Analysis of anisotropic nonlocal operators and jump processes

Universität Bielefeld
Fakultät für Mathematik

Dissertation

zur Erlangung des akademischen Grades
DOKTOR DER MATHEMATIK (DR. MATH.)

eingereicht von
M.Sc. Jamil Chaker
am 1. September 2017

Contents

1. Introduction

Anisotropies and discontinuities are phenomena of great interest that arise in several natural and financial models. Within this thesis we bring together these two subjects and study anisotropic nonlocal operators from a probabilistic and an analytic perspective. On the one hand, we investigate the solvability of systems of stochastic differential equations driven by pure jump Lévy processes with anisotropic and singular Lévy measures. On the other hand, we study regularity properties of weak solutions to a class of integro-differential equations determined by nonlocal operators whose kernels are singular and anisotropic.

Stochastic models where the underlying stochastic process is a Lévy processes with jumps, are increasingly important. Since discontinuities do naturally occur, stochastic models with jumps are in certain circumstances more suitable to capture empirical features than diffusion models. Lévy processes with jumps have become an important tool in financial mathematics. For instance, Merton derives in [Mer76] an option pricing formula when the underlying stock return is generated by a mixture of a continuous and a jump process. It has most features of the Black-Scholes formula, see [BS12], but outperforms it in some empirical facts such as the Volatility Smile Fitting in option pricing. Furthermore, Lévy jump processes as tempered stable processes are used for option pricing. For details on Lévy processes in finance, we refer the reader to [Sch03].

The analytic pendant to Lévy processes with jumps are nonlocal operators. They appear for example as generators of Lévy or Lévy-type processes. To explain nonlocality of operators let us first give the definition of local operators. Let V, W be function spaces. A linear operator $\mathcal{A} : V \to W$ is called local, if

$$\operatorname{supp}(\mathcal{A}f) \subset \operatorname{supp}(f) \qquad \text{for all } f \in V.$$

An operator is called nonlocal, if it is not a local operator.

Let us give some examples of local and nonlocal operators. Let $a_i : \mathbb{R}^d \to \mathbb{R}$ be a family of continuous and bounded functions. We define the local operator

$$\mathcal{A} : C^1(\mathbb{R}^d) \to C(\mathbb{R}^d), f \mapsto \sum_{i=1}^{d} a_i(\cdot) \frac{\partial f}{\partial x_i}(\cdot).$$

Another important example for a local operator is the Laplace operator, which is defined as follows

$$\Delta : C^2(\mathbb{R}^d) \to C(\mathbb{R}^d), \quad f \mapsto \operatorname{div}(\nabla f(\cdot)) = \operatorname{tr}(D^2 f(\cdot)) = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}(\cdot).$$

The operator $\Delta u(x)$ can be evaluated if u is known on a small neighborhood of x. This follows immediately from the definition of the derivative. Nonlocal operators do not have this property.

Consider for $\alpha \in (0,2)$ the operator on $\mathcal{S}(\mathbb{R}^d)$ of the form

$$(-\Delta)^{\alpha/2}u(x) = c(d,\alpha) \lim_{\epsilon \to 0} \int_{\mathbb{R}^d \setminus B_\epsilon(x)} (u(x) - u(y))|x-y|^{-d-\alpha} \, \mathrm{d}y,$$

where $c(d,\alpha)$ is a normalizing constant, chosen such that

$$\mathcal{F}((-\Delta)^{\alpha/2}u)(\xi) = |\xi|^\alpha \mathcal{F}u(\xi).$$

It is known that this constant behaves like $\alpha(2-\alpha)$ for $\alpha \nearrow 2$ and $\alpha \searrow 0$. This operator is called fractional Laplacian of order $\alpha/2$ and is a nonlocal operator. If one considers the transition semigroup of the isotropic α-stable Lévy process on $C_0(\mathbb{R}^d)$, then the fractional Laplacian is the infinitesimal generator on $C_0^2(\mathbb{R}^d)$ of to the semigroup. Although $((-\Delta)^{\alpha/2})_{\alpha \in (0,2)}$ is a family of nonlocal operators, it converges for all $u \in \mathcal{S}(\mathbb{R}^d)$ to the Laplace operator, a local operator. That is, for every $x \in \mathbb{R}^d$

$$\lim_{\alpha \nearrow 2} (-\Delta)^{\alpha/2}u(x) = (-\Delta u(x)).$$

This convergence result can easily be proven by rewriting $(-\Delta)^{\alpha/2}$ in terms of second order differences or alternately using Fourier analysis.

Partial differential equations involving nonlocal operators arise in various contexts such as continuum mechanics, population dynamics and game theory. For example, in [CV11] the authors consider a porous medium equation with nonlocal diffusion effects which arises in population dynamics. Another interesting application of nonlocal operators appears in image processing, see e.g. [GO08].

Anisotropy is the property of being directionally dependent and is a natural phenomenon. For example, the intensities of a light emitted by a fluorescence are not equal along different axes of polarization. This phenomenon is known as fluorescence anisotropy. A detailed exposition can be found for instance in [Lak06].
Crystals are solids, whose physical properties depend on the spatial direction. Therefore they are anisotropic. Another interesting example is given by liquid crystals. These substances have liquid and crystal properties; On the one hand the state of matter is more or less fluid as a liquid and on the other hand it has crystal-like properties such as birefringence, which is anisotropic. They are used for example in liquid crystal displays (LCD). For details on liquid crystals, see [Bli11].

Systems of stochastic differential equations

The first topic that we treat in this thesis is the solvability of systems of stochastic differential equations driven by pure jump Lévy processes. Consider for $d \in \mathbb{N}$, $d \geq 2$

the stochastic process $Z_t = (Z_t^1, \dots, Z_t^d)$, which consists of d independent symmetric one-dimensional stable processes with stability indices that may differ. The system we are investigating in this part is

$$\begin{aligned} dX_t^i &= \sum_{j=1}^{d} A_{ij}(X_{t-}) dZ_t^j, \qquad \text{for } i \in \{1, \dots, d\}, \\ X_0 &= x_0, \end{aligned} \tag{1.0.1}$$

where $A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a matrix-valued function, which is pointwise non-degenerate and has bounded continuous entries. We prove existence of solutions to the system (1.0.1) and uniqueness of solutions under the additional assumption that A is diagonal.

The case where Z_t is consisting of d independent copies of one-dimensional symmetric α-stable Lévy processes with $\alpha \in (0, 2)$ has been studied in [BC06]. Furthermore, if Z_t consists of independent copies of one-dimensional Brownian motions, which are 2-stable Lévy processes, the process Z_t is a d-dimensional Brownian motion and this corresponds to the well-known case of diffusion processes.

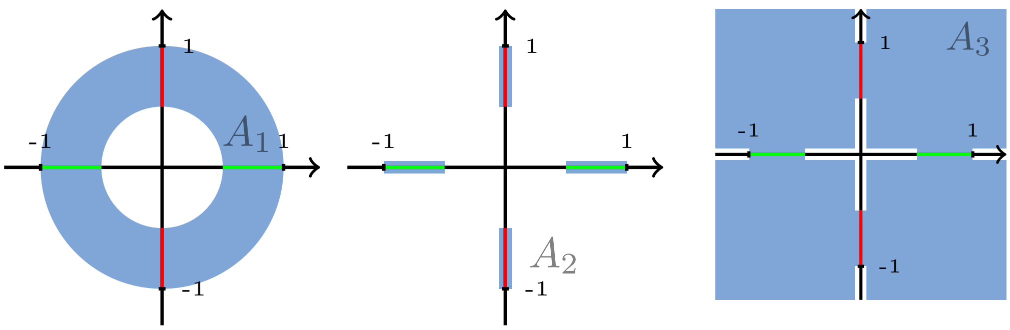
Let $(Z_t^i)_{t\geq 0}$ be a one-dimensional pure jump Lévy process with Lévy measure $c_{\alpha_i}|h|^{-1-\alpha_i}\, \mathrm{d}h$ for $\alpha_i \in (0, 2)$, where c_{α_i} is a normalizing constant. If we assume there are $i, j \in \{1, \dots, d\}$, $i \neq j$ such that $\alpha_i \neq \alpha_j$, then the resulting process $(Z_t)_{t\geq 0}$ is a Lévy process whose Lévy measure is concentrated on the union of the coordinate axes and weights different directions differently. Thus, this stochastic process has an anisotropic Lévy measure of the form

$$\nu(\mathrm{d}h) = \sum_{i=1}^{d} \left(\frac{c_{\alpha_i}}{|h_i|^{1+\alpha_i}}\, \mathrm{d}h_i \left(\prod_{j\neq i} \delta_{\{0\}}(\mathrm{d}h_j) \right) \right). \tag{1.0.2}$$

For illustrative purposes assume for the moment $d = 2$. Then

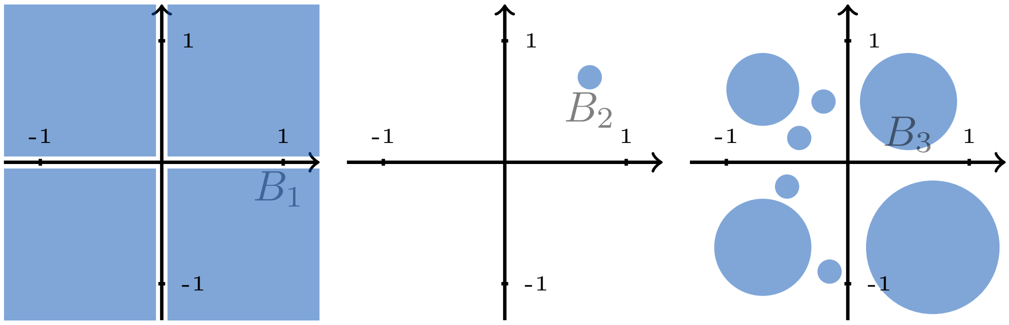
$$\nu(\mathrm{d}h) = \frac{c_{\alpha_1}}{|h_1|^{1+\alpha_1}}\, \mathrm{d}h_1 \left(\delta_{\{0\}}(\mathrm{d}h_2)\right) + \frac{c_{\alpha_2}}{|h_2|^{1+\alpha_2}}\, \mathrm{d}h_2 \left(\delta_{\{0\}}(\mathrm{d}h_1)\right).$$

From this definition we see that $\nu(A) > 0$ if and only if the measurable set A has an intersection with a coordinate axis whose one-dimensional Lebesgue measure is positive.



This figure shows three sets A_k, $k \in \{1,2,3\}$ for which the measure $\nu(\mathrm{d}h)$ assigns the same value. The reason for that is that the intersections with the coordinate axes coincide. Note that although the green and red colored lines in the figure intersect the respective axis on the same sections, the measure ν weights the green and the red colored lines differently, if $\alpha_1 \neq \alpha_2$.

Let us show some examples of sets with measure zero that have empty intersection with all coordinate axes.



We use the martingale problem method to study solvability of the system of stochastic differential equations, which goes back to the work of Stroock and Varadhan from 1969, see [SV69]. The celebrated martingale problem provides an equivalent concept of existence and uniqueness in law for weak solutions to stochastic differential equations. The authors study elliptic operators in nondivengence form given by

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial f}{\partial x_i}(x),$$

where $a : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is bounded, continuous and strictly elliptic and b is measurable and bounded. A probability measure $\mathbb{P}^x$ on $C([0,\infty))$ is called a solution to the martingale problem for L started at $x \in \mathbb{R}^d$ if $\mathbb{P}^x(X_0 = x) = 1$ and for any $f \in C_b^2(\mathbb{R}^d)$

$$M_t = f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,\mathrm{d}s$$

is a $\mathbb{P}^x$-martingale with respect to the filtration $(\sigma(X_s; s \leq t))_{t\geq 0}$, where $X_t(\omega) = \omega(t)$ are the coordinate maps on $C([0,\infty);\mathbb{R}^d)$. If for every starting point $x \in \mathbb{R}^d$ the solution to the martingale problem is unique, then the martingale problem for L is called well-posed.
Stroock and Varadhan show that existence and well-posedness of the martingale problem for L is equivalent to existence and uniqueness of weak solutions to the stochastic differential equation

$$dX_t = \sigma(X_t)\,\mathrm{d}W_t + b(X_t)\,\mathrm{d}t,$$

where W_t is a standard d-dimensional Brownian motion and σ is the Lipschitz square root of a, i.e. $a = \sigma\sigma^T$.

The existence of a solution $\mathbb{P}^x$ to the martingale problem for L implies the existence of a solution to the Cauchy problem for the operator L with initial data in $C^2(\mathbb{R}^d)$, that is for $f \in C^2(\mathbb{R}^d)$

$$u(x,t) = \mathbb{E}^x(f(X_t))$$

is a solution to

$$\begin{cases} \partial_t u = Lu & \text{in } \mathbb{R}^d \times (0,\infty), \\ u_0 = f & \text{for } t = 0, \end{cases}$$

where $\mathbb{E}^x$ is the expectation with respect to $\mathbb{P}^x$. In the case of well-posedness of the martingale problem the solution $(X, \mathbb{P}^x, x \in \mathbb{R}^d)$ is a strong Markov family.
An overview of the martingale problem for elliptic operators in non-divergence form can be found in [SV79, Chapter 6] or [Bas98, Chapter VI].

It is clear that the space of continuous functions is not suitable to study the martingale problem for nonlocal operators, since the corresponding stochastic process does not have continuous sample paths. The appropriate space in which to study the martingale problem for jump-type process is the Skorohod space, that is the space of right-continuous functions that have left limits, endowed with an appropriate topology.

In the 1970's the martingale problem for nonlocal operators has been studied, among others, by Komatsu in [Kom73], by Stroock in [Str75] and by Lepeltier and Marchal in [LM76]. Up to the present day, the martingale problem is still an intensely studied topic. For instance, in [AK09] unique solvability of the Cauchy problem for a class of integro-differential operators is shown to imply the well-posedness of the martingale problem for the corresponding operator. In [CZ16b] the authors study well-posedness of the martingale problem for a class of stable-like operators and in [Pri15] the author considers degenerate stochastic differential equations and proves weak uniqueness of solutions using the martingale problem. In [Kü17], existence and uniqueness for stochastic differential equations driven by Lévy processes and stable-like processes with unbounded coefficients are studied. For an overview of the martingale problem on the Skorohod space, see [Jac05, Chapter 4] and the references therein.

We now want to comment on known results and formulate the main results on this part.

In [Bas88], Bass considers nonlocal operators of the form

$$Lf(x) = \int_{\mathbb{R}\setminus\{0\}} (f(x+h) - f(x) - f'(x)h\mathbb{1}_{[-1,1]}(h))\, \nu(x, \mathrm{d}h)$$

for $f \in C_b^2(\mathbb{R})$ and gives sufficient conditions on ν for the existence and uniqueness of a solution to the martingale problem for L. Further, the author proves that the associated stochastic process X_t is a Feller process with respect to the unique solution $\mathbb{P}^x$ to the martingale problem for L started at $x \in \mathbb{R}$. One example in the paper is $\nu(x, \mathrm{d}h) \asymp |h|^{-1-\alpha(x)}\mathrm{d}h$ with $0 < \inf\{\alpha(x) \colon x \in \mathbb{R}\} \le \sup\{\alpha(x) \colon x \in \mathbb{R}\} < 2$. In this case the Dini continuity of $\alpha : \mathbb{R} \to (0,2)$ is a sufficient condition for well-posedness of the martingale problem. Although the results in the paper were proven in one spatial dimension they can be extended to higher dimensions. In [SW13], the authors present sufficient conditions for the transience and the existence of local times for Feller processes. The studies contain the class of stable-like processes of [Bas88]. With a different method the authors prove a transience criterion and the existence of local times for these kind of processes in d dimensions for $d \in \mathbb{N}$.

In [Hoh94], Hoh considers operators of a similar form, but the starting point is a different representation of the operator. The author studies the operator as a pseudo-differential operator of the form

$$Lf(x) = -p(x, D)f(x) = -(2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi} p(x, \xi) \cdot \widehat{f}(\xi) \, \mathrm{d}\xi$$

for $f \in C_0^\infty(\mathbb{R}^d)$, where for any fixed $x \in \mathbb{R}^d$, $p(x, \cdot)$ is negative definite. In the paper, uniqueness of the martingale problem for pseudo-differential operators with the symbol $p(x, \xi)$ of the form

$$p(x, \xi) = -\sum_{i=1}^{d} b_i(x) a_i(\xi)$$

is studied, where b_i, $i \in \{1, \dots, d\}$ are non-negative, bounded and $d + m$ times continuously differentiable for some $m \in \mathbb{N}$ and a_i, $i \in \{1, \dots, d\}$ are continuous non-negative definite with $a_i(0) = 0$. This covers for example symbols of the type

$$\sum_{i=1}^{d} b_i(x) |\xi_j|^{\alpha_j}$$

for $\alpha_j \in (0, 2]$ with the already mentioned conditions on b_i.

In [BC06] the authors study the system of stochastic differential equations of the form (1.0.1), where the driving process $Z_t = (Z_t^1, \dots, Z_t^d)$ consists of d independent copies of a one-dimensional symmetric stable process of index $\alpha \in (0, 2)$. Hence the order of differentiability is the same in every direction, which is the main difference to our model. In the article the authors prove existence and well-posedness of the martingale problem for

$$\mathcal{L}f(x) = \sum_{j=1}^{d} \int_{\mathbb{R}\setminus\{0\}} \left(f(x + a_j(x)w) - f(x) - w \mathbb{1}_{|w|\leq 1} \nabla f(x) \cdot a_j(x) \right) \frac{c_\alpha}{|w|^{1+\alpha}} \, \mathrm{d}w,$$

where $f \in C_b^2(\mathbb{R}^d)$ and $a_j(x)$ denotes the j^{th} column of $A(x)$ and show that this is equivalent to existence and uniqueness of solutions to (1.0.1). The proof of the existence of a weak solution to (1.0.1) is not problematic. The hard part is the proof of the well-posedness of the martingale problem for $\mathcal{L}$. This operator is a pseudo-differential operator of the form

$$\mathcal{L}f(x) = -\int_{\mathbb{R}^d} p(x, \xi) e^{-ix\cdot\xi} \widehat{f}(\xi) \, \mathrm{d}\xi,$$

where

$$p(x, \xi) = -c_\alpha \sum_{i=1}^{d} |\xi \cdot a_j(x)|^\alpha.$$

Because of the lack of differentiability, this symbol does not fit into the set-up of [Hoh94]. The authors' central idea is the usage of a perturbation argument as in [SV79]. The main part of the paper is the proof of an L^p-boundedness result of pseudo-differential operators whose symbols have the form

$$-\frac{|a(x)\cdot\xi|^\alpha}{\sum_{i=1}^d |\xi_i|^\alpha},$$

where $a:\mathbb{R}^d\to\mathbb{R}^d$ is continuous and bounded with respect to the Euclidean norm from above and below by positive constants. The proof of this L^p-boundedness follows from a result by Calderon and Zygmund, see [CZ56]. Therefore, the main difficulty is to show that the operator fits into the set-up of [CZ56], which is done with the method of rotations.

In [BC10], Bass and Chen study the same system of stochastic differential equations and prove Hölder regularity of harmonic functions with respect to $\mathcal{L}$. Furthermore, they give a counter example and thus show that the Harnack inequality for harmonic functions is not fulfilled.

Our consideration of the system (1.0.1) driven by the anisotropic Lévy process leads to the operator

$$\mathcal{L}f(x) = \sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} \left(f(x+a_j(x)w) - f(x) - w\mathbb{1}_{|w|\leq 1}\nabla f(x)\cdot a_j(x)\right) \frac{c_{\alpha_j}}{|w|^{1+\alpha_j}}\,\mathrm{d}w. \tag{1.0.3}$$

The first main result in this thesis is the existence of solutions to (1.0.1). We prove the existence of weak solutions without adding any additional assumptions on the coefficients.

Theorem (c.f. Theorem 5.0.3)**.** *Let $x\mapsto A_{ij}(x)$ be bounded and continuous for every $i,j\in\{1,\dots,d\}$. Then there exists a weak solution to* (1.0.1).

In order to prove the uniqueness we restrict ourselves to matrices whose entries are zero outside the diagonal. Thus we have to prove an L^p-boundedness result for pseudo-differential operators $\mathcal{B}$ of the form

$$\mathcal{B}f(x) = \int_{\mathbb{R}^d} \left(\sum_{k=1}^d \frac{c_{\alpha_k}|A_{kk}(x)\xi_k|^{\alpha_k}}{\sum_{i=1}^d |C_i\xi_i|^{\alpha_i}}\right) e^{-ix\cdot\xi}\widehat{f}(\xi)\,\mathrm{d}\xi$$

for some constants $C_i\neq 0$ to apply the perturbation argument as in [SV79].

The main ingredient in the proof of the L^p-bound for the perturbation operator $\mathcal{B}$ is a Fourier multiplier theorem which goes back to Bañuelos and Bogdan, see [BnB07, Theorem 1]. To apply this multiplier theorem we have to show that the perturbation operator $\mathcal{B}$ is an operator on $L^2(\mathbb{R}^d)$ with

$$\widehat{\mathcal{B}f}(\xi) = \frac{\int_{\mathbb{R}^d}(\cos(\xi\cdot z)-1)\phi(z)\,V(\mathrm{d}z)}{\int_{\mathbb{R}^d}(\cos(\xi\cdot z)-1)\,V(\mathrm{d}z)}\widehat{f}(\xi)$$

for a measurable and bounded function $\phi : \mathbb{R}^d \to \mathbb{C}$ and a positive Lévy measure V. This allows us to prove the well-posedness of the martingale problem for $\mathcal{L}$.

Theorem (c.f. Theorem 6.0.1)**.** *Suppose A satisfies $A_{ij} \equiv 0$ for $i \neq j$, $x \mapsto A_{jj}(x)$ is bounded continuous for all $j \in \{1, \dots, d\}$ and $A(x)$ is non-degenerate for any $x \in \mathbb{R}^d$. For every $x_0 \in \mathbb{R}^d$, there is a unique solution to the martingale problem for $\mathcal{L}$ started at $x_0 \in \mathbb{R}^d$.*

This anisotropic system (1.0.1) has been studied in [Cha16], where harmonic functions are shown to satisfy a Hölder estimate.

Regularity estimates for anisotropic nonlocal equations

The second subject we treat in this thesis is the study of regularity estimates for a class of nonlocal operators whose kernels are anisotropic.

For given $\alpha_1, \dots, \alpha_d \in (0, 2)$ we consider the family of measures $\mu_{\text{axes}}(x, \cdot)$ defined as follows

$$\mu_{\text{axes}}(x, \mathrm{d}y) = \sum_{k=1}^{d} \left(\alpha_k (2 - \alpha_k) |x_k - y_k|^{-1-\alpha_k} \, \mathrm{d}y_k \prod_{i \neq k} \delta_{\{x_i\}}(\mathrm{d}y_i) \right), \quad x \in \mathbb{R}^d.$$

In the case $x = 0$ the measure $\mu_{\text{axes}}(0, \mathrm{d}y)$ coincides with the Lévy measure $\nu(\mathrm{d}y)$ from (1.0.2) up to the constants. This family plays the role of the reference family for our considerations. In order to prove local results we need to define an appropriate metric on $\mathbb{R}^d$ such that the different orders of differentiability along the coordinate axes get compensated. We define a metric on $\mathbb{R}^d$

$$d(x, y) := \sup_{k \in \{1, \dots, d\}} \left\{ |x_k - y_k|^{\alpha_k/2} \mathbb{1}_{\{|x_k - y_k| \le 1\}}(x, y) + \mathbb{1}_{\{|x_k - y_k| > 1\}}(x, y) \right\}.$$

For radii $r \in (0, 1]$ and $x \in \mathbb{R}^d$, balls in the metric space $(\mathbb{R}^d, d)$ have the form

$$B_r^d(x) = \{y \in \mathbb{R}^d \colon d(x, y) < r\} = \bigtimes_{k=1}^{d} \left(x_k - r^{\frac{2}{\alpha_k}}, x_k + r^{\frac{2}{\alpha_k}} \right) =: M_r(x).$$

We use for brevity the notation $M_r := M_r(0)$.

Consider a family of measures $\mu(x, \cdot)$, $x \in \mathbb{R}^d$ with certain properties which we will not discuss at this point. The detailed assumptions on the family of measures can be found in Chapter 7. The families $(\mu(x, \cdot))_{x \in \mathbb{R}^d}$ and $(\mu_{\text{axes}}(x, \cdot))_{x \in \mathbb{R}^d}$ are supposed to have the following relation.

For every $r \in (0, 1]$, $x_0 \in M_1$ and $u \in L^2(M_r(x_0))$ with

$$\int_{M_r(x_0)} \int_{M_r(x_0)} (u(x) - u(y))^2 \, \mu(x, \mathrm{d}y) \, \mathrm{d}x \\ + 2 \int_{M_r(x_0)^c} \int_{M_r(x_0)} (u(x) - u(y))^2 \, \mu(x, \mathrm{d}y) \, \mathrm{d}x < \infty$$

there is a constant $c_1 \geq 1$, independent of r, x_0, u and $\alpha_1, \dots, \alpha_d$, such that

$$c_1^{-1} \mathcal{E}^{\mu}_{M_r(x_0)}(u,u) \leq \mathcal{E}^{\mu_{\text{axes}}}_{M_r(x_0)}(u,u) \leq c_1 \mathcal{E}^{\mu}_{M_r(x_0)}(u,u),$$

where

$$\mathcal{E}^{\mu}_{M_r(x_0)}(u,u) = \int_{M_r(x_0)} \int_{M_r(x_0)} (u(x) - u(y))^2 \, \mu(x, \mathrm{d}y) \, \mathrm{d}x.$$

We consider operators of the form

$$\mathcal{L}u(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^d \setminus B_\epsilon(x)} (u(y) - u(x)) \, \mu(x, \mathrm{d}y),$$

where u is chosen from a suitable function space. The aim is to study weak solutions to

$$\mathcal{L}u = f \qquad \text{in } M_r(x) \tag{1.0.4}$$

for sufficiently smooth functions f and prove Hölder continuity for weak solutions to $\mathcal{L}u = 0$ in M_1.

Let us first put the problem in historical context and refer to some selected results in the literature.

In the nineteen fifties, De Giorgi [DG57] and Nash [Nas57] independently prove an a priori Hölder estimate for weak solutions u to second order equations of the form

$$\operatorname{div}(A(x) \nabla u(x)) = 0$$

for uniformly elliptic and measurable coefficients A. In [Mos61], Moser proves Hölder continuity of weak solutions and gives a proof of an elliptic Harnack inequality for weak solutions to this equation. This article provides a new technique of how to derive an a priori Hölder estimate from the Harnack inequality. For a large class of local operators, the Hölder continuity can be derived from the Harnack inequality, see for instance [GT01]. For a comprehensive introduction into Harnack inequalities, we refer the reader e.g. to [Kas07b].

The corresponding case of operators in non-divergence form is treated in [KS79]. The authors develop a technique for proving Hölder regularity and the Harnack inequality for harmonic functions corresponding to non-divergence form elliptic operators. They take a probabilistic point of view and make use of the martingale problem to prove regularity estimates of harmonic functions. The main tool is a support theorem, which gives information about the topological support for a solution to the martingale problem associated to the operator.

For the Harnack inequality there are significant differences between the case of local and nonlocal operators. In the case of nonlocal operators, the Harnack inequality does not hold under just local assumptions on the function. Kaßmann shows in [Kas07a] that one needs to assume nonnegativity of the function on the whole space to prove the Harnack inequality.

The study of regularity estimates for harmonic functions corresponding to nonlocal operators is an intensely studied topic. There has been great progress in the last decades.

We start by referring to regularity estimates for integro-differential operators in non-divergence form.

In [BL02a] Bass and Levin consider operators of the form

$$Lu(x) = \int_{\mathbb{R}^d\setminus\{0\}} (u(x+h) - u(x) - \mathbb{1}_{|h|\leq 1} h \cdot \nabla u(x))a(x,h)\, \mathrm{d}h. \tag{1.0.5}$$

They study harmonic functions with respect to the operator $\mathcal{L}$, provided that $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is symmetric in the second variable and $a(x,h) \asymp |h|^{-d-\alpha}$ for all $x, h \in \mathbb{R}^d$, where $\alpha \in (0,2)$. Using probabilistic techniques the authors prove a Harnack inequality for nonnegative bounded harmonic functions. Furthermore, they derive Hölder continuity for bounded and harmonic functions from the Harnack inequality. The results of this work have been extended to more general kernels by many authors. For instance, in [BK05b] the authors establish a Hölder estimate for harmonic functions to the operator L, where they replace the jump measure $a(x,h)\,\mathrm{d}h$ by $n(x,\mathrm{d}h)$, which is not required to have a density with respect to the Lebesgue meaure. Furthermore, in [BK05a] a scale dependent Harnack inequality for harmonic functions to the operator L is proven, where again no density with respect to the Lebesgue measure is required. Song and Vondraček extend in [SV04] the method of [BL02a] to prove the Harnack inequality for more general classes of Markov processes. Silvestre provides in [Sil06] a purely analytical proof of Hölder continuity for harmonic functions with respect to a class of integro differential equations given by (1.0.5), where no symmetry on the kernel a is assumed. Caffarelli and Silvestre study in [CS09] viscosity solutions to fully nonlinear integro-differential equations and prove a nonlocal version of the Aleksandrov-Bakelman-Pucci estimate, a Harnack inequality and a Hölder estimate. There are many important results concerning Hölder estimates and the Harnack inequality for integro-differential equations in non-divergence form including [BCI11], [LD14], [KRS14] and [WZ15]. Heat kernel estimates to nonlocal operators including perturbation of lower order can be found in [CZ16a].

Because we consider in this thesis nonlocal operators in divergence form, we keep the survey on non-divergence form operators short. It contains just a few references and is not complete at all. For details to the results, we refer the reader to the respective articles.

Let us now turn to some results on nonlocal operators in divergence form.

Bass and Levin obtain in [BL02b] sharp transition probability estimates for Markov chains on the integer lattice and prove a Harnack inequality. Chen and Kumagai provide a general approach in [CK03]. They extend the results of [BL02b] to d-sets (F,π), which is a general class of state spaces. The authors consider Dirichlet forms on $L^2(F,\pi)$ and show that there are associated Feller processes. Moreover, they establish estimates on hitting probabilities and prove Hölder continuity of the transition density functions. Barlow, Bass, Chen and Kassmann study in [BKK10] rather general symmetric pure jump Markov processes with the corresponding Dirichlet forms $(\mathcal{E},\mathcal{F})$ and prove a scale dependent parabolic Harnack inequality for nonnegative functions solving the heat equation with respect to $\mathcal{E}$. In the spirit of De Giorgi's approach, Caffarelli, Chan and Vasseur establish in [CCV11] regularity results of weak solutions for nonlocal evolutionary equations. Further contributions to the theory have been made in [CK10], [CKK11] and [CS16]. Note that heat kernel estimates, regularity results and Harnack inequalities have been studied for quite general Dirichlet forms in metric measure spaces. We refer to [BGK09], [GHL15] and [CKW17] for further references.

As already mentioned, one needs to assume nonnegativity of the function on the whole space in order to prove the Harnack inequality. It is an interesting question if one can prove a modified Harnack inequality without the assumption on the nonnegativity on the whole space. In [Kas11] the author introduces a new formulation of the Harnack inequality, where one does not assume nonnegativity on the whole space but needs to add a natural tail term on the right hand side, which compensates for the nonlocality of the operator.

We now discuss the main results and techniques used in Part III of this thesis. Kaßmann extends in [Kas09] the De Giorgi-Nash-Moser theory to nonlocal integro-differential operators given by

$$Lu(x) = 2 \lim_{\epsilon \to 0} \int_{\mathbb{R}^d \setminus B_\epsilon(x)} (u(y) - u(x))\, k(x,y)\, \mathrm{d}y$$

for a nonnegative kernel $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ and $\alpha \in (0,2)$. The author assumes for $|x-y| \leq 1$ that $k(x,y) \asymp |x-y|^{-d-\alpha}$ and establishes a Moser iteration scheme leading to a weak Harnack inequality and Hölder regularity estimates with purely analytic methods. In the article [DK15] Kaßmann and Dyda follow the approach of [Kas09] and provide a general tool for the derivation of a priori Hölder estimates for weak solutions with the help of the weak Harnack inequality. The authors study weak solutions to a large class of nonlocal equations, that allows to consider operators of the form

$$Lu(x) = 2 \lim_{\epsilon \to 0} \int_{\mathbb{R}^d \setminus B_\epsilon(x)} (u(y) - u(x))\, \nu(x, \mathrm{d}y),$$

where $\nu(x,\cdot)$ is a family of measures, that does not even have to posses a density with respect to the Lebesgue measure. One main assumption is a local comparability condition of the corresponding energy forms for L and the fractional Laplacian of order $\alpha/2$ for some $\alpha \in (0,2)$. This assumption is quite general and allows for instance, families as $\mu_{\text{axes}}(x,\cdot)$ for $\alpha_1 = \cdots = \alpha_d = \alpha \in (0,2)$.

In this thesis we follow the strategy of [DK15] and derive an a priori Hölder estimate for weak solutions to (1.0.4) using the weak Harnack inequality. Let $\Omega \subset \mathbb{R}^d$ be open. To put the problem into a functional analytic framework, we need to define appropriate function spaces. We define weak solutions with the help of symmetric nonlocal bilinear forms. The space of test functions consists of all functions $u \in L^2(\Omega)$ with $u \equiv 0$ on Ω^c and

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(x))^2\, \mu(x, \mathrm{d}y)\, \mathrm{d}x < \infty.$$

This space is denoted by $H^{\mu}_{\Omega}(\mathbb{R}^d)$. Solutions are defined on the space $V^{\mu}(\Omega|\mathbb{R}^d)$ which consist of all functions $u \in L^2(\Omega)$ with

$$\int_{\Omega}\int_{\mathbb{R}^d} (u(y)-u(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x < \infty.$$

To obtain the weak Harnack inequality, we have to derive some functional inequalities for our bilinear forms such as a localized Sobolev-type inequality and a Poincaré inequality for functions from the space of solutions. The result that we obtain is the weak Harnack inequality for weak supersolutions to (1.0.4).

Theorem (c.f. Theorem 8.1.5)**.** *Let $f \in L^q(M_1)$ for some $q > \max\{2, \sum_{k=1}^d \frac{1}{\alpha_k}\}$. Let $u \in V^{\mu}(M_1\big|\mathbb{R}^d)$, $u \geq 0$ in M_1 satisfy*

$$\mathcal{E}(u,\phi) \geq (f,\phi) \quad \text{for every non-negative } \phi \in H^{\mu}_{M_1}(\mathbb{R}^d). \tag{1.0.6}$$

Then there exists $p_0 \in (0,1)$, $c_1 > 0$, independent of u, such that

$$\inf_{M_{\frac14}} u \geq c_1 \left(\fint_{M_{\frac12}} u(x)^{p_0}\,\mathrm{d}x \right)^{1/p_0} - \sup_{x\in M_{\frac{15}{16}}} 2\int_{\mathbb{R}^d\setminus M_1} u^-(z)\mu(x,\mathrm{d}z) - \|f\|_{L^q(M_{\frac{15}{16}})}.$$

The two main ingredients in the proof of the a priori Hölder estimate for weak solutions are the weak Harnack inequality and a decay of oscillation for weak solutions. From these two estimates we can deduce the Hölder estimate for weak solutions, which is the main result in this part of the thesis.

Theorem (c.f. Theorem 9.0.3)**.** *Assume $u \in V^{\mu}(M_1\big|\mathbb{R}^d)$ satisfies*

$$\mathcal{E}(u,\phi) = 0 \quad \text{for every non-negative } \phi \in H^{\mu}_{M_1}(\mathbb{R}^d).$$

Then there are $c_1 \geq 1$ and $\delta \in (0,1)$, independent of u, such that the following Hölder estimate holds for almost every $x, y \in M_{\frac12}$

$$|u(x)-u(y)| \leq c_1\|u\|_{\infty}|x-y|^{\delta}. \tag{1.0.7}$$

Outline

The thesis is divided into three parts.

The first part consists of Chapter 2 and Chapter 3. In Chapter 2 we summarize the required facts on integrable spaces. Chapter 3 reviews the relevant material on probability theory. In this part we omit proofs and give detailed references to the literature. We focus on the results which are important in the scope of this thesis.

The system of stochastic differential equations is investigated in Part II. This part is divided into three chapters. The first chapter provides a detailed exposition of the objects in this part and contains proofs of some auxiliary results. In the second chapter we prove the existence of a solution to the system. The uniqueness of solutions to the system is proved in the third chapter.

In Part III regularity of weak solutions to a class of nonlocal equations is studied. The part is split into three chapters and one appendix chapter. The first chapter contains important definitions and auxiliary results. In the second chapter the weak Harnack inequality for supersolutions to a inhomogenuous integro-differential equation is derived and in the third chapter we prove an a priori Hölder estimate for weak solutions. Appendix A contains some examples of families of measures, which are defined in Part III.

Notation

Unless otherwise specified, we will use $\mathbb{R}^d$ to denote the d-dimensional Euclidean space, equipped with the scalar product $\cdot$ and the Euclidean norm $|\cdot|$. The space is endowed with the Borel σ-field $\mathcal{B}(\mathbb{R}^d)$ and the Lebesgue measure $\mathrm{d}x$. For a set $A \subset \mathbb{R}^d$ we denote its closure by $\overline{A}$ and use $|A|$ to denote its Lebesgue measure. The complement of a set $A \subset \mathbb{R}^d$ is denoted by $\mathbb{R}^d \setminus A$. The characteristic function of a set $A \subset \mathbb{R}^d$ is symbolized by $\mathbb{1}_A$. Furthermore, we denote the ball in $\mathbb{R}^d$ with center $x \in \mathbb{R}^d$ and radius $r > 0$ by $B_r(x)$ and in the case $x = 0$, for abbreviation we set $B_r(0) = B_r$.

The ball in a metric space (M, d) with center $x \in M$ and radius $r > 0$ is denoted by $B_r^d(x)$. Again, we write $B_r^d(0) = B_r^d$.

Let V be a vector space. We denote its dual space by V^*. For $x \in V$ and $y \in V^*$ we denote the dual pairing by $\langle x, y \rangle := y(x)$.

Let $\Omega \subset \mathbb{R}^d$. For two functions $f, g : \Omega \to \mathbb{R}_+$, we write $f \asymp g$, if there is a constant $c > 0$ such that

$$\frac{1}{c} f(x) \le g(x) \le c f(x) \quad \text{for all } x \in \Omega.$$

For $\alpha \in \mathbb{N}_0^d$ let

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} \dots \partial^{\alpha_n}},$$

where $|\alpha| = \alpha_1 + \dots + \alpha_d$.

Let $\Omega \subset \mathbb{R}^d$ be open and $k \in \mathbb{N}$. We define

$$\begin{aligned}
C(\Omega) &:= \{f : \Omega \to \mathbb{R} \colon f \text{ is continuous } \},\\
C^k(\Omega) &:= \{f : \Omega \to \mathbb{R} \colon \partial^\alpha f \in C(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \le k\},\\
C_b(\Omega) &:= \{f : \Omega \to \mathbb{R} \colon f \text{ is continuous and bounded}\},\\
C_b^k(\Omega) &:= \{f : \Omega \to \mathbb{R} \colon \partial^\alpha f \in C_b(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \le k\},\\
C_0(\Omega) &:= \{f : \Omega \to \mathbb{R} \colon f \in C(\Omega) \text{ and } \forall \epsilon > 0 \exists K \subset \Omega \text{ compact s.t. } |f(x)| < \epsilon \ \forall x \in K^c\},\\
C_0^k(\Omega) &:= \{f : \Omega \to \mathbb{R} \colon \partial^\alpha f \in C_0(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \le k\},\\
C_c(\Omega) &:= \{f : \Omega \to \mathbb{R} \colon f \text{ is continuous and has compact support}\},\\
C_c^k(\Omega) &:= \{f : \Omega \to \mathbb{R} \colon \partial^\alpha f \in C_c(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \le k\}.
\end{aligned}$$

Let

$$C^\infty(\Omega) = \bigcap_{k=0}^\infty C^k(\Omega) \quad \text{and} \quad C_c^\infty(\Omega) = \bigcap_{k=0}^\infty C_c^k(\Omega).$$

A function $f \in C^\infty(\Omega)$, is called Schwartz function, if for any $\alpha, \beta \in \mathbb{N}_0^d$

$$\rho_{\alpha,\beta}(f) := \sup_{x \in \Omega} |x^\alpha \partial^\beta f(x)| < \infty,$$

where $x^\alpha = \prod_{i=1}^d x_i^{\alpha_i}$. We denote the space of all Schwartz functions on Ω by $\mathcal{S}(\Omega)$. The space $\mathcal{S}(\mathbb{R}^d)$ is called Schwartz space. Note that by definition all derivatives of such functions multiplied with any polynomial stay bounded.

For an integrable function $f : \mathbb{R}^d \to \mathbb{R}$, we define its Fourier transform $\widehat{f} = \mathcal{F}f$ by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{ix\cdot\xi} \, \mathrm{d}x. \quad \xi \in \mathbb{R}^d.$$

We denote the inverse Fourier transform of f by $\mathcal{F}^{-1} f$.

We use the letter c with subscripts for positive constants whose exact values are not important and we write $c_i = c_i(\cdot)$ if we want to highlight all the quantities the constant depends on.

Abgrenzung des eigenen Beitrags gemäß §10(2) der Promotionsordnung

Lemma 7.1.1 was developed in collaboration between the author and his supervisor. The proof of Lemma 7.1.1 will appear in the published version of [DK15].

Acknowledgements

I would like to thank my advisor Prof. Dr. Moritz Kaßmann for his confidence, encouragement, and persistent help. It is always a joy to have mathematical and non-mathematical conversations with him. Without his guidance this thesis would not have been materialized.

Special thanks also go to my colleagues Dr. Timothy Candy, Dr. Bartłomiej Dyda, Dr. Martin Friesen, Andrea Nickel, Tim Schulze, Dr. Karol Szczypkowski, Dr. Paul Voigt and André Wilke for many stimulating conversations and a pleasant working environment. Their advices and comments have always been a great help.

I would like to thank Dr. Armin Schikorra for stimulating conversations concerning Sobolev-type inequalities.

Special thanks go to my family and my girlfriend Anastasia Kerbs for their constant encouragement, love and support.

Financial support by the German Science Foundation DFG (SFB 701) is gratefully acknowledged.

Part I.

Basics

Structure of Part I

This part reviews some of the standard facts from analysis and probability theory. We set down our notation and introduce the basic vocabulary, which will be needed in the scope of this thesis.

The aim of this part is to acquaint the reader with the required definitions and facts, which will be needed in this thesis. It is not our purpose to give a complete theory. We only touch specific aspects of the theories and restrict our attention to those results, which are relevant for this thesis. In this part we will omit proofs but give references to the literature in the beginning of each section.

The part is divided into two chapters. Chapter 2 consists of four sections and reviews facts on spaces of integrable functions. In Chapter 3 we present results from basic probability theory and the theory of stochastic integration. It is split into three sections.

2. Analytic Basics

2.1. Lebesgue spaces

Let $(M, \mathcal{M}, \mu)$ be a measure space. A function $f : M \to \mathbb{R}$ is called measurable if for all $c \in \mathbb{R}$

$$\{x \in M \colon f(x) \leq c\} \in \mathcal{M}.$$

For $p \in [1, \infty]$ let

$$\mathcal{L}^p(M, \mu) = \{f : M \to \mathbb{R} \colon f \text{ is measurable and } \|f\|_{L^p(M,\mu)} < \infty\},$$

where

$$\begin{aligned} \|f\|_{L^p(M,\mu)} &:= \|f(x)\|_{L^p_x(M,\mu)} := \left(\int_M |f(x)|^p \, \mu(\mathrm{d}x)\right)^{1/p}, \qquad \text{if } 1 \leq p < \infty, \\ \|f\|_{L^\infty(M,\mu)} &:= \inf\{c \geq 0 \colon |f(x)| \leq c \text{ for almost every } x \in M\}, \end{aligned} \tag{2.1.1}$$

where the integral in (2.1.1) is the Lebesgue integral.

Note that $(\mathcal{L}^p(M, \mu), \|\cdot\|_{L^p(M,\mu)})$ is not a normed vector space, since $\|f\|_{L^p(M,\mu)} = 0$ does not imply $f \equiv 0$. In fact, $(\mathcal{L}^p(M, \mu), \|\cdot\|_{L^p(M,\mu)})$ is a seminormed vector space due to Minkowski's inequality

$$\|f + g\|_{L^p(M,\mu)} \leq \|f\|_{L^p(M,\mu)} + \|g\|_{L^p(M,\mu)}.$$

Since for measurable functions $\|f\|_p = 0$ if and only if $f = 0$ almost everywhere, it would be desirable to identify two function f and g if they coincide almost everywhere. To do this, we consider the quotient space.

Definition 2.1.1. *Let $p \in [1, \infty]$. The Lebesgue space $L^p(M, \mu)$ is the quotient space*

$$L^p(M, \mu) := \mathcal{L}^p(M, \mu)/\ker = \mathcal{L}^p(M, \mu)/\{f \in \mathcal{L}^p(M, \mu) \colon \|f\|_{L^p(M,\mu)} = 0\},$$

endowed with the Lebesgue norm $\|\cdot\|_{L^p(M,\mu)}$ from (2.1.1).

The space $(L^p(M, \mu), \|\cdot\|_{L^p(M,\mu)})$ is a normed vector space and by the Riesz–Fischer theorem it is complete for every $1 \leq p \leq \infty$.

We simply write $L^p(M)$ or L^p instead of $L^p(M, \mu)$ and $\|\cdot\|_p$ or $\|\cdot\|_{L^p(M)}$ instead of $\|\cdot\|_{L^p(M,\mu)}$ when no confusion can arise.

Consider for the moment the special case $(M, \mathcal{M}, \mu) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathrm{d}x)$ and let $\Omega \subset \mathbb{R}^d$ be open. The space $L^2(\Omega)$ is a separable Hilbert space with the inner product

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x) g(x) \,\mathrm{d}x.$$

Recall for $f \in L^1(\mathbb{R}^d)$ the definition of the Fourier transform

$$\mathcal{F}f(\xi) := \widehat{f}(\xi) := \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(x) \,\mathrm{d}x, \quad \xi \in \mathbb{R}^d.$$

The next theorem is an extension result for the Fourier transform to $L^2(\mathbb{R}^d)$.

Theorem 2.1.2 (Plancherel's theorem)**.** *Let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Then $\widehat{f} \in L^2(\mathbb{R}^d)$ and*

$$\|\widehat{f}\|_2 = \|f\|_2.$$

Further the mapping $f \mapsto \widehat{f}$ has a unique extension to a linear isometric map from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. For $f, g \in L^2(\mathbb{R}^d)$

$$(f, g)_{L^2(\mathbb{R}^d)} = (\widehat{f}, \widehat{g})_{L^2(\mathbb{R}^d)}. \tag{2.1.2}$$

Equation (2.1.2) is called Parseval's identity.

We summarize without proofs some important properties on $L^p(\mathbb{R}^d)$-spaces. For the proofs see for instance [AE01] or [Gra14a].

One of the most important inequalities in the theory of L^p-spaces is due to Hölder.

Theorem 2.1.3 (Hölder's inequality)**.** *Let $p, q \in [1, \infty]$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{2.1.3}$$

Let $f \in L^p(M)$ and $g \in L^q(M)$. Then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

A generalization of Hölder's inequality is given by the next corollary.

Corollary 2.1.4 (Generalized Hölder's inequality)**.** *Let $n \in \mathbb{N}$. Let $r \in [1, \infty)$ and $p_1, \dots, p_n \in [1, \infty]$ such that*

$$\sum_{k=1}^n \frac{1}{p_k} = \frac{1}{r}.$$

Let $f_k \in L^{p_k}(M)$ for all $k \in \{1, \dots, n\}$. Then $\left(\prod_{k=1}^d f_k\right) \in L^r(M)$ and

$$\left\| \prod_{k=1}^n f_k \right\|_r \leq \prod_{k=1}^n \|f_k\|_{p_k}.$$

Given $p, q \in [1, \infty]$ such that (2.1.3) holds, then q is said to be the Hölder conjugate of p and vice versa. In the special case $p = q = 2$ Hölder's inequality gives the integral formulation of the Cauchy-Schwarz inequality. Using Hölder's inequality, one can deduce that the dual space of $L^p(M)$ for $1 \le p < \infty$ is given by $L^q(M)$, where q is the Hölder conjugate of p. Note that this is not true for $p = \infty$.

Let $f \in L^p(M)$. If M has finite measure, Hölder's inequality implies

$$\|f\|_p \le \mu(M)^{\frac{1}{p}-\frac{1}{q}} \|f\|_q,$$

i.e. $L^p \subset L^q$, whenever $1 \le q \le p$ and $\mu(M) < \infty$.

Another conclusion of Hölder's inequality is the following interpolation inequality.

Corollary 2.1.5 (Lyapunov's inequality)**.** *Let $1 \le q \le r \le p$ such that $p \neq q$ and define*

$$\theta := \frac{q(p-r)}{r(p-q)}.$$

Let $f \in L^p(M) \cap L^q(M)$. Then $f \in L^r(M)$ and

$$\|f\|_r \le \|f\|_p^{1-\theta} \|f\|_q^{\theta}.$$

We now define the weak $L^p(M, \mu)$-space as follows.

Definition 2.1.6. *Let $p \in [1, \infty)$. The space*

$$L^p_w(M, \mu) := \{f : M \to \mathbb{R} \colon f \textit{ is measurable and } \|f\|_{p,weak} < \infty\}$$

is called weak L^p-space, where

$$\|f\|_{p,weak} := \sup_{t>0} \left(t\mu(\{x \in M \colon |f(x)| > t\})^{1/p} \right).$$

We emphasize that $\|\cdot\|_{p,\text{weak}}$ is not a norm, since the triangle inequality does not hold. Moreover, for any $f \in L^p(M, \mu)$

$$\|f\|_{p,\text{weak}} \le \|f\|_p,$$

which implies $L^p(M, \mu) \subset L^p_w(M, \mu)$.

2.2. Lorentz spaces

In the section we will introduce a generalization of the Lebesgue spaces. This presentation summarizes the definitions and results in [Gra14a, Section 1.4.2] and [Tar07] .

We start with the notion of the decreasing rearrangement function. Let $f : M \to \mathbb{R}$ be a measurable function. Its decreasing rearrangement f^* is defined as

$$f^* : [0, \infty) \to [0, \infty], \quad f^*(t) = \inf\{s \ge 0 \colon d_f(s) \le t\},$$

where $d_f(s) = \mu(\{x \in M\colon |f(x)| > s\})$ is the distribution function of f.

One important result for decreasing rearrangement functions is due to Hardy and Littlewood.

Theorem 2.2.1 (Hardy–Littlewood inequality)**.** *Let $f, g \in C_0(\Omega)$ be nonnegative functions. Then*

$$\int_{\mathbb{R}^d} f(x)g(x)\,\mathrm{d}x \le \int_0^\infty f^*(t)g^*(t)\,\mathrm{d}t.$$

Definition 2.2.2. *Let $p \in [1,\infty]$ and $q \in [1,\infty]$. We define the Lorentz space as follows*

$$L^{p,q}(M,\mu) := \{f : M \to \mathbb{R}\colon f \textit{ is measurable and } \|f\|_{L^{p,q}(M,\mu)} < \infty\},$$

where

$$\begin{aligned}
\|f\|_{L^{p,q}(M)} &= \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q}, && \textit{if } p, q < \infty,\\
\|f\|_{L^{p,\infty}(M)} &= \sup_{t>0} \left(t^{1/p} f^*(t)\right), && \textit{if } p < \infty, q = \infty,\\
\|f\|_{L^{\infty,\infty}(M)} &= \|f\|_{L^\infty(M)}.
\end{aligned}$$

Again, we identify two functions if they coincide almost everywhere. We should emphasize that $(L^{p,q}(M,\mu), \|\cdot\|_{L^{p,q}(M,\mu)})$ is not a normed space, whenever $p \neq q$. From Cavalieri's principle one can deduce, that for any $p \in [1,\infty]$, $L^{p,p}(M,\mu) = L^p(M,\mu)$. Furthermore, it is easy to see from the definition that for any $p \in [1,\infty)$, $L^{p,\infty}(M,\mu) = L^p_w(M,\mu)$.

In the following we will summarize some required properties of Lorentz spaces.

To shorten notation, we omit the regarding measure space in the designation of Lorentz spaces if no confusion can arise. We write $\|\cdot\|_{p,q}$ instead of $\|\cdot\|_{L^{p,q}(M,\mu)}$ if the underlying space is clearly known.

Applying the Hardy-Littlewood inequality and Hölder's inequality for Lebesgue functions, we achieve the following Hölder's inequality for Lorentz functions.

Theorem 2.2.3 (Hölder's inequality)**.** *Let $p, q \in [1,\infty]$ and p', q' their Hölder conjugates. If $f \in L^{p,q}(M,\mu)$ and $g \in L^{p',q'}(M,\mu)$, then $(fg) \in L^1(M,\mu)$ and*

$$\|fg\|_{L^1(M,\mu)} \le \|f\|_{L^{p,q}(M,\mu)} \|g\|_{L^{p',q'}(M,\mu)}.$$

Furthermore, we have the following embedding result for Lorentz spaces.

Theorem 2.2.4. *Let $p, q, r \in [1,\infty]$. Assume $q < r$ and $f \in L^{p,q}(M,\mu)$. Then $f \in L^{p,r}(M,\mu)$ and*

$$\|f\|_{L^{p,r}(M,\mu)} \le \left(\frac{q}{p}\right)^{1/q-1/r} \|f\|_{L^{p,q}(M,\mu)}.$$

Alternatively, Lorentz spaces can also be defined as real interpolation spaces between the spaces $L^1(M)$ and $L^\infty(M)$, see for instance [Tar07, Lemma 22.6]. This approach yields the following result.

Theorem 2.2.5 ([Sch12, Proposition 2.2])**.** *Let $p \in (2,\infty)$ and $q \in [1,\infty]$. Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then there is a $c_1 = c_1(p) > 0$ such that*

$$\|\mathcal{F}^{-1} f\|_{p,q} \leq c_1 \|f\|_{p',q}, \tag{2.2.1}$$

where p' is the Hölder conjugate of p.

Choosing $f = \widehat{u}$ and using the fact that the Fourier transform is an automorphism on the Schwartz space, (2.2.1) implies

$$\|u\|_{p,q} \leq c_1 \|\widehat{u}\|_{p',q},$$

where $p \in (2,\infty)$ and $q \in [1,\infty]$.

2.3. Sobolev spaces

This section is a quick review on Sobolev spaces. We consider the Euclidean space endowed with the Lebesgue measure, that is $M = \mathbb{R}^d, \mathcal{M} = \mathcal{B}(\mathbb{R}^d)$ and $\mu(\mathrm{d}x) = \mathrm{d}x$. Let $\Omega \subset \mathbb{R}^d$ be open. For a deeper discussion on the general theory of Sobolev spaces we refer the reader to [Maz11]. See [DNPV12] for a treatment of Sobolev spaces of fractional order.

