Actions of Hochschild Cohomology and Local Duality in Modular Representation Theory

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Abstract

We develop an extension of the theory of idempotents in tensor tringulated category to deal with action of tensor triangulated category on another triangulated category. As an application, we obtain some actions of Hochschild cohomology ring on some triangulated categories. Let A be a finite dimensional self-injective algebra over a field $\Bbbk$ and A^e the enveloping algebra of A over $\Bbbk$. We prove that the homotopy category $\mathcal{K}(\operatorname{Inj} A^e)$ of injective modules over A^e is tensor triangulated with tensor product $\otimes_A$ and tensor unit the injective resolution $\mathbf{i}A$ of A as A^e-modules, and there is an action of $\mathcal{K}(\operatorname{Inj} A^e)$ on the homotopy category $\mathcal{K}(\operatorname{Inj} A)$ of injective modules over A. This yields a central ring action of the Hochschild cohomology ring $\mathrm{HH}^*(A/\Bbbk)$ on $\mathcal{K}(\operatorname{Inj} A)$, by identifying $\mathrm{HH}^*(A/\Bbbk)$ with the graded endomorphism ring $\operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A^e)}(\mathbf{i}A)$. Moreover, this ring action on $\mathcal{K}(\operatorname{Inj} A)$ extend the one on the derived category $\mathcal{D}(A)$.

In the special case that A has a Hopf algebra structure we show that the canonical action of the cohomology ring $\mathrm{H}^*(A, \Bbbk)$ on $\mathcal{K}(\operatorname{Inj} A)$ factors through the action of the Hochschild cohomology above. This is also a consequence of the theory of idempotents and actions. Using this factorization and a recent result of Benson, Iyengar, Krause and Pevtsova, we prove that if A is also symmetric, for example when A is the group algebra $\Bbbk G$ of a finite group G, the category $\mathcal{K}(\operatorname{Inj} A)$ is Gorenstein as a $\mathrm{HH}^*(A/\Bbbk)$-linear triangulated category. As a corollary, the bounded derived category $\mathcal{D}^b(\operatorname{mod} A)$ of finite dimensional modules, which can be identified as the full subcategory of compact objects of $\mathcal{K}(\operatorname{Inj} A)$, has a local Serre duality as $\mathrm{HH}^*(A/\Bbbk)$-linear triangulated category.

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Contents

Chapter 1

Introduction

Background and main results

The notion of Gorenstein and local duality are ubiquitous, for example they appear in commutative algebra [Bas63], representation theory of finite dimensional algebra [AR91, Hap91], algebraic topology [DGI06], homotopy theory [BHV18], differential graded algebra [FJ03, FIJ03], modular representation theory [Ben01, BG08], and representation theory of finite group scheme [BIKP16].

For triangulated categories, the Gorenstein property is introduced by Benson, Iyengar, Krause, and Pevtsova in [BIKP16] as a generalization of two results in commutative algebra and modular representation theory. The setting is that the triangulated category is compactly generated and there is an *action* of a graded-commutative noetherian ring; see Section 2.4 for details. In such situation, Benson, Iyengar, and Krause have constructed local cohomology functors in [BIK08], generalizing the notion in commutative algebra. As in the classical case, the definition of Gorenstein for triangulated categories is connected to these generalized local cohomology functors. We emphasize again that these notions depend on an action of a graded-commutative noetherian ring. We recall briefly what this means.

Let $\mathcal{K}$ be a triangulated category and R a graded-commutative noetherian ring; thus R is $\mathbb{Z}$-graded and satisfies $r \cdot s = (-1)^{|r||s|} s \cdot r$ for each pair of homogeneous elements $r, s \in R$. An *action* of R on $\mathcal{K}$ is a homomorphism $\phi\colon R \longrightarrow \mathrm{Z}^*(\mathcal{T})$ of graded rings, where $\mathrm{Z}^*(\mathcal{T})$ is the graded centre of $\mathcal{T}$; see Section 2.3. This yields for each object $M \in \mathcal{K}$ a homomorphism $\phi_M\colon R \longrightarrow \mathrm{End}^*_{\mathcal{K}}(M)$ of graded rings such that for all objects $M, N \in \mathcal{K}$ the R-module structures on the graded abelian group

$$\mathrm{Hom}^*_{\mathcal{K}}(M, N) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{K}}(M, \Sigma^n N)$$

induced by ϕ_X and ϕ_Y agree, up to the usual sign rule, namely $r \cdot f = (-1)^{|r||f|} f \cdot r$ for homogeneous elements $r \in R$ and $f \in \mathrm{Hom}^*_{\mathcal{K}}(M, N)$. Therefore, we also say that $\mathcal{K}$ is an *R-linear* triangulated category when R acts on $\mathcal{T}$.

In some cases, ring actions are induced by actions of *tensor triangulated categories*. By this we mean a triangulated category $\mathcal{T}$ with a monoidal or tensor structure $(\otimes, \mathbb{1})$ which is compatible with the triangulation of $\mathcal{T}$. If $(\mathcal{T}, \otimes, \mathbb{1})$ is a tensor triangulated category, an *action* of $\mathcal{T}$ on a triangulated category $\mathcal{K}$ is a bifunctor $*\colon \mathcal{T} \times \mathcal{K} \longrightarrow \mathcal{K}$ satisfying some coherence axioms; see Section 2.3 for details. It then follows from these axioms, that an action of $\mathcal{T}$ on $\mathcal{K}$ induces a ring action of the graded endomorphism ring $\operatorname{End}^*_{\mathcal{T}}(\mathbb{1})$ of the tensor unit $\mathbb{1}$ on $\mathcal{K}$ given by $- * M\colon \operatorname{End}^*_{\mathcal{T}}(\mathbb{1}) \longrightarrow \operatorname{End}^*_{\mathcal{K}}(M)$ for all $M \in \mathcal{K}$. Stevenson developed a support theory based on an action of a *symmetric* tensor triangulated category [Ste11] and the Balmer spectrum $\operatorname{Spc} \mathcal{T}^c$ of the full subcategory of compact object in $\mathcal{T}$ [Bal05]. For our purposes, we only need the induced ring action.

In the modular representation theory of finite group, the main triangulated categories of interest are the derived category $\mathcal{D}(\Bbbk G)$ and the stable module category $\underline{\operatorname{Mod}}\, \Bbbk G$ of the group algebra $\Bbbk G$ for a finite group G and field $\Bbbk$ of characteristic dividing the order of G. Krause investigated in [Kra05] the homotopy category $\mathcal{K}(\operatorname{Inj} \Bbbk G)$ of injective modules, which 'glues' both categories together (see Section 2.1 and Section 2.2 for the definition of these categories). Moreover, $\mathcal{K}(\operatorname{Inj} \Bbbk G)$ has tensor product of complexes $\otimes$, taken over $\Bbbk$, coming from the Hopf algebra structure on $\Bbbk G$ with unit the injective resolution $\mathbf{i}\Bbbk$ of the trivial $\Bbbk G$-module $\Bbbk$ [BK08]. This induces a canonical action of the group cohomology ring $\operatorname{H}^*(G, \Bbbk)$ on $\mathcal{K}(\operatorname{Inj} \Bbbk G)$ as there is an isomorphism $\operatorname{H}^*(G, \Bbbk) = \operatorname{Ext}^*_A(k, k) \cong \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} \Bbbk G)}(\mathbf{i}\Bbbk)$.

For an arbitrary finite dimensional algebra A there is no obvious choice of a tensor triangulated category acting on $\mathcal{K}(\operatorname{Inj} A)$. We develop a machinary using *idempotents* in tensor triangulated categories which gives us a tensor triangulated category acting on $\mathcal{K}(\operatorname{Inj} A)$, at least if A is self-injective. In [Ric97, BF11, BD14, Hog17], idempotents are used to study various tensor triangulated categories, and we extend slightly this theory to include an actions on another triangulated category (see Chapter 3).

