivalent:

1. ϕ minimizes (5.5).
2. $\phi \in W(\xi,Y)$ and

$$\int_Y \left(\sum_{i=1}^d \frac{\partial f}{\partial \xi_i}(x,\nabla\phi)\partial_j v \right) \mathrm{d}x = 0 \tag{5.6}$$

   for all $v \in H^1_\sharp(Y)$, the space of all Y-periodic functions in $H^1(\mathbb{R}^d)$.

3. $\phi \in H^1_{\text{loc}}(\mathbb{R}^d)$, $\nabla\phi$ is Y-periodic, $\int_Y \nabla\phi = \xi$ and
$$\operatorname{div}(f_\xi(y, \nabla\phi)) = 0 \tag{5.7}$$
on $\mathbb{R}^d$ in the weak sense.

For a proof of this we refer to [DM93, Prop. 25.1, Prop 25.3].

From this we obtain the homogenization result as follows. For $\xi \in \mathbb{R}^d$ let $w_\xi \in H^1_{\text{loc}}(\mathbb{R}^d)$ be a solution to (5.7). It is easy to check that w_ξ is unique up to an additive constant.

Writing $\xi = \sum_{i=1}^d \xi_i e_i$, we obtain
$$w_\xi(y) = \xi_1 w_{e_1}(y) + \xi_2 w_{e_2}(y) + \cdots + \xi_d w_{e_d}(y) + c$$
for almost every $y \in \mathbb{R}^d$. By (5.6) we obtain for every $\xi \in \mathbb{R}^d$
$$f_0(\xi) = \int_Y \left(\sum_{i,j=1}^d a_{ij} \partial_i w_\xi \partial_j w_\xi \right) \mathrm{d}y = \sum_{k,l=1}^d a^*_{kl} \xi_k \xi_l,$$
where
$$a^*_{kl} = \int_Y \left(\sum_{i,j=1}^d a_{ij} \partial_i w_{e_k} \partial_j w_{e_l} \right) \mathrm{d}y. \tag{5.8}$$
Thus the computation of f_0 is reduced to the solution of d boundary value problems for $\xi = e_1, .., e_d$. (5.8) is called the corrector equation for the effective coefficients.

Let (u_ε) be a sequence of solutions to (5.3) and thus a minimizing sequence of $\widetilde{F}_\varepsilon(u)$. Since the functional $\widetilde{F}_\varepsilon$ Γ-converge to the functional
$$\widetilde{F}_0(u) = \begin{cases} \int_\Omega a^*_{ij} \partial_i u \partial_j u \,\mathrm{d}x - \int_\Omega u(x) f(x) \,\mathrm{d}x & \text{if } u \in H^1_0(\Omega), \\ +\infty & \text{else,} \end{cases}$$
we obtain that u_ε converges to the unique solution u_0 of (5.4) in $L^2(\Omega)$.

5.2. Homogenization for elliptic nonlocal operators

In this section we define a setup for the homogenization of nonlocal equations in divergence form. To prove a homogenization result we try to adapt the approach introduced in Section 5.1 to the nonlocal setting. Using the representation result by Beurling-Deny for regular Dirichlet forms we obtain a representation of the limit functional on the level of subsequences. Nevertheless we are not able to deduce a homogenization formula. Since the order $\alpha \in (0, 2)$ of the appearing operator in this section is fixed, we suppress the norming factor $\alpha(2 - \alpha)$, which is substantial in the previous chapters, in all computations.

Throughout this section we assume that Ω is an open, bounded and smooth subset of $\mathbb{R}^d$. We consider the equation
$$\mathcal{L}u = f \quad \text{in } \Omega, \tag{5.9a}$$
$$u = 0 \quad \text{on } \Omega^c, \tag{5.9b}$$

where $\mathcal{L}$ is given by (5.1). We assume that the kernel $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ is measurable, symmetric and satisfies for almost every $x, y \in \mathbb{R}^d$ the pointwise comparability condition

$$\lambda \, |x-y|^{-d-\alpha} \leq k(x,y) \leq \lambda^{-1} \, |x-y|^{-d-\alpha} \tag{5.10}$$

for some $\lambda > 0$ and $\alpha \in (0, 2)$.

Further we suppose that we can write $k(x, y) = K(x, x - y)$ and that K is one-periodic in the first argument. From this we define kernels K_ε by

$$K_\varepsilon(x,h) = \varepsilon^{-d-\alpha} K(\frac{x}{\varepsilon}, \frac{h}{\varepsilon}).$$

In the same way we introduce

$$k_\varepsilon(x,y) = \varepsilon^{-d-\alpha} k(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}).$$

Depending on the context, we either use the notation with k_ε or the notation with K_ε. Note that by construction k_ε is ε-periodic in both variables.

Now for $\varepsilon > 0$ we consider the equation

$$\mathcal{L}^\varepsilon u_\varepsilon = f \quad \text{ in } \Omega, \tag{5.11a}$$
$$u_\varepsilon = 0 \quad \text{ on } \Omega^c, \tag{5.11b}$$

where

$$\begin{aligned} \mathcal{L}^\varepsilon u(x) &= P.V. \int_{\mathbb{R}^d} (u(x) - u(x+h)) K_\varepsilon(x,h) \, \mathrm{d}h \\ &= P.V. \int_{\mathbb{R}^d} (u(x) - u(y)) k_\varepsilon(x,y) \, \mathrm{d}y \, . \end{aligned}$$

Note that for simplicity we consider only homogeneous boundary data equal zero.

For any $\varepsilon > 0$ (5.11) has a weak solution by Proposition 3.9, i.e. for any $f \in L^2(\Omega)$ there is $u_\varepsilon \in V_0^{\alpha/2}(\Omega|\mathbb{R}^d)$, such that

$$\iint_{(\Omega^c \times \Omega^c)^c} (u(x) - u(y))(v(x) - v(y)) k_\varepsilon(x,y) \, \mathrm{d}y \, \mathrm{d}x = \int_\Omega v(x) f(x) \, \mathrm{d}x$$

for all $v \in V_0^{\alpha/2}(\Omega|\mathbb{R}^d)$. Note that since the function u and v vanish outside Ω we can replace the integration on the left-hand side by $\mathbb{R}^d \times \mathbb{R}^d$.

Question: What happens if $\varepsilon \to 0$? Does u_ε converge to some u_0 and does u_0 solve some effective equation?

The idea is, to use the approach of Γ-convergence as outlined above in Section 5.1. Unfortunately, due to the nonlocal character of the operator, some of the above techniques can not be applied. For example the localization technique does not provide an accurate integral representation theorem for the limit functional. The work of Mosco [Mos94] gives an alternative approach using the theory of Dirichlet forms and in particular the representation theory for regular Dirichlet forms due to Beurling and Deny.

Let us consider the energy functionals $\widetilde{F}_\varepsilon^0 : L^2(\mathbb{R}^d) \to [0, \infty]$ associated to (5.11) defined by

$$\widetilde{F}_\varepsilon^0(u) = \begin{cases} \frac{1}{4} \iint\limits_{(\Omega^c\times\Omega^c)^c} (u(x) - u(y))^2 k_\varepsilon(x, y)\,\mathrm{d}y\,\mathrm{d}x - \int\limits_\Omega u(x)f(x)\,\mathrm{d}x, & \text{if } u \in V_0^{\alpha/2}(\Omega|\mathbb{R}^d), \\ +\infty, & \text{else.} \end{cases}$$

As proved in Lemma 4.4 a minimizer of this functionals is a weak solution to (5.11). Note that we can replace $(\Omega^c \times \Omega^c)^c$ by $\mathbb{R}^d \times \mathbb{R}^d$ in the energy functional, since $u = 0$ on Ω^c almost everywhere. As in the local case we can neglect the forcing term due to Proposition A.3 and consider instead the functionals $F_\varepsilon^0 : L^2(\mathbb{R}^d) \to [0, \infty]$ defined by

$$F_\varepsilon^0(u) = \begin{cases} \frac{1}{4} \iint\limits_{(\Omega^c\times\Omega^c)^c} (u(x) - u(y))^2 k_\varepsilon(x, y)\,\mathrm{d}y\,\mathrm{d}x, & \text{if } u \in V_0^{\alpha/2}(\Omega|\mathbb{R}^d), \\ +\infty, & \text{else.} \end{cases}$$

We use Γ-convergence with respect to the strong topology of $L^2(\mathbb{R}^d)$. This implies the precompactness of a minimizing sequence u_ε, since the minimizers of F_ε^0 are uniformly bounded in $V_0^{\alpha/2}(\Omega|\mathbb{R}^d)$, which follows from Corollary 2.24 and (5.12) below.

First we consider the functional without boundary condition, i.e.

$$F_\varepsilon(u) = \begin{cases} \frac{1}{4} \iint\limits_{(\Omega^c\times\Omega^c)^c} (u(x) - u(y))^2 k_\varepsilon(x, y)\,\mathrm{d}y\,\mathrm{d}x, & \text{if } u \in V^{\alpha/2}(\Omega|\mathbb{R}^d), \\ +\infty, & \text{else.} \end{cases}$$

Analogously to Theorem 4.8 we can recover the Γ-convergence of the restricted functionals F_ε^0 afterwards.

The compactness property of Γ-convergence, Proposition A.9, implies that there is a sequence (ε_k) and a functional F_0 on $L^2(\mathbb{R}^d)$ such that

$$F_0(u) = \Gamma - \lim_{k\to\infty} F_{\varepsilon_k}(u).$$

5.2.1. An application of Beurling Deny

To obtain an integral representation of the limit functional we use the above mentioned result of Mosco concerning the Γ-convergence of Dirichlet forms. He proves that the Γ-limit of a sequence of Dirichlet form is again a Dirichlet form. Further, if the sequence satisfies some lower and upper bounds, so does the limit form.

To apply the theory of Dirichlet forms, we consider the functionals F_ε as bilinear forms,

$$F_\varepsilon(u) = \mathcal{E}_\varepsilon(u, u),$$

where the bilinear form $\mathcal{E}_\varepsilon$ is defined as

$$\mathcal{E}_\varepsilon(u, v) = \frac{1}{4} \iint\limits_{(\Omega^c\times\Omega^c)^c} (u(x) - u(y))(v(x) - v(y))k_\varepsilon(x, y)\,\mathrm{d}y\,\mathrm{d}x.$$

Due to the inequality (5.10)

$$\lambda \mathcal{E}^\alpha(u,u) \le \mathcal{E}_\varepsilon(u,u) \le \lambda^{-1}\mathcal{E}^\alpha(u,u), \tag{5.12}$$

where the reference form $\mathcal{E}^\alpha$ is defined as

$$\mathcal{E}^\alpha(u,v) = \iint\limits_{(\Omega^c\times\Omega^c)^c} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x.$$

By (5.10) $(\mathcal{E}^\alpha, V^{\alpha/2}(\Omega|\mathbb{R}^d))$ and thus by (5.12) also $(\mathcal{E}_\varepsilon, V^{\alpha/2}(\Omega|\mathbb{R}^d))$ are regular Dirichlet forms on $L^2(\mathbb{R}^d)$.

