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Abstract

In this paper we develop an economic growth model that includes anthropogenic climate change. We explicitly include a research sector that creates new technologies and simultaneously expands productivities of existing technologies. The environment is affected by R&D activities both negatively, through increase of output from productivity growth, as well as positively, as new technologies are less harmful for the environment. We consider three different versions of the model. In the first version, there are no constraints with respect to research spending, while the R&D sector affects the rest of the economy. In this version, the environmental damages are lower than in the model with simple exogenous technical change. Next, we consider the research dynamics with a constant R&D budget set by the agent. We find that there exist two different steady states of this economy, one with higher productivities and less new technologies being developed, and the other with more technologies being created. In the last version, finally, we allow for dynamic R&D spending. For this version, it is demonstrated that sustained economic growth with preservation of the environment is possible given certain conditions as regards the technology and R&D spending.

Keywords: Climate change, green growth, vertical and horizontal innovations, optimal control

JEL classification: C61, O31, O44

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1 Introduction

In this paper we develop a simple growth model of the world economy which takes into account environmental damages. There exist quite a many such models in the economics literature, starting with the seminal paper (Nordhaus 2007). Some of those models are of the integrated assessment type (IAM) and employ a detailed description of the economy under consideration together with many sectors and parameters which are estimated empirically. Other types of models are of a simpler structure and are used to study some new approaches as regards the modelling of the natural environment in growth theory. In this second strand of literature, one can find two different approaches of how the enviroment affects the economic subsystem: one approach includes environmental quality in the utility function of the optimizing agent, as e.g. in (Ligthart and Van Der Ploeg 1994), whereas the second approach assumes a productivity decrease due to the environmental degradation. An example for the second approach may be (Lans Bovenberg and Smulders 1995), where the assumption of pollution-augmenting technical progress is adopted. Our paper belongs to this second type of models.

The main focus of our paper is the influence of different forms of technical change on the evolution of the economy and the environment. Hence, the main departure from the majority of the literature on economic growth with environmental degradation concerns the way how the technological change is modelled.

Often, the technology in environmental-economic models is modelled as an exogenous process of accumulation of knowledge according to some given function, without any influence from the part of the optimizing agent. However, there is a number of papers where the environmental variables are controlled by the agent, together with the technology. These papers build upon two well-known models of endogenous growth, namely (Romer 1990) and (Aghion and Howitt 1992). As an example for an endogenous growth model, based on variety expansion, including environmental damages, one may take the model by (Barbier 1999), while the papers by Grimaud (Grimaud 1999; Grimaud and Rougé 2003; Grimaud and Rougé 2005) are based on the model of vertical innovations by Aghion&Howitt. These and similar papers do not take into account the environmental friendliness of technologies being developed and only consider productivity growth. At the same time, there is a discussion in the literature about the possibility of ‘Green Growth’, where the productivity increase of the economy does not go along with environmental damages. In recent years, endogenous growth models have bee constructed in which a distinction is made between ‘clean’ and ‘dirty’ technologies or R&D sectors. This type of modelling bears the name of directed technical change and the most recent example of such an approach is (Acemoglu, Aghion, Bursztyn, and Hemous 2012). In our paper we do not limit ourselves to the two technologies carbon-intensive and clean, as in the cited paper. Instead, we model the R&D sector of the economy as a generator of new technologies on a continuous basis. Each of these new technologies has zero productivity at the time of its invention and this productivity is subject to additional investments. In such a way, the technological progress is described by two dimensions: the creation of new and cleaner technologies and the productivity growth of existing technologies.

The overall productivity of the economy depends on the productivity of all the technologies being invented up to time t, while the intensity of emissions is defined by the cleanliness of all these technologies. In the first version of the model, R&D expenditures are assumed to be determined exogenously and the technology investments are determined from an optimal control problem, while the optimizing agent treats them as given in its optimization of capital investment and consumption. Thus, the direction of technical change is independent of the consumption decisions of the agent, but it influences the overall state of the environment. In this model, environmental damages are smaller than in the case of an exogenously given technology, but are still rising. In the second version, we assume a constant R&D budget that imposes a constraint on the agent in determining optimal technology investments. With such a constraint, there exist two different steady state levels of research, one with a smaller range of technologies being developed and higher overall productivity and the other with a maximal range of technologies and lower overall productivity. It is shown that the R&D budget constraint plays the key role in determining the actual steady state of the economy. However, in this setting permanent growth is impossible since newer technologies require more R&D spending and this spending is constant. These two versions of the model are described in section 2.

At last, we allow for a dynamic R&D budget to be set optimally by the agent. In this version of the model, both positive and environmentally clean growth is possible along the balanced growth path provided the growth rates of the overall productivity and the number of technologies follow some special patterns. Our claim is that the 'Green Growth' hypothesis can hold, but only if the evolution of the technology is rich enough to allow for different and dynamically adjusting directions of technical change and if R&D financing is constantly growing. This model is presented in section 3 of the paper.

2 The model with exogenous R&D

To start with, we describe the basic growth model featuring climate change.

2.1 The basic growth model including climate change

The economy is described from the point of view of the optimizing agent in the same way as in (Bréchet, Camacho, and Veliov 2011). We modify the basic model by allowing for technical change to depend on R&D expenditures. The economy contains a range of productivity levels and emissions technologies and associated sectors of the economy. However, to keep things simple, we assume that capital is distributed evenly across these sectors.

With the assumption of evenly distributed capital across all existing sectors, the technical change may be disembodied from the capital of each individual sector:

$$Y(t) = \phi(\tau(t))\Big(\frac{k(t)}{n(t)}\Big)^{\alpha} \cdot A^E(t) \text{ with } A^E(t) = \int_0^{n(t)} q(i,t)di, \tag{1}$$

where $A^E(t)$ denotes overall productivity of all the sectors of the economy.

The economy is affected by R&D through 3 different channels: first, overall R&D investments, $R(t)$, are deduced from the income of the agent; second, total productivity of capital in the economy grows as a result of aggregate R&D, $A^E(t)$; third, the intensity of emissions from output gradually reduces due to the adoption of cleaner technologies, $e(t)$.

As a result, one arrives at the following optimization problem for the agent:

$$
\begin{aligned}
J^E = & \max_{c,a} \left\{ \int_0^\infty \mathrm{e}^{-rt} \left[\frac{\left[c(t) A^E(t) \phi\left(\tau(t)\right) \left(\frac{k(t)}{n(t)}\right)^\alpha \right]^{1-\gamma}}{1-\gamma} \right] dt \right\} \\
& \text{s.t.} \\
& \dot{k}(t) = -\delta k(t) + [1 - c(t) - f_1(a(t) - f_2(R(t))] A^E(t) \phi(\tau(t)) \left(\frac{k(t)}{n(t)}\right)^\alpha; \\
& \dot{\tau}(t) = -\lambda(m(t))\tau(t) + d(m(t)); \\
& \dot{m}(t) = -\nu m(t) + (1 - a(t)) e(t) A^E(t) \phi(\tau(t)) \left(\frac{k(t)}{n(t)}\right)^\alpha .
\end{aligned} \tag{2}
$$

This problem contains three exogenously given functions of time, which determine the total productivity, the emissions intensity and R&D investments. The notation is as follows:

- J^E is the objective functional of the agent;
- r is the rate of time preference;
- $c(t)$ is the consumption rate per capita;
- $Y(t)$ is the total output per capita;
- $k(t)$ is the total capital per capita;
- δ is the depreciation rate of capital;
- $a(t)$ is the abatement rate;
- $\tau(t)$ is the average temperature relative to the pre-industrial level;
- λ is the rate of temperature decrease due to emissions into space;
- $m(t)$ is the greenhouse gas (GHG) concentration in the atmosphere of the earth;

- ν is the rate of recovery of the atmosphere due to natural decay;
- $e(t)$ is the reduction of the emissions intensity (due to cleaner technologies);
- $\phi(\tau(t))$ is the damage function depending on the temperature increase;
- α is the parameter of capital productivity;
- γ is the inverse of the inter-temporal elasticity of substitution of consumption;
- $R(t)$ are total R&D investments;
- $q(i,t)$ is the productivity of technology i at time t;
- $n(t)$ is the total spectrum of technologies available at time t.

