Endogenous growth and structural change through vertical and horizontal innovations

Anton Bondarev   **Alfred Greiner**

Endogenous growth and structural change through vertical and horizontal innovations

Anton Bondarev   Alfred Greiner*

Abstract

   This paper combines horizontal and vertical innovations to build an endogenous growth model that allows for structural change. Older technologies are continuously replaced by newer ones due to creative destruction and new technologies appear as a result of horizontal innovations and as a result of consumers' preferences for variety. We assume fixed operational costs for the manufacturing sector and an endogenously determined price of the patent for each new technology. The duration of a patent is not limited but every industry is profitable only for a certain period of time, thus making the effective time of existence of the technology endogenous and finite. We demonstrate that such an economy exhibits constant growth rates that are proportional to the average productivity growth, despite the ongoing disappearance of older technologies from the industry.

---

*Department of Business Administration and Economics, Bielefeld University, Universitätstraße 25, 33615 Bielefeld, Germany, e-mail: abondarev@wiwi.uni-bielefeld.de, agreiner@wiwi.uni-bielefeld.de

1 Introduction

This paper proposes a new way of thinking about the role of technical change in economic growth. We combine horizontal and vertical innovations in the spirit of Peretto and Connolly (2007) but we do not assume them to be a primary source of growth. Rather, innovations serve as a generator of endogenous structural change in the economy, thus modelling creative destruction. It is this structural change, which is described as a continuous emergence and disappearance of sectors in the economy, which generates sustained growth in our model.

The endogenous growth literature has a long tradition of identifying technical change as the primary source of permanent economic growth, dating back to the seminal papers by Romer (1990a) and Aghion and Howitt (1992). In those models, economic growth stems from either horizontal or vertical innovations (creative destruction). In the more recent contribution by Peretto and Connolly (2007), the attempt to unify both types of technical change has been made. In that paper, however, horizontal innovations are limited due to the presence of fixed costs and growth occurs mainly because of a permanent development of already existing technologies (productivity growth). In our paper, we make a similar attempt and extend the analysis to include patents for new technologies in the same way as in Romer (1990a). This gives rise to continuous sustained horizontal innovations and to productivity growth of the existing sectors. The competitive nature of the R&D sector leads to the gradual disappearance of older technologies from the economy. This effect is new in the economics literature and allows to discuss the determinants of endogenous structural change in the economy.

The literature on structural change is not very extensive. The majority of models discuss the reallocation of productive factors from some sectors of the economy to others, but the number of sectors is assumed to be constant, like in Meckl (2002), Huntington (2010), Laitner (2000). However, the rapid technical change leads not only to the overall productivity growth but, rather, to the structural transformation of the economy destroying older sectors and creating newer ones, as broadly discussed already by Schumpeter (1942) and formally treated in Boucekkine et al. (2005). It turns out

that, in order to model such a dynamic transformation of the economy, one has to take into account the formation of patent prices as well as R&D behaviour in the spirit of Nordhaus (1967).

In our model, monopolistic competition in the manufacturing sector together with free entry in technology affects the patent prices in such a way, that excessive monopolistic profits are not used not for capital accumulation, but rather for the development of newer technologies. The overall structure of the R&D sector in our model resembles venture capital firms: new technology is invented by the R&D firms with the intention of its further development up to the point when it becomes productive and the only stimulus for such a development is the revenue resulting from patent payments from the manufacturing sector. The patent itself is of unlimited duration, but the endogenous emergence of new technologies limits the time of its use. Thus, the infinite duration of patents in this setting does not create obstacles for technological advances, because the technology itself becomes outdated at some point. The overall life-cycle of each technology resembles the cycles being mentioned already in Albernathy and Utterback (1985) but at an economy-wide level and in a more formal setting.

The setup of the R&D sector resembles the structure used in optimal control models dealing with an endogenously determined domain of heterogeneity, developed mainly in Belyakov et al. (2011) and it is close to the homogeneous version of the multi-product monopolist in Bondarev (2012). Horizontal and vertical innovations are interrelated, with profitability of vertical innovations being the stimulus for inventing new technologies and the scope of horizontal innovations limiting the speed of vertical innovations. The structure of the R&D process is the main interesting point in our model, which is very simplistic and standard in all other respects from the structural point of view.

Hence, the main goal of this paper is to build a formal model of an economy that endogenously grows through structural change. New technologies arrive at some constant speed because of the symmetry of all new technologies. This symmetry leads to equal profitability of newer technologies and to equal incentives to innovate for all incumbent R&D firms. At the same time, all technologies are developed through optimal investment plans that are also identical across the range of technologies. As a result, the

productivity of the economy grows proportionally to the accumulated capital, making the model close to traditional endogenous growth models. However, the productivity of new sectors grows faster than that of old ones, since the abundance of accumulated capital is higher at the time when a new technology is invented and is to be developed. In order to achieve this effect, we assume that there is some depreciation of productivities, which reflects the fact that any technology needs a certain maintenance to be of use. The faster growth of newer technologies attracts capital into their development being drawn from older technologies development, thus creating structural change. In the end, the profits of the manufacturing sector are dwindling for older technologies because labour is reallocated towards newer sectors and these older sectors disappear from the economy. However, the total range of sectors in the economy is constant due to the linear expansion of the variety. This greatly simplifies the analysis and allows an analytical solution of the model.

The rest of the paper is structured as follows. Section 2 introduces the structure of the model. Section 3 provides the results and the analysis. Section 4 concludes with some discussion of possible future extensions. Some of the more tedious mathematical proofs are given in the Appendix.

2 The Model Setup and Preliminary Results

There are three types of economic agents: Households supply labour and capital and consume manufactured goods. Goods producers sell manufactured goods, employ labour and buy technologies. Finally, the R&D sector sells technologies and employs capital. There is a continuum of goods indexed by i with an endogenous spectrum. This spectrum can be extended by horizontal innovations. Each good i is provided by a single monopolistic producer.

The model has two distinctive features. The first is an endogenously determined patent length. Firms exploiting a technology have to pay the R&D sector for the invention, but only for a certain endogenously determined time period. Since the R&D sector cannot earn any more income with a specific technology after its economic

expiration, it stops developing it. The last and most important feature is the presence of fixed operating costs for production. Due to this fixed cost element and to the fact that the technology stops to be further developed upon its economic expiration, older sectors (technologies) disappear from the economy as soon as the demand for the sector's product decreases to the point where revenues are not sufficient to cover fixed costs. It should be noted that for the obsolescence of sectors both finite patents and fixed operating costs are necessary.