Let A be a finite dimensional $\Bbbk$-algebra and $A^e = A \otimes A^{\mathrm{op}}$ its enveloping algebra. Consider the homotopy category $\mathcal{K}(A^e)$ of A^e-modules. It has a natural tensor structure, where the tensor product is $\otimes_A$ and the unit is just A, viewed as a complex concentrated in degree 0. There is an action $\mathcal{K}(A^e) \times \mathcal{K}(A) \xrightarrow{-\otimes_A -} \mathcal{K}(\operatorname{Inj} A)$ given by tensor product over A. Our first result is that, if A is self-injective, then the natural map $A \xrightarrow{\eta} \mathbf{i}A$, where $\mathbf{i}A$ is the injective resolution of A as A^e-module, is a unital idempotent arrow in $\mathcal{K}(A^e)$, in the sense that the induced map

$$A \otimes_A X \xrightarrow{\eta \otimes X} \mathbf{i}A \otimes_A X \quad \text{and} \quad X \otimes_A A \xrightarrow{X \otimes \eta} X \otimes_A \mathbf{i}A$$

are isomorphisms. We compute, up to equivalence, that the induced subcategories

$$\mathbf{i}A \cdot \mathcal{K}(A^e) \cdot \mathbf{i}A = \{X \in \mathcal{K}(A^e) \mid \mathbf{i}A \otimes_A X \otimes_A \mathbf{i}A \cong X\}$$

is the homotopy category $\mathcal{K}(\operatorname{Inj} A^e)$ of injective A^e-modules, and that

$$\mathbf{i}A \cdot \mathcal{K}(A) = \{Y \in \mathcal{K}(A) \mid \mathbf{i}A \otimes_A Y \cong Y\}$$

is $\mathcal{K}(\operatorname{Inj} A)$. Recall also that the Hochschild cohomology ring of A, first introduced by Hochschild in [Hoc45], is defined by

$$\operatorname{HH}^*(A/\Bbbk) = \operatorname{Ext}^*_{A^e}(A, A) \cong \operatorname{End}^*_{\mathcal{D}(A^e)}(A) \cong \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A^e)}(\mathbf{i}A).$$

Applying our machinery in Section 3.2, we obtain the following theorem.

Theorem (Theorem 4.3.7)**.** *Let A be a finite dimensional self-injective algebra over $\mathcal{K}$. Then the tensor triangulated category $\mathcal{K}(\operatorname{Inj} A^e)$ acts on $\mathcal{K}(\operatorname{Inj} A)$ via tensor product of complexes over A. In particular, we get an action of* $\operatorname{HH}^*(A/\Bbbk)$ *on $\mathcal{K}(\operatorname{Inj} A)$ given by*

$$\operatorname{HH}^*(A/\Bbbk) \cong \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A^e)}(\mathbf{i}A) \xrightarrow{-\otimes_A M} \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A)}(M)$$

for all $M \in \mathcal{K}(\operatorname{Inj} A)$

The above theorem is an extension of the actions of the Hochschild cohomology ring on finitely generated A-modules and on the bounded derived category $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)$, which were studied in [SS04] and [Sol06] to develop support varieties for modules and complexes. This theory is advanced in [EHT$^+$04] for the special case of finite dimensional self-injective algebra. Let us note that there is also an action of $\operatorname{HH}^*(A/\Bbbk)$ on $\mathcal{D}(A)$ induced by the action of the tensor triangulated category $(\mathcal{D}(A^e), \otimes^{\mathbf{L}}_A, A)$ on $\mathcal{D}(A)$. Again, Theorem 4.3.7 extends this in the sense that the embedding $\mathcal{D}(A) \hookrightarrow \mathcal{K}(\operatorname{Inj} A)$ is $\operatorname{HH}^*(A/\Bbbk)$-linear.

In the case of group algebra, or more generally finite dimensional Hopf algebra, there is a relation between the action of Hochschild cohomology with the action of group cohomology. Let A be a finite dimensional Hopf algebra. In particular, A is a Frobenius algebra [Par71], i.e. A and its dual $DA = \operatorname{Hom}_k(A, k)$ are isomorphic as (left) A-modules, and hence self-injective. Thus $\mathcal{K}(\operatorname{Inj} A)$ becomes a $\operatorname{HH}^*(A/\Bbbk)$-linear triangulated category, as described above. As in the group algebra case, $(\mathcal{K}(\operatorname{Inj} A), \otimes, \mathbf{i}\Bbbk)$ is a tensor triangulated category. Letting $\mathcal{K}(\operatorname{Inj} A)$ acts on itself, we obtain ring action of $\operatorname{H}^*(A, \Bbbk) \cong \operatorname{Ext}^*_A(k, k)$ on $\mathcal{K}(\operatorname{Inj} A)$ since there is an isomorphism $\operatorname{H}^*(A, \Bbbk) \cong \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A)}(\mathbf{i}\Bbbk)$ of graded rings.

Theorem (Theorem 4.4.11 and Theorem 5.1.3)**.** *The canonical ring action of the ordinary cohomology* $\operatorname{H}^*(A, \Bbbk)$ *on $\mathcal{K}(\operatorname{Inj} A)$ factors through the action of the Hochschild cohomology* $\operatorname{HH}^*(A/\Bbbk)$ *via a finite map. More precisely, there is a finite homomorphism* $\alpha\colon \operatorname{H}^*(A, \Bbbk) \longrightarrow \operatorname{HH}^*(A/\Bbbk)$ *of graded rings such that for each complex M in $\mathcal{K}(\operatorname{Inj} A)$*

the diagram

$$\begin{array}{ccccc} \mathrm{H}^*(A,\Bbbk) & \xrightarrow{\cong} & \mathrm{End}^*_{\mathcal{K}(A)}(\mathbf{i}\Bbbk) & \xrightarrow{-\otimes M} & \mathrm{End}^*_{\mathcal{K}(\mathrm{Inj}\,A)}(M) \\ & & \Big\downarrow{\scriptstyle\alpha} & & \Big\| \\ \mathrm{HH}^*(A/\Bbbk) & \xrightarrow{\cong} & \mathrm{End}^*_{\mathcal{K}(A^e)}(\mathbf{i}A) & \xrightarrow{-\otimes_A M} & \mathrm{End}^*_{\mathcal{K}(\mathrm{Inj}\,A)}(M) \end{array}$$

is commutative.

The above theorem is a slight extension of a result of Pevtsova and Witherspoon [PW09]; see also [CI17] for an application of this. We mention that the theorem is also an application of our machinary of idempotents and actions in Section 3.3.

Now we explain local cohomology functors and duality for triangulated categories. Recall that an object $\mathcal{C}$ in a triangulated category $\mathcal{T}$ which admits small coproducts is *compact* if the functor $\mathrm{Hom}_{\mathcal{T}}(C,-)$ commutes with coproduct. We denote the full subcategory of compact object of $\mathcal{T}$ by $\mathcal{T}^{\mathrm{c}}$. The category $\mathcal{T}$ is called *compactly generated* when there exists a set of compact object $\mathcal{G} \subset \mathcal{T}^{\mathrm{c}}$ such that $\mathcal{T}$ is the smallest subcategory containing $\mathcal{G}$ and closed under coproduct. All the triangulated categories that are mentioned before, namely $\underline{\mathrm{Mod}}\,\Bbbk G$, $\mathcal{D}(R)$ and $\mathcal{K}(\mathrm{Inj}\,A)$, where G is a finite group, R is any ring and A is a finite dimensional algebra, are compactly generated.

Let R be a graded-commutative noetherian ring and $\mathcal{T}$ a compactly generated R-linear triangulated category. For each $\mathfrak{p} \in \mathrm{Spec}\,R$, there is a functor $\varGamma_{\mathfrak{p}}\colon \mathcal{T} \longrightarrow \mathcal{T}$ constructed in [BIK08] as an analogue of the local cohomology functor in commutative algebra. The second important functor is $T_{\mathfrak{p}}\colon \mathcal{T}^{\mathrm{c}} \longrightarrow \mathcal{T}$ constructed using Brown representability (see [Nee96], or [Kra02]) which satisfies

$$\mathrm{Hom}_R(\mathrm{Hom}^*_{\mathcal{T}}(C,-), I(\mathfrak{p})) \cong \mathrm{Hom}_{\mathcal{T}}(-, T_{\mathfrak{p}}(C))$$

for each $C \in \mathcal{T}^{\mathrm{c}}$, where $I(\mathfrak{p})$ denotes the injective hull of $R/\mathfrak{p}$ as a graded R-module. We say that $\mathcal{T}$ is *Gorenstein* with respect to the action of R if there is an R-linear triangle equivalence $F\colon \mathcal{T}^{\mathrm{c}} \xrightarrow{\simeq} \mathcal{T}^{\mathrm{c}}$ and for every $\mathfrak{p}$ in $\mathrm{Spec}(R)$ there is an integer $d(\mathfrak{p})$ and a natural isomorphism

$$\varGamma_{\mathfrak{p}} \circ F \cong \Sigma^{d(\mathfrak{p})} \circ T_{\mathfrak{p}}$$

of functors $\mathcal{T}^{\mathrm{c}} \longrightarrow \mathcal{T}$. For example, a commutative ring A is Gorenstein if and only if the derived category $\mathcal{D}(A)$ is Gorenstein as an A-linear triangulated category (see, for example, [BH93]. The second example is that $\underline{\mathrm{Mod}}\,\Bbbk G$ is Gorenstein with respect to the canonical action of $\mathrm{H}^*(G,\Bbbk)$ [BG08, Ben08].