Let us assume that there is an exogenous R&D sector, which may be constrained or unconstrained in its activities by the total of resources R the agents decides to spend for research activities.

The R&D sector continuously develops new technologies, $i \in \mathbf{I}$, from the potential spectrum of technologies. These technologies differ from each other in the extent of environmental damage from them with newer technologies yielding less damages (the function of the damages to be defined). This results in a gradual reduction of the emissions intensity, $e(t)$, as new technologies are adopted. At the same time, the agent develops the productivity of all these new technologies (changing according to some investment efficiency function), which in turn governs aggregate productivity dynamics.

2.2 The dynamics of R&D investments

The structure of the R&D sector is close to the one in the model by (Acemoglu, Aghion, Bursztyn, and Hemous 2012), but with a continuum of possible technologies instead of only two (dirty and clean). The dynamic model follows the lines of (Belyakov, Tsachev, and Veliov 2011), where the mathematical foundations for this type of models are discussed, but is more closely to the stylized model in (Bondarev 2012). The R&D spending is determined from an optimization problem of maximizing the productivity of all new technologies subject to laws of motion of productivity increase for each technology and and subject to the process of variety expansion of technologies. It is assumed that each new technology has zero productivity at the time it is invented,

$$q(i, t_i(0)) = 0, \tag{3}$$

which makes sense from an economic point of view and where $t_i(0)$ denotes the time of invention of the technology i. The time of invention of the technology, $t_i(0)$, is the inverse function of the process of variety expansion, $n(t)$:

$$t_i(0) = f^{-1}(n(t))|_{n(t)=i}. \tag{4}$$

Thus, the objective functional to be maximized is written as:

$$J^{Tech} \stackrel{\text{def}}{=} \max_{u(\cdot),g(\cdot)} \int_0^\infty \mathrm{e}^{-rt} \left(\int_0^{n(t)} \left[q(i,t) - \frac{1}{2} g(i,t)^2 \right] di - \frac{1}{2} u(t)^2 \right) dt. \tag{5}$$

with:

- $u(t)$ - investments into variety expansion;
- $g(i, t)$ - investments into the productivity growth of technology i at time t;
- $n(t)$ - level of variety of technologies being achieved;
- $q(i, t)$ - productivity of technology i at time t.

The dynamics of productivity for technologies and of the variety of technologies are given by:

$$\begin{aligned} &\dot{n}(t) = \xi u(t); \\ &\dot{q}(i,t) = \psi(i) g(i,t) - \beta q(i,t), \ \forall i \in [0;1] = \mathbf{I} \subset \mathbb{R}. \end{aligned} \tag{6}$$

with:

- $\psi(i)$ - efficiency of investments $g(i, t)$ into the productivity growth of technology i;
- β - decay rate of productivity of technology i in the absence of investments, similar across technologies;

- ξ - efficiency of investments into the expansion of variety of technologies.

One has also the number of static constraints on controls and states:

$$q(i,t)|_{i=n(t)} = 0;\ 0 \leq n(t) \leq 1;\ q(i,t) \geq 0;\ u(t) \geq 0;\ g(i,t) \geq 0. \tag{7}$$

From (6), (7) it can be seen, that the productivity of each technology declines over time ($q(i,t)$ may decrease), but the technology itself, once invented, cannot be forgotten ($n(t)$ cannot decrease). The spectrum of technologies is bounded by some positive value N which is normalized to one. In such a framework the number of sectors in the economy grows in time, but there is no real structural change, since older sectors do not disappear from the system.

For simplicity we first assume that the technical change is completely exogenous to the rest of the economy. In this case, the research sector problem may be solved separately from the optimization problem 2. To solve the R&D problem described by (5), (6) we make use of the maximum principle. The details can be found in the Appendix A.

We derive the analytic solution for the infinite time horizon only. The estimation of the dynamics for limited time horizons has been done with the help of asymptotic methods as in (Reid 1980). It appears to be qualitatively the same and bounded by the dynamics of the infinite time version of the model. Following the arguments from the Appendix A, the productivity growth of every new technology is given by the equation:

$$\dot{q}(i,t) = \frac{\psi(i)^2}{r+\beta} - \beta q(i,t). \tag{8}$$

which is a linear equation in $q(i,t)$ for each i.

The optimal investments into productivities of new technologies are constant in time and vary only across technologies for the infinite time horizon:

$$g(i)^* = \frac{\psi(i)}{r+\beta}. \tag{9}$$

Assuming decreasing efficiency of investments and constant decay rates of productivity across technologies,

$$\psi(i) = \psi \cdot \sqrt{1-i};\ \beta(i) = \beta. \tag{10}$$

we obtain the explicit solution for the productivity evolution for each i:

$$q^*(i,t) = \frac{\psi^2(1-i)}{(\beta+r)\beta} \cdot (1 - \mathrm{e}^{-\beta(t-t_i(0))}). \tag{11}$$

The time of emergence $t_i(0)$ depends on the variety expansion speed of new technologies, $n(t)$. The latter is obtained as the solution for the ODE system (A.11) derived in the Appendix A. The associated optimal investments into variety expansion are the function of shadow price of variety expansion only. The resulting variety expansion is:

$$n^*(t) = 1 + (n_0 - 1) \cdot \mathrm{e}^{-\frac{1}{2}A_1 t}, \text{ with } A_1 = \frac{\sqrt{r^4 + 2r^3\beta + r^2\beta^2 + 2\xi^2\gamma^2} - (r^2 + r\beta)}{r+\beta} > 0. \tag{12}$$

which defines investments into variety expansion as a function of time and of the initial conditions:

$$u^*(t) = \frac{(1-n_0)}{2\xi^2} A_1 \mathrm{e}^{-\frac{1}{2}A_1}. \tag{13}$$

The overall dynamics of the R&D sector is described by productivities in the form (11) and by the expansion of the spectrum of technologies by (12). Figure 1 provides the 3-d reconstruction of the qualitative dynamics. The main features of the technology evolution

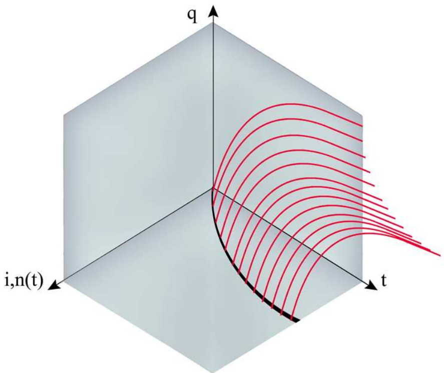


Figure 1: Reconstruction of exogenous unrestricted technological change

are summarized in the Proposition 1.

Proposition 1 (Technology evolution with independent R&D) *With no constraints on the evolution of technologies, they are evolving in the following fashion:*

1. *Each new technology is developed up to some maximal productivity level, $\bar{q}(i)$ and then is supported on this level;*

2. *Every technology has a different maximal productivity level, which decreases across technologies if the function $\psi(i)$ is decreasing and vice versa;*

3. *Variety of technologies, $n(t)$, grows over time with declining speed for a decreasing $\psi(i)$ function and with a rising speed for an increasing function;*

4. *Constant growth rates of the variety are only possible for homogeneous technologies (linear growth).*

2.3 Aggregate dynamics

The overall productivity of the economy, $A^E(t)$, as defined in equation 1, is given by the total of productivities of all technologies in the existing range at a given time t. Since these productivities are known, this is an explicit function of time:

$$A^E(t) = \int_0^{n(t)} q^*(i,t)di = F_1(t), \text{ with } F_1(t) = B_1 \cdot \mathbf{e}^{-\beta t} - B_2 \cdot \mathbf{e}^{-\frac{3}{2}A_1 t} + B_3, \qquad (14)$$

where B_1, B_2, B_3 are functions of constant parameters n_0, ξ, β, r.

