

Public debt and aggregate stability with endogenous growth and a state-dependent consumption tax

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Abstract

We analyze how different budgetary rules affect the stability of an economy in a basic endogenous growth model with public debt and a state-dependent consumption tax rate. We show that a discretionary policy implies that the government violates its inter-temporal budget constraint along a balanced growth path, whereas a balanced budget rule guarantees that the economy is stable. A rule based debt policy gives rise to stability if the reaction of the primary surplus to higher public debt is sufficiently large. Further, in case of a strongly regressive consumption tax rate over a certain range, multiple balanced growth paths may emerge. The main results can be generalized to hold for any endogenous growth model with infinitely lived households.

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Keywords: Budget rules, public debt, inter-temporal budget constraint, stability, endogenous growth

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1 Introduction

The current financial and public debt crisis drastically illustrates that excessive borrowing can endanger the stability of economies. Particularly in the euro zone, some economies got into severe troubles and national bankruptcies could only be avoided by creating the European Financial Stability Facility (EFSF) in 2010, replaced by the European Stability Mechanism (ESM) in 2012, that lends money to heavily indebted countries. This shows that the question of how public debt and deficit policies affect stability of market economies is not only of theoretical interest but has also practical relevance. With this contribution we intend to this line of research by analyzing the effects of different budgetary rules within an endogenous growth model where we allow for a state-dependent consumption tax rate.

It is well known that in theoretical growth models governments can essentially determine the dynamics of economies by its debt and deficit policies. For example, Schmidt-Grohe and Uribe (1997) demonstrate that a balanced government budget may lead to indeterminacy if the distortionary income tax rate is used to balance the government budget for a given level of public expenditures. The reason for that outcome is that there is a negative relation between aggregate activity and the income tax rate. If economic agents expect that the after-tax return rises they will increase their supply of production factors leading to a rise in the tax revenue. If public spending is fixed a balanced government budget, thus, leads to a lower tax rate such that the initial expectations will be fulfilled. As a consequence, there exist multiple equilibria, implying that the steady state is indeterminate.

However, that result heavily depends on the assumption that the income tax rate is set such that the government budget is balanced. Thus, Guo and Harrison (2004) can demonstrate that the result derived by Schmidt-Grohe and Uribe (1997) does not hold any longer when the income tax rate is fixed and public spending is adjusted in order to balance the budget of the government at each point in time. Then, the equilibrium is

unique and saddle point stable. Thus, it is not the fact that the government budget is balanced but rather that the adjustment is made through variations in the income tax rate that generates the outcome of Schmidt-Grohe and Uribe (1997). The contribution of Guo and Harrison (2004) is interesting because if one takes seriously the tax smoothing rule derived by Barro (1979), the question of arises why the government should balance its budget through adjustments in the tax rates when non-constant tax rates lead to an excess-burden that can be avoided otherwise.

Another possibility to avoid an indeterminate equilibrium path is to assume that it is the consumption tax that is adjusted in a way such that the government budget is balanced at each point in time, as demonstrated by Giannitsarou (2007). There, it is shown that the economy is always characterized by a unique saddle-point stable growth path. The latter result, however, is not robust. Thus, Nourry et al. (2013) present a growth model where the consumption tax rate is a function of the level of consumption and the tax rate be progressive or regressive. These authors demonstrate that a strongly regressive consumption tax rate may give rise to indeterminate equilibrium paths.

All of those contributions have in common that they neglect economic growth as well as the destabilizing effects arising from the accumulation of public debt. Looking at the current financial and public debt crisis in the real world, however, one realizes that the decisive aspect about public debt is that high deficits tend to make the debt to GDP ratio explosive, thus, jeopardizing stability of the whole economy. When the evolution of public debt and endogenous growth are explicitly taken into account, it turns out that a balanced government budget tends to stabilize the economy in the sense that the economy is saddle point stable and converges to the balanced growth path (see e.g. Greiner, 2008, 2011). In the case of permanent deficits, however, convergence to the balanced growth path and, hence, stability is guaranteed only if the government puts a sufficiently high weight on stabilizing the public debt to GDP ratio. Otherwise, the economy may become unstable and for a certain range of the parameter, determining the reaction of the government to

higher debt, the economy may be characterized by persistent endogenous growth cycles (see Greiner, 2007).

