Universität Bielefeld

Faculty of Business Administration and Economics

Working Papers in Economics and Management

No. 04-2014
February 2014

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March 1, 2014

Abstract

This paper discusses the role of technological spillovers and technological races in dynamic strategic interactions setup. Two multiproduct regional monopolies invest simultaneously into new products creation and into further development of the quality of these products. They can benefit from the costless technological spillover in the development of each such a product in the case the other's quality is higher. At the same time they cooperate in the joint creation of new products. The dynamic interaction in this setting previously considered only the situation with one of the monopolies being the constant technological leader. In this paper the full spectrum of possible equilibria of the game is studied. It is demonstrated, that the closer in terms of technology two firms are to each other, the more intense is the competition and overall research output is threatened, resembling inverted-U relationship between technology and competition.

Keywords: heterogeneous innovations, technological race, technology spillovers, distributed control, differential games

JEL codes: C02, L0, O31.

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1 Introduction

In this paper the dynamic model of strategic R&D interactions between two multiproduct regional monopolies is developed. The differential game includes only the interactions in the area of generation of innovations without market competition. For this the assumption of linear profit function as of Dawid et al. (2010) is used.

Thus the two agents being considered may be seen as perfectly separated regional monopolies, which invest into joint R&D ventures and develop the resulting new products simultaneously and independently. The interaction between these monopolies is realized by two channels. First, they both cooperate in the joint creation of new technologies, described as the horizontal innovations process and second, they can imitate each other in the area of subsequent development of new products. Both types of innovations require specific investments represented as subsequent controls of both monopolies, following the methodology of Rosenkranz (2003).

An important simplification is that there are no constraints on the resources being used for investments by both firms. In reality investments are constrained by the capital market, but this is not modelled in the paper. Resource-constrained version of the model may be considered in the spirit of Baveja et al. (2000) and other perimeter-type constrained models.

The framework of heterogeneous innovation is a relatively new attempt of unification of vertical and horizontal innovations. The main difference from earlier approaches, like that of Peretto (1998) and Peretto and Connolly (2007) is in relaxing the assumption of similar and symmetric nature of all the newer technologies being generated by the horizontal innovations process. Instead the differences in efficiency of development across technologies and across firms play the crucial role in the current analysis. Thus the paper may be seen as a contribution to the field of literature on horizontal and vertical innovations, like Aghion and Howitt (1992), Romer (1990), Aghion et al. (2005) in the spirit of creative destruction notion dating back to Schumpeter (1942).

However the current model is of partial equilibrium type and concentrates on the single industry, rather than economy-wide effects of technical change. In this respect the paper is an extension of results on R&D competition and cooperation in industries, like in papers on multi-product innovative monopolies, Lambertini and Orsini (2001), Lambertini (2003), Lambertini (2009), papers on R&D cooperation, D'Aspremont and Jacquemin (1988), Navas and Kort (2007) and papers on dynamic strategic interactions in the field of R&D, as Reinganum (1982), Hartwick (1984), Judd (2003), Cellini and Lambertini (2002), and Lambertini and Mantovani (2010).

The paper does not include any type of uncertainties and thus patent races as in, for example, Denicolo (1996) are impossible in this setting. Adding uncertainties is possible but would lead into the field of stochastic differential games, where closed-form solutions are difficult to obtain. At the same time the patent races are excluded because of the joint R&D venture on the level of horizontal innovations, being financed by both firms.

The framework of heterogeneous innovations has been formulated in the papers Belyakov et al. (2011) and Bondarev (2012), with the first being of more rigorous nature while the other concentrates on the effect of heterogeneity of special type on the dynamic behaviour of the monopolist. Current paper develops the multi-agent setting in the heterogeneous innovations framework, being addressed in the recent Bondarev (2014). In this last paper only the situation with constant technological leadership has been analyzed. The current paper builds on the analysis of the paper Bondarev (2014) by considering general configuration of parameters. This allows to discuss not only the constant leadership situation, but also the symmetric outcome (where no leadership in technology occurs) and the regime of leap-frogging. This last is the most complicated and interesting one. In the suggested framework the technology leap-frogging occurs because of the differences in innovation efficiencies between the agents. Such a leap-frogging may occur only once and the conditions under which it may occur may be interpreted as the conditions on the technological distance between the players. In this sense the current paper is an extension of well-known result in Aghion et al. (2005) on inverted-U relationship between technological distance and competition, but in the multidimensional context of heterogeneous vertical and horizontal innovations.

There are three main contributions in this paper. First, it shows the benefits from technological spillovers for both firms stemming from the inclusion of both vertical and horizontal innovations into analysis thus generalizing the result of the constant technological leadership setting of Bondarev (2014). Second, it extends the analysis of the interrelationship between technological competition and generation of innovations from Aghion et al. (2005) in multi-dimensional dynamic setup. Third, the effect of technological leap-frogging is captured within the differential game structure and the time of the leap is endogenously defined for every technology by relative efficiencies of investments of both agents. The relationship between technological distance between firms and total value of R&D activities is established. The maximum technological gain for both firms and for the industry is obtained under constant technological leadership of one of the players, while under technology leap-frogging the later in time this leap frogging occurs, the less is the total technological output of the industry. The time of the switch in leadership reflects the intensity of competition between firms in the development of each subsequent product and it may be demonstrated that constant leadership gives rise to sustainable cooperation and endogenous specialization. At the same time technological race in the form of leap-frogging deteriorates both he value of R&D activity for both competing agents and reduces the total generation of innovations in the industry.

The rest of the paper is organized as follows. The next Section 2 introduces the formal model. The solution is separated into the one for vertical innovations and for horizontal ones. They both are presented in the Section 3. Main results of the paper are formulated as propositions in Section 4. Section 5 with discussion concludes the paper. The majority of mathematical proofs may be found in the Appendix.

2 Model

Assume there are two firms in an industry. These firms may invest into common R&D activities for creation of new versions for the basic industry-defining product and into the development of quality (productivity) of already introduced products. The range of products available for introduction is bounded from above by the real positive number N and investments into creation of new products and into development of existing products (specific and different for each such a product) are nonnegative. The firm, which has the lower level of technology for product i at time t benefits from the technology spillover from the leader, which boosts the follower's innovations for this product. Naturally in the symmetric situation with both firms having equal levels of development of the given product, no spillovers will occur. I also allow for the possibility of changing leadership. This situation is labelled as catching-up and results in the firm which have been the follower in product i development for some time to become the technological leader due to accumulated benefits from technological spillovers. However the position of the leader has its own advantages, since the follower in vertical innovations would invest more into creation of newer products (with zero investments of the leader in the closed-loop solution). The model setup follows the same lines as the model of Bondarev (2014) where some more detailed discussion on the nature of strategic interactions between firms may be found. This paper is used as a benchmark for reference. Here only formal structure necessary for further exposition is given.

2.1 Formal framework

Assuming the same linear profit functions and fixed costs for mature products as in the benchmark paper. Objectives of both firms are given as:

$$J_j \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=} \max_{u_j(\bullet),g_j(\bullet)} \int_0^\infty \mathrm{e}^{-rt} \left[\int_0^{n(t)} \left(q_j(i,t) - \frac{1}{2} g_j(i,t)^2 \right) di - \frac{1}{2} u_j(t)^2 \right] dt. \tag{1}$$

Where:

- j is the label of the firm, $j \in [1,2]$;
- $q_j(i,t)$ is the quality of product i at time t for the firm j;
- $g_j(i,t)$ are investments into the development of product i at time t being made by the firm j;
- $n(t)$ is the common for both firms achieved level of the variety of products at time t;
- $u_j(t)$ are investments of the firm j into the expansion of variety of products at time t.

The basic intuition behind (1) is clear: every firm is maximizing the effect from its quality innovations for each of the introduced products at every time t. The range of products, being introduced till time t is given by $n(t)$. The objective of the firm is thus to maximize quality innovations for all the products within this range. At the same time the expansion of the range of products requires specific type of investments u_j. Observe that the introduction of the new product per se does not bring the increase in the value for the firm, since it is assumed that such a product has zero level of technology.

Assume the process of development of products is continuous in time and yields new products proportionally to joint investments of both firms with possibly different efficiencies of investments across firms. The range of these new products is limited from above by some $N \in \mathbb{R}^+$:

$$\dot{n}(t) = \alpha_j u_j(t) + \alpha_{-j} u_{-j}(t),\ n(t) \in [0;N] \subset \mathbb{R}^+. \tag{2}$$

Where α_j, α_{-j} are investment efficiencies for variety expansion for both firms respectively.