The emissions intensity reduction is obtained in the same manner as this is the aggregate of emissions intensities of all the technologies being developed up to the time t. To define the emissions intensity it is necessary to specify how the carbon-intensity changes across technologies. We assume that each next technology is cleaner than the preceding one. The simplest way to define this is to assume a linear decrease of the emissions intensity from 1 to zero:

$$\iota(i) = 1 - i. \qquad (15)$$

With such a specification, the economy starts with the most dirty technology which emits one unit of GHGs per unit of output, while at the end of the possible spectrum are technologies which do not contribute to the emissions at all. However, these technologies are very hard to develop due to a decreasing efficiency of investments into their productivity growth, defined by the $\psi(i)$ function above. The emissions intensity reduction function is explicitly given by:

$$e^E(t) = \frac{\int\limits_0^{n(t)} \iota(i)di}{n(t)} = F_2(t), \text{ with } F_2(t) = 1 - \frac{1}{2}n(t). \tag{16}$$

The overall amount of R&D investments is given by the quantity

$$R(t) = \int\limits_0^{n(t)} g(i,t)di + u(t), \tag{17}$$

where investments are given by the optimal values from the R&D optimization problem above. This quantity for known variety expansion may be expressed as a function of time and of parameters only:

$$R(t) = \frac{1}{2}\left(G_1 - G_2 \mathrm{e}^{-A_1 t} - G_3 \mathrm{e}^{-\frac{1}{2}A_1 t}\right). \tag{18}$$

where G_1, G_2, G_3 again are functions of the constant parameters n_0, ξ, ψ, β, r.

2.4 Resource-constrained exogenous research sector

The model with exogenous R&D expenditures has one strong weakness: the R&D influences the economy and the environment, but the economy does not influence R&D. Since we have a continuum of possible technologies of heterogeneous nature, it would be difficult to include the optimization of R&D expenditures of all types (for every i) into the problem of the agent. Therefore, we assume that the agent only controls the overall level of R&D expenditures by setting the resource constraint as concerns R&D investments:

$$R = u(t) + \int\limits_0^{n(t)} g(i,t)di. \tag{19}$$

With such a constraint, the optimization problem becomes a resource-constrained one. It has the same objective, (5), and the same dynamic constraints, (6), but it includes an additional path constraint for the controls (19). We again use the maximum principle to solve the problem. The main difference in the solution procedure consists in the usage of an augmented Hamiltonian function instead of the usual one. The exact formal treatment of this problem can be found in the Appendix B. Here, we only point out that the inclusion of the path constraint into the model does not allow to achieve an explicit closed-form solution, even for infinite time horizon case. However, due to the special structure of the problem, the steady state dynamics of variety expansion do not depend on the productivity of technologies and may be analysed by conventional methods.

We start with deriving the dynamical system for the constrained variety expansion which is given by:

$$\dot{n}(t) = \frac{\xi}{1+n(t)} \left(R + \xi\lambda(t)n(t) + \frac{2}{3}\psi \frac{(\sqrt{1-n(t)} - n(t)\sqrt{1-n(t)} - 1)}{r+\beta} \right);$$

$$\dot{\lambda}(t) = \Lambda_1(n(t))\lambda(t) - \Lambda_2(n(t))\lambda^2(t) + \Lambda_3(n(t)), \tag{20}$$

where $\Lambda_1(\cdot), \Lambda_2(\cdot), \Lambda_3(\cdot)$ are functions of variety expansion and of parameters.

This non-linear system has at most 2 steady states depending on the value of the resource constraint R. With a tight constraint, there are no steady states at all while with a very relaxed constraint the only steady state is the maximum level of variety allowed, $\bar{n} = 1$. However, with R in between these two extremes there exist two different steady states. In one of these steady states the variety of technologies is not maximal, but is limited to roughly (1/2) of the range, as Figure 2 illustrates.

There is also the possibility for the existence of only one steady state of the second type, which corresponds to a lower efficiency of investments into variety expansion. However, we are mainly interested in the situation with multiple steady states, as this provides a possibility to discuss the technology lock-in phenomena.

Next, we consider the dynamics of productivity for technology i:

$$\begin{aligned}\dot{q}(i,t) = {} & \frac{1}{(r+\beta)(1+n(t))} \cdot \\ & \cdot \left(\frac{2}{3}\psi^2\sqrt{1-i}(1-n(t))^{3/2} + 3\psi\sqrt{1-i}((r+\beta)R - \frac{2}{3}\psi - \xi(r+\beta)\lambda(t) \right) - \\ & - \frac{3}{r+\beta}\left(\beta(r+\beta)q(i,t) - \psi^2(1-i)\right) q(i,t).\end{aligned} \tag{21}$$

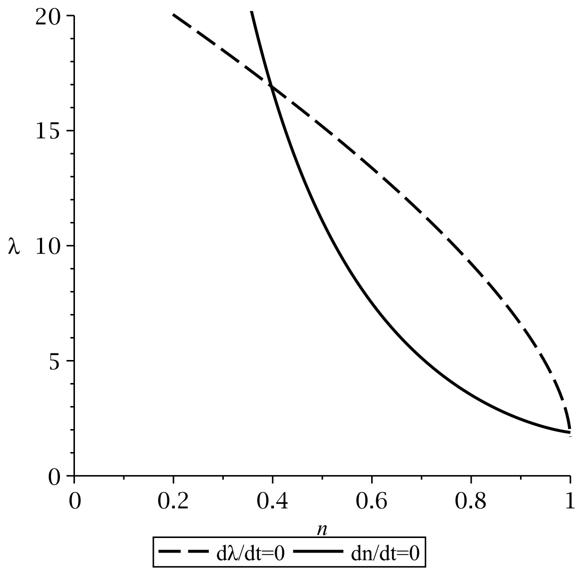


Figure 2: Multiplicity of steady states

This equation is a function of the variety expansion and of its shadow price. Hence, the steady state value of the productivity of each technology is a function of the steady state values of the variety expansion and of the shadow price of investments into it:

$$\bar{q}(i) = \frac{\psi^2(1-i)}{\beta(r+\beta)} + \frac{2}{3}\psi^2\sqrt{1-i}\frac{(1-\bar{n})^{3/2}}{(1+\bar{n})} + \psi\frac{\sqrt{1-i}}{(1+\bar{n})}(R(r+\beta) - \frac{2}{3}\psi - \xi(r+\beta)\bar{\lambda}). \quad (22)$$

Since there are two possible steady sates for the system (λ, n), every technology has two possible steady state levels of productivity. This differs from the unconstrained technology case considered above where only one steady state of the system exists, namely the one with maximum variety expansion $\bar{n} = 1$.