A function $f : \Omega \to \mathbb{R}$ is called locally integrable, if

$$\int_\Omega |f\phi| \,\mathrm{d}x < \infty \quad \text{for all } \phi \in C_c^\infty(\Omega).$$

The set of all locally integrable functions on Ω is denoted by $L^1_{\mathrm{loc}}(\Omega)$. Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ and f be a locally integrable function on Ω. A function v is called weak derivative of f of order α, if

$$\int_\Omega f(x)\partial^\alpha \phi(x) \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega \phi(x) v(x) \,\mathrm{d}x \qquad \text{for all } \phi \in C_c^\infty(\Omega).$$

We denote the function v by $\partial^\alpha f$.

Definition 2.3.1. *Let $k \in \mathbb{N}_0$ and $p \in [1,\infty]$. We define the Sobolev space $W^{k,p}(\Omega)$ of integer order by*

$$W^{k,p}(\Omega) := \left\{ f \in L^p(\Omega) \colon \|f\|_{W^{k,p}(\Omega)} < \infty \right\},$$

where

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|\partial^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} \qquad \textit{for } p < \infty,$$
$$\|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^\infty(\Omega)}.$$

The space $W^{k,p}(\Omega)$ endowed with the norm $\|f\|_{W^{k,p}(\Omega)}$ is a Banach space for $1 \le p \le \infty$.

As mentioned in Section 2.1 $L^2(\Omega)$ is a separable Hilbert space. The same is true in the case of Sobolev spaces. We set for $k \in \mathbb{N}_0$

$$H^k(\Omega) = W^{k,2}(\Omega)$$

and define the inner product

$$(f,g)_{H^k(\Omega)} = \sum_{|\alpha|\le k} (\partial^\alpha f, \partial^\alpha g)_{L^2(\Omega)}.$$

This inner product on $H^k(\Omega)$ is induced by the inner product on $L^2(\Omega)$. This implies that the space $H^k(\Omega)$ is, endowed with $(\cdot,\cdot)_{H^k(\Omega)}$, a separable Hilbert space.

Let $W_0^{k,p}(\Omega)$ be the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ and as before, we set $H_0^k(\Omega) = W_0^{k,2}(\Omega)$. The dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$ and is a Banach space endowed with the norm

$$\|f\|_{H^{-1}(\Omega)} = \sup\{\langle f, u\rangle \colon u \in H_0^1(\Omega), \|u\|_{H^1(\Omega)}\}.$$

We proceed by defining Sobolev spaces of fractional order and explain their connection to a class of nonlocal operators. For a thorough treatment we refer the reader to [DNPV12].

Definition 2.3.2. *Let $s \in (0,1)$. We define the fractional Sobolev space $W^{s,p}(\Omega)$ by*

$$W^{s,p}(\Omega) = \left\{ f \in L^p(\Omega) \colon \frac{|f(x)-f(y)|}{|x-y|^{d/p+s}} \in L^p(\Omega\times\Omega) \right\}$$

endowed with the norm

$$\|f\|_{W^{s,p}(\Omega)} = \left(\|f\|^p_{L^p(\Omega)} + c(d,s) \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+ps}} \,\mathrm{d}x\,\mathrm{d}y \right)^{1/p}. \tag{2.3.1}$$

We set $H^s(\Omega) = W^{s,2}(\Omega)$ and define the scalar product on $H^s(\Omega)$ by

$$(f,g)_{H^s(\Omega)} = c(d,s) \int_\Omega \int_\Omega \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{d+2s}} \,\mathrm{d}x\,\mathrm{d}y.$$

For $f \in C_c^\infty(\mathbb{R}^d)$ we define

$$(-\Delta)^s f(x) := c(d,s) \lim_{\epsilon\to 0} \int_{\mathbb{R}^d \setminus B_\epsilon(x)} \frac{f(x)-f(y)}{|x-y|^{d+2s}} \,\mathrm{d}y, \qquad x \in \mathbb{R}^d.$$

This operator is called fractional Laplacian and is a nonlocal operator. This operator appears for instance naturally as the generator of the isotropic rotationally symmetric α-stable Lévy processes, where $\alpha = 2s$. The constant $c(d,s)$ is defined is defined such that the symbol of the fractional Laplacian $(-\Delta)^s$ is $|\xi|^{2s}$, that is for $u \in \mathcal{S}(\mathbb{R}^d)$

$$\mathcal{F}((-\Delta)^s f)(\xi) = |\xi|^{2s}\mathcal{F}(f).$$

Moreover, the fractional Laplacian is linked to Sobolev spaces of fractional order as follows

$$(f,f)_{H^s(\mathbb{R}^d)} = ((-\Delta)^s f, f)_{L^2(\mathbb{R}^d)}, \quad f \in C_c^\infty(\mathbb{R}^d)$$

and has the following asymptotics for $f \in C_c^\infty(\mathbb{R}^d)$

$$\lim_{s \searrow 0} (-\Delta)^s f = f \quad \text{and} \quad \lim_{s \nearrow 1} (-\Delta)^s f = (-\Delta) f.$$

For a proof we refer the reader to [DNPV12, Proposition 4.4].

We close this section by stating the Sobolev inequality on balls in the euclidean norm. It follows immediately from [BBM02, Theorem 1] by scaling.

Theorem 2.3.3. *Let $d \in \mathbb{N}, d \geq 2$, $R > 0$ and $\alpha_0 \in (0,2)$. There is a constant $c_1 > 0$ such that for all $\alpha \in (\alpha_0, 2)$, $r \in (0,R)$ and $u \in H^s(B_r)$*

$$\left(\int_{B_r} |u(x)|^{\frac{2d}{(d-\alpha)}} \, \mathrm{d}x \right)^{(d-\alpha)/d} \leq c_1 \int_{B_r} \int_{B_r} \frac{(u(y)-u(x))^2}{|x-y|^{d+\alpha}} \, \mathrm{d}y \, \mathrm{d}x + c_1 r^{-\alpha} \int_{B_r} u(x)^2 \, \mathrm{d}x.$$

2.4. John-Nirenberg's lemma for doubling measures

In this section we introduce doubling metric measure spaces and give John-Nirenberg's lemma for doubling measures. For more details on doubling measure spaces and spaces of bounded mean oscillation, see [BB11] and [HKM06] and the references given therein.

Let (M,d) be a metric space. For $x \in M$ and $r > 0$, let $B_r^d(x)$ denote the ball with respect to the metric d with center x and radius r, that is

$$B_r^d(x) = \{y \in M \colon d(x,y) < r\}.$$

Note that a ball in a metric space does in general not have a unique radius and center. As an example consider the metric space $((-1,1),d)$ where d is the Euclidean metric, i.e. $d(x,y) = |x-y|$. Then $B_1^d(-1) = B_{3/4}^d(-3/4)$ or $B_2(x) = B_3(y)$ for any $x, y \in (-1,1)$.

Definition 2.4.1. *A measure μ on (M,d) is called doubling measure if there exists a constant $c \geq 1$ such that*

$$0 < \mu(B_{2r}^d(x)) \leq c\mu(B_r^d(x)) < \infty \quad \textit{for all } x \in M, r > 0.$$

A metric space endowed with a doubling measure is called doubling space.

Let $f : M \to \mathbb{R}$ be locally integrable. We define the mean of f over a relatively compact set $A \subset M$ by

$$[f]_A := \fint_A f(x) \, \mu(\mathrm{d}x) := \frac{1}{\mu(A)} \int_A f(x) \, \mu(\mathrm{d}x).$$

Definition 2.4.2. *Let $\Omega \subset M$ be open. We define the class of functions of bounded mean oscillation by*

$$\mathrm{BMO}(\Omega, \mu) = \{f \in L^1_{loc}(\Omega) \colon \|f\|_{BMO(\Omega,\mu)} < \infty\},$$

where

$$\|f\|_{BMO(\Omega,\mu)} := \sup \left\{ \fint_B |f(x) - [f]_B| \, \mu(\mathrm{d}x) \colon B \subset \Omega, B \text{ is a ball in } (M,d) \right\}.$$

By definition, the space $\mathrm{BMO}(\Omega, \mu)$ is a subset of $L^1_{\mathrm{loc}}(\Omega, \mu)$. If we exclude the constant functions, then $\| \cdot \|_{\mathrm{BMO}(\Omega,\mu)}$ is a norm.

For a ball $B = B^d_r(x)$, we denote the ball with the same center and radius $2r$ by $2B := B^d_{2r}(x)$.

Let us state the John-Nirenberg inequality for doubling metric measure spaces. For a comprehensive proof we refer the reader to [HKM06, Theorem 19.5] or [BB11, Theorem 3.15].

Lemma 2.4.3. *Let $\Omega \subset \mathbb{R}^d$, d be a metric on $\mathbb{R}^d$ and μ be a doubling measure on (Ω, d). Let $\Omega \subset \mathbb{R}^d$ be open. A function $f : \Omega \to \mathbb{R}$ is in $BMO(\Omega, \mu)$ if and only if for every ball B such that $2B \subset \Omega$ and for every $t > 0$ there are $c_1, c_2 > 0$ such that*

$$\mu(\{x \in B \colon |f(x) - [f]_B| > t\}) \leq c_1 e^{-c_2 t} \mu(B).$$

The positive constants c_1, c_2 and the BMO norm $\|f\|_{\mathrm{BMO}(\Omega,\mu)}$ depend only on each other, the dimension d and the doubling constant.

3. Probabilistic Basics

3.1. Preliminaries

In this section we summarize some basic definitions and review standard facts from probability theory. For detailed discussions on general probability theory, we refer the reader to [Dur10] and [Str11].

In the following let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. We call a measurable map $X : \Omega \to \mathbb{R}^d$ random variable and a measurable set $A \in \mathcal{F}$ event.

For an integrable random variable $X : \Omega \to \mathbb{R}$ we define its expectation $\mathbb{E}[X]$ by

$$\mathbb{E}[X] = \int_\Omega X(\omega)\,\mathrm{d}\mathbb{P}(\omega)$$

and for $F \in \mathcal{F}$ we set $\mathbb{E}[X; F] = \mathbb{E}[X\mathbb{1}_F]$. Let $X : \Omega \to \mathbb{R}^d$ be a random variable and $f : \mathbb{R}^d \to \mathbb{R}$ an integrable map. For $x \in \mathbb{R}^d$, let

$$\mathbb{E}^x[f(X)] := \mathbb{E}[f(X + x)].$$

A fundamental inequality for convex functions is Jensen's inequality. Due to its applications, it is of great importance in modern mathematics and appears in several forms in the literature. We will give the probabilistic form of the inequality.

Theorem 3.1.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function and $X : \Omega \to \mathbb{R}$ a random variable. Assume $\phi \circ X$ and X are integrable. Then*

$$\phi(\mathbb{E}[X]) \leq \mathbb{E}[\phi(X)].$$

Every random vector $X : \Omega \to \mathbb{R}^d$ induces in a natural way a probability measure on $\mathbb{R}^d$, the so-called distribution $\mathbb{P}_X$, by

$$\mathbb{P}_X(A) = \int_A \mathrm{d}\mathbb{P}_X(x) := \mathbb{P}(X \in A), \quad A \in \mathcal{B}(\mathbb{R}^d). \tag{3.1.1}$$

If two random variables X, Y have the same distribution, we write $X \overset{d}{=} Y$.

The expectation of the composition of a function and a random vector can be calculated, using distributions, as follows.

Theorem 3.1.2. *Let $X : \Omega \to \mathbb{R}^d$ be a random variable and $f : \mathbb{R}^d \to \mathbb{R}$ a bounded or non-negative function. Then*

$$\mathbb{E}[f(X)] = \int_\Omega f(X(\omega))\, \mathrm{d}\mathbb{P}(\omega) = \int_{\mathbb{R}^d} f(x)\, \mathrm{d}\mathbb{P}_X(x).$$

Suppose $\mathcal{G} \subset \mathcal{F}$ is a σ-field and $X : \Omega \to \mathbb{R}$ is an integrable $\mathcal{F}$-measurable random variable. The conditional expectation of X given $\mathcal{G}$ is defined as any $\mathcal{G}$-measurable random variable $\mathbb{E}[X|\mathcal{G}]$ with

$$\int_A X\, \mathrm{d}\mathbb{P} = \int_A \mathbb{E}[X|\mathcal{G}]\, \mathrm{d}\mathbb{P} \qquad \text{for all } A \in \mathcal{G}.$$

We emphasize that $\mathbb{E}[X|\mathcal{G}]$ exists, whenever X is integrable.
Let $B \in \mathcal{F}$. The conditional probability of B, given $\mathcal{G}$, is given by the conditional expectation

$$\mathbb{E}[\mathbb{1}_B|\mathcal{G}]. \tag{3.1.2}$$

We have the following properties for conditional expectations:

Theorem 3.1.3.

1. *The conditional expectation $\mathbb{E}[X|\mathcal{G}]$ is $\mathbb{P}$-almost surely uniquely defined.*
2. *Let $X, Y : \Omega \to \mathbb{R}$ be integrable random variables and $a, b \in \mathbb{R}$. Then*

$$\mathbb{E}[aX + bY|\mathcal{G}] = a\mathbb{E}[X|\mathcal{G}] + b\mathbb{E}[Y|\mathcal{G}].$$

3. *Let $\mathcal{E} \subset \mathcal{G} \subset \mathcal{F}$ be σ-fields and $X : \Omega \to \mathbb{R}$ integrable. Then*

$$\mathbb{E}[X|\mathcal{E}] = \mathbb{E}\left[\mathbb{E}[X|\mathcal{G}]\,|\,\mathcal{E}\right] \quad \textit{and} \quad \mathbb{E}[X|\mathcal{E}] = \mathbb{E}\left[\mathbb{E}[X|\mathcal{E}]\,|\,\mathcal{G}\right].$$

4. *Let $X : \Omega \to \mathbb{R}$ be measurable with respect to $\mathcal{F}$ and $Y : \Omega \to \mathbb{R}$ be measurable with respect to $\mathcal{G}$. Assume X and XY are integrable. Then*

$$\mathbb{E}[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}].$$

Let $B \in \mathcal{B}(\mathbb{R})$. Using (3.1.2), the conditional probability of $X \in B$, given $\mathcal{G}$, is given by

$$\mathbb{P}(\omega, B) := \mathbb{E}[\mathbb{1}_{\{\omega\in\Omega:\, X(\omega)\in B\}}|\mathcal{G}] = \mathbb{E}[\mathbb{1}_B(X)|\mathcal{G}].$$

We would like to point out the following. $\mathbb{P}(\cdot, B)$ is almost surely uniquely defined for $B \in \mathcal{B}(\mathbb{R})$ and can be modified on a set of probability zero. Furthermore, for any sequence $(B_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathbb{R})$

$$\mathbb{P}\left(\omega, \bigcup_{n\in\mathbb{N}} B_n\right) = \sum_{n\in\mathbb{N}} \mathbb{P}(\omega, B_n) \tag{3.1.3}$$

for all $\omega \in \Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$. Suppose there is a version $\mathbb{P}'$ of $\mathbb{P}$, such that (3.1.3) is true for all $\omega \in \Omega$ and any sequence $(B_n) \subset \mathcal{B}(\mathbb{R})$. In general such a version may not exist, since the set Ω' depends on (B_n). Hence there is a uncountable number of such collections of sets and therefore the corresponding exceptional sets, could add up to a non-measurable set or a set of positive probability.

Hence, we will define a more subtle concept to overcome these difficulties.

Definition 3.1.4. *Let $\mathcal{G} \subset \mathcal{F}$ be σ-fields. A regular conditional probability for $\mathbb{E}[\cdot|\mathcal{G}]$ is a map $\mathcal{Q} : \Omega \times \mathcal{G} \to [0, 1]$ such that*

1. *for each $\omega \in \Omega$, $\mathcal{Q}(\omega, \cdot)$ is a probability measure on $(\Omega, \mathcal{F})$,*
2. *for each $A \in \mathcal{F}$, $\mathcal{Q}(\cdot, A)$ is a $\mathcal{G}$-measurable random variable,*
3. *for each $A \in \mathcal{F}$ and each $B \in \mathcal{G}$*

$$\int_B \mathcal{Q}(\omega, A) \, \mathrm{d}\mathbb{P}(\omega) = \mathbb{P}(A \cap B).$$

Regular conditional probabilities exist whenever the space Ω is a complete and separable metric space, c.f. [Bas98, Theorem I.5.2].
One big advantage of regular conditional probabilities will be their usage in the theory of stochastic differential equations. Using the martingale problem, regular conditional probabilities will allow us to extend local unique weak solutions to stochastic differential equations to unique global weak solutions.

Next, we introduce independence of random variables. For this purpose, we first need the definition of independence of two events and the independence of two random variables and the definition of the σ-field generated by a random variable. Let $A, B \in \mathcal{F}$. We call two events A, B independent, if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Definition 3.1.5. *Let $X : \Omega \to \mathbb{R}^d$ be a random variable and set*

$$\sigma(X) = \{\{X \in A\} \colon A \in \mathcal{B}(\mathbb{R}^d)\}.$$

We call $\sigma(X)$ the σ-field generated by X.

Using Definition 3.1.5, we can define the independence of two random variables.

Definition 3.1.6. *Two random variables $X : \Omega \to \mathbb{R}^d$ and $Y : \Omega \to \mathbb{R}^d$ are called independent, if for every choice $A \in \sigma(X)$, $B \in \sigma(Y)$ the events A, B are independent .*

One useful calculation rule for independent random variables is the following:

Theorem 3.1.7. *Let $X, Y : \Omega \to \mathbb{R}$ be independent random variables such that X, Y and XY are integrable. Then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.*

A family of random variables $X_t : \Omega \to \mathbb{R}^d$, $t \in [0, \infty)$ is called a stochastic process. We use the common notation for stochastic processes $\{X_t \colon t \geq 0\} = (X_t)_{t \geq 0} = (X_t)_t$. A process $(X_t)_{t \geq 0}$ is called adapted, if X_t is $\mathcal{F}_t$-measurable for any $t \geq 0$ and it is called predictable, if it is measurable with respect to the σ-Algebra generated by all left-continuous adapted processes. For fixed $\omega \in \Omega$ the mapping $t \mapsto X_t(\omega)$ is called sample path or trajectory. It is a single outcome of a stochastic process and can be understood as a chronological ordered sequence of random events.

Let $X_t : \Omega \to \mathbb{R}^d$, $t \in [0, \infty)$ be a stochastic process and let $(\mathbb{R}^d)^{[0,\infty)}$ be the set of all functions from $[0, \infty)$ to $\mathbb{R}^d$. Then for each $\omega \in \Omega$ the path $(X_t)_{t\geq 0}(\omega)$ is an element of $(\mathbb{R}^d)^{[0,\infty)}$. We denote for each $\omega \in \Omega$ the functions $t \mapsto X_t(\omega)$ by $X(\omega)$. Then $X : \Omega \to (\mathbb{R}^d)^{[0,\infty)}$ is measurable with respect to

$$\mathcal{B}(\mathbb{R}^d)^{[0,\infty)} = \bigotimes_{t\in[0,\infty)} \mathcal{B}(\mathbb{R}^d).$$

Since the stochastic process can be seen as a random variable X that takes values in $(\mathbb{R}^d)^{[0,\infty)}$, the law (or distribution) of $(X_t)_{t\geq 0}$ is defined as the probability measure $\mathbb{P} \circ X^{-1}$ on $((\mathbb{R}^d)^{[0,\infty)}, \mathcal{B}(\mathbb{R}^d)^{[0,\infty)})$.

Definition 3.1.8. *Two stochastic processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are called independent, if for any $n \in \mathbb{N}$ the random variables*

$$X = (X_{t_1}, \dots, X_{t_n}) \quad \text{and} \quad Y = (Y_{t_1}, \dots, Y_{t_n})$$

are independent for any $t_1, \dots, t_n \in [0, \infty)$.

An object of great importance in this thesis is the class of martingales. The best general references are [RW00a] and [RW00b].

Definition 3.1.9. *Let $M_t : \Omega \to \mathbb{R}, t \geq 0$ be an integrable, adapted stochastic process.*

1. *$(M_t)_{t\geq 0}$ is called supermartingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, if $\mathbb{E}[M_t|\mathcal{F}_s] \leq M_s$ for $s \leq t$.*
2. *$(M_t)_{t\geq 0}$ is called submartingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, if $\mathbb{E}[M_t|\mathcal{F}_s] \geq M_s$ for $s \leq t$.*
3. *$(M_t)_{t\geq 0}$ is called martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, if $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ for $s \leq t$.*

Martingales are a natural model of fair games. By definition these processes exclude the possibility of winning strategies based on game history.

The next theorem, Doob's martingale inequality, gives an uniform L^p-bound for a martingale on a compact time interval by the L^p-norm of the end-value.

Theorem 3.1.10. *Let $(M_t)_{t\geq 0}$ be a martingale or a non-negative submartingale with right continuous sample paths. Further let $p > 1$. Then for all $T > 0$*

$$\left\| \sup_{s\leq T} |M_s| \right\|_{L^p} \leq \frac{p}{p-1} \|M_T\|_{L^p}.$$

Let $(M_t)_{t\geq 0}$ be a martingale with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$. A nice property of martingales is that the expected value of a martingale at any time $t \geq 0$ is equal to the expected value of its initial value. If we set $s = 0$ in the definition above and take expectations,

$$\mathbb{E}[M_t] = \mathbb{E}[M_0]. \tag{3.1.4}$$

An interesting question is, if one can replace t in (3.1.4) by a random variable. We introduce a class of suitable random variables.

Definition 3.1.11. *A random variable $\mathcal{T} : \Omega \to [0, \infty)$ is called stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, if for all $t \geq 0$*

$$\{\omega \in \Omega \colon \mathcal{T}(\omega) \leq t\} \in \mathcal{F}_t.$$

Stopping times can be understood as a kind of random time. It can be interpreted as waiting time until an event happens.

To shorten notation, we will omit in the designation of martingales and stopping times the regarding filtration. We give two important examples of stopping times. Let $A \subset \mathbb{R}^d$ be a measurable set.

1. The first exit time τ_A for a process $(X_t)_{t\geq 0}$ from A is defined by

$$\tau_A(\omega) := \inf\{t \in [0, \infty) \colon X_t(\omega) \notin A\}. \tag{3.1.5}$$

2. The first hitting time T_A for a process $(X_t)_{t\geq 0}$ of A is defined by

$$T_A(\omega) := \inf\{t \in [0, \infty) \colon X_t(\omega) \in A\}. \tag{3.1.6}$$

From now on τ_A and T_A denote the first exit time resp. first hitting time.

Associated with a stopping time $\mathcal{T}$, we define the stopping time σ-field $\mathcal{F}_{\mathcal{T}}$ by

$$\mathcal{F}_{\mathcal{T}} := \{A \in \mathcal{F} \colon A \cap \{\omega \colon T(\omega) \leq t\} \text{ for all } t \geq 0\}.$$

A stopping time $\mathcal{T}$ is called bounded, if there is a $K > 0$ such that $\mathcal{T}(\omega) \leq K$ for all $\omega \in \Omega$.

Using stopping times, we can formulate a theorem which gives an analogous statement as (3.1.4) for bounded stopping times under certain conditions. This theorem is known as Doob's optional stopping theorem.

Theorem 3.1.12. *Let $\mathcal{T}$ be a bounded stopping time and $(M_t)_{t\geq 0}$ a martingale. Then* $\mathbb{E}[M_{\mathcal{T}}] = \mathbb{E}[M_0]$.

We next use stopping times to extend the class of martingales by the class of so-called local martingales.

Definition 3.1.13. *Let $(M_t)_{t\geq 0}$ be an integrable, adapted stochastic process. If there exists a sequence of stopping times $(\mathcal{T}_k)_{k\in\mathbb{N}}$ such that*

$$\mathcal{T}_k < \mathcal{T}_{k+1}\ \mathbb{P}\text{-almost surely} \quad \text{and} \quad \mathcal{T}_k \overset{k\to\infty}{\longrightarrow} \infty\ \mathbb{P}\text{-almost surely}$$

and the stopped process $(M_t^{\mathcal{T}_k})_{t\geq 0} := (M_{t\wedge\mathcal{T}_k})_{t\geq 0}$ is a uniformly integrable martingale for each $k \in \mathbb{N}$, then $(M_t)_{t\geq 0}$ is called local martingale.

Recall that every martingale is a local martingale and every bounded local martingale is a martingale.

We will end our review by recalling the concept of convergence in distribution and Skorohod's representation theorem. Since this theorem will be used in a quite general framework, we will formulate it in the set-up of metric measure spaces. Let (S, ρ) be a metric space and $\mathcal{S}$ the Borel σ-field on S. Most definitions above can be adjusted by just replacing $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ by $(S, \mathcal{S})$.

Definition 3.1.14.

1. *Let $\mu_n, n \in \mathbb{N}$ be a sequence of measures on $(S, \mathcal{S})$ and μ a measure on $(S, \mathcal{S})$. We say that $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to μ, if*

$$\lim_{n\to\infty} \int_S f \ \mathrm{d}\mu_n = \int_S f \ \mathrm{d}\mu \quad \text{for all} \ \ f \in C_b(S).$$

   In this case, we write: $\mu_n \overset{w}{\longrightarrow} \mu$.

2. *Let $(X_t)_{n\in\mathbb{N}}$ be a family of S-valued random variables. We say X_n convergences in distribution to a random variable X, if the sequence of distributions $\mathbb{P}_{X_n}$ converges weakly to $\mathbb{P}_X$.*
   In this case we write $X_n \overset{d}{\longrightarrow} X$.

Note that by Theorem 3.1.2, $X_n \overset{d}{\longrightarrow} X$ is equivalent to

$$\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)] \quad \text{for all bounded, continuous functions on } S.$$

We have seen that, given a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a random variable X, the distribution induces a probability measure $\mathbb{P}_X$ on $\mathbb{R}^d$ in a natural way by (3.1.1).

Vice versa, given a probability measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, we can choose the probability space $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu)$ and define the random variable $X(\omega) = \omega$ for all $\omega \in \Omega = \mathbb{R}^d$. Then μ is the distribution of X with respect to $\mathbb{P}$.

The next theorem shows that a weakly convergent sequence of probability measures with a limit that has a separable support, can be represented as the distribution of a pointwise convergent sequence of random variables. It is known as Skorohod's representation theorem.

Theorem 3.1.15 ([Bil99, Theorem 6.7]). *Let μ_n, $n \in \mathbb{N}$ be a sequence of probability measures on $(S, \mathcal{S})$ and μ a probability measure on $(S, \mathcal{S})$. Suppose that $\mu_n \xrightarrow{w} \mu$ and μ has a separable support. Then there exist random elements X_n and X, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $\mathbb{P}_{X_n} = \mu_n$, $\mathbb{P}_X = \mu$, and the sequence X_n converges $\mathbb{P}$-almost surely to X.*

3.2. Lévy Processes

In this section we summarize some important facts on Lévy processes. For a more complete theory we refer the reader to [App09], [Ber96] and [Sat13]. For simplicity of notation, we write $(\Omega, \mathcal{F}, \mathbb{P})$ instead of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ for a filtered probability space. Unless otherwise specified, we will always work with the minimal augmented filtration $(\mathcal{F}_t)_{t\geq 0}$, that is

$$\mathcal{F}_t = \sigma(X_s; s \leq t).$$

This filtration has at time $t \geq 0$ full information about the process in the past up to time t. We will first start by an important class of functions in the theory of Lévy processes.

Definition 3.2.1. *A function $f : [0, \infty) \to \mathbb{R}^d$ is called càdlàg, if it is right continuous on $[0, \infty)$ and has left limits for all $t \in (0, \infty)$.*
The space $\mathbb{D}([0, \infty))$ of all càdlàg functions endowed with the Skorohod topology is called Skorohod space.

If f is càdlàg, we will denote the left limit at each point $t \in (0, \infty)$ by

$$f(t-) = \lim_{s \nearrow t} f(s).$$

Clearly, any continuous function is càdlàg and a càdlàg function f is continuous at t if and only if $f(t) = f(t-)$. The jump of f at $t \in (0, \infty)$ will be denoted by $\Delta f(t) = f(t) - f(t-)$.

Definition 3.2.2. *A $\mathbb{R}^d$-valued stochastic process $L = (L_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called Lévy process, if*

1. $L_0 = 0 \quad \mathbb{P}-a.s.$,
2. *L has stationary increments, i.e. for any $s, t \geq 0$ we have $L_{t+s} - L_t \overset{d}{=} L_s$,*
3. *L has independent increments, i.e. for every choice of $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \cdots < t_n$ the random variables $L_{t_n} - L_{t_{n-1}}, L_{t_{n-1}} - L_{t_{n-2}}, \ldots, L_{t_1} - L_{t_0}$ are independent,*
4. *L is stochastically continuous, i.e. for any $\epsilon > 0$ and $t_0 \geq 0$, $\lim_{t \to t_0} \mathbb{P}(|L_t - L_{t_0}| > \epsilon) = 0$.*

An important property of Lévy processes is that one can construct a version of a Lévy process whose paths are càdlàg. From now on we assume every Lévy process L to have càdlàg paths.

Lévy processes form an important class of stochastic processes, which play a significant role in many fields like financial stock prices or population models. They represent the motion of a point whose successive displacements are random and independent, and statistically identical over different time intervals of the same length. Thus, they may be viewed as the continuous-time analog of a random walk.

From Definition 3.2.2 one can easily deduce that any Lévy process $(L_t)_{t\geq 0}$ is a semi-martingale. Due to the càdlàg property, the amount of jumps of a Lévy process is at most countable and by stochastic continuity

$$\text{for any fixed } t \in (0,\infty),\, \Delta L_t = 0 \quad \mathbb{P} - \text{almost surely.} \tag{3.2.1}$$

Adding two independent Lévy processes gives again a Lévy process.

Lemma 3.2.3. *Let $(L_t^1)_{t\geq 0}$ and $(L_t^2)_{t\geq 0}$ be two independent Lévy processes. Then $(L_t^1 + L_t^2)_{t\geq 0}$ is a Lévy process.*

Let us give two important examples of Lévy processes. They are significant building blocks of general Lévy processes:

Definition 3.2.4.

1. *A $\mathbb{N}$-valued Lévy process $N = (N_t)_{t\geq 0}$ is called Poisson process with intensity $\lambda > 0$, if it has Poisson distribution, i.e.*

$$\mathbb{P}(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \qquad \text{for all } k \in \mathbb{N} \cup \{0\}, t \geq 0.$$

2. *A real-valued Lévy process $B = (B_t)_{t\geq 0}$ is called (standard) Brownian motion in $\mathbb{R}$, if the trajectories $t \mapsto B_t$ are $\mathbb{P}$-a.s. continuous and $B_{t+s} - B_s$ has normal distribution with mean zero and variance t, i.e.*

$$\mathbb{P}(B_t \in A) = \int_A \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right) dy \quad \text{for all } s,t \geq 0 \text{ and } A \in \mathcal{B}(\mathbb{R}).$$

Let $B_t^1, \ldots, B_t^d$ be d independent one-dimensional Brownian motions, defined as above. Then $B = (B_t^1, \ldots, B_t^d)$ is called $d-$dimensional standard Brownian motion. The sample paths of Brownian motions are continuous, but they are nowhere differentiable. The canonical space for the sample paths is the space of the continuous real-valued functions $C([0,\infty))$, endowed with the topology of locally uniform convergence, which is induced by the metric

$$d(f,g) = \sum_{k\geq 1} \frac{1}{2^k} \left(\left(\sup_{x\in[0,k]} |f(x) - g(x)| \right) \wedge 1 \right),$$

where $a \wedge b := \min\{a,b\}$. This metric topology is complete and separable. Since general Lévy processes have only càdlàg paths, instead of continuous sample paths, it is clear that the space of continuous functions is not suitable for the description of processes with jumps such as Lévy processes. Hence it makes sense to consider processes on the Skorohod space from Definition 3.2.1. This space equipped with the topology of locally uniform convergence is still complete, but not separable. The non-separability of the space causes well-known problems of measurability in the theory of weak convergence of measures on this space. One possibility to make this space also separable, is to weaken the topology. There is a metrizable topology, the so-called Skorohod topology, such that $\mathbb{D}([0,\infty))$ is complete and separable. Let Λ be the set of all continuous and strictly increasing functions $\lambda : [0,\infty) \to [0,\infty)$ with $\lambda(0) = 0$ and $\lambda(t) \to \infty$ as $t \to \infty$. For $\lambda \in \Lambda$ we define

$$l(\lambda) := \sup_{s<t} \left| \log \left(\frac{\lambda(t) - \lambda(s)}{t-s} \right) \right|.$$

For $f, g \in \mathbb{D}([0,\infty))$ let

$$\delta(f,g) := \inf_{\lambda \in \Lambda} \left(l(\lambda) + \sup_{t \in [0,\infty)} |f(t) - g(\lambda(t))| \right).$$

Then δ is a metric and the space $\mathbb{D}([0,\infty))$ equipped with this metric becomes a complete and separable metric space. The topology generated by this metric is called Skorohod topology. The convergence on this topology is characterized as follows:

Let $f, f_n \in \mathbb{D}([0,\infty))$ for all $n \in \mathbb{N}$. We say f_n convergences to f, if there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in Λ such that

$$\sup_{s\in[0,\infty)} |\lambda_n(s) - s| \overset{n\to\infty}{\longrightarrow} 0$$

and

$$\sup_{s\in[0,K]} |f_n(\lambda_n(s)) - f(s)| \overset{n\to\infty}{\longrightarrow} 0 \text{ for all } K \in \mathbb{N}.$$

For more details on the Skorohod space, see [Bil99].

It is well known that the properties of stationary and independent increments imply that every Lévy process satisfies the Markov property. Furthermore Lévy processes are even strong Markov processes since the sample paths are càdlàg.

Theorem 3.2.5. *Let $L = (L_t)_{t\geq 0}$ be a Lévy process and $\mathcal{T}$ be a stopping time. On the set $\{\mathcal{T} < \infty\}$ the post–$\mathcal{T}$ process $(L_{\mathcal{T}+s} - L_{\mathcal{T}})_{s\geq 0}$ has the same distribution as L and is independent of the pre-$\mathcal{T}$ information $\mathcal{F}_{\mathcal{T}}$.*

Let $k \in \mathbb{N}$. The k^{th} moment of a random variable X is given by $\mathbb{E}[|X|^k]$, if it exists. The next theorem tells us that Lévy processes with bounded jumps have finite moments of all order.

Theorem 3.2.6. *Let $L = (L_t)_{t\geq 0}$ be a Lévy process such that there is a $K > 0$ with*

$$\sup_{t\geq 0} |\Delta L_t| \leq K \quad \mathbb{P}\text{-almost surely.}$$

Then $\mathbb{E}[|X|^k] < \infty$ for all $k \in \mathbb{N}$.

Let us recall the definition of characteristic functions.

Definition 3.2.7. *Let X be a random variable. The characteristic function $\phi_X : \mathbb{R}^d \to \mathbb{C}$ of X is defined as the Fourier transform of its distribution, i.e.*

$$\phi_X(\xi) = \int_{\mathbb{R}^d} e^{i\xi\cdot x}\, \mathbb{P}(\mathrm{d}x).$$

By Theorem 3.1.2, the characteristic function can be written as

$$\phi_X(\xi) = \mathbb{E}[e^{u\xi\cdot X}].$$

If X and Y are independent random variables, then one can easily see that also $e^{i\xi\cdot X}$ and $e^{i\xi\cdot Y}$ are independent and hence by Theorem 3.1.7

$$\phi_{X+Y}(\xi) = \phi_X(\xi)\phi_Y(\xi). \tag{3.2.2}$$

The characteristic function of any $\mathbb{R}^d$-valued random variable completely defines its probability distribution.

The characteristic function of Lévy processes can be represented by the so-called Lévy–Khintchine triplet (A, γ, ν), where A is a symmetric non-negative-definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ and ν is a measure on $\mathbb{R}^d$ such that

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (|h|^2 \wedge 1)\, \nu(\mathrm{d}h) < \infty.$$

Any Lévy process is fully determined by its Lévy–Khintchine triplet (A, γ, ν) in the following way, known as Lévy–Khintchine formula.

Theorem 3.2.8. *Let $L = (L_t)_{t\geq 0}$ be a Lévy processes. Then for any $t \geq 0$, the characteristic function is given by*

$$\phi_{L_t}(\xi) = \exp\left(-\frac{1}{2}\xi\cdot(A\xi) + i\gamma\cdot\xi + \int_{\mathbb{R}^d} (e^{i\xi\cdot x} - 1 - i\xi\cdot x\mathbb{1}_{B_1}(x)\, \nu(\mathrm{d}x)\right) = \exp(-\psi_t(\xi)).$$

The function $\psi_t : \mathbb{R}^d \to \mathbb{C}$ is called the characteristic exponent of μ. A is called Gaussian covariance matrix, $\gamma \in \mathbb{R}^d$ is named drift parameter and ν is the Lévy measure of the associated Lévy process. It is a Radon measure, which can be described in terms of jumps.

By Theorem 3.2.8 and (3.2.2), a Lévy process can be seen as a stochastic process consisting of three independent components:
A Brownian motion, represented by A, a drift term expressed by γ and a pure jump part represented by ν. This decomposition of a Lévy process is known as Lévy-Itô decomposition. It states that the jump part consists of a compound Poisson process and a square integrable pure jump martingale. In particular, if we fix a positive number $\kappa > 0$ and

decompose the jump part of a Lévy process $(L_t)_{t\geq 0}$ into jumps of size less or equal κ and jumps of size greater κ, then the process

$$Y_t = \sum_{0\leq s\leq t} \Delta L_s \mathbb{1}_{\{|\Delta L_s|\leq \kappa\}} \quad \text{is a square integrable martingale,} \tag{3.2.3}$$

that is $(Y_t)_{t\geq 0}$ is a martingale such that $\mathbb{E}[|M_t|^2] < \infty$ for all $t \geq 0$.

For details on the Lévy–Khintchine formula and the Lévy-Itô decomposition, see e.g. [Sat13].

As a last point of our review on Lévy processes, we will study the generator of a Lévy process. Essentially, this consideration follows [Sat13, Section 6.31].

Let $(L_t)_{t\geq 0}$ be a Lévy processes with triplet (A, γ, ν) with $A = (A_{jk})$. Further let $(P_t)_{t\geq 0}$ be the transition semigroup of $(L_t)_{t\geq 0}$ on $C_0(\mathbb{R}^d)$ that is

$$P_t f(x) := \mathbb{E}^x[f(L_t)] := \mathbb{E}[f(L_t + x)], \quad x \in \mathbb{R}^d.$$

Theorem 3.2.9. *$(P_t)_{t\geq 0}$ is a strongly continuous semigroup on $C_0(\mathbb{R}^d)$ with norm $\|P_t\| = 1$. Let $\mathcal{L}$ be its infinitesimal generator with domain $\mathcal{D}(\mathcal{L})$. Then $C_c^\infty(\mathbb{R}^d)$ is a core of $\mathcal{L}$, $C_0^2(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{L})$, and*

$$\mathcal{L}f(x) = \frac{1}{2}\sum_{j,k=1}^{d} A_{jk}\frac{\partial^2 f(x)}{\partial x_j \partial x_k} + \gamma \cdot \nabla f(x) + \int_{\mathbb{R}^d} \left(f(x+y) - f(x) - \mathbb{1}_{B_1}(y) y \cdot \nabla f(x)\right) \nu(\mathrm{d}y) \tag{3.2.4}$$

for $f \in C_0^2(\mathbb{R}^d)$.

For instance, the d-dimensional isotropic α-stable process, $\alpha \in (0,2)$, is a Lévy process with triplet $(0, 0, c(d,\alpha)|h|^{-d-\alpha}\mathrm{d}h)$, where $c(d,\alpha)$ is an appropriate chosen constant. Its infinitesimal generator on $C_0^2(\mathbb{R}^d)$ is given by the so-called fractional Laplacian,

$$-(-\Delta)^{-\alpha/2} = \int_{\mathbb{R}^d} \left(f(x+y) - f(x) - \mathbb{1}_{B_1}(y) y \cdot \nabla f(x)\right) \frac{c(d,\alpha)}{|y|^{d+\alpha}}\,\mathrm{d}y.$$

The formula in (3.2.4) is well-defined for $f \in C_b^2(\mathbb{R}^d)$ and has the following property on $C_b^2(\mathbb{R}^d)$, which is known as Dynkin's formula.

Theorem 3.2.10. *Let $(L_t)_{t\geq 0}$ be a Lévy process and let the operator $\mathcal{L}$ be defined as in (3.2.4) on $C_b^2(\mathbb{R}^d)$. Then*

$$M_t := f(L_t) - f(L_0) - \int_0^t \mathcal{L}f(L_s)\,\mathrm{d}s$$

is a local martingale for all $f \in C_b^2(\mathbb{R}^d)$.

3.3. Stochastic calculus

We want to close the preliminary chapter, by giving two crucial facts on stochastic integration with respect to semimartingales. Since it would go beyond the capacity of this thesis we will not introduce the theory of stochastic integration with respect to semimartingales, but set up the relevant notation and terminology. For a thorough treatment of the theory, we refer the reader to [Pro05], [Kle12] and [JS03].

The following review follows [Pro05, Chapter II].
Let us start with the definition of the quadratic variation process of a semimartingale. We will simply write X for a stochastic process $(X_t)_{t\geq 0}$ when no confusion can arise.

Definition 3.3.1. *Let X, Y be two semimartingales.*

1. *The quadratic variation process $[X,X] = ([X,X]_t)_{t\geq 0}$ of X is defined by*

$$[X,X]_t := X_t^2 - 2\int_0^t X_{s-}\,\mathrm{d}X_s.$$

2. *The quadratic covariation process $[X,Y] = ([X,Y]_t)_{t\geq 0}$ of X, Y is defined by*

$$[X,Y]_t := X_tY_t - \int_0^t X_{s-}\,\mathrm{d}Y_s - \int_0^t Y_{s-}\,\mathrm{d}X_s.$$

One can easily show, by using basic properties of the stochastic integral, the following rule for the jumps of the quadratic variation process

$$\Delta[X,X]_t = (\Delta X_t)^2.$$

Moreover $([X,X])_{t\geq 0}$ is an adapted and increasing process with càdlàg paths. The quadratic variation of a square integrable martingale M has the property

$$M_t^2 - [M,M]_t \text{ is a martingale.}$$

An adapted stochastic process with càdlàg paths is called finite variation process if the paths have $\mathbb{P}$-almost surely finite variation on every compact interval of $[0,\infty)$. The quadratic variation of a finite variation process X is given by the sum of the squares of its jumps, i.e.

$$[X,X]_t = \sum_{0<s\leq t} (\Delta X_s)^2 \tag{3.3.1}$$

An important process is the predictable quadratic variation process of a semimartingale.

Definition 3.3.2. *Let X be a semimartingale. The unique predictable stochastic process $\langle X\rangle = (\langle X\rangle_t)_{t\geq 0}$ that makes $[X,X]_t - \langle X\rangle_t$ a local martingale is called the predictable quadratic variation process (or sharp bracket process) of X.*

The predictable quadratic variation process is by definition the compensator of the quadratic variation process of a semimartingale. Its existence follows from the Doob-Meyer decomposition, c.f. [Kle12, Section 8.9]. For a treatment of compensators in general, we refer the reader to [Pro05, Section III.5].

We briefly study compensators of square integrable martingales in the next theorem.

Theorem 3.3.3. *[Kle12, Theorem 8.24] Let M be a square integrable martingale. Then the sharp bracket process of M is the unique predictable increasing process such that*

$$M_t^2 - \langle M \rangle_t \text{ is a martingale.}$$

For a locally square integrable martingale M, we have that $\langle M \rangle_t$ is the unique process such that $M_t^2 - \langle M \rangle_t$ is a local martingale.

By Theorem 3.3.3, we have for a square integrable martingale with $M_0 = 0$

$$\mathbb{E}\left[M_t^2\right] = \mathbb{E}\left[[M,M]_t\right] = \mathbb{E}\left[\langle M \rangle_t\right].$$

We like to emphasize two technical calculation rules. The following is the Itô isometry for local martingales.

Theorem 3.3.4. *Let $M = (M_t)_{t\geq 0}$ be a local martingale and $(H_t)_{t\geq 0}$ a predictable process such that*

$$\mathbb{E}\left[\int_0^t H_s^2 \, \mathrm{d}\langle M \rangle_s\right] < \infty \quad \text{for any } t \geq 0.$$

Then

$$\int_0^t H_s \, dM_s \quad \text{is a square integrable martingale}$$

and

$$\mathbb{E}\left[\left(\int_0^t H_s \, \mathrm{d}M_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 \, \mathrm{d}\langle M \rangle_s\right].$$

The second important result we want to mention is Itô's formula, which is a kind of change of variables rule for semimartingales. For a d-dimensional stochastic process X we use $(X_t)_{t\geq 0} = ((X_t^1, \ldots, X_t^d))_{t\geq 0}$ to denote its coordinates.

Theorem 3.3.5. *Let $M = (M_t)_{t\geq 0}$ be a semimartingale and $f \in C^2(\mathbb{R}^d)$. Then $(f(M_t))_{t\geq 0}$ is a semimartingale and*

$$\begin{aligned} f(M_t) =& f(M_0) + \sum_{i=1}^d \int_0^t \frac{\partial f(M_{s-})}{\partial x_i} dM_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f(M_{s-})}{\partial x_i \partial x_j} \, \mathrm{d}[M^i, M^j]_s \\ &+ \sum_{s \leq t} \left(\Delta f(M_s) - \sum_{i=1}^d \frac{\partial f(M_{s-})}{\partial x_i} \Delta M_s^i - \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 f(M_{s-})}{\partial x_i \partial x_j} \Delta M_s^i \Delta M_s^j \right). \end{aligned}$$

Part II.

Systems of stochastic differential equations

Structure of Part II

In this part we study systems of stochastic differential equations driven by a class of anisotropic Lévy processes. The aim is to prove that solutions to the system exist and are unique under an additional condition.

This part is divided into three chapters:
In Chapter 4 we introduce a class of anisotropic Lévy processes and present a system of stochastic differential equations driven by these processes. We also constitute sufficient preparation by proving auxiliary results.
Chapter 5 is devoted to the study of existence of solutions to the system of stochastic differential equations. The proof is, roughly speaking, divided into two parts. In the first part we prove existence of solutions for sufficiently smooth, namely Lipschitz continuous, coefficients. The proof is based on the Picard iteration method. In the second part we prove by approximation the existence of solutions to the system of stochastic differential equations for continuous and bounded coefficients. In order to do so we have to prove that solutions are preserved in the limit.
In Chapter 6 we prove uniqueness of solutions to the system. The main idea is to prove uniqueness of the resolvent operator belonging to weak solutions to the system, which provides uniqueness of solutions to the corresponding martingale problem and therefore uniqueness of weak solutions to the system of stochastic differential equations. We will show that the resolvent operator can be expressed as a sum consisting of the resolvent operator for solutions to the system with fixed coefficients and the corresponding perturbation integral operator. Showing that these operators have a unique bounded linear extension to $L^p(\mathbb{R}^d)$ will be crucial to prove uniqueness of weak solutions.

4. Preliminaries

The aim of this chapter is to provide a detailed exposition of the system of stochastic differential equations.

A one-dimensional pure jump Lévy process $(L_t)_{t\geq 0}$ is called symmetric stable process of order $\gamma \in (0,2)$ if the Lévy–Khintchine triplet is given by $(0,0,c_\gamma |h|^{-1-\gamma}\,\mathrm{d}h)$, where the constant $c_\gamma = 2^\gamma \Gamma((1+\gamma)/2)/\,|\Gamma(-\gamma/2)|$ is chosen such that

$$c_\gamma \int_{\mathbb{R}\setminus\{0\}} (1-\cos(w))\frac{1}{|w|^{1+\gamma}}\,\mathrm{d}w = 1. \tag{4.0.1}$$

This leads to the fact that the Fourier symbol of the generator of $(L_t)_{t\geq 0}$ is given by $|\xi|^\gamma$, c.f. Lemma 6.1.1. For this reason, the characteristic function of a one-dimensional symmetric stable process of order $\gamma \in (0,2)$ is given by

$$\mathbb{E}e^{i\xi L_t} = e^{-t|\xi|^\gamma} \qquad \text{for all } \xi \in \mathbb{R}.$$

A comprehensive computation of the constant and proofs of its behavior can be found in [Fel13, Section 2.4].

For a one-dimensional symmetric stable process of order γ we have the following Lévy system formula. A good reference for the Lévy system formula is e.g. [CF12, Appendix A.3.4].

Let $(L_t)_{t\geq 0}$ be a one-dimensional symmetric stable process of order γ.

Lemma 4.0.1. *Let $\mathcal{F}_t = \sigma(L_s; s \leq t)$ and $F(x,t)(\omega)$ be a stochastic process that is jointly measurable with respect to $\mathcal{B}(\mathbb{R}) \times \mathcal{F}_t$. Suppose*

$$\int_0^t \int_{\mathbb{R}} |F(h,s)| c_\gamma |h|^{-1-\gamma}\,\mathrm{d}h\,\mathrm{d}s < \infty.$$

Then

$$\sum_{s\leq t} F(\Delta L_s, s) - \int_0^t \int_{\mathbb{R}} F(h,s) c_\gamma |h|^{-1-\gamma}\,\mathrm{d}h\,\mathrm{d}s$$

is a local martingale.

Let $M > 0$ be fixed and $(L_t)_{t\geq 0}$ be a one-dimensional symmetric stable process of order γ. We define $(\overline{L}_t)_{t\geq 0}$ by

$$\overline{L}_t = L_t - \sum_{s\leq t} \Delta L_s \mathbb{1}_{\{|\Delta L_s| > M\}},$$

which is the process $(L_t)_{t\geq 0}$ with all jumps of size larger than M removed.

Since $\overline{L}$ is a Lévy process with bounded jumps, by (3.2.3) we know that it is a square integrable martingale. The predictable quadratic variation process $(\langle \overline{L}\rangle_t)_{t\geq 0}$ is an increasing process with zero initial value and thus by Theorem 3.3.3, $(\overline{L}_t)^2 - \langle \overline{L}\rangle_t$ is a martingale. Furthermore, from the Lévy system formula for $F(h,s) = |h|^2$ and the fact that the quadratic variation process of a finite variation process can be expressed by the squares of its sums, we see that for $t \geq 0$,

$$\langle \overline{L}\rangle_t = \int_0^t \left(\int_{|h|\leq M} \frac{c_\gamma h^2}{|h|^{1+\gamma}} \,\mathrm{d}h \right) \mathrm{d}s = \frac{2c_\gamma M^{2-\gamma}}{2-\gamma} t. \tag{4.0.2}$$

Let $d \in \mathbb{N}, d \geq 2$ and let $(Z_t^i)_{t\geq 0}, i = 1,\dots,d$ be one-dimensional symmetric stable processes of order $\alpha_i \in (0,2)$. We assume that the processes Z^i, $i = 1,\dots,d$, are independent and set

$$Z = (Z_t)_{t\geq 0} = (Z_t^1,\dots,Z_t^d)_{t\geq 0}.$$

Let $(\breve{Z}_t^i)_{t\geq 0}$ be the d-dimensional Lévy process, defined by $\breve{Z}_t^i = Z_t^i e_i$, where e_i is the i^{th} standard coordinate vector. Then $(\breve{Z}_t^i)_{t\geq 0}$ is a Lévy process with Lévy–Khintchine triplet $(0,0,\breve{\nu}_i(\mathrm{d}h))$, where $\breve{\nu}_i$ is given by

$$\breve{\nu}_i(\mathrm{d}w) = \frac{c_{\alpha_i}}{|w_i|^{1+\alpha_i}} \,\mathrm{d}w_i \left(\prod_{j\neq i} \delta_{\{0\}}(\mathrm{d}w_j) \right).$$

Obviously, $(Z_t)_{t\geq 0}$ is the sum of the d independent Lévy processes $\breve{Z}_t^i$, $i = 1,\dots,d$ and hence by Lemma 3.2.3 a Lévy process itself.

Using the independence of the $\breve{Z}_t^i$'s, the Lévy-measure of $(Z_t)_t$ is given as the sum of the $\breve{\nu}_i$'s, i.e.

$$\nu(\mathrm{d}w) = \sum_{i=1}^d \left(\frac{c_{\alpha_i}}{|w_i|^{1+\alpha_i}} \,\mathrm{d}w_i \left(\prod_{j\neq i} \delta_{\{0\}}(\mathrm{d}w_j) \right) \right).$$

The support of this measure is the union of the coordinate axes. Hence $\nu(A) = 0$ for every set $A \subset \mathbb{R}^d$, which has an empty intersection with the coordinate axes.

The process Z_t makes a jump into the i^{th} direction of the coordinate axis, whenever Z_t^i makes a jump. Let a time $t \geq 0$ be given such that $\Delta Z_t^i \neq 0$. By stochastic continuity of Lévy processes, $\Delta Z_t^j = 0$ holds for all $j \neq i$ at this given time t. Hence the process Z_t jumps along the direction of the coordinate axes.

We now present the system of stochastic differential equations and make the assumptions on the coefficients.

Let $x_0 \in \mathbb{R}^d$ and $A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be a matrix-valued function. We consider the system of stochastic differential equations

$$\begin{aligned} dX_t^i &= \sum_{j=1}^{d} A_{ij}(X_{t-})dZ_t^j, \qquad \text{for } i \in \{1,\dots,d\}, \\ X_0 &= x_0, \end{aligned} \tag{SDE}$$

where $x_0 = (x_0^1, \dots, x_0^d) \in \mathbb{R}^d$.

An equivalent formulation of (SDE) is the following

$$X_t^i = x_0^i + \sum_{j=1}^{d} \int_0^t A_{ij}(X_{s-})dZ_s^j, \quad i = 1,\dots,d. \tag{4.0.3}$$

For $f \in C_b^2(\mathbb{R}^d)$ let

$$\mathcal{L}f(x) = \sum_{j=1}^{d} \int_{\mathbb{R}\setminus\{0\}} (f(x + a_j(x)h) - f(x) - h\mathbb{1}_{\{|h|\le 1\}}\nabla f(x)\cdot a_j(x))\frac{c_{\alpha_j}}{|h|^{1+\alpha_j}}\mathrm{d}h, \tag{4.0.4}$$

where $a_j(x)$ denotes the jth column of the matrix $A(x)$. In Proposition 4.0.4 we will show that $\mathcal{L}$ fulfills Dynkin's formula for any weak solution to (SDE).

Let us first recall the concept of weak solutions:
We say that a probability measure $\mathbb{P}$ is a weak solution to the system (4.0.3), starting at x_0, if there exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ and stochastic processes $(X_t^1, \dots, X_t^d)$ and $(Z_t^1, \dots, Z_t^d)$ such that (4.0.3) holds and the processes $(Z_t^i)_{i=1,\dots,d}$ are independent one-dimensional symmetric stable processes of index α_i under $\mathbb{P}$. In particular, the existence of a unique weak solution to (4.0.3) implies that the law of $(X)_{t\ge 0}$ is uniquely determined.