With this contribution we intend to generalize the results of the aforementioned contributions by analyzing a basic endogenous growth model with public debt and a state-dependent consumption tax rate. The consumption tax rate depends on consumption relative to capital may be progressive or regressive, as in Nourry et al. (2013). We can show that a discretionary policy where the government sets the primary surplus arbitrarily implies that the inter-temporal budget constraint of the government is violated. On the other hand, a balanced budget gives rise to a stable economy in the sense that convergence to the balanced growth path is always given. In case of a strong regressive consumption tax rate, multiple balanced growth paths may exist and a locally indeterminate equilibrium can arise. When the government adopts a rule based policy where the primary surplus rises as public debt increases, relative to GDP, stability of the economy is assured for a sufficiently high strong increase of the primary surplus. Again, for a strong regressive consumption tax rate multiple balanced growth paths and local indeterminacy may occur.

The rest of the paper is organized as follows. In the next section we present the structure of our endogenous growth model allowing for public debt and a state-dependent consumption tax rate. Section 3 analyzes stability of the model depending on different budgetary policies and section 4, finally, concludes.

2 The model with a state-dependent consumption tax rate

The structure of our model is basically the same as in Guo and Harrison (2004) except that we allow for a state-dependent tax rate as in Nourry et al. (2013). The economy consists of three sectors: A household sector which receives labour income and income

from its saving, a productive sector and the government. First, we describe the household sector.

2.1 The household sector

The household sector is represented by one household which maximizes the discounted stream of utility arising from per-capita consumption, $C(t)$, and from leisure, $L^m - L(t)$, over an infinite time horizon subject to its budget constraint, taking factor prices as given. L^m denotes the maximum available amount of time and $L(t)$ is the actual labor input. The maximization problem of the household can be written as¹

$$\max_{C,L} \int_0^{\infty} e^{-\rho t} (\ln C - L^{1+\gamma}/(1+\gamma)) dt, \quad (1)$$

subject to

$$(wL + rK + r_p B + \pi) = \dot{W} + (1 + \tau_c(c))C + \delta K, \quad (2)$$

with $\rho \in (0, 1)$ the household's rate of time preference, $\gamma \geq 0$ is the inverse of the elasticity of labour supply and $\delta \in (0, 1)$ is the depreciation rate of capital. The variable w denotes the wage rate and r is the return to capital and r_p is the interest rate on government bonds. $W \equiv B + K$ gives wealth which is equal to public debt, B , and capital, K , and π gives possible profits of the productive sector, the household takes as given in solving its optimization problem. Finally, $\tau_c(c) > 0$, with $c := C/K$, is the consumption tax rate that depends on consumption relative to capital and will be discussed in detail in the section describing the government sector.² Following Nourry et al. (2013) we assume that the household treats the consumption tax rate as a parameter, i.e. it does not take into account that it is a function of consumption relative to capital. The dot gives the derivative with respect to time.

¹From now on we omit the time argument t if no ambiguity arises.

²The function $\tau_c(c)$ is assumed to be continuous and C^1 .

A no-arbitrage condition requires that the return to capital equals the return to government bonds yielding $r_p = r - \delta$. Thus, the budget constraint of the household can be written as

$$\dot{W} = (wL + rW + \pi) - \delta W - C(1 + \tau_c(c)). \quad (3)$$

The current-value Hamiltonian for this optimization problem is written as

$$\mathcal{H} = \ln C - L^{1+\gamma}/(1 + \gamma) + \lambda((wL + rW + \pi) - \delta W - C(1 + \tau_c(c))), \quad (4)$$

where λ is the co-state variable or the shadow price of wealth.

Necessary optimality conditions are given by

$$C(1 + \tau_c(c)) = wL^{-\gamma} \quad (5)$$

$$\dot{C} = Cr - C(\rho + \delta) \quad (6)$$

If the transversality condition $\lim_{t \rightarrow \infty} e^{-\rho t} W/C = 0$ holds, which is fulfilled for a time path on which assets grow at the same rate as consumption, the necessary conditions are also sufficient.

2.2 The productive sector

The productive sector is represented by one firm which behaves competitively and which maximizes static profits. The production function of the firm is given by,

$$Y = K^{1-\alpha} \bar{K}^\xi L^\beta, \quad (7)$$

with $(1 - \alpha) \in (0, 1)$ the capital share, $\beta \in (0, 1)$ the labour share and $(1 - \alpha) + \beta \leq 1$. Y is output and \bar{K} represents the average economy-wide level of capital and we assume constant returns to capital in the economy, that is $(1 - \alpha) + \xi = 1$.

Using $(1 - \alpha) + \xi = 1$ and that $K = \bar{K}$ in equilibrium, profit maximization gives

$$r = (1 - \alpha)L^\beta \quad (8)$$

$$w = \beta L^{\beta-1} K \quad (9)$$

2.3 The Government

The government in our economy receives tax revenues from income taxation and has revenues from issuing government bonds. As concerns public spending, C_p , we again follow Guo and Harrison (2004) and Nourry et al. (2013) and assume that it is a mere waste of resources that is neither productive nor yields utility for the household. The main reason for that assumption is that we intend to focus on the effects of public debt policies per se so that we neglect possible distortions resulting from public spending.