It can be demonstrated analytically that the steady state productivity level associated with the second non-maximal steady state of $n(t)$ is lower than the one associated with the steady state with maximal variety expansion. To see this, observe first that the shadow price of variety expansion, λ, are zero in the maximal steady state and non-zero in the other one (see Fig. 2):

$$\bar{\lambda}_{MAX} = 0 < \bar{\lambda}_{MIN} > 0. \quad (23)$$

Now, consider the derivative of the steady state productivity with respect to λ:

$$\frac{\partial \bar{q}(i)}{\partial \bar{\lambda}} = -\frac{\xi\psi\sqrt{1-i}}{\beta(1+\bar{n})} < 0. \tag{24}$$

This holds for any n. Since in the maximal steady state the value of λ is lower than in the other one, we may conclude that

$$\bar{q}(i)_{MAX} > \bar{q}(i)_{MIN}. \tag{25}$$

Hence, depending on the choice of the constraint R, there are two steady states denoted by MAX and MIN, respectively. These steady states are such that:

$$\bar{q}(i)_{MAX} > \bar{q}(i)_{MIN}, \forall i \leq 1, \quad \bar{n}_{MAX} > \bar{n}_{MIN}. \tag{26}$$

Now, we compare the steady state levels of output and of emissions associated with the two different states of R&D, MIN and MAX. A priori, one would expect higher productivity, output and fewer emissions and, therefore, less environmental degradation for the MAX steady state with more and cleaner technologies. However, due to the fact of a decreasing efficiency of investments into newer technologies, the capital is distributed to a smaller extent and its concentration is higher. That is why the output is higher for the MIN steady state although the aggregate productivity is lower:

$$\bar{Y}_{MIN} > \bar{Y}_{MAX}, \quad \bar{A}_{MIN} < \bar{A}_{MAX}. \tag{27}$$

At the same time, the emissions intensity is higher for the MIN case since there are less technologies and all of them are dirtier than for the MAX case:

$$e(t) = 1 - \frac{1}{2}n(t), \quad \bar{e}_{MIN} = 1 - \frac{1}{2}\bar{n}_{MIN} > \bar{e}_{MAX} = \frac{1}{2} \tag{28}$$

As a result, the total impact of the economy on the environment for the MIN steady state is more damaging than for the other one, but the economic performance is better. Thus, we obtain the well-known result as regards technology lock-in: there exist two steady states of the economy and, in the absence of outside stimuli, it will always converge to the dirtier one with fewer new technologies. This is summarized in Proposition 2.

Proposition 2 (Economy with technology lock-in) *In the presence of a constraint as concerns R&D, there are two possible steady states of the economy, MIN and MAX. In the first steady state, the total output of the economy is higher, but not all available technologies are developed because capital is concentrated in older more productive sectors, which are more productive. In the second steady state, the environmental degradation is slower, but output is lower than in the first steady state since capital is shifted to newer and less productive sectors. However, the total productivity of the economy in this case is higher due to a greater number of sectors being present. Other things equal, only the MIN steady state realizes since there are no stimuli to produce newer technologies.*

However, due to the introduction of heterogeneity of technologies and due to a continuous spectrum of them being available, there are more options available to influence the outcome than environmental taxation. This possibility lies in the resource constraint for the R&D sector: with this constraint being weak enough, the MIN steady state of the economy disappears.

It is straightforward to notice that the amount of resources being allocated to R&D positively affects the speed of variety expansion and of the productivity growth of all technologies. Consider first the variety expansion:

$$\frac{\partial \dot{n}(t)}{\partial R} = \frac{\xi}{1+n(t)} > 0. \tag{29}$$

Thus, the more resources are devoted to R&D, the higher is the variety expansion speed. This influence declines with an increasing available range of technologies.

The productivity development of every introduced technology is also influenced by the resource constraint:

$$\frac{\partial \dot{q}(i,t)}{\partial R} = \frac{\psi\sqrt{1-i}}{1+n(t)} > 0. \tag{30}$$

The influence declines across technologies and with a rising spectrum of available technologies. Now, one may ask whether additional resources in R&D will increase productivities (and thus emissions and output) or whether the spectrum of technologies (and thus emissions) declines to a greater extent in relative measure. To answer this question, we compare the derivatives given by (29) and (30). The productivity growth has to be accounted for across all the available technologies so that we need an integral expression:

$$\frac{\partial}{\partial R}\int\limits_{0}^{n(t)}\dot{q}(i,t)di = \int\limits_{0}^{n(t)}\frac{\partial\dot{q}(i,t)}{\partial R}di = \frac{2}{3}\frac{\psi}{(1+n(t))}\cdot\Big(1-(1-n(t))^{2/3}\Big), \tag{31}$$

which represents the total change of the productivity growth of all technologies. Then, the relative measure of resource allocation is determined by the following relationship:

$$\frac{\xi}{\psi} \lesseqgtr \frac{2}{3}\Big(1-(1-n(t))^{2/3}\Big). \tag{32}$$

This expression depends on the current value of the spectrum, $n(t)$. The closer the current spectrum of technologies is to its limit, 1, the higher the chances that additional resources will be allocated more to the productivity growth of existing technologies, rather than to the introduction of new technologies.

However, this may be true only for a specific relation between investment efficiencies. Assume, for example, $\xi = \psi$, that is similar efficiency of investments into productivities and into the increase of the range of technologies. In this case, the ratio on the left hand side of (32) will be higher than the expression on the right hand side for any value of $n(t)$ since this latter is limited by 2/3. In such a case, new additional resources will always be devoted to the development of new technologies rather than to the productivity growth, independent oo the already available range.

As an alternative, consider that the efficiency of investments into variety expansion is twice as low as the aggregate efficiency for productivity improvement, $\xi = 0.5 \cdot \psi$. In this case, starting from $n = 0.6$ more resources will go to the productivity increase rather than to variety expansion.

Hence, the type of innovations (vertical versus horizontal) can be influenced through the appropriate choice of the resource constraint: the higher is the R&D budget, the faster is the addition of new technologies into the pool and, therefore, the slower is the environmental degradation. At the same time, an increase in the R&D budget would raise productivities of all of the existing technologies, thus, increasing emissions from the output. Consequently, limiting R&D to stop the development of new technologies will also result in lower productivity growth and in fewer emissions.

One can conclude that the relaxation of the budget constraint for R&D may lead to the better steady state of the economy. However, this will be true only if:

- Two steady states exist and not only one and
- The efficiency of investments into productivities of existing technologies is not very high.

If the first condition is violated, the additional resources will be deliberately spent on productivity growth, thus, increasing the environmental threat. If the second condition is not met, some new technologies will appear, reducing the capital concentration on older dirtier sectors. But this will be offset by the increase in output of these old technologies and the environment will again suffer. These results are summarized in Proposition 3.

Proposition 3 (On the selection of the steady state with limited R&D) *An increase in the R&D budget may lead to a variety expansion of technologies and, thus, to a slowdown of environmental degradation. For this, two conditions are necessary:*

1. *The resources for R&D are limited so that two steady states of the economy exist;*
2. *The efficiency of investments into variety of technologies is not less than the overall efficiency for productivity investments, $\xi \geq \psi$.*

Given these conditions, a rise of resources dedicated to R&D leads to selection of the maximal (environmentally friendly) steady state.

We conclude that in the case of a low efficiency of investments into variety expansion, increasing resources spent for R&D are necessary. In this case, the agent would be able to increase funding at initial stages in order to boost the creation of new technologies and reduce it afterwards to slow down productivity growth. This would be equivalent to a fine tuning of the direction of technical change in time, and not only indirectly influencing it through discretionary static changes in R&D spending.

3 The model with endogenous R&D

In this section we relax the assumption of a constant and static budget constraint as concerns R&D spending in the economy. Now, it is assumed that the agent optimally chooses the amount of resources spent for overall R&D as an aggregate. The optimization problem of the agent is again given by Eqs. (2), with the additional constraint Eq. (19). The only difference is that this constraint is now allowed to be a function of time, $R = f(t)$. Still, the amount of R&D that is devoted to vertical and to horizontal innovations, respectively, is the decision of a separate optimization problem. It turns out, that in a dynamic setting both vertical and horizontal R&D investments as well as the evolution of technologies are functions of the dynamic R&D budget constraint.