Henceforth A is a finite dimensional cocommutative Hopf algebra over $\mathcal{K}$. The ring $\mathrm{HH}^*(A/\Bbbk)$ is finitely generated by [FS97] and $\mathcal{K}(\mathrm{Inj}\,A)$ is a compactly generated $\mathrm{HH}^*(A/\Bbbk)$-linear triangulated category. Thus there are functors $\varGamma_{\mathfrak{q}}\colon \mathcal{K}(\mathrm{Inj}\,A) \to \mathcal{K}(\mathrm{Inj}\,A)$ and $T_{\mathfrak{q}}\colon \mathcal{K}^c(\mathrm{Inj}\,A) \to \mathcal{K}(\mathrm{Inj}\,A)$ for each $\mathfrak{q} \subset \mathrm{HH}^*(A/\Bbbk)$. Our main theorem of this thesis is that when A is also symmetric, the category $\mathcal{K}(\operatorname{Inj} A)$, viewed as a $\mathrm{HH}^*(A/\Bbbk)$-linear triangulated category, is Gorenstein.

Theorem (Theorem 5.3.3). *Let A be a finite dimensional symmetric cocommutative Hopf algebra over a field $\Bbbk$. The category $\mathcal{K}(\operatorname{Inj} A)$ viewed as an $\mathrm{HH}^*(A/\Bbbk)$-linear triangulated category is Gorenstein, where the global Serre functor is just the identity functor on $\mathcal{K}^{\mathrm{c}}(\operatorname{Inj} A)$ and $d(\mathfrak{q}) = \dim \mathrm{HH}^*(A/\Bbbk)/\mathfrak{q}$ for each prime $\mathfrak{q}$ in $\operatorname{Spec} \mathrm{HH}^*(A/\Bbbk)$. More precisely, there is a natural isomorphism*

$$\varGamma_{\mathfrak{q}} \cong \Sigma^{d(\mathfrak{q})} \circ T_{\mathfrak{q}}$$

of functors $\mathcal{K}^{\mathrm{c}}(\operatorname{Inj} A) \longrightarrow \mathcal{K}(\operatorname{Inj} A)$.

We mention that the proof of Thereom 5.3.3 above consists of two main ingredients and uses a transfer of Gorenstein property argument. The first one is that $\mathcal{K}(\operatorname{Inj} A)$ is Gorenstein with respect to the action of the ordinary cohomology ring $\mathrm{H}^*(A, \Bbbk)$ [BIKP16] and the second one is Theorem 4.4.11.

Let us explain some consequences of the above theorem. Let A be a finite dimensional symmetric cocommutative Hopf algebra over $\Bbbk$, for instance the group algebra $\Bbbk G$ for a finite group G. First, via the embedding $\underline{\operatorname{Mod}}\, A \to \mathcal{K}(\operatorname{Inj} A)$ and $\mathcal{D}(A) \to \mathcal{K}(\operatorname{Inj} A)$ (see, Section 2.2), the category $\underline{\operatorname{Mod}}\, A$ and $\mathcal{D}(A)$ are also Gorenstein with respect to the action of $\mathrm{HH}^*(A/\Bbbk)$.

Now consider the full subcategory of compact object $\mathcal{K}^{\mathrm{c}}(\operatorname{Inj} A)$ of $\mathcal{K}(\operatorname{Inj} A)$. In [Kra05], it is showed that $\mathcal{K}^{\mathrm{c}}(\operatorname{Inj} A)$ is equivalent to the bounded derived category $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)$ of finite dimensional A-modules. Thus, we may view $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)$ as a $\mathrm{HH}^*(A/\Bbbk)$-linear triangulated category via restriction. For each $\mathfrak{q} \in \operatorname{Spec} \mathrm{HH}^*(A/\Bbbk)$, we consider the triangulated category $\gamma_{\mathfrak{p}}(\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A))$ that is obtained from $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)$ by localising the graded morphisms at $\mathfrak{q}$ and then taking the full subcategory of objects such that the graded endomorphisms are $\mathfrak{q}$-torsion; see Section 2.4 for details.

Corollary (Corollary 5.4.2). *Let A be a finite dimensional symmetric cocommutative Hopf algebra over a field $\Bbbk$. The category $\mathcal{K}^{\mathrm{c}}(\operatorname{Inj} A) \simeq \mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)$, as a $\mathrm{HH}^*(A/\Bbbk)$-linear category, satisfies local Serre duality, in the sense that for each prime $\mathfrak{q}$ in $\operatorname{Spec} \mathrm{HH}^*(A/\Bbbk)$ there is a natural isomorphism*

$$\operatorname{Hom}_{\mathrm{HH}^*(A/\Bbbk)}(\operatorname{Hom}^*_{\gamma_{\mathfrak{p}}(\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A))}(X, Y), I(\mathfrak{q})) \cong \operatorname{Hom}_{\gamma_{\mathfrak{p}}(\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A))}(Y, \Sigma^{-d(\mathfrak{q})} X),$$

for X, Y in $\gamma_{\mathfrak{p}}(\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A))$.

There is a notion of Serre functor for $\Bbbk$-linear triangulated category, introduced by Bondal and Kapranov in [BK90], which generalizes Serre duality in algebraic geometry. The definition of (local) Serre duality in the above corollary is an analogue of this for a triangulated category with an action of graded commutative noetherian local ring.

The next corollary establishes the existence of Auslander-Reiten triangles (or AR-triangles, for short), introduced by Happel [Hap88] as an analogue of AR-sequence in module categories. A connection between Serre functor and AR-triangles for triangulated categories that are Hom-finite over a field was shown by Reiten and Van den Bergh [RVdB02]. In fact, their proof also work in our setting (see [BIKP16]), which gives us the following.

Corollary (Corollary 5.4.3)**.** *Let A be a finite dimensional symmetric cocommutative Hopf algebra over a field $\mathcal{K}$. The category $(\Gamma_{\mathfrak{q}}\mathcal{K}(\operatorname{Inj} A))^{c}$ has AR-triangles for $\mathfrak{q}$ in* $\operatorname{Spec}\operatorname{HH}^*(A/\Bbbk)$.

Outline

This thesis is organized as follows. In Chapter 2, we review basic notions, definitions and known results that are needed in the following chapter; for example, we explain the notion of a ring action on a triangulated category and the Gorenstein property with respect to such action. In the next chapter, we discuss the abstract theory of idempotents in tensor triangulated category. Applications of the idempotent theory to obtain actions of Hochschild cohomology on various categories is discussed next in Chapter 4. In Chapter 5, we prove the main theorem of this thesis, namely the Gorenstein property of the homotopy category of injectives of a symmetric cocommutative Hopf algebra with respect to an action of its Hochschild cohomology ring. In the last chapter we collect some problems that we are unable to solve.

Notations and conventions

Throughout the thesis, $\Bbbk$ denotes an arbitrary field. If M, N are $\Bbbk$-modules then $M \otimes N$ means $M \otimes_{\Bbbk} M$ and $\operatorname{Hom}(M, N)$ means $\operatorname{Hom}_{\Bbbk}(M, N)$. These also hold when M, N are modules over a $\Bbbk$-algebra A. Following the usual custom, we also use $\otimes$ to denote an 'abstract' tensor product in a general tensor category, but the meaning should be clear from the context.