The accounting identity describing the accumulation of public debt is given by:

$$\dot{B} = r_p B - S, \quad (10)$$

where S is the government surplus exclusive of net interest payments that is given by $S = \tau_c C - C_p$. As mentioned above, the consumption tax rate is a function that depends on consumption relative to capital. For growing economies it is more appropriate to define the tax base relative to capital or output. Otherwise, the tax rate would be permanently growing which would cause analytical problems and that would not be realistic either.

When the consumption tax rate is exogenously determined, it is a constant parameter, $\tau'_c = 0$, and we have the usual case typically analyzed in the economics literature. When the consumption tax rate rises as consumption relative to capital increases, i.e. if $\tau'_c > 0$ holds, we say that the consumption tax rate is progressive and it is regressive in case the tax rate declines as consumption relative to capital increases, i.e. for $\tau'_c < 0$.³ Further, we distinguish between a weakly regressive consumption tax and a strongly regressive one. The consumption tax is weakly regressive if the marginal tax revenue relative to capital, $T_c = C\tau_c/K$, declines by less than one as consumption relative to capital rises, i.e. if $-1 < (\partial T_c / \partial c) < 0$ holds, and the consumption tax is strongly regressive if the marginal tax revenue declines by more than one, i.e. when $(\partial T_c / \partial c) < -1$. Or, defining

³A progressive (regressive) tax implies that the tax rate declines (rises) with a higher investment share and, thus, with a higher growth rate.

$\eta := (\partial T_c / \partial c)(c/T_c)$ as the elasticity of the total consumption tax revenue relative to capital, the consumption tax rate is weakly (strongly) regressive if the elasticity of the total tax revenue relative to capital is larger (smaller) than the negative of the inverse of the consumption tax rate, that is when $\eta > (<) -1/\tau_c$ holds. This implies that the elasticity of consumption multiplied by one plus the tax rate, relative to capital, is positive (negative) in case of a weakly (strongly) regressive consumption tax rate. It will turn out that in some cases this distinction is decisive as regards the existence and stability of balanced growth paths.

The inter-temporal budget constraint of the government is fulfilled if

$$\lim_{t \rightarrow \infty} e^{-\int_0^t r_p(\mu) d\mu} B(t) = 0 \quad (11)$$

holds, which is the no-Ponzi game condition.

As concerns the behaviour of the government we consider three budgetary rules: First, a discretionary fiscal policy where the government arbitrarily sets public spending, given its tax revenues. Second, a balanced budget rule, where the government sets public spending such that the public deficit equals zero implying that public debt remains constant. And, third, a rule based policy where the primary surplus relative to GDP is a positive function of the debt to GDP ratio and of an exogenously given term. The economic rationale behind that policy is that the government has to run primary surpluses in the future, when it incurs deficits today, in order to avoid playing a Ponzi game. Further, there is very strong empirical evidence that governments indeed follow this rule, see e.g. Bohn (1998) for the USA and Greiner et al. (2007) for European economies. Formally, for the rule based policy the primary surplus relative to GDP is given by the following expression:

$$\frac{S}{Y} = \phi + \psi \left(\frac{B}{Y} \right), \quad \text{with } \phi \in \mathbb{R}, \psi \in \mathbb{R}_{++} \quad (12)$$

The coefficient ψ will be called the reaction coefficient since it determines how strong the primary surplus rises as public debt increases, relative to GDP respectively.

3 Analysis of the model

Before we analyze our model we give the definition of an equilibrium and of a balanced growth path. An equilibrium allocation for our economy is defined as follows.

Definition 1 *An equilibrium is a sequence of variables $\{C(t), K(t), B(t)\}_{t=0}^{\infty}$ and a sequence of prices $\{w(t), r(t)\}_{t=0}^{\infty}$ such that, given prices and fiscal rules, the firm maximizes profits, the household solves (1) subject to (2) and the budget constraint of the government (10) is fulfilled.*

In definition 2 we define a balanced growth path.

Definition 2 *A balanced growth path (BGP) is a path such that the economy is in equilibrium and such that consumption and capital grow at the same strictly positive constant growth rate, that is $\dot{C}/C = \dot{K}/K = \dot{C}_p/C_p = g$, $g > 0$, $g = \text{constant}$, and either (i) $\dot{B}/B = g$ or (ii) $\dot{B} = 0$.*

Definition 2 shows that we consider two different situations. Case (i) describes a situation which is characterized by public deficits where the government debt grows at the same rate as all other endogenous variables in the long-run. Case (ii) gives the balanced budget rule. This rule guarantees zero public deficits and implies that the level of public debt is constant such that the debt to GDP ratio converges to zero in the long-run in case of ongoing growth.