The evolution of each technology (its quality) is separated for every firms and is subject to optimally controlled technology-specific innovations and potential technological spillover effect:

$$\dot{q}_j(i,t) = \frac{\partial q_j(i,t)}{\partial t} = \gamma_j \sqrt{N-i} g_j(i,t) - \beta_j q_j(i,t) + \theta \cdot \max\{0, (q_{-j}(i,t) - q_j(i,t))\};$$

(3)

$$\dot{q}_{-j}(i,t) = \frac{\partial q_{-j}(i,t)}{\partial t} = \gamma_{-j} \sqrt{N-i} g_{-j}(i,t) - \beta_{-j} q_{-j}(i,t) + \theta \cdot \max\{0, (q_j(i,t) - q_{-j}(i,t))\};$$

$$q_{j,-j}(i,t) \mid_{t_0(i):i=n(t)} = 0;$$
$$q_{j,-j}(i,0) = 0, \forall i \in [0;N] \subset \mathbb{R}^+;$$

(4)

Where:

- γ_j are efficiencies of investments into the quality of product i for the firm $\{1,2\}$, constant across products and time;
- β_j are technology decay rates in the absence of investments for both agents;
- θ is the speed of technological spillover, equal for both agents;
- $t_0(i)$ is the time of emergence of technology i.

Observe the special role of the term $\gamma_j \sqrt{N-i}$. This is the measure of heterogeneity across technologies. This special form of difference is taken both from economic and mathematical considerations. From economic viewpoint the $\gamma(\bullet)$ is a decreasing function of the index of the technology i, thus making it harder to develop newer products for both firms. From mathematical viewpoint the $\sqrt{N-i}$ functional form helps to linearize the resulting differential system, making it possible to obtain the closed-form solution for the game. Observe that the structure of the game and conceptual findings are independent of this particular choice of the functional form. However, results may be substantially different if either different functional forms would be assumed for both players or the efficiency would be an increasing function of the technology index i. Thus denote the choice of this functional form as a particular Assumption:

Assumption 1 *The efficiency of investments into vertical innovations is a monotonically decreasing function in the space of technologies and is independent of time for any player j and has the functional form:*

(5) $$\gamma_j(i) = \gamma_j \sqrt{N-i}.$$

For any product i development of quality starts without any spillovers and this spillover may appear only eventually due to different speeds of development of this product. Conditions for one or another firm to be the leader in the development of quality of a given product i are obtained as a part of the solution procedure. The model itself allows for 3 different cases for every product development, depending on the configuration of parameters. These are:

- One of the firms is the constant leader in quality development of product i;
- Both firms have equal levels of quality development and no technological spillover occur;
- Initially one of the firms is the leader in quality development, but after some time the other firm catches up with the leader and becomes the leader itself.

The benchmark model Bondarev (2014) tackles only with the first case, while the present paper concentrates on two other subcases in an effort to obtain the full characterisation of possible outcomes of the game. To solve this model I decompose the problem given by Eqs. (1), (2), (3), (4) into the problem of maximizing the development of every given technology i given by (3) and into the problem of maximizing value from creation of new technologies from (2) along the same lines as in the benchmark case. The full characterization of the solution is first given for vertical innovations and then these results are used for solving the problem of horizontal innovations.

3 Solutions

Following the decomposition procedure as in the benchmark paper, one may write down the value function for vertical (quality) innovations in each product i for each player j as following:

$$V_j(i) = \max_{g_j(i)} \int_{t_0(i)}^{\infty} \mathrm{e}^{-r(t-t(0)_i)} \left\{ q_j(i,t) - \frac{1}{2} g_j(i,t)^2 \right\} dt; \quad j \in \{1,2\}, \forall i :\in [0;N] \subset \mathbb{R}^+. \tag{6}$$

where $t_0(i)$ is the time of emergence of the product i, which is defined from the dynamics of the variety expansion process and is similar for both agents due to the form of dynamic constraints (2), (3).

Proposition 1 *Value functions of quality innovations management game for each separate product i for both firms, $V_j(q(i))$ are invariant to the state of the variety expansion game, $n(t)$ except for the emergence time of this product, $t_0(i)$.*

The more detailed idea of the proof of this Proposition may be found in the benchmark paper. With value functions being defined in this form it is straightforward to notice that for each technology i the solution is given by a pair of HJB equations, which are independent of other technologies evolution and on the variety expansion process $n(t)$. Without restrictions of the benchmark case both players may be the follower, the leader or they may have symmetric strategies, depending on the configuration of parameters. The general form of HJB equations for both players is:

$$
\begin{aligned}
& rV_j(i) = \\
& \max_{g_j(\bullet)}\Bigg\{ q_j(i,t) - \frac{1}{2}g_j(i,t)^2 + \\
& + \frac{\partial V_j(i)}{\partial q_j(i,t)}\left(\gamma_j\sqrt{(N-i)}g_j(i,t) - \beta_j q_j(i,t) + \max\{0, \theta\big(q_{-j}(i,t) - q_j(i,t)\big)\}\right) \\
& + \frac{\partial V_j(i)}{\partial q_{-j}(i,t)}\left(\gamma_{-j}\sqrt{(N-i)}g_{-j}(i,t) - \beta_{-j} q_{-j}(i,t) + \max\{0, \theta\big(q_j(i,t) - q_{-j}(i,t)\big)\}\right)\Bigg\};
\end{aligned}
$$

$$
\begin{aligned}
& rV_{-j}(i) = \\
& \max_{g_{-j}(\bullet)}\Bigg\{ q_{-j}(i,t) - \frac{1}{2}g_{-j}(i,t)^2 + \\
& \frac{\partial V_{-j}(i)}{\partial q_j(i,t)}\left(\gamma_j\sqrt{(N-i)}g_j(i,t) - \beta_j q_j(i,t) + \max\{0, \theta\big(q_{-j}(i,t) - q_j(i,t)\big)\right) \\
& + \frac{\partial V_{-j}(i)}{\partial q_{-j}(i,t)}\left(\gamma_{-j}\sqrt{(N-i)}g_{-j}(i,t) - \beta_{-j} q_{-j}(i,t) + \max\{0, \theta\big(q_j(i,t) - q_{-j}(i,t)\big)\}\right)\Bigg\}.
\end{aligned}
\tag{7}
$$

The realisation of the max term in both equations determines relative follower and leader at each point in time. It is important to stress that the leader-follower configuration of players is preserved for all technologies as long as the function of efficiencies, $\gamma(\bullet)$ is a monotonic function of i which is indeed the case, (5).

With this observation in mind for each technology i there are three cases as it has been pointed out:

1. Constant leadership of one of the players is described in subsection 3.1;

2. Symmetric outcome of the game is analyzed in the subsection 3.2;

3. The case of switch in leadership (catching-up case) is analyzed in subsection 3.3.

3.1 Constant leadership in technology

In the benchmark paper it is assumed that constant leadership regime occurs under the condition

$$\gamma_j > \gamma_{-j},\ \beta_j = \beta_{-j}. \tag{8}$$

taking firm j as a leader for simplicity.

It is characterized by constant domination of the quality of one of the players above the other, thus creating the technological spillover effect to the benefit of the follower (superscript F denotes from now on the quantities for the following firm and L quantities for the leading firm):

$$\forall t \in [t_0(i), \infty) : q^L(i,t) > q^F(i,t). \tag{9}$$

However if one allows for differences in decay parameters β_j this is not the only case where constant leadership takes place. The above condition defines the situation with unique equilibrium of the game, while with differences in decay rates two equilibria might exist on both sides of the diagonal $q_j(i,t) = q_{-j}(i,t)$. However if the constant leadership regime arises in the game, the solution resembles one in the benchmark case.

Proposition 2 (Solution for constant leadership case)
For the constant leadership case the optimal strategies of the leader, g^L and of the follower g^F are respectively:

$$g^L(i,t) = \begin{cases} \frac{\gamma^L\sqrt{(N-i)}}{r+\beta^L}, t : i \geq n(t); \\ 0, t : i < n(t). \end{cases} \tag{10}$$

and

$$g^F(i,t) = \begin{cases} \frac{\gamma^F\sqrt{(N-i)}}{r+\beta^F+\theta}, t : i \geq n(t); \\ 0, t : i < n(t). \end{cases} \tag{11}$$

yielding qualities evolution for both firms as functions of the position of the product, i and time t:

$$\dot{q}^L(i,t) = \frac{(\gamma^L)^2(N-i)}{\beta^L + r} - \beta^L q^L(i,t), \tag{12}$$

$$\dot{q}^F(i,t) = \frac{(\gamma^F)^2(N-i)}{\beta^F + r + \theta} + \theta q^L(i,t) - (\beta^F + \theta) q^F(i,t). \tag{13}$$

The derivation of these strategies amounts to solving the pair of HJB equations (7) with setting one of the players as the follower and assuming linear value functions of both players. Details and proof may be found in the benchmark model. One may immediately note that with this constant strategies the leadership may be preserved only as long as Eq. (9) holds.

However if $\beta^L > \beta^F$ this is not necessarily the case. One can analyze the steady states of the system (12),(13) by setting derivatives to zero. It turns out, that depending on the parameter configurations the steady state value of the state variable for the leader might be lower than that of the follower, thus creating the multiplicity of equilibria of this game. This multiplicity in turn creates the opportunity for the follower to catch up with the leader. This situation is studied in subsection 3.3 and the complete characterization of conditions for different regimes to arise is postponed till Section 4.

