Catching-up and falling behind: Effects of learning in an R&D differential game with spillovers

Anton Bondarev   **Alfred Greiner**

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Abstract

In this paper we analyze the dynamics of an R&D differential game allowing for technological spillovers and sigmoid learning functions of multiproduct oligopolies. We demonstrate how the presence of learning together with spillovers may generate a rich set of outcomes, varying from constant leadership to catching-up and falling behind as well as from technology lock-in to a situation with a large number of high quality products. These types of outcomes are qualitatively different both from the single firm dynamics with learning and from the duopoly case with spillovers and without learning.

Keywords: endogenous market structure; learning by doing; technological spillovers; heterogeneous innovations; differential games

JEL classification: C61, C73, L13, L16, O32, O33

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1   Introduction

Technical progress is a key factor of economic growth in market economies. Both the generation of new capital goods as well as the improvement of existing methods of production raise efficiency. The current debate on ways how to realize the transition to a green economy involves the discussion of an optimal industry structure to speed up the introduction of new technologies (such as the transition to renewable energy sources). Yet, it is still not clear whether existing large multi-product firms are more active in R&D, as it is claimed in (Acemoglu and Cao 2015) for example, or whether new entrants invest more. It may be the case that monopolistic firms are prone to preserve existing technologies rather than inventing new ones, leading to a technology lock-in, as discussed in (Krysiak 2011) and later on in (Greiner and Bondarev 2017), while competition may stimulate the industry to move forward to new technologies. One important aspect in this transformation of the market is the role of technological leadership.

This paper deals with the question of whether a constant technological leadership is beneficial for the generation of new technologies or whether the taking over by an initial follower can speed up the introduction of new technologies. A closely related question is whether the competition in R&D is beneficial in terms of helping to avoid a technology lock-in and what the limits for these benefits are.

To answer these questions we analyze a model of multiple innovating firms with doubly-differentiated innovations (vertical and horizontal) where we take into consideration R&D spillovers. The leader is defined as the firm with the highest developed technology i, but, the leadership may change over time[1], either temporarily or permanently. We obtain three qualitatively different scenarios: a permanent technological leadership of one of the companies, a catching-up by the initial follower with a constant leadership afterwards, or a temporary loss of the leading position. The outcome depends on the technological distance between the leader and the followers: once the distance is high, the leader is permanent as in baby food industry with the permanent Nestle leadership, for example. When the distance is lower, there may be a catching-up situation as it happened with computer hardware manufacturing, which experienced the loss of AMD leadership that has not been recovered (yet). The third type of dynamics may happen if the distance

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[1]this notion of leader has nothing in common with the Stackelberg notion: all firms act simultaneously and the leader does not have any advantage except for a higher technological level.

is non-monotonic across applications and the follower can have an advantage in some specific range of technologies, but not in others.

Empirical studies stress the importance of spillovers for an industry, like (Henderson 1993), (Bos, Economidou, and Sanders 2013) with the former finding that leading firms invest more into incremental R&D while followers (new entrants) concentrate on fundamental discoveries and the latter studying the non-monotonicity of relationships across types of innovations over the industry life-cycle. The role of spillovers in R&D has been empirically studied in (Henderson and Cockburn 1996), (Jaffe 1986), (Bernstein and Nadiri 1989) and recently in (Bloom, Schankerman, and Van Reenen 2013), among others, where an ambiguous effect of spillovers with respect to the generation of innovations has been found.

There is a substantial literature on multi-product innovative monopolies, see e.g. (Lambertini and Orsini 2001), (Lambertini 2003), (Lambertini 2009). In these papers multi-product monopolies and oligopolies are considered, but no full dynamical analysis is performed there. Further, these papers do not account for the possibility of both constant leadership and leapfrogging. The papers on R&D cooperation, such as (D'Aspremont and Jacquemin 1988), (Navas and Kort 2007), study the benefits of R&D cartels, but do not allow for costless spillovers (imitations). Further, there exist papers on dynamic strategic interactions in the field of R&D, such as (Reinganum 1982), (Hartwick 1984), (Judd 2003), (Cellini and Lambertini 2002), and (Lambertini and Mantovani 2010), but those do not allow for the multi-dimensionality of interactions, i.e. they do not consider the simultaneous existence of cooperation and imitation.

The framework of heterogeneous innovations has been set forth in the papers (Belyakov, Tsachev, and Veliov 2011), (Bondarev 2012), (Belyakov, Haunschmied, and Veliov 2014), with the first being of a more rigorous nature while the second concentrates on the effect of heterogeneity of a special type on the dynamic behaviour of the monopolist and the last describing an OLG-type economy with heterogeneous products. Later on, that approach has been extended to a setting with multi-agents in (Bondarev 2014).

Problems of market dynamics and oligopoly dynamics have been studied by Carl Chiarella, too, who contributed a lot to the field of economics and finance. Especially in the early stage of his career he analyzed problems of that subject. In (Chiarella, Kemp, and van Long 1989) the authors develop a leader-follower model that analyzes the interaction of R&D, the leakage of knowledge and product pricing. The leader invests

in R&D and sets the product price. Newly developed technologies become available to the follower with a time delay. The paper, then, derives the optimal trajectories and derives several results with respect to comparative statics. The paper by (Chiarella 1991) studies a Cournot oligopoly model with time delays in production and information. It is demonstrated that a Hopf bifurcation can occur giving rise to limit cycles and the conditions leading to that phenomenon are characterized. Imperfect competition on the product and factor markets is allowed for in the model by (Chiarella and Okuguchi 1995). There, the existence and stability of a Cournot duopoly with those characteristics are analyzed and it is demonstrated that antisymmetric equilibria may exist and conditions for local and global stability are derived.

In this paper we combine the two approaches presented in (Bondarev 2016) and in (Greiner and Bondarev 2017). Our goal is to demonstrate how the shape of the learning functions determine whether there is a constant leader or whether the follower catches-up and the leader falls behind. Further, we demonstrate that the sigmoid learning functions may give rise to technological lock-ins implying that the initial condition with respect to the number of products produced is decisive whether the economy ends up in a situation with a small number of low quality goods or whether there are many products produced at a high quality. In addition, we analyze how the number of competitors affects the results of our model.

The rest of the paper is organized as follows. The next section 2 presents the basic model and section 3 derives the optimality conditions and the differential equations describing the dynamics. Section 4 analyzes the dynamics of the model and section 5, finally, concludes.

2   The basic model

We consider an economy with N multiproduct firms engaged in R&D. The firms invest both in horizontal innovations as well as in the improvement of the qualities of existing products, i.e. in vertical innovations. As regards the structure of the R&D competition, we assume that the firms cooperate in the development of new products while the quality improvements of the products are undertaken by each firm separately where the followers can benefit from the technological leader through spillovers. The reason for this model structure is that it may be beneficial for firms to cooperate when the development of

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new products is expensive even if they are competitors otherwise. Further, there exists empirical evidence for such a behavior and in the literature one sometimes refers to such a situation as co-opetition, i.e. the simultaneous existence of cooperation and competition. (Gnyawali and Park 2009) find evidence that in particular small and medium sized enterprises pursue such strategies because they face very large challenges in their attempt to realize technological innovations and co-opetition is an effective means to achieve innovations. But large firms often cooperate, too, in particular in high technology sectors because of several challenges such as shortened product life cycles, the need for large investments in R&D and the importance of technological standards, for example, see (Gnyawali and Park 2011).

Thus, the formal intertemporal optimization problem of the firms can be written as,

$$\forall k \in N \subset \mathbb{R}_{++} : \ J_k \overset{\text{def}}{=} \max_{u(\cdot),g(\cdot)} \int_0^\infty e^{-rt} \left(\int_0^{n(t)} \left[q_k(i,t) - \frac{1}{2} g_k(i,t)^2 \right] di - \frac{1}{2} u_k(t)^2 \right) dt. \quad (1)$$

As mentioned above, the evolution of the variety of products (technologies) is governed by joint investments of the firms whereas the development of the quality of each new product is individual and subject to the imitation effect:

$$\dot{n}(t) = \sum_{k=1}^N u_k(t) - \delta n(t),$$

$$\dot{q}_k(i,t) = \psi_k(i) g_k(i,t) - \beta_k q_k(i,t) + \theta \max\left\{ 0, q_{-k}(i,t) - q_k(i,t) \right\}, \ \forall i \in \mathbf{I} \subset \mathbb{R}_+. \quad (2)$$

The last term in the second equation of (2) gives the spillover effect, where θ determines its magnitude.

We next define what is called leadership in this differential game. To this end we restrict our attention to the case $\beta_k = \beta$ so that the leadership in development of each technology i cannot change in time. Then we can define[2]

Definition 1 (Leader and followers).
The player k is the leader in development of technology i as long as this firm has maximal efficiency of investments into i:

$$\psi_k(i) > \psi_{/\!\!k}(i) \quad (3)$$

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[2]under different β_k leadership may change for each i, as shown in (Bondarev 2016), for constant β across players the leader in each technology remains constant as shown in (Bondarev 2014).

Further, we impose a number of state and control constraints:

$$\forall k \in N :$$

$$\forall i \in \mathbf{I}, \forall t \geq 0 : \left. g_k(i,t) \right|_{i>n(t)} = 0; \tag{4}$$

$$\forall i \in \mathbf{I}, \forall t \geq 0 : q_k(i,t) \geq 0; \tag{5}$$

$$0 \leq g_k(i,t) < \infty, \ 0 \leq u_k(t) < \infty; \tag{6}$$

$$n(t) \in \mathbf{I}. \tag{7}$$

Condition (4) states that each non-existent technology has zero investments while it is out of the market. This makes sense from an economic point of view because it states that there is a difference between the invention of a product and its innovation. Condition (5) states that level of each technology cannot be negative. Condition (6) imposes non-negativity constraints and the boundedness of investments, condition (7) constrains the variety to be positive real range and maximum principle is understood in the sense of (Skritek, Stachev, and Veliov 2014), (Aseev and Veliov 2015)[3].

The functions $\psi_k(i)$ determine the efficiency of R&D in quality innovations so that we refer to them as efficiency functions. These are functions of the variety already invented and we will allow for learning by doing effects implying that $\psi(\cdot)$ is an increasing function. The relevance of learning by doing has been well explored in economics over the last 50 years so that we do not justify that assumption any further. In addition, we posit that $\psi(\cdot)$ features non-linearities and assume that the efficiency function displays a convex-concave shape. This assumption seems to be plausible because at early stages of development, learning effects rise more than proportional since the stock of knowledge is still low such that it is relatively easy to acquire new additional knowledge. However, that cannot go on to infinity because the more knowledge has already been acquired, the more difficult it becomes to acquire additional knowledge. Hence, once a certain level of knowledge has been reached, the efficiency gain of additional knowledge becomes smaller. Therefore, sooner or later the function takes a concave form and converges to a finite value.