To study our model, we note that it is completely described by the following differential equations,

$$\frac{\dot{C}}{C} = (1 - \alpha)\omega (C/K)^{-\beta/(1-\beta+\gamma)} h(c)^{-\beta/(1-\beta+\gamma)} - (\rho + \delta), \quad C(0) > 0, \quad (13)$$

$$\frac{\dot{K}}{K} = \omega (C/K)^{-\beta/(1-\beta+\gamma)} h(c)^{-\beta/(1-\beta+\gamma)} - (C/K) - (C_p/K) - \delta, \quad K_0 > 0, \quad (14)$$

$$\frac{\dot{B}}{B} = (1 - \alpha)\omega (C/K)^{-\beta/(1-\beta+\gamma)} h(c)^{-\beta/(1-\beta+\gamma)} - \frac{S}{B} - \delta, \quad B_0 > 0, \quad (15)$$

with $\omega := \beta^{\beta/(1-\beta+\gamma)}$ and where we used $r_p = r - \delta$. The initial conditions with respect to capital and public debt are assumed to be given while consumption can be chosen by the household at time $t = 0$. The exact specification of the growth rates of physical capital and of public debt depends on whether the government pursues a discretionary policy, the balanced budget rule or the rule based policy.

To analyze our economy around a BGP we use $c = C/K$ and define the variable $b := B/K$. Differentiating these variables with respect to time leads to a two dimensional system of differential equations given by

$$\dot{c} = c \left(\frac{\dot{C}}{C} - \frac{\dot{K}}{K} \right), \quad (16)$$

$$\dot{b} = b \left(\frac{\dot{B}}{B} - \frac{\dot{K}}{K} \right). \quad (17)$$

A solution of $\dot{c} = \dot{b} = 0$ with respect to c, b gives a BGP for our model and the corresponding ratios b^*, c^* on the BGP.⁴

In the next subsection, we first study our model assuming that the government pursues a discretionary fiscal policy.

3.1 Discretionary fiscal policy

Proposition 1 below demonstrates that the government violates its inter-temporal budget constraint if it adopts a discretionary policy along a BGP.

Proposition 1 *If the government follows a discretionary fiscal policy, its inter-temporal budget constraint is violated along a BGP.*

Proof: See the appendix.

This proposition shows that any fiscal policy that does not react to the evolution of public debt, by following some sort of budgetary rule for example, implies that the inter-temporal budget constraint of the government is violated. That results holds because in a

⁴The * denotes BGP values.

growing economy the debt to GDP ratio becomes explosive when the government sets the primary surplus arbitrarily. A permanently rising debt to GDP ratio, however, implies that the present value of public debt does not converge to zero. The latter holds because a continuously increasing debt to GDP ratio would require a permanently rising primary surplus relative to GDP, so that the inter-temporal budget constraint can hold, which, however, is not possible. This is not possible since the primary surplus must be financed out of GDP such that the ratio of the primary surplus relative to GDP is necessarily bounded from above.

Next, we analyze the balanced budget rule.

3.2 The balanced budget rule

To model the balanced budget rule, we set $\phi = 0$ and $\psi = r - \delta = r_p$ in equation (12). From equation (10) one immediately realizes that this implies $\dot{B} = 0$, that is a balanced budget and, thus, a constant level of public debt. Proposition 2 gives results with respect to the existence and stability of a BGP for our economy assuming a balanced government budget and a progressive consumption tax rate.

Proposition 2 *There exists a unique saddle point stable BGP if the government runs a balanced budget and the consumption tax rate is constant, progressive or weakly regressive.*

Proof: See the appendix.

This proposition demonstrates that the economy is characterized by a unique saddle point stable balanced growth path when the government runs a balanced budget and with a progressive consumption tax rate.⁵ Proposition 2 also holds for a regressive consumption tax rate as long as the elasticity of the total tax revenue relative to capital is not lower than the negative of the inverse of the consumption tax rate, i.e. as long as $\eta > -1/\tau_c$

⁵Strictly speaking, we can only show that there exists a rest point for (16)-(17) but not that it implies a positive g . However, for a sufficiently small ρ and δ the balanced growth rate will always be positive.

holds. For a strongly regressive tax scheme over a certain interval $c \in [\underline{c}, \bar{c}]$, multiple BGPs may arise. Proposition 3 gives the result.

Proposition 3 *Assume that the government runs a balanced budget and the consumption tax rate is constant, progressive or weakly regressive for $0 < c < \underline{c}$ and $\bar{c} < c < \infty$ and strongly regressive for $\underline{c} \leq c \leq \bar{c}$. Then, multiple (three) BGPs can exist. The first and the third BGP are locally saddle point stable and the second BGP is locally indeterminate.*

Proof: See the appendix.