3.2 Symmetric game with no technological leader

The symmetric outcome of the game is characterized by the condition

$$\forall t \in [t_0(i), \infty) : q_j(i,t) = q_{-j}(i,t). \tag{14}$$

thus cancelling the imitation effect. However due to the potential benefit from technological spillover while investing less into technology there are two different possible outcomes in this regime.

First observe that the symmetric regime in the quality innovations game arises only when both efficiency of investments and decay rates of players are equal,

$$\gamma_j = \gamma_{-j}, \beta_j = \beta_{-j}. \tag{15}$$

In this case the value functions of players are no longer continuously differentiable due to the spillover term in HJB equations (7). The solution may be found by making use of the continuity of value functions alone the line $q_j(i,t) = q_{-j}(i,t)$. The range of available for both players strategies is the spectrum between follower's strategy, (11) and leader's strategy, (10). The selection of the optimal pair is based on the comparison of values obtained while pursuing one or the other strategy. It turns out, that two different symmetric outcomes my occur:

Proposition 3 (Solution for the symmetric case)
If (15) *holds, both players choose identical strategies in the symmetric outcome of the game.*

If spillover parameter is zero, $\theta = 0$, *both players choose the leadership strategy,*

$$g_j^0(i,t) = g_{-j}^0(i,t) = \frac{\gamma\sqrt{(N-i)}}{r+\beta}, t : i \leq n(t) \tag{16}$$

If spillover parameter is positive, $\theta > 0$ *both players behave themselves as followers,*

$$g_j^\theta(i,t) = g_{-j}^\theta(i,t) = \frac{\gamma\sqrt{(N-i)}}{r+\beta+\theta}, t : i \leq n(t). \tag{17}$$

The subsequent qualities evolve as:

$$\dot{q}^0(i,t) = \frac{\gamma^2(N-i)}{(r+\beta)^2} - \beta q^0(i,t), t : i \leq n(t), \tag{18}$$

$$\dot{q}^\theta(i,t) = \frac{\gamma^2(N-i)}{(r+\beta+\theta)^2} - \beta q^\theta(i,t), t : i \leq n(t). \tag{19}$$

The derivation of optimal strategies may be found in the Appendix B. The potential presence of the technological spillover stimulates both players to reduce their own investments in the hope to benefit from the development of the other player. As a result, the level of development of technology is lower for both players. If the technological spillover is not possible, there is no strategic interaction between players at all and they both behave themselves as isolated monopolies in the area of vertical innovations. Patent races will not occur in this situation because of joint research efforts in variety expansion: both firms acquire the new technology at the same time $t_0(i)$.

3.3 Catching-up in the space of technologies

The catching-up situation is characterized by the conditions

$$\begin{aligned} &\forall t \in [t_0(i), t_i^{TRIG}) : q_j(i,t) > q_{-j}(i,t), \\ &t_i^{TRIG} : q_j(i, t_i^{TRIG}) = q_{-j}(i, t_i^{TRIG}), \\ &\forall t \in (t_i^{TRIG}, \infty) : q_j(i,t) < q_{-j}(i,t), \\ &\forall j \in [1,2]. \end{aligned} \tag{20}$$

meaning that one of the players starts investments as being the technological leader, but with time another player catches up with the leader due to benefits from spillover effect and (possibly) lower decay rates. Then at some trigging time their levels of development equalize and from that time on another player $-j$ becomes the technological leader in quality.

For this scenario to realize some special combination of parameters has to hold to ensure multiple steady state levels of technology to co-exist. This is discussed in the Section 4 further on. For the moment assume this combination holds. The solution procedure in this case is more complicated than for constant leadership. The concept is worked out in several steps.

First establish the auxiliary Proposition 4

Proposition 4 (Uniqueness of the catching-up event)
If the parameters of both firms allow for the catching-up, at most one trigger point, $\{q_i^{TRIG}, t_i^{TRIG}\}$ for each i exists. It is defined by the condition

$$t_0(i) < t = t_i^{TRIG} : q_j(i,t) = q_{-j}(i,t) = q_i^{TRIG} \tag{21}$$

Proof of this proposition amounts to the observation that in constant leadership case strategies are constant, (10), (11). Thus even if parameters of the game allow for the eventual catching-up of technologies at zero technology level, the strategies of players after this event are formulated in the same form as for constant leadership but with the after-trigger non-zero technology level. Since strategies are linear combinations of parameters, they allow for at most one point of equal qualities except the $t_0(i)$. It follows, that if the game allows for the catching-up, it can happen only once for each product i, changing relative positions of the leader and the follower. Thus one may speak about situations before the catching-up and after the catching-up as uniquely defined.

Before the catching-up both players optimize somewhat different value functions than in constant leadership situation, since they both possess perfect information about each other and about the time when the catching-up will happen. This last is endogenously defined, since the higher is investments rate of the follower relative to that of the leader the sooner the technology of the follower will catch-up with that of the leader. However this trigger time is subject to change due to the efforts of both players and thus it is not controlled by only one of them. Both players however know that after the trigger they will play constant strategies given by (10) and (11) (since no other trigger may occur) and will receive respective payoffs defined by value functions of the constant leadership game described in subsection 3.1. These values are used by players as estimation of the salvage value of the finite-time game they play before the trigger. This is formulated by Lemma 17 in Appendix C.

Before the trigger both players maximize their respective value functions over the finite time period $[t_0(i), t_i^{TRIG}]$ using value functions of the rest of the game (being in constant leadership mode) as salvage functions. Thus the objective of a firm, which is the initial leader (and the follower after catching-up occurs) is a combination of (6) and the value of being the follower for the rest of the game.

$$V_j^{LF}(i) =$$

(22)

$$= \max_{g_j(i)} \left\{ \int_{t_0(i)}^{\infty} \mathrm{e}^{-r(t-t(0)_i)} \left\{ q_j(i,t) - \frac{1}{2} g_j(i,t)^2 \right\} dt + \mathrm{e}^{-rt_i^{TRIG}} V_j^F(i)|_{q_j(i,t)=q_{-j}(i,t)=q(i,t_i^{TRIG})} \right\};$$

$j \in \{1,2\}, \forall i :\in [0;N] \subset \mathbb{R}^+$.

The objective of a firm, which is the initial follower and the leader after the catching-up is respectively:

$$V_j^{FL}(i) =$$

(23)

$$= \max_{g_j(i)} \left\{ \int_{t_0(i)}^{\infty} \mathrm{e}^{-r(t-t(0)_i)} \left\{ q_j(i,t) - \frac{1}{2} g_j(i,t)^2 \right\} dt + \mathrm{e}^{-rt_i^{TRIG}} V_j^L(i)|_{q_j(i,t)=q_{-j}(i,t)=q(i,t_i^{TRIG})} \right\};$$

$j \in \{1,2\}, \forall i :\in [0;N] \subset \mathbb{R}^+$.

where superscripts LF and FL denote the initial leader and subsequent follower and vice versa quantities and $V_j^F(i), V_j^L(i)$ may be obtained from (6) by substituting for optimal strategies and quality dynamics in the same way as in the benchmark case. They are given in Appendix A.

The solution is found by application of Maximum Principle (which is equivalent to HJB approach for the case of piecewise-constant strategies being considered here) to the system given by objectives (22), (23) and dynamics of quality given by (3).

Proposition 5 (Solution of the quality game before the catching-up)
When parameters of the model allow for the catching-up to occur, the solution of the game before the catching-up is maximization of (22) *and* (23) *w. r. t.* (3) *via Maximum principle. The strategies of the players before the catching-up are given by linear functions of shadow costs of increasing own direct investments* ψ_j^j*:*

(24) $$g_j^{LF} = \gamma_j \sqrt{N-i}\,\psi_j^j(t,\eta),$$

(25) $$g_{-j}^{FL} = \gamma_{-j} \sqrt{N-i}\,\psi_{-j}^{-j}(t,\eta)..$$

and evolution of qualities is given by the dynamical system

$$\dot{q}_j(i,t) = \gamma_j^2(N-i)\psi_j^j(t,\eta) - \beta_j q_j(i,t);$$
$$\dot{q}_{-j}(i,t) = \gamma_{-j}^2(N-i)\psi_{-j}^{-j}(t,\eta) - \beta_{-j} q_{-j}(i,t) + \theta(q_j(i,t) - q_{-j}(i,t));$$
$$q_j(i,t_0(i)) = q_{-j}(i,t_0(i)) = 0. \quad (26)$$

where η are the opportunity costs of changing the time of a trigger, t_i^{TRIG}.

The sketch of the formal proof is given in the Appendix C. Note that the co-state functions ψ_j^j are defined as functions of the opportunity costs of moving the trigger time. These are constant in time and depend only on the time of the trigger itself. By definition

$$\eta : q_j(i,t_i^{TRIG}) = q_{-j}(i,t_i^{TRIG}). \quad (27)$$

Provided the solution for qualities evolution before the trigger time this last may be obtained from equalizing opportunity costs η across players. For this one make use of Proposition 4 above.