The optimization problem of the agent can be written as follows:

$$
\begin{aligned}
J^G = \max_{c(\cdot),a(\cdot),R(\cdot)} & \left\{ \int_0^\infty \mathrm{e}^{-rt} \cdot \left[\frac{[c(t)Y(t)]^{1-\gamma}}{1-\gamma} \right] dt \right\} \\
& \text{s.t.} \\
& \dot{k}(t) = -\delta k(t) + [1 - c(t) - f_1(a(t)) - f_2(R(t))]Y(t); \\
& \dot{\tau}(t) = -\lambda(m(t))\tau(t) + d(m(t)); \\
& \dot{m}(t) = -\nu m(t) + (1 - a(t))e^E(t)Y(t); \\
& Y(t) = \phi(\tau(t)) \int_0^{n(t)} q(i,t)k(i,t)^\alpha di; \\
& k(i,t) = \frac{k(t)}{n(t)},
\end{aligned}
\tag{33}
$$

where $f_2(R(t))$ replaces total exogenous R&D investments from the previous version. Indeed, under a dynamic budget constraint all controls of the R&D optimization problem are functions of the dynamic $R(t)$. To see this, consider the optimization problem with respect to R&D subject to the resource constraint:

$$J^R = \max_{g(\cdot),u(\cdot)} \int_0^{\infty} \mathrm{e}^{-rt} \int_0^{n(t)} q(i,t) - \frac{1}{2}g^2(i,t)di - \frac{1}{2}u^2(t)dt \tag{34}$$

$$\text{s.t.} \tag{35}$$

$$\dot{n}(t) = \xi u(t); \tag{36}$$

$$\dot{q}(i,t) = \psi(i)g(i,t) - \beta q(i,t), \forall i \in [0,..,1]; \tag{37}$$

$$\int_0^{n(t)} g(i,t) + u(t) \leq R(t); \tag{38}$$

$$q(i,t)|_{i=n(t)} = 0. \tag{39}$$

This problem yields optimal controls for vertical and horizontal innovations as functions of $R(t)$:

$$u^*(t) = \xi\lambda_n(t) - \iota(n(t), \lambda_n(t), R(t)); \tag{40}$$

$$g^*(i,t) = \psi(i)\lambda_q(i,t) - \iota(n(t), \lambda_n(t), R(t)), \tag{41}$$

with

$$\iota = \frac{\xi\lambda(t) + \int_0^{n(t)} \lambda_q(i,t)\psi(i)di}{1+n} - \frac{R(t)}{1+n(t)}, \tag{42}$$

where $\lambda(t)$ is the shadow price of investments into variety expansion and $\lambda_q(i,t)$ are shadow prices of investments into productivity of technology i. The ι function is the Lagrange multiplier for the resource constraint.

We can solve independently for the co-state of the technology and, then, for the productivity of each technology, which is a function of the shadow price of variety expansion. The whole R&D dynamics is defined by the pair of equations for variety expansion and its shadow price. These are given in Appendix C.

Next, we impose the balanced growth path (BGP) condition:

$$\frac{\dot{n}(t)}{n(t)} = g_n. \tag{43}$$

Thus, we have a constant growth of variety expansion, independent of the efficiency function:

$$\bar{n}(t) = g_n \mathbf{e}^{g_n t}. \tag{44}$$

The co-state variable on the BGP is a function of the BGP growth rate and of $R(t)$:

$$\bar{\lambda}(t) = \frac{g_n \bar{n}(t) - \frac{\xi R(t)}{1+\bar{n}(t)} + \frac{\xi g_n \bar{n}(t) \int\limits_0^{\bar{n}(t)} \psi(i) di}{(r+\beta)(1+\bar{n}(t))^2}}{\xi^2 (1 - \frac{1}{1+\bar{n}(t)})}. \tag{45}$$

Now, the dynamics of every technology along the BGP is a function of time and of $R(t)$ only. We need, however, the function $\int\limits_0^{n(t)} q(i,t) di = f(R(t), t)$ to determine optimal controls for the problem formulated in 33. For that we need another BGP condition, namely,

$$\frac{d/dt \left(\int\limits_0^{n(t)} q(i,t) di \right)}{\int\limits_0^{n(t)} q(i,t) di} = g_Q \tag{46}$$

Using this condition we can obtain the function of integral productivity from the R&D expenditures (see Appendix C). With $\bar{Q} = f(R,t), \bar{n}(t) = f(t)$ at hand we can determine the optimal R&D budget for the agent.

Next, we can set up the Hamiltonian for the agent's optimization problem and derive the associated FOCs. However, the expressions are very cumbersome and an explicit solution for the dynamic problem of the agent cannot be obtained. The derivations are given in the Appendix C. We make only some qualitative conclusions here.

With the requirement of constant growth rates, the investments into variety expansion are also fully defined by the growth rates:

$$u^*(t) = g_n/\xi \cdot n(t); \tag{47}$$

At the same time, the resource constraint for R&D requires

$$R(t) - g_n/\xi \cdot n(t) = \int\limits_0^{n(t)} g(i,t) di. \tag{48}$$

With a constant R as before, this would mean declining investments into productivities since the variety grows. However, with a dynamic R both a positive g_n and g_Q are possible if the growth of $R(t)$ is positive and greater then g_n. In this situation one can obtain the relationship between the variety growth rate g_n and $R(t)$:

$$\begin{aligned} R(t) &= \\ =&1/2\, e^{gr\,t} \left(\frac{(2\,(r+\beta-\alpha)\,gr+\alpha)\, e^{2\,gr\,t}}{\alpha\,(1+e^{gr\,t})^2\,(r+\beta)\,(-1+e^{gr\,t})} + \frac{((\alpha+2\,r+2\,\beta)\,gr-\alpha)\, e^{3\,gr\,t}}{\alpha\,(1+e^{gr\,t})^2\,(r+\beta)\,(-1+e^{gr\,t})} \right) + \\ +&1/2\, e^{gr\,t} \left(\frac{2\,(\alpha-(r+\beta)\,gr)\, e^{gr\,t}}{\alpha\,(1+e^{gr\,t})^2\,(r+\beta)\,(-1+e^{gr\,t})} - \frac{2gr(r+\beta)}{\alpha\,(1+e^{gr\,t})^2\,(r+\beta)\,(-1+e^{gr\,t})} \right) \end{aligned} \tag{49}$$

This relationship yields growing or decreasing R&D expenditures, depending on the required growth rate of variety expansion. If a low variety expansion rate is preferred, one has a decrease of expenditures and vice versa. The output growth rate is a function of the growth rates of other variables:

$$\begin{aligned} g_Y = \dot{Y}/Y = \frac{(\phi(\tau(t))\dot{Q}(k/n)^\xi)x}{(\phi(\tau(t))Q(k/n)^\xi)} = \\ \frac{((g_k-g_n)\alpha+g_Q-2g_\tau)\tau(t)^2+(g_k-g_n)\alpha+g_Q}{1+\tau(t)^2} \end{aligned} \tag{50}$$

At the same time, in the absence of environmental spillovers, this would be:

$$g_Y^0 = (g_k-g_n)\alpha+g_Q. \tag{51}$$

Thus, positive output growth rates are possible, if:

1. The growth rate of capital is higher than the growth rate of variety of technologies, $g_k > g_n$ and
2. The aggregate growth rate in the absence of environmental spillovers is higher than the first term, provided it is negative.

This first term of the numerator in (50) may be positive if $g_Q > 2g_\tau$. But this case is counterintuitive: the presence of environmental spillovers yields higher growth rates than the model economy that neglects the environment. For that, one must have a high enough g_Q. If the overall productivity grows because of growing productivities of older technologies, this will increase the temperature growth rate. However, if this overall productivity growth occurs because of newer technologies, since $g_Q = f(g_n, R)$, both environmentally friendly and higher growth is possible.

The output growth rate for the economy with environmental damages would be the same as that of the unconstrained economy, if the growth rate of temperature was zero:

$$g_\tau = 0: \ \frac{((g_k - g_n)\alpha + g_Q)\tau(t)^2 + (g_k - g_n)\alpha + g_Q}{1 + \tau(t)^2} = (g_k - g_n)\alpha + g_Q. \tag{52}$$

This growth rate is positive if $g_k > g_n$ or if $g_k < g_n$ and $(g_k - g_n)\alpha < g_Q$ holds. Taking the extreme case with $g_k = 0$ (thus no emissions from output) we must have $g_Q > -\alpha g_n$ which is always true if both are positive.