The following result yields that the Γ-limit of a sequence of Dirichlet forms is again a Dirichlet form. Note that Mosco actually proves that the Γ-limit of a sequence of Markovian forms is a Dirichlet form, since closedness of the limit form follows from the lower semi-continuous of the Γ-limit.

Theorem 5.1 (Thm. 2.8.1,[Mos94])**.** *Let (a_n) be a sequence of Dirichlet forms on a measure space H. Then there is a Dirichlet form a_∞ on H and a subsequence (n_k) such that (a_{n_k}) Γ-converge to a_∞ in H as $k\to\infty$.*

Further the following comparison criteria hold for the Γ-converging forms.

Proposition 5.2 (Cor.2.10.3, [Mos94])**.** *Let (a_n) be a sequence of Dirichlet forms in H, $\widetilde{a}$ a Dirichlet form in H and $0<\lambda\le\Lambda$ constants such that*

$$\lambda\widetilde{a}(u,u) \le a_n(u,u) \le \Lambda\widetilde{a}(u,u)$$

for every $n\in\mathbb{N}$ and every $u\in H$. Moreover let (a_n) Γ-converge to a Dirichlet form a_∞ in H as $n\to\infty$. Then

$$\lambda\widetilde{a}(u,u) \le a_\infty(u,u) \le \Lambda\widetilde{a}(u,u)$$

for every $u\in H$. Further $\mathcal{D}(a_\infty)=\mathcal{D}(\widetilde{a})$.

Due to this the limit functional F_0 can be identified with a Dirichlet form satisfying the same bounds as F_ε.

Corollary 5.3. *Let ε_n be a sequence converging to 0, such that*

$$F_0(u) = \Gamma - \lim_{n\to\infty} F_{\varepsilon_n}(u) = \Gamma - \lim_{n\to\infty} \mathcal{E}_{\varepsilon_n}(u,u).$$

Then $F_0(u)=\mathcal{E}_0(u,u)$ for all $u\in L^2(\mathbb{R}^d)$, where $\mathcal{E}_0$ is a regular Dirichlet form on $L^2(\mathbb{R}^d)$ satisfying the bounds

$$\lambda\mathcal{E}^\alpha(u,u) \le \mathcal{E}_0(u,u) \le \Lambda\mathcal{E}^\alpha(u,u).$$

Proof. The corollary follows directly from (5.12), Theorem 5.1 and Proposition 5.2 for $H = L^2(\mathbb{R}^d)$. □

The representation theory for regular Dirichlet forms introduced by Beurling and Deny allows us to characterize the limit form. Further, using the comparison criteria of Proposition 5.2 we can deduce that the limit form is again a pure nonlocal form.

Proposition 5.4. *Let $u, v \in C_c(\Omega) \cap V^{\alpha/2}(\Omega|\mathbb{R}^d)$. The Dirichlet form $\mathcal{E}_0$ on $L^2(\mathbb{R}^d)$ can be expressed as*

$$\mathcal{E}_0(u,v) = \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))(v(x)-v(y))J(\,\mathrm{d}x,\,\mathrm{d}y).$$

where J is a Radon measure on $\mathbb{R}^d \times \mathbb{R}^d \setminus \mathrm{diag}$. Here $\mathrm{diag} = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \,|x = y\}$.

Proof. By the formula of Beurling-Deny [FŌT94, Thm. 3.2.1] for $u, v \in C_c(\mathbb{R}^d)$ the limit form $\mathcal{E}_0$ can be expressed as

$$\begin{aligned}\mathcal{E}_0(u,v) =& \mathcal{E}^{(c)}(u,v) + \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} (u(x)-u(y))(v(x)-v(y))J(\,\mathrm{d}y,\,\mathrm{d}x) \\ &+ \int\limits_{\mathbb{R}^d} u(x)v(x)k(\,\mathrm{d}x).\end{aligned}$$

Here J is a positive symmetric Radon measure on $\mathbb{R}^d \times \mathbb{R}^d \setminus \mathrm{diag}$, k is a positive Radon measure on $\mathbb{R}^d$ and $\mathcal{E}^{(c)}$ is the local part of the form. It follows from Corollary 5.3 and [Lej78, Prop. 1.5.5] that the local part is zero.

Next we prove that also the killing measure k is zero. W.l.o.g. we assume that $0 \in \Omega$. Consider for $R > 0$ the function $u_R(x) = \mathbb{1}_{B_R(0)} \in L^2(\mathbb{R}^d)$ and assume that $\Omega \subset B_{R/2}(0)$. Then

$$\mathcal{E}^\alpha(u,u) \leq \iint\limits_{B_R^c\,\Omega} \frac{1}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x \leq C\,|\Omega|\,R^{-\alpha}.$$

Now we obtain

$$\int\limits_{\Omega} u_R^2(x)k(\,\mathrm{d}x) \leq \mathcal{E}_0(u_R,u_R) \leq \lambda^{-1}C\,|\Omega|\,R^{-\alpha} \to 0$$

as $R \to \infty$ and thus $k \equiv 0$. □

Remark 5.5.
1. Since the integrand in the double integral term of $\mathcal{E}_0$ is symmetric, we can assume that also the measure $J(\,\mathrm{d}x,\,\mathrm{d}y)$ is symmetric. By the comparability to $\mathcal{E}^\alpha$ we obtain that

   $$\iint\limits_{K\,\mathbb{R}^d} \left(1 \wedge |x-y|^2\right) J(\,\mathrm{d}x,\,\mathrm{d}y) < \infty \text{ for all compact } K \subset \mathbb{R}^d.$$

   Then by disintegration (cf. [Kal02, Chap. 5]) and by the symmetry of J we find a family of measures $j(x,\cdot)$, such that

   $$\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} u(x)-u(y))(v(x)-v(y))J(\,\mathrm{d}x,\,\mathrm{d}y) = \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} u(x)-u(y))(v(x)-v(y))j(x,\,\mathrm{d}y)\,\mathrm{d}x,$$

   cf. [SU12, Sec. 2].
2. Note that the integration in the double integral in the definition of $\mathcal{E}_0$ is over $\mathbb{R}^d \times \mathbb{R}^d$, which is in contrast to $\mathcal{E}_\varepsilon$. Since we restrict ourself to the case of boundary data equal zero, we can disregard this. When we want to consider nonzero boundary data, a more accurate study of the support of J is needed.

To connect the limit functional to the solution of a boundary value problem with homogeneous boundary data zero, we have to prove that the Γ-limit is stable under the restriction to functions that vanish outside Ω. This can be proved exactly the same way as Theorem 4.8.

Proposition 5.6. *Let ε_n be a sequence converging to* 0*, such that*

$$F_0 = \Gamma - \lim_{n\to\infty} F_{\varepsilon_n}.$$

Then

$$F_0^0(u) = \Gamma - \lim_{n\to\infty} F_{\varepsilon_n}^0(u),$$

where

$$F_0^0(u) = \begin{cases} \mathcal{E}_0(u,u), & \text{if } u \in V_0^{\alpha/2}(\Omega|\mathbb{R}^d) \\ +\infty, & \text{else.} \end{cases}$$

Proof. The proof is analogous to the proof of Theorem 4.8. □

Remark 5.7. Note that we do not use the periodicity of k_ε in the whole section. On this account we only obtain the existence of an abstract limit functional, that can be represented by a Dirichlet form. Nevertheless we can not connect the abstract jumping measure J to a given family k_ε.

5.2.2. Additivity and translation invariance of localized functionals

In this section we introduce a localization method for the nonlocal setting. We prove subadditivity of the localized limit functionals for disjoint open sets. Using the periodicity of K_ε we prove the translation invariance of the localized functionals, which is the crucial step to identify an explicit formula for the limit functionals in the local setting, see Step 2 of Section 5.1. Nevertheless we are not able to connect the integral representation obtained in Subsection 5.2.1 to the family of kernels k_ε.

Let us consider the localized functionals $F_\varepsilon : L^2(\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,\infty]$ defined by

$$F_\varepsilon(u,A) = \begin{cases} \iint\limits_{A\,\mathbb{R}^d} (u(x)-u(y))^2 K_\varepsilon(x,h)\,\mathrm{d}h\,\mathrm{d}x, & \text{if } u \in V^{\alpha/2}(A|\mathbb{R}^d), \\ +\infty, & \text{else.} \end{cases}$$

As in the local case we can assume that there is a countable family $\mathcal{V} \in \mathcal{A}(\Omega)$ and a sequence (ε_n), such that

$$F(u,A) = \Gamma - \lim_{n\to\infty} F_{\varepsilon_n}(u,A)$$

exists for all $A \in \mathcal{V}$ using the compactness of Γ-convergence. For example we can take $\mathcal{V}$ as the family of all open polyrectangles with rational vertices.

Note that superadditivity is preserved by Γ-convergence, cf. [DM93, Prop. 16.12]. The idea to prove subadditivity is the following: Let $A, B \in \mathcal{A}(\Omega)$ and $u \in L^2(\mathbb{R}^d)$. Without loss of generality we can assume that $F(u,A)$ and $F(u,B)$ are finite. By the definition of Γ-convergence we find sequences $(v_j) \in L^2(\mathbb{R}^d)$ and $(w_j) \in L^2(\mathbb{R}^d)$, such that

$$F(u,A) = \lim_{j\to\infty} F_j(v_j, A), \quad F(u,B) = \lim_{j\to\infty} F_j(w_j, B).$$

Now the goal is to construct a sequence (u_j), such that

$$F_j(u_j, A \cup B) \leq F_j(v_j, A) + F_j(w_j, B) + R_j \quad \text{where } R_j \overset{j\to\infty}{\longrightarrow} 0.$$

In the local case one uses the sequence $u_j = \phi v_j + (1-\phi)w_j$, where $\phi \in C_0^\infty(A)$, $\phi = 1$ on $A' \Subset A$.