The solution for the game after the trigger is obtained alone the same lines as for the constant leadership case, subsection 3.1. However, the level of technology at which this part of the game starts is not zero, but equal to the trigger level, $q_j(i,t) = q_{-j}(i,t) = q(i,t_i^{TRIG})$, so instead of (4) different initial condition is used:

$$q_{j,-j}(i,t_i^{TRIG}) = q_i^{TRIG}. \quad (28)$$

The solution of the game after the catching-up amounts to solving the dynamical system (3) with initial condition (28) with strategies of players given by (10) and (11) for initial follower and initial leader respectively. The complete solution is formulated piecewise as following.

Proposition 6 (Complete solution of the quality game for catching-up case) *When parameters of the model allow for the catching-up to occur, the solution for the quality game is:*

1. *A pair of strategies* (24), (25) *and dynamical system* (26) *before the catching-up time,*

2. *A pair of strategies* (11) *for player j and* (10) *for player $-j$ and dynamical system* (3) *with initial condition* (28) *after catching-up time.*

The next subsection deals with solution for the variety expansion part of the whole innovations game. It turns out that despite the presence of three qualitatively different cases for the vertical innovations part, the horizontal innovations part may be characterized in a unified and simple manner.

3.4 Solution for variety expansion

The second part of the value generated by innovations is the expansion of variety of technologies which is subject to joint R&D efforts of both firms. According to the decomposition of the overall problem given for both players by Eq. (1) the objective of the variety expansion differential game for any player j is:

$$\begin{aligned} & V_j(n) = \\ (29) \qquad & = \max_{u_j} \int_0^\infty \mathrm{e}^{-rt} \Big((\alpha_j u_j(t) + \alpha_{-j} u_{-j}(t)) \cdot V_j(q(i)) \mid_{i=n(t)} - \frac{1}{2} u_j(t)^2 \Big) dt, \\ & j \in \{1,2\}, \forall i :\in [0;N] \subset \mathbb{R}^+. \end{aligned}$$

subject to the common process of horizontal innovations investments, (2). The problem of variety expansion is symmetric for both players except for the term $V_j(q(i)) \mid_{i=n(t)}$ which denotes the value of the development of next to be invented technology, $i = n(t)$. Thus the overall incentive for horizontal innovations for both firms is the potential benefits from further development of new technology. These benefits depend on the regime which will realize in the quality game for the next product.

However one can solve the variety expansion game parametrically and the solution form will be the same for any regime of the associated quality game. This is possible due to the fact that value function of the vertical innovations for any product i is linear in i and thus the value of vertical innovations for the boundary product is linear in $n(t)$ irrespective of the regime of the quality game.

Both value functions from quality innovations game are linear in $n(t)$, as the inspection of (A.2), (B.6) and (C.9) shows. For variety expansion game only the potential value is relevant, that is the value functions estimated at zero quality levels. Denote the constant part of quality game value functions by C_j, C_{-j}:

$$\begin{aligned} V_j(0) &= C_j \cdot (N - n(t)); \\ (30) \qquad V_{-j}(0) &= C_{-j} \cdot (N - n(t)). \end{aligned}$$

where $C_j, C_{-j} \geq 0$ are given by state-independent parts of value functions of the quality game. From this observation one may derive optimal strategies in the variety expansion game for both players in the same way as in the benchmark model. I state this result without proof, which is fully analogous to the one in the benchmark model:

Proposition 7 (Solution of variety expansion game)
Solution for variety expansion game is given by:

1. *Optimal investment policies,* $u_j^*(t), u_{-j}^*(t)$*, given by* (D.4)*, which are feedback policies and decrease in time and over the achieved variety;*

2. *Associated common variety level, $n(t)$, given by* (D.3)*, which has decreasing growth rate and reaches maximal level N at infinity.*

Variety expansion speed depends positively on value derived by both players from quality innovations into the boundary product $i = n(t)$.

Sketch of formal treatment may be found in Appendix D.

The properties of solutions and conditions for different regimes of the game to realize are discussed further in Section 4.

4 Analysis and results

4.1 Steady states of vertical innovations game

Observe that the set of solutions for constant leadership case yields two different steady-state levels of quality for every player, one corresponding to being the leader and the other with being the follower. The first steady-state level of quality is not always higher then the second one, but the actual steady state in which trajectories of qualities will be eventually depends on the relation between efficiency parameters of the players. To be more specific for any player steady-state level of quality is given by (j player is taken for certainty):

$$q_j^{STL}(i) = \frac{\gamma_j^2(N-i)}{\beta_j(\beta_j+r)};$$

$$q_j^{STF}(i) = \frac{\gamma_{-j}^2\theta^2 + \gamma_{-j}^2\theta(\beta_j+r) + \gamma_j^2\beta_{-j}(\beta_{-j}+r)}{\beta_{-j}(\beta_j+\theta)(r+\theta+\beta_j)(\beta_{-j}+r)} \times (N-i). \tag{31}$$

where superscripts STL, STF denote the steady state level of quality while being the leader and the follower respectively. The same form with change of j by $-j$ and vice versa holds for the other player.

In contrast to the benchmark model, it may be the case that for the given player the steady-state level of quality in the follower mode is higher then of being in the leader mode giving rise to multiple equilibria of the game. I first establish the conditions for the existence and realization of the unique steady state of the game.

Proposition 8 (Uniqueness of the steady state)

1. For the game of quality development to have unique non-symmetric steady state it is necessary, that

$$q_j^{STL}(i) > q_{-j}^{STL}(i). \tag{32}$$

Then player j is the final leader in this game for any i.

For the game to realize in constant leadership mode, it is sufficient:

$$\begin{aligned} & q_j^{STL}(i) > q_j^{STF}(i), \\ & q_{-j}^{STF}(i) > q_{-j}^{STL}(i). \end{aligned} \tag{33}$$

Then there is the unique steady state in quality game, $\left\{q_j^{STL}(i), q_{-j}^{STF}(i)\right\}$ *which is realized through constant leadership strategies* (10), (11).

2. In the case of symmetric outcome there is always the unique steady state, associated with equal qualities of both players, realized through either (17) *or* (16) *for both players.*

However, these opportunities do not exhaust the set of possible outcomes of the quality game. It might be the case, that two potential steady states of the system exist. This is the case when $q_{-j}^{STF}(i) < q_{-j}^{STL}(i)$ and the player $-j$ is better off as the leader also. In this situation the catching up of quality levels may occur.

Proposition 9 (Existence of multiple steady states) *For two steady states of the quality game to exist, the following conditions have to hold:*

1. Both players are better off in the leadership mode,

$$\begin{aligned} & q_j^{STL}(i) > q_j^{STF}(i), \\ & q_{-j}^{STL}(i) > q_{-j}^{STF}(i), \end{aligned} \tag{34}$$

2. For both players in the leadership mode quality level is higher, than for the other player in the follower mode:

$$\begin{aligned} & q_j^{STL}(i) > q_{-j}^{STF}(i), \\ & q_{-j}^{STL}(i) > q_j^{STF}(i). \end{aligned} \tag{35}$$

than there are two steady states in the non-symmetric quality game, given by $\left\{q_j^{STL}(i), q_{-j}^{STF}(i)\right\}$ *and* $\left\{q_{-j}^{STL}(i), q_j^{STF}(i)\right\}$.

The first condition is straightforward. The second condition is the requirement, that the leader's quality is higher than that of the follower, otherwise no spillover may occur.

Now observe, that from (31) the conditions for multiplicity of steady states may be re-expressed in terms of parameters of technologies, $\gamma_{j,-j}, \beta_{j,-j}$. Thus, the phenomenon of catching up is the direct consequence of *technological distance* between the players.

Formal conditions for the leadership may be easily derived from the steady states. The player, whose steady-state level of quality while being the leader is higher becomes the final leader in the game. This is equivalent to the statement, that this player's effective investments rate is higher in the leadership mode, than that of the other player:

$$\frac{\gamma_j^2}{\beta_j(\beta_j+r)} > \frac{\gamma_{-j}^2}{\beta_{-j}(\beta_{-j}+r)}. \tag{36}$$

This is the condition for the system to end up in the steady state with player j being the leader in quality ($q_j(i) > q_{-j}(i)$) and is the function of both players' investment efficiencies. If additionally,

$$\frac{\gamma_j^2}{\beta_j(\beta_j+r+\theta)} > \frac{\gamma_{-j}^2}{\beta_{-j}(\beta_{-j}+r+\theta)}, \tag{37}$$

then player j is always the leader and no catching up may occur (the unique steady state exists).