Unless otherwise specified, modules always mean left modules. We identify right modules with modules over the opposite ring or algebras. Thus, for a ring A, the category of left (resp. right) modules over A is denoted $\operatorname{Mod} A$ (resp. $\operatorname{Mod} A^{\mathrm{op}}$). Similarly, for $\Bbbk$-algebra A and B, an (A, B)-bimodule M will be identified with a (left) module over $A \otimes B^{\mathrm{op}}$ with action $(a \otimes b)m = amb$ for all $a \in A$, $b \in B$ and $m \in M$. For example, an (A, A)-bimodule will be identified with a module over the enveloping algebra $A^{e} = A \otimes A^{\mathrm{op}}$ of A.

We index our complexes cohomologically, so a complex X of A-modules is of the form

$$\cdots \longrightarrow X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \longrightarrow \cdots.$$

Note that this is called a cochain complex or cocomplex in some literatures.

Let R be a graded-commutative ring. We write $\operatorname{Spec} R$ for the set of all homogeneous (or graded) prime ideals of R. Similarly, we write $\operatorname{Mod} R$ and $\operatorname{Inj} R$ for the category of graded modules over R and the full subcategory of injective objects in $\operatorname{Mod} R$; for non-graded ring A, the notations $\operatorname{Mod} A$ and $\operatorname{Inj} A$ denote the corresponding categories of (non-graded) A-modules

For a $\Bbbk$-algebra A, we write $\mathrm{HH}^*(A/\Bbbk; M)$, instead of $\mathrm{HH}^*(A; M)$, for the Hochschild cohomology group with coefficient in an A^e-module M, to emphasize that it is computed over $\Bbbk$. More precisely, we use the enveloping algebra $A^e = A \otimes A^{\mathrm{op}}$ of A over $\Bbbk$ and define $\mathrm{HH}^*(A/\Bbbk; M) = \operatorname{Ext}^*_{A^e}(A, M)$.

The shift or suspension in a triangulated category will always be denoted by Σ; we use this notation for all triangulated categories concerned, but this should not cause any confusion. A distinguished triangle in a triangulated category is also called an exact triangle or just triangle for short; we do not consider non-distinguished triangle. Similarly, a triangulated functor between triangulated categories is also called an exact functor.

For a bifunctor $F\colon \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$ and a morphism $\alpha\colon X \to Y$ in $\mathcal{A}$, we often write $F(\alpha, Z)$ for $F(\alpha, \mathrm{Id}_Z)\colon F(X, Z) \longrightarrow F(Y, Z)$. For example, we write $\alpha \otimes Z$ to denote $\alpha \otimes \mathrm{Id}_Z$.

We say that a diagram of functors is commutative to mean that it is commutative up to a natural isomorphism. For example,

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ \downarrow{\scriptstyle G'} & & \downarrow{\scriptstyle G} \\ \mathcal{C} & \xrightarrow{F'} & \mathcal{D} \end{array}$$

is commutative if there is a natural isomorphism $\varphi\colon GF \xrightarrow{\cong} F'G'$. Given a functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$, we also write $F\colon \operatorname{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \operatorname{Hom}_{\mathcal{B}}(FX, FY)$ for the map $\alpha \mapsto F\alpha$. Then we say that the square above induce a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{A}}(X, Y) & \xrightarrow{F} & \operatorname{Hom}_{\mathcal{B}}(FX, FY) \\ \downarrow{\scriptstyle G'} & & \downarrow{\scriptstyle G} \\ \operatorname{Hom}_{\mathcal{C}}(G'X, G'Y) & \xrightarrow{F'} & \operatorname{Hom}_{\mathcal{D}}(GFX, GFY), \end{array}$$

eventhough the bottom row is not really F'.

Chapter 2

Preliminaries

The purpose of this chapter is to make preparations for the following chapters. In the first section, we discuss basic homological algebra, in the language of homotopy and derived categories. In the next section, we discuss the homotopy category of injective modules studied in [Kra05] and its relation with the derived category. In Section 2.3, we explain the abstract theory of tensor triangulated category and its action on another triangulated category. Such action induce a ring action on the target category. The machinary of local cohomology and support introduced in [BIK08], which depends on ring actions will be discussed next in Section 2.4. We also explain the Gorenstein property and local duality for triangulated categories following [BIKP16], which generalize the notions in commutative algebra. In the last section, we discuss a class of algebras that we are interested in, namely Hopf algebras.

2.1 Homological algebra

In this section, we summarize the required basic homological algebra facts using the setting of homotopy and derived categories. We refer to [Ver96], [GM03], and [KS06] for details.

Throughout this section A is a ring and $\mathcal{B}$ an additive subcategory of $\operatorname{Mod} A$. For example $\mathcal{B}$ may be the full subcategories $\operatorname{Proj} A$, $\operatorname{Inj} A$, $\operatorname{mod} A$ of projective, injective, finitely generated A-modules, respectively.

Complexes

A *complex* X of A-modules is a sequence

$$\cdots \longrightarrow X^{i-1} \xrightarrow{d^{i-1}} X^{i} \xrightarrow{d^{i}} X^{i+1} \longrightarrow \cdots$$

of A-modules and A-linear map such that $d^i \circ d^{i-1} = 0$ for all $i \in \mathbb{Z}$. Given complexes X and Y, a *homogeneous map* $f\colon X \longrightarrow Y$ of degree n is an element of

$\prod_{i\in\mathbb{Z}} \operatorname{Hom}_A(X^i, Y^{i+n})$, and may be pictured as follows

$$\begin{array}{ccccccccc} \cdots \longrightarrow X^{i-1} & \xrightarrow{d^{i-1}} & X^i & \xrightarrow{d^i} & X^{i+1} & \xrightarrow{d^{i+1}} & \cdots \\ & \searrow^{f^{i-1}} & & \searrow^{f^i} & & \searrow^{f^{i+1}} & \\ \cdots \longrightarrow & Y^{i-1+n} & \longrightarrow & Y^{i+n} & \longrightarrow & Y^{i+1+n} & \longrightarrow \cdots . \end{array}$$

A homogenous map f of degree n is said to be a *chain map* if $d^{i+n} \circ f^i = (-1)^n f^{i+1} \circ d^i$ for $i \in \mathbb{Z}$. Thus, the differential $d = (d^i)_{i\in\mathbb{Z}}$ of a complex X is a chain map of degree 1. We may view a complex as a pair $\big(\{X^i\}_{i\in\mathbb{Z}}, d\big)$ where d is a homogeneous map of degree 1 satisfying $d \circ d = 0$.

The complexes of objects in $\mathcal{B}$ together with chain maps of degree 0 form an additive category denoted by $\mathcal{C}(\mathcal{B})$. It is a full subcategory of the abelian category $\mathcal{C}(A) = \mathcal{C}(\operatorname{Mod} A)$.

For each complex $X \in \mathcal{C}(A)$ and each $n \in \mathbb{Z}$ we define the *n-th cohomology group* to be

$$\mathrm{H}^n(X) = \operatorname{Ker}(d^i)/\operatorname{Im}(d^{i+1}).$$

This gives us a functor $\mathrm{H}^n \colon \mathcal{C}(A) \longrightarrow \operatorname{Mod} A$ for each $n \in \mathbb{Z}$. A complex X is called *acyclic* if $\mathrm{H}^n(X) = 0$ for all $n \in \mathbb{Z}$.

For complexes X and Y of A-modules (with differential d_X and d_Y, respectively), we can form a complex $\operatorname{Hom}_A(X, Y)$ of abelian groups whose component in degree n is the set of all homogenous maps of degree n, i.e.

$$(\operatorname{Hom}_A(X,Y))^n = \prod_{i\in\mathbb{Z}} \operatorname{Hom}_A(X^i, Y^{i+n})$$

and differential $d_{\operatorname{Hom}_A(X,Y)}$ defined by

$$d_{\operatorname{Hom}_A(X,Y)}(f) = d_Y \circ f - (-1)^n f \circ d_X.$$

We observe that a homogeneous map f of degree n is a chain map if and only if it is a *cycle* in $\operatorname{Hom}_A(X, Y)$, that is $d^n_{\operatorname{Hom}_A(X,Y)}(f) = 0$.