However, along the BGP we need to consider $g_m = \dot{m}/m$, which gives us the relationship between g_Q and g_n, g_m. If we require $g_m = 0$ on the BGP, this gives a relation between g_Q and g_n only. This expression is:

$$g_Q = \frac{X(g_n)e(n)(1-a)}{\nu e^{\lambda\tau/\kappa}} - \beta. \tag{53}$$

Since both g_Q, g_n are constant on the BGP, this implies some specific relations between the temperature and the overall productivity level X. In particular, the relation (53) requires that accumulated damages (given by the denominator of the right hand side) should not be too high. If the denominator goes to infinity (in the case of a large temperature increase being already achieved), the productivity growth, g_Q, will tend to zero, if one followed a green growth policy. Otherwise, positive emissions and further temperature increases would be inevitable. This follows from the fact that all of the R&D are financed from the R&D budget and, thus, require more output (other things equal) that leads to a more GHG emissions. To offset these additional damages, a higher efficiency X is required.

However, the requirement of constant growth rates both for variety of technologies and for the total productivity growth are essential requirements for the form of the efficiency function $\gamma(i)$, since the productivity growth depends on it.

Now, take closer look on the dynamics of productivities given (C.3), derived in Appendix C. It turns out that, depending on the term in brackets, productivity may grow or decrease for each technology. Denote

$$\xi\lambda(t) + \frac{1}{r+\beta}\int\limits_0^{n(t)} \psi(i)di + R(t) = \Gamma. \tag{54}$$

If $\Gamma > 0$, productivity growth for each technology is lower than for the case of unconstrained R&D. If it is negative, the growth is higher. To obtain the same rates of growth as for the version of unconstrained R&D, we set $\Gamma = 0$ and assume the same efficiency function, $\psi(i)$. In this case, one can obtain the dynamics of variety expansion by substituting for $\lambda(t)$ in (C.4):

$$\dot{n}(t) = -\xi\left(R(t) + \frac{1}{r+\beta}\int\limits_0^{n(t)} \psi(i)di\right)\left(1 - \frac{1}{1+n(t)}\right) + \xi\frac{R(t)}{1+n(t)} - \frac{\xi}{r+\beta}\frac{\int_0^{n(t)} \psi(i)di}{1+n(t)} =$$

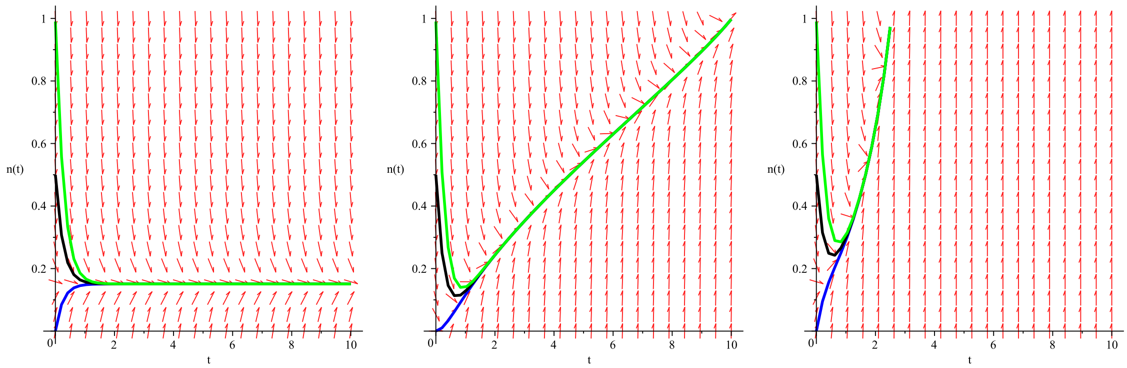
$$= R(t)(2\xi/(1+n(t)) - \xi) - \frac{\xi}{r+\beta}\int\limits_0^{n(t)} \psi(i)di. \tag{55}$$

This equation may be estimated for different time paths of R&D expenditures: for constant R&D, for linearly rising R&D or for some other dynamics of the R&D budget. It turns out that the evolution of the variety expansion and its speed are fully determined by the dynamics of the research expenditures, as productivity investments are defined by the form of the efficiency function and by Γ. Consider, for example, the variety expansion dynamics for constant, for linearly rising and for exponentially increasing research expenditures, shown in Figure 3.

With a linear growth of R&D expenditures, there is the possibility for constant variety expansion, g_n, as it is required by (43). Thus, with a linear growth of the research expenditures the green growth scenario is possible, even for a declining efficiency of the technologies, defined by $\psi(i) = \psi\sqrt{(1-i)}$. We conclude this section with the following Proposition.

Proposition 4 (On the possibility of green growth) *The green growth scenario (simultaneous output growth and preservation of the climate) is possible if:*

1. *There exists a significant multiplicity of technologies (heterogeneous spectrum);*



(a) Constant R&D (b) Linearly rising R&D (c) Expon. rising R&D

Figure 3: Dynamics of variety expansion, $n(t)$, as a function of time, t.

2. *Research spending is controlled by the agent and is at least linearly increasing;*

3. *Efficiency of newer technologies compared to older ones if does not decrease faster than the square root, in case the efficiency declines.*

In particular, a constant linear variety expansion and a constant output growth are achievable under linearly rising research expenditures and decreasing efficiency of new technologies together with a non-increasing temperature.

It would be of interest to consider the situation of zero overall productivity growth since, in this case, condition (53) is particularly simple and it might be possible to explicitly derive an optimal research expenditures rule. However, for productivity growth to be zero and variety expansion being positive, one needs finite life cycles of all the technologies. This is considered to be an expansion of the current work.

4 Conclusion

In this paper we have developed a model of economic growth featuring climate change, where growth is driven by simultaneous vertical and horizontal innovations. The key feature of these innovations is that all of the new technologies are different from each other in productivity and with respect to their GHG emissions. It turns out that possibilities for environmentally friendly economic growth can be increased if the multiplicity of technologies is taken into account.

First, we have considered the scenario with exogenously given R&D expenditures that are not subject to a constraint. For this model version, we could identify two countervailing forces affecting the economic performance and the state of the environment: The introduction of newer and cleaner technologies decreases the intensity of emissions from output, whereas higher productivities of all existing technologies raise emissions. Further, due to the presence of a continuum of technologies, there exists the effect of capital blurring that increases the effect of cleaner technologies: capital is redirected from older industries towards newer ones since there are more opportunities for investments in these new industries than in older (and more productive) ones. As a result, the negative effect of productivity growth on the environment is smaller than in the standard exogenous growth model or in the model of directed growth with a limited number of sectors as in (Acemoglu, Aghion, Bursztyn, and Hemous 2012).

Second, we allowed for a static budget constraint as concerns the R&D expenditures in the economy. In this situation, the economy may be characterized by multiple steady states, similar to (Greiner, Grüne, and Semmler 2010). This multiplicity is the result of an uneven distribution of investments between the introduction of new technologies and the development of older ones. With limited research expenditures it is likely that the majority of resources will be spent on the development of existing technologies, rather than on the introduction of newer ones. This will lead to the technology lock-in phenomenon, as described in the economics literature, when newer technologies are underdeveloped or even non-existent. However, due to the structure of the R&D process considered in our paper, this lock-in may be overcome by an increase in research spending if certain conditions as regards the parameters are fulfilled. We also argue that in the case of a single steady state (with sufficiently low research spending) such a once-and-for-all adjustment cannot overcome the technology lock-in and a rising R&D budget has to be considered as an alternative.

We, then, developed the model version with rising R&D expenditures in the last section of the paper. For that model it was not possible to obtain the closed-form solution for the control variables. However, the analysis along the balancd growth path of the economy has demonstrated that the growth rates of the variety of technologies (and, thus, of environmental degradation and of output) depend on the distribution of efficiencies across the whole spectrum of technologies and on the level of the research spending. The latter determines the shape of the variety expansion on the balanced growth path: with constant research expenditures the variety is constant, too, with linearly rising expenditures the variety expansion is also linear and so on. It turns out that a linear variety expansion is sufficient to obtain a green growth scenario: growing output without a rising greenhouse gas concentration in the atmosphere. The feasibility of this scenario depends on the structure of the technology space (defined by the differences in investment efficiencies across technologies) and on the accumulated environmental damages. The higher is the already accumulated greenhouse gas concentration and, thus, the damage, the harder it is to provide positive overall productivity growth. Since we are in the situation of such accumulated damages, the scenario of zero productivity growth should be considered. Under such a scenario each technology has a finite life cycle and, therefore, the economy would switch towards cleaner technologies over time.