We use the following Ansatz: Define $u_{i,n}$ by

$$u_{i,n} = u + \phi_i(v_n - u) + \psi_i(w_n - u)$$

where $\phi_i \in C_c^\infty(A_i)$, $A_i \Subset A$ and $\psi_i \in C_c^\infty(B_i)$, $B_i \Subset B$.
Since $u_{i,n} \to u$ in $L^2(\mathbb{R}^d)$ we obtain

$$F(u, A \cup B) \leq \liminf_{n\to\infty} F_n(u_{i,n}, A \cup B). \tag{5.13}$$

Further for $\varepsilon > 0$ there is $N \in \mathbb{N}$, such that for all $n \geq N$:

$$F(u, A) + \varepsilon \geq F_n(v_n, A) \tag{5.14}$$
$$F(u, B) + \varepsilon \geq F_n(w_n, B). \tag{5.15}$$

If the left-hand side is finite this implies that for some $N \in \mathbb{N}$ and all $n \geq N$ there is $K \geq 1$, such that

$$\iint\limits_{A\,\mathbb{R}^d} (v_n(x) - v_n(y))^2 \, |x-y|^{-d-\alpha} < K \tag{5.16}$$
$$\iint\limits_{B\,\mathbb{R}^d} (w_n(x) - w_n(y))^2 \, |x-y|^{-d-\alpha} < K. \tag{5.17}$$

Now we are in the position to prove the subadditivity.

Lemma 5.8. *Let $A, B \in \mathcal{A}(\Omega)$ with $A \cap B = \emptyset$. Then*

$$F(u, A \cup B) \leq F(u, A) + F(u, B)$$

Proof. First we set $k_{\varepsilon_n} = k_n$ and

$$G_n(u, A) = \left(\iint\limits_{A\,\mathbb{R}^d} (u(x) - u(y))^2 k_n(x, y) \, \mathrm{d}y \, \mathrm{d}x \right)^{1/2}.$$

Let $K_1 \Subset A$ and $K_2 \Subset B$ compact. Set $\delta = \min\{\mathrm{dist}(\partial A, K_1), \mathrm{dist}(\partial B, K_2)\}$. We define for $\nu > 0$:

$$D_0 = K_1, \qquad D_i = \{x \in R^d : \mathrm{dist}(x, K_1) < \frac{i\delta}{\nu}\}, \quad i = 1, ..., \nu.$$

$$E_0 = K_2, \qquad E_i = \{x \in R^d : \mathrm{dist}(x, K_2) < \frac{i\delta}{\nu}\}, \quad i = 1, ..., \nu.$$

On these sets we define cut-off functions $\phi_i, ..., \phi_\nu$, $\psi_1, ..., \psi_\nu$ by:

$$\begin{cases} \phi_i \in C_0^1(D_i), & 0 \le \phi(x) \le 1 \, \forall x \in D_i, \\ \phi_i(x) = 1, & \forall x \in D_{i-1}, \\ |D\phi_i| \le \frac{\nu+1}{\delta} \end{cases} \tag{5.18}$$

and

$$\begin{cases} \psi_i \in C_0^1(E_i), & 0 \le \psi(x) \le 1 \, \forall x \in E_i, \\ \psi_i(x) = 1, & \forall x \in E_{i-1}, \\ |D\psi_i| \le \frac{\nu+1}{\delta}. \end{cases} \tag{5.19}$$

We set for all $n \in \mathbb{N}$, $i = 1, ..., \nu$:

$$u_{i,n} = u + (v_n - u)\phi_i + (w_n - u)\psi_i.$$

Let $\varepsilon > 0$. By (5.13), (5.14) and (5.15) we find $N \in \mathbb{N}$ such that for all $n \ge N$

$$\begin{aligned} F(u, A \cup B) - (F(u, A) + F(u, B)) - \varepsilon &\le F_n(u_{i,n}, A \cup B) - F_n(v_n, A) - F_n(w_n, B) \\ &\le F_n(u_{i,n}, A \cup B) - F_n(v_n, D_i) - F_n(w_n, E_i) \\ &\le F_n(u_{i,n}, (A \cup B) \setminus (D_i \cup E_i)) \\ &\quad + F_n(u_{i,n}, D_i) + F_n(u_{i,n}, E_i) - F_n(v_n, D_i) - F_n(w_n, E_i) \\ &= I + II \end{aligned}$$

We set $(A \cup B) \setminus (D_i \cup E_i) = \mathcal{B}_i$. First we estimate I:

$$\begin{aligned} F_n(u_{i,n}, \mathcal{B}_i) &= \int_{\mathcal{B}_i} \Gamma_{k_n}(u + (v_n - u)\phi_i + (w_n - u)\psi_i) \,\mathrm{d}x \\ &\le 3 \int_{\mathcal{B}_i} \Gamma_{k_n}(u) + \Gamma_{k_n}((v_n - u)\phi_i) + \Gamma_{k_n}((w_n - u)\psi_i) \,\mathrm{d}x \\ &\le 3 \left(\int_{\mathcal{B}_i} \Gamma_{k_n}(u) \,\mathrm{d}x + \int_{\mathcal{B}_i} \Gamma_{k_n}((v_n - u)\phi_i) \,\mathrm{d}x + \int_{\mathcal{B}_i} \Gamma_{k_n}((w_n - u)\psi_i) \,\mathrm{d}x \right) \end{aligned}$$

By (5.10) the first part can be estimated independent of n and i:

$$3 \int_{\mathcal{B}_i} \Gamma_{k_n}(u) \,\mathrm{d}x \le C \iint_{(A \cup B) \setminus (K_1 \cup K_2)\, \mathbb{R}^d} (u(x) - u(y))^2 \, |x - y|^{-d-\alpha} \,\mathrm{d}y \,\mathrm{d}x.$$

For the second (and analogously for the third term) we obtain independent of i

$$\begin{aligned} \int_{\mathcal{B}_i} \Gamma_{k_n}((v_n - u)\phi_i) \,\mathrm{d}x &= \int_{\mathcal{B}_i} \int_{\mathbb{R}^d} (((v_n - u)\phi_i)(y))^2 k_n(x, y) \,\mathrm{d}y \,\mathrm{d}x \\ &= \int_{\mathcal{B}_i} \int_{\mathbb{R}^d} (v_n(y) - u(y))^2 (\phi_i(y) - \underbrace{\phi_i(x)}_{=0})^2 k_n(x, y) \,\mathrm{d}y \,\mathrm{d}x \end{aligned}$$

$$\leq \int_{\mathbb{R}^d} (v_n(y) - u(y))^2 \int_{(A \cup B) \setminus (K_1 \cup K_2)} (\phi_i(y) - \phi_i(x))^2 k_n(x,y) \, \mathrm{d}x \, \mathrm{d}y$$
$$\leq \left(\frac{\nu + 1}{\delta} \right)^2 C \left\| v_n - u \right\|^2_{L^2(\mathbb{R}^d)} .$$

Now we estimate the differences II. Using Cauchy-Schwarz on the product space, we obtain

$$F_n(D_i, u_{i,n}) - F_n(D_i, v_n) \leq G_n(D_i, u_{i,n} - v_n) G_n(D_i, u_{i,n} + v_n) \tag{5.20}$$

and

$$F_n(E_i, u_{i,n}) - F_n(E_i, w_n) \leq G_n(E_i, u_{i,n} - w_n) G_n(E_i, u_{i,n} + w_n). \tag{5.21}$$

First we estimate the first term on the right-hand side of (5.20):

$$\begin{aligned}
G_n(D_i, u_{i,n} - v_n) &= \left(\int_{D_i} \Gamma_{k_n}(u_{i,n} - v_n) \, \mathrm{d}x \right)^{1/2} \\
&\leq \sqrt{2} \left(\int_{D_i} \Gamma_{k_n}((1 - \phi_i)(u - v_n)) \, \mathrm{d}x \right)^{1/2} + \sqrt{2} \left(\int_{D_i} \Gamma_{k_n}(\psi_i(w_n - u)) \, \mathrm{d}x \right)^{1/2} \\
&\leq \sqrt{2} \left(\int_{D_i \setminus D_{i-1}} \Gamma_{k_n}((1 - \phi_i)(u - v_n)) \, \mathrm{d}x \right)^{1/2} + \sqrt{2} \left(\int_{D_{i-1}} \Gamma_{k_n}((1 - \phi_i)(u - v_n)) \, \mathrm{d}x \right)^{1/2} \\
&+ \sqrt{2} \left(\int_{D_i} \Gamma_{k_n}(\psi_i(w_n - u)) \, \mathrm{d}x \right)^{1/2} \\
&= I + II + III.
\end{aligned}$$

Now we look at II:

$$\begin{aligned}
II &= \left(\iint_{D_{i-1} \, D_{i-1}^c} ((1 - \phi_i(y))(v_n(y) - u(y)))^2 \, k_n(x,y) \, \mathrm{d}y \, \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq \left(\int_{D_{i-1}^c} (v_n(y) - u(y))^2 \int_{D_{i-1}} (\underbrace{1}_{=\phi_i(x)} - \phi_i(y))^2 k_n(x,y) \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}} \\
&\leq \left(\int_{D_{i-1}^c} (v_n(y) - u(y))^2 \int_{D_{i-1}} (\phi_i(x) - \phi_i(y))^2 k_n(x,y) \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}} \\
&\leq C \left(\frac{\nu + 1}{\delta} \right) \left(\int_{D_{i-1}^c} (v_n(y) - u(y))^2 \int_{D_{i-1}} |x - y|^{-d-\alpha+2} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq C\left(\frac{\nu+1}{\delta}\right)\|v_n-u\|_{L^2(D_{i-1})}$$
$$\leq C\left(\frac{\nu+1}{\delta}\right)\|v_n-u\|_{L^2(\mathbb{R}^d)}.$$

Set $D_i\setminus D_{i-1}=\mathcal{D}_i$. Choose $R>0$ such that $\Omega\subset B_{R-1}(0)$. Now by (5.10):

$$\begin{aligned}
I &\leq 2\left(\iint\limits_{\mathcal{D}_i\,\mathbb{R}^d}\big[(1-\phi_i(x))(u_n(x)-u(x))-(1-\phi_i(y))(u_n(y)-u(y))\big]^2k_n(x,y)\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\\
&\leq\left(\iint\limits_{\mathcal{D}_i\,\mathbb{R}^d}\big[\phi_i(y)-\phi_i(x)\big]^2\big[(u_n(x)-u(x))+(u_n(y)-u(y))\big]^2k_n(x,y)\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\\
&\quad+\left(\iint\limits_{\mathcal{D}_i\,\mathbb{R}^d}\big[(u_n(x)-u(x))-(u_n(y)-u(y))\big]^2\big[(1-\phi_i(x))+(1-\phi_i(y))\big]^2k_n(x,y)\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\\
&\leq C\left(\frac{\nu+1}{\delta}\right)\lambda\left(\int\limits_{\mathbb{R}^d}\big[u_n(x)-u(x)\big]^2\int\limits_{B_R}|x-y|^{-d-\alpha+2}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\\
&\quad+C\lambda\left(\int\limits_{\mathcal{D}_i}\big[(u_n(x)-u(x)\big]^2\int\limits_{B_R^c}|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\\
&\quad+C\lambda\left(\frac{\nu+1}{\delta}\right)\left(\int\limits_{\mathbb{R}^d}\big[u_n(y)-u(y)\big]^2\int\limits_{\mathcal{D}_i}|x-y|^{-d-\alpha+2}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}\\
&\quad+C\lambda\left(\iint\limits_{\mathcal{D}_i\,\mathbb{R}^d}\big[(u_n(x)-u(x))-(u_n(y)-u(y))\big]^2|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\\
&\leq C(\lambda,R)\left(\frac{\nu+1}{\delta}+1\right)\|u_n-u\|_{L^2(\mathbb{R}^d)}\\
&\quad+C\lambda\left(\iint\limits_{\mathcal{D}_i\,\mathbb{R}^d}\big[(u_n(x)-u(x))-(u_n(y)-u(y))\big]^2|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}.
\end{aligned}$$