An equivalent formulation of the concept of existence and uniqueness of weak solutions to stochastic differential equations is given by the so-called martingale problem method. Both concepts are fully equivalent (c.f. [RW00b, Section V.4]), but the martingale problem method has some advantages which will be discussed later.
Recall that the function space $\mathbb{D}([0,\infty)$ is the space of all right-continuous $\mathbb{R}^d$-valued functions on $[0,\infty)$ with left limits endowed with the Skorohod topology.

Definition 4.0.2. *Let $\mathcal{L}$ be an operator whose domain includes $C_b^2(\mathbb{R}^d)$. Let $(X_t)_{t\ge 0}$ be the coordinate maps on $\Omega = \mathbb{D}([0,\infty))$, that is $X_t(\omega) = \omega(t)$ and $(\mathcal{F}_t)_{t\ge 0}$ the filtration generated by the cylindrical sets. We say a probability measure $\mathbb{P}$ is a solution to the martingale problem for $\mathcal{L}$, started at x_0, if the following two conditions hold:*

1. $\mathbb{P}(X_0 = x_0) = 1$.
2. *For each $f \in C_b^2(\mathbb{R}^d)$*

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\mathrm{d}s$$

is a $\mathbb{P}$-martingale.

By focusing our attention on probability measures, the martingale problem formulation gives us the following three powerful techniques (c.f. [RW00b, Chapter V, (19.8)]):

1. the theory of weak convergence,
2. the theory of regular conditional probabilities,
3. localization.

These three techniques will be of great importance in this part of this thesis. In the application the advantages of these techniques will become evident.

We make the following assumptions on the coefficients to the system (SDE):

Assumption 1 ([BC06, Assumption 2.1.]).

1. *For every $x \in \mathbb{R}^d$ the matrix $A(x)$ is non-degenerate, that is*
$$\inf_{u\in\mathbb{R}^d\colon |u|=1} |A(x)u| > 0.$$
2. *The functions $x \mapsto A_{ij}(x)$ are continuous and bounded for all $1 \le i, j \le d$.*

The system (SDE) has been studied in the case $\alpha_1 = \alpha_2 = \cdots = \alpha_d = \alpha \in (0, 2)$ by Bass and Chen in their articles [BC06], [BC10]. In [BC06] the authors proved with the help of the martingale problem the existence and uniqueness of a weak solution to (SDE) in the case $\alpha_1 = \alpha_2 = \cdots = \alpha_d = \alpha \in (0, 2)$. A main tool to obtain uniqueness is by using a bound on the L^p-operator norm of the corresponding perturbation integral operator. In order to do so the authors use the method of rotations, which is not applicable in the case of different indices.
In [BC10] the authors study bounded harmonic functions for the corresponding integral operator and show that such harmonic functions are Hölder continuous. This result has been extended in [Cha16] for the case where the α_i's are allowed to be different.

Note that by symmetry of the density for any $j \in \{1, \dots, d\}$
$$\begin{aligned}
&\int_{\mathbb{R}\setminus\{0\}} (f(x + a_j(x)h) - 2f(x) + f(x - a_j(x)h)) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}}\,\mathrm{d}h \\
&\quad= \frac{1}{2}\int_{\mathbb{R}\setminus\{0\}} (f(x + a_j(x)h) - f(x) - h\mathbb{1}_{\{|h|\le 1\}}\nabla f(x)\cdot a_j(x)) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}}\,\mathrm{d}h \\
&\qquad+ \frac{1}{2}\int_{\mathbb{R}\setminus\{0\}} (f(x - a_j(x)h) - f(x) + h\mathbb{1}_{\{|h|\le 1\}}\nabla f(x)\cdot a_j(x)) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}}\,\mathrm{d}h \\
&\quad= \frac{1}{2}\int_{\mathbb{R}\setminus\{0\}} (f(x + a_j(x)h) - 2f(x) + f(x - a_j(x)h)) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}}\,\mathrm{d}h,
\end{aligned}$$

which allows us to first write the integro-differential operator $\mathcal{L}$ with weighted second order differences.

Definition 4.0.3. *Let $f \in C_b^2(\mathbb{R}^d)$, then*

$$\mathcal{L}f(x) = \frac{1}{2}\sum_{j=1}^{d}\int_{\mathbb{R}\setminus\{0\}} (f(x + a_j(x)h) - 2f(x) + f(x - a_j(x)h))\frac{c_{\alpha_j}}{|h|^{1+\alpha_j}}\,\mathrm{d}h.$$

Next we will show that weak solutions to the system (SDE) are naturally connected to the operator $\mathcal{L}$. Let us assume for now that there is a weak solution $\mathbb{P}$ to (SDE). The existence of a weak solution will be shown in Chapter 5.

Proposition 4.0.4. *Suppose A is bounded and measurable. Let $\mathbb{P}$ be a weak solution to* (SDE)*. If $f \in C_b^2(\mathbb{R}^d)$, then*

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\,\mathrm{d}s$$

is a $\mathbb{P}$-martingale.

Proof. The proof is adapted from [BC06, Proposition 4.1.]. Let $\overline{Z}^k$ be Z^k with all jumps larger than one in absolute value removed. Since $\overline{Z}_t^k$ is a Lévy process with bounded jumps, by Theorem 3.2.6 it has finite moments of all order and by (3.2.3) it is a martingale. Using Itô's formula

$$\begin{aligned}
f(X_t) - f(X_0) &= \int_0^t \nabla f(X_{s-})\cdot dX_s + \sum_{s\le t}(f(X_s) - f(X_{s-}) - \nabla f(X_{s-})\cdot\Delta X_s)\\
&= \int_0^t \nabla f(X_{s-})A(X_{s-})\,\mathrm{d}Z_s\\
&\quad + \sum_{s\le t}(f(X_{s-} + A(X_{s-})\Delta Z_s) - f(X_{s-}) - \nabla f(X_{s-})A(X_{s-})\Delta Z_s)\\
&= \int_0^t \nabla f(X_{s-})A(X_{s-})\,\mathrm{d}\left(\overline{Z}_s + \sum_{u\le s}\Delta Z_u \mathbb{1}_{\{|\Delta Z_u|>1\}}\right)\\
&\quad + \sum_{s\le t}(f(X_{s-} + A(X_{s-})\Delta Z_s) - f(X_{s-}) - \nabla f(X_{s-})A(X_{s-})\Delta Z_s)\\
&= \int_0^t \nabla f(X_{s-})A(X_{s-})\,\mathrm{d}\overline{Z}_s\\
&\quad + \sum_{s\le t}\big(f(X_{s-} + A(X_{s-})\Delta Z_s) - f(X_{s-}) - \nabla f(X_{s-})A(X_{s-})\Delta Z_s\mathbb{1}_{\{|\Delta Z_s|\le 1\}}\big)\\
&= \int_0^t \nabla f(X_{s-})A(X_{s-})\,\mathrm{d}\overline{Z}_s\\
&\quad + \sum_{k=1}^{d}\sum_{s\le t}\Big(f(X_{s-} + a_j(X_{s-})\Delta Z_s^k) - f(X_{s-}) - \nabla f(X_{s-})a_j(X_{s-})\Delta\overline{Z}_s^j\Big)
\end{aligned}$$

$$=: \int_0^t \nabla f(X_{s-})A(X_{s-})\,\mathrm{d}\overline{Z}_s + \sum_{j=1}^d \beta_t^j.$$

By Lemma 4.0.1 for any $j \in \{1, \dots, d\}$

$$\beta_t^j - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (f(X_{s-}+a_j(X_{s-})h) - f(X_{s-}) - h\mathbb{1}_{\{|h|\le 1\}}\nabla f(X_{s-})\cdot a_j(X_{s-}))\frac{c_{\alpha_j}}{|h|^{1+\alpha_j}}\,\mathrm{d}h\,\mathrm{d}s$$

is a $\mathbb{P}$-martingale. The càdlàg property implies that $X_s = X_{s-}$ holds $\mathbb{P}$ almost surely. Hence

$$\sum_{j=1}^d \beta_t^j - \int_0^t \mathcal{L}f(X_{s-})ds = \sum_{j=1}^d \beta_t^j - \int_0^t \mathcal{L}f(X_s)ds$$

is a $\mathbb{P}$-martingale. Since $\overline{Z}_t^j$ is a martingale and the stochastic integral with respect to a martingale is itself again a martingale, the assertion follows. □

We have shown in Proposition 4.0.4 that any weak solution $\mathbb{P}$ to (SDE) is a solution to the martingale problem for the operator $\mathcal{L}$.

An important property to show existence and uniqueness of solutions to the system of stochastic differential equations will be the tightness of a sequence of solutions to the martingale problem. In the following, we will discuss some auxiliary result related to the tightness of sequences of probability measures. We start by recalling the definition of tightness and relatively compactness.

Definition 4.0.5. *Let (S, ρ) be a metric space and $\mathcal{S}$ the Borel σ-field on S. A sequence of probability measures $(P_n)_{n\in\mathbb{N}}$ on S is called tight, if for every $\epsilon > 0$ there is $N \in \mathbb{N}$ and a compact set $K \in \mathcal{S}$ such that*

$$\mathbb{P}_n(K) > 1 - \epsilon \quad \textit{for all } n \ge N.$$

The sequence $(P_n)_{n\in\mathbb{N}}$ is called relatively compact, if any sequence of its elements contains a weakly convergent subsequence. The limiting probability measure might be different for different subsequences.

The following theorem is known as Prokhorov's theorem and provides an equivalence of tightness and weak convergence of probability measures.

Theorem 4.0.6. *If a family of probability measures on $(S, d, \mathcal{S})$ is tight, then it is relatively compact.*
If the space is separable and complete and a family of probability is relatively compact, then it is tight.

For a deeper discussion of weak convergence and tightness, we refer the reader to [Bil99].

Let

$$\tau_\eta := \inf\{s : |X_s - X_0| \ge \eta\} \quad \text{for } \eta \in (0,1). \tag{4.0.5}$$

In [Bas88], the author examines sufficient conditions to derive existence and well-posedness for the martingale problem associated to the integro-differential operator

$$Af(x) = \int_{\mathbb{R}} [f(x+h) - f(x) - f'(x)h\mathbb{1}_{[-1,1]}(h)]\, v(x, \mathrm{d}h)$$

on $C^2(\mathbb{R})$, where $v(x, \mathrm{d}h)$ satisfies the Lévy condition uniformly in x. To this end a tightness criterion for families of solutions is proven. We close this chapter by quoting two propositions from [Bas88].

Proposition 4.0.7 ([Bas88], Proposition 3.1)**.** *Suppose* $\mathbb{P}(X_0 = x_0) = 1$ *and that for every* $f \in C_b^2(\mathbb{R})$ *there exists a constant* c_f *depending only on* $\|f\|_\infty$ *and* $\left\|\frac{\partial^2 f}{\partial x_i \partial x_j}\right\|_\infty$ *such that* $f(X_t) - f(X_0) - c_f t$ *is a supermartingale. Then there exists a constant* $c_1 > 0$*, independent of* x_0*, such that*

$$\mathbb{P}(\tau_\eta \leq t) \leq \frac{c_1 t}{\eta^2}.$$

Note that the constant c_1 in Proposition 4.0.7 is independent of the probability measure $\mathbb{P}$.

The following result is a tightness criterion for a sequence of distributions. Using the Aldous criterion (see [Ald78]), which is a sufficient condition for the tightness of a sequence of stochastic processes in terms of the behavior after stopping times, the following proposition is proved.

Proposition 4.0.8 ([Bas88], Proposition 3.2)**.** *Suppose for each* n *that* $\mathbb{P}_n(X_0 = x_0) = 1$ *and that for every* $f \in C_b^2$ *there exists a constant* c_f *depending only on* $\|f\|_\infty$ *and* $\left\|\frac{\partial^2 f}{\partial x_i \partial x_j}\right\|_\infty$ *such that* $f(X_t) - f(X_0) - c_f t$ *is a* $\mathbb{P}_n$*-supermartingale. Then the sequence* $\mathbb{P}_n$ *is tight on* $\mathbb{D}([0, t_0])$ *for each* t_0*.*

Although the proofs of the propositions in [Bas88] are one-dimensional they do easily extend to d-dimensional case without any significant changes.

5. Existence

The aim of this chapter is to prove the existence of a weak solution to the system (SDE). We first prove the existence of a weak solution to (SDE) for Lipschitz continuous coefficients by using the method of Picard iteration. Afterwards, we approximate the continuous coefficient matrix A by a sequence of Lipschitz continuous coefficient matrices A^n, $n \in \mathbb{N}$ and show that the resulting weak solutions $\mathbb{P}_n$ converge to a measure $\mathbb{P}$ which is a weak solution to (SDE).

Let us first recall Gronwall's inequality.

Theorem 5.0.1 ([Pro05, Theorem V.68]). *Let $h : [0, \infty) \to [0, \infty)$ be such that*

$$h(s) \leq A + B \int_0^s h(r) \,\mathrm{d}r < \infty \quad \text{for } 0 \leq s < t.$$

Then $h(t) \leq Ae^{Bt}$. Moreover if $A = 0$, then h vanishes identically.

The next theorem shows the existence of weak solutions to (SDE), when the coefficients are Lipschitz continuous. The proof follows the proof sketch of [BC06, Proposition 4.2] and the ideas of [Bas98, Theorem I.3.1]. In [Bas98] the author proves the existence of a pathwise solution to the stochastic differential equation

$$dX_t = \sigma(X_t)\mathrm{d}W_t + b(X_t)\mathrm{d}t, \quad X_0 = x,$$

where $(W_t)_{t\geq 0}$ is a one-dimensional Brownian motion and the coefficients σ and b are Lipschitz continuous.

Proposition 5.0.2. *Let $x \mapsto A_{ij}(x)$ be bounded and Lipschitz continuous for every $i, j \in \{1, \ldots, d\}$. Then there exists a weak solution to the system* (SDE).

Proof. Let $M \geq 1$ be fixed. For each $j \in \{1, \ldots, d\}$ we decompose the process Z_t^j into the processes $\overline{Z}_t^j$ and $\widetilde{Z}_t^j$, where $\overline{Z}_t^j$ is Z_t^j with all jumps of size larger than M in absolute value removed. Then $\overline{Z}_t^j$ is a Lévy process with Lévy measure $c_{\alpha_j} |h|^{1+\alpha_j} \mathbb{1}_{\{|w| \leq M\}} \,\mathrm{d}w$. Moreover, by (3.2.3) each $\overline{Z}_t^j$ is a square integrable martingale and by Theorem 3.2.6 it has finite moments of all orders.

Since each A_{ij} is Lipschitz continuous, there is a $L_{ij} > 0$ such that $|A_{ij}(x) - A_{ij}(y)| \leq L_{ij}|x - y|$ for all $x, y \in \mathbb{R}^d$. We define

$$L := \max\{L_{ij} \colon i, j \in \{1, \ldots, d\}\}.$$

Let $i \in \{1, \dots, d\}$ be fixed but arbitrary. We will now use the Picard iteration method to show existence and uniqueness of a weak solution to the system

$$\begin{aligned} dX_t^i(M) &= \sum_{j=1}^d A_{ij}(X_{t-}(M))\,\mathrm{d}\overline{Z}_t^j, \quad i = 1, \dots, d, \\ X_0(M) &= x_0. \end{aligned} \tag{5.0.1}$$

Define $X_t^{(0)}(M) = x_0$ and inductively for $k \in \mathbb{N}_0$ the sequence of processes $X_t^{(k)}(M) = (X_t^{1,(k)}(M), \dots, X_t^{d,(k)}(M))$ by

$$X_t^{i,(k+1)}(M) = x_0^i + \int_0^t \sum_{j=1}^d A_{ij}(X_{s-}^{(k)}(M))\,\mathrm{d}\overline{Z}_s^j$$

for $i = 1, \dots, d$. For each $k \in \mathbb{N}$ and $i \in \{1, \dots, d\}$ we have

$$X_t^{i,(k+1)}(M) - X_t^{i,(k)}(M) = \int_0^t \sum_{j=1}^d (A_{ij}(X_{s-}^{(k)}(M)) - A_{ij}(X_{s-}^{(k-1)}(M)))\,\mathrm{d}\overline{Z}_s^j,$$

which is a martingale. Hence

$$\begin{aligned} |X_t^{(k+1)}(M) - X_t^{(k)}(M)|^2 &= \sum_{i=1}^d |X_t^{i,(k+1)}(M) - X_t^{i,(k)}(M)|^2 \\ &= \sum_{i=1}^d \left| \int_0^t \sum_{j=1}^d [A_{ij}(X_{s-}^{i,(k)}(M)) - A_{ij}(X_{s-}^{i,(k-1)}(M))]\,\mathrm{d}\overline{Z}_s^j \right|^2. \end{aligned} \tag{5.0.2}$$

For each $k \in \mathbb{N}_0$ we define the function $g_k : [0, \infty) \to [0, \infty)$ by

$$g_k(t) := \mathbb{E}\left[\sup_{s \le t} |X_s^{(k+1)}(M) - X_s^{(k)}(M)|^2\right]. \tag{5.0.3}$$

Hence by the sub-additivity of the supremum, Doob's martingale inequality and (5.0.2),

$$g_k(t) \le 4 \sum_{i=1}^d \mathbb{E}\left[\left| \int_0^t \sum_{j=1}^d [A_{ij}(X_{s-}^{(k)}(M)) - A_{ij}(X_{s-}^{(k-1)}(M))]\,\mathrm{d}\overline{Z}_s^j \right|^2\right].$$

Using the Itô isometry, (4.0.2) and the Lipschitz continuity of the coefficients, we can find a constant $c_1 > 0$, such that for any $k \in \mathbb{N}$

$$g_k(t) \le 4 \sum_{i=1}^d \mathbb{E}\left[\left| \int_0^t \sum_{j=1}^d [A_{ij}(X_{s-}^{(k)}(M)) - A_{ij}(X_{s-}^{(k-1)}(M))]\,\mathrm{d}\overline{Z}_s^j \right|^2\right]$$

$$
\begin{aligned}
&\le 4d \sum_{i=1}^{d} \mathbb{E}\left[\sum_{j=1}^{d} \left|\int_0^t [A_{ij}(X_{s-}^{(k)}(M)) - A_{ij}(X_{s-}^{(k-1)}(M))]\,\mathrm{d}\overline{Z}_s^j\right|^2\right] \\
&= 4d \sum_{i=1}^{d} \mathbb{E}\left[\sum_{j=1}^{d} \frac{2c_{\alpha_j} M^{2-\alpha_j}}{2-\alpha_j} \int_0^t |A_{ij}(X_{s-}^{(k)}(M)) - A_{ij}(X_{s-}^{(k-1)}(M))|^2\,\mathrm{d}s\right] \\
&\le 8dL^2 \mathbb{E}\left[\sum_{j=1}^{d} \frac{2c_{\alpha_j} M^{2-\alpha_j}}{2-\alpha_j} \int_0^t |X_{s-}^{(k)}(M) - X_{s-}^{(k-1)}(M)|^2\,\mathrm{d}s\right] \\
&\le c_1 \mathbb{E}\left[\int_0^t |X_{s-}^{(k)}(M) - X_{s-}^{(k-1)}(M)|^2\,\mathrm{d}s\right] \\
&= c_1 \mathbb{E}\left[\int_0^t |X_{s}^{(k)}(M) - X_{s}^{(k-1)}(M)|^2\,\mathrm{d}s\right] \\
&\le c_1 \int_0^t \mathbb{E}\left[\sup_{u\le s} |X_{u}^{(k)}(M) - X_{u}^{(k-1)}(M)|^2\right]\,\mathrm{d}s \\
&= c_1 \int_0^t g_{k-1}(s)\,\mathrm{d}s.
\end{aligned}
$$

For $k = 0$, we have

$$
g_0(t) := \mathbb{E}\left[\sup_{s\le t} |X_s^{(1)}(M) - X_s^{(0)}(M)|^2\right] = \mathbb{E}\left[\sup_{s\le t}\left(\sum_{i=1}^{d}\left|\int_0^t \sum_{j=1}^{d} (A_{ij}(x_0))\,\mathrm{d}\overline{Z}_s^j\right|^2\right)\right].
$$

Since A_{ij} is bounded for any $i, j \in \{1, \dots, d\}$ there is a constant $c_2 > 0$

$$
g_0(t) \le 4d \sum_{i=1}^{d} \mathbb{E}\left[\sum_{j=1}^{d} \frac{2c_{\alpha_j}}{2-\alpha_j} \int_0^t |A_{ij}(x_0)|^2\,\mathrm{d}s\right] \le c_2 t.
$$

Clearly $g_k(0) = 0$ for every $k \in \{1, \dots, d\}$. We will show by induction that for all $k \in \mathbb{N}$

$$
g_k(t) \le c_1^k c_2 \frac{t^{k+1}}{(k+1)!}, \tag{5.0.4}
$$

where $c_1, c_2 > 0$ are the constants from above. We have

$$
g_1(t) \le c_1 \int_0^t g_0^i(s)\,\mathrm{d}s \le c_1 \int_0^t c_2 s\,\mathrm{d}s = c_1 c_2 \frac{t^2}{2} = c_1 c_2 \frac{t^2}{2!}.
$$

Using the induction hypothesis we get

$$
g_{k+1}(t) \le c_1 \int_0^t g_k(s)\,\mathrm{d}s \le c_1 \int_0^t c_1^k c_2 \frac{s^{k+1}}{(k+1)!}\,\mathrm{d}s = c_1^{k+1} c_2 \frac{t^{k+2}}{(k+2)!},
$$

which proves (5.0.4). Hence

$$\sum_{k=0}^{\infty} g_k(t) = \sum_{k=0}^{\infty} c_1^k c_2 \frac{t^{k+1}}{(k+1)!} = \frac{c_2}{c_1} \sum_{k=0}^{\infty} \frac{(c_1 t)^{k+1}}{(k+1)!} < \infty$$

for any fixed $t \geq 0$. We define for fixed $t \geq 0$ the norm

$$\|Y_t\|_1 := \left(\mathbb{E}\left[\sup_{s \leq t} |Y_s|^2 \right] \right)^{1/2}. \tag{5.0.5}$$

The space of all stochastic processes with càdlàg sample paths such that the norm $\|\cdot\|_1$ is finite is a Banach space. Let $n > m$, then

$$\begin{aligned} \|X_t^{(n)}(M) - X_t^{(m)}(M)\|_1 &= \left(\mathbb{E}\left[\sup_{s \leq t} |X_s^{(n)}(M) - X_s^{(m)}(M)|^2 \right] \right)^{1/2} \\ &= \left(\mathbb{E}\left[\sup_{s \leq t} \left| \sum_{k=m}^{n-1} X_s^{(k+1)}(M) - X_s^{(k)}(M) \right|^2 \right] \right)^{1/2} \\ &\leq \left(\mathbb{E}\left[\sum_{k=m}^{n-1} \sup_{s \leq t} |X_s^{(k+1)}(M) - X_s^{(k)}(M)|^2 \right] \right)^{1/2} \\ &= \left(\sum_{k=m}^{n-1} g_k(t) \right)^{1/2} \longrightarrow 0, \qquad \text{as } n, m \to \infty, \end{aligned}$$

which means that $(X_t^{(n)}(M))_n$ is a Cauchy sequence with respect to $\|\cdot\|_1$ for every fixed $t \geq 0$. Therefore there exists a process $X_t(M)$ such that

$$\|X_t^{(n)}(M) - X_t(M)\|_1 \to 0 \quad \text{for } n \to \infty. \tag{5.0.6}$$

Next we need to show that $X_t(M)$ indeed solves (5.0.1). For each $t \geq 0$ and $i \in \{1, \ldots, d\}$ there is a subsequence such that

$$\begin{aligned} &\left| \int_0^t \sum_{j=1}^d A_{ij}(X_{s-}(M)) d\overline{Z}_s^j - \int_0^t \sum_{j=1}^d A_{ij}(X_{s-}^{(n_l)}(M)) \, \mathrm{d}\overline{Z}_s^j \right| \\ &\quad = \left| \sum_{j=1}^d \int_0^t A_{ij}(X_{s-}^i(M)) - A_{ij}(X_{s-}^{i,(n_l)}(M)) \, \mathrm{d}\overline{Z}_s^j \right| \to 0 \quad \text{as } l \to \infty, \end{aligned}$$

which implies that $X_t(M)$ is a solution to (5.0.1).

Now, we show that the solution $X_t(M)$ is a unique solution to (5.0.1). Suppose $X_t(M)$ and $Y_t(M)$ solve (5.0.1). For each $t \geq 0$ define

$$h(t) = \mathbb{E}\left[\sup_{s \leq t} |X_s(M) - Y_s(M)|^2 \right].$$

As above we get

$$h(t) \leq c_4 \int_0^t h(s)\,\mathrm{d}s.$$

Gronwall's Lemma gives $h(t) = 0$, which implies the uniqueness.

Hence we have a unique solution to (SDE) up to the first time, where one of the Z_t^j's makes a jump of size M or larger.

We now use a piecing-together method to construct the solution for all $t \geq 0$.

Define $M_0 = M$ and the stopping time τ_0 by

$$\tau_0 = \inf\{t \geq 0 \colon |\Delta Z_t^j| \geq M_0 \text{ for a } j \in \{1, \dots, d\}\}.$$

We denote by M_1 the jump of Z_t^j at time τ_0.

We define $X_t = X_t(M_0) = X_t(M) = (X_t^1(M), \dots, X_t^d(M))$ for all $t \in [0, \tau_0)$. By the uniqueness of (5.0.1) X_t is a unique solution of (SDE) on $[0, \tau_0)$.

Set

$$\tau_2 = \inf\{t \geq \tau_0 \colon |\Delta Z_t^j| \geq M_1 \text{ for a } j \in \{1, \dots, d\}\}.$$

Again, we can find a unique solution to

$$X_t^i(M_1) = X_{\tau_0-}^i(M_1) + \int_{\tau_0-}^t \sum_{j=1}^d A_{ij}(X_{s-}(M_1))\,\mathrm{d}\overline{Z}_s^j,$$

on $[\tau_0, \tau_1)$. We define $X_t = X_t(M_1)$ on $[\tau_0, \tau_1)$.

Iterating this method countably many times, we get a solution to (SDE). Note that we used the fact that Z_t^j, $j \in \{1, \dots, d\}$ are Lévy process and therefore only make countably many jumps.

We choose $\Omega = \mathbb{D}([0, \infty))$ and $\mathcal{F}$ to be the σ-Algebra generated by all left-continuous adapted processes. Furthermore, let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by X_t.

If we define $\mathbb{P}'$ as the law of $(X_t)_{t \geq 0}$, then by definition $\mathbb{P}'$ is a weak solution to (SDE). □

We will now strengthen the previous result and prove the main result of this chapter.

Theorem 5.0.3. *Let $x \mapsto A_{ij}(x)$ be bounded and continuous for every $i, j \in \{1, \dots, d\}$. Then there exists a weak solution to* (SDE).

Proof. We follow the proof of [BC06, Theorem 4.3].

Let A_{ij}^n be a sequence of Lipschitz continuous and bounded functions such that for any $i, j \in \{1, \dots, d\}$

$$\lim_{n \to \infty} A_{ij}^n \to A_{ij}$$

uniformly on compact sets. For $n \in \mathbb{N}$ by Proposition 5.0.2 there is a weak solution to the SDE

$$\begin{cases} dX_t^i = \sum_{j=1}^d A_{ij}^n(X_{t-}^i)\,\mathrm{d}Z_t^j, \\ X_0 = x_0, \end{cases} \tag{5.0.7}$$

where $A^n(\cdot) = (A_{ij}^n(\cdot))_{1\le i,j\le d}$. Let us denote the corresponding probability measure to the weak solution of (5.0.7) by $\mathbb{P}_n$. We denote the expectation with respect to $\mathbb{P}_n$ by $\mathbb{E}_n$.

For $f \in C_b^2(\mathbb{R}^d)$ let

$$\mathcal{L}^n f(x) = \sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} (f(x + a_j^n(x)h) - f(x) - h\mathbb{1}_{\{|h|\le 1\}}\nabla f(x)\cdot a_j^n(x))\frac{c_{\alpha_j}}{|h|^{1+\alpha_j}}\mathrm{d}h,$$

where $a_j^n(x)$ is the j-th column of $A^n(x)$. By Proposition 5.0.2 and Proposition 4.0.4 for all $f \in C_b^2(\mathbb{R}^d)$

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}^n f(X_s)\,\mathrm{d}s$$

is a $\mathbb{P}_n$-martingale. We will first show that $(f(X_t) - f(X_0) - c_2 t)_{t\ge 0}$ is a supermartingale under $\mathbb{P}_n$. Decomposing the operator as follows

$$\begin{aligned} |\mathcal{L}^n f(x)| \le &\sum_{k=1}^d \int_{\{|h|>1\}} |f(x + e_k\xi_k^n(x)h) - f(x)|\frac{c_{\alpha_k}}{|h|^{1+\alpha_k}}\,\mathrm{d}h \\ &+ \sum_{k=1}^d \int_{\{|h|\le 1\}} |f(x + e_k\xi_k^n(x)h) - f(x) - h\partial_k f(x)\xi_k^n(x)|\frac{c_{\alpha_k}}{|h|^{1+\alpha_k}}\,\mathrm{d}h \end{aligned}$$

and using a Taylor series expansion on the second integrand, there is a constant $c_1 > 0$ such that

$$|f(x + a_j^n(x)h) - f(x) - h\mathbb{1}_{\{|h|\le 1\}}\nabla f(x)\cdot a_j^n(x)| \le c_1 h^2 \max_{1\le i,j\le d}\left\|\frac{\partial^2 f}{\partial x_i \partial x_j}\right\|_\infty.$$

Hence,

$$|\mathcal{L}^n f(x)| \le c_1\|f\|_\infty + c_1\left\|\frac{\partial^2 f}{\partial x_i\partial x_j}\right\|_\infty := c_2,$$

where $c_2 > 0$ is independent of n. Therefore, integrating leads to

$$\int_s^t |\mathcal{L}^n f(X_u)|\,\mathrm{d}u \le c_2(t-s) \quad \text{for any } t > s \ge 0.$$

Thus,

$$\begin{aligned} \mathbb{E}_n\left[f(X_t) - f(X_0) - c_2 t|\mathcal{F}_s\right] &= \mathbb{E}_n\left[f(X_t) - f(X_0) - c_2(t-s)|\mathcal{F}_s\right] - c_2 s \\ &\le \mathbb{E}_n\left[f(X_t) - f(X_0) - \int_s^t |\mathcal{L}^n f(X_u)|\,\mathrm{d}u\middle|\mathcal{F}_s\right] - c_2 s \end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{E}_n\left[f(X_t) - f(X_0) - \int_s^t \mathcal{L}^n f(X_u)\,\mathrm{d}u \middle| \mathcal{F}_s\right] - c_2 s\\
&= \mathbb{E}_n\left[f(X_t) - f(X_0) - \int_0^t \mathcal{L}^n f(X_u)\,\mathrm{d}u \middle| \mathcal{F}_s\right] + \int_0^s \mathcal{L}^n f(X_u)\,\mathrm{d}u - c_2 s\\
&= f(X_s) - f(X_0) - \int_0^s \mathcal{L}^n f(X_u)\,\mathrm{d}u + \int_0^s \mathcal{L}^n f(X_u)\,\mathrm{d}u - c_2 s\\
&= f(X_s) - f(X_0) - c_2 s.
\end{aligned}
$$

We conclude $(f(X_t) - f(X_0) - c_2 t)_{t\geq 0}$ is a supermartingale under $\mathbb{P}_n$. Hence by Proposition 4.0.8 the sequence $(\mathbb{P}_n)_n$ is tight in the Skorohod space $\mathbb{D}([0,\infty))$. Therefore, by Prokhorov's theorem we can find a subsequence $(\mathbb{P}_{n_j})$ such that $\mathbb{P}_{n_j}$ converges weakly to a limit, which we call $\mathbb{P}$. We decompose the big jumps of Z_t^i as follows

$$
U_t^{i,+} := \sum_{s\leq t} \Delta Z_s^i \mathbb{1}_{\{\Delta Z_s^i > 1\}} \quad \text{and} \quad U_t^{i,-} := -\sum_{s\leq t} \Delta Z_s^i \mathbb{1}_{\{\Delta Z_s^i < -1\}}.
$$

Moreover let

$$
\overline{Z}_t^i := Z_t^i - (U_t^{i,+} - U_t^{i,-}).
$$

Since $\overline{Z}_t^i$ is a Lévy process with bounded jumps, it is a square integrable martingale and has finite moments of all orders. Moreover $U_t^{i,+}$ and $U_t^{i,-}$ are independent and increasing processes. Define

$$
U_t^+ := (U_t^{1,+}, \dots, U_t^{d,+}) \quad \text{and} \quad U_t^- := (U_t^{1,-}, \dots, U_t^{d,-}).
$$

By the Skorohod representation theorem we can find a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and processes $\{(X_t^n, \overline{Z}_t^n, V_t^{n,+}, V_t^{n,-}), n \geq 1\}$ and $(\mathcal{X}_t, \overline{\mathcal{Z}}_t', \mathcal{U}_t^+, \mathcal{U}_t^-)$ such that

1. the law of $(\mathcal{X}_t, \overline{\mathcal{Z}}_t', \mathcal{U}_t^+, \mathcal{U}_t^-)$ under $\mathbb{P}'$ is the same as the law of $(X_t, \overline{Z}_t, U_t^+, U_t^-)$ under $\mathbb{P}$,
2. the law of (X_t^n, Z_t^n) under $\mathbb{P}'$ is the same as the law of $(X_t, (\overline{Z}_t + U_t^+ - U_t^-))$ under $\mathbb{P}_n$ for every $n \in \mathbb{N}$,
3. $(X_\cdot^{n_j}, \overline{Z}_\cdot^{n_j}, V_\cdot^{n_j,+}, V_\cdot^{n_j,-})$ converges almost surely to $(\mathcal{X}_\cdot, \overline{\mathcal{Z}}_\cdot, \mathcal{U}_\cdot^+, \mathcal{U}_\cdot^-)$ in the Skorohod space $\mathbb{D}([0,\infty))$, where for $Z_t^n = (Z_t^{1,n}, \dots, Z_t^{d,n})$,

$$
V_t^{i,n,+} := \sum_{s\leq t} \Delta Z_s^{i,n} \mathbb{1}_{\{\Delta Z_s^{i,n} > 1\}} \quad \text{and} \quad V_t^{i,n,-} := -\sum_{s\leq t} \Delta Z_s^{i,n} \mathbb{1}_{\{\Delta Z_s^{i,n} < -1\}}
$$

and

$$
V_t^{n,+} = (V_t^{1,n,+}, \dots, V_t^{d,n,+}) \quad \text{and} \quad V_t^{n,-} = (V_t^{1,n,-}, \dots, V_t^{d,n,-}).
$$

For notational simplification, we take n_j to be n. Note that

$$
\mathcal{Z}_t := \overline{\mathcal{Z}}_t, \mathcal{U}_t^+, \mathcal{U}_t^-
$$

consists of d independent one-dimensional stable processes of index $\alpha_j \in (0,2)$ for $j \in \{1, \dots, d\}$.

Now we want to show that $(\mathcal{X}, \mathcal{Z})$ solves (SDE).

Since for any ω, $(X^n_\cdot(\omega), V^{n,+}_\cdot(\omega), V^{n,-}_\cdot(\omega))$ converges almost surely to $(\mathcal{X}_\cdot(\omega), \mathcal{U}^+_\cdot(\omega), \mathcal{U}^-_\cdot(\omega))$ in the space $\mathbb{D}([0,\infty))$ by the definition of convergence in the Skorohod space there is a sequence of increasing Lipschitz continuous functions $\{\lambda_n(t) : n \geq 1\}$, depending on ω, such that

$$\lim_{n\to\infty} \sup_{s\leq T} |\lambda_n(t) - t| = 0 \quad \text{for every } T > 0 \tag{5.0.8}$$

and such that the uniform distance between $(X^n_t(\omega), V^{n,+}_t(\omega), V^{n,-}_t(\omega))$ and $(\mathcal{X}_{\lambda_n(t)}(\omega), \mathcal{U}^+_{\lambda_n(t)}(\omega), \mathcal{U}^-_{\lambda_n(t)}(\omega))$ on each compact time interval goes to zero as $n \to \infty$. Together with the convergence of A^n to A on each compact set, we have for any $T > 0$

$$\begin{aligned} \lim_{n\to\infty} \sup_{t\leq T} \Big| \sum_{j=1}^d \int_0^t & A^n_{ij}(X^n_{s-}(\omega))\, \mathrm{d}\left(V^{i,n,+}_s(\omega) - V^{i,n,-}_s(\omega)\right) \\ & - \sum_{j=1}^d \int_0^{\lambda_n(t)} A^n_{ij}(\mathcal{X}_{s-}(\omega))\, \mathrm{d}\left(\mathcal{U}^{j,+}_s(\omega) - \mathcal{U}^{j,-}_s(\omega)\right) \Big| = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{j=1}^d \int_0^t & A^n_{ij}(X^n_{s-}(\omega))\, \mathrm{d}\left(V^{i,n,+}_s(\omega) - V^{i,n,-}_s(\omega)\right) \\ &= \sum_{j=1}^d \int_0^{\lambda_n(t)} A^n_{ij}(\mathcal{X}_{s-}(\omega))\, \mathrm{d}\left(\mathcal{U}^{j,+}_s(\omega) - \mathcal{U}^{j,-}_s(\omega)\right) \end{aligned}$$

at any continuity point t for the right hand side. Using the Itô isometry, (4.0.2) and the boundedness of the $A'_{ij}s$, we can find a constant $c_3 > 0$ such that

$$\begin{aligned} \mathbb{E} & \left[\left| \sum_{j=1}^d \int_0^t A^n_{ij}(X^n_{s-})\, \mathrm{d}\overline{Z}^{j,n}_s - \sum_{j=1}^d \int_0^t A_{ij}(\mathcal{X}_{s-})\, \mathrm{d}\overline{\mathcal{Z}}^j_s \right|^2 \right] \\ &= \mathbb{E}\left[\left| \sum_{j=1}^d \int_0^t \left(A^n_{ij}(X^n_{s-}) - A_{ij}(\mathcal{X}_{s-})\right) d\overline{Z}^{j,n}_s + \sum_{j=1}^d \int_0^t A_{ij}(\mathcal{X}_{s-})\, \mathrm{d}\left(\overline{Z}^{j,n}_s - \overline{\mathcal{Z}}^j_s\right) \right|^2 \right] \\ &\leq 2d \sum_{j=1}^d \mathbb{E}\left[\left| \sum_{j=1}^d \int_0^t \left(A^n_{ij}(X^n_{s-}) - A_{ij}(\mathcal{X}_{s-})\right) \mathrm{d}\overline{Z}^{j,n}_s \right|^2 \right] \\ &\quad + 2d \sum_{j=1}^d \mathbb{E}\left[\left| \int_0^t A_{ij}(\mathcal{X}_{s-})\, \mathrm{d}\left(\overline{Z}^{j,n}_s - \overline{\mathcal{Z}}^j_s\right) \right|^2 \right] \\ &= 2d \sum_{j=1}^d \mathbb{E} \int_0^t \left(A^n_{ij}(X^n_{s-}) - A_{ij}(\mathcal{X}_{s-})\right)^2 \mathrm{d}\langle \overline{Z}^{j,n} \rangle_s \end{aligned}$$

$$+ 2d\sum_{j=1}^{d}\mathbb{E}\int_0^t A_{ij}(\mathcal{X}_{s-})^2\,\mathrm{d}\langle\overline{Z}^{j,n}-\overline{\mathcal{Z}}^j\rangle_s$$

$$\leq c_3\sum_{j=1}^{d}\mathbb{E}\int_0^t \left(A_{ij}^n(X_{s-}^n)-A_{ij}(\mathcal{Z}_{s-})\right)^2\,\mathrm{d}s + c_3\sum_{j=1}^{d}\mathbb{E}\left(\overline{Z}_t^{j,n}-\overline{\mathcal{Z}}_t^j\right)^2.$$

Since $\overline{\mathcal{Z}}_t^j$ is a Lévy process with bounded jumps, it has finite moments of all order. Using the fact that X_t^n and $\overline{Z}_t^n$ converge almost surely to $\mathcal{X}_t$ and $\overline{\mathcal{Z}}_t$, respectively, in the space $\mathbb{D}([0,\infty))$, we have

$$\lim_{n\to\infty}\mathbb{E}\left(\overline{Z}_t^{j,n}-\overline{\mathcal{Z}}_t^j\right)^2 = 0.$$

Further for any $\varepsilon > 0$, there exists $l > 0$ such that

$$\mathbb{P}\left(\sup_{s\leq t}|X_s^n| \leq l \text{ for all } n \geq 1\right) > 1-\varepsilon.$$

Since $A^n(\cdot)$ converges uniformly to $A(\cdot)$ on the set $[-l,l]$, there is a constant $c_4 > 0$, independent of n, such that

$$\lim_{n\to\infty}\sum_{j=1}^{d}\mathbb{E}\int_0^t \left(A_{ij}^n(X_{s-}^n)-A_{ij}(\mathcal{X}_{s-})\right)^2\,\mathrm{d}s \leq c_4\epsilon,$$

which implies that the limit is zero, because ϵ can be chosen arbitrarily small. Thus we conclude

$$\lim_{n\to\infty}\sum_{j=1}^{d}\int_0^t A_{ij}^n(X_{s-}^n)\,\mathrm{d}Z_s^{j,n} = \sum_{j=1}^{d}\int_0^t A_{ij}(\mathcal{X}_{s-})\,\mathrm{d}\mathcal{Z}_s^j$$

almost surely for each fixed $t \geq 0$ and thus

$$\mathcal{X}_t = x_0 + \int_0^t A(\mathcal{X}_{s-})d\mathcal{Z}_s \quad \text{a.s.}$$

for every fixed $t \geq 0$. Since the process on the left and right hand side of the last display are right continuous with left limits, we see that $(\mathcal{X},\mathcal{Z})$ solves (SDE). □

6. Uniqueness

This chapter is devoted to prove uniqueness of solutions to the martingale problem for $\mathcal{L}$ started at $x_0 \in \mathbb{R}^d$. This leads to uniqueness of weak solutions to (SDE). For this purpose we prove some estimates on the operator $\mathcal{L}$ and the resolvent operators of the solutions.

We will add the following assumption on the coefficients of (SDE):

Assumption 2. *Suppose $A_{ij}(x) = 0$ for all $x \in \mathbb{R}^d$, whenever $i \neq j$.*

Note that Assumption 2 simply means that the matrix A is diagonal. For abbreviation, we will denote the entries $A_{jj}(\cdot)$ by $A_j(\cdot)$. Hence the matrix is given by

$$A(x) = \operatorname{diag}(A_1(x), \dots, A_d(x)) = \begin{pmatrix} A_1(x) & 0 & \dots & 0 \\ 0 & A_2(x) & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & A_d(x) \end{pmatrix},$$

where for $j \in \{1, \dots, d\}$, $A_j : \mathbb{R}^d \to \mathbb{R}$ is bounded and continuous and the matrix A is non-degenerate.

The aim of this chapter is to prove the following theorem:

Theorem 6.0.1. *Suppose A satisfies Assumption 2, $x \mapsto A_j(x)$ is bounded continuous for all $j \in \{1, \dots, d\}$ and $A(x)$ is non-degenerate for any $x \in \mathbb{R}^d$. For every $x_0 \in \mathbb{R}^d$, there is a unique solution to the martingale problem for $\mathcal{L}$ started at $x_0 \in \mathbb{R}^d$.*

For this purpose we first study the system (SDE), where the coefficients are fixed. This leads to an affine transformation of the Lévy process $(Z_t)_{t \geq 0}$. We will study this process in detail in Section 6.1.

In Section 6.2 we study the resolvent operator of solutions to (SDE) and show that they are bounded in $L^p(\mathbb{R}^d)$.

Finally in Section 6.3 we prove uniqueness of weak solutions to (SDE) if the quantity

$$\sup_{j \in \{1, \dots, d\}} \| |A_j(\cdot)|^{\alpha_j} - |A_j(x_0)|^{\alpha_j} \|_{L^\infty(\mathbb{R}^d)}$$

is sufficiently small. Using this result, we prove Theorem 6.0.1 in Section 6.4 with the help of regular conditional probabilities.

We want to emphasize that as soon as Theorem 6.0.1 is proved, by standard regular condition probability arguments it immediately follows that $\{X_t, \mathbb{P}^x, x \in \mathbb{R}^d\}$ is a family of Markov processes, see for instance [Bas98]Theorem VI.5.1.

6.1. Perturbation

Let $x_0 = (x_0^1, \dots, x_0^d) \in \mathbb{R}^d$ be a fixed point. We define the process $(U_t)_{t\geq 0}$ by

$$U_t = U_0 + A(x_0)Z_t.$$

Note that $(U_t)_{t\geq 0}$ is an affine transformation of a Lévy process and has stationary and independent increments and càdlàg paths. Hence $(U_t - U_0)_{t\geq 0}$ is a Lévy process. For $f \in C_b^2(\mathbb{R}^d)$ consider the operator

$$\mathcal{L}_0 f(x) = \sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} (f(x + e_j A_j(x_0)h) - f(x) - h\mathbb{1}_{\{|h|\leq 1\}} \partial_j f(x) A_j(x_0)) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}} \,\mathrm{d}h. \tag{6.1.1}$$

By (3.2.4) and Theorem 3.2.10 this operator fulfills Dynkin's formula.

For $f \in L^1(\mathbb{R}^d)$, we define its Fourier transform by

$$\mathcal{F}f(\xi) := \widehat{f}(\xi) := \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(x) \,\mathrm{d}x, \quad \xi \in \mathbb{R}^d.$$

Using Plancherel's theorem $\|\mathcal{F}f\|_{L^2} = (2\pi)^d \|f\|_{L^2}$ and the fact that compactly supported smooth functions are integrable and dense in $L^2(\mathbb{R}^d)$, we can extend $\mathcal{F}$ to a linear bijection on $L^2(\mathbb{R}^d)$. The definition of the Fourier transform implies $\widehat{f}(\xi + a) = e^{ia\cdot\xi}\widehat{f}(\xi)$ for any $a \in \mathbb{R}^d$. Classical references on Fourier analysis are [Gra14a] and [Gra14b].

For $j \in \{1, \dots, d\}$, let

$$\mathcal{I}_j f(x) := \int_{\mathbb{R}\setminus\{0\}} (f(x + e_j A_j(x_0)h) - f(x) - h\mathbb{1}_{\{|h|\leq 1\}} \partial_j f(x) A_j(x_0)) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}} \,\mathrm{d}h. \tag{6.1.2}$$

Lemma 6.1.1. *Let $f \in C_c^\infty(\mathbb{R}^d)$. Then*

$$\widehat{\mathcal{L}_0 f}(\xi) = -\sum_{j=1}^d |\xi_j A_j(x_0)|^{\alpha_j} \widehat{f}(\xi).$$

Proof. By the substitution $w = (\xi_j A_j(x_0))h$ and Fubini's theorem we get

$$\begin{aligned}
\widehat{\mathcal{I}_j f}(\xi) &= \frac{1}{2} \int_{\mathbb{R}^d} e^{i\xi\cdot x} \int_{\mathbb{R}\setminus\{0\}} (f(x + e_j A_j(x_0)h) - 2f(x) + f(x - e_j A_j(x_0)h)) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}} \,\mathrm{d}h \,\mathrm{d}x \\
&= \frac{1}{2} \widehat{f}(\xi) \int_{\mathbb{R}\setminus\{0\}} (e^{ih\xi_j A_j(x_0)} - 2 + e^{-ih\xi_j A_j(x_0)}) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}} \,\mathrm{d}h \\
&= \widehat{f}(\xi) \int_{\mathbb{R}\setminus\{0\}} (\cos(h\xi_j A_j(x_0)) - 1) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}} \,\mathrm{d}h \\
&= -\widehat{f}(\xi) \int_{\mathbb{R}\setminus\{0\}} (1 - \cos(h\xi_j A_j(x_0))) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}} \,\mathrm{d}h
\end{aligned}$$

$$= -\widehat{f}(\xi)|(\xi_j A_j(x_0))|^{\alpha_j} \int_{\mathbb{R}\setminus\{0\}} (1-\cos(w)) \frac{c_{\alpha_j}}{|w|^{1+\alpha_j}} \, \mathrm{d}w.$$

The choice of c_{α_j} yields

$$\int_{\mathbb{R}\setminus\{0\}} (1-\cos(w)) \frac{c_{\alpha_j}}{|w|^{1+\alpha_j}} \, dw = 1,$$

which proves the assertion. □

Note that by Lemma 6.1.1 the characteristic function of $(U_t^j)_{t\geq 0}$ is given by

$$\mathbb{E}\left(e^{i\xi\cdot U_t^j}\right) := \exp(-t\Psi_j(\xi_j)) = \exp\left(-t|\xi_j A_j(x_0)|^{\alpha_j}\right).$$

Since the Z_t^i's are independent, the U_t^i's are also independent. Therefore the characteristic function of $(U_t)_{t\geq 0}$ is the product of the characteristic functions of $(U_t^j)_{t\geq 0}$, i.e.

$$\mathbb{E}\left(e^{i\xi\cdot U_t}\right) = \exp\left(-t\sum_{j=1}^{d} \Psi_j(\xi_j)\right) =: \exp(-t\Psi(\xi)). \tag{6.1.3}$$

Next we will show a scaling result for the transition density function of $(U_t)_{t\geq 0}$. It is reasonable to first study the transition density of Z_t, since $(U_t)_{t\geq 0}$ is given as an affine transformation of $(Z_t)_{t\geq 0}$.

By Lemma 6.1.1, we deduce that the characteristic function of $(Z_t^j)_{t\geq 0}$ is given by

$$\mathbb{E}\left(e^{i\xi\cdot Z_t^j}\right) := \exp(-t\psi_j(\xi_j)) = \exp\left(-t|\xi_j|^{\alpha_j}\right).$$

Note that, since $\exp(-\psi_j(\cdot)) \in L^1(\mathbb{R})$, the inverse Fourier-Transform exists and therefore the transition density function q_t^j of Z_t^j exists and is given by

$$q_t^j(x_j) = \mathcal{F}^{-1}\left(e^{-t\psi_j}\right)(x_j) = \frac{1}{(2\pi)}\left(\int_{\mathbb{R}} e^{-ix_j y_j} e^{-t\psi_j(y_j)} \, \mathrm{d}y\right). \tag{6.1.4}$$

Hence

$$\int_{\mathbb{R}} e^{ix\xi_j} q_t^j(x) \, \mathrm{d}x = \exp(-t|\xi_j|^{\alpha_j}). \tag{6.1.5}$$

Since the processes Z_t^j, $j \in \{1,\ldots,d\}$ are independent and $Z_t = (Z_t^1,\ldots,Z_t^d)$, the transition density function of $(Z_t)_{t\geq 0}$ is given by

$$q_t(x) = \prod_{j=1}^{d} q_t^j(x_j). \tag{6.1.6}$$

The scaling property for one-dimensional symmetric α_j-stable processes states $q_t^j(x_j) = t^{-1/\alpha_j} q_1(t^{-1/\alpha_j} x_j)$. See e.g. [Ber96, Chapter 8] for more details.

Using this scaling property of the one-dimensional processes and (6.1.6) we get the following scaling property for the transition density of $(Z_t)_{t\geq 0}$

$$q_t(x) = \prod_{j=1}^{d} q_t^j(x_j) = \prod_{j=1}^{d} t^{-1/\alpha_j} q_1^j(t^{-1/\alpha_j} x_j) = t^{-\sum_{k=1}^{d} 1/\alpha_k} q_1(t^{-1/\alpha_1} x_1, \ldots, t^{-1/\alpha_d} x_d).$$

We next deduce a scaling property for the transition density of $(U_t)_{t\geq 0}$.
Let $B \in \mathcal{B}(\mathbb{R}^d)$. Then by using the substitution $z = A(x_0)x + U_0$

$$\begin{aligned}\mathbb{P}(U_t \in B) &= \mathbb{P}(U_0 + A(x_0)Z_t \in B) = \mathbb{P}(A(x_0)Z_t \in B - U_0) = \mathbb{P}(Z_t \in A(x_0)^{-1}(B - U_0)) \\ &= \int_{\mathbb{R}^d} q_t(x)\, \mathbb{1}_{A(x_0)^{-1}(B-U_0)}(x)\, dx \\ &= \frac{1}{\det(A(x_0))} \int_B q_t(A(x_0)^{-1}(z - U_0))\, dz.\end{aligned}$$

Set

$$p_t^j(x) = \frac{1}{\sqrt[d]{\det(A(x_0))}} q_t^j((A(x_0)^{-1}(x - U_0))_j). \tag{6.1.7}$$

Then the transition density $p_t(x)$ of $(U_t)_{t\geq 0}$ is given by

$$\begin{aligned}\frac{1}{\det(A(x_0))} q_t(A(x_0)^{-1}(x - U_0)) &= \frac{1}{\det(A(x_0))} \prod_{j=1}^{d} q_t^j((A(x_0)^{-1}(x - U_0))_j) \\ &= \prod_{j=1}^{d} p_t^j(x) = p_t(x).\end{aligned} \tag{6.1.8}$$

Moreover, we have

$$p_t(x) = \frac{t^{-\sum_{k=1}^{d} 1/\alpha_k}}{\det(A(x_0))} q_1(\Xi(t)(A(x_0)^{-1}(x - U_0))), \tag{6.1.9}$$

where

$$\Xi(t) = \operatorname{diag}(t^{1/\alpha_1}, \ldots, t^{1/\alpha_d}) = \begin{pmatrix} t^{1/\alpha_1} & 0 & \ldots & 0 \\ 0 & t^{1/\alpha_2} & \ldots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \ldots & 0 & t^{1/\alpha_d} \end{pmatrix}.$$

By the convolution theorem for the Fourier-transform we get

$$\begin{aligned} p_t * p_s &= \mathcal{F}^{-1}\left(\mathcal{F}(p_t)\mathcal{F}(p_s)\right) = \mathcal{F}^{-1}\left(\mathcal{F}(\mathcal{F}^{-1}(e^{-t\Psi}))\mathcal{F}(\mathcal{F}^{-1}(e^{-s\Psi}))\right) \\ &= \mathcal{F}^{-1}\left(e^{-t\Psi}e^{-s\Psi}\right) = \mathcal{F}^{-1}\left(e^{-(t+s)\Psi}\right) = p_{t+s}. \end{aligned} \tag{6.1.10}$$

We now define the transition semigroup $(P_t)_{t\geq 0}$ of $(U_t)_{t\geq 0}$ and $(Q_t)_{t\geq 0}$ of $(Z_t)_{t\geq 0}$ on $C_0(\mathbb{R}^d)$ by

$$P_t f(x) = \int_{\mathbb{R}^d} p_t(x - y) f(y)\, \mathrm{d}y = \mathbb{E}[f(U_t + x)] =: \mathbb{E}^x[f(U_t)]$$

and

$$Q_t f(x) = \int_{\mathbb{R}^d} q_t(x-y) f(y)\,\mathrm{d}y = \mathbb{E}[f(Z_t + x)] =: \mathbb{E}^x[f(Z_t)].$$

By e.g. [Sat13, Theorem 31.5] the operators $\{P_t : t \geq 0\}$ indeed define a strongly continuous semigroup on $C_0(\mathbb{R}^d)$ with operator norm $\|P_t\| = 1$.