Proposition 3 demonstrates that in the case of a strongly regressive consumption tax rate over a certain range of the consumption share, such that the total tax revenue declines as consumption rises, multiple BGPs may arise. The BGP that is associated with the lowest consumption share, that implies the highest balanced growth rate, and the BGP that goes along with the highest consumption share, that yields the lowest long-run growth rate, are locally saddle point stable. The medium BGP, that gives a balanced growth rate lower than the first BGP but higher than the third BGP, is locally indeterminate since the Jacobian evaluated at that rest point has two negative eigenvalues, implying that this rest point is a stable node. Implicitly, proposition 3 also shows that there exists no BGP if the consumption tax rate is strongly regressive everywhere. There must be at least some range for which the consumption tax rate is constant, progressive or weakly regressive.

The emergence of indeterminacy can be explained as follows. When agents expect that the return to labour input rises as they supply more labour, the higher labour supply will lead to higher consumption relative to capital and to a lower consumption tax rate if the consumption tax rate is strongly regressive. That holds because with a strongly regressive consumption tax rate, labour supply and consumption relative to capital are positively correlated in optimum, which is seen from equations (5) and (9). Thus, there exist self-fulfilling expectations in the case of a strongly regressive consumption tax. When the consumption tax is weakly regressive, labour supply and consumption relative to capital

are negatively correlated, just as for a constant and for a progressive tax rate, so that the expectation of a higher labour supply leading to higher after-tax consumption will not be fulfilled.

In the next subsection, we analyze our model assuming that the government runs permanent deficits but sticks to the budget rule specified in equation (12).

3.3 The rule-based policy

When the government sticks to the rule specified in equation (12), the differential equations describing the economy around a BGP are given by,

$$\dot{c} = c \left(ch(c) - (\alpha + \phi)\omega (ch(c))^{-\beta/(1-\beta+\gamma)} - \rho - \psi b \right), \quad (18)$$

$$\dot{b} = b \left(ch(c) - (\alpha + \phi)\omega (ch(c))^{-\beta/(1-\beta+\gamma)} - \psi(1 + b) - \phi\omega (ch(c))^{-\beta/(1-\beta+\gamma)} / b \right) \quad (19)$$

In that case, a constant or progressive consumption tax rate gives rise to unique BGP that is saddle point stable if the reaction of the primary surplus to higher debt, modelled by the parameter ψ , is sufficiently large. Proposition 4 gives the result.

Proposition 4 *Assume that the government sticks to the primary surplus rule (12) and the consumption tax rate is constant or progressive. Then, there exists a unique BGP that is saddle point stable for $\psi > r_p - g$ and unstable for $\psi < r_p - g$.*

Proof: See the appendix.

Proposition 4 demonstrates that the BGP is unique for a constant or progressive consumption tax rate, as for the case of balanced budget. However, in contrast to the balanced budget case, the BGP is stable if and only if the reaction coefficient that determines how strong the primary surplus increases as public debt rises, relative to GDP respectively, is sufficiently high, that is if it exceeds the difference between the interest rate on public debt and the growth rate. That result makes sense from an economic point of view. When the debt to GDP ratio rises, this implies that the interest payments rise

by r_p . At the same time, the primary surplus rises at the rate g since the primary surplus grows at the same rate as all other variables on the BGP. Since $r_p > g$ holds on a BGP, the debt to GDP ratio will become explosive unless the government raises the primary surplus at least by that difference, which implies in our model that the public spending will be reduced.

Proposition 5 *Assume that government sticks to the primary surplus rule (12) and the consumption tax rate is constant or progressive for $0 < c < \underline{c}$ and $\bar{c} < c < \infty$ and strongly regressive for $\underline{c} \leq c \leq \bar{c}$. Then, multiple (three) BGPs can exist. The first and the third BGP are locally saddle point stable for $\psi > r_p - g$ and and unstable for $\psi < r_p - g$. The second BGP is locally saddle point stable for $\psi < r_p - g$ and indeterminate for $\psi > r_p - g$ and $\alpha + \phi > 0$.*

Proof: See the appendix.

Proposition 5 shows that multiple BGPs can arise if the consumption tax rate is strongly regressive over a certain range, as in the case of the balanced budget scenario. As concerns stability, proposition 5 demonstrates that a high value for the reaction coefficient ψ tends to stabilize the economy, in the sense that it raises the number of negative eigenvalues. For example, the Jacobian evaluated at second BGP has one negative eigenvalue for $\psi < r_p - g$, implying saddle point stability, and two negative eigenvalues for $\psi > r_p - g$, implying that it is a stable focus.⁶ In the latter case, there exists a continuum of initial values for c such that convergence to the BGP is given.