Formally, the shape of the efficiency functions $\psi_k(i)$ is described by the following properties for all $k \in N$:

1. $\psi_k(i)$ is continuously differentiable (at least C^2 class)

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[3]this last does not require compactness of the state space, transversality conditions converge to standard ones with constant discount rate

2. $\frac{\partial \psi_k(i)}{\partial i} > 0$;

3. $\forall k \in N : \exists! i_k : \frac{\partial^2 \psi_k(i)|_{i < i_k}}{\partial i^2} > 0, \frac{\partial^2 \psi_k(i)|_{i > i_k}}{\partial i^2} < 0, \frac{\partial^2 \psi_k(i)|_{i = i_k}}{\partial i^2} = 0.$

One particular specification following this assumption is:

$$\psi_k(i) = a\, e^{-b \cdot e^{-d \cdot i}}, \; a, b, d > 0. \tag{8}$$

This functional form captures the non-monotonic learning by doing effect. In particular, due to learning by doing, the efficiency of investments increase across products at a rising speed, then it declines for later products and at last converges to a constant for $i \to \infty$. The latter reflects the fact that learning effects are bounded such that the change in productivity is not an ever accelerating process for a given level of R&D investment. A different function that displays the same shape would be the logistic function. The qualitative analysis, however, would not change with that function just as with any other increasing sigmoid function.

We thus may have catching-up in the space of technologies, but not within each technology development, but instead leadership may change across technologies. We thus define the leader of the game at time t as follows:

Definition 2 (Leader of the game).
At time t the player k is called the leader of the game if this player's efficiency of investments into the next technology to be invented, $n(t)$, is maximal:

$$\psi_k(n(t)) > \psi_{k\!\!/}(n(t)) \tag{9}$$

Given the assumption of a sigmoid efficiency function, one obtains already three qualitatively separate cases for $N = 2$: One with a strict domination of one of the players, $\forall i : \psi_k(i) > \psi_{-k}(i)$, a second one where the initial leader is caught-up and falls behind for $i > i^o$, $\psi_k(i) > (<,=)\psi_{-k}(i)$, for $i < (>,=)i^o$, and, third, the situation where the follower catches-up and the leader falls behind but catches-up for its part and takes the lead again, $\psi_k(i) < \psi_{-k}(i)$, for $i_1^o < i < i_2^o$, and $\psi_k(i) > \psi_{-k}(i)$, for $i < i_1^o$ and for $i > i_2^o$, with equality of the efficiency functions for $i = i_1^o$, $i = i_2^o$. These different cases are illustrated in Figure 1, where we denote the situation with a one-time change of the leadership as the contested leadership.

Now, observe that with convex-concave efficiency functions of the same type there may exist multiple regimes of this differential game (and up to three steady-states for

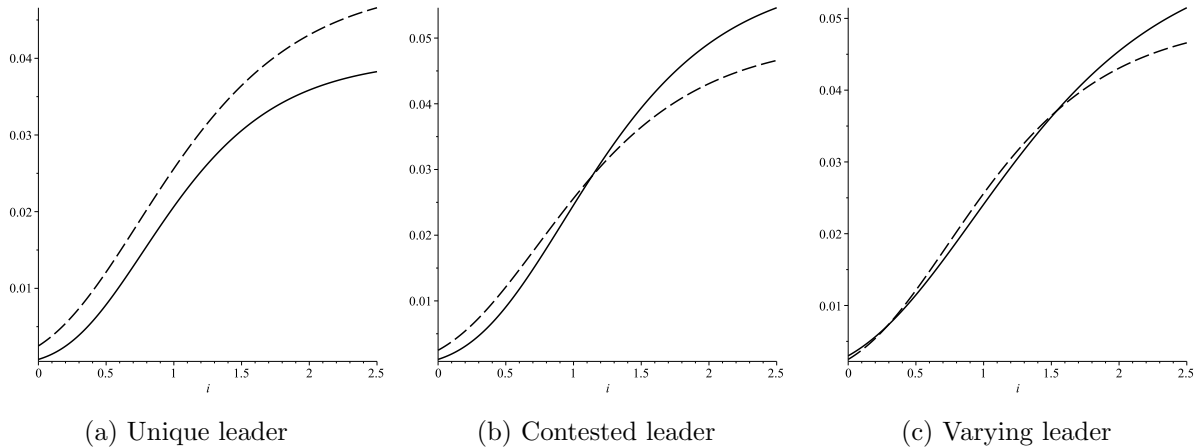|		
(a) Unique leader	(b) Contested leader	(c) Varying leader

Figure 1: Learning dynamics.

each regime), with low and high variety levels. If no intersection of efficiency functions exist, the game has a constant leader (in the sense of (9)) and the dynamical system possesses up to three steady-states, $\bar{n}_1 < \bar{n}_2 < \bar{n}_3$. In the low steady-state no or only few new technologies with low quality are developed, in the high steady-state new technologies are developed and they have higher quality for both firms[4]. If there exists an intersection, as in Figure 1b, one player is the leader in the development of technologies up to the point at which the efficiency functions intersect where the leadership changes. This piecewise-defined system is more complicated and may again be characterized by three steady-states. If a double intersection exists, there will be three different regimes in such a system with up to three steady-states[5].

In the following sections we formally characterize these scenarios, derive the optimality conditions for the three regimes characterized by Figures 1a, 1b, 1c and we discuss the characterization of the steady-states and their dynamics.

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[4]we denote the steady-state with the lowest (largest) value of n as the low (high) steady-state.

[5]In fact, for both latter scenarios each of the regimes possesses up to three steady-states, but some of them may lie above or at the boundary i^o, i_1^o, i_2^o. These are called virtual or boundary equilibria respectively of the underlying system following (Di Bernardo, Budd, Champneys, Kowalczyk, Nordmark, Olivar, and Piiroinen 2008).

3 Optimality conditions and system dynamics

Observe that due to the presence of the imitation term in the quality dynamics, i.e. due to the spillover effect, it is not straightforward to implement the Maximum Principle to the original problem given by (1) subject to (2) since the leadership may switch across players, depending on the regime of the game. Therefore, we have to consider three different cases.

3.1 Unique constant leader

We start with the simplest situation of a constant leadership across products, defined by

$$\forall i \in \bar{\mathbf{I}} \ \exists! k : \ \psi_k(i) > \psi_{-k}(i) \tag{10}$$

In this case, the problem of the leader coincides with the single agent optimization problem and the problems of all the followers (imitators) are similar. Thus, we can set without loss of generality $N = 2$. The current-value Hamiltonians for both players, labeled from now on leader and follower, are:

$$\mathcal{H}_L = \int_0^{n(t)} \Big[q_L(i,t) - \frac{1}{2} g_L(i,t)^2 \Big] di - \frac{1}{2} u_L(t)^2 + \lambda_n^L \cdot (u_L(t) + u_F(t) - \delta\, n(t)) +$$

$$+ \int_0^{n(t)} \lambda_q^L(i,t) \cdot \Big(\psi_L(i) g_L(i,t) - \beta q_L(i,t) \Big) di,$$

$$\mathcal{H}_F = \int_0^{n(t)} \Big[q_F(i,t) - \frac{1}{2} g_F(i,t)^2 \Big] di - \frac{1}{2} u_F(t)^2 + \lambda_n^F \cdot (u_L(t) + u_F(t) - \delta\, n(t)) +$$

$$+ \int_0^{n(t)} \lambda_q^F(i,t) \cdot \Big(\psi_F(i) g_F(i,t) - (\beta + \theta) q_F(i,t) + \theta q_L(i,t) \Big) di \tag{11}$$

with L, F marking leader and follower quantities.

The first order conditions for this problem are given by,

$$u_k(t) = \lambda_n^k(t); \tag{12}$$

$$g_k(i,t) = \psi_k(i)\lambda_q^k(i,t); \tag{13}$$

and the differential equation system for the co-state variables is:

$$\forall k \in N \subset \mathbb{R}_+ :$$

$$\dot{\lambda}_n^k(t) = r\lambda_n^k(t) - \frac{\partial \mathcal{H}_k}{\partial n} =$$

$$= (r+\delta)\lambda_n^k(t) + \frac{1}{2} g_k(n(t),t)^2 - \lambda_q^k(n(t),t)\,\psi_k(n(t))\,g_k(n(t),t); \tag{14}$$

$$\forall i \le n(t) : \dot{\lambda}_q^L(i,t) = r\lambda_q^L(i,t) - \frac{\partial \mathcal{H}_L}{\partial q_L} = (r+\beta)\lambda_q^L(i,t) - 1; \tag{15}$$

$$\dot{\lambda}_q^F(i,t) = r\lambda_q^F(i,t) - \frac{\partial \mathcal{H}_F}{\partial q_F} = (r+\beta+\theta)\lambda_q^F(i,t) - 1. \tag{16}$$

Proposition 1 (Dynamics of the R&D with constant leader).
The dynamics of the R&D are completely described by the following autonomous differential equation system[6],

$$\dot{u}_L(t) = (r+\delta)\,u_L(t) - \frac{\psi_L(n(t))^2}{2\,(r+\beta)^2}; \tag{17}$$

$$\dot{u}_F(t) = (r+\delta)\,u_F(t) - \frac{\psi_F(n(t))^2}{2\,(r+\beta+\theta)^2}\;; \tag{18}$$

$$\dot{n}(t) = \sum_{k=1}^{N} u_k(t) - \delta\,n(t),\ n(0) = n_0; \tag{19}$$

$$\forall i \le n(t) : q_L(i,0) = q_F(i,0) = q_0(i),$$

$$\dot{q}_L(i,t) = \frac{\psi_L(i)^2}{(r+\beta)} - \beta\,q_L(i,t); \tag{20}$$

$$\forall k \ne L : \dot{q}_F(i,t) = \frac{\psi_F(i)^2}{(r+\beta+\theta)} - (\beta+\theta)\,q_F(i,t) + \theta q_L(i,t). \tag{21}$$

Proof. The system (17)-(21) follows from application of the Maximum principle to the problem described by Hamiltonians (11) and thus completely describes the resulting dynamics. $\qquad\square$

We next define the new variable, which gives the total sum of variety investments:

$$U \stackrel{def}{=} \sum_{k}^{N} u_k \rightarrow \dot{U} = (r+\delta)U - \left(\sum_{k=1,k\ne L}^{N-1} \frac{\psi_k(n(t))^2}{2\,(r+\beta+\theta)^2}\right) - \frac{\psi_L(n(t))^2}{2\,(r+\beta)^2} \tag{22}$$

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[6]we always require the transversality conditions $\lim_{t\to\infty} e^{-rt}\lambda_k(t) = 0, k = L, F$, with $\lambda_k(t) = u_k(t)$, to be met without explicitly mentioning them from now on.

The evolution of the whole system is then defined by the couple $(U(t), n(t))$ and could be analyzed through conventional methods. It is important to notice that $u_k(t)$ are smoothly differentiable functions only if the constant leadership condition (10) holds.