We start with the description of the households, followed by the manufacturing sector and by R&D activities.

2.1 Households

Households are modeled in a similar way as in Peretto and Connolly (2007). The amount of labor is constant and distributed across the range of existing final sectors:

$$L = \int_{N_{min}}^{N_{max}} L(i)di;$$

$$N_{min} < N_{max} < N(t) \tag{1}$$

Here L is the total labour in the economy (equal to population), $L(i)$ is the employment in sector i, $N(t)$ is the number of technologies (range) being invented up to time t, N_{max} is the range of manufacturing sectors with positive operating profit (any new technology does not immediately yield positive productivity) and $N_{min}(t)$ is the range of sectors, which disappeared from the economy up to time t. The range of developed sectors is growing over time reflecting the expansion in the variety of products. However, the range of existing sectors, given by $N_{max}(t) - N_{min}(t)$, may grow decrease or stay constant in time, depending on the characteristics of the process of variety expansion of technologies, $\dot{N}$. The labor employed by an individual sector is not constant. It is redistributed from older sectors to newer ones.

The objective function of the household is

$$J^H = \int_0^\infty \mathbf{e}^{-\rho t} U(C) dt \ . \tag{2}$$

with $U(C) = \ln C$ being the utility function. The representative household is maximizing utility from consumption C over a continuum of differentiated products from existing sectors

$$C = \left[\int_{N_{min}}^{N_{max}} C_i^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \ , \tag{3}$$

with ε the elasticity of substitution between goods.

The flow budget constraint of the household is

$$\dot{K} = rK + wL - E \ , \tag{4}$$

where K is capital and r is the interest rate. We assume zero depreciation rate of capital for simplicity. Positive depreciation will not essentially change the results of the paper. The wage rate is assumed to be the numeraire, $w = 1$, and labour L is constant. Consumption expenditures E are defined as

$$E = \int_{N_{min}}^{N_{max}} P_i C_i di \ . \tag{5}$$

along the same range of existing sectors.

The accumulation of capital comes from the difference between consumption expenditures and income of the household, which is the sum of capital and labor income.

Consumption of the individual good i is given by (see Appendix A),

$$C_i = \frac{E \cdot P_i^{-\varepsilon}}{\int_{N_{min}}^{N_{max}} P_j^{1-\varepsilon} dj} \ . \tag{6}$$

The standard Euler equation implies that the optimal growth rate for expenditure is given by

$$\frac{\dot{E}}{E} = r - \rho \ . \tag{7}$$

2.2 Goods Producers

Goods producers employ labor and buy technology from the R&D sector. With these inputs they produce the goods they sell to the consumers. The output of good i is given by:

$$Y_i = A_i^\alpha L_i \ . \tag{8}$$

The profit of firm i is

$$\Pi_i = P_i Y_i - L_i - \Psi \ , \tag{9}$$

where Ψ is a fixed operating cost.

The only use for output is consumption, so that $C_i = Y_i$. Firm i, therefore, sets its price to (see Appendix A),

$$P_i = \frac{\varepsilon}{\varepsilon - 1} A_i^{-\alpha} \ . \tag{10}$$

Inserting (6) and (10) into (8) yields labor demand as,

$$L_i^D = \frac{\epsilon - 1}{\epsilon} \cdot \frac{E \cdot A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj} \ . \tag{11}$$

Thus, labour employed in sector i is a function of the relative productivity of labour in sector i. However, since we have fixed operating costs the profit, π, is nonnegative not immediately from the time of invention of technology i, but after some time. At the same time after the product of the given sector i becomes outdated, the demand for it will decrease to the point where no positive profits can be made. At this point, labour in the sector becomes zero again. Thus, the fully specified labour sectoral demand has

a piecewise form:

$$
L^D(i) = \begin{cases} 0, \, t < \tau_{max}(i), \tau_{max}(i) : \pi_i = 0, \dot{\pi}_i > 0; \\[2mm] \dfrac{\epsilon-1}{\epsilon} E \dfrac{A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj}, \, \tau_{max}(i) < t \leq \tau_{min}(i), \tau_{min}(i) : \pi_i = 0, \dot{\pi}_i < 0; \\[2mm] 0, t > \tau_{min}(i). \end{cases}
$$

Here, and further throughout the paper we denote

- $\tau_{min} = N_{min}^{-1}(i)$, the time when product (technology) i becomes outdated and the profit of manufacturing decreases below zero;

- $\tau_{max} = N_{max}^{-1}(i)$, the time when product (technology) i becomes profitable and the manufacturing sector starts the production of a positive amounts;

- $\tau_0 = N^{-1}(i)$, the time when product (technology) i is invented through horizontal innovations.

Technology is acquired by the good producers in the form of a patent. Pricing for this patent follows Nordhaus (1967), Romer (1990b) and Grimaud and Rouge (2004): The price for the patent equals the total value of profits which can be derived from it. Formally, the price is defined as,

$$
p_A(i) \stackrel{def}{=} \int\limits_{\tau_{max}}^{\tau_{min}} e^{-r(t-\tau_0)} \pi_i dt. \tag{12}
$$

The date at which patent i starts, τ_{max}, is endogenously determined through the process of horizontal innovations and the effective duration of the patent is endogenously determined through the demand for the manufactured product i, τ_{min}.

Since positive profits may be extracted by the manufacturing firms only during a limited period of time, the price of the patent is also defined only over a limited interval. After the patent has expired, the technology is freely available for production for everyone, in principle. However, since technologies become outdated, older technologies are not used in production despite their zero price.

A first result we can derive is that the patent price is independent of time. It only depends on the ratio of the level of productivity in sector i, which accumulates over the entire lifespan of the patent and total productivity of the economy.

Proposition 1 *The price of the patent $p_A(i)$ is not a function of time.*

A proof of this can be found in the Appendix.

2.3 The R&D Sector

The general structure of R&D sector follows the lines of the paper Bondarev (2012) with homogeneous technologies. There are two types of R&D: Productivity-improving (vertical) innovations and variety-expanding (horizontal) innovations.