III can be estimated as follows:

$$\frac{1}{\sqrt{2}}III=\left(\iint\limits_{D_i\,\mathbb{R}^d}\Big(\underbrace{\psi_i(x)}_{=0}(w_n(x)-u(x))-\psi_i(y)(w_n(y)-u(y))\Big)^2k_n(x,y)\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{2}}$$

$$= \left(\iint_{D_i\, \mathbb{R}^d} (\psi_i(y)(w_n(y) - u(y)))^2 \, k_n(x,y) \, \mathrm{d}y \, \mathrm{d}x \right)^{\frac{1}{2}}$$

$$= \left(\iint_{D_i\, \mathbb{R}^d} (\underbrace{\psi(x)}_{=0} - \psi_i(y))^2 \, ((w_n(y) - u(y)))^2 \, k_n(x,y) \, \mathrm{d}y \, \mathrm{d}x \right)^{\frac{1}{2}}$$

$$= \left(\int_{\mathbb{R}^d} ((w_n(y) - u(y)))^2 \int_{D_i} (\psi(x) - \psi_i(y))^2 k_n(x,y) \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}}$$

$$\leq \left(\frac{\nu + 1}{\delta} \right) \left(\int_{\mathbb{R}^d} ((w_n(y) - u(y)))^2 \int_{D_i} |x-y|^2 \, k_n(x,y) \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}}$$

$$\leq C \left(\frac{\nu + 1}{\delta} \right) \left(\int_{\mathbb{R}^d} ((w_n(y) - u(y)))^2 \int_{D_i} |x-y|^{-d-\alpha+2} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}}$$

$$\leq C \left(\frac{\nu + 1}{\delta} \right) \|w_n - u\|_{L^2(\mathbb{R}^d)}$$

Next we prove that the second term on the right-hand side of (5.20) is uniformly bounded.

$$G_n(D_i, u_{i,n} + v_n) \leq G_n(D_i, u) + G_n(D_i, v_n) + G_n(D_i, \phi_i(v_n - u)) + G_n(D_i, \psi_i(w_n - u))$$
$$= a + b + c + d$$

Now $d = III$ from above, a and b are bounded by $\sqrt{K}$ due to (5.16). Further we have

$$c = \left(\int_{D_i} \Gamma_{k_n}(\phi_i(v_n - u)) \, \mathrm{d}x \right)^{1/2}$$

$$\leq \left(\int_{D_i} \Gamma_{k_n}(v_n - u) \, \mathrm{d}x \right)^{1/2}$$

$$+ \frac{1}{2} \left(\iint_{D_i\, \mathbb{R}^d} ((v_n - u)(x) + (v_n - u)(y))^2 (\phi(x) - \phi(y))^2 k_n(x,y) \, \mathrm{d}y \, \mathrm{d}x \right)^{1/2}$$

$$\leq C(K) + \left(\int_{D_i} (v_n - u)(x)^2 \int_{\mathbb{R}^d} (\phi(x) - \phi(y))^2 k_n(x,y) \, \mathrm{d}y \, \mathrm{d}x \right)^{1/2}$$

$$+ \left(\int_{\mathbb{R}^d} (v_n - u)(y)^2 \int_{D_i} (\phi(x) - \phi(y))^2 k_n(x,y) \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2}$$

$$\leq C(K) + C \left(\frac{\nu + 1}{\delta} \right) \|v_n - u\|_{L^2(\mathbb{R}^d)} .$$

Altogether we obtain

$$\begin{aligned}
G_n(D_i, u_{i,n} - v_n) G_n(D_i, u_{i,n} + v_n) \leq C \left(\frac{\nu+1}{\delta}\right)^2 & \left(\|v_n - u\|^2_{L^2(\mathbb{R}^d)} + \|w_n - u\|^2_{L^2(\mathbb{R}^d)}\right) \\
&+ C(K) \left(\|v_n - u\|_{L^2(\mathbb{R}^d)} + \|w_n - u\|_{L^2(\mathbb{R}^d)}\right) \\
&+ C(K) \left(\int\limits_{D_i \setminus D_{i-1}} \Gamma_\alpha(v_n - u) \,\mathrm{d}x \right)^{1/2} \\
&+ C(K) \left(\|v_n - u\|_{L^2(\mathbb{R}^d)} + \|w_n - u\|_{L^2(\mathbb{R}^d)}\right),
\end{aligned}$$

where we have used

$$\left(\int\limits_{D_i \setminus D_{i-1}} \Gamma(v_n - u) \,\mathrm{d}x \right)^{1/2} \leq C(K).$$

Analogously one estimates the right-hand side of (5.21):

$$\begin{aligned}
G_n(E_i, u_{i,n} - w_n) G_n(E_i, u_{i,n} + w_n) \leq C \left(\frac{\nu+1}{\delta}\right)^2 & \left(\|v_n - u\|^2_{L^2(\mathbb{R}^d)} + \|w_n - u\|^2_{L^2(\mathbb{R}^d)}\right) \\
&+ C(K) \left(\|v_n - u\|_{L^2(\mathbb{R}^d)} + \|w_n - u\|_{L^2(\mathbb{R}^d)}\right) \\
&+ C(K) \left(\int\limits_{E_i \setminus E_{i-1}} \Gamma_\alpha(w_n - u) \,\mathrm{d}x \right)^{1/2} \\
&+ C(K) \left(\|v_n - u\|_{L^2(\mathbb{R}^d)} + \|w_n - u\|_{L^2(\mathbb{R}^d)}\right).
\end{aligned}$$

Thus for all i we have

$$\begin{aligned}
F(u, A \cup B) - (F(u,A) + F(u,B)) - \varepsilon < C_1 & \iint\limits_{(A\cup B)\setminus(K_1 \cup K_2)\, \mathbb{R}^d} (u(x) - u(y))^2 \, |x-y|^{-d-\alpha} \,\mathrm{d}y \,\mathrm{d}x \\
&+ C_2 \left(\frac{\nu+1}{\delta}\right)^2 \left(\|v_n - u\|^2_{L^2(\mathbb{R}^d)} + \|w_n - u\|^2_{L^2(\mathbb{R}^d)}\right) \\
&+ C_3 \left(\|v_n - u\|_{L^2(\mathbb{R}^d)} + \|w_n - u\|_{L^2(\mathbb{R}^d)}\right) \\
&+ C_4 \left(\iint\limits_{D_i \setminus D_{i-1}\, \mathbb{R}^d} ((v_n(x) - u(x)) - (v_n(y) - u(y)))^2 \, |x-y|^{-d-\alpha} \,\mathrm{d}y \,\mathrm{d}x \right)^{1/2} \\
&+ C_5 \left(\iint\limits_{E_i \setminus E_{i-1}\, \mathbb{R}^d} ((w_n(x) - u(x)) - (w_n(y) - u(y)))^2 \, |x-y|^{-d-\alpha} \,\mathrm{d}y \,\mathrm{d}x \right)^{1/2},
\end{aligned}$$

where the constant depend only on K. Summing up this inequality for all $i = 1, ..., \nu$, and dividing by ν yields

$$\begin{aligned}
F(u,A\cup B)-(F(u,A)+F(u,B))-\varepsilon < C_1 &\iint\limits_{(A\cup B)\setminus(K_1\cup K_2)\,\mathbb{R}^d} (u(x)-u(y))^2\,|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x \\
&+ C_2\left(\frac{\nu+1}{\delta}\right)^2\left(\|v_n-u\|^2_{L^2(\mathbb{R}^d)}+\|w_n-u\|^2_{L^2(\mathbb{R}^d)}\right)\\
&+ C_3\sqrt{K}\left(\|v_n-u\|_{L^2(\mathbb{R}^d)}+\|w_n-u\|_{L^2(\mathbb{R}^d)}\right)\\
&+ C_4\frac{1}{\sqrt{\nu}}\left(\iint\limits_{A\,\mathbb{R}^d}((v_n(x)-u(x))-(v_n(y)-u(y)))^2\,|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\\
&+ C_5\frac{1}{\sqrt{\nu}}\left(\iint\limits_{B\,\mathbb{R}^d}((w_n(x)-u(x))-(w_n(y)-u(y)))^2\,|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\\
&= \clubsuit+\spadesuit+\bigstar,
\end{aligned}$$

where we have used the inequality $\sum_{i=1}^n\sqrt{a_i}\le\sqrt{n\sum_{i=1}^n a_i}$.
Choosing $K_1\Subset A$ und $K_2\Subset B$, such that $\clubsuit<\frac{\varepsilon}{3}$, then ν, such that $\bigstar<\frac{\varepsilon}{3}$ and finally n such that $\spadesuit<\frac{\varepsilon}{3}$ the right-hand side can be made arbitrary small. Thus

$$F(u,A\cup B)\le F(u,A)+F(u,B).$$

□

By the pointwise estimate (5.10) we obtain that the localized functionals $F_0(u,A)$ satisfy the growth estimate

$$\lambda\iint\limits_{A\,\mathbb{R}^d}\frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x\le F_0(u,A)\le\lambda^{-1}\iint\limits_{A\,\mathbb{R}^d}\frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x. \tag{5.22}$$

Now, for sufficiently smooth u, we can apply a lemma taken from [DG75, Lem. 1] to obtain an integral representation for F_0:

Lemma 5.9. *Let $\varphi',\varphi'':\mathbb{R}^d\to\mathbb{R}$ be continuous, nonnegative and let τ be a set function on $\mathcal{A}(\Omega)$. Let*

$$\int\limits_B\varphi'(x)\,\mathrm{d}x\le\tau(B)\le\int\limits_B\varphi''(x)\,\mathrm{d}x$$

for all $B\in\mathcal{A}(\Omega)$. Moreover, assume that for all polyrectangles A,B,C in $\mathcal{A}(\Omega)$ with

$$A\cap B=\emptyset,\quad A\cup B\subset C\quad\text{and}\ |C\setminus(A\cup B)|=0$$

it holds

$$\tau(C)=\tau(A)+\tau(B).$$

Then, there is a measurable function φ, which satisfies, for every $x \in \mathbb{R}^d$, $\varphi'(x) \le \varphi(x) \le \varphi''(x)$ and such that

$$\tau(B) = \int\limits_B \varphi(x)\,\mathrm{d}x$$

for every $B \in \mathcal{A}(\Omega)$.