3.2 Contested leadership

If condition (10) fails, the Hamiltonians cannot be written down in a smooth way, since there exists at least one point of intersection of the efficiency functions, indexed by i^o, such that for all technologies beyond this point the other firm becomes the leader in innovations. We first assume that there is only one point of intersection,

$$\exists! i^o : \forall i < i^o : \psi_k(i) > \psi_{-k}(i), \forall i \geq i^o : \psi_k(i) \leq \psi_{-k}(i) \tag{23}$$

Then, we may still construct Hamiltonians using the bounded domain of $n(t)$ as follows[7]:

$$\mathcal{H}_{LF} = \int_0^{n(t)} \left[q_{LF}(i,t) - \frac{1}{2} g_{LF}(i,t)^2 \right] di - \frac{1}{2} u_{LF}(t)^2 + \lambda_n^{LF} \cdot (u_{LF}(t) + u_{FL}(t) - \delta n(t)) +$$

$$+ \int_0^{n(t)} \lambda_q^{LF}(i,t) \cdot \left(\psi_{LF}(i) g_{LF}(i,t) - \beta q_{LF}(i,t) \right) di +$$

$$\int_0^{i^o} \lambda_q^{LF}(i,t) \cdot 0 \, di + \int_{i^o}^{\bar{\mathbf{I}}} \lambda_q^{LF}(i,t) \cdot \theta(q_{FL}(i,t) - q_{LF}(i,t)) di,$$

$$\mathcal{H}_{FL} = \int_0^{n(t)} \left[q_{FL}(i,t) - \frac{1}{2} g_{FL}(i,t)^2 \right] di - \frac{1}{2} u_{FL}(t)^2 + \lambda_n^{FL} \cdot (u_{LF}(t) + u_{FL}(t) - \delta n(t)) +$$

$$+ \int_0^{n(t)} \lambda_q^{FL}(i,t) \cdot \left(\psi_{FL}(i) g_{FL}(i,t) - \beta q_{FL}(i,t) \right) di +$$

$$\int_0^{i^o} \lambda_q^{FL}(i,t) \cdot \theta(q_{LF}(i,t) - q_{FL}(i,t)) di + \int_{i^o}^{\bar{\mathbf{I}}} \lambda_q^{FL}(i,t) \cdot 0 \, di \tag{24}$$

where FL, LF marks initial follower and subsequent leader and vice versa quantities. As long as $n(t) \in \bar{\mathbf{I}}$, at any time t the Hamiltonian function is continuous in the states. However, there is a switching point $t^o : n(t^o) = i^o$, where the players change their relative positions.

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[7]observe that the restriction $N = 2$ becomes essential in this case: with $N > 2$ more than one intersection may exist.

First-order necessary conditions as expressed through costates remain the same, (12), (13), but the quality co-state equations now differ across the range of products:

$$\dot{\lambda}_q^{LF}(i,t) = \begin{cases} (r+\beta)\lambda_q^{LF}(i,t) - 1, i < i^o \\ (r+\beta+\theta)\lambda_q^{LF}(i,t) - 1, i > i^o \end{cases} \tag{25}$$

$$\dot{\lambda}_q^{FL}(i,t) = \begin{cases} (r+\beta+\theta)\lambda_q^{FL}(i,t) - 1, i < i^o \\ (r+\beta)\lambda_q^{FL}(i,t) - 1, i > i^o \end{cases} \tag{26}$$

This implies that for technologies before i^o one player acts as a leader and for technologies beyond i^o the other player becomes the leader in qualities development. Still, for every given i the co-state variable $\lambda(t)_q^k(i,t)$ is a continuous function of time.

For the index i^o none of the firms is the leader and the symmetric special regime occurs. In that regime we have[8]:

$$i = i^o : \dot{\lambda}_q^{LF}(i,t) = (r+\beta+\theta)\lambda_q^{LF}(i,t) - 1, \ \dot{\lambda}_q^{FL}(i,t) = (r+\beta+\theta)\lambda_q^{FL}(i,t) - 1 \tag{27}$$

As a consequence, the differential equations giving the evolution of the co-states of variety expansion are also defined piecewise:

$$\dot{\lambda}_n^k(t) \overset{(14)}{=} (r+\delta)\lambda_n^k(t) - \frac{1}{2}\left(\lambda_q^k(n(t),t)\,\psi_k(n(t))\right)^2;$$

$$\dot{\lambda}_n^{FL}(t) = \begin{cases} (r+\delta)\lambda_n^{FL}(t) - \frac{1}{2}\frac{\psi_{FL}(n(t))^2}{(r+\beta+\theta)^2}, n(t) \leq i^o \\ (r+\delta)\lambda_n^{FL}(t) - \frac{1}{2}\frac{\psi_{FL}(n(t))^2}{(r+\beta)^2}, n(t) > i^o \end{cases} \tag{28}$$

$$\dot{\lambda}_n^{LF}(t) = \begin{cases} (r+\delta)\lambda_n^{LF}(t) - \frac{1}{2}\frac{\psi_{LF}(n(t))^2}{(r+\beta)^2}, n(t) < i^o \\ (r+\delta)\lambda_n^{LF}(t) - \frac{1}{2}\frac{\psi_{LF}(n(t))^2}{(r+\beta+\theta)^2}, n(t) \geq i^o \end{cases} \tag{29}$$

with the $n(t)$ dynamics being the same as above.

Thus, we can summarize our results in the following proposition:

Proposition 2 (Dynamics of the R&D with a contested leadership).

The dynamics of the R&D are completely described by the following differential equation

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[8]it has been shown in (Bondarev 2016) that for $\theta > 0$ both players behave as followers in symmetric regime and as leaders if $\theta = 0$. We thus assume $\theta > 0$ everywhere.

system,

$$\dot{u}_{FL}(t) = \begin{cases} (r+\delta)\,u_{FL}(t) - \frac{1}{2}\frac{\psi_{FL}(n(t))^2}{(r+\beta+\theta)^2}, n(t) \leq i^o \\ (r+\delta)u^{FL}(t) - \frac{1}{2}\frac{\psi_{FL}(n(t))^2}{(r+\beta)^2}, n(t) > i^o \end{cases} \quad ; \tag{30}$$

$$\dot{u}_{LF}(t) = \begin{cases} (r+\delta)u^{LF}(t) - \frac{1}{2}\frac{\psi_{LF}(n(t))^2}{(r+\beta)^2}, n(t) < i^o \\ (r+\delta)u^{LF}(t) - \frac{1}{2}\frac{\psi_{LF}(n(t))^2}{(r+\beta+\theta)^2}, n(t) \geq i^o \end{cases} \quad ; \tag{31}$$

$$\dot{n}(t) = \sum_{k=1}^{N} u_k(t) - \delta\,n(t), \ n(0) = n_0; \tag{32}$$

$$\forall i \leq n(t) : q_{LF}(i,0) = q_{FL}(i,0) = q_0(i),$$

$$\dot{q}_{FL}(i,t) = \begin{cases} \frac{\psi_{FL}(i)^2}{(r+\beta+\theta)} - (\beta+\theta)q_{FL}(i,t) + \theta q_{LF}(i,t), i \leq i^o \\ \frac{\psi_{FL}(i)^2}{r+\beta} - \beta q_{FL}(i,t), i > i^o \end{cases}$$

$$\dot{q}_{LF}(i,t) = \begin{cases} \frac{\psi_{LF}(i)^2}{(r+\beta)} - \beta q_{LF}(i,t), i < i^o \\ \frac{\psi_{LF}(i)^2}{r+\beta+\theta} - (\beta+\theta)q_{LF}(i,t) + \theta q_{FL}(i,t), i \geq i^o \end{cases} \quad . \tag{33}$$

Proof. The same as for Proposition 1. $\qquad\square$

The total variety investments now follow the dynamic law:

$$\dot{U} = \begin{cases} (r+\delta)U - \frac{1}{2}\left(\frac{\psi_{FL}(n(t))^2}{(r+\beta+\theta)^2} + \frac{\psi_{LF}(n(t))^2}{(r+\beta)^2}\right), n(t) < i^o \\ (r+\delta)U - \frac{1}{2}\left(\frac{\psi_{FL}(n(t))^2}{(r+\beta+\theta)^2} + \frac{\psi_{LF}(n(t))^2}{(r+\beta+\theta)^2}\right), n(t) = i^o \\ (r+\delta)U - \frac{1}{2}\left(\frac{\psi_{LF}(n(t))^2}{(r+\beta+\theta)^2} + \frac{\psi_{FL}(n(t))^2}{(r+\beta)^2}\right), n(t) > i^o \end{cases} \tag{34}$$

Observe that at $n(t) = i^o$ the so-called special regime occurs and both firms act as followers.

3.3 Varying leadership

Finally, we consider the case of a varying leaderships as illustrated in Figure 1c (with $N = 2$ as before) where a double intersection of the efficiency functions exists. The formal condition for that case is:

$$\exists i^1 : \forall i < i^1 : \psi_{-k}(i) > \psi_k(i); \tag{35}$$

$$\exists i^2 > i^1 : \forall i^1 < i < i^2 : \psi_k(i) > \psi_{-k}(i); \tag{36}$$

$$\forall i > i^2 : \psi_{-k}(i) > \psi_k(i). \tag{37}$$

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We can use the boundedness of the state space to construct Hamiltonians for both players in this case:

$$\mathcal{H}_{LL} = \int_0^{n(t)} \left[q_{LL}(i,t) - \frac{1}{2} g_{LL}(i,t)^2 \right] di - \frac{1}{2} u_{LL}(t)^2 + \lambda_n^{LL} \cdot (u_{LL}(t) + u_{FF}(t) - \delta\, n(t)) +$$

$$+ \int_0^{n(t)} \lambda_q^{LL}(i,t) \cdot \left(\psi_{LL}(i) g_{LL}(i,t) - \beta q_{LL}(i,t) \right) di +$$

$$+ \int_{i^1}^{i^2} \lambda_q^{LL}(i,t) \cdot \theta(q_{FF}(i,t) - q_{LL}(i,t)) di,$$

$$\mathcal{H}_{FF} = \int_0^{n(t)} \left[q_{FF}(i,t) - \frac{1}{2} g_{FF}(i,t)^2 \right] di - \frac{1}{2} u_{FF}(t)^2 + \lambda_n^{FF} \cdot (u_{LL}(t) + u_{FF}(t) - \delta\, n(t)) +$$

$$+ \int_0^{n(t)} \lambda_q^{FF}(i,t) \cdot \left(\psi_{FF}(i) g_{FF}(i,t) - \beta q_{FF}(i,t) \right) di +$$

$$\int_0^{i^1} \lambda_q^{FF}(i,t) \cdot \theta(q_{LL}(i,t) - q_{FF}(i,t)) di + \int_{i^2}^{\bar{\mathbf{I}}} \lambda_q^{FF}(i,t) \cdot \theta(q_{LL}(i,t) - q_{FF}(i,t)) di \qquad (38)$$

where LL marks initial and final leader and FF marks initial and final follower.

The dynamics of the co-states system is now characterized by three different regimes:

$$\dot{\lambda}_q^{LL}(i,t) = \begin{cases} (r+\beta)\lambda_q^{LL}(i,t) - 1, i < i^1 \\ (r+\beta+\theta)\lambda_q^{LL}(i,t) - 1, i^2 \geq i \geq i^1 \\ (r+\beta)\lambda_q^{LL}(i,t) - 1, i > i^2 \end{cases} \qquad (39)$$

$$\dot{\lambda}_q^{FF}(i,t) = \begin{cases} (r+\beta+\theta)\lambda_q^{FF}(i,t) - 1, i \leq i^1 \\ (r+\beta)\lambda_q^{FF}(i,t) - 1, i^2 > i > i^1 \\ (r+\beta+\theta)\lambda_q^{FF}(i,t) - 1, i \geq i^2 \end{cases} \qquad (40)$$

and the same is true for the variety expansion dynamics.

From an economic point of view, the existence of two intersection points of the efficiency functions implies that the initial leader is caught-up by the follower which then becomes the leader. However, the new follower catches-up for its part and becomes again the leader once the second intersection point of the efficiency functions is reached. Then, the initial follower, which temporarily was the leader, becomes again the follower and the initial leader takes its leading position again. The following Proposition 3 summarizes the equations describing the dynamics in this situation.