Note that $P_t f$ is also well-defined for $f \in C_b(\mathbb{R}^d)$ and the family has, thanks to the Chapman-Kolmogorov identity of the transition function, the semigroup property

$$P_t P_s f = P_{t+s} f \quad \text{for } f \in C_b(\mathbb{R}^d). \tag{6.1.11}$$

But $(P_t)_{t\geq 0}$ is not strongly continuous on $C_b(\mathbb{R}^d)$.

We now state an important result on the limit behavior of the semigroup.

Theorem 6.1.2. *Let $f \in C_0(\mathbb{R}^d)$. Then*

$$\lim_{t\to\infty} \|P_t f\|_\infty = 0. \tag{6.1.12}$$

Proof. Let $\epsilon > 0$. Choose $R > 1$ such that $\|f\|_\infty \leq \epsilon/2$ on $\mathbb{R}^d \setminus B_R(0)$. Then

$$\begin{aligned} P_t f(x) &= \int_{\mathbb{R}^d} p_t(x-y) f(y)\,\mathrm{d}y = \int_{B_R(0)} p_t(x-y) f(y)\,\mathrm{d}y + \int_{\mathbb{R}^d \setminus B_R(0)} p_t(x-y) f(y)\,\mathrm{d}y \\ &:= (I) + (II). \end{aligned}$$

For each $t \geq 0$, we have

$$(II) \leq \frac{\epsilon}{2} \int_{\mathbb{R}^d \setminus B_R(0)} p_t(x-y)\,\mathrm{d}y \leq \frac{\epsilon}{2} \underbrace{\int_{\mathbb{R}^d} p_t(x-y)\,\mathrm{d}y}_{=1} = \frac{\epsilon}{2}.$$

Moreover by (6.1.9), we know there is a consant $c_1 > 0$ such that

$$p_t(x) \leq c_1 \frac{t^{-\sum_{k=1}^d 1/\alpha_k}}{\det(A(x_0))}.$$

Thus,

$$\begin{aligned} (I) &= \int_{B_R(0)} p_t(x-y) f(y)\,\mathrm{d}y \leq c_1 \frac{t^{-\sum_{k=1}^d 1/\alpha_k}}{\det(A(x_0))} \int_{B_R(0)} f(y)\,\mathrm{d}y \\ &\leq c_1 \|f\|_\infty \frac{t^{-\sum_{k=1}^d 1/\alpha_k}}{\det(A(x_0))} |B_R(0)|. \end{aligned}$$

Choose $t_0 \geq 0$ such that for all $t \geq t_0$

$$\|f\|_\infty \frac{t^{-\sum_{k=1}^d 1/\alpha_k}}{\det(A(x_0))} |B_R(0)| \leq \frac{\epsilon}{2}.$$

Note that the choice of t_0 is independent of x. Hence the assertion follows. □

Next we will introduce some important operators associated to the family of operators $(P_t)_{t\geq 0}$ on $C_b(\mathbb{R}^d)$. From (6.1.4) and (6.1.7) we immediately see $p_t(z) = p_t(-z)$ for every $z \in \mathbb{R}^d$.
Hence there exists a positive and symmetric potential density function r_λ with respect to $(U_t)_{t\geq 0}$, that is

$$0 < r_\lambda(y-x) = r_\lambda(x-y) := \int_0^\infty e^{-\lambda t} p_t(x-y)\,\mathrm{d}t. \tag{6.1.13}$$

Let $f \in C_b(\mathbb{R}^d)$. For $\lambda > 0$ we define the λ-resolvent operator of $(U_t)_{t\geq 0}$ by

$$R_\lambda f(x) := \int_{\mathbb{R}^d} f(y) r_\lambda(x-y)\,\mathrm{d}y = \int_0^\infty e^{-\lambda t} P_t f(x)\,\mathrm{d}t = \mathbb{E}^x\left[\int_0^\infty e^{-\lambda t} f(U_t)\,\mathrm{d}t\right]. \tag{6.1.14}$$

The resolvent operator describes the distribution of the process evaluated at independent exponential times. That is, if $\tau = \tau(\lambda)$ has exponential law with parameter $\lambda > 0$ and is independent of $(U_t)_{t\geq 0}$, then

$$\mathbb{E}[f(U_\tau)] = \lambda R_\lambda f.$$

It is often more convenient to work with the resolvent operators than with the semigroup, thanks to the smoothing effect of the Laplace transform and to the lack of memory of exponential laws.

To study these objects in detail, we first have to define for $\lambda \geq 0$ the λ-potential measures $V^\lambda(x,\cdot)$, $x \in \mathbb{R}^d$ on $\mathcal{B}(\mathbb{R}^d)$ by

$$V^\lambda(x,B) = \mathbb{E}^x\left[\int_0^\infty e^{-\lambda t} \mathbb{1}_{\{U_t \in B\}}\,\mathrm{d}t\right] \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d). \tag{6.1.15}$$

Note that V^λ is obviously well-defined for all $\lambda > 0$. Since $U_t(\omega)$ is measurable for all $(t,\omega) \in (0,\infty) \times \Omega$, the application of Fubini's theorem implies

$$\begin{aligned} V^\lambda(B) = \mathbb{E}\left[\int_0^\infty e^{-\lambda t} \mathbb{1}_B(U_t)\,\mathrm{d}t\right] &= \int_0^\infty e^{-\lambda t} \mathbb{E}\left[\mathbb{1}_B(U_t)\right]\,\mathrm{d}t \\ &= \int_0^\infty e^{-\lambda t} \mathbb{P}(U_t \in B)\,\mathrm{d}t \leq \int_0^\infty e^{-\lambda t}\,\mathrm{d}t = \frac{1}{\lambda}. \end{aligned}$$

Clearly, this argument is not valid for $\lambda = 0$. This case will be studied separately at a later point. The 0-potential measure will be denoted by $V(x,B)$ and is called potential measure. By (6.1.14) and the definition of V^λ we obtain an additional representation of the λ-resolvent operator on $C_b(\mathbb{R}^d)$ by

$$R_\lambda f(x) = \int_{\mathbb{R}^d} f(y) V^\lambda(x,\,\mathrm{d}y).$$

Lemma 6.1.3. *Let $f \in C_b(\mathbb{R}^d)$. Then we have the following properties.*

1. $R_\lambda f - R_\mu f = (\mu - \lambda) R_\lambda R_\mu f$ *for* $\lambda, \mu > 0$,
2. $R_\lambda R_\mu f = R_\mu R_\lambda f$.

Proof. The first statement of the Lemma is called the Resolvent identity. Since the second statement follows immediately by Fubini's theorem, we skip it.

Let $f \in C_b(\mathbb{R}^d)$. By the semigroup property of $(P_t)_{t\geq 0}$, (6.1.11) and Fubini's theorem we get

$$\begin{aligned} R_\lambda R_\mu f(x) &= \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\mu s} P_t P_s f(x)\, \mathrm{d}s\, \mathrm{d}t = \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\mu s} P_{t+s} f(x)\, \mathrm{d}s\, \mathrm{d}t \\ &= \int_0^\infty e^{-(\lambda-\mu)t} \int_t^\infty e^{-\mu s} P_s f(x)\, \mathrm{d}s\, \mathrm{d}t \\ &= \int_0^\infty \int_0^s e^{-(\lambda-\mu)t} e^{-\mu s} P_s f(x)\, \mathrm{d}t\, \mathrm{d}s \\ &= \int_0^\infty \frac{-1 + e^{-(\lambda-\mu)s}}{\mu - \lambda} e^{-\mu s} P_s f(x)\, \mathrm{d}s \\ &= \frac{1}{\mu - \lambda} \left(\int_0^\infty e^{-\lambda s} P_s f(x)\, \mathrm{d}s - \int_0^\infty e^{-\mu s} P_s f(x)\, \mathrm{d}s \right) = \frac{R_\lambda f(x) - R_\mu f(x)}{\mu - \lambda}. \end{aligned}$$

□

Let us define

$$\beta = \sum_{j=1}^d \frac{1}{\alpha_j}. \tag{6.1.16}$$

We will prove some elementary facts about R_λ and P_t.

Proposition 6.1.4. *Let* $\lambda > 0$ *and* $f \in C_b(\mathbb{R}^d)$.

1. *If* $p \in [1, \infty]$, *then*
$$\|R_\lambda f\|_p \leq \frac{\|f\|_p}{\lambda}.$$
2. *If* $p \in (1, \infty]$, *then*
$$|P_t f(x)| \leq \frac{t^{-\beta/p}}{\det(A(x_0))} \|p_1(\cdot)\|_q \|f\|_p, \tag{6.1.17}$$
where q *is the conjugate exponent to* p.
3. *If* $p > \beta$, *then*
$$|R_\lambda f(x)| \leq c_1 \|f\|_p,$$
where $c_1 = \frac{\|p_1(\cdot)\|_q}{\det(A(x_0))} \int_0^\infty e^{-\lambda t} t^{-\beta/p} dt$.

Proof. The idea of the proof goes back to [BC06, Proposition 2.2].
Without loss of generality we can assume $f \in L^p(\mathbb{R}^d)$. Otherwise the assertions are trivially true.

1. By Young's inequality and the conservativeness of p_t, we have

$$\|P_t f\|_p \le \|p_t\|_1 \|f\|_p = \|f\|_p.$$

Therefore by Minkowski's inequality

$$\|R_\lambda f\|_p \le \int_0^\infty e^{-\lambda t} \|P_t f\|_p \,\mathrm{d}t \le \frac{1}{\lambda}\|f\|_p.$$

2. By Hölder's inequality,

$$|P_t f(x)| = \left| \int_{\mathbb{R}^d} p_t(x-y) f(y) \,\mathrm{d}y \right| \le \|f\|_p \|p_t(x-\cdot)\|_q = \|f\|_p \|p_t(\cdot)\|_q.$$

Using the scaling property for p, we get

$$p_t(x) = \frac{t^{-\beta}}{\det(A(x_0))} q_1(\Xi(t)(A(x_0)^{-1}(x-U_0))).$$

Hence

$$\begin{aligned}
\|p_t(\cdot)\|_q &= \|\frac{t^{-\beta}}{\det(A(x_0))} q_1(\Xi(t)(A(x_0)^{-1}(\cdot - U_0)))\|_q \\
&= \frac{t^{-\beta}}{\det(A(x_0))} \|q_1(\Xi(t)(A(x_0)^{-1}(\cdot - U_0)))\|_q \\
&= \frac{t^{-\beta}}{\det(A(x_0))} \left(\det(\Xi(t))^{-1}\right)^{1/q} \|q_1((A(x_0)^{-1}(\cdot - U_0)))\|_q \\
&= \frac{t^{-\beta} t^{\beta/q}}{\det(A(x_0))} \|q_1((A(x_0)^{-1}(\cdot - U_0)))\|_q \\
&= \frac{t^{-\beta \frac{q-1}{q}}}{\det(A(x_0))} \|p_1(\cdot)\|_q = \frac{t^{-\beta/p}}{\det(A(x_0))} \|p_1(\cdot)\|_q.
\end{aligned}$$

3. Using the previous estimate,

$$|R_\lambda f(x)| = \left| \int_{\mathbb{R}^d} e^{-\lambda t} P_t f(x) \,\mathrm{d}t \right| \le \frac{\|p_1(\cdot)\|_q}{\det(A(x_0))} \left(\int_0^\infty e^{-\lambda t} t^{-\beta/p} \,\mathrm{d}t \right) \|f\|_p.$$

□

Let $\mathcal{A}$ be the infinitesimal generator of the semigroup P_t on $C_0(\mathbb{R}^d)$ with domain $D(\mathcal{A})$. Note that $\mathcal{L}_0 = \mathcal{A}$ on $C_0^2(\mathbb{R}^d)$.

We now study λ-potential measures for the case $\lambda = 0$. First, we give the definition of the potential operator.

Definition 6.1.5. *The potential operator* $(N, D(N))$ *for* $(P_t)_{t\geq 0}$ *is the operator on* $C_0(\mathbb{R}^d)$*, defined by*

$$Nf(x) = \lim_{t\to\infty} \int_0^t P_s f(x)\ \mathrm{d}s,$$

where $f \in D(N) := \{f \in C_0(\mathbb{R}^d)\colon Nf \text{ exists in } C_0(\mathbb{R}^d)\}$.

Next we state a proposition, which shows that $(N, D(N))$ plays the role of an "inverse" operator to $-\mathcal{L}_0$. Let $R(N)$ denote the range of the operator N.

Proposition 6.1.6 ([BF75, Proposition 11.9.])**.** *The following three conditions are equivalent:*

(i) $D(N)$ *is dense in* $C_0(\mathbb{R}^d)$,

(ii) $R(N)$ *is dense in* $C_0(\mathbb{R}^d)$,

(iii) $\lim_{t\to\infty} P_t f = 0$ *for all* $f \in C_0(\mathbb{R}^d)$.

When conditions (i)-(iii) are fulfilled the potential operator is a densely defined, closed operator in $C_0(\mathbb{R}^d)$*, and the infinitesimal generator* A *is injective and satisfies*

$$N = -\mathcal{A}^{-1} \quad \textit{and} \quad \mathcal{A} = -N^{-1}.$$

An important object will be the 0-resolvent operator, that is the limit

$$R_0 f := \lim_{\lambda\to 0} R_\lambda f,$$

where $f \in D(R_0) := \{f \in C_0(\mathbb{R}^d)\colon R_0 f \text{ exists in } C_0(R^d)\}$. By [BF75, Proposition 11.15] $R_0 = N$ if $\lim_{t\to\infty} P_t f = 0$ for all $f \in C_0(\mathbb{R}^d)$, which is fulfilled by Theorem 6.1.2. Moreover, by Proposition 6.1.6 R_0 is well-defined and a densely defined and closed operator in $C_0(\mathbb{R}^d)$.

Lemma 6.1.7. *Let* $f \in D(R_0)$ *and* $\lambda > 0$*. Then*

$$R_\lambda f - R_0 f = -\lambda R_\lambda R_0 f.$$

Proof. Let $f \in D(R_0)$. By Lemma 6.1.3 for $\lambda, \mu > 0$ we have

$$R_\lambda f - R_\mu f = (\mu - \lambda) R_\lambda R_\mu f. \tag{6.1.18}$$

Since $f \in D(R_0)$ the limit $\lim_{\mu\to 0} R_\mu f$ exists in $C_0(\mathbb{R}^d)$. Thus the result follows by taking the limit $\mu \to 0$ in (6.1.18). □

Next we want to study the long-time behavior of the process $(U_t)_{t\geq 0}$ in terms of the potential measure, c.f. [Ber96].

Definition 6.1.8. *We say that a Lévy process is transient if the potential measures are Radon measures, that is, for every compact set $K \subset \mathbb{R}^d$*

$$V(x, K) < \infty, \quad x \in \mathbb{R}^d.$$

For $z \in \mathbb{C}$, we write $\mathcal{R}(z)$ for the real part of z. One method to verify transience of a Lévy process is the following.

Theorem 6.1.9 ([Ber96, Theorem 17])**.** *Let $(L_t)_{t\geq 0}$ be a Lévy process with characteristic exponent Ψ. If for some $r > 0$*

$$\int_{B_r} \mathcal{R}\left(\frac{1}{\Psi(\xi)}\right) \mathrm{d}\xi < \infty, \tag{6.1.19}$$

then $(L_t)_{t\geq 0}$ is transient.

Note that Definition 6.1.8 and Theorem 6.1.9 also apply for shifted Lévy processes, i.e. Lévy processes whose initial value is not zero.

We will next show that $(U_t)_{t\geq 0}$ is transient by verifying (6.1.19).

Proposition 6.1.10. $(U_t)_{t\geq 0}$ *is transient.*

Proof. By Theorem 6.1.9 and (6.1.3), if

$$\exists r > 0 : \int_{B_r} \frac{1}{\sum_{j=1}^d |A_j(x_0)\xi_j|^{\alpha_j}} \mathrm{d}\xi < \infty,$$

then $(U_t)_{t\geq 0}$ is transient. Let $r < 1$ such that for $\xi \in B_r$. Then $|\xi_j| < 1$ for any $j \in \{1, \dots, d\}$. Let $c_1 := \min\{|A_j(x_0)| \colon j \in \{1, \dots, d\}\}$ and $\alpha_{\max} = \max\{\alpha_j \colon j \in \{1, \dots, d\}\}$. Then

$$\begin{aligned}\sum_{j=1}^d |A_j(x_0)\xi_j|^{\alpha_j} &\geq c_1 \sum_{j=1}^d |\xi_j|^{\alpha_{\max}} \geq c_1 \max_{j\in\{1,\dots,d\}} \{|\xi_j|^{\alpha_{\max}}\} \\ &= c_1 \left(\max_{j\in\{1,\dots,d\}} \{|\xi_j|^2\}\right)^{\alpha_{\max}/2} \geq \frac{c_1}{d} \left(\sum_{j=1}^d |\xi_j|^2\right)^{\alpha_{\max}/2} = c_2|\xi|^{\alpha_{\max}}\end{aligned}$$

Hence

$$\begin{aligned}\int_{B_r} \frac{1}{\psi(\xi)} \mathrm{d}\xi &= \int_{B_r} \frac{1}{\sum_{j=1}^d |A_j(x_0)\xi_j|^{\alpha_j}} \mathrm{d}\xi \leq c_3 \int_{B_\epsilon} \frac{1}{|\xi|^{\alpha_{\max}}} \mathrm{d}\xi \\ &= c_4 \int_0^r s^{d-1} s^{-\alpha} \,\mathrm{d}s = c_4 \int_0^r s^{d-1-\alpha} \,\mathrm{d}s < \infty,\end{aligned}$$

since $d \geq 2$ and $\alpha_{\max} \in (0, 2)$ and therefore $d - \alpha_{\max} = \delta > 0$. □

Because of the transience of the Lévy process U_t we have $R_0 f(x) \to 0$ as $|x| \to \infty$ for $f \in C_c(\mathbb{R}^d)$, see [Sat13, Exercise 39.14]. Furthermore is easy to see that $R_0 f$ for $f \in C_c(\mathbb{R}^d)$ is continuous by dominated convergence theorem. Hence for $f \in C_c(\mathbb{R}^d)$ we know $Rf \in C_0(\mathbb{R}^d)$. We have

$$R_0 f(x) = \mathbb{E}^x \int_0^\infty f(U_s)\,\mathrm{d}s \quad f \in C_c(\mathbb{R}^d). \tag{6.1.20}$$

Different to R_λ the operator R_0 is not well-defined on $C_b(\mathbb{R}^d)$. For instant let f be a non-zero constant function. Then by the representation (6.1.20) of R_0, one can easily see that $R_0 f$ is infinite.

In the next step, we will use Definition 4.0.3 to write the operator $\mathcal{L}_0$ on $f \in C_b^2(\mathbb{R}^d)$ with respect to a density.

Lemma 6.1.11. *Let $f \in C_b^2(\mathbb{R}^d)$, then*

$$\mathcal{L}_0 f(x) = \frac{1}{2}\sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} (f(x+e_j h) - 2f(x) + f(x-e_j h)) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}} |A_j(x_0)|^{\alpha_j} \mathrm{d}h.$$

Proof. By Definition 4.0.3

$$\mathcal{L}_0 f(x) = \frac{1}{2}\sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} (f(x+e_j A_j(x_0)h) - 2f(x) + f(x-e_j A_j(x_0)h)) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}}\,\mathrm{d}h.$$

Using the substitution $t = A_j(x_0)h$ for each summand, we get

$$\begin{aligned}
\mathcal{L}_0 f(x) &= \frac{1}{2}\sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} (f(x+e_j A_j(x_0)h) - 2f(x) + f(x-e_j A_j(x_0)h)) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}}\,\mathrm{d}h \\
&= \frac{1}{2}\sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} (f(x+e_j t) - 2f(x) + f(x-e_j t)) \frac{c_{\alpha_j}}{|A_j(x_0)^{-1} t|^{1+\alpha_j}} |A_j(x_0)|^{-1}\,\mathrm{d}t \\
&= \frac{1}{2}\sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} (f(x+e_j t) - 2f(x) + f(x-e_j t)) \frac{c_{\alpha_j}}{|t|^{1+\alpha_j}} |A_j(x_0)|^{\alpha_j}\,\mathrm{d}t,
\end{aligned}$$

which proves the assertion. □

Next we give a Fourier multiplier theorem, which goes back to [BnB07].

Given $p \in (1, \infty)$, let

$$p^* := \max\left\{p, \frac{p}{p-1}\right\} \quad \Longleftrightarrow \quad p^* - 1 = \max\left\{(p-1), (p-1)^{-1}\right\}.$$

Let $\Pi \geq 0$ be a symmetric Lévy measure on $\mathbb{R}^d$ and ϕ a complex-valued, Borel-measurable and symmetric function with $|\phi(z)| \leq 1$ for all $z \in \mathbb{R}^d$.

Theorem 6.1.12 ([BnB07, Theorem 1]). *The Fourier multiplier with the symbol*

$$M(\xi) = \frac{\int_{\mathbb{R}^d} (\cos(\xi \cdot z) - 1)\phi(z)\,\Pi(\mathrm{d}z)}{\int_{\mathbb{R}^d} (\cos(\xi \cdot z) - 1)\,\Pi(\mathrm{d}z)} \tag{6.1.21}$$

is bounded in $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, with the norm at most $p^* - 1$. That is, if we define the operator $\mathcal{M}$ on $L^2(\mathbb{R}^d)$ by

$$\widehat{\mathcal{M}f}(\xi) = M(\xi)\widehat{f}(\xi),$$

then $\mathcal{M}$ has a unique linear extension to $L^p(\mathbb{R}^d)$, $1 < p < \infty$, and

$$\|\mathcal{M}f\|_p \le (p^* - 1)\|f\|_p.$$

For $j \in \{1, \dots, d\}$, let

$$\mathcal{M}_j f(x) = \int_{\mathbb{R}\setminus\{0\}} (f(x + e_j t) - 2f(x) + f(x - e_j t)) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}}\,\mathrm{d}h.$$

Our aim is to show that this operator fits into the set-up of Theorem 6.1.12. Let $f \in L^2(\mathbb{R}^d)$. Then

$$\begin{aligned}
\widehat{\mathcal{M}_j f}(\xi) &= \int_{\mathbb{R}^d} e^{ix\cdot\xi} \int_{\mathbb{R}\setminus\{0\}} (f(x + e_j t) - 2f(x) + f(x - e_j t)) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}}\,\mathrm{d}h\,\mathrm{d}x \\
&= \int_{\mathbb{R}\setminus\{0\}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} (f(x + e_j t) - 2f(x) + f(x - e_j t)) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}}\,\mathrm{d}x\,\mathrm{d}h \\
&= \int_{\mathbb{R}\setminus\{0\}} \widehat{f}(\xi)(e^{ihe_j\cdot\xi} - 2 + e^{-ihe_j\cdot\xi}) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}}\,\mathrm{d}h \\
&= 2\int_{\mathbb{R}\setminus\{0\}} \widehat{f}(\xi)(\cos(h\xi_j) - 1) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}}\,\mathrm{d}h.
\end{aligned}$$

For $f \in C_c^2(\mathbb{R}^d)$, $R_0 f$ is well-defined and therefore, by the previous calculation

$$\widehat{\mathcal{M}_j R_0 f}(\xi) = -2\frac{\int_{\mathbb{R}\setminus\{0\}} (\cos(h\xi_j) - 1)\frac{c_{\alpha_j}}{|h|^{1+\alpha_j}}\,\mathrm{d}h}{\sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} (\cos(h\xi_j) - 1)\frac{c_{\alpha_j}}{|h|^{1+\alpha_j}} |A_j(x_0)|^{\alpha_j}\,\mathrm{d}h} \widehat{f}(\xi).$$

If we define for $z = (z_1, \dots, z_d) \in \mathbb{R}^d$

$$\begin{aligned}
\Pi(\mathrm{d}z) &= \sum_{j=1}^d |A_j(x_0)|^{\alpha_j} \frac{c_{\alpha_j}}{|z_j|^{1+\alpha_j}}\,\mathrm{d}z_j \prod_{i\neq j} \delta_{\{0\}}(\mathrm{d}z_i), \\
\phi(z) &= \mathbb{1}_{\{z = e_j u:\ u\in\mathbb{R}\}}(z)|A_j(x_0)|^{-\alpha_j},
\end{aligned} \tag{6.1.22}$$

then we can write

$$\widehat{\mathcal{M}_j R_0 f}(\xi) = -2\frac{\int_{\mathbb{R}^d} (\cos(z\cdot\xi) - 1)\phi(z)\,\Pi(\mathrm{d}z)}{\int_{\mathbb{R}^d} (\cos(z\cdot\xi) - 1)\,\Pi(\mathrm{d}z)} \widehat{f}(\xi).$$

Therefore by Theorem 6.1.12 for $f \in C_c^2(\mathbb{R}^d)$

$$\|\widehat{\mathcal{M}_j R_0} f\|_p \le 2a(p^* - 1)\|f\|_p, \tag{6.1.23}$$

where

$$a = \max\{|A_1(x_0)|^{-\alpha_1}, \dots, |A_d(x_0)|^{-\alpha_d}\}. \tag{6.1.24}$$

Let us define the perturbation operator on $C_b^2(\mathbb{R}^d)$ by

$$\mathcal{B}f(x) = \mathcal{L}f(x) - \mathcal{L}_0 f(x).$$

Set

$$\eta := \sup_{j \in \{1,\dots,d\}} \| |A_j(\cdot)|^{\alpha_j} - |A_j(x_0)|^{\alpha_j} \|_{L^\infty(\mathbb{R}^d)}. \tag{6.1.25}$$

We assume

$$\eta \le \eta_0 := \frac{1}{4da(p^* - 1)}, \tag{Loc}$$

where a is defined as in (6.1.24).

Proposition 6.1.13. *Let $f \in C_0(\mathbb{R}^d)$ such that $R_0 f \in C_b^2(\mathbb{R}^d)$. Let $p \in (1, \infty)$ and assume η satisfies* (Loc)*. Then*

$$\|\mathcal{B}R_0 f\|_p \le \frac{1}{4}\|f\|_p.$$

Proof. Without loss of generality, we can assume $f \in L^p(\mathbb{R}^d)$. Otherwise the right-hand side of the assertion is infinite and the statement is trivially true. Using Hölder's inequality, we get

$$\begin{aligned}
\|\mathcal{B}R_0 f\|_p &= \|(\mathcal{L} - \mathcal{L}_0)R_0 f\|_p \\
&= \Bigg\| \frac{1}{2}\sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} (R_0 f(\cdot + e_j t) - 2R_0 f(\cdot) + R_0 f(\cdot - e_j t)) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}} \,\mathrm{d}h \\
&\qquad \times (|A_j(\cdot)|^{\alpha_j} - |A_j(x_0)|^{\alpha_j}) \Bigg\|_p \\
&\le \sum_{j=1}^d \Bigg\| \frac{1}{2} \int_{\mathbb{R}\setminus\{0\}} (R_0 f(\cdot + e_j t) - 2R_0 f(\cdot) + R_0 f(\cdot - e_j t)) \frac{c_{\alpha_j}}{|h|^{1+\alpha_j}} \,\mathrm{d}h \\
&\qquad \times (|A_j(\cdot)|^{\alpha_j} - |A_j(x_0)|^{\alpha_j}) \Bigg\|_p \\
&\le \frac{1}{2}\eta \sum_{j=1}^d \|\mathcal{M}_j R_0 f\|_p.
\end{aligned}$$

Using (6.1.23), $\|\mathcal{M}_j R_0 f\|_p \le 2a(p^* - 1)\|f\|_p$. Hence by the definition of η

$$\|\mathcal{B}R_0 f\|_p \le \eta d a(p^* - 1)\|f\|_p \le \frac{1}{4}\|f\|_p.$$

□

6.2. Boundedness of the Resolvent

The aim of this section is to prove that the resolvent operator for any weak solution to (SDE) is bounded for any $f \in C_b^2(\mathbb{R}^d)$ by the L^p-norm of f.
More precisely, assume that $\mathbb{P}$ is a solution to the martingale problem for $\mathcal{L}$ started at x_0, see Definition 4.0.2 and $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. Let

$$S_\lambda f = \mathbb{E}\left[\int_0^\infty e^{-\lambda t} f(X_t)\,\mathrm{d}t\right], \quad f \in C_b(\mathbb{R}^d). \tag{6.2.1}$$

We want to show that under the assumption (Loc) there is a constant $c_1 > 0$ such that

$$|S_\lambda f| \leq c_1 \|f\|_p. \tag{6.2.2}$$

For each $n \in \mathbb{N}$ we first define the truncated process

$$Y_t^n = \sum_{k=0}^\infty X_{k/2^n} \mathbb{1}_{\{\frac{k}{2^n} \leq t < \frac{k+1}{2^n}\} \cap \{t \leq n\}} + X_n \mathbb{1}_{\{t > n\}} \tag{6.2.3}$$

and U_t^n as the solution to the system of stochastic differential equations

$$dU_t^n = A(Y_{t-}^n) dZ_t, \; U_0^n = x_0, \tag{6.2.4}$$

where $x_0 \in \mathbb{R}^d$ is as in (SDE). Since Y_t^n is piecewise constant and constant after time n, for every $n \in \mathbb{N}$, there is a unique solution U_t^n to (6.2.4). Let

$$V_\lambda^n f := \mathbb{E}\left[\int_0^\infty e^{-\lambda t} f(U_t^n)\,\mathrm{d}t.\right], \quad f \in C_0(\mathbb{R}^d). \tag{6.2.5}$$

First we will show that $U_t^n(\omega)$ converges to $X_t(\omega)$ for all $\omega \in \Omega$ and $t \geq 0$ and the resolvent operator $V_\lambda^n f$ converges to $S_\lambda f$ for any $f \in C_b(\mathbb{R}^d)$ and $\lambda > 0$ fixed.

Lemma 6.2.1. *We have*

$$\lim_{n\to\infty} U_t^n = X_t \quad \textit{for all } \omega \in \Omega \textit{ and } t \geq 0$$

and for all $f \in C_b(\mathbb{R}^d)$

$$\lim_{n\to\infty} V_\lambda^n f = S_\lambda f.$$

Proof. Note that

$$\lim_{n\to\infty} Y_{t-}^n = X_{t-}.$$

Using dominated convergence

$$\lim_{n\to\infty} U_t^n = \lim_{n\to\infty}\left(x_0 + \int_0^t A(Y_{s-}^n)\,\mathrm{d}Z_s\right) = x_0 + \int_0^t A(X_{s-})\,\mathrm{d}Z_s = X_t.$$

Again by dominated convergence we get

$$\lim_{n\to\infty} V_\lambda^n f = \lim_{n\to\infty} \mathbb{E}\left[\int_0^\infty e^{-\lambda t} f(U_t^n)\,\mathrm{d}t\right] = \mathbb{E}\left[\int_0^\infty e^{-\lambda t} f(X_t)\,\mathrm{d}t\right] = S_\lambda f.$$

□

The following Lemma gives an L^p-bound for V_λ^n, depending on n. Since we want to consider the limit of V_λ^n for $n \to \infty$, the statement of this lemma is not sufficient for our purposes. Thus we have to improve the result to an uniform L^p-bound independent of n afterwards. This will be done in Theorem 6.2.4.

Lemma 6.2.2. *Let $p > \beta$, $n \in \mathbb{N}$ and $\lambda > 0$. There is a constant $c_1 > 0$, depending on n, such that for all $f \in C_b(\mathbb{R}^d)$*

$$|V_\lambda^n f| \le c_1 \|f\|_p.$$

Proof. We follow the proof of [BC06, Lemma 5.1].

Let $n \in \mathbb{N}$ be fixed. In the time interval $[0, \frac{1}{2^n}]$ we have $Y_t^n = x_0$ and therefore $U_t^n \overset{d}{=} U_t = x_0 + A(x_0)Z_t$. Recall that the resolvent operator of $(U_t)_{t\ge 0}$ was denoted by R_λ. Therefore

$$\begin{aligned}\left|\mathbb{E}\left[\int_0^{1/2^n} e^{-\lambda t} f(U_t^n)\,\mathrm{d}t\right]\right| &= \left|\mathbb{E}\left[\int_0^{1/2^n} e^{-\lambda t} f(U_t)\,\mathrm{d}t\right]\right| \\ &\le \mathbb{E}\left[\int_0^{1/2^n} e^{-\lambda t} |f(U_t)|\,\mathrm{d}t\right] \\ &\le R_\lambda(|f|)(x_0).\end{aligned}$$

By Proposition 6.1.4 there exists $c_2 > 0$, independent of x, such that

$$R_\lambda(|f|)(x) \le c_2 \|f\|_p.$$

Let $k \in \mathbb{N}$. In the time interval $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ we have

$$U_t^n = U_{(k/2^n)-}^n + A(X_{k/2^n}) Z_{t-k/2^n}. \tag{6.2.6}$$

That is an affine transformation of a Lévy process and therefore a Markov process. Let R_λ^n denote the resolvent for U_t^n as in (6.2.6). By shifting the integral and using the Markov property of $(U_t^n)_{t\ge 0}$, we get

$$\begin{aligned}\left|\mathbb{E}\left[\int_{k/2^n}^{(k+1)/2^n} e^{-\lambda t} f(U_t^n)\,\mathrm{d}t\right]\right| &= \left|\mathbb{E}\left[\int_0^{1/2^n} e^{-\lambda(t+k/2^n)} f(U_{t+k/2^n}^n)\,\mathrm{d}t\right]\right| \\ &= e^{-\lambda k/2^n}\left|\mathbb{E}\left[\int_0^{1/2^n} e^{-\lambda t} f(U_{t+k/2^n}^n)\,\mathrm{d}t\right]\right| \\ &= e^{-\lambda k/2^n}\left|\mathbb{E}\left[\mathbb{E}^{U_{k/2^n}^n}\left[\int_0^{1/2^n} e^{-\lambda t} f(U_t^n)\,\mathrm{d}t\right]\right]\right| \\ &\le e^{-\lambda k/2^n} \sup_{z\in\mathbb{R}^d} \mathbb{E}^z\left[\int_0^{\infty} e^{-\lambda t} |f(U_t^n)|\,\mathrm{d}t\right] \\ &= e^{-\lambda k/2^n} \sup_{z\in\mathbb{R}^d} R_\lambda^n(|f|)(z).\end{aligned}$$

Again by Proposition 6.1.4, there exists a constant $c_2 > 0$ such that

$$e^{-\lambda k/2^n} \sup_{z\in\mathbb{R}^d} R^n_\lambda(|f|)(z) \le c_2 e^{-\lambda k/2^n}\|f\|_p.$$

Using the triangle inequality we get

$$\begin{aligned}|V^n_\lambda f| = \left|\mathbb{E}\left[\int_0^\infty e^{-\lambda t} f(U^n_t)\,\mathrm{d}t\right]\right| &\le \sum_{k=0}^\infty \left|\mathbb{E}\left[\int_{k/2^n}^{(k+1)/2^n} e^{-\lambda t} f(U^n_t)\,\mathrm{d}t\right]\right| \\ \le c_2\|f\|_p \sum_{k=0}^\infty e^{-\lambda k/2^n} &= \frac{c_2}{1-e^{-\lambda/2^n}}\|f\|_p = c_1\|f\|_p,\end{aligned}$$

which finishes the proof. □

Next we want to prove for every $\lambda > 0$ a uniform bound in n for $\sup\limits_{\|f\|_p\le 1} V^n_\lambda f$. For this purpose, we need to define auxiliary functions.
Let $n \in \mathbb{N}$. For $s \ge 0$ and $\omega \in \Omega$ we define

$$\widetilde{A^n_j}(s,\omega) = \sum_{k=0}^\infty A_j(X_{\frac{k}{2^n}-}(\omega))\mathbb{1}_{\{\frac{k}{2^n}\le s<\frac{k+1}{2^n}\}\cap\{s\le n\}} + A_j(X_{n-}(\omega))\mathbb{1}_{\{s\ge n\}}$$

and for $f \in C^2_b(\mathbb{R}^d)$

$$\widetilde{\mathcal{L}_n} f(x,s,\omega) := \sum_{j=1}^d \int_{\mathbb{R}\setminus\{0\}} [f(x+e_j\widetilde{A^n_j}(s,\omega)h) - f(x) - h\mathbb{1}_{\{|h|\le1\}}\partial_j f(x)\widetilde{A^n_j}(s,\omega)]\frac{c_{\alpha_j}}{|h|^{1+\alpha_k}}\,\mathrm{d}h.$$

Moreover, let

$$\widetilde{\mathcal{B}_n} f(x,s,\omega) = \widetilde{\mathcal{L}_n} f(x,s,\omega) - \mathcal{L}_0 f(x).$$

Note that for each $j \in \{1,\dots,d\}$ by continuity it holds

$$\lim_{n\to\infty} \widetilde{A^n_j}(s,\omega) = A_j(X_{s-}(\omega))$$

and by dominated convergence we have for all $f \in C^2_b(\mathbb{R}^d)$

$$\lim_{n\to\infty} \widetilde{\mathcal{L}_n} f(x,s,\omega) = \mathcal{L}f(X_{s-}(\omega)).$$

Proposition 6.2.3. *Let $f \in C_0(\mathbb{R}^d)$ such that $R_0 f \in C^2_b(\mathbb{R}^d)$. Let $p \in (1,\infty)$ and assume that η defined in* (6.1.25) *satisfies* (Loc)*. Then*

$$\|\widetilde{\mathcal{B}_n} R_0 f\| \le \frac{1}{4}\|f\|_p.$$

Proof. Note that we can rewrite the operator $\widetilde{\mathcal{B}_n}$ by Definition 4.0.3 with weighted second order differences.

Since
$$\sup_{j\in\{1,\dots,d\}} \sup_{(s,\omega)\in[0,\infty)\times\Omega} \left| |\widetilde{A_j^n}(s,\omega)|^{\alpha_j} - |A_j(x_0)|^{\alpha_j} \right| \le \eta,$$
we also have (Loc) if we take $\widetilde{A_j^n}$ instead of A_j.
The proof of Proposition 6.2.3 is now similar to the proof of Proposition 6.1.13. □

We will now improve the result of Lemma 6.2.2 by proving that there is an upper bound independent of n such that the result holds. At first we prove the result for compactly supported functions and afterwards, in Corollary 6.2.5, we show by an elementary limit argument that the result is true for functions in $C_b^2(\mathbb{R}^d)$.

Theorem 6.2.4. *Suppose $p > \beta$ and $\lambda > 0$. There exists a constant $c_1 > 0$, such that for all $n \in \mathbb{N}$ and $g \in C_c^2(\mathbb{R}^d)$*
$$|V_\lambda^n g| \le c_1 \|g\|_p.$$

Proof. The idea of the proof is to write $V_\lambda^n f$ in terms of R_λ and $\widetilde{B}R_\lambda$ and use Proposition 6.2.3 to get an upper bound independent of n. It is based on the proof of [BC06, Theorem 5.3].

Let $\overline{Z}^j$ be Z^j with all jumps larger than one in absolute value removed. Suppose $f \in C_b^2(\mathbb{R}^d)$ and apply Itô's formula to $(U_t^n)_{t\ge 0}$:

$$\begin{aligned}
f(U_t^n) - f(U_0^n) &= \int_0^t \nabla f(U_{s-}^n)\cdot \mathrm{d}U_s^n + \sum_{s\le t}(f(U_s^n) - f(U_{s-}^n) - \nabla f(U_{s-}^n)\cdot \Delta U_s^n) \\
&= \int_0^t \nabla f(U_{s-}^n) A(Y_{s-}^n)\,\mathrm{d}Z_s + \sum_{s\le t}(f(U_{s-}^n + A(Y_{s-}^n)\Delta Z_s) \\
&\qquad - f(U_{s-}^n) - \nabla f(U_{s-}^n)A(Y_{s-}^n)\Delta Z_s) \\
&= \int_0^t \nabla f(U_{s-}^n) A(Y_{s-}^n)\,\mathrm{d}\overline{Z}_s + \sum_{s\le t}(f(U_{s-}^n + A(Y_{s-}^n)\Delta Z_s) \\
&\qquad - f(U_{s-}^n) - \mathbb{1}_{\{|\Delta Z_s|\le 1\}}\nabla f(U_{s-}^n)A(Y_{s-}^n)\Delta Z_s) \\
&= \sum_{j=1}^d \int_0^t \partial_j f(U_{s-}^n)\widetilde{A_j^n}(s,\omega)\,\mathrm{d}\overline{Z}_s^j + \sum_{j=1}^d\sum_{s\le t}(f(U_{s-}^n + e_j\widetilde{A_j^n}(s,\omega)\Delta Z_s^j) \\
&\qquad - f(U_{s-}^n) - \mathbb{1}_{\{|\Delta Z_s^j|\le 1\}}\partial_j f(U_{s-}^n)\widetilde{A_j^n}(s,\omega)\Delta Z_s^j).
\end{aligned}$$

By the Lévy system formula

$$f(U_t^n) - f(U_0^n) + \sum_{j=1}^d \int_0^t \int_{\mathbb{R}\setminus\{0\}} (f(U_{s-}^n + e_j\widetilde{A_j}(s,\omega)u) - f(U_{s-}^n)$$

$$- u\mathbb{1}_{\{|u|\le 1\}}\partial_j f(U^n_{s-})\widetilde{A_j}(s,\omega))\frac{c_{\alpha_j}}{|u|^{1+\alpha_j}}\,\mathrm{d}u\,\mathrm{d}s$$

is a $\mathbb{P}$-martingale.

Taking the expectation with respect to $\mathbb{P}$, we get

$$\mathbb{E}\left[f(U^n_t)\right] = \mathbb{E}\left[f(U^n_0)\right] + \mathbb{E}\left[\int_0^t \widetilde{\mathcal{L}_n} f(U^n_{s-}(\omega), s, \omega)\,\mathrm{d}s\right].$$

Multiplying both sides of the equation by $e^{-\lambda t}$ and integrating over t from 0 to ∞, we obtain

$$\begin{aligned} V^n_\lambda f &= \frac{1}{\lambda}\mathbb{E}[f(U^n_0)] + \mathbb{E}\left[\int_0^\infty e^{-\lambda t}\int_0^t \widetilde{\mathcal{L}_n} f(U^n_s(\omega), s, \omega)\,\mathrm{d}s\,\mathrm{d}t\right] \\ &= \frac{1}{\lambda}\mathbb{E}[f(U^n_0)] + \mathbb{E}\left[\int_0^\infty \widetilde{\mathcal{L}_n} f(U^n_s(\omega), s, \omega)\int_s^\infty e^{-\lambda t}\,\mathrm{d}t\,\mathrm{d}s\right] \\ &= \frac{1}{\lambda}\mathbb{E}[f(U^n_0)] + \frac{1}{\lambda}\mathbb{E}\left[\int_0^\infty e^{-\lambda s}\widetilde{\mathcal{L}_n} f(U^n_s(\omega), s, \omega)\,\mathrm{d}s\right]. \end{aligned} \tag{6.2.7}$$

Let $g \in C^2_c(\mathbb{R}^d)$. Then

$$(\lambda - \mathcal{L}_0)R_\lambda g(x) = g(x) \iff \mathcal{L}_0 R_\lambda g(x) = -g(x) + \lambda R_\lambda g(x).$$

Hence

$$\widetilde{\mathcal{L}_n} R_\lambda g(x, s, \omega) = \widetilde{\mathcal{B}_n} R_\lambda g(x, s, \omega) - g(x) + \lambda R_\lambda g(x). \tag{6.2.8}$$

Let $f = R_\lambda g$. By translation invariance $f \in C^2_b(\mathbb{R}^d)$. Plugging (6.2.8) into (6.2.7) for $f = R_\lambda g$ yields

$$V^n_\lambda R_\lambda g = \frac{1}{\lambda}\mathbb{E}[R_\lambda g(U^n_0)] + \frac{1}{\lambda}\mathbb{E}\left[\int_0^\infty e^{-\lambda s}\widetilde{\mathcal{B}_n} R_\lambda g(U^n_s(\omega), s, \omega)\,\mathrm{d}s\right] - \frac{1}{\lambda}V^n_\lambda g + V^n_\lambda R_\lambda g,$$

which is equivalent to

$$V^n_\lambda g = \mathbb{E}[R_\lambda g(U^n_0)] + \mathbb{E}\left[\int_0^\infty e^{-\lambda s}\widetilde{\mathcal{B}_n} R_\lambda g(U^n_s(\omega), s, \omega)\,\mathrm{d}s\right].$$

Let $h = g - \lambda R_\lambda g$. Then by Lemma 6.1.7

$$R_0 h = R_0(g - \lambda R_\lambda g) = R_0 g - \lambda R_0 R_\lambda = R_\lambda g.$$

Thus $R_0 h \in C^2_b(\mathbb{R}^d)$. Using the triangle inequality and Proposition 6.1.4, we get

$$\|h\|_p \le \|g\|_p + \|\lambda R_\lambda g\|_p \le 2\|g\|_p.$$

Note

$$\left|\mathbb{E}\left[\int_0^\infty e^{-\lambda s}\widetilde{\mathcal{B}_n} R_0 h(U^n_s(\omega), s, \omega)\,\mathrm{d}s\right]\right| \le \mathbb{E}\left[\int_0^\infty e^{-\lambda s}|\widetilde{\mathcal{B}_n} R_0 h(U^n_s(\omega), s, \omega)|\,\mathrm{d}s\right]$$

$$= V_\lambda^n(|\widetilde{\mathcal{B}_n} R_0 h(U_s^n(\omega), s, \omega)|).$$

We define

$$\Theta_n := \sup_{\|g\|_p \le 1} |V_\lambda^n g|.$$

By Lemma 6.2.2 there is a $c_2 > 0$, depending on n, but being independent of g, such that $|V_\lambda^n g| \le c_2 \|g\|_p$. Hence $\Theta_n \le c_2 < \infty$.

Now we need to find a constant, independent of n, such that the assertion holds. Note that we have show $h \in C_0(\mathbb{R}^d)$ with $R_0 h \in C_b^2(\mathbb{R}^d)$ which allows us to apply Proposition 6.2.3 on h. By Proposition 6.1.4 and Proposition 6.2.3 there exists a $c_3 > 0$, independent of n, such that

$$\begin{aligned} |V_\lambda^n g| &= \left| R_\lambda g(x_0) + \mathbb{E}\left[\int_0^\infty e^{-\lambda s} \widetilde{\mathcal{B}_n} R_0 h(U_s^n(\omega), s, \omega) \,\mathrm{d}s \right] \right| \\ &\le |R_\lambda g(x_0)| + \left| \mathbb{E}\left[\int_0^\infty e^{-\lambda s} \widetilde{\mathcal{B}_n} R_0 h(U_s^n(\omega), s, \omega) \,\mathrm{d}s \right] \right| \\ &\le c_3 \|g\|_p + V_\lambda^n(|\widetilde{\mathcal{B}_n} R_0 h(U_s^n(\omega), s, \omega)|) \\ &\le c_3 \|g\|_p + \Theta_n(\|\widetilde{\mathcal{B}_n} R_0 h(U_s^n(\omega), s, \omega)\|_p) \\ &\le c_3 \|g\|_p + \Theta_n \left(\frac{1}{4} \|h\|_p \right) \le \|g\|_p \left(c_3 + \frac{1}{2} \Theta_n \right). \end{aligned}$$

Taking the supremum over all $g \in C_c^2(\mathbb{R}^d)$ with $\|g\|_p \le 1$, we get

$$\Theta_n \le c_3 + \frac{1}{2}\Theta_n \iff \Theta_n \le 2c_3 < \infty,$$

which proves the assertion for $g \in C_c^2(\mathbb{R}^d)$. □

Note that we had to take $g \in C_c^2(\mathbb{R}^d)$ in the proof of Theorem 6.2.4 so that the expressions in the proof are well-defined. By a standard limit argument we conclude.

Corollary 6.2.5. *Suppose $p > \beta$ and $\lambda > 0$. There exists a constant $c_1 > 0$, such that for all $n \in \mathbb{N}$ and $f \in C_b^2(\mathbb{R}^d)$*

$$|V_\lambda^n f| \le c_1 \|f\|_p.$$

Proof. By Theorem 6.2.4 we already know the result holds on $C_c^2(\mathbb{R}^d)$. The assertion on $C_b^2(\mathbb{R}^d)$ follows by dominated convergence. Let $f \in C_b^2(\mathbb{R}^d)$. Without loss of generality, we can assume $f \in L^p(\mathbb{R}^d)$. Otherwise the right hand side of the assertion is infinite and the statement trivially holds true. Let $g_m \in C_c^2(\mathbb{R}^d)$ such that $g_m = f$ on B_m and $\operatorname{supp}(g_m) \subset B_{m+1}$. Then $g_m \to f$ as $m \to \infty$. Since $f \in L^p(\mathbb{R}^d)$, by dominated convergence g_m also converges to f in L^p. Moreover, since g_m and f are bounded, we have

$$\lim_{m\to\infty} |V_\lambda^n g_m(x)| = |V_\lambda^n \lim_{m\to\infty} g_m(x)|,$$

which finishes the proof. □

Finally, we can prove the desired result.

Corollary 6.2.6. *Suppose $p > \beta$. There exists a constant $c_1 > 0$, such that for all $f \in C_b^2(\mathbb{R}^d)$*

$$|S_\lambda f| \le c_1 \|f\|_p.$$

Proof. By Lemma 6.2.1 and Corollary 6.2.5 we get

$$|S_\lambda f| = \lim_{n\to\infty} |V_\lambda^n f| \le \lim_{n\to\infty} c_1 \|f\|_p = c_1 \|f\|_p,$$

where we have used the fact that c_1 is independent of n. □

The next proposition gives a representation of $S_\lambda f$. We follow the proof of [BC06, Proposition 6.1].

Proposition 6.2.7. *Let $f \in C_b^2(\mathbb{R}^d)$ and $\lambda > 0$. Then*

$$S_\lambda f = R_\lambda f(x_0) + S_\lambda \mathcal{B} R_\lambda f.$$

Proof. Let $\mathbb{P}$ be a solution to the martingale problem for $\mathcal{L}$ started at x_0. Then, if $g \in C_b^2(\mathbb{R}^d)$

$$g(X_t) - g(X_0) - \int_0^t \mathcal{L} f(X_s)\,\mathrm{d}s \qquad \text{is a } \mathbb{P}\text{-martingale.}$$

Hence

$$\mathbb{E}[g(X_t)] = \mathbb{E}[g(X_0)] - \mathbb{E}\left[\int_0^t \mathcal{L} g(X_s)\,\mathrm{d}s\right].$$

Multiplying both sides by $e^{-\lambda t}$ and integratinging over t from 0 to ∞ we get

$$\begin{aligned}
S_\lambda g &= \frac{1}{\lambda} g(x_0) + \mathbb{E}\left[\int_0^\infty \int_0^t e^{-\lambda t} \mathcal{L} g(X_s)\,\mathrm{d}s\,\mathrm{d}t\right] \\
&= \frac{1}{\lambda} g(x_0) + \mathbb{E}\left[\int_0^\infty \mathcal{L} g(X_s) \int_s^\infty e^{-\lambda t}\,\mathrm{d}t\,\mathrm{d}s\right] \\
&= \frac{1}{\lambda} g(x_0) + \frac{1}{\lambda}\mathbb{E}\left[\int_0^\infty e^{-\lambda s} \mathcal{L} g(X_s)\,\mathrm{d}s\right] \\
&= \frac{1}{\lambda} g(x_0) + \frac{1}{\lambda} S_\lambda \mathcal{L} g.
\end{aligned}$$

Set $g = R_\lambda f$ for $f \in C_c^2(\mathbb{R}^d)$. Then

$$\mathcal{L} R_\lambda f = \mathcal{L}_0 R_\lambda f + \mathcal{B} R_\lambda f = \lambda R_\lambda f - f + \mathcal{B} R_\lambda f.$$

Hence

$$S_\lambda g = S_\lambda R_\lambda f = \frac{1}{\lambda} R_\lambda g(x_0) + \frac{1}{\lambda} S_\lambda \mathcal{L} R_\lambda f$$

$$= \frac{1}{\lambda} R_\lambda g(x_0) + S_\lambda R_\lambda f - \frac{1}{\lambda} S_\lambda f + \frac{1}{\lambda} S_\lambda \mathcal{B} R_\lambda f,$$

which is equivalent to

$$S_\lambda f = R_\lambda f(x_0) + S_\lambda \mathcal{B} R_\lambda f.$$

By the same argument as in the proof of Corollary 6.2.5, we get the assertion for $f \in C_b^2(\mathbb{R}^d)$. □

6.3. Auxiliary results

In this section we will give some auxiliary results, which will be important to prove Theorem 6.0.1.

The following theorem gives a sufficient condition for uniqueness of solutions to the martingale problem for $\mathcal{L}$ started at x_0.

Theorem 6.3.1. *Let $\mathbb{P}_1, \mathbb{P}_2$ be two solutions to the martingale problem for $\mathcal{L}$ started at x_0. Suppose for all $x \in \mathbb{R}^d$, $\lambda > 0$ and $f \in C_b^2(\mathbb{R}^d)$,*

$$\mathbb{E}_1 \left[\int_0^\infty e^{-\lambda t} f(X_t) \, \mathrm{d}t \right] = \mathbb{E}_2 \left[\int_0^\infty e^{-\lambda t} f(X_t) \, \mathrm{d}t \right].$$

Then for each $x_0 \in \mathbb{R}^d$ the solution to the martingale problem for $\mathcal{L}$ has a unique solution.

A proof of this theorem can be found e.g. in [Bas98, Theorem V.3.2]. Although the author studies the martingale problem for the elliptic operator $\mathcal{A}$ in nondivergence form given on $C^2(\mathbb{R}^d)$ by

$$\mathcal{A} f(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial f(x)}{\partial x_i},$$

where a_{ij} and b_i are bounded and measurable, the proof of Theorem 6.3.1 does not significantly change and does apply for a large class of operators. Hence, we will not give the proof and refer the reader to [Bas98, Theorem V.3.2].

Recall that assumption (Loc) states

$$\sup_{j \in \{1,\dots,d\}} \| |A_j(\cdot)|^{\alpha_j} - |A_j(x_0)|^{\alpha_j} \|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{4da(p^*-1)},$$

where $a = \max\{|A_1(x_0)|^{-\alpha_1}, \dots, |A_d(x_0)|^{-\alpha_d}\}$ and $p^* - 1 = \max\left\{(p-1), (p-1)^{-1}\right\}$. By Proposition 6.1.13 this assumption implies $\|\mathcal{B} R_0 h\|_p \leq \frac{1}{4} \|h\|_p$ for $h \in C_c^2(\mathbb{R}^d)$.

We first prove uniqueness of solutions to the martingale problem for $\mathcal{L}$ under the assumption (Loc).

Proposition 6.3.2. *Let $x_0 \in \mathbb{R}^d$ and assume* (Loc) *holds for the coefficients of $\mathcal{L}$. Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ are two solutions to the martingale problem for $\mathcal{L}$ started at x_0. Then $\mathbb{P}_1 = \mathbb{P}_2$.*

Proof. We follow the proof of [BC06, Proposition 6.2].

