PhD Thesis

Path by Path Uniqueness for Stochastic Differential Equations in Infinite Dimensions

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Table of Contents

Introduction

In this thesis we study the stochastic differential equation (SDE)

$$\begin{cases} \mathrm{d}X_t = -AX_t\mathrm{d}t + f(t, X_t)\mathrm{d}t + \mathrm{d}W_t \\ X_0 = x_0 \in H, \end{cases} \tag{SDE}$$

in an infinite-dimensional separable Hilbert space H driven by a cylindrical Wiener process W with a bounded Borel measurable drift f and deterministic initial condition $x_0 \in H$.

In the above equation (SDE) $A\colon D(A) \longrightarrow H$ is a positive definite, self-adjoint, closed, densely defined linear operator such that the trace of its inverse A^{-1} is finite. On the one hand the operator A pushes the solution X towards zero. On the other hand the drift term f is only bounded and Borel measurable. Furthermore, we assume the components $(f_n)_{n\in\mathbb{N}}$ of f to decay quite rapidly as $n \to \infty$. However, no assumptions on the drift f with regards to regularity are required.

We show in this thesis that already these conditions imply the existence of a path-by-path unique solution (in the sense of A. M. Davie (see [Dav07])) to the above equation, which extends Davie's theory of path-by-path uniqueness to abstract infinite-dimensional Hilbert spaces.

SDEs have been a very active research topic in the last decades. Several approaches and notions of solutions were developed. For example the pathwise approach, where a solution X to the above equation is interpreted as a stochastic process, weak solutions, where one essentially studies the laws of the solutions, and the mild approach where a solution to the above equation (SDE) is a function X solving the mild integral equation

$$X_t(\omega) = e^{-tA}x_0 + \int_0^t e^{-(t-s)A} f(s, X_s(\omega))\ \mathrm{d}s + Z_t^A(\omega), \tag{IE$_\omega$}$$

where

$$Z_t^A := \int_0^t e^{-(t-s)A}\ \mathrm{d}W_s$$

and e^{-tA} for $t \geq 0$ denotes the semigroup with generator $-A$.

In this thesis we consider the so-called path-by-path approach, where equation (SDE) is not considered as a stochastic differential equation. In the path-by-path picture we first plug in an $\omega \in \Omega$ into the corresponding mild integral equation of (SDE) and try for every $\omega \in \Omega$ to find a (unique) continuous function $X(\omega)\colon [0, T] \longrightarrow H$ satisfying equation (IE$_\omega$), which can now be considered as an ordinary integral equation (IE), that is perturbed by an Ornstein–Uhlenbeck path $Z^A(\omega)$. If such a function can be found for almost all $\omega \in \Omega$, the map $\omega \longmapsto X(\omega)$ is called a path-by-path solution to the equation (SDE). For path-by-path *uniqueness* we require that there exists a set $\Omega_0 \subseteq \Omega$ with $\mathbb{P}[\Omega_0] = 1$ such that all solutions coincide on Ω_0.

Naturally, this notion of uniqueness is much stronger than the notion of pathwise uniqueness for solutions to stochastic differential equations. Nevertheless, we prove that equation (SDE) even admits a path-by-path unique solution. We want to emphasize that pathwise uniqueness implies that for any two solutions X and Y of equation (SDE) a set Ω_0 of full measure can be found such that X and Y coincide on Ω_0. In general, however, this set Ω_0 will depend on both X and Y. The notion of uniqueness in the path-by-path approach is much stronger, i.e. a set Ω_0 of full measure can be found such that all solutions coincide for all $\omega \in \Omega_0$, which is what we shall prove in this thesis.

The main theorem of this thesis states that there exists a *unique* mild solution to equation (SDE) in the path-by-path sense.

Theorem (Main result)

Assume that A and f fulfill Assumption 1.1.2 in Chapter 1 below. Given *any* filtered stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty[}, \mathbb{P}, (W_t)_{t\in[0,\infty[})$ there exists $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}[\Omega_0] = 1$ such that for every $\omega \in \Omega_0$ we have

$$\# \left\{ g \in \mathcal{C}([0,T], H) \mid g \text{ solves } (\mathrm{IE})_\omega \right\} = 1.$$

Since we obtain a unique solution for almost all Wiener paths $W(\omega)$, this result can also be interpreted as a uniqueness theorem for randomly perturbed ordinary differential equations (ODEs), more precisely integral equations. We refer to [Fla11] for an in-depth discussion about the various notions of uniqueness for SDEs and perturbed ODEs.

The Story in a Nutshell

Let W be a $\mathbb{R}^d$-valued Wiener process. The question whether for $f \in \mathcal{C}_b(\mathbb{R}^d, \mathbb{R}^d)$ the integral equation

$$x_t(\omega) = x_0 + \int_0^t f(x_s(\omega)) \, \mathrm{d}s + W_t(\omega)$$

has at most one solution for almost all ω has been first posed by N. Krylov and was mentioned by V. Bogachev as an open problem in [Bog95, 7.1.7]. Through I. Gyöngy the question found its way to A. M. Davie, who gave an affirmative answer in [Dav07]. Indeed, let $f\colon [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a bounded, measurable, but not necessarily continuous function. Then for almost all ω the equation

$$x_t(\omega) = x_0 + \int_0^t f(s, x_s(\omega)) \, \mathrm{d}s + W_t(\omega)$$

has at most one solution.

This result can be understood as a "regularization by noise effect" since in the absence of noise the above integral equation can admit more than one solution.

In 2011 A. M. Davie even extended upon this and proved that path-by-path uniqueness holds in the non-degenerate multiplicative noise case (see [Dav11]). Let $b\colon [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ be a non-degenerate (i.e. $b(t,x)$ is invertible) map and let the components b_{ij} be differentiable in x with $\frac{\partial b_{ij}}{\partial x_k}$ being locally Hölder continuous in (t,x), then the equation

$$x_t(\omega) = x_0 + \int_0^t f(s, x_s(\omega))\ \mathrm{d}s + \int_0^t b(s, x_s(\omega))\ \mathrm{d}W_s(\omega)$$

has at most one solution for almost all ω. Here, the second integral is defined in the sense of rough paths.

By virtue of these results, the question arose what kind of stochastic processes have such a "regularizing property".

In 2012 R. Catellier and M. Gubinelli answered this question in [CG12] by proving that fractional Brownian motion in $\mathbb{R}^d$ also possesses this "regularizing property". Let B^H be a fractional Brownian motion with Hurst parameter H. Then the equation

$$x_t(\omega) = x_0 + \int_0^t f(s, x_s(\omega))\ \mathrm{d}s + B_t^H(\omega)$$

has for almost all ω a unique solution as long as $f \in \mathcal{B}^{\alpha+1}$ with $\alpha > 1 - \frac{1}{2H}$, where $\mathcal{B}^\alpha$ denotes the Besov–Hölder space of order α. Note that for $\alpha < 0$ elements in $\mathcal{B}^\alpha$ are no longer functions. In this case the integral is defined as

$$\int_0^t f(s, x_s(\omega))\ \mathrm{d}s := \lim_{\varepsilon\to 0} \int_0^t (\rho_\varepsilon * f)(s, x_s(\omega))\ \mathrm{d}s,$$

where ρ_ε are suitable mollifiers. This result by R. Catellier and M. Gubinelli does not generalize Davie's result, because setting $H = \frac{1}{2}$ implies that the drift f is required to be in $\mathcal{C}^{1+\varepsilon}$ for an $\varepsilon > 0$. However, this suggests that on the one hand there seems to be a tradeoff between the regularity of the drift f and the regularizing effect of the noise B^H and on the other hand fractional Brownian motion becomes more regularizing the smaller H gets. For example, if $H < \frac{1}{4}$, path-by-path uniqueness holds for Schwartz distributions $f \in \mathcal{C}^{-\varepsilon}$ for sufficiently small $\varepsilon > 0$.

In 2014, by a completely different approach L. Beck, F. Flandoli, M. Gubinelli and M. Maurelli prove in [BFGM14] that path-by-path uniqueness does not only hold for SDEs, but also for SPDEs. If $f \in L^q([0,T], L^p(\mathbb{R}^d, \mathbb{R}^d))$ with

$$\frac{d}{p} + \frac{2}{q} < 1$$

(the so-called Krylov–Röckner condition), then the stochastic continuity equation

$$\mathrm{d}x_t + \mathrm{div}(f(x_t))\mathrm{d}t + \sigma(x_t \circ \mathrm{d}W_t) = 0$$

exhibits path-by-path uniqueness. Furthermore, if the Krylov–Röckner condition holds for $\mathrm{div}\, f$ then the stochastic transport equation

$$\mathrm{d}x + f\cdot\nabla x\,\mathrm{d}t + \sigma\nabla x \circ \mathrm{d}W_t = 0$$

exhibits path-by-path uniqueness as well.

Later that year remarkable simplifications to the original proof of A. M. Davie have been made by A. Shaposhnikov in [Sha14]. One of the most important inequalities ([Dav07, Proposition 2.2]), which heavily relied on explicit Gaussian calculations, has been proven in a much more abstract setting using time-reversal as well as H. Föllmer, P. Protter and A. Shiryaev's Itô-formula for time-reversed Brownian motion. This opened the door to analyze the question of path-by-path uniqueness for much more complicated noises namely stochastic processes, that are strong solutions to an SDE as long as the coefficients of the SDE fulfill some quite mild conditions.

One year later, in 2015, E. Priola proved in [Pri15] that the Brownian motion W of A. M. Davie can be replaced by a Lévy process L if the Lévy measure ν of L fulfills the condition

$$\int_{|x|>1} |x|^{\theta} \, \nu(\mathrm{d}x) < \infty$$

for some $\theta > 0$. This shows that continuity of the noise term is not a requirement for path-by-path uniqueness (or Davie type uniqueness, a term coined by E. Priola) to hold.

Finally, in 2016 O. Butkovsky and L. Mytnik showed in [BM16] that path-by-path uniqueness holds for the stochastic heat equation

$$\frac{\partial x}{\partial t} = \frac{1}{2}\frac{\partial^2 x}{\partial z^2} + b(x(t,z)) + \dot{W}(t,z),$$

where $b \in \mathcal{B}(\mathbb{R},\mathbb{R})$ is just bounded, measurable and $\dot{W}$ denotes space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$. In there article it turns out that the smoothing of the Laplace operator and the "regularization effect" of space-time white noise is sufficient to proof path-by-path uniqueness if b is only bounded. Furthermore, they showed that the set Ω_0, on which all solutions coincide, is independent of the initial condition as long as the initial condition belongs to a specific class.

In conclusion the initial result of A. M. Davie in [Dav07] has been widely extended. However, there are still a lot of open questions.

- Does the result hold if the noise is not of Gaussian nature and does not contain a Gaussian component?
- Is there a tradeoff between the size of the drift and regularizing effect of the noise? If yes, is the result in [BFGM14] the sub-critical case?
- Is there a concrete counter-example, where pathwise uniqueness holds, but path-by-path uniqueness fails to hold?
- Is there an explicit analytic description of the null-set of paths, which has to be excluded?
- Is the regularization phenomenon due to the finite-dimensionality of the state-space or are there non-trivial examples in infinite-dimensional Hilbert or Banach spaces?

In this thesis we enlarge the set of the results of path-by-path uniqueness by giving an affirmative answer to the last question.

Our Contribution

We present a general framework for the analysis of path-by-path uniqueness for equations of type (SDE). We introduce the effective dimension (see Definition 3.1.2) of a space which, similarly to the Kolmogorov ε-entropy, measures the size of an (infinite-dimensional) totally bounded set. For a given set $Q \subseteq \mathbb{R}^{\mathbb{N}}$ the effective dimension is a sequence $(\mathrm{ed}(Q)_m)_{m\in\mathbb{N}}$ taking values in $\mathbb{N} \cup \{\infty\}$ that measures the size of the said set.

For a given drift f in equation (SDE) we associate such a set Q to f. Our framework allows to handle arbitrary sets Q as long as they are of finite *effective* dimension (see Definition 3.1.2).

Moreover, given a set $Q \subseteq \mathbb{R}^{\mathbb{N}}$ as above we introduce regularizing noises (see Definition 5.1.1). We define regularizing noises as stochastic processes obeying certain regularity assumptions, which depend on the set $Q \subseteq \mathbb{R}^{\mathbb{N}}$. Examples of regularizing noises are Brownian motion in $\mathbb{R}^d$ (where $Q \subseteq \mathbb{R}^d$ can be any bounded set) and Ornstein–Uhlenbeck processes on a Hilbert space H.

Given the effective dimension of a set Q and a regularization noise X we present estimates for the map

$$\varphi_{n,k} \colon x \longmapsto \int_{k2^{-n}}^{(k+1)2^{-n}} f(s, X_s(\omega) + x) - f(s, X_s(\omega)) \,\mathrm{d}s$$

(and more complicated expressions involving $\varphi_{n,k}$) which are essential for our analysis.

On this abstract level we prove that, if the non-linearity f is Q-valued, *every* Q-regularizing noise X (with certain index and order (as defined in Definition 5.1.1) depending on the effective dimension of Q) regularizes our SDE in the sense that path-by-path uniqueness holds.

This means that given the SDE

$$\mathrm{d}Y_t = AY_t\mathrm{d}t + f(t, Y_t)\mathrm{d}t + \mathrm{d}Z_t,$$

where Z is some stochastic process, and setting

$$X_t := \int_0^t e^{-(t-s)A} \,\mathrm{d}Z_s$$

then if the non-linearity f of the SDE is Q-valued and X is a Q-regularizing noise, the above SDE admits a path-by-path unique solution as long as $h \geq \frac{1}{2}$ and

$$\frac{1-h}{h} < \frac{2\alpha\gamma}{2+\alpha+2\gamma} \leq \frac{1}{h}.$$

Here, h is the index, α the order of the regularizing noise X and γ (see Definition 3.2.1) measures the effective dimension of Q.

On a concrete level, taking $H := \mathbb{R}^d$ (and hence $\gamma = \infty$ since the effective dimension of $\mathbb{R}^d$ is trivial see Proposition 3.1.5 below), $A := 0$ and Z to be Brownian motion on $\mathbb{R}^d$, which is a regularizing noise with $h = \frac{1}{2}$ and $\alpha = 2$, then the above condition is fulfilled and we therefore recover A. M. Davie's result of [Dav07].

If H is a separable Hilbert space, A as in the beginning of this introduction (i.e. such that $\operatorname{trace} A^{-1} < \infty$) and Z a cylindrical Wiener process, then X is an H-valued Ornstein–Uhlenbeck process. We prove that such an Ornstein–Uhlenbeck process is a regularizing noise with $h = \frac{1}{2}$ and $\alpha = 2$ (see Corollary 5.2.3). The above condition is, therefore, fulfilled for every $\gamma > 2$, so that the above equation (SDE) has a path-by-path unique solution for all Q-valued non-linearities f as long as the effective dimension of Q is bounded by

$$\operatorname{ed}(Q)_m \leq C(\ln(m+1))^{1/\gamma}.$$

This improves a result previously obtained by the author in [Wre17].

Outline of the Proof

First, we observe that the main result would be trivial if f were Lipschitz continuous in the spatial variable. Let x and y be two solutions of $(\mathrm{IE})_\omega$. We then have

$$|x(t) - y(t)|_H = \left| \int_0^t e^{-(t-s)A} \left(f(s, x(s)) - f(s, y(s)) \right) \, \mathrm{d}s \right|_H \leq \operatorname{Lip}(f) \int_0^t |x(s) - y(s)|_H \, \mathrm{d}s.$$

So, by Gronwall's Inequality we obtain $x = y$.

In the case when f is not Lipschitz continuous in the second parameter, we have to analyze the equation more carefully. Introducing the variable $u := x - y$, the above expression then reads

$$|u(t)|_H = \left| \int_0^t e^{-(t-s)A} \left(f(s, u(s) + y(s)) - f(s, y(s)) \right) \, \mathrm{d}s \right|_H .$$

In our analysis we show that for $x, x' \in H$ we have the estimate

$$\left| \int_0^t e^{-(t-s)A} \left(f(s, x + y(s)) - f(s, x' + y(s)) \right) \, \mathrm{d}s \right|_H \lesssim |x - x'|_H + \delta_t,$$

where $\delta_t > 0$ is a number which can be made arbitrarily small by letting $t \to 0$ and $\lesssim$ means that the left-hand is bounded by the right-hand up to a multiplicative constant C_ε on a set Ω_ε with $\mathbb{P}[\Omega_\varepsilon^c] \leq \varepsilon$. This estimate acts as a substitute for the Lipschitz continuity of f.

Since y is a solution to (SDE) in the mild sense we have

$$y(t) = \int_0^t e^{-(t-s)A} f(s, y(s)) \, \mathrm{d}s + Z_t^A,$$

where Z^A is an H-valued Ornstein–Uhlenbeck process with drift term A driven by a cylindrical Brownian motion and for simplicity we assume the initial condition to be zero. Since f is bounded, Novikov's condition is fulfilled so that by Girsanov's Transformation Theorem we can find a new measure ν, which is equivalent to our original measure $\mathbb{P}$, so that y becomes an Ornstein–Uhlenbeck process $\tilde{Z}^A := y$. Under this measure our equation for u reads

$$|u(t)|_H = \left| \int_0^t e^{-(t-s)A} \left(f(s, u(s) + \tilde{Z}_s^A) - f(s, \tilde{Z}_s^A) \right) \, \mathrm{d}s \right|_H . \tag{1}$$

The aim is now to analyze the regularity of the right-hand side in order to obtain an estimate, which can be used to obtain a Gronwall-type estimate. Here, we have to exploit the effect of the noise $\tilde{Z}^A$. The idea is that the noise not only provides additional regularity in expectation (which would only be enough to prove merely pathwise uniqueness), but the path $t \longmapsto \tilde{Z}_t^A(\omega)$ itself already regularize the equation enough, so that it is possible to obtain regularizing behavior for a large class of $\omega \in \Omega$.

To see this ω-wise regularizing behavior let us consider the one-dimensional case when f is time-independent and the noise is a standard Brownian motion, i.e.

$$u(t) = \int_0^t f(u(s) + B_s(\omega)) \, \mathrm{d}s.$$

Since u is a Lipschitz continuous function and B is only β-Hölder continuous for $\beta < \frac{1}{2}$, we expect that the oscillations of B are faster than the oscillations of u. Therefore, for small times it is not unreasonable to expect that

$$u(t) \approx \int_0^t f(u + B_s(\omega)) \, \mathrm{d}s,$$

where $u := u(s)$ for some fixed $s \in [0, t]$. We now rewrite the expression using the occupation measure L of B as follows

$$\int_{\mathbb{R}} f(u + x) L([0, t], B(\omega), \mathrm{d}x).$$

Recall that the occupation measure of a Brownian motion in one dimension has a density α w.r.t. Lebesgue measure, so that we can simplify the above to

$$\int_{\mathbb{R}} f(u + x) \alpha([0, t], B(\omega), x) \, \mathrm{d}x.$$

Since we integrate over the whole space w.r.t. Lebesgue measure we can identify the integral as a convolution $f \star \alpha$ between f and α. In conclusion we have

$$u(t) \approx (f \star \alpha([0, t], B(\omega), \cdot))(u).$$

Due to the fact that α is for almost all ω Hölder continuous of order β for $\beta < \frac{1}{2}$ we effectively have replaced the original drift f by the much more regular $f \star \alpha$. Note that the entire argument has to be ω-wise since we are interested in ω-wise regularization. To establish pathwise uniqueness one could obtain a stronger regularizing effect by using for example that the probability density function of B is of class C^∞. However, since we establish path-by-path uniqueness we have to use the somewhat “deeper” path properties of the noise.

Let us now go back to equation (1)

$$|u(t)|_H = \left| \int_0^t e^{-(t-s)A}(f(s,u(s)+\tilde{Z}_s^A) - f(s,\tilde{Z}_s^A)) \,\mathrm{d}s \right|_H$$

and analyze the right-hand side for small times. For $n \in \mathbb{N}$, $k \in \{0, ..., 2^n - 1\}$ and $x \in H$ we set

$$\varphi_{n,k}(f;x,\omega) := \int_{k2^{-n}}^{(k+1)2^{-n}} e^{-(t-s)A}(f(s,x+\tilde{Z}_s^A) - f(s,\tilde{Z}_s^A)) \,\mathrm{d}s.$$

For convenience we usually write $\varphi_{n,k}(x)$ instead of $\varphi_{n,k}(f;x,\omega)$. We want to prove that the map $x \longmapsto \varphi_{n,k}(x)$ exhibits some kind of regularity due to the noise. We obviously have $|\varphi_{n,k}(x)|_H \lesssim 2^{-n}$, however we would like to prove something along the lines of $|\varphi_{n,k}(x)|_H \lesssim |x|_H^\beta$ for some $\beta > 0$. Our approach is the following: First assume that f is continuously differentiable in the spatial variable with derivative f'. In this case we can consider

$$Y := \left| \int_0^1 f'(s,\tilde{Z}_s^A) \,\mathrm{d}s \right|_H .$$

We prove that the random variable Y is exponentially square-integrable (see Theorem 4.2.2), i.e. there exists $\alpha > 0$ such that

$$\mathbb{E}\left[\alpha Y^2\right] < \infty. \tag{2}$$

To prove (2) we follow A. Shaposhnikov's approach (see [Sha14]) who proved a similar result for finite-dimensional Brownian motions. His idea is as follows: Consider the process

$$[0,1] \ni t \longmapsto \int_0^t f'(s,\tilde{Z}_s^A) \,\mathrm{d}s.$$

This process can be decomposed as the sum of a forward and backward semi-martingale. Furthermore, these semi-martingales can be identified as forward and backward Itô integrals, so that we obtain

$$\int_0^t f'(s,\tilde{Z}_s^A) \,\mathrm{d}\langle \tilde{Z}^A \rangle_s = \int_0^t f(s,\tilde{Z}_s^A) \,\mathrm{d}^*\tilde{Z}_s^A - \int_0^t f(s,\tilde{Z}_s^A) \,\mathrm{d}\tilde{Z}_s^A, \tag{3}$$

where $\mathrm{d}\tilde{Z}_s$ denotes the Itô integral and $\mathrm{d}^*\tilde{Z}_s$ the backwards Itô integral w.r.t. $\tilde{Z}_s$. The backwards integrals can be rewritten as an Itô forward integral by employing the time-reversed process $t \longmapsto \tilde{Z}_{1-t}$. Since an Ornstein–Uhlenbeck process is an Itô diffusion process with particularly nice coefficients, the time-reversed process is again an Itô diffusion process and the coefficients can be explicitly calculated. Using semi-martingale decomposition and the Burkholder–Davis–Gundy Inequality, we can estimate the right-hand side of (3) by the bracket processes of the two integrals. Since f is bounded and $\langle \tilde{Z} \rangle_t = t$ we complete the proof of (2).

Using Chebychev's Inequality we easily obtain a concentration of measure result, namely

$$\mathbb{P}\left[\left|\int_0^t f(\tau,\tilde{Z}_\tau^A+x)-f(\tau,\tilde{Z}_\tau^A)\,\mathrm{d}\tau\right|_H > \eta\sqrt{t}|x|_H\right] \leq Ce^{-c\eta^2}$$

and by using that $\tilde{Z}^A$ is a Markov process we even obtain (see Theorem 5.2.2)

$$\mathbb{P}\left[\left|\int_s^t f(\tau,\tilde{Z}_\tau^A+x)-f(\tau,\tilde{Z}_\tau^A)\,\mathrm{d}s\right|_H > \eta\sqrt{t-s}|x|_H \middle| \mathcal{G}_r\right] \leq Ce^{-c\eta^2}$$

for $r<s<t$, where $\tilde{Z}^A$ is adapted to the filtration $(\mathcal{G}_r)_{r\in[0,\infty[}$. In conclusion we have

$$\mathbb{P}[|\varphi_{n,k}(x)|_H > \eta 2^{-n/2}|x|_H] \leq Ce^{-c\eta^2}.$$

Here, we see that we lost some time regularity, since we only have $2^{-n/2}$ instead of 2^{-n}, however we gained regularity in space.

In order to get a "$\mathbb{P}$-a.s. version" of this estimate we use this faster than exponential decay to prove a uniform estimate of the following kind

$$\mathbb{P}\left[\bigcup_{n\in\mathbb{N}}\bigcup_{k=0}^{2^n-1}\bigcup_x\left\{|\varphi_{n,k}(x)|_H > \tilde{\eta}2^{-n/2}|x|_H\right\}\right] \stackrel{\tilde{\eta}\to\infty}{\longrightarrow} 0,$$

where x runs through a countable, dense subset of H. This countable, dense subset will be the union of nested finite lattices in the space where f takes its values in. These approximating lattices are fine-tuned, so that we obtain

$$|\varphi_{n,k}(x)|_H \lesssim n^{\frac{1}{2}+\frac{1}{\gamma}}2^{-n/2}\left(|x|_H+2^{-2^n}\right), \tag{φ-1}$$

(see Theorem 6.1.5) where $\gamma>0$ is a parameter controlling the decay of $(f_n)_{n\in\mathbb{N}}$, i.e. the components of the drift f of (SDE). Here, by going to a "$\mathbb{P}$-a.s. version" we loose some time as well as space regularity. The term $n^{\frac{1}{2}}2^{-n/2}$ seems unavoidable since for just Brownian motion the increments of length h are, according to Lévy's modulus of continuity theorem, of size $\sqrt{2h\ln(1/h)}$, so for $h=2^{-n}$ we obtain precisely the same. The term $n^{\frac{1}{\gamma}}$ reflects the fact that we work in infinite dimensions. This term is only of polynomial order due to the fact that we assume a very fast decay of $(f_n)_{n\in\mathbb{N}}$ (see Assumption 1.1.2 below). The term 2^{-2^n} is artificially created. The actual estimate is of order

$$|x|_H\ln(1-\ln(|x|_H))^{1/\gamma}.$$

However, it is much easier to estimate this by $|x|_H+2^{-2^n}$ and manipulate each term separately than dealing with iterated logarithms.

Furthermore, we obtain the following estimate (see Theorem 6.2.1) for two points x and y

$$|\varphi_{n,k}(x)-\varphi_{n,k}(y)|_H \lesssim \sqrt{n}2^{-\delta n}|x-y|_H+2^{-2^{\theta_\delta n}}, \tag{φ-2}$$

where $0<\delta<\frac{1}{2}$ and θ_δ depends only on δ and γ (see Theorem 6.2.1 for the definition of θ_δ). However, θ_δ vanishes when $\delta\to\frac{1}{2}$ and goes to $\frac{\gamma}{\gamma+2}$ for $\delta\to 0$. We therefore have a tradeoff between the two terms. The reason why this estimate is weaker than the previous one is due to the fact that we have to consider the event

$$\bigcup_{n\in\mathbb{N}} \bigcup_{k=0}^{2^n-1} \bigcup_x \bigcup_y \left\{ |\varphi_{n,k}(x) - \varphi_{n,k}(y)|_H > \tilde{\eta} 2^{-n/2} |x-y|_H \right\},$$

where *both* x and y run through a countable, dense set increasing the probability (especially in infinite dimensions) of the above event vastly.

Since our estimates only hold on a dense subset we have to prove that $x \longmapsto \varphi_{n,k}(x)$ is continuous in a suitable topology. In fact, we also need to prove that the map

$$h \longmapsto \int_0^1 f(s, \tilde{Z}_s^A + h(s)) - f(s, \tilde{Z}_s^A) \, \mathrm{d}s$$

is continuous for a sufficiently large class of h (see Theorem 7.2.1). If f were continuous this would trivially follow from Lebesgue's Dominated Convergence Theorem. However, since we do not assume any regularity for f we have to approximate f by continuous functions $(f_m)_{m\in\mathbb{N}}$ and estimate

$$\int_0^1 f(s, \tilde{Z}_s^A + h(s)) - f_m(s, \tilde{Z}_s^A + h(s)) \, \mathrm{d}s.$$

We construct f_m so that the set $\{f \neq f_m\}$ is open and of small mass w.r.t. the measure $\mathrm{d}t \otimes \mathbb{P}(Z_t^A)$, i.e. the product between the one-dimensional Lebesgue measure and the image measure of Z_t^A under $\mathbb{P}$. We, therefore, have to prove that

$$\int_0^1 \mathbb{1}_U(s, \tilde{Z}_s^A + h(s)) \, \mathrm{d}s \leq \varepsilon$$

uniformly for all h (see Lemma 7.1.4). Since U is open, $\mathbb{1}_U$ is lower semi-continuous and hence we are allowed to approximate h from below by piecewise constant functions h_n. For these h_n we can use our previous estimate (φ-2) to obtain the required result and therefore extend estimates (φ-1) as well as (φ-2) to the whole space.

It turns out that in the final part of the proof we have to consider terms of type

$$\sum_{q=1}^{N} |\varphi_{n,k+q}(x_{q+1}, x_q)|_H,$$

where

$$\varphi_{n,k+q}(x,y) := \varphi_{n,k+q}(x) - \varphi_{n,k+q}(y)$$

for a sequence of points $\{x_q \in H | q = 1, ..., N\}$. Using just the estimate (φ-2) and obtaining an estimate of order $\sqrt{n}2^{-\delta n}N$ for each term under the sum is, unfortunately, insufficient to prove the final theorem since N will later be chosen to be of order 2^n. The technique to overcome this is two-fold:

On the one hand the $\varphi_{n,k+q}$-terms have to "work together" to achieve an expression of order N. However, since $\{\varphi_{n,k+q}(x_q) \mid q = 1, ..., N\}$ are "sufficiently uncorrelated" the law of large numbers tells us to expect on average an estimate of order $\sqrt{N}$.

On the other hand in later applications (see Lemma 9.2.3) x_q will be samples from the solution of the integral equation (IE$_\omega$), so that it is reasonable to assume that $|x_{q+1} - x_q|_H \approx |\varphi_{n,k+q}(x_q)|_H$. Exploiting this enables us to use *both* of our previous established estimates for every term $|\varphi_{n,k+q}(x_{q+1}, x_q)|_H$.

Using both techniques, we end up with an estimate of order $\mathcal{O}(2^{-n}N)$ (see Corollary 8.2.2). More precisely, we obtain

$$\sum_{q=1}^{N} |\varphi_{n,k+q}(x_{q+1}, x_q)|_H \leq C \left[2^{-n} \sum_{q=0}^{N} |x_q|_H + 2^{-\delta n} \sum_{q=0}^{N} |\gamma_{n,k,q}|_H + 2^{-3n/4} |x_0|_H + N 2^{-2^{\theta} \delta n} \right]. \quad (\varphi\text{-}3)$$

Here, $\gamma_{n,k,q}$ is defined as

$$\gamma_{n,k,q} := x_{q+1} - x_q - \varphi_{n,k+q}(x_q),$$

i.e. the error between x_{q+1} and the Euler approximation of x_{q+1} given x_q.

With all this technical machinery at our disposal we can now elaborate the main proof. Recall that our aim is to prove that given a function u solving

$$u(t) = \int_0^t e^{-(t-s)A} \left(f(s, \tilde{Z}_s^A(\omega) + u(s)) - f(s, \tilde{Z}_s^A(\omega)) \right) \, \mathrm{d}s$$

we have to show that $u \equiv 0$. First, observe that for integers $n \in \mathbb{N}$ and $k \in \{0, ..., 2^n - 1\}$ we obtain

$$\begin{aligned} |u((k+1)2^{-n}) - u(k2^{-n})|_H &\approx \left| \int_0^t e^{-(t-s)A} \left(f(s, \tilde{Z}_s^A(\omega) + u(s)) - f(s, \tilde{Z}_s^A(\omega)) \right) \, \mathrm{d}s \right|_H \\ &= |\varphi_{n,k}(u(\,\cdot\,))|_H. \end{aligned}$$

Let u_ℓ be the sequence of functions, which are constant on the dyadic intervals $[k2^{-\ell}, (k+1)2^{-\ell}[$, converge to u and fulfill the property

$$u(k2^{-\ell}) = u_\ell(k2^{-\ell}).$$

Using the above mentioned approximation result (Theorem 7.2.1), we obtain that

$$|u((k+1)2^{-n}) - u(k2^{-n})|_H \approx \lim_{\ell \to \infty} |\varphi_{n,k}(u_\ell(\,\cdot\,))|_H.$$

Rewriting the limit as a telescoping sum we can express the above by

$$|\varphi_{n,k}(u_n(\,\cdot\,))|_H + \sum_{\ell=n}^{\infty} |\varphi_{n,k}(u_{\ell+1}(\,\cdot\,), u_\ell(\,\cdot\,))|_H.$$

Splitting the integrals and using the property $u(k2^{-\ell}) = u_\ell(k2^{-\ell})$ on each of the dyadic intervals of size $2^{-\ell}$ we can rewrite this is in the somewhat more complicated form (see Lemma 9.1.9)

$$|\varphi_{n,k}(u(k2^{-n}))|_H + \sum_{\ell=n}^{\infty} \sum_{r=k2^{\ell+1-n}}^{(k+1)2^{\ell+1-n}} \left| \varphi_{\ell,r} \left(u((r+1)2^{-\ell-1}), u(r2^{-\ell-1}) \right) \right|_H.$$

For the first expression we use the estimate (φ-1) (see Lemma 9.2.1) to get an expression of order (in the sense of $\lesssim$)

$$n^{\frac{1}{2}+\frac{1}{\gamma}} 2^{-n/2} \left(|u(k2^{-n})|_H + 2^{-2^n} \right).$$

We split the second sum into two cases. If ℓ is "large" (i.e. $\ell \geq N$), we use estimate (φ-2) (see Lemma 9.2.2) to obtain

$$\sum_{\ell=N}^{\infty} \sum_{r=k2^{\ell+1-n}}^{(k+1)2^{\ell+1-n}} \sqrt{\ell} 2^{-\delta\ell} |u((r+1)2^{-\ell-1}) - u(r2^{-\ell-1})|_H + 2^{-2^{\theta\delta\ell}}.$$

and use that u is Lipschitz continuous, which yields an estimate of order (in the sense of $\lesssim$)

$$\sum_{\ell=N}^{\infty} \sum_{r=k2^{\ell+1-n}}^{(k+1)2^{\ell+1-n}} \sqrt{\ell} 2^{-\delta\ell} 2^{-\ell} \leq \sum_{\ell=N}^{\infty} \sqrt{\ell} 2^{-\delta\ell} 2^{-n} \leq \sum_{\ell=N}^{\infty} 2^{-\delta\ell/2} 2^{-n} = 2^{-\delta N/2} 2^{-n}.$$

For small ℓ we use the estimate (φ-3) (see Lemma 9.2.3) to obtain

$$\sum_{\ell=n}^{N} \left[2^{-\ell} \sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}} |u(r2^{-\ell})|_H + 2^{-\delta\ell} \sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}} |\gamma_{\ell,r}|_H + 2^{-\ell/2} |u(k2^{-n})|_H + 2^{-2^{\theta\delta\ell}} \right].$$

Here, $\gamma_{\ell,r}$ is the error between $u((r+1)2^{-\ell})$ and the Euler approximation of $u((r+1)2^{-\ell})$ given $u(r2^{-\ell})$. We can express this as

$$\gamma_{\ell,r} = u((r+1)2^{-\ell}) - u(r2^{-\ell}) - \varphi_{\ell,r}(u(r2^{-\ell})).$$

We note that we have already established that

$$|u((r+1)2^{-\ell} - u(r2^{-\ell})|_H \approx |\varphi_{\ell,r}(u(r2^{-\ell}))|_H + \sum_{\ell'=\ell}^{\infty} \sum_{r'=r2^{\ell+1-n}}^{(r+1)2^{\ell+1-n}} \left| \varphi_{\ell',r'} \left(u((r'+1)2^{-\ell'-1}), u(r'2^{-\ell'-1}) \right) \right|_H,$$

so that it is natural to estimate the Euler approximation error in the following way

$$\begin{aligned} |\gamma_{\ell,r}|_H &= |u((r+1)2^{-\ell}) - u(r2^{-\ell}) - \varphi_{\ell,r}(u(r2^{-\ell}))|_H \\ &\approx \sum_{\ell'=\ell}^{\infty} \sum_{r'=r2^{\ell+1-n}}^{(r+1)2^{\ell+1-n}} \left| \varphi_{\ell',r'} \left(u((r'+1)2^{-\ell'-1}), u(r'2^{-\ell'-1}) \right) \right|_H. \end{aligned}$$

The right-hand side is similar to the expression we would like to estimate. Hence, by assuming that n (and therefore ℓ) is sufficiently large, the term in front of the Euler error $2^{-\delta\ell}$ can be made smaller than $\frac{1}{2}$, so that the Error term $|\gamma_{\ell,r}|_H$ is $\frac{1}{2}$ times a term, that is already on the left-hand side of the inequality. From this we deduce that the second term of (4) for small ℓ is bounded by

$$\sum_{\ell=n}^{N} \left[2^{-\ell} \sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}} |u(r2^{-\ell})|_H + 2^{-\ell/2} |u(k2^{-n})|_H + 2^{-2^{\theta\delta\ell}} \right].$$

We are left with estimating the term

$$\sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}} |u(r2^{-\ell})|_H.$$

For this expression we use the following trick (see Lemma 9.1.7):

$$\sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}} |u(r2^{-\ell})|_H \leq 2 \sum_{r=k2^{\ell-1-n}}^{(k+1)2^{\ell-1-n}} |u(r2^{-(\ell-1)})|_H + \sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}} |u((r+1)2^{-\ell}) - u(r2^{-\ell})|_H.$$

The terms in the second sum can be rewritten as $|\varphi_{\ell,r}(u(r2^{-\ell}))|_H$ and estimated in a similar way as before. For the first sum we can perform the same trick as before $(\ell - n)$-times until we are left with a sum containing only the single term $|u(k2^{-n})|_H$, which is fine as long as the growth of the constant in front of this term is controlled.

Altogether we obtain the estimate

$$|u((k+1)2^{-n}) - u(k2^{-n})|_H \lesssim 2^{-n}|u(k2^{-n})|_H \left[n^{\frac{1}{2}+\frac{1}{\gamma}}2^{n/2} + N\right],$$

where N is the threshold that controls our cases, i.e. if $\ell \geq N$, we consider ℓ to be "large", and ℓ to be "small", if $\ell < N$.

By letting N be of the order $\ln(1/|u(k2^{-n})|_H)$ and using that $n^{\frac{1}{2}+\frac{1}{\gamma}}2^{n/2} \leq N$ (which requires $|u(k2^{-n})|_H$ to be sufficiently small) we establish (see Theorem 9.2.4) that

$$|u((k+1)2^{-n}) - u(k2^{-n})|_H \lesssim 2^{-n}|u(k2^{-n})|_H \ln(1/|u(k2^{-n})|_H).$$

From this we use a discrete log-type Grownwall inequality (see Lemma 2.2.1) to deduce that u must vanish at all dyadic points and hence by continuity vanish everywhere. Since u is defined as the difference of two solutions this established path-by-path uniqueness and completes the proof of the main result.

We note that for the simplification of the above exposition we have only described the case when the noise term inside $\varphi_{n,k}$ is an H-valued Ornstein–Uhlenbeck process $\tilde{Z}^A$. In the following chapters we, of course, consider the general case when $\tilde{Z}^A$ is replaced by a general regularizing noise X. The special Ornstein–Uhlenbeck case then follows by setting $h = \frac{1}{2}$ and $\alpha = 2$.

Structure of the Thesis

This thesis consists of three parts. In the first part we introduce the Girsanov transformation (Proposition 1.2.1) used to reduce the problem at hand by a slightly simpler one in Chapter 1.
In Chapter 2 we discuss Gronwall inequalities namely linear (Lemma 2.1.1) and logarithmic ones (Lemma 2.2.1). A logarithmic Gronwall inequality is one of the main ingredients used in the proof of the main result (Corollary 9.2.5).
In Chapter 3 we introduce the effective dimension of cuboids (Definition 3.1.2), which are infinite Cartesian products of intervals. Here, we also introduce the set Q^γ (Definition 3.2.1) and calculate its effective dimensions (Lemma 3.2.2).

In the second part we focus on Hilbert space-valued Ornstein–Uhlenbeck processes. In Chapter 4 we show the exponential integrability of certain random variables (Theorem 4.2.2), where a Hilbert space-valued Ornstein–Uhlenbeck process is the source of noise. This extends the results of A. M. Davie and A. Shaposhnikov (see [Dav07] and [Sha14]) from Brownian motion to Ornstein–Uhlenbeck processes. Here, we first consider one-dimensional Ornstein–Uhlenbeck processes (Proposition 4.1.3) and reduce the Hilbert space case to the one-dimensional case.
In Chapter 5 we introduce so-called "regularizing noises" (Definition 5.1.1) and by using the results obtained in the previous chapter show that Hilbert space-valued Ornstein–Uhlenbeck processes are regularizing noises (Corollary 5.2.3).

In the third part we consider an abstract noise, which is "regularizing" as defined in Chapter 5, and prove the main result.
In Chapter 6 we introduce the maps $\varphi_{n,k}$ (Definition 6.1.1) for a given regularizing noise and prove two estimates (Theorem 6.1.5 and 6.2.1) for this map.
We use these estimates in Chapter 7 to show that each $\varphi_{n,k}$ is a continuous map w.r.t. a certain topology (Theorem 7.2.1). The obtained continuity of $\varphi_{n,k}$ is used to extend the estimates obtained in Chapter 6 to a larger space (Corollary 7.2.2).
In Chapter 8 we consider sums of terms in $\varphi_{n,k}$. We introduce Euler approximation sequences and prove an estimate for these kinds of terms if the argument forms an Euler approximation sequence (Lemma 8.1.3). In the second section of Chapter 8 we approximate general sequences by an Euler approximation sequence to obtain a general result for sums of terms in $\varphi_{n,k}$ (Theorem 8.2.1).
In Chapter 9, the last chapter required to prove the main result of this thesis, we use the results of Chapter 6 to 8 to prove a logarithmic Gronwall type estimate (Theorem 9.2.4) for the reduced problem, obtained by the Girsanov transformation in Chapter 1 (see Proposition 1.2.1), and thus a simple application of the results obtained in Chapter 2 (see Corollary 2.2.2) completes the proof of the main result (Corollary 9.2.5).
Finally, we formulate several corollaries of the main result in Chapter 10.

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Part I

Preliminaries

1 Girsanov Transformation

In this chapter we introduce the precise setting, state the main result (Theorem 1.1.3) and reduce the main result to a slightly simpler problem using a Girsanov transformation (see Proposition 1.2.1).

1.1 Framework & Main result

Let H be a separable Hilbert space over $\mathbb{R}$ and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty[}, \mathbb{P})$ be a filtered stochastic basis with sigma-algebra $\mathcal{F}$, a right-continuous, normal filtration $\mathcal{F}_t \subseteq \mathcal{F}$ and a probability measure $\mathbb{P}$. Let $(W_t)_{t\in[0,\infty[}$ be a cylindrical $\mathcal{F}_t$-Wiener process taking values in $\mathbb{R}^{\mathbb{N}}$. Let $A\colon D(A) \longrightarrow H$ be a positive definite, self-adjoint, linear operator such that A^{-1} is trace-class with trivial kernel. Hence, there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H and a sequence of positive numbers $(\lambda_n)_{n\in\mathbb{N}}$ such that

$$Ae_n = \lambda_n e_n, \qquad \lambda_n > 0,\ \forall n \in \mathbb{N}.$$

Furthermore, we define

$$\sum_{n\in\mathbb{N}} \lambda_n^{-1} =: \Lambda < \infty. \tag{1.1.1}$$

By fixing this basis $(e_n)_{n\in\mathbb{N}}$ we identify H with ℓ^2, so that $H \cong \ell^2 \subseteq \mathbb{R}^{\mathbb{N}}$.

We study the following stochastic differential equation (SDE)

$$\begin{cases} \mathrm{d}x(t) = -Ax(t)\,\mathrm{d}t + f(t, x(t))\,\mathrm{d}t + \mathrm{d}W_t \\ x(0) = x_0, \end{cases} \tag{SDE}$$

where $f\colon [0,T]\times H \longrightarrow H$ is a bounded, Borel measurable function and $x_0 \in H$. We consider the mild form for a given $\omega \in \Omega$ of the above SDE i.e. a solution x satisfies $\mathbb{P}$-a.s.

$$x(t) = e^{-tA}x_0 + \int_0^t e^{-(t-s)A} f(s, x(s))\,\mathrm{d}s + \left(\int_0^t e^{-(t-s)A}\,\mathrm{d}W_s\right)(\omega), \qquad \forall t \in [0,T]. \tag{IE$_\omega$}$$

where e^{-tA} denotes the semigroup of the operator $-A$ at time $t \geq 0$. For a given cylindrical Wiener process $(W_t)_{t\in[0,\infty[}$ we define the H ($\cong \ell^2$)-valued Ornstein–Uhlenbeck $(Z_t^A)_{t\in[0,\infty[}$ with drift term A by

$$Z_t^A := \int_0^t e^{-(t-s)A}\,\mathrm{d}W_s,$$

Note that for almost all $\omega \in \Omega$ the sample paths of Z^A are continuous and that we have $Z_0^A = 0$. Furthermore, notice that Z^A is a mild solutions to the following stochastic differential equation.

$$\mathrm{d}Z_t^A = -AZ_t^A \mathrm{d}t + \mathrm{d}W_t.$$

Additionally, we define the projections

$$\pi_t(f) := f(t), \qquad \forall f \in \mathcal{C}([0,\infty[, H),\ t \in [0,\infty[,$$

which come with their canonical filtration

$$\overline{\mathcal{G}}_t := \sigma(\pi_s | s \leq t) \tag{1.1.2}$$

and we set

$$\mathcal{G}_t := \{(Z^A)^{-1}(F) | F \in \overline{\mathcal{G}}_t\}$$

as the initial sigma-algebra of Z^A, so that Z^A becomes $\mathcal{G}_t/\overline{\mathcal{G}}_t$-measurable.

Remark 1.1.1 (Existence of weak solutions)

Using Girsanov's Theorem (see e.g. [LR15, Theorem I.0.2]) we can construct a filtered stochastic basis as above and an $(\mathcal{F}_t)_{t\in[0,\infty[}$-adapted stochastic process $(X_t)_{t\in[0,T[}$ with $\mathbb{P}$-a.s. continuous sample paths in H which solves (SDE). I.e. we have

$$\begin{cases} \mathrm{d}X_t = -AX_t \,\mathrm{d}t + f(t, X_t)\,\mathrm{d}t + \mathrm{d}W_t \\ \ X_0 = x_0. \end{cases}$$

On an *arbitrary* filtered stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty[}, \mathbb{P}, (W_t)_{t\in[0,\infty[})$, as above, it is a priori not clear whether it carries a solution $(X_t)_{t\in[0,T]}$ as in Remark 1.1.1.

Let us now state the assumptions on the drift f and the main result.

Assumption 1.1.2

Let $f\colon [0,1]\times H \longrightarrow H$ be a Borel measurable map with components $f = (f_n)_{n\in\mathbb{N}}$ w.r.t. our fixed basis $(e_n)_{n\in\mathbb{N}}$ satisfying the following conditions

$$\|f\|_{\infty,A} := \sup_{t\in[0,1],x\in H} \left(\sum_{n\in\mathbb{N}} \lambda_n e^{2\lambda_n} f_n(t,x)^2\right)^{1/2} < \infty$$

and

$$\|f_n\|_\infty = \sup_{t\in[0,1],x\in H} |f_n(t,x)| \leq \exp\left(-e^{c_\gamma n^\gamma}\right)$$

for an $\gamma > 2$ and $c_\gamma > 0$.

Theorem 1.1.3 (Main result)

Let A and f be as above and assume that f fulfills Assumption 1.1.2. Given *any* filtered stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty[}, \mathbb{P}, (W_t)_{t\in[0,\infty[})$ there exists $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}[\Omega_0] = 1$ such that for every $\omega \in \Omega_0$ we have

$$\#\{g \in \mathcal{C}([0,T], H) \mid g \text{ solves } (\mathrm{IE})_\omega\} = 1,$$

i.e. (SDE) has a path-by-path unique mild solution.