4 Conclusion

When the medium- to long-term effects of public debt with respect to stability are analyzed, the inter-temporal budget constraint and the debt to GDP ratio play the decisive

⁶The condition $\alpha + \phi > 0$ is not strict and indeterminacy is expected to hold also without it.

role. Hence, theoretical models dealing with this research question should explicitly take into account the differential equation describing the evolution of public debt as well as ongoing economic growth.

In this paper we have analyzed how different budgetary rules affect the stability of endogenously growing economies with a state-dependent consumption tax rate. We could show that a discretionary policy implies that the government violates its inter-temporal budget constraint along a balanced growth path. It should be noted that this result holds for any endogenous growth model with infinitely lived households, in particular also for models with a more general utility function than the logarithmic function such as functions that are characterized by a constant relative or absolute risk aversion.⁷

Further, the balanced budget rule raises the stability of the economy in the sense that it increases the number of negative real eigenvalues of the Jacobian by one. This is due to the fact that the debt to GDP ratio, or relative to capital, converges to zero asymptotically under the balanced budget rule. The question of whether the economy converges to the balanced growth path, then, depends on the structure of the endogenous growth model under consideration. For our basic endogenous growth model, convergence to the balanced growth path is always given and locally indeterminate equilibrium paths may occur, depending on the number of balanced growth paths which, for its part, depends on the consumption tax system.

For the rule based policy, we have seen that in the case of a unique BGP the model is stable if the reaction of the primary surplus to variations in public debt is sufficiently strong. Only if the government puts a high importance on controlling public debt such that the debt to GDP ratio does not become explosive, convergence to a balanced growth path can be assured. Generally speaking, higher values of the reaction coefficient raise

⁷In an endogenous growth model, a logarithmic utility function over consumption is the only formulation that is consistent with a stationary labour supply if labour and consumption are separable and when the production function is Cobb-Douglas, cf. Benhabib and Farmer (1994).

stability in the sense that they increase the number of negative eigenvalues which implies that indeterminate equilibrium paths may arise.

We conjecture that the latter result holds in general, as long as the transformation of the original system can be made by simply analyzing ratios where all variables are divided by the physical capital stock. However, the stability properties may be independent of the reaction coefficient when a more complicate transformation is necessary to obtain a dynamic system whose rest point determines the long-run balanced growth path (see for example Greiner, 2013). Hence, this result is not as general as the ones with respect to the discretionary policy and with respect to the balanced government budget that hold independent of the way how the original system is transformed.

The emergence of multiple balanced growth paths crucially depends on the way how the consumption tax rate depends on the consumption share. Only in the case of a strongly regressive consumption tax rate over a certain range, multiple balanced growth paths may occur. Then, indeterminate equilibrium paths can emerge, depending on the public debt and deficit policy of the government. The economic mechanism behind that outcome is that a strongly regressive consumption tax rate implies a positive correlation between optimal labour supply and consumption, that can give rise to self-fulfilling expectations and locally indeterminate equilibrium paths. If the consumption tax rate is progressive or exogenously given, the long-run balanced growth path is always unique and either unstable or stable, where indeterminate equilibrium paths are excluded in the latter case.

5 Appendix

Proof of proposition 1

On a BGP, the debt to private capital ratio evolves according to

$$\dot{b} = b r_p - \frac{S(0)}{K_0} - b \left(\frac{\dot{K}}{K} \right) = b \rho - \frac{S(0)}{K_0},$$

where we used $r_p = r - \delta$ and $\dot{K}/K = r - (\rho + \delta)$ on a BGP. Since the primary surplus is not a function of public debt, neither the growth rate of capital nor that of consumption depend on public debt. Further, taking into account that r and S/K are constant on a BGP, the solution to the differential equation is given by,

$$b(t) = \frac{S(0)/K_0}{\rho} + e^{\rho t} \left(b_0 - \frac{S(0)/K_0}{\rho} \right).$$

Noting that $\dot{K}/K = g = r - (\rho + \delta)$ holds on a BGP, we get for the level of public debt

$$B(t) = K_0 e^{gt} (C_1 + C_2 e^{\rho t}),$$

with $C_1 = (S(0)/K_0)/\rho$ and $C_2 = b_0 - (S(0)/K_0)/\rho$.