Appendices

A Derivation of optimal controls and of the dynamics for the model with no constraints on R&D

In this case, the maximum principle yields the following Hamiltonian for the control problem (5), (6), (7):

$$\mathcal{H}^F = \int_0^{n(t)} \left[q(i,t) - \frac{1}{2} g(i,t)^2 \right] di - \frac{1}{2} u(t)^2 + \lambda_n(t) \cdot (\xi u(t)) +$$
$$\int_0^{n(t)} \lambda_q(i,t) \cdot (\psi(i) g(i,t) - \beta q(i,t)). \tag{A.1}$$

with superscript F denoting the version of the model with exogenous R&D. The first-order conditions for optimal controls:

$$u(t) = \xi \lambda_n^F(t), \quad g(i,t) = \psi(i) \lambda_q^F(i,t). \tag{A.2}$$

The co-state system is:

$$\dot{\lambda}_n^F(t) = r \lambda_n^F(t) + \frac{1}{2} g(n(t),t)^2 - \psi(n(t)) \lambda_q^F(n(t),t) g(n(t),t) - \lambda_q^F(t) g(n(t),t);$$

$$\dot{\lambda}_q^F(i,t) = (r+\beta) \lambda_q^F(i,t) - 1, \tag{A.3}$$

where

- in the first equation we make use of the condition $q(i,t)|_{i=n(t)} = 0$,
- $\psi(n(t)) = \psi(i)|_{i=n(t)}$ is the value of investments efficiency function at the boundary of variety expansion at the moment t,
- $g(n(t),t) = g(i),t)|_{i=n(t)}$ is the value of investments into the productivity of the next technology to be invented,
- $\lambda_q^F(n(t),t) = \lambda_q^F(i,t)_{i=n(t)}$ is the shadow price of investments into the boundary technology productivity.

We make use of the standard transversality conditions for finite T, which are replaced for infinite T by the catching-up version of them (with $\lim\limits_{t\to\infty}$ replacing $\lim\limits_{t\to T}$):

$$\lambda_n^F(T) = 0, \quad \lambda_q^F(i,T) = 0. \tag{A.4}$$

since the salvage value is zero. The co-state for each technology may be obtained independently:

$$\lambda_q^F(i,t) = \frac{1 - e^{(r+\beta)(t-T)}}{(r+\beta)}. \tag{A.5}$$

Observe that with infinite time horizon, $T \to \infty$ this yields a constant shadow price in time for each technology:

$$\lambda_q^F(i,t)|_{T\to\infty} = \frac{1}{(r+\beta)}. \tag{A.6}$$

The co-state gives optimal investments for each technology as a function of investment efficiency and of time only (non-feedback strategy):

$$g^F(i,t) = \psi(i) \cdot \frac{1 - e^{(r+\beta)(t-T)}}{(r+\beta)}, \quad g^F(i,t)_{T\to\infty} = \frac{\psi(i)}{(r+\beta)}. \tag{A.7}$$

The optimal productivity for each technology is then obtained from (6):

$$\begin{aligned}
&\dot{q}(i,t) = \psi(i) \cdot \left(\psi(i) \frac{1 - e^{(r+\beta)(t-T)}}{(r+\beta)}\right) - \beta \cdot q(i,t);\\
&q^F(i,t) = \frac{\psi^2(i)}{(2\beta + r)(\beta + r)\beta} \cdot \\
&\cdot \left(\beta e^{(2+r)t_i(0) - \beta t - (r+\beta)T} - \beta e^{(r+\beta)(t-T)} + (2\beta + r)(1 - e^{-\beta(t - t_i(0))})\right);\\
&q^F(i,t)_{T\to\infty} = \frac{\psi^2(i)}{(\beta + r)\beta} \cdot (1 - e^{-\beta(t-t_i(0))}).
\end{aligned} \tag{A.8}$$

The system for variety expansion is:

$$\begin{aligned}
&\dot{\lambda}_n(t) = r\lambda_n(t) - \frac{1}{2} \cdot \psi(n(t))^2 \cdot \left(\frac{1 - e^{(r+\beta)(t-T)}}{(r+\beta)}\right)^2;\\
&\dot{n}(t) = \xi^2 \lambda_n(t).
\end{aligned} \tag{A.9}$$

In order to obtain any (even numeric) solution, one need to specify the function $\psi(n(t))$ explicitly. We choose the function with a decreasing efficiency across technologies, which linearises the system above:

$$\psi(i) = \psi \cdot \sqrt{(1-i)};$$
$$\dot{\lambda}_n(t) = r\lambda_n(t) - \frac{1}{2} \cdot \psi^2 \cdot (1-n(t)) \cdot \left(\frac{1-e^{(r+\beta)(t-T)}}{(r+\beta)}\right)^2;$$
$$\dot{n}(t) = \xi^2 \lambda_n(t). \tag{A.10}$$

This system has no analytic solution in elementary functions, but for the infinite time horizon case it has one and the finite time solution is bounded from above by the infinite time solution since the time-dependent term is a contraction operator, cf. (Reid 1980):

$$\dot{\lambda}_n(t)|_{T\to\infty} = r\lambda_n(t) - \frac{1}{2(r+\beta)^2} \cdot \psi^2 \cdot (1-n(t));$$
$$\dot{n}(t)|_{T\to\infty} = \alpha^2 \lambda_n(t)|_{T\to\infty};$$
$$n^F(t)|_{T\to\infty} = 1 + (n_0 - 1) \cdot e^{-\frac{1}{2}A_1 t};$$
$$A_1 = \frac{\sqrt{r^4 + 2r^3\beta + r^2\beta^2 + 2\alpha^2\psi^2} - (r^2 + r\beta)}{r+\beta} > 0. \tag{A.11}$$

The explicit solution can be obtained only for the infinite time horizon version of the model. Next, define the time of emergence of each new technology as a function of the position of the technology in the spectrum:

$$t_i(0) = f^{-1}(n)|_{n(t)=i} = -2 \cdot \frac{1}{a_1} \cdot \ln\left(\frac{1-i}{1-n_0}\right) > 0. \tag{A.12}$$

Thus, the productivity growth function for the infinite time horizon for each technology i is fully defined as a function of time and of the index only:

$$q^F(i,t)_{T\to\infty} = \frac{\psi^2(1-i)}{(\beta+r)\beta} \cdot \left(1 - \left(\frac{1-i}{1-n_0}\right)^{-\frac{\beta}{a_1}} \cdot e^{-\beta t}\right) \tag{A.13}$$

B Derivation of optimal controls and of the dynamics for the resource-constrained R&D model

The dynamic optimization problem for this case is to maximize (5) with respect to (6), (7) and (19).

The control equality constraint of the type (19) is added to the Hamiltonian (A.1) to obtain the augmented Hamiltonian of the problem:

$$\mathcal{H}^R = \int_0^{n(t)} \Big[q(i,t) - \frac{1}{2}g(i,t)^2\Big] di - \frac{1}{2}u(t)^2 + \lambda_n(t) \cdot (\xi u(t)) +$$
$$\int_0^{n(t)} \lambda_q(i,t) \cdot (\psi(i)g(i,t) - \beta q(i,t)) + l(t) \cdot \Big(R - u(t) - \int_0^{n(t)} g(i,t)di\Big). \tag{B.1}$$

where $l(t)$ is the time-varying Lagrange multiplier for the resource constraint.