Theorem 1.1.3 follows from the following

Proposition 1.1.4

Let A and f be as in Theorem 1.1.3. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty[}, \mathbb{P}, (W_t)_{t\in[0,\infty[})$ be a filtered stochastic basis and $(X_t)_{t\in[0,\infty[}$ be a solution of (SDE) (as in Remark 1.1.1). Then path-by-path uniqueness holds, i.e. there exists $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}[\Omega_0] = 1$ such that

$$\#\{g \in \mathcal{C}([0,T], H) \mid g \text{ solves } (\mathrm{IE})_\omega\} = 1$$

holds for every $\omega \in \Omega_0$.

Proof (of Theorem 1.1.3)

Take an arbitrary filtered probability space and let $((X_t^1)_{t\in[0,\infty[}, (W_t)_{t\in[0,\infty[})$ and $((X_t^2)_{t\in[0,\infty[}, (W_t)_{t\in[0,\infty[})$ be two weak solutions driven by the same cylindrical $(\mathcal{F}_t)_{t\in[0,\infty[}$-Wiener process motion. Then by Proposition 1.1.4 it follows that path-by-path uniqueness, and hence pathwise uniqueness, holds i.e. $X^1 = X^2$ $\mathbb{P}$-a.s. Hence the Yamada–Watanabe Theorem (see [RSZ08, Theorem 2.1]) implies that there exists even a *strong* solution for equation (SDE). In conclusion, by invoking Proposition 1.1.4 again, this proves the existence *and* path-by-path uniqueness of solutions on *every* filtered stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty[}, \mathbb{P}, (W_t)_{t\in[0,\infty[})$.

□

Remark 1.1.5

Set $\Omega := L^2([0,T], H)$ and $\mathbb{P}$ such that the projection $\pi_t(\omega) := \omega(t)$ is a cylindrical Brownian motion. As in the introduction consider the map

$$Z^A \colon L^2([0,T], H) \longrightarrow \mathcal{C}([0,T], H), \qquad \omega \longmapsto \left(t \mapsto \int_0^t e^{-(t-s)A} \, \mathrm{d}\omega(s) \right).$$

Note that due to [DZ92, Theorem 5.2] $\left(\mathbb{P} \circ Z^A\right)^{-1}$ equals $N(0, K)$, the Gaussian measure on $L^2([0,T], H)$ with covariance operator K defined by

$$(K\varphi)(t) = \int_0^T k(t,s)\varphi(s) \, \mathrm{d}s,$$

where

$$k(t,s) = \int_0^{t\wedge s} e^{-(t-r)A} \left(e^{-(s-r)A}\right)^\star \,\mathrm{d}r$$

and $N(0,K)[Z^A(\Omega)] = 1$. Note that, since Z^A is injective, Kuratowski's Theorem (see [Kal97, Theorem A1.7]) implies that $Z^A(\Omega)$ is a Borel set.

Let f be as in Assumption 1.1.2 then path-by-path uniqueness holds for the SDE

$$\mathrm{d}x_t = -Ax_t\mathrm{d}t + f(t,x_t)\mathrm{d}t + \omega(t).$$

I.e. there exists $\Omega_0 \subseteq \mathcal{C}([0,T],H)$ with $\mathbb{P}[\Omega_0] = 1$ such that for every $\omega \in \Omega_0$ there exists a unique function $g \in \mathcal{C}([0,T],H)$ solving the above equation.

1.2 Reduction via Girsanov Transformation

Proposition 1.2.1 (Reduction via Girsanov's Theorem)

Let $f\colon [0,T]\times H \longrightarrow H$ be a bounded Borel measurable function. Assume that for every process $(\tilde{Z}_t^A)_{t\in[0,\infty[}$ on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,\infty[})$ with $\tilde{Z}_0^A = 0$, which is an Ornstein–Uhlenbeck process with drift term A w.r.t. some measure $\tilde{\mathbb{P}} \approx \mathbb{P}$ on $(\Omega,\mathcal{F})$, there exists a set $\Omega'_{\tilde{Z}^A} \subseteq \Omega$ with $\tilde{\mathbb{P}}[\Omega'_{\tilde{Z}^A}] = 1$ such that for all fixed $\omega \in \Omega'_{\tilde{Z}^A}$ the only function $u \in \mathcal{C}([0,T],H)$ solving

$$u(t) = \int_0^t e^{-(t-s)A}\left(f(s,\tilde{Z}_s^A(\omega)+u(s)) - f(s,\tilde{Z}_s^A(\omega))\right)\,\mathrm{d}s \tag{1.2.1.1}$$

for all $t\in[0,T]$ is the trivial solution $u\equiv 0$, then the assertion of Proposition 1.1.4 holds with $\Omega_0 := \Omega'_{\tilde{Z}^A}$, where $\tilde{Z}_t^A := X_t - e^{-tA}x_0$ with X being a solution of (SDE). Recall that X is an Ornstein–Uhlenbeck process under a measure $\tilde{\mathbb{P}}$ obtained via Girsanov transformation.

Remark 1.2.2 (Dependence of Ω_0)

The set of "good omegas" Ω_0 of the main result 1.1.3 therefore depends solely on the strong solution X, the initial condition x_0 and the drift f.

Proof

Let $(X_t)_{t\in[0,T]}$ be a solution to (SDE). We set $\tilde{Z}_t^A := X_t - e^{-tA}x_0$ so that $\tilde{Z}^A$ is an Ornstein–Uhlenbeck process with drift term A starting in 0 under a measure $\tilde{\mathbb{P}} \approx \mathbb{P}$ obtained by Girsanov's Theorem as mentioned in Remark 1.1.1.

Then, by assumption there is a set $\Omega'_{\tilde{Z}^A}$ with $\mathbb{P}[\Omega'_{\tilde{Z}^A}] = \tilde{\mathbb{P}}[\Omega'_{\tilde{Z}^A}] = 1$ such that for all $\omega \in \Omega'_{\tilde{Z}^A}$ every solution u to equation (1.2.1.1) is trivial.

Let $\omega \in \Omega'_{\tilde{Z}^A}$ and $x \in \mathcal{C}([0,T],H)$ be a solution to $(\mathrm{IE})_\omega$. We then have

$$x_t = e^{-tA}x_0 + \int_0^t e^{-(t-s)A} f(s,x_s)\,\mathrm{d}s + \left(\int_0^t e^{-(t-s)A}\,\mathrm{d}W_s\right)(\omega).$$

Setting $u_t := x_t - X_t(\omega)$ yields that

$$\begin{aligned} u_t &= \int_0^t e^{-(t-s)A} f(s,x_s)\,\mathrm{d}s - \int_0^t e^{-(t-s)A} f(s,X_s(\omega))\,\mathrm{d}s \\ &= \int_0^t e^{-(t-s)A}(f(s,u_s + X_s(\omega)) - f(s,X_s(\omega)))\,\mathrm{d}s. \end{aligned}$$

By plugging in the definition of $\tilde{Z}^A$ and by setting

$$\tilde{f}_{x_0}(t,z) := f(t, z + e^{-tA}x_0)$$

we rewrite the above equation to

$$u_t = \int_0^t e^{-(t-s)A}(\tilde{f}_{x_0}(s,u_s + \tilde{Z}^A_s(\omega)) - \tilde{f}_{x_0}(s,\tilde{Z}^A_s(\omega)))\,\mathrm{d}s.$$

Since $\tilde{Z}^A$ is an Ornstein–Uhlenbeck process under $\tilde{\mathbb{P}}$ starting at zero and $\omega \in \Omega'_{\tilde{Z}^A}$ we conclude that $u \equiv 0$ and henceforth $x_t = X_t(\omega)$. Analogously, we obtain for any other solution x' that $x'_t = X_t(\omega) = x_t$ so that all solutions of $(\mathrm{IE})_\omega$ coincide on $\Omega'_{\tilde{Z}^A}$ and are therefore unique. □

2 Gronwall Inequalities

Recall from the last chapter (see Proposition 1.2.1) that our main aim is to prove that for almost all Ornstein–Uhlenbeck paths $Z^A(\omega)$ every function u satisfying

$$u(t) = \int_0^t e^{-(t-s)A} \left(f(s, Z_s^A(\omega) + u(s)) - f(s, Z_s^A(\omega)) \right) \, \mathrm{d}s$$

is the trivial function $u \equiv 0$. If we discretize the problem we expect that

$$u((k+1)2^{-n}) - u(k2^{-n}) \approx \int_{k2^{-n}}^{(k+1)2^{-n}} f(s, Z_s^A(\omega) + u(k2^{-n})) - f(s, Z_s^A(\omega)) \, \mathrm{d}s$$

for all $k \in \{0, ..., 2^n - 1\}$ and sufficiently large $n \in \mathbb{N}$. So that, if we assume f to be Lipschitz continuous in the spatial variable, we obviously obtain

$$|u((k+1)2^{-n}) - u(k2^{-n})|_H \leq \mathrm{Lip}(f) 2^{-n} |u(k2^{-n})|_H. \tag{2.1}$$

We therefore obtain the growth condition

$$|u((k+1)2^{-n})|_H \leq (1 + \mathrm{Lip}(f)2^{-n})|u(k2^{-n})|_H.$$

Using a standard linear discrete Gronwall Inequality we obtain

$$|u((k+1)2^{-n})|_H \leq |u(0)|_H \exp(\mathrm{Lip}(f)) = 0.$$

We therefore deduce that u must be the zero function. In the non-Lipschitz case we can not hope to prove an estimate like (2.1), however, we can prove an estimate along the lines of

$$|u((k+1)2^{-n})|_H \leq (1 + C2^{-n})|u(k2^{-n})|_H \log(1/|u(k2^{-n})|_H), \tag{2.2}$$

where we, of course, have to impose the somewhat technical condition $|u(k2^{-n})|_H \neq 0$. In this chapter we develop the necessary tools to establish that u is trivial from an inequality similar to estimate (2.2).

2.1 Linear Gronwall Inequalities

Lemma 2.1.1 (Gronwall)

Let $\alpha \geq 0$, $r \in \mathbb{N}$ and for every $q \in \{0, ..., r-1\}$ we have $\beta_q \geq 0$ and $x_q \geq 0$ satisfying

$$x_q \leq (1+\alpha)x_{q-1} + \beta_{q-1}.$$

We then have

$$x_q \leq (1+\alpha)^q \left(x_0 + \sum_{q'=0}^{q-1} \beta_{q'} \right)$$

for every $q \in \{1, ..., r\}$.

Proof

The assertion is trivial for $q = 1$. For $q > 1$ the assertion follows via induction in the following manner

$$\begin{aligned} x_{q+1} &\leq (1+\alpha)x_q + \beta_q \leq (1+\alpha)(1+\alpha)^q \left(x_0 + \sum_{q'=0}^{q-1} \beta_{q'} \right) + \beta_q \\ &= (1+\alpha)^{q+1} \left(x_0 + \sum_{q'=1}^{q-1} \beta_{q'} \right) + \beta_q \leq (1+\alpha)^{q+1} \left(x_0 + \sum_{q'=0}^{q} \beta_{q'} \right). \end{aligned}$$

□

Corollary 2.1.2 (Gronwall)

If, additionally to the above situation of Lemma 2.1.1, we have $\alpha \leq \frac{1}{r}$. We obtain

$$x_q \leq e \left(x_0 + \sum_{q'=0}^{r-1} \beta_{q'} \right).$$

for every $q \in \{1, ..., r\}$, where $e := \exp(1)$.

Proof

Using the Lemma 2.1.1 and the assumption $\alpha \leq \frac{1}{r}$ we obtain

$$x_q \leq (1+\alpha)^r \left(x_0 + \sum_{q'=0}^{q-1} \beta_{q'} \right) \leq \left(1 + \frac{1}{r}\right)^r \left(x_0 + \sum_{q'=0}^{r-1} \beta_{q'} \right).$$

Since

$$\left(1 + \frac{1}{r}\right)^r \overset{r\to\infty}{\longrightarrow} e$$

in an increasing way we have the following estimate for x_q

$$x_q \leq e \left(x_0 + \sum_{q'=0}^{r-1} \beta_{q'} \right).$$

□

2.2 Log-Linear Gronwall Inequalities

Lemma 2.2.1 (log-Gronwall Inequality cf. [Wre17, Lemma 6.1])

Let $K > 0$, $m \in \mathbb{N}$ "sufficiently big" i.e. $K \leq \ln(2)2^m$ and $0 < \beta_0, ..., \beta_{2^m} < 1$ and assume that

$$\Delta\beta_j \leq K2^{-m}\beta_j \log_2(1/\beta_j), \qquad \forall j \in \{0, ..., 2^m - 1\}$$

holds, where $\Delta\beta_j := \beta_{j+1} - \beta_j$. Then, we have

$$\beta_j \leq \exp\left(\log_2(\beta_0)e^{-2K-1}\right), \qquad \forall j \in \{0, ..., 2^m\}.$$

Proof

For every $j \in \{0, ..., 2^m\}$ we define

$$\gamma_j := \log_2(1/\beta_j).$$

By assumption we have

$$\begin{aligned}\gamma_{j+1} &= -\log_2(\beta_{j+1}) \geq -\log_2(\beta_j + K2^{-m}\beta_j\gamma_j) \\ &= -\log_2(\beta_j) - \log_2(1 + K2^{-m}\gamma_j) = \gamma_j - \frac{1}{\ln 2}\ln(1 + K2^{-m}\gamma_j).\end{aligned}$$

Using the inequality $\ln(1 + x) \leq x$ the above, and hence γ_{j+1}, is larger than

$$\gamma_j\left(1 - \frac{K}{\ln 2}2^{-m}\right).$$

By induction on $j \in \{0, ..., 2^m\}$ we obtain

$$\gamma_j \geq \gamma_0\left(1 - \frac{K}{\ln 2}2^{-m}\right)^j.$$

Since, by assumption, m is "sufficiently big" the term inside the brackets is in the interval $[0, 1]$ so that γ_j is bounded from below by

$$\gamma_0\left(1 - \frac{K}{\ln 2}2^{-m}\right)^{2^m} \geq \gamma_0 e^{-K/\ln(2)-1} \geq \gamma_0 e^{-2K-1}.$$

Plugging in the definition of γ_j implies that

$$\log_2(1/\beta_j) \geq \log_2(1/\beta_0)e^{-2K-1}.$$

Isolating β_j yields

$$\beta_j \leq \exp\left(\log_2(\beta_0)e^{-2K-1}\right).$$

$\square$

Corollary 2.2.2

Let $f\colon [0,1] \longrightarrow H$ be a continuous function with $f(0) = 0$. If there exist constants $m_0 \in \mathbb{N}$ and $K > 0$ so that for all $m \geq m_0$ there exist $0 < \alpha_m < \alpha'_m < 1$ with $\lim\limits_{m\to\infty} \alpha'_m = 0$ satisfying

$$\frac{\ln \alpha_m}{\ln \alpha'_m} \geq \frac{\ln 2}{e^{-2K-1}} \tag{2.2.2.1}$$

for all $m \geq m_0$ and such that for all $\beta_m \in [\alpha_m, \alpha'_m]$ we have

$$|f(j2^{-m})|_H \leq \beta_m \qquad \Longrightarrow \qquad |f((j+1)2^{-m})|_H \leq \beta_m(1 + K2^{-m}\log_2(1/\beta_m))$$

for all $j \in \{0, ..., 2^m - 1\}$ then $f \equiv 0$.

Proof

Let f, α_m and α'_m be as in the assertion. For sufficiently large $m \in \mathbb{N}$ (i.e. $K \leq \ln(2)2^m$ and $m \geq m_0$) we set

$$\beta_m^{(0)} := \alpha_m$$

and define

$$\beta_m^{(j+1)} := \beta_m^{(j)}(1 + K2^{-m}\log_2(1/\beta_m^{(j)}))$$

for $j \in \{0, ..., 2^m - 1\}$. By the very definition we have

$$\beta_m^{(j+1)} - \beta_m^{(j)} = K2^{-m}\beta_m^{(j)}\log_2(1/\beta_m^{(j)})$$

for every $j \in \{0, ..., 2^m - 1\}$. Hence, Lemma 2.2.1 is applicable which implies that

$$\begin{aligned}\beta_m^{(j)} &\leq \exp\left(\log_2(\beta_m^{(0)})e^{-2K-1}\right) = \exp\left(\log_2(\alpha_m)e^{-2K-1}\right) \\ &= \exp\left(\ln(\alpha_m)\frac{e^{-2K-1}}{\ln 2}\right) \overset{(2.2.2.1)}{\leq} \exp\left(\ln(\alpha'_m)\right) = \alpha'_m.\end{aligned}$$

Together with the fact that $\beta_m^{(j)}$ is increasing we have

$$\alpha_m \leq \beta_m^{(j)} \leq \alpha'_m, \qquad \forall j \in \{0, ..., 2^m\}. \tag{2.2.2.2}$$

Since $f(0) = 0$ we have $|f(0)|_H \leq \beta_m^{(0)}$. Due to inequality (2.2.2.2) and the assumption for $j = 0$ we conclude that

$$|f(2^{-m})|_H \leq \beta_m^{(0)}(1 + K2^{-m}\log_2(1/\beta_m^{(0)})) = \beta_m^{(1)}.$$

Via an induction on j and again inequality (2.2.2.2) we obtain

$$|f(j2^{-m})|_H \leq \beta_m^{(j)} \leq \alpha'_m, \qquad \forall j \in \{0, ..., 2^m\}.$$

By letting $m \to \infty$ and using that $\lim\limits_{m\to\infty} \alpha'_m = 0$, we deduce that f vanishes at all dyadic points. By continuity of f it follows $f \equiv 0$.

□

3 Approximation Lattices

In this chapter we introduce the effective dimension of a set (see Definition 3.1.2). The effective dimension measures the size of set in a similar way than Kolmogorov's ε-entropy. That idea is that given a set $B \subseteq \mathbb{R}^{\mathbb{N}}$ we look at the "size" of the lattices $B \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$ for every $m \in \mathbb{N}$. The sets $B \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$ are the so-called approximating lattices of B. For every $m \in \mathbb{N}$ we obtain a number describing the "size" of $B \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$. Encapsulating these number in a sequence yields the effective dimension of the set B. This sequence generalizes the typical notion of the dimension (see Definition 3.1.5).

In the second section of this chapter we look at a specific set $Q^\gamma \subseteq \mathbb{R}^{\mathbb{N}}$, which is used in the proof of the main result. Here in this chapter, we estimate the effective dimension of the set Q^γ.

3.1 The effective Dimension of a Cuboid

Definition 3.1.1 (Cuboid)

Let $B \subseteq \mathbb{R}^{\mathbb{N}}$ if there are sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ with $a_n \leq 0 \leq b_n$ for all $n \in \mathbb{N}$ such that

$$B = \prod_{n\in\mathbb{N}} [a_n, b_n]$$

we say that B is a *cuboid*. If $a_n < b_n$ for only finitely many $n \in \mathbb{N}$ we say that the cuboid B has finite dimension. Otherwise we call B an infinite-dimensional cuboid.

Definition 3.1.2 (Effective dimension)

Let $B \subseteq \mathbb{R}^{\mathbb{N}}$ be a cuboid. For points $x \in B$ we write $(x_n)_{n\in\mathbb{N}}$ for the components of x. For every $m \in \mathbb{N}$ we set

$$d_m(B) := \sup_{x \in B \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}} \inf \{n \mid x_{n'} = 0 \ \forall n' \geq n\} \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}.$$

I.e. given any point $(x_n)_{n\in\mathbb{N}}$ in the set $B \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$, all components x_n are zero for $n \geq d_m(B)$ and $d_m(B)$ is the smallest integer with this property.

We define the *effective dimension* of a set $B \subseteq \mathbb{R}^{\mathbb{N}}$ by

$$\mathrm{ed}\colon \{B \subseteq \mathbb{R}^{\mathbb{N}} | B \text{ is a cuboid}\} \longrightarrow \overline{\mathbb{N}}^{\mathbb{N}}$$

$$B \longmapsto \mathrm{ed}(B) := (d_m(B))_{m\in\mathbb{N}}.$$

Furthermore, B is called *effectively finite-dimensional* if

$$\mathrm{ed}(B)_m < \infty, \qquad \forall m \in \mathbb{N}.$$

Definition 3.1.3 (Effectively equivalent)

Let $|\cdot|_1$ and $|\cdot|_2$ be two norm on a cuboid B. $|\cdot|_1$ and $|\cdot|_2$ are called *effectively equivalent* if for every $m \in \mathbb{N}$ they are equivalent on the restricted domain $B \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$. I.e. for every $m \in \mathbb{N}$ there exist constants $c_m, C_m \in \mathbb{R}$ such that

$$c_m|x|_1 \leq |x|_2 \leq C_m|x|_1, \qquad \forall x \in B \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}.$$

Example 3.1.4 (Hilbert cube)

Let

$$\mathbb{H} := \prod_{n=1}^{\infty} \left[0, \frac{1}{n}\right]$$

be the Hilbert cube. We have $\mathrm{ed}(\mathbb{H})_m < 2^m$, because let $m \in \mathbb{N}$ and $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{H}\cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$ then for every $n > 2^m$ we can write $x_n = k2^{-m}$, where k is an integer, but on the other hand we have $x_n \leq 1/n$. We therefore conclude that $k \leq 2^m/n < 1$ and hence $k = 0$ which shows that the sequence $(x_n)_{n\in\mathbb{N}}$ is trivial after the 2^m-th element.

Proposition 3.1.5

Let $B \subseteq \mathbb{R}^{\mathbb{N}}$ be a cuboid. The following properties holds

(i) B is finite-dimensional iff $\lim \mathrm{ed}(B) < \infty$.

(ii) B is infinite-dimensional iff $\lim \mathrm{ed}(B) = \infty$.

Note that B is finite-dimensional iff there exists a $d \in \mathbb{N}$ such that there is a bijection $B \cong \mathbb{R}^d$ and B is infinite-dimensional iff there is no such bijection for any $d \in \mathbb{N}$.

Proof

(i) Let $B \subseteq \mathbb{R}^d \subseteq \mathbb{R}^{\mathbb{N}}$ be a cuboid. Then for every $m \in \mathbb{N}$ we obviously have $\mathrm{ed}(B)_m \leq d$ and hence

$$\lim \mathrm{ed}(B) = \lim_{m\to\infty} \mathrm{ed}(\mathbb{R}^d) \leq d.$$

On the other hand, if $d := \lim \mathrm{ed}(B) < \infty$, then by using that B is a cuboid of the form

$$B = \prod_{n\in\mathbb{N}} [a_n, b_n]$$

the following property holds

$$\forall n \geq d \colon \forall x \in \bigcup_{m\in\mathbb{N}} 2^{-m}\mathbb{Z} \cap [a_n, b_n] = \{0\},$$

We therefore conclude that for every $n \geq d$ we have $a_n = b_n = 0$. Note that $d = \lim \mathrm{ed}(B)$ is not the dimension of the space B, but merely an upper bound for the dimension of B.

Part (ii) follows by logical contraposition of (i).

□

Proposition 3.1.6 (Cf. [Wre17, Proposition 2.3])

Let $B \subseteq \mathbb{R}^{\mathbb{N}}$ be an effectively finite-dimensional cuboid then the norm $|\cdot|_2$ and the maximum norm $|\cdot|_\infty$ are effectively equivalent. More precisely, we have

$$|x|_2 \leq \sqrt{\mathrm{ed}(B)_m}|x|_\infty, \qquad m \in \mathbb{N},\ x \in B \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$$

and

$$|x|_\infty \leq |x|_2, \qquad m \in \mathbb{N},\ x \in B \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}.$$

Proof

Let $m \in \mathbb{N}$. For every $x \in B \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$ we have

$$|x|_2^2 = \sum_{n=1}^{\infty} |x_n|^2 = \sum_{n=1}^{\mathrm{ed}(B)_m} |x_n|^2 \leq \mathrm{ed}(B)_m |x|_\infty^2$$

and

$$|x|_\infty^2 \leq \sum_{n=1}^{\infty} |x_n|^2 = |x|_2.$$

□

3.2 The effective Dimension of the Set Q^γ

Definition 3.2.1 (The set Q^γ)

For any $\gamma > 0$ and $c_\gamma > 0$ we define

$$Q^\gamma := \{x \in \mathbb{R}^{\mathbb{N}} \colon |x_n| \leq \exp\left(-e^{c_\gamma n^\gamma}\right),\ x = (x_n)_{n\in\mathbb{N}}\}.$$

Additionally, for $r \in \mathbb{N}$ we set

$$Q_r^\gamma := \{x \in Q^\gamma \colon |x|_\infty \leq 2^{-r}\},$$

so that $Q_0^\gamma = Q^\gamma$. Note that for $m \in \mathbb{N}$ the lattice $Q^\gamma \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$ is the set of all points $x \in Q^\gamma$, where the components x_n of x can be written as

$$x_n = k_n 2^{-m}$$

with certain $k_n \in \mathbb{Z}$ for every $n \in \mathbb{N}$.

Lemma 3.2.2 (Cf. [Wre17, Lemma 2.4])

For $r, m \in \mathbb{N}$ we have

$$\mathrm{ed}(Q_r^\gamma)_m \leq c_\gamma^{-1/\gamma}(\ln(m+1))^{1/\gamma}.$$

Note that this implies that Q_r^γ is effectively finite-dimensional for every $r \in \mathbb{N}$.

Proof

Let $x \in Q_r^\gamma \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$. Observe that every component x_n is of the form $x_n = k_n 2^{-m}$ with

$$k_n \in \{-2^{m-r}, ..., 2^{m-r}\}.$$

Set

$$d_m := c_\gamma^{-1/\gamma}(\ln(m+1))^{1/\gamma}.$$

We are going to show that $k_n = 0$ holds for every $n \geq d_m$.

$$|x_n| = |k_n|2^{-m} \leq \exp\left(-e^{c_\gamma n^\gamma}\right) \Rightarrow |k_n| \leq 2^{m+1}\exp\left(-e^{c_\gamma n^\gamma}\right),$$

which implies that

$$|k_n| \leq 2^{m+1}\exp\left(-e^{c_\gamma n^\gamma}\right) \leq e^{\ln(2)(m+1)}\exp\left(-\exp\left(c_\gamma (d_m)^\gamma\right)\right) = e^{\ln(2)(m+1)-\exp(c_\gamma(d_m)^\gamma)}$$

$$= e^{\ln(2)(m+1)-(m+1)} = e^{(\ln(2)-1)(m+1)} \leq e^{\ln(2)-1} < 1.$$

In conclusion, $|k_n| = 0$ for all $n \geq d_m$ and hence we have

$$\mathrm{ed}(Q_r)_m \leq d_m \leq c_\gamma^{-1/\gamma}(\ln(m+1))^{1/\gamma}.$$

□

Theorem 3.2.3 (Cf. [Wre17, Theorem 2.5])

Let $r \in \mathbb{N}$ and $m \in \mathbb{N}$. The number of points in the m-lattice of Q_r^γ can be estimated as follows

$$\#(Q_r^\gamma \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}) \leq (2 \cdot 2^{m-r}+1)^{\mathrm{ed}(Q_r^\gamma)_m}.$$

Proof

Let $m \in \mathbb{N}$ and $x \in Q_r^\gamma \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$ and note that as in the last proof every component x_n is of the form $x_n = k_n 2^{-m}$ with

$$k_n \in \{-2^{m-r}, ..., 2^{m-r}\}.$$

k_n can take at most $2 \cdot 2^{m-r}+1$ different values in the dimensions $1 \leq n < \mathrm{ed}(Q_r^\gamma)_m$, so that the total number of points $x \in Q_r^\gamma \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$ can be estimated by

$$(2 \cdot 2^{m-r}+1)^{\mathrm{ed}(Q_r^\gamma)_m}.$$

Note that $k_n = 0$ for $n \geq \mathrm{ed}(Q_r^\gamma)_m$.

□

Proposition 3.2.4

Let $Q \subseteq \mathbb{R}^{\mathbb{N}}$ be a cuboid, $C_Q \in \mathbb{R}$ be a constant and $\gamma > 0$ such that

$$\mathrm{ed}(Q)_m \leq C_Q(\ln(m+1))^{1/\gamma}.$$

holds for all $m \in \mathbb{N}$ then there exists a constant c_γ (dependent on C_Q and γ) such that

$$\forall (x_m)_{m\in\mathbb{N}} \in Q\colon |x_m| \leq \exp\left(-e^{c_\gamma m^\gamma}\right).$$

Proof

Let $Q \subseteq \mathbb{R}^{\mathbb{N}}$ and $C_Q \in \mathbb{R}$ as above. Since Q is a cuboid there exist two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that

$$B = \prod_{n\in\mathbb{N}} [a_n, b_n].$$

Let $m' \in \mathbb{N}$ be arbitrary and let $x \in Q \cap 2^{-m'}\mathbb{Z}^{\mathbb{N}}$ then by the definition of the effective dimension we have

$$x_n = 0, \qquad \forall n \geq \mathrm{ed}(Q)_{m'}.$$

and hence we obtain

$$|a_n| < 2^{-m'} \qquad \text{and} \qquad |b_n| < 2^{-m'}$$

for all $n \geq \mathrm{ed}(Q)_{m'}$. Setting

$$n := \lceil C_Q(\ln(m'+1))^{1/\gamma} \rceil \geq \mathrm{ed}(Q)_{m'}$$

yields

$$\left| a_{\lceil C_Q(\ln(m'+1))^{1/\gamma}\rceil} \right| < 2^{-m'}.$$

Since $m' \in \mathbb{N}$ was chosen arbitrary this expression holds for any $m' \in \mathbb{N}$, so that by setting

$$m' := \left\lfloor \exp\left(\left(\frac{m}{C_Q}\right)^\gamma\right) - 1 \right\rfloor$$

for some $m \in \mathbb{N}$ we obtain

$$|a_m| < 2^{-\lfloor \exp\left((m/C_Q)^\gamma\right) - 1 \rfloor} \leq 2^{-\exp\left((m/C_Q)^\gamma\right) + 2}$$

and hence we can find a $0 < c_\gamma < C_Q^{-\gamma}$ such that

$$|a_m| < \exp\left(-\exp\left(c_\gamma m^\gamma\right)\right)$$

and analogously we obtain the same estimate where a_m is replaced with b_m, which completes the proof.

□

Corollary 3.2.5 (Cf. [Wre17, Corollary 2.6])

Let B be a cuboid such that $B \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$ is a finite set for every $m \in \mathbb{N}$. Set for every $r \in \mathbb{N}$

$$B_r := \{x \in B \colon |x|_\infty \leq 2^{-r}\}.$$

Then, for every $m \in \mathbb{N}$ there exists a map

$$\pi_m^{(r)} \colon B_r \longrightarrow B_r \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$$

with the property that

$$|x - \pi_m^{(r)}(x)|_\infty \leq 2^{-m}$$

and

$$|x - y|_\infty \leq |x - \pi_m^{(r)}(x)|_\infty \quad \Rightarrow \quad y = \pi_m^{(r)}(x)$$

holds for all $x \in B_r$, $y \in B_r \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$, $m \in \mathbb{N}$ and $r \in \mathbb{Z}$.

Proof

Let $r \in \mathbb{N}$ and $m \in \mathbb{N}$. Since $B_r \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$ is a finite set we can write

$$B_r \cap 2^{-m}\mathbb{Z}^{\mathbb{N}} = \{y_1, ..., y_N\},$$

where $N \in \mathbb{N}$ is some number depending on both r and m. For every $x \in B_r$ we set

$$\mathcal{I}(x) := \left\{ i \in \{1, ..., N\} : |x - y_i|_\infty = \min_{1 \leq j \leq N} |x - y_j|_\infty \right\}.$$

Furthermore, we define

$$\pi_m^{(r)}(x) := y_{\min \mathcal{I}(x)}.$$

Observe that the map $\pi_m^{(r)}$ fulfills all the required properties.

□

Definition 3.2.6 (Dyadic point)

We set

$$\mathbb{D} := \left\{ (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \middle| \forall n \in \mathbb{N},\ \exists m_n \in \mathbb{N},\ x_n \in 2^{-m_n}\mathbb{Z}^{\mathbb{N}} \right\}.$$

We say that $x \in \mathbb{R}^{\mathbb{N}}$ is a *dyadic point* if $x \in \mathbb{D}$.

Part II

Hilbert space-valued Ornstein–Uhlenbeck Processes

4 Probabilistic Regularization by Noise

Let $b\colon [0,1] \times H \longrightarrow H$ be a bounded and Borel measurable function, which is smooth in the spatial variable and Z^A an Hilbert-space valued Ornstein–Uhlenbeck process on a given filtered stochastic basis. In this chapter we show that the random variable

$$Y := \left| \int_0^1 b'(s, Z_s^A) \, \mathrm{d}s \right|_H .$$

is exponentially square-integrable in the sense that there exists an $\alpha > 0$ such that

$$\mathbb{E}\left[\alpha Y^2\right] < \infty. \tag{4.1}$$

Here, b' denotes the derivative of b w.r.t. the spatial variable.

We split the proof of the above result into two sections. In the first section we consider the case of a one-dimensional Ornstein–Uhlenbeck process. In the second section we reduce the infinite-dimensional case to the one-dimensional case.

In the first section, where Z^A is just a simple one-dimensional Ornstein–Uhlenbeck process Z^λ, i.e. a solution to

$$\begin{cases} \mathrm{d}Z_t^\lambda = -\lambda Z_t^\lambda \mathrm{d}t + \mathrm{d}B_t, \\ \quad Z_0^\lambda = 0. \end{cases}$$

with $\lambda > 0$ and $(B_t)_{t\in[0,\infty[}$ is a one-dimensional Brownian motion. We will notice that the α from inequality (4.1) depends on λ (Proposition 4.1.3). Since we want to extend this to the infinite-dimensional setting in the second section, we have to control the mapping $\lambda \longmapsto \alpha$. We prove that for λ approaching infinity we have

$$\alpha\lambda e^{-2\lambda}.$$

This enables us to show the above mentioned result in the Hilbert space setting with α replaced by

$$\inf_{n\in\mathbb{N}} \alpha_{\lambda_n} e^{2\lambda_n} \lambda_n^{-1},$$

where $\lambda_n > 0$ are the eigenvalues of the operator A, the drift term of the Hilbert space-valued Ornstein–Uhlenbeck process Z^A.

4.1 One-dimensional Ornstein–Uhlenbeck Processes

The following lemma is needed in the one, as well as, the infinite-dimensional case. To simplify the exposition, we will prove it here solely for the infinite-dimensional case which directly implies the one-dimensional case.

Lemma 4.1.1

Let $(Z^{A,(n)})_{n\in\mathbb{N}}$ be the components of an $\ell^2 \cong H$-valued Ornstein–Uhlenbeck process with drift term A driven by the cylindrical Wiener process $(B^{(n)})_{n\in\mathbb{N}}$. Then, there exists a cylindrical Wiener process $(\tilde{B}^{(n)})_{n\in\mathbb{N}}$ such that

$$Z_t^{A,(n)} = (2\lambda_n)^{-1/2} e^{-\lambda_n t} \tilde{B}^{(n)}_{e^{2\lambda_n t}-1}$$

holds for every $n \in \mathbb{N}$ and $t \geq 0$, where $(\lambda_n)_{n\in\mathbb{N}}$ are the eigenvalues of the operator A.

Proof

Let

$$Z_t^A = (Z_t^{A,(n)})_{n\in\mathbb{N}} \in \ell^2 \cong H$$

be the components of $(Z_t^A)_{t\in[0,\infty[}$ and $(\lambda_n)_{n\in\mathbb{N}}$ be the eigenvalues of A w.r.t. the basis $(e_n)_{n\in\mathbb{N}}$. Note that every component $Z^{A,(n)}$ is a one-dimensional Ornstein–Uhlenbeck process with drift term $\lambda_n > 0$ driven by the one-dimensional Wiener process $B^{(n)}$. Define $\tilde{B}^{(n)}$ by

$$\tilde{B}_t^{(n)} := \int_0^{\gamma^{(n)}(t)} \sqrt{c^{(n)}(s)}\, \mathrm{d}B_s^{(n)}, \qquad \forall t \in [0,1],$$

where

$$\gamma^{(n)}(t) := (2\lambda_n)^{-1} \ln(t+1) \qquad \text{and} \qquad c^{(n)}(t) := (2\lambda_n) e^{2\lambda_n t}.$$

Observe that

$$\left(\gamma^{(n)}(t)\right)' = \frac{1}{c^{(n)}\left(\gamma^{(n)}(t)\right)}$$

and, hence, by [Øks10, Theorem 8.5.7] $(\tilde{B}_t^{(n)})_{t\in[0,\infty[}$ is a Brownian motion for every $n \in \mathbb{N}$. The conclusion now follows from this simple calculation

$$(2\lambda_n)^{-1/2} \tilde{B}^{(n)}_{e^{2\lambda_n t}-1} = (2\lambda_n)^{-1/2} \int_0^t (2\lambda_n)^{1/2} e^{\lambda_n s}\, \mathrm{d}B_s^{(n)} = \int_0^t e^{\lambda_n s}\, \mathrm{d}B_s^{(n)} = Z_t^{A,(n)} e^{\lambda_n t}.$$

□

Proposition 4.1.2

Let $b\colon [0,1] \times H \longrightarrow H$ be a Borel measurable function with components $b = b^{(n)}$ w.r.t. our fixed basis $(e_n)_{n\in\mathbb{N}}$ such that

$$\|b\|_{\infty,A} := \sup_{t\in[0,1], x\in H} \left(\sum_{n\in\mathbb{N}} \lambda_n e^{2\lambda_n} |b^{(n)}(t,x)|^2 \right)^{1/2} < \infty$$

then

$$\|b\|_{\infty} := \sup_{t\in[0,1], x\in H} \left(\sum_{n\in\mathbb{N}} |b^{(n)}(t,x)|^2 \right)^{1/2} < \infty,$$

where $(\lambda_n)_{n\in\mathbb{N}}$ are the eigenvalues of the operator A as mentioned in the introduction.

Proof

Let b be as in the assumption. Set

$$M := \{n \in \mathbb{N} \mid \lambda_n e^{2\lambda_n} < 1\}$$

Since $\lambda_n \longrightarrow \infty$ for n approaching infinity we obviously have $\#(M) < \infty$ so that

$$\begin{aligned}\|b\|_\infty^2 &= \sup_{t\in[0,1],x\in H} \sum_{n\in\mathbb{N}} |b^{(n)}(t,x)|^2 \\ &\leq \underbrace{\sup_{t\in[0,1],x\in H} \sum_{n\in M} |b^{(n)}(t,x)|^2}_{<\infty} + \sup_{t\in[0,1],x\in H} \sum_{n\in\mathbb{N}\setminus M} \underbrace{\lambda_n e^{2\lambda_n}}_{\geq 1} |b^{(n)}(t,x)|^2.\end{aligned}$$

Using the assumption on b completes the proof.

□

Proposition 4.1.3 (Cf. [Wre16, Proposition 2.1])

There exists and absolute constant $C \in \mathbb{R}$ and a non-increasing map

$$\begin{aligned}\alpha\colon]0,\infty[&\longrightarrow]0,\infty[\\ \lambda &\longmapsto \alpha_\lambda\end{aligned}$$

with

$$\alpha_\lambda e^{2\lambda}\lambda^{-1} \geq \frac{e}{1152}, \qquad \forall \lambda > 0.$$

such that for all one-dimensional Ornstein–Uhlenbeck processes $(Z_t^\lambda)_{t\in[0,\infty[}$ with drift term $\lambda > 0$, i.e.

$$\begin{cases} \mathrm{d}Z_t^\lambda = -\lambda Z_t^\lambda \mathrm{d}t + \mathrm{d}B_t, \\ \quad Z_0^\lambda = 0. \end{cases}$$

where $(B_t)_{t\in[0,\infty[}$ is a one-dimensional Brownian motion and for all Borel measurable functions $b\colon [0,1]\times\mathbb{R} \longrightarrow H$, which are in the second component twice continuously differentiable with

$$\|b\|_\infty := \sup_{t\in[0,1],x\in\mathbb{R}} |b(t,x)|_H < \infty.$$

The following inequality

$$\mathbb{E}\exp\left(\frac{\alpha_\lambda}{\|b\|_\infty^2}\left|\int_0^1 b'(t,Z_t^\lambda)\ \mathrm{d}t\right|_H^2\right) \leq C \leq 3$$

holds, where b' denotes the first derivative of b w.r.t. the second variable x.

Proof

Sketch of the proof:

Note that the bracket process $\langle Z^\lambda \rangle$ of an Ornstein–Uhlenbeck process is just $\langle Z^\lambda \rangle_t = t$. Hence, we have

$$\int_0^1 b'(t, Z_t^\lambda)\ \mathrm{d}t = \int_0^1 b'(t, Z_t^\lambda)\ \mathrm{d}\langle Z^\lambda \rangle_t.$$

The integral on the right-hand side looks like an Itô correction term and can therefore be rewritten as the following difference of a backwards and forward Itô integral.

$$\int_0^1 b(s, Z_s^\lambda)\ \mathrm{d}^* Z_s^\lambda - \int_0^1 b(s, Z_s^\lambda)\ \mathrm{d}Z_s^\lambda,$$

where d^* denotes the backwards Itô integral. Let us denote with $\overleftarrow{\cdot}$ the time-reversal operator of a stochastic process. The above expression can then be expressed as two forward Itô integrals as follows

$$-\int_0^1 b(1-s, \overleftarrow{Z^\lambda}_s)\ \mathrm{d}\overleftarrow{Z^\lambda}_s - \int_0^1 b(s, Z_s^\lambda)\ \mathrm{d}Z_s^\lambda.$$

Since Z is an Itô diffusion process with a "nice" drift the time-reversed process $\overleftarrow{Z}$ can be explicitly calculated to be of the form

$$\overleftarrow{Z^\lambda}_t = \overleftarrow{Z^\lambda}_0 + \int_0^t \overleftarrow{Z^\lambda}_s \left(\lambda - \frac{2\lambda}{1 - e^{2\lambda(s-1)}} \right)\ \mathrm{d}s + \tilde{W}_t,$$

where $\tilde{W}$ is a new Brownian motion. We can therefore decompose Z^λ as well as the semi-martingale $\overleftarrow{Z^\lambda}$ into a martingale part and a part of bounded variation. Plugging this decomposition into

$$\int_0^1 b'(t, Z_t^\lambda)\ \mathrm{d}t = -\int_0^1 b(1-s, \overleftarrow{Z^\lambda}_s)\ \mathrm{d}\overleftarrow{Z^\lambda}_s - \int_0^1 b(s, \overleftarrow{Z^\lambda}_s)\ \mathrm{d}Z_s^\lambda$$

we are left with estimating various integrals. For the stochastic integrals we use the Burkholder–Davis–Gundy Inequality and for the deterministic integrals we develop a bound by quite explicit calculations. In the end our bounds are strong enough to deduce that the random variable

$$\alpha_\lambda \int_0^1 b'(t, Z_t^\lambda)\ \mathrm{d}t$$

is exponentially square-integrable as long as $\alpha_\lambda > 0$ is small enough.

Beginning of the proof:

Let $(Z_t^\lambda)_{t\in[0,\infty[}$ be a one-dimensional Ornstein–Uhlenbeck process, i.e. a strong solution to

$$\mathrm{d}Z_t^\lambda = -\lambda Z_t^\lambda \mathrm{d}t + \mathrm{d}B_t,$$

where $\lambda > 0$, $Z_0^\lambda = 0$ and let $b\colon [0,1]\times\mathbb{R} \longrightarrow H$ be as in the assertion. Define

$$Y_s := b(s, Z_s^\lambda), \qquad \forall s \in [0,1]$$

and denote by $(Y^n)_{n\in\mathbb{N}}$ the components of Y. Then by [BJ97, Remark 2.5] we have for every $n \in \mathbb{N}$

$$\langle Y^n, Z^\lambda\rangle_1 = \int_0^1 b_n'(s, Z_s^\lambda)\ \mathrm{d}\langle Z^\lambda\rangle_s = \int_0^1 b_n'(s, Z_s^\lambda)\ \mathrm{d}s,$$

where b_n is the n-th component of b and the quadratic covariation $\langle Y^n, Z^\lambda\rangle_t$ is the uniform in probability limit of

$$\sum_{\substack{t_i,t_{i+1}\in \mathrm{D}_m\\ 0\le t_i\le t}} \left[Y_{t_{i+1}}^n - Y_{t_i}^n\right]\cdot\left[Z_{t_{i+1}}^\lambda - Z_{t_i}^\lambda\right],$$

where D_m is a sequence of partitions of $[0,t]$ with a mesh converging to 0 as m approaches infinity.

Moreover, applying [BJ97, Corollary 2.3] results in

$$\int_0^1 b_n'(s, Z_s^\lambda)\ \mathrm{d}s = \langle Y^n, Z^\lambda\rangle_1 = \int_0^1 Y_s^n\ \mathrm{d}^* Z_s^\lambda - \int_0^1 Y_s^n\ \mathrm{d}Z_s^\lambda, \tag{4.1.3.1}$$

where the backward integral is defined as

$$\int_0^t Y_s^n\ \mathrm{d}^* Z_s^\lambda := -\int_{1-t}^1 \overleftarrow{Y}_s^n\ \mathrm{d}\overleftarrow{Z}_s^\lambda, \qquad \forall t\in[0,1] \tag{4.1.3.2}$$

and

$$\overleftarrow{X}_s := X_{1-s}, \qquad\qquad \forall s\in[0,1]$$

denotes the time-reversal of a generic stochastic process X. Since identity (4.1.3.1) holds for all components $n\in\mathbb{N}$ we also have

$$\int_0^1 b'(s, Z_s^\lambda)\ \mathrm{d}s = \langle Y, Z^\lambda\rangle_1 = \int_0^1 Y_s\ \mathrm{d}^* Z_s^\lambda - \int_0^1 Y_s\ \mathrm{d}Z_s^\lambda, \tag{4.1.3.3}$$

where $\langle Y, Z^\lambda\rangle$ is defined as $(\langle Y^n, Z^\lambda\rangle)_{n\in\mathbb{N}}$.