Observe that if one of the players is selected as a final leader, (36) implies also:

$$\frac{\gamma_j^2}{\beta_j(\beta_j+r)} > \frac{\gamma_{-j}^2}{\beta_{-j}(\beta_{-j}+r+\theta)}. \tag{38}$$

If both players may be final leaders, than the condition for multiplicity of steady states is reduced to the pair of inequalities:

$$\begin{aligned} \frac{\gamma_j^2}{\beta_j(\beta_j+r)} &> \frac{\gamma_{-j}^2}{\beta_{-j}(\beta_{-j}+r+\theta)}, \\ \frac{\gamma_{-j}^2}{\beta_{-j}(\beta_{-j}+r)} &> \frac{\gamma_j^2}{\beta_j(\beta_j+r+\theta)}. \end{aligned} \tag{39}$$

Now to define, which of the steady states will actually realise, one has to define a couple of additional conditions. The first one is (36), defining which player actually will be the leader in the end. The other one is the requirement, that investments of this final leader in the follower regime are less efficient, than that of the final follower:

$$\frac{\gamma_j^2}{\beta_j(\beta_j+r+\theta)} > \frac{\gamma_{-j}^2}{\beta_{-j}(\beta_{-j}+r+\theta)}. \tag{40}$$

Now we are in position to define the state of the game with catching up. It is done in the following Proposition.

Proposition 10 (Realisation of catching-up scenario of the game)
For catching up scenario to realize, the following conditions have to be fulfilled:

1. *Two steady states of the game exist. For this* (39) *have to hold;*
2. *One of the players has higher final quality level in the leadership mode,* (36)*;*
3. *This player has also higher efficiency of investments in the follower mode, than the other player,* (40).

Under these conditions, player j is the final leader of the game, but initially is the follower.

The meaning of these conditions is rather straightforward: player j will end up as a leader if his steady-state level of quality as a leader is higher then that of the player $-j$, (hence the second condition). Then the steady state with him as a leader is eventually selected. At the same time this does not mean per se that the catching-up will occur. For that it is also necessary that two steady states exist in the system, hence the first condition. The last condition denotes that investments of the player j in the follower mode are higher then those of player $-j$ in the follower mode. Then the system will end up in the steady-state with player j as a leader but there is some initial period of time, when qualities are low enough, and player j's investment are lower then those of player $-j$.

Observe that the catching-up may occur only if the investment efficiency of one player is higher then that of the other (so he is initially the leader in quality) but his rate of decay is sufficiently higher then that of the follower. Then the follower will eventually catch-up with the leader because of accumulated quality level. The rate of decay becomes more significant then efficiency of investments while follower's growth rate is higher due to the imitation effect.

4.2 Full characterization of the vertical innovations game with catching-up

It has been seen that depending on the configuration of parameters, the pair of innovating firms may end up in one of three different regimes of investments. There is a symmetric regime with equal investments of both players, the regime of constant technological leadership of one of the players and the possibility of the change in technological leadership. The natural question is under which of the regimes the generation of innovations is the highest for both firms.

Proposition 11 (Ordering of innovations across regimes)

The presence of technological spillover threatens vertical innovations for both players under symmetric play, but boosts innovations for both players under constant leadership mode and in catching-up mode. The effect of technological spillover is lower under catching-up than under constant leadership and thus generation of innovations is slower. In particular the following holds:

1. *For any parameters' configuration it holds that*

$$q^0(i,t) > q^\theta(i,t), \tag{41}$$

2. *if* $\gamma_j = \gamma_{-j} = \frac{1}{2}(\gamma^L + \gamma^F), \beta_j = \beta_{-j} = \frac{1}{2}(\beta^L + \beta^F)$ *in symmetric mode, then*

$$t \in (t_0(i), \infty) : q^L(i,t) > q^F(i,t) > q^0(i,t) > q^\theta(i,t), \tag{42}$$

3. *if* $\gamma_j = \gamma_{-j} = \frac{1}{2}(\gamma^{LF} + \gamma^{FL}), \beta_j = \beta_{-j} = \frac{1}{2}(\beta^{LF} + \beta^{FL})$ *in symmetric mode, then*

$$t \in (t_0(i), \infty) : q^{FL}(i,t) > q^{LF}(i,t) > q^0(i,t) > q^\theta(i,t), \tag{43}$$

4. *with* $\gamma^F = \gamma^{FL}, \beta^F = \beta^{FL}$ *the follower is better off in constant leadership mode*

$$q^F(,t) > q^{FL}(i,t); \tag{44}$$

5. *with* $\gamma^L = \gamma^{LF}, \beta^L = \beta^{LF}$ *the leader is better off in constant leadership mode*

$$q^L(,t) > q^{LF}(i,t); \tag{45}$$

for any $i \leq n(t)$.

Proof Amounts to the observation that the technological spillover if realized has always a positive effect on the generation of vertical innovations for the follower but has detrimental effect if not realized. In particular, point 1 follows from comparison of (18) and (19); point 2 follows from comparison of (12),(13) with symmetric cases. Point 3 may be proved observing the evolution of qualities in catching-up case. Both players benefit from technological spillovers, one before the trigger point and the other afterwards. In symmetric regime this does not happen, thus with average efficiencies as defined the resulting quality evolution will be slower. Point 4 is done by variation of parameters of one of the players in constant leadership regime while keeping other player's parameters intact. It follows that the player who is the constant leader, may benefit from technological spillover in catching-up regime after the switch, but the realized steady state will be lower, since Proposition 9. The player, who is the follower in constant leadership regime is better off because of constant benefit from technological spillover while in catching-up the benefit is limited to the time before the trigger point.

The immediate consequence is that the overall generation of vertical innovations under symmetric regime is slower than under constant leadership modes,

Corollary 12 *The sum of vertical innovations generated by both firms is higher in constant leadership mode than in any of the two symmetric regimes and it is higher in catching-up mode than under symmetric regime with potential spillover:*

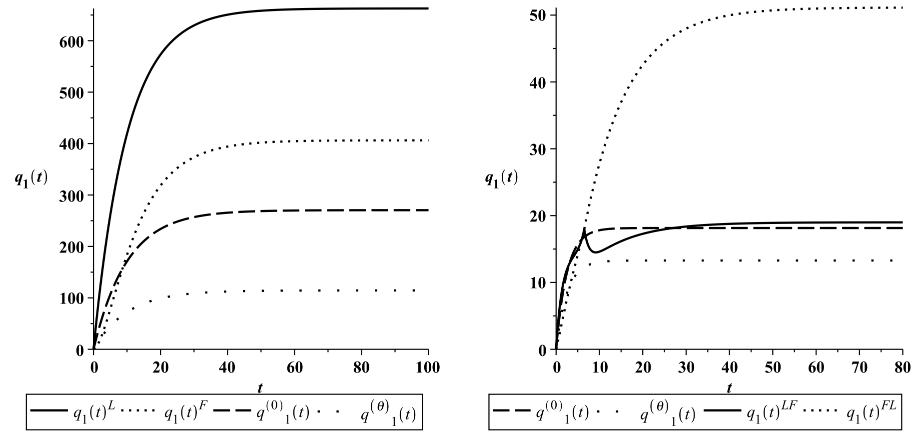
$$q^L(i,t)+q^F(i,t) > 2q^0(i,t) > q^{LF}(i,t)+q^{FL}(i,t) > 2q^\theta(i,t). \tag{46}$$

The comparison of vertical innovations under different regimes is provided at Figure 1. The parameters being used for this illustration are presented in the Table 1. The efficiency of investments for symmetric players is taken as an average of efficiencies of asymmetric players and decay rates are assumed to be equal for both players.

Table 1: Parameters values used in Figure 1.

Parameter	Constant leadership	Symmetric play for constant leadership	Catching-up	Symmetric play for catching-up
N	10	10	10	10
r	0.01	0.01	0.01	0.01
i	1	1	1	1
θ	0.15	0.15	0.15	0.15
γ_1	0.9	0.575	0.9	0.575
γ_2	0.25	0.575	0.25	0.575
β_1	0.1	0.1	0.7	0.4
β_2	0.1	0.1	0.1	0.4

From this figure it can be seen that the asymmetry in parameters of players has a positive effect on the output of innovations. This reflects the positive effect of technological spillover. At the same time, if the potential for technological spillover is present, but players are symmetric, their total generation of innovations is slower, than with the absence of such a spillover for any investments efficiency γ. Thus technological spillover may play both positive and negative role in innovations generation.