2.3.1 Horizontal innovations

The process of creation of new technologies follows the setup of Peretto and Connolly (2007). As in that paper, we assume that new technologies appear due to knowledge creation that is governed by private initiatives of R&D firms. New technologies are created through investments, u, chosen by the firm, into this kind of innovations:

$$\dot{N} = \delta \, u(t), \; \delta > 0 \tag{13}$$

The incentive for horizontal innovations is the potential profit from selling the new technology to manufacturing firms. Since horizontal innovations have zero productivity at the time when they are invented,[1] the value of horizontal R&D consists in expected future profits from vertical innovations:

$$V_N = \max_{u(\cdot)} \int\limits_0^\infty \mathrm{e}^{-rt} \left(\delta \pi^R(i,t)|_{i=N} u(t) - \frac{1}{2} u^2(t) \right) dt \tag{14}$$

where the profit of developing technology i is just the price of the patent, (12), minus R&D costs to be determined later on. The amount of optimal investments into horizontal innovations is given by Proposition 2.

---

[1] In this sense, we differ between the invention of a new technology and its economic use.

Proposition 2 *With the value of the horizontal innovations given by (14), the optimal investments are proportional to the expected profits from the development of the next invented product,*

$$u^*(t) = \delta \pi^R(i,t)|_{i=N}. \tag{15}$$

The proof amounts to constructing the standard Hamilton-Jacobi-Bellman (HJB) equation for this problem. This can be found in the Appendix. It is equivalent to the HJB problem for homogeneous products in Bondarev (2012).

In this way, the horizontal expansion will be a function of the profits resulting from the development of the next-to-be-invented product:

$$N(t) = \delta^2 \pi^R(i,t)|_{i=N} t + N_0. \tag{16}$$

At the same time, both horizontal and vertical R&D are competitive in using capital accumulated by households. Thus, the capital market clearing condition has to hold:

$$K^S(t) = K^D(t), \ \forall t; \tag{17}$$

which gives, together with equation (15),

$$\int_{N_{min}(t)}^{N(t)} g(i,t)di + \delta \pi^R(i,t)|_{i=N} = K^S(t) \tag{18}$$

This can be used to obtain total investments into productivities as:

$$\int_{N_{min}(t)}^{N(t)} g(i,t)di = G(t);$$

$$G(t) = K(t) - u(t). \tag{19}$$

where $g(i,t)$ are the investments into the improvement of productivity in sector i and $G(t)$ is the aggregate capital available for vertical innovations.

2.3.2 Vertical innovations

Productivity-improving innovations (vertical innovations) are given as a continuum of products, which have zero productivity upon their invention. This productivity may be developed through specific investments for every product. Profits from R&D result from selling patents to manufacturing firms for each new technology A_i (the incentive to create new technologies) and investments into the development of each new technology (vertical innovations) are financed from this patent payment. Costs of R&D are the costs of the development of the productivity through technology-specific investments g_i. Thus, the profit of the R&D firm developing technology i is:

$$\pi^R(i,t) = p_A(i) - \frac{1}{2} \int_{\tau_0}^{\tau_{min}} e^{-r(t-\tau_0)} g^2(i,t) dt, \tag{20}$$

with investments going into the increase of productivity:

$$\dot{A}(i,t) = \gamma g(i,t) - \beta A(i,t). \tag{21}$$

where $\gamma > 0$ is the efficiency of investments into the productivity increase (equal for all sectors) and $\beta > 0$ is the cost of supporting the productivity on the current level.

At any time, there exist $N(t) - N_{min}(t)$ range of new technologies and, thus, exactly this range of firms in vertical R&D. It should be noted that the range of firms in the manufacturing sector is different and is given by $N_{max}(t) - N_{min}(t)$. Hence, the aggregate vertical R&D is obtained by solving a dynamic problem of optimal investment plans subject to the availability of resources (for the moment we assume that the price of capital, r, is the same and constant in the economy, a proof of this will be seen later).

The aggregate problem for vertical R&D reads:

$$V = \max_{g(\cdot)} \int_0^\infty e^{-rt} \int_{N_{min}(t)}^{N(t)} p_A(i) - \frac{1}{2}g^2(i,t)didt; \tag{22}$$

$$s.t. \tag{23}$$

$$\dot{A}(i,t) = \gamma g(i,t) - \beta A(i,t), \ \forall i \in [N_{min}, N] \tag{24}$$

$$\int_{N_{min}(t)}^{N(t)} g(i,t)di = G(t). \tag{25}$$

For those technologies, which are outside of the operating range, $i > N_{min}$, there is no development since the price of the patent, p_A, is paid only for the development of the technology during its operational time, i.e. for $t \in [\tau_{max}(i), \tau_{min}(i)]$.

Using the Maximum Principle, we derive optimal investments for each R&D firm as a function of the shadow costs of investments, $\psi(i,t)$, this last being a function of price of the patent:

$$\dot{\psi}(i,t) = r\psi(i,t) - \frac{\partial p_A(i)}{\partial A(i)},$$

$$g^*(i,t) = \gamma\psi(i,t) - \frac{\int_{N_{min}(t)}^{N(t)} \gamma\psi(i,t)di - G(t)}{N(t) - N_{min}(t)}. \tag{26}$$

We can now derive Proposition 3 which will help us to obtain the symmetric solution of the model:

Proposition 3 *The effect of a rise in productivity with respect to the price of the patent is the same for all technologies,*

$$\frac{\partial p_A(i)}{\partial A_i} = c, \ \forall i \tag{27}$$

The proof can be found in the Appendix.

Using the fact that derivative of the patent price is not a function of time (Proposition 1) and Proposition 3, it may be demonstrated that the shadow costs of investments

12

are the same across all existing technologies:

$$\psi^*(i,t) = \frac{c}{r+\beta} \tag{28}$$

Then, investments into productivities of all the existing technologies are symmetric:

$$g^*(t) = \frac{G(t)}{N(t) - N_{min}(t)}, \tag{29}$$

as well as dynamics of the productivity levels:

$$\dot{A}(i,t) = \frac{G(t)}{N(t) - N_{min}(t)} - \beta A(i,t). \tag{30}$$

2.4 Market Clearing

Final goods and capital markets

Now we can demonstrate that total expenditures per capita do not grow over time and are constant. For this, we consider the expression for expenditures:

$$E(t) = \int_{N_{min}}^{N_{max}} P(i,t)C(i,t)di = \int_{N_{min}}^{N_{max}} P(i,t)Y(i,t)di = \frac{\epsilon}{\epsilon-1} \int_{N_{min}}^{N_{max}} L(i,t)di = \frac{\epsilon}{\epsilon-1}L, \tag{31}$$

since technology cancels out from product prices. Total labour is assumed to be constant so that the total expenditures are also constant. From this we define the interest rate:

$$\frac{\dot{E}}{E} = r - \rho = 0 \rightarrow r = \rho, \tag{32}$$

which should be expected in equilibrium. Thus, the optimal evolution of capital can also be found by solving (4) for K with $\dot{E} = 0$. Further, using $E = L\epsilon/(\epsilon-1)$ and noting that the wage rate is the numeraire, we get the evolution of capital as,

$$\dot{K} = rK - \frac{1}{\epsilon-1}L, \tag{33}$$

which can be solved yielding,

$$K(t) = e^{rt} \left(K_0 - \frac{1}{(\epsilon - 1)r} L \right) + \frac{1}{r(\epsilon - 1)} L. \tag{34}$$