In addition to this, Z^λ is an Itô diffusion process with generator

$$L_t = a(t,x)\nabla_x + \frac{1}{2}\sigma(t,x)\Delta_x = -\lambda x\nabla_x + \frac{1}{2}\Delta_x.$$

I.e. $a(t,x) = -\lambda x$ and $\sigma(t,x) = 1$. The probability density of Z_t^λ w.r.t. Lebesgue measure is

$$p_t(x) = \sqrt{\frac{\lambda}{\pi(1-e^{-2\lambda t})}} e^{-\lambda x^2/(1-e^{-2\lambda t})}.$$

Observe that a and σ fulfill the conditions of [MNS89, Theorem 2.3], hence, the drift term $\overleftarrow{a}$ and diffusion term $\overleftarrow{\sigma}$ of the generator $\overleftarrow{L}_t$ of the time-reversed process $\overleftarrow{Z}^\lambda$ are given by

$$\overleftarrow{a}(t,x) = -a(1-t,x) + \frac{1}{p_{1-t}(x)} \nabla_x \left(\sigma(1-t,x) p_{1-t}(x)\right) = \left(\lambda - \frac{2\lambda}{1-e^{2\lambda(t-1)}}\right) x$$

and

$$\overleftarrow{\sigma}(t,x) = \sigma(1-t,x) = 1.$$

Therefore (see [BR07, Remark 2.4]), we obtain

$$\overleftarrow{Z}_t^\lambda = \overleftarrow{Z}_0^\lambda + \overleftarrow{W}_t + \int_0^t \overleftarrow{Z}_s^\lambda \left(\lambda - \frac{2\lambda}{1-e^{2\lambda(s-1)}}\right) \,\mathrm{d}s, \tag{4.1.3.4}$$

where $\overleftarrow{W}_t$ is a new Brownian motion defined by this equation. Set

$$\mathcal{G}_t^0 := \sigma\left(\overleftarrow{W}_s - \overleftarrow{W}_t, t \le s \le 1\right)$$

and let $\tilde{\mathcal{G}}_t$ be the completion of $\mathcal{G}_t^0$. Define

$$\mathcal{G}_t := \sigma\left(\tilde{\mathcal{G}}_{1-t} \cup \sigma(Z_1^\lambda)\right)$$

then $\overleftarrow{W}_t$ is a $\mathcal{G}_t$-Brownian motion (see [Par86]). In conclusion we have by combining equation (4.1.3.3) with (4.1.3.2)

$$-\int_0^1 b'(s, Z_s^\lambda) \,\mathrm{d}s = \int_0^1 b(1-s, \overleftarrow{Z}_s^\lambda) \,\mathrm{d}\overleftarrow{Z}_s^\lambda + \int_0^1 b(s, Z_s^\lambda) \,\mathrm{d}Z_s^\lambda.$$

By plugging in (4.1.3.4) this is equal to

$$\underbrace{\int_0^1 b(1-s, \overleftarrow{Z}_s^\lambda) \,\mathrm{d}\overleftarrow{W}_s}_{=:I_1} + \underbrace{\int_0^1 b(1-s, \overleftarrow{Z}_s^\lambda) \overleftarrow{Z}_s^\lambda \left(\lambda - \frac{2\lambda}{1-e^{2\lambda(s-1)}}\right) \,\mathrm{d}s}_{=:I_2} + \underbrace{\int_0^1 b(s, Z_s^\lambda) \,\mathrm{d}Z_s^\lambda}_{=:I_3}$$

$$= I_1 + I_2 + I_3 =: I.$$

Observe that by (4.1.3.4) and the Yamada–Watanabe Theorem (see [RSZ08, Theorem 2.1]) $\overleftarrow{Z}_t^\lambda$ is a strong solution of an SDE driven by the noise $\overleftarrow{W}_t$, hence, $\overleftarrow{Z}_t^\lambda$ is $\mathcal{G}_t$-measurable so that the stochastic integral I_1 makes sense. In conclusion we get

$$\mathbb{E} \exp\left(\alpha_\lambda \left| \int_0^1 b'(t, Z_t^\lambda) \,\mathrm{d}t \right|_H^2\right) = \mathbb{E}\exp(\alpha_\lambda |I|_H^2) = \mathbb{E}\exp(\alpha_\lambda |I_1 + I_2 + I_3|_H^2), \tag{4.1.3.5}$$

for α_λ to be defined later. We will estimate the terms I_1, I_2 and I_3 separately.

Estimate for I_1:

Define

$$M_t := \int_0^t b(1-s, \overleftarrow{Z}_s^\lambda)\, \mathrm{d}\overleftarrow{W}_s, \qquad \forall t \in [0,1].$$

Observe that $(M_t)_{t\in[0,1]}$ is a $(\mathcal{G}_t)_{t\in[0,1]}$-martingale with $M_0 = 0$. Also note the following estimate for the quadratic variation of M

$$0 \le |\langle M\rangle_t|_H \le \int_0^t \|b\|_\infty^2 \,\mathrm{d}s \le \|b\|_\infty^2, \qquad \forall t \in [0,1].$$

In the next step we use the Burkholder–Davis–Gundy Inequality for time-continuous martingales with the optimal constant. In the celebrated paper [Dav76, Section 3] it is shown that the optimal constant in our case is the largest positive root of the Hermite polynomial of order $2k$. We refer to the appendix of [Ose12] for a discussion of the asymptotic of the largest positive root. See also [Kho14, Appendix B], where a self-contained proof of the Burkholder–Davis–Gundy Inequality with asymptotically optimal constant can be found for the one-dimensional case. A proof for H-valued martingales can be obtained by a slight modification of [Kho14, Theorem B.1] to $\mathbb{R}^d$-valued martingales and by projecting H onto $\mathbb{R}^d$. The optimal constants in different cases is discussed in the introduction of [Wan91].

Also note that the H-valued case can simply reduced to the two-dimensional case by enlargement of filtrations. Given an H-valued martingale M one can construct a $\mathbb{R}^2$-valued martingale N such that $|M_t|_H = |N_t|$ and $\langle M\rangle_t = \langle N\rangle_t$ (see [KS91]).

We have

$$\mathbb{E}|I_1|_H^{2k} = \mathbb{E}|M_1|_H^{2k} \le 2^{2k}(2k)^k \underbrace{\mathbb{E}|\langle M\rangle_1|_H^k}_{\le \|b\|_\infty^{2k}} \le 2^{3k} \underbrace{k^k}_{\le 2^{2k}k!} \|b\|_\infty^{2k} \le 2^{5k} k! \|b\|_\infty^{2k}.$$

Choosing $\alpha_1 = \frac{1}{64}$ we obtain

$$\mathbb{E}\exp\left(\frac{\alpha_1}{\|b\|_\infty^2}|I_1|_H^2\right) = \mathbb{E}\sum_{k=0}^\infty \frac{\alpha_1^k |I_1|_H^{2k}}{\|b\|_\infty^{2k} k!} = \sum_{k=0}^\infty \frac{\alpha_1^k \mathbb{E}|I_1|_H^{2k}}{\|b\|_\infty^{2k} k!} \le \sum_{k=0}^\infty 2^{-k} = 2 =: C_1.$$

Estimate for I_2:

We have for any $\alpha_2^{(\lambda)} > 0$ to be specified later

$$\begin{aligned}\mathbb{E}\exp\left(\frac{\alpha_2^{(\lambda)}}{\|b\|_\infty^2}|I_2|_H^2\right) &= \mathbb{E}\exp\left(\frac{\alpha_2^{(\lambda)}}{\|b\|_\infty^2}\left|\int_0^1 b(1-t, \overleftarrow{Z}_t^\lambda)\overleftarrow{Z}_t^\lambda \lambda\left(1 - \frac{2}{1-e^{2\lambda(t-1)}}\right) \mathrm{d}t\right|_H^2\right) \\ &\le \mathbb{E}\exp\left(\frac{\alpha_2^{(\lambda)}}{\|b\|_\infty^2}\left|\int_0^1 \underbrace{|b(1-t, \overleftarrow{Z}_t^\lambda)|_H}_{\le \|b\|_\infty} |\overleftarrow{Z}_t^\lambda| \lambda \frac{1+e^{2\lambda(t-1)}}{1-e^{2\lambda(t-1)}} \,\mathrm{d}t\right|^2\right)\end{aligned}$$

$$= \mathbb{E} \exp \left(\frac{\alpha_2^{(\lambda)}}{\|b\|_\infty^2} \left| \int_0^1 \|b\|_\infty \frac{|\overleftarrow{Z_t^\lambda}|}{\sqrt{e^{2\lambda(1-t)}-1}} \lambda \underbrace{\left(e^{2\lambda(1-t)}-1\right) \frac{1+e^{2\lambda(t-1)}}{1-e^{2\lambda(t-1)}}}_{=e^{-2\lambda(t-1)}+1} \frac{\mathrm{d}t}{\sqrt{e^{2\lambda(1-t)}-1}} \right|^2 \right)$$

$$\leq \mathbb{E} \exp \left(\alpha_2^{(\lambda)} \left| \int_0^1 \frac{|\overleftarrow{Z_t^\lambda}|}{\sqrt{e^{2\lambda(1-t)}-1}} \lambda (e^{2\lambda(1-t)}+1) \frac{\mathrm{d}t}{\sqrt{e^{2\lambda(1-t)}-1}} \right|^2 \right).$$

Setting

$$D_\lambda := \int_0^1 \frac{\mathrm{d}t}{\sqrt{e^{2\lambda(1-t)}-1}} = \frac{\arctan\left(\sqrt{e^{2\lambda}-1}\right)}{\lambda} < \infty,$$

the above term can be written as

$$\mathbb{E} \exp \left(\alpha_2^{(\lambda)} \left| \int_0^1 \frac{|\overleftarrow{Z_t^\lambda}|}{\sqrt{e^{2\lambda(1-t)}-1}} \lambda \left(e^{2\lambda(1-t)}+1\right) D_\lambda \frac{\mathrm{d}t}{D_\lambda \sqrt{e^{2\lambda(1-t)}-1}} \right|^2 \right).$$

Applying Jensen's Inequality w.r.t. the probability measure $\frac{\mathrm{d}t}{D_\lambda \sqrt{e^{2\lambda(1-t)}-1}}$ and the convex function

$$x \longmapsto \exp\left(\alpha_2^{(\lambda)} |x|^2\right)$$

results in the above being bounded by the following

$$\mathbb{E} \int_0^1 \exp \left[\alpha_2^{(\lambda)} \left| \frac{|\overleftarrow{Z_t^\lambda}|}{\sqrt{e^{2\lambda(1-t)}-1}} \lambda \left(e^{2\lambda(1-t)}+1\right) D_\lambda \right|^2 \right] \frac{\mathrm{d}t}{D_\lambda \sqrt{e^{2\lambda(1-t)}-1}}$$

$$= \mathbb{E} \int_0^1 \exp \left[\alpha_2^{(\lambda)} \frac{|Z_{1-t}^\lambda|^2}{e^{2\lambda(1-t)}-1} \lambda^2 \left(e^{2\lambda(1-t)}+1\right)^2 D_\lambda^2 \right] \frac{\mathrm{d}t}{D_\lambda \sqrt{e^{2\lambda(1-t)}-1}}.$$

Setting $\alpha_2^{(\lambda)} := \frac{1}{4\lambda(e^{2\lambda}+1)D_\lambda^2}$ and applying Fubini's Theorem the above term can be estimated by

$$\int_0^1 \mathbb{E} \exp \left(\frac{1}{4} \frac{\lambda(e^{2\lambda(1-t)}+1)|Z_{1-t}^\lambda|^2}{e^{2\lambda(1-t)}-1} \right) \frac{\mathrm{d}t}{D_\lambda \sqrt{e^{2\lambda(1-t)}-1}}. \tag{4.1.3.6}$$

Using Lemma 4.1.1 we have

$$Z_{1-t}^\lambda = (2\lambda)^{-1/2} e^{-\lambda(1-t)} \overline{B}_{e^{2\lambda(1-t)}-1},$$

where $\overline{B}$ is another Brownian motion. Plugging this into (4.1.3.6) we get the following bound for (4.1.3.6)

$$\int_0^1 \mathbb{E}\exp\left(\frac{1}{8}\frac{\overbrace{(e^{2\lambda(1-t)}+1)e^{-2\lambda(1-t)}}^{\leq 2}\overline{B}^2_{e^{2\lambda(1-t)}-1}}{e^{2\lambda(1-t)}-1}\right)\frac{\mathrm{d}t}{D_\lambda\sqrt{e^{2\lambda(1-t)}-1}}$$

$$\leq \int_0^1 \underbrace{\mathbb{E}\exp\left(\frac{1}{4}\frac{\overline{B}^2_{e^{2\lambda(1-t)}-1}}{e^{2\lambda(1-t)}-1}\right)}_{=\sqrt{2}}\frac{\mathrm{d}t}{D_\lambda\sqrt{e^{2\lambda(1-t)}-1}}$$

$$=\sqrt{2}\underbrace{\int_0^1 \frac{\mathrm{d}t}{D_\lambda\sqrt{e^{2\lambda(1-t)}-1}}}_{=1}=\sqrt{2}=:C_2.$$

Estimate for I_3:

Recall that

$$\mathbb{E}|I_3|_H^{2k}=\mathbb{E}\left|\int_0^1 b(s,Z_s^\lambda)\ \mathrm{d}Z_s^\lambda\right|_H^{2k}. \tag{4.1.3.7}$$

Plugging in

$$Z_t^\lambda=-\lambda\int_0^t Z_s^\lambda\ \mathrm{d}s+B_t$$

into Equation (4.1.3.7) results in

$$\mathbb{E}|I_3|_H^{2k}\leq 2^{2k}\mathbb{E}\left|\int_0^1 b(s,Z_s^\lambda)\lambda Z_s^\lambda\ \mathrm{d}s\right|_H^{2k}+2^{2k}\mathbb{E}\left|\int_0^1 b(s,Z_s^\lambda)\ \mathrm{d}B_s\right|_H^{2k}.$$

For the first term on the right-hand side we use Jensen's Inequality and for the second term a similar calculation as for the estimate of I_1 yields that the above is smaller than

$$2^{2k}\mathbb{E}\int_0^1 \underbrace{\|b\|_\infty^{2k}}_{\leq\|b\|_\infty^{2k}}\lambda^{2k}|Z_s^\lambda|^{2k}\ \mathrm{d}s+2^{2k}2^{5k}k!\|b\|_\infty^{2k}.$$

Using Fubini's Theorem we estimate this by

$$2^{2k}\|b\|_\infty^{2k}\lambda^{2k}\int_0^1 \mathbb{E}|Z_s^\lambda|^{2k}\ \mathrm{d}s+2^{2k}2^{5k}k!\leq 2^{2k}\lambda^{2k}\max_{s\in[0,1]}\mathbb{E}|Z_s^\lambda|^{2k}+2^{2k}2^{5k}k!\|b\|_\infty^{2k}.$$

Again, with the help of Lemma 4.1.1 we have

$$Z_s^\lambda=(2\lambda)^{-1/2}e^{-\lambda s}\overline{B}_{e^{2\lambda s}-1},$$

where $\overline{B}$ is another Brownian motion. Estimating the $2k$-moments yields

$$
\begin{aligned}
\mathbb{E}|Z_s^\lambda|^{2k} &= (2\lambda)^{-k} e^{-\lambda 2ks} \mathbb{E}\left|\overline{B}_{e^{2\lambda s}-1}\right|^{2k} \\
&= (2\lambda)^{-k} \underbrace{e^{-\lambda 2ks}\left|e^{2\lambda s}-1\right|^{k}}_{\leq 1} 2^k \pi^{-1/2} \Gamma\left(k+\frac{1}{2}\right) \\
&\leq \lambda^{-k}\pi^{-1/2}\Gamma\left(k+\frac{1}{2}\right) \leq \lambda^{-k} k!, \qquad \forall s \in [0,1].
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\mathbb{E}|I_3|_H^{2k} &\leq 2^{2k}\lambda^{2k}\|b\|_\infty^{2k} \max_{s\in[0,1]} \mathbb{E}|Z_s^\lambda|^{2k} + 2^{2k}2^{5k}\|b\|_\infty^{2k} k! \\
&\leq 2^{2k}\lambda^{2k}\lambda^{-k}\|b\|_\infty^{2k} k! + 2^{2k}2^{5k}\|b\|_\infty^{2k} k! = 2^{2k}\lambda^{k}\|b\|_\infty^{2k} k! + 2^{2k}2^{5k}\|b\|_\infty^{2k} k!.
\end{aligned}
$$

Choosing $\alpha_3^{(\lambda)} = 2^{-6}\min\left(\lambda^{-1}, 2^{-2}\right)$ we obtain

$$
\mathbb{E}\exp\left(\frac{\alpha_3^{(\lambda)}}{\|b\|_\infty^2}|I_3|_H^2\right) = \mathbb{E}\sum_{k=0}^\infty \frac{\left|\alpha_3^{(\lambda)}\right|^k |I_3|_H^{2k}}{\|b\|_\infty^{2k} k!} = \sum_{k=0}^\infty \frac{\left|\alpha_3^{(\lambda)}\right|^k \mathbb{E}|I_3|_H^{2k}}{\|b\|_\infty^{2k} k!} \leq \sum_{k=0}^\infty 2\cdot 2^{-k} = 4 =: C_3.
$$

Final estimate:

We are now ready to plug in all previous estimates to complete the proof. Setting

$$
\alpha_\lambda := \frac{1}{9}\min(\alpha_1, \alpha_2^{(\lambda)}, \alpha_3^{(\lambda)})
$$

we conclude

$$
\begin{aligned}
\mathbb{E}\exp\left(\frac{\alpha_\lambda}{\|b\|_\infty^2}|I|_H^2\right) &= \mathbb{E}\exp\left(\frac{\alpha_\lambda}{\|b\|_\infty^2}|I_1+I_2+I_3|_H^2\right) \\
&\leq \mathbb{E}\exp\left(3\frac{\alpha_\lambda}{\|b\|_\infty^2}|I_1|_H^2 + 3\frac{\alpha_\lambda}{\|b\|_\infty^2}|I_2|_H^2 + 3\frac{\alpha_\lambda}{\|b\|_\infty^2}|I_3|_H^2\right) \\
&= \mathbb{E}\exp\left(3\frac{\alpha_\lambda}{\|b\|_\infty^2}|I_1|_H^2\right)\exp\left(3\frac{\alpha_\lambda}{\|b\|_\infty^2}|I_2|_H^2\right)\exp\left(3\frac{\alpha_\lambda}{\|b\|_\infty^2}|I_3|_H^2\right).
\end{aligned}
$$

We apply the Young Inequality to split the three terms

$$
\mathbb{E}\frac{\exp\left(3\frac{\alpha_\lambda}{\|b\|_\infty^2}|I_1|_H^2\right)^3}{3} + \mathbb{E}\frac{\exp\left(3\frac{\alpha_\lambda}{\|b\|_\infty^2}|I_2|_H^2\right)^3}{3} + \mathbb{E}\frac{\exp\left(3\frac{\alpha_\lambda}{\|b\|_\infty^2}|I_3|_H^2\right)^3}{3}
$$

and using the estimates for I_1, I_2 and I_3 results in the following bound

$$
\mathbb{E}\frac{\exp\left(\frac{\alpha_1}{\|b\|_\infty^2}|I_1|_H^2\right)}{3} + \mathbb{E}\frac{\exp\left(\frac{\alpha_2^{(\lambda)}}{\|b\|_\infty^2}|I_2|_H^2\right)}{3} + \mathbb{E}\frac{\exp\left(\frac{\alpha_3^{(\lambda)}}{\|b\|_\infty^2}|I_3|_H^2\right)}{3} \leq \frac{1}{3}(C_1+C_2+C_3) = \frac{6+\sqrt{2}}{3} \leq 3.
$$

We still need to show that the map α fulfills the claimed properties.

Simplification of α_λ:

Recall that

$$\alpha_\lambda = \frac{1}{9}\min(\alpha_1, \alpha_2^{(\lambda)}, \alpha_3^{(\lambda)}) = \frac{1}{9}\min\left(\frac{1}{256}, \frac{1}{4\lambda(e^{2\lambda}+1)D_\lambda^2}, \frac{1}{64\lambda}\right)$$

and

$$D_\lambda = \frac{\arctan\left(\sqrt{e^{2\lambda}-1}\right)}{\lambda}.$$

First, we want to prove that α_λ is the same as

$$\frac{1}{9}\min\left(\frac{1}{256}, \frac{1}{4\lambda(e^{2\lambda}+1)D_\lambda^2}\right).$$

I.e. $\alpha_3^{(\lambda)}$ is always larger than α_1 or $\alpha_2^{(\lambda)}$. Note that for $\lambda \in]0,4]$ $\alpha_3^{(\lambda)}$ is obviously larger than α_1, hence it is enough to show that $\alpha_3^{(\lambda)} \geq \alpha_2^{(\lambda)}$ for all $\lambda > 4$. We have

$$2\lambda^2 + 2\lambda - \frac{10}{3\pi}\sqrt{16}\lambda + 2 \geq 0, \qquad \forall \lambda \in \mathbb{R},$$

which implies

$$\frac{10}{3\pi}\sqrt{16}\lambda \leq 2 + 2\lambda + 2\lambda^2 \leq \sqrt{e^{2\lambda}+1}, \qquad \forall \lambda > 4.$$

Reordering and using that arctan is an increasing function leads us to

$$\sqrt{16}\lambda \leq \sqrt{e^{2\lambda}+1}\frac{3\pi}{10} = \sqrt{e^{2\lambda}+1}\arctan\left(\sqrt{1+\frac{2}{\sqrt{5}}}\right)$$

$$\leq \sqrt{e^{2\lambda}+1}\arctan\left(\sqrt{e^2-1}\right) \leq \sqrt{e^{2\lambda}+1}\arctan\left(\sqrt{e^{2\lambda}-1}\right)$$

for all $\lambda > 4$. Therefore we obtain

$$16\lambda^2 \leq \left(e^{2\lambda}+1\right)\arctan^2\left(\sqrt{e^{2\lambda}-1}\right),$$

which finally implies

$$\alpha_3^{(\lambda)} = \frac{1}{64\lambda} \geq \frac{\lambda}{4\left(e^{2\lambda}+1\right)\arctan^2\left(\sqrt{e^{2\lambda}-1}\right)} = \alpha_2^{(\lambda)}.$$

In conclusion we proved that

$$\alpha_\lambda = \frac{1}{9}\min\left(\frac{1}{256}, \frac{1}{4\lambda(e^{2\lambda}+1)D_\lambda^2}\right).$$

Asymptotic behavior of α_λ:

Let us now analyze $\alpha_2^{(\lambda)}$. Set

$$f(\lambda) := \alpha_2^{(\lambda)} e^{2\lambda}\lambda^{-1} = \frac{e^{2\lambda}}{4\lambda^2(e^{2\lambda}+1)D_\lambda^2} = \frac{e^{2\lambda}}{4(e^{2\lambda}+1)\arctan^2(\sqrt{e^{2\lambda}-1})}.$$

We obviously have

$$\frac{e^{2\lambda}}{e^{2\lambda}+1} \overset{\lambda\to\infty}{\longrightarrow} 1$$

and

$$\arctan\left(\sqrt{e^{2\lambda}-1}\right) \overset{\lambda\to\infty}{\longrightarrow} \frac{\pi}{2}.$$

Therefore,

$$f(\lambda) = \frac{e^{2\lambda}}{4(e^{2\lambda}+1)\arctan^2(\sqrt{e^{2\lambda}-1})} \overset{\lambda\to\infty}{\longrightarrow} \frac{1}{\pi^2}$$

holds. We want to show that f is monotonically decreasing and hence the above limit is a lower bound for f. To this end we calculate the first derivative of f

$$f'(\lambda) = -\frac{e^{2\lambda}\left(e^{2\lambda}+1-2\arctan\left(\sqrt{e^{2\lambda}-1}\right)\sqrt{e^{2\lambda}-1}\right)}{4\arctan^3\left(\sqrt{e^{2\lambda}-1}\right)\sqrt{e^{2\lambda}-1}\,(e^{2\lambda}-1)^2}.$$

since the denominator is clearly positive, we have to show that

$$e^{2\lambda}+1-2\arctan\left(\sqrt{e^{2\lambda}-1}\right)\sqrt{e^{2\lambda}-1} > 0, \qquad \forall\lambda > 0.$$

Substituting $x := \sqrt{e^{2\lambda}-1}$ leads to

$$x^2+2 > 2x\arctan(x), \qquad \forall x > 0. \tag{4.1.3.8}$$

We prove this inequality in two steps. First note that

$$x^2 - \frac{10\pi}{12}x + 2 > 0, \qquad \forall x > 0$$

holds, so that for all x with $0 < x \le 2+\sqrt{3}$ we have the estimate

$$x^2+2 > 2x\frac{5\pi}{12} = 2x\arctan(2+\sqrt{3}) \ge 2x\arctan(x)$$

and, on the other hand, for $x \ge 2+\sqrt{3}$ we obtain

$$x^2+2 \ge (2+\sqrt{3})x+2 > (2+\sqrt{3})x \ge \pi x = 2x\frac{\pi}{2} > 2x\arctan(x).$$

In conclusion (4.1.3.8) holds, so that $f' < 0$ and therefore

$$f(\lambda) \ge \frac{1}{\pi^2}, \qquad \forall\lambda > 0.$$

All together this yields

$$\alpha_\lambda e^{2\lambda}\lambda^{-1} = \frac{1}{9}\min\left(\frac{1}{256}e^{2\lambda}\lambda^{-1}, \alpha_2^{(\lambda)}e^{2\lambda}\lambda^{-1}\right) \ge \frac{1}{9}\min\left(\frac{e}{128}, \frac{1}{\pi^2}\right) = \frac{e}{1152}.$$

α_λ is constant on $[0,1]$:

Claim:

$$\alpha_2^{(\lambda)} \geq \frac{1}{256}, \qquad \forall \lambda \in [0,1].$$

Let $\lambda \in [0,1]$ and set

$$g(\lambda) := \frac{(e^{2\lambda}+1)(e^{2\lambda}-1)}{\lambda}.$$

g has the first derivative

$$g'(\lambda) = \frac{1-(1-4\lambda)e^{4\lambda}}{\lambda^2}.$$

We want to show that $1-(1-4\lambda)e^{4\lambda}$ is non-negative and thus prove that g is an non-decreasing function. To this end observe that

$$(1-4\lambda)e^{4\lambda}$$

is a decreasing function on $[0,\infty[$, since the derivative $-16\lambda e^{4\lambda}$ is clearly non-positive, so that

$$(1-4\lambda)e^{4\lambda} \leq 1$$

holds for all $\lambda \geq 0$. This leads to

$$1-(1-4\lambda)e^{4\lambda} \geq 0, \qquad \forall \lambda \geq 0.$$

This proves that g is non-decreasing. Using this we can easily conclude

$$\max_{\lambda\in[0,1]} g(\lambda) \leq g(1) = (e^2+1)(e^2-1) \leq 64$$

and hence

$$g(\lambda) = \frac{(e^{2\lambda}+1)(e^{2\lambda}-1)}{\lambda} \leq \frac{256}{4}, \qquad \forall \lambda \in [0,1].$$

Taking the reciprocal on both sides yields

$$\alpha_2^{(\lambda)} = \frac{\lambda}{4(e^{2\lambda}+1)(e^{2\lambda}-1)} \geq \frac{1}{256}, \qquad \forall \lambda \in [0,1]. \qquad (4.1.3.9)$$

Note that

$$\arctan(x) \leq x, \qquad \forall x \in \mathbb{R}_+.$$

This can be proved by calculating the Taylor-polynomial up to the first order and dropping the remainder term which is always negative on $\mathbb{R}_+$. Using this on our above estimate (4.1.3.9) we obtain

$$\frac{\lambda}{4(e^{2\lambda}+1)\arctan^2\left(\sqrt{e^{2\lambda}-1}\right)} \geq \frac{1}{256}, \qquad \forall \lambda \in [0,1].$$

This implies that α_λ is constant on the interval $[0,1]$.

α_λ is non-increasing:

By the previous part we can assume that $\lambda \geq 1$. We have to show that $\alpha_2^{(\lambda)}$ is non-increasing on the interval $[1, \infty[$. We do this by showing that the derivative of $\alpha_2^{(\lambda)}$

$$\left(\alpha_2^{(\lambda)}\right)' = -\frac{\overbrace{2\lambda}^{=:\,p_1} - \overbrace{\arctan\left(\sqrt{e^{2\lambda}-1}\right)\sqrt{e^{2\lambda}-1}}^{=:\,n_1}}{4\arctan^3\left(\sqrt{e^{2\lambda}-1}\right)\sqrt{e^{2\lambda}-1}(e^{2\lambda}+1)^2}$$

$$-\frac{\overbrace{2\lambda e^{2\lambda}}^{=:\,p_2} - \overbrace{\arctan\left(\sqrt{e^{2\lambda}-1}\right)\sqrt{e^{2\lambda}-1}e^{2\lambda}}^{=:\,n_2} + \overbrace{2\lambda\arctan\left(\sqrt{e^{2\lambda}-1}\right)\sqrt{e^{2\lambda}-1}e^{2\lambda}}^{=:\,p_3}}{4\arctan^3\left(\sqrt{e^{2\lambda}-1}\right)\sqrt{e^{2\lambda}-1}(e^{2\lambda}+1)^2}.$$

is non-positive. So, to simplify notation we have to show that

$$p_1 - n_1 + p_2 - n_2 + p_3 \geq 0, \qquad \forall\lambda \geq 1 \tag{4.1.3.10}$$

holds. Note that for $\lambda \geq 1$

$$p_3 - n_1 - n_2 \geq \arctan\left(\sqrt{e^{2\lambda}-1}\right)\sqrt{e^{2\lambda}-1}e^{2\lambda}\,(2\lambda - 2) \geq 0,$$

so that (4.1.3.10) holds, which finishes the proof that $\alpha_2^{(\lambda)}$ is non-increasing on $[1, \infty[$. Together with the previous established result that α is constant on $[0, 1]$ this completes the proof that α_λ is non-increasing on $\mathbb{R}_+$.

□

4.2 Hilbert space-valued Ornstein–Uhlenbeck Processes

In this section we consider an H-valued Ornstein–Uhlenbeck process Z^A with drift term A and prove in Theorem 4.2.2 a similar result as Proposition 4.1.3 of the previous section. The key ingredient is the following lemma, which is used to reduce the Hilbert space case to the one-dimensional case.

Lemma 4.2.1 (Cf. [Wre16, Lemma 2.2])

Let $(Z_t^A)_{t\in[0,\infty[}$ be an H-valued Ornstein–Uhlenbeck process with drift term A, i.e. a solution to

$$\begin{cases} \mathrm{d}Z_t^A = -AZ_t^A\mathrm{d}t + \mathrm{d}B_t, \\ \;Z_0^A = 0. \end{cases}$$

Let $(\lambda_n)_{n\in\mathbb{N}}$ be the eigenvalues of A. Let $C \in \mathbb{R}$ and the map α be as in Proposition 4.1.3. Then for all Borel measurable functions $b\colon [0,1] \times H \longrightarrow H$, which are in the second component twice continuously differentiable with

$$\|b\|_\infty := \sup_{t\in[0,1],x\in\mathbb{R}} |b(t,x)|_H \in]0, \infty[$$

we have

$$\mathbb{E}\exp\left(\frac{\alpha_{\lambda_i}}{\|b\|_\infty^2}\left|\int_0^1 \partial_{x_i} b(t, Z_t)\ \mathrm{d}t\right|_H^2\right) \le C \le 3 \qquad \forall i \in \mathbb{N},$$

where $\partial_{x_i} b$ denotes the derivative of b w.r.t. the i-th component of the second parameter x.

Proof

Let us define the mapping

$$\begin{aligned} \varphi_A \colon \mathcal{C}([0,\infty[, H) &\longrightarrow \mathcal{C}([0,\infty[, H) \\ f = (f^{(n)})_{n\in\mathbb{N}} &\longmapsto \left(t \longmapsto \left((2\lambda_n)^{-1/2} e^{-\lambda_n t} f^{(n)}(e^{2\lambda_n t} - 1)\right)_{n\in\mathbb{N}}\right). \end{aligned}$$

φ_A is bijective and we have used that $\mathcal{C}([0,\infty[, \mathbb{R}^{\mathbb{N}}) \cong \mathcal{C}([0,\infty[, \mathbb{R})^{\mathbb{N}}$ as topological spaces. By definition of the product topology φ_A is continuous if and only if $\pi_n \circ \varphi_A$ is continuous for every $n \in \mathbb{N}$.

$$\begin{array}{ccc} \mathcal{C}([0,\infty[, H) & \xrightarrow{\ \varphi_A\ } & \mathcal{C}([0,\infty[, H) \\ & \searrow^{\varphi_A^{(n)} := \pi_n \circ \varphi_A} & \downarrow \pi_n \\ & & \mathcal{C}([0,\infty[, \mathbb{R}) \end{array}$$

Here, π_n denotes the projection to the n-th component. The above mapping φ_A is continuous and, therefore, measurable w.r.t. the Borel sigma-algebra. Using this transformation, the Ornstein–Uhlenbeck measure $\mathbb{P}_A$, as defined in the introduction, can be written as

$$\mathbb{P}_A[F] = Z^A(\mathbb{P})[F] = (\varphi_A \circ \tilde{B})(\mathbb{P})[F] = \varphi_A(\mathcal{W})[F], \qquad \forall F \in \mathcal{B}\left(\mathcal{C}\left([0,\infty[, H\right)\right),$$

because of

$$Z_t^A = \varphi_A \circ \tilde{B}_t.$$

Hence, we have

$$\mathbb{P}_A = \varphi_A(\mathcal{W}) = \varphi_A\left(\bigotimes_{n\in\mathbb{N}} \mathcal{W}^{(n)}\right) = \bigotimes_{n\in\mathbb{N}} \varphi_A^{(n)}\left(\mathcal{W}^{(n)}\right), \tag{4.2.1.1}$$

where $\mathcal{W}^{(n)}$ is the projection of $\mathcal{W}$ to the n-th coordinate and the last equality follows from

$$\int_F \mathrm{d}\varphi_A\left(\bigotimes_{n\in\mathbb{N}} \mathcal{W}^{(n)}\right) = \prod_{n\in\mathbb{N}} \int_{\pi_n\left(\varphi_A^{-1}(F)\right)} \mathrm{d}\mathcal{W}^{(n)} = \prod_{n\in\mathbb{N}} \int_{(\varphi_A^{(n)})^{-1}(\pi_n(F))} \mathrm{d}\mathcal{W}^{(n)} = \left(\bigotimes_{n\in\mathbb{N}} \varphi_A^{(n)}\left(\mathcal{W}^{(n)}\right)\right)[F].$$

Starting from the left-hand side of the assertion we have

$$\mathbb{E}\exp\left(\frac{\alpha_{\lambda_i}}{\|b\|_\infty^2}\left|\int_0^1 \partial_{x_i} b(t, Z_t^A)\ \mathrm{d}t\right|_H^2\right).$$

Using Equation (4.2.1.1) we can write this as

$$\int_{\mathcal{C}([0,\infty[,\mathbb{R})^{\mathbb{N}}} \exp\left(\frac{\alpha_{\lambda_i}}{\|b\|_\infty^2}\left|\int_0^1 \partial_{x_i} b\left(t, ((\varphi_A^{(n)} \circ f_n)(t))_{n\in\mathbb{N}}\right)\, \mathrm{d}t\right|_H^2\right) \mathrm{d}\bigotimes_{n\in\mathbb{N}} \mathcal{W}^{(n)}(f_n),$$

where $(f_n)_{n\in\mathbb{N}}$ are the components of f. Using Fubini's Theorem we can perform the i-th integral first and obtain

$$\int_{\mathcal{C}([0,\infty[,\mathbb{R})^{\mathbb{N}\setminus\{i\}}} \int_{\mathcal{C}([0,\infty[,\mathbb{R})} \exp\left(\frac{\alpha_{\lambda_i}}{\|b\|_\infty^2}\left|\int_0^1 \partial_{x_i} b\left(t, ((\varphi_A^{(n)} \circ f_n)(t))_{n\in\mathbb{N}}\right)\, \mathrm{d}t\right|_H^2\right) \mathrm{d}\mathcal{W}^{(i)}(f_i)\mathrm{d}\bigotimes_{\substack{n\in\mathbb{N}\\ n\neq i}} \mathcal{W}^{(n)}(f_n).$$

Since $\varphi_A^{(i)} \circ f_i$ is under $\mathcal{W}^{(i)}$ distributed as $Z^{A,(i)}$ under $\mathbb{P}$. By Proposition 4.1.3 the inner integral is smaller than C, so that the entire expression is smaller than

$$\int_{\mathcal{C}([0,\infty[,\mathbb{R})^{\mathbb{N}\setminus\{i\}}} C \,\mathrm{d}\bigotimes_{\substack{n\in\mathbb{N}\\ n\neq i}} \mathcal{W}^{(n)}(f_n) = C,$$

where in the last step we used that $\mathcal{W}^{(n)}$ are probability measures.

□

Theorem 4.2.2 (Cf. [Wre16, Theorem 2.3])

Let $\ell \in]0,1]$ and $(Z_t^{\ell A})_{t\in[0,\infty[}$ be an H-valued Ornstein–Uhlenbeck process with drift term ℓA, i.e.

$$\begin{cases} \mathrm{d}Z_t^{\ell A} = -\ell A Z_t^{\ell A}\mathrm{d}t + \mathrm{d}B_t, \\ Z_0^{\ell A} = 0. \end{cases}$$

There exists an absolute constant $C \in \mathbb{R}$ (independent of A and ℓ) such that for all Borel measurable functions $b\colon [0,1] \times H \longrightarrow H$ with

$$\|b\|_{\infty,A} := \sup_{t\in[0,1],x\in H} \left(\sum_{n\in\mathbb{N}} \lambda_n e^{2\lambda_n} b_n(t,x)^2\right)^{1/2} < \infty.$$

The following inequality

$$\mathbb{E}\exp\left(\frac{\beta_{A,b}}{\|h\|_\infty^2}\left|\int_0^1 b(t, Z_t^{\ell A} + h(t)) - b(t, Z_t^{\ell A})\, \mathrm{d}t\right|_H^2\right) \leq C \leq 3,$$

where

$$\beta_{A,b} := \frac{1}{4}\Lambda^{-2}\|b\|_{\infty,A}^{-2} \inf_{n\in\mathbb{N}} \alpha_{\lambda_n} e^{2\lambda_n}\lambda_n^{-1} > 0$$

(w.l.o.g. $\|b\|_{\infty,A} > 0$) holds uniformly for all bounded, measurable functions $h\colon [0,1] \longrightarrow H$ with

$$\|h\|_\infty := \sup_{t\in[0,1]} |h(t)|_H \in]0,\infty[$$

and

$$\sum_{n\in\mathbb{N}} |h_n(t)|^2 \lambda_n^2 < \infty, \qquad \forall t \in [0,1].$$

Recall that Λ is defined in equation (1.1.1) and α is the map from Proposition 4.1.3.

Proof

Step 1: The case for twice continuously differentiable b.

Let $Z^{\ell A}$ be an H-valued Ornstein–Uhlenbeck process, $b\colon [0,1]\times H \longrightarrow H$ a bounded, Borel measurable function which is twice continuously differentiable in the second variable with $\|b\|_{\infty,A} < \infty$ (and hence $\|b\|_\infty < \infty$ by Proposition 4.1.2), and $h\colon [0,1] \longrightarrow H$ a bounded, measurable function with $\|h\|_\infty \neq 0$. Let α and C be as in Proposition 4.1.3. Recall that Λ is defined as

$$\Lambda = \sum_{n\in\mathbb{N}} \lambda_n^{-1} < \infty.$$

Note that by Proposition 4.1.3 $\beta_{A,b} > 0$. By the Fundamental Theorem of Calculus we obtain

$$\begin{aligned}
&\mathbb{E}\exp\left(\frac{4\beta_{A,b}}{\|h\|_\infty^2}\left|\int_0^1 b(t, Z_t^{\ell A} + h(t)) - b(t, Z_t^{\ell A})\ \mathrm{d}t\right|_H^2\right)\\
&= \mathbb{E}\exp\left(\frac{4\beta_{A,b}}{\|h\|_\infty^2}\left|\int_0^1 b(t, Z_t^{\ell A} + \theta h(t))\Big|_{\theta=0}^{\theta=1}\ \mathrm{d}t\right|_H^2\right)\\
&= \mathbb{E}\exp\left(\frac{4\beta_{A,b}}{\|h\|_\infty^2}\left|\int_0^1\int_0^1 b'(t, Z_t^{\ell A} + \theta h(t))h(t)\ \mathrm{d}\theta\mathrm{d}t\right|_H^2\right),
\end{aligned}$$

where b' denotes the Fréchet derivative of b w.r.t. x. Using Fubini's Theorem we can switch the order of integration, so that the above equals

$$\begin{aligned}
&\mathbb{E}\exp\left(4\beta_{A,b}\left|\int_0^1\int_0^1 b'(t, Z_t^{\ell A} + \theta h(t))\frac{h(t)}{\|h\|_\infty}\ \mathrm{d}t\mathrm{d}\theta\right|_H^2\right)\\
&= \mathbb{E}\exp\left(4\beta_{A,b}\left|\int_0^1\int_0^1 \sum_{i\in\mathbb{N}} \underbrace{b'(t, Z_t^{\ell A} + \theta h(t))e_i}_{=\partial_{x_i} b(t, Z_t^{\ell A}+\theta h(t))} \frac{h_i(t)}{\|h\|_\infty}\ \mathrm{d}t\mathrm{d}\theta\right|_H^2\right)\\
&= \mathbb{E}\exp\left(4\beta_{A,b}\left|\int_0^1\int_0^1 \sum_{i\in\mathbb{N}} \frac{h_i(t)}{\|h\|_\infty}\sum_{j\in\mathbb{N}} \partial_{x_i} b_j(t, Z_t^{\ell A} + \theta h(t))e_j\ \mathrm{d}t\mathrm{d}\theta\right|_H^2\right)
\end{aligned}$$

$$= \mathbb{E} \exp \left(4\beta_{A,b} \left| \int_0^1 \int_0^1 \sum_{i \in \mathbb{N}} \lambda_i^{-1/2} \partial_{x_i} \underbrace{\frac{h_i(t)}{\|h\|_\infty} \lambda_i^{1/2} \sum_{j \in \mathbb{N}} b_j(t, Z_t^{\ell A} + \theta h(t)) e_j}_{= e^{-\lambda_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A})} \, \mathrm{d}t \mathrm{d}\theta \right|_H^2 \right), \quad (4.2.2.1)$$

where

$$\tilde{b}_{h,\theta,i}(t,x) := e^{\lambda_i} \frac{h_i(t)}{\|h\|_\infty} \lambda_i^{1/2} \sum_{j \in \mathbb{N}} b_j(t, x + \theta h(t)) e_j.$$

Note that $\|\tilde{b}_{h,\theta,i}\|_\infty \leq 1$ because for all $(t,x) \in [0,1] \times H$ we have

$$\begin{aligned}
|\tilde{b}_{h,\theta,i}(t,x)|_H &= \underbrace{\frac{|h_i(t)|}{\|h\|_\infty}}_{\leq 1} \lambda_i^{1/2} e^{\lambda_i} \left| \sum_{j \in \mathbb{N}} b_j(t, x + \theta h(t)) e_j \right|_H \\
&\leq \lambda_i^{1/2} e^{\lambda_i} \left(\sum_{j \in \mathbb{N}} \lambda_j^{-1} e^{-2\lambda_j} \lambda_j e^{2\lambda_j} b_j(t, x + \theta h(t))^2 \right)^{1/2} \\
&\leq \underbrace{\lambda_i^{1/2} e^{\lambda_i} \sup_{j \in \mathbb{N}} \lambda_j^{-1/2} e^{-\lambda_j}}_{\leq 1} \underbrace{\left(\sum_{j \in \mathbb{N}} \lambda_j e^{2\lambda_j} b_j(t, x + \theta h(t))^2 \right)^{1/2}}_{\leq \|b\|_{\infty,A}} \leq \|b\|_{\infty,A}.
\end{aligned}$$

Using Jensen's Inequality and again Fubini's Theorem the expression (4.2.2.1) is bounded from above by

$$\int_0^1 \mathbb{E} \exp \left(4\beta_{A,b} \left| \sum_{i \in \mathbb{N}} \lambda_i^{-1/2} \int_0^1 e^{-\lambda_i} \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) \, \mathrm{d}t \right|_H^2 \right) \mathrm{d}\theta.$$

Applying Hölder Inequality we can split the sum and estimate this from above by

$$\begin{aligned}
&\int_0^1 \mathbb{E} \exp \left(4\beta_{A,b} \underbrace{\sum_{i \in \mathbb{N}} \lambda_i^{-1}}_{=\Lambda} \sum_{i \in \mathbb{N}} \left| \int_0^1 e^{-\lambda_i} \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) \, \mathrm{d}t \right|_H^2 \right) \mathrm{d}\theta \\
&= \int_0^1 \mathbb{E} \exp \left(4\beta_{A,b} \Lambda \sum_{i \in \mathbb{N}} \left| \int_0^1 e^{-\lambda_i} \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) \, \mathrm{d}t \right|_H^2 \right) \mathrm{d}\theta \\
&= \int_0^1 \mathbb{E} \prod_{i \in \mathbb{N}} \exp \left(4\beta_{A,b} \Lambda \left| \int_0^1 e^{-\lambda_i} \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) \, \mathrm{d}t \right|_H^2 \right) \mathrm{d}\theta.
\end{aligned}$$

Young's Inequality with $p_i := \lambda_i \Lambda$ leads us to the upper bound

$$\int_0^1 \mathbb{E} \sum_{i\in\mathbb{N}} \frac{1}{p_i} \exp\left(4\beta_{A,b}\Lambda p_i \left| \int_0^1 e^{-\lambda_i} \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) \, \mathrm{d}t \right|_H^2 \right) \mathrm{d}\theta.$$

$$= \int_0^1 \sum_{i\in\mathbb{N}} \frac{1}{p_i} \mathbb{E} \exp\left(4\beta_{A,b}\Lambda^2 \lambda_i \left| \int_0^1 e^{-\lambda_i} \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) \, \mathrm{d}t \right|_H^2 \right) \mathrm{d}\theta. \tag{4.2.2.2}$$

Recall that

$$\beta_{A,b} = \frac{1}{4} \Lambda^{-2} \|b\|_{\infty,A}^{-2} \inf_{n\in\mathbb{N}} \alpha_{\lambda_n} e^{2\lambda_n} \lambda_n^{-1},$$

hence, we can estimate (4.2.2.2) from above by

$$\int_0^1 \sum_{i\in\mathbb{N}} \frac{1}{p_i} \mathbb{E} \exp\left(\alpha_{\lambda_i} e^{2\lambda_i} \left| \int_0^1 e^{-\lambda_i} \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) \, \mathrm{d}t \right|_H^2 \right) \mathrm{d}\theta.$$

$$= \int_0^1 \sum_{i\in\mathbb{N}} \frac{1}{p_i} \mathbb{E} \exp\left(\alpha_{\lambda_i} \left| \int_0^1 \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) \, \mathrm{d}t \right|_H^2 \right) \mathrm{d}\theta.$$

Since $\ell \in \,]0,1]$ and α is non-increasing by Proposition 4.1.3 the above is smaller than

$$\int_0^1 \sum_{i\in\mathbb{N}} \frac{1}{p_i} \mathbb{E} \exp\left(\alpha_{\ell\lambda_i} \left| \int_0^1 \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) \, \mathrm{d}t \right|_H^2 \right) \mathrm{d}\theta.$$

Applying Lemma 4.2.1 for every $\theta \in [0,1]$ and $i \in \mathbb{N}$ results in the estimate

$$\int_0^1 \underbrace{\sum_{i\in\mathbb{N}} \frac{1}{p_i}}_{=1} C \, \mathrm{d}\theta = C.$$

Step 2: The general case: Non-smooth b.