For a complex X of A-modules and a complex Y of A^{op}-module we can form a complex $X \otimes_A Y$ of abelian groups whose component in degree n is

$$(X \otimes_A Y)^n = \bigoplus_{i\in\mathbb{Z}} X^i \otimes_A Y^{n-i}$$

and differential

$$d_{X\otimes_A Y}(x \otimes y) = d_X(x) \otimes y + (-1)^i x \otimes d_Y(y)$$

for all $x \in X^i$ and $y \in Y^j$.

The homotopy category

Let X, Y be complexes of A-modules. An homogeneous map $f\colon X \longrightarrow Y$ is called *null-homotopic* if it is a *boundary* in the complex $\operatorname{Hom}_A(X,Y)$, that is $f = d_{\operatorname{Hom}_A(X,Y)}(s)$ for some homogeneous map s. In particular, every null-homotopic map is a chain map, and a chain map of degree 0 is null-homotopic if and only if $f = d_Y \circ s + s \circ d_X$ for some homogeneous map s of degree -1.

The class of null-homotopic chain maps of degree 0 in $\mathcal{C}(\mathcal{B})$ form an ideal in the sense that $f \circ g$ is null-homotopic if f or g is null-homotopic. The *homotopy category* $\mathcal{K}(\mathcal{B})$ of $\mathcal{B}$ is defined to be the quotient of $\mathcal{C}(\mathcal{B})$ with respect to this ideal. More precisely, the objects of $\mathcal{K}(\mathcal{B})$ are complexes of objects in $\mathcal{B}$ and the morphisms are given by

$$\operatorname{Hom}_{\mathcal{K}(\mathcal{B})}(X,Y) = \operatorname{Hom}_{\mathcal{C}(\mathcal{B})}(X,Y)/I(X,Y),$$

where $I(X,Y)$ is the set of null-homotopic morphisms from X to Y. It follows, by definition, that we have natural isomorphism

$$\operatorname{Hom}_{\mathcal{K}(\mathcal{B})}(X,Y) \simeq \mathrm{H}^0(\operatorname{Hom}_A(X,Y)),$$

since the morphisms of complexes are the cycles and the null-homotopic maps are the boundaries in degree 0 of the complex $\operatorname{Hom}_A(X,Y)$.

The homotopy category $\mathcal{K}(\mathcal{B})$ have a natural triangulated structure. The translation Σ is given by

$$(\Sigma X)^n = X^{n+1}, \quad d_{\Sigma X} = -d_X$$

for $X \in \mathcal{K}(\mathcal{B})$. Given a degree 0 chain map $f\colon X \longrightarrow Y$, the *mapping cone* of f is the complex $\operatorname{cone}(f)$, with

$$\operatorname{cone}(f)^n = Y^n \oplus X^{n+1}, \quad d_{\operatorname{cone}(f)} = \begin{pmatrix} d_Y & f \\ 0 & -d_X \end{pmatrix}.$$

A triangle in $\mathcal{K}(\mathcal{B})$ is *distinguished* or *exact* if it is isomorphic, in $\mathcal{K}(\mathcal{B})$, to a *mapping cone sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} \operatorname{cone}(f) \xrightarrow{h} \Sigma X,$$

where

$$g = \begin{pmatrix} \mathrm{Id}_Y \\ 0 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 0 & -\mathrm{Id}_X \end{pmatrix}$$

The derived category

We consider $\mathcal{B} = \operatorname{Mod} A$ and write $\mathcal{K}(A)$ for $\mathcal{K}(\operatorname{Mod} A)$. A morphism σ in $\mathcal{K}(A)$ is called a *quasi-isomorphism* if $H^n(\sigma)$ is an isomorphism for all integer n, or equivalently, if $\operatorname{cone}(\sigma)$ is an acylic complex. The *derived category* $\mathcal{D}(A)$ of A is the localization, in the sense of [GZ67], of $\mathcal{K}(A)$ with respect to the class of all quasi-isomorphism. This means that $\mathcal{D}(A)$ is a category, together with a functor $Q\colon \mathcal{K}(A) \longrightarrow \mathcal{D}(A)$ satisfying the following two properties:

(1) The morphism $Q(\sigma)$ is an isomorphism in $\mathcal{D}(A)$ for all quasi-isomorphism σ in $\mathcal{K}(A)$;

(2) Let $F\colon \mathcal{D}(A) \longrightarrow \mathcal{C}$ be a functor such that $F(\sigma)$ is an isomorphism for all quasi-isomorphism σ. Then F factors uniquely through Q.

Following the standard construction, the objects of $\mathcal{D}(A)$ may be chosen to be the same as $\mathcal{K}(A)$, and the functor Q is identity on objects. See [Ver96] for details.

The category $\mathcal{D}(A)$ have a natural triangulated structure such that the canonical functor Q is triangulated, and any triangulated functor $F\colon \mathcal{D}(A) \longrightarrow \mathcal{D}$, which sends quasi-isomorphisms to isomorphisms factors through Q.

Resolution

We say that a complex X in $\mathcal{K}(A)$ is called *K-projective* if $\mathrm{Hom}_{\mathcal{K}(A)}(X, Z) = 0$ for all acyclic complexes Z. A *K-projective resolution* of a complex M is a K-projective complex $\mathbf{p}M$ together with a quasi-isomorphism $\mathbf{p}M \xrightarrow{\sim} M$. Dually, X is said to be *K-injective* if $\mathrm{Hom}_{\mathcal{K}(A)}(Z, X) = 0$ for all acyclic complexes Z, and a *K-injective resolution* of M is a complex $\mathbf{i}M$ together with a quasi-isomorphism $M \xrightarrow{\sim} \mathbf{i}M$. The full subcategory of K-projective (resp., K-injective) complexes in $\mathcal{K}(A)$ is denoted by $\mathcal{K}_{\mathrm{proj}}(A)$ (resp., $\mathcal{K}_{\mathrm{inj}}(A)$). They are triangulated subcategories of $\mathcal{K}(A)$ which are closed under direct summands.

Example 2.1.1. (1) A bounded above complex of projective modules is K-projective. In particular, the usual definition of projective resolution of an A-module is a K-projective resolution.

(2) The subcategory $\mathcal{K}_{\mathrm{proj}}(A)$ is closed under coproducts. In particular, a complex of projective modules with zero differential is K-projective.

(3) Dual to (1), a bounded below complex of injective modules is K-injective, and the usual injective resolution of an A-module is a K-injective resolution.

(4) The subcategory $\mathcal{K}_{\mathrm{inj}}(A)$ is closed under products. In particular, a complex of injective modules with zero differential is K-injective.

The following theorem, due to Spaltenstein, asserts that K-projective and K-injective resolution in $\mathcal{K}(A)$ always exist.

Theorem 2.1.2 ([Spa88, Kel94, BN93])**.** *For each complex X in $\mathcal{K}(A)$, there exist (exact) triangles*

$$\mathbf{p}X \xrightarrow{\varepsilon_X} X \longrightarrow X' \longrightarrow \Sigma\mathbf{p}X$$

and

$$X'' \longrightarrow X \xrightarrow{\eta_X} \mathbf{i}X \longrightarrow \Sigma X''$$

in $\mathcal{K}(A)$ *with* X *and* X'' *acyclic. In particular,* $\mathbf{p}X \xrightarrow{\varepsilon_X} X$ *and* $X \xrightarrow{\eta_X} \mathbf{i}X$ *are K-projective and K-injective resolution of* X*, respectively.*

Remark 2.1.3. The assignment $X \longmapsto \mathbf{p}X$ induces an exact functor $\mathbf{p}\colon \mathcal{K}(A) \longmapsto \mathcal{K}(A)$ which sends quasi-isomorphisms to isomorphisms. Thus, $\mathbf{p}$ factors through $\mathcal{D}(A)$. The induced functor $\mathcal{D}(A) \longrightarrow \mathcal{K}(A)$ will also be denoted by $\mathbf{p}$. Similarly, we have a functor $\mathbf{i}\colon \mathcal{D}(A) \longrightarrow \mathcal{K}(A)$. It is a formal consequences of Theorem 2.1.2 that $\mathbf{p}$ and $\mathbf{i}$ are fully faithful left and right adjoints of the canonical functor $\mathcal{K}(A) \longrightarrow \mathcal{D}(A)$. Thus,

$$\mathrm{Hom}_{\mathcal{K}(A)}(\mathbf{p}X, Y) \cong \mathrm{Hom}_{\mathcal{D}(A)}(X, Y) \cong \mathrm{Hom}_{\mathcal{K}(A)}(X, \mathbf{i}Y).$$

Next, we state some simple properties of K-projective and K-injective complexes.