This shows that capital accumulation is positive as long as the initial assets of households are sufficiently large:

$$K_0 > \frac{1}{\epsilon - 1} \frac{1}{r} L. \tag{35}$$

Labour market clearing

The condition for labour market clearing is obtained from the fact that labour in the economy is constant:

$$\int_{N_{min}}^{N_{max}} L^D(i,t)di = L = L \int_{N_{min}}^{N_{max}} \frac{A_i^{-\alpha(1-\epsilon)}}{\int_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)}dj} di;$$

$$\int_{N_{min}}^{N_{max}} \frac{A_i^{-\alpha(1-\epsilon)}}{\int_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)}dj} di = \frac{\int_{N_{min}}^{N_{max}} A_i^{-\alpha(1-\epsilon)}di}{\int_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)}dj} = 1. \tag{36}$$

But this last condition is automatically satisfied, hence the labour market is cleared.

3   Solution of the Model and its Steady State

To finally solve for vertical innovations as well as for the range of existing sectors of the economy we need the previously obtained results:

- Expenditures are constant;

- The evolution of capital is given by (34);

- Horizontal innovations are linear, given by (15).

3.1 Variety Expansion

Now we can derive the range of the outdated sectors and proceed with the solution for the steady state. It should be noted that eq. (30) can be explicitly solved only after $N_{min}(t)$ has been obtained. This is the range of outdated sectors at time t. This quantity is obtained from the zero profit condition of the manufacturing sector with this index:

$$N_{min}(t): \frac{1}{\epsilon - 1} L \frac{A_{N_{min}}^{\alpha(\epsilon-1)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{\alpha(\epsilon-1)} dj} - \Psi = 0; \qquad (37)$$

The determination of $N_{max}(t)$ follows the same procedure with the only difference that this is the number of sectors which enter the market:

$$N_{max}(t): \frac{1}{\epsilon - 1} L \frac{A_{N_{max}}^{\alpha(\epsilon-1)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{\alpha(\epsilon-1)} dj} - \Psi = 0; \qquad (38)$$

With this we can derive the next Proposition.

Proposition 4 *The productivity of the oldest operational sector, $A_{N_{min}}$, is equal to the productivity of the newest operational sector, $A_{N_{max}}$, at time when the first is leaving the economy and the latter is entering its operational phase:*

$$A_{N_{min}} = \left((\Psi/L)(\epsilon - 1) \int\limits_{N_{min}}^{N_{max}} A_j^{\alpha(\epsilon-1)} dj \right)^{1/\alpha(\epsilon-1)} = A_{N_{max}}. \qquad (39)$$

At the same time, the productivity of each sector grows within its operational phase,

$$A_i(\tau_{min}(i)) > A_i(\tau_{max}(i)) \qquad (40)$$

For any sector i these two relations are fulfilled at times $\tau_{min}(i)$ and $\tau_{max}(i)$, respectively, determining the time of the disappearance of the sector and the time of its appearance in the economy. At both points in time, the profit of the sector is zero

but the overall accumulated productivity is different. At $t = \tau_{max}$ the profit of sector i grows, $\dot{\pi}(i) > 0$, while at $t = \tau_{min}$ the profit decreases, $\dot{\pi}(i) < 0$. This makes the difference between N_{max} and N_{min}.

It can be shown, that the sign of the derivative of the profit function depends on the following relation,

$$\dot{\pi}(i) \lesseqqgtr 0 \Leftrightarrow \frac{\dot{A}(i)}{A(i)} - \left(\int_{N_{min}}^{N_{max}} \frac{\dot{A}(j)}{A(j)} dj \right) \lesseqqgtr 0. \tag{41}$$

Note that this implies $\dot{N}_{max} - \dot{N}_{min} = 0$. At the same time, the profit evolves differently for these two technologies:

$$\dot{\pi}(N_{min}) < 0, \dot{\pi}(N_{max}) > 0. \tag{42}$$

We first compute the derivative of the profit for an arbitrary technology:

$$\dot{\pi}(i) = \frac{A(i,t)^{\alpha(\epsilon-1)}}{\int_{N_{min}}^{N_{max}} A_j^{\alpha(\epsilon-1)} dj} \left(\alpha(\epsilon-1) \left(\frac{\dot{A}(i,t)}{A(i,t)} - \int_{N_{min}}^{N_{max}} \frac{\dot{A}(j,t)}{A(j,t)} dj \right) \right) +$$

$$\frac{A(i,t)^{\alpha(\epsilon-1)}}{\int_{N_{min}}^{N_{max}} A_j^{\alpha(\epsilon-1)} dj} \left(\frac{A(N_{max},t)\dot{N}_{max} - A(N_{min},t)\dot{N}_{min}}{\int_{N_{min}}^{N_{max}} A_j^{\alpha(\epsilon-1)} dj} \right) \tag{43}$$

Making use of (39), we get

$$\dot{\pi}(i) = \frac{A(i,t)^{\alpha(\epsilon-1)}}{\int_{N_{min}}^{N_{max}} A_j^{\alpha(\epsilon-1)} dj} \left(\alpha(\epsilon-1) \left(\frac{\dot{A}(i,t)}{A(i,t)} - \int_{N_{min}}^{N_{max}} \frac{\dot{A}(j,t)}{A(j,t)} dj \right) + (\Psi/L)(\epsilon-1)(\dot{N}_{max} - \dot{N}_{min}) \right)$$

$$\tag{44}$$

Noting that the maximum profit for any sector i is reached at the point when $\dot{\pi}(i) = 0$,

16

it follows that the growth of N_{min} and N_{max} is given by:

$$\dot{\pi}(i) = 0 \Leftrightarrow \left(\frac{\dot{A}(i,t)}{A(i,t)} - \int_{N_{min}}^{N_{max}} \frac{\dot{A}(j,t)}{A(j,t)} dj \right) = (1/\alpha)(\Psi/L)(\dot{N}_{max} - \dot{N}_{min}) \qquad (45)$$

The bracket in the lefthand side has to be equal to zero because the productivity growth rate of sector i and the average productivity growth rate in the economy are identical. Since all the technologies are symmetric, except for the time of their invention, it is straightforward to state that the maximum profit for the industry is reached at the point when its productivity grows at the average rate of the economy. Otherwise, there will still be room for improvements of the technology or the technology is already outdated. These considerations demonstrate that $\dot{N}_{max} - \dot{N}_{min} = 0$ holds. This proves the conjectured equation (41) and is stated in the following Proposition.