Let $b\colon [0,1] \times H \longrightarrow H$ be a bounded, Borel measurable function with $\|b\|_{\infty,A} < \infty$ (and hence $\|b\|_\infty < \infty$ by Proposition 4.1.2), and $h\colon [0,1] \longrightarrow H$ a bounded, Borel measurable function with $0 \neq \|h\|_\infty < \infty$ and

$$\sum_{n\in\mathbb{N}} |h_n(t)|^2 \lambda_n^2 < \infty \qquad \forall t \in [0,1].$$

Let $\beta_{A,b}$ and C be the constants from Step 1. Set $\varepsilon := \exp \frac{-64\beta_{A,b}}{\|h\|_\infty^2}$ as well as

$$\mu_0 := \mathrm{d}t \otimes Z_t^{\ell A}[\mathbb{P}],$$
$$\mu_h := \mathrm{d}t \otimes (Z_t^{\ell A} + h(t))[\mathbb{P}].$$

Note that the measure $Z_t^{\ell A}[\mathbb{P}]$ is equivalent to the invariant measure $N(0, \frac{1}{2\ell}A^{-1})$ due to [DZ92, Theorem 11.13] and analogously $(Z_t^{\ell A} + h(t))[\mathbb{P}]$ to $N(h(t), \frac{1}{2\ell}A^{-1})$. Furthermore, $h(t)$ is in the domain of A for every $t \in [0,1]$ because of

$$\sum_{n\in\mathbb{N}} \langle h(t), e_n\rangle^2 \lambda_n^2 \leq \sum_{n\in\mathbb{N}} |h_n(t)|^2 \lambda_n^2 < \infty.$$

We set

$$g(t) := 2\ell A h(t).$$

Observe that $g(t) \in H$ for every $t \in [0,1]$ because of

$$|g(t)|_H^2 = 4\ell^2 \sum_{n\in\mathbb{N}} \lambda_n^2 |h_n(t)|^2 < \infty.$$

Hence, [Bog98, Corollary 2.4.3] is applicable i.e. $N(0, \frac{1}{2\ell}A^{-1})$ and $(Z_t^{\ell A} + h(t))[\mathbb{P}]$ are equivalent measures. By the Radon–Nikodym Theorem there exist a density ρ so that

$$\frac{\mathrm{d}\mu_h}{\mathrm{d}\mu_0} = \rho.$$

Furthermore, there exists $\delta > 0$ such that

$$\int_A \rho \ \mathrm{d}\mu_0(t,x) \leq \frac{\varepsilon}{2}, \tag{4.2.2.3}$$

for all measurable sets $A \subseteq [0,1] \times H$ with $\mu_0[A] \leq \delta$. Set

$$\overline{\delta} := \min\left(\delta, \frac{\varepsilon}{2}\right). \tag{4.2.2.4}$$

By Lusin's Theorem (see [Tao11, Theorem 1.3.28]) there exist a closed set $K \subseteq [0,1] \times H$ with $\mu_0[K] \geq 1 - \overline{\delta}$ such that the restriction

$$b\,|_K \colon K \longrightarrow H, \qquad (t,x) \longmapsto b(t,x)$$

is continuous. Note that

$$(\mu_0 + \mu_h)[K^c] = \underbrace{\mu_0[K^c]}_{\leq \overline{\delta} \leq \frac{\varepsilon}{2}} + \mu_h[K^c] \leq \frac{\varepsilon}{2} + \underbrace{\int_{K^c} \rho \ \mathrm{d}\mu_0(t,x)}_{\leq \frac{\varepsilon}{2} \text{ by (4.2.2.4) and (4.2.2.3)}} \leq \varepsilon. \tag{4.2.2.5}$$

Applying Dugundji's Extension Theorem (see [Dug51, Theorem 4.1]) to the function $b|_K$ guarantees that there exists a continuous function $\overline{b}\colon [0,1] \times H \longrightarrow H$ with $\|\overline{b}\|_\infty \leq \|b\|_\infty$ and $\|\overline{b}\|_{\infty,A} \leq \|b\|_{\infty,A}$ which coincides with b on K. Starting from the left-hand side of the assertion we have

$$\mathbb{E}\exp\left(\frac{\beta_{A,b}}{\|h\|_\infty^2}\left|\int_0^1 b(t, Z_t^{\ell A} + h(t)) - b(t, Z_t^{\ell A}) \ \mathrm{d}t\right|_H^2\right).$$

Adding and subtracting $\overline{b}$ and using that $b - \overline{b} = 0$ on K yields that the above equals

$$\mathbb{E} \exp \left(\frac{\beta_{A,b}}{\|h\|_\infty^2} \left| \int_0^1 \mathbb{1}_{K^c}(t, Z_t^{\ell A} + h(t)) \, [\underbrace{b(t, Z_t^{\ell A} + h(t)) - \bar{b}(t, Z_t^{\ell A} + h(t))}_{\in[-2,2]}] \right. \right.$$

$$- \mathbb{1}_{K^c}(t, Z_t^{\ell A}) \, [\underbrace{b(t, Z_t^{\ell A}) - \bar{b}(t, Z_t^{\ell A})}_{\in[-2,2]}] \; \mathrm{d}t$$

$$\left. \left. + \int_0^1 \bar{b}(t, Z_t^{\ell A} + h(t)) - \bar{b}(t, Z_t^{\ell A}) \; \mathrm{d}t \right|_H^2 \right).$$

Applying the fact that $(a+b)^2 \leq 2a^2 + 2b^2$ we estimate from above by

$$\mathbb{E} \exp \left(\frac{8\beta_{A,b}}{\|h\|_\infty^2} \left(\int_0^1 \mathbb{1}_{K^c}(t, Z_t^{\ell A} + h(t)) + \mathbb{1}_{K^c}(t, Z_t^{\ell A}) \; \mathrm{d}t \right)^2 \right.$$

$$\left. + \frac{2\beta_{A,b}}{\|h\|_\infty^2} \left| \int_0^1 \bar{b}(t, Z_t^{\ell A} + h(t)) - \bar{b}(t, Z_t^{\ell A}) \; \mathrm{d}t \right|_H^2 \right)$$

$$= \mathbb{E} \exp \left(\frac{8\beta_{A,b}}{\|h\|_\infty^2} \left(\int_0^1 \mathbb{1}_{K^c}(t, Z_t^{\ell A} + h(t)) + \mathbb{1}_{K^c}(t, Z_t^{\ell A}) \; \mathrm{d}t \right)^2 \right)$$

$$\cdot \; \exp \left(\frac{2\beta_{A,b}}{\|h\|_\infty^2} \left| \int_0^1 \bar{b}(t, Z_t^{\ell A} + h(t)) - \bar{b}(t, Z_t^{\ell A}) \; \mathrm{d}t \right|_H^2 \right)$$

and using Young's Inequality this is bounded by

$$\underbrace{\frac{1}{2} \mathbb{E} \exp \left(\frac{16\beta_{A,b}}{\|h\|_\infty^2} \left| \int_0^1 \mathbb{1}_{K^c}(t, Z_t^{\ell A} + h(t)) + \mathbb{1}_{K^c}(t, Z_t^{\ell A}) \; \mathrm{d}t \right|_H^2 \right)}_{=: A_1}$$

$$+ \underbrace{\frac{1}{2} \mathbb{E} \exp \left(\frac{4\beta_{A,b}}{\|h\|_\infty^2} \left| \int_0^1 \bar{b}(t, Z_t^{\ell A} + h(t)) - \bar{b}(t, Z_t^{\ell A}) \; \mathrm{d}t \right|_H^2 \right)}_{=: A_2}.$$

Let us estimate A_1 first

$$\begin{aligned}
A_1 &= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{16\beta_{A,b}}{\|h\|_\infty^2} \right)^k \mathbb{E} \left| \int_0^1 \mathbb{1}_{K^c}(t, Z_t^{\ell A} + h(t)) + \mathbb{1}_{K^c}(t, Z_t^{\ell A}) \, \mathrm{d}t \right|_H^{2k} \\
&\leq 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{16\beta_{A,b}}{\|h\|_\infty^2} \right)^k 2^{2k} \underbrace{(\mu_h[K^c] + \mu_0[K^c])}_{\leq \varepsilon \text{ by (4.2.2.5)}} \leq 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{64\beta_{A,b}}{\|h\|_\infty^2} \right)^k \varepsilon \\
&\leq 1 + \exp\left(\frac{64\beta_{A,b}}{\|h\|_\infty^2} \right) \varepsilon = 1 + 1 = 2.
\end{aligned}$$

This concludes the estimate for A_1. Let us now estimate A_2. Since $\overline{b}$ is continuous there exists a sequence $\overline{b}^{(m)} \colon [0,1] \times H \longrightarrow H$ of functions with $\|\overline{b}^{(m)}\|_\infty < \infty$ and $\|\overline{b}^{(m)}\|_{\infty,A} < \infty$ which are smooth in the second variable (i.e. twice continuously differentiable) such that $\overline{b}^{(m)}$ converges to $\overline{b}$ everywhere, i.e.

$$\overline{b}^{(m)}(t,x) \overset{m\to\infty}{\longrightarrow} \overline{b}(t,x), \qquad \forall t \in [0,1],\ \forall x \in H.$$

Using the above considerations A_2 equals

$$\mathbb{E} \exp\left(\frac{4\beta_{A,b}}{\|h\|_\infty^2} \left| \int_0^1 \lim_{m\to\infty} \overline{b}^{(m)}(t, Z_t^{\ell A} + h(t)) - \overline{b}^{(m)}(t, Z_t^{\ell A}) \, \mathrm{d}t \right|_H^2 \right),$$

which in turn can be bounded using Fatou's Lemma by

$$\liminf_{m\to\infty} \mathbb{E} \exp\left(\frac{4\beta_{A,b}}{\|h\|_\infty^2} \left| \int_0^1 \overline{b}^{(m)}(t, Z_t^{\ell A} + h(t)) - \overline{b}^{(m)}(t, Z_t^{\ell A}) \, \mathrm{d}t \right|_H^2 \right). \tag{4.2.2.6}$$

Applying Step 1 with b replaced by $\overline{b}^{(m)}$ yields that (4.2.2.6) and henceforth A_2 is bounded by C, so that in conclusion we have

$$\mathbb{E} \exp\left(\frac{\beta_{A,b}}{\|h\|_\infty^2} \left| \int_0^1 b(t, Z_t^{\ell A} + h(t)) - b(t, Z_t^{\ell A}) \, \mathrm{d}t \right|_H^2 \right) \leq \frac{1}{2} A_1 + \frac{1}{2} A_2 \leq 1 + \frac{C}{2} \leq 3,$$

which completes the proof.

□

5 A Concentration of Measure Result

In this chapter we introduce our definition of a regularizing noise and show that an Ornstein–Uhlenbeck process (in the same setting as in the previous chapter) is a regularizing noise according to our definition. We call a stochastic process $X\colon [0,1]\times\Omega \longrightarrow H$ a regularizing noise if it fulfills certain conditions (see Definition 5.1.1) which are derived from the main estimate of the previous chapter (see Theorem 4.2.2). Furthermore, we describe a regularizing noise with three parameters: (Q, h, α).

With $Q \subseteq H$ we denote the subspace of H in which X behaves in a regularizing fashion. In applications this will be a much smaller space than H itself. This also encodes *how* regularizing X is, since usually for stochastic differential equations there is a trade-off between the size of the non-linearity and the "regularizing power" of a noise term. In applications the non-linearity of a stochastic differential equation will be required to take values in the smaller space Q.

h (called the index of fractionality or just index) on the other hand encodes the time-regularization of the noise. For Brownian motion (and Ornstein–Uhlenbeck process since Ornstein–Uhlenbeck processes are driven by a Brownian motion in additive form) this will be $\frac{1}{2}$. However, for fractional Brownian motion we expect $h = 1 - H$, where H is the Hurst parameter of the fractional Brownian motion. Notice that the noise becomes *more* regularizing the irregular (in terms of path-regularity) it is.

Lastly, α (called order) is used to capture the decay of the tail of the noise. For Brownian motion and Ornstein–Uhlenbeck processes this will be simply be 2 since the probability density function of the noise behaves like $\sim e^{-|x|^2}$ for $|x|$ approaching infinity. In general, we expect $\alpha = 2$ for Gaussian noises.

This chapter contains two sections. In the first section we consider an abstract regularizing noise X and prove a concentration of measure result and tail estimate for these regularizing noises.

In the last section we use the estimate established in the previous chapter to prove that a Hilbert space-valued Ornstein–Uhlenbeck process is indeed a regularizing noise according to our definition in the first section. Henceforth, the concentration results of the first section are automatically established for Ornstein–Uhlenbeck processes.

5.1 Regularizing Noises

Definition 5.1.1 (Regularizing noise)

Let $X\colon [0,1]\times\Omega \longrightarrow H$ be a stochastic process adapted to a filtration $(\mathcal{F}_t)_{t\in[0,1]}$ and $Q \subseteq \mathbb{R}^{\mathbb{N}}$. We call X a *Q-regularizing noise of order $\alpha > 0$ with index $h \in]0,1[$* if the following conditions are fulfilled

(i)

$$Q \subseteq \ell^2 \cong H$$

(ii)

$$\mathbb{P}\left[\left|\int_s^t b(s,X_s+x)-b(s,X_s+y)\,\mathrm{d}s\right|_H > \eta|t-s|^h|x-y|_H \,\middle|\, \mathcal{F}_r\right] \leq Ce^{-c\eta^\alpha}$$

for all $0 \leq r < s < t \leq 1$, all Borel measurable functions $b\colon [0,1]\times H \longrightarrow Q$ and $x, y \in 2Q$ for some constants $C, c > 0$ (independent of r, s, t, x, y, but not b!).

(iii) For every Borel measurable function $f\colon [0,1] \longrightarrow Q$ the image measure $(X_t + f(t))[\mathbb{P}]$ is equivalent to $X_t[\mathbb{P}]$ for every $t \in [0,1]$.

Remark 5.1.2

If X is a self-similar process of index $h \in]0,1[$ i.e. for every $a \geq 0$ we have

$$\{X_{at} \mid t \geq 0\} \overset{\text{dist}}{=} \{a^h X_t \mid t \geq 0\}$$

and X is a regularizing noise then the index h in Definition 5.1.1 is precisely the index of self-similarity of the process X as the following Proposition 5.1.6 shows.

Notation 5.1.3

We define

$$\mathrm{reg}(Q,h,\alpha) := \{X\colon [0,1]\times\Omega \longrightarrow H | X \text{ is a } Q\text{-regularizing noise of order } \alpha \text{ with index } h\}.$$

Proposition 5.1.4

For all $Q \subseteq Q' \subseteq \mathbb{R}^{\mathbb{N}}$ we have

$$\mathrm{reg}(Q',h,\alpha) \subseteq \mathrm{reg}(Q,h,\alpha).$$

Proof

Let $X \in \mathrm{reg}(Q',h,\alpha)$. Since Q is smaller than Q' Condition (i) and (iii) of Definition 5.1.1 are trivially fulfilled for (X, Q') and since every function $b\colon [0,1]\times H \longrightarrow Q$ can be considered as a function $b\colon [0,1]\times H \longrightarrow Q'$ so is Condition (ii).

□

Proposition 5.1.5

For all $\alpha < \alpha'$ we have

$$\mathrm{reg}(Q,h,\alpha') \subseteq \mathrm{reg}(Q,h,\alpha).$$

Proof

Let $X \in \operatorname{reg}(Q,h,\alpha')$ and c be the constant from Condition (ii) of Definition 5.1.1 of the regularizing noise X. Let $\eta > 0$. We set

$$c' := \max_{0<x<1} x^{\alpha} - x^{\alpha'} > 0.$$

If $\eta \geq 1$ we obviously have $\eta^{\alpha} \leq \eta^{\alpha'}$. If, on the other hand, $\eta < 1$ then $\eta^{\alpha} - c' \leq \eta^{\alpha'}$. We therefore obtain

$$e^{-c\eta^{\alpha'}} \leq e^{-cc'\eta^{\alpha}+cc'} = e^{cc'} e^{-c\eta^{\alpha}},$$

which implies that Condition (ii) Definition 5.1.1 is fulfilled and therefore $X \in \operatorname{reg}(Q,h,\alpha)$ which completes the proof.

□

Proposition 5.1.6

Let X be a self-similar Markov process of index $h \in]0,1[$. Assume that Condition (ii) of Definition 5.1.1 is fulfilled for the case $s=0$, $t=1$ i.e. we have

$$\mathbb{P}\left[\left|\int_0^1 b(s,X_s+x) - b(s,X_s+y)\ \mathrm{d}s\right|_H > \eta|x-y|_H\right] \leq Ce^{-c\eta^{\alpha}}$$

for some b, c, C, α, all $x, y \in H$ and every $\eta > 0$. Then Condition (ii) of Definition 5.1.1 holds for all $0 \leq r \leq s < t \leq 1$ for the same b, x, y, c, C, α i.e. we have

$$\mathbb{P}\left[\left|\int_s^t b(r,X_r+x) - b(r,X_r+y)\ \mathrm{d}r\right|_H > \eta|t-s|^h|x-y|_H \,\middle|\, \mathcal{F}_r\right] \leq Ce^{-c\eta^{\alpha}}$$

for all $\eta > 0$.

Proof

Let $((X_t)_{t\in[0,\infty[}, (\mathcal{F}_t)_{t\in[0,\infty[}), r, s, t, b, x$ and y be as in the assertion. In order to complete the proof we have to bound the expression

$$\mathbb{P}^{(\mathrm{d}\omega)}\left[\left|\int_s^t b(s,X_r+x) - b(s,X_r+y)\ \mathrm{d}r\right|_H > \eta|t-s|^h|x-y|_H \,\middle|\, \mathcal{F}_r\right].$$

For the reader's convenience we added the integration variable as a superscript to the respective measure which we integrate against. Fix an $\omega' \in \Omega$. Using the transformation $r' := \ell^{-1}(r-s)$, where $\ell := |t-s|$ this equals

$$\mathbb{P}^{(\mathrm{d}\omega)}\left[\ell\left|\int_0^1 b(\ell s'+r, X_{\ell s'+r}+x) - b(\ell s'+r, X_{\ell s'+r}+y)\,\mathrm{d}s'\right|_H > \eta\ell^h|x-y|_H \,\middle|\, \mathcal{F}_r\right](\omega').$$

We define

$$\tilde{b}(t,z) := b(\ell t+r, \ell^h z),$$
$$\tilde{x} := \ell^{-h}x,$$
$$\tilde{y} := \ell^{-h}y,$$

Furthermore, we define the image measure

$$\mathbb{P}_x := \mathbb{P}\circ X(\,\cdot\,,x)^{-1}, \qquad \forall x\in H,$$

where $X(t,x)$ is the stochastic process X started in x at time t. Hence, the above expression simplifies to

$$\mathbb{P}^{(\mathrm{d}\omega)}_{X_r(\omega')}\left[\ell\left|\int_0^1 b(\ell s'+r, X_{\ell s'}+x) - b(\ell s'+r, X_{\ell s'}+y(\ell s'+r))\,\mathrm{d}s'\right|_H > \eta\ell^h|x-y|_H\right].$$

$$\mathbb{P}^{(\mathrm{d}\omega)}_{X_r(\omega')}\left[\left|\int_0^1 \tilde{b}(s,\ell^{-h}X_{\ell s}+\tilde{x}) - \tilde{b}(s,\ell^{-h}X_{\ell s}+\tilde{y})\,\mathrm{d}s\right|_H > \eta|\tilde{x}-\tilde{y}|_H\right].$$

Since X is by assumption self-similar of index h this is the same as

$$\mathbb{P}^{(\mathrm{d}\omega)}_{X_r(\omega')}\left[\left|\int_0^1 \tilde{b}(s,X_r+\tilde{x}) - \tilde{b}(s,X_s+\tilde{y})\,\mathrm{d}s\right|_H > \eta|\tilde{x}-\tilde{y}|_H\right].$$

Note that $\tilde{b}$ is a Borel measurable functions and takes values in the same space as b. By assumption the above is therefore smaller than

$$Ce^{-c\eta^\alpha},$$

which completes the proof.

□

Example 5.1.7 (Brownian motion in $\mathbb{R}^d$)

Let $H := \mathbb{R}^d$,

$$Q := \{x\in\mathbb{R}^d : |x|\le 1\}$$

and $X\colon [0,1]\times\Omega \longrightarrow \mathbb{R}^d$ be a Brownian motion. Then X is a Q-regularizing noise with order $\alpha = 2$ and index $h = \frac{1}{2}$.

Condition (i) of Definition 5.1.1 is trivially fulfilled. Likewise, Condition (iii) since we are in a finite-dimensional space. Condition (ii) has been proven by A. Davie in [Dav07, Corollary 2.6].

Corollary 5.1.8 (Cf. [Wre16, Corollary 3.2])

Let X be a Q-regularizing noise of order α with index h. There exists a constant $C_X > 0$ so that for all $0 \le r \le s < t \le 1$ and for every Borel measurable function $b\colon [s,t]\times H \longrightarrow Q$ and for all $\mathcal{F}_r$-measurable random variables $x, y\colon \Omega \longrightarrow 2Q$. We have for all $p \in \mathbb{N}$

$$\mathbb{E}\left[\left|\int_s^t b(s, X_s + x) - b(s, X_s + y)\,\mathrm{d}s\right|_H^p \middle| \mathcal{F}_r\right] \le C_X^p p^{p/2} |t-s|^{hp} |x-y|_H^p,$$

where $C_X > 0$ only depends on the constants (C, c, α) in Definition 5.1.1 of the regularizing noise X.

Proof

Let $0 \le r \le s < t \le 1$ and b, p as in the assertion.

Step 1: Deterministic x, y

Let $x, y \in H$ be non-random with $x \neq y$. We set

$$S := |t-s|^{-h} |x-y|_H^{-1} \left|\int_s^t b(s, X_s + x) - b(s, X_s + y)\,\mathrm{d}s\right|_H$$

and calculate

$$\mathbb{E}\left[S^p \middle| \mathcal{F}_r\right] = \mathbb{E}\left[\int_0^\infty \mathbb{1}_{\{S>\eta\}} p\eta^{p-1}\,\mathrm{d}\eta \middle| \mathcal{F}_r\right].$$

Notice that the above is valid since S is a non-negative random variable. Using Fubini's Theorem the above equals

$$\int_0^\infty p\eta^{p-1} \mathbb{P}\left[S > \eta \middle| \mathcal{F}_r\right]\,\mathrm{d}\eta.$$

Plugging in the definition of S the above line reads

$$\int_0^\infty p\eta^{p-1} \mathbb{P}\left[\left|\int_s^t b(s, X_s + x) - b(s, X_s + y)\,\mathrm{d}s\right|_H > \eta |t-s|^h |x-y|_H \middle| \mathcal{F}_r\right]\,\mathrm{d}\eta.$$

We estimate the probability inside the integral by using the fact that X is a regularizing noise (more precisely Condition (ii) of Definition 5.1.1). Therefore, the above expression is smaller than

$$C\int_0^\infty p\eta^{p-1}e^{-c\eta^\alpha}\ \mathrm{d}\eta = C\frac{c^{1-\frac{p}{\alpha}}}{c\alpha}p\int_0^\infty \eta'^{\frac{p}{\alpha}-1}e^{-\eta'}\ \mathrm{d}\eta'$$
$$= \frac{C}{\alpha}c^{-\frac{p}{\alpha}}p\Gamma\left(\frac{p}{2}\right).$$

Using Stirling's formula this is bounded from above by

$$\frac{C}{\alpha}c^{-\frac{p}{\alpha}}\underbrace{p\sqrt{\frac{4\pi}{p}}2^{-p/2}e^{-p/2}e^{\frac{1}{6p}}}_{\leq\sqrt{2\pi}e^{-1/2}e^{\frac{1}{6}}}p^{p/2} \leq \frac{2C}{\alpha}c^{-\frac{p}{\alpha}}p^{p/2},$$

which proves that $\mathbb{E}[S^p|\mathcal{F}_r] \leq C_X^p p^{p/2}$, concluding the assertion in the case that x and y are deterministic.

Step 2: Random x, y

Let $x, y\colon \Omega \longrightarrow 2Q$ be $\mathcal{F}_r$ measurable random variables of the form

$$x = \sum_{i=1}^n \mathbb{1}_{A_i}x_i, \qquad y = \sum_{i=1}^n \mathbb{1}_{A_i}y_i,$$

where $x_i, y_i \in H$ and $(A_i)_{1\leq i\leq n}$ are pairwise disjoint sets in $\mathcal{F}_r$. Notice that due to the disjointness we have

$$b\left(t, X_t + \sum_{i=1}^n \mathbb{1}_{A_i}x_i\right) - b\left(t, X_t + \sum_{i=1}^n \mathbb{1}_{A_i}y_i\right) = \sum_{i=1}^n \mathbb{1}_{A_i}\left[b(t, X_t + x_i) - b(t, X_t + y_i)\right].$$

Let p be a positive integer. Starting from the left-hand side of the assertion and using the above identity yields

$$\mathbb{E}\left[\left\|\int_s^t b(t, X_t + x) - b(t, X_t + y)\ \mathrm{d}t\right\|_H^p \middle| \mathcal{F}_r\right]$$
$$= \sum_{i=1}^n \mathbb{E}\left[\mathbb{1}_{A_i}\left\|\int_s^t b(t, X_t + x_i) - b(t, X_t + y_i)\ \mathrm{d}t\right\|_H^p \middle| \mathcal{F}_r\right].$$

Since $A_i \in \mathcal{F}_r$ this can be expressed as

$$\sum_{i=1}^n \mathbb{1}_{A_i}\mathbb{E}\left[\left\|\int_s^t b(t, X_t + x_i) - b(t, X_t + y_i)\ \mathrm{d}t\right\|_H^p \middle| \mathcal{F}_r\right]$$

and by invoking Step 1 this is bounded from above by

$$C_X^p p^{p/2}|t-s|^{hp}\sum_{i=1}^n \mathbb{1}_{A_i}|x_i-y_i|_H^p = C_X^p p^{p/2}|t-s|^{hp}|x-y|_H^p.$$

In conclusion we obtained the result for step functions x, y. The result for general $\mathcal{F}_r$ measurable random variables x, y now follows by approximation via step functions and taking limits.

□

5.2 The Ornstein–Uhlenbeck Process is a Regularizing Noise

In this section we establish that a Hilbert space-values Ornstein–Uhlenbeck process (as studied in the previous chapter) is a regularizing noise (see Corollary 5.2.3) as defined in the previous section.

Definition 5.2.1

Let $(Z_t^A)_{t\in[0,\infty[}$ be an Ornstein–Uhlenbeck process with drift term A and $C_A > 0$. We set

$$Q^A := \left\{ x = (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \,\middle|\, \sum_{n\in\mathbb{N}} \lambda_n e^{2\lambda_n}|x_n|^2 < C_A \right\},$$

where $(\lambda_n)_{n\in\mathbb{N}}$ are the eigenvalues of the operator A.

Theorem 5.2.2 (Cf. [Wre16, Corollary 3.1])

For every Borel measurable function $b\colon [0,1]\times H \longrightarrow Q^A$ there exists a constant $\beta_{A,b} > 0$ (depending only on the drift term A of the Ornstein–Uhlenbeck process $(Z_t^A)_{t\in[0,\infty[}$ and the function b) such that for all $0 \le r \le s < t \le 1$, all Borel measurable functions $h_1, h_2\colon [s,t] \longrightarrow 2Q^A$ and for any $\eta \ge 0$ the inequality

$$\mathbb{P}\left[\left| \int_s^t b(s, Z_s^A + h_1(s)) - b(s, Z_s^A + h_2(s))\,\mathrm{d}s \right|_H > \eta\ell^{1/2}\|h_1-h_2\|_\infty \,\middle|\, \mathcal{G}_r \right] \le 3e^{-\beta_{A,b}\eta^2}$$

holds, where $\ell := t-s$. Recall that $\mathcal{G}_t := \{(Z^A)^{-1}(F) | F \in \overline{\mathcal{G}}_t\}$ is the filtration of $(Z_t^A)_{t\in[0,\infty[}$, where $\overline{\mathcal{G}}_t := \sigma(\pi_s | s \le t)$ and π_s are the projections $\pi_s\colon f \longmapsto f(s)$.

Proof

Let r, s, t, ℓ, b, h_1 and h_2 be as in the assertion. Note that the assertion is trivial if $h_1 = h_2$, hence w.lo.g. we assume $h_1 \neq h_2$. Furthermore, note that since $b\colon [0,1]\times H \longrightarrow Q^A$ we have $\|b\|_{\infty,A} < \infty$ (as will be later be required by Theorem 4.2.2) and $\|b\|_\infty < \infty$ due to Proposition 4.1.2.

Let $((B_t)_{t\in[0,\infty[}, (\mathcal{F}_t)_{t\in[0,\infty[})$ be the Wiener process of the Ornstein–Uhlenbeck process $(Z_t^A)_{t\in[0,\infty[}$. We define the stochastic processes $\tilde{Z}_t^{\ell A} := \ell^{-1/2} Z_{\ell t}^A$ and $\tilde{B}_t := \ell^{-1/2}B_{\ell t}$. Note that $\tilde{B}$ is again a Brownian motion w.r.t. the normal, right-continuous filtration $(\tilde{\mathcal{F}}_t^\ell)_{t\in[0,\infty[} := (\mathcal{F}_{\ell t})_{t\in[0,\infty[}$. Additionally, we have

$$\tilde{Z}_t^{\ell A} = \ell^{-1/2} Z_{\ell t}^A = \ell^{-1/2} \int_0^{\ell t} e^{(\ell t - s)A} \, \mathrm{d}B_s$$
$$= \int_0^{\ell t} e^{\ell\left(t-\frac{s}{\ell}\right)A} \ell^{-1/2} \, \mathrm{d}B_{\frac{s}{\ell}\ell} = \int_0^t e^{(t-s')\ell A} \, \mathrm{d}\tilde{B}_{s'}.$$

Hence, $\tilde{Z}^{\ell A}$ is an Ornstein–Uhlenbeck process with drift term ℓA.

For the reader's convenience we add the integration variable as a superscript to the respective measure which we integrate against, hence the left-hand side of the claim reads

$$\mathbb{P}^{(\mathrm{d}\omega)}\left[\left|\int_r^u b(s, Z_s^A(\omega) + h_1(s)) - b(s, Z_s^A(\omega) + h_2(s)) \, \mathrm{d}s\right|_H > \eta \ell^{1/2} \|h_1 - h_2\|_\infty \middle| \mathcal{G}_r\right].$$

Fix an $\omega' \in \Omega$. Using the transformation $s' := \ell^{-1}(s-r)$ the above equals

$$\mathbb{P}^{(\mathrm{d}\omega)}\left[\ell\left|\int_0^1 b(\ell s' + r, Z_{\ell s'+r}^A(\omega) + h_1(\ell s' + r)) - b(\ell s' + r, Z_{\ell s'+r}^A(\omega) + h_2(\ell s' + r)) \, \mathrm{d}s'\right|_H > \eta \ell^{1/2} \|h_1 - h_2\|_\infty \middle| \mathcal{G}_r\right](\omega').$$

Furthermore, we define the image measure

$$\mathbb{P}_x := \mathbb{P} \circ Z^A(\,\cdot\,, x)^{-1}, \qquad \forall x \in H,$$

where

$$Z^A(t,x) := Z_t^A + e^{-tA}x, \qquad \forall x \in H,\ t \in [0,\infty[.$$

Recall the definitions of π_t and $\overline{\mathcal{G}_t}$ in the statement of this theorem. Since $\mathcal{G}_t$ is the initial sigma-algebra of $\overline{\mathcal{G}_t}$ w.r.t. Z^A we have

$$\mathbb{E}\left[\pi_t \circ Z^A | \mathcal{G}_r\right](\omega') = \mathbb{E}_0\left[\pi_t | \overline{\mathcal{G}_r}\right](Z^A(\omega')),$$

where $\mathbb{E}_0$ denotes the expectation w.r.t. the measure $\mathbb{P}_0$. Applying this to the above situation we obtain that the left-hand side of the assertion reads

$$\mathbb{P}_0^{(\mathrm{d}\omega)}\left[\left|\int_0^1 b(\ell s + r, \pi_{\ell s+r}(\omega) + h_1(\ell s + r)) - b(\ell s + r, \pi_{\ell s+r}(\omega) + h_2(\ell s + r)) \, \mathrm{d}s\right|_H > \eta \ell^{-1/2} \|h_1 - h_2\|_\infty \middle| \overline{\mathcal{G}_r}\right](Z^A(\omega')),$$

Applying the universal Markov property (see [Bau96, Equation (42.18)] or [Jac05, Equation (3.108)]) we have

$$= \mathbb{P}^{(\mathrm{d}\omega)}_{\pi_r(Z^A(\omega'))}\left[\left|\int_0^1 b(\ell s + r, \pi_{\ell s}(\omega) + h_1(\ell s + r)) - b(\ell s + r, \pi_{\ell s}(\omega) + h_2(\ell s + r))\,\mathrm{d}s\right|_H > \eta \ell^{-1/2}\|h_1 - h_2\|_\infty\right]. \tag{5.2.2.1}$$

We define

$$\begin{aligned}\tilde{b}(t,x) &:= b(\ell t + r, \ell^{1/2}x),\\ \tilde{h}_1(t) &:= \ell^{-1/2}h_1(\ell t + r),\\ \tilde{h}_2(t) &:= \ell^{-1/2}h_2(\ell t + r),\end{aligned}$$

so that expression (5.2.2.1) simplifies to

$$\mathbb{P}^{(\mathrm{d}\omega)}_{\pi_r(Z^A(\omega'))}\left[\left|\int_0^1 \tilde{b}(s, \ell^{-1/2}\pi_{\ell s} + \tilde{h}_1(s)) - \tilde{b}(s, \ell^{-1/2}\pi_{\ell s} + \tilde{h}_2(s))\,\mathrm{d}s\right|_H > \eta\left\|\tilde{h}_1 - \tilde{h}_2\right\|_\infty\right].$$

Note that $\tilde{b}$, $\tilde{h}_1$, $\tilde{h}_2$ are all bounded Borel measurable functions, $\tilde{b}$ is Q^A-valued since b is Q^A-valued and $\|\tilde{b}\|_\infty = \|b\|_\infty$ as well as $\|\tilde{b}\|_{\infty,A} = \|b\|_{\infty,A}$. Plugging in the definition of $\mathbb{P}_x$ the above reads

$$(\mathbb{P}\circ Z^A(\,\cdot\,, Z^A_r(\omega'))^{-1})^{(\mathrm{d}\omega)}\left[\left|\int_0^1 \tilde{b}(s, \ell^{-1/2}\pi_{\ell s}(\omega) + \tilde{h}_1(s)) - \tilde{b}(s, \ell^{-1/2}\pi_{\ell s}(\omega) + \tilde{h}_2(s))\,\mathrm{d}s\right|_H > \eta\left\|\tilde{h}_1 - \tilde{h}_2\right\|_\infty\right]$$

$$= \mathbb{P}\left[\left|\int_0^1 \tilde{b}_{\omega',\tilde{h}_2}(s, \underbrace{\ell^{-1/2}Z^A(\ell s, Z^A_r(\omega')) - \ell^{-1/2}e^{-\ell s A}Z^A_r(\omega')}_{=\ell^{-1/2}Z^A_{\ell s} = \tilde{Z}^{\ell A}_s} + \tilde{h}_1(s) - \tilde{h}_2(s)) - \tilde{b}_{\omega',\tilde{h}_2}(s, \underbrace{\ell^{-1/2}Z^A(\ell s, Z^A_r(\omega')) - \ell^{-1/2}e^{-\ell s A}Z^A_r(\omega')}_{=\ell^{-1/2}Z^A_{\ell s} = \tilde{Z}^{\ell A}_s})\,\mathrm{d}s\right|_H > \eta\left\|\tilde{h}_1 - \tilde{h}_2\right\|_\infty\right],$$

where $\tilde{b}_{\omega',\tilde{h}_2}(t,x) := \tilde{b}(t, x + \ell^{-1/2}e^{-\ell t A}Z^A_r(\omega') + \tilde{h}_2(t))$. Recall that $\tilde{Z}^{\ell A}$ is an Ornstein–Uhlenbeck process which starts in 0. By Theorem 4.2.2 there exist constants $\beta_{A,b}$ (depending on the drift term A and b, but independent of ℓ since $\ell \in \,]0,1]$) and an absolute constant $0 < C \leq 3$ such that the conclusion of Theorem 4.2.2 holds for every Ornstein–Uhlenbeck process $\tilde{Z}^{\ell A}$ with the same constants $\beta_{A,b}$ and C. Since $\exp(\beta_{A,b}|\cdot|^2)$ is increasing on $\mathbb{R}_+$ the above equals

$$\mathbb{P}\left[\exp\left(\frac{\beta_{A,b}}{\left\|\tilde{h}_1-\tilde{h}_2\right\|_\infty^2}\left|\int_0^1 \tilde{b}_{\omega',\tilde{h}_2}(s,\tilde{Z}_s^{\ell A}+\tilde{h}_1(s)-\tilde{h}_2(s))-\tilde{b}_{\omega',\tilde{h}_2}(s,\tilde{Z}_s^{\ell A})\,\mathrm{d}s\right|_H^2\right)>\exp\left(\beta_{A,b}\eta^2\right)\right]$$

and by Chebyshev's Inequality this can be estimated from above by

$$e^{-\beta_{A,b}\eta^2}\mathbb{E}\exp\left(\frac{\beta_{A,b}}{\left\|\tilde{h}_1-\tilde{h}_2\right\|_\infty^2}\left|\int_0^1 \tilde{b}_{\omega',\tilde{h}_2}(s,\tilde{Z}_s^{\ell A}+\tilde{h}_1(s)-\tilde{h}_2(s))-\tilde{b}_{\omega',\tilde{h}_2}(s,\tilde{Z}_s^{\ell A})\,\mathrm{d}s\right|_H^2\right).$$

Since $\|\tilde{b}_{\omega',\tilde{h}_2}\|_\infty = \|\tilde{b}\|_\infty$ as well as $\|\tilde{b}_{\omega',\tilde{h}_2}\|_{\infty,A} = \|\tilde{b}\|_{\infty,A}$ hold, the conclusion of Theorem 4.2.2 implies that the above expression is smaller than

$$Ce^{-\beta_{A,b}\eta^2}.$$

□

Corollary 5.2.3 ($Z^A \in \operatorname{reg}(Q^A, 1/2, 2)$)

Let $(Z_t^A)_{t\in[0,\infty[}$ be an Ornstein–Uhlenbeck process with drift term A with filtration $(\mathcal{G}_t)_{t\in[0,\infty[}$ as defined in the previous Theorem 5.2.2. Let $Q^A \subseteq H$ be as in Definition 5.2.1 (for arbitrary $C_A > 0$) we then have

$$Z^A \in \operatorname{reg}\left(Q^A, \frac{1}{2}, 2\right).$$

Proof

We have to show that $(Z_t^A)_{t\in[0,\infty[}$ fulfills the three conditions of Definition 5.1.1 with $(Q,h,\alpha) = (Q^A, \frac{1}{2}, 2)$. By Definition 5.2.1 Q^A obviously fulfills $Q^A \subseteq \ell^2 \cong H$. Condition (i) is therefore fulfilled.

Let $b\colon [0,1]\times H \longrightarrow Q^A$ be a Borel measurable function and $x, y \in 2Q^A$ be given. Then, invoking Theorem 5.2.2 with the constant functions $h_1 \equiv x$ and $h_2 \equiv y$ proves Condition (ii). Notice here that we consider the Ornstein–Uhlenbeck process $(Z_t^A)_{t\in[0,\infty[}$ under the filtration $(\mathcal{G}_t)_{t\in[0,\infty[}$ as defined in Chapter 1 and the statement of Theorem 5.2.2.

Let $f\colon [0,1] \longrightarrow Q^A$ be a Borel measurable function. For Condition (iii) to be fulfilled we have to show that the image measure of Z^A shifted by f is equivalent to image measure of Z^A under $\mathbb{P}$.

Note that the measure $Z_t^A[\mathbb{P}]$ is equivalent to the invariant measure $N(0, \frac{1}{2}A^{-1})$ due to [DZ92, Theorem 11.13] and analogously $(Z_t^A + f(t))[\mathbb{P}]$ to $N(f(t), \frac{1}{2}A^{-1})$. Furthermore, $f(t)$ is in the domain of A for every $t \in [0,1]$ because f takes values in Q^A and due to

$$\sum_{n\in\mathbb{N}}\langle f(t), e_n\rangle^2\lambda_n^2 \le \sum_{n\in\mathbb{N}}|f_n(t)|^2\lambda_n^2 < \infty.$$

We set

$$g(t) := 2Af(t).$$

Observe that $g(t) \in H$ for every $t \in [0,1]$ because of

$$|g(t)|_H^2 = 4 \sum_{n \in \mathbb{N}} \lambda_n^2 |f_n(t)|^2 < \infty.$$

Hence, [Bog98, Corollary 2.4.3] is applicable i.e. $N(0, \frac{1}{2}A^{-1})$ and $(Z_t^A + f(t))[\mathbb{P}]$ are equivalent measures.

In conclusion all three conditions are met and the Ornstein–Uhlenbeck process $(Z_t^A)_{t \in [0,\infty[}$ is therefore a regularizing noise w.r.t. the space Q^A with index $\frac{1}{2}$ and order 2.

$\square$

Part III

Regularization by Noise

6 Pathwise Regularization by Noise

Let $X\colon [0,1]\times\Omega \longrightarrow H$ be a stochastic process adapted to a filtration $(\mathcal{F}_t)_{t\in[0,1]}$. We assume furthermore that X is a Q-regularizing noise for a cuboid Q (see Definition 3.1.1) of order $\alpha > 0$ with index $h \in]0,1[$ in the sense of Definition 5.1.1.

Let, additionally, $b\colon [0,1]\times H \longrightarrow Q$ be a Borel measurable map, $n\in\mathbb{N}$ and $k\in\{0,...,2^n-1\}$. In this chapter we analyze the mapping

$$H \ni x \longmapsto \left| \int_{k2^{-n}}^{(k+1)2^{-n}} b(s, X_s(\omega)+x) - b(s, X_s(\omega))\, \mathrm{d}s \right|_H$$

as well as

$$H\times H \ni (x,y) \longmapsto \left| \int_{k2^{-n}}^{(k+1)2^{-n}} b(s, X_s(\omega)+x) - b(s, X_s(\omega)+y)\, \mathrm{d}s \right|_H$$

for a given path $t \longmapsto X_t(\omega)$ for a *fixed* $\omega\in\Omega$.

In this first section we show that the first mapping is bounded by

$$Cn^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{-hn} \left(|x|_\infty + 2^{-2^n}\right).$$

Here, h is the index and α the order of the regularizing noise X. $\gamma > 0$ controls the size of the cuboid Q in terms of its effective dimension. If we formally put $\gamma = +\infty$ we are in the finite-dimensional case and generalize A. M. Davie's estimate (see [Dav07]) i.e. we obtain the same estimate since $\mathbb{R}^d$-valued Brownian motion is a regularizing noise with $(h,\alpha) = (\frac{1}{2}, 2)$ (see Example 5.1.7).

In the second section we show in a similar way that the second mapping is bounded from above by

$$C\left(n^{\frac{1}{\alpha}} 2^{-\delta n}|x-y|_\infty + 2^{-2^{\theta_\delta n}}\right).$$

Here, there is a tradeoff between the regularity we obtain in time $\delta\in[0,h]$ and the residual term $\theta_\delta := (h-\delta)\frac{2\alpha\gamma}{2+\alpha+2\gamma}$.

Compared to the estimates obtained in the previous part, all estimate hold for all ω in a set $A_{\varepsilon,b}$ with $\mathbb{P}[A^c_{\varepsilon,b}] \le \varepsilon$, where $\varepsilon > 0$ can be taken arbitrary small. However, the constants C (later denoted by C_ε) depend crucially on ε and explode as ε approaches 0.

6.1 Estimate for $x \longmapsto \varphi_{n,k}(x)$

Definition 6.1.1

Let $b\colon [0,1] \times H \longrightarrow Q \subseteq H$ be a Borel measurable function. For $n \in \mathbb{N} \setminus \{0\}$, $k \in \{0, ..., 2^n - 1\}$ and $x \in H$ we define

$$\varphi_{n,k}\colon H \times \Omega \longrightarrow Q$$

by

$$\varphi_{n,k}(b; x, \omega) := \int_{k2^{-n}}^{(k+1)2^{-n}} b(s, X_s(\omega) + x) - b(s, X_s(\omega)) \, \mathrm{d}s.$$

Usually we drop the b and ω and just write $\varphi_{n,k}(x)$ instead of $\varphi_{n,k}(b; x, \omega)$. Additionally, we set

$$\varphi_{n,k}(x, y) := \int_{k2^{-n}}^{(k+1)2^{-n}} b(s, X_s(\omega) + x) - b(s, X_s(\omega) + y) \, \mathrm{d}s.$$

Remark 6.1.2

Note that for fixed $n \in \mathbb{N}$, $k \in \{0, ..., 2^n - 1\}$ and $\omega \in \Omega$ the map

$$|\varphi_{n,k}(\,\cdot\,,\,\cdot\,)|_H\colon H \times H \longrightarrow \mathbb{R}_+, \qquad (x, y) \longmapsto |\varphi_{n,k}(x, y)|_H$$

is a pseudometric on H.

Lemma 6.1.3 (Cf. [Wre17, Lemma 3.3])

For $r, m \in \mathbb{N}$ and $\gamma \geq 1$ we have

$$\ln(r + m + 1)^{1/\gamma} \leq \ln(r + 1)^{1/\gamma} + \ln(m + 1)^{1/\gamma}.$$

Proof

Let $r, m \in \mathbb{N}$. We have

$$r + m + 1 \leq rm + r + m + 1 = (r + 1) \cdot (m + 1),$$

which implies that

$$\ln(r + m + 1) \leq \ln((r + 1) \cdot (m + 1)) = \ln(r + 1) + \ln(m + 1).$$

Since $\frac{1}{\gamma} \leq 1$ we immediately obtain

$$\ln(r + m + 1)^{1/\gamma} \leq \ln(r + 1)^{1/\gamma} + \ln(m + 1)^{1/\gamma}$$

due to the fact that $x \longmapsto x^{1/\gamma}$ is concave which completes the proof.

$\square$

Definition 6.1.4 (The usual assumptions)

Let $(X_t)_{t\in[0,1]}$ be a stochastic process adapted to a filtration $(\mathcal{F}_t)_{t\in[0,1]}$ and $Q \subseteq \mathbb{R}^{\mathbb{N}}$ a cuboid (see Definition 3.1.1) or a subset of a cuboid. We say that the tuple $((X_t)_{t\in[0,1]}, (\mathcal{F}_t)_{t\in[0,1]}, Q)$ fulfills the *usual assumptions* if

(i) $X \in \operatorname{reg}(Q, h, \alpha)$. I.e. X is a Q-regularizing noise of order α with index h.

(ii) There exists $C_Q > 0$ and $\gamma \geq 1$ such that $\operatorname{ed}(Q)_m \leq C_Q(\ln(m+1))^{1/\gamma}$ for all $m \in \mathbb{N}$, i.e. the effective dimension of Q grows at most like $\ln(m)^{1/\gamma}$.

From now on we will always assume that we are working with a tuple $((X_t)_{t\in[0,1]}, (\mathcal{F}_t)_{t\in[0,1]}, Q)$, which fulfills the usual assumptions.

Theorem 6.1.5 (Cf. [Wre17, Theorem 3.4])

Assume that the usual assumptions (see Definition 6.1.4) are fulfilled. For every $\varepsilon > 0$ there exists $C_\varepsilon \in \mathbb{R}$ such that for every Borel measurable function $b\colon [0,1] \times H \longrightarrow Q$, $n \in \mathbb{N} \setminus \{0\}$ and $k \in \{0, ..., 2^n - 1\}$ there exists a measurable set $A_{\varepsilon,b,n,k} \in \mathcal{F}_{(k+1)2^{-n}} \subseteq \Omega$ with $\mathbb{P}[A_{\varepsilon,b,n,k}] \leq \frac{\varepsilon}{3} e^{-n}$ such that on $A^c_{\varepsilon,b,n,k}$

$$|\varphi_{n,k}(x)|_H \leq C_\varepsilon n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{-hn} \left(|x|_\infty + 2^{-2^n}\right)$$

holds for all points $x \in 2Q \cap \mathbb{D}$ (see Definition 3.2.6 for the definition of the set $\mathbb{D}$).

Remark 6.1.6

Note that the constant C_ε depends on ε and γ, but not on b. Conversely, the set of "good omegas" $A^c_{\varepsilon,b,n,k}$ depends on ε, b, n and k.