Using the periodicity of k_ε, we can prove that the localized limit functional is translation invariant.

Proposition 5.10. *Let (ε_n) be a sequence converging to zero such that*

$$\Gamma - \lim_{n\to\infty} F_{\varepsilon_n}(u, A) = F_0(u, A)$$

exists for all $A \in \mathcal{V}$. Let $\rho \in \mathbb{R}^d$ and assume that $\tau_\rho A \in \Omega$. Then

$$F_0(u, A) = F_0(\tau_\rho u, \tau_\rho A)\,.$$

The proof follows the arguments of the first part of the proof of [DM93, Thm. 24.1].

Proof. Let $\rho \in \mathbb{R}^d$ and let $F'' : L^2(\mathbb{R}^d) \times \mathcal{A}(\Omega)$ be the increasing functional

$$F''(\cdot, A) = \Gamma - \limsup_{n\to\infty} F_{\varepsilon_n}(\cdot, A)$$

for every $A \in \mathcal{A}(\Omega)$. Fix $u \in L^2(\mathbb{R}^d)$.We have to prove that $F_0(u, A) = F_0(\tau_\rho u, \tau_\rho A)$. By interchanging y with $-y$ it is sufficient to prove that $F_0(u, A) \ge F_0(\tau_\rho u, \tau_\rho A)$. F_0 is the inner regular envelope of F'', cf. [DM93, Rem. 16.3]. Thus it suffices to prove that

$$F''(\tau_\rho u, \tau_\rho A') \le F_0(u, A)$$

for every $A' \in \mathcal{A}(\Omega)$ with $A' \Subset A$. Now for fixed A' we choose $A'' \in \mathcal{A}(\Omega)$, such that

$$A' \Subset A'' \Subset A.$$

By Definition A.4 there is a sequence $(u_n) \in L^2(\mathbb{R}^d)$ such that

$$F_0(u, A) \ge F''(u, A'') = \limsup_{n\to\infty} F_{\varepsilon_n}(u_n, A'').$$

Now, since $\varepsilon_n \to 0$, there exists a sequence $(z_n) \in \mathbb{Z}^d$ such that $\varepsilon_n z_n \overset{n\to\infty}{\longrightarrow} y$, (set for example $z_n = \lfloor \frac{y}{\varepsilon_n} \rfloor$). Set $y_n = \varepsilon_n z_n$. By the continuity of the shift, it follows that $\tau_{\rho_n} u \overset{n\to\infty}{\longrightarrow} \tau_\rho u$ in $L^2(\mathbb{R}^d)$. Since $u_n \in V^{\alpha/2}(A''|\Omega)$ if and only if $\tau_{\rho_n} u_n \in V^{\alpha/2}(A''_{\tau_{\rho_n}}|\Omega)$, in this case we obtain

$$\begin{aligned} F_{\varepsilon_n}(u_n, A'') &= \iint\limits_{A''\,\mathbb{R}^d} (u(x) - u(y))^2 k_{\varepsilon_n}(x, y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int\limits_{A''}\int\limits_{\mathbb{R}^d} (u(x) - u(y))^2 k_{\varepsilon_n}(x + \rho_n, y + \rho_n)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int\limits_{\tau_{\rho_n} A''}\int\limits_{\mathbb{R}^d} (u(x - \rho_n) - u(y - \rho_n))^2 k_{\varepsilon_n}(x, y)\,\mathrm{d}y\,\mathrm{d}x = F_{\varepsilon_n}(\tau_{\rho_n} u_n, \tau_{\rho_n} A''), \end{aligned}$$

where we have used the ε_n-periodicity of k_{ε_n}. Now there is $N \in \mathbb{N}$, such that for all $n > N$ $\tau_\rho A' \subset \tau_{\rho_n} A''$ and thus

$$F_{\varepsilon_n}(u_n, A'') = F_{\varepsilon_n}(\tau_{\rho_n} u_n, \tau_{\rho_n} A'') \geq F_{\varepsilon_n}(\tau_{\rho_n} u_n, \tau_\rho A').$$

Using again the definition of the Γ-upper limit, we finally obtain

$$\begin{aligned} F''(\tau_\rho u, \tau_\rho A') &\leq \limsup_{n\to\infty} F_{\varepsilon_n}(\tau_{\rho_n} u_n, \tau_r hoA') \\ &\leq \limsup_{n\to\infty} F_{\varepsilon_n}(u_n, A'') \leq F_0(u, A). \end{aligned}$$

□

5.2.3. Open questions in the nonlocal case

In this section we explain where the methods known from the homogenization of second order equations break down when considering nonlocal operators, starting from what we have proved in the previous sections.

Let us recall what we proved so far. For any given sequences $\varepsilon_n \to 0$, there exists a subsequence ε_n (not relabeled), such that the Γ-limit of the functionals F_{ε_n} exists. Further the limit is again a Dirichlet form and can be represented for $u \in V_0^{\alpha/2}(\Omega|\mathbb{R}^d)$ as

$$F_0(u) = \mathcal{E}_0(u, u) = \iint\limits_{(\Omega^c \times \Omega^c)^c} (u(x) - u(y))^2 J(\,\mathrm{d}x,\,\mathrm{d}y).$$

For fixed $u \in L^2(\mathbb{R}^d)$ we have a local integral representation of the limit functional due to Lemma 5.9. Thus for $A \in \mathcal{A}(\Omega)$

$$F_0(u, A) = \int\limits_A \phi_u(x)\,\mathrm{d}x$$

for some function ϕ_u, depending on u.

One approach in the local setting is to use the linear function $u_\xi(x) = \xi \cdot x$ to prove that ϕ_u can be rewritten as a function f depending on x and ∇u independent of $u \in H^1(\Omega)$. Note that $\nabla u(x) = \xi$ for all $x \in \mathbb{R}^d$. A second approach is to approximate a given function u with polynomials. This part can not be transfered to the nonlocal setting by mainly two reasons:

- Even for linear functions u_ξ the nonlocal analogue of $\nabla u_\xi(x)$ is not simply characterized by the vector ξ.
- It is unclear, how to approximate functions in the nonlocal setting via affine functions or polynomials.

Note that so far we just consider the Γ-convergence of an arbitrary subsequence. Key to prove the homogenization formula in the local setting is to connect the limit functional with the family f_ε and to express f_0 as a minimizer of a cell problem, i.e., to solve the periodic problem on one periodicity cell Y. From this one concludes that the limit does not depend on the convergent subsequence and thus the Urysohn property of Γ-convergence implies the Γ-convergence of any sequence $\varepsilon_n \to 0$.

There are at least two different approaches to obtain this. The first one uses again the linear function $u_\xi(x) = \xi \cdot x$. Note that in the local case $f(\nabla u) = f(\xi)$, see [DM93, Thm. 24.1]. The second approach uses the fact that f_0 is H^1-quasiconvex and thus can be rewritten as a minimum problem

$$f_0(\xi) = \min \left\{ \int_Y f(\xi + \nabla v(y)) \, \mathrm{d}y | v \in H_0^1(Y) \right\},$$

see [BD98, Prop. 4.3].

To obtain such a representation in the nonlocal setting seems to be quite challenging due to the nonlocal character of the substitute of f_0. This would depend, instead of a single vector, on any value of the differences $(u(x) - u(y))$ for $x, y \in \mathbb{R}^d$.

A. Γ-Convergence

Γ-convergence was first introduced by DeGiorgi in the early 1970s to solve asymptotic problems in the calculus of variation. Since then Γ-convergence has become a tool in a wide range of applications connected to the calculus of variation and PDE's.

We give a short summery on Γ-convergence and repeat some basic results and features of Γ-convergence that are instrumental in our applications. We follow the illustration in [Bra02] and [BD98] and restrict ourself to case of Γ-convergence on metric spaces.

Our short survey should answer the following question: Suppose $F_n : X \to [-\infty, \infty]$ is a sequence of functionals on X and (x_n) is a sequence of minimizers. Does (x_n) converge to any $x \in X$ and does x minimize any functional F? The answer to this question is that Γ-convergence is the right notion of convergence, such that convergence of F_n to some F implies the convergence of minimizers.

A.1. Definition and basic properties

There are many equivalent definitions of Γ-convergence, which are useful in different contexts. We give the sequential definition of Γ-convergence and afterwards we prove an equivalent definition in terms of the topology of X. Afterwards we recall some basic properties of Γ-convergence, in particular the convergence of minimizers and compactness of Γ-convergence.

Definition A.1. Let (X, d) be a metric space, $F, F_n : X \to [-\infty, +\infty]$. We say that F_n $\Gamma(d)-$converges to F (or that F_n Γ-converges in the topology of d to F) in $x \in X$, if

(i) (**l**iminf inequality) for every sequence (x_n) with $x_n \xrightarrow{d} x$

$$F(x) \le \liminf_{n\to\infty} F_n(x_n),$$

(ii) (**l**imsup inequality) there exists a sequence (x_n) with $x_n \xrightarrow{d} x$, such that

$$F(x) \ge \limsup_{n\to\infty} F_n(x_n).$$

We say that F_n $\Gamma(d)$-converges to F on X, if F_n $\Gamma(d)$-converge to F for all $x \in X$. If F_n Γ-converges to F in $x \in X$, we will write

$$F(x) = \Gamma(d) - \lim_{n\to\infty} F_n(x).$$

Another way of stating ((ii)) is to say: There is a sequence (x_n) $x_n \xrightarrow{d} x$, such that

$$\lim_{n\to\infty} F_n(x_n) = F(x).$$

This sequence is called a recovery sequence. If there is no risk of confusion, we will omit the metric d in the notion of Γ-convergence. To prove same essential properties of Γ-convergence, it is useful to have a definition directly in terms of the topology of X. For $x \in X$ let us denote by $\mathcal{N}(x)$ the family of all neighborhoods of x. An equivalent formulation of Definition A.1 is:

Proposition A.2. *[Bra02, Thm. 1.17] Let (X, d) be a metric space and $F_n, F : X \to [-\infty, \infty]$. Then the following is equivalent:*

(i) F_n *Γ-converges to* F *in* $x \in X$.

(ii) $F(x) = \sup_{U \in \mathcal{N}(x)} \liminf_{n\to\infty} \inf_{y \in U} F_n(y)$.