Proposition 5 *New sectors emerge at the same speed as older sectors disappear from the economy, $\dot{N}_{max} - \dot{N}_{min} = 0$.*

Moreover, it shows that the range of existing sectors in the economy is constant if $\dot{N}_{max} = \dot{N}_{min} = \dot{N}$. However, that result has yet to be proven.

Indeed, for the economy to be dynamically consistent it is necessary that older sectors do not disappear faster, than newer sectors appear. This is given by the condition $\dot{N}_{max} - \dot{N}_{min} = 0$. At the same time, for all the productivities to grow at the same rate it is necessary that the range $N - N_{min}$ stays constant. Otherwise, condition (39) would be violated since newer technologies would grow faster or slower than older ones if the range is not constant. It should be noted that this would be not the case with heterogeneous technologies. However, since in this paper we treat technologies as of equal efficiency the constancy of their range is necessary for balanced growth. Thus, we have

Proposition 6 *The variety expansion of technologies is linear and equals the rate of structural change, making the existing range of sectors of the economy constant,*

$$\dot{N}_{max} = \dot{N}_{min} = \dot{N} = \delta \pi^R. \qquad (46)$$
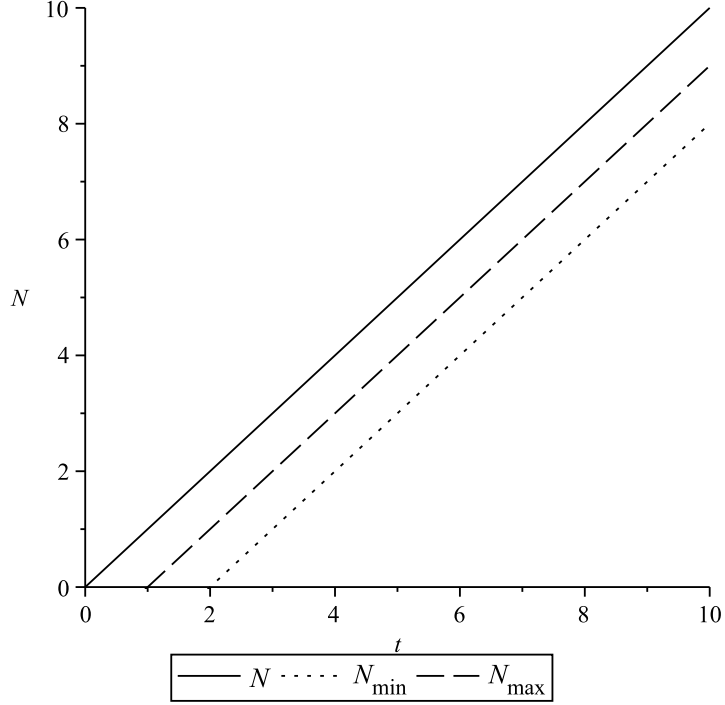
Figure 1 illustrates the result of Proposition 6.

Figure 1: Linear variety expansion

3.2   Productivity Growth

With the results of the last subsection, we can now determine the time $\tau_{max}(i)$ when sector i enters the market. The latter is obtained from the following two conditions:

$$\tau_{max}(i): \frac{1}{\epsilon-1}L\frac{A(i,\tau_{max})^{\alpha(\epsilon-1)}}{\int\limits_{N_{min}(\tau_{max})}^{N_{max}(\tau_{max})} A(j,\tau_{max})^{\alpha(\epsilon-1)}dj} - \Psi = 0; \tag{47}$$

$$\frac{\dot{A}(i,t)}{A(i,t)} - \int\limits_{N_{min}}^{N_{max}} \frac{\dot{A}(j,t)}{A(j,t)}dj > 0. \tag{48}$$

18

while the time of disappearance of the sector is determined by the pair:

$$\tau_{min}(i) : \frac{1}{\epsilon - 1} L \frac{A(i, \tau_{min})^{\alpha(\epsilon-1)}}{\int_{N_{min}(\tau_{min})}^{N_{max}(\tau_{min})} A(j, \tau_{min})^{\alpha(\epsilon-1)} dj} - \Psi = 0; \tag{49}$$

$$\frac{\dot{A}(i,t)}{A(i,t)} - \int_{N_{min}}^{N_{max}} \frac{\dot{A}(j,t)}{A(j,t)} dj < 0. \tag{50}$$

Comparing these two conditions (for the same technology i) one may see that the growth of each technology *within the time of operation*, $t \in [\tau_{max}(i), \tau_{min}(i)]$, is the same:

$$\frac{A(i, \tau_{max}(i))^{\alpha(\epsilon-1)}}{A(i, \tau_{min}(i))^{\alpha(\epsilon-1)}} = \frac{\int_{N_{min}(\tau_{max}(i))}^{N_{max}(\tau_{max}(i))} A(j, \tau_{max}(i))^{\alpha(\epsilon-1)} dj}{\int_{N_{min}(\tau_{min}(i))}^{N_{max}(\tau_{min}(i))} A(j, \tau_{min}(i))^{\alpha(\epsilon-1)} dj}, \quad \forall i \in [N_{min}, N_{max}] \tag{51}$$

This means the productivity growth of each technology is monotonic and proportional to all the others since the technology becomes profitable. At the same time, equation (39) holds. Together with equation (51) this implies that the average productivity grows over time and the growth rates for all operating technologies are identical. Thus, one gets positive output growth despite a constant range of operating sectors. This is known as the magistrale property of the dynamic system: from the time $\tau_{max}(i)$ onwards each individual technology growth is independent of its time of invention. Proposition 7 states this result.