Proof

Sketch of the proof:

The idea of the proof is to first of all consider the event

$$E_{n,k,x} := \{\omega \in \Omega\colon |\varphi_{n,k}(x)|_H > \eta_{\varepsilon,b,n} |x|_\infty 2^{-hn}\},$$

where $x \in 2Q$. Since $X \in \operatorname{reg}(Q, h, \alpha)$ the probability of the above event is bounded from above by

$$Ce^{c\eta^\alpha_{\varepsilon,b,n}}.$$

However, since we have to prove an estimate *uniformly* in $x \in 2Q \cap \mathbb{D}$ we actually have to consider the event

$$\bigcup_{x \in 2Q\cap\mathbb{D}} E_{n,k,x}$$

and therefore we obtain an estimate for the probability of this event of the form

$$\sum_{x\in 2Q\cap\mathbb{D}} Ce^{c\eta^{\alpha}_{\varepsilon,b,n}}.$$

Since we want the sum to be convergent (and moreover arbitrary small for sufficiently small $\varepsilon > 0$) this would require $\eta_{\varepsilon,b,n}$ to be dependent on x which is undesirable. A way out of this dilemma is dissect the set Q as follows

$$\bigcup_{x\in 2Q\cap\mathbb{D}} E_{n,k,x} = \bigcup_{m=0}^{\infty} \bigcup_{x\in 2Q\cap 2^{-m}\mathbb{Z}^{\mathbb{N}}} E_{n,k,x}$$

and choose $\eta_{\varepsilon,b,n}$ (and therefore $E_{n,k,x}$) dependent on the newly introduced variable m.

However, since we are only really interested in the case when x is "small" we can do even better! We introduce the new variable $r \in \mathbb{N}$ and set

$$Q_r := \{x \in Q \colon |x|_\infty \leq 2^{-r}\}.$$

Then Q_r is the set Q "localized around zero". Since we, obviously, always have $|\varphi_{n,k}(x)|_H \leq C2^{-n}$ it is enough to consider r from zero to some large number N and dissect the set Q as follows

$$\bigcup_{x\in 2Q\cap\mathbb{D}} E_{n,k,x} = \bigcup_{r=0}^{2^n}\bigcup_{m=r}^{\infty} \bigcup_{x\in 2Q_r\cap 2^{-m}\mathbb{Z}^{\mathbb{N}}} E_{n,k,x}.$$

Here, we have chosen $N = 2^n$. By letting $\eta_{\varepsilon,b,n}$ furthermore depend on m and r, more precisely we will set

$$\eta_{\varepsilon,b,n} = \eta_{\varepsilon,b,n,m,r} \sim (\ln(1/\varepsilon))^{1/\alpha}\, n^{1/\alpha}(m-r)^{1/\alpha}\,\mathrm{ed}(Q_r)_m^{\frac{1}{\alpha}+\frac{1}{2}}$$

we can show that the probability of the event is uniformly small while still obtaining a strong estimate.

In spite of all this effort we still have the problem that in the end we obtain an estimate of the form

$$|\varphi_{n,k}(x)|_H \leq C\,(\ln(1/\varepsilon))^{1/\alpha}\, n^{1/\alpha}(m-r)^{1/\alpha}\,\mathrm{ed}(Q_r)_m^{\frac{1}{\alpha}+\frac{1}{2}}|x|_\infty 2^{-hn}.$$

We can control r since first of all $r \leq 2^n$ and secondly r is of the order $\log_2(|x|_\infty^{-1})$. Nevertheless, for a given x the variable $m \geq r$ depends on which level m in the lattice $Q_r \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$ the point x lives in. We therefore modify our approach in the following way:

Consider the set

$$E_{n,k,x,y,m,r} := \{\omega \in \Omega \colon |\varphi_{n,k}(x,y)|_H > \eta_{\varepsilon,b,n,m,r}|x-y|_\infty 2^{-hn}\}$$

together with the dissection

$$\bigcup_{x,y\in 2Q\cap\mathbb{D}} E_{n,k,x,y,m,r} = \bigcup_{r=0}^{2^n}\bigcup_{m=r}^{\infty} \bigcup_{x,y\in 2Q_r\cap 2^{-m}\mathbb{Z}^{\mathbb{N}}} E_{n,k,x,y,m,r}.$$

One would assume that with our current approach the probability of this event increased by a lot, since we are now consider all pairs (x, y), but it actually only increased by a factor 2 in one of the exponents. Following this modified approach yields an estimate of the form

$$|\varphi_{n,k}(x,y)|_H \leq C\left(\ln(1/\varepsilon)\right)^{1/\alpha} n^{1/\alpha}(m-r)^{1/\alpha}\operatorname{ed}(Q_r)_m^{\frac{1}{\alpha}+\frac{1}{2}}|x-y|_\infty 2^{-hn}.$$

We are now able to circumvent our previous issue with the following construction: For a given $x \in 2Q$ we construct for every $m \in \{r, r+1, ...\}$ a point $x_m \in 2Q_r \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$ (i.e. x_m is a lattice point on the m-th level) in such a way that x_m is close to x. Since $|\varphi_{n,k}(\,\cdot\,,\,\cdot\,)|_H$ is a pseudometric (as mentioned in Remark 6.1.2) we can use the triangle inequality

$$|\varphi_{n,k}(x)|_H \leq |\varphi_{n,k}(x_r,0)|_H + \sum_{m=r}^{\infty}|\varphi_{n,k}(x_{m+1},x_m)|_H$$

and use our estimate from above for $|\varphi_{n,k}(x,y)|_H$. Note, that this time we *are* able to estimate the variable m as follows: for the first term on the right-hand side and for the first term under the sum we have $m = r$. For all other terms we estimate $|x_{m+1}-x_m|_\infty \leq 2^{-m}$ and show that the sum is dominated by the first term. Let us now pursue the approach in detail.

Beginning of the proof:

Step 1:

Let $\varepsilon > 0$. For $r \geq 0$ we set (similar as in Definition 3.2.1) $Q_r := \{x \in Q : |x|_\infty \leq 2^{-r}\}$. Let m be an integer with $m \geq r$ and $x, y \in 2Q_r \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$. We are going to estimate the probability of the event $\{|\varphi_{n,k}(x,y)|_H > \eta\}$ for a suitable $\eta \geq 0$. To this end let (C, c) be the constants from Definition 5.1.1 and we set

$$\eta_\varepsilon := \left(\ln\left(\frac{24C}{\varepsilon}\right)\right)^{1/\alpha}. \tag{6.1.6.1}$$

W.l.o.g. we assume that ε is sufficiently small so that η_ε is, first of all, well-defined and furthermore $\eta_\varepsilon \geq 1$. Let us consider the following probability.

$$\mathbb{P}\left[|\varphi_{n,k}(x,y)|_H > c^{-1/\alpha}\eta_\varepsilon(1+2n+5(1+m-r))^{1/\alpha}\operatorname{ed}(2Q_r)_m^{\frac{1}{\alpha}+\frac{1}{2}}|x-y|_\infty 2^{-hn}\right].$$

Since $x, y \in 2Q_r \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$ and $|\cdot|_\infty$, $|\cdot|_2$ are effectively equivalent norms i.e. $|\cdot|_2 \leq \sqrt{\operatorname{ed}(2Q_r)_m}|\cdot|_\infty$ (see Proposition 3.1.6) the above expression is smaller than

$$\mathbb{P}\left[|\varphi_{n,k}(x,y)|_H > c^{-1/\alpha}\eta_\varepsilon(1+2n+5(1+m-r))^{1/\alpha}\operatorname{ed}(2Q_r)_m^{\frac{1}{\alpha}}|x-y|_2 2^{-hn}\right].$$

Since X is a regularizing noise this probability is smaller than

$$Ce^{-\eta_\varepsilon^\alpha \operatorname{ed}(2Q_r)_m}e^{-\eta_\varepsilon^\alpha(2n+5(1+m-r))\operatorname{ed}(2Q_r)_m}.$$

Using that $\eta_\varepsilon \geq 1$ and $\operatorname{ed}(2Q_r)_m \geq 1$ the above is bounded from above by

$$Ce^{-\eta_\varepsilon^\alpha}e^{-(2n+5(1+m-r))\operatorname{ed}(2Q_r)_m} = Ce^{-\eta_\varepsilon^\alpha}e^{-2n}e^{-5(1+m-r)\operatorname{ed}(2Q_r)_m}.$$

In order to get a uniform bound we calculate

$$\mathbb{P}\left[\bigcup_{r=0}^{2^n}\bigcup_{m=r}^{\infty}\bigcup_{\substack{x,y\in\\ 2Q_r\cap 2^{-m}\mathbb{Z}^{\mathbb{N}}}}\left\{|\varphi_{n,k}(x,y)|_H > c^{-1/\alpha}\eta_\varepsilon(1+2n+5(1+m-r))^{1/\alpha}\operatorname{ed}(2Q_r)_m^{\frac{1}{\alpha}+\frac{1}{2}}|x-y|_\infty 2^{-hn}\right\}\right]$$

$$\leq C\sum_{r=0}^{2^n}\sum_{m=r}^{\infty}\sum_{\substack{x,y\in\\ 2Q_r\cap 2^{-m}\mathbb{Z}^{\mathbb{N}}}} e^{-\eta_\varepsilon^\alpha}e^{-2n}e^{-5(1+m-r)\operatorname{ed}(2Q_r)_m}$$

$$= Ce^{-\eta_\varepsilon^\alpha}\sum_{r=0}^{2^n}\sum_{m=r}^{\infty}\#\{(x,y)\mid x,y\in 2Q_r\cap 2^{-m}\mathbb{Z}^{\mathbb{N}}\}e^{-2n}e^{-5(1+m-r)\operatorname{ed}(2Q_r)_m}.$$

Using the usual assumptions (see Definition 6.1.4) and Proposition 3.2.4 we can invoke Theorem 3.2.3, which results in

$$\#\{x\mid x\in 2Q_r\cap 2^{-m}\mathbb{Z}^{\mathbb{N}}\}\leq 2\exp\left(2(1+m-r)\operatorname{ed}(2Q_r)_m\right).$$

Hence, we can bound the above probability by

$$4Ce^{-\eta_\varepsilon^\alpha}\sum_{r=0}^{2^n}\sum_{m=r}^{\infty}\exp\left(4(1+m-r)\operatorname{ed}(2Q_r)_m\right)e^{-2n}e^{-5(1+m-r)\operatorname{ed}(2Q_r)_m}$$

$$= 4Ce^{-\eta_\varepsilon^\alpha}e^{-2n}\sum_{r=0}^{2^n}\sum_{m=r}^{\infty}\exp\left(-(1+m-r)\operatorname{ed}(2Q_r)_m\right).$$

Note that the last sum converges since $\operatorname{ed}(2Q_r)_m\geq 1$ and because of

$$\sum_{m=r}^{\infty}\exp\left(-(1+m-r)\operatorname{ed}(2Q_r)_m\right)\leq\sum_{m=0}^{\infty}\exp\left(-(1+m)\right)\leq 1$$

the above is smaller than

$$4Ce^{-\eta_\varepsilon^\alpha}\sum_{r=0}^{2^n}e^{-2n} = 4Ce^{-\eta_\varepsilon^\alpha}(2^n+1)e^{-2n}\leq 8Ce^{-\eta_\varepsilon^\alpha}e^{-n}.$$

Plugging in Definition (6.1.6.1) of η_ε the above is smaller than $\frac{\varepsilon}{3}e^{-n}$. In conclusion there exists a measurable set $A_{\varepsilon,b,n,k}\subseteq\Omega$ with $\mathbb{P}[A_{\varepsilon,b,n,k}]\leq\frac{\varepsilon}{3}e^{-n}$ such that on $A^c_{\varepsilon,b,n,k}$ we have

$$\begin{aligned}|\varphi_{n,k}(x,y)|_H &\leq c^{-1/\alpha}\eta_\varepsilon\left(1+2n+5(1+m-r)\right)^{1/\alpha}\operatorname{ed}(Q_r)_m^{\frac{1}{\alpha}+\frac{1}{2}}|x-y|_\infty 2^{-hn}\\ &\leq\frac{10}{\alpha c^{1/\alpha}}\eta_\varepsilon(n^{1/\alpha}+(1+m-r)^{1/\alpha})\operatorname{ed}(Q_r)_m^{\frac{1}{\alpha}+\frac{1}{2}}|x-y|_\infty 2^{-hn}\end{aligned}\tag{6.1.6.2}$$

for $n\geq 1$, $k\in\{0,...,2^n-1\}$, $r\in\{0,...,2^n\}$, $m\geq r$ and $x,y\in 2Q_r\cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$.

Step 2:

Claim: For every dyadic number $x\in 2Q_r$ with $r\in\{0,...,2^n\}$ and $n\geq 1$, $k\in\{0,...,2^n-1\}$ we have

$$|\varphi_{n,k}(x)|_H\leq C_\varepsilon\eta_\varepsilon 2^{-hn}2^{-r}n^{1/\alpha}(\ln(r+2))^{\frac{2+\alpha}{2\alpha\gamma}}.\tag{6.1.6.3}$$

on $A^c_{\varepsilon,b,n,k}$. Indeed, let x be a dyadic number such that $x \in 2Q_r$ with $r \in \{0, ..., 2^n\}$. Recall Corollary 3.2.5. For every $m \in \mathbb{N}$ with $m \geq r$ we set

$$x_m := 2\pi^{(r)}_{m+1}\left(\frac{x}{2}\right) \in 2Q_r \cap 2^{-m}\mathbb{Z}^{\mathbb{N}},$$

where $\pi^{(r)}_m$ is the map from Corollary 3.2.5. I.e. $|x - x_m|_\infty \leq 2^{-m}$. By the triangle inequality (see Remark 6.1.2) and $\varphi_{n,k}(x) = \varphi_{n,k}(x, 0)$ we immediately get

$$|\varphi_{n,k}(x)|_H \leq |\varphi_{n,k}(x_r, 0)|_H + \sum_{m=r}^{\infty} |\varphi_{n,k}(x_{m+1}, x_m)|_H.$$

Note that the sum on the right-hand side is actually a finite sum, because x is dyadic, so that $x_m = x$ for m sufficiently large. Note that $x_m, x_{m+1} \in 2^{-(m+1)}\mathbb{Z}^{\mathbb{N}}$ hence, by using inequality (6.1.6.2), the above expression is bounded from above by

$$\frac{10}{\alpha c^{1/\alpha}}\eta_\varepsilon \left(n^{1/\alpha} + (1 + r - r)^{1/\alpha}\right) \mathrm{ed}(2Q_r)_r^{\frac{1}{\alpha}+\frac{1}{2}} |x_r|_\infty 2^{-hn}$$
$$+\frac{10}{\alpha c^{1/\alpha}}\eta_\varepsilon \sum_{m=r}^{\infty}(n^{1/\alpha} + (1 + (m+1) - r)^{1/\alpha})\, \mathrm{ed}(2Q_r)_{m+1}^{\frac{1}{\alpha}+\frac{1}{2}} |x_{m+1} - x_m|_\infty 2^{-hn}.$$

Using the definition of x_m and $|x_{m+1} - x_m|_\infty \leq |x_{m+1} - x|_\infty + |x_m - x|_\infty \leq 2^{-m+1}$ this can be estimated from above by

$$\frac{10}{\alpha c^{1/\alpha}}\eta_\varepsilon \left(n^{1/\alpha} + 1\right) \mathrm{ed}(2Q_r)_r^{\frac{1}{\alpha}+\frac{1}{2}} 2^{-r} 2^{-hn}$$
$$+\frac{40}{\alpha c^{1/\alpha}} c^{-1/\alpha}\eta_\varepsilon \sum_{m=r}^{\infty}(n^{1/\alpha} + (1 + (m+1) - r)^{1/\alpha}) 2^{-(m+1)}\, \mathrm{ed}(2Q_r)_{m+1}^{\frac{1}{\alpha}+\frac{1}{2}} 2^{-hn}$$
$$\leq \frac{40}{\alpha c^{1/\alpha}}\eta_\varepsilon 2^{-hn} \sum_{m=r}^{\infty} \left(n^{1/\alpha} + (1 + m - r)^{1/\alpha}\right) \mathrm{ed}(2Q_r)_m^{\frac{1}{\alpha}+\frac{1}{2}} 2^{-m}.$$

By the usual assumptions we have that $\mathrm{ed}(Q_r)_m \leq C_Q(\ln(m+1))^{1/\gamma}$, where $\gamma > 1$ and $C_Q > 0$ are the constants from Definition 6.1.4. Using this we can further estimate the above expression by

$$\frac{40 C_Q^{\frac{1}{\alpha}+\frac{1}{2}}}{\alpha c^{1/\alpha}}\eta_\varepsilon 2^{-hn} \sum_{m=r}^{\infty} \left(n^{1/\alpha} + (1 + m - r)^{1/\alpha}\right) (\ln(m+1))^{(\frac{1}{\alpha}+\frac{1}{2})\frac{1}{\gamma}} 2^{-m}$$
$$\leq \frac{40 C_Q^{\frac{1}{\alpha}+\frac{1}{2}}}{\alpha c^{1/\alpha}}\eta_\varepsilon 2^{-hn} \sum_{m=0}^{\infty} \left(n^{1/\alpha} + (1 + m)^{1/\alpha}\right) (\ln(r + m+1))^{\frac{2+\alpha}{2\alpha\gamma}} 2^{-m-r}.$$

Using Lemma 6.1.3 and putting all constants into the new constant C_ε for the sake of readability the above is smaller than

$$C_\varepsilon 2^{-hn} 2^{-r} \sum_{m=0}^{\infty} \left(n^{1/\alpha} + (1+m)^{1/\alpha}\right) \left((\ln(r+1))^{\frac{2+\alpha}{2\alpha\gamma}} + \ln(m+2)^{\frac{2+\alpha}{2\alpha\gamma}}\right) 2^{-m}$$
$$\leq C_\varepsilon 2^{-hn} 2^{-r} \left[n^{1/\alpha}(\ln(r+1))^{\frac{2+\alpha}{2\alpha\gamma}} \sum_{m=0}^{\infty} 2^{-m} + n^{1/\alpha} \sum_{m=0}^{\infty} (\ln(m+2))^{\frac{2+\alpha}{2\alpha\gamma}} 2^{-m} \right.$$

$$+(\ln r)^{\frac{2+\alpha}{2\alpha\gamma}}\sum_{m=0}^{\infty}(1+m)^{1/\alpha}2^{-m}+\sum_{m=0}^{\infty}(1+m)^{1/\alpha}(\ln(m+2))^{\frac{2+\alpha}{2\alpha\gamma}}2^{-m}\Bigg].$$

Since $\gamma \geq 1$ we can estimate $(\ln(m+2))^{\frac{2+\alpha}{2\alpha\gamma}} \leq 2^{m/2}$. The above expression is therefore bounded by

$$\begin{aligned} &C_\varepsilon 2^{-hn}2^{-r}\left[2n^{1/\alpha}(\ln r)^{\frac{2+\alpha}{2\alpha\gamma}}+n^{1/\alpha}\sum_{m=0}^{\infty}2^{-m/2}+4(\ln r)^{\frac{2+\alpha}{2\alpha\gamma}}+\sum_{m=0}^{\infty}(1+m)^{1/\alpha}2^{-m/2}\right]\\ &\qquad\leq C_\varepsilon 2^{-hn}2^{-r}\left[2n^{1/\alpha}(\ln r)^{1/\gamma}+4n^{1/\alpha}+4(\ln r)^{\frac{2+\alpha}{2\alpha\gamma}}+6\right].\end{aligned}$$

And since we have $1 \leq (\ln(r+3))^{\frac{2+\alpha}{2\alpha\gamma}}$, we obtain

$$|\varphi_{n,k}(x)|_H \leq C_\varepsilon 2^{-hn}2^{-r}n^{1/\alpha}(\ln(r+3))^{\frac{2+\alpha}{2\alpha\gamma}},$$

which proves Claim (6.1.6.3).

Step 3:

For a fixed $n \in \mathbb{N}$ let $x \in 2Q \cap \mathbb{D}$ such that $|x|_\infty > 2^{-2^n}$. We set

$$r := \lfloor \log_2 |x|_\infty^{-1} \rfloor \leq \lfloor 2^n \rfloor \leq 2^n.$$

And hence we have

$$2^{-r} = 2^{-\log_2 \lfloor |x|_\infty^{-1} \rfloor} \leq 2^{-\log_2 |x|_\infty^{-1}+1} = 2|x|_\infty.$$

Additionally, we have $r \in \{-2, ..., 2^n\}$ and $x \in 2Q_r$, because of the fact that

$$|x|_\infty = 2^{-\log_2 |x|_\infty^{-1}} \leq 2^{-r}.$$

Hence, we can apply Step 2 (6.1.6.3) to obtain

$$\begin{aligned}|\varphi_{n,k}(x)|_H &\leq C_\varepsilon 2^{-r}n^{1/\alpha}2^{-hn}(\ln(r+3))^{\frac{2+\alpha}{2\alpha\gamma}}\\ &\leq C_\varepsilon n^{1/\alpha}2^{-hn}|x|_\infty\left(\log_2\left(2^{3n}\right)\right)^{\frac{2+\alpha}{2\alpha\gamma}} \leq C_\varepsilon n^{1/\alpha}(3n)^{\frac{2+\alpha}{2\alpha\gamma}}2^{-hn}|x|_\infty.\end{aligned}$$

Step 4:

Conversely to Step 3, for fixed $n \in \mathbb{N}$ let $x \in 2Q \cap \mathbb{D}$ such that $|x|_\infty \leq 2^{-2^n}$. Then $x \in Q_r$ with $r = 2^n$ so that by Invoking Step 2 (i.e. Inequality (6.1.6.3)) we have

$$\begin{aligned}|\varphi_{n,k}(x)|_H &\leq C_\varepsilon 2^{-r}n^{1/\alpha}2^{-hn}(\ln(r+3))^{\frac{2+\alpha}{2\alpha\gamma}} \leq C_\varepsilon 2^{-2^n}n^{1/\alpha}2^{-n/2}\left(\log_2\left(2^{3n}\right)\right)^{1/\gamma}\\ &\leq C_\varepsilon n^{1/\alpha}2^{-hn}2^{-2^n}(3n)^{\frac{2+\alpha}{2\alpha\gamma}}.\end{aligned}$$

This concludes the proof.

□

6.2 Estimate for $(x, y) \longmapsto \varphi_{n,k}(x, y)$

Let us now prove an estimate for the term $|\varphi_{n,k}(x, y)|_H$. Since, due to technicalities in the proof of Theorem 6.1.5, we were forced to prove an estimate for $|\varphi_{n,k}(x, y)|_H$ in the previous section the proof in this section will mainly follow along the same lines as the previous section.

Theorem 6.2.1 (Cf. [Wre17, Theorem 3.6])

Assume that the usual assumptions (see Definition 6.1.4) are fulfilled. For every $\varepsilon > 0$ there exists $C_\varepsilon \in \mathbb{R}$ such that for every Borel measurable function $b\colon [0, 1] \times H \longrightarrow Q$ satisfying Assumption 1.1.2 there exists a measurable set $A_{\varepsilon,b} \subseteq \Omega$ with $\mathbb{P}[A_{\varepsilon,b}] \leq \varepsilon$ such that on $A^c_{\varepsilon,b}$ and for every $0 < \delta < h$ we have

$$|\varphi_{n,k}(x, y)|_H \leq C_\varepsilon \left[n^{\frac{1}{\alpha}} 2^{-\delta n} |x - y|_\infty + 2^{-2^{\theta_\delta n}} \right]$$

for all points $x, y \in Q \cap \mathbb{D}$ with $|x - y|_\infty \leq 1$, $n \geq 1$, $k \in \{0, ..., 2^n - 1\}$ where $\theta_\delta := (h - \delta)\frac{2\alpha\gamma}{2+\alpha+2\gamma}$.

Remark 6.2.2

Note that the constant C_ε depends on ε and γ, but not on b. Conversely, the set of "good omegas" $A^c_{\varepsilon,b}$ depends on both, ε and b.

Proof

Sketch of the proof:

Since this proof is similar to the proof of Theorem 6.1.5 in the previous section we merely point out the differences.

We would like to obtain an estimate which is "good" when $|x - y|_\infty$ is small instead of just $|x|_\infty$ being small, hence, the "localization trick" from the previous proof does not work anymore. Compared to the last proof there is no variable r anymore, so that we have to dissect the set $2Q \cap \mathbb{D}$ in the lattices $2Q \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$ as described in the beginning of the sketch of the proof of Theorem 6.1.5. Following the approach of the last proof with this setup, results in the estimate

$$|\varphi_{n,k}(x, y)|_H \leq C n^{\frac{1}{\alpha}} 2^{-hn} m^{\frac{1}{\alpha}+\frac{2+\alpha}{2\alpha\gamma}} 2^{-m},$$

where $m \in \mathbb{N}$ is chosen such that $2^{-m-1} \leq |x - y|_\infty \leq 2^{-m}$. We then proceed to bound the term $m^{\frac{1}{\alpha}+\frac{2+\alpha}{2\alpha\gamma}}$ by giving up a little bit of 2^{-hn} term. This yields the following bound

$$|\varphi_{n,k}(x, y)|_H \leq C \left[n^{\frac{1}{\alpha}} 2^{-\delta n} 2^{-m} + 2^{-2^{\theta_\delta n}} \right].$$

Here, $\delta > 0$ is an arbitrary number smaller than h and $\theta_\delta > 0$ a number depending on δ as well as on α and γ. The result then follows by bounding 2^{-m} by $2|x - y|_\infty$.

Beginning of the proof:

Step 1:

Let $m \in \mathbb{N}$, $0 < \delta < h$ and $x, y \in Q \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$. Let (c, C) be the constants from Definition 5.1.1 for the regularizing noise X and $\varepsilon > 0$. We set

$$\eta_\varepsilon := \left(\ln\left(\frac{C}{\varepsilon}\right)\right)^{1/\alpha}.$$

We again assume that $\varepsilon > 0$ is small enough so that the above is well-defined and $\eta_\varepsilon \geq 1$.

Analogously to the previous proof we estimate

$$\mathbb{P}\left[|\varphi_{n,k}(x,y)|_H > c^{-1/\alpha}\eta_\varepsilon(1+2n+5(1+m))^{1/\alpha}\,\mathrm{ed}(Q)_m^{\frac{1}{\alpha}+\frac{1}{2}}|x-y|_\infty 2^{-hn}\right],$$

Since $x, y \in Q \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$ and $|\cdot|_\infty$, $|\cdot|_2$ are effectively equivalent norms i.e. $|\cdot|_2 \leq \sqrt{\mathrm{ed}(Q)_m}|\cdot|_\infty$ (see Proposition 3.1.6) the above expression is smaller than

$$\mathbb{P}\left[|\varphi_{n,k}(x,y)|_H > c^{-1/\alpha}\eta_\varepsilon(1+2n+5(1+m))^{1/\alpha}(\mathrm{ed}(Q)_m)^{\frac{1}{\alpha}}|x-y|_2 2^{-hn}\right].$$

Due to the fact that X is a regularizing noise this expression is bounded by

$$Ce^{-\eta_\varepsilon^\alpha\,\mathrm{ed}(Q)_m}e^{-\eta_\varepsilon^\alpha(2n+5(1+m))\,\mathrm{ed}(Q)_m}$$

and since $\eta_\varepsilon \geq 1$ as well as $\mathrm{ed}(Q)_m \geq 1$ the above expression can be estimated from above by

$$e^{-\eta_\varepsilon^\alpha}e^{-(2n+5(1+m))\,\mathrm{ed}(Q)_m} \leq e^{-\eta_\varepsilon^\alpha}e^{-2n}e^{-5(1+m)\,\mathrm{ed}(Q)_m}.$$

Using this, we estimate the following probability

$$\mathbb{P}\left[\bigcup_{n=1}^{\infty}\bigcup_{m=0}^{\infty}\bigcup_{\substack{x,y\in\\ Q\cap 2^{-m}\mathbb{Z}^{\mathbb{N}}}}\bigcup_{k=0}^{2^n-1}|\varphi_{n,k}(x,y)|_H > c^{-1/\alpha}\eta_\varepsilon\left(1+2n+5(1+m)\right)^{1/\alpha}\mathrm{ed}(Q)_m^{\frac{1}{\alpha}+\frac{1}{2}}|x-y|_\infty 2^{-hn}\right]$$

$$\leq C\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\substack{x,y\in\\ Q\cap 2^{-m}\mathbb{Z}^{\mathbb{N}}}}\sum_{k=0}^{2^n-1}e^{-\eta_\varepsilon^\alpha}e^{-2n}e^{-5(1+m)\,\mathrm{ed}(Q)_m}$$

$$\leq Ce^{-\eta_\varepsilon^\alpha}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\#\{(x,y)|x,y\in Q\cap 2^{-m}\mathbb{Z}^{\mathbb{N}}\}2^n e^{-2n}e^{-5(1+m)\,\mathrm{ed}(Q)_m}.$$

Using the usual assumptions (see Definition 6.1.4) and Proposition 3.2.4 we can invoke Theorem 3.2.3 for $r = 0$ so that we have

$$\#\{(x,y)\mid x,y\in Q\cap 2^{-m}\mathbb{Z}^{\mathbb{N}}\} \leq \exp\left(4(1+m)\,\mathrm{ed}(Q)_m\right).$$

So that we can bound the above probability by

$$Ce^{-\eta_\varepsilon^\alpha}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\exp\left(4(1+m)\,\mathrm{ed}(Q)_m\right)2^n e^{-2n}e^{-5(1+m)\,\mathrm{ed}(Q)_m}$$

$$\leq Ce^{-\eta_\varepsilon^\alpha} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} 2^n e^{-2n} \exp\left(-(1+m)\,\mathrm{ed}(Q)_m\right).$$

Note that the last sum converges since $\mathrm{ed}(Q)_m \geq 1$. Hence, the above is bounded from above by

$$Ce^{-\eta_\varepsilon^\alpha} \sum_{n=1}^{\infty} 2^n e^{-2n} \underbrace{\sum_{m=0}^{\infty} \exp\left(-(1+m)\right)}_{\leq 1}$$

so that, in conclusion, we have estimated the above probability by

$$Ce^{-\eta_\varepsilon^\alpha} \sum_{n=1}^{\infty} 2^n e^{-2n} \leq Ce^{-\eta_\varepsilon^\alpha} = \varepsilon.$$

Therefore, we obtain

$$\begin{aligned} |\varphi_{n,k}(x,y)|_H &\leq c^{-1/\alpha}\eta_\varepsilon \left(1 + 2n + 5(1+m)\right)^{1/\alpha} \mathrm{ed}(Q)_m^{\frac{1}{\alpha}+\frac{1}{2}} |x-y|_\infty 2^{-hn} \\ &\leq \frac{10}{\alpha c^{1/\alpha}} \eta_\varepsilon \left(n^{1/\alpha} + (1+m)^{1/\alpha}\right) \mathrm{ed}(Q)_m^{\frac{1}{\alpha}+\frac{1}{2}} |x-y|_\infty 2^{-hn} \end{aligned} \tag{6.2.2.1}$$

for $n \geq 1$, $k \in \{0, ..., 2^n - 1\}$, $m \in \mathbb{N}$ and for all $x, y \in Q \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$ on a set $A_{\varepsilon,b}^c \subseteq \Omega$ with $\mathbb{P}[A_{\varepsilon,b}] \leq \varepsilon$.

Step 2:

Claim: For all points $x, y \in Q \cap \mathbb{D}$, with $|x-y|_\infty \leq 1$, $n \geq 1$ and $k \in \{0, ..., 2^n - 1\}$ we have

$$|\varphi_{n,k}(x)|_H \leq C_\varepsilon n^{\frac{1}{\alpha}} \left[2^{-\delta n}|x-y|_\infty + 2^{-2^{\theta\delta n}}\right]. \tag{6.2.2.2}$$

on $A_{\varepsilon,b}^c$. Indeed, let $x, y \in Q$ be two dyadic points in Q with $|x-y|_\infty \leq 1$. W.l.o.g. we assume $x \neq y$. Fix $r \in \mathbb{N}$ be so that $2^{-r-1} \leq |x-y|_\infty \leq 2^{-r}$. Note that this implies that $r \geq 0$. Using Corollary 3.2.5 for every $m \in \mathbb{N}$ with $m \geq r$ we set

$$\begin{aligned} x_m &:= \pi_m^{(0)}(x) \in Q \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}, \\ y_m &:= \pi_m^{(0)}(y) \in Q \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}. \end{aligned}$$

By the triangle inequality (see Remark 6.1.2) we immediately get

$$|\varphi_{n,k}(x,y)|_H \leq |\varphi_{n,k}(x_r, y_r)|_H + \sum_{m=r}^{\infty} |\varphi_{n,k}(x_{m+1}, x_m)|_H + \sum_{m=r}^{\infty} |\varphi_{n,k}(y_{m+1}, y_m)|_H.$$

Note that both sums on the right-hand side are actually a finite sums, because x and y are dyadic points. Also note that $x_m, x_{m+1}, y_m, y_{m+1} \in 2^{-(m+1)}\mathbb{Z}^{\mathbb{N}}$, so that by using inequality (6.2.2.1) the above expression is bounded from above by

$$\begin{aligned} &5c^{-1/\alpha}\eta_\varepsilon \left(n^{1/\alpha} + (1+r)^{1/\alpha}\right) \mathrm{ed}(Q)_r^{\frac{1}{\alpha}+\frac{1}{2}} |x_r - y_r|_\infty 2^{-hn} \\ &+10c^{-1/\alpha}\eta_\varepsilon \sum_{m=r}^{\infty} (n^{1/\alpha} + (m+2)^{1/\alpha})\, \mathrm{ed}(Q)_{m+1}^{\frac{1}{\alpha}+\frac{1}{2}} 2^{-(m-1)} 2^{-hn}, \end{aligned}$$

where we have used that by the definition of x_r we have $|x_{m+1} - x_m|_\infty \leq |x_{m+1} - x|_\infty + |x_m - x|_\infty \leq 2^{-(m-1)}$ and an analogous calculation for $|y_{m+1} - y_m|_\infty$. Since $|x_r - y_r|_\infty \leq |x_r - x|_\infty + |x - y|_\infty + |y - y_r|_\infty \leq 2^{-(r-2)}$ this can be further estimated from above by

$$40c^{-1/\alpha}\eta_\varepsilon \sum_{m=r}^{\infty}(n^{1/\alpha} + (m+1)^{1/\alpha})\,\mathrm{ed}(Q)_m^{\frac{1}{\alpha}+\frac{1}{2}} 2^{-m}2^{-hn}.$$

By the usual assumptions we have that $\mathrm{ed}(Q_r)_m \leq C_Q(\ln(m+1))^{1/\gamma}$, where $\gamma > 1$ and $C_Q > 0$ are the constants from Definition 6.1.4. Using this we can further estimate the above expression by

$$\frac{40C_Q^{\frac{1}{\alpha}+\frac{1}{2}}}{\alpha c^{1/\alpha}}\eta_\varepsilon \sum_{m=r}^{\infty}(n^{1/\alpha} + (m+1)^{1/\alpha})(\ln(m+1))^{(\frac{1}{\alpha}+\frac{1}{2})\frac{1}{\gamma}} 2^{-m}2^{-hn}$$

and since $n^{1/\alpha} + (m+1)^{1/\alpha} \leq 2(n(m+1))^{1/\alpha}$ this is bounded by

$$\frac{80C_Q^{\frac{2+\alpha}{2\alpha\gamma}}}{\alpha c^{1/\alpha}}\eta_\varepsilon \sum_{m=r}^{\infty} n^{1/\alpha}(m+1)^{1/\alpha}(\ln(m+1))^{\frac{2+\alpha}{2\alpha\gamma}} 2^{-m}2^{-hn}.$$

By performing an index shift this can be written as

$$\frac{80C_Q^{\frac{2+\alpha}{2\alpha\gamma}}}{\alpha c^{1/\alpha}}\eta_\varepsilon n^{1/\alpha}2^{-hn}2^{-r} \sum_{m=0}^{\infty}(m+r+1)^{1/\alpha}(\ln(m+r+1))^{\frac{2+\alpha}{2\alpha\gamma}} 2^{-m}.$$

We use $(m+r+1)^{1/\alpha} \leq \frac{2}{\alpha}\left((m+1)^{1/\alpha} + r^{1/\alpha}\right)$ and invoke Lemma 6.1.3 to estimate this further from above by

$$\frac{160C_Q^{\frac{2+\alpha}{2\alpha\gamma}}}{\alpha^2 c^{1/\alpha}}\eta_\varepsilon n^{1/\alpha}2^{-hn}2^{-r} \sum_{m=0}^{\infty}((m+1)^{1/\alpha} + r^{1/\alpha})\left((\ln(m+1))^{\frac{2+\alpha}{2\alpha\gamma}} + (\ln(r+1))^{\frac{2+\alpha}{2\alpha\gamma}}\right) 2^{-m}.$$

Expanding the terms yields

$$\frac{160C_Q^{\frac{2+\alpha}{2\alpha\gamma}}}{\alpha^2 c^{1/\alpha}}\eta_\varepsilon n^{1/\alpha}2^{-hn}2^{-r} \sum_{m=0}^{\infty}\Big[(m+1)^{1/\alpha}(\ln(m+1))^{\frac{2+\alpha}{2\alpha\gamma}} + (m+1)^{1/\alpha}(\ln(r+1))^{\frac{2+\alpha}{2\alpha\gamma}}$$
$$+ r^{1/\alpha}(\ln(m+1))^{\frac{2+\alpha}{2\alpha\gamma}} + r^{1/\alpha}(\ln(r+1))^{\frac{2+\alpha}{2\alpha\gamma}}\Big] 2^{-m}.$$

Plugging in $(\ln(r+1))^{\frac{2+\alpha}{2\alpha\gamma}} \leq 2^{r/2}$ and evaluating the sum term by term leads us to the following upper bound

$$\frac{160C_Q^{\frac{2+\alpha}{2\alpha\gamma}}}{\alpha^2 c^{1/\alpha}}\eta_\varepsilon n^{1/\alpha}2^{-hn}2^{-r}\left[\sum_{m=0}^{\infty}(m+1)^{1/\alpha}2^{-m/2} + (\ln(r+1))^{\frac{2+\alpha}{2\alpha\gamma}}\sum_{m=0}^{\infty}(m+1)^{1/\alpha}2^{-m}\right.$$
$$\left. + r^{1/\alpha}\sum_{m=0}^{\infty}2^{-m/2} + r^{1/\alpha}(\ln(r+1))^{\frac{2+\alpha}{2\alpha\gamma}}\sum_{m=0}^{\infty}2^{-m}\right]$$
$$\leq \frac{160C_Q^{\frac{2+\alpha}{2\alpha\gamma}}}{\alpha^2 c^{1/\alpha}}\eta_\varepsilon n^{1/\alpha}2^{-hn}2^{-r}\left[6 + 3(\ln(r+1))^{1/\gamma} + 4r^{1/\alpha} + 2r^{1/\alpha}(\ln(r+1))^{\frac{2+\alpha}{2\alpha\gamma}}\right]$$

$$\leq \frac{1680 C_Q^{\frac{2+\alpha}{2\alpha\gamma}}}{\alpha^2 c^{1/\alpha}} \eta_\varepsilon n^{\frac{1}{\alpha}} 2^{-hn} (\ln(r+2))^{\frac{2+\alpha}{2\alpha\gamma}} (r+1)^{1/\alpha} 2^{-r}.$$

$$\leq \frac{3360 C_Q^{\frac{2+\alpha}{2\alpha\gamma}}}{\alpha^2 c^{1/\alpha}} \eta_\varepsilon n^{\frac{1}{\alpha}} 2^{-hn} (r+1)^{\frac{2+\alpha}{2\alpha\gamma}} (r+1)^{1/\alpha} 2^{-(r+1)}.$$

In conclusion we finally obtained

$$|\varphi_{n,k}(x,y)|_H \leq \frac{3360 C_Q^{\frac{2+\alpha}{2\alpha\gamma}}}{\alpha^2 c^{1/\alpha}} \eta_\varepsilon n^{1/\alpha} 2^{-hn} (r+1)^{\frac{2+\alpha}{2\alpha\gamma}} r^{1/\alpha} 2^{-(r+1)}. \tag{6.2.2.3}$$

We are going to estimate this further using the following claim:

Set $\theta_\delta := (h-\delta)\frac{2\alpha\gamma}{2+\alpha+2\gamma} > 0$.

Claim:

There is a constant $C_{\alpha,\gamma} > 0$ (independent of n and m) such that

$$n^{\frac{1}{\alpha}} r^{\frac{1}{\alpha}+\frac{2+\alpha}{2\alpha\gamma}} 2^{-r} 2^{-hn} \leq n^{\frac{1}{\alpha}} 2^{-\delta n} 2^{-r} + C_{\alpha,\gamma} 2^{-2^{\theta_\delta n}} \qquad \forall n, r \in \mathbb{N} \tag{6.2.2.4}$$

holds.

Proof of Claim (6.2.2.4)**:**

Case 1: $r \leq 2^{1+\theta_\delta n}$

$$n^{\frac{1}{\alpha}} r^{\frac{1}{\alpha}+\frac{2+\alpha}{2\alpha\gamma}} 2^{-hn} \leq n^{\frac{1}{\alpha}} 2^{(1+\theta_\delta n)(\frac{1}{\alpha}+\frac{2+\alpha}{2\alpha\gamma})} 2^{-hn} = n^{\frac{1}{\alpha}} 2^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} \underbrace{2^{\theta_\delta \frac{2+\alpha+2\gamma}{2\alpha\gamma} n}}_{=2^{(h-\delta)n}} 2^{-hn} = C^{(1)}_{\alpha,\gamma} n^{\frac{1}{\alpha}} 2^{-\delta n}.$$

Case 2: $2^{1+\theta_\delta n} < r$

$$\underbrace{n^{\frac{1}{\alpha}} 2^{-hn}}_{\leq 1} r^{\frac{1}{\alpha}+\frac{2+\alpha}{2\alpha\gamma}} 2^{-r} \leq \underbrace{r^{\frac{1}{\alpha}+\frac{2+\alpha}{2\alpha\gamma}} 2^{-r/2}}_{\leq C^{(2)}_{\alpha,\gamma}} 2^{-r/2} \leq C^{(2)}_{\alpha,\gamma} 2^{-2^{\theta_\delta n}}.$$

This ends the proof of Claim (6.2.2.4). Using (6.2.2.4) and inequality (6.2.2.3) we conclude that

$$|\varphi_{n,k}(x,y)|_H \leq \frac{3360 C_{\alpha,\gamma} C_Q^{\frac{2+\alpha}{2\alpha\gamma}}}{\alpha^2 c^{1/\alpha}} \eta_\varepsilon \left[2 \cdot n^{\frac{1}{\alpha}} 2^{-\delta n} 2^{-r} + 16 \cdot 2^{-2^{\theta_\delta n}}\right].$$

Recall that $2^{-r-1} \leq |x-y|_\infty$ so that the above is smaller than

$$C_\varepsilon \eta_\varepsilon \left[n^{\frac{1}{\alpha}} 2^{-\delta n} |x-y|_\infty + 2^{-2^{\theta_\delta n}}\right],$$

which finishes the proof of Claim (6.2.2.2).

□

7 Continuity of $\varphi_{n,k}$

Let as in the previous chapter X be a Q-regularizing noise and $b\colon [0,1] \times H \longrightarrow Q$ be a Borel measurable map. In this chapter we show that the map

$$\varphi_{n,k}\colon x \longmapsto \int\limits_{k2^{-n}}^{(k+1)2^{-n}} b(s, X_s(\omega) + x) - b(s, X_s(\omega))\ \mathrm{d}s,$$

as defined in the last chapter, is continuous. We even show that for sequences of functions $h_m\colon [0,1] \longrightarrow Q$ living in small set Φ, which converge pointwise to a limiting function $h \in \Phi$ we have

$$\int\limits_{k2^{-n}}^{(k+1)2^{-n}} b(s, X_s(\omega) + h_m(s)) - b(s, X_s(\omega))\ \mathrm{d}s \overset{m\to\infty}{\longrightarrow} \int\limits_{k2^{-n}}^{(k+1)2^{-n}} b(s, X_s(\omega) + h(s)) - b(s, X_s(\omega))\ \mathrm{d}s$$

with probability 1. Using this result for constant functions we can extend the two estimates from the previous chapter (Theorem 6.1.5 and Theorem 6.2.1) from $Q \cap \mathbb{D}$ to Q.

Since the proof of the above mentioned result is split into two steps, this chapter is split into two sections as well. The idea is to construct a *continuous* function $\bar{b}\colon [0,1] \times H \longrightarrow Q$ which coincides with b on a large set. If we replace b by $\bar{b}$ above, the result follows from Lebesgue's dominated convergence Theorem.

In the first section we show that for every $\varepsilon > 0$ and any sufficiently "small" set $U \subseteq [0,1] \times H$ we have

$$\int\limits_0^1 \mathbb{1}_U(s, X_s(\omega) + h(s)))\ \mathrm{d}s \leq \varepsilon$$

uniformly for all $h \in \Phi$. Here, U acts as the set $\{(t,x) \in [0,1] \times H \mid b(t,x) \neq \bar{b}(t,x)\}$, where b and $\bar{b}$ do not coincide.

In the second section we construct the function $\bar{b}$ and carry out the proof of the above mentioned result in Theorem 7.2.1. We, moreover, extend the estimates obtained in the previous chapter to the set Q in Corollary 7.2.2.

7.1 A Uniform Bound for Regularizing Noises

Definition 7.1.1

Let $L > 0$. We define

$$\Phi := \{h\colon [0,1] \longrightarrow 2Q\colon |h(s) - h(t)|_\infty \le L|s-t|,\ \forall s,t \in [0,1]\},$$
$$\Phi_n := \left\{ h\colon [0,1] \longrightarrow 2Q \cap \mathbb{D} \,\middle|\, \begin{array}{l} \forall 0 \le k < 2^n\colon \forall s,t \in [k2^{-n},(k+1)2^{-n}[\colon h(s) = h(t) \text{ and} \\ \forall m,\ell \in \mathbb{Z}\cap[0,2^n]\colon |h(m2^{-n}) - h(\ell 2^{-n})|_\infty \le L|m-\ell|2^{-n} \end{array} \right\},$$
$$\Phi^* := \Phi \cup \bigcup_{n\in\mathbb{N}} \Phi_n.$$

Remark 7.1.2

Note that elements in Φ are continuous, since functions in Φ are Lipschitz continuous (with Lipschitz constant at most L). Φ_n will be used to approximate elements in Φ. Also note that Φ and Φ_n are separable w.r.t. the maximum norm and hence Φ^* is separable.

Observe that the above spaces are constructed in such a way that the assumptions we impose on f (see Assumption 1.1.2) implies that the function u from Proposition 1.2.1 is in the space Φ. In other words, the difference of two solutions of (1.2.1.1) always lives in the space Φ due to Assumption 1.1.2.

Lemma 7.1.3 (Cf. [Wre17, Lemma 4.4])

Let $h \in \Phi^*$ and $n \in \mathbb{N}$. We then have

$$\sum_{k=0}^{2^n-1} \left|h((2k+1)2^{-(n+1)}) - h(2k2^{-(n+1)})\right|_\infty \le \frac{L}{2}.$$

Proof

Let $h \in \Phi^*$ and $n \in \mathbb{N}$ be as in the assertion. If $h \in \Phi$ the inequality follows immediately from the Lipschitz continuity of h. Let $h \in \Phi_m$ for some $m \in \mathbb{N}$.