Proposition 3 (Dynamics of the R&D with a varying leadership).

The dynamics of the R&D are completely described by the following differential equation system,

$$\dot{u}_{FF}(t) = \begin{cases} (r+\delta)\,u_{FF}(t) - \frac{1}{2}\frac{\psi_{FF}(n(t))^2}{(r+\beta+\theta)^2}, n(t) \leq i^1 \\ (r+\delta)u_{FF}(t) - \frac{1}{2}\frac{\psi_{FF}(n(t))^2}{(r+\beta)^2}, i^2 > n(t) > i^1 \\ (r+\delta)\,u_{FF}(t) - \frac{1}{2}\frac{\psi_{FF}(n(t))^2}{(r+\beta+\theta)^2}, n(t) \geq i^2 \end{cases} \quad ; \tag{41}$$

$$\dot{u}_{LL}(t) = \begin{cases} (r+\delta)u_{LL}(t) - \frac{1}{2}\frac{\psi_{LL}(n(t))^2}{(r+\beta)^2}, n(t) < i^1 \\ (r+\delta)u_{LL}(t) - \frac{1}{2}\frac{\psi_{LL}(n(t))^2}{(r+\beta+\theta)^2}, i^2 \geq n(t) \geq i^1 \\ (r+\delta)u_{LL}(t) - \frac{1}{2}\frac{\psi_{LL}(n(t))^2}{(r+\beta)^2}, n(t) > i^2 \end{cases} \quad ; \tag{42}$$

$$\dot{n}(t) = \sum_{k=1}^{N} u_k(t) - \delta\,n(t), \; n(0) = n_0; \tag{43}$$

$$\forall i \leq n(t) : q_{LL}(i,0) = q_{FF}(i,0) = q_0(i),$$

$$\dot{q}_{FF}(i,t) = \begin{cases} \frac{\psi_{FF}(i)^2}{(r+\beta+\theta)} - (\beta+\theta)q_{FF}(i,t) + \theta q_{LL}(i,t), i \leq i^1 \\ \frac{\psi_{FF}(i)^2}{r+\beta} - \beta q_{FF}(i,t), i^2 > i > i^1 \\ \frac{\psi_{FF}(i)^2}{(r+\beta+\theta)} - (\beta+\theta)q_{FF}(i,t) + \theta q_{LL}(i,t), i \geq i^2 \end{cases}$$

$$\dot{q}_{LL}(i,t) = \begin{cases} \frac{\psi_{LL}(i)^2}{(r+\beta)} - \beta q_{LL}(i,t), i < i^1 \\ \frac{\psi_{LL}(i)^2}{r+\beta+\theta} - (\beta+\theta)q_{LL}(i,t) + \theta q_{FF}(i,t), i^2 \geq i \geq i^1 \\ \frac{\psi_{LL}(i)^2}{(r+\beta)} - \beta q_{LL}(i,t), i > i^2 \end{cases} \quad . \tag{44}$$

Proof. The same as for Proposition 1. □

The total variety investments now follow the dynamic law:

$$\dot{U} = \begin{cases} (r+\delta)U - \frac{1}{2}\left(\frac{\psi_{FF}(n(t))^2}{(r+\beta+\theta)^2} + \frac{\psi_{LL}(n(t))^2}{(r+\beta)^2}\right), n(t) < i^1 \\ (r+\delta)U - \frac{1}{2}\left(\frac{\psi_{LL}(n(t))^2}{(r+\beta+\theta)^2} + \frac{\psi_{FF}(n(t))^2}{(r+\beta)^2}\right), i^2 > n(t) > i^1 \\ (r+\delta)U - \frac{1}{2}\left(\frac{\psi_{FF}(n(t))^2}{(r+\beta+\theta)^2} + \frac{\psi_{LL}(n(t))^2}{(r+\beta)^2}\right), n(t) > i^2 \\ (r+\delta)U - \frac{1}{2}\left(\frac{\psi_{FF}(n(t))^2}{(r+\beta+\theta)^2} + \frac{\psi_{LL}(n(t))^2}{(r+\beta+\theta)^2}\right), n(t) = i^1; n(t) = i^2 \end{cases} \tag{45}$$

Observe that in the case of a varying leadership, the total investments have two special regimes where both firms act as followers: for $n(t) = i^1$ and for $n(t) = i^2$.

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4 Analysis of the dynamics of the model

We now turn to the study of the long-term dynamics of the model given in Propositions 1, 2 and 3.

4.1 Constant leader

If one of the firms has higher investment efficiency than the other for any i, the dynamics of the system is fully characterized by the pair of equations (19), (22)[9]. This is a 2-dimensional autonomous ODE system with steady-state equilibria characterized by[10]

$$\dot{n} = 0, \ \dot{U} = 0. \tag{46}$$

We start with the description of the global dynamics of the game in this simplest case. It is given by the following Proposition, where the $^-$ denotes steady-state values:

Proposition 4 (Global dynamics of the game with constant leader).
The differential game characterized by (19), (22) has the following global properties:

1. *It may have up to three open-loop equilibria, with $\bar{n}_1 < \bar{n}_2 < \bar{n}_3$ and $\bar{U}_1 < \bar{U}_2 < \bar{U}_3$;*

2. *Only situations with one equilibrium ($\{\bar{n}_1, \bar{U}_1\}$ or $\{\bar{n}_3, \bar{U}_3\}$) or three equilibria are structurally stable;*

3. *If one equilibrium exists, it is saddle-point stable;*

4. *If three equilibria exist, $\{\bar{n}_1, \bar{U}_1\}, \{\bar{n}_3, \bar{U}_3\}$ are saddle-point stable and $\{\bar{n}_2, \bar{U}_2\}$ is either an unstable node or an unstable focus.*

5. *If $\{\bar{n}_2, \bar{U}_2\}$ is an unstable focus and no heteroclinic connections exist across two other equilibria, there exist thresholds n_s^k, $k = L, F$, such that the industry ends up at $\{\bar{n}_1, \bar{U}_1\}$ if $n_0 < n_s^k$ and at $\{\bar{n}_3, \bar{U}_3\}$ if $n_0 > n_s^k$.*

The proof involves several Lemmas, which can be found together with their proofs in the Appendix.

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[9] since $q(i, t)$ dynamics is fully defined by these.

[10] this is indeed the case as long as $\forall k, \forall t : u_k(t) \geq 0$ as required by (7).

Proposition 4 demonstrates that multiple steady-states can exist. If the system converges to the higher steady-state, new technologies are developed and the refinement of existing technologies is going on. The leader invests more into the refinement of existing technologies and the follower invests more into the expansion of the variety of technologies up to $\bar{n}_3$. The resulting strategic behaviour is qualitatively similar to the one described in (Bondarev 2014), but with different investment efficiency functions of the players. It should also be noted that there can exist two such thresholds, one for the leader and one for the follower, since this is a game.

If the system converges to the low steady-state, new technologies are not developed at all, and some existing technologies are scrapped. All firms invest less than necessary to support the existing research infrastructure up to the point when the variety of technologies stabilizes at the $\bar{n}_1$ level. The quality of those technologies which are scrapped is stopped to be improved as soon as $i > n(t)$ and declines to zero. Only those technologies for which $i \in \bar{n}_1$ are improved up to the steady-state levels $\bar{q}_1^k(i)$, different for all players. As long as $\beta_k = \beta_{-k}$ the leader's quality in the steady-state is always higher than that of the follower(s) and the imitation effect is present.

Since optimal investments are given by (13) for all players it follows that the leader invests more than followers in all technologies (either long-term surviving or those to be scrapped) and the followers invest more into the support of the current variety, making the convergence to the low steady-state slower than for the single monopolist case.

Proposition 5 (The role of competitive fringe with a constant leader).
If there exists a constant technological leader in the industry ((10) holds) and multiple steady-states exist (Prop. 4 holds), then the following holds true:

1. *If $n_0 > n_s$ the speed of variety expansion is increasing in the number of followers N;*

2. *If $n_0 < n_s$ the speed of variety reduction is the smaller the higher is the number of followers N;*

3. *The profit of the leader is increasing with the number of followers N in both cases.*

Proof. The proof of points 1 and 2 amounts to a direct comparison of investments $u_k(t)$ of followers with that of the leader and from the observation that $U(t)$ is monotonically increasing in N. Point 3 follows from the fact that the longer the given technology i is present on the market, the higher is the profit the leader draws from it and the span of time

technology i exists positively depends on the speed of variety expansion (i.e. negatively on the speed of degradation). □

Thus, we see that the presence of followers is beneficial for the leading firm, both in the case of convergence to the low steady-state and in case the industry converges to the high steady-state. This outcome is the result of joint efforts to expand the number of products $n(t)$ and from the endogenous specialization of investments. The multiplicity of possible steady-states is the consequence of the assumed learning dynamics.

In order to illustrate the existence of a threshold, we next present an example where we numerically calculate the value of the thresholds. It should be recalled that there exist two such thresholds, one for the leader and one for the follower. Further, the thresholds are either determined by the bounding trajectory of the unstable focus or by that value of initial technologies where convergence to the low steady-state yields the same value for the functional (1) as convergence to the high steady-state, in case when the starting value of the technologies lies in the range that allows convergence to either the low or to the high steady-state. In the latter case, one often denotes the threshold as a Skiba point.

As mentioned above, in principle the analysis is the same as the one in (Greiner and Bondarev 2017). However, due to the fact that this is a game, the threshold of the initial state of technologies n_s^k cannot be determined by using (46) alone, but one needs in addition the value of u_L for the leader and of $u_F = U - u_L$ for the follower, respectively. To determine the threshold we compute the difference of the maximized Hamiltonians as a function of the initial state and of the initial controls, $\Delta H^0 = H^0(u_j^1(0), U^1(0), n_0, \cdot)/r - H^0(u_j^2(0), U^2(0), n_0, \cdot)/r$, $j = L, F$, with the superscript [1] and [2] denoting the inital controls such that the system converges to the high and to the low steady-state, respectively. Depending on whether the difference of the maximized Hamiltonians is positive or negative for a certain $\{u_j(0), U(0)\}$, $j = L, F$, and for a given n_0, convergence to the high or to the low steady-state is optimal. The threshold n_s^k is reached when the maximized Hamiltonian for the combination $\{u_j(0), U(0)\}$ leading to the low steady-state takes the same value as for that combination converging to the high steady-state or when the bounding trajectory is reached.

For the numerical example, we choose the following parameter values. The parameter in the leader's efficiency function $\psi_L(\cdot)$ are set to $a_L = 0.047$, $b_L = 3$ and $d_L = 1.5$ and the parameter in the follower's efficiency function $a_F = 0.04$, $b_F = 4$ and $d_F = 1.8$. The

other parameter values are $r = 0.035$, $\beta = 0.05$, $\theta = 0.2$ and $\delta = 0.21$. The corresponding steady-states are obtained as $\{\bar{n} = 7.98 \cdot 10^{-3}, \bar{U} = 1.68 \cdot 10^{-3}, \bar{u}_L = 1.66 \cdot 10^{-3}\}$, $\{\bar{n} = 1.229, \bar{U} = 0.258, \bar{u}_L = 0.241\}$ and $\{\bar{n} = 2.949, \bar{U} = 0.619, \bar{u}_L = 0.241\}$, where the first and the third are saddle-point stable while the second is an unstable focus.