Thus, for $g > 0$ the limit of the present value of public debt is given by,

$$\lim_{t \rightarrow \infty} e^{(-r+\delta)t} B(t) = K_0 C_2.$$

This shows that the limit of the present value of public debt is positive (negative) if the initial primary surplus falls short (exceeds) the initial level of public debt multiplied by the difference between the interest rate on public debt and the balanced growth rate, i.e. if $S(0) < (>) B_0 (r_p - g)$ holds. If the limit of the present value is negative, the private sector is a borrower and fails to pay back its (gross) debt. The case $S(0) = B_0 (r_p - g)$ has Lebesgue measure zero and, therefore, is not generic. If the government sets $S(0)$ intentionally such that the latter equality holds, the fiscal policy cannot be termed a discretionary one. In endogenous growth models with an exogenously given labour supply and a utility function with a constant absolute or relative risk aversion, $1/\sigma$, the balanced growth rate is given by $g = (r_p - \rho)/\sigma >$. Proceeding as above shows that the present value of public debt either converges to a finite value, for $r_p > g$, or it diverges to infinity, for $r_p < g$. \square

Proof of proposition 2

To prove proposition 2, we first note that the balanced budget rule implies $b^* = 0$ since public debt is constant while the capital stock grows over time. Further, $\phi = 0$ and $\psi = r - \delta$ hold in this case. Using this, the equation \dot{c}/c can be written as

$$\dot{c}/c = (f_1(c) - \alpha\omega f_1(c)^{-\beta/(1-\beta+\gamma)} - \rho) \quad (A.1)$$

with $f_1(c) := ch(c) = c(1 + \tau_c(c))$. It is easily seen that for $\tau'_c(c) \geq 0$ we get,

$$\lim_{c \rightarrow 0} (\dot{c}/c) = -\infty, \quad \lim_{c \rightarrow \infty} (\dot{c}/c) = +\infty, \quad \partial(\dot{c}/c)/\partial c > 0.$$

This proves the existence of a unique c^* which solves $\dot{c}/c = 0$.

The Jacobian matrix for (16)-(17) is given by

$$J = \begin{bmatrix} a_{11} & \partial\dot{c}/\partial b \\ 0 & -g \end{bmatrix},$$

with c and b evaluated at the rest point $\{c^*, 0\}$ and $\partial\dot{K}/\partial K = g$. The term a_{11} is

$$a_{11} = c(1 + \alpha\omega(\beta/(1-\beta+\gamma))f_1(c)^{-\beta/(1-\beta+\gamma)})f'_1(c).$$

The eigenvalues of the Jacobian are given by $\{a_{11}, -g\}$. A progressive tax rate implies $f'_1(c) > 0$ and, thus, $a_{11} > 0$ so that there is one positive and one negative eigenvalue. \square

Proof of proposition 3

A BGP is again obtained for a rest point of \dot{c}/c (equation (A.1) in the proof of proposition 2). For $f'_1 < 0 \leftrightarrow \eta < -1/\tau_c$, the curve \dot{c}/c is a decreasing function in c for $c \in [\underline{c}, \bar{c}]$, whereas it is increasing in c for $f'_1 > 0 \leftrightarrow \eta > -1/\tau_c$. Therefore, \dot{c}/c first rises, then declines for $c \in [\underline{c}, \bar{c}]$ and monotonically rises for $c > \bar{c}$ such that three BGPs can exist. The eigenvalues of the Jacobian are given by $\{a_{11}, -g\}$ (see the proof of proposition 2). Since $f'_1 > 0$ holds for the first and for the third BGP, a_{11} is positive for those two BGPs, whereas a_{11} is negative at the second BGP. \square

Proof of proposition 4

To prove this proposition we set $\dot{C}/C = \dot{B}/B$, which must hold on a BGP, giving

$$\phi \omega f_1(c)^{-\beta/(1-\beta+\gamma)} b^{-1} = \rho - \psi \quad (A.2)$$

with $f_1(c) := ch(c)$. Substituting this relation in \dot{b}/b gives,

$$\dot{b}/b = f_1(c) - \omega(f_1(c))^{-\beta/(1-\beta+\gamma)} (\alpha + \rho \phi / (\rho - \psi)) - \rho$$

From (A.2) we know that $b^* > 0$ implies that ϕ and $\rho - \psi$ have the same sign so that $\phi / (\rho - \psi) > 0$ holds. With this, it is easily seen that for $f_1'(c) > 0$, the following relations hold,

$$\lim_{c \rightarrow 0} (\dot{b}/b) = -\infty, \quad \lim_{c \rightarrow \infty} (\dot{b}/b) = +\infty, \quad \partial(\dot{b}/b)/\partial c > 0.$$

This proves the existence of a unique c^* which solves $\dot{b}/b = 0$.