Case 1: $m \ge n+1$

We have

$$\sum_{k=0}^{2^n-1} \left|h((2k+1)2^{-(n+1)}) - h(2k2^{-(n+1)})\right|_\infty$$
$$= \sum_{k=0}^{2^n-1} \left|h((2k+1)2^{m-(n+1)}2^{-m}) - h(2k2^{m-(n+1)}2^{-m})\right|_\infty .$$

Using the assumption that $h \in \Phi_m$ by definition of Φ_m the above expression is bounded from above by

$$\sum_{k=0}^{2^n-1} L 2^{m-(n+1)} 2^{-m} = \frac{L}{2}.$$

Case 2: $m < n+1$

Since $h \in \Phi_m$ is constant on all intervals of the form $[k2^{-m}, (k+1)2^{-m}[$ the sum simplifies to

$$\sum_{k=0}^{2^n-1} \left| h((2k+1)2^{-(n+1)}) - h(2k2^{-(n+1)}) \right|_\infty = \sum_{k=0}^{2^{m-1}-1} \left| h((2k+1)2^{-m}) - h(2k2^{-m}) \right|_\infty .$$

And using the definition of Φ_m the above sum is bounded by

$$\sum_{k=0}^{2^{m-1}-1} L 2^{-m} = \frac{L}{2}.$$

□

Lemma 7.1.4 (Cf. [Wre17, Lemma 4.5])

Assume that the usual assumptions (see Definition 6.1.4) are fulfilled. For every $\varepsilon > 0$ there exist $\delta > 0$ such that for every open set $U \subseteq [0,1] \times H$ with mass $\mu[U] \leq \delta$, where $\mu = \mathrm{d}t \otimes X_t[\mathbb{P}]$, then there is a measurable set $\Omega_{\varepsilon,U} \subseteq \Omega$ with

$$\mathbb{P}[\Omega \setminus \Omega_{\varepsilon,U}] \leq \varepsilon$$

such that the inequality

$$\int_0^1 \mathbb{1}_U(s, X_s + h(s)) \,\mathrm{d}s \leq \varepsilon$$

holds on $\Omega_{\varepsilon,U}$ uniformly for any $h \in \Phi^*$.

Proof

Sketch of the proof:

First, note that since the set $U \subseteq [0,1] \times H$ is open the mapping $(t,x) \longmapsto \mathbb{1}_U(t,x)$ is lower-semicontinuous. This implies that for a given function $h \in \Phi^*$ we have

$$\int_0^1 \mathbb{1}_U(t, X_t(\omega) + h(t)) \,\mathrm{d}t \leq \lim_{n\to\infty} \int_0^1 \mathbb{1}_U(t, X_t(\omega) + h_n(t)) \,\mathrm{d}t \tag{7.1.4.1}$$

by Lebesgues dominated convergence Theorem for any sequence $(h_n)_{n\in\mathbb{N}}$ as long as $(h_n)_{n\in\mathbb{N}}$ converges pointwise to h. We therefore do only need to prove the assertion for a countable and dense subset of Φ^*.

For every $n \in \mathbb{N}$ we consider the sets

$$\left\{h \in \Phi^*\colon \mathbb{1}_{[k2^{-n},(k+1)2^{-n}[}h(s) = \mathbb{1}_{[k2^{-n},(k+1)2^{-n}[}h(t) \in 2^{-n}\mathbb{Z}^{\mathbb{N}},\ \forall s,t \in [0,1],\ k \in \{0,...,2^n-1\}\right\}$$

i.e. the sets in which the functions $h \in \Phi^*$ are $2^{-n}\mathbb{Z}^{\mathbb{N}}$-valued and constant on all dyadic intervals $[k2^{-n},(k+1)2^{-n}[$. Notice, that the union of these sets form a dense and countable subset of Φ^*. The strategy of the proof is then as follows.

For a given $h \in \Phi^*$ we construct a sequence of functions $(h_n)_{n\in\mathbb{N}}$, which lies in the set introduced above and converges pointwise to h. We rewrite the above limit (7.1.4.1) as

$$\int_0^1 \mathbb{1}_U(t, X_t(\omega) + h(t))\ \mathrm{d}t$$

$$\leq \int_0^1 \mathbb{1}_U(t, X_t(\omega) + h_m(t))\ \mathrm{d}t + \sum_{n=m}^{\infty} \int_0^1 \mathbb{1}_U(t, X_t(\omega) + h_{n+1}(t)) - \mathbb{1}_U(t, X_t(\omega) + h_n(t))\ \mathrm{d}t$$

for a suitable (i.e. sufficiently large $m \in \mathbb{N}$). We split the second integral into the dyadic intervals $[k2^{-(n+1)},(k+1)2^{-(n+1)}[$. Since the functions h_{n+1} and h_n are constant on these intervals, we can rewrite the above with the help of our function $\varphi_{n,k}$ (see Definition 6.1.1) so that we end up with

$$\int_0^1 \mathbb{1}_U(t, X_t(\omega)+h_m(t))\ \mathrm{d}t + \sum_{n=m}^{\infty} \sum_{k=0}^{2^{n+1}-1} \varphi_{n+1,k}\left(\mathbb{1}_U; h_{n+1}(k2^{-(n+1)})\right) - \varphi_{n+1,k}\left(\mathbb{1}_U; h_n(k2^{-(n+1)})\right).$$

Using Theorem 6.2.1 we can bound this from above by

$$\int_0^1 \mathbb{1}_U(t, X_t(\omega) + h_m(t))\ \mathrm{d}t + \sum_{n=m}^{\infty} \sum_{k=0}^{2^{n+1}-1} \left(n^{\frac{1}{\alpha}} 2^{-\delta n} |h_{n+1}(k2^{-(n+1)}) - h_n(k2^{-(n+1)})|_\infty + 2^{-2^{\theta\delta}}\right).$$

Since functions $h \in \Phi^*$ are either Lipschitz continuous or dyadic approximations of Lipschitz continuous functions we have $|h_{n+1}(k2^{-(n+1)}) - h_n(k2^{-(n+1)})|_\infty \approx 2^{-(n+1)}$ (see Lemma 7.1.3). Hence, by virtue of the term $2^{-\delta n}$, the sum over n converges and becomes arbitrarily small as m gets large. We are left with estimating the integral

$$\int_0^1 \mathbb{1}_U(t, X_t(\omega) + h_m(t))\ \mathrm{d}t,$$

which is comparable with the situation we started with. However, we have to prove that this integral is small for only *finitely* many functions h_m! Henceforth, by requiring that the set U is sufficiently small, the above integral is smaller than any given $\varepsilon > 0$ uniformly for finitely many functions h_m and we therefore conclude the proof.

Beginning of the proof:

Let $\varepsilon > 0$ and let $C_{\varepsilon/2}$ be the constant from Theorem 6.2.1, where we set $\delta := \frac{h}{2}$, so that $\theta := \theta_\delta = \frac{h}{2}\frac{2\alpha\gamma}{2+\alpha+2\gamma}$. Choose $m \in \mathbb{N}$ sufficiently large, i.e. choose $m \in \mathbb{N}$ so that

$$(4+L)C_{\varepsilon/2}\sum_{n=m}^{\infty} n^{\frac{1}{\alpha}}2^{-hn/2} \leq \frac{\varepsilon}{2} \qquad \text{and} \qquad m \geq \frac{4}{\theta^2 \ln(2)^2}. \tag{7.1.4.2}$$

holds. Here $L > 0$ is the constant from Definition 7.1.1. Set $\mathcal{N}_m := Q \cap 2^{-m}\mathbb{Z}^{\mathbb{N}}$ and note that $\mathcal{N}_m$ is a finite 2^{-m}-net of Q w.r.t. the maximum norm.

We set

$$\begin{aligned} \mu &:= \mathrm{d}t \otimes X_t[\mathbb{P}], \\ \mu_z &:= \mathrm{d}t \otimes (X_t + z)[\mathbb{P}] \end{aligned}$$

for all $z \in Q$. Since X is a regularizing noise, we can use Condition (iii) in Definition 5.1.1 to conclude with the help of the Radon–Nikodyn Theorem that there exist densities ρ_z so that

$$\frac{\mathrm{d}\mu_z}{\mathrm{d}\mu} = \rho_z.$$

Furthermore, the family $\{\rho_z | z \in \mathcal{N}_m\}$ is uniformly integrable, since $\mathcal{N}_m$ is finite. Hence, there exists $\delta > 0$ such that

$$\int_A \rho_z(t,x)\,\mathrm{d}\mu(t,x) \leq \frac{\varepsilon^2}{4 \cdot 2^m \#(\mathcal{N}_m)}, \qquad \forall z \in \mathcal{N}_m \tag{7.1.4.3}$$

for every measurable set $A \subseteq \Omega$ with $\mu[A] \leq \delta$. Let $U \subseteq [0,1] \times H$ be open with mass $\mu[U] \leq \delta$. Then, by invoking Theorem 6.2.1 for the function $\mathbb{1}_U$ with the constant $C_{\varepsilon/2}$ and $\delta := h/2$, there exists a measurable set $A_{\varepsilon,U} \subseteq \Omega$ with $\mathbb{P}[A_{\varepsilon,U}] \leq \frac{\varepsilon}{2}$ such that

$$\left| \int_{k2^{-n}}^{(k+1)2^{-n}} \mathbb{1}_U(t, X_t + x) - \mathbb{1}_U(t, X_t + y)\,\mathrm{d}t \right| \leq C_{\varepsilon/2}\left(n^{\frac{1}{\alpha}} 2^{-hn/2}|x-y|_\infty + 2^{-2^{\theta n}} \right).$$

holds for every $n \geq 1$, $k \in \{0, ..., 2^n - 1\}$ and $x, y \in Q \cap \mathbb{D}$ on $A^c_{\varepsilon,U}$. Furthermore, we define the events $B_{\varepsilon,U}$ by

$$B_{\varepsilon,U} := \bigcup_{z \in \mathcal{N}_m} \left\{ \int_0^1 \mathbb{1}_U(s, X_s + z)\,\mathrm{d}s > \frac{\varepsilon}{2 \cdot 2^m} \right\}.$$

We then have

$$
\begin{aligned}
\mathbb{P}[B_{\varepsilon,U}] &= \mathbb{P}\left[\bigcup_{z\in\mathcal{N}_m}\left\{\int_0^1 \mathbb{1}_U(s, X_s+z)\,\mathrm{d}s > \frac{\varepsilon}{2\cdot 2^m}\right\}\right] \\
&\le \sum_{z\in\mathcal{N}_m}\mathbb{P}\left[\int_0^1 \mathbb{1}_U(s,X_s+z)\,\mathrm{d}s > \frac{\varepsilon}{2\cdot 2^m}\right] \le \frac{2\cdot 2^m}{\varepsilon}\sum_{z\in\mathcal{N}_m}\mathbb{E}\int_0^1 \mathbb{1}_U(s,X_s+z)\,\mathrm{d}s \\
&= \frac{2\cdot 2^m}{\varepsilon}\sum_{z\in\mathcal{N}_m}\int_{[0,1]\times H}\mathbb{1}_U(s,x)\,\mathrm{d}\mu_z(s,x) = \frac{2\cdot 2^m}{\varepsilon}\sum_{z\in\mathcal{N}_m}\int_U \rho_z(s,x)\,\mathrm{d}\mu(s,x).
\end{aligned}
$$

Since $\mu[U]\le\delta$ using inequality (7.1.4.3) the above is bounded from above by

$$\frac{2\cdot 2^m}{\varepsilon}\#(\mathcal{N}_m)\frac{\varepsilon^2}{4\cdot 2^m\#(\mathcal{N}_m)} = \frac{\varepsilon}{2}.$$

In conclusion we proved that we have $\mathbb{P}[B_{\varepsilon,U}]\le\frac{\varepsilon}{2}$ and therefore obtained that

$$\mathbb{P}[A^c_{\varepsilon,U}\cap B^c_{\varepsilon,U}] \ge 1-\varepsilon.$$

For every $h\in\Phi$ and $n\in\mathbb{N}$ we define

$$h_n(t) := \sum_{k=0}^{2^n-1}\mathbb{1}_{[k2^{-n},(k+1)2^{-n}[}(t)\frac{\lfloor 2^n h(k2^{-n})\rfloor}{2^n} \in \underbrace{Q\cap 2^{-n}\mathbb{Z}^{\mathbb{N}}}_{=\mathcal{N}_n}, \qquad \forall t\in[0,1], \tag{7.1.4.4}$$

where $\lfloor\cdot\rfloor$ denotes the componentwise floor function. Note that h_n is Q-valued since h is Q-valued. Furthermore, $h_n(t)$ is a dyadic number for all $t\in[0,1]$. Also note that h_n converges to h for $n\to\infty$.

Now, let

$$E_{\varepsilon,U} := \bigcap_{h\in\Phi^*}\left\{\int_0^1 \mathbb{1}_U(t, X_t+h(t))\,\mathrm{d}t \le \varepsilon\right\}.$$

We are going to prove that $A^c_{\varepsilon,U}\cap B^c_{\varepsilon,U}\subseteq E_{\varepsilon,U}$ holds. To this end let $\omega\in A^c_{\varepsilon,U}\cap B^c_{\varepsilon,U}$. Using that $\omega\in B^c_{\varepsilon,U}$ we have

$$\left|\int_0^1 \mathbb{1}_U(t,X_t(\omega)+h_m(t))\,\mathrm{d}t\right| \le \sum_{k=0}^{2^m-1}\left|\int_{k2^{-m}}^{(k+1)2^{-m}}\mathbb{1}_U(t,X_t(\omega)+\underbrace{h_m(t)}_{\in\mathcal{N}_m})\,\mathrm{d}t\right| \le \sum_{k=0}^{2^m-1}\frac{\varepsilon}{2\cdot 2^m} = \frac{\varepsilon}{2}.$$

And since $\omega\in A^c_{\varepsilon,U}$ we obtain for $n\ge m$

$$
\begin{aligned}
&\left|\int_0^1 \mathbb{1}_U(t,X_t(\omega)+h_{n+1}(t)) - \mathbb{1}_U(t,X_t(\omega)+h_n(t))\,\mathrm{d}t\right| \\
&\le \sum_{k=0}^{2^{n+1}-1}\left|\int_{k2^{-(n+1)}}^{(k+1)2^{-(n+1)}}\mathbb{1}_U(t,X_t(\omega)+\underbrace{h_{n+1}(t)}_{\in Q\cap\mathbb{D}}) - \mathbb{1}_U(t,X_t(\omega)+\underbrace{h_n(t)}_{\in Q\cap\mathbb{D}})\,\mathrm{d}t\right|
\end{aligned}
$$

$$\leq \sum_{k=0}^{2^{n+1}-1} C_{\varepsilon/2}\left(n^{\frac{1}{\alpha}}2^{-hn/2}|h_{n+1}(k2^{-n-1})-h_n(k2^{-n-1})|_\infty+2^{-2^{\theta n}}\right)$$

$$\leq C_{\varepsilon/2}\left[2^{n+1}2^{-2^{\theta n}}+n^{\frac{1}{\alpha}}2^{-hn/2}\sum_{k=0}^{2^{n+1}-1}|h_{n+1}(k2^{-n-1})-h_n((k/2)2^{-n})|_\infty\right].$$

Note that since h_n is constant on intervals of the form $[k2^{-n},(k+1)2^{-n}[$ we have $h_n((k/2)2^{-n}) = h_n(\lfloor k/2\rfloor 2^{-n})$, so that the above equals

$$C_{\varepsilon/2}\left[2^{n+1}2^{-2^{\theta n}}+n^{\frac{1}{\alpha}}2^{-hn/2}\sum_{k=0}^{2^{n+1}-1}|h_{n+1}(k2^{-n-1})-h_n(\lfloor k/2\rfloor 2^{-n})|_\infty\right].$$

Plugging in Definition (7.1.4.4) yields that the above expression can be written as

$$C_{\varepsilon/2}\left[2^{n+1-2^{\theta n}}+n^{\frac{1}{\alpha}}2^{-hn/2}\sum_{k=0}^{2^{n+1}-1}2^{-n-1}\left|\lfloor 2^{n+1}h(k2^{-n-1})\rfloor-2\left\lfloor 2^n h\left(\lfloor k/2\rfloor 2^{-n}\right)\right\rfloor\right|_\infty\right]$$

$$\leq C_{\varepsilon/2}\left[2^{n-\frac{1}{2}\theta^2\ln(2)^2n^2}+n^{\frac{1}{\alpha}}2^{-hn/2}\sum_{k=0}^{2^{n+1}-1}2^{-n-1}\underbrace{\left|\lfloor 2^{n+1}h(k2^{-n-1})\rfloor-2^{n+1}h(k2^{-n-1})\right|_\infty}_{\leq 1}\right.$$

$$+n^{\frac{1}{\alpha}}2^{-hn/2}\sum_{k=0}^{2^{n+1}-1}\left|h(k2^{-n-1})-h\left(\lfloor k/2\rfloor 2^{-n}\right)\right|_\infty$$

$$\left.+n^{\frac{1}{\alpha}}2^{-hn/2}\sum_{k=0}^{2^{n+1}-1}2^{-n}\underbrace{\left|2^n h\left(\lfloor k/2\rfloor 2^{-n}\right)-\left\lfloor 2^n h\left(\lfloor k/2\rfloor 2^{-n}\right)\right\rfloor\right|_\infty}_{\leq 1}\right]$$

$$\leq C_{\varepsilon/2}\left[2^{n-\frac{1}{2}\theta^2\ln(2)^2mn}+3n^{\frac{1}{\alpha}}2^{-hn/2}+n^{\frac{1}{\alpha}}2^{-hn/2}\sum_{k=0}^{2^{n+1}-1}\left|h\left(k2^{-(n+1)}\right)-h\left(2\lfloor k/2\rfloor 2^{-(n+1)}\right)\right|_\infty\right].$$

Since $k=2\lfloor k/2\rfloor$ in case k is even the sum can be restricted to k of the form $k=2k'+1$ for $k'\in\{0,...,2^n-1\}$. with the help of (7.1.4.2) the above is bounded by

$$C_{\varepsilon/2}\left[2^{n-2n}+3n^{\frac{1}{\alpha}}2^{-hn/2}+n^{\frac{1}{\alpha}}2^{-hn/2}\sum_{k'=0}^{2^n-1}\left|h\left((2k'+1)2^{-(n+1)}\right)-h\left(2k'2^{-(n+1)}\right)\right|_\infty\right].$$

Using Lemma 7.1.3 we can further estimate the above sum by $\frac{L}{2}$, where $L>0$ is the constant from Definition 7.1.1, so that in conclusion we obtain

$$\left|\int_0^1 \mathbb{1}_U(t,X_t(\omega)+h_{n+1}(t))-\mathbb{1}_U(t,X_t(\omega)+h_n(t))\,\mathrm{d}t\right|\leq(4+L)C_{\varepsilon/2}n^{\frac{1}{\alpha}}2^{-hn/2}.$$

Therefore as long as $\omega \in A^c_{\varepsilon,U} \cap B^c_{\varepsilon,U}$ we have by Lebesgue's dominated convergence Theorem, the lower semi-continuity of $\mathbb{1}_U$ and by the above calculation

$$\int_0^1 \mathbb{1}_U(t, X_t(\omega) + h(t)) \, \mathrm{d}t \leq \lim_{n\to\infty} \int_0^1 \mathbb{1}_U(t, X_t(\omega) + h_n(t)) \, \mathrm{d}t$$

$$= \int_0^1 \mathbb{1}_U(t, X_t(\omega) + h_m(t)) \, \mathrm{d}t + \sum_{n=m}^{\infty} \int_0^1 \mathbb{1}_U(t, X_t(\omega) + h_{n+1}(t)) - \mathbb{1}_U(t, X_t(\omega) + h_n(t)) \, \mathrm{d}t$$

$$\leq \frac{\varepsilon}{2} + (4+L)C_{\varepsilon/2} \sum_{n=m}^{\infty} n^{\frac{1}{\alpha}} 2^{-hn/2} \overset{(7.1.4.2)}{\leq} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

In conclusion we have proven that $A^c_{\varepsilon,U} \cap B^c_{\varepsilon,U} \subseteq E_{\varepsilon,U}$ and hence $\mathbb{P}[E_{\varepsilon,U}] \geq 1 - \varepsilon$ which completes the proof.

□

7.2 The Approximation Theorem and Consequences thereof

Let us now proceed to prove the main theorem of this chapter.

Theorem 7.2.1 (Approximation Theorem cf. [Wre17, Theorem 4.6])

Assume that the usual assumptions (see Definition 6.1.4) are fulfilled. Let $b\colon [0,1] \times H \longrightarrow Q$ be a Borel measurable function. Then there exists a measurable set $\Omega' \subseteq \Omega$ with $\mathbb{P}[\Omega'] = 1$ such that for every sequence $(h_m)_{m\in\mathbb{N}} \subseteq \Phi^*$ which converges pointwise to a function $h \in \Phi^*$ i.e. $\lim_{m\to\infty} |h(t) - h_m(t)|_H = 0$ we have

$$\lim_{m\to\infty} \int_0^1 b(s, X_s(\omega) + h_m(s)) \, \mathrm{d}s = \int_0^1 b(s, X_s(\omega) + h(s)) \, \mathrm{d}s$$

for all $\omega \in \Omega'$.

Proof

Let b be as in the assertion. For $\ell \in \mathbb{N}$ let $\varepsilon_\ell := 2^{-\ell}$. By Lemma 7.1.4 for every ε_ℓ there exists a δ_ℓ such that for every pair $(\varepsilon_\ell, \delta_\ell)$ the conclusions of Lemma 7.1.4 holds. Applying Lusin's Theorem to the pair (b, δ_ℓ) yields for every $\ell \in \mathbb{N}$ a closed set $K_\ell \subseteq [0,1] \times H$ with $\mu[K^c_\ell] \leq \delta_\ell$, where $\mu := \mathrm{d}t \otimes X_t[\mathbb{P}]$, so that

$$b\,|_{K_\ell}\colon K_\ell \longrightarrow H, \qquad (t,x) \longmapsto b(t,x)$$

is continuous. By Dugundji's Extension Theorem (see [Dug51, Theorem 4.1]) (applied to the above maps) there exist functions $\overline{b}_\ell\colon [0,1] \times H \longrightarrow H$ such that

$$b(t,x) = \overline{b}_\ell(t,x), \qquad \forall (t,x) \in K_\ell,$$

$$\|\overline{b}_\ell\|_\infty = \|b\|_\infty$$

and

$$\overline{b}_\ell \text{ is continuous.}$$

Then, by invoking Lemma 7.1.4 for $(\varepsilon_\ell, \delta_\ell, K_\ell^c)$ we obtain for every $\ell \in \mathbb{N}$ a measurable set Ω'_ℓ with $\mathbb{P}[\Omega'_\ell] \geq 1 - \varepsilon_\ell$ such that for any $\omega \in \Omega'_\ell$ and $h \in \Phi^*$

$$\int_0^1 \mathbb{1}_{K_\ell^c}(s, X_s(\omega) + h(s)) \, \mathrm{d}s \leq \varepsilon_\ell$$

holds. Let

$$\Omega' := \liminf_{\ell \to \infty} \Omega'_\ell.$$

Since we have

$$\sum_{\ell \in \mathbb{N}} \mathbb{P}[\Omega_\ell'^c] \leq \sum_{\ell \in \mathbb{N}} \varepsilon_\ell = \sum_{\ell \in \mathbb{N}} 2^{-\ell} < \infty$$

the Borel–Canteli Lemma implies that

$$\mathbb{P}[\limsup_{\ell \to \infty} \Omega_\ell'^c] = 0 \quad \Rightarrow \quad \mathbb{P}[\Omega'] = 1.$$

Let $\omega \in \Omega'$ be fixed. Then, there is an $N(\omega) \in \mathbb{N}$ such that for all $\ell > N(\omega)$ we have $\omega \in \Omega_\ell$ and therefore for all $m \in \mathbb{N}$ we obtain

$$\left| \int_0^1 \mathbb{1}_{K_\ell^c}(s, X_s(\omega) + h_m(s)) \, \mathrm{d}s \right| \leq \varepsilon_\ell. \tag{7.2.1.1}$$

Note that inequality (7.2.1.1) also holds if we replace h_m by h, since $h \in \Phi^*$ by assumption. The assertion now follows easily by the following calculation

$$\left| \int_0^1 b(s, X_s(\omega) + h_m(s)) - \overline{b}_\ell(s, X_s(\omega) + h_m(s)) \, \mathrm{d}s \right|_H$$

$$\leq \int_0^1 \mathbb{1}_{K_\ell^c}(s, X_s(\omega) + h_m(s)) \underbrace{\left| b(s, X_s(\omega) + h_m(s)) - \overline{b}_\ell(s, X_s(\omega) + h_m(s)) \right|_H}_{\leq 2} \, \mathrm{d}s$$

$$\leq 2 \underbrace{\int_0^1 \mathbb{1}_{K_\ell^c}(s, X_s(\omega) + h_m(s)) \, \mathrm{d}s}_{\leq \varepsilon_\ell \text{ by } (7.2.1.1)}.$$

In conclusion we have

$$\lim_{m \to \infty} \left| \int_0^1 b(s, X_s(\omega) + h_m(s)) - b(s, X_s + h(s)) \, \mathrm{d}s \right|_H$$

$$\leq \lim_{m\to\infty} \left| \int_0^1 b(s, X_s(\omega) + h_m(s)) - \overline{b}_\ell(s, X_s + h_m(s)) \,\mathrm{d}s \right.$$

$$\left. + \int_0^1 \overline{b}_\ell(s, X_s(\omega) + h_m(s)) - b(s, X_s(\omega) + h(s)) \,\mathrm{d}s \right|_H .$$

Using the above calculation this is bounded from above by

$$2\varepsilon_\ell + \lim_{m\to\infty} \left| \int_0^1 \overline{b}_\ell(s, X_s(\omega) + h_m(s)) - b(s, X_s(\omega) + h(s)) \,\mathrm{d}s \right|_H .$$

Since $\overline{b}_\ell$ is continuous and h_m converges pointwise to h this is the same as

$$2\varepsilon_\ell + \left| \int_0^1 \overline{b}_\ell(s, X_s(\omega) + h(s)) \,\mathrm{d}s - b(s, X_s(\omega) + h(s)) \,\mathrm{d}s \right|_H$$

$$\leq 2\varepsilon_\ell + \int_0^1 \mathbb{1}_{K_\ell^c}(s, X_s(\omega) + h(s)) \underbrace{\left|\overline{b}_\ell(s, X_s(\omega) + h(s)) - b(s, X_s(\omega) + h(s))\right|_H}_{\leq 2} \,\mathrm{d}s \leq 4\varepsilon_\ell,$$

where the last inequality follows by invoking inequality (7.2.1.1) for h_m replaced by h. Taking the limit $\ell \to \infty$ completes the proof of the assertion, since the left-hand side is independent of ℓ.

□

Using the above Approximation Theorem (Theorem 7.2.1) we can now extend the estimates obtained in Chapter 6 (Theorem 6.1.5 and 6.2.1) to the whole space Q as the following Corollary shows.

Corollary 7.2.2 (Cf. [Wre17, Corollary 4.7])

Assume that the usual assumptions (see Definition 6.1.4) are fulfilled. For every $\varepsilon > 0$ there exists $C_\varepsilon \in \mathbb{R}$ such that for every function $b\colon [0,1] \times H \longrightarrow Q$, $n \in \mathbb{N} \setminus \{0\}$ and $k \in \{0, ..., 2^n - 1\}$ there exists a measurable set $A_{\varepsilon,b,n,k} \in \mathcal{F}_{(k+1)2^{-n}} \subseteq \Omega$ with $\mathbb{P}[A_{\varepsilon,b,n,k}] \leq \frac{\varepsilon}{3} e^{-n}$ such that

$$\mathbb{1}_{A^c_{\varepsilon,b,n,k}} |\varphi_{n,k}(x)|_H \leq C_\varepsilon n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{-hn} \left(|x|_\infty + 2^{-2^n}\right)$$

holds for every $x \in 2Q$ and by setting

$$A_{\varepsilon,b} := \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{2^n-1} A_{\varepsilon,b,n,k}$$

we have $\mathbb{P}[A_{\varepsilon,b}] \leq \varepsilon$ with the property that

$$\mathbb{1}_{A_\varepsilon^c} |\varphi_{n,k}(x,y)|_H \leq C_\varepsilon \left[n^{\frac{1}{\alpha}} 2^{-\delta n} |x-y|_\infty + 2^{-2^{\theta_\delta n}} \right]$$

holds for all $x, y \in 2Q$, $n \geq 1$ and $k \in \{0, ..., 2^n - 1\}$, where $\theta_\delta := (h-\delta)\frac{2\alpha\gamma}{2+\alpha+2\gamma}$.

Proof

The first inequality follows from Theorem 6.1.5 for all points $x \in 2Q \cap \mathbb{D}$. For general points $x \in 2Q$ this follows by approximating $2Q \cap \mathbb{D} \ni x_n \longrightarrow x$ and using Theorem 7.2.1.

The second inequality follows in the same way by combining Theorem 6.2.1 and Theorem 7.2.1. Note that the estimate can be trivially extended from points $x, y \in Q$ to $x, y \in 2Q$ by changing the constant C_ε and using that $\varphi_{n,k}$ is a pseudometric (see Remark 6.1.2).

Observe that one can choose (C_ε / $A_{\varepsilon,b}$), so that the conclusion of Theorem 6.1.5 and 6.2.1 hold (with the same constant / one the same set).

□

8 Long-Time Regularization using Euler Approximations

In this chapter we will prove estimates for terms of the type

$$\sum_{q=1}^{N} |\varphi_{n,k+q}(x_{q+1}, x_q)|_H.$$

We will first prove a concentration of measure result for the above term in Lemma 8.1.3. Using this we prove a $\mathbb{P}$-a.s. sure version of this estimate in Theorem 8.2.1. However, this estimate only holds for medium-sized N (i.e. $N = 2^{\varepsilon n}$ for some $\varepsilon \in]0, 1[$). By splitting the sum and using Theorem 8.2.1 repetitively we conclude the full estimate in Corollary 8.2.2.

Note that applying our previous estimate for $\varphi_{n,k}$ (Corollary 7.2.2) to every term under the sum would result in an estimate of order $n^{\frac{1}{\alpha}} 2^{-\delta n} N$. Since N will later be chosen to be of order 2^n this is of no use. The technique to overcome this is two-fold:

On the one hand the $\varphi_{n,k+q}$ terms have to "work together" to achieve an expression of order N. However, since $\{\varphi_{n,k+q}(x_q) \mid q = 1, ..., N\}$ are "sufficiently uncorrelated" the law of large numbers tells us to expect on average an estimate of order $\sqrt{N}$.

On the other hand, in later applications x_q will be values taken from the solution of the integral equation $(\text{IE})_\omega$, so that it is reasonable to assume that $|x_{q+1} - x_q|_H \approx |\varphi_{n,k+q}(x_q)|_H$. Exploiting this enables to use *both* of our previous established estimates (Corollary 7.2.2) for every $|\varphi_{n,k+q}(x_{q+1}, x_q)|_H$ term.

Using both techniques we end up with an estimate of order $2^{-n}N$ (see Corollary 8.2.2).

We split this chapter into two sections. In the first we only consider the case when $x_{q+1} = x_q + \varphi_{n,k+q}(x_q)$ (a so-called Euler approximation).

In the second section we consider general points $x_q \in Q \subseteq \mathbb{R}^{\mathbb{N}}$. We choose z_0 close to x_0 and define the Euler approximation z_q of x_q by

$$z_{q+1} := z_q + \varphi_{n,k+q}(z_q)$$

and proceed to estimate the above sum in terms of the difference

$$\gamma_q := x_{q+1} - x_q - \varphi_{n,k+q}(x_q)$$

between x_{q+1} and the Euler approximation of x_{q+1} given x_q.

8.1 Euler Approximations

In this section we concentrate on the case when for a given $x_0 \in Q \subseteq \mathbb{R}^{\mathbb{N}}$ we have $x_{q+1} = x_q + \varphi_{n,k+q}(x_q)$. A sequence $(x_q)_{q=1,\dots,N}$ of this form is called an Euler approximation sequence.

Theorem 8.1.1 (Burkholder–Davis–Gundy Inequality)

Let $(M_n, \mathcal{F}_n)_{n\in\mathbb{N}}$ be a real-valued martingale. For $2 \leq p < \infty$ we have

$$(\mathbb{E}|M_n|^p)^{1/p} \leq p(\mathbb{E}|\langle M\rangle_n^{p/2})^{1/p}. \tag{8.1.1.1}$$

Proof

In the celebrated paper [Dav76, Section 3] it is shown that the optimal constant in the case of discrete Martingales is the largest positive root of the Hermite polynomial of order $2p$. We refer to the appendix of [Ose12] for a discussion of the asymptotic of the largest positive root. See also [Kho14, Appendix B], where a self-contained proof of the Burkholder–Davis–Gundy Inequality with asymptotically optimal constant can be found for the one-dimensional case.

Lemma 8.1.2 (Cf. [Wre17, Lemma 5.2])

Let $(M_n, \mathcal{F}_n)_{n\in\mathbb{N}}$ be a real-valued martingale of the form

$$M_r := \sum_{k=1}^{r} X_k$$

with $\mathbb{E}[X_k^p] \leq C^p p^p$ for all $k \in \mathbb{N}$ and $p \in [1, \infty[$ then

$$\mathbb{E}\left[\exp\left(\frac{1}{8}\left(\frac{M_r}{C\sqrt{r}}\right)^{1/2}\right)\right] \leq 2$$

holds for all $r \in \mathbb{N}$.

Proof

Let $(M_n, \mathcal{F}_n)_{n\in\mathbb{N}}$ be a martingale. Using the Burkholder–Davis–Gundy Inequality (8.1.1.1) for every $r, p \in \mathbb{N}$ with $p \geq 2$ we have

$$\begin{aligned}\mathbb{E}[M_r^p] &\leq p^p \mathbb{E}[\langle M\rangle_r^{p/2}] = p^p \mathbb{E}\left[\left(\sum_{k=1}^{r} X_k^2\right)^{p/2}\right] \\ &\leq p^p r^{p/2-1} \mathbb{E}\left[\sum_{k=1}^{r} X_k^p\right] \leq p^p r^{p/2-1} r C^p p^p = C^p r^{p/2} p^{2p}.\end{aligned}$$

In conclusion we obtain

$$\mathbb{E}[M_r^p] \leq C^p r^{p/2} p^{2p} \tag{8.1.2.1}$$

for every $p \geq 2$. Furthermore, using Inequality (8.1.2.1) for $p = 2$, we trivially have by Jensen's Inequality

$$\mathbb{E}[M_r^{1/2}] \leq \mathbb{E}[M_r^2]^{1/4} \leq C^{1/2} r^{1/4} 2, \tag{8.1.2.2}$$

$$\mathbb{E}[M_r^1] \leq \mathbb{E}[M_r^2]^{1/2} \leq C r^{1/2} 2^2 \tag{8.1.2.3}$$

and

$$\mathbb{E}[M_r^{3/2}] \leq \mathbb{E}[M_r^2]^{3/4} \leq C^{3/2} r^{3/4} 2^3. \tag{8.1.2.4}$$

Hence, starting from the left-hand side of the assertion we obtain

$$\mathbb{E}\left[\exp\left(\frac{1}{8}\left(\frac{M_r}{C\sqrt{r}}\right)^{1/2}\right)\right] = \sum_{p=0}^{\infty} 8^{-p} \frac{\mathbb{E}[M_r^{p/2}]}{p! C^{p/2} r^{p/4}}.$$

We split the sum for different p and use the above inequalities (8.1.2.2), (8.1.2.3), (8.1.2.4) and (8.1.2.1) to bound the above expression by

$$1 + \underbrace{8^{-1}\frac{C^{1/2} r^{1/4} 2}{C^{1/2} r^{1/4}}}_{=4^{-1}} + \underbrace{8^{-2}\frac{C r^{1/2} 2^2}{C r^{1/2}}}_{=4^{-2}} + \underbrace{8^{-3}\frac{C^{3/2} r^{3/4} 2^3}{C^{3/2} r^{3/4}}}_{=4^{-3}} + \sum_{p=4}^{\infty} \underbrace{8^{-p}\frac{(p/2)^p}{p!}}_{=4^{-p}\frac{p^p}{p!}} \leq 1 + \sum_{p=1}^{\infty} 4^{-p}\frac{p^p}{p!}.$$

Using Stirling's Formula for $p \geq 1$

$$3p^p e^{-p} \leq e^{\frac{1}{12p+1}} \sqrt{2\pi p} p^p e^{-p} \leq p!$$

and the above calculation we finally calculate

$$\mathbb{E}\left[\exp\left(\frac{1}{8}\left(\frac{M_r}{C\sqrt{r}}\right)^{1/2}\right)\right] \leq 1 + \frac{1}{3}\sum_{p=1}^{\infty} 4^{-p} e^p \leq 2.$$

□

Lemma 8.1.3 (Cf. [Wre17, Lemma 5.3])

Assume that the usual assumptions (see Definition 6.1.4) are fulfilled. Let $\varepsilon > 0$, $(b_q)_{q\in\mathbb{N}}$ be a sequence of functions $b_q \colon [0,1] \times H \longrightarrow Q$, then there exists a measurable set $A_{\varepsilon,b} := A_{\varepsilon,(b_q)_{q\in\mathbb{N}}} \subseteq \Omega$, a constant $C \in \mathbb{R}$ and $N_\varepsilon \in \mathbb{N}$ such that for all $x_0 \in Q$, all $n \in \mathbb{N}$ with $n \geq N_\varepsilon$, all $r \in \mathbb{N}$ with $r \leq 2^{hn/2}$, $k \in \{0, ..., 2^n - r - 1\}$ and for every $\eta > 0$ we have

$$\mathbb{P}\left[\mathbb{1}_{A^c_{\varepsilon,b}} \sum_{q=1}^{r} |\varphi_{n,k+q}(b_q; x_{q-1}, x_q)|_H > \eta C\left(2^{-2hn}\sqrt{r}|x_0|_H + \sqrt{r}2^{-2^n}\right) + C2^{-2hn}\sum_{q=0}^{r-1}|x_q|_H\right] \leq 4e^{-\eta^{1/2}},$$

where $x_{q+1} := x_q + \varphi_{n,k+q}(b_q; x_q)$ for $q \in \{0, ..., r-1\}$ is the Euler approximation sequence for x_0.

Proof

Sketch of the proof:

The idea of the proof is to use the identity

$$\begin{aligned}\underbrace{|\varphi_{n,k+q}(b_q; x_{q-1}, x_q)|_H}_{=:Y_q} &= \underbrace{|\varphi_{n,k+q}(b_q; x_{q-1}, x_q)|_H}_{=Y_q} - \underbrace{\mathbb{E}[|\varphi_{n,k+q}(b_q; x_{q-1}, x_q)|_H|\mathcal{F}_{(k+q)2^{-n}}]}_{=:Z_q} \\ &+ \underbrace{\mathbb{E}[|\varphi_{n,k+q}(b_q; x_{q-1}, x_q)|_H|\mathcal{F}_{(k+q)2^{-n}}]}_{=Z_q} - \underbrace{\mathbb{E}[\mathbb{E}[|\varphi_{n,k+q}(b_q; x_{q-1}, x_q)|_H|\mathcal{F}_{(k+q)2^{-n}}]|\mathcal{F}_{(k+q-1)2^{-n}}]}_{=:V_q} \\ &+ \underbrace{\mathbb{E}[\mathbb{E}[|\varphi_{n,k+q}(b_q; x_{q-1}, x_q)|_H|\mathcal{F}_{(k+q)2^{-n}}]|\mathcal{F}_{(k+q-1)2^{-n}}]}_{=V_q}.\end{aligned}$$

By defining

$$X_q := Y_q - Z_q,$$
$$W_q := Z_q - V_q$$

this can be rewritten in the less explicit form

$$Y_q = X_q + W_q + V_q.$$

We now sum over q to obtain

$$\sum_{q=1}^{r} Y_q = \sum_{q=1}^{r} X_q + \sum_{q=1}^{r} W_q + \sum_{q=1}^{r} V_q.$$

Note that since X_q (respectively W_q) is a random variable minus its conditional expectation, hence, the maps

$$r \longmapsto \sum_{q=1}^{r} X_q,$$
$$r \longmapsto \sum_{q=1}^{r} W_q$$

are martingales w.r.t. the filtration $(\mathcal{F}_{(k+r)2^{-n}})_{r\in\mathbb{N}}$ and $(\mathcal{F}_{(k+r-1)2^{-n}})_{r\in\mathbb{N}}$. These two martingales can be estimated by their bracket process using the Burkholder–Davis–Gundy Inequality (see the previous Lemma 8.1.2 for details). We will then calculate the bracket process and use our previous developed estimates for $\varphi_{n,k}$ (Theorem 6.1.5 in the form of Corollary 7.2.2) as well as Corollary 5.1.8.

This leaves us with estimating the residual term

$$V_q = \mathbb{E}[\mathbb{E}[|\varphi_{n,k+q}(b_q; x_{q-1}, x_q)|_H|\mathcal{F}_{(k+q)2^{-n}}]|\mathcal{F}_{(k+q-1)2^{-n}}].$$

Since we are dealing with the conditional expectation we can use the tail estimate for $\varphi_{n,k}$ (Corollary 5.1.8) for $p = 1$. Proceeding in this manner, we obtain an upper bound containing $2^{-hn}|x_{q-1} - x_q|_H$. Since x_q is an Euler approximation sequence we have

$$|x_{q-1} - x_q|_H = |\varphi_{n,k+q}(x_q)|_H,$$

so that we can apply Corollary 5.1.8 again to obtain an upper bound of the order $2^{-2hn}|x_{q-1}|_H$.

There is a technical problem with this approach however. Theorem 6.1.5 only holds on a set $A^c_{\varepsilon,b_q,n,k+q} \subseteq \Omega$. We therefore, in order to resolve this issue, modify the Euler approximation sequence x_q so that $x_q = 0$ if x_q is outside of the set $A^c_{\varepsilon,b_q,n,k+q}$. Since we would like to use the above mentioned martingale estimate (Lemma 8.1.2) we have to modify x_q in such a way that x_q is still measurable w.r.t. $\mathcal{F}_{(k+q)2^{-n}}$.

Beginning of the proof:

Let $\varepsilon > 0$, $n \in \mathbb{N} \setminus \{0\}$ and $b_q\colon [0,1] \times H \longrightarrow Q$ be as in the assertion. Using Corollary 7.2.2 there exists $C_\varepsilon \in \mathbb{R}$ and $A_{\varepsilon,b_q,n,k+q} \in \mathcal{F}_{(k+1)2^{-n}}$ with $\mathbb{P}[A_{\varepsilon,b_q,n,k+q}] \leq \frac{1}{3}e^{-n}\varepsilon$ such that for all $x \in 2Q$ we have

$$|\varphi_{n,k+q}(b_q;x)|_H \leq C_\varepsilon n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{-hn} \left(|x|_H + 2^{-2^n}\right). \tag{8.1.3.1}$$

on $A^c_{\varepsilon,b_q,n,k+q}$. Note that x is allowed to be a random variable and we have used that $|\cdot|_\infty \leq |\cdot|_H$. We now set

$$N_\varepsilon := \min\left\{n \in \mathbb{N} \setminus \{0\} | C_\varepsilon n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} \leq 2^{hn/2}\right\}.$$

Let, as in the assertion, be $n \in \mathbb{N}$ with $n \geq N_\varepsilon$, $r \leq 2^{hn/2}$, $k \in \{0, ..., 2^n - r - 1\}$ and $x_0 \in Q$. Additionally, let $x_{q+1} := x_q + \varphi_{n,k+q}(b_q; x_q)$ be the Euler approximation sequence defined for $q \in \{0, ..., r-1\}$. We write $x_q = (x_q^{(i)})_{i \in \mathbb{N}}$ for the components of x_q and for $q \in \{1, ..., r\}$ we calculate

$$|x_q^{(i)}| \leq |x_{q-1}^{(i)}| + \left| \int\limits_{(k+q)2^{-n}}^{(k+q+1)2^{-n}} b_q^{(i)}(s, X_s + x_q) - b_q^{(i)}(s, X_s)\ \mathrm{d}s \right| \leq |x_{q-1}^{(i)}| + 2\|b_q^{(i)}\|_\infty 2^{-n}.$$

Via induction on q we deduce

$$|x_q^{(i)}| \leq |x_0^{(i)}| + 2q2^{-n}\|b_q^{(i)}\|_\infty \leq |x_0^{(i)}| + \underbrace{2r2^{-n}}_{\leq 1}\|b_q^{(i)}\|_\infty.$$

and since both $x_q \in Q$ and, by assumption, b_q takes values in Q we conclude that $x_q \in 2Q$ for all $q \in \{1, ..., r\}$. Note that x_q is $\mathcal{F}_{(k+q)2^{-n}}$-measurable. Due to the fact that inequality (8.1.3.1) only holds on $A^c_{\varepsilon,b_q,n,k} \subseteq \Omega$ we modify x_q in the following way

$$\begin{aligned} \hat{x}_0 &:= x_0, \\ \hat{x}_{q+1} &:= \hat{x}_q + \mathbb{1}_{A^c_{\varepsilon,b_q,n,k+q}} \varphi_{n,k+q}(\hat{x}_q). \end{aligned}$$

Observe that we lose the property that $x_{q+1} - x_q = \varphi_{n,k+q}(b_q; x_q)$, but we still have $\hat{x}_q \in 2Q$ and

$$|\hat{x}_{q+1} - \hat{x}_q|_H \leq |\varphi_{n,k+q}(b_q; \hat{x}_q)|_H. \tag{8.1.3.3}$$

Most importantly, the modified Euler approximation sequence $(\hat{x}_q)_{q=0,...,r}$ is still $\mathcal{F}_{(k+q)2^{-n}}$-measurable. We set

$$A_{\varepsilon,b} := A_{\varepsilon,(b_q)_{q\in\mathbb{N}}} := \bigcup_{n\in\mathbb{N}} \bigcup_{k=0}^{2^n-1} \bigcup_{q\in\mathbb{N}} A_{\varepsilon,b_q,n,k}$$

in a similar way as in Corollary 7.2.2. We obviously have $\mathbb{P}[A_{\varepsilon,b}] \leq \varepsilon$ and for the modified Euler approximation we obtain for every $q \in \{0,...,r-1\}$

$$|\hat{x}_{q+1}|_H = |\hat{x}_q + \mathbb{1}_{A^c_{\varepsilon,b_q,n,k+q}} \varphi_{n,k+q}(b_q;\hat{x}_q)|_H \leq |\hat{x}_q|_H + \mathbb{1}_{A^c_{\varepsilon,b_q,n,k+q}} |\varphi_{n,k+q}(b_q;\hat{x}_q)|_H$$

and using inequality (8.1.3.1) for x replaced by $\hat{x}_q$ and $C_\varepsilon n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} \leq 2^{hn/2}$ this is bounded from above by

$$|\hat{x}_q|_H + C_\varepsilon n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{-hn} \left(|\hat{x}_q|_H + 2^{-2^n}\right) \overset{(8.1.3.2)}{\leq} (1+2^{-hn/2})|\hat{x}_q|_H + 2^{-hn/2}2^{-2^n}.$$

By applying the discrete Gronwall inequality (see Corollary 2.1.2) with $\alpha = 2^{-hn/2}$ and $\beta = 2^{-hn/2}2^{-2^n}$ (or via induction over $q \in \{0,...,r\}$ and using that $q \leq r \leq 2^{hn/2}$) we have

$$\begin{aligned} |\hat{x}_q|_H &\leq (1+2^{-hn/2})^q |\hat{x}_0|_H + \sum_{\ell=0}^{q-1} (1+2^{-hn/2})^\ell 2^{-hn/2} 2^{-2^n} \\ &\leq \underbrace{(1+2^{-hn/2})^{2^{hn/2}}}_{\leq e} |x_0|_H + \underbrace{r2^{-hn/2}}_{\leq 1} \underbrace{(1+2^{-hn/2})^{2^{hn/2}}}_{\leq e} 2^{-2^n}. \end{aligned}$$

In conclusion we obtain

$$|\hat{x}_q|_H \leq e\left(|x_0|_H + 2^{-2^n}\right). \tag{8.1.3.4}$$

for all $q \in \{0,...,r\}$.