With these parameter values, we then numerically solved the differential equation system given by (17), (19) and (22) and computed the difference of the maximized Hamiltonians $\Delta H^0(\cdot)$ for the leader and for the follower. This shows that for $n = n_s^L = 0.565$ the maximized Hamiltonians for the leader take the same value independent of whether the initial controls are set such that the system converges to the low or to the high steady-state. Doing the same for the follower shows that convergence to the high steady-state always gives a larger value for the objective functional (1) than convergence to the low steady-state in the range between the bounding trajectories of the unstable focus, that is in the range where the system could converge to either the low or to the high steady-state. In that case, the bounding trajectory represents the threshold which is given by $n_s^F = 0.366$.[11]

These computations show that the system converges to the low steady-state if the initial number of technologies n_0 is smaller than $n_s^F = 0.366$ and it converges to the high steady-state for $n_0 > n_s^L = 0.565$. But, this result implies that for initial technologies $n_0 \in (n_s^F = 0.366, n_s^L = 0.565)$ it cannot be determined to which steady-state the system converges. In that case the outcome of the game depends on which of the two players can enforce his strategy. If the leader dominates, the system will converge to the low steady-state, if the follower dominates the system will converge to the high steady-state.

The following figure 2 illustrates the result in the $(n - u_L - U)$ phase space where we assumed that the leader is the dominating player[12]. The two grey trajectories marked with arrows indicate the optimal paths converging to the low and to the high steady-state, respectively, depending on whether the industry starts at a value below or above the threshold n_s^L. The spirals in the phase space give the trajectories of the unstable focus, i.e. of the middle steady-state, with the grey spirals showing the projection onto the (n, u_L) plane.

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[11]However, the difference between the two Hamiltonians for that value is very small ($1.9 \cdot 10^{-4}$).

[12]we multiplied the values of u_L by two for reasons of clearness of the graphic.
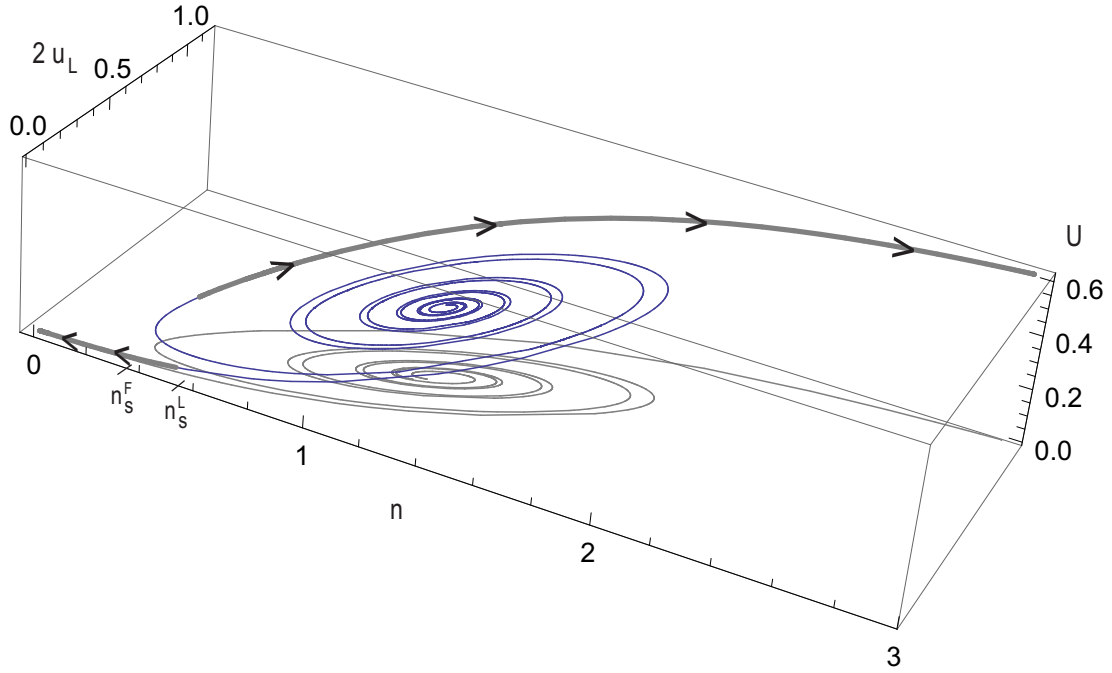
Figure 2: Thresholds n_s^F, n_s^L and optimal paths in the $(n - u_L - U)$ phase space when the leader dominates.

4.2 Regime switching with changing leadership

Next, we analyze the cases when the leadership changes. Conditions on efficiency functions for it to happen are given by (23) and by (35)-(37), defining two and three different regions of the state space, respectively. These situations are illustrated in Figures 1b and 1c. Assuming that such intersections exist, the initial range of technologies (which is the same for all players) will define the regime of the game and whether a high or a low variety of technologies will realize.

We start with the definition of regimes of the game.

Definition 3 (Regimes of the game).
The system (32), (34) is in the lower regime as long as $n(t) < i^o$ holds. It is in the upper regime as long as $n(t) > i^o$ holds.

For the varying leadership case we have a lower, an upper and a medium regime which appears for $i_2^o > n(t) > i_1^o$.

4.2.1 Contested leadership

First, we study the case of the contested leadership as the dynamics of the varying leadership is in most cases qualitatively the same. Assume that $n_0 < i^o$. Then, the initial dynamics of the system is given by:

$$\dot{U}_O(t) = (r + \delta)U - \frac{1}{2}\left(\frac{\psi_{FL}(n(t))^2}{(r + \beta + \theta)^2} + \frac{\psi_{LF}(n(t))^2}{(r + \beta)^2}\right),$$
$$\dot{n}_O(t) = U(t) - \delta\, n(t), \tag{47}$$

where the subscript O denotes the lower regime.

For $n(t^0) = i^o$ the symmetric special regime is obtained, where both firms act as followers. The dynamic system for this regime reads as

$$\dot{U}_S(t^0) = (r + \delta)U - \frac{1}{2}\left(\frac{\psi_{FL}(i^o)^2}{(r + \beta + \theta)^2} + \frac{\psi_{LF}(i^o)^2}{(r + \beta + \theta)^2}\right),$$
$$\dot{n}_S(t^0) = U - \delta\, i^o, \tag{48}$$

where the subscript S denotes the special regime.

If the resulting dynamics of $n(t)$ is positive, the system will leave the special regime in the direction of the upper regime. There, the dynamics is given by:

$$\dot{U}_U(t) = (r + \delta)U - \frac{1}{2}\left(\frac{\psi_{LF}(n(t))^2}{(r + \beta + \theta)^2} + \frac{\psi_{FL}(n(t))^2}{(r + \beta)^2}\right),$$
$$\dot{n}_U(t) = U(t) - \delta\, n(t), \tag{49}$$

where the subscript U denotes the upper regime.

Due to the sigmoid learning function this system can have three steady-states and may by characterized by a threshold that separates the two basins of attraction. It should be noted that, as in the case with a unique leader, there can exist one threshold for the leader and one for the follower. However, only the dominant player is decisive and it is this threshold that determines the outcome of the game.

Every regime in the absence of the switch may have up to three steady-state equilibria of the same type as the game with the constant leader. We denote by $\{\bar{n}, \bar{U}\}_{1,2,3}$ and $\{\hat{n}, \hat{U}\}_{1,2,3}$ steady-state equilibria of the lower and of the upper regime, respectively. We distinguish then between regular, virtual and border equilibria following (Di Bernardo, Budd, Champneys, Kowalczyk, Nordmark, Olivar, and Piiroinen 2008):

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Definition 4 (Types of equilibria)**.**
The steady-state equilibrium $\{\bar{n}, \bar{U}\}_j, j \in \{1, 2, 3\}$, of the lower regime is regular, if $\bar{n}_j < i^o$.
The steady-state equilibrium of the upper regime is regular if $\hat{n}_j > i^o$.

The steady-state equilibrium j of the lower (upper) regime is virtual, if $\bar{n}_j > i^o (\hat{n}_j < i^o)$.

The steady-state equilibrium of the lower (upper) regime is a border equilibrium, if $\bar{n}_j = i^o (\hat{n}_j = i^o)$.

From the shape of the learning curves and due to the fact that they intersect once it follows that at least one of the equilibria of every regime is a border or a virtual equilibrium, i.e. it cannot be the case that all equilibria of one regime are situated at the same side of the intersection[13] i^o. Hence, it is always possible to find an initial value n_0 such that the system will move to the switching manifold $\Sigma_o : n(t) = i^o$ at least once.

Further, in analogy to the case of a permanent leadership, the situation with a contested leadership is expected to be characterized by a threshold that separates the basins of attraction. We denote that threshold for the contested leadership case by n_{as}. In Lemma 1 we give conditions such that the systems changes from one regime to the other.

Lemma 1. *As long as initial conditions are such that either $i^o > n_0 > n_{as}$ or $i^o < n_0 < n_{as}$ the game will reach the switching manifold Σ_o at least once.*

Proof. By definition of the threshold level n_{as}. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now, assume that the system starts in the lower regime and that the initial number of technologies n_0 is larger than the threshold n_{as} but lower than i^o. Then, investments are made such that the number of technologies rises over time and reaches the special regime at $t = t^o$ that is characterized by $n(t^o) = i^o$. If the dynamics of $n(t)$ is positive, the system will leave the special regime in the direction of the upper regime and converge to the high steady-state $\{\hat{n}, \hat{U}\}_3$. If the initial number of technologies n_0 falls short of the threshold n_{as}, the system converges to the low steady-state $\{\bar{n}, \bar{U}\}_1$ and no regime change occurs.

As soon as a regime change is possible, three distinct types of dynamics are feasible[14]:

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[13]The intersection $n = i^o$ is known as the switching manifold that we denote by Σ_o following the literature on piece-wise smooth systems.

[14]here, we adopt the more general definition from (Di Bernardo, Budd, Champneys, Kowalczyk, Nordmark, Olivar, and Piiroinen 2008) to our specific system.

Definition 5. *The dynamical system* (32), (34) *is said to be in the transversal mode if* $\{\dot{n}_O \cdot \dot{n}_U\}_{n=i^o} > 0$ *holds and it is in the sliding mode for* $\{\dot{n}_O \cdot \dot{n}_U\}_{n=i^o} < 0$. *It is in the tangent mode if either* $\dot{n}_O = 0$ *or* $\dot{n}_U = 0$ *at* $n = i^o$.

Intuitively, the system is in transversal mode if the vector fields on both sides of the i^o line have the same algebraic sign and it is in the sliding mode if the signes differ.

Thus, the game may enter the so-called sliding mode, when it stays along the line $n = i^o$ where the special dynamics arise. However, due to the special structure of the switching manifold and the dynamical system at hand, we can demonstrate that the sliding mode of the first order[15] cannot occur:

Lemma 2. *The system* (32), (34) *crosses the switching manifold* Σ_o *transversally everywhere except for the point* $n = i^o, U = \delta i^o$, *where the tangent mode occurs.*

Proof. Follows immediately from the observation that $\dot{n}_O = \dot{n}_U$. □

Therefore, we neglect the possible tangent mode in our numerical simulations we present below to illustrate the dynamics of the contested leadership case.

Further, it is important to note that the optimal trajectory is continuous with a kink at the boundary i^o in the case of a regime change. That holds because the continuation of the stable manifold of the upper (lower) regime does not coincide with the trajectory of the unstable focus of the lower (upper) regime. We summarize our results in the following Proposition 6.