Proposition 7 *The productivities of all technologies grow at the same average speed during the time period of operational activity of the technology,*

$$\dot{A}(i) = \dot{Q} = \frac{G}{N - N_{min}} - \beta Q, \quad \forall i \in [N_{min}, N_{max}], \forall t \in [\tau_{max}(i), \tau_{min}(i)] \tag{52}$$

with Q denoting the average level of productivity. Since $N - N_{min} = const$ and we know G, the evolution of each productivity within the operating time-range can be obtained. Figure 2 illustrates the evolution of the technologies.
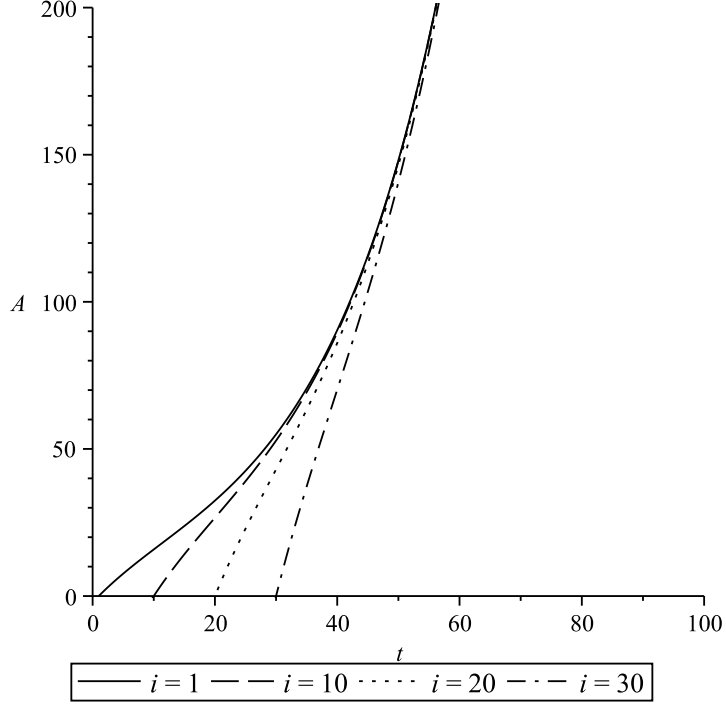
19

Figure 2: Convergence of productivities to the magistrale

3.3 Output Growth

In order to obtain the output growth rate we first note that aggregate output is given by:

$$Y = \int\limits_{N_{min}}^{N_{max}} \left(\frac{A(i,t)^{\alpha\epsilon}}{\int\limits_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)}} \right) di = \frac{\int\limits_{N_{min}}^{N_{max}} A(i,t)^{\alpha\epsilon} di}{\int\limits_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)} dj} \tag{53}$$

The growth of output, then, is:

$$\dot{Y} = \frac{d/dt\left(\int\limits_{N_{min}}^{N_{max}} A(i,t)^{\alpha\epsilon} di\right)\int\limits_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)} dj - \int\limits_{N_{min}}^{N_{max}} A(i,t)^{\alpha\epsilon} di(d/dt)\left(\int\limits_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)} dj\right)}{\left(\int\limits_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)} dj\right)^2}$$

$$\tag{54}$$

With this result, we can state our last Proposition:

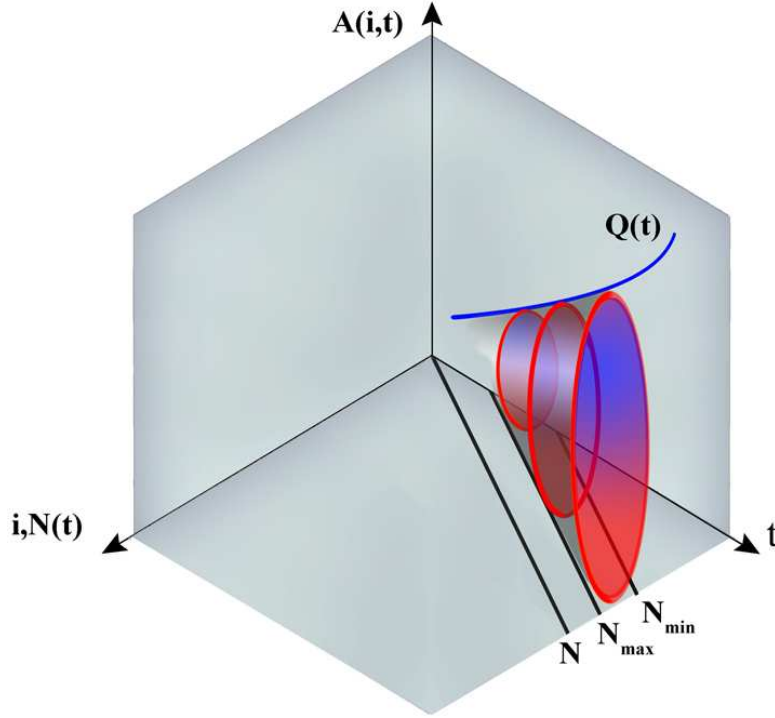**Proposition 8** *The growth rate of the economy is constant and proportional to the*

20

Figure 3: Evolution of the economy with Q denoting the average productivity.

growth rate of the productivities of operational technologies times the range of existing sectors,

$$\frac{\dot{Y}}{Y} = \alpha \left(N_{max} - N_{min}\right) \frac{\dot{Q}}{Q}. \tag{55}$$

The proof is obtained by direct computation and can be found in the Appendix.

This shows that the economy with a constant range of changing technologies exhibits a positive growth rate that is proportional to the growth rate of the average productivities of operating technologies, Q. The latter is always positive and proportional to the capital used to generate vertical innovations, G. Thus, the growth rate of the economy is constant for a constant range of sectors. The overall evolution of this economy is illustrated by the 3-d representation in Figure 3.

4 Conclusion

In this paper we have constructed an endogenous growth model allowing for endogenous structural change resulting from technologies becoming obsolete. The latter results

from fixed operational costs, limited labour supply and, mainly, from the dynamic interrelationship between vertical and horizontal innovations.

The overall R&D investments are driven by profits from selling patents to final producers in the manufacturing sector. These patents transform monopolistic profits of the manufacturing sector into resources being used for innovative activity in the spirit of Romer (1990a). However, the inclusion of productivity growth for all new technologies allows to account for the endogenous process of sectors and associated technologies becoming obsolete. This result is possible due to the fact that the price of the patent for a technology equals total additional profits of the manufacturing sector, and not just the increase in productivity as in Peretto and Connolly (2007). The evolution of the economy is proportional to the productivity growth in the same way as in the aforementioned paper, but we are able to model structural change as an endogenous phenomenon. The key assumption for sustainable growth in our framework are potentially unlimited horizontal innovations and the fact that all technologies are symmetric and homogeneous. It would be of interest to extend the model to allow for heterogeneous technologies. This would allow for non-constant growth rates which would depend on the structure of the space of ideas and on the speed of horizontal innovations.