To study stability, we compute the Jacobian matrix evaluated at the rest point of (18)-(19). The Jacobian is given by

$$J = \begin{bmatrix} a_{11} & -\psi c \\ a_{21} & \phi \omega f_1(c)^{-\beta/(1-\beta+\gamma)} b^{-1} - \psi b \end{bmatrix},$$

with c and b evaluated at the rest point $\{c^*, b^*\}$ and a_{11} and a_{21} given by

$$\begin{aligned} a_{11} &= c f_1'(c) \left(1 + (\phi + \alpha) \omega (\beta / (1 - \beta + \gamma)) f_1(c)^{-1-\beta/(1-\beta+\gamma)} \right) \\ a_{21} &= b f_1'(c) \left(1 + (\phi(1 + b^{-1}) + \alpha) \omega (\beta / (1 - \beta + \gamma)) f_1(c)^{-1-\beta/(1-\beta+\gamma)} \right) \end{aligned}$$

Using $\psi + \phi \omega f_1(c)^{-\beta/(1-\beta+\gamma)} b^{-1} = \rho$ from (A.2), the determinant of the Jacobian matrix can be computed as

$$\begin{aligned} \det J &= c f_1'(c) \left((\rho - \psi) + (\rho - \psi) \omega (\beta / (1 - \beta + \gamma)) f_1(c)^{-1-\beta/(1-\beta+\gamma)} \alpha \right) + \\ &\quad \psi \phi \omega (\beta / (1 - \beta + \gamma)) f_1(c)^{-1-\beta/(1-\beta+\gamma)} \end{aligned}$$

Using again (A.2) we can rewrite the determinant as

$$\begin{aligned}\det J &= cf_1'(c)(\rho - \psi) \left(1 + \omega(\beta/(1 - \beta + \gamma))f_1(c)^{-1-\beta/(1-\beta+\gamma)}\alpha\right) + \\ &cf_1'(c)(\rho - \psi) \rho b(\beta/(1 - \beta + \gamma))f_1(c)^{-1}\end{aligned}\quad (A.3)$$

Thus, $f_1'(c) > 0$ and $b > 0$ implies $\det J < (>)0 \leftrightarrow \psi > (<)\rho$.

For $\psi < \rho$ the determinant is positive. To show that the BGP is unstable we have to compute the trace of the Jacobian, $\text{tr } J$, which is given by,

$$\begin{aligned}\text{tr } J &= cf_1'(c) \left(1 + (\beta/(1 - \beta + \gamma))\omega f_1(c)^{-1-\beta/(1-\beta+\gamma)}(\phi + \alpha)\right) - \psi b \\ &+ \phi \omega b^{-1} f_1(c)^{-\beta/(1-\beta+\gamma)}\end{aligned}\quad (A.4)$$

To see that $\text{tr } J$ is positive we first note that a positive value of b^* implies $\phi > 0$ for $\psi < \rho$. Further, we use $f_1'(c) = ch'(c) + h(c)$ and from $\dot{c}/c = 0$ we get $ch(c) - \psi b = \rho + f_1(c)^{-\beta/(1-\beta+\gamma)}\omega(\phi + \alpha) > 0$, so that the trace of the Jacobian can be written as

$$\text{tr } J = ch'(c)(c + C_3) + h(c)C_3 + \phi \omega b^{-1} f_1(c)^{-\beta/(1-\beta+\gamma)} + \rho + f_1(c)^{-\beta/(1-\beta+\gamma)}\omega(\phi + \alpha),$$

with $C_3 = c(\beta/(1 - \beta + \gamma))\omega f_1(c)^{-1-\beta/(1-\beta+\gamma)}(\phi + \alpha)$. Thus, the trace and the determinant are both positive, so that the BGP is unstable for $\psi < \rho$. Noting that $g = -\rho + r_p$ proves the proposition. \square

Proof of proposition 5

A BGP is again obtained for a rest point of \dot{b}/b , as in the proof of proposition 4. For $f_1' < 0$, the curve \dot{b}/b is a decreasing function in c for $c \in [\underline{c}, \bar{c}]$, whereas it is increasing in c for $f_1' > 0$. Therefore, \dot{b}/b first rises, then declines for $c \in [\underline{c}, \bar{c}]$ and monotonically rises for $c > \bar{c}$ such that three BGPs can exist. The determinant and the trace of the Jacobian are given by (A.3) and (A.4) (in the proof of proposition 4). Thus, it is immediately seen that the stability properties are the same as in the proof of proposition 4 for $f_1'(c) > 0$.

With $f_1'(c) < 0$, the determinant is negative for $\psi < \rho$ and positive for $\psi > \rho$. Further, with $\psi > \rho$, a positive b^* implies $\phi < 0$ so that the trace is negative for $\alpha + \phi > -f_1(c)^{1+\beta/(1-\beta+\gamma)}\omega^{-1}\beta/(1-\beta+\gamma)$ (the last condition is sufficient but not necessary for the negativity of the trace). \square

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