The FOCs for this problem read as:

$$\begin{aligned} u(t) &= \xi\lambda_n(t) - l(t); \\ g(i,t) &= \psi(i)\lambda_q(i,t) - l(t); \\ R - u(t) - &\int_0^{n(t)} g(i,t)di = 0. \end{aligned} \tag{B.2}$$

The latter yields $l(t)$ as:

$$l(t) = \frac{1}{3}\frac{2\psi - 2\psi(1-n(t))^{3/2} + 3(r+\beta)(\xi\lambda_n(t) - R)}{(r+\beta)(1+n(t))}, \tag{B.3}$$

which is a function of the variety expansion and of its co-state variable. The differential equations for all system variables, then, are obtained in the same way as above but with the substitution for controls of (B.2) with $l(t)$ defined in (B.3). The resulting productivities, $q(i,t)$, are functions of the variety of technologies, $n(t)$, and of the resource constraint R which enters optimal investments, (B.2), through the Lagrange multiplier. Because of this, the analytic solution cannot be obtained as in the exogenous R&D model version in Appendix A.

C Derivation of optimal controls and of the dynamics for the model with a dynamic R&D constraint

Now, we have

$$\dot{\lambda}_q^R(i,t) = (r+\beta)\lambda_q^R(i,t) + 1; \tag{C.1}$$

$$\lambda_q^R(i,t) = \frac{1}{r+\beta} = \lambda_q^R \, \forall i, \tag{C.2}$$

where the superscript R denotes the version with endogenous R&D. The technology dynamics is:

$$\dot{q}(i,t) = \psi(i)\Big(\psi(i)\frac{1}{r+\beta} - \frac{\xi\lambda_n^R(t) + \frac{1}{r+\beta}\int\limits_0^{n(t)}\psi(i)di}{1+n(t)} - \frac{R(t)}{1+n(t)}\Big) - \beta q(i,t). \tag{C.3}$$

The variety expansion dynamics is:

$$\dot{n}(t) = \xi^2\lambda_n^R(t)\Big(1 - \frac{1}{1+n(t)}\Big) + \xi R(t)\frac{1}{1+n(t)} - \xi\frac{\int\limits_0^{n(t)}\psi(i)di}{(1+n(t))(r+\beta)}. \tag{C.4}$$

The associated co-state dynamics is pretty complex:

$$\begin{aligned}\dot{\lambda}_n^R(t) = {} & -1/2\,\frac{(\lambda_n^R(t))^2\xi^2}{(1+n(t))^2} + \frac{\Big(-\int_0^{n(t)}\psi(i)di\,\xi + r(r+\beta)(n(t))^2\Big)}{(r+\beta)(1+n(t))^2} + \\ & + \frac{(\psi(n(t))\xi + 2\,r(r+\beta))\,n(t) + \psi(n(t))\xi + (r+R(t)\xi)(r+\beta)\lambda_n^R(t)}{(r+\beta)(1+n(t))^2} - \\ & - 1/2\,\frac{((r+\beta)R(t)+\psi(n(t)))(-2\int_0^{n(t)}\psi(i)di + (r+\beta)R(t) + \psi(n(t))(1+2\,n(t)))}{(1+n(t))^2(r+\beta)^2}\end{aligned} \tag{C.5}$$

but depends only on $n(t)$ and on $R(t)$. The evolution of each technology $q(i,t)$ is fully defined by the pair $n(t) - \lambda_n^R(t)$ which are defined by the level of R&D expenditures, $R(t)$.

With a constant growth rate of the variety, we obtain the aggregate productivity as a function of research expenditures only:

$$\frac{d}{dt}\left(\int\limits_0^{n(t)} q(i,t)di\right) \overset{q(n,t)=0}{=} \int\limits_0^{n(t)} \dot{q}(i,t)di = X - \beta \int\limits_0^{n(t)} q(i,t)di; \tag{C.6}$$

$$\bar{Q} = \int\limits_0^{\bar{n}(t)} q(i,t)di = \frac{X}{g_Q + \beta}; \tag{C.7}$$

$$X = \int\limits_0^{\bar{n}(t)} \left[\psi(i)\Big(\psi(i)\frac{1}{r+\beta} - \frac{\xi\bar{\lambda}_R(t) + \frac{1}{r+\beta}\int\limits_0^{\bar{n}(t)}\psi(i)di}{1+\bar{n}(t)} - \frac{R(t)}{1+\bar{n}(t)}\Big)\right] di \tag{C.8}$$

Now, the Hamiltonian for the optimization problem (33) is written as:

$$\mathcal{H}^G = \frac{[c(t)Y(t)]^{1-\gamma}}{1-\gamma} + \lambda_k(-\delta k(t) + [1 - c(t) - f_1(a(t)) - f_2(R(t))]Y(t)) +$$
$$+ \lambda_m(-\nu m(t) + (1 - a(t))e^E(t)Y(t)) + \lambda_\tau(-\lambda(m(t))\tau(t) + d(m(t))). \tag{C.9}$$

The FOC for $R(t)$ should be:

$$\frac{\partial \mathcal{H}^G}{\partial R} = \frac{\partial U}{\partial R} + \lambda_k\Big(-\frac{\partial f_2(R)}{\partial R}Y(R) + [1 - c - f_1(a) - f_2(R)]\frac{\partial Y(R)}{\partial R}\Big) +$$
$$+ \lambda_m\Big((1-a)e(n)\frac{\partial Y(R)}{\partial R}\Big); \tag{C.10}$$
$$\frac{\partial U}{\partial R} = c^{1-\gamma}Y(R)^{-\gamma}\frac{\partial Y}{\partial R}; \tag{C.11}$$
$$\frac{\partial Y(R)}{\partial R} = \frac{\partial X}{\partial R}\frac{(\frac{k}{n})^\alpha}{g_Q + \beta_R}; \tag{C.12}$$
$$\frac{\partial f_2(R)}{\partial R} = 0.05\frac{1}{(R-1)^2}; \tag{C.13}$$
$$\frac{\partial X}{\partial R} = \frac{1-n}{1+n}n\int\limits_0^n \psi^R(i)di. \tag{C.14}$$

With the same definition of the efficiency function, $\gamma_R(i) = \gamma_R\sqrt{1-i}$ this last derivative is:

$$\frac{\partial X}{\partial R} \overset{\psi^R(i)=\psi(1-i)^{1/2}}{=} \frac{1}{2}\cdot\psi^R\cdot\left(\frac{2-3n+n^2}{(1+n)n}\right) = D_1(\bar{n}) \tag{C.15}$$

that is constant over R. The total productivity is a linear function of R:

$$X(R) = Q_1(\bar{n}, g_n) + Q_2(\bar{n}, g_n)R, \tag{C.16}$$

and the expression for the gradient over the research expenditures will be:

$$\frac{\partial \mathcal{H}^G}{\partial R} = c^{1-\gamma}((Q_1(\bar{n}, g_n) + Q_2(\bar{n}, g_n)R)\left(\frac{k}{n}\right)^{\alpha})^{-\gamma} D_1(\bar{n})\frac{(\frac{k}{n})^{\alpha}}{g_Q+\beta} + \lambda_k \cdot$$
$$\cdot \Big(-\frac{0.05}{(R-1)^2}(Q_1(\bar{n}, g_n) + Q_2(\bar{n}, g_n)R)\left(\frac{k}{n}\right)^{\alpha} + [1 - c - f_1(a) - \frac{0.05R}{(1-R)}]D_1(\bar{n})\frac{\left(\frac{k}{n}\right)^{\alpha}}{g_Q+\beta}\Big)$$
$$+ \lambda_m\Big((1-a)e(n)D_1(\bar{n})\frac{\left(\frac{k}{n}\right)^{\alpha}}{g_Q+\beta}\Big) = 0. \tag{C.17}$$

The other two FOCs for consumption and for abatement are of the same type as in the previous version of the model. Inserting the expressions for consumption and abatement into the FOC for R, one can get an explicit expression for R as a function of the co-states and of the states only.

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