Let $p > \beta$. Moreover let S^1_λ and S^2_λ be defined as above with respect to $\mathbb{P}_1$ and $\mathbb{P}_2$ respectively. Set

$$S^\Delta_\lambda g := S^1_\lambda g - S^2_\lambda g,$$

where $g \in C^2_b(\mathbb{R}^d)$ and

$$\Theta = \sup_{\|g\|_p \le 1} |S^\Delta_\lambda g|.$$

By Corollary 6.2.6, we have $\Theta < \infty$. Proposition 6.2.7 implies

$$S^\Delta_\lambda f = S^\Delta_\lambda \mathcal{B} R_\lambda f, \quad f \in C^2_b(\mathbb{R}^d).$$

Let $f \in C^2_c(\mathbb{R}^d)$ and define $h := f - \lambda R_\lambda f$. As in the proof of Theorem 6.2.4 we conclude $h \in C_0(\mathbb{R}^d)$ and $R_0 h = R_\lambda f \in C^2_b(\mathbb{R}^d)$. By Proposition 6.1.13, we have

$$\|BR_0 h\|_p \le \frac{1}{4}\|h\|_p.$$

Furthermore, $\|h\|_p \le 2\|f\|_p$ and thus Therefore

$$|S^\Delta_\lambda \mathcal{B} R_\lambda f| = |S^\Delta_\lambda \mathcal{B} R_0 h| \le= \Theta \|\mathcal{B} R_0 h\|_p \le \frac{1}{4}\Theta\|h\|_p \le \frac{1}{2}\Theta\|f\|_p.$$

Taking the supremum over $f \in C^2_c(\mathbb{R}^d)$ with $\|f\|_p \le 1$, we have $\Theta \le \frac{1}{2}\Theta$ and since Θ is finite we have $\Theta = 0$ by Corollary 6.2.6. The result for $f \in C^2_b(\mathbb{R}^d)$ follows as in Corollary 6.2.5.

Therefore for any $f \in C^2_b(R^d)$ and $\lambda > 0$, $S^1_\lambda f = S^2_\lambda f$. The assertion follows from Theorem 6.3.1. ☐

Next we prove a tightness estimate, which will be of great interest in the study of the uniqueness of weak solutions to (SDE).

Corollary 6.3.3. *Let $\mathbb{P}$ be a weak solution to* (SDE)*. Then there exists a constant $c_1 > 0$ depending only on the upper bounds of $|A_{ij}(x)|$ for $1 \le i, j \le d$ and the dimension d, such that for every $\delta > 0$ and $t \ge 0$*

$$\mathbb{P}\left(\sup_{s \le t} |X_s - X_0| > \delta\right) \le ct/\delta^2.$$

Proof. As in the proof of Theorem 5.0.3, if $f \in C^2_b(\mathbb{R}^d)$

$$f(X_t) - f(X_0) - c_f t$$

is a $\mathbb{P}$-supermartingale. Hence the assertion follows by Proposition 4.0.7 for $\eta = \delta$. ☐

Corollary 6.3.3 immediately implies that for any weak solution $\mathbb{P}$ to (SDE)

$$\lim_{\lambda\to\infty} \mathbb{P}\left(\sup_{s\le t} |X_s - X_0| > \lambda\right) = 0. \tag{6.3.1}$$

In the following we show that regular conditional probabilities for solutions to the martingale problem are also solutions to the martingale problem.

Let Θ_t be the shift operator on $\mathbb{D}([0,\infty))$ that is $f(s)\circ\Theta_t = f(s+t)$. Recall the definition of the first exit time:

$$\tau := \tau_{B_r(x_0)} := \inf\{t \ge 0\colon |X_t - x_0| \ge r\}.$$

We define

$$\mathbb{P}_\tau(A) = \mathbb{P}_i(A\circ\Theta_\tau)$$

and let $\mathbb{E}_\tau$ be the expectation with respect to $\mathbb{P}_\tau$.

Lemma 6.3.4. *Let $\mathbb{P}$ be a solution to the martingale problem for $\mathcal{L}$ started at $x_0 \in \mathbb{R}^d$ and $\mathbb{Q}(\cdot,\cdot)$ be a regular conditional probability for $\mathbb{E}[\cdot\,|\mathcal{F}_\tau]$. Then $\mathbb{Q}(\omega,\cdot)$ is $\mathbb{P}$-almost surely a solution to the martingale problem for $\mathcal{L}$ started at $X_\tau(\omega)$.*

Proof. The proof is based on the proof of [Bas98, Proposition VI.2.1].

Let $B(\omega) = \{\omega' \in \Omega\colon X_0(\omega') = X_\tau(\omega))\}$. We first show $\mathbb{Q}(\omega, B(\omega)) = 1$. Let $A \in \mathcal{F}_\tau$. Then

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}\left(X_\tau = X_\tau; A\right) \\ &= \mathbb{E}\left[\mathbb{P}\left(X_\tau = X_0\circ\Theta_\tau|\mathcal{F}_\tau\right); A\right] \\ &= \mathbb{E}\left[\mathbb{Q}(\omega, B(\omega)); A\right]. \end{aligned} \tag{6.3.2}$$

Hence $\mathbb{Q}(\omega, B(\omega)) = 1$, since (6.3.2) holds for all $A \in \mathcal{F}_\tau$.

Let $f \in C_b^2(\mathbb{R}^d)$. It remains to show that

$$M_t := f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\,ds$$

is a $\mathbb{Q}(\omega,\cdot)$-martingale for almost every $\omega \in \Omega$.

By definition, we can write for any $s \ge 0$, $B \in \mathcal{F}_t$ and $A \in \mathcal{F}_\tau$

$$\mathbb{E}_\tau((M_s\mathbb{1}_B); A) = \mathbb{E}((M_s\mathbb{1}_B)\circ\Theta_\tau; A) = \mathbb{E}(M_s\circ\Theta_\tau; B\circ\Theta_\tau\cap A). \tag{6.3.3}$$

It is sufficient to show for $u > t \ge 0$

$$\mathbb{E}\left(M_t\circ\Theta_\tau; B\circ\Theta_\tau\cap A)\right] = \mathbb{E}\left(M_u\circ\Theta_\tau; B\circ\Theta_\tau\cap A\right), \tag{6.3.4}$$

where $B \in \mathcal{F}_t$ and $A \in \mathcal{F}_\tau$. By definition this is

$$\mathbb{E}_\mathbb{Q}(M_u; A) = \mathbb{E}_\mathbb{Q}(M_t; A)$$

for all $A \in \mathcal{F}_t$, where $\mathbb{E}_{\mathbb{Q}}$ is the expectation with respect to $\mathbb{Q}(\omega, \cdot)$. Therefore

$$\mathbb{E}_{\mathbb{Q}}(M_u|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(M_t|\mathcal{F}_t) = M_t,$$

which finishes the proof.

Thus we have to show (6.3.4). First note that

$$\begin{aligned} M_t \circ \Theta_\tau &= f(X_{t+\tau}) - f(X_\tau) - \int_\tau^{t+\tau} \mathcal{L}f(X_s)\, ds \\ &= \left(f(X_{t+\tau}) - f(X_0) - \int_0^{t+\tau} \mathcal{L}f(X_s)\, ds \right) \\ &\qquad - \left(f(X_\tau) - f(X_0) - \int_0^{\tau} \mathcal{L}f(X_s)\, ds \right) \\ &= M_{t+\tau} - M_\tau. \end{aligned}$$

Hence, $M_t \circ \Theta_\tau$ is a martingale with respect to $\mathcal{F}_{t+\tau}$. Let $u > t \geq 0$. Using the martingale property we get

$$\begin{aligned} \mathbb{E}\left(M_t \circ \Theta_\tau; B \circ \Theta_\tau \cap A\right)] &= \mathbb{E}\left(M_{t+\tau} - M_\tau; B \circ \Theta_\tau \cap A\right) \\ &= \mathbb{E}\left(M_{t+\tau}; B \circ \Theta_\tau \cap A\right) - \mathbb{E}\left(M_\tau; B \circ \Theta_\tau \cap A\right) \\ &= \mathbb{E}\left(\mathbb{E}\left(M_{u+\tau}|\mathcal{F}_{t+\tau}\right); B \circ \Theta_\tau \cap A\right) - \mathbb{E}\left(M_\tau; B \circ \Theta_\tau \cap A\right) \\ &= \mathbb{E}\left(M_{u+\tau}; B \circ \Theta_\tau \cap A\right) - \mathbb{E}\left(M_\tau; B \circ \Theta_\tau \cap A\right) \\ &= \mathbb{E}\left(M_{u+\tau} - M_\tau; B \circ \Theta_\tau \cap A\right) \\ &= \mathbb{E}\left(M_u \circ \Theta_\tau; B \circ \Theta_\tau \cap A\right), \end{aligned}$$

where $B \in \mathcal{F}_t$ and $A \in \mathcal{F}_\tau$. This proves (6.3.4). □

6.4. Proof of the uniqueness for solutions to the system of stochastic differential equations

Proof of Theorem 6.0.1. The proof follows the idea of the proof of [Bas98, Theorem VI.3.6]

Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be two solutions to the martingale problem for $\mathcal{L}$ started at $x_0 \in \mathbb{R}^d$. Recall that we consider the canonical process $(X_t)_{t\geq 0}$ on the Skorohod space $\Omega = \mathbb{D}([0,\infty);\mathbb{R}^d)$, i.e. $X_t(\omega) = \omega(t)$ and $(\mathcal{F}_t)$ is the minimal augmented filtration with respect to the process $(X_t)_t$. We denote the σ-field of the probability space by $\mathcal{F}_\infty$.

For $N \in \mathbb{N}$ let

$$\rho_N := \tau_{\overline{B_N}} := \inf\{t \geq 0 : |X_t - x_0| > N\}.$$

Since càdlàg functions are locally bounded the process X_t does not explode in finite time. Further by the transience of Z_t, we have for $i = 1, 2$,

$$\rho_N \to \infty \quad \mathbb{P}_i\text{-a.s.} \quad \text{as } N \to \infty. \tag{6.4.1}$$

To prove $\mathbb{P}_1 = \mathbb{P}_2$, we have to show that all finite dimensional distributions of X_t under $\mathbb{P}_1$ and $\mathbb{P}_2$ are the same. By (6.4.1) it is sufficient to show that there is a $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$

$$\mathbb{P}_1\big|_{\mathcal{F}_{\rho_N}} = \mathbb{P}_2\big|_{\mathcal{F}_{\rho_N}}.$$

Choose $N_0 = \lfloor |x_0| \rfloor + 1$ and let $N \geq N_0$ be arbitrary. Set

$$\|A\|_\infty := \max_{1\leq j\leq d} \sup_{x\in\mathbb{R}^d} |A_j(x)|.$$

Since $A(x)$ is non-degenerate at each point $x \in \mathbb{R}^d$,

$$\mu_1(A, N) := \inf_{x\in B_N} \inf_{\substack{u\in\mathbb{R}^d \\ \|u\|=1}} |A(x)u| > 0.$$

Let $\widetilde{\eta_0} := \frac{\eta_0}{2}$, where η_0 is defined as in (Loc). Since A is continuous on $\mathbb{R}^d$, it is uniformly continuous on $\overline{B_{N+1}}$. Hence there is a $r \in (0, 1)$ such that

$$\sup_{1\leq j\leq d} |A_j(x) - A_j(y)| < \frac{\widetilde{\eta_0}}{2} \qquad \text{for } x, y \in B_{N+1}, |x - y| < r.$$

Let $\widetilde{A} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be diagonal such that $\widetilde{A} = A$ on B_r and the functions $x \mapsto \widetilde{A_j}(x)$ on the diagonal are continuous and bounded for all $j \in \{1, \ldots, d\}$. Moreover let $\widetilde{A}$ be uniformly non-degenerate such that

$$\mu_{1,1}(\widetilde{A}) = \inf_{u\in\mathbb{R}^d : |u|=1} \inf_{x\in\mathbb{R}^d} |\widetilde{A}(x)u| > \frac{\mu_1(A, N)}{2}$$

and

$$\sup_{j\in\{1,\ldots,d\}} |\widetilde{A}_j(\cdot) - \widetilde{A}_j(x_0)|_{L^\infty(B_r)} < \widetilde{\eta_0}.$$

Let $\widetilde{\mathcal{L}}$ be defined as $\mathcal{L}$ with A replaced by $\widetilde{A}$. By Proposition 6.3.2, there is a unique solution of the martingale problem for $\widetilde{\mathcal{L}}$ started at any $x_0 \in \mathbb{R}^d$. We call this solution $\widetilde{\mathbb{P}}$.

Let $\widetilde{\mathbb{Q}}(\cdot, \cdot)$ be a regular conditional probability for $\widetilde{\mathbb{E}}[\cdot\,|\mathcal{F}_\tau]$ By Lemma 6.3.4 $\widetilde{\mathbb{Q}}(\omega, \cdot)$ is $\widetilde{\mathbb{P}}$-almost surely a solution to the martingale problem for $\widetilde{\mathcal{L}}$ started at $X_\tau(\omega)$. For abbreviation we denote this measure by $\widetilde{\mathbb{Q}}$.

Define the measure on $(\mathcal{F}_\infty \circ \Theta_\tau) \cap \mathcal{F}_\tau$ by

$$\overline{\mathbb{P}}(A \cap B \circ \Theta_\tau) := \int_A \widetilde{\mathbb{Q}}(B)\, \mathrm{d}\mathbb{P}_i, \quad A \in \mathcal{F}_\tau,\ B \in \mathcal{F}_\infty,$$

which represents the process behaving according to $\mathbb{P}_i$ up to time τ and afterwards according to $\widetilde{\mathbb{P}}$.

We now show that $\overline{\mathbb{P}}_i$ solves the martingale problem for $\widetilde{\mathcal{L}}$ started at x_0.

Clearly $\overline{\mathbb{P}}_i(X_0 = x_0) = \mathbb{P}_i(X_0 = x_0) = 1$.
Let $f \in C_b^2(\mathbb{R}^d)$. Then

$$\begin{aligned} M_t &= f(X_{t\wedge\tau}) - f(X_0) - \int_0^{t\wedge\tau} \widetilde{\mathcal{L}} f(X_s)\,\mathrm{d}s \\ &= f(X_{t\wedge\tau}) - f(X_0) - \int_0^{t\wedge\tau} \mathcal{L} f(X_s)\,\mathrm{d}s \end{aligned}$$

is $\mathcal{F}_\tau$ measurable for each $t \geq 0$ and by assumption a $\mathbb{P}_i$-martingale. Therefore $(M_t)_{t\geq 0}$ a $\overline{\mathbb{P}}_i$-martingale. Further

$$N_t = f(X_{t+\tau}) - f(X_\tau) - \int_\tau^{t+\tau} \widetilde{\mathcal{L}}\,\mathrm{d}s$$

is a $\overline{\mathbb{P}}_i$-martingale by Lemma 6.3.4.

Hence $\overline{\mathbb{P}}_i$, $i = 1, 2$, is a solution to the martingale problem for $\widetilde{\mathcal{L}}$ started at x_0. By definition of $\widetilde{\mathcal{L}}$ the coefficients satisfy the assumptions of Proposition 6.3.2 and therefore $\overline{\mathbb{P}}_1 = \overline{\mathbb{P}}_2$, which implies $\mathbb{P}_1\big|_{\mathcal{F}_\tau} = \mathbb{P}_2\big|_{\mathcal{F}_\tau}$.

We define the sequence of exit times $(\tau_k)_{k\in\mathbb{N}}$ as follows

$$\tau_1 := \tau \quad \text{and} \quad \tau_{k+1} = \inf\{t > \tau_k \colon |X_t - X_{\tau_k}| > r\} \wedge \rho_N.$$

Iterating the piecing-together method from before, we get $\mathbb{P}_1 = \mathbb{P}_2$ on $\mathcal{F}_{\tau_k}$ for all $k \in \mathbb{N}$. By Corollary 6.3.3 it holds that $\tau_k \to \rho_N$ as $k \to \infty$ and hence we get $\mathbb{P}_1 = \mathbb{P}_2$ on $\mathcal{F}_{\rho_N}$.

This finishes the proof. $\square$

Part III.

Regularity estimates for anisotropic nonlocal equations

Structure of Part III

In this part we study regularity estimates of weak solutions to nonlocal equations. We consider linear nonlocal operators of the form

$$\mathcal{L}u(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^d \setminus B_\epsilon(x)} (u(y) - u(x))\, \mu(x, \mathrm{d}y), \tag{6.4.2}$$

where $(\mu(x, \cdot))_{x \in \mathbb{R}^d}$ is a family of measures whose precise properties will be formulated in the first chapter of this part. Weak solutions are defined with the help of nonlocal bilinear forms. The aim of this part is to show that weak solutions to

$$\mathcal{L}u = 0 \quad \text{in } (-1,1)^d, \tag{6.4.3}$$

satisfy an a priori Hölder estimate.

This part consists of three chapters and an appendix chapter. Chapter 7 provides a detailed exposition of weak solutions. It is intended to familiarize the reader with nonlocal bilinear forms and contains important definitions. Moreover, in this chapter a Sobolev type inequality on the whole of $\mathbb{R}^d$ and a localized version is proven.
The aim Chapter 8 is to develop important properties of weak supersolutions to

$$\mathcal{L}u = f \quad \text{in } M_r(x),$$

where $M_r(x)$ are balls in an appropriate chosen metric and f is a sufficiently regular function. Furthermore, we prove the weak Harnack inequality of weak supersolutions to the foregoing differential equation. This is done by Moser iteration.
In Chapter 9 an a priori Hölder estimate for weak solutions to (6.4.3) is proven using the weak Harnack inequality for weak supersolutions and a decay of oscillation estimate.
Appendix A contains some examples of permissible families of measures $\mu(x, \cdot)$, $x \in \mathbb{R}^d$.

7. Nonlocal equations and weak solutions

This chapter contains a comprehensive exposition of the main objects of Part III. We introduce function spaces, state the assumptions on our model and define the notion of weak solutions.

Let $d \in \mathbb{N}, d \geq 3$ and $\alpha_1, \dots, \alpha_d \in (0,2)$. We set

$$\beta = \sum_{k=1}^{d} \frac{1}{\alpha_k}. \tag{7.0.1}$$

Throughout this part the dimension d is always assumed to be greater or equal 3.

Let $\operatorname{diag} := \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \colon x = y\}$ and let $(\mu(x,\cdot))_{x\in\mathbb{R}^d}$ be a family of measures with the following symmetry property:

Assumption 1. *For every* $(A \times B) \in ((\mathbb{R}^d \times \mathbb{R}^d) \setminus \operatorname{diag})$

$$\int_A \int_B \mu(x, \mathrm{d}y)\, \mathrm{d}x = \int_B \int_A \mu(x, \mathrm{d}y)\, \mathrm{d}x.$$

Furthermore, let $(\mu(x,\cdot))_{x\in\mathbb{R}^d}$ satisfy the following uniform Lévy condition:

Assumption 2. *We have*

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (|x-y|^2 \wedge 1)\mu(x, \mathrm{d}y) < \infty.$$

Later, we will add two more assumptions on the family $(\mu(x,\cdot))_{x\in\mathbb{R}^d}$.

Let $\mathcal{L}$ be a nonlocal operator of the form

$$\mathcal{L}u(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^d \setminus B_\epsilon(x)} (u(y) - u(x))\, \mu(x, \mathrm{d}y)$$

and let $\Omega \subset \mathbb{R}^d$ be an open and bounded set. To put the nonlocal problem

$$\begin{aligned} \mathcal{L}u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{in } \mathbb{R}^d \setminus \Omega, \end{aligned} \tag{7.0.2}$$

for appropriate functions f, g, into a functional analytic framework, we need to define function spaces which provide regularity across the boundary.

For this approach, we first recall the definition of weak solutions to elliptic operators in divergence form.
Let $\Omega \subset \mathbb{R}^d$ be open. Further let $a : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be a measurable and uniformly elliptic function, i.e. there is $\Lambda > 0$ such that

$$\Lambda^{-1}|\xi|^2 \leq (a(x)\xi)\cdot\xi \leq \Lambda|\xi|^2 \quad \text{for all } x, \xi \in \mathbb{R}^d.$$

Let $f \in L^2(\Omega)$ and $g \in H^1(\Omega)$. A function $u \in H^1(\Omega)$ is called weak solution of

$$\begin{aligned} \operatorname{div}(a\nabla u) &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned}$$

if $u - g \in H_0^1(\Omega)$ and for every $\phi \in H_0^1(\Omega)$

$$\int_\Omega (a\nabla u)\cdot\nabla\phi = \int_\Omega f\phi.$$

The existence of a unique weak solutions is a direct consequence of the Lax-Milgram Lemma.

It is clear, that $H^1(\Omega)$ and $H_0^1(\Omega)$ are not appropriate spaces to study weak solutions to (7.0.2). We need to replace them by function space which encode information on $\mathbb{R}^d \setminus \Omega$ in an expedient way, such that the expressions are meaningful.

We will introduce a class of functions which are suitable in the nonlocal context and give a definition of weak solutions to (7.0.2). The following definitions go back to [FKV15].

Definition 7.0.1. *Let $\Omega \subset \mathbb{R}^d$ open and bounded. We define the function space*

$$V^\mu(\Omega|\mathbb{R}^d) := \left\{ u : \mathbb{R}^d \to \mathbb{R} : \ u\big|_\Omega \in L^2(\Omega), (u,u)_{V^\mu(\Omega|\mathbb{R}^d)} < \infty \right\}, \tag{7.0.3}$$

where

$$(u,v)_{V^\mu(\Omega|\mathbb{R}^d)} := \int_\Omega \int_{\mathbb{R}^d} (u(x)-u(y))(v(x)-v(y))\, \mu(x,\mathrm{d}y)\,\mathrm{d}x.$$

Moreover, let

$$H_\Omega^\mu(\mathbb{R}^d) := \left\{ u : \mathbb{R}^d \to \mathbb{R} : \ u \equiv 0 \text{ on } \mathbb{R}^d \setminus \Omega, \|u\|_{H_\Omega^\mu(\mathbb{R}^d)} < \infty \right\}, \tag{7.0.4}$$

where

$$\|u\|^2_{H_\Omega^\mu(\mathbb{R}^d)} := \|u\|^2_{L^2(\Omega)} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(y)-u(x))^2 \mu(x,\mathrm{d}y)\,\mathrm{d}x$$

Note that for $u \in H_\Omega^\mu(\mathbb{R}^d)$, by the symmetry of μ in the sense of Assumption 1 and since $u \equiv 0$ on $\mathbb{R}^d \setminus \Omega$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(y)-u(x))^2 \mu(x,\mathrm{d}y)\,\mathrm{d}x$$

$$= \int_\Omega \int_\Omega (u(y) - u(x))^2 \mu(x, \mathrm{d}y)\,\mathrm{d}x + 2 \int_{\Omega^c} \int_\Omega (u(y) - u(x))^2 \mu(x, \mathrm{d}y)\,\mathrm{d}x.$$

The spaces $V^\mu(\Omega|\mathbb{R}^d)$ and $H^\mu_\Omega(\mathbb{R}^d)$ provide regularity across the boundary and will be the replacement of $H^1(\Omega)$ resp. $H^1_0(\Omega)$ in the definition of weak solutions to (7.0.2).

Let $u, v : \mathbb{R}^d \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^d$ be open and bounded. We define

$$\mathcal{E}^\mu_\Omega(u, v) = \int_\Omega \int_\Omega (u(y) - u(x))(v(y) - v(x))\, \mu(x, \mathrm{d}y)\,\mathrm{d}x$$

and

$$\mathcal{E}^\mu(u, v) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(x))(v(y) - v(x))\mu(x, \mathrm{d}y)\mathrm{d}x, \tag{7.0.5}$$

whenever the quantities are finite. Let $\Omega \subset \mathbb{R}^d$ be open. Note that for $u \in V^\mu(\Omega|\mathbb{R}^d)$ and $v \in H^\mu_\Omega(\mathbb{R}^d)$ the quantity in (7.0.5) is finite.

Definition 7.0.2. *Let $\Omega \subset \mathbb{R}^d$ be bounded and open. Let $u, g \in V^\mu(\Omega|\mathbb{R}^d)$ and $f \in L^q(\mathbb{R}^d)$ for some $q \geq 2$. We call u weak solution to* (7.0.2)*, if $u - g \in H^\mu_\Omega(\mathbb{R}^d)$ and for every $\phi \in H^\mu_\Omega(\mathbb{R}^d)$*

$$\mathcal{E}^\mu(u, \phi) = (f, \phi)_{L^2}.$$

Note that the existence of a unique weak solution to (7.0.2) follows again by the Lax-Milgram Lemma and the fact that for open and bounded sets $L^q(\Omega) \subset L^2(\Omega) \subset (H_\Omega(\mathbb{R}^d))^*$ for any $q \geq \max\{2, \beta\}$. For a deeper discussion, see [FKV15].

It is not our purpose to study the existence of weak solutions, rather regularity of weak solutions if they exist.

Our aim is to study operators whose kernels might be anisotropic. For this purpose, we will define a family of measures that will play the role of the reference family. Let $\alpha_1, \ldots, \alpha_d \in (0, 2)$ be given.

Let $(\mu_{\text{axes}}(x, \cdot))_{x \in \mathbb{R}^d}$ be a family of measures on $\mathbb{R}^d$ given by

$$\mu_{\text{axes}}(x, \mathrm{d}y) = \sum_{k=1}^d \left(\alpha_k (2 - \alpha_k) |x_k - y_k|^{-1-\alpha_k}\, \mathrm{d}y_k \prod_{i \neq k} \delta_{\{x_i\}}(\mathrm{d}y_i) \right). \tag{7.0.6}$$

Given $x \in \mathbb{R}^d$, the measure $\mu_{\text{axes}}(x, \cdot)$ only charges distances that occur along the axes

$$\{x + te_k : \, t \in \mathbb{R}\}, \quad k \in \{1, \ldots, d\},$$

whose union is an one-dimensional subset of $\mathbb{R}^d$. Note that in the case $x = 0$ the measure $\mu_{\text{axes}}(0, \mathrm{d}y)$ coincides, up to constants, with the Lévy-measure $\nu(\mathrm{d}y)$ of the pure jump Lévy process $(Z_t)_{t \geq 0} = (Z^1_t, \ldots, Z^d_t)_{t \geq 0}$ from Part II. The point $x \in \mathbb{R}^d$ indicates the center at which the axes of the support of $\mu_{\text{axes}}(x, \cdot)$ intersect. The normalizing constant c_{α_j} of a symmetric stable process of order α_j has the property $c_{\alpha_j} \asymp \alpha_j (2 - \alpha_j)$ for

$\alpha_j \searrow 0$ and $\alpha_j \nearrow 2$. Since the explicit value of α_j plays a minor role in the purpose of this part, we replaced it by $\alpha_j(2-\alpha_j)$. The quantity $(c_{\alpha_j})^{-1}\alpha_j(2-\alpha_j)$ stays bounded as $\alpha_j \nearrow 2$ and $\alpha_j \searrow 0$.

In order to deal with the anisotropy of the measures we consider a corresponding scale of cubes.

Definition 7.0.3. *Let $r > 0$ and $x \in \mathbb{R}^d$. We define*

$$M_r(x) := \bigtimes_{k=1}^{d} \left(x_k - r^{\frac{2}{\alpha_k}}, x_k + r^{\frac{2}{\alpha_k}} \right).$$

For abbreviation, we set $M_r(0) = M_r$ and $M_r^c := \mathbb{R}^d \setminus M_r$.

Let d be a metric on $\mathbb{R}^d$ defined by

$$d(x,y) := \sup_{k \in \{1,\dots,d\}} \left\{ |x_k - y_k|^{\alpha_k/2} \mathbb{1}_{\{|x_k-y_k| \leq 1\}}(x,y) + \mathbb{1}_{\{|x_k-y_k|>1\}}(x,y) \right\}. \tag{7.0.7}$$

Note that a ball of radius $r \leq 1$ in the metric space $(\mathbb{R}^d, d)$ is given by the set of Definition 7.0.3, i.e. for $r \in (0,1]$

$$B_r^d(x) = \{y \in \mathbb{R}^d \colon d(x,y) < r\} = \bigtimes_{k=1}^{d} \left(x_k - r^{\frac{2}{\alpha_k}}, x_k + r^{\frac{2}{\alpha_k}} \right) =: M_r(x).$$

This metric reflects the different differentiability orders in each direction and compensates their behavior in a suitable way, which will be discussed later. Let $\Omega \subset \mathbb{R}^d$ and consider the metric space (Ω, d). Note that by the definition of d for all $x \in \Omega$ we have $B_r(x) = \Omega$, whenever $r > 1$.

We will now formulate two important assumptions on the measures $\mu(x,\cdot)$ with regard to $\mu_{\text{axes}}(x,\cdot)$. For examples of families $\mu(x,\cdot)$, $x \in \mathbb{R}^d$ satisfying these assumptions we refer the reader to Appendix A. The first assumption is a local comparability assumption of the energies.

Assumption 3. *Let $\rho \in (0,1]$, $x_0 \in M_1$ and $u \in H^{\mu_{axes}}_{M_\rho(x_0)}(\mathbb{R}^d)$. There is a constant $C > 1$, independent of $u, \alpha_1, \dots, \alpha_d, \rho$ and x_0, such that*

$$C^{-1}\mathcal{E}^{\mu}_{M_\rho(x_0)}(u,u) \leq \mathcal{E}^{\mu_{axes}}_{M_\rho(x_0)}(u,u) \leq C\mathcal{E}^{\mu}_{M_\rho(x_0)}(u,u).$$

Note that Assumption 3 does not require the measures $\mu(x,\mathrm{d}y)$ to be supported on the union of the coordinate axes. We prove for $d = 2$ in Theorem A.0.3 the inequality $C^{-1}\mathcal{E}^{\mu}_{M_\rho(x_0)}(u,u) \leq \mathcal{E}^{\mu_{\text{axes}}}_{M_\rho(x_0)}(u,u)$ for families of measures which have a density with respect to the Lebesgue measure.

The last assumption on $\mu(x,\cdot)$, $x \in \mathbb{R}^d$, is a uniform upper bound assumption, which will imply the existence of suitable cut-off functions for the energies $\mathcal{E}^{\mu}$.

Assumption 4. *Let $x_0 \in M_1$, $r \in (0,1]$ and $\lambda > 1$. Let $\tau \in C^1(\mathbb{R}^d)$ such that*

$$\begin{cases} supp(\tau) \subset M_{\lambda r}(x_0), \\ \|\tau\|_\infty \leq 1, \\ \tau \equiv 1 \text{ on } M_r(x_0), \\ \|\partial_k \tau\|_\infty \leq \frac{2}{(\lambda^{2/\alpha_k}-1)r^{2/\alpha_k}} \text{ for all } k \in \{1,\dots,d\}. \end{cases} \tag{7.0.8}$$

We assume there is a $c_1 > 0$, independent of $x_0, \lambda, r, \alpha_1, \dots, \alpha_d$ and τ, such that

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2 \mu(x,\mathrm{d}y) \leq c_1 \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2 \mu_{axes}(x,\mathrm{d}y). \tag{7.0.9}$$

From now on we assume that the family $\mu(x,\cdot), x \in \mathbb{R}^d$ always satisfies the four assumptions.

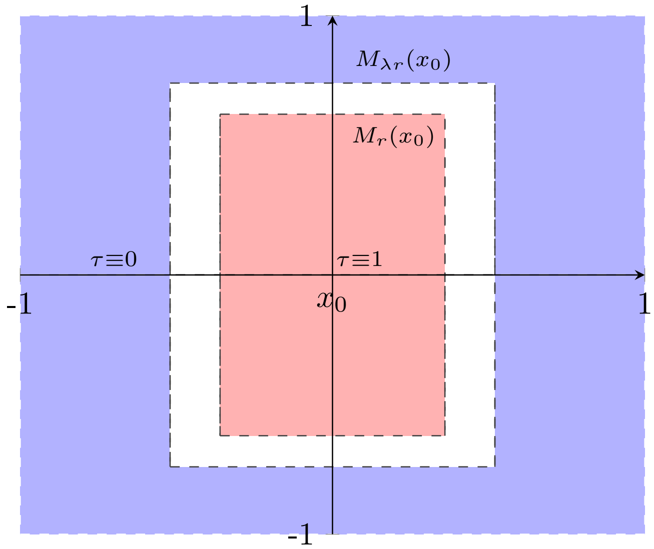


Figure 7.1.: Illustration of the areas of τ for $r = 0.4$, $\lambda = 2$, $\alpha_1 = 0.7$ and $\alpha_2 = 1.5$.

Considering $\mu^\alpha(x,\mathrm{d}y) := c(d,\alpha)|x-y|^{-d-\alpha}$ for $\alpha \in (0,2)$ and the cut-off function $\tau(x) = \Big(0 \vee (1 + (0 \wedge \frac{r-|x-x_0|}{(\lambda-1)r}\Big)$, leads to the existence of a constant c_1, independent of r, λ, x_0, τ and α, such that

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2 |x-y|^{-d-\alpha}\,\mathrm{d}y \leq c_1(\lambda-1)^{-\alpha} r^{-\alpha}.$$

Taking the limit $\alpha \nearrow 2$ leads to $|\nabla\tau|^2 \leq c_1(\lambda-1)^{-2}r^{-2}$. Hence Assumption 4 can be understood as an adjusted version of an anisotropic gradient estimate for operators with fractional derivatives along the coordinate axes.

Note that functions τ as in Assumption 4 are obviously elements in $H^{\mu}_{M_{\lambda r}(x_0)}(\mathbb{R}^d)$. We start our studies with a result that provides the existence of appropriate cut-off functions.

Its worth mentioning that the power (-2) in Lemma 7.0.4 arises because of the choice 2 in the numerator of the definition of M_r. This number could be replaced in the definition of M_r and in the denominator of the power in the metric d by any number greater or equal than $\max\{\alpha_k \colon k \in \{1, \dots, d\}\}$.

Lemma 7.0.4. *Let $x_0 \in M_1$, $r \in (0, 1]$ and $\lambda > 1$. Moreover let $\tau \in C^1(\mathbb{R}^d)$ satisfy (7.0.8). Then there is a constant $c_1 > 0$, independent of $x_0, \lambda, r, \alpha_1, \dots, \alpha_d$ and τ, such that*

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \mu(x, \mathrm{d}y) \leq c_1 r^{-2} \left(\sum_{k=1}^{d} (\lambda^{2/\alpha_k} - 1)^{-\alpha_k} \right).$$

Proof. Set

$$I_k = \left(x_k - (\lambda^{2/\alpha_k} - 1) r^{2/\alpha_k}, x_k + (\lambda^{2/\alpha_k} - 1) r^{2/\alpha_k} \right).$$

Then we have for any $x \in \mathbb{R}^d$

$$\begin{aligned}
\int_{\mathbb{R}^d} (\tau(x) - \tau(y))^2 \mu_{\text{axes}}(x, \mathrm{d}y) &= \sum_{k=1}^{d} \int_{\mathbb{R}} (\tau(x) - \tau(x - e_k x_k + e_k y_k))^2 \frac{\alpha_k (2 - \alpha_k)}{|x_k - y_k|^{1+\alpha_k}} \, \mathrm{d}y_k \\
&\leq \sum_{k=1}^{d} \left[\|\partial_k \tau\|_\infty^2 \int_{I_k} \frac{\alpha_k(2 - \alpha_k)}{|h|^{1+\alpha_k - 2}} \, \mathrm{d}h + \int_{\mathbb{R} \setminus I_k} \frac{\alpha_k(2 - \alpha_k)}{|h|^{1+\alpha_k}} \, \mathrm{d}h \right] \\
&\leq \sum_{k=1}^{d} \left[\frac{2}{\left((\lambda^{2/\alpha_k} - 1) r^{2/\alpha_k} \right)^2} \int_{I_k} \frac{\alpha_k(2 - \alpha_k)}{|h|^{1+\alpha_k - 2}} \, \mathrm{d}h + \int_{\mathbb{R} \setminus I_k} \frac{\alpha_k(2 - \alpha_k)}{|h|^{1+\alpha_k}} \, \mathrm{d}h \right] \\
&= \sum_{k=1}^{d} \Bigg[\frac{4}{\left((\lambda^{2/\alpha_k} - 1) r^{2/\alpha_k} \right)^2} \int_0^{(\lambda^{2/\alpha_k} - 1) r^{2/\alpha_k}} \frac{\alpha_k(2 - \alpha_k)}{|h|^{1+\alpha_k - 2}} \, \mathrm{d}h \\
&\qquad\qquad + 2 \int_{(\lambda^{2/\alpha_k} - 1) r^{2/\alpha_k}}^{\infty} \frac{\alpha_k(2 - \alpha_k)}{|h|^{1+\alpha_k}} \, \mathrm{d}h \Bigg] \\
&= \sum_{k=1}^{d} \left[4\alpha_k \left((\lambda^{2/\alpha_k} - 1) r^{2/\alpha_k} \right)^{-\alpha_k} + 2(2 - \alpha_k) \left(\lambda^{2/\alpha_k} - 1) r^{2/\alpha_k} \right)^{-\alpha_k} \right] \\
&\leq 8 r^{-2} \left(\sum_{k=1}^{d} (\lambda^{2/\alpha_k} - 1)^{-\alpha_k} \right).
\end{aligned}$$

Using Assumption 4, the assertion follows. □

Note that since $\mathrm{supp}(\tau) \subset M_{\lambda r}(x_0)$ and $M_{\lambda r}(x_0)$ is an open set, we have

$$\mathrm{dist}(\mathrm{supp}(\tau), \mathbb{R}^d \setminus M_{\lambda r}(x_0)) > 0.$$

Corollary 7.0.5. *Let $x_0 \in M_1$, $r \in (0,1]$ and $\lambda > 1$. Let $\tau : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function satisfying (7.0.8). Then there is a constant $c_1 > 0$, independent of $u, x_0, \lambda, r, \alpha_1, \ldots, \alpha_d$, such that for any $u \in V^{\mu}(M_{\lambda r}(x_0)|\mathbb{R}^d)$*

$$\int_{M_{\lambda r}(x_0)} \int_{\mathbb{R}^d \setminus M_{\lambda r}(x_0)} u(x)^2 \tau(x)^2 \,\mu(x, \mathrm{d}y)\,\mathrm{d}x \le c_1 r^{-2} \left(\sum_{k=1}^{d} (\lambda^{2/\alpha_k} - 1)^{-\alpha_k} \right) \|u\|^2_{L^2(M_{\lambda r}(x_0))}.$$

Proof. We have,

$$\begin{aligned}
&\int_{M_{\lambda r}(x_0)} \int_{\mathbb{R}^d \setminus M_{\lambda r}(x_0)} u(x)^2 \tau(x)^2 \,\mu(x, \mathrm{d}y)\,\mathrm{d}x \\
&\qquad = \int_{M_{\lambda r}(x_0)} u(x)^2 \int_{\mathbb{R}^d \setminus M_{\lambda r}(x_0)} (\tau(x) - \tau(y))^2 \,\mu(x, \mathrm{d}y)\,\mathrm{d}x \\
&\qquad \le \int_{M_{\lambda r}(x_0)} u(x)^2 \left(\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(x) - \tau(y))^2 \,\mu(x, \mathrm{d}y) \right) \mathrm{d}x \\
&\qquad = \|u\|^2_{L^2(M_{\lambda r}(x_0))} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(x) - \tau(y))^2 \,\mu(x, \mathrm{d}y) \\
&\qquad \le c_1 r^{-2} \left(\sum_{k=1}^{d} (\lambda^{2/\alpha_k} - 1)^{-\alpha_k} \right) \|u\|^2_{L^2(M_{\lambda r}(x_0))},
\end{aligned}$$

where we use Lemma 7.0.4 □

One important tool in our studies will be a Sobolev-type inequality. We will use Fourier analysis to give an elementary proof.

We start our investigations with a comparability result, which gives a representation of $(u,u)_{V^{\mu_{\text{axes}}}(\mathbb{R}^d|\mathbb{R}^d)}$ in terms of the Fourier transform of u.

Lemma 7.0.6. *Let $u \in V^{\mu_{axes}}(\mathbb{R}^d|\mathbb{R}^d)$. Then*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))^2 \mu_{axes}(x, \mathrm{d}y)\,\mathrm{d}x \asymp \left\| \widehat{u}(\xi) \left(\sum_{k=1}^{d} |\xi_k|^{\alpha_k} \right)^{\frac{1}{2}} \right\|^2_{L^2_\xi(\mathbb{R}^d)},$$

where the comparability constants only depends on the dimension d.

Proof. By Fubini's and Plancherel's theorem,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))^2 \mu_{\text{axes}}(x, \mathrm{d}y)\,\mathrm{d}x \\
&\quad = \sum_{k=1}^{d} \int_{\mathbb{R}^d} \int_{\mathbb{R}} (u(x) - u(x - e_k x_k + e_k y_k))^2 \frac{\alpha_k (2 - \alpha_k)}{|x_k - y_k|^{1+\alpha_k}} \,\mathrm{d}y_k\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{d} \int_{\mathbb{R}^d} \int_{\mathbb{R}} (u(x) - u(x - e_k h_k))^2 \frac{\alpha_k(2-\alpha_k)}{|h_k|^{1+\alpha_k}} \, \mathrm{d}h_k \, \mathrm{d}x \\
&= \sum_{k=1}^{d} \int_{\mathbb{R}} \|u(x) - u(x - e_k h_k)\|^2_{L^2_x(\mathbb{R}^d)} \frac{\alpha_k(2-\alpha_k)}{|h_k|^{1+\alpha_k}} \, \mathrm{d}h_k \\
&= \sum_{k=1}^{d} \int_{\mathbb{R}} \|\mathcal{F}[u(x) - u(x - e_k h_k)]\|^2_{L^2_x(\mathbb{R}^d)} \frac{\alpha_k(2-\alpha_k)}{|h_k|^{1+\alpha_k}} \, \mathrm{d}h_k \\
&= \sum_{k=1}^{d} \int_{\mathbb{R}} \|\widehat{u}(\xi)(1 - e^{i\xi_k h_k})\|^2_{L^2_\xi(\mathbb{R}^d)} \frac{\alpha_k(2-\alpha_k)}{|h_k|^{1+\alpha_k}} \, \mathrm{d}h_k \\
&= \sum_{k=1}^{d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} |\widehat{u}(\xi)|^2 (1 - e^{i\xi_k h_k})^2 \, \mathrm{d}\xi \, \frac{\alpha_k(2-\alpha_k)}{|h_k|^{1+\alpha_k}} \, \mathrm{d}h_k \\
&= \sum_{k=1}^{d} \alpha_k(2-\alpha_k) \int_{\mathbb{R}^d} |\widehat{u}(\xi)|^2 \int_{\mathbb{R}} \frac{(1 - e^{i\xi_k h_k})^2}{|h_k|^{1+\alpha_k}} \, \mathrm{d}h_k \, \mathrm{d}\xi.
\end{aligned}
$$

By the asymptotic behavior of the normalizing constant of the symmetric stable process, there is a constant $c_1 \geq 1$, independent of $\alpha_1, \dots, \alpha_d$, such that for any $k \in \{1, \dots, d\}$

$$
c_1^{-1} |\xi_k|^{\alpha_k} \leq \alpha_k(2-\alpha_k) \int_{\mathbb{R}} \frac{(1 - e^{i\xi_k h_k})^2}{|h_k|^{1+\alpha_k}} \, \mathrm{d}h_k \leq c_1 |\xi_k|^{\alpha_k}.
$$

Hence

$$
\begin{aligned}
&\sum_{k=1}^{d} \alpha_k(2-\alpha_k) \int_{\mathbb{R}^d} |\widehat{u}(\xi)|^2 \int_{\mathbb{R}} \frac{(1 - e^{i\xi_k h_k})^2}{|h_k|^{1+\alpha_k}} \, \mathrm{d}h_k \, \mathrm{d}\xi \\
&\qquad \asymp \sum_{k=1}^{d} \int_{\mathbb{R}^d} |\widehat{u}(\xi)|^2 |\xi_k|^{\alpha_k} \, \mathrm{d}\xi = \left\| \widehat{u}(\xi) \left(\sum_{k=1}^{d} |\xi_k|^{\alpha_k} \right)^{\frac{1}{2}} \right\|^2_{L^2_\xi(\mathbb{R}^d)}.
\end{aligned}
$$

□

In the following theorem we prove a Sobolev-type inequality on the whole of $\mathbb{R}^d$. Note that the case $\alpha_1 = \dots = \alpha_d = \alpha \in (0,2)$ leads to $\beta = d/\alpha$ and therefore

$$
\Theta = \frac{2\beta}{\beta - 1} = \frac{2d}{d - \alpha}
$$

in Theorem 7.0.7, which is the exponent for the fractional Sobolev inequality for the Gagliardo-seminorm, see Theorem 2.3.3.

Theorem 7.0.7. *Let $\Theta = 2\beta/(\beta-1)$. There exists a constant $c_1 = c_1(d, \Theta) > 0$ such that for every compactly supported $u \in V^{\mu_{axes}}(\mathbb{R}^d|\mathbb{R}^d)$*

$$
\|u\|^2_{L^{\frac{2\beta}{\beta-1}}(\mathbb{R}^d)} \leq c_1 \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))^2 \, \mu_{axes}(x, \mathrm{d}y) \, \mathrm{d}x \right).
$$

Proof. Let $\Theta \geq 2$. We denote its Hölder conjugate by Θ'.

Using Theorem 2.2.4, Theorem 2.2.5 and the Hölder-inequality, see [Sch12, Proposition 2.2], there are $c_2, c_3 > 0$, depending only on Θ, such that

$$\begin{aligned} \|u\|_{L^{\Theta}(\mathbb{R}^d)} &= \|u\|_{L^{\Theta,\Theta}(\mathbb{R}^d)} \leq c_2 \|u\|_{L^{\Theta,2}(\mathbb{R}^d)} \leq c_3 \|\widehat{u}\|_{L^{\Theta',2}(\mathbb{R}^d)} \\ &\leq c_3 \left\| \left(\sum_{k=1}^{d} |\xi_k|^{\alpha_k} \right)^{-\frac{1}{2}} \right\|_{L_\xi^{2\Theta'/(2-\Theta'),\infty}(\mathbb{R}^d)} \left\| \left(\sum_{k=1}^{d} |\xi_k|^{\alpha_k} \right)^{\frac{1}{2}} \widehat{u}(\xi) \right\|_{L_\xi^2(\mathbb{R}^d)} . \end{aligned} \tag{7.0.10}$$

Our aim is to show

$$K(\xi) = \left(\sum_{k=1}^{d} |\xi_k|^{\alpha_k} \right)^{-\frac{1}{2}} \in L^{2\Theta'/(2-\Theta'),\infty}(\mathbb{R}^d),$$

which implies the assertion by Lemma 7.0.6, where

$$\Theta := \frac{2\beta}{\beta - 1}.$$

Let $\xi \in \mathbb{R}^d$. Then there is obviously an index $i \in \{1, \dots, d\}$ such that

$$|\xi_i|^{\alpha_i} \geq |\xi_j|^{\alpha_j} \quad \text{for all } j \neq i.$$

Thus there is a $c_4 \geq 1$, depending only on d, such that

$$c_4^{-1} |\xi_i|^{-\alpha_i/2} \leq \left(\sum_{k=1}^{d} |\xi_k|^{\alpha_k} \right)^{-1/2} = \left(|\xi_i|^{\alpha_i} \left(1 + \sum_{k \neq i} \frac{|\xi_k|^{\alpha_k}}{|\xi_i|^{\alpha_i}} \right) \right)^{-1/2} \leq c_4 |\xi_i|^{-\alpha_i/2}.$$

Hence

$$\begin{aligned} |\{|K(\xi) \geq t\}| &= \left| \left\{ \left| \left(\sum_{k=1}^{d} |\xi_k|^{\alpha_k} \right)^{-1/2} \right| \geq t \right\} \right| \\ &\leq \sum_{i=1}^{d} \left| \{ (|\xi_i|^{-\alpha_i/2} \geq t) \wedge (|\xi_i|^{\alpha_i} \geq |\xi_j|^{\alpha_j}) \text{ for all } j \neq i \} \right| \\ &= \sum_{i=1}^{d} \left| \{ (|\xi_i| \leq t^{-2/\alpha_i}) \wedge (|\xi_j| \leq |\xi_i|^{\alpha_i/\alpha_j}) \text{ for all } j \neq i \} \right| =: c_4 \sum_{i=1}^{d} \eta_i. \end{aligned}$$

For each $i \in \{1, \dots, d\}$, we have

$$\eta_i = 2^d \int_0^{t^{-2/\alpha_i}} \left(\prod_{j \neq i} \int_0^{\xi_i^{\alpha_i/\alpha_j}} \mathrm{d}\xi_j \right) \mathrm{d}\xi_i = 2^d \int_0^{t^{-2/\alpha_i}} \xi_i^{\sum_{j \neq i} \frac{\alpha_i}{\alpha_j}} \, \mathrm{d}\xi_i = \frac{2^d}{\sum_{j \neq i} \frac{\alpha_i + \alpha_j}{\alpha_j}} t^{-\frac{2}{\alpha_i} \left(\sum_{j \neq i} \frac{\alpha_i}{\alpha_j} + 1 \right)}$$

$$\leq \frac{2^d}{d-1} t^{-2\left(\sum_{j=1}^{d} \frac{1}{\alpha_j}\right)} = c_5 t^{-2\beta}.$$

Hence, we have $K \in L^{2\beta,\infty}$, if

$$\frac{2\Theta'}{2-\Theta'} = 2\beta \iff \frac{2-\Theta'}{\Theta'} = \frac{1}{\beta} \iff \frac{1}{\Theta} = \frac{1}{2} - \frac{1}{2\beta} = \frac{1}{2}\left(\frac{\beta-1}{\beta}\right) \iff \Theta = \frac{2\beta}{\beta-1},$$

from which the assertion follows. □

Next we prove a localized Sobolev inequality on the sets $M_r(x_0)$.

Theorem 7.0.8. *Let $x_0 \in M_1$, $r \in (0,1]$ and $\lambda > 1$. Let $u \in V^{\mu_{axes}}(M_{\lambda r}(x_0)|\mathbb{R}^d)$. Let $\Theta = 2\beta/(\beta-1)$. Then there is a constant $c_1 > 0$, independent of $x_0, \lambda, r, \alpha_1, \dots, \alpha_d$ and u, but depending on d, Θ, such that*

$$\begin{aligned} \|u\|^2_{L^{\frac{2\beta}{\beta-1}}(M_r(x_0))} \leq c_1 \Bigg(& \int_{M_{\lambda r}(x_0)} \int_{M_{\lambda r}(x_0)} (u(x)-u(y))^2 \, \mu_{axes}(x, \mathrm{d}y) \, \mathrm{d}x \\ & + r^{-2} \left(\sum_{k=1}^{d} (\lambda^{2/\alpha_k} - 1)^{-\alpha_k} \right) \|u\|^2_{L^2(M_{\lambda r}(x_0))} \Bigg). \end{aligned} \tag{7.0.11}$$

Proof. Let $\tau : \mathbb{R}^d \to \mathbb{R}$ be as in (7.0.8), that is $\tau \in C^1(\mathbb{R}^d)$ such that

- $\mathrm{supp}(\tau) \subset M_{\lambda r}(x_0)$,
- $\|\tau\|_\infty \leq 1$,
- $\tau \equiv 1$ on $M_r(x_0)$,
- $\|\partial_k \tau\|_\infty \leq \dfrac{2}{(\lambda^{2/\alpha_k} - 1) r^{2/\alpha_k}}$.

For simplicity of notation we write $M_r = M_r(x_0)$. Let $v \in L^2(\mathbb{R}^d)$ such that $v \equiv u$ on $M_{\lambda r}$ and $\mathcal{E}(v,v) < \infty$.

By Theorem 7.0.7 there is a $c_2 = c_2(d, \Theta) > 0$ such that

$$\begin{aligned} \|v\tau\|^2_{L^\Theta(\mathbb{R}^d)} &\leq c_2 \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (v(x)\tau(x) - v(x)\tau(y))^2 \, \mu_{\mathrm{axes}}(x, \mathrm{d}y) \, \mathrm{d}x \right) \\ &= c_2 \Bigg(\int_{M_{\lambda r}} \int_{M_{\lambda r}} (v(x)\tau(x) - v(x)\tau(y))^2 \, \mu_{\mathrm{axes}}(x, \mathrm{d}y) \, \mathrm{d}x \\ &\qquad + 2 \int_{M_{\lambda r}} \int_{(M_{\lambda r})^c} (v(x)\tau(x) - v(x)\tau(y))^2 \, \mu_{\mathrm{axes}}(x, \mathrm{d}y) \, \mathrm{d}x \Bigg) \\ &=: c_2 (I_1 + 2I_2). \end{aligned}$$

We have

$$I_1 = \int_{M_{\lambda r}} \int_{M_{\lambda r}} (v(x)\tau(x) - v(x)\tau(y))^2 \, \mu_{\mathrm{axes}}(x, \mathrm{d}y) \, \mathrm{d}x$$

$$
\begin{aligned}
&= \frac{1}{4}\int_{M_{\lambda r}}\int_{M_{\lambda r}} [(v(y)-v(x))(\tau(x)+\tau(y)) \\
&\qquad + (v(x)+v(y))(\tau(x)-\tau(y))]^2\, \mu_{\text{axes}}(x,\mathrm{d}y)\,\mathrm{d}x \\
&\le \frac{1}{4}\Bigg(\int_{M_{\lambda r}}\int_{M_{\lambda r}} 2[(v(y)-v(x))(\tau(x)+\tau(y))]^2\, \mu_{\text{axes}}(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad + \int_{M_{\lambda r}}\int_{M_{\lambda r}} 2[(v(x)+v(y))(\tau(x)-\tau(y))]^2\, \mu_{\text{axes}}(x,\mathrm{d}y)\,\mathrm{d}x\Bigg) \\
&= \frac{1}{2}(J_1+J_2),
\end{aligned}
$$

Using $(\tau(x)+\tau(y)) \le 2$ for all $x, y \in M_{\lambda r}$ leads to

$$
\begin{aligned}
J_1 &= \int_{M_{\lambda r}}\int_{M_{\lambda r}} [(v(y)-v(x))(\tau(x)+\tau(y))]^2\, \mu_{\text{axes}}(x,\mathrm{d}y)\,\mathrm{d}x \\
&\le 4\int_{M_{\lambda r}}\int_{M_{\lambda r}} (v(y)-v(x))^2\, \mu_{\text{axes}}(x,\mathrm{d}y)\,\mathrm{d}x \\
&= 4\int_{M_{\lambda r}}\int_{M_{\lambda r}} (u(y)-u(x))^2\, \mu_{\text{axes}}(x,\mathrm{d}y)\,\mathrm{d}x.
\end{aligned}
$$

By $(v(x)+v(y))^2(\tau(x)-\tau(x))^2 \le 2v(x)^2(\tau(x)-\tau(x))^2 + 2v(y)^2(\tau(x)-\tau(x))^2$ and Lemma 7.0.4, we have

$$
\begin{aligned}
J_2 &\le 4\|v\|^2_{L^2(M_{\lambda r})} \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2 \mu_{\text{axes}}(x,\mathrm{d}y) \\
&\le c_3 r^{-2}\left(\sum_{k=1}^{d}(\lambda^{2/\alpha_k}-1)^{-\alpha_k}\right)\|u\|^2_{L^2(M_{\lambda r})}.
\end{aligned}
$$

It remains to estimate I_2. We have by Corollary 7.0.5

$$
\begin{aligned}
I_2 &= \int_{M_{\lambda r}}\int_{(M_{\lambda r})^c} (v(x)\tau(x)-v(x)\tau(y))^2\, \mu_{\text{axes}}(x,\mathrm{d}y)\,\mathrm{d}x \\
&= \int_{M_{\lambda r}}\int_{(M_{\lambda r})^c} (v(x)\tau(x))^2\, \mu_{\text{axes}}(x,\mathrm{d}y)\,\mathrm{d}x \\
&\le c_4 r^{-2}\left(\sum_{k=1}^{d}(\lambda^{2/\alpha_k}-1)^{-\alpha_k}\right)\|u\|^2_{L^2(M_{\lambda r})}.
\end{aligned}
$$

Hence there is a constant c_1, independent of $x_0, \lambda, r, \alpha_1, \ldots, \alpha_d$ and u, such that

$$
\begin{aligned}
&\|u\|^2_{L^\Theta(M_r)} = \|v\|^2_{L^\Theta(M_r)} = \|v\tau\|^2_{L^\Theta(M_r)} \le \|v\tau\|^2_{L^\Theta(\mathbb{R}^d)} \\
&\le c_1\left(\int_{M_{\lambda r}}\int_{M_{\lambda r}} (u(x)-u(y))^2\, \mu_{\text{axes}}(x,\mathrm{d}y)\ \mathrm{d}x + r^{-2}\left(\sum_{k=1}^{d}(\lambda^{2/\alpha_k}-1)^{-\alpha_k}\right)\|u\|^2_{L^2(M_{\lambda r})}\right).
\end{aligned}
$$

□

From this theorem we immediately deduce the following corollary.