Proposition 2.1.4. *Let* X *be K-projective complexes. Then the following holds.*

(1) *The functor* $\mathrm{Hom}_A(X, -)$ *preserves acyclic complexes, or equivalently preserves quasi-isomorphisms. In particular, if* $\sigma\colon Z \to Z'$ *is a quasi-isomorphism, then*

$$\mathrm{Hom}_{\mathcal{K}(A)}(X, \sigma)\colon \mathrm{Hom}_{\mathcal{K}(A)}(X, Z) \longrightarrow \mathrm{Hom}_{\mathcal{K}(A)}(X, Z')$$

is an isomorphism.

(2) *If* $\sigma\colon X \longrightarrow Y$ *is a quasi-isomorphism with* Y *K-projective, then* σ *is an isomorphism.*

Proposition 2.1.5. *Let* X *be K-injective complexes. Then the following holds.*

(1) *The functor* $\mathrm{Hom}_A(-, X)$ *preserves acyclic complexes, or equivalently preserves quasi-isomorphisms. In particular, if* $\sigma\colon Z \to Z'$ *is a quasi-isomorphism, then*

$$\mathrm{Hom}_{\mathcal{K}(A)}(\sigma, X)\colon \mathrm{Hom}_{\mathcal{K}(A)}(Z', X) \longrightarrow \mathrm{Hom}_{\mathcal{K}(A)}(Z', X)$$

is an isomorphism.

(2) *If* $\sigma\colon X \longrightarrow Y$ *is a quasi-isomorphism with* Y *K-injective, then* σ *is an isomorphism.*

Extension groups

Let M and N be A-modules. We view them as complexes in $\mathcal{K}(A)$ in the usual way. The *n-th extension group* of N by M is by definition the cohomology of the complex $\mathrm{Hom}_A(M, \mathbf{i}N)$, i.e.

$$\mathrm{Ext}^n_A(M, N) = H^n(\mathrm{Hom}_A(M, \mathbf{i}N)) \cong \mathrm{Hom}_{\mathcal{K}(A)}(M, \Sigma^n \mathbf{i}N)).$$

The natural map $\eta_M \colon M \longrightarrow \mathbf{i}M$ induce a natural isomorphism

$$\operatorname{Ext}^n_A(M,N) \cong \operatorname{Hom}_{\mathcal{K}(A)}(\mathbf{i}M, \Sigma^n \mathbf{i}N)) \xrightarrow{\simeq} \operatorname{Hom}_{\mathcal{K}(A)}(M, \Sigma^n \mathbf{i}N)).$$

Regarding the above isomorphism as identification, a product

$$\operatorname{Ext}^n_{\mathcal{A}}(M,N) \times \operatorname{Ext}^m_{\mathcal{A}}(N,L) \longrightarrow \operatorname{Ext}^{m+n}_{\mathcal{A}}(M,L)$$

called the Yoneda product, is simply given by composition and shift in $\mathcal{K}(A)$, namely $(f,g) \longmapsto (\Sigma^n g) \circ f$. In this way, we obtain a graded ring

$$\operatorname{Ext}^*_A(M,M) = \operatorname{End}^*_{\mathcal{K}(A)}(\mathbf{i}M) = \coprod_{n\in\mathbb{Z}} \operatorname{Hom}^n_{\mathcal{K}(A)}(\mathbf{i}M, \Sigma^n \mathbf{i}M).$$

Note that one get similar result by using projective resolution:

$$\operatorname{Ext}^n_A(M,N) \cong \operatorname{Hom}_{\mathcal{K}(A)}(\mathbf{p}M, \Sigma^n N) \cong \operatorname{Hom}_{\mathcal{K}(A)}(\mathbf{p}M, \mathbf{p}\Sigma^n N).$$

2.2 Homotopy category of injectives

Let A be a noetherian ring, for instance finite dimensional algebra over a field $\Bbbk$. In this section, we discuss the main category that we are interested in, namely the homotopy category $\mathcal{K}(\operatorname{Inj} A)$ of injective A-modules. This category was studied extensively in [Kra05] in the generality of locally noetherian Grothendieck category. Note that $\operatorname{Inj} A$ is an additive subcategory of $\operatorname{Mod} A$ which is closed under coproducts since A is noetherian. In particular, $\mathcal{K}(\operatorname{Inj} A)$ has coproducts.

One motivation to study $\mathcal{K}(\operatorname{Inj} A)$ is the fact that it is the 'compactly generated' completion of $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)$, see Section 2.4 and Example 2.4.1. Another one is the following special case of self-injective algebra. Let A be a finite dimensional self-injective algebra over $\Bbbk$, for instance the group algebra $\Bbbk G$ of a finite group G. In this case, the full subcategories $\operatorname{Proj} A$ and $\operatorname{Inj} A$ coincide. The triangulated category of interest is the stable module category $\underline{\operatorname{Mod}}\, A$ of A. Its objects are A-modules and the morphism spaces are given by

$$\underline{\operatorname{Hom}}(X,Y) = \operatorname{Hom}_A(X,Y)/\operatorname{PHom}(X,Y),$$

where $\operatorname{PHom}(X,Y)$ is the subspace of morphisms that factors through a projective module.

Let $\Omega^{-1} \colon \underline{\operatorname{Mod}}\, A \longrightarrow \underline{\operatorname{Mod}}\, A$ be the cosyzygy functor; it is defined by the exact sequence

$$0 \longrightarrow X \longrightarrow E \longrightarrow \Omega^{-1} X \longrightarrow 0$$

in $\operatorname{Mod} A$, where E is an injective module. Every exact sequence

$$0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow 0$$

fits into a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & \Omega^{-1}X & \longrightarrow & 0. \end{array}$$

The category $\underline{\operatorname{Mod}}\, A$ has a triangulated structure where the suspension Σ is given by $\Sigma = \Omega^{-1}$, and a triangle is exact if it is isomorphic to a triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

as above; see [Hap88] for details.

There is a relation between $\underline{\operatorname{Mod}}\, A$ and the full subcategory $\mathcal{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ of acyclic complexes in $\mathcal{K}(\operatorname{Inj} A)$, namely there is an equivalence $\underline{\operatorname{Mod}}\, A \xrightarrow{\simeq} \mathcal{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ of triangulated categories, which sends a module to its complete resolution, that is the acyclic complex obtained by 'splicing' the projective and injective resolution of the module.

Let $I\colon \mathcal{K}(\operatorname{Inj} A) \xhookrightarrow{I} \mathcal{K}(A)$ denotes the canonical inclusion and Q the composite

$$\mathcal{K}(\operatorname{Inj} A) \xhookrightarrow{I} \mathcal{K}(A) \xrightarrow{\text{can}} \mathcal{D}(A).$$

The following result of Krause states that $\mathcal{K}(\operatorname{Inj} A)$ is the result of 'gluing' $\underline{\operatorname{Mod}}\, A$ and $\mathcal{D}(A)$ together.

Theorem 2.2.1 ([Kra05, Corollary 4.3]). *The pair of canonical functors*

$$\mathcal{K}_{\mathrm{ac}}(\operatorname{Inj} A) \xhookrightarrow{I} \mathcal{K}(\operatorname{Inj} A) \xrightarrow{Q} \mathcal{D}(A)$$

induces a recollement, in the sense of [BBD82]

$$\mathcal{K}_{\mathrm{ac}}(\operatorname{Inj} A) \;\underset{\overset{\longleftarrow}{I_\rho}}{\overset{\overset{I_\lambda}{\longleftarrow}}{\xrightarrow{\;I\;}}}\; \mathcal{K}(\operatorname{Inj} A) \;\underset{\overset{\longleftarrow}{Q_\rho}}{\overset{\overset{Q_\lambda}{\longleftarrow}}{\xrightarrow{\;Q\;}}}\; \mathcal{D}(A).$$

More precisely, the functors I and Q admit left adjoints $I\lambda$ and $Q\lambda$ as well as right adjoints I_ρ and Q_ρ such that the following adjunction morphisms

$$I_\lambda \circ I \xrightarrow{\cong} \mathrm{Id}_{\mathcal{K}_{\mathrm{ac}}(\operatorname{Inj} A)} \xrightarrow{\cong} I_\rho \circ I$$

and

$$Q \circ Q_\rho \xrightarrow{\cong} \mathrm{Id}_{\mathcal{D}(A)} \xrightarrow{\cong} Q \circ Q_\lambda$$

are isomorphisms.