References

Aghion, P. and Howitt, P. (1992). A model of growth through creative destruction. *Econometrica*, 60(2):323–351.

Albernathy, W. and Utterback, J. (1985). Mapping the winds of creative destruction. *Research Policy*, 14(1).

Belyakov, A., Tsachev, T., and Veliov, V. (2011). Optimal control of heterogeneous systems with endogenous domain of heterogeneity. *Applied Mathematics and Optimization*, 64:287–311.

Bondarev, A. (2012). The long-run dynamics of product and process innovations for a

multi-product monopolist. *Economics of Innovation and New Technology*, 21(8):775–799.

Boucekkine, R., del Rio, F., and Licandro, O. (2005). Obsolescence and modernization in the growth process. *Journal of Development Economics*, 77(1):153–171.

Grimaud, A. and Rouge, L. (2004). Polluting non-renewable resources, tradeable permits and endogenous growth. *International journal of global environmental issues*, 4(1):38–57.

Huntington, H. (2010). Structural change and us energy use: Recent patterns. *The Energy journal*, 31(3):25–39.

Laitner, J. (2000). Structural change and economic growth. *Review of Economic Studies*, 67(3):545–561.

Meckl, J. (2002). Structural change and generalized balanced growth. *Journal of Economics*, 77(3):241–266.

Nordhaus, W. D. (1967). The optimal life of a patent. *Cowles Foundation discussion paper #241,*.

Peretto, P. F. and Connolly, M. (2007). The manhattan metaphor. *Journal of Economic Growth*, 12(4):329–350.

Romer, P. (1990a). Endogenous technological change. *The Journal of Political Economy*, 98(5):S71–S102.

Romer, P. M. (1990b). Endogenous technological change. *Journal of Political Economy*, 98(5):pp. S71–S102.

Schumpeter, J. (1942). *Capitalism, Socialism and Democracy.* Harper & Row, New York.

Appendices

A   Households and firms optimality conditions

A.1   Derivations for the household

The derivation of equation (6): The Lagrangian for the household is

$$L = \left[\int_{N_{min}}^{N_{max}} C_i^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} - \lambda \left(\int_{N_{min}}^{N_{max}} P_i C_i di - rK + \dot{K} + W \right) . \tag{A.1}$$

The first order condition (FOC) for the consumption good i is

$$C_i^{-\frac{1}{\varepsilon}} C^{-\frac{1}{\varepsilon}} = \lambda P_i . \tag{A.2}$$

Taking the FOC for i and for j and substituting yields

$$C_i = C_j \left(\frac{P_i}{P_j} \right)^{-\varepsilon} . \tag{A.3}$$

Substituting this back into the equation for expenditures, equation (5), yields

$$C_j \left(\frac{1}{P_j} \right)^{-\varepsilon} \int_{N_{min}}^{N_{max}} P_i^{1-\varepsilon} di = E , \tag{A.4}$$

which can be rearranged to yield expression (6).

A.2   Derivations for the manufacturing sector

The derivation of equation (10): The output by an individual firm Y_i equals the consumption of that good C_i so that we can insert equation (6) into the profit function:

$$\begin{aligned}
\pi_i &= P_i Y_i - L_i - \Psi \tag{A.5} \\
&= P_i Y_i - Y_i A_i^{-\alpha} - \Psi \tag{A.6} \\
&= P_i E \frac{P_i^{-\varepsilon}}{\int_{N_{min}}^{N_{max}} P_j^{1-\varepsilon} dj} - E \frac{P_i^{-\varepsilon}}{\int_{N_{min}}^{N_{max}} P_j^{1-\varepsilon} dj} A_i^{-\alpha} - \Psi . \tag{A.7}
\end{aligned}$$

Maximizing this with respect to the price yields

$$\frac{\partial \pi_i}{\partial P_i} = \frac{E}{\int_{N_{min}}^{N_{max}} P_j^{1-\varepsilon} dj}(1-\varepsilon)P_i^{-\varepsilon} - \frac{E}{\int_{N_{min}}^{N_{max}} P_j^{1-\varepsilon} dj}P_i^{-\varepsilon-1}(-\varepsilon)A_i^{-\alpha} = 0 \ . \tag{A.8}$$

Thus, the price is

$$P_i = \frac{\varepsilon}{\varepsilon - 1}A_i^{-\alpha} \ . \tag{A.9}$$

B   Proofs of propositions

B.1   Proof for Proposition 1

1. Using equation (12) we can write the price of a patent i as

$$p_A(i) = \int_{N^{-1}(i)}^{\infty} e^{-r(t-N^{-1}(i))}\pi_i dt \tag{B.1}$$

   where $N^{-1}(i)$ is the time when technology i is invented.

2. Denote $\tau_0(i) = N^{-1}(i), \tau_{max}(i) = N_{max}^{-1}(i), \tau_{min}(i) = N_{min}^{-1}(i)$, as the time of the invention of a technology, of it becoming profitable and of it going out of production, respectively.

3. Note that $N \geq N_{max} \geq N_{min}$ implies $\tau_0(i) \leq \tau_{max}(i) \leq \tau_{min}(i)$ as long as $N(t)$ is a non-decreasing function. This last is true as long as $u(t) \geq 0$, which is required by the formulation of the horizontal innovations problem, (13).

4. The profit of a manufacturing firm in sector i is given by (9). Substituting for prices, labour and technology in it, one gets:

$$\begin{aligned} \pi_i &= P_i Y_i - L_i - \Psi = \frac{\epsilon}{\epsilon - 1}A_i^{-\alpha}A_i^{\alpha}L_i - L_i - \Psi = \left(\frac{\epsilon}{\epsilon - 1} - 1\right)L_i - \Psi = \\ &\frac{1}{\epsilon}E\frac{A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)}dj} - \Psi. \end{aligned} \tag{B.2}$$

5. The profit is nonnegative only within the interval $t \in [\tau_{max}(i), \tau_{min}(i)]$ such that the patent price is defined also for that interval.