Corollary 7.0.9. *Let $x_0 \in M_1$ and $r \in (0,1)$. Let $\lambda \in (1, r^{-1}]$ and $u \in V^{\mu}(M_{\lambda r}(x_0)|\mathbb{R}^d)$. Let $\Theta = 2\beta/(\beta-1)$. Then there is a $c_1 > 0$, independent of $x_0, \lambda, r, \alpha_1, \ldots, \alpha_d$ and u, but depending on d, Θ, such that*

$$\begin{aligned} \|u\|^2_{L^{\frac{2\beta}{\beta-1}}(M_r(x_0))} \leq c_1 \Bigg(&\int_{M_{\lambda r}(x_0)} \int_{M_{\lambda r}(x_0)} (u(x)-u(y))^2 \, \mu(x, \mathrm{d}y) \, \mathrm{d}x \\ &+ r^{-2} \left(\sum_{k=1}^{d} (\lambda^{2/\alpha_k} - 1)^{-\alpha_k} \right) \|u\|^2_{L^2(M_{\lambda r}(x_0))} \Bigg). \end{aligned} \tag{7.0.12}$$

Proof. Since by assumption $\rho := \lambda r \leq 1$, the assertion follows immediately by Theorem 7.0.8 and Assumption 3. □

We close this chapter with two important definitions and some basic properties.

Given $r > 0$ and $x \in \mathbb{R}^d$, we define the subspace $\mathcal{S}_{x,r}$ of $V^{\mu}(M_r(x)|\mathbb{R}^d)$ of all weak solutions to the equation $\mathcal{L}u = 0$ in $M_r(x)$ by

$$\mathcal{S}_{x,r} = \left\{ u \in V^{\mu}(M_r(x)|\mathbb{R}^d) \colon \mathcal{E}^{\mu}(u, \Phi) = 0 \text{ for every } \Phi \in H^{\mu}_{M_r(x)}(\mathbb{R}^d) \right\}. \tag{7.0.13}$$

The following Lemma gives some basic properties of the space of solutions.

Lemma 7.0.10. *Let $x, y \in \mathbb{R}^d$, $r, s > 0$ and $a \in \mathbb{R}$.*

1. *If $u \in \mathcal{S}_{x,r}$, then $au \in \mathcal{S}_{x,r}$ and $u + a \in S_{x,r}$.*
2. *If $M_r(x) \subset M_s(y)$, then $\mathcal{S}_{y,s} \subset \mathcal{S}_{x,r}$.*

Proof. Let us first prove the first statement. We obviously have

$$\begin{aligned} (au, au)_{V^{\mu}(M_r(x)|\mathbb{R}^d)} &= a^2 (u,u)_{V^{\mu}(M_r(x)|\mathbb{R}^d)} < \infty, \\ (u+a, u+a)_{V^{\mu}(M_r(x)|\mathbb{R}^d)} &= (u,u)_{V^{\mu}(M_r(x)|\mathbb{R}^d)} < \infty. \end{aligned}$$

Hence $(au) \in V^{\mu}(M_r(x)|\mathbb{R}^d)$ and $(u+a) \in V^{\mu}(M_r(x)|\mathbb{R}^d)$ for any $a \in \mathbb{R}$. Moreover, for $\Phi \in H^{\mu}_{M_r(x)}(\mathbb{R}^d)$

$$\mathcal{E}^{\mu}(au, \Phi) = a\mathcal{E}^{\mu}(u, \Phi) = 0 \quad \text{and} \quad \mathcal{E}^{\mu}(u+a, \Phi) = \mathcal{E}^{\mu}(u, \Phi) = 0,$$

which proves the first assertion.

Not let $M_r(x) \subset M_s(y)$ and $u \in \mathcal{S}_{y,s}$. Then for every $\Phi \in H^{\mu}_{M_s(y)}(\mathbb{R}^d)$, we have $\mathcal{E}^{\mu}(u, \Phi) = 0$. To show that $u \in \mathcal{S}_{x,r}$. But this is true, since $M_r(x) \subset M_s(y)$ implies $H^{\mu}_{M_r(x)}(\mathbb{R}^d) \subset H^{\mu}_{M_s(y)}(\mathbb{R}^d)$. □

Given $x \in \mathbb{R}^d$ and $r > 0$, we define a measure $\nu_{x,r}$ on $\mathcal{B}(\mathbb{R}^d \setminus \{x\})$, which has singularity around x. It is essentially a scaled description of $\mu_{\text{axes}}(x, \cdot)$.

Definition 7.0.11. *Given $x \in \mathbb{R}^d$ and $r > 0$. We define a measure on $\mathcal{B}(\mathbb{R}^d \setminus \{x\})$ by*

$$\nu_{x,r}(A) = r \sum_{k=1}^{d} \alpha_k(2-\alpha_k) \int_{\{h \in \mathbb{R}\colon\, e_k h \in A\}} |x_k - h|^{-1-\alpha_k} \,\mathrm{d}h.$$

For $0 < r < R$ let

$$A_{r,R}(x) = M_R(x) \setminus M_r(x).$$

The measure in Definition 7.0.11 has the following helpful properties.

Lemma 7.0.12. *Let $x \in \mathbb{R}^d$, $0 < r < R$, $\sigma, \Theta > 1$ and $j \in \mathbb{N}$. Then*

1. $\nu_{x,r}\left(\mathbb{R}^d \setminus M_{r\Theta^j}(x)\right) = \sum_{k=1}^{d} 2(2-\alpha_k)\Theta^{-2j}$,
2. $\nu_{x,r}\left(A_{r\Theta^j, r\Theta^{j+1}}(x)\right) \le \sum_{k=1}^{d} 2(2-\alpha_k)\Theta^{-2j}$,
3. *If $y \in M_{\frac{r}{\sigma}}(x)$, then*

$$\nu_{y,r}\left(A_{r\Theta^j, r\Theta^{j+1}}(x)\right) \le \sum_{k=1}^{d} 2(2-\alpha_k)\left(\frac{\sigma}{\sigma-1}\right)^2 \Theta^{-2j}. \tag{7.0.14}$$

Proof. The first two assertions follow by direct calculations. We have

$$\begin{aligned}
\nu_{x,r}\left(\mathbb{R}^d \setminus M_{r\Theta^j}(x)\right) &= r^2 \sum_{k=1}^{d} 2\alpha_k(2-\alpha_k) \int_{x_k + (r\Theta^j)^{2/\alpha_k}}^{\infty} |x_k - h|^{-1-\alpha_k} \,\mathrm{d}h \\
&= r^2 \sum_{k=1}^{d} 2\alpha_k(2-\alpha_k) \int_{(r\Theta^j)^{2/\alpha_k}}^{\infty} h^{-1-\alpha_k} \,\mathrm{d}h \\
&= r^2 \sum_{k=1}^{d} 2(2-\alpha_k)(r\Theta^j)^{-2} = \sum_{k=1}^{d} 2(2-\alpha_k)\Theta^{-2j}
\end{aligned}$$

and

$$\nu_{x,r}\left(A_{r\Theta^j, r\Theta^{j+1}}(x)\right) \le \nu_{x,r}\left(\mathbb{R}^d \setminus M_{r\Theta^j}(x)\right).$$

It remains to prove the third assertion.

Let $y \in M_{\frac{r}{\sigma}}(x)$ and $r' := (1 - \frac{1}{\sigma})r$. Moreover, let $z \in M_{r'\Theta^j}(y)$. We have for any $k \in \{1, \dots, d\}$

$$\begin{aligned}
\left(\frac{r}{\sigma}\right)^{2/\alpha_k} + (r'\Theta^j)^{2/\alpha_k} &= \left(\frac{r}{\sigma}\right)^{2/\alpha_k} + \left(\left(r - \frac{r}{\sigma}\right)\Theta^j\right)^{2/\alpha_k} \\
&\le \left(\frac{r}{\sigma} + r\Theta^j - \frac{r}{\sigma}\Theta^j\right)^{2/\alpha_k}
\end{aligned}$$

$$\leq \left(\frac{r}{\sigma}\Theta^j + r\Theta^j - \frac{r}{\sigma}\Theta^j\right)^{2/\alpha_k} = (r\Theta^j)^{2/\alpha_k}.$$

This implies, $M_{r'\Theta^j}(y) \subset M_{r\Theta^j}(x)$ or equivalently $\mathbb{R}^d \setminus M_{r\Theta^j}(x) \subset \mathbb{R}^d \setminus M_{r'\Theta^j}(y)$. Hence

$$\nu_{y,r'}\left(A_{r\Theta^j,r\Theta^{j+1}}(x)\right) \leq \nu_{y,r'}\left(\mathbb{R}^d \setminus M_{r\Theta^j}(x)\right) \leq \nu_{y,r'}\left(\mathbb{R}^d \setminus M_{r'\Theta^j}(y)\right) = \sum_{k=1}^d 2(2-\alpha_k)\Theta^{-j}.$$

The assertion follows now by the trivial fact

$$\nu_{y,r'}\left(A_{r\Theta^j,r\Theta^{j+1}}(x)\right) = \left(1-\frac{1}{\sigma}\right)^2 \nu_{y,r}\left(A_{r\Theta^j,r\Theta^{j+1}}(x)\right).$$

□

7.1. An algebraic inequality

In this section we derive a formula that is needed when working with the localized Moser iteration for negative exponents. The aim of this section is to prove Lemma 7.1.1. This result can be found in the published version of [DK15].

Lemma 7.1.1. *There exist positive constants $c_1, c_2 > 0$ such that for every $a, b > 0$, $p > 1$ and $0 \leq \tau_1, \tau_2 \leq 1$ the following is true:*

$$(b-a)(\tau_1^2 a^{-p} - \tau_2^2 b^{-p}) \geq c_1\left(\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}}\right)^2 - \frac{c_2 p}{p-1}(\tau_1-\tau_2)^2(b^{-p+1}+a^{-p+1}).$$

The above result is nothing but a discrete version of

$$\left(\nabla u, \nabla(\tau^2 u^{-p})\right) \geq c_1(p)|\nabla\left(\tau u^{\frac{-p+1}{2}}\right)|^2 - c_2(p)|\nabla\tau|^2 u^{-p+1},$$

where u, τ are positive functions. The term $\left(\nabla u, \nabla(\tau^2 u^{-p})\right)$ appears naturally if one chooses $\varphi = \tau^2 u^{-p}$ in the bilinear form $\int(\nabla u, \nabla\varphi)$. Most often, τ is chosen as a cut-off function with values in $[0,1]$ and v is a weak (super-)solution to some equation.

Lemma 7.1.1 is needed when using test functions of the form $\tau_1^2 u^{-p}$ for a localization function τ, a number $p > 1$ and a function u satisfying

$$\begin{aligned} \mathcal{E}(u,\phi) &\geq (f,\phi) \quad \text{for any nonnegative } \phi \in H^{\mu}_{M_{\lambda r}(x_0)}(\mathbb{R}^d), \\ u(x) &\geq \epsilon \qquad \text{for a.a. } x \in M_{\lambda r}(x_0) \text{ and some } \epsilon > 0. \end{aligned}$$

7.1.1. Some technical observations

In this subsection we will prove some technical lemmata, which we need to prove Lemma 7.1.1.

Lemma 7.1.2. *Assume* $\tau_1, \tau_2 \geq 0$ *and* $\frac{\tau_1}{\tau_2} \in [\frac{1}{2}, 2]$. *Then*

$$\frac{\tau_1^2 + \tau_2^2}{|\tau_1^2 - \tau_2^2|} \geq \frac{5}{3}.$$

Proof. Note that

$$\frac{\tau_1^2 + \tau_2^2}{|\tau_1^2 - \tau_2^2|} = \frac{\frac{\tau_1^2}{\tau_2^2} + 1}{|\frac{\tau_1^2}{\tau_2^2} - 1|} = \frac{t+1}{|t-1|},$$

where $t = \frac{\tau_1^2}{\tau_2^2}$. There are three cases:

1. If $t = 1$, then $\frac{t+1}{t-1} = +\infty$ and the assertion is true.
2. If $t > 1$, then $\frac{t+1}{|t-1|} = \frac{t+1}{t-1}$. Note that $\frac{t+1}{t-1} \geq \frac{5}{3}$ holds true iff $t + 1 \geq \frac{5}{3}t - \frac{5}{3}$ $\iff t \leq 4 \iff \frac{\tau_1}{\tau_2} \leq 2$.
3. If $t < 1$, then $\frac{t+1}{|t-1|} = \frac{t+1}{-t+1}$. Note that $\frac{t+1}{-t+1} \geq \frac{5}{3}$ holds true iff $t + 1 \geq -\frac{5}{3}t + \frac{5}{3}$ $\iff t \geq \frac{1}{4} \iff \frac{\tau_1}{\tau_2} \geq \frac{1}{2}$.

□

Lemma 7.1.3. *Assume* $p > 1$ *and* $\eta \in (1, \frac{5}{3})$. *Set* $\lambda = \left(\frac{\eta-1}{1+\eta}\right)^{1/p}$. *Assume* $a, b > 0$ *and* $\frac{b}{a} \notin (\lambda, \frac{1}{\lambda})$. *Then*

$$\frac{a^{-p} + b^{-p}}{|a^{-p} - b^{-p}|} \leq \eta.$$

Proof. Set $t = \left(\frac{b}{a}\right)^p$. Then

$$\frac{a^{-p} + b^{-p}}{|a^{-p} - b^{-p}|} = \frac{\left(\frac{a}{b}\right)^{-p} + 1}{|\left(\frac{a}{b}\right)^{-p} - 1|} = \frac{t+1}{|t-1|}.$$

Now there are two cases:

1. Case 1: $t > 1$.

$$\frac{t+1}{|t-1|} \leq \eta \iff \frac{t+1}{t-1} \leq \eta \iff t \geq \frac{1+\eta}{\eta - 1} \iff \frac{b}{a} \geq \left(\frac{1+\eta}{\eta-1}\right)^{1/p}.$$

2. Case 2: $t < 1$.

$$\frac{t+1}{|t-1|} \leq \eta \iff \frac{t+1}{-t+1} \leq \eta \iff t \leq \frac{\eta - 1}{1+\eta} \iff \frac{b}{a} \leq \left(\frac{\eta-1}{1+\eta}\right)^{1/p}.$$

□

Lemma 7.1.4. *There is $c_1 > 0$ such that for $p > 1$, $\lambda = \left(\frac{1}{7}\right)^{1/p}$ and $a, b > 0$ with $\frac{b}{a} \in (\lambda, \frac{1}{\lambda})$ the following is true:*

$$\frac{|b-a|(a^{-p}+b^{-p})^2}{|a^{-p}-b^{-p}|} \leq \frac{c_1}{p}(b^{-p+1}+a^{-p+1}).$$

Proof. Set $\frac{b}{a} = \xi \in (\lambda, \frac{1}{\lambda})$. Then

$$\begin{aligned} & \frac{|b-a|(a^{-p}+b^{-p})^2}{|a^{-p}-b^{-p}|} \leq \frac{c_1}{p}(b^{-p+1}+a^{-p+1}) \\ \iff & \frac{|a||\xi-1|a^{-2p}(1+\xi^{-p})^2}{|\xi^{-p}-1|a^{-p}} \leq \frac{c_1}{p}a^{-p+1}(\xi^{-p+1}+1) \\ \iff & \frac{|\xi-1|(1+\xi^{-p})^2}{|\xi^{-p}-1|} \leq \frac{c_1}{p}(\xi^{-p+1}+1) \\ \iff & \frac{|\xi-1|(1+\xi^{-p})^2}{|\xi^{-p}-1|(\xi^{-p+1}+1)} \leq \frac{c_1}{p}. \end{aligned} \tag{7.1.1}$$

Let us prove (7.1.1). Note that

$$\frac{|\xi-1|(1+\xi^{-p})^2}{|\xi^{-p}-1|(\xi^{-p+1}+1)} \leq \frac{|\xi-1|(1+7)^2}{|\xi^{-p}-1|} = 64\frac{|\xi-1|}{|\xi^{-p}-1|}.$$

We want to apply the mean value theorem to the function $\xi \mapsto g(\xi) = \xi^{-p}$.
Then $g'(\xi) = (-p)\xi^{-(p+1)}$. The mean value theorem implies

$$\frac{|\xi^{-p}-1|}{|\xi-1|} = \frac{|g(\xi)-1|}{|\xi-1|} = |g(x)| = px^{-(p+1)} \quad \text{for some } x \in (\xi,1)\cup(1,\xi).$$

Thus,

$$\frac{|\xi^{-p}-1|}{|\xi-1|} \geq p\left(\frac{1}{\lambda}\right)^{-(p+1)} = p\left(7^{1/p}\right)^{-(p+1)} = p7^{-1-\frac{1}{p}},$$

from which we deduce

$$\frac{|\xi-1|(1+\xi^{-p})^2}{|\xi^{-p}-1|(\xi^{-p+1}+1)} \leq 64\frac{7^{1+\frac{1}{p}}}{p} \leq \frac{64\cdot 49}{p} = \frac{c_1}{p}.$$

□

Lemma 7.1.5. *For $p > 1$ and $a, b > 0$ the following is true:*

$$(b-a)(a^{-p}-b^{-p}) \geq \frac{2}{p-1}(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}})^2.$$

The proof of the above lemma is simple and can be found in several places, e.g., in [Kas09].

Lemma 7.1.6. *Assume* $p > 1$, $a, b > 0$ *and* $\tau_1, \tau_2 \geq 0$. *Then*

$$(\tau_1 + \tau_2)^2 \left(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}}\right)^2 \geq 2\left(\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}}\right)^2 - 2(\tau_1 - \tau_2)^2(a^{-p+1} + b^{-p+1})$$

Proof. Note

$$2\left(\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}}\right) = (\tau_1 - \tau_2)(a^{\frac{-p+1}{2}} + b^{\frac{-p+1}{2}}) + (\tau_1 + \tau_2)(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}}).$$

From this equality we obtain the assertion as follows:

$$\begin{aligned} &4\left(\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}}\right)^2 \\ &\leq 2(\tau_1 - \tau_2)^2(a^{\frac{-p+1}{2}} + b^{\frac{-p+1}{2}})^2 + 2(\tau_1 + \tau_2)^2(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}})^2 \\ &\leq 4(\tau_1 - \tau_2)^2(a^{-p+1} + b^{-p+1}) + 2(\tau_1 + \tau_2)^2(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}})^2. \end{aligned}$$

□

7.1.2. Proof of the inequality

Let us prove the main result of this section.

Proof. Note,

$$\begin{aligned} -(\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 &= -\tau_1^2 a^{-p+1} - \tau_2^2 b^{-p+1} + 2\tau_1 a^{\frac{-p+1}{2}} \tau_2 b^{\frac{-p+1}{2}} \\ &\geq -\tau_1^2 a^{-p+1} - \tau_2^2 b^{-p+1}. \end{aligned}$$

Let us first consider the case $\frac{\tau_1}{\tau_2} \notin (\frac{1}{2}, 2)$. In this case

$$\max\{\tau_1, \tau_2\} \leq 2|\tau_1 - \tau_2|. \tag{7.1.2}$$

Thus, we obtain

$$\begin{aligned} &(b - a)(\tau_1^2 a^{-p} - \tau_2^2 b^{-p}) \\ &\geq -\tau_1^2 a^{-p+1} - \tau_2^2 b^{-p+1} + (\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 - (\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 \\ &\geq (\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 - 2\tau_1^2 a^{-p+1} - 2\tau_2^2 b^{-p+1} \\ &\geq (\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 - 2\max\{\tau_1, \tau_2\}^2 a^{-p+1} - 2\max\{\tau_1, \tau_2\}^2 b^{-p+1} \\ &\geq (\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 - 8(\tau_1 - \tau_2)^2(a^{-p+1} + b^{-p+1}). \end{aligned}$$

The proof in the case $\frac{\tau_1}{\tau_2} \notin (\frac{1}{2}, 2)$ is complete.

Let us now assume $\frac{\tau_1}{\tau_2} \in [\frac{1}{2}, 2]$. A general observation is

$$(b-a)(\tau_1^2 a^{-p} - \tau_2^2 b^{-p}) = \underbrace{\frac{1}{2}(b-a)(\tau_1^2 - \tau_2^2)(a^{-p} + b^{-p})}_{:=P} + \underbrace{\frac{1}{2}(b-a)(\tau_1^2 + \tau_2^2)(a^{-p} - b^{-p})}_{:=G} .$$

By Lemma 7.1.5

$$\frac{1}{2}(b-a)(a^{-p} - b^{-p}) \geq \frac{1}{p-1}(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}})^2 . \tag{7.1.3}$$

Choose $\eta = \frac{4}{3}$ and $\lambda = \left(\frac{1}{7}\right)^{1/p}$. Let us consider two sub-cases.

1. Case: $\frac{b}{a} \in (\lambda, \frac{1}{\lambda})$, $\frac{\tau_1}{\tau_2} \in [\frac{1}{2}, 2]$. In this case

$$\begin{aligned} |P| &= \left[\frac{1}{4}(\tau_1 + \tau_2)|b-a|^{1/2}|a^{-p} - b^{-p}|^{1/2}\right] \\ &\qquad \times \left[[2|\tau_1 - \tau_2||a^{-p} - b^{-p}|^{-1/2}|b-a|^{1/2}(a^{-p} + b^{-p})\right] \\ &\leq \frac{1}{16}(\tau_1 + \tau_2)^2(b-a)(a^{-p} - b^{-p}) + 4(\tau_1 - \tau_2)^2 \underbrace{\frac{(b-a)(a^{-p} + b^{-p})^2}{(a^{-p} - b^{-p})}}_{:=F} . \end{aligned}$$

Because of Lemma 7.1.4, we know that there is $c_5 > 0$ such that $|F| \leq \frac{c_5}{p}(b^{-p+1} + a^{-p+1})$. Altogether, we obtain

$$\begin{aligned} &(b-a)(\tau_1^2 a^{-p} - \tau_2^2 b^{-p}) \\ &= \frac{1}{2}(b-a)(\tau_1^2 - \tau_2^2)(a^{-p} + b^{-p}) + \frac{1}{2}(b-a)(\tau_1^2 + \tau_2^2)(a^{-p} - b^{-p}) \\ &\geq \frac{1}{2}(b-a)(\tau_1^2 - \tau_2^2)(a^{-p} + b^{-p}) + \frac{1}{4}(b-a)(\tau_1 + \tau_2)^2(a^{-p} - b^{-p}) \\ &\geq -\frac{1}{16}(\tau_1 + \tau_2)^2(b-a)(a^{-p} - b^{-p}) - 4(\tau_1 - \tau_2)^2 \frac{(b-a)(a^{-p} + b^{-p})^2}{(a^{-p} - b^{-p})} \\ &\quad + \frac{1}{4}(b-a)(\tau_1 + \tau_2)^2(a^{-p} - b^{-p}) \\ &= \frac{3}{16}(\tau_1 + \tau_2)^2(b-a)(a^{-p} - b^{-p}) - 4(\tau_1 - \tau_2)^2 \frac{(b-a)(a^{-p} + b^{-p})^2}{(a^{-p} - b^{-p})} \\ &\geq \frac{3}{16(p-1)}(\tau_1 + \tau_2)^2(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}})^2 - \frac{4c_5}{p}(\tau_1 - \tau_2)^2(b^{-p+1} + a^{-p+1}) \\ &\geq \frac{6}{16(p-1)}(\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 - \left(\frac{4c_5}{p} + \frac{6}{16(p-1)}\right)(\tau_1 - \tau_2)^2(b^{-p+1} + a^{-p+1}), \end{aligned}$$

where applied Lemma 7.1.6.

2. Case: $\frac{b}{a} \notin (\lambda, \frac{1}{\lambda})$, $\frac{\tau_1}{\tau_2} \in [\frac{1}{2}, 2]$. Then Lemma 7.1.2 and Lemma 7.1.3 imply

$$P \geq -|P| = -\frac{1}{2}|b-a||\tau_1^2 - \tau_2^2|(a^{-p} + b^{-p}) \geq -\frac{3}{10}|b-a|(\tau_1^2 + \tau_2^2)(a^{-p} + b^{-p})$$

$$\begin{aligned} &\geq -\frac{3}{10} \cdot \frac{4}{3} |b-a| (\tau_1^2 + \tau_2^2) |a^{-p} - b^{-p}| = -\frac{2}{5} (b-a)(\tau_1^2 + \tau_2^2)(a^{-p} - b^{-p}) \\ &= -\frac{4}{5} G. \end{aligned}$$

Thus, due to Lemma 7.1.5, we obtain

$$\begin{aligned} (b-a)(\tau_1^2 a^{-p} - \tau_2^2 b^{-p}) &= P + G \geq \frac{1}{5} G \geq \frac{1}{5(p-1)} (\tau_1^2 + \tau_2^2)(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}})^2 \\ &\geq \frac{1}{10(p-1)} (\tau_1 + \tau_2)^2 (a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}})^2 \\ &\geq \frac{1}{5(p-1)} (\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 - \frac{1}{5(p-1)} (\tau_1 - \tau_2)^2 (b^{-p+1} + a^{-p+1}). \end{aligned}$$

The proof in the case $\frac{\tau_1}{\tau_2} \in [\frac{1}{2}, 2]$ is complete, which finishes the proof of Lemma 7.1.1, choosing c_1 and c_2 appropriately. □

8. Properties of weak supersolutions

In this chapter we establish the weak Harnack inequality for functions $u \in V^{\mu}(M_1|\mathbb{R}^d)$ satisfying

$$\mathcal{E}(u,\phi) \geq (f,\phi) \quad \text{for every nonnegative } \phi \in H^{\mu}_{M_1}(\mathbb{R}^d) \tag{8.0.1}$$

using the Moser iteration technique, where $f \in L^q$ for some $q > \max\{2,\beta\}$ and β is defined as in (7.0.1).

For this purpose we have to prove a Poincaré inequality and show that the logarithm of a weak supersolutions are functions of bounded mean oscillation.

We first prove an auxiliary scaling result for the energy forms with respect to the reference family $\mu_{\text{axes}}(x,\cdot)$, $x \in \mathbb{R}^d$. Recall the definition

$$\beta = \sum_{k=1}^{d} \frac{1}{\alpha_k}.$$

Lemma 8.0.1. *Let $\lambda > 0$, $\Omega \subset \mathbb{R}^d$ be open, $u \in V^{\mu_{axes}}(\Omega|\mathbb{R}^d)$ and let $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism defined by*

$$\Psi(x) = \begin{pmatrix} \lambda^{\frac{2}{\alpha_1}} & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & \lambda^{\frac{2}{\alpha_d}} \end{pmatrix} x.$$

Then

$$\mathcal{E}^{\mu_{axes}}_{\Omega}(u \circ \Psi, u \circ \Psi) = \lambda^{2-2\beta} \mathcal{E}^{\mu_{axes}}_{\Psi(\Omega)}(u,u).$$

Proof. Change of variables gives us

$$\begin{aligned}
&\int_\Omega \int_\Omega (u(\Psi(x)) - u(\Psi(y)))^2 \, \mu_{\text{axes}}(x, \mathrm{d}y) \, \mathrm{d}x \\
&= \int_\Omega \sum_{k=1}^{d} \int_\Omega (u(\Psi(x)) - u(\Psi(y)))^2 \alpha_k (2-\alpha_k) |x_k - y_k|^{-1-\alpha_k} \, \mathrm{d}y_k \prod_{i \neq k} \delta_{\{x_i\}}(\mathrm{d}y_i) \, \mathrm{d}x \\
&= \int_\Omega \sum_{k=1}^{d} \int_{\mathbb{R}} \mathbb{1}_{\{y_k \in \mathbb{R} :\, (x_1,\dots,x_{k-1},y_k,x_{k+1},\dots,x_d) \in \Omega\}} \\
&\qquad (u(\Psi(x)) - u(\Psi(x_1,\dots,x_{k-1},y_k,x_{k+1},\dots,x_d)))^2 \alpha_k (2-\alpha_k) |x_k - y_k|^{-1-\alpha_k} \, \mathrm{d}y_k \, \mathrm{d}x
\end{aligned}$$

$$= \lambda^{-2\beta} \int_{\Psi(\Omega)} \sum_{k=1}^{d} \int_{\mathbb{R}} \mathbb{1}_{\left\{y_k \in \mathbb{R}\colon\, \Psi^{-1}(\widetilde{x}_1,\dots,\widetilde{x}_{k-1},\lambda^{2/\alpha_k} y_k,\widetilde{x}_{k+1},\dots,\widetilde{x}_d)\in\Omega\right\}}$$
$$(u(\widetilde{x}) - u(\widetilde{x}_1,\dots,\widetilde{x}_{k-1},\lambda^{2/\alpha_k} y_k,\widetilde{x}_{k+1},\dots,\widetilde{x}_d))^2 \alpha_k(2-\alpha_k) |\lambda^{-\frac{2}{\alpha_k}}\widetilde{x}_k - y_k|^{-1-\alpha_k} \, \mathrm{d}y_k \, \mathrm{d}\widetilde{x}$$
$$= \lambda^{2-2\beta} \int_{\Psi(\Omega)} \sum_{k=1}^{d} \int_{\mathbb{R}} \mathbb{1}_{\{\widetilde{y}_k \in \mathbb{R}\colon\, \Psi^{-1}(\widetilde{x}_1,\dots,\widetilde{x}_{k-1},\widetilde{y}_k,\widetilde{x}_{k+1},\dots,\widetilde{x}_d)\in\Omega\}}$$
$$(u(\widetilde{x}) - u(\Psi(\widetilde{x}_1,\dots,\widetilde{x}_{k-1},\widetilde{y}_k,\widetilde{x}_{k+1},\dots,\widetilde{x}_d)))^2 \alpha_k(2-\alpha_k) |(\widetilde{x}_k - \widetilde{y}_k)|^{-1-\alpha_k} \, \mathrm{d}\widetilde{y}_k \, \mathrm{d}\widetilde{x}$$
$$= \lambda^{2-2\beta} \int_{\Psi(\Omega)} \sum_{k=1}^{d} \int_{\mathbb{R}} \mathbb{1}_{\{\widetilde{y}_k \in \mathbb{R}\colon\, (\widetilde{x}_1,\dots,\widetilde{x}_{k-1},\widetilde{y}_k,\widetilde{x}_{k+1},\dots,\widetilde{x}_d)\in\Psi(\Omega)\}}$$
$$(u(\widetilde{x}) - u(\Psi(\widetilde{x}_1,\dots,\widetilde{x}_{k-1},\widetilde{y}_k,\widetilde{x}_{k+1},\dots,\widetilde{x}_d)))^2 \alpha_k(2-\alpha_k) |(\widetilde{x}_k - \widetilde{y}_k)|^{-1-\alpha_k} \, \mathrm{d}\widetilde{y}_k \, \mathrm{d}\widetilde{x}$$
$$= \lambda^{2-2\beta} \int_{\Psi(\Omega)} \sum_{k=1}^{d} \int_{\Psi(\Omega)} (u(\widetilde{x}) - u(\widetilde{y}))^2 \alpha_k(2-\alpha_k) |\widetilde{x}_k - \widetilde{y}_k|^{-1-\alpha_k} d\widetilde{y}_k \prod_{i\neq k} \delta_{\{\widetilde{x}_i\}}(\mathrm{d}\widetilde{y}_i) \, \mathrm{d}\widetilde{x}$$
$$= \lambda^{1-\beta} \int_{\Psi(\Omega)} \int_{\Psi(\Omega)} (u(x)-u(y))^2 \mu_{\text{axes}}(x,\mathrm{d}y) \, \mathrm{d}x.$$

□

Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set. For $f \in L^1(\Omega)$ let

$$[f]_\Omega := \fint_\Omega f(x)\,\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega f(x)\,\mathrm{d}x.$$

Furthermore, let

$$\gamma = \max\left\{(\alpha_k(2-\alpha_k))^{-1}\colon k \in \{1,\dots,d\}\right\}.$$

Lemma 8.0.2. *Let $r \in (0,1]$ and $x_0 \in M_1$. Assume $v \in V^\mu(M_r(x_0)|\mathbb{R}^d)$. There exists a constant $c_1 > 0$, independent of x_0, r and v, such that*

$$\|v - [v]_{M_r(x_0)}\|^2_{L^2(M_r(x_0))} \le c_1 r^2 \mathcal{E}^\mu_{M_r(x_0)}(v,v).$$

Proof. For simplicity of notation we assume $x_0 = 0$ and write $M_r = M_r(x_0)$ and $M_{\lambda r} = M_{\lambda r}(x_0)$. The proof for general $x_0 \in \mathbb{R}^d$ works analogously. By Jensen's inequality

$$\begin{aligned} \|v - [v]_{M_r}\|^2_{L^2(M_r)} &= \int_{M_r} \left(v(x) - \fint_{M_r} v(y)\,\mathrm{d}y\right)^2 \mathrm{d}x \\ &= \int_{M_r} \left(v(x) - \frac{1}{|M_r|}\int_{M_r} v(y)\,\mathrm{d}y\right)^2 \mathrm{d}x \\ &= \int_{M_r} \left(\frac{1}{|M_r|}\int_{M_r} (v(x)-v(y))\,\mathrm{d}y\right)^2 \mathrm{d}x \end{aligned}$$

$$\leq \frac{1}{|M_r|} \int_{M_r} \int_{M_r} (v(x) - v(y))^2 \, \mathrm{d}y \, \mathrm{d}x := J.$$

We define a polygonal chain $\ell = (\ell_0(x,y), \ldots, \ell_d(x,y))$ connecting x and y and whose line segments are parallel to the coordinate axes:

$$\ell_k(x,y) = (l_1^k, \ldots, l_d^k), \quad \text{where} \begin{cases} l_j^k = y_j, & \text{if } j \leq k, \\ l_j^k = x_j, & \text{if } j > k. \end{cases}$$

Thus

$$J \leq \frac{d}{|M_r|} \sum_{k=1}^{d} \int_{M_r} \int_{M_r} (v(\ell_{k-1}(x,y)) - v(\ell_k(x,y)))^2 \, \mathrm{d}y \, \mathrm{d}x := \frac{d}{|M_r|} \sum_{k=1}^{d} I_k.$$

Let us fix $k \in \{1, \ldots, d\}$ and set $w = \ell_{k-1}(x,y) = (y_1, \ldots, y_{k-1}, x_k, \ldots, x_d)$. Moreover let $z := x + y - w = (x_1, \ldots, x_{k-1}, y_k, \ldots, y_d)$. Then $\ell_k(x,y) = w + e_k(z_k - w_k) = (y_1, \ldots, y_k, x_{k+1}, \ldots, x_d)$. By Fubini's Theorem

$$\begin{aligned}
I_k &= \int_{-r^{2/\alpha_1}}^{r^{2/\alpha_1}} \cdots \int_{-r^{2/\alpha_d}}^{r^{2/\alpha_d}} \int_{-r^{2/\alpha_1}}^{r^{2/\alpha_1}} \cdots \int_{-r^{2/\alpha_d}}^{r^{2/\alpha_d}} (v(\ell_{k-1}(x,y)) - v(\ell_k(x,y)))^2 \\
&\qquad\qquad \mathrm{d}y_d \cdots \mathrm{d}y_1 \, \mathrm{d}x_d \cdots \mathrm{d}x_1 \\
&= \left(\prod_{j \neq k} \int_{-r^{2/\alpha_j}}^{r^{2/\alpha_j}} \mathrm{d}z_j \right) \int_{-r^{2/\alpha_1}}^{r^{2/\alpha_1}} \cdots \int_{-r^{2/\alpha_d}}^{r^{2/\alpha_d}} \int_{-r^{2/\alpha_k}}^{r^{2/\alpha_k}} (v(w) - v(w + e_k(z_k - w_k)))^2 \\
&\qquad\qquad \mathrm{d}z_k \, \mathrm{d}w_d \cdots \mathrm{d}w_1 \\
&= 2^{d-1} r^{\sum_{j \neq k} 2/\alpha_j} \int_{M_r} \int_{-r^{2/\alpha_k}}^{r^{2/\alpha_k}} (v(w) - v(w + e_k(z_k - w_k)))^2 \, \mathrm{d}z_k \, \mathrm{d}w \\
&= 2^{d-1} r^{\sum_{j \neq k} 2/\alpha_j} \int_{M_r} \int_{-r^{2/\alpha_k}}^{r^{2/\alpha_k}} (v(w) - v(w + e_k(z_k - w_k)))^2 \frac{|x_k - z_k|^{1+\alpha_k}}{|x_k - z_k|^{1+\alpha_k}} \, \mathrm{d}z_k \, \mathrm{d}w \\
&\leq 2^{d-1} r^{\sum_{j \neq k} 2/\alpha_j} \int_{M_r} \int_{-r^{2/\alpha_k}}^{r^{2/\alpha_k}} (v(w) - v(w + e_k(z_k - w_k)))^2 \frac{2^{1+\alpha_k} r^{2/\alpha_k + 2}}{|x_k - z_k|^{1+\alpha_k}} \, \mathrm{d}z_k \, \mathrm{d}w \\
&\leq 4 2^d r^{2\beta} r^2 \int_{M_r} \int_{-r^{2/\alpha_k}}^{r^{2/\alpha_k}} (v(w) - v(w + e_k(z_k - w_k)))^2 \frac{1}{|x_k - z_k|^{1+\alpha_k}} \, \mathrm{d}z_k \, \mathrm{d}w \\
&\leq \frac{4r^2}{\alpha_k(2-\alpha_k)} |M_r| \int_{M_r} \int_{-r^{2/\alpha_k}}^{r^{2/\alpha_k}} (v(w) - v(w + e_k(z_k - w_k)))^2 \frac{\alpha_k(2-\alpha_k)}{|x_k - z_k|^{1+\alpha_k}} \, \mathrm{d}z_k \, \mathrm{d}w \\
&\leq 4r^2 \gamma |M_r| \int_{M_r} \int_{-r^{2/\alpha_k}}^{r^{2/\alpha_k}} (v(w) - v(w + e_k(z_k - w_k)))^2 \frac{\alpha_k(2-\alpha_k)}{|x_k - z_k|^{1+\alpha_k}} \, \mathrm{d}z_k \, \mathrm{d}w.
\end{aligned}$$

Hence there are $c_1, c_2 > 0$, independent of ρ, v and x_0, but depending on d and γ, such that

$$\|v - [v]_{M_r}\|^2_{L^2(M_r)}$$

$$\begin{aligned}&\leq c_2 r^2 \sum_{k=1}^{d} \int_{M_r} \int_{-r^{2/\alpha_k}}^{r^{2/\alpha_k}} (v(w) - v(w + e_k(z_k - w_k)))^2 \frac{\alpha_k(2-\alpha_k)}{|x_k - z_k|^{1+\alpha_k}} \,\mathrm{d}z_k \,\mathrm{d}w \\ &= c_2 r^2 \mathcal{E}^{\mu_{\text{axes}}}_{M_r}(v,v) \leq c_1 r^2 \mathcal{E}^{\mu}_{M_r}(v,v),\end{aligned}$$

where we used Assumption 3 in the last inequality. □

The next lemma provides a Morrey-Besov-type inequality for $\log u$.

Lemma 8.0.3. *Let $x_0 \in M_1$, $r \in (0,1]$ and $\lambda > 1$. Assume $f \in L^q(M_{\lambda r}(x_0))$ for some $q > 2$. Assume $u \in V^{\mu}(M_{\lambda r}(x_0)|\mathbb{R}^d)$ is nonnegative in $\mathbb{R}^d$ and satisfies*

$$\begin{aligned}\mathcal{E}^{\mu}(u,\phi) &\geq (f,\phi) \quad \text{for any nonnegative } \phi \in H^{\mu}_{M_{\lambda r}(x_0)}(\mathbb{R}^d), \\ u(x) &\geq \epsilon \qquad \text{for almost all } x \in M_{\lambda r}(x_0) \text{ and some } \epsilon > 0.\end{aligned} \tag{8.0.2}$$

There exists a constant $c_1 > 0$, independent of $x_0, \lambda, r, \alpha_1, \ldots, \alpha_d$ and u, such that

$$\begin{aligned}&\int_{M_r(x_0)} \int_{M_r(x_0)} \left(\sum_{k=1}^{\infty} \frac{(\log u(y) - \log u(x))^{2k}}{(2k)!} \right) \mu(x, \mathrm{d}y) \,\mathrm{d}x \\ &\quad \leq c_1 \left(\sum_{k=1}^{d} \left(\lambda^{2/\alpha_k} - 1\right)^{-\alpha_k} \right) r^{-2} |M_{\lambda r}(x_0)| + \epsilon^{-1} \|f\|_{L^q(M_{\lambda r}(x_0))} |M_{\lambda r}(x_0)|^{\frac{q}{q-1}}.\end{aligned}$$

Proof. We follow the lines of [DK15, Lemma 4.4].

Let $\tau : \mathbb{R}^d \to \mathbb{R}$ be as in (4). Then by Lemma 7.0.4, there is $c_2 > 0$, such that

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \mu(x, \mathrm{d}y) \leq c_2 r^{-2} \left(\sum_{k=1}^{d} (\lambda^{2/\alpha_k} - 1)^{-\alpha_k} \right).$$

For abbreviation, we write $M_{\lambda r}(x_0) = M_{\lambda r}$ and $M_r(x_0) = M_r$. By definition of τ and symmetry of the measures in the sense of Assumption 1,

$$\begin{aligned}&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \mu(x, \mathrm{d}y) \,\mathrm{d}x \\ &\quad = \int_{M_{\lambda r}} \int_{M_{\lambda r}} (\tau(y) - \tau(x))^2 \mu(x, \mathrm{d}y) \,\mathrm{d}x + 2 \int_{M_{\lambda r}} \int_{M^c_{\lambda r}} (\tau(y) - \tau(x))^2 \mu(x, \mathrm{d}y) \,\mathrm{d}x \\ &\quad \leq 2 \int_{M_{\lambda r}} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \mu(x, \mathrm{d}y) \,\mathrm{d}x \\ &\quad \leq 2 |M_{\lambda r}| \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \mu(x, \mathrm{d}y) \\ &\quad \leq c_3 \left(\sum_{k=1}^{d} (\lambda^{2/\alpha_k} - 1)^{-\alpha_k} \right) r^{-2} |M_{\lambda r}|.\end{aligned} \tag{8.0.3}$$

Let $\phi(x) = -\tau^2(x)u^{-1}(x) \leq 0$. By (8.0.2)

$$\begin{aligned}
(f,\phi) &\geq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(y)-u(x))\left(\tau^2(x)u^{-1}(x) - \tau^2(y)u^{-1}(y)\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= \int_{M_{\lambda r}}\int_{M_{\lambda r}} \tau(x)\tau(y)\left(\frac{\tau(x)u(y)}{\tau(y)u(x)} + \frac{\tau(y)u(x)}{\tau(x)u(y)} - \frac{\tau(y)}{\tau(x)} - \frac{\tau(x)}{\tau(y)}\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad + 2\int_{M_{\lambda r}}\int_{M_{\lambda r}^c} (u(y)-u(x))\left(\tau^2(x)u^{-1}(x) - \tau^2(y)u^{-1}(y)\right)\ \mu(x,\mathrm{d}y)\,\mathrm{d}x.
\end{aligned}$$

Setting $A(x,y) = \frac{u(y)}{u(x)}$ and $B(x,y) = \frac{\tau(y)}{\tau(x)}$ we obtain for the first expression on the right-hand-side of the foregoing inequality

$$\begin{aligned}
&\int_{M_{\lambda r}}\int_{M_{\lambda r}} \tau(x)\tau(y)\left(\frac{A(x,y)}{B(x,y)} + \frac{B(x,y)}{A(x,y)} - B(x,y) - \frac{1}{B(x,y)}\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= \int_{M_{\lambda r}}\int_{M_{\lambda r}} \tau(x)\tau(y)\left[\left(\frac{A(x,y)}{B(x,y)} + \frac{B(x,y)}{A(x,y)} - 2\right) - \left(\sqrt{B(x,y)} - \frac{1}{\sqrt{B(x,y)}}\right)^2\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= \int_{M_{\lambda r}}\int_{M_{\lambda r}} \tau(x)\tau(y)\left[\left(\frac{(A(x,y)-B(x,y))^2}{A(x,y)B(x,y)}\right) - \left(\sqrt{B(x,y)} - \frac{1}{\sqrt{B(x,y)}}\right)^2\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= \int_{M_{\lambda r}}\int_{M_{\lambda r}} \tau(x)\tau(y)\left(2\sum_{k=1}^{\infty}\frac{(\log A(x,y) - \log B(x,y))^{2k}}{(2k)!}\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad - \int_{M_{\lambda r}}\int_{M_{\lambda r}} \tau(x)\tau(y)\left(\sqrt{B(x,y)} - \frac{1}{\sqrt{B(x,y)}}\right)^2\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= \int_{M_{\lambda r}}\int_{M_{\lambda r}} \tau(x)\tau(y)\left(2\sum_{k=1}^{\infty}\frac{\left(\log\frac{u(y)}{\tau(y)} - \log\frac{u(x)}{\tau(x)}\right)^{2k}}{(2k)!}\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad - \int_{M_{\lambda r}}\int_{M_{\lambda r}} (\tau(x)-\tau(y))^2\ \mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\geq \int_{M_r}\int_{M_r}\left(2\sum_{k=1}^{\infty}\frac{(\log u(y) - \log u(x))^{2k}}{(2k)!}\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad - \int_{M_{\lambda r}}\int_{M_{\lambda r}} (\tau(x)-\tau(y))^2\ \mu(x,\mathrm{d}y)\,\mathrm{d}x,
\end{aligned}$$

where we applied that for $a, b \geq 0$

$$\frac{(a-b)^2}{ab} = (a-b)(b^{-1}-a^{-1}) = 2\sum_{k=1}^{\infty}\frac{(\log a - \log b)^{2k}}{(2k)!}$$

and

$$\left(\sqrt{B(x,y)} - \frac{1}{\sqrt{B(x,y)}}\right)^2 = \frac{(\tau(y)-\tau(x))^2}{\tau(y)\tau(x)}.$$

Altogether, we obtain

$$\begin{aligned}(f,\phi) \geq &\int_{M_r}\int_{M_r} \left(2\sum_{k=1}^{\infty} \frac{(\log u(y) - \log u(x))^{2k}}{(2k)!}\right) \mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &- \int_{M_{\lambda r}}\int_{M_{\lambda r}} (\tau(x)-\tau(y))^2\, \mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &+ 2\int_{M_{\lambda r}}\int_{M_{\lambda r}^c} (u(y)-u(x))\left(\tau^2(x)u^{-1}(x) - \tau^2(y)u^{-1}(y)\right)\, \mu(x,\mathrm{d}y)\,\mathrm{d}x.\end{aligned}$$

The third term on the right-hand side can be estimated as follows:

$$\begin{aligned}&2\int_{M_{\lambda r}}\int_{M_{\lambda r}^c} (u(y)-u(x))\left(\tau^2(x)u^{-1}(x) - \tau^2(y)u^{-1}(y)\right)\, \mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &= 2\int_{M_{\lambda r}}\int_{M_{\lambda r}^c} (u(y)-u(x))\left(-\tau^2(x)u^{-1}(x)\right)\, \mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &= 2\int_{M_{\lambda r}}\int_{M_{\lambda r}^c} \frac{\tau^2(x)}{u(x)} u(y)\, \mu(x,\mathrm{d}y)\,\mathrm{d}x - 2\int_{M_{\lambda r}}\int_{M_{\lambda r}^c} \tau^2(x)\, \mu(x,\mathrm{d}y)\ \mathrm{d}x \\ &\geq -2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2\, \mu(x,\mathrm{d}y)\,\mathrm{d}x,\end{aligned}$$

where we used nonnegativity of u in $\mathbb{R}^d$. Therefore, using Hölder's inequality and $|u^{-1}| \leq \epsilon^{-1}$

$$\begin{aligned}&\int_{M_r}\int_{M_r} \left(2\sum_{k=1}^{\infty} \frac{(\log u(y) - \log u(x))^{2k}}{(2k)!}\right) \mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &\leq 3\int_{M_{\lambda r}}\int_{\mathbb{R}^d} (\tau(x)-\tau(y))^2\, \mu(x,\mathrm{d}y)\,\mathrm{d}x + (f, -\tau^2 u^{-1}) \\ &\leq c_1 \left(\sum_{k=1}^{d} \left(\lambda^{2/\alpha_k} - 1\right)^{-\alpha_k}\right) r^{-2}|M_{\lambda r}| + \|f\|_{L^q(M_{\lambda r})} \|u^{-1}\|_{L^{q/(q-1)}(M_{\lambda r})} \\ &\leq c_1 \left(\sum_{k=1}^{d} \left(\lambda^{2/\alpha_k} - 1\right)^{-\alpha_k}\right) r^{-2}|M_{\lambda r}| + \epsilon^{-1}\|f\|_{L^q(M_{\lambda r})} |M_{\lambda r}|^{q/(q-1)}.\end{aligned} \tag{8.0.4}$$

□

Recall the definition

$$\gamma := \max\{(\alpha_k(2-\alpha_k))^{-1} \colon k \in \{1,\dots,d\}\}.$$

The next result is a direct consequence of Lemma 8.0.2 for $v = \log(u)$ and Lemma 8.0.3. It implies that $\log(u)$ is in the BMO space, if u is a weak supersolution.

Corollary 8.0.4. *Let $x_0 \in M_1$, $r \in (0,1]$ and $\lambda \in [\frac{5}{4}, 2]$. Moreover, let $f \in L^q(M_{2r}(x_0))$ for some $q > 2$. Assume $u \in V^\mu(M_{\lambda r}(x_0) | \mathbb{R}^d)$ is nonnegative in $\mathbb{R}^d$ and satisfies*

$$\begin{aligned} \mathcal{E}^\mu(u,\phi) &\geq (f,\phi) \quad \text{for any nonnegative } \phi \in H^\mu_{M_{\lambda r}(x_0)}(\mathbb{R}^d), \\ u(x) &\geq \epsilon \qquad\quad \text{for almost all } x \in M_{2r} \text{ and } \epsilon > r\|f\|_{L^q(M_{\lambda r}(x_0))}. \end{aligned} \tag{8.0.5}$$

Then there exists a constant $c_1 > 0$, independent of x_0, r and u, but depending on d and γ, such that

$$\|\log u - [\log u]_{M_r(x_0)}\|^2_{L^2(M_r(x_0))} \leq c_2 |M_r(x_0)|. \tag{8.0.6}$$

Proof. Set $M_r = M_r(x_0)$ and $M_{\lambda r} = M_{\lambda r}(x_0)$. Note

$$\begin{aligned} |M_{\lambda r}| &= \left(\prod_{k=1}^d 2(\lambda r)^{2/\alpha_k} \right) = \lambda^{2\beta} 2^d r^{2\beta} = \lambda^{2\beta} |M_r| \leq 2^{2\beta} |M_r|, \\ |M_{\lambda r}|^{\frac{q}{q-1}} &= \lambda^{2\beta \frac{q}{q-1}} 2^{d\frac{q}{q-1}} r^{2\beta\frac{q}{q-1}} \leq \lambda^{4\beta} 2^{2d} r^{2\beta} = 2^{4\beta+d} |M_r|, \end{aligned} \tag{8.0.7}$$

where we used the facts $\max\{x/(x-1) \colon x \geq 2\} = 2$, $\lambda \leq 2$ and $r \leq 1$.
Using Lemma 8.0.2 for $v := \log(u)$, Lemma 8.0.3 and (8.0.7), we observe

$$\begin{aligned} &\|\log u - [\log u]_{M_r}\|^2_{L^2(M_r)} \leq c_1 r^2 \mathcal{E}_{M_r(x_0)}(\log u, \log u) \\ &\leq 2c_1 r^2 \int_{M_r} \int_{M_r} \left(\sum_{k=1}^\infty \frac{(\log u(y) - \log u(x))^{2k}}{(2k)!} \right) \mu(x, \mathrm{d}y)\, \mathrm{d}x \\ &\leq 2c_1 r^2 \left(c_3 \left(\sum_{k=1}^d \left(\lambda^{\frac{2}{\alpha_k}} - 1 \right)^{-\alpha_k} \right) r^{-2} |M_{\lambda r}| + \epsilon^{-1} \|f\|_{L^q(M_{\lambda r})} |M_{\lambda r}|^{q/(q-1)} \right) \\ &\leq 2c_1 r^2 \left(c_3 \left(\sum_{k=1}^d \left(\left(\frac{5}{4}\right)^{\frac{2}{\alpha_k}} - 1 \right)^{-\alpha_k} \right) r^{-2} |M_{\lambda r}| + r^{-1} |M_{\lambda r}|^{q/(q-1)} \right) \\ &\leq 2c_1 \left(c_3 16 d |M_{\lambda r}| + 2^{4\beta+d} |M_r| \right) \\ &= 2c_1 \left(c_3 16 d 2^{2\beta} |M_r| + 2^{4\beta+d} |M_r| \right) = c_1(d,\beta) |M_r|. \end{aligned}$$

Here we have used $\max\{(5/4)^{2/x} - 1)^{-x} \colon x \in (0,2]\} = 16$. □

Finally, we can prove the following result.