(a) Constant leadership and symmetric play

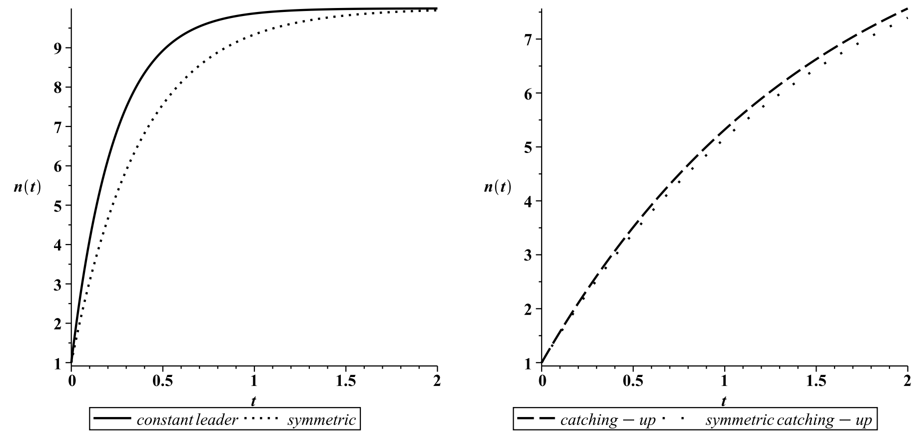
(b) Catching-up and symmetric play

Figure 1: Vertical innovations in different regimes

4.3 Horizontal innovations and persistence of endogenous specialization effect

To acquire the ordering of horizontal innovations analogous to Proposition 11 observe that the investments of both players and overall generation of horizontal innovations has the same form independent of the regime realised in vertical innovations game. Thus the overall intensity of investments of both players and generation of innovations is defined by the relative sizes of potential value in different regimes given by of (E.1), (E.2), (E.3), (E.4), (E.5) and (E.6). It is straightforward to notice, that potential value of innovations is lower in symmetric regimes than in constant leadership modes. Direct calculation of value functions for catching-up case and symmetric case with parameters from Table 1 above reveals that value under catching-up is higher for both players than under symmetric play. Then it comes with no surprise that investments of both players in horizontal innovations under symmetric play in qualities are the lowest across three possible regimes. The Figure 2 illustrates the dynamics with parameter values from Table 1 and setting $\alpha_j = \alpha_{-j} = 0.5, N = 10$.

The relationship between horizontal innovations under constant leadership and catching-up regimes is of the most interest. First observe that the player which ends up as a follower in vertical innovations is specializing in horizontal innovations, since value of vertical innovations for this firm is higher, $V^{LF}(i) > V^{FL}(i)$. Thus the effect of endogenous specialization of innovations, described in Bondarev (2014) survives for catching-up regime. However, since benefits from tech-



(a) Constant leadership and symmetric play

(b) Catching-up and symmetric play

Figure 2: Horizontal innovations in different regimes

nological spillover in catching-up case are lower, the horizontal innovations investments for both players are lower than in the constant leadership case. I characterize the dynamics of horizontal innovations as following:

Proposition 13 (Characterisation of horizontal innovations)

1. *The beneficial effect of technological spillover propagates to the horizontal innovations and symmetric players generate the lowest amount of discoveries;*
2. *In asymmetric play the player whose potential benefit from the vertical innovations is higher invests more into horizontal innovations;*
3. *Under catching-up the specialization of the potential follower in horizontal innovations is weaker than under constant leadership;*
4. *Under catching-up in qualities generation of horizontal innovations is slower than under constant leadership.*

In the end one may observe that under catching-up both horizontal and vertical innovations are generated at a slower pace than under constant leadership of one of the players. The 3d-reconstruction of dynamics of the model at Figure 3 helps to illustrate the argument.

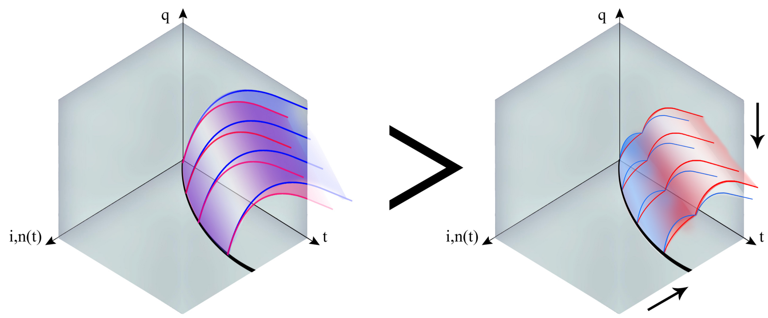


Figure 3: Reconstruction of two regimes of the model

4.4 Technological distance and output of innovations

Now we can establish the main result of the paper. From the analysis of steady states of vertical innovations game above it is straightforward to note that the closer are technological parameters of the firms to each other the higher is the chance of the catching-up to take place. Thus the difference between parameters plays the role of technological distance, analogous to the distance to frontier in Amable et al. (2007), but this distance is relative and reflects the differences between competing firms. This distance may be viewed upon as the inverse of intensity of competition in technologies between firms. Define overall technological efficiency of each firm as

$$e_j = \gamma_j - \beta_j \tag{47}$$

The technological distance between firms may be expressed as the difference between their technological efficiencies,

$$\mathtt{D}_{j,-j} \overset{def}{=} e_j - e_{-j} \tag{48}$$

This distance is zero for symmetric players, increases for the case of constant leadership and then decreases for the case of catching-up. However since there are two different cases of symmetric play in the game, one needs to account for the effect of technological spillover also. Modify the technological distance defined above by this term:

$$\mathtt{D}^*_{j,-j} = e_j - e_{-j} + \frac{1}{\theta} \tag{49}$$

Such defined quantity yields infinity for symmetric play without technological spillover and some finite number for symmetric game with such a spillover. The inverse of θ is taken to define intensity of competition between firms: it may be represented as the inverse of this modified distance and for symmetric firms has to be zero (since there is no interactions between firms at vertical innovations level in this case). There are no other effects on the ordering of innovations of $\frac{1}{\theta}$ inclusion instead of θ into (49). Define intensity of technological competition as

$$\mathtt{I}_{j,-j} = (\mathtt{D}^*)^{-1}. \tag{50}$$

Using the results of the Propositions 11 and 13 one may state:

Proposition 14 (Technological distance and output of innovations)
The relationship between modified technological distance $\mathtt{D}^*$ *and total generation of vertical innovations follows the U-relationship. It is equivalent to inverted-U relationship between intensity of competition* $\mathtt{I}$ *and output of innovations.*

This result stresses the importance of technological spillovers in the model: it can be beneficial for both firms, if the distance between firms is high, but it can be detrimental for innovative activity, if firms are close in their overall technological efficiency. Observe that this result cannot be obtained if horizontal innovations are neglected. They provide an incentive for the leading player to continue vertical innovations instead of seeking the benefit of technological spillover. Without the endogenous specialization which is the result of combining both types of innovations in the model the costless dynamic technological spillover would only to negative effects. Denote

$$\mathtt{Q} = \int_{n_0}^{n(t)} \left(q_j(i,t) + q_{-j}(i,t)\right) di \tag{51}$$

total generation of innovations by both players at any time t. Figure 4 illustrates the inverted -U relationship between $\mathtt{Q}$ and $\mathtt{I}$ across regimes of the quality game.

5 Discussion

In this paper I generalize the result of endogenous specialization of innovative activity to a more general framework which allows for catching-up in technologies as well as for symmetric investments of both firms. The positive effect of technological spillover is confirmed for the generalized model, but the catching-up situation substantially enriches the set of conclusions and policy implications from the model.

The result on inverse-U relationship between intensity of technological competition (or race) and generation of innovations of both types means that technological spillover effect may be both beneficial and detrimental both for competing

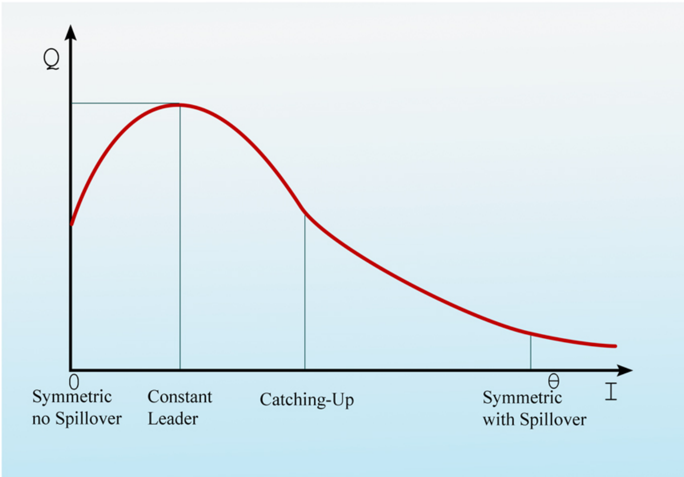


Figure 4: Relationship between technological competition and generation of innovations.

firms and the industry as a whole. If the differences between firms in terms of their efficiency in technology management is substantial, the technological spillover leads to the boost in innovations of both applied and fundamental types. However if these firms are close to each other and catching-up in applied technology occurs, the benefit of technological spillover reduces for both of them. As a result the endogenous specialization is still observed but at a smaller scale than for constant leadership. In the case firms are very close to each other (symmetric efficiencies) the potential for technological spillover may lead to reduction of generation of innovations in both the space of vertical innovations and on the fundamental level. Thus the suggested model supports the claim of inverted-U relationship between intensity of competition (taken as the inverse of technological distance between firms) and output of innovations. This conclusion is stronger than the original of Aghion et al. (2005), since multidimensional and fully dynamic framework of strategic interactions is considered. At the same time the model demonstrates the persistence of cooperation at the fundamental level of research which is the basis for sustainable solution in both areas of innovative activity.