Remark 2.2.2. Since A is self-injective, the left adjoint Q_λ is essentially given by taking K-projective resolution with projective-injective components (see [BK08]). More precisely, the functor $\mathbf{p}\colon \mathcal{D}(A) \longrightarrow \mathcal{K}(A)$ factors as

$$\mathcal{D}(A) \xrightarrow{Q_\lambda} \mathcal{K}(\operatorname{Inj} A) \overset{I}{\hookrightarrow} \mathcal{K}(\operatorname{Inj} A).$$

2.3 Tensor triangulated categories and actions

Triangulated categories were introduced independently by Verdier in [Ver96] and by Dold and Puppe in [DP61]. Since then, they become important tools in many branch of mathematics, in particular in commutative algebra and modular representation theory. In both areas, triangulated categories often admits additional structures. One that is important to us is a monoidal or tensor structure. Two standard examples are $\mathcal{D}(R)$ for a commutative ring R and $\underline{\operatorname{Mod}}\, \Bbbk G$ for a finite group G. In some situations, these categories acts on other triangulated categories. These actions may become useful tools to study the latter categories in terms of the former. Stevenson initiated this point of view in [Ste11, Ste13] and defined support theory using the spectrum of the acting tensor triangulated category. Buan, Krause, Snashall, and Solberg developed in [BKSS15] an 'axiomatic' support theory using ring action induced by action of tensor triangulated category.

Tensor triangulated categories

We begin with some basic definitions following [BKSS15].

Definition 2.3.1. A *tensor triangulated category* is a triangulated category $\mathcal{T}$ with an additional structure $(\otimes, \mathbb{1}, a, l, r, \lambda, \rho)$, where $- \otimes - \colon \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ is a bifunctor which is exact in each variable, $\mathbb{1}$ is an object in $\mathcal{T}$, and a, l, r are natural isomorphisms

$$a_{X,Y,Z}\colon (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z),$$

$$l_X\colon \mathbb{1} \otimes X \longrightarrow X, \quad r_X\colon X \otimes \mathbb{1} \longrightarrow X,$$

such that the following diagrams commute for all objects X, Y, Z, W in $\mathcal{T}$:

$$\begin{array}{ccc}
((X \otimes Y) \otimes Z) \otimes W & \xrightarrow{a_{X\otimes Y,Z,W}} & (X \otimes Y) \otimes (Z \otimes W) \\
\downarrow{\scriptstyle a_{X,Y,Z}\otimes W} & & \\
(X \otimes (Y \otimes Z)) \otimes W & & \downarrow{\scriptstyle a_{X,Y,Z\otimes W}} \\
\downarrow{\scriptstyle a_{X,Y\otimes Z,W}} & & \\
X \otimes ((Y \otimes Z) \otimes W) & \xrightarrow{X\otimes a_{Y,Z,W}} & X \otimes (Y \otimes (Z \otimes W))
\end{array}$$

and

$$\begin{array}{ccc} (X \otimes \mathbb{1}) \otimes Y & \xrightarrow{a_{X,\mathbb{1},Y}} & X \otimes (\mathbb{1} \otimes Y) \\ & {\scriptstyle r_X \otimes Y} \searrow \quad \swarrow {\scriptstyle X \otimes l_Y} & \\ & X \otimes Y. & \end{array}$$

Furthermore, λ and ρ are natural isomorphisms

$$\lambda_{X,Y} \colon X \otimes \Sigma Y \longrightarrow \Sigma(X \otimes Y)$$
$$\rho_{X,Y} \colon \Sigma X \otimes Y \longrightarrow \Sigma(X \otimes Y)$$

making the following diagrams commutative

$$\begin{array}{ccc} \mathbb{1} \otimes \Sigma X & \xrightarrow{l_{\Sigma X}} & \Sigma X \\ \downarrow{\scriptstyle \lambda_{\mathbb{1},X}} & & \| \\ \Sigma(\mathbb{1} \otimes X) & \xrightarrow{\Sigma(l_X)} & \Sigma X \end{array} \qquad \begin{array}{ccc} \Sigma X \otimes \mathbb{1} & \xrightarrow{r_{\Sigma X}} & \Sigma X \\ \downarrow{\scriptstyle \rho_{X,\mathbb{1}}} & & \| \\ \Sigma(X \otimes \mathbb{1}) & \xrightarrow{\Sigma(r_X)} & \Sigma X \end{array}$$

and the following diagram anti-commutative

$$\begin{array}{ccc} \Sigma X \otimes \Sigma Y & \xrightarrow{\rho_{X,\Sigma Y}} & \Sigma(X \otimes \Sigma Y) \\ \downarrow{\scriptstyle \lambda_{\Sigma X,Y}} & & \downarrow{\scriptstyle \Sigma(\lambda_{X,Y})} \\ \Sigma(\Sigma X \otimes Y) & \xrightarrow{\Sigma(\rho_{X,Y})} & \Sigma^2(X \otimes Y) \end{array}$$

for all objects X and Y in $\mathcal{T}$. In this situation, we call $\otimes$ the *tensor product* functor for $\mathcal{T}$ and $\mathbb{1}$ the *tensor unit* object of $\mathcal{T}$. The natural isomorphisms a, l, r, λ, ρ are called the *coherence isomorphisms* of $\mathcal{T}$.

Remark 2.3.2. For simplicity, we will often follow the custom of suppressing the coherence isomorphisms in tensor triangulated categories, and write simply $(\mathcal{T}, \otimes, \mathbb{1})$ for $(\mathcal{T}, \otimes, \mathbb{1}, a, l, r, \lambda, \rho)$.

Remark 2.3.3. A tensor triangulated categories $(\mathcal{T}, \otimes, \mathbb{1})$ is called *symmetric* if there is a natural isomorphism $\tau_{X,Y} \colon X \otimes Y \longrightarrow Y \otimes X$ satisfying some compatibility axiom (see [ML98]). We do not discuss this in detail because we will mainly consider non-symmetric tensor product, and we do not use any advantage of having a symmetric tensor product.

Let $\mathcal{T}$ be a triangulated category and X an object in $\mathcal{T}$. The graded endomorphism ring of X is, by definition, the graded abelian group

$$\operatorname{End}^*_{\mathcal{T}}(X) := \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^i X)$$

with multiplication induced by shift and composition, that is $\alpha \cdot \beta = (\Sigma^n \alpha) \circ \beta$ for homogeneous elements $\alpha\colon X \to \Sigma^m X$, and $\beta\colon X \to \Sigma^n X$,

Recall that a graded ring R is called graded-commutative if $r \cdot s = (-1)^{mn} s \cdot r$ for homogeneous elements $r, s \in R$ of degree m, n, respectively.

Theorem 2.3.4 ([SA04, Theorem 1.7]). *Let $\mathcal{T}$ be a tensor triangulated category with unit $\mathbb{1}$. Then the graded endomorphism ring $\operatorname{End}^*_{\mathcal{T}}(\mathbb{1})$ of the unit is graded-commutative.*

Remark 2.3.5. The main idea of the proof of Theorem 2.3.4 is to relate the multiplication in $\operatorname{End}^*_{\mathcal{T}}(\mathbb{1})$ defined using composition and shift with the multiplication defined using tensor products: for $\alpha\colon \mathbb{1} \to \Sigma^m \mathbb{1}$ and $\beta\colon \mathbb{1} \to \Sigma^n \mathbb{1}$, both $(-1)^{mn} \alpha \cdot \beta$ and $\beta \cdot \alpha$ equals the composite

$$\mathbb{1} \xrightarrow{\cong} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\alpha \otimes \beta} \Sigma^m \mathbb{1} \otimes \Sigma^n \mathbb{1} \xrightarrow{\cong} \Sigma^m (\mathbb{1} \otimes \Sigma^n \mathbb{1}) \xrightarrow{\cong} \Sigma^{m+n} (\mathbb{1} \otimes \mathbb{1}) \xrightarrow{\cong} \Sigma^{m+n} \mathbb{1},$$

where $\cong$ denotes various coherence isomorphisms that we suppressed.