For the next step we define

$$\begin{aligned} Y_q &:= |\varphi_{n,k+q}(b_q;\hat{x}_{q-1},\hat{x}_q)|_H, \\ Z_q &:= \mathbb{E}[Y_q|\mathcal{F}_{(k+q)2^{-n}}] = \mathbb{E}[|\varphi_{n,k+q}(b_q;\hat{x}_{q-1},\hat{x}_q)|_H|\mathcal{F}_{(k+q)2^{-n}}], \\ X_q &:= Y_q - Z_q, \end{aligned}$$

as well as

$$M_\tau := \sum_{q=1}^{r\wedge\tau} X_q.$$

with $\tau \in \mathbb{N}$. Note that M_τ is a $\mathcal{F}_{(k+\tau+1)2^{-n}}$-martingale with $M_0 = 0$. Furthermore, for every $p \in \mathbb{N}$ we have the following bound of the increments of M

$$\mathbb{E}[|X_q|^p] \leq 2^{p-1}\mathbb{E}[|Y_q|^p + |Z_q|^p] \leq 2^p \mathbb{E}[|\varphi_{n,k+q}(b_q;\hat{x}_{q-1},\hat{x}_q)|^p_H].$$

Using Corollary 5.1.8 and inequality (8.1.3.3) this is bounded by

$$C_X^p p^{p/2} 2^{-hpn} \mathbb{E}[|\hat{x}_{q-1} - \hat{x}_q|^p_H] \leq C_X^p p^{p/2} 2^{-hpn} \mathbb{E}[|\varphi_{n,k+q-1}(b_{q-1};\hat{x}_{q-1})|^p_H].$$

Using Corollary 5.1.8 again this is bounded by

$$C_X^{2p} p^p 2^{-2hpn} \mathbb{E}[|\hat{x}_{q-1}|_H^p] \leq C_X^{2p} p^p 2^{-2hpn} \mathbb{E}[|\hat{x}_{q-1}|_H^p].$$

Applying inequality (8.1.3.4) yields

$$\mathbb{E}[|X_q|_H^p] \leq C_X^{2p} p^p 2^{-2hpn} e^p \left(|x_0|_H + 2^{-2^n}\right)^p.$$

Note that x_0 is deterministic. Using this bound we invoke Lemma 8.1.2 with

$$C := C_X^2 2^{-2hn} \left(|x_0|_H + 2^{-2^n}\right)$$

and hence we obtain the following bound for the martingale $(M_\tau)_{\tau \in \mathbb{N}}$

$$\mathbb{E}\left[\exp\left(\frac{1}{8}\left(\frac{r^{-1/2} 2^{2hn} M_r}{C_X^2 \left(|x_0|_H + 2^{-2^n}\right)}\right)^{1/2}\right)\right] \leq 2. \tag{8.1.3.5}$$

In a similar way as (X_q, Y_q, Z_q, M_τ) we define

$$\begin{aligned} V_q &:= \mathbb{E}[Z_q | \mathcal{F}_{(k+q-1)2^{-n}}], \\ W_q &:= Z_q - V_q, \end{aligned}$$

and

$$M'_\tau := \sum_{\tau=1}^{r \wedge \tau} W_q.$$

Observe that M'_τ is a $\mathcal{F}_{(k+\tau)2^{-n}}$-martingale and in a completely analogous way as above we obtain

$$\mathbb{E}\left[\exp\left(\frac{1}{8}\left(\frac{r^{-1/2} 2^{2hn} M'_r}{C_X^2 \left(|x_0|_H + 2^{-2^n}\right)}\right)^{1/2}\right)\right] \leq 2. \tag{8.1.3.6}$$

Let us now consider the term V_q

$$V_q = \mathbb{E}[Z_q | \mathcal{F}_{(k+q-1)2^{-n}}] = \mathbb{E}[\mathbb{E}[|\varphi_{n,k+q}(b_q; \hat{x}_{q-1}, \hat{x}_q)|_H | \mathcal{F}_{(k+q)2^{-n}}] | \mathcal{F}_{(k+q-1)2^{-n}}]$$

Using Corollary 5.1.8 for $p = 1$ and inequality (8.1.3.3) this is bounded by

$$C_X 2^{-hn} \mathbb{E}[|\hat{x}_{q-1} - \hat{x}_q|_H | \mathcal{F}_{(k+q-1)2^{-n}}] \leq C_X 2^{-hn} \mathbb{E}[|\varphi_{n,k+q-1}(b_{q-1}; \hat{x}_{q-1})|_H | \mathcal{F}_{(k+q-1)2^{-n}}].$$

Invoking Corollary 5.1.8 again this can be further bounded from above by

$$C_X^2 2^{-2hn} \mathbb{E}[|\hat{x}_{q-1}|_H | \mathcal{F}_{(k+q-1)2^{-n}}] = C_X^2 2^{-2hn} |\hat{x}_{q-1}|_H.$$

This leads us to

$$\sum_{q=1}^{r} V_q \leq C_X^2 2^{-2hn} \sum_{q=0}^{r-1} |\hat{x}_q|_H. \tag{8.1.3.7}$$

For notational ease we set $C' := \sqrt{8} C_X^2$. Finally, starting from the left-hand side of the assertion and using $Y_q = X_q + W_q + V_q$ we get for every $\eta > 0$

$$\mathbb{P}\left[\mathbb{1}_{A^c_{\varepsilon,b}}\sum_{q=1}^{r}|\varphi_{n,k+q}(b_q;x_{q-1},x_q)|_H > \eta C'\left(2^{-2hn}\sqrt{r}|x_0|_H+\sqrt{r}2^{-2^n}\right)+C'2^{-2hn}\sum_{q=0}^{r-1}|x_q|_H\right]$$

$$\leq \mathbb{P}\left[\sum_{q=1}^{r}\mathbb{1}_{A^c_{\varepsilon,b}}\underbrace{|\varphi_{n,k+q}(b_q;\hat{x}_{q-1},\hat{x}_q)|_H}_{=Y_q=X_q+W_q+V_q} > \eta C'\left(2^{-2hn}\sqrt{r}|x_0|_H+\sqrt{r}2^{-2^n}\right)+C'2^{-2hn}\sum_{q=0}^{r-1}|x_q|_H\right]$$

$$\leq \underbrace{\mathbb{P}\left[\sum_{q=1}^{r}V_q > C'2^{-2hn}\sum_{q=0}^{r-1}|x_q|_H\right]}_{=0 \text{ by } (8.1.3.7)} + \mathbb{P}\left[\sum_{q=1}^{r}X_q+W_q > \eta C'\sqrt{r}\left(2^{-2hn}|x_0|_H+2^{-2^n}\right)\right]$$

$$\leq \mathbb{P}\left[\underbrace{\sum_{q=1}^{r}X_q}_{=M_r} > C'\eta\sqrt{r}\left(2^{-2hn}|x_0|_H+2^{-2^n}\right)\right]+\mathbb{P}\left[\underbrace{\sum_{q=1}^{r}W_q}_{=M'_r} > C'\eta\sqrt{r}\left(2^{-2hn}|x_0|_H+2^{-2^n}\right)\right]$$

$$= \mathbb{P}\left[\frac{r^{-1/2}2^{2hn}}{C'\left(|x_0|_H+2^{-2^n}\right)}M_r > \eta\right]+\mathbb{P}\left[\frac{r^{-1/2}2^{2hn}}{C'\left(|x_0|_H+2^{-2^n}\right)}M'_r > \eta\right].$$

By applying the increasing function $x \mapsto \exp(x^{1/2})$ to both sides and using Chebyshev's Inequality this can be bounded from above by

$$\exp(-\eta^{1/2})\mathbb{E}\left[\exp\left(\frac{r^{-1/2}2^{2hn}}{C'\left(|x_0|_H+2^{-2^n}\right)}M_r\right)^{1/2}+\exp\left(\frac{r^{-1/2}2^{2hn}}{C'\left(|x_0|_H+2^{-2^n}\right)}M'_r\right)^{1/2}\right].$$

Using inequality (8.1.3.5) and (8.1.3.6) we can conclude that

$$\mathbb{P}\left[\mathbb{1}_{A^c_{\varepsilon,b}}\sum_{q=1}^{r}|\varphi_{n,k+q}(b_q;x_{q-1},x_q)|_H > \eta C'\left(2^{-2hn}\sqrt{r}|x_0|_H+\sqrt{r}2^{-2^n}\right)+C'2^{-2hn}\sum_{q=0}^{r-1}|x_q|_H\right] \leq 4e^{-\eta^{1/2}},$$

which completes the proof.

□

8.2 Long-Time Regularization

In this section we now consider the case of arbitrary $(x_q)_{q=0,\dots,r}$. Given $x_0 \in Q$ we construct a $z_0 \in Q$ which is "close" to x_0 and consider the Euler approximation sequence $(z_q)_{q=0,\dots,r}$ by setting

$$z_{q+1} = z_q + \varphi_{n,k+q}(z_q).$$

z_q then acts as an approximation of x_q. By controlling the error between x_q and z_q we are able to prove the following theorem.

Theorem 8.2.1 (Cf. [Wre17, Theorem 5.4])

Assume that the usual assumptions (see Definition 6.1.4) are fulfilled. For every $\varepsilon > 0$ there exist $C_\varepsilon \in \mathbb{R}$, $\Omega_{\varepsilon,b} \subseteq \Omega$ with $\mathbb{P}[\Omega^c_{\varepsilon,b}] \le \varepsilon$ and $N_\varepsilon \in \mathbb{N}$ such that for all sequences $(b_q)_{q\in\mathbb{N}}$ of functions $b_q\colon [0,1] \times H \longrightarrow Q$, all $n \in \mathbb{N}$ with $n \ge N_\varepsilon$, $k \in \{0, ..., 2^n - r - 1\}$, $\delta \in]0, h[$ and for all $y_0, ..., y_r \in Q$ we have

$$\sum_{q=1}^{r} |\varphi_{n,k+q}(b_q; y_{q-1}, y_q)|_H \le C_\varepsilon \left[2^{-2hn} \max\left(r, n^{2+\frac{2}{\gamma}}\sqrt{r}\right) |y_0|_H + 2^{-\delta n/4} \sum_{q=0}^{r-1} |\gamma_{n,k,q}|_H + r2^{-2^{\min(\theta_\delta,1)n}}\right],$$

on $\Omega_{\varepsilon,b}$ for $1 \le r \le 2^{\delta n/4}$, where $\gamma_{n,k,q} := y_{q+1} - y_q - \varphi_{n,k+q}(b_q; y_q)$ for $q \in \{0, ..., r-1\}$ is the error between y_q and the Euler approximation of y_q given y_{q-1} (i.e. $y_q + \varphi_{n,k+q}(b_q; y_q)$) and $\theta_\delta = (h - \delta)\frac{2\alpha\gamma}{2+\alpha+2\gamma}$ as in Theorem 6.2.1.

Proof

Sketch of the proof:

In order to prove the theorem we first have to get a "$\mathbb{P}$-a.s. version" of Lemma 8.1.3. This is done by considering the event

$$B_{\varepsilon/2,b,n,r,k,x_0} := \left\{\sum_{q=1}^{r} |\varphi_{n,k+q}(x_{q-1}, x_q)|_H > \eta C\sqrt{r}\left(2^{-2hn}|x_0|_H + 2^{-2^n}\right) + C2^{-2hn}\sum_{q=0}^{r-1}|x_q|_H\right\}.$$

By Lemma 8.1.3 the probability of the above event $B_{\varepsilon/2,b,n,r,k}$ is bounded from above by $4e^{-\eta^{1/2}}$. In the first step of the proof we show that by setting $\eta \approx n^{1+\frac{1}{\gamma}}(\log(1/\varepsilon))^2$ the probability of the event

$$\bigcup_{n=N_\varepsilon}^{\infty} \bigcup_{r=0}^{2^{\delta n/4}} \bigcup_{k=0}^{2^n-r-1} \bigcup_{s=0}^{2^{2n}} \bigcup_{x_0 \in Q_s \cap 2^{-(s+n)}\mathbb{Z}^{\mathbb{N}}} B_{\varepsilon/2,b,n,r,k,x_0}$$

is bounded by ε. Here, $Q_s := \{x \in Q\colon |x|_\infty \le 2^{-s}\}$ as in the proof of Theorem 6.1.5.

In the second step for a given a sequence $y_0, ..., y_r$ we construct a $z_0 \in Q_s \cap 2^{-(s+n)}\mathbb{Z}^{\mathbb{N}}$ with $s \in \{0, ..., 2^{2n}\}$ such that z_0 is "close" to y_0. By defining $z_{q+1} := z_q + \varphi_{n,k+q}(z_q)$ the sequence $(z_q)_{q=0,...,r}$ is then an Euler approximating sequence. Hence, we can use the above "$\mathbb{P}$-a.s. version" of Lemma 8.1.3 for x_q replaced by z_q and therefore we obtain an estimate for the expression

$$\sum_{q=1}^{r} |\varphi_{n,k+q}(z_{q-1}, z_q)|_H.$$

In the third step we estimate $|\varphi_{n,k+q}(z_q, y_q)|_H$. Using the triangle inequality

$$|z_{q+1} - y_{q+1}|_H \le |z_{q+1} - z_q + y_q - y_{q+1}|_H + |z_q - y_q|_H \le |\varphi_{n,k+q}(z_q, y_q)|_H + |\gamma_{n,k,q}|_H + |z_q - y_q|_H$$

together with the fact that $|y_0 - z_0|_H$ is small (by construction of z_0) we can perform an induction over q to obtain the required estimate.

In the final step we use the identity

$$y_{q-1} - y_q = y_{q-1} - z_{q-1} + z_{q-1} - z_q + z_q - y_q$$

to deduce that

$$|\varphi_{n,k+q}(y_{q-1}, y_q)|_H \leq |\varphi_{n,k+q}(y_{q-1}, z_{q-1})|_H + |\varphi_{n,k+q}(z_{q-1}, z_q)|_H + |\varphi_{n,k+q}(z_q, y_q)|_H.$$

Applying the estimates obtained in the previous steps concludes to proof.

Beginning of the proof:

Step 1:

Let $\varepsilon > 0$ and $C_{\varepsilon/2}$ the constant from Corollary 7.2.2. Similar to the proof of Lemma 8.1.3 we set

$$N_\varepsilon := \min\left\{n \in \mathbb{N} \setminus \{0\} | C_{\varepsilon/2} n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} \leq 2^{\delta n/4}\right\}. \tag{8.2.1.1}$$

For the sake of readability we write $b = (b_q)_{q \in \mathbb{N}}$. By Lemma 8.1.3 there is $A_{\varepsilon/2,b} \subseteq \Omega$ with $\mathbb{P}[A_{\varepsilon/2,b}] \leq \frac{\varepsilon}{2}$ and a constant $C \in \mathbb{R}$ such that for $x_{q+1} := x_q + \varphi_{n,k+q}(b_q; x_q)$ and any $x_0 \in Q$ we have

$$\mathbb{P}\Bigg[\underbrace{\mathbb{1}_{A^c_{\varepsilon/2,b}} \sum_{q=1}^{r} |\varphi_{n,k+q}(b_q; x_{q-1}, x_q)|_H > \eta C \sqrt{r}\left(2^{-2hn}|x_0|_H + 2^{-2^n}\right) + C 2^{-2hn} \sum_{q=0}^{r-1} |x_q|_H}_{=:B_{\varepsilon/2,b,n,r,k,x_0}}\Bigg] \leq 4e^{-\eta^{1/2}} \tag{8.2.1.2}$$

for all $\eta > 0$. We note that $r \leq 2^{\delta n/4} \leq 2^{hn/4}$. In order to obtain an almost sure bound we define

$$B_{\varepsilon/2,b} := \bigcup_{n=N_\varepsilon}^{\infty} \bigcup_{r=0}^{2^{\delta n/4}} \bigcup_{k=0}^{2^n - r - 1} \bigcup_{s=0}^{2^{2n}} \bigcup_{x_0 \in Q_s \cap 2^{-(s+n)}\mathbb{Z}^{\mathbb{N}}} B_{\varepsilon/2,b,n,r,k,x_0}.$$

W.l.o.g. we assume that $\varepsilon > 0$ is sufficiently small so that

$$\tilde{\eta}_\varepsilon := \log \frac{40}{\varepsilon} \geq 1$$

and applying Lemma 8.1.3 in the form of inequality (8.2.1.2) with $\eta := (1 + 2(3n)^{1+\frac{1}{\gamma}})^2 \tilde{\eta}_\varepsilon^2$ yields

$$\begin{aligned}\mathbb{P}\left[B_{\varepsilon/2,b}\right] &\leq 4 \sum_{n=N_\varepsilon}^{\infty} \sum_{r=0}^{2^{\delta n/4}} \sum_{k=0}^{2^n-r-1} \sum_{s=0}^{2^{2n}} \sum_{x_0 \in Q_s \cap 2^{-(s+n)}\mathbb{Z}^{\mathbb{N}}} e^{-\eta^{1/2}} \\ &\leq 4 \sum_{n=N_\varepsilon}^{\infty} 2^{\delta n/4} 2^n \sum_{s=0}^{2^{2n}} \#(Q_s \cap 2^{-(s+n)}\mathbb{Z}^{\mathbb{N}}) e^{-2(3n)^{1+\frac{1}{\gamma}}} e^{-\tilde{\eta}_\varepsilon}.\end{aligned}$$

Using the usual assumptions (see Definition 6.1.4) and Proposition 3.2.4 we can use Theorem 3.2.3 to estimate the above expression by

$$4e^{-\tilde{\eta}_\varepsilon}\sum_{n=N_\varepsilon}^{\infty} 2^{2n}\sum_{s=0}^{2^{2n}}(2\cdot 2^n+1)^{\mathrm{ed}(Q_s)_{s+n}}e^{-2(3n)^{1+\frac{1}{\gamma}}}$$

By the usual assumptions we have $\mathrm{ed}(Q_s)_{s+n} \leq C_Q(\ln(s+n+1))^{1/\gamma}$, where $\gamma \geq 1$ and $C_Q > 0$ are the constants from Definition 6.1.4. Henceforth, the above sum can is bounded by

$$4e^{-\tilde{\eta}_\varepsilon}\sum_{n=N_\varepsilon}^{\infty} 2^{2n}\sum_{s=0}^{2^{2n}}(2\cdot 2^n+1)^{\ln(s+n+1)^{1/\gamma}}e^{-2(3n)^{1+\frac{1}{\gamma}}}$$

$$\leq 4e^{-\tilde{\eta}_\varepsilon}\sum_{n=N_\varepsilon}^{\infty} 2^{4n}(2\cdot 2^n+1)^{\ln(1+2^{2n}+n)^{1/\gamma}}e^{-2(3n)^{1+\frac{1}{\gamma}}} \leq 4e^{-\tilde{\eta}_\varepsilon}\sum_{n=N_\varepsilon}^{\infty} 2^{4n}(3^n)^{(3n)^{1/\gamma}}e^{-2(3n)^{1+\frac{1}{\gamma}}}$$
$$= 4e^{-\tilde{\eta}_\varepsilon}\sum_{n=N_\varepsilon}^{\infty} 2^{4n}\underbrace{3^{(3n)^{\frac{1}{\gamma}}}e^{-(3n)^{1+\frac{1}{\gamma}}}}_{\leq 1}e^{-(3n)^{1+\frac{1}{\gamma}}} \leq 4e^{-\tilde{\eta}_\varepsilon}\underbrace{\sum_{n=N_\varepsilon}^{\infty} 2^{4n}e^{-3n}}_{\leq 5} \leq 20e^{-\tilde{\eta}_\varepsilon} = \frac{\varepsilon}{2}.$$

Henceforth, $\mathbb{P}[B_{\varepsilon/2,b}] \leq \frac{\varepsilon}{2}$. We set $\Omega_{\varepsilon,b} := A^c_{\varepsilon/2,b} \cap B^c_{\varepsilon/2,b}$. Note that $\mathbb{P}[\Omega^c_{\varepsilon,b}] \leq \varepsilon$.

In conclusion there exists $C_\varepsilon \in \mathbb{R}$ such that for all $n \geq N_\varepsilon$, $r \leq 2^{\delta n/4}$, $k \in \{0, ..., 2^n - r - 1\}$ and $x_0 \in Q_s \cap 2^{-(s+n)}\mathbb{Z}^{\mathbb{N}}$ with $s \in \{0, ..., 2^{2n}\}$

$$\sum_{q=1}^{r}|\varphi_{n,k+q}(b_q; x_{q-1}, x_q)|_H \leq C_\varepsilon\left[n^{2+\frac{2}{\gamma}}2^{-2hn}\sqrt{r}|x_0|_H + 2^{-2hn}\sum_{q=0}^{r-1}|x_q|_H + r2^{-2^{\theta_\delta n}}\right] \quad (8.2.1.3)$$

holds on $\Omega_{\varepsilon,b}$ with $x_q := x_q + \varphi_{n,k+q}(x_q)$. Recall that $\theta_\delta := (h-\delta)\frac{2\alpha\gamma}{2+\alpha+2\gamma}$.

Step 2:

Let n, k, $r \in \mathbb{N}$ and y_0, ..., $y_r \in Q$ be as in the statement of this theorem. From now on fix an $\omega \in \Omega_{\varepsilon,b}$. Let s be the largest integer in $\{0, ..., 2^{2n}\}$ such that

$$|y_0|_H \leq 2^{-s}$$

holds. This implies that $y_0 \in Q_s$. Since s is maximal with the above property we have

$$2^{-(s+1)} < |y_0|_H \qquad \text{or} \qquad |y_0|_H \leq 2^{-s} = 2^{-2^{2n}}$$

and hence

$$2^{-s} \leq \max(2|y_0|_H, 2^{-2^{2n}}) \leq 2|y_0|_H + 2^{-2^{2n}}. \quad (8.2.1.4)$$

Since $y_0 \in Q_s$ we can construct $z_0 \in Q_s \cap 2^{-(s+n)}\mathbb{Z}^{\mathbb{N}}$, which is close to y_0, in the following way:

Recall that since we assume that the usual assumptions (see Definition 6.1.4) hold Proposition 3.2.4 implies that there is a constant $c_\gamma > 0$ such that

$$|x_n| \leq \exp\left(-e^{c_\gamma n^\gamma}\right), \qquad \forall (x_n)_{n\in\mathbb{N}} \in Q. \quad (8.2.1.5)$$

Set $d := (\ln(c_\gamma^{-1}(2s+2n)))^{1/\gamma}$. For the components $i < d$ we choose $z_0 = (z_0^{(i)})_{i\in\mathbb{N}}$ so that

$$|y_0^{(i)} - z_0^{(i)}| \leq 2^{-s-n}, \tag{8.2.1.6}$$

and $z_0^{(i)} := 0$ for all $i \geq d$. The distance between y_0 and z_0 is therefore bounded by

$$|y_0 - z_0|_H^2 = \sum_{0\leq i<d} |y_0^{(i)} - z_0^{(i)}|^2 + \sum_{d\leq i<\infty} |y_0^{(i)}|^2.$$

Using the above inequality (8.2.1.6) and the fact that, since $y_0 \in Q$, the components of y_0 satisfy the bound (8.2.1.5) this can be estimated by

$$d2^{-2s-2n} + \sum_{i=d}^{\infty} \exp\left(-2e^{c_\gamma i^\gamma}\right) \leq d2^{-2s-2n} + \underbrace{\exp\left(-e^{c_\gamma d^\gamma}\right)}_{=e^{-2s-2n}} \underbrace{\sum_{i=0}^{\infty} \exp\left(-e^{c_\gamma i^\gamma}\right)}_{=:C_\gamma^2<\infty},$$

where we have used $\exp(-2e^{c_\gamma i^\gamma}) \leq \exp(-e^{c_\gamma d^\gamma})\exp(-e^{c_\gamma i^\gamma})$ in the last step. Therefore, we get

$$|y_0 - z_0|_H \leq C_\gamma \sqrt{\ln(2s+2n)^{1/\gamma}} 2^{-s-n}$$

and hence by inequality (8.2.1.4) we obtain

$$|y_0 - z_0|_H \leq C_\gamma \sqrt{\ln(2s+2n)^{1/\gamma}} \left(2^{1-n}|y_0|_H + 2^{-n}2^{-2^{2n}}\right)$$

$$\leq 2C_\gamma \sqrt{\ln(2^{2n+1}+2n)^{1/\gamma}} \left(2^{-n}|y_0|_H + 2^{-n}2^{-2^{2n}}\right) \leq 2C_\gamma \sqrt{\ln(2^{4n})^{1/\gamma}} \left(2^{-n}|y_0|_H + 2^{-n}2^{-2^{2n}}\right)$$

$$= 2C_\gamma \sqrt{\frac{(\log_2(2^{4n}))^{1/\gamma}}{\ln(2)^{1/\gamma}}} \left(2^{-n}|y_0|_H + 2^{-n}2^{-2^{2n}}\right) = 2C_\gamma \sqrt{\frac{(4n)^{1/\gamma}}{\ln(2)^{1/\gamma}}} \left(2^{-n}|y_0|_H + 2^{-n}2^{-2^{2n}}\right).$$

In conclusion we have

$$|u_0|_H = |y_0 - z_0|_H \leq \tilde{C}_\gamma \left(n^{\frac{1}{2\gamma}} 2^{-n}|y_0|_H + 2^{-2^{2n}}\right). \tag{8.2.1.7}$$

We define $z_1, ..., z_r$ recursively by

$$z_{q+1} := z_q + \varphi_{n,k+q}(b_q; z_q).$$

Note that $z_0, ..., z_q$ are deterministic since we have fixed ω. Using the definition of z_q we have

$$|z_{q+1}|_H \leq |z_q|_H + |\varphi_{n,k+q}(b_q; z_q)|_H.$$

Recall that $\omega \in \Omega_{\varepsilon,b} \subseteq A_{\varepsilon/2,b}^c$ and hence we can invoke the conclusion of Corollary 7.2.2, so that the above expression is bounded from above by

$$|z_q|_H + C_{\varepsilon/2} n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{-hn}(|z_q|_H + 2^{-2^n}) \leq (1+2^{-hn/2})|z_q|_H + 2^{-hn/2}2^{-2^n},$$

where we have used Definition (8.2.1.1) to conclude that $C_{\varepsilon/2} n^{\frac{1}{\alpha}+\frac{1}{\gamma}} 2^{-hn} \leq 2^{-hn/2}$. By applying the discrete Gronwall inequality (see Corollary 2.1.2) with $\alpha = 2^{-hn/2}$ and $\beta = 2^{-hn/2} 2^{-2^n}$ (or via induction over $q \in \{0, ..., r\}$ and using that $q \leq r \leq 2^{\delta n/2} \leq 2^{hn/2}$) we have

$$\begin{aligned}|z_q|_H &\leq (1+2^{-hn/2})^q |z_0|_H + \sum_{\ell=0}^{q-1} (1+2^{-hn/2})^\ell 2^{-hn/2} 2^{-2^n} \\ &\leq \underbrace{(1+2^{-hn/2})^r}_{\leq e} |z_0|_H + \underbrace{(1+2^{-hn/2})^r}_{\leq e} \underbrace{r 2^{-hn/2}}_{\leq 1} 2^{-2^n} \leq e\left(|z_0|_H + 2^{-2^n}\right).\end{aligned}$$

Since z_0, ..., z_r is by definition an Euler approximation and $z_0 \in Q_s \cap 2^{-(s+n)}\mathbb{Z}^{\mathbb{N}}$ the conclusion of Step 1 (inequality (8.2.1.3)) with x_q replaced by z_q holds and we obtain that

$$\begin{aligned}\sum_{q=1}^{r} |\varphi_{n,k+q}(z_{q-1}, z_q)|_H &\leq C_\varepsilon \left[n^{2+\frac{2}{\gamma}} 2^{-2hn} \sqrt{r} |z_0|_H + r 2^{-2^{\theta_\delta n}} + 2^{-2hn} \sum_{q=0}^{r-1} |z_q|_H \right] \\ &\leq C_\varepsilon \left[n^{2+\frac{2}{\gamma}} 2^{-2hn} \sqrt{r} |z_0|_H + r 2^{-2^{\theta_\delta n}} + 2^{-2hn} \sum_{q=0}^{r-1} e(|z_0|_H + 2^{-2^n}) \right] \\ &= C_\varepsilon \left[n^{2+\frac{2}{\gamma}} 2^{-2hn} \sqrt{r} |z_0|_H + r 2^{-2^{\theta_\delta n}} + 2^{-2hn} r e \left(|z_0|_H + 2^{-2^n}\right) \right] \\ &\leq 2eC_\varepsilon \left[\max\left(n^{2+\frac{2}{\gamma}} \sqrt{r}, r\right) 2^{-2hn} |z_0|_H + r 2^{-2^{\min(\theta_\delta,1)n}} \right] \\ &\leq 2eC_\varepsilon \left[\max\left(n^{2+\frac{2}{\gamma}} \sqrt{r}, r\right) 2^{-2hn} (|y_0|_H + |y_0 - z_0|_H) + r 2^{-2^{\min(\theta_\delta,1)n}} \right].\end{aligned}$$

Applying inequality (8.2.1.7) yields that the above expression is bounded from above by

$$\begin{aligned}&2eC_\varepsilon \left[\max\left(n^{2+\frac{2}{\gamma}} \sqrt{r}, r\right) 2^{-2hn} \left(|y_0|_H + \tilde{C}_\gamma \left(\underbrace{n^{\frac{1}{2\gamma}} 2^{-n}}_{\leq 1} |y_0|_H + 2^{-2^{2n}} \right) \right) + r 2^{-2^{\min(\theta_\delta,1)n}} \right] \\ &\qquad \leq C_{\varepsilon,\gamma} \left[\max\left(n^{2+\frac{2}{\gamma}} \sqrt{r}, r\right) 2^{-2hn} |y_0|_H + r 2^{-2^{\min(\theta_\delta,1)n}} \right],\end{aligned}$$

where $C_{\varepsilon,\gamma} := 4eC_\varepsilon \tilde{C}_\gamma$. Therefore we obtain

$$\sum_{q=1}^{r} |\varphi_{n,k+q}(z_{q-1}, z_q)|_H \leq C_{\varepsilon,\gamma} \left[\max\left(n^{2+\frac{2}{\gamma}} \sqrt{r}, r\right) 2^{-2hn} |y_0|_H + r 2^{-2^{\min(\theta_\delta,1)n}} \right]. \qquad (8.2.1.8)$$

Step 3:

Claim:

$$\sum_{q=1}^{r} |\varphi_{n,k+q}(z_q, y_q)|_H \leq C'_\varepsilon \left[r 2^{-n} |y_0|_H + r 2^{-2^{\theta_\delta n}} + 2^{-\delta n/4} \sum_{q=0}^{r-1} |\gamma_{n,k,q}|_H \right]. \qquad (8.2.1.9)$$

Proof of (8.2.1.9):

We set $u_q := z_q - y_q$ for $q \in \{0, ..., r\}$ and bound the increments of u_q in the following way.

$$\begin{aligned} |u_{q+1} - u_q|_H &= |z_{q+1} - y_{q+1} - z_q + y_q|_H = |\varphi_{n,k+q}(b_q; z_q) - y_{q+1} + y_q|_H \\ &\le |\varphi_{n,k+q}(b_q; z_q) - y_{q+1} + y_q + \gamma_{n,k,q}|_H + |\gamma_{n,k,q}|_H \\ &= |\varphi_{n,k+q}(b_q; z_q) - \varphi_{n,k+q}(b_q; y_q)|_H + |\gamma_{n,k,q}|_H . \\ &= |\varphi_{n,k+q}(b_q; z_q, y_q)|_H + |\gamma_{n,k,q}|_H \end{aligned}$$

We therefore deduce that

$$|u_{q+1}|_H \le |u_{q+1} - u_q|_H + |u_q|_H \le |\varphi_{n,k+q}(b_q; z_q, y_q)|_H + |\gamma_{n,k,q}|_H + |u_q|_H .$$

By the conclusion of Corollary 7.2.2 and Definition (8.2.1.1) this is bounded by

$$C_{\varepsilon/2}\left(n^{\frac{1}{\alpha}} 2^{-\delta n} \underbrace{|z_q - y_q|_H}_{=|u_q|_H} + 2^{-2^{\theta_\delta n}} \right) + |\gamma_q|_H + |u_q|_H \overset{(8.2.1.1)}{\le} (1+2^{-\delta n/2})|u_q|_H + C_{\varepsilon/2} 2^{-2^{\theta_\delta n}} + |\gamma_{n,k,q}|_H .$$

Using again the discrete Gronwall inequality (Corollary 2.1.2) this time with $\alpha = 2^{-\delta n/2}$ and $\beta = 2^{-hn/2} 2^{-2^{\theta_\delta n}}$ (or via induction over $q \in \{0, ..., r\}$ and using that $q \le r \le 2^{\delta n/2}$) we have

$$\begin{aligned} |u_q|_H &\le C_{\varepsilon/2} \underbrace{(1 + 2^{-\delta n/2})^r}_{\le e} \left(|u_0|_H + r 2^{-2^{\theta_\delta n}} + \sum_{q=0}^{r-1} |\gamma_{n,k,q}|_H \right) \\ &\le e C_{\varepsilon/2} \left(|u_0|_H + r 2^{-2^{\theta_\delta n}} + \sum_{q=0}^{r-1} |\gamma_{n,k,q}|_H \right) . \end{aligned}$$

Using the above calculation together with inequality (8.2.1.7) yields

$$|u_q|_H \le e C_{\varepsilon/2} \left(\tilde{C}_\gamma n^{\frac{1}{2\gamma}} 2^{-n} |y_0|_H + 2r 2^{-2^{\min(\theta_\delta, 1)n}} + \sum_{q=0}^{r-1} |\gamma_{n,k,q}|_H \right) \tag{8.2.1.10}$$

and hence, by combining this estimate with Corollary 7.2.2 we have

$$\begin{aligned} |\varphi_{n,k+q}(z_q, y_q)|_H &\overset{(8.2.1.1)}{\le} C_{\varepsilon/2} \left(n^{\frac{1}{\alpha}} 2^{-\delta n} |z_q - y_q|_H + 2^{-2^n} \right) \le C_{\varepsilon/2} \left(2^{-\delta n/2} |u_q|_H + 2^{-2^n} \right) \\ &\overset{(8.2.1.10)}{\le} e C_{\varepsilon/2}^2 2^{-\delta n/2} \left(\tilde{C}_\gamma n^{\frac{1}{2\gamma}} 2^{-n} |y_0|_H + 2r 2^{-2^{\min(\theta_\delta, 1)n}} + \sum_{q=0}^{r-1} |\gamma_{n,k,q}|_H \right) + C_{\varepsilon/2} 2^{-2^n} . \end{aligned}$$

In conclusion since $r \le 2^{\delta n/4}$ we obtain

$$|\varphi_{n,k+q}(z_q, y_q)|_H \le C'_\varepsilon \left[2^{-n} |y_0|_H + 2^{-2^{\min(\theta_\delta, 1)n}} + 2^{-\delta n/2} \sum_{q=0}^{r-1} |\gamma_{n,k,q}|_H \right]$$

and hence summing over $q = 1, ..., r$ and using again that $r \le 2^{\delta n/4}$ completes the proof of Claim (8.2.1.9).

Step 4:

Finally, using the identity $y_{q-1} - y_q = y_{q-1} - z_{q-1} + z_{q-1} - z_q + z_q - y_q$ the left-hand side of the assertion can be bounded as follows

$$\sum_{q=1}^{r} |\varphi_{n,k+q}(b_q; y_{q-1}, y_q)|_H \leq \sum_{q=1}^{r} |\varphi_{n,k+q}(b_q; y_{q-1}, z_{q-1})|_H + |\varphi_{n,k+q}(b_q; z_{q-1}, z_q)|_H + |\varphi_{n,k+q}(b_q; z_q, y_q)|_H.$$

Applying inequalities (8.2.1.8), (8.2.1.9) and (8.2.1.9) with z_q, y_q replaced by z_{q-1}, y_{q-1} respectively yields that this is bounded by

$$C''_\varepsilon \left[2^{-2hn} \max\left(r, n^{2+\frac{2}{\gamma}}\sqrt{r}\right) |y_0|_H + r2^{-2^{\min(\theta_\delta,1)n}} + 2^{-\delta n/4} \sum_{q=0}^{r-1} |\gamma_{n,k,q}|_H \right].$$

□

Corollary 8.2.2 (Cf. [Wre17, Corollary 5.5])

Assume that the usual assumptions (see Definition 6.1.4) are fulfilled. For every $\varepsilon > 0$ there exists $C_\varepsilon \in \mathbb{R}$ and $N_\varepsilon \in \mathbb{N}$ such that for every sequence $(b_q)_{q\in\mathbb{N}}$ of Borel measurable functions $b_q\colon [0,1] \times H \longrightarrow Q$ there exists a measurable set $\Omega_{\varepsilon,(b_q)_{q\in\mathbb{N}}} \subseteq \Omega$ with $\mathbb{P}[\Omega^c_{\varepsilon,(b_q)_{q\in\mathbb{N}}}] \leq \varepsilon$ such that for all $n \in \mathbb{N}$ with $n \geq N_\varepsilon$, all $N \in \mathbb{N}$ with $N \leq 2^n$, $k \in \{0, ..., 2^n - N\}$, $\delta \in]0, h[$ and for all $x_q \in Q$ for $q \in \{0, ..., N\}$ we have

$$\sum_{q=0}^{N-1} |\varphi_{n,k+q}(b_q; x_{q+1}, x_q)|_H$$

$$\leq C_\varepsilon \left[2^{-2hn} \sum_{q=0}^{N} |x_q|_H + 2^{-(2h-\delta/4)n}|x_0|_H + 2^{-\delta n/4} \sum_{q=0}^{N-1} |\gamma_{n,k,q}|_H + N2^{-2^{\min(\theta_\delta,1)n}} \right],$$

on $\Omega_{\varepsilon,b}$, where $\gamma_{n,k,q} := x_{q+1} - x_q - \varphi_{n,k+q}(b_q; x_q)$ is the error between x_{q+1} and the Euler approximation of x_{q+1} given x_q and $\theta_\delta := (h-\delta)\frac{2\alpha\gamma}{2+\alpha+2\gamma}$.

Proof

We define $r := \lfloor 2^{\delta n/4} \rfloor$. For the sake of notional ease we set $x_{q'} = 0$ whenever $q' > N$. In order to estimate the left-hand side of the assertion we will use Theorem 8.2.1. To this end we split the sum into s pieces of size r. Choose $i \in \{0, ..., r-1\}$ such that

$$\sum_{t=0}^{\lfloor r^{-1}N \rfloor} |x_{i+tr}|_H \leq \frac{1}{r} \sum_{q=0}^{r-1} \sum_{t=0}^{\lfloor r^{-1}N \rfloor} |x_{q+tr}|_H$$

holds. Since we calculate the mean of $\sum_{t=0}^{\lfloor r^{-1}N \rfloor} |x_{q+tr}|_H$ on the right-hand side, it is clear that such an i always exists. Set $s := \lfloor r^{-1}(N-i) \rfloor$ and note that $s \leq \lfloor r^{-1}N \rfloor$. Using this we have

$$\sum_{t=0}^{s} |x_{i+tr}|_H \leq \frac{1}{r} \sum_{q=0}^{r-1} \sum_{t=0}^{\lfloor r^{-1}N \rfloor} |x_{q+tr}|_H.$$

Hence, we obtain

$$\sum_{t=0}^{s} |x_{i+tr}|_H \leq r^{-1} \sum_{q=0}^{N-1} |x_q|_H. \tag{8.2.2.1}$$

Starting with the left-hand side of the assertion we split the sum into three parts. The first part contains the terms x_q for $q = 0$ to $q = i$. Since $i \leq r \leq 2^{\delta n/4}$ this can be handled by applying Theorem 8.2.1 directly. The second part contains s sums of size r. Here, Theorem 8.2.1 is applicable for every term of the outer sum running over t. The last part can be handled, in the same way as the first part, by directly applying Theorem 8.2.1. This strategy yields

$$\begin{aligned}\sum_{q=0}^{N-1} |\varphi_{n,k+q}(b_q; x_{q+1}, x_q)|_H &= \sum_{q=0}^{i-1} |\varphi_{n,k+q}(b_q; x_{q+1}, x_q)|_H \\ &+ \sum_{t=0}^{s-1} \sum_{q=0}^{r-1} |\varphi_{n,k+i+tr+q}(b_q; x_{q+1+i+tr}, x_{q+i+tr})|_H \\ &+ \sum_{q=0}^{N-i-rs-1} |\varphi_{n,k+i+sr+q}(b_q; x_{q+1+i+sr}, x_{q+i+sr})|_H\end{aligned}$$

$$\begin{aligned}&\leq C_\varepsilon \left[2^{-2hn} \max\left(r, n^{2+\frac{2}{\gamma}}\sqrt{r}\right) |x_0|_H + 2^{-\delta n/4} \sum_{q=0}^{i-1} |\gamma_{n,k,q}|_H + r2^{-2^{\min(\theta_\delta,1)n}} \right] \\ &\quad + C_\varepsilon \sum_{t=0}^{s-1} \left[2^{-2hn} r |x_{i+tr}|_H + 2^{-\delta n/4} \sum_{q=0}^{r-1} |\gamma_{n,k,i+tr+q}|_H + r2^{-2^{\min(\theta_\delta,1)n}} \right] \\ &\quad + C_\varepsilon \left[2^{-2hn} \max\left(r, n^{2+\frac{2}{\gamma}}\sqrt{r}\right) |x_{i+sr}|_H + 2^{-\delta n/4} \sum_{q=0}^{N-i-rs-1} |\gamma_{n,k,i+sr+q}|_H + r2^{-2^{\min(\theta_\delta,1)n}} \right].\end{aligned}$$

$$\leq C_\varepsilon \left[2^{-2hn} r |x_0|_H + 2^{-2hn} r \sum_{t=0}^{s} |x_{i+tr}|_H + 2^{-\delta n/4} \sum_{q=0}^{N-1} |\gamma_{n,k,q}|_H + (s+2) r 2^{-2^{\min(\theta_\delta,1)n}} \right].$$

Estimating this further by using inequality (8.2.2.1) and $r \leq 2^{\delta n/4}$ yields the following bound

$$2C_\varepsilon \left[2^{-(2h-\delta/4)n} |x_0|_H + 2^{-2hn} \sum_{q=0}^{N-1} |x_q|_H + 2^{-\delta n/4} \sum_{q=0}^{N-1} |\gamma_{n,k,q}|_H + N2^{-2^{\min(\theta_\delta,1)n}} \right],$$

which completes the proof.

□

9 Proof of the Main Result

In this chapter let $X\colon [0,1]\times\Omega \longrightarrow H$ be a stochastic process adapted to a filtration $(\mathcal{F}_t)_{t\in[0,1]}$. We assume furthermore that X is a Q-regularizing noise of order $\alpha > 0$ with index $h \in [\frac{1}{2}, 1[$ in the sense of Definition 5.1.1. Assume that the *usual assumptions* (see Definition 6.1.4) hold, i.e. there is a constant $C_Q \in \mathbb{R}$ is such that $\mathrm{ed}(Q)_m \leq C_Q(\ln(m+1))^{1/\gamma}$ for $\gamma \geq 1$.

In this chapter we assume that

$$\frac{1-h}{h} < \frac{2\alpha\gamma}{2+\alpha+2\gamma} \leq \frac{1}{h} \tag{9.1}$$

Note that in case $h = \frac{1}{2}$ this condition simplifies to

$$2+\alpha+2\gamma < 2\alpha\gamma \leq 4+2\alpha+4\gamma \tag{9.2}$$

and if additionally $\alpha = 2$, this reduces simply to $\gamma > 2$.

9.1 Preparation

Let $f\colon [0,1]\times H \longrightarrow Q$ be a Borel measurable map. For the sake of notional ease let us define the function $b_{n,k}$ as follows.

Definition 9.1.1

Let $A\colon D(A) \longrightarrow H$ be a positive definite, self-adjoint, closed, densely defined linear operator such that the trace of its inverse A^{-1} is finite. For all $n \in \mathbb{N}$ and $k \in \{0, ..., 2^n - 1\}$ we define

$$b_{n,k}(t,x) := e^{-((k+1)2^{-n}-t)A} f(t,x), \qquad \forall t \in [k2^{-n}, (k+1)2^{-n}],\ x \in H.$$

For given $\varepsilon > 0$ let, furthermore, $\Omega_{\varepsilon,f} \subseteq \Omega$ be such that Theorem 7.2.1, Corollary 7.2.2 and Corollary 8.2.2 hold for all $\omega \in \Omega_{\varepsilon,f}$ and all $(b_{n,k})_{(n,k)\in\mathbb{N}\times\{0,\dots,2^n-1\}}$ with $\mathbb{P}[\Omega_{\varepsilon,f}^c] \leq \varepsilon$ with the same constant C_ε. This can always be achieved since for a given Borel measurable function $f\colon [0,1]\times H \longrightarrow Q$ there are only countable many $(b_{n,k})_{(n,k)\in\mathbb{N}\times\{0,\dots,2^n-1\}}$.

From now on fix an $\omega \in \Omega_{\varepsilon,f}$. In this chapter we consider functions $u \in \mathcal{C}([0,1],H)$, which are solutions to the following integral equation

$$u(t) = \int_0^t e^{-(t-s)A}\left(f(s, X_s(\omega)+u(s)) - f(s, X_s(\omega))\right)\, \mathrm{d}s, \qquad \forall t \in [0,1]. \tag{9.1.1}$$

We prove that u fulfills a log-type Gronwall inequality (see Theorem 9.2.4) and conclude that u must be trivial in order to be a solution to the above integral equation (see Corollary 9.2.5).

Remark 9.1.2

Since we have already established that, as long Q is small enough, an Ornstein–Uhlenbeck process is a regularizing noise (see Corollary 5.2.3) we are able to conclude from the above mentioned result that equation (1.2.1.1) has only the trivial solution $u \equiv 0$. Henceforth, from Proposition 1.2.1 the main result of this thesis follows. Details of this argument can be found in Chapter 10 below.

From this point onwards let $u \in \mathcal{C}([0,1], H)$ be a solution of equation (9.1.1) and assume that $u \in \Phi$ (see Definition 7.1.1 for the definition of the set Φ) with Lipschitz constant $L \geq 1$.

Definition of δ

Let $0 < \delta < h$ such that

$$\frac{1-h}{h-\delta} < \frac{2\alpha\gamma}{2+\alpha+2\gamma}.$$

Recall that h is the index of the regularizing noise and that we imposed the condition

$$\frac{1-h}{h} < \frac{2\alpha\gamma}{2+\alpha+2\gamma},$$

which guarantees the existence of such a δ.

Definition of ζ

Recall that θ_δ was defined as

$$\theta_\delta = (h-\delta)\frac{2\alpha\gamma}{2+\alpha+2\gamma}.$$

By our definition of δ we have that

$$1-h < \theta_\delta.$$

Hence, there exists $\zeta > 0$ such that

$$1-h+\zeta < \theta_\delta.$$

Definition of m_0

Let $m_0 \in \mathbb{N}$ be the smallest number such that the following inequalities are fulfilled

$$\begin{aligned}
m_0^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} &\leq h2^{\zeta m_0}, \\
2^{-hm_0/2} &\leq \frac{1}{2}, \\
2^{-\delta m_0/4} &\leq \frac{1}{2}, \\
3h2^{-m_0}2^{\theta_\delta m_0} &\leq \frac{1}{2}, \\
2^{(1-h+\zeta)m_0} &< 2^{\theta_\delta m_0} - m_0 \leq 2^{m_0}, \\
24CC_\varepsilon 2^{-\delta m_0/4} &\leq \frac{1}{2},
\end{aligned} \tag{9.1.2.1}$$

where $C \geq 1$ is a constant such that

$$\left|e^{-2^{-n}A} - 1\right|_{\text{op}} \leq C2^{-n}$$

for all $n \in \mathbb{N}$.

Note that our assumption $\frac{2\alpha\gamma}{2+\alpha+2\gamma} \leq \frac{1}{h}$ implies that

$$\theta_\delta = (h - \delta)\frac{2\alpha\gamma}{2+\alpha+2\gamma} \overset{(9.1)}{\leq} 1 - \frac{\delta}{h} < 1, \tag{9.1.2.2}$$

which guarantees the existence of such a $m_0 \in \mathbb{N}$.