Appendices

A Value functions of the quality game under constant leadership

Under constant leadership value functions of both players are linear in both states. Namely, they have the form

$$V_j(i) = A_1^j q_j(i,t) + A_2^j q_{-j}(i,t) + A_3^j, \tag{A.1}$$

for both players. Solving HJB equations (7) with a constant leader one arrives to a system of algebraic equation on coefficients of both value functions. Solution yields value functions of the following form (for L denoting the leader and F denoting the follower quantities):

$$V^L(i) = \frac{q^L(i,t)}{r+\beta^L} + \frac{1}{2}\frac{(\gamma^L)^2}{r(r+\beta^L)^2}(N-i);$$

$$V^F(i) = \frac{q^F(i,t)}{\theta+r+\beta^L} + \frac{\theta}{r+\beta^L}\cdot\frac{q^L(i,t)}{\theta+r+\beta^F} +$$

$$+\left(\frac{1}{2}\frac{(\gamma^L)^2}{(r+\beta^F+\theta)^2 r} + \frac{\theta\,(\gamma^L)^2}{(r+\beta^F+\theta)\,r\,(\beta^L+r)^2}\right)(N-i). \tag{A.2}$$

B Derivation of optimal strategies for symmetric game of vertical innovations

First observe, that functions (7) are non-differentiable alone the line $q_j(i) = q_{-j}(i)$. Thus the optimal strategies of players cannot be found through usual first order conditions on HJB equations. To derive strategies (16) and (17) first establish the following:

Lemma 15 *In the symmetric case,* (14)*, one-sided derivatives of value functions of players are not equal to each other,*

$$\begin{aligned} &\frac{\partial V_j(i)}{\partial q_j(i)}\Big|_{q_j(i)\xrightarrow{+}q_{-j}(i)} < \frac{\partial V_j(i)}{\partial q_j(i)}\Big|_{q_j(i)\xrightarrow{-}q_{-j}(i)};\\ &\frac{\partial V_j(i)}{\partial q_{-j}(i)}\Big|_{q_j(i)\xrightarrow{+}q_{-j}(i)} > \frac{\partial V_j(i)}{\partial q_{-j}(i)}\Big|_{q_j(i)\xrightarrow{-}q_{-j}(i)}. \end{aligned} \tag{B.1}$$

and thus a continuous spectrum of possible strategies for both players exist, with limits being given by leader's strategy (10) *and follower's strategy* (11).

The proof amounts to direct computation of derivatives of value functions, given that value functions for both players are (A.2). For every player j the derivative is computed using the leader's value function for $q_j(i) \overset{+}{\to} q_{-j}(i)$ and the follower's value function for $q_j(i) \overset{-}{\to} q_{-j}(i)$.

Given (B.1) it is straightforward to conclude, that candidates for optimal strategies for both players are indeed in the above mentioned spectrum. The problem of finding the optimal strategy in symmetric game thus amounts to the problem of selection of a pair of candidates from the spectrum (10), (11). For this establish

Lemma 16 *The pair of optimal controls in symmetric case is a unique one.*

Observe that value functions have to be continuous alone the line $q_j(i) = q_{-j}(i)$. Thus they are unique and have to be generated by a unique pair of optimal controls. Continuity of value functions requires that value fucntino should converge to the same value for $q(i)$ values lower and higher than the symmetric one, $q^{SYM}(i) = q_j(i) = q_{-j}(i)$. This is given by

$$\text{(B.2)} \qquad \lim_{q_j(i)\overset{+}{\to}q^{SYM}(i)} V_j(i) = \lim_{q_j(i)\overset{-}{\to}q^{SYM}(i)} V_j(i) = V^{SYM}(i),$$

with $V^{SYM}(i)$ denoting the symmetrical value of vertical innovations for both players. Assuming linear value functions and constant strategies as for the constant leader-follower case one can compute this limits as:

(B.3)

$$\lim_{q_j(i)\overset{+}{\to}q^{SYM}(i)} V_j(i) = \frac{1}{r+\beta}q^0(i) + \frac{1}{2}\frac{1}{r(r+\beta)}\gamma^2(N-i);$$

(B.4)

$$\lim_{q_j(i)\overset{-}{\to}q^{SYM}(i)} V_j(i) = \frac{\beta+r+\theta}{(r+\beta)(r+\beta+\theta)}q^{SYM}(i) + \frac{1}{2}\frac{(r^2+\beta^2)+2(r+\theta)(\beta+\theta)}{r(r+\beta)^2(r+\beta+\theta)^2}\gamma^2(N-i).$$

These two expression coincide only for $\theta = 0$. Thus we obtain the first pair of symmetric strategies, (16) which correspond to the absence of the spillover effect.

Now consider the case of $\theta > 0$. Value functions of both players ar eno longer continuous alone the symmetry line. However since constant strategies correspond to the open-loop equilibrium, players cannot later their behaviour after choosing their strategies at the point $q_j(i) = q_{-j}(i) = 0$. At this point both players have the incentive to minimize their investments in an effort to benefit from the technological spillover. At the same time the set of possible strategies is given by the range (10), (11). Observe that the minimum of investments is reached for any player j when following (11). Thus with positive technological spillover in symmetric case both players choose the strategy (17).

Value functions of both players under the symmetric regime resembles the one of the leader in constant leadership mode:

$$V^0(i) = \frac{1}{r+\beta} q^0(i) + \frac{1}{2} \frac{1}{r(r+\beta)} \gamma^2 (N-i), \tag{B.5}$$

$$V^\theta(i) = \frac{q^\theta(i,t)}{\theta + r + \beta} \cdot \left(1 + \frac{\theta}{r+\beta}\right) +$$

$$+ \left(\frac{1}{2} \frac{(\gamma)^2}{(r+\beta+\theta)^2 r} + \frac{\theta (\gamma)^2}{(r+\beta+\theta)\, r\, (\beta+r)^2}\right)(N-i). \tag{B.6}$$

C Derivation of optimal strategies for catching-up game of vertical innovations

Lemma 17 (Boundary conditions for value functions) *The player, which is the leader before the trigger, receives the value of the follower in constant leadership game as a payoff after the trigger and the player which is the follower before the trigger, receives the value of the leader in constant leadership game as a payoff after the trigger. They are boundary conditions for value functions of both players before the trigger point:*

$$V_j^{LF}(i)|_{t=t_i^{TRIG}} = V_j^F(i)_{t=t_i^{TRIG}}, \tag{C.1}$$

$$V_{-j}^{FL}(i)|_{t=t_i^{TRIG}} = V_{-j}^L(i)_{t=t_i^{TRIG}}. \tag{C.2}$$

With this salvage values the objectives of both firms before the trigger are (22) and (23). The solution is found by the application of the Maximum Principle. Hamiltonians of both players before the trigger are (with player j being the leader in technology before the trigger for certainty):

$$\mathscr{H}_j = q_j(i,t) - \frac{1}{2}g_j(i,t)^2 + \\ \psi_j^j(i,t)\Big(\gamma_j\sqrt{N-i}g_j(i,t) - \beta_j q_j(t)\Big) + \\ \psi_{-j}^j(i,t)\Big(\gamma_{-j}\sqrt{N-i}g_{-j}^*(i,t) - \beta_{-j}q_{-j}(i,t) + \theta(q_j(i,t) - q_{-j}(i,t))\Big);$$

$$\mathscr{H}_{-j} = q_{-j}(i,t) - \frac{1}{2}g_{-j}(i,t)^2 + \\ \psi_j^{-j}(i,t)\Big(\gamma_j\sqrt{N-i}g_j^*(i,t) - \beta_j q_j(i,t)\Big) + \\ \psi_{-j}^{-j}(i,t)\Big(\gamma_{-j}\sqrt{N-i}g_{-j}(i,t) - \beta_{-j}q_{-j}(i,t) + \theta(q_j(i,t) - q_{-j}(i,t))\Big). \tag{C.3}$$

The time and state constraints of this problem are taken into account by the augmented hamiltonians:

$$\mathscr{L}_j = \mathscr{H}_j + \mu_j(i,t)(q_j(i,t) - q_{-j}(i,t)) + \eta_j(i,t_i^{TRIG})(q_j(i,t_i^{TRIG}) - q_{-j}(i,t_i^{TRIG}));$$

(C.4)
$$\mathscr{L}_{-j} = \mathscr{H}_{-j} + \mu_{-j}(i,t)(q_j(i,t) - q_{-j}(i,t)) + \eta_{-j}(i,t_i^{TRIG})(q_{-j}(i,t_i^{TRIG}) - q_{-j}(i,t_i^{TRIG})).$$

Multipliers $\mu_j(i,t), \mu_{-j}(i,t)$ are zero as long as the j's quality is higher than that of the $-j$ player. This holds during the before the trigger phase always, thus,

$$\mu_j(i,t) = \mu_{-j}(i,t) = 0. \tag{C.5}$$

Since the trigger point is unique, the lagrange multipliers defining this point are similar for both players:

$$\eta_j(i,t_i^{TRIG}) = \eta_{-j}(i,t_i^{TRIG}) = \eta(t_i^{TRIG}), \tag{C.6}$$

Now taking F.O.C. w. r. t. controls of (C.4) one arrives to the formulation of optimal controls as of (24) and (25). Substituting them into (3) and taking j as a leader one arrives to (26).