We omit the proof and refer to [BD98, Prop. 7.5/7.6]. An important property of Γ-convergence is, that it is stable under continuous perturbation. This allows us to neglect any d-continuous functional in the computation of the $\Gamma(d)$-limit.

Proposition A.3. *Let (X, d) be a metric space, $F_n, F : X \to [-\infty, \infty]$ and $G : X \to \mathbb{R}$ d-continuous. Further let F_n Γ-converge to F on X. Then*

$$\Gamma - \lim_{n\to\infty} (F_n + G)(x) = F(x) + G(x)$$

for all $x \in X$.

Proof. The proposition is a direct consequence of the above definition. Let $x_n \xrightarrow{d} x$, then

$$F(x) + G(x) \le \liminf_{n\to\infty} F_n(x_n) + \lim_{n\to\infty} G(x_n) \le \liminf_{n\to\infty} (F_n(x_n) + G(x_n)).$$

On the other hand let x_n be a recovery sequence. Then

$$F(x) + G(x) = \lim_{n\to\infty} F_n(x_n) + \lim_{n\to\infty} G(x_n) = \lim_{n\to\infty} (F_n(x_n) + G(x_n)).$$ □

As for usual limits, we define an upper and lower Γ-limit, as two quantities that always exist. Of course, then the existence of the Γ-limit is equivalent to the equality of the upper and lower limit.

Definition A.4. Let (X, d) be a metric space, $F_n : X \to [-\infty, +\infty]$ a sequence of functionals on X. Then the quantity

$$\Gamma - \liminf_{n\to\infty} F_n(x) = \inf\{\liminf_{n\to\infty} F_n(x_n) | d(x_n, x) \to 0\}$$

is called the Γ-lower limit of the sequence F_n. Analogously the quantity

$$\Gamma - \limsup_{n\to\infty} F_n(x) = \inf\{\limsup_{n\to\infty} F_n(x_n) | d(x_n, x) \to 0\}$$

is called Γ-upper limit of the sequence F_n.

Corollary A.5. *The infimum in the above terms can be replaced by the minimum, i.e.*

$$\Gamma - \liminf_{n\to\infty} F_n(x) = \min\{\liminf_{n\to\infty} F_n(x_n) | d(x_n, x) \to 0\}$$

and

$$\Gamma - \limsup_{n\to\infty} F_n(x) = \min\{\limsup_{n\to\infty} F_n(x_n) | d(x_n, x) \to 0\}.$$

Furthermore the Γ-lower and Γ-upper limit define lower semicontinuous functions.

We follow the arguments of [BD98, Prop.7.6] and [BD98, Rem. 7.3].

Proof. We prove the first statement for the Γ-upper limit. Let us assume that $\Gamma - \limsup F_n(x) < +\infty$, otherwise the statement is trivially satisfied. Let $\Gamma - \limsup F_n(x) > -\infty$, w.l.o.g. let $\Gamma - \limsup F_n(x) = 0$. Then for $k \in \mathbb{N}$, let $x_n^k \overset{n\to\infty}{\longrightarrow} x$ and

$$\limsup_{n\to\infty} F_n(x_n^k) \leq \frac{1}{k}.$$

We define $\sigma_0 = 0$ and for $k \in \mathbb{N}$

$$\sigma_k = \min\left\{h > \sigma_{k-1} | F_n(x_n^k) \leq \frac{2}{k}, d(x_n^k, x) \leq \frac{1}{k} \forall n \geq h\right\}.$$

Then taking $x_n = x_n^k$ for $\sigma_k \leq n < \sigma_{k+1}$, we obtain $\limsup_{n\to\infty} F_n(x_n) = 0$. The case $\Gamma - \limsup F_n(x) = -\infty$ is proven in the same way.

We prove the second statement for the Γ-lower limit. Denote $\Gamma - \liminf_{n\to\infty} F_n(x) = F_\infty(x)$ and let $x^k \overset{k\to\infty}{\longrightarrow} x$. By the first part of the corollary for each $k \in \mathbb{N}$ we find a sequence (x_j^k) with $x_j^k \overset{j\to\infty}{\longrightarrow} x^k$ and $\liminf_{j\to\infty} F_j(x_j^k) \leq F_\infty(x^k)$. We define $\sigma_0 = 0$ and and for $i \in \mathbb{N}$

$$\sigma_i = \min\left\{h > \sigma_{i-1} | \left|F_\infty(x^i) - F_j(x_j^i)\right| \leq \frac{1}{i}, d(x_j^i, x^i) \leq \frac{1}{i} \forall i \geq h\right\}.$$

Set $x_j = x_j^k$ for $\sigma_k \leq j < \sigma_{k+1}$. Then we have $x_j \overset{j\to\infty}{\longrightarrow} x$ and by the definition of the Γ-lower limit

$$F_\infty(x) \leq \liminf_{j\to\infty} F_j(x_j) = \liminf_{k\to\infty} F_\infty(x_k). \qquad \square$$

Remark A.6. The lower semi-continuity of the Γ-upper limit allows us to reduce its computation to a dense subset $\mathcal{D} \subset X$. Let d' be a metric inducing a topology which is not weaker than the topology induced by d, i.e. $d'(x_n, x) \to 0$ implies $d(x_n, x) \to 0$. Let $F : X \to [0, \infty]$ be continuous with respect to d and set $X_0 = \{x \in X | F(x) < \infty\}$.

If we assume that

1. $\mathcal{D}$ is dense in X_0 with respect to d',
2. $\Gamma - \limsup_{n\to\infty} F_n(x) \leq F(x)$ for all $x \in \mathcal{D}$,

then we have $\Gamma - \limsup_{n\to\infty} F_n(x) \leq F(x)$ on X. Note that if $x \in X \setminus X_0$, there is nothing to prove. Let $(x_k) \in \mathcal{D}$ and $x \in X_0$ such that $d'(x_k, x) \to 0$. This implies $d(x_k, x) \to 0$ and thus by the lower semi continuity and (2) we obtain

$$\begin{aligned}\Gamma - \limsup_{n\to\infty} F_n(x) &\leq \liminf_{k\to\infty} \left(\Gamma - \limsup_{n\to\infty} F_n(x_k)\right) \\ &\leq \liminf_{k\to\infty} F(x_k) = F(x).\end{aligned}$$

A.2. Convergence of minimizers and compactness of Γ-convergence

To obtain the convergence of minimizers we need an *equi-coerciveness* conditions on the underlying functionals F_n, so that the sequence of minimizers is precompact in X.

Definition A.7. A sequence (F_n) of functions $F_n : X \to \bar{\mathbb{R}}$ is called *equi-coercive* on X, if for all $t \in \mathbb{R}$ there exists a compact set K_t such that $\{F_n \leq t\} \subset K_t$.

It is equivalent to say that for all $n \in \mathbb{N}$, there exists a compact set K such that

$$\inf\{F_n(x)|x \in X\} = \inf\{F_n(x)|x \in K\}.$$

Now we can state one of the main properties of Γ-convergence.

Theorem A.8. *([Bra02, Thm. 1.21]) Let (X, d) be a metric space and let (F_n) be a sequence of equi-coercive functions on X and let $F = \Gamma - \lim_{n\to\infty} F_n$. Then*

$$\exists \min_X F = \lim_{j\to\infty} \inf_X F_j. \tag{1.1}$$

Moreover, if (x_j) is a precompact sequence such that $\lim_{j\to\infty} F_j(x_j) = \lim_{j\to\infty} \inf_X F_j$ then every limit point of a subsequence of (x_j) is a minimum point of F.

The proof follows the lines of [BD98, Thm. 7.1]

Proof. Let n_k be a sequence of indices such that $\lim\limits_{k\to\infty} F_{n_k} = \liminf\limits_{n\to\infty} F_n$. By the equi-coerciveness of (F_n) there exists $K \subset X$ compact, such that $\inf\{F_n(x)|x \in X\} = \inf\{F_n(x)|x \in K\}$ for all $n \in \mathbb{N}$ and a sequence (x_k) in K, such that

$$\lim_{k\to\infty} F_{n_k}(x_k) = \liminf_{k\to\infty} F_{n_k} = \liminf_{n\to\infty} F_n.$$

Since K is compact, after possibly passing to a subsequence of (x_k) (again denoted by (x_k)), we can assume that $x_k \stackrel{k\to\infty}{\longrightarrow} x_0$. Now from the definition of Γ-convergence we obtain

$$F(x_0) \leq \liminf_{k\to\infty} F_{n_k}(x_k) = \liminf_{n\to\infty} F_n$$

and

$$\inf_X F \leq \inf_K F \leq \liminf_{n\to\infty} F_n.$$

By ((ii)) of Definition A.1 for all $x \in X$ there is a sequence (x_n), such that $x_n \stackrel{n\to\infty}{\longrightarrow} x$,

$$F(x) \geq \limsup_{n\to\infty} F_n(x_n)$$

and thus

$$\limsup_{n\to\infty} \inf_X F_n \leq F(x).$$

Hence

$$\limsup_{n\to\infty} \inf_X F_n \leq \inf_X F.$$

From this (1.1) follows. Let $x_n \overset{n\to\infty}{\longrightarrow} x_0$ such that

$$\lim_{n\to\infty} F_n(x_n) = \lim_{n\to\infty} \inf_X F_n.$$

From this and (1.1) the second assertion follows.

□

Note that a metric in which one proves Γ-convergence is not given beforehand, rather the metric should be chosen in a way that the equi-coerciveness of the functionals follows directly. A stronger metric implies a stronger convergence result, but a sequence of minimizers may not be precompact any more.

The last property we want to address is the compactness of Γ-convergence.

Proposition A.9. *([Bra02, Prop. 1.42]) Let (X, d) be a metric space, (F_n) a sequence of functionals, with $F_n : X \to [-\infty, \infty]$ for all $n \in \mathbb{N}$. Then there exists a subsequence F_{n_k} such that*

$$\Gamma - \lim_{k\to\infty} F_{n_k}(x)$$

exists for all $x \in X$.