Action of tensor triangulated categories

Definition 2.3.6. An *action* of a tensor triangulated category $(\mathcal{T}, \otimes, \mathbb{1}, a, l, r, \lambda, \rho)$ on a triangulated $\mathcal{K}$ is a bifunctor $- * -\colon \mathcal{T} \times \mathcal{K} \longrightarrow \mathcal{K}$ which is exact in each variable together with natural isomorphisms a', l', λ', ρ'

$$a'_{X,Y,M}\colon (X \otimes Y) * M \longrightarrow X * (Y * M), \quad l'_M\colon \mathbb{1} * M \longrightarrow M,$$

$$\lambda'_{X,M}\colon X * \Sigma M \longrightarrow \Sigma(X * M), \quad \rho'_{X,M}\colon \Sigma X * M \longrightarrow \Sigma(X * M)$$

such that the following diagrams are commutative:

$$\begin{array}{ccc}
((X \otimes Y) \otimes Z) * M & \xrightarrow{a'_{X\otimes Y,Z,M}} & (X \otimes Y) * (Z * M) \\
\downarrow{\scriptstyle a_{X,Y,Z} * M} & & \\
(X \otimes (Y \otimes Z)) * M & & \downarrow{\scriptstyle a'_{X,Y,Z*M}} \\
\downarrow{\scriptstyle a'_{X,Y\otimes Z,M}} & & \\
X * ((Y \otimes Z) * M) & \xrightarrow{X * a'_{Y,Z,M}} & X * (Y * (Z * M))
\end{array}$$

$$\begin{array}{ccc}
(X \otimes \mathbb{1}) * M & \xrightarrow{a'_{X,\mathbb{1},M}} & X * (\mathbb{1} * M) \\
& {\scriptstyle r_X * M} \searrow \quad \swarrow {\scriptstyle X * l'_M} & \\
& X * M &
\end{array}$$

and

$$\begin{array}{ccc} (\mathbb{1} \otimes X) * M & \xrightarrow{a'_{\mathbb{1},X,M}} & \mathbb{1} * (X * M) \\ & {\scriptstyle l_{X*M}} \searrow \quad \swarrow {\scriptstyle l'_{X*M}} & \\ & X * M. & \end{array}$$

Furthermore, λ' and ρ' make the following diagram commutative

$$\begin{array}{ccc} \mathbb{1} * \Sigma M & \xrightarrow{l'_{\Sigma M}} & \Sigma M \\ \downarrow {\scriptstyle \lambda' M} & & \| \\ \Sigma(\mathbb{1} * M) & \xrightarrow{\Sigma(l'_M)} & \Sigma M, \end{array}$$

and the following diagram anti-commutative

$$\begin{array}{ccc} \Sigma X * \Sigma M & \xrightarrow{\rho'_{X,\Sigma M}} & \Sigma(X * \Sigma M) \\ \downarrow {\scriptstyle \lambda'_{\Sigma X,M}} & & \downarrow {\scriptstyle \Sigma(\lambda'_{X,M})} \\ \Sigma(\Sigma X * M) & \xrightarrow{\Sigma(\rho'_{X,M})} & \Sigma^2(X * M) \end{array}$$

for all $X \in \mathcal{T}$ and $M \in \mathcal{K}$. We call the natural isomorphisms a', l', λ', ρ' the *coherence isomorphisms* for the action.

Remark 2.3.7. Let $(\mathcal{T}, \otimes, \mathbb{1}, a, l, r, \lambda, \rho)$ be a tensor triangulated category. Then it follows directly from the definition, that there is an action of $\mathcal{T}$ on itself by taking $* = \otimes$, $a' = a$, $l' = l$, $\lambda' = \lambda$ and $\rho' = \rho$.

Remark 2.3.8. Similar to Remark 2.3.2, we will often suppress the coherence isomorphisms, and write $\mathcal{T} \times \mathcal{K} \xrightarrow{*} \mathcal{K}$ to mean $(*, a', l', \lambda', \rho')$.

The induced central ring action

Let $\mathcal{K}$ be a triangulated category. The *graded center* of $\mathcal{K}$ is the graded-commutative ring

$$\mathrm{Z}^*(\mathcal{K}) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Z}^n(\mathcal{K})$$

whose component in degree $n \in \mathbb{Z}$ is

$$\mathrm{Z}^n(\mathcal{K}) = \{\eta \colon \mathrm{Id}_{\mathcal{T}} \longrightarrow \Sigma^n \mid \eta\Sigma = (-1)^n \Sigma\eta\}.$$

While $\mathrm{Z}(\mathcal{K})$ may not be a set, this is not a problem because we will only consider the image of genuine rings in the graded center.

Let R be a graded-commutative ring. A *central ring action* (or just action for short) of R on $\mathcal{K}$ is a homomorphism $\phi\colon R \longrightarrow \mathrm{Z}^*(\mathcal{K})$ of graded rings. Such an action induce homomorphism $\phi_M : R \longrightarrow \mathrm{End}^*_{\mathcal{K}}(M)$ for each $M \in \mathcal{K}$ by evaluation.

An action of a tensor triangulated category on a triangulated category induces a ring action of the graded endomorphism ring of the tensor unit.

Proposition 2.3.9. *An action $\mathcal{T} \times \mathcal{K} \xrightarrow{*} \mathcal{K}$ of $(\mathcal{T}, \otimes, \mathbb{1})$ on $\mathcal{K}$ induce a ring action $\phi : \mathrm{End}^*_{\mathcal{T}}(\mathbb{1}) \longrightarrow \mathrm{Z}^*(\mathcal{K})$ such that the map ϕ_M sends $\mathbb{1} \xrightarrow{r} \Sigma^n \mathbb{1}$ to the composite*

$$M \xrightarrow{\cong} \mathbb{1} * M \xrightarrow{r*M} \Sigma^n \mathbb{1} * M \xrightarrow{\cong} \Sigma^n(\mathbb{1} * M) \xrightarrow{\cong} \Sigma^n M.$$

Proof. See, for example, [Ste11, Proposition 2.1.7]. □

Remark 2.3.10. Note that apart from the coherence isomorphisms, the only morphism in the above composition is the map $- * M$. Therefore we will often write the map ϕ_M as $- * M\colon \mathrm{End}^*_{\mathcal{T}}(\mathbb{1}) \longrightarrow \mathrm{End}^*_{\mathcal{K}}(M)$.

For $M, N \in \mathcal{K}$, the graded abelian group

$$\mathrm{Hom}^*_{\mathcal{K}}(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{K}}(M, \Sigma^i N)$$

is a $(\mathrm{End}^*_{\mathcal{K}}(N), \mathrm{End}^*_{\mathcal{K}}(M))$-bimodule with left and right multiplication induced by shift and composition. Via ϕ_M and ϕ_N, $\mathrm{Hom}^*_{\mathcal{K}}(M, N)$ becomes a graded symmetric (R, R)-bimodule, in the sense that

$$r \cdot \beta = (-1)^{|r||\beta|} \beta \cdot r$$

for homogeneous elements $r \in R$ and $\beta \in \mathrm{Hom}^*_{\mathcal{K}}(M, N)$. Thus, we will also say that $\mathcal{K}$ is an *R-linear* triangulated category to mean that $\mathcal{K}$ is a triangulated category with an action of R.

Linear functors

Let R be a graded-commutative ring and $\mathcal{K}, \mathcal{L}$ be R-linear triangulated categories. We say that a functor $F\colon \mathcal{K} \longrightarrow \mathcal{L}$ is *R-linear* if it is an exact functor such that for each $X \in \mathcal{K}$ the following diagram is commutative:

$$\begin{array}{ccc} R & \xrightarrow{\phi_X} & \mathrm{End}^*_{\mathcal{K}}(X) \\ & \searrow^{\psi_{FX}} & \downarrow F \\ & & \mathrm{End}^*_{\mathcal{L}}(FX), \end{array}$$

where ϕ and ψ denote the action of R on $\mathcal{K}$ and $\mathcal{L}$, respectively.

Lemma 2.3.11 ([BIK12, Lemma 7.1])**.** *Let R and $\mathcal{K}, \mathcal{L}$ be as above and let $F\colon \mathcal{K