Theorem 8.0.5. *Assume $x_0 \in M_1$, $r \in (0,1]$ and $f \in L^q(M_{\frac{5}{4}r}(x_0))$ for some $q > 2$. Assume $u \in V^\mu(M_{\frac{5}{4}r}(x_0) | \mathbb{R}^d)$ is nonnegative in $\mathbb{R}^d$ and satisfies*

$$\begin{aligned} \mathcal{E}^\mu(u,\phi) &\geq (f,\phi) \quad \text{for any nonnegative } \phi \in H^\mu_{M_{\frac{5}{4}r}(x_0)}(\mathbb{R}^d), \\ u(x) &\geq \epsilon \qquad\quad \text{for almost all } x \in M_{\frac{5}{4}r} \text{ and some } \epsilon > r\|f\|_{L^q(M_{\frac{5}{4}r}(x_0))}. \end{aligned}$$

Then there exist $\overline{p} \in (0,1)$ *and* $c_1 > 0$*, independent of* x_0, r, u *and* ϵ*, such that*

$$\left(\fint_{M_r(x_0)} u(x)^{\overline{p}} \,\mathrm{d}x \right)^{1/\overline{p}} \,\mathrm{d}x \leq c_1 \left(\fint_{M_r(x_0)} u(x)^{-\overline{p}} \,\mathrm{d}x \right)^{-1/\overline{p}}. \tag{8.0.8}$$

Proof. This proof follows the proof of [DK15, Lemma 4.5].

The main idea is to prove $\log u \in \mathrm{BMO}(M_r(x_0))$ and use the John-Nirenberg inequality for doubling metric measure spaces.
Recall that for $r \in (0,1]$ the set $M_r(x_0)$ is a ball in the metric space $(\mathbb{R}^d, d)$ defined in (7.0.7). Endowed with the Lebesgue measure, the metric measure space $(M_1(x_0), d, \mathrm{d}x)$ is a doubling space, since for all $x \in \mathbb{R}^d$ and $r \in (0, 1/2]$

$$|B^d_{2r}(x)| = |M_{2r}(x)| = 2^{2\beta} |M_r(x)| = 2^{2\beta} B^d_r(x).$$

Choose $z_0 \in M_r(x_0)$ and $\rho > 0$ such that $M_{2\rho}(z_0) \subset M_r(x_0)$. Note that by (8.0.7) $|M_{2\rho}|^{\frac{q}{q-1}} \leq 2^{4\beta+d} |M_\rho|$.
Corollary 8.0.4 and the Hölder inequality imply

$$\begin{aligned}\int_{M_\rho(z_0)} \Big| \log u(x) - [\log u]_{M_\rho(z_0)} \Big| \,\mathrm{d}x &\leq \| \log u - [\log u]_{M_\rho(z_0)} \|_{L^2(M_\rho(z_0))} \sqrt{|M_\rho|} \\ &\leq c_2 |M_\rho|.\end{aligned}$$

This shows $\log u \in \mathrm{BMO}(M_r(x_0))$. Lemma 2.4.3 states, that $\log u \in \mathrm{BMO}(M_r(x_0))$, iff for each $M_\rho \Subset M_r(x_0)$ and $\kappa > 0$

$$|\{x \in M_\rho \colon | \log u(x) - [\log u]_{M_\rho}| > \kappa\}| \leq c_3 e^{-c_4 \kappa} |M_\rho|, \tag{8.0.9}$$

where the positive constants c_3, c_4 and the BMO norm depend only on each other, the dimension d and the doubling constant.
By Cavalieri's principle, we have for $h : M_R(x_0) \to [0, \infty]$, using the change of variable $t = e^\kappa$, that

$$\begin{aligned}\fint_{M_r(x_0)} e^{h(x)} \,\mathrm{d}x &= \frac{1}{|M_r|} \int_0^\infty |\{x \in M_r(x_0) \colon e^{h(x)} > t\}| \,\mathrm{d}t \\ &= \frac{1}{|M_r|} \left(\int_0^1 |\{x \in M_r(x_0) \colon e^{h(x)} > t\}| \,\mathrm{d}t \right. \\ &\qquad \left. + \int_1^\infty |\{x \in M_r(x_0) \colon e^{h(x)} > t\}| \,\mathrm{d}t \right) \\ &\leq 1 + \frac{1}{|M_r|} \int_0^\infty e^\kappa |\{x \in M_r(x_0) \colon h(x) > \kappa\}| \,\mathrm{d}\kappa.\end{aligned}$$

Let $\overline{p} \in (0,1)$ be chosen such that $\overline{p} < c_4$. The application of (8.0.9) implies

$$\fint_{M_r(x_0)} \exp\left(\overline{p} | \log u(y) - [\log u]_{M_r(x_0)}| \right) \,\mathrm{d}y$$

$$
\begin{aligned}
&\le 1 + \int_0^\infty e^\kappa \frac{|\{x \in M_r(x_0)\colon |\log u(x) - [\log u]_{M_r(x_0)}| > \kappa/\overline{p}\}|}{|M_r|} \,\mathrm{d}\kappa \\
&\le 1 + \int_0^\infty e^\kappa \frac{c_3 e^{-c_4\kappa/\overline{p}} |M_r(x_0)|}{|M_r|} \,\mathrm{d}\kappa \\
&\le 1 + c_3 \int_0^\infty e^{(1-c_4/\overline{p})\kappa} \,\mathrm{d}\kappa \\
&= 1 + \frac{c_3}{c_4/\overline{p} - 1} = \frac{c_4 - \overline{p} + c_3\overline{p}}{c_4 - \overline{p}} =: c_5 < \infty.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left(\fint_{M_r(x_0)} u(y)^{\overline{p}} \,\mathrm{d}y \right) \left(\fint_{M_r(x_0)} u(y)^{-\overline{p}} \,\mathrm{d}y \right) \\
&= \left(\fint_{M_r(x_0)} e^{\overline{p}(\log u(y) - [\log u]_{M_r})} \,\mathrm{d}y \right) \left(\fint_{M_r(x_0)} e^{-\overline{p}(\log u(y) - [\log u]_{M_r})} \,\mathrm{d}y \right) \le c_5^2 = c_1,
\end{aligned}
$$

which implies the assertion. □

8.1. The weak Harnack inequality

In this section we prove the weak Harnack inequality for functions u satisfying (8.1.8). The main technique is the Moser iteration for negative exponents.

We start with an easy auxiliary result.

Lemma 8.1.1. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $q > \beta$ and $f \in L^{\frac{\beta}{\beta-1}}(\Omega)$. Then for any $a > 0$*

$$
\|f\|_{L^{\frac{q}{q-1}}(\Omega)} \le \frac{\beta}{q} a \|f\|_{L^{\frac{\beta}{\beta-1}}(\Omega)} + \frac{q-\beta}{q} a^{-\beta/(q-\beta)} \|f\|_{L^1(\Omega)} .
$$

Proof. First, note that for $q > \beta$:

$$
1 \le \frac{q}{q-1} < \frac{\beta}{\beta-1}, \quad \text{because} \quad \frac{\beta}{\beta-1} = 1 + \frac{1}{\beta-1}.
$$

Set

$$
\nu = \frac{1}{\frac{q}{q-1}} \cdot \frac{\frac{\beta}{\beta-1} - \frac{q}{q-1}}{\frac{\beta}{\beta-1} - 1} = \frac{q-\beta}{q}.
$$

Lyapunov's inequality implies

$$
\|f\|_{L^{\frac{q}{q-1}}(\Omega)} \le \|f\|_{L^{\frac{\beta}{\beta-1}}(\Omega)}^{1-\nu} \|f\|_{L^1(\Omega)}^{\nu} = \|f\|_{L^{\frac{\beta}{\beta-1}}(\Omega)}^{\beta/q} \|f\|_{L^1(\Omega)}^{(q-\beta)/q} .
$$

Using Young's inequality, we get for any $a > 0$

$$\begin{aligned}
\|f\|^{\beta/q}_{L^{\frac{\beta}{\beta-1}}(\Omega)} \|f\|^{(q-\beta)/q}_{L^1(\Omega)} &= a^{\beta/q} \|f\|^{\beta/q}_{L^{\frac{\beta}{\beta-1}}(\Omega)} a^{-\beta/q} \|f\|^{(q-\beta)/q}_{L^1(\Omega)} \\
&\le \frac{\beta}{q} \left(a^{\beta/q} \|f\|^{\beta/q}_{L^{\frac{\beta}{\beta-1}}(\Omega)} \right)^{q/\beta} + \frac{q-\beta}{q} \left(a^{-\beta/q} \|f\|^{(q-\beta)/q}_{L^1(\Omega)} \right)^{q/(q-\beta)} \\
&= \frac{\beta}{q} a \|f\|_{L^{\frac{\beta}{\beta-1}}(\Omega)} + \frac{q-\beta}{q} a^{-\beta/(q-\beta)} \|f\|_{L^1(\Omega)} .
\end{aligned}$$

□

Note that in the case $\alpha_1 = \cdots = \alpha_d = \alpha \in (0,2)$ the condition $q > \beta = d/\alpha$ is identical to the condition in the case of $\mu^\alpha(x, \mathrm{d}y) = c(d,\alpha)|x-y|^{-d-\alpha}$, see [DK15, Theorem 1.1]. The next result allows us to apply Moser's iteration technique for negative exponents. It is a purely local result although the energy form is nonlocal.

Lemma 8.1.2. *Assume $x_0 \in M_1$ and $r \in [0,1)$. Moreover, let $\lambda \in (1, \min\{r^{-1}, \sqrt{2}\})$ and $f \in L^q(M_{\lambda r}(x_0))$ for some $q > \max\{2, \beta\}$. Assume $u \in V^\mu(M_{\lambda r}(x_0)|\mathbb{R}^d)$ satisfies*

$$\begin{aligned}
\mathcal{E}(u,\phi) &\ge (f,\phi) \quad \text{for any nonnegative } \phi \in H^\mu_{M_{\lambda r}(x_0)}(\mathbb{R}^d), \\
u(x) &\ge \epsilon \qquad \text{for a.a. } x \in M_{\lambda r}(x_0) \text{ and some } \epsilon > \|f\|_{L^q(M_{\lambda r}(x_0))} r^{(q-\beta)/q}.
\end{aligned}$$

Then for $p > 1$, there is a $c_1 > 0$ independent of $u, x_0, r, p, \alpha_1, \ldots, \alpha_d$ and ϵ, such that

$$\left\|u^{-1}\right\|^{p-1}_{L^{(p-1)\frac{\beta}{\beta-1}}(M_r(x_0))} \le c_1 \frac{p}{p-1} \left(\sum_{k=1}^d (\lambda^{2/\alpha_k} - 1)^{-\alpha_k} \right) r^{-2} \left\|u^{-1}\right\|^{p-1}_{L^{p-1}(M_{\lambda r}(x_0))} .$$

Proof. Let $\tau : \mathbb{R}^d \to \mathbb{R}$ be as in Assumption 4. We follow the idea of the proof of [DK15, Lemma 4.6].

For abbreviation let $M_r = M_r(x_0)$. Since $\mathcal{E}(u,\phi) \ge (f,\phi)$ for any nonnegative $\phi \in H_{M_r}(\mathbb{R}^d)$ we get

$$\mathcal{E}(u, -\tau^2 u^{-p}) \le (f, -\tau^2 u^{-p}).$$

Furthermore, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(y)-u(x)) \left(\tau(x)^2 u(x)^{-p} - \tau(y)^2 u(y)^{-p}\right) \mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad = \int_{M_{\lambda r}}\int_{M_{\lambda r}} (u(y)-u(x)) \left(\tau(x)^2 u(x)^{-p} - \tau(y)^2 u(y)^{-p}\right) \mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad + 2\int_{M_{\lambda r}}\int_{M^c_{\lambda r}} (u(y)-u(x)) \left(\tau(x)^2 u(x)^{-p} - \tau(y)^2 u(y)^{-p}\right) \mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad =: J_1 + 2J_2.
\end{aligned}$$

We first rewrite and estimate J_2, using Lemma 7.0.4,

$$
\begin{aligned}
J_2 &= \int_{M_{\lambda r}} \int_{M_{\lambda r}^c} (u(y)-u(x)) \left(\tau(x)^2 u(x)^{-p} - \tau(y)^2 u(y)^{-p}\right) \mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= \int_{M_{\lambda r}} \int_{M_{\lambda r}^c} (u(y)-u(x))\, \tau(x)^2 u(x)^{-p} \mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= \int_{M_{\lambda r}} \int_{M_{\lambda r}^c} u(y)\tau(x)^2 u(x)^{-p} \mu(x,\mathrm{d}y)\,\mathrm{d}x - \int_{M_{\lambda r}} \int_{M_{\lambda r}^c} \tau(x)^2 u(x)^{-p+1} \mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\geq - \int_{M_{\lambda r}} \int_{M_{\lambda r}^c} \tau(x)^2 u(x)^{-p+1} \mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= - \int_{M_{\lambda r}} \int_{M_{\lambda r}^c} (\tau(x)-\tau(y))^2 u(x)^{-p+1} \mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\geq -\|u^{-p+1}\|_{L^1(M_{\lambda r})} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2 \mu(x,\mathrm{d}y) \\
&\geq -\|u^{-1}\|_{L^{p-1}(M_{\lambda r})}^{p-1} c_2 r^{-2} \left(\sum_{k=1}^d \left(\lambda^{2/\alpha_k} - 1\right)^{-\alpha_k} \right).
\end{aligned}
$$

Now we estimate J_1. Applying Lemma 7.1.1 for $a = u(x), b = u(y), \tau_1 = \tau(x), \tau_2 = \tau(y)$ on J_1, there exist $c_3, c_4 > 0$ such that

$$
\begin{aligned}
J_1 &= \int_{M_{\lambda r}} \int_{M_{\lambda r}} (u(y)-u(x)) \left(\tau(x)^2 u(x)^{-p} - \tau(y)^2 u(y)^{-p}\right) \mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\geq c_3 \int_{M_{\lambda r}} \int_{M_{\lambda r}} \left(\tau(x)u(x)^{\frac{-p+1}{2}} - \tau(x)u(x)^{\frac{-p+1}{2}}\right)^2 \mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad - c_4 \frac{p}{p-1} \int_{M_{\lambda r}} \int_{M_{\lambda r}} (\tau(y)-\tau(x))^2 (u(y)^{-p+1} + u(x)^{-p+1}) \mu(x,\mathrm{d}y)\,\mathrm{d}x.
\end{aligned}
$$

Hence

$$\begin{aligned}
&\int_{M_{\lambda r}}\int_{M_{\lambda r}} \left(\tau(x)u(x)^{\frac{-p+1}{2}} - \tau(x)u(x)^{\frac{-p+1}{2}}\right)^2 \mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad \le \frac{1}{c_3}J_1 + c_4\frac{p}{p-1}\int_{M_{\lambda r}}\int_{M_{\lambda r}} (\tau(y)-\tau(x))^2(u(y)^{-p+1}+u(x)^{-p+1})\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad = \frac{1}{c_3}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(y)-u(x))\left(\tau(x)^2u(x)^{-p} - \tau(y)^2u(y)^{-p}\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x - \frac{2}{c_3}J_2 \\
&\qquad + c_4\frac{p}{p-1}\int_{M_{\lambda r}}\int_{M_{\lambda r}} (\tau(y)-\tau(x))^2(u(y)^{-p+1}+u(x)^{-p+1})\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad \le \frac{1}{c_3}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(y)-u(x))\left(\tau(x)^2u(x)^{-p} - \tau(y)^2u(y)^{-p}\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad + \frac{16}{c_3}\|u^{-p+1}\|_{L^1(M_{\lambda r})}r^{-2}\left(\sum_{k=1}^d\left(\lambda^{2/\alpha_k}-1\right)^{-\alpha_k}\right) \\
&\qquad + c_4\frac{p}{p-1}\int_{M_{\lambda r}}\int_{M_{\lambda r}} (\tau(y)-\tau(x))^2(u(y)^{-p+1}+u(x)^{-p+1})\mu(x,\mathrm{d}y)\,\mathrm{d}x.
\end{aligned} \tag{8.1.1}$$

We now will derive the assertion from (8.1.1).

The first term of the right-hand-side of (8.1.1) can be estimated with the help of Lemma 8.1.1 as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(y)-u(x))\left(\tau(x)^2u(x)^{-p} - \tau(y)^2u(y)^{-p}\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x = \mathcal{E}(u,-\tau^2u^{-p}) \\
&\quad \le (f,-\tau^2u^{-p}) \le \epsilon^{-1}|(f,-\tau^2u^{-p+1})| = \epsilon^{-1}|(\tau f,\tau u^{-p+1})| \\
&\quad \le \epsilon^{-1}\|\tau f\|_{L^q(\mathbb{R}^d)}\|\tau u^{-p+1}\|_{L^{\frac{q}{q-1}}(\mathbb{R}^d)} \\
&\quad \le \epsilon^{-1}\|\tau f\|_{L^q(\mathbb{R}^d)}\left(\frac{\beta}{q}a\|\tau u^{-p+1}\|_{L^{\frac{\beta}{\beta-1}}(\mathbb{R}^d)} + \frac{q-\beta}{q}a^{\frac{-\beta}{q-\beta}}\|\tau u^{-p+1}\|_{L^1(\mathbb{R}^d)}\right) \\
&\quad \le \epsilon^{-1}\|f\|_{L^q(M_{\lambda r})}\left(\frac{\beta}{q}a\|\tau u^{-p+1}\|_{L^{\frac{\beta}{\beta-1}}(\mathbb{R}^d)} + \frac{q-\beta}{q}a^{\frac{-\beta}{q-\beta}}\|\tau u^{-p+1}\|_{L^1(\mathbb{R}^d)}\right) \\
&\quad \le r^{(\beta-q)/q}\left(\frac{\beta}{q}a\|\tau u^{-p+1}\|_{L^{\frac{\beta}{\beta-1}}(\mathbb{R}^d)} + \frac{q-\beta}{q}a^{\frac{-\beta}{q-\beta}}\|\tau u^{-p+1}\|_{L^1(\mathbb{R}^d)}\right),
\end{aligned}$$

where $a>0$ can be chosen arbitrarily. Set

$$a = r^{(\beta-q)/q}\omega$$

for some $\omega>0$. Since $1<\lambda\le\sqrt{2}$ we have in particular for all $k\in\{1,\dots,d\}$

$$\lambda \le (2^{1/\alpha_k}+1)^{\alpha_k/2} \iff (\lambda^{2/\alpha_k}-1)^{-\alpha_k} \ge \frac{1}{2}.$$

Thus

$$\left(\sum_{k=1}^d(\lambda^{2/\alpha_k}-1)^{-\alpha_k}\right) \ge 1.$$

Hence, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(y)-u(x))\left(\tau(x)^2u(x)^{-p}-\tau(y)^2u(y)^{-p}\right)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad\le \frac{q}{\beta}\omega\frac{\beta}{q}\|\tau u^{-p+1}\|_{L^{\frac{\beta}{\beta-1}}(\mathbb{R}^d)} + \frac{q-\beta}{\beta}r^{-1}\omega^{\frac{-\beta}{q-\beta}}\|\tau u^{-p+1}\|_{L^1(\mathbb{R}^d)} \\
&\qquad\le \frac{q}{\beta}\omega\|u^{-p+1}\|_{L^{\frac{\beta}{\beta-1}}(M_{\lambda r})} + \frac{q-\beta}{\beta}r^{-1}\omega^{\frac{-\beta}{q-\beta}}\|\tau u^{-p+1}\|_{L^1(M_{\lambda r})} \\
&\qquad= \frac{q}{\beta}\omega\|u^{-1}\|^{p-1}_{L^{(p-1)\frac{\beta}{\beta-1}}(M_{\lambda r})} + \frac{q-\beta}{\beta}r^{-1}\omega^{\frac{-\beta}{q-\beta}}\|u^{-1}\|^{p-1}_{L^{p-1}(M_{\lambda r})} \\
&\qquad\le \frac{q}{\beta}\omega\|u^{-1}\|^{p-1}_{L^{(p-1)\frac{\beta}{\beta-1}}(M_{\lambda r})} \\
&\qquad\qquad + \frac{q-\beta}{\beta}r^{-2}\left(\sum_{k=1}^{d}(\lambda^{2/\alpha_k}-1)^{-\alpha_k}\right)\omega^{\frac{-\beta}{q-\beta}}\|u^{-1}\|^{p-1}_{L^{p-1}(M_{\lambda r})}.
\end{aligned}
\tag{8.1.2}
$$

The third term of the right-hand-side of (8.1.1) can be estimated as follows:

$$
\begin{aligned}
&\int_{M_{\lambda r}}\int_{M_{\lambda r}} (\tau(y)-\tau(x))^2(u(y)^{-p+1}+u(x)^{-p+1})\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad= 2\int_{M_{\lambda r}}\int_{M_{\lambda r}} (\tau(y)-\tau(x))^2(u(x)^{-p+1})\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad\le 2\|u^{-p+1}\|_{L^1(M_{\lambda r})}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}(\tau(y)-\tau(x))^2\mu(x,\mathrm{d}y) \\
&\qquad\le c_5r^{-2}\left(\sum_{k=1}^{d}\left(\lambda^{2/\alpha_k}-1\right)^{-\alpha_k}\right)\|u^{-p+1}\|_{L^1(M_{\lambda r})} \\
&\qquad= c_5r^{-2}\left(\sum_{k=1}^{d}\left(\lambda^{2/\alpha_k}-1\right)^{-\alpha_k}\right)\|u^{-1}\|^{p-1}_{L^{p-1}(M_{\lambda r})}.
\end{aligned}
$$

The left-hand-side can be estimated from below with Corollary 7.0.9

$$
\begin{aligned}
&\int_{M_{\lambda r}}\int_{M_{\lambda r}} \left(\tau(x)u(x)^{\frac{-p+1}{2}}-\tau(x)u(x)^{\frac{-p+1}{2}}\right)^2\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad\ge c_6\|\tau u^{\frac{-p+1}{2}}\|^2_{L^{\frac{2\beta}{\beta-1}}(M_r)} - r^{-2}\left(\sum_{k=1}^{d}(\lambda^{2/\alpha_k}-1)^{-\alpha_k}\right)\|\tau u^{\frac{-p+1}{2}}\|^2_{L^2(M_{\lambda r})} \\
&\qquad\ge c_6\|u^{\frac{-p+1}{2}}\|^2_{L^{\frac{2\beta}{\beta-1}}(M_r)} - r^{-2}\left(\sum_{k=1}^{d}(\lambda^{2/\alpha_k}-1)^{-\alpha_k}\right)\|u^{\frac{-p+1}{2}}\|^2_{L^2(M_{\lambda r})} \\
&\qquad= c_6\|u^{-1}\|^{p-1}_{L^{(p-1)\frac{\beta}{\beta-1}}(M_r)} - r^{-2}\left(\sum_{k=1}^{d}(\lambda^{2/\alpha_k}-1)^{-\alpha_k}\right)\|u^{-1}\|^{p-1}_{L^{p-1}(M_{\lambda r})}.
\end{aligned}
$$

Combining these estimates there exists a constant $c_1 > 0$, independent of x_0, r, λ, $\alpha_1, \dots, \alpha$ and u, but depending on d and $2\beta/(\beta-1)$, such that

$$\|u^{-1}\|^{p-1}_{L^{(p-1)\frac{\beta}{\beta-1}}(M_r)} \leq c_1(\omega^{\frac{-\beta}{q-\beta}} + \frac{p}{p-1})\left(\sum_{k=1}^{d}(\lambda^{2/\alpha_k}-1)^{-\alpha_k}\right) r^{-2}\|u^{-1}\|^{p-1}_{L^{p-1}(M_{\lambda r})} + \frac{1}{c_2 c_4}\omega\|u^{-1}\|_{L^{(p-1)\frac{\beta}{\beta-1}}(M_{\lambda r})}.$$

Choosing ω small enough, this proves the assertion. □

The following result is one of the main ingredient to prove the weak Harnack inequality.

Lemma 8.1.3. *Assume $x_0 \in M_1$ and $r \in [0,1)$. Moreover, let $\lambda \in (1, \min\{r^{-1}, \sqrt{2}\})$. Assume $f \in L^q(M_{\lambda r}(x_0))$ for some $q > \max\{2, \beta\}$ and let $u \in V^\mu(M_{\lambda r}(x_0)|\mathbb{R}^d)$ satisfy*

$$\mathcal{E}(u,\phi) \geq (f,\phi) \quad \text{for any nonnegative } \phi \in H^\mu_{M_{\lambda r}}(\mathbb{R}^d),$$
$$u(x) \geq \epsilon \qquad \text{for almost all } x \in M_{\lambda r} \text{ and some } \epsilon > \|f\|_{L^q(M_{\lambda r}(x_0))} r^{(q-\beta)/q}.$$

Then for any $p_0 > 0$ there is a constant $c_1 > 0$, independent of $u, x_0, \lambda, r, \epsilon$ and $\alpha_1, \dots, \alpha_d$, such that

$$\inf_{x \in M_r(x_0)} u(x) \geq c_1 \left(\fint_{M_{2r}(x_0)} u(x)^{-p_0}\, \mathrm{d}x \right)^{-1/p_0}. \tag{8.1.3}$$

Proof. We set $M_r = M_r(x_0)$. For $n \in \mathbb{N}_0$ we define the sequences

$$r_n = \left(\frac{n+2}{n+1}\right) r \quad \text{and} \quad p_n = p_0 \left(\frac{\beta}{\beta-1}\right)^n.$$

Then $r_0 = 2r$, $r_k > r_{k+1}$ for all $k \in \mathbb{N}_0$ and $r_n \searrow r$ as $n \to \infty$. Note

$$r_n = \frac{(n+2)^2}{(n+1)(n+3)} r_{n+1} =: \lambda_n r_{n+1}.$$

Moreover $p_0 = p_0$, $p_k < p_{k+1}$ for all $k \in \mathbb{N}_0$ and $p_n \nearrow +\infty$ as $n \to \infty$.
Using

$$\frac{-2}{p_n} - \frac{2\beta}{p_{n+1}} = \frac{-2\beta}{p_n},$$

we have

$$\frac{r_{n+1}^{-2/p_n}}{|M_{r_{n+1}}|^{1/p_{n+1}}} = \frac{\lambda_n^{2/p_n} r_n^{-2/p_n}}{2^{d/p_{n+1}} \lambda_n^{-2\beta/p_{n+1}} r_n^{2\beta/p_{n+1}}} = 2^{d/(\beta p_n)} \lambda_n^{2\beta/p_n} 2^{-d/p_n} r_n^{-2\beta/p_n} = \frac{2^{d/(\beta p_n)} \lambda_n^{2\beta/p_n}}{|M_{r_n}|^{1/p_n}}.$$

Moreover, by Lemma 8.1.2, we have for $p = p_n + 1$

$$\|u^{-1}\|_{L^{p_{n+1}}(M_{r_{n+1}})} = \|u^{-1}\|_{L^{p_n \frac{\beta}{\beta-1}}(M_{r_{n+1}})}$$
$$\leq c_2^{1/p_n} \left(\frac{p_n+1}{p_n}\right)^{1/p_n} \left(\sum_{k=1}^{d} (\lambda_n^{2/\alpha_k} - 1)^{-\alpha_k}\right)^{1/p_n} r_{n+1}^{-2/p_n} \|u^{-1}\|_{L^{p_n}(M_{r_n})}.$$

This yields

$$\left(\fint_{M_{r_{n+1}}} (u^{-1})^{p_{n+1}}\right)^{1/p_{n+1}} = \frac{1}{|M_{r_{n+1}}|^{1/p_{n+1}}} \left(\int_{M_{r_{n+1}}} (u^{-1})^{p_{n+1}}\right)^{1/p_{n+1}}$$
$$= \frac{1}{|M_{r_{n+1}}|^{1/p_{n+1}}} \|u^{-1}\|_{L^{p_{n+1}}(M_{r_{n+1}})}$$
$$\leq \frac{r_{n+1}^{-2/p_n}}{|M_{r_{n+1}}|^{1/p_{n+1}}} c_2^{1/p_n} \left(\frac{p_n+1}{p_n}\right)^{1/p_n} \left(\sum_{k=1}^{d} (\lambda_n^{2/\alpha_k} - 1)^{-\alpha_k}\right)^{1/p_n} \|u^{-1}\|_{L^{p_n}(M_{r_n})}$$
$$= \frac{2^{d/(\beta p_n)} \lambda_n^{2\beta/p_n}}{|M_{r_n}|^{1/p_n}} c_2^{1/p_n} \left(\frac{p_n+1}{p_n}\right)^{1/p_n} \left(\sum_{k=1}^{d} (\lambda_n^{2/\alpha_k} - 1)^{-\alpha_k}\right)^{1/p_n} \|u^{-1}\|_{L^{p_n}(M_{r_n})}$$
$$= 2^{d/(\beta p_n)} \lambda_n^{2\beta/p_n} c_2^{1/p_n} \left(\frac{p_n+1}{p_n}\right)^{1/p_n} \left(\sum_{k=1}^{d} (\lambda_n^{2/\alpha_k} - 1)^{-\alpha_k}\right)^{1/p_n} \left(\fint_{M_{r_n}} (u^{-1})^{p_n}\right)^{1/p_n}.$$

which is equivalent to

$$\left(\fint_{M_{r_n}} u^{-p_n}\right)^{-1/p_n} \leq 2^{d/(\beta p_n)} \lambda_n^{2\beta/p_n} c_2^{1/p_n} \left(\frac{p_n+1}{p_n}\right)^{1/p_n}$$
$$\times \left(\sum_{k=1}^{d} (\lambda_n^{2/\alpha_k} - 1)^{-\alpha_k}\right)^{1/p_n} \left(\fint_{M_{r_{n+1}}} u^{-p_{n+1}}\right)^{-1/p_{n+1}}. \tag{8.1.4}$$

Iterating (8.1.4) leads to

$$\left(\fint_{M_{r_0}} u^{-p_0}\right)^{-1/p_0} \leq \left(\prod_{j=0}^{n} 2^{d/(\beta p_j)}\right) \left(\prod_{j=0}^{n} \lambda_j^{2\beta/p_j}\right) \left(\prod_{j=0}^{n} c_2^{1/p_j}\right) \left(\prod_{j=0}^{n} \left(\frac{p_j+1}{p_j}\right)^{1/p_j}\right)$$
$$\times \left(\prod_{j=0}^{n} \left(\sum_{k=1}^{d} (\lambda_j^{2/\alpha_k} - 1)^{-\alpha_k}\right)^{1/p_j}\right) \left(\fint_{M_{r_{n+1}}} u^{-p_{n+1}}\right)^{-1/p_{n+1}}. \tag{8.1.5}$$

We continue by showing that the expressions on the right-hand-side of (8.1.5) are well-defined. We have

- $\prod_{j=0}^{n} 2^{d/(\beta p_j)} = 2^{\sum_{j=0}^{n} d/(\beta p_j)} = 2^{(d/(p_0\beta) \sum_{j=0}^{n} ((\beta-1)/\beta)^j} \overset{n\to\infty}{\longrightarrow} 2^{d/p_0} < \infty,$

- $\displaystyle\prod_{j=0}^{n} \lambda_j^{2\beta/p_j} = \exp\left(\frac{2\beta}{p_j}\log(\lambda_j)\right) = \exp\left(\sum_{j=0}^{n}\frac{2\beta}{p_0}\left(\frac{\beta-1}{\beta}\right)^j \log\left(\frac{(j+2)^2}{(j+1)(j+3)}\right)\right)$
$\displaystyle\le \exp\left(\sum_{j=0}^{n}\frac{2\beta}{p_0}\left(\frac{\beta-1}{\beta}\right)^j\right) \overset{n\to\infty}{\longrightarrow} \exp(2\beta^2/p_0) < \infty,$
- $\displaystyle\prod_{j=0}^{n} c_2^{1/p_j} = c_2^{\sum_{j=0}^{n} 1/p_j} = c_2^{\frac{1}{p_0}\sum_{j=0}^{n}((\beta-1)/\beta)^j} \overset{n\to\infty}{\longrightarrow} c_2^{\beta/p_0} < \infty,$
- $\displaystyle\left(\prod_{j=0}^{n}\left(1+\frac{1}{p_j}\right)^{1/p_j}\right) = \exp\left(\sum_{j=0}^{n}\frac{1}{p_j}\log\left(1+\frac{1}{p_j}\right)\right) \le \exp\left(\sum_{j=0}^{n}\frac{1}{p_j}\log\left(1+\frac{1}{p_0}\right)\right)$
$\displaystyle= \exp\left(\sum_{j=0}^{n}\left(\frac{\beta-1}{\beta}\right)^j\log\left(1+\frac{1}{p_0}\right)\right) \overset{n\to\infty}{\longrightarrow} \exp\left(\beta\log\left(1+\frac{1}{p_0}\right)\right) < \infty.$

It remains to show

$$\left(\prod_{j=0}^{n}\left(\sum_{k=1}^{d}(\lambda_j^{2/\alpha_k}-1)^{-\alpha_k}\right)^{1/p_j}\right) \to c_5 < \infty \quad \text{as } n\to\infty. \tag{8.1.6}$$

Note

$$\begin{aligned}
\left(\sum_{k=1}^{d}(\lambda_n^{2/\alpha_k}-1)^{-\alpha_k}\right)^{1/p_n} &= \left(\sum_{k=1}^{d}\left(\left(\frac{(n+2)^2}{(n+1)(n+3)}\right)^{2/\alpha_k}-1)\right)^{-\alpha_k}\right)^{1/p_n}\\
&\le \left(\sum_{k=1}^{d}\left(\left(\frac{(n+2)^2}{(n+1)(n+3)}\right)-1)\right)^{-\alpha_k}\right)^{1/p_n}\\
&= \left(\sum_{k=1}^{d}\left(\frac{(n+2)^2-(n+1)(n+3)}{(n+1)(n+3)})\right)^{-\alpha_k}\right)^{1/p_n}\\
&= \left(\sum_{k=1}^{d}\left(\frac{(n+1)(n+3)}{(n+2)^2-(n+1)(n+3)})\right)^{\alpha_k}\right)^{1/p_n}\\
&\le \left(d\left(\frac{(n+1)(n+3)}{(n+2)^2-(n+1)(n+3)})\right)^{2}\right)^{1/p_n}
\end{aligned}$$

and

$$\begin{aligned}
&\prod_{j=1}^{n+1}\left(d\left(\frac{(j+1)(j+3)}{(j+2)^2-(j+1)(j+3})\right)^2\right)^{1/p_j}\\
&\qquad = \exp\left(\sum_{j=1}^{n+1}\frac{2}{p_j}\left(\log\left(d\frac{(j+1)(j+3)}{(j+2)^2-(j+1)(j+3)})\right)\right)\right).
\end{aligned}$$

Let $h : (0,\infty) \to \mathbb{R}$ be defined as follows

$$x \mapsto h(x) = \frac{x+1)(x+3)}{(x+2)^2 - (x+1)(x+3)} = x^2 + 4x + 3$$

and choose $b \in \left(1, \frac{\beta}{\beta-1}\right)$. Because of the quadratic growth of h there obviously exists a natural number N such that for all $j \geq N$

$$\log\left(d\frac{(j+1)(j+3)}{(j+2)^2 - (j+1)(j+3)})\right) \leq b^j.$$

Since $b\frac{\beta-1}{\beta} < 1$, we have

$$\begin{aligned}
&\exp\left(\sum_{j=N}^{n+1} \frac{2}{p_j}\left(\log\left(d\frac{(j+1)(j+3)}{(j+2)^2-(j+1)(j+3)})\right)\right)\right) \\
&\leq \exp\left(\sum_{j=N}^{n+1} \frac{2}{p_j}\left(b^j\right)\right) \\
&= \exp\left(\sum_{j=N}^{n+1} 2p_0\left(\frac{\beta-1}{\beta}\right)^j b^j\right) \\
&= \exp\left(\sum_{j=N}^{n+1} 2p_0\left(b\frac{\beta-1}{\beta}\right)^j\right) \to c_4 < \infty \quad \text{as } n \to \infty,
\end{aligned}$$

which implies (8.1.6), i.e.

$$\prod_{j=1}^{n+1}\left(\sum_{k=1}^{d}(\lambda_j^{2/\alpha_k} - 1)^{-\alpha_k}\right)^{1/p_j} \to c_5 < \infty \quad \text{as } n \to \infty.$$

Since

$$\lim_{n\to\infty}\left(\fint_{M_{r_n}} u^{-p_n}\right)^{-1/p_n} = \inf_{x\in M_r} u(x),$$

taking the limit $n \to \infty$ in (8.1.5), proves the assertion. □

From Lemma 8.1.3 and Theorem 8.0.5 we immediately conclude the following result.

Corollary 8.1.4. *Let $f \in L^q(M_1)$ for some $q > \max\{2,\beta\}$. There are $p_0, c_1 > 0$ such that for every $u \in V^\mu(M_1|\mathbb{R}^d)$ with $u \geq 0$ in $\mathbb{R}^d$ and*

$$\mathcal{E}(u,\phi) \geq (f,\phi) \quad \text{for every nonnegative } \phi \in H^\mu_{M_1}(\mathbb{R}^d),$$

the following holds

$$\inf_{M_{\frac{1}{4}}} u \geq c_1 \left(\fint_{M_{\frac{1}{2}}} u(x)^{p_0}\,\mathrm{d}x\right)^{1/p_0} - \|f\|_{L^q(M_{\frac{15}{16}})}.$$

Proof. This proof follows the proof of [DK15, Theorem 4.1] Define $v = u + \|f\|_{L^q(M_{\frac{15}{16}})}$. Then for any nonnegative $\phi \in H^{\mu}_{M_1}(\mathbb{R}^d)$, one obviously has

$$\mathcal{E}(u,\phi) = \mathcal{E}(v,\phi).$$

By Theorem 8.0.5 there are a $c_2 > 0$, $p_0 \in (0,1)$ such that

$$\left(\fint_{M_{\frac{1}{2}}} v(x)^{p_0} \, \mathrm{d}x \right)^{1/p_0} \mathrm{d}x \leq c_2 \left(\fint_{M_{\frac{1}{2}}} v(x)^{-p_0} \, \mathrm{d}x \right)^{-1/p_0} . \tag{8.1.7}$$

Moreover, by Lemma 8.1.3 there is a $c_3 > 0$ such that for $r = \frac{1}{2}$ and p_0 as in (8.1.7)

$$\begin{aligned} \inf_{x \in M_{\frac{1}{4}}} v(x) &\geq c_3 \left(\fint_{M_{\frac{1}{2}}} v(x)^{-p_0} \, \mathrm{d}x \right)^{-1/p_0} \geq \frac{c_3}{c_2} \left(\fint_{M_{\frac{1}{2}}} v(x)^{p_0} \, \mathrm{d}x \right)^{1/p_0} \\ &\geq \frac{c_3}{c_2} \left(\fint_{M_{\frac{1}{2}}} u(x)^{p_0} \, \mathrm{d}x \right)^{1/p_0} . \end{aligned}$$

which is equivalent to

$$\inf_{M_{\frac{1}{4}}} u \geq c_1 \left(\fint_{M_{\frac{1}{2}}} u(x)^{p_0} \, \mathrm{d}x \right)^{1/p_0} - \|f\|_{L^q(M_{\frac{15}{16}})}.$$

□

Finally, we have all tools to prove the weak Harnack inequality for weak supersolutions to $\mathcal{L}u = f$.

For a function $u : \mathbb{R}^d \to \mathbb{R}$ let

$$\begin{aligned} u^+(x) &:= \max\{u(x), 0\}, \\ u^-(x) &:= -\min\{u(x), 0\}. \end{aligned}$$

Theorem 8.1.5. *Let $f \in L^q(M_1)$ for some $q > \max\{2, \beta\}$. Let $u \in V^{\mu}(M_1 | \mathbb{R}^d)$, $u \geq 0$ in M_1 satisfy*

$$\mathcal{E}(u,\phi) \geq (f,\phi) \quad \textit{for every nonnegative } \phi \in H^{\mu}_{M_1}(\mathbb{R}^d). \tag{8.1.8}$$

Then there exists $p_0 \in (0,1)$, $c_1 > 0$, independent of u, such that

$$\inf_{M_{\frac{1}{4}}} u \geq c_1 \left(\fint_{M_{\frac{1}{2}}} u(x)^{p_0} \, \mathrm{d}x \right)^{1/p_0} - \sup_{x \in M_{\frac{15}{16}}} 2 \int_{\mathbb{R}^d \setminus M_1} u^-(z) \mu(x, \mathrm{d}z) - \|f\|_{L^q(M_{\frac{15}{16}})}.$$

Proof. By (8.1.8), for any nonnegative $\phi \in H^{\mu}_{M_1}(\mathbb{R}^d)$

$$\mathcal{E}(u^+,\phi) = \mathcal{E}(u,\phi) + \mathcal{E}(u^-,\phi) \geq (f,\phi) + \mathcal{E}(u^-,\phi). \tag{8.1.9}$$

Since $\phi \in H^{\mu}_{M_1}(\mathbb{R}^d)$ and $u^- \equiv 0$ on M_1 , we have

$$(f,\phi) = \int_{\mathbb{R}^d} f(x)\phi(x)\,\mathrm{d}x = \int_{M_1} f(x)\phi(x)\,\mathrm{d}x$$

and

$$\begin{aligned}
\mathcal{E}(u^-,\phi) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u^-(y)-u^-(x))(\phi(y)-\phi(x))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= \int_{M_1}\int_{M_1} (u^-(y)-u^-(x))(\phi(y)-\phi(x))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad + 2\int_{M_1}\int_{M_1^c} (u^-(y)-u^-(x))(\phi(y)-\phi(x))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad + \int_{(M_1)^c}\int_{(M_1)^c} (u^-(y)-u^-(x))(\phi(y)-\phi(x))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= 2\int_{M_1}\int_{(M_1)^c} (u^-(y))(-\phi(x))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= -2\int_{M_1}\int_{(M_1)^c} u^-(y)\phi(x)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x.
\end{aligned}$$

Hence, we get from (8.1.9)

$$\mathcal{E}(u^+,\phi) \geq \int_{M_1} \phi(x)\left(f(x) - 2\int_{(M_1)^c} u^-(y)\,\mu(x,\mathrm{d}y)\right)\mathrm{d}x.$$

Therefore, u^+ satisfies all assumptions of Corollary 8.1.4 with $q = +\infty$ and $\widetilde{f} : M_1 \to \mathbb{R}$, defined by

$$\widetilde{f}(x) = f(x) - 2\int_{\mathbb{R}^d\setminus M_1} u^-(y)\,\mu(x,\mathrm{d}y).$$

If $\sup\limits_{x\in M_{\frac{15}{16}}} \int_{\mathbb{R}^d\setminus M_1} u^-(z)\mu(x,\mathrm{d}z) = \infty$, then the assertion of the theorem is obviously true. Thus we can assume this quantity to be finite. Applying Corollary 8.1.4 and Hölder's inequality

$$\begin{aligned}
\inf_{M_{\frac14}} u &\geq c_1 \left(\fint_{M_{\frac12}} u(x)^{p_0}\,\mathrm{d}x\right)^{1/p_0} - \|\widetilde{f}\|_{L^q(M_{\frac{15}{16}})} \\
&= c_1 \left(\fint_{M_{\frac12}} u(x)^{p_0}\,\mathrm{d}x\right)^{1/p_0} - \|f\|_{L^q(M_{\frac{15}{16}})} - 2\left\|\int_{\mathbb{R}^d\setminus M_1} u^-(y)\,\mu(x,\mathrm{d}y)\right\|_{L^q(M_{\frac{15}{16}})}
\end{aligned}$$

$$\geq c_1 \left(\fint_{M_{\frac{1}{2}}} u(x)^{p_0} \,\mathrm{d}x \right)^{1/p_0} - \|f\|_{L^q(M_{\frac{15}{16}})} - \sup_{x \in M_{\frac{15}{16}}} 2 \int_{\mathbb{R}^d \setminus M_1} u^-(y)\, \mu(x, \mathrm{d}y).$$

□

An immediate consequence of Theorem 8.1.5 is the following result. It follows from Theorem 8.1.5 via scaling and translation.

Corollary 8.1.6. *Let $x_0 \in M_1$, $r \in (0,1]$. Assume $u \in V^{\mu}(M_r(x_0)|\mathbb{R}^d)$ satisfies $u \geq 0$ in $M_r(x_0)$ and $\mathcal{E}(u, \Phi) \geq 0$ for every $\Phi \in H^{\mu}_{M_r(x_0)}(\mathbb{R}^d)$. Then there exists $p_0 \in (0,1)$, $c_1 > 0$, independent of u, x_0 and r, such that*

$$\inf_{M_{\frac{1}{4}r}(x_0)} u \geq c_1 \left(\fint_{M_{\frac{1}{2}r}(x_0)} u(x)^{p_0} \,\mathrm{d}x \right)^{1/p_0} - r \sup_{x \in M_{\frac{15}{16}r}(x_0)} 2 \int_{\mathbb{R}^d \setminus M_r(x_0)} u^-(z) \mu(x, \mathrm{d}z).$$

9. Hölder regularity for weak solutions

In this chapter we prove the main result of this part of the thesis, that is an a priori Hölder estimate for weak solutions to $\mathcal{L}u = 0$ in M_1. For this purpose, we first prove a decay of oscillation, which implies together with the weak Harnack inequality the desired Hölder estimate. [DK15] provides a general scheme for the derivation of a priori Hölder estimates from the weak Harnack inequality. We apply this scheme in our setup.

Theorem 9.0.1. *Let $x_0 \in \mathbb{R}^d$, $r_0 \in (0,1]$ and $\lambda > 1$, $\sigma > 1$, $\Theta > 1$. Let $r \in (0, r_0]$. Assume there is a constant $c_a > 0$ such that for $u \in \mathcal{S}_{x_0,r}$ and $u \geq 0$ in $M_r(x_0)$*

$$\left(\fint_{M_{\frac{r}{\lambda}}(x_0)} u(x)^p \,\mathrm{d}x \right)^{1/p} \leq c_a \left(\inf_{M_{\frac{r}{\Theta}}(x_0)} u + \sup_{x \in M_{\frac{r}{\sigma}}(x_0)} \int_{\mathbb{R}^d} u^-(z) \nu_{x,r}(\mathrm{d}z) \right). \tag{9.0.1}$$

Then there exists $\delta \in (0,1)$ such that for $r \in (0, r_0]$, $u \in \mathcal{S}_{x_0,r}$

$$\operatorname*{osc}_{M_\rho(x_0)} u \leq 2\Theta^\delta \|u\|_\infty \left(\frac{\rho}{r} \right)^\delta, \quad 0 < \rho \leq r.$$

Proof. This proof follows the proof of [DK15, Theorem 1.4].

In the following, we write M_r instead of $M_r(x_0)$ for $r > 0$.
Fix $r \in (0, r_0)$ and $u \in \mathcal{S}_{x_0,r}$. Let c_a be the constant from (9.0.1). Set $\kappa = (2c_a 2^{1/p})^{-1}$ and

$$\delta = \frac{\log\left(\frac{2}{2-\kappa}\right)}{\log(\Theta)} \quad \Longrightarrow 1 - \frac{\kappa}{2} = \Theta^{-\delta}. \tag{9.0.2}$$

Set $M_0 = \|u\|_\infty$, $m_0 = \inf\{u(x) \colon x \in \mathbb{R}^d\}$ and $M_{-n} = M_0, m_{-n} = m_0$ for $n \in \mathbb{N}$.

Our aim is to construct an increasing sequence $(m_n)_{n\in\mathbb{Z}}$ and a decreasing sequence $(M_n)_{n\in\mathbb{Z}}$ such that for all $n \in \mathbb{Z}$

$$\begin{cases} m_n \leq u(z) \leq M_n \\ M_n - m_n \leq K\Theta^{-n\delta} \end{cases} \tag{9.0.3}$$

for almost all $z \in M_{r\Theta^{-n}}$, where $K = M_0 - m_0 \in [0, 2\|u\|_\infty]$.

Before we prove (9.0.3), we show how (9.0.3) implies the assertion. Let $\rho \in (0, r]$. There is $j \in \mathbb{N}_0$ such that

$$r\Theta^{-j-1} \leq \rho \leq r\Theta^{-j}.$$

Note, that this implies in particular

$$\Theta^{-j} \le \frac{\rho}{r}\Theta.$$

From (9.0.3), we achieve

$$\begin{aligned}\operatorname*{osc}_{M_\rho} u &\le \operatorname*{osc}_{M_{r\Theta^{-j}}} u \le M_j - m_j \le K\Theta^{-\delta j} \le 2\|u\|_\infty \Theta^{-\delta j} \\ &= 2\|u\|_\infty (\Theta^{-j})^\delta \le 2\|u\|_\infty \Theta^\delta \left(\frac{\rho}{r}\right)^\delta .\end{aligned}$$

Hence it remains to show (9.0.3). Assume there is $k \in \mathbb{N}$ and there are M_n, m_n, such that (9.0.3) holds true for $n \le k-1$. We need to choose M_k, m_k such that (9.0.3) still holds for $n = k$. For $z \in \mathbb{R}^d$ set

$$v(z) = \left(u(z) - \frac{M_{k-1} + m_{k-1}}{2}\right) \frac{2\Theta^{(k-1)\delta}}{K}.$$

Then $v \in \mathcal{S}_{x_0,r}$ and $|v(z)| \le 1$ for almost every $z \in M_{r\Theta^{-(k-1)}}$.
Let $z \in \mathbb{R}^d$ be such that $d(z,x_0) \ge r\Theta^{-k+1}$. Choose $j \in \mathbb{N}$ such that

$$r\Theta^{-k+j} \le d(z,x_0) \le r\Theta^{-k+j+1}.$$

For such z and j, we conclude

$$\begin{aligned}\frac{K}{2\Theta^{(k-1)\delta}} v(z) &= u(z) - \frac{M_{k-1} + m_{k-1}}{2} \\ &\ge m_{k-1} - \frac{M_{k-1} + m_{k-1}}{2} \\ &\ge m_{k-j-1} - \frac{M_{k-1} + m_{k-1}}{2} \\ &= m_{k-j-1} + M_{k-j-1} - M_{k-j-1} - \frac{M_{k-1} + m_{k-1}}{2} \\ &\ge -(M_{k-j-1} - m_{k-j-1}) + \frac{M_{k-1} - m_{k-1}}{2} \\ &\ge -K\Theta^{-(k-j-1)\delta} + \frac{K}{2}\Theta^{-(k-1)\delta}.\end{aligned}$$

Thus

$$v(z) \ge 1 - 2\Theta^{j\delta} \ge 1 - 2\left(\frac{\Theta d(z,x_0)}{r\Theta^{-k+1}}\right)^\delta . \tag{9.0.4}$$

Moreover,

$$\begin{aligned}\frac{K}{2\Theta^{(k-1)\delta}} v(z) &= u(z) - \frac{M_{k-1} + m_{k-1}}{2} \\ &\le M_{k-j-1} - \frac{M_{k-1} + m_{k-1}}{2}\end{aligned}$$

$$\leq (M_{k-j-1} - m_{k-j-1}) - \frac{M_{k-1} - m_{k-1}}{2}$$
$$\leq K\Theta^{-(k-j-1)\delta} - \frac{K}{2}\Theta^{-(k-1)\delta}.$$

Hence

$$v(z) \leq 2\Theta^{j\delta} - 1 \leq 2\left(\frac{\Theta d(z,x_0)}{r\Theta^{-k+1}}\right)^{\delta} - 1. \tag{9.0.5}$$

We will distinguish two cases.

1. First assume

$$|\{x \in M_{\frac{r\Theta^{-k+1}}{\lambda}} : v(x) \leq 0\}| \geq \frac{1}{2}|M_{\frac{r\Theta^{-k+1}}{\lambda}}|. \tag{9.0.6}$$

Our aim is to show that in this case

$$v(z) \leq 1 - \kappa \quad \text{for almost every } z \in M_{r\Theta^{-k}}. \tag{9.0.7}$$

We will first show that this implies (9.0.3). Recall, that by the induction hypothesis (9.0.3) holds true for $n \leq k-1$. Hence we need to find m_k, M_k satisfying (9.0.3). Assume (9.0.7) holds.

Then for almost any $z \in M_{r\Theta^{-k}}$

$$\begin{aligned} u(z) &= \frac{K}{2\Theta^{(k-1)\delta}} v(z) + \frac{M_{k-1} + m_{k-1}}{2} \\ &\leq \frac{K}{2\Theta^{(k-1)\delta}}(1-\kappa) + \frac{M_{k-1} + m_{k-1}}{2} \\ &= \frac{K}{2\Theta^{(k-1)\delta}}(1-\kappa) + \frac{M_{k-1} - m_{k-1}}{2} + m_{k-1} \\ &\leq m_{k-1} + \left(1 - \frac{\kappa}{2}\right) K\Theta^{-(k-1)\delta} \\ &\leq m_{k-1} + K\Theta^{-k\delta}. \end{aligned}$$

If we now set $m_k = m_{k-1}$ and $M_k = m_k + K\Theta^{-k\delta}$, then by the induction hypothesis $u(z) \geq m_{k-1} = m_k$ and by the previous calculation $u(z) \leq M_k$. Hence (9.0.3) follows.