Proposition 6 (Dynamics for the contested leadership case)**.**
The optimal trajectories of the contested leadership are generically continuous functions of time with a kink at the boundary i^o in case of regime switching.

Assume there are no heteroclinic connections across $\{\bar{n}, \bar{U}\}_1, \{\hat{n}, \hat{U}\}_3$. *Then, there exists a threshold* n_{as} *such that the low steady-state of the lower regime* $\{\bar{n}, \bar{U}\}_1$ *realizes if* $n_0 < n_{as}$. *The high steady-state of the upper regime* $\{\hat{n}, \hat{U}\}_3$ *realizes otherwise.*

Proof. The first claim follows from the fact that the sliding mode can be excluded for the model except for the tangency point, Lemma 2. The second claim follows from the

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[15]sliding mode of the first order occurs if first directional derivatives of the flow on both sides of the switching manifold do not agree in sign. Higher order sliding mode may happen if one or both derivatives are zero, but higher order derivatives are not. We leave this opportunity for now.

definition of the threshold: for all values below it the low steady-state realizes as an equilibrium of the game. $\qquad\square$

Remark: The assumption of the absence of heteroclinic connections is essential since, otherwise, the system may leave one of the saddle-point steady-states and reach the other one.

Hence, the player that is more efficient in quality improvements of the existing variety of products ends up as the leader of the game. Thus, we can observe an endogenous change in leadership: if the initial follower has an advantage in developing some future technologies, this firm will eventually become the leader of the game provided the initially known range of technologies is sufficiently high.

Next, we illustrate the regime change for the contested leadership case with the help of a numerical example. To do so choose the following parameter values $r = 0.035$, $\delta = 0.22$, $\beta = 0.05$ and $\theta = 0.2$ which are basically identical to those from the last section. Further, we define

$$\Psi_1 := \frac{\psi_{FL}(\cdot)^2 (r+\beta)^2}{(r+\beta+\theta)^2} + \psi_{LF}(\cdot)^2, \ \Psi_2 := \frac{\psi_{LF}(\cdot)^2 (r+\beta)^2}{(r+\beta+\theta)^2} + \psi_{FL}(\cdot)^2 \tag{50}$$

As regards the functions Ψ_1 and Ψ_2 we assume that they are given by $\Psi_1 = a_1 \, e^{-b_1 \cdot e^{-d_1 \cdot i}}$, with $a_1 = 0.05$, $b_1 = 3$, $d_1 = 1.5$, and $\Psi_2 = a_2 \, e^{-b_2 \cdot e^{-d_2 \cdot i}}$, with $a_2 = 0.06$, $b_2 = 3.75$, $d_2 = 1.2$. With these parameter values, the efficiency functions ψ_{LF} and ψ_{FL} can be obtained from (50). The point of intersection i^o of ψ_{LF} and ψ_{FL} is given by $i^o = 2.047$. The functions ψ_{LF} and ψ_{FL} are illustrated in Figure 3.

With the definition of Ψ_1 and Ψ_2, the dynamics of the model is described by,

$$\dot{U}(t) = (r+\delta)U - \frac{1}{2}\left(\frac{\Psi_1(n(t))^2}{(r+\beta)^2}\right), \text{ for } n(t) < i^o,$$

$$\dot{U}(t) = (r+\delta)U - \frac{1}{2}\left(\frac{\Psi_2(n(t))^2}{(r+\beta)^2}\right), \text{ for } n(t) > i^o,$$

$$\dot{n}(t) = U(t) - \delta \, n(t), \, n(0) = n_0 \tag{51}$$

Given our parameter values, we can solve the differential equation system (51) numerically. The next Figure 4 gives a picture of the global dynamics with the $\dot{n} = 0$ isocline and the $\dot{U} = 0$ isocline that has a kink at $n = i^o = 2.047$. The trajectory of U is continuous at that point but not differentiable. We should like to point out that we did not compute the
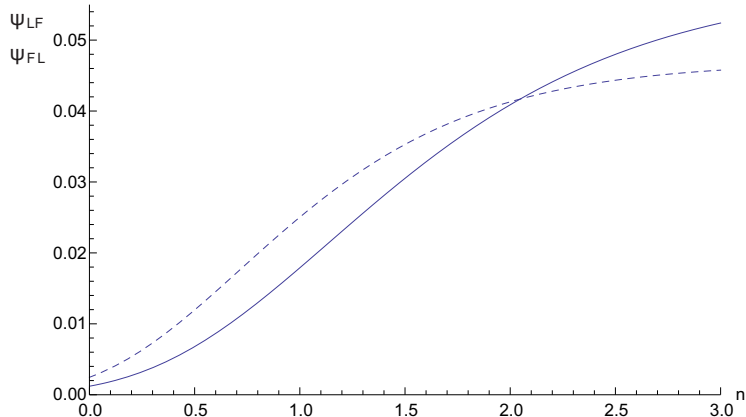
Figure 3: Efficiency functions ψ_{LF} (dashed) and ψ_{FL} (solid) for the catching-up case.

exact value of the threshold for that system since the calculations are extremely complex because a regime change can occur that must be taken into account in the computations.

In Figure 4 there is one trajectory going to the upper regime's steady-state $\{\hat{n}, \hat{U}\}_3$, which starts at the lower regime and the other one leading to the lower regime's steady-state $\{\bar{n}, \bar{U}\}_1$. It can be seen, that the vector fields of the upper regime and of the lower regime show the same sign over the switching manifold everywhere except for the tangency at the $\dot{n} = 0$ and Σ_o intersection point. The $\dot{U} = 0$ isocline has a kink at the switching manifold. Observe that there is only one trajectory in the lower regime which converges to the stable manifold of the upper steady-state. It is obtained by integrating backward in time the trajectory from the point at the switching manifold, where the upper regime's optimal trajectory converging to the stable manifold starts.

Next, we want to analyze how the emergence of a competitor that becomes the leader at a certain point in time affects the outcome of our model. To do so we consider the case of the contested leadership, as illustrated in Figure 4 for example, and compare it to the hypothetical situation where the leader of the lower regime is the permanent leader. The threshold for the case with the leader of the lower regime as the permanent leader is denoted by n_s. We limit our considerations to a situation as shown in Figure 4 where n_s and n_{as} are to the left of the switching manifold Σ_o. With this assumption, Lemma 3 gives the location of the threshold n_{as} relative to n_s.

Lemma 3. *Assume that the system is in the transversal mode and thresholds n_s, n_{as} to the left of the switching manifold Σ_o exist. Then, the inequality $n_{as} < n_s$ holds.*
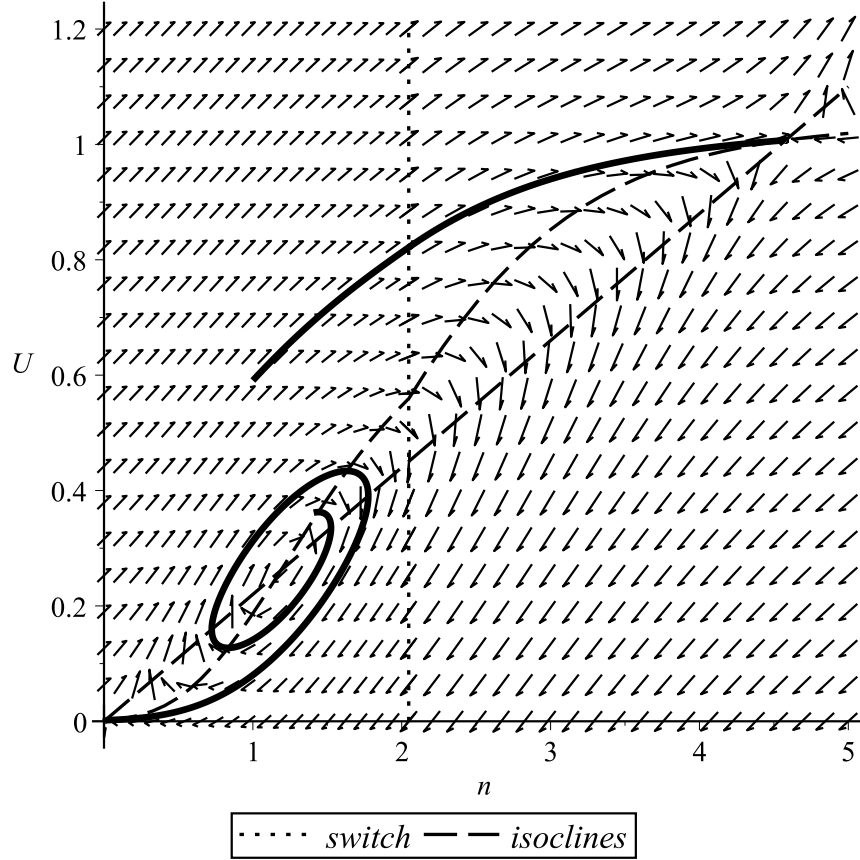
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Figure 4: Global dynamics with a change in leadership in the $(n - U)$ phase diagram.

The proof can be found in Appendix B.

Denote n_{ss} the threshold for hypothetical situation where the leader of the upper regime is the permanent leader. Then we observe the following:

Corollary 1. *Assume $\bar{n}_2 < i^o$. Under assumptions of Lemma 3 $n_{ss} \leq i^o$ holds.*

Proof. This is a simple consequence of the configuration of steady states of the system: once $\bar{n}_2 < i^o$ it follows that $\hat{n}_2 \leq i^o$ implying $n_{ss} \leq i^o$ (since both unstable steady states lie at one side of the switching manifold by construction of Ψ functions). $\square$

From Lemma 3 it follows, that the presence of the competitor reduces the range of initial variety values, n_0, that leads to the technology lock-in, where the system converges to the low steady state. For the history in the range (n_{as}, n_s) the contested leader regime

avoids the technology lock-in and the best possible equilibrium (in terms of the range and development of technologies) realizes, whereas the system would go to the low variety outcome if the initial leader was the constant leader. In this sense we can claim that the presence of competing R&D projects leading to a change in leadership is beneficial for the industry. However, we note that this argument has its limits.

Replicating the analysis above for an arbitrary finite number of players N we note the following: for every single crossing of efficiency functions the results above hold. Thus, the N-player situation may be divided into a sequence of 2-player games with the leader and the closest follower being of relevance. If more than one efficiency function intersect at a given i_k^o, the firm with the higher efficiency after i_k^o is selected as the leader. The ordering of steady-states follows the same way as for the 2-player case for each closest competitor. The difference to the situation with 2 firms lies in the fact that there are several switching manifolds and, thus, many possible tangent modes where the special regime can occur.

Before we summarize our results in the next Proposition we need the following Definition.