Proof. Let (U_k) be a countable base of open sets generating the topology of X. Since $\overline{\mathbb{R}}$ is compact, there is a sequence of integers $(\sigma_j^0)_j$ such that

$$\lim_{j\to\infty} \inf_{y\in U_0} F_{\sigma_j^0}(y)$$

exists. Now we define $(\sigma_j^1)_j$ as any subsequence of $(\sigma_j^0)_j$ along which the limit

$$\lim_{j\to\infty} \inf_{y\in U_1} F_{\sigma_j^1}(y)$$

exists. Recursively for $k > 1$ we define $(\sigma_j^k)_j$ as any subsequence of $(\sigma_j^{k-1})_j$ along which the limit

$$\lim_{j\to\infty} \inf_{y\in U_k} F_{\sigma_j^k}(y)$$

exists. The diagonal sequence σ_k^k has the property that the limit

$$\lim_{k\to\infty} \inf_{y\in U_i} F_{\sigma_k^k}(y)$$

exists for all $i \in \mathbb{N}$. Since the limit exists this implies

$$\liminf_{k\to\infty} \inf_{y\in U_i} F_{\sigma_k^k}(y) = \limsup_{k\to\infty} \inf_{y\in U_i} F_{\sigma_k^k}(y)$$

for all $i \in \mathbb{N}$. The assertion follows from Proposition A.2. □

B. Dirichlet forms

Let us summarize the main definitions and properties of Dirichlet forms we use in the scope of this thesis. For a more detailed introduction to Dirichlet forms we refer to [MR92] and [FU12].

Consider the Hilbert space $L^2(X,\mu)$, where X is a locally compact separable metric space and μ is a positive Radon measure such that $\text{supp}(\mu) = X$. On $L^2(X,\mu)$ we consider a bilinear form $\mathcal{E} : L^2(X,\mu) \times L^2(X,\mu) \to [-\infty,\infty]$ and set

$$\mathcal{D}(\mathcal{E}) = \left\{ u \in L^2(X,\mu) | \mathcal{E}(u,u) < \infty \right\}.$$

Now we are able to define the main objects of this chapter.

Definition B.1. A bilinear form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$, $\mathcal{D}(\mathcal{E}) \subset L^2(X,\mu)$ is called lower bounded Dirichlet form if the following properties hold: There is $\gamma > 0$ and $C > 0$ such that

1. $\mathcal{E}(u,u) \geq \gamma(u,u)_{L^2(X,\mu)}$ for all $u \in \mathcal{D}(\mathcal{E})$.
2. For all $u \in \mathcal{D}(\mathcal{E})$ one has $\mathcal{E}(u,v) \leq C\, \mathcal{E}_\gamma(u,u)^{1/2} \mathcal{E}_\gamma(v,v)^{1/2}$.
3. $\mathcal{D}(\mathcal{E})$ is a closed dense subspace of $L^2(X,\mu)$.
4. For all $u \in \mathcal{D}(\mathcal{E})$ one has $(u \vee 0) \wedge 1 \in \mathcal{D}(\mathcal{E})$ and

$$\begin{aligned} \mathcal{E}(u + (u \vee 0) \wedge 1, u - (u \vee 0) \wedge 1) &\geq 0 \\ \mathcal{E}(u - (u \vee 0) \wedge 1, u + (u \vee 0) \wedge 1) &\geq 0 \end{aligned}$$

If $\mathcal{E}$ is symmetric, i.e. $\mathcal{E}(u,v) = \mathcal{E}(v,u)$ for all $u,v \in \mathcal{D}(\mathcal{E})$ the last point is equivalent to

$$\mathcal{E}((u \vee 0) \wedge 1, (u \vee 0) \wedge 1) \geq \mathcal{E}(u,u)$$

and $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is called symmetric Dirichlet form.

Here we set

$$\mathcal{E}_\gamma(u,u) = \mathcal{E}(u,u) + \gamma(u,u)_{L^2(X,\mu)}.$$

To give an representation formula for a general symmetric Dirichlet form, we need the following definition about the richness of $\mathcal{D}(\mathcal{E})$.

Definition B.2. A symmetric Dirichlet form is called regular if $C_c(X) \cap \mathcal{D}(\mathcal{E})$ is dense $\mathcal{D}(\mathcal{E})$ with respect to the $\mathcal{E}_1$-norm and dense in $C_c(X)$ with respect to the uniform norm.

Now we can state the representation theorem due to Beurling and Deny.

Theorem B.3. *Let $\mathcal{E}$ be a regular symmetric Dirichlet form on $L^2(X,\mu)$. Then for all $u,v \in C_0(X)\cap\mathcal{D}(\mathcal{E})$ $\mathcal{E}$ can be expressed as:*

$$\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v) + \int\limits_{X\times X\setminus d} ((u(x)-u(y)(v(x)-v(y)))\, J(\,\mathrm{d}y,\,\mathrm{d}x) + \int_X u(x)v(x)k(\,\mathrm{d}x).$$

Here $\mathcal{E}^{(c)}(u,v)$ is a strongly local form. J is a symmetric positive Radon measure on $X\times X\setminus d$, where d denotes the diagonal $x=y$ and k is a positive Radon measure on X.

We omit the proof and refer to [FU12, Thm. 3.2.1]. A form a is called strongly local if $a(u,v)=0$ for all v that are constant in a neighborhood of $\operatorname{supp}(u)$. The local part is called the diffusion part of the form, while the other two parts are called the jumping and killing part of the form.

C. Definitions and auxiliary results

C.1. Domains

We denote the boundary of a arbitrary open set $\Omega \subset \mathbb{R}^d$ by

$$\partial\Omega = \overline{\Omega} \cap \left(\mathbb{R}^d \setminus \Omega\right).$$

A domain Ω is any open connected subset of $\mathbb{R}^d$.

Definition C.1. Let $\Omega \subset \mathbb{R}^d$ be open.

1. We say that Ω is a Lipschitz hypograph if there is a Lipschitz function $\xi : \mathbb{R}^{d-1} \to \mathbb{R}$, such that
$$\Omega = \left\{x \in \mathbb{R}^d | x_n < \xi(x') \text{ for all } x' = (x_1, ..., x_{d-1}) \in \mathbb{R}^{d-1}\right\}.$$
2. We say that Ω is a Lipschitz domain if its boundary $\partial\Omega$ is compact and there exists finite families $\{B_j\}$ of open balls and open bounded sets $\{\Omega_j\}$, such that
   a) The family $\{B_j\}$ is a finite cover of $\partial\Omega$, i.e. $\partial\Omega \subset \bigcup_j B_j$.
   b) Each Ω_j can be transformed to a Lipschitz hypograph by rotation and translation.
   c) The set Ω satisfies $B_j \cap \Omega = B_j \cap \Omega_j$.
3. Let $k \in \mathbb{N}$. We say that Ω is a C^k domain if the function ξ from above is C^k and Lipschitz is substituted by C^k in (2).

For C^1 domains we can change coordinates locally by a mapping Ψ, such that the boundary is flatten out near a point $x \in \partial\Omega$ and $\Psi(B_r(x) \cap \Omega) \subset \mathbb{R}^d_+$ for some $r > 0$. For the sake of completeness we will give the construction of Ψ. Let Ω be an open C^1 domain and $x \in \partial\Omega$. Then according to Definition C.1 there is a ball $B_r(x)$ and a C^1 function $\xi : \mathbb{R}^{d-1} \to \mathbb{R}$, such that (after relabeling the coordinate axes)

$$\Omega \cap B_r(x) = \left\{x \in \mathbb{R}^d | x_n < \xi(x') \text{ for all } x' = (x_1, ..., x_{d-1}) \in \mathbb{R}^{d-1}\right\},$$
$$\Omega^c \cap B_r(x) = \left\{x \in \mathbb{R}^d | x_n \geq \xi(x') \text{ for all } x' = (x_1, ..., x_{d-1}) \in \mathbb{R}^{d-1}\right\}.$$

We define

$$\Psi(x) = \begin{cases} x_i, & \text{for } i = 1, .., d-1, \\ x_d - \xi(x_1, ..., x_{d-1}). \end{cases}$$

Then Ψ is a C^1-diffeomorphism and with $\Psi(B_r(x)) = K$ we obtain

$$\Psi(B_r(x) \cap \Omega) = K \cap \mathbb{R}^d_+,$$
$$\Psi(B_r(x) \cap \Omega^c) = K \cap \mathbb{R}^d_-.$$

C.2. Auxiliary computations

We define the carré du champ Γ_{k^α} for a general kernel k^α as

$$\begin{aligned}\Gamma_{k^\alpha}(f,g)(x) &= \frac{1}{2}\Big(\mathcal{L}^{k^\alpha}(fg) - f\mathcal{L}^{k^\alpha}(g) - \mathcal{L}^{k^\alpha}(f)g\Big)\\ &= \int\limits_{\mathbb{R}^d} (f(x)-f(y))(g(x)-g(y))k^\alpha(x,y)\,\mathrm{d}y.\end{aligned}$$

For the sake of brevity we write $\Gamma_{k^\alpha}(f,f)(x) = \Gamma_{k^\alpha}(f)(x)$.

Lemma C.2. *Let $f,g \in V^{\alpha/2}(\Omega|\mathbb{R}^d)$. Then*

(i)

$$\Gamma_{k^\alpha}(f,g) \le \sqrt{\Gamma_{k^\alpha}(f)}\sqrt{\Gamma_{k^\alpha}(g)}$$

(ii)

$$\Gamma_{k^\alpha}(f+g) \le 2\Gamma_{k^\alpha}(f) + 2\Gamma_{k^\alpha}(g)$$

(iii)

$$\begin{aligned}\Gamma_{k^\alpha}(f\cdot g) \le &\frac{1}{2}\int\limits_{\mathbb{R}^d}(f(x)-f(y))^2(g(x)+g(y))^2k^\alpha(x,y)\,\mathrm{d}y\\ &+\frac{1}{2}\int\limits_{\mathbb{R}^d}(f(x)+f(y))^2(g(x)-g(y))^2k^\alpha(x,y)\,\mathrm{d}y\end{aligned}$$

Proof. (i) follows directly from the Cauchy-Schwarz inequality.

(ii) follows directly from Youngs inequality.

(iii) follows from the identity

$$ab - cd = \frac{1}{2}\Big((a-c)(b+d) + (a+c)(b-d)\Big)$$

□

Lemma C.3. *Let $A \subset \mathbb{R}^d$. If $A_1 \dot{\cup} A_2$, then $(A^c \times A^c)^c = (A_1^c \times A_1^c)^c \cup (A_2^c \times A_2^c)^c$.*

Proof. Let $(x_1,x_2) \in (A^c \times A^c)^c$, w.l.o.g. we can assume that $x_1 \in A$. Thus

$$x_1 \in A_1 \vee x_1 \in A_2.$$

But this implies that $(x_1,x_2) \in (A_1^c \times A_1^c)^c$ or $(x_1,x_2) \in (A_2^c \times A_2^c)^c$. Now w.l.o.g. let $(x_1,x_2) \in (A_1^c \times A_1^c)$, then x_1 or x_2 are in A_1 and thus $(x_1,x_2) \in (A^c \times A^c)^c$. □

Lemma C.4. *Let $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ be a smooth mollifier. Let $(u_n), u \in L^2(\mathbb{R}^d)$ with $\|u_n - u\|_{L^2(\mathbb{R}^d)} \to 0$. Set $u_n^\varepsilon = \eta_\varepsilon * u_n$ and $u^\varepsilon = \eta_\varepsilon * u$. Then*

$$u_n^\varepsilon \overset{n\to\infty}{\longrightarrow} u^\varepsilon \quad \text{in } C^k(K)$$

for all $k \in \mathbb{N}$ and any compact set $K \subset \mathbb{R}^d$.