Assumption on $\boldsymbol{\beta}$ / Definition of $\boldsymbol{N}$

Fix a $m \in \mathbb{N}$ with $m \geq m_0$ and $j \in \{0, ..., 2^m - 1\}$ and assume that there exists $\beta \in \mathbb{R}$ satisfying

$$2^{m-2^{\theta_\delta m}} \leq \beta \leq 2^{-2^{(1-h+\zeta)m}} \quad \text{and} \quad |u(j2^{-m})|_H \leq \beta. \tag{9.1.2.3}$$

We additionally set

$$N := \lceil 3h \log_2(1/\beta) \rceil.$$

Remark 9.1.3 (Existence of β)

The interval in which β lives is non-empty since by our previous definitions we have

$$1 - h + \zeta < \theta_\delta$$

from which we deduce that

$$2^{(1-h+\zeta)m} < 2^{\theta_\delta m}.$$

Since $m \geq m_0$ the definition of m_0 guarantees even that

$$2^{(1-h+\zeta)m} < 2^{\theta_\delta m} - m$$

so that

$$2^{-2^{(1-h+\zeta)m}} > 2^{m-2^{\theta_\delta m}}.$$

Lemma 9.1.4

With everything as above we have

$$2^{(1+\delta/4-2h)m} \leq m^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)m} \leq N \leq 3h \cdot 2^{\theta_\delta m} < \frac{1}{2} \cdot 2^m. \tag{9.1.4.1}$$

Proof

Starting from left to right we have

$$1 + \frac{\delta}{4} - 2h \leq 1 - h + \underbrace{\delta - h}_{\leq 0} \leq 1 - h$$

since $\delta < h$. Therefore,

$$2^{(1+\delta/4-2h)m} \leq 2^{(1-h)m}$$

and since $m \geq m_0 \geq 1$ we have $m^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} \geq 1$.

The second inequality follows from $N = \lceil 3h \log_2(1/\beta) \rceil$ and

$$N \geq 3h \log_2(1/\beta) \overset{(9.1.2.3)}{\geq} 3h2^{(1-h+\zeta)m} = h2^{\zeta m}2^{(1-h)m}$$

and since $h2^{\zeta m} \geq m^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}}$ by (9.1.2.1)

$$N \geq m^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)m}.$$

The third inequality holds since

$$N = \lceil 3h \log_2(1/\beta) \rceil \leq 3h \log_2(1/\beta) + 1 \overset{(9.1.2.3)}{\leq} 3h(2^{\theta_\delta m} - m) + 1 \leq 3h2^{\theta_\delta m}.$$

Finally, the last inequality follow from (9.1.2.1) and

$$3h2^{-m}2^{\theta_\delta m} \leq 3h2^{-m_0}2^{\theta_\delta m_0} \overset{(9.1.2.1)}{\leq} \frac{1}{2}.$$

This implies that

$$3h2^{\theta_\delta m} \leq \frac{1}{2} \cdot 2^m,$$

which completes the proof.

□

Definition 9.1.5

Let $A > 0$ be the smallest positive real number satisfying

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u((k+1)2^{-n}) - u(k2^{-n})|_H \leq A2^{-m} \left[N + n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)n} \right], \quad \forall n \in \{m, ..., N\} \tag{9.1.5.1}$$

i.e.

$$A := \max_{m \leq n \leq N} \frac{2^m}{N + n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)n}} \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u((k+1)2^{-n}) - u(k2^{-n})|_H.$$

Definition 9.1.6

For fixed $m \in \mathbb{N}$ and $j \in \{0, ..., 2^m - 1\}$ as before we define for every $n \in \mathbb{N}$ with $n \geq m$

$$\psi_n := \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u(k2^{-n})|_H.$$

Lemma 9.1.7

Let everything be as described above. We then have

$$\psi_n \leq 2 \cdot 2^{n-m} \left[\beta + \frac{A}{2}\right], \qquad \forall n \in \{m, ..., N\}.$$

Proof

For $n = m$ we have $\psi_n = |u(j2^{-m})|_H$ which is smaller than β by (9.1.2.3). Let $n \in \{m+1, ..., N\}$. By splitting the sum in two sums, one where k is even and one where k is odd, we can estimate ψ_n by ψ_{n-1} in the following way

$$\psi_n = \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u(k2^{-n})|_H = \sum_{\substack{k=j2^{n-m} \\ 2|k}}^{(j+1)2^{n-m}-1} |u(k2^{-n})|_H + \sum_{\substack{k=j2^{n-m} \\ 2\nmid k}}^{(j+1)2^{n-m}-1} |u(k2^{-n})|_H$$

$$\leq \sum_{\substack{k=j2^{n-m} \\ 2|k}}^{(j+1)2^{n-m}-1} |u(k2^{-n})|_H + \sum_{\substack{k=j2^{n-m} \\ 2\nmid k}}^{(j+1)2^{n-m}-1} |u(k2^{-n}) - u((k-1)2^{-n})|_H + |u((k-1)2^{-n})|_H + |u((k+1)2^{-n}) - u(k2^{-n})|_H.$$

Since $k-1$ is even whenever k is odd, rewriting the term $|u((k-1)2^{-n})|_H$ yields that the above equals

$$\sum_{\substack{k=j2^{n-m} \\ 2|k}}^{(j+1)2^{n-m}-1} \left(|u(k2^{-n})|_H + |u(k2^{-n})|_H\right)$$

$$+ \sum_{\substack{k=j2^{n-m} \\ 2\nmid k}}^{(j+1)2^{n-m}-1} |u(k2^{-n}) - u((k-1)2^{-n})|_H + |u((k+1)2^{-n}) - u(k2^{-n})|_H$$

$$= 2 \sum_{k=j2^{n-m-1}}^{(j+1)2^{n-m-1}-1} |u(k2^{-n+1})|_H + \sum_{\substack{k=j2^{n-m} \\ 2\nmid k}}^{(j+1)2^{n-m}-1} |u(k2^{-n}) - u((k-1)2^{-n})|_H + |u((k+1)2^{-n}) - u(k2^{-n})|_H$$

$$= \underbrace{2 \sum_{k=j2^{n-1-m}}^{(j+1)2^{n-1-m}-1} |u(k2^{-(n-1)})|_H}_{=\psi_{n-1}} + \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u((k+1)2^{-n}) - u(k2^{-n})|_H.$$

Since $n \in \{m+1, ..., N\}$ we can use inequality (9.1.5.1) from Definition 9.1.5 to estimate the second sum by $A2^{-m}[N + A2^{-m} n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)n}]$ so that, henceforth, the above is bounded from above by

$$\psi_n \leq 2\psi_{n-1} + A2^{-m}N + A2^{-m} n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)n}.$$

By invoking the discrete Gronwall inequality (Lemma 2.1.1) with $\alpha = 1$ or by induction on n we deduce

$$\psi_n \leq 2^{n-m}\psi_m + \sum_{\ell=m+1}^{n} A2^{n-\ell-m}N + \sum_{\ell=m+1}^{n} A2^{n-\ell-m} \ell^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)\ell}$$

$$\leq 2^{n-m}|u(j2^{-m})|_H + A2^{n-m}N \sum_{\ell=m+1}^{n} 2^{-\ell} + A2^{n-m} \sum_{\ell=m+1}^{n} 2^{-h\ell/2}, \qquad \forall n \in \{m, ..., N\}.$$

We use $|u(j2^{-m})|_H \leq \beta$ (see (9.1.2.3) for the definition of β) to bound the above expression by

$$2^{n-m}\left[\beta + AN \sum_{\ell=m+1}^{n} 2^{-\ell} + A \sum_{\ell=m+1}^{n} 2^{-h\ell/2}\right] \leq 2^{n-m}\left[\beta + A2^{-m}N + A2^{-hm/2}\right].$$

This can, moreover, be simplified so that ψ_n is bounded by

$$2^{n-m}\left[\beta + 2A\max(2^{-m}N, 2^{-hm/2})\right].$$

In summary we obtain

$$\psi_n \leq 2 \cdot 2^{n-m}\left[\beta + A\max(2^{-m}N, 2^{-hm/2})\right], \qquad \forall n \in \{m, ..., N\}.$$

By Lemma 9.1.4 we have $2^{-m}N \leq \frac{1}{2}$ and since $m \geq m_0$ we have by the definition of m_0 (see (9.1.2.1)) that the $2^{-hm/2} \leq 2^{-hm_0/2} \leq \frac{1}{2}$ and therefore the above maximum is bounded by $\frac{1}{2}$. In conclusion we deduce that

$$\psi_n \leq 2 \cdot 2^{n-m}\left[\beta + \frac{A}{2}\right], \qquad \forall n \in \{m, ..., N\},$$

which completes the proof.

□

Definition 9.1.8

For every $n \in \{m, ..., N\}$ and $\ell \in \mathbb{N}$ with $\ell \geq n$ we set

$$\Lambda_\ell := \sum_{r=j2^{\ell+1-m}}^{(j+1)2^{\ell+1-m}-2} \left|\varphi_{\ell+1,r+1}\left(b_{n,\lfloor r2^{n-\ell-1}\rfloor}; u\left((r+1)2^{-(\ell+1)}\right), u\left(r2^{-(\ell+1)}\right)\right)\right|_H$$

Lemma 9.1.9

Let everything be defined as above. There exists a non-negative constant $C \in \mathbb{R}$ such that for all $n \in \{m, ..., N\}$

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u((k+1)2^{-n}) - u(k2^{-n}) - \varphi_{n,k}(b_{n,k}; u(k2^{-n}))|_H \leq C2^{-n}\psi_n + \sum_{\ell=n}^{\infty} \Lambda_\ell$$

holds.

Proof

Recall from the beginning of this chapter that $u \in \Phi$ is a solution to equation (9.1.1). We set

$$u_n(t) := \sum_{k=0}^{2^n-1} \mathbb{1}_{[k2^{-n},(k+1)2^{-n}[}(t) u(k2^{-n}).$$

Note that u_n converges pointwise to u on $[0,1[$ and $u_n \in \Phi^*$ (see Definition 7.1.1 for the definition of the set Φ^*) by construction and since $u \in \Phi$. We have

$$\left|u((k+1)2^{-n}) - u(k2^{-n}) - \varphi_{n,k}(b_{n,k}; u(k2^{-n}))\right|_H$$

$$= \left|u((k+1)2^{-n}) - u(k2^{-n}) - \int_{k2^{-n}}^{(k+1)2^{-n}} b_{n,k}(t, X_t(\omega) + u(k2^{-n})) - b_{n,k}(t, X_t(\omega))\ \mathrm{d}t\right|_H$$

Since u solves equation (9.1.1) the above equals

$$\left|\int_0^{(k+1)2^{-n}} e^{-((k+1)2^{-n}-t)A}(f(t, X_t(\omega) + u(t)) - f(t, X_t(\omega)))\ \mathrm{d}t\right.$$

$$- \int_0^{k2^{-n}} e^{-(k2^{-n}-t)A}(f(t, X_t(\omega) + u(t)) - f(t, X_t(\omega)))\ \mathrm{d}t$$

$$\left.- \int_{k2^{-n}}^{(k+1)2^{-n}} b_{n,k}(t, X_t(\omega) + u(k2^{-n})) - b_{n,k}(t, X_t(\omega)))\ \mathrm{d}t\right|_H$$

$$= \Bigg| \int_{k2^{-n}}^{(k+1)2^{-n}} e^{-((k+1)2^{-n}-t)A}(f(t,X_t(\omega)+u(t)) - f(t,X_t(\omega)))\,\mathrm{d}t$$

$$- \int_{k2^{-n}}^{(k+1)2^{-n}} b_{n,k}(t,X_t(\omega)+u(k2^{-n})) - b_{n,k}(t,X_t(\omega)))\,\mathrm{d}t$$

$$+ \int_0^{k2^{-n}} \left(e^{-((k+1)2^{-n}-t)A} - e^{-(k2^{-n}-t)A}\right) \cdot (f(t,X_t(\omega)+u(t)) - f(t,X_t(\omega)))\,\mathrm{d}t\Bigg|_H .$$

Using the definition of $b_{n,k}$ (see Definition 9.1.1) this can be simplified and bounded by

$$\left| \int_{k2^{-n}}^{(k+1)2^{-n}} b_{n,k}(t,X_t(\omega)+u(t)) - b_{n,k}(t,X_t(\omega)+u(k2^{-n}))\,\mathrm{d}t\right|_H$$

$$+ \underbrace{\left|e^{-2^{-n}A} - 1\right|_{\mathrm{op}}}_{\leq C2^{-n}} \cdot \left| \int_0^{k2^{-n}} e^{-(k2^{-n}-t)A}(f(t,X_t(\omega)+u(t)) - f(t,X_t(\omega)))\,\mathrm{d}t\right|_H .$$

Since u_n is constant on $[k2^{-n},(k+1)2^{-n}[$ and using again that u solves equation (9.1.1) we can estimate this by

$$\left| \int_{k2^{-n}}^{(k+1)2^{-n}} b_{n,k}(t,X_t(\omega)+u(t)) - b_{n,k}(t,X_t(\omega)+u(k2^{-n}))\,\mathrm{d}t\right|_H + C2^{-n}|u(k2^{-n})|_H.$$

By invoking Theorem 7.2.1 this can be rewritten as

$$\lim_{\ell\to\infty} \left| \int_{k2^{-n}}^{(k+1)2^{-n}} b_{n,k}(t,X_t(\omega)+u_\ell(t)) - b_{n,k}(t,X_t(\omega)+u_n(t))\,\mathrm{d}t\right|_H + C2^{-n}|u(k2^{-n})|_H$$

$$\leq C2^{-n}|u(k2^{-n})|_H + \sum_{\ell=n}^{\infty} \left| \int_{k2^{-n}}^{(k+1)2^{-n}} b_{n,k}(t,X_t(\omega)+u_{\ell+1}(t)) - b_{n,k}(t,X_t(\omega)+u_\ell(t))\,\mathrm{d}t\right|_H .$$

$$\leq C2^{-n}|u(k2^{-n})|_H + \sum_{\ell=n}^{\infty} \sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}-1} \left| \int_{2r2^{-\ell-1}}^{(2r+2)2^{-\ell-1}} b_{n,k}(t,X_t(\omega)+u_{\ell+1}(t)) - b_{n,k}(t,X_t(\omega)+u_\ell(t))\,\mathrm{d}t\right|_H$$

$$= C2^{-n}|u(k2^{-n})|_H + \sum_{\ell=n}^{\infty} \sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}-1} \left| \int_{2r2^{-\ell-1}}^{(2r+1)2^{-\ell-1}} \underbrace{b_{n,k}(t,X_t(\omega)+u(2r2^{-\ell-1})) - b_{n,k}(t,X_t(\omega)+u(r2^{-\ell}))}_{=0}\,\mathrm{d}t\right|_H$$

$$+\sum_{\ell=n}^{\infty}\sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}-1}\left|\int_{(2r+1)2^{-\ell-1}}^{(2r+2)2^{-\ell-1}} b_{n,k}(t,X_t(\omega)+u((2r+1)2^{-\ell-1}))-b_{n,k}(t,X_t(\omega)+u(r2^{-\ell}))\,\mathrm{d}t\right|_H$$

$$=C2^{-n}|u(k2^{-n})|_H+\sum_{\ell=n}^{\infty}\sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}-1}|\varphi_{\ell+1,2r+1}\left(b_{n,k};u\left((2r+1)2^{-\ell-1}\right),u\left(r2^{-\ell}\right)\right)|_H.$$

Summing over $k\in\{j2^{n-m},...,(j+1)2^{n-m}-1\}$ leads us to

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1}|u((k+1)2^{-n})-u(k2^{-n})-\varphi_{n,k}(b_{n,k};u(k2^{-n}))|_H$$

$$\leq\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1}\left(C2^{-n}|u(k2^{-n})|_H+\sum_{\ell=n}^{\infty}\sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}-1}|\varphi_{\ell+1,2r+1}\left(b_{n,k};u\left((2r+1)2^{-\ell-1}\right),u\left(r2^{-\ell}\right)\right)|_H\right).$$

$$=\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1}C2^{-n}|u(k2^{-n})|_H+\sum_{\ell=n}^{\infty}\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1}\sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}-1}|\varphi_{\ell+1,2r+1}\left(b_{n,k};u\left((2r+1)2^{-\ell-1}\right),u\left(r2^{-\ell}\right)\right)|_H$$

$$=\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1}C2^{-n}|u(k2^{-n})|_H+\sum_{\ell=n}^{\infty}\sum_{r=j2^{\ell-m}}^{(j+1)2^{\ell-m}-1}|\varphi_{\ell+1,2r+1}\left(b_{n,\lfloor r2^{n-\ell}\rfloor};u\left((2r+1)2^{-\ell-1}\right),u\left(r2^{-\ell}\right)\right)|_H$$

$$=\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1}C2^{-n}|u(k2^{-n})|_H+\sum_{\ell=n}^{\infty}\sum_{r=j2^{\ell+1-m}}^{(j+1)2^{\ell+1-m}-2}|\varphi_{\ell+1,r+1}\left(b_{n,\lfloor r2^{n-\ell-1}\rfloor};u\left((r+1)2^{-\ell-1}\right),u\left(r2^{-\ell-1}\right)\right)|_H.$$

In conclusion we obtain

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1}\left|u((k+1)2^{-n})-u(k2^{-n})-\varphi_{n,k}(b_{n,k};u(k2^{-n}))\right|_H\leq\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1}C2^{-n}|u(k2^{-n})|_H+\sum_{\ell=n}^{\infty}\Lambda_\ell$$

and with the help of Definition 9.1.6 we rewrite this as

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1}\left|u((k+1)2^{-n})-u(k2^{-n})-\varphi_{n,k}(b_{n,k};u(k2^{-n}))\right|_H\leq C2^{-n}\psi_n+\sum_{\ell=n}^{\infty}\Lambda_\ell,$$

which concludes the proof.

□

9.2 The Main Proof

The idea of the proof is the following: We use the reversed triangle inequality together with Lemma 9.1.9 to isolate the term $|u((k+1)2^{-n}) - u(k2^{-n})|_H$. On the right-hand side we have the $C2^{-n}\psi_n$ term and two sums. For the first sum (the one involving the term $|\varphi_{n,k}(b_{n,k}; u(k2^{-n}))|_H$) we simply use Theorem 6.1.5 (in the form of Corollary 7.2.2) to obtain the estimate in Lemma 9.2.1. We will split the second sum (the one involving the term Λ_ℓ) in the cases $\ell < N$ and $N \leq \ell$. In the first case we use Corollary 8.2.2, which will lead us to the inequality in Lemma 9.2.3. For the second case we have to do a more direct computation, which heavily relies on the fact that u is Lipschitz continuous which is executed in Lemma 9.2.2.

Combining all of this will result the final bound (9.2.4.1). Using the knowledge of the already established estimate in Lemma 9.1.9 and the Definition of A (see Definition 9.1.5) we will be able to estimate A in terms of β (inequality (9.2.4.2)). Feeding this back into inequality (9.1.5.1) for $n = m$ completes the proof.

Lemma 9.2.1

Let everything be defined as above. There exists a constant $\tilde{C}_\varepsilon \in \mathbb{R}$ such that for every $n \in \{m, ..., N\}$ we have

$$C2^{-n}\psi_n + \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |\varphi_{n,k}(b_{n,k}; u(k2^{-n}))|_H \leq 8CC_\varepsilon n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)n} 2^{-m} \left[\beta + \frac{A}{2}\right],$$

where $C > 0$ is the constant from Lemma 9.1.9.

Proof

Starting from the left-hand side of the assertion we apply Corollary 7.2.2 to obtain

$$\begin{aligned} &C2^{-n}\psi_n + \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |\varphi_{n,k}(b_{n,k}; u(k2^{-n}))|_H \\ &\leq C2^{-n}\psi_n + \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} C_\varepsilon n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{-hn} \left(\left|u(k2^{-n})\right|_H + 2^{-2^n}\right) \end{aligned}$$

and since $n \geq m$ this is smaller than

$$\begin{aligned} &C2^{-n}\psi_n + C_\varepsilon n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{-hn} \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} \left(|u(k2^{-n})|_H + 2^{-2^m}\right) \\ &= C2^{-n}\psi_n + C_\varepsilon n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{-hn} \left(2^{n-m}2^{-2^m} + \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u(k2^{-n})|_H\right). \end{aligned}$$

Again, using that $n \in \{m, ..., N\}$ and the definition of ψ_n (Definition 9.1.6) this can be written as

$$C2^{-n}\psi_n + C_\varepsilon n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{-hn}\left(2^{n-m}2^{-2^m} + \psi_n\right).$$

Since $2^{-n} \le n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{-hn}$ the above is furthermore bounded from above by

$$2CC_\varepsilon n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{-hn}\left(2^{n-m}2^{-2^m} + \psi_n\right).$$

Using Lemma 9.1.7 this can be further estimated by

$$2CC_\varepsilon n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{-hn}\left(2^{n-m}2^{-2^m} + 2\cdot 2^{n-m}\left[\beta + \frac{A}{2}\right]\right)$$

$$\le 4CC_\varepsilon n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)n}2^{-m}\left(2^{-2^m} + \beta + \frac{A}{2}\right)$$

Since $2^{-2^m} \le \beta$ by (9.1.2.1) and (9.1.2.3) this can be further estimate from above by

$$8CC_\varepsilon n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)n}2^{-m}\left[\beta + \frac{A}{2}\right].$$

Hence, we obtain for all $n \in \{m, ..., N\}$

$$C2^{-n}\psi_n + \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |\varphi_{n,k}(b_{n,k}; u(k2^{-n}))|_H \le 8CC_\varepsilon n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)n}2^{-m}\left[\beta + \frac{A}{2}\right].$$

□

Lemma 9.2.2

With everything as defined above we have

$$\sum_{\ell=N}^{\infty} \Lambda_\ell \le 3LC_\varepsilon 2^{-m}2^{-hN/3}.$$

Proof

By applying Corollary 7.2.2 with $\delta := \frac{h}{2}$ we obtain

$$\sum_{\ell=N}^{\infty}\Lambda_\ell = \sum_{\ell=N}^{\infty}\sum_{r=j2^{\ell+1-m}}^{(j+1)2^{\ell+1-m}-2} \left|\varphi_{\ell+1,r+1}\left(b_{n,\lfloor r2^{n-\ell-1}\rfloor}; u\left((r+1)2^{-(\ell+1)}\right), u\left(r2^{-(\ell+1)}\right)\right)\right|_H.$$

$$\le \sum_{\ell=N}^{\infty}\sum_{r=j2^{\ell+1-m}}^{(j+1)2^{\ell+1-m}-2} C_\varepsilon\left((\ell+1)^{\frac{1}{\alpha}}2^{-h\ell/2}\left|u\left((r+1)2^{-(\ell+1)}\right) - u\left(r2^{-(\ell+1)}\right)\right|_\infty + 2^{-\ell}\right).$$

Using the Lipschitz continuity of u (recall that L is the Lipschitz constant of u) this is smaller than

$$LC_\varepsilon\sum_{\ell=N}^{\infty}(\ell+1)^{\frac{1}{\alpha}}2^{-h\ell/2}\sum_{r=j2^{\ell+1-m}}^{(j+1)2^{\ell+1-m}-1}\left(|(r+1)2^{-\ell-1} - r2^{-\ell-1}| + 2^{-\ell}\right)$$

$$= LC_\varepsilon \sum_{\ell=N}^{\infty} (\ell+1)^{\frac{1}{\alpha}} 2^{-h\ell/2} \sum_{r=j2^{\ell+1-m}}^{(j+1)2^{\ell+1-m}-1} (2^{-\ell-1} + 2^{-\ell})$$
$$= 3LC_\varepsilon \sum_{\ell=N}^{\infty} 2^{\ell-m} (\ell+1)^{\frac{1}{\alpha}} 2^{-h\ell/2} 2^{-\ell} = 3LC_\varepsilon 2^{-m} \sum_{\ell=N}^{\infty} (\ell+1)^{\frac{1}{\alpha}} 2^{-h\ell/2} \leq 3LC_\varepsilon 2^{-m} 2^{-hN/3}.$$

And hence we obtain

$$\sum_{\ell=N}^{\infty} \Lambda_\ell \leq 3LC_\varepsilon 2^{-m} 2^{-hN/3}.$$

□

Lemma 9.2.3

Let everything be as above and recall that $h \geq \frac{1}{2}$, which is crucial for this lemma. We then have

$$\sum_{\ell=n}^{N-1} \Lambda_\ell \leq 24 C_\varepsilon C 2^{-m} N \left[\beta + \frac{A}{2}\right] + \frac{1}{2} \sum_{\ell=n}^{\infty} \Lambda_\ell.$$

Proof

For $n \leq \ell < N$ we define

$$\gamma_{\ell,r} := u((r+1)2^{-\ell}) - u(r2^{-\ell}) - \varphi_{\ell,r}(b_{n,\lfloor r2^{n-\ell} \rfloor}; u(r2^{-\ell})), \qquad \forall r \in \{0, ..., 2^\ell - 1\}$$

and note that due to Lemma 9.1.9 we have

$$\sum_{r=j2^{\ell-m}}^{(j+1)2^{\ell-m}-1} |\gamma_{\ell,r}|_H \leq C 2^{-\ell} \psi_\ell + \sum_{\ell'=\ell}^{\infty} \Lambda_{\ell'}. \tag{9.2.3.1}$$

Recall the definition of Λ_ℓ (see Definition 9.1.8):

$$\Lambda_\ell = \sum_{r=j2^{\ell+1-m}}^{(j+1)2^{\ell+1-m}-2} \left| \varphi_{\ell+1,r+1} \left(b_{n,\lfloor r2^{n-\ell-1} \rfloor}; u\left((r+1)2^{-(\ell+1)}\right), u\left(r2^{-(\ell+1)}\right)\right) \right|_H.$$

Using Corollary 8.2.2 yields that this is bounded from above by

$$C_\varepsilon \left[2^{-2h\ell} \sum_{r=j2^{\ell+1-m}}^{(j+1)2^{\ell+1-m}-1} |u(r2^{-(\ell+1)})|_H + 2^{-\delta\ell/4} \sum_{r=j2^{\ell+1-m}}^{(j+1)2^{\ell+1-m}-1} |\gamma_{\ell+1,r}|_H + 2^{(\delta/4-2h)\ell} |u(j2^{-m})|_H + 2^{\ell+1-m} 2^{-2^{\theta_\delta \ell}} \right],$$

where we used that $\theta_\delta \leq 1$ (which follows from inequality (9.1.2.2)). Plugging in Definition 9.1.6 this can be simplified to

$$C_\varepsilon \left[2^{-2h\ell}\psi_{\ell+1} + 2^{-\delta\ell/4} \sum_{r=j2^{\ell+1-m}}^{(j+1)2^{\ell+1-m}-1} |\gamma_{\ell+1,r}|_H + 2^{(\delta/4-2h)\ell}|u(j2^{-m})|_H + 2^{\ell+1-m}2^{-2^{\theta\delta\ell}} \right].$$

Summing over $\ell \in \{n, ..., N-1\}$ then yields

$$\sum_{\ell=n}^{N-1} \Lambda_\ell \le C_\varepsilon \sum_{\ell=n}^{N-1} \left[2^{-2h\ell}\psi_{\ell+1} + 2^{-\delta\ell/4} \sum_{r=j2^{\ell+1-m}}^{(j+1)2^{\ell+1-m}-1} |\gamma_{\ell+1,r}|_H + 2^{(\delta/4-2h)\ell}|u(j2^{-m})|_H + 2^{\ell+1-m}2^{-2^{\theta\delta\ell}} \right].$$

$$\le C_\varepsilon \left[\sum_{\ell=n}^{N-1} \left(2^{-2h\ell}\psi_{\ell+1} + 2^{-\delta\ell/4} \left(C2^{-\ell}\psi_{\ell+1} + \sum_{\ell'=\ell}^{\infty} \Lambda_{\ell'} \right) \right) + 2^{(\delta/4-2h)n}|u(j2^{-m})|_H + 2^{n+1-m}2^{-2^{\theta\delta n}} \right],$$

where we used inequality (9.2.3.1) from above and exploiting that $2^{-\ell} \le 2^{-2h\ell}$ yields that the above expression is bounded by

$$C'_\varepsilon \left[\sum_{\ell=n}^{N-1} 2^{-2h\ell}\psi_{\ell+1} + 2^{-\delta n/4} \sum_{\ell'=n}^{\infty} \Lambda_{\ell'} + 2^{(\delta/4-2h)n}|u(j2^{-m})|_H + 2^{n+1-m}2^{-2^{\theta\delta n}} \right],$$

where $C'_\varepsilon := 2C_\varepsilon C$. Since $\ell < N$ we can use Lemma 9.1.7 to estimate the term $\psi_{\ell+1}$ and the assumption $|u(j2^{-m})|_H \le \beta$ (see (9.1.2.3)) to obtain the following estimate

$$\sum_{\ell=n}^{N-1} \Lambda_\ell \le 4C'_\varepsilon \left[\sum_{\ell=n}^{N-1} 2^{-2h\ell}2^{\ell-m} \left[\beta + \frac{A}{2} \right] + 2^{-\delta n/4} \sum_{\ell'=n}^{\infty} \Lambda_{\ell'} + 2^{(\delta/4-2h)n}\beta + 2^{n-m}2^{-2^{\theta\delta n}} \right].$$

and since $m \le n$ this is smaller than

$$4C'_\varepsilon \left[\sum_{\ell=n}^{N-1} 2^{(1-2h)\ell}2^{-m} \left[\beta + \frac{A}{2} \right] + 2^{-\delta m/4} \sum_{\ell'=n}^{\infty} \Lambda_{\ell'} + 2^{(\delta/4-2h)m}\beta + 2^{-2^{\theta\delta m}} \right].$$

Recall that $h \ge \frac{1}{2}$ and hence $2^{(1-2h)\ell} \le 1$. Therefore, the above is bounded from above by

$$4C'_\varepsilon \left[2^{-m}N \left[\beta + \frac{A}{2} \right] + 2^{-\delta m/4} \sum_{\ell'=n}^{\infty} \Lambda_{\ell'} + 2^{(\delta/4-2h)m}\beta + 2^{-2^{\theta\delta m}} \right].$$

Using that $2^{(\delta/4-2h)m} = 2^{-m}2^{(1+\delta/4-2h)m} \le 2^{-m}N$ (see Lemma 9.1.4.1), the above expression can be again bounded by

$$4C'_\varepsilon \left[2^{-m}N \left[\beta + \frac{A}{2} \right] + 2^{-\delta m/4} \sum_{\ell=n}^{\infty} \Lambda_\ell + 2^{-m}N\beta + 2^{-2^{\theta\delta m}} \right].$$

Recall that by (9.1.2.3) we have $2^{-2^{\theta\delta m}} \le 2^{-m}\beta$, so that in conclusion we deduce

$$\sum_{\ell=n}^{N-1} \Lambda_\ell \le 12C'_\varepsilon \left[2^{-m}N \left[\beta + \frac{A}{2} \right] + 2^{-\delta m/4} \sum_{\ell=n}^{\infty} \Lambda_\ell \right],$$

Since $m \ge m_0$ and m_0 is defined (see (9.1.2.1)) such that $12C'_\varepsilon 2^{-\delta m_0/4} \le \frac{1}{2}$ we have

$$\sum_{\ell=n}^{N-1} \Lambda_\ell \leq 24 C_\varepsilon C 2^{-m} N \left[\beta + \frac{A}{2}\right] + \frac{1}{2} \sum_{\ell=n}^{\infty} \Lambda_\ell .$$

□

From this point onwards we can forget all the definitions of this chapter.

Theorem 9.2.4 (Cf. [Wre17, Theorem 6.2])

Assume that the usual assumptions (see Definition 6.1.4) are fulfilled and that additionally $h \geq \frac{1}{2}$ and (9.1) i.e.

$$\frac{1-h}{h} < \frac{2\alpha\gamma}{2+\alpha+2\gamma} \leq \frac{1}{h}$$

holds. Let $\varepsilon > 0$ and $f\colon [0,1] \times H \longrightarrow Q$ then there exist $\Omega_{\varepsilon,f} \subseteq \Omega$ with $\mathbb{P}[\Omega_{\varepsilon,f}^c] \leq \varepsilon$, $K = K(\varepsilon) > 0$ and $m_0 = m_0(\varepsilon) \in \mathbb{N}$, $\delta \in]0, h[$ and $\zeta > 0$ such that for any function $u \in \Phi$ being a solution of equation (9.1.1) for a fixed $\omega \in \Omega_{\varepsilon,f}$, for all integers m with $m \geq m_0$, $j \in \{0, ..., 2^m - 1\}$ and β satisfying

$$2^{m-2^{\theta_\delta m}} \leq \beta \leq 2^{-2^{(1-h+\zeta)m}} \quad \text{and} \quad |u(j2^{-m})|_H \leq \beta,$$

where $\theta_\delta = (h-\delta)\frac{2\alpha\gamma}{2+\alpha+2\gamma}$ we have

$$|u((j+1)2^{-m})|_H \leq \beta \left(1 + K 2^{-m} \log_2(1/\beta)\right).$$

Proof

Let $\varepsilon > 0$ and f be as in the assertion. Let $b_{n,k}$ be as in Definition 9.1.1. Note that for all $n \in \mathbb{N}$ and $k \in \{0, ..., 2^n - 1\}$ $b_{n,k}$ is Q-valued since f is Q-valued. Let $\Omega_{\varepsilon,f}, \delta, \zeta, N$ be as in the beginning of this chapter.

Putting together both estimates Lemma 9.2.2 and Lemma 9.2.3 for Λ_ℓ we have

$$\sum_{\ell=n}^{\infty} \Lambda_\ell = \sum_{\ell=n}^{N-1} \Lambda_\ell + \sum_{\ell=N}^{\infty} \Lambda_\ell \leq 24 C_\varepsilon C 2^{-m} N \left[\beta + \frac{A}{2}\right] + \frac{1}{2} \sum_{\ell=n}^{\infty} \Lambda_\ell + 3 L C_\varepsilon 2^{-m} 2^{-hN/3}.$$

Henceforth, we deduce

$$\sum_{\ell=n}^{\infty} \Lambda_\ell \leq 48 C_\varepsilon C 2^{-m} N \left[\beta + \frac{A}{2}\right] + 6 L C_\varepsilon 2^{-m} 2^{-hN/3}$$

and since $N = \lceil 3h \log_2(1/\beta) \rceil$ this expression is bounded by

$$54 L C_\varepsilon C 2^{-m} N \left[\beta + \frac{A}{2}\right].$$

Therefore, we have

$$\sum_{\ell=n}^{\infty} \Lambda_\ell \leq 54 L C_\varepsilon C 2^{-m} N \left[\beta + \frac{A}{2}\right]. \tag{9.2.4.1}$$

From Lemma 9.1.9 and the reversed triangle inequality we deduce

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u((k+1)2^{-n}) - u(k2^{-n})|_H \le C2^{-n}\psi_n + \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |\varphi_{n,k}(b_{n,k}; u(k2^{-n}))|_H + \sum_{\ell=n}^{\infty} \Lambda_\ell.$$

With the help of Lemma 9.2.1 and (9.2.4.1), we estimate this by

$$\begin{aligned}\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u((k+1)2^{-n}) - u(k2^{-n})|_H &\le C2^{-n}\psi_n + \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |\varphi_{n,k}(b_{n,k}; u(k2^{-n}))|_H + \sum_{\ell=n}^{\infty} \Lambda_\ell \\ &\le 8C_\varepsilon C n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)n} 2^{-m} \left[\beta + \frac{A}{2}\right] + 54LC_\varepsilon C 2^{-m} N \left[\beta + \frac{A}{2}\right] \\ &\le 54LC_\varepsilon C2^{-m} \left[n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)n} + N\right] \cdot \left[\beta + \frac{A}{2}\right].\end{aligned}$$

Note that the above argument holds for all $n \in \{m, ..., N\}$. Hence, by the minimality of A (i.e. Definition 9.1.5) and inequality (9.1.5.1) we have

$$A2^{-m} \left[n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)n} + N\right] \le 54LC_\varepsilon C2^{-m} \left[n^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)n} + N\right] \cdot \left[\beta + \frac{A}{2}\right]$$

for all $n \in \{m, ..., N\}$. This implies that

$$A \le 54LC_\varepsilon C \left[\beta + \frac{A}{2}\right].$$

We deduce from this that

$$A \le 108LC_\varepsilon C\beta. \tag{9.2.4.2}$$

Setting $n = m$ in (9.1.5.1) of Definition 9.1.5 reads

$$|u((j+1)2^{-m}) - u(j2^{-m})|_H \overset{(9.1.5)}{\le} A2^{-m} \left[m^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)m} + N\right].$$

Putting $|u(j2^{-m})|_H$ to the right-hand side we deduce that

$$|u((j+1)2^{-m})|_H \le |u(j2^{-m})|_H + A2^{-m} \left[m^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)m} + N\right]$$

and since we have $A \le 108LC_\varepsilon C\beta$ (see inequality (9.2.4.2)) as well as $m^{\frac{2+\alpha+2\gamma}{2\alpha\gamma}} 2^{(1-h)m} \le N$ by Lemma 9.1.4 yields that the above expression is smaller than

$$\beta + 108LC_\varepsilon C\beta 2^{-m} N = \beta \left(1 + 108LC_\varepsilon C2^{-m} \lceil 3h \log_2(1/\beta) \rceil\right) \le \beta \left(1 + K2^{-m} \log_2(1/\beta)\right),$$

where the constant is defined as $K := 648LhC_\varepsilon C$ which completes the proof.

□

Corollary 9.2.5 (Cf. [Wre17, Corollary 6.3])

Assume that the usual assumptions (see Definition 6.1.4) are fulfilled and that additionally $h \geq \frac{1}{2}$ and (9.1) i.e.

$$\frac{1-h}{h} < \frac{2\alpha\gamma}{2+\alpha+2\gamma} \leq \frac{1}{h}$$

holds. Let $f\colon [0,1] \times H \longrightarrow Q$, then there exists a set $N_f \subseteq \Omega$ with $\mathbb{P}[N_f] = 0$ such that for all $\omega \in N_f^c$ every $u \in \mathcal{C}([0,1], H)$ is a solution to

$$u(t) = \int_0^t e^{-(t-s)A}(f(s, u(s) + X_s(\omega)) - f(s, X_s(\omega)))\,\mathrm{d}s, \qquad \forall t \in [0,1]. \qquad (9.2.5.1)$$

then $u \equiv 0$.

Proof

Step 1:

Let $\varepsilon > 0$ and $\Omega_{\varepsilon,f}$ be the of set of Theorem 9.2.4. Fix $\omega \in \Omega_{\varepsilon,f}$ and let u, as stated in the assertion, be a solution to the above equation (9.2.5.1). Since $\|f\|_\infty < \infty$ the function u is in the set Φ with $L := 2\|f\|_\infty$ (see Definition 7.1.1).

Applying Theorem 9.2.4 gives us a $K > 0$ and $m_0 \in \mathbb{N}$ as well as $\delta > 0$ and $\zeta > 0$ such that $1 + h + \zeta < \theta_\delta$. For sufficiently large $m \in \mathbb{N}$ (i.e. $K \leq \ln(2)2^m$ and $m \geq m_0$) we define

$$\alpha_m := 2^{m - 2^{\theta_\delta m}}$$

and

$$\alpha'_m := 2^{-2^{(1-h+\zeta)m}}.$$

Furthermore, Theorem 9.2.4 implies that for all $\beta \in [\alpha_m, \alpha'_m]$ we have

$$|u(j2^{-m})|_H \leq \beta \qquad \Longrightarrow \qquad |u((j+1)2^{-m})|_H \leq \beta(1 + K2^{-m}\log_2(1/\beta))$$

for all $j \in \{0, ..., 2^m - 1\}$.

A simple calculations yields

$$\frac{\ln \alpha_m}{\ln \alpha'_m} = \frac{2^{\theta_\delta m} - m}{2^{(1-h+\zeta)m}} \overset{m\to\infty}{\longrightarrow} \infty,$$

so that we are able to invoke Corollary 2.2.2 which implies that $u \equiv 0$.

Step 2:

Let $k \in \mathbb{N}$. By setting $\varepsilon := 1/k$ in Step 1 we conclude that there is $\Omega_{k,f} \subseteq \Omega$ with $\mathbb{P}[\Omega_{k,f}^c] \leq 1/k$ such that $u \equiv 0$ for all $\omega \in \Omega_{k,f}$. By defining

$$N_f := \bigcap_{k=1}^\infty \Omega_{k,f}^c$$

we have $\mathbb{P}[N_f] = 0$ and $u \equiv 0$ for all $\omega \in N_f^c$ which concludes the proof.

□

10 Applications

In this chapter we apply the result of the previous chapter to deduce path-by-path uniqueness for several stochastic differential equations (Theorem 10.1.1) using the language of effective dimension and regularizing noises we developed in the previous chapters.

As a simple Corollary we obtain a proof of the main result 1.1.3 (see Corollary 10.2.1).

10.1 Proof of the main result in abstract form

Theorem 10.1.1

Given *any* filtered stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty[}, \mathbb{P}, (W_t)_{t\in[0,\infty[})$. Let $X \in \operatorname{reg}(Q, h, \alpha)$ with h $\geq \frac{1}{2}$ and $C_Q > 0$ such that

$$\operatorname{ed}(Q)_m \leq C_Q(\ln(m+1))^{1/\gamma}, \qquad \forall m \in \mathbb{N}$$

for some $\gamma \geq 1$ and

$$\frac{1-h}{h} < \frac{2\alpha\gamma}{2+\alpha+2\gamma} \leq \frac{1}{h} \tag{10.1.1.1}$$

then the for every Borel measurable function $f\colon [0,T] \times H \longrightarrow Q$ the stochastic differential equation

$$\mathrm{d}Y_t = -AY_t\mathrm{d}t + f(t, Y_t)\mathrm{d}t + \mathrm{d}W_t,$$

where $(W_t)_{t\in[0,\infty[}$ is a cylindrical Wiener process and

$$X_t = \int_0^t e^{-(t-s)A}\, \mathrm{d}W_s$$

has a path-by-path unique solution in the mild sense.

Proof

By Proposition 1.2.1 it is sufficient to show that there exists a measurable set $\Omega_0 \subseteq \Omega$ with $\mathbb{P}[\Omega_0] = 1$ such that for all $\omega \in \Omega_0$ the only function $u \in \mathcal{C}([0,1], H)$ solving

$$u(t) = \int_0^t e^{-(t-s)A} \left(f(s, X_s(\omega) + u(s)) - f(s, X_s(\omega))\right)\, \mathrm{d}s$$

is the trivial function $u \equiv 0$. Since by assumption $X \in \operatorname{reg}(Q, h, \alpha)$ and the usual assumptions (Definition 6.1.4) are fulfilled, this follows simply by invoking of Corollary 9.2.5.

□

10.2 Proof of the main result

In this section we prove the main result (Theorem 1.1.3). Recall that in Chapter 1 we have reduced Theorem 1.1.3 to Proposition 1.1.4 and via a Girsanov Transformation to Proposition 1.2.1. For the reader's convenience we restate the main result.

Corollary 10.2.1

Given *any* filtered stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty[}, \mathbb{P}, (W_t)_{t\in[0,\infty[})$. Let $f\colon [0,T]\times H \longrightarrow H$ such that f fulfills Assumption 1.1.2. Then the stochastic differential equation

$$\mathrm{d}Y_t = -AY_t\mathrm{d}t + f(t, Y_t)\mathrm{d}t + \mathrm{d}W_t,$$

where $(W_t)_{t\in[0,\infty[}$ is a cylindrical Wiener process has a path-by-path unique solution in the mild sense.

Proof

Recall that by Assumption 1.1.2 we have $\|f\|_{\infty,A} < \infty$, so that we can set

$$Q := \left\{x \in \mathbb{R}^{\mathbb{N}}\colon \sum_{n\in\mathbb{N}} \lambda_n e^{2\lambda_n}|x_n|^2 \leq \|f\|_{\infty,A}\right\} \cap \{x \in \mathbb{R}^{\mathbb{N}}\colon |x_n| < \exp\left(-e^{c_\gamma n^\gamma}\right)\}, \quad (10.2.1.1)$$

where $\gamma > 2$ and $c_\gamma > 0$ are the constants from Assumption 1.1.2. We note that Assumption 1.1.2 implies that f is Q-valued.

Recall the definition of the set Q^A (see Definition 5.2.1) and Q^γ (see Definition 3.2.1). Setting $C_A := \|f\|_{\infty,A}$ in the definition of the set Q^A we can rewrite (10.2.1.1) to

$$Q = Q^A \cap Q^\gamma.$$

We conclude that

$$Z_t^A := \int_0^t e^{-(t-s)A}\, \mathrm{d}W_s,$$

together with the filtration $(\mathcal{G}_t)_{t\in[0,\infty[}$ as defined in (1.1.2), is by Corollary 5.2.3 a Q^A-regularizing noise with $h = \frac{1}{2}$ and $\alpha = 2$. Invoking Proposition 5.1.4 yields that Z^A is a Q-regularizing noise with the same h and α.

Furthermore, since $Q \subseteq Q^\gamma$ we have $\mathrm{ed}(Q)_m \leq \mathrm{ed}(Q^\gamma)_m \leq C_Q(\ln(m+1))^{1/\gamma}$ for all $m \in \mathbb{N}$ by Proposition 3.2.2 and henceforth the usual assumptions (Definition 6.1.4) are fulfilled. Moreover, condition (10.1.1.1) of Theorem 10.1.1 for $h = \frac{1}{2}$ and $\alpha = 2$ reads

$$1 < \frac{4\gamma}{4+2\gamma} \leq 2.$$

This is obviously fulfilled since by Assumption 1.1.2 we have $\gamma > 2$. Therefore, invoking Theorem 10.1.1 with $X := Z^A$ completes the proof.

□

10.3 Finite-dimensional case

Let $H = \mathbb{R}^d$ and $Q := \{x \in \mathbb{R}^d \colon |x| \leq C\}$ for a constant $C > 0$. Then by Proposition 3.1.5 we have $\sup_{m\in\mathbb{N}} \mathrm{ed}(Q)_m < \infty$ so that

$$\mathrm{ed}(Q)_m \leq C_Q(\ln(m+1))^{1/\gamma}, \qquad \forall m \in \mathbb{N}$$

is fulfilled for any $\gamma > 0$. Recall that (see Example 5.1.7) for any $\mathbb{R}^d$-valued Brownian motion B we have $B \in \mathrm{reg}(Q, \frac{1}{2}, 2)$.

Corollary 10.3.1 (Cf. [Dav07])

Let $X \in \mathrm{reg}(Q, h, \alpha)$ with $h \geq \frac{1}{2}$ for any bounded set $Q \subseteq \mathbb{R}^d$ such that

$$\frac{1-h}{h} < \alpha \leq \frac{1}{h},$$

then for every bounded Borel measurable function $f \colon [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ the stochastic differential equation

$$\mathrm{d}Y_t = f(t, Y_t)\mathrm{d}t + \mathrm{d}X_t$$

has a path-by-path unique solution.

Proof

Let $C := \|f\|_\infty$ then by setting $Q := \{x \in \mathbb{R}^d \colon |x| \leq C\}$ as above, the function f is Q-valued. In order to invoke Theorem 10.1.1 condition (10.1.1.1) i.e.

$$\frac{1-h}{h} < \frac{2\alpha\gamma}{2+\alpha+2\gamma} \leq \frac{1}{h}$$

has to be fulfilled for some $\gamma \geq 1$. Note that the mapping

$$\gamma \longmapsto \frac{2\alpha\gamma}{2+\alpha+2\gamma}$$

is increasing and

$$\lim_{\gamma\to\infty} \frac{2\alpha\gamma}{2+\alpha+2\gamma} = \alpha.$$

Hence, we can always find a $\gamma \geq 1$ satisfying condition (10.1.1.1), since we have, by assumption, that

$$\frac{1-h}{h} < \alpha \leq \frac{1}{h}.$$

Invoking Theorem 10.1.1 therefore completes the proof.

$\square$

Part IV

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