6. Inserting this into the patent price one gets:

$$p_A(i) = \int\limits_{\tau_{max}(i)}^{\tau_{min}(i)} e^{-r(t-\tau_0(i))} \left(\frac{1}{\epsilon} E \frac{A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj} - \Psi \right) dt; \tag{B.3}$$

7. Formally, taking the definite integral amounts to the difference between two values of the antiderivative:

$$p_A(i) = \mathbf{F}|_{t=\tau_{min}(i)} \left(e^{-r(t-\tau_0(i))} \left(\frac{1}{\epsilon} E \frac{A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj} - \Psi \right) \right) -$$

$$- \mathbf{F}|_{t=\tau_{max}(i)} \left(e^{-r(t-\tau_0(i))} \left(\frac{1}{\epsilon} E \frac{A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj} - \Psi \right) \right) \tag{B.4}$$

8. Without explicit computation of this expressions it is straightforward to see that the resulting patent price is not a function of time, but a difference of two values of such a function at fixed points in time:

$$p_A(i) = \mathbf{F}(i, \tau_{min}(i), \tau_{max}(i)) \neq f(t). \tag{B.5}$$

.

B.2 Proof of Proposition 2

The HJB equation for the problem given by Eqs. (14), (13) is:

$$rV = \max_{u(\cdot)} \left\{ \delta \pi^R(i,t)|_{i=N} u(t) - \frac{1}{2} u^2(t) + \frac{\partial V}{\partial N} \delta u(t) \right\} \tag{B.6}$$

Using FOC we have

$$u^* = \delta \pi^R(i,t)|_{i=N} + \delta \frac{\partial V}{\partial N},$$

$$\text{(B.7)}$$

Substituting back into the HJB equation, we find that it can be satisfied only for $V = const.$, as long as $\pi^R(i,t)|_{i=N}$ is constant.

This last has to be constant, since there is a free entry condition for vertical innovations: if some of the technologies yielded higher profits, all of the resources would go into the development of only those more profitable technologies. However, the investments are symmetric, thus, requiring constant and equal profits across technologies. Hence, we have

$$u^* = \delta \pi^R(i,t)|_{i=N} = \delta \pi^R.$$

$$\text{(B.8)}$$

.

B.3 Proof of Proposition 3

Using Fubini's theorem from equation (B.3) we can put the differentiation sign under the integrating term:

$$\frac{\partial p_A(i)}{\partial A_i} = \frac{\partial \left(\int_{\tau_{max}(i)}^{\tau_{min}(i)} e^{-r(t-\tau_0(i))} \left(\frac{1}{\epsilon} E \frac{A_i^{-\alpha(1-\epsilon)}}{\int_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj} - \Psi \right) dt \right)}{\partial A_i} =$$

$$\int_{\tau_{max}(i)}^{\tau_{min}(i)} e^{-r(t-\tau_0(i))} \left(\frac{E}{\epsilon} \frac{\frac{\partial A_i^{-\alpha(1-\epsilon)}}{\partial A_i}}{\int_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj} \right) dt = - \int_{\tau_{max}(i)}^{\tau_{min}(i)} e^{-r(t-\tau_0(i))} \left(\frac{E}{\epsilon} \frac{\alpha(1-\epsilon) A_i^{1-\alpha(1-\epsilon)}}{\int_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj} \right) dt$$

$$\text{(B.9)}$$

Taking the integral in the same way as in Proposition 1, we have

$$
\frac{\partial p_A(i)}{\partial A_i} = \mathbf{F}|_{t=\tau_{max}(i)} \left(e^{-r(t-\tau_0(i))} \frac{1}{\epsilon} E \frac{\alpha(1-\epsilon)}{\int_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj} A_i^{1-\alpha(1-\epsilon)} \right) -
$$

$$
- \mathbf{F}|_{t=\tau_{min}(i)} \left(e^{-r(t-\tau_0(i))} \frac{1}{\epsilon} E \frac{\alpha(1-\epsilon)}{\int_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj} A_i^{1-\alpha(1-\epsilon)} \right) \tag{B.10}
$$

This amounts to some function of the increase in productivity A_i from time $\tau_{max}(i)$ until $\tau_{min}(i)$. With symmetric technologies this growth would be the same for all i, although the points in time $\tau_{min}(i), \tau_{max}(i)$ will be different. Note that this expression does not depend on the variable A_i, but only on the level of it at two fixed points in time. This proves that $\partial p_A(i)/\partial A_i = c$.

B.4   Proof of Proposition 8

The direct calculation of output growth rates yields

$$
\frac{\dot{Y}}{Y} = \frac{d/dt \left(\int_{N_{min}}^{N_{max}} A(i,t)^{\alpha\epsilon} di \right)}{\int_{N_{min}}^{N_{max}} A(i,t)^{\alpha\epsilon} di} - \frac{d/dt \left(\int_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)} dj \right)}{\int_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)} dj} \; \dot{N}_{min} \overset{=}{\underset{=}{}} \dot{N}_{max}
$$

$$
= \frac{\int_{N_{min}}^{N_{max}} \left(\dot{A}(i,t)^{\alpha\epsilon} di \right)}{\int_{N_{min}}^{N_{max}} A(i,t)^{\alpha\epsilon} di} - \frac{\int_{N_{min}}^{N_{max}} \left(\dot{A}(j,t)^{\alpha(\epsilon-1)} \right) dj}{\int_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)} dj} =
$$

$$
= \alpha\epsilon \frac{\int_{N_{min}}^{N_{max}} A(i,t)^{\alpha\epsilon-1} \dot{A}(i,t) di}{\int_{N_{min}}^{N_{max}} A(i,t)^{\alpha\epsilon} di} - \alpha(\epsilon-1) \frac{\int_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)-1} \dot{A}(j,t) dj}{\int_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)} dj} =
$$

$$
= \alpha\epsilon \int_{N_{min}}^{N_{max}} \frac{\dot{A}(i,t)}{A(i,t)} di - \alpha(\epsilon-1) \int_{N_{min}}^{N_{max}} \frac{\dot{A}(j,t)}{A(j,t)} dj = \alpha \int_{N_{min}}^{N_{max}} \frac{\dot{A}(i,t)}{A(i,t)} di
$$

Using $\dot{A}(i,t) = G/(N - N_{min}) - \beta A(i,t)$ and $A(i,t) = Q$ we get,

$$\frac{\dot{Y}}{Y} = \alpha \int_{N_{min}}^{N_{max}} \frac{\dot{A}(i,t)}{A(i,t)} di = \alpha \frac{G}{N - N_{min}} \int_{N_{min}}^{N_{max}} A(i,t)^{-1} di - \alpha \int_{N_{min}}^{N_{max}} \beta di =$$

$$= \alpha \frac{G}{N - N_{min}} \frac{N_{max} - N_{min}}{Q} - \alpha\beta(N_{max} - N_{min}) = \alpha \left(N_{max} - N_{min}\right) \frac{\dot{Q}}{Q} \quad \text{(B.11)}$$