Proof. Let $\alpha \in \mathbb{N}_0^d$. Using the properties of the convolution we obtain

$$\begin{aligned} \|\partial^\alpha u_n^\varepsilon - \partial^\alpha u^\varepsilon\|_\infty &= \|(u_n - u) * \partial^\alpha \eta_\varepsilon\|_\infty \\ &\le \|u_n - u\|_{L^2(\mathbb{R}^d)} \|\partial^\alpha \eta_\varepsilon\|_{L^2(\mathbb{R}^d)} \\ &\le C(\varepsilon) \|u_n - u\|_{L^2(\mathbb{R}^d)} \overset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

□

Bibliography

[AF03] Robert A. Adams and John J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[AK09] Gilles Aubert and Pierre Kornprobst. Can the nonlocal characterization of Sobolev spaces by Bourgain et al. be useful for solving variational problems? *SIAM J. Numer. Anal.*, 47(2):844–860, 2009.

[All92] Grégoire Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518, 1992.

[AM10] Burak Aksoylu and Tadele Mengesha. Results on nonlocal boundary value problems. *Numer. Funct. Anal. Optim.*, 31(12):1301–1317, 2010.

[AP09] B. Aksoylu and M. L. Parks. Variational Theory and Domain Decomposition for Nonlocal Problems. *ArXiv e-prints*, September 2009.

[AVMRTM10] Fuensanta Andreu-Vaillo, José M. Mazón, Julio D. Rossi, and J. Julián Toledo-Melero. *Nonlocal diffusion problems*, volume 165 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[Bak97] John A. Baker. Integration over spheres and the divergence theorem for balls. *Amer. Math. Monthly*, 104(1):36–47, 1997.

[BBM01] Jean Bourgain, Haïm Brezis, and Petru Mironescu. Another look at Sobolev spaces, 2001. Original research article appeared at in Optimal Control and Partial Differential Equations IOS Press ISBN 1 58603 096 5.

[BBM02] Jean Bourgain, Haïm Brezis, and Petru Mironescu. Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications. *J. Anal. Math.*, 87:77–101, 2002. Dedicated to the memory of Thomas H. Wolff.

[BD98] Andrea Braides and Anneliese Defranceschi. *Homogenization of multiple integrals*, volume 12 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998.

[Bra02] Andrea Braides. *Γ-convergence for beginners*, volume 22 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2002.

[DCKP14] Agnese Di Castro, Tuomo Kuusi, and Giampiero Palatucci. Nonlocal Harnack inequalities. *J. Funct. Anal.*, 267(6):1807–1836, 2014.

[DG75] Ennio De Giorgi. Sulla convergenza di alcune successioni d'integrali del tipo dell'area. *Rend. Mat. (6)*, 8:277–294, 1975. Collection of articles dedicated to Mauro Picone on the occasion of his ninetieth birthday.

[DGL77] E. De Giorgi and G. Letta. Une notion générale de convergence faible pour des fonctions croissantes d'ensemble. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(1):61–99, 1977.

[DGLZ12] Qiang Du, Max Gunzburger, R. B. Lehoucq, and Kun Zhou. Analysis and approximation of nonlocal diffusion problems with volume constraints. *SIAM Rev.*, 54(4):667–696, 2012.

[DGLZ13] Qiang Du, Max Gunzburger, R. B. Lehoucq, and Kun Zhou. A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws. *Math. Models Methods Appl. Sci.*, 23(3):493–540, 2013.

[DK11] B. Dyda and M. Kassmann. Regularity estimates for elliptic nonlocal operators. *ArXiv e-prints*, September 2011.

[DM93] Gianni Dal Maso. *An introduction to* Γ*-convergence*. Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston, Inc., Boston, MA, 1993.

[DPV11] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *ArXiv e-prints*, April 2011.

[DRV14] S. Dipierro, X. Ros-Oton, and E. Valdinoci. Nonlocal problems with Neumann boundary conditions. *ArXiv e-prints*, July 2014.

[Eva10] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[Fel13] Matthieu Felsinger. *Parabolic equations associated with symmetric nonlocal operators*. PhD thesis, 2013.

[FKV14] Matthieu Felsinger, Moritz Kassmann, and Paul Voigt. The dirichlet problem for nonlocal operators. *Mathematische Zeitschrift*, pages 1–31, 2014.

[Foc09] Matteo Focardi. Homogenization of random fractional obstacle problems via Γ-convergence. *Comm. Partial Differential Equations*, 34(10-12):1607–1631, 2009.

[FŌT94] Masatoshi Fukushima, Yōichi Ōshima, and Masayoshi Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.

[FRS16] J. Fernandez Bonder, A. Ritorto, and A. M. Salort. $H-$convergence result for nonlocal elliptic-type problems via Tartar's method. *ArXiv e-prints*, May 2016.

[FU12] Masatoshi Fukushima and Toshihiro Uemura. Jump-type Hunt processes generated by lower bounded semi-Dirichlet forms. *Ann. Probab.*, 40(2):858–889, 2012.

[GO08] Guy Gilboa and Stanley Osher. Nonlocal operators with applications to image processing. *Multiscale Model. Simul.*, 7(3):1005–1028, 2008.

[GT77] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Springer-Verlag, Berlin, 1977. Grundlehren der Mathematischen Wissenschaften, Vol. 224.

[HJ96] Walter Hoh and Niels Jacob. On the Dirichlet problem for pseudodifferential operators generating Feller semigroups. *J. Funct. Anal.*, 137(1):19–48, 1996.

[Kal02] Olav Kallenberg. *Foundations of modern probability.* Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[KD16] M. Kassmann and B. Dyda. Function spaces and extension results for nonlocal Dirichlet problems. *ArXiv e-prints*, December 2016.

[KM97] Juha Kinnunen and Olli Martio. Hardy's inequalities for Sobolev functions. *Math. Res. Lett.*, 4(4):489–500, 1997.

[Lej78] Yves Lejan. Mesures associées à une forme de dirichlet. applications. *Bulletin de la Société Mathématique de France*, 106:61–112, 1978.

[Lev04] S. Z. Levendorskiĭ. Pricing of the American put under Lévy processes. *Int. J. Theor. Appl. Finance*, 7(3):303–335, 2004.

[LS11] Giovanni Leoni and Daniel Spector. Characterization of Sobolev and BV spaces. *J. Funct. Anal.*, 261(10):2926–2958, 2011.

[LS14] Giovanni Leoni and Daniel Spector. Corrigendum to "Characterization of Sobolev and BV spaces" [J. Funct. Anal. 261 (10) (2011) 2926–2958]. *J. Funct. Anal.*, 266(2):1106–1114, 2014.

[LU68] Olga A. Ladyzhenskaya and Nina N. Ural′tseva. *Linear and quasilinear elliptic equations.* Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis. Academic Press, New York-London, 1968.

[McL00] William McLean. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, Cambridge, 2000.

[MD13] Tadele Mengesha and Qiang Du. Analysis of a scalar nonlocal peridynamic model with a sign changing kernel. *Discrete Contin. Dyn. Syst. Ser. B*, 18(5):1415–1437, 2013.

[MD15] Tadele Mengesha and Qiang Du. On the variational limit of a class of nonlocal functionals related to peridynamics. *Nonlinearity*, 28(11):3999–4035, 2015.

[Men12] Tadele Mengesha. Nonlocal Korn-type characterization of Sobolev vector fields. *Commun. Contemp. Math.*, 14(4):1250028, 28, 2012.

[Mos94] Umberto Mosco. Composite media and asymptotic Dirichlet forms. *J. Funct. Anal.*, 123(2):368–421, 1994.

[MR92] Zhi Ming Ma and Michael Röckner. *Introduction to the theory of (nonsymmetric) Dirichlet forms.* Universitext. Springer-Verlag, Berlin, 1992.

[MS02a] V. Maz′ya and T. Shaposhnikova. On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces. *J. Funct. Anal.*, 195(2):230–238, 2002.

[MS02b] V. Maz′ya and T. Shaposhnikova. On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces. *J. Funct. Anal.*, 195(2):230–238, 2002.

[MS15] Tadele Mengesha and Daniel Spector. Localization of nonlocal gradients in various topologies. *Calc. Var. Partial Differential Equations*, 52(1-2):253–279, 2015.

[MT97] François Murat and Luc Tartar. H-convergence. In *Topics in the mathematical modelling of composite materials*, volume 31 of *Progr. Nonlinear Differential Equations Appl.*, pages 21–43. Birkhäuser Boston, Boston, MA, 1997.

[Pon04a] Augusto C. Ponce. An estimate in the spirit of Poincaré's inequality. *J. Eur. Math. Soc. (JEMS)*, 6(1):1–15, 2004.

[Pon04b] Augusto C. Ponce. A new approach to Sobolev spaces and connections to Γ-convergence. *Calc. Var. Partial Differential Equations*, 19(3):229–255, 2004.

[Sch03] Wim Schoutens. *Lévy processes in finance.* Wiley series in probability and statistics. Wiley, Hoboken, N.J. [u.a.], 2003.

[Sch10] Russell W. Schwab. Periodic homogenization for nonlinear integro-differential equations. *SIAM J. Math. Anal.*, 42(6):2652–2680, 2010.

[Sch13] Russell W. Schwab. Stochastic homogenization for some nonlinear integro-differential equations. *Comm. Partial Differential Equations*, 38(2):171–198, 2013.

[Sil00] S. A. Silling. Reformulation of elasticity theory for discontinuities and long-range forces. *J. Mech. Phys. Solids*, 48(1):175–209, 2000.

[Spa68] S. Spagnolo. Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche. *Ann. Scuola Norm. Sup. Pisa (3) 22 (1968), 571-597; errata, ibid. (3)*, 22:673, 1968.

[SU12] René L. Schilling and Toshihiro Uemura. On the structure of the domain of a symmetric jump-type Dirichlet form. *Publ. Res. Inst. Math. Sci.*, 48(1):1–20, 2012.

[SW11] R. L. Schilling and J. Wang. Lower bounded semi-dirichlet forms associated with Lévy type operators. *ArXiv e-prints*, 2011.

[Tri06] Hans Triebel. *Theory of function spaces. III*, volume 100 of *Monographs in Mathematics.* Birkhäuser Verlag, Basel, 2006.

[Wlo87] J. Wloka. *Partial Differential Equations.* Cambridge University Press, 1987.