Definition 6. *The steady-state equilibrium $\tilde{U}$, $\tilde{n}$ is called a pseudoequilibrium if it is an equilibrium of the sliding mode (special regime).*

Denote the thresholds of the system with player k being the (virtual constant) leader as $\{n_s^k\}$, $k \in N$, by $\{n_{as}^k\}$ thresholds for the overall system with switching leadership[16] and associated steady-states $\bar{n}_{1,2,3}^k$. We can now state the following result:

Proposition 7 (The role of competitive fringe with a contested leader)**.**
If there is a contested leadership situation in the industry ((23) holds for each i_k^o) with the assumptions underlying Proposition 6, Lemma 3 and Corollary 1 for every next closest follower, then the following holds true:
For $N < N^* < \infty$:

1. *If $n_0 < n_s^1$, the system may converge either to the low or to the high steady state. If additionally $n_0 < n_{as}^1$ the system converges to $\bar{n}_1^1$.*

2. *If $n_0 > n_s^1$, the highest steady-state $\bar{n}_3^N$ realizes;*

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[16]thus $n_s^1 = n_s, n_s^2 = n_{ss}$, $n_{as}^1 = n_{as}$ for $N = 2$

3. *For $N^* < N \to \infty$ the game consists of special regimes only and the pseudoequilibrium $\{\tilde{n}, \tilde{U}\}$: $\dot{n_S}|_{n=\tilde{n}} = 0, \dot{U}_S|_{U=\tilde{U}} = 0$ realizes with certainty.*

Proof. **Point 1** follows immediately from the analysis of the 2-firms case (Lemma 3): $n_0 < n_s^1$ is not the condition for convergence to the low steady state, but rather $n_0 < n_{as}^1 < n_s^1$. Still n_{as}^k is difficult if possible to compute and we refer rather to n_s^k thresholds (which could be computed).

Point 2 follow from Corollary 1: once $n_0 > n_s^1$ holds, it follows that $n_0 > n_{as}^1$ and the switch in leadership to player 2 occurs (and the next closest follower). This would define a new n_{as}^2, but $n_s^2 < i_1^o$, and thus $n_{as}^2 < i_1^o$ via Lemma 3, thus the optimal trajectory cannot go back into the leadership of the player 1 (in the absence of heteroclinic connections, Prop. 6). It would go via transversal mode to the leadership of player 3, since $\bar{n}_3^3 > i_2^o$ otherwise there is no switch. Repeating this argument for every next leader-follower pair we obtain point 2.

Point 3: In case the number of firms approaches infinity, there are infinitely many switching manifolds of type Σ_o^k. The pseudoequilibrium can realize only if the trajectory enters the switching manifold exactly at the point where $\dot{n}_S|_{n=\tilde{n}} = 0$, $\dot{U}_S|_{U=\tilde{U}} = 0$, since no sliding motion is possible. However, since $N \to \infty$, the trajectory recovered by stitching different regimes consists only of points at different switching manifolds. However, it is continuous as long as the point lies in the transversal regime. At some point this trajectory will thus hit the pseudoequilibrium.

At last take N^* as any arbitrary large natural number and use continuum hypothesis (so the case $N > N^*$ realizes only if $N \subset \mathbb{R}_+$). $\qquad\square$

Observe that the pseudoequilibrium, point 3 in Proposition 7, is an inefficient one, since all the firms try to imitate each other with no avail so that underinvestments occur for $\theta > 0$. The exact variety of technologies and their qualities achieved depend on the distribution of the efficiency functions of all players. This requires a more detailed analysis left for future research.

Thus, we have found both positive and negative impacts of competition and of a change in leadership on the R&D output in the contested leadership case: on the one hand, the increase in the number of followers/imitators can help to avoid a technology lock-in with a low variety of technologies. On the other hand, once the number of competitors is very high, another lock-in appears in the form of a pseudoequilibrium. Again, what the

exact number N^* is when such a situation may occur, depends on the distribution of the efficiency functions and how close they are to each other.

4.2.2   Regime switching in the varying leadership case

We now briefly address the case where conditions (35)-(37) hold. For the case of a varying leadership we have three regimes, but in case of 2 firms two of them coincide with the low regime of the contested leader case, (47), and the medium regime is the same as the upper regime of the contested leader case, (49). There are two switching manifolds, denoted by $\Sigma_1 : n = i_1^o$ and $\Sigma_2 : n = i_2^o$ where the dynamics is given by the special regime (48). Again, at the boundaries, both players act as followers.

Depending on the location of the steady-states of both subsystems there are more opportunities than for the contested leader case, namely:

1. Cases $\bar{n}_{1,2} < i_1^o, \hat{n}_{1,2} < i_1^o$, $\bar{n}_3 > i_1^o, \hat{n}_3 > i_1^o$ and $\bar{n}_1 < i_1^o, \hat{n}_1 < i_1^o$, $\bar{n}_{2,3} > i_2^o, \hat{n}_{2,3} > i_2^o$ correspond to the switching mode of the contested leadership case;

2. Case $\bar{n}_1 < i_1^o, \hat{n}_1 < i_1^o, \bar{n}_3 > i_2^o, \hat{n}_3 > i_2^o, i_1^o < \bar{n}_2, \hat{n}_2 < i_2^o$ is specific for the varying leadership case.

In case 2 more than one regime change can occur. This can be observed when the system starts in the lower regime, passes through the middle regime and enters the upper regime where it converges to the high steady-state. Again, the optimal trajectory is continuous with kinks at the boundaries i_1^o and i_2^o. Depending on the parameters of the model, the optimal trajectories may visit all three regions of the phase space recurrently making multiple leadership changes possible. Still, the final result will be the convergence to one of the saddle-point stable steady-states in the lower or upper regime. Leaving out structurally unstable cases, we claim that the varying leadership case is qualitatively similar to the contested one in this respect.

5   Conclusions

In this paper we study an R&D differential game of a standard type, like in (Ben Youssef and Zaccour 2014), (Bondarev 2014), (Bondarev 2016), (Dawid, Greiner, and Zou 2010) and take into account learning by doing of the participating firms with the learning curves

displaying a sigmoid shape. It turns out that the form of the learning functions is crucial as regards the outome of the model, in particular with respect to the question of whether there is a constant leader or whether we can observe a catching-up of the follower or a varying leadership. In addition, the sigmoid form of the learning curve may give rise to the technology lock-in phenomenon.

The R&D multiproduct duopoly considered in the paper allows for different qualitative regimes. As long as one of the players has a higher efficiency of quality investments into all the technologies, no switch in technological leadership may occur, but we still can observe the technology lock-in situation due to the potential multiplicity of steady-states of the governing dynamic system. Depending on the initial range of technologies at the disposal of both players, the system may end up in the steady-state with a low variety of underdeveloped technologies or in the steady-state with the high variety of fully developed technologies. This finding is in line with those of (Acemoglu, Gancia, and Zilibotti 2012), (Krysiak 2011), but for a oligopoolistic setup.

When the learning curves of the players intersect, we observe a catch-up of the initial follower that becomes the leader for its part while the initial leader falls behind. In case there is one point of intersection, this situation is perpetuated meaning that the new leader remains in that position. If there are two intersection points of the learning curves, the new leader will fall behind again and the initial leader that became the follower temporarily catches-up and takes the lead again.

We extended our findings to an arbitrary finite number of R&D firms and found that the increase in competition in terms of the number of firms can help to avoid a technology lock-in that may arise without a change in leadership. However, once the number of firms approaches infinity another lock-in appears and the game ends up in the pseudoequilibrium. This one is an improvement compared to the initial low variety equilibrium, but does not utilize all possible technologies. Thus, competition has an ambiguous effect in our model.

The suggested model has immediate policy implications. Once the situation with multiple equilibria in R&D leadership arises, there exist thresholds separating the basins of attraction of the low and high steady-states of the system. Thus, it suffices to design a lump-sum tax/subsidy for one of the firms to avoid the technology lock-in of the first type (i.e. the low variety equilibrium). However, this is not the case for the second type (i.e. for the pseudoequilibrium), since any subsidy will move the system to a different (albeit

higher) pseudoequilibrium. Hence, it is important to take into account the number of R&D firms on the market and the distance between their investment efficiencies when designing the appropriate R&D policy for a certain industry.

A   Proof of Proposition 4

To prove **Point 1** we note that the dynamics of U is given by the (finite) sum of Gompertz functions, which are known to be sigmoid. It follows that the sum of these functions is also a sigmoid function. Therefore, the differential equations (19), (22) have up to three steady-states which is the direct consequence of the sigmoid shape of the efficiency function being involved. Which of the steady-states would realize depends on the initial stock of technologies, n_0. This result is equivalent to the one in (Greiner and Bondarev 2017) albeit for the differential game with N players.

To prove **Point 2** to **Point 5** we first state the following three Lemmas.

Lemma 4. *As long as three steady-states exist in the system* (19)*,* (22)*, the following holds:*

1. *Steady states $\{\bar{n}_1, \bar{U}_1\}$ and $\{\bar{n}_3, \bar{U}_3\}$ are saddle-point stable*

2. *The steady-state $\{\bar{n}_2, \bar{U}_2\}$ is unstable*

3. *No limit nor heteroclinic cycles nor homoclinic loops occur around any of these steady-states.*

Proof. Consider the Jacobian matrix of the system (19), (22) and its eigenvalues. There are two eigenvalues of the form:

$$\lambda_{1,2} = \frac{1}{2} \frac{r^3 + (\theta + 2\beta)r^2 + (\beta + \theta)\beta r}{(r + \beta + \theta)(r + \beta)} \pm \frac{\sqrt{F(\psi_k(n(t)), \psi_{-k}(n(t)))}}{(r + \beta + \theta)(r + \beta)} \quad \text{(A.1)}$$

The first term is always positive and real as long as $\{r, \beta, \theta\} \geq 0$, and square root yields positive or complex value depending on the sign of $F(\psi_k(n(t)), \psi_{-k}(n(t)))$. Thus the only possible combinations of eigenvalues for any point are:

- For positive $F(\psi_k(n(t)), \psi_{-k}(n(t)))$:

- $-\frac{1}{2}(r^3+(\theta+2\beta)r^2+(\beta+\theta)\beta r) > \sqrt{F(\psi_k(n(t)),\psi_{-k}(n(t)))} > 0$ then $\Re(\lambda_{1,2}) > 0$: unstable node equilibrium;

- $-\frac{1}{2}(r^3+(\theta+2\beta)r^2+(\beta+\theta)\beta r) = \sqrt{F(\psi_k(n(t)),\psi_{-k}(n(t)))} > 0$ then $\Re(\lambda_1) = 0, \Re(\lambda_2) > 0$: saddle-node point;

- $\sqrt{F(\psi_k(n(t)),\psi_{-k}(n(t)))} > \frac{1}{2}(r^3+(\theta+2\beta)r^2+(\beta+\theta)\beta r) > 0$ then $\Re(\lambda_1) > 0, \Re(\lambda_2) < 0$: saddle-point equilibrium;

- For non-positive $F(\psi_k(n(t)),\psi_{-k}(n(t)))$:

  - $F(\psi_k(n(t)),\psi_{-k}(n(t))) = 0$ then $\Re(\lambda_{1,2}) > 0$: unstable node equilibrium;

  - $F(\psi_k(n(t)),\psi_{-k}(n(t))) < 0$ then $\Re(\lambda_{1,2}) > 0$ and $\Im(\lambda_{1,2}) > 0$: unstable focus equilibrium.