The system of co-states evolution for player j is the boundary value problem due to Lemma 17:

$$\begin{aligned}
&\frac{\partial}{\partial t}\psi_j^j(i,t) = (r+\beta_j)\psi_j^j(i,t) + \theta\psi_{-j}^j(t) - 1;\\
&\frac{\partial}{\partial t}\psi_{-j}^j(i,t) = (r+\beta_{-j}+\theta)\psi_{-j}^j(i,t);\\
&\psi_j^j(i,T(i)) = \frac{\partial V_j^F(i)|_{t=t_i^{TRIG}}}{\partial q_j(i,T(i))} + \eta(t_i^{TRIG});\\
&\psi_{-j}^j(i,T(i)) = \frac{\partial V_j^F(i)|_{t=t_i^{TRIG}}}{\partial q_{-j}(i,T(i))} - \eta(t_i^{TRIG})
\end{aligned} \tag{C.7}$$

yielding co-sates as functions of η and parameters of the model. The same type of dynamical system holds for player $-j$:

$$\begin{aligned}
&\frac{\partial}{\partial t}\psi_j^{-j}(i,t) = (r+\beta_j)\psi_j^{-j}(i,t) - \theta\psi_{-j}^{-j}(t);\\
&\frac{\partial}{\partial t}\psi_{-j}^{-j}(i,t) = (r+\beta_{-j}+\theta)\psi_{-j}^{-j}(i,t) - 1;\\
&\psi_j^{-j}(i,T(i)) = \frac{\partial V_{-j}^L(i)|_{t=t_i^{TRIG}}}{\partial q_j(i,T(i))} + \eta(t_i^{TRIG});\\
&\psi_{-j}^{-j}(T_i) = \frac{\partial V_{-j}^L(i)|_{t=t_i^{TRIG}}}{\partial q_{-j}(i,T(i))} - \eta(t_i^{TRIG})
\end{aligned} \tag{C.8}$$

with derivatives of value functions computed from (A.2). As these derivatives are constant, co-sates for both players may be expressed as functions of parameters and η only. Thus the system (26) may be solved w. r. t. to η and trigger time.

Finally the trigger time t_i^{TRIG} is defined from the condition (28).

Value functions of both players in the catching-up mode correspond to the optimized Hamiltonians (C.3), since the optimal co-states $\psi_{j,-j}^{j,-j}$ already include the future value of the evolution after the catching-up which are in turn linear functions of i:

$$\begin{aligned}
&V^{LF}(i) = \mathscr{H}^{LF}(g_{opt}^{LF}, g_{opt}^{FL}, q_{opt}^{LF}, q_{opt}^{FL}, \eta(t_i^{TRIG})) =\\
&= f_1^{LF}(\gamma_{j,-j},\beta_{j,-j},\theta,r)q^{LF}(i,t) + f_2^{LF}(\gamma_{j,-j},\beta_{j,-j},\theta,r)q^{FL}(i,t) + f_3^{LF}(\gamma_{j,-j},\beta_{j,-j},\theta,r)(N-i),\\
&V^{FL}(i) = \mathscr{H}^{LF}(g_{opt}^{LF}, g_{opt}^{FL}, q_{opt}^{LF}, q_{opt}^{FL}, \eta(t_i^{TRIG})) =\\
&= f_1^{FL}(\gamma_{j,-j},\beta_{j,-j},\theta,r)q^{LF}(i,t) + f_2^{FL}(\gamma_{j,-j},\beta_{j,-j},\theta,r)q^{FL}(i,t) + f_3^{FL}(\gamma_{j,-j},\beta_{j,-j},\theta,r)(N-i).
\end{aligned} \tag{C.9}$$

D Optimal strategies and solution for the problem of variety expansion

Only the sketch of solution is presented. More detailed formal treatment is fully analogous to the benchmark paper. Hamiltonian functions for both firms:

$$\mathscr{H}_j(n,\lambda_j,u_j,u_{-j}) =$$

(D.1)

$$= \Big((\alpha_j u_j(t)+\alpha_{-j}u_{-j}(t))C_j(N-n(t)) - \frac{1}{2}u_j(t)^2\Big) + \lambda_j(t)(\alpha_j u_j(t)+\alpha_{-j}u_{-j}(t));$$

$$\mathscr{H}_{-j}(n,\lambda_j,u_j,u_{-j}) =$$

(D.2)

$$= \Big((\alpha_j u_j(t)+\alpha_{-j}u_{-j}(t))C_{-j}(N-n(t)) - \frac{1}{2}u_{-j}(t)^2\Big) + \lambda_{-j}(t)(\alpha_j u_j(t)+\alpha_{-j}u_{-j}(t)).$$

The first-order conditions on the investments into the variety expansion for both firms define investments as functions of co-state variables. Substituting these into Hamiltonian functions and writing down co-state equations yield the canonical system for the variety expansion game. Inserting optimal controls into the dynamic constraint for variety expansion together with canonical system constitutes the system of 3 linear ODEs with one initial condition and two boundary conditions (transversal ones) which is then solved. The solution for variety expansion is:

$$n^*(t) = N-(N-n_0)\mathrm{e}^{\frac{1}{2}(r-\sqrt{r(r+4\alpha_j^2C_j+4\alpha_{-j}^2C_{-j})})t}. \tag{D.3}$$

Then I use (D.3) to define explicitly investments of both firms as functions of time:

$$u_j^*(t) = \frac{\alpha_j C_j(r+\sqrt{r(r+4\alpha_j^2C_j+4\alpha_{-j}^2C_{-j})})}{2\alpha_j^2C_j+2\alpha_{-j}^2C_{-j}+r+\sqrt{r(r+4\alpha_j^2C_j+4\alpha_{-j}^2C_{-j})}}\cdot \cdot(N-n_0)\mathrm{e}^{\frac{1}{2}(r-\sqrt{r(r+4\alpha_j^2C_j+4\alpha_{-j}^2C_{-j})})t};$$

$$u_{-j}^*(t) = \frac{\alpha_{-j} C_{-j}(r+\sqrt{r(r+4\alpha_j^2C_j+4\alpha_{-j}^2C_{-j})})}{2\alpha_j^2C_j+2\alpha_{-j}^2C_{-j}+r+\sqrt{r(r+4\alpha_j^2C_j+4\alpha_{-j}^2C_{-j})}}\cdot \cdot(N-n_0)\mathrm{e}^{\frac{1}{2}(r-\sqrt{r(r+4\alpha_j^2C_j+4\alpha_{-j}^2C_{-j})})t}. \tag{D.4}$$

Eqs. (D.3) and (D.4) give Proposition 7.

E Potential values of the quality game for product $i = n(t)$

They are given by setting $q_j(i) = q_{-j}(i) = 0$ in (A.2), (B.6), (C.9):

(E.1)

$$V^L(i)|_{q_j(i)=q_{-j}(i)=0, i=n(t)} = \frac{1}{2}\frac{(\gamma^L)^2}{r(r+\beta^L)^2}(N-n(t));$$

(E.2)

$$V^F(i)|_{q_j(i)=q_{-j}(i)=0, i=n(t)} = \left(\frac{1}{2}\frac{(\gamma^L)^2}{(r+\beta^F+\theta)^2 r} + \frac{\theta\,(\gamma^L)^2}{(r+\beta^F+\theta)\,r\,(\beta^L+r)^2}\right)(N-n(t);$$

(E.3)

$$V^0(i)|_{q_j(i)=q_{-j}(i)=0, i=n(t)} = \frac{1}{2}\frac{1}{r(r+\beta)}\gamma^2(N-n(t)),$$

(E.4)

$$V^\theta(i)|_{q_j(i)=q_{-j}(i)=0, i=n(t)} = \left(\frac{1}{2}\frac{(\gamma)^2}{(r+\beta+\theta)^2 r} + \frac{\theta\,(\gamma)^2}{(r+\beta+\theta)\,r\,(\beta+r)^2}\right)(N-n(t));$$

(E.5)

$$V^{LF}(i)|_{q_j(i)=q_{-j}(i)=0, i=n(t)} = \frac{1}{r}\left(\frac{1}{2}(\gamma^{LF})^2(\psi^{LF}_{LF}(0))^2 + (\gamma^{FL})^2\psi^{LF}_{FL}(0)\psi^{FL}_{FL}(0)\right)(N-n(t));$$

(E.6)

$$V^{FL}(i)|_{q_j(i)=q_{-j}(i)=0, i=n(t)} = \frac{1}{r}\left(\frac{1}{2}(\gamma^{FL})^2(\psi^{FL}_{FL}(0))^2 + (\gamma^{LF})^2\psi^{FL}_{LF}(0)\psi^{LF}_{LF}(0)\right)(N-n(t)).$$

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