It remains to show $v(z) \leq 1-\kappa$ for almost every $z \in M_{r\Theta^{-k}}$.
Consider $w = 1 - v$ and note $w \in \mathcal{S}_{x_0, r\Theta^{-(k-1)}}$ and $w \geq 0$ in $M_{r\Theta^{-(k-1)}}$. By assumption (9.0.1) of the theorem for $r_1 = r\Theta^{-k+1} \in (0, r]$ we obtain

$$\left(\fint_{M_{\frac{r_1}{\lambda}}} w(x)^p \,\mathrm{d}x\right)^{1/p} \leq c_a \left(\inf_{M_{\frac{r_1}{\Theta}}} w + \sup_{x \in M_{\frac{r_1}{\sigma}}} \int_{\mathbb{R}^d} w^-(z)\nu_{x,r_1}(\mathrm{d}z)\right).$$

Using assumption (9.0.6) the left hand side can be estimated as follows

$$\left(\fint_{M_{\frac{r\Theta^{-(k-1)}}{\lambda}}} w(x)^p \,\mathrm{d}x\right)^{1/p} \geq \left(\fint_{M_{\frac{r\Theta^{-(k-1)}}{\lambda}}} w(x)^p \mathbb{1}_{\{v(x)\leq 0\}} \,\mathrm{d}x\right)^{1/p}$$

$$= \left(\frac{|\{x \in M_{\frac{r\Theta^{-k+1}}{\lambda}} : v(x) \leq 0\}|}{|M_{\frac{r\Theta^{-k+1}}{\lambda}}|} \right)^{1/p}$$
$$\geq \left(\frac{\frac{1}{2}|M_{\frac{r\Theta^{-k+1}}{\lambda}}|}{|M_{\frac{r\Theta^{-k+1}}{\lambda}}|} \right)^{1/p} = \frac{1}{2^{1/p}}.$$

Moreover by (9.0.5)

$$(1 - v(z))^- \leq (1 - 2\Theta^{j\delta} - 1)^- = 2\Theta^{j\delta} - 2. \tag{9.0.8}$$

Consequently by (9.0.1) and (9.0.8)

$$\begin{aligned}
\inf_{M_{r\Theta^{-k}}} w &\geq \frac{c_a^{-1}}{2^{1/p}} - \sup_{x \in M_{\frac{r\Theta^{-(k-1)}}{\sigma}}} \int_{\mathbb{R}^d} w^-(z) \nu_{x, r\Theta^{-(k-1)}}(\mathrm{d}z) \\
&= (c_a 2^{1/p})^{-1} - \sup_{x \in M_{\frac{r\Theta^{-(k-1)}}{\sigma}}} \sum_{j=1}^{\infty} \int \mathbb{1}_{A_{r\Theta^{-k+j}, r\Theta^{-k+j+1}}(x_0)} (1 - v(z))^- \\
&\qquad \times \nu_{x, r\Theta^{-(k-1)}}(\mathrm{d}z) \\
&\geq (c_a 2^{1/p})^{-1} - \sum_{j=1}^{\infty} \sup_{x \in M_{\frac{r\Theta^{-(k-1)}}{\sigma}}} \int \mathbb{1}_{A_{r\Theta^{-k+j}, r\Theta^{-k+j+1}}(x_0)} (1 - v(z))^- \\
&\qquad \times \nu_{x, r\Theta^{-(k-1)}}(\mathrm{d}z) \\
&\geq (c_a 2^{1/p})^{-1} - \sum_{j=1}^{\infty} (2\Theta^{j\delta} - 2) \eta_{x_0, r, \Theta, j, k},
\end{aligned}$$

where

$$\eta_{x_0, r, \Theta, j, k} = \sup_{x \in M_{\frac{r\Theta^{-(k-1)}}{\sigma}}} \nu_{x, r\Theta^{-(k-1)}}(A_{r\Theta^{-k+j}, r\Theta^{-k+j+1}}(x_0)).$$

Lemma 7.0.12 implies

$$\eta_{x_0, r, \Theta, j, k} \leq 4d \frac{\sigma}{\sigma} \Theta^{-j-1} = c_2 \Theta^{-j}.$$

Thus

$$\inf_{M_{r\Theta^{-k}}} w \geq (c_a 2^{1/p})^{-1} - 2c_2 \sum_{j=1}^{\infty} (\Theta^{j\delta} - 1) \Theta^{-j-1}.$$

If $\delta \in (0, 1)$ is small enough, the sum on the right-hand side is finite. There is a number $l \in \mathbb{N}$ such that

$$\sum_{j=l+1}^{\infty} (\Theta^{j\delta} - 1) \Theta^{-j-1} \leq \sum_{j=l+1}^{\infty} \Theta^{-j(1-\delta)-2} \leq (16c_a)^{-1}.$$

Given this $l \in \mathbb{N}$, choose $\delta \in (0,1)$ small such that

$$\sum_{j=1}^{l} (\Theta^{j\delta} - 1)\Theta^{-j-1} \le (16c_a)^{-1},$$

where δ depends only on c_a, c_2 and Θ. Thus

$$w \ge \inf_{M_{r\Theta^{-k}}} w \ge \kappa \quad \text{on } M_{r\Theta^{-k}},$$

or equivalently $v \le 1 - \kappa$ on $M_{r\Theta^{-k}}$.

2. Now, we assume

$$|\{x \in M_{\frac{r\Theta^{-k+1}}{\lambda}} : v(x) > 0\}| \ge \frac{1}{2}|M_{\frac{r\Theta^{-k+1}}{\lambda}}|. \tag{9.0.9}$$

Our aim is to show that in this case

$$v(z) \ge -1 + \kappa \quad \text{for almost every } M_{r\Theta^{-k}}.$$

Similar to the first case, this implies for almost every $z \in M_{r\Theta^{-k}}$

$$\begin{aligned} u(z) &= \frac{K}{2\Theta^{(k-1)\delta}} v(z) + \frac{M_{k-1} + m_{k-1}}{2} \\ &\ge \frac{K}{2\Theta^{(k-1)\delta}}(-1+\kappa) + \frac{M_{k-1} + m_{k-1}}{2} \\ &= \frac{K}{2\Theta^{(k-1)\delta}}(-1+\kappa) - \frac{M_{k-1} - m_{k-1}}{2} + M_{k-1} \\ &\ge M_{k-1} - \left(1 - \frac{\kappa}{2}\right) K\Theta^{-(k-1)\delta} \\ &\ge M_{k-1} - K\Theta^{-k\delta}. \end{aligned}$$

Choosing $M_k = M_{k-1}$ and $m_k = M_{k-1} - K\Theta^{-k\delta}$, then by the induction hypothesis $u(z) \le M_{k-1} = M_k$ and by the previous calculation $u(z) \ge m_k$. Hence (9.0.3) follows.

It remains to show in this case $v(z) \le -1 + \kappa$ for almost every $z \in M_{r\Theta^{-k}}$.

Consider $w = 1 + v$ and note $w \in \mathcal{S}_{x_0, r\Theta^{-(k-1)}}$ and $w \ge 0$ in $M_{r\Theta^{-(k-1)}}$. Then the desired statement follows analogously to Case 1.

□

Recall the definition of the metric d,

$$d(x,y) := \sup_{k \in \{1,\dots,d\}} \left\{ |x_k - y_k|^{\alpha_k/2} \mathbb{1}_{\{|x_k - y_k| \le 1\}}(x,y) + \mathbb{1}_{\{|x_k - y_k| > 1\}}(x,y) \right\}.$$

Let M be a ball with respect to the metric d, i.e. $M = M_r(z)$ for some $z \in \mathbb{R}^d$ and $r \in (0,1]$. We denote its radius with respect to the metric d by

$$\text{radius}(M) = r.$$

Corollary 9.0.2. *Let $x_0 \in M_1$ and $r_0 \in (0,1]$. Let $\sigma, \Theta, \lambda > 1$. Assume there is a $c_a > 0$ such that for $r \in (0, r_0]$ and $x \in \mathbb{R}^d$ with $u \in \mathcal{S}_{x,r}$ and $u \geq 0$ in $M_r(x)$*

$$\left(\fint_{M_{\frac{r}{\lambda}}(x)} u(z)^p \,\mathrm{d}z \right)^{1/p} \leq c_a \left(\inf_{M_{\frac{r}{\Theta}}(x)} u + \sup_{\xi \in M_{\frac{r}{\sigma}}(x)} \int_{\mathbb{R}^d} u^-(z) \nu_{\xi,r}(\mathrm{d}z) \right). \tag{9.0.10}$$

Then there exist $\delta \in (0,1)$ such that for every $u \in \mathcal{S}_{x_0,r_0}$ and almost every $x, y \in M_{r_0}(x_0)$

$$|u(x) - u(y)| \leq 16\Theta^\delta \|u\|_\infty \left(\frac{d(x,y)}{\max\{d(x,(M_{r_0}(x_0))^c), d(y,(M_{r_0}(x_0))^c)\}} \right)^\delta .$$

Proof. We follow the proof of [DK15, Corollary 5.2]. Recall that balls in the metric d are given by the sets M_r. Within this proof balls have to be understood with respect to the metric d.

We assume without loss of generality $s := d(y,(M_{r_0}(x_0))^c) \geq d(x,(M_{r_0}(x_0))^c)$.

We first prove the assertion for $x, y \in M_{r_0}(x_0)$ such that $d(x,y) \geq s/8$. In this case, we have

$$\left(\frac{d(x,y)}{\max\{d(x,(M_{r_0}(x_0))^c), d(y,(M_{r_0}(x_0))^c)\}} \right)^\delta \geq \left(\frac{1}{8} \right)^\delta$$

and therefore the assertion directly follows by Theorem 9.0.1.

Now assume $d(x,y) < s/8$. We fix a number $\rho \in (0, r_0/4)$ and consider all $x, y \in M_{r_0}(x_0)$ such that

$$\frac{\rho}{2} \leq d(x,y) \leq \rho. \tag{9.0.11}$$

We cover $M_{r_0-4\rho}(x_0)$ by a countable family of balls $(\widetilde{M^i_\rho})_i$, where $\widetilde{M^i_\rho}$ are, as usual, balls with radius ρ.

Without loss of generality, we assume $M^i_\rho \cap M_{r_0-4\rho}(x_0) \neq \emptyset$. Let M^i denote the balls with the same center as $\widetilde{M^i_\rho}$ and radius twice as big, that is $M^i := \widetilde{M^i_{2\rho}}$. Furthermore, let $M^{i,*}$ be the balls with the same center as as $\widetilde{M^i_\rho}$ but maximal radius such that $M^{i,*}_\rho \subset M_{r_0}(x_0)$. Let $x, y \in M_{r_0}(x_0)$ satisfy (9.0.11). Then $s > 8d(x,y) \geq 4\rho$ implies $y \in M_{r_0.4\rho}(x_0)$ and therefore $y \in \widetilde{M^i_\rho}$ for some index i. Hence $x, y \in M^i$.

Applying Theorem 9.0.1 to x_0 and r_0 being the center and radius to $M^{i,*}$, respectively, and obtain

$$\operatorname*{osc}_{M^i} u \leq 2\Theta^\delta \|u\|_\infty \left(\frac{\text{radius}(M^i)}{\text{radius}(M^{i,*}))} \right)^\delta \leq 2\Theta^\delta \|u\|_\infty \left(\frac{\rho}{s-\rho} \right)^\delta \leq \frac{16}{3} \|u\|_\infty \Theta^\delta \left(\frac{d(x,y)}{s} \right)^\delta .$$

Hence the corollary is proved, provided x and y satisfy $|u(x) - u(y)| \leq \operatorname*{osc}_{M^i} u$.

Consider now $\rho = r_0 2^{-j}$ for $j \in \mathbb{N}$, $j \geq 3$ in (9.0.11). Then the assertion holds for almost all x and y such that $d(x,y) \leq r_0/8$, which finishes the proof. □

We finish this part of the thesis with the main result.

Theorem 9.0.3. *Assume* $u \in V^{\mu}(M_1 | \mathbb{R}^d)$ *satisfies*

$$\mathcal{E}(u,\phi) = 0 \quad \textit{for every nonnegative } \phi \in H^{\mu}_{M_1}(\mathbb{R}^d).$$

Then there are $c_1 \geq 1$ *and* $\delta \in (0,1)$*, independent of* u*, such that the following Hölder estimate holds for almost every* $x, y \in M_{\frac{1}{2}}$

$$|u(x) - u(y)| \leq c_1 \|u\|_\infty |x-y|^{\delta}. \tag{9.0.12}$$

Proof. The proof follows from Corollary 8.1.6 and Corollary 9.0.2. It is complete if we can apply Corollary 9.0.2 for $x_0 = 0$ and $r_0 = \frac{1}{2}$. Assume $r \in (0, r_0]$ and $M_r \subset M_{\frac{1}{2}}$. Let $\alpha_{\min} := \min\{\alpha_1, \dots, \alpha_d\}$. First note that for all $x, y \in M_{\frac{1}{2}}$

$$\begin{aligned} d(x,y) &= \sup_{k \in \{1,\dots,d\}} \left\{ |x_k - y_k|^{\alpha_k/2} \right\} \leq \sup_{k \in \{1,\dots,d\}} \left\{ |x_k - y_k|^{\alpha_{\min}/2} \right\} \\ &= \left(\sup_{k \in \{1,\dots,d\}} \{|x_k - y_k|\} \right)^{\alpha_{\min}/2} \leq \left(\frac{1}{d} |x-y| \right)^{\alpha_{\min}/2}. \end{aligned} \tag{9.0.13}$$

Let $u \in \mathcal{S}_{x,r}$ and $u \geq 0$ in $M_r(x)$. Then by Corollary 8.1.6 there are $c_2 \geq 1$ and $p_0 \in (0,1)$ such that

$$\inf_{M_{\frac{1}{4}r}(x)} u \geq c_2 \left(\fint_{M_{\frac{1}{2}r}(x)} u(x)^{p_0} \, \mathrm{d}x \right)^{1/p_0} - r \sup_{x \in M_{\frac{15}{16}r}(x)} 2 \int_{\mathbb{R}^d \setminus M_r(x)} u^{-}(z) \mu(x, \mathrm{d}z).$$

Setting $\lambda = 4, \Theta = 2$ and $\sigma = \frac{16}{15}$ and applying Corollary 9.0.2 for $x_0 = 0$, $r_0 = \frac{1}{2}$, we obtain

$$|u(x) - u(y)| \leq 16 \Theta^{\delta} \|u\|_\infty \left(\frac{d(x,y)}{\max\{d(x, M^c_{1/2}), d(y, M^c_{1/2})\}} \right)^{\delta}.$$

Using (9.0.13) proves the assertion. □

A. Examples

This appendix contains some examples of permissible families of measures, which satisfy the assumptions in Part III. Although $\mu_{\text{axes}}(x, \cdot)$, $x \in \mathbb{R}^d$ is supported on the union of the coordinate axes, the family $\mu(x, \cdot)$, $x \in \mathbb{R}^d$ might be have a density with respect to the Lebesgue measure as we will see within this Appendix. In the following we write (Ak), $k \in \{1, \dots, 4\}$ for the respective assumption.

We start with two simple examples.

Example 1. *Let $\mu(x, \mathrm{d}y)$ be a family that satisfies $(A1), (A2), (A3)$ and $(A4)$. Let $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function with $1 \le a(x, y) \le 2$ and $a(x, y) = a(y, x)$ for all $x, y \in \mathbb{R}^d$. Then*

$$\widetilde{\mu}(x, \mathrm{d}y) = a(x, y)\mu(x, \mathrm{d}y)$$

clearly fulfills $(A1), (A2), (A3)$ and $(A4)$.

Example 2. *Let $\mu_1(x, \mathrm{d}y), \mu_2(x, \mathrm{d}y)$, $x \in \mathbb{R}^d$, be two families such that $(A1), (A2), (A3)$ and $(A4)$ hold for each family. Furthermore, let $a, b : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be measurable functions with $1 \le a(x, y), b(x, y) \le 2$ and $a(x, y) = a(y, x)$ and $b(x, y) = b(y, x)$ for all $x, y \in \mathbb{R}^d$. Then clearly*

$$\mu(x, \mathrm{d}y) = a(x, y)\mu_1(x, \mathrm{d}y) + b(x, y)\mu_2(x, \mathrm{d}y)$$

satisfies $(A1), (A2), (A3)$ and $(A4)$.

In the same spirit as [DK15, Example 2], one can consider the following example.

Example 3. *Let $\alpha_1, \dots, \alpha_d \in (0, 2)$ be given and let $\beta_1 \in (0, \alpha_1]$, $\beta_2 \in (0, \alpha_2], \dots, \beta_d \in (0, \alpha_d]$. Then*

$$\mu(x, \mathrm{d}y) = \sum_{k=1}^{d} \left((\alpha_k(2 - \alpha_k)) \left[|x_k - y_k|^{-1-\alpha_k} + |x_k - y_k|^{-1-\beta_k} \right] \mathrm{d}y_k \prod_{i \neq k} \delta_{\{x_i\}}(\mathrm{d}y_i) \right)$$

satisfies Assumption 3, since

$$|x_k - y_k|^{-1-\alpha_k} \le |x_k - y_k|^{-1-\alpha_k} + |x_k - y_k|^{-1-\beta_k} \le 2|x_k - y_k|^{-1-\alpha_k}$$

for any $x_0 \in M_1$ and all $x, y \in M_1(x_0)$.

Next, we give a non-trivial example for a lower bound in Assumption 3.

We first start by proving some properties of the measure $\mu_{\text{axes}}(0, \cdot)$.

Lemma A.0.1. *For every $r > 0$,*

$$\int_{M_r} |z|^2 \, \mu_{axes}(0, \mathrm{d}z) = 2 \sum_{k=1}^{d} r^{-2+4/\alpha_k}.$$

Proof. This follows by an easy calculation. We have for every $r > 0$

$$\begin{aligned}
\int_{M_r} |z|^2 \, \mu_{\text{axes}}(0, \mathrm{d}z) &= \int_{M_r} (z_1^2 + \cdots + z_d^2) \sum_{k=1}^{d} \left((2-\alpha_k)|z_k|^{-1-\alpha_k} \, \mathrm{d}z_k \prod_{i \neq k} \delta_{\{0\}}(\mathrm{d}z_i) \right) \\
&= \sum_{k=1}^{d} \int_{-r^{2/\alpha_k}}^{r^{2/\alpha_k}} (2-\alpha_k) z_k^2 |z_k|^{-1-\alpha_k} \, \mathrm{d}z_k = \sum_{k=1}^{d} 2 \int_0^{r^{2/\alpha_k}} (2-\alpha_k) z_k^{1-\alpha_k} \, \mathrm{d}z_k \\
&= \sum_{k=1}^{d} \frac{2(2-\alpha_k)}{(2-\alpha_k)} \left(r^{2/\alpha_k} \right)^{2-\alpha_k} = 2 \sum_{k=1}^{d} r^{-2+4/\alpha_k}.
\end{aligned}$$

□

Lemma A.0.2. *For every $r > 0$,*

$$\int_{M_r^c} \mu_{axes}(0, \mathrm{d}z) = \left(\sum_{k=1}^{d} \frac{2(2-\alpha_k)}{\alpha_k} \right) r^{-2}.$$

Proof. The result follows by a direct calculation. We have for every $r > 0$

$$\begin{aligned}
\int_{M_r^c} \mu_{\text{axes}}(0, \mathrm{d}z) &= \int_{M_r^c} \sum_{k=1}^{d} \left((2-\alpha_k)|z_k|^{-1-\alpha_k} \, \mathrm{d}z_k \prod_{i \neq k} \delta_{\{0\}}(\mathrm{d}z_i) \right) \\
&= \sum_{k=1}^{d} \int_{(-\infty, -r^{2/\alpha_k}) \cup (r^{2/\alpha_k}, \infty)} (2-\alpha_k)|z_k|^{-1-\alpha_k} \, \mathrm{d}z_k \\
&= \sum_{k=1}^{d} 2 \int_{r^{2/\alpha_k}}^{\infty} (2-\alpha_k) z_k^{-1-\alpha_k} \, \mathrm{d}z_k \\
&= \left(\sum_{k=1}^{d} \frac{2(2-\alpha_k)}{\alpha_k} \right) r^{-2}
\end{aligned}$$

□

For simplicity, we consider in the following $d = 2$ and $x_0 = 0$.
Let $b = (b_1, b_2) \in \mathbb{R}^2$ be defined by

$$b_1 = \frac{1}{1+\gamma-\alpha_1} \quad \text{and} \quad b_2 = \frac{1}{1+\gamma-\alpha_2}, \tag{A.0.1}$$

where

$$\gamma = \frac{|\alpha_1 - \alpha_2| + \alpha_1\alpha_2}{\min\{\alpha_1, \alpha_2\}}. \tag{A.0.2}$$

Note that $\gamma > 0$ is clear, since $\alpha_1, \alpha_2 > 0$. Moreover, let

$$\Gamma(z) = \{(x_1, x_2) \in \mathbb{R}^2 \colon |x_2 - z_2| \leq |x_1 - z_1|^{1/b_1} \text{ or } |x_1 - z_1| \leq |x_2 - z_2|^{1/b_2}\}$$

and for brevity we write

$$\Gamma := \Gamma(0) = \{(x_1, x_2) \in \mathbb{R}^2 \colon |x_2| \leq |x_1|^{1/b_1} \text{ or } |x_1| \leq |x_2|^{1/b_2}\}.$$

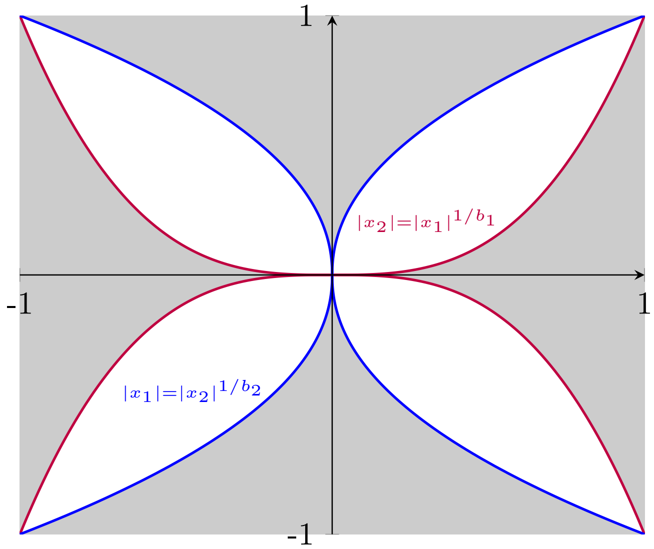


Figure A.1.: Example of $\Gamma \cap M_1$ for $\alpha_1 = 0.7$ and $\alpha_2 = 1.5$.

We define a kernel $k : \mathbb{R}^d \to [0, \infty]$,

$$k(z) = \left(1 - \gamma + \frac{1}{\min\{b_1, b_2\}}\right) \mathbb{1}_{\Gamma \cap M_1} |z|^{-2-\gamma} \tag{A.0.3}$$

and a family of measures

$$\mu(x, \mathrm{d}y) = k(x - y)\, \mathrm{d}y.$$

Note that in the case $\alpha_1 = \alpha_2 = \alpha \in (0, 2)$ we have

$$b_1 = b_2 = 1,\ \gamma = \alpha,\ \Gamma = \mathbb{R}^2, \text{ and } k(z) = (2 - \alpha)\mathbb{1}_{M_1}|z|^{-2-\alpha}.$$

The authors in [DK15] have shown, that the energies for μ_{axes} with $\alpha_1 = \cdot = \alpha_d = \alpha \in (0, 2)$ and μ^α are locally comparable. Thus, the case where the α_i's are the same recovers the case $\mu^\alpha(x, \mathrm{d}y) = (2 - \alpha)|x - y|^{-2-\alpha}\, \mathrm{d}y$. With regard to the known theory, this observation strengthens the hypothesis, that this definition is a good candidate for a measure satisfying Assumption 3.

Our aim is to prove the following theorem.

Theorem A.0.3. *There exists a constant $c_1 > 0$ such that for every $r \in (0,1)$ and every $v \in V^{\mu_{axes}}(M_r|\mathbb{R}^d)$*

$$\mathcal{E}^{\mu}_{M_r}(v,v) \leq c_1 \mathcal{E}^{\mu_{axes}}_{M_r}(v,v).$$

For this purpose we first prove some auxiliary results.

Lemma A.0.4. *Let γ be defined as in* (A.0.2)*. Then $\gamma \geq \alpha_1$ and $\gamma \geq \alpha_2$.*

Proof. We assume without loss of generality $\alpha_1 \leq \alpha_2$. The other case follows by symmetry.

$$\gamma - \alpha_2 = \frac{\alpha_2 - \alpha_1 + \alpha_1\alpha_2}{\alpha_1} - \alpha_2 = \frac{\alpha_2 - \alpha_1 + \alpha_1\alpha_2 - \alpha_1\alpha_2}{\alpha_1} = \frac{\alpha_2 - \alpha_1}{\alpha_1} \geq 0.$$

Hence $\gamma \geq \alpha_2 \geq \alpha_1$, which proves the assertion. □

Note that by Lemma A.0.4 the quantities b_1, b_2 are well-defined and non-negative.

Lemma A.0.5. *Let $\alpha_1, \alpha_2 \in (0,2)$, γ as in* (A.0.2) *and b_1, b_2 as in* (A.0.1)*. Then*

$$b_1(-1-\gamma) \geq -1 - \alpha_2$$

and

$$b_2(-1-\gamma) \geq -1 - \alpha_1.$$

Proof. We assume without loss of generality $\alpha_1 \geq \alpha_2$. We have

$$\begin{aligned}
\gamma &= \frac{\alpha_1 - \alpha_2 + \alpha_1\alpha_2}{\alpha_2} = \frac{\alpha_1}{\alpha_2} - 1 + \alpha_1, \\
b_1 &= \frac{1}{1+\gamma-\alpha_1} = \frac{1}{1+\frac{\alpha_1}{\alpha_2} - 1 + \alpha_1 - \alpha_1} = \frac{\alpha_2}{\alpha_1}, \\
b_2 &= \frac{1}{1+\gamma-\alpha_2} = \frac{1}{1+\frac{\alpha_1}{\alpha_2} - 1 + \alpha_1 - \alpha_2} \leq \frac{\alpha_2}{\alpha_1}.
\end{aligned}$$

Hence

$$b_1(-1-\gamma) = \frac{\alpha_2}{\alpha_1}\left(-\frac{\alpha_1}{\alpha_2} - \alpha_1\right) = -1 - \alpha_2$$

and

$$b_2(-1-\gamma) = \frac{\alpha_2}{\alpha_1}\left(-\frac{\alpha_1}{\alpha_2} - \alpha_1\right) = -1 - \alpha_2 \geq -1 - \alpha_1.$$

□

We define

$$A(r) := \{(x_1, x_2) \in M_r \colon 0 < x_1 \text{ and } 0 < x_2 < x_1^{1/b_1}\},$$
$$B(r) := \{(x_1, x_2) \in M_r \colon 0 < x_2 \text{ and } 0 < x_1 < x_2^{1/b_2}\}.$$

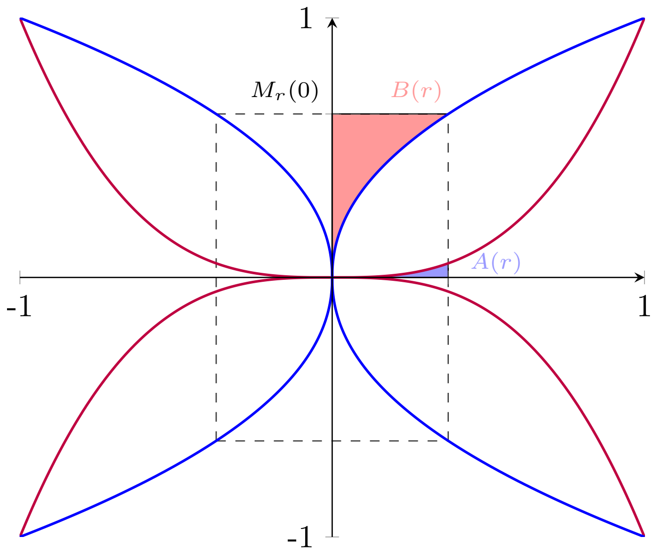


Figure A.2.: Example of $\Gamma \cap A(r)$ and $\Gamma \cap B(r)$ for $r = 0.7$, $\alpha_1 = 0.7$ and $\alpha_2 = 1.5$.

Now, we show that $k(y)\,\mathrm{d}y$ is a Lévy measure that has the same behavior in r on the sets M_r and $(M_r^c$ as $\mu(0, \mathrm{d}y)$.

Lemma A.0.6. *For every $r \in (0, 1)$*

$$\int_{M_r} |z|^2\, k(z)\,\mathrm{d}z \leq 4(r^{-2+4/\alpha_1} + r^{-2+4/\alpha_2}).$$

Proof. By symmetry

$$\int_{M_r} |z|^2\, k(z)\,\mathrm{d}z = 4\int_{A(r)} |z|^2\, k(z)\,\mathrm{d}z + 4\int_{B(r)} |z|^2\, k(z)\,\mathrm{d}z.$$

A direct calculation shows

$$\begin{aligned}\int_{A(r)} |z|^2\, k(z)\,\mathrm{d}z &= \int_{A(r)} \left(1 - \gamma + \frac{1}{\min\{b_1, b_2\}}\right) \mathbb{1}_{\Gamma\cap M_1} |z|^{-\gamma}\,\mathrm{d}z \\ &= \left(1 - \gamma + \frac{1}{\min\{b_1, b_2\}}\right) \int_0^{r^{2/\alpha_1}} \int_0^{x^{1/b_1}} (x^2 + y^2)^{-\gamma/2}\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\leq \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right)\int_0^{r^{2/\alpha_1}}\int_0^{x^{1/b_1}} x^{-\gamma}\,\mathrm{d}y\,\mathrm{d}x\\
&= \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right)\int_0^{r^{2/\alpha_1}} x^{-\gamma+1/b_1}\,\mathrm{d}x\\
&= \frac{\left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right)}{\left(1-\gamma+\frac{1}{b_1}\right)} r^{2(1-\gamma+1/b_1)/\alpha_1}\\
&\leq r^{2(1-\gamma+1+\gamma-\alpha_1)/\alpha_1} = r^{2(2-\alpha_1)/\alpha_1} = r^{-2+4/\alpha_1}.
\end{aligned}
$$

Analogously, we observe

$$\int_{B(r)} |z|^2\, k(z)\,\mathrm{d}z \leq r^{-2+4/\alpha_2}.$$

Hence

$$\int_{M_r} |z|^2\, k(z)\,\mathrm{d}z \leq 4(r^{-2+4/\alpha_1} + r^{-2+4/\alpha_2}),$$

which finishes the proof. □

Lemma A.0.7. *There exists a $c_1 > 0$, such that for every $r \in (0,1)$*

$$\int_{M_r^c} k(z)\,\mathrm{d}z \leq c_1 r^{-2}.$$

Proof. We assume without loss of generality $\alpha_1 \leq \alpha_2$. Then we have $r^{2/\alpha_1} \leq r^{2/\alpha_2}$. Note that the corner points of $\overline{M_r}$ are elements of $\{(|x|^{1/b_2}, |x|) \in \mathbb{R}^2 : x \in [-1,1]\}$, since

$$\frac{2}{\alpha_1} b_2 = \frac{2}{\alpha_1}\frac{1}{1+\frac{\alpha_2-\alpha_1+\alpha_1\alpha_2}{\alpha_1}-\alpha_2} = \frac{2}{\alpha_1}\frac{1}{1+\frac{\alpha_2}{\alpha_1}-1+\alpha_2-\alpha_2} = \frac{2}{\alpha_2}.$$

We define

$$
\begin{aligned}
C(r) &:= \{(x_1,x_2) \in M_1 \setminus M_r : 0 < x_1 \text{ and } 0 < x_2 < x_1^{1/b_1}\},\\
D(r) &:= \{(x_1,x_2) \in M_1 \setminus M_r : 0 < x_2 \text{ and } 0 < x_1 < x_2^{1/b_2}\}.
\end{aligned}
$$

Again by symmetry

$$\int_{M_r^c} k(z)\,\mathrm{d}z = 4\int_{C(r)} k(z)\,\mathrm{d}z + 4\int_{D(r)} k(z)\,\mathrm{d}z.$$

We have

$$
\begin{aligned}
\int_{C(r)} k(z)\,\mathrm{d}z &= \int_{C(r)} \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right) \mathbb{1}_{\Gamma\cap M_1} |z|^{-2-\gamma}\,\mathrm{d}z\\
&\leq c_2 \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right)\int_{r^{2/\alpha_1}}^1 \int_0^{s^{(\frac{1}{b_1}-1)}} s^{-1-\gamma}\,\mathrm{d}\phi\,\mathrm{d}s
\end{aligned}
$$

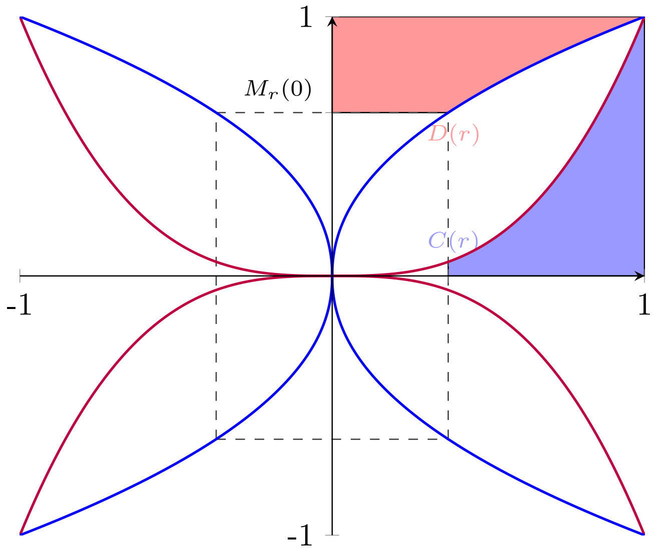


Figure A.3.: Example of $C(r)$ and $D(r)$ for $r = 0.7$, $\alpha_1 = 0.7$ and $\alpha_2 = 1.5$.

$$\begin{aligned}
&= c_2 \left(1 - \gamma + \frac{1}{\min\{b_1, b_2\}}\right) \int_{r^{2/\alpha_1}}^{1} s^{-1-\gamma+\frac{1}{b_1}-1} \, \mathrm{d}s \\
&= c_2 \left(1 - \gamma + \frac{1}{\min\{b_1, b_2\}}\right) \int_{r^{2/\alpha_1}}^{1} s^{-1-\gamma+1+\gamma-\alpha_1-1} \, \mathrm{d}s \\
&= c_2 \left(1 - \gamma + \frac{1}{\min\{b_1, b_2\}}\right) \int_{r^{2/\alpha_1}}^{1} s^{-1-\alpha_1} \, \mathrm{d}s \\
&= c_3 \frac{\left(1 - \gamma + \frac{1}{\min\{b_1,b_2\}}\right)}{\alpha_1} r^{-2\alpha_1/\alpha_1} = c_4 r^{-2}.
\end{aligned}$$

By definition of b_2, we observe

$$\begin{aligned}
\frac{-2\alpha_2}{\alpha_1} b_2 &= \frac{-2\alpha_2}{\alpha_1} \frac{1}{1+\gamma-\alpha_2} = \frac{-2\alpha_2}{\alpha_1} \frac{1}{1 + \frac{|\alpha_1-\alpha_2|+\alpha_1\alpha_2}{\min\{\alpha_1,\alpha_2\}} - \alpha_2} \\
&= \frac{-2\alpha_2}{\alpha_1} \frac{1}{1 + \frac{\alpha_2-\alpha_1+\alpha_1\alpha_2}{\alpha_1} - \alpha_2} = \frac{-2\alpha_2}{\alpha_1} \frac{1}{1 + \frac{\alpha_2}{\alpha_1} - 1 + \alpha_2 - \alpha_2} = -2.
\end{aligned} \tag{A.0.4}$$

Using (A.0.4), we have

$$\begin{aligned}
\int_{D(r)} k(z) \, \mathrm{d}z &= \int_{D(r)} \left(1 - \gamma + \frac{1}{\min\{b_1, b_2\}}\right) \mathbb{1}_{\Gamma \cap M_1} |z|^{-2-\gamma} \, \mathrm{d}z \\
&\le c_5 \left(1 - \gamma + \frac{1}{\min\{b_1, b_2\}}\right) \int_{r^{(2/\alpha_1)b_2}}^{1} \int_0^{s^{(\frac{1}{b_2}-1)}} s^{-1-\gamma} \, \mathrm{d}\phi \, \mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&= c_5 \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right) \int_{r^{(2/\alpha_1)b_2}}^1 s^{-1-\gamma+\frac{1}{b_2}-1}\,\mathrm{d}s \\
&= c_5 \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right) \int_{r^{(2/\alpha_1)b_2}}^1 s^{-1-\gamma+1+\gamma-\alpha_2-1}\,\mathrm{d}s \\
&= c_5 \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right) \int_{r^{(2/\alpha_1)b_2}}^1 s^{-1-\alpha_2}\,\mathrm{d}s \\
&\leq c_6 r^{-\frac{2\alpha_2}{\alpha_1}b_2} = c_6 r^{-2},
\end{aligned}$$

which finishes the proof. □

We immediately conclude the following property of $k(z)\,\mathrm{d}z$.

Corollary A.0.8. *The measure $\nu : \mathcal{B}(\mathbb{R}^2) \to [0,\infty)$, defined by*

$$\nu(A) = \int_A k(z)\,\mathrm{d}z$$

is a Lévy measure.

Now we prove Theorem A.0.3.

Proof of Theorem A.0.3. Let us fix $r \in (0,1)$ and write for brevity $M = M_r$. We have

$$\begin{aligned}
\mathcal{E}^\mu_M(u,u) &= \int_M \int_M (u(x)-u(y))^2\, k(x-y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_M \int_M (u(x)-u(y))^2 \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right) \mathbb{1}_{\Gamma\cap M_1}(x-y)|x-y|^{-2-\gamma}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_M \int_{M\cap\Gamma(x)} (u(x)-u(y))^2 \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right) |x-y|^{-2-\gamma}\,\mathrm{d}y\,\mathrm{d}x \\
&\leq 2\int_M \int_{M\cap\Gamma(x)} \left((u(x)-u(x_1,y_2))^2 + (u(x_1,y_2)-u(y))^2\right) \\
&\qquad\qquad \times \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right) |x-y|^{-2-\gamma}\,\mathrm{d}y\,\mathrm{d}x \\
&= 2(I_1+I_2).
\end{aligned}$$

Hence, it remains to estimate I_1 and I_2. Let us start with I_1.

$$\begin{aligned}
I_1 &= \int_M \int_{M\cap\Gamma(x)} (u(x)-u(x_1,y_2))^2 \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right) |x-y|^{-2-\gamma}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_M \int_{M\cap\Gamma(x)} (u(x)-u(x+(y_2-x_2)e_2))^2 \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right) |x-y|^{-2-\gamma}\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right) \int_M \int_{x_2-2r^{2/\alpha_2}}^{x_2+2r^{2/\alpha_2}} (u(x)-u(x+(y_2-x_2)e_2))^2
\end{aligned}$$

$$\times 2\Bigg(\underbrace{\int_{x_1}^{x_1+|y_2-x_2|^{1/b_2}} |x-y|^{-2-\gamma}\,\mathrm{d}y_1}_{=A} + \underbrace{\int_{x_1+|y_2-x_2|^{b_1}}^{x_1+1} |x-y|^{-2-\gamma}\,\mathrm{d}y_1}_{=B}\Bigg)\,\mathrm{d}y_2\,\mathrm{d}x.$$

Next, we need to estimate A and B. By setting $h_1 = y_1 - x_1$ and $h_2 = y_2 - x_2$, we get

$$\begin{aligned}
A &= \int_{x_1}^{x_1+|y_2-x_2|^{1/b_2}} |x-y|^{-2-\gamma}\,\mathrm{d}y_1 = \int_0^{|h_2|^{1/b_2}} |h|^{-2-\gamma}\,\mathrm{d}h_1 \\
&\leq |h_2|^{1/b_2} \sup_{h_1\in[0,|h_2|^{1/b_2}]} \left(|h_1|^2+|h_2|^2\right)^{(-2-\gamma)/2} \\
&= |h_2|^{-2-\gamma+1/b_2} = |h_2|^{-1-\alpha_2} = |x_2-y_2|^{-1-\alpha_2}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
B &= \int_{x_1+|y_2-x_2|^{b_1}}^{x_1+1} |x-y|^{-2-\gamma}\,\mathrm{d}y_1 = \int_{|h_2|^{b_1}}^{1} |h|^{-2-\gamma}\,\mathrm{d}h_1 \\
&\leq \int_{|h_2|^{b_1}}^{1} |h_1|^{-2-\gamma}\,\mathrm{d}h_1 = \int_{|h_2|^{b_1}}^{1} h_1^{-2-\gamma}\,\mathrm{d}h_1 \\
&= \frac{1}{1+\gamma}\left(|h_2|^{b_1(-1-\gamma)}-1\right) \leq \frac{1}{1+\gamma}|h_2|^{b_1(-1-\gamma)} \\
&\leq |h_2|^{-1-\alpha_2} = |x_2-y_2|^{-1-\alpha_2},
\end{aligned}$$

where we used Lemma A.0.5. Now we study the term I_2. The estimates are similar to the ones for I_1.

$$\begin{aligned}
I_2 &= \int_M \int_{M\cap\Gamma(x)} (u(x_1,y_2)-u(y))^2 \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right) |x-y|^{-2-\gamma}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_M \int_{M\cap\Gamma(x)} (u(x_1,y_2)-u((x_1,y_2)+(y_1-x_1)e_1)^2 \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right) |x-y|^{-2-\gamma}\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \int_M \int_{M\cap\Gamma(x)} (u(x)-u(x+(y_1-x_1)e_1))^2 \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right) |x-y|^{-2-\gamma}\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \left(1-\gamma+\frac{1}{\min\{b_1,b_2\}}\right) \int_M \int_{x_1-2r^{2/\alpha_1}}^{x_1+2r^{2/\alpha_1}} (u(x)-u(x+(y_1-x_1)e_1))^2 \\
&\qquad \times 2\Bigg(\underbrace{\int_{x_2}^{x_2+|y_1-x_1|^{1/b_1}} |x-y|^{-2-\gamma}\,\mathrm{d}y_2}_{=C} + \underbrace{\int_{x_2+|y_1-x_1|^{b_2}}^{x_2+1} |x-y|^{-2-\gamma}\,\mathrm{d}y_2}_{=D}\Bigg)\,\mathrm{d}y_1\,\mathrm{d}x
\end{aligned}$$

By setting $h_1 = y_1 - x_1$ and $h_2 = y_2 - x_2$, we get

$$\begin{aligned}
C &= \int_{x_2}^{x_2+|y_1-x_1|^{1/b_1}} |x-y|^{-2-\gamma}\,\mathrm{d}y_2 = \int_0^{|h_1|^{1/b_1}} |h|^{-2-\gamma}\,\mathrm{d}h_2 \\
&\leq |h_1|^{1/b_1} \sup_{h_1\in[0,|h_2|^{1/b_2}]} \left(|h_1|^2+|h_2|^2\right)^{(-2-\gamma)/2}
\end{aligned}$$

$$= |h_1|^{-2-\gamma+1/b_1} = |h_1|^{-1-\alpha_1} = |y_1 - x_1|^{-1-\alpha_1}.$$

Moreover,

$$\begin{aligned} D &= \int_{x_2+|y_1-x_1|^{b_2}}^{x_2+1} |x-y|^{-2-\gamma}\, dy_2 = \int_{|h_1|^{b_2}}^{1} |h|^{-2-\gamma}\, \mathrm{d}h_2 \\ &\leq \int_{|h_1|^{b_2}}^{1} |h_2|^{-2-\gamma}\, dh_2 = \int_{|h_1|^{b_2}}^{1} h_2^{-2-\gamma}\, \mathrm{d}h_2 \\ &= \frac{1}{1+\gamma}\left(|h_1|^{b_2(-1-\gamma)} - 1\right) \leq \frac{1}{1+\gamma}|h_1|^{b_2(-1-\gamma)} \\ &\leq |h_1|^{-1-\alpha_1} = |x_1 - y_1|^{-1-\alpha_1}. \end{aligned}$$

Putting these estimates together proves the assertion. □

Bibliography

[AE01] Herbert Amann and Joachim Escher. *Analysis. III.* Grundstudium Mathematik. [Basic Study of Mathematics]. Birkhäuser Verlag, Basel, 2001.

[AK09] Helmut Abels and Moritz Kassmann. The Cauchy problem and the martingale problem for integro-differential operators with non-smooth kernels. *Osaka J. Math.*, 46(3):661–683, 2009.

[Ald78] David Aldous. Stopping times and tightness. *Ann. Probability*, 6(2):335–340, 1978.

[App09] David Applebaum. *Lévy processes and stochastic calculus*, volume 116 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2009.

[Bas88] Richard F. Bass. Uniqueness in law for pure jump Markov processes. *Probab. Theory Related Fields*, 79(2):271–287, 1988.

[Bas98] Richard F. Bass. *Diffusions and elliptic operators*. Probability and its Applications (New York). Springer-Verlag, New York, 1998.

[BB11] Anders Björn and Jana Björn. *Nonlinear potential theory on metric spaces*, volume 17 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2011.

[BBM02] Jean Bourgain, Haïm Brezis, and Petru Mironescu. Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications. *J. Anal. Math.*, 87:77–101, 2002. Dedicated to the memory of Thomas H. Wolff.

[BC06] Richard F. Bass and Zhen-Qing Chen. Systems of equations driven by stable processes. *Probab. Theory Related Fields*, 134(2):175–214, 2006.

[BC10] Richard F. Bass and Zhen-Qing Chen. Regularity of harmonic functions for a class of singular stable-like processes. *Math. Z.*, 266(3):489–503, 2010.

[BCI11] Guy Barles, Emmanuel Chasseigne, and Cyril Imbert. Hölder continuity of solutions of second-order non-linear elliptic integro-differential equations. *J. Eur. Math. Soc. (JEMS)*, 13(1):1–26, 2011.

[Ber96] Jean Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[BF75] Christian Berg and Gunnar Forst. *Potential theory on locally compact abelian groups*. Springer-Verlag, New York-Heidelberg, 1975. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 87.

[BGK09] Martin T. Barlow, Alexander Grigor'yan, and Takashi Kumagai. Heat kernel upper bounds for jump processes and the first exit time. *J. Reine Angew. Math.*, 626:135–157, 2009.

[Bil99] Patrick Billingsley. *Convergence of probability measures.* Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[BK05a] Richard F. Bass and Moritz Kassmann. Harnack inequalities for non-local operators of variable order. *Trans. Amer. Math. Soc.*, 357(2):837–850, 2005.

[BK05b] Richard F. Bass and Moritz Kassmann. Hölder continuity of harmonic functions with respect to operators of variable order. *Comm. Partial Differential Equations*, 30(7-9):1249–1259, 2005.

[BKK10] Richard F. Bass, Moritz Kassmann, and Takashi Kumagai. Symmetric jump processes: localization, heat kernels and convergence. *Ann. Inst. Henri Poincaré Probab. Stat.*, 46(1):59–71, 2010.

[BL02a] Richard F. Bass and David A. Levin. Harnack inequalities for jump processes. *Potential Anal.*, 17(4):375–388, 2002.

[BL02b] Richard F. Bass and David A. Levin. Transition probabilities for symmetric jump processes. *Trans. Amer. Math. Soc.*, 354(7):2933–2953, 2002.

[Bli11] Lev M. Blinov. *Structure and Properties of Liquid Crystals.* Springer Science+Business Media B.V., Dordrecht, 2011.

[BnB07] Rodrigo Bañuelos and Krzysztof Bogdan. Lévy processes and Fourier multipliers. *J. Funct. Anal.*, 250(1):197–213, 2007.

[BS12] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities [reprint of J. Polit. Econ. 81 (1973), no. 3, 637–654]. In *Financial risk measurement and management*, volume 267 of *Internat. Lib. Crit. Writ. Econ.*, pages 100–117. Edward Elgar, Cheltenham, 2012.

[CCV11] Luis Caffarelli, Chi Hin Chan, and Alexis Vasseur. Regularity theory for parabolic nonlinear integral operators. *J. Amer. Math. Soc.*, 24(3):849–869, 2011.

[CF12] Zhen-Qing Chen and Masatoshi Fukushima. *Symmetric Markov processes, time change, and boundary theory*, volume 35 of *London Mathematical Society Monographs Series.* Princeton University Press, Princeton, NJ, 2012.

[Cha16] Jamil Chaker. Regularity of solutions to anisotropic nonlocal equations. *ArXiv e-prints*, Jul 2016.

[CK03] Zhen-Qing Chen and Takashi Kumagai. Heat kernel estimates for stable-like processes on d-sets. *Stochastic Process. Appl.*, 108(1):27–62, 2003.

[CK10] Zhen-Qing Chen and Takashi Kumagai. A priori Hölder estimate, parabolic Harnack principle and heat kernel estimates for diffusions with jumps. *Rev. Mat. Iberoam.*, 26(2):551–589, 2010.

[CKK11] Zhen-Qing Chen, Panki Kim, and Takashi Kumagai. Global heat kernel estimates for symmetric jump processes. *Trans. Amer. Math. Soc.*, 363(9):5021–5055, 2011.

[CKW17] Zhen-Qing Chen, Takashi Kumagai, and Jian Wang. Elliptic harnack inequalities for symmetric non-local dirichlet forms. *ArXiv e-prints*, Mar 2017.

[CS09] Luis Caffarelli and Luis Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.*, 62(5):597–638, 2009.

[CS16] Luis Caffarelli and Pablo Raúl Stinga. Fractional elliptic equations, Caccioppoli estimates and regularity. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(3):767–807, 2016.

[CV11] Luis Caffarelli and Juan Luis Vazquez. Nonlinear porous medium flow with fractional potential pressure. *Arch. Ration. Mech. Anal.*, 202(2):537–565, 2011.

[CZ56] Alberto P. Calderón and Antoni Zygmund. On singular integrals. *Amer. J. Math.*, 78:289–309, 1956.

[CZ16a] Zhen-Qing Chen and Xicheng Zhang. Heat kernels and analyticity of non-symmetric jump diffusion semigroups. *Probab. Theory Related Fields*, 165(1-2):267–312, 2016.

[CZ16b] Zhen-Qing Chen and Xicheng Zhang. Uniqueness of stable-like processes. *ArXiv e-prints*, Apr 2016.

[DG57] Ennio De Giorgi. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)*, 3:25–43, 1957.

[DK15] Bartłomiej Dyda and Moritz Kassmann. Regularity estimates for elliptic nonlocal operators. *ArXiv e-prints*, Sep 2015.

[DNPV12] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[Dur10] Rick Durrett. *Probability: theory and examples*, volume 31 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, fourth edition, 2010.

[Fel13] Matthieu Felsinger. *Parabolic equations associated with symmetric nonlocal operators*. PhD thesis, Bielefeld University, 2013.

[FKV15] Matthieu Felsinger, Moritz Kassmann, and Paul Voigt. The Dirichlet problem for nonlocal operators. *Math. Z.*, 279(3-4):779–809, 2015.

[GHL15] Alexander Grigor'yan, Jiaxin Hu, and Ka-Sing Lau. Generalized capacity, Harnack inequality and heat kernels of Dirichlet forms on metric measure spaces. *J. Math. Soc. Japan*, 67(4):1485–1549, 2015.

[GO08] Guy Gilboa and Stanley Osher. Nonlocal operators with applications to image processing. *Multiscale Model. Simul.*, 7(3):1005–1028, 2008.

[Gra14a] Loukas Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[Gra14b] Loukas Grafakos. *Modern Fourier analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[GT01] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[HKM06] Juha Heinonen, Tero Kilpeläinen, and Olli Martio. *Nonlinear potential theory of degenerate elliptic equations*. Dover Publications, Inc., Mineola, NY, 2006. Unabridged republication of the 1993 original.

[Hoh94] Walter Hoh. The martingale problem for a class of pseudo-differential operators. *Math. Ann.*, 300(1):121–147, 1994.

[Jac05] Niels Jacob. *Pseudo differential operators and Markov processes. Vol. III*. Imperial College Press, London, 2005. Markov processes and applications.

[JS03] Jean Jacod and Albert N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003.

[Kas11] Moritz Kassmann. Harnack inequalities and hölder regularity estimates for nonlocal operator revisited. *SFB 701-preprint No. 11015*, 20011.

[Kas07a] Moritz Kassmann. The classical Harnack inequality fails for nonlocal operators. *SFB 611-preprint No. 360*, 2007.

[Kas07b] Moritz Kassmann. Harnack inequalities: an introduction. *Bound. Value Probl.*, pages Art. ID 81415, 21, 2007.

[Kas09] Moritz Kassmann. A priori estimates for integro-differential operators with measurable kernels. *Calc. Var. Partial Differential Equations*, 34(1):1–21, 2009.

[Kle12] Fima C. Klebaner. *Introduction to stochastic calculus with applications*. Imperial College Press, London, third edition, 2012.

[Kom73] Takashi Komatsu. Markov processes associated with certain integro-differential operators. *Osaka J. Math.*, 10:271–303, 1973.

[KRS14] Moritz Kassmann, Marcus Rang, and Russell W. Schwab. Integro-differential equations with nonlinear directional dependence. *Indiana Univ. Math. J.*, 63(5):1467–1498, 2014.

[KS79] Nicolai V. Krylov and Mikhail V. Safonov. An estimate for the probability of a diffusion process hitting a set of positive measure. *Dokl. Akad. Nauk SSSR*, 245(1):18–20, 1979.

[Kü17] Franziska Kühn. On martingale problems and feller processes. *ArXiv e-prints*, Jun 2017.

[Lak06] Joseph R. Lakowicz. *Principles of Fluorescence Spectroscopy.* Springer Science+Business Media, LLC, Boston, MA, third edition edition, 2006.

[LD14] Héctor Chang Lara and Gonzalo Dávila. Regularity for solutions of non local parabolic equations. *Calc. Var. Partial Differential Equations*, 49(1-2):139–172, 2014.

[LM76] Jean-Pierre Lepeltier and Bernard Marchal. Problème des martingales et équations différentielles stochastiques associées à un opérateur intégro-différentiel. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 12(1):43–103, 1976.

[Maz11] Vladimir Maz'ya. *Sobolev spaces with applications to elliptic partial differential equations*, volume 342 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer, Heidelberg, augmented edition, 2011.

[Mer76] Robert C. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3(1):125 – 144, 1976.

[Mos61] Jürgen Moser. On Harnack's theorem for elliptic differential equations. *Comm. Pure Appl. Math.*, 14:577–591, 1961.

[Nas57] John Nash. Parabolic equations. *Proc. Nat. Acad. Sci. U.S.A.*, 43:754–758, 1957.

[Pri15] Enrico Priola. On weak uniqueness for some degenerate SDEs by global L^p estimates. *Potential Anal.*, 42(1):247–281, 2015.

[Pro05] Philip E. Protter. *Stochastic integration and differential equations*, volume 21 of *Stochastic Modelling and Applied Probability.* Springer-Verlag, Berlin, 2005. Second edition. Version 2.1, Corrected third printing.

[RW00a] Leonard C. G. Rogers and David Williams. *Diffusions, Markov processes, and martingales. Vol. 1.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Foundations, Reprint of the second (1994) edition.

[RW00b] Leonard C. G. Rogers and David Williams. *Diffusions, Markov processes, and martingales. Vol. 2.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Itô calculus, Reprint of the second (1994) edition.

[Sat13] Ken-iti Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2013.

[Sch03] Wim Schoutens. *Lévy processes in finance.* Wiley series in probability and statistics. Wiley, Hoboken, N.J. [u.a.], 2003.

[Sch12] Armin Schikorra. Regularity of $n/2$-harmonic maps into spheres. *J. Differential Equations*, 252(2):1862–1911, 2012.

[Sil06] Luis Silvestre. Hölder estimates for solutions of integro-differential equations like the fractional Laplace. *Indiana Univ. Math. J.*, 55(3):1155–1174, 2006.

[Str75] Daniel W. Stroock. Diffusion processes associated with Lévy generators. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 32(3):209–244, 1975.

[Str11] Daniel W. Stroock. *Probability theory.* Cambridge University Press, Cambridge, second edition, 2011. An analytic view.

[SV69] Daniel W. Stroock and Srinivasa R. S. Varadhan. Diffusion processes with continuous coefficients. I. *Comm. Pure Appl. Math.*, 22:345–400, 1969.

[SV79] Daniel W. Stroock and Srinivasa R. S. Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin-New York, 1979.

[SV04] Renming Song and Zoran Vondraček. Harnack inequality for some classes of Markov processes. *Math. Z.*, 246(1-2):177–202, 2004.

[SW13] René L. Schilling and Jian Wang. Some theorems on Feller processes: transience, local times and ultracontractivity. *Trans. Amer. Math. Soc.*, 365(6):3255–3286, 2013.

[Tar07] Luc Tartar. *An introduction to Sobolev spaces and interpolation spaces*, volume 3 of *Lecture Notes of the Unione Matematica Italiana*. Springer, Berlin; UMI, Bologna, 2007.

[WZ15] Linlin Wang and Xicheng Zhang. Harnack inequalities for SDEs driven by cylindrical α-stable processes. *Potential Anal.*, 42(3):657–669, 2015.