Thus, no asymptotically stable steady-states may exist, and no limit cycles are possible (this requires both eigenvalues to have zero real parts). Next we know that $\psi_k(n(t)),\psi_{-k}(n(t))$ are sigmoid functions, and thus the function $F(\psi_k(n(t)),\psi_{-k}(n(t)))$ is non-monotonic over n, as Figure 5 illustrates.
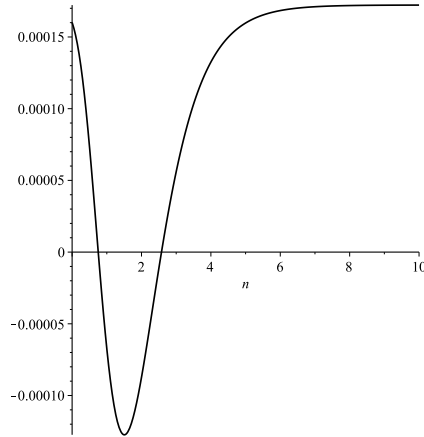

Figure 5: Regular form of the $F(\psi_k(n(t)),\psi_{-k}(n(t)))$ function.

We have at most two points with $\Re(\lambda_1) > 0, \Re(\lambda_2) < 0$ and one point with $\Re(\lambda_{1,2}) > 0$. This proves major part of the Lemma. Absence of heteroclinic cycles and homoclinic loops follows from Poincare-Bendixson theorem in the same way as shown in (Wagener 2003). $\qquad\square$

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Next, observe that only situations with 1 and 3 steady-states are generic. To see that, consider changes in the parameters a, b, d that determine the efficiency functions for all players. It turns out that only parameter a influences the number of steady-states, while others influence only the exact location of $\bar{n}, \bar{U}$. We thus make a small exercise and run a bifurcation analysis for a value in $N = 2$ case for the leader. We have the following result:

Lemma 5 (Saddle-node bifurcations).

The system (19), (22) *undergoes 2 saddle-node bifurcations with respect to parameter a_k: there exist a_k^0, a_k^1 such that saddle-node bifurcation points exist: steady-states $\{\bar{n}_2, \bar{U}_2\}$, $\{\bar{n}_3, \bar{U}_3\}$ merge into the saddle-node for $a_k = a_k^0$, and for $a_k = a_k^1$ steady-states $\{\bar{n}_2, \bar{U}_2\}$, $\{\bar{n}_1, \bar{U}_1\}$ merge into the saddle-node. For $a_k^1 > a_k > a_k^0$ there exist three steady-states. For $a_k < a_k^0$ there exists only $\{\bar{n}_1, \bar{U}_1\}$ and for $a_k > a_k^1$ there exists only $\{\bar{n}_3, \bar{U}_3\}$.*

Proof. The saddle-node bifurcation point is characterized by one of eigenvalues being zero. In our case this can be only if $\Re(\lambda_1) = 0, \Re(\lambda_2) > 0$, and no imaginary parts are present, making the bifurcation point a node. Hence this is an unstable point, as one of eigenvalues is positive. $\qquad\square$

Figure 6 illustrates the argument: plotting the steady-state curve in n variable we find steady-states as intersections of this curve with n axis. Varying parameter a_k we observe the saddle-node bifurcation: two steady-states of the system merge art a_k^0 and disappear afterwards. We thus may without loss of generality consider the system to have either 1 or 3 steady-states.

As long as multiple (3) steady-states are present in the system, it is the bistable system[17]. Thus, we may expect the existence of the so-called Skiba point(s) in this setup. Necessary conditions for that are established in the following Lemma.

Lemma 6. *The system* (19),(22) *possesses a threshold for a generic parameter range in between possible heteroclinic connections if:*

1. *It has 3 steady-states (it is bistable);*

2. *The unstable steady-state is a focus, i.e. $\Im(\lambda_{1,2}^{\bar{n}_2}) \neq 0$*

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[17]the dynamical system is called bistable if it has two stable equlibria and one unstable in between the stable ones.
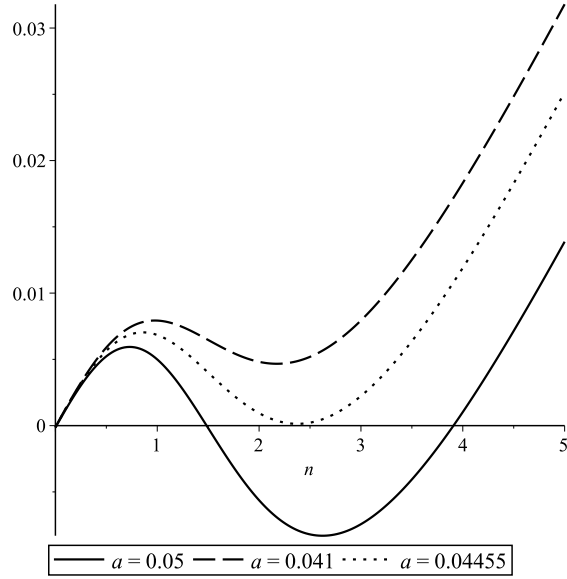
Figure 6: Number of steady-states under variation of a_k parameter.

Proof. Largely follows that of (Wagener 2003), we have only to show that the system has cusp bifurcation, which is granted by Lemma 5. □

Now, we can easily prove the rest of the Proposition.

Proof. **Point 2** to **Point 5** are shown as follows:

- **Point 2**: Structural stability means the quantity of equilibria remains the same under local perturbations. The situation with two equilibria arises only under saddle-node bifurcations, as Lemma 5 demonstrates, thus it is structurally unstable.

- **Points 3,4**: Follow from Lemma 4.

- **Point 5**: Follows from Lemma 6.

This completes the proof of Proposition 4. □

B   Proof of Lemma 3.

Proof. First, we make statements about the location of the optimal trajectories. We denote by n_{ss} the threshold for the upper system without a regime change in the case of

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a constant leadership of the leader in the upper regime. The trajectory optimal for the upper regime converges to the stable manifold of $\{\hat{n}, \hat{U}\}_3$. Fix such a trajectory starting from n_{ss} in the case of constant leadership of the leader in the upper regime. Fix the point (i^o, U^*) on the switching manifold where this trajectory intersects Σ_o. Now, take the trajectory optimal for the lower regime alone (i.e. without a regime change) converging to $\{\bar{n}, \bar{U}\}_3$ from n_s in the case of constant leadership of the leader of the lower regime. Fix the point of its intersection with the switching manifold and denote it by (i^o, U^o). From $\{\hat{n}, \hat{U}\}_3 > \{\bar{n}, \bar{U}\}_3$ it follows that $U^* > U^o$. Since there is only one optimal trajectory for any n_0 (including n_s, n_{ss}) converging to the upper steady-state for constant leadership of both players, it follows that continuation of the upper regime's optimal trajectory backwards from the point U^* is higher than the optimal trajectory for the lower regime.

Next, consider the overall optimal trajectory (i.e. in the case of a regime change) converging to $\{\hat{n}, \hat{U}\}_3$. Denote its intersection with the switching manifold by (i^o, U^{**}). In the upper regime, this trajectory coincides with the optimal trajectory when there is no regime change (i.e. when the leader of the upper regime is the permanent leader), thus $U^{**} = U^* > U^o$. Now, note that to the left of the switching manifold Σ_o the efficiency of investments is smaller for the upper regime than for the lower regime (i.e. $\Psi_2 < \Psi_1$, for $n < i^o$), due to the definition of the switching mode. This implies that the growth of investments, $\dot{U}$, is smaller for the upper regime than for the lower regime (to the left of i^o), implying that the optimal trajectory of the system with a regime change is above the optimal trajectory for the system when the leader of the upper regime is the permanent leader.

Figure 7 shows the optimal trajectories indicated by arrows, where the dotted trajectory gives the optimal trajectory for the case of a regime change.

Next, we compare the maximized Hamiltonians in the case of a regime change and in the case where the initial leader is the permanent leader. Note that the optimal trajectories converging to the low steady-state are identical in these two cases. This implies that the maximized Hamiltonians converging to the low steady-state take the same value.

Recall that the threshold of the system without a regime change (where the leader of the lower regime is the permanent leader) is denoted by n_s and the maximized Hamiltonian of the lower regime with no regime change is denoted by $H_{nc}^0(n_s, \cdot)$, for $n(0) = n_s$. Now, consider the maximized virtual Hamiltonian, $H_v^0(n_s, \cdot)$, for $n(0) = n_s$, in the lower regime
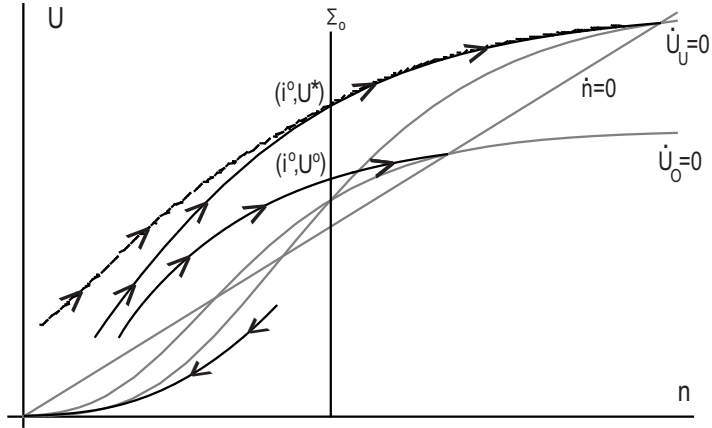
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Figure 7: Global dynamics with and without a change in leadership.

with $U(0)$ on the optimal trajectory of the model with a regime change but that does not converge to $\{\hat{n}, \hat{U}\}_3$, i.e. that has no kink at $n = i^o$. Obviously, $U(0)$ is larger for $H_v^0(n_s, \cdot)$ than for $H_{nc}^0(n_s, \cdot)$. Since a higher $U(0)$ implies a higher $u_L(0)$ and a higher $u_F(0)$ and since the maximized Hamiltonian rises with $u_L(0)$, $u_F(0)$, we get $H_v^0(n_s, \cdot) > H_{nc}^0(n_s, \cdot)$.

Next, we compare the maximized virtual Hamiltonian $H_v^0(n_s, \cdot)$ with the maximized Hamiltonian for the case of a regime change, $H_{rc}^0(n_s, \cdot)$, where the kink in the trajectory must be taken into account. The maximized Hamiltonian for the system with a regime change is given by

$$H_{rc}^0 = H_O^0(n_s, \Psi_1, \cdot) - \frac{H_O^0(i^o, \Psi_1, \cdot)}{e^{rt^o}} + \frac{H_U^0(i^o, \Psi_2, \cdot)}{e^{rt^o}},$$

where the subscript O (U) denotes values of the lower (upper) regime with a regime change and t^o is the point in time when the system reaches the switching manifold Σ_o. The maximized virtual Hamiltonian is

$$H_v^0 = H_O^0(n_s, \Psi_1, \cdot) - \frac{H_O^0(i^o, \Psi_1, \cdot)}{e^{rt^o}} + \frac{H_O^0(i^o, \Psi_1, \cdot)}{e^{rt^o}}.$$

From our considerations above we know that $U_O(t^o) = U_U(t^o)$ and $u_{O,L}(t^o) = u_{U,L}(t^o)$. Further, $\Psi_2 \geq \Psi_1$ for $n \geq i^o$ implies $H_U^0(i^o, \Psi_2, \cdot) \geq H_O^0(i^o, \Psi_1, \cdot)$. This shows that $H_{rc}^0 \geq H_v^0$ holds.

Thus, we obtain $H_{rc}^0 \geq H_v^0 > H_{nc}^0$. Since the maximized Hamiltonian converging to the higher steady-state is increasing in n, the inequality $H_{rc}^0 > H_{nc}^0$ implies $n_{as} < n_s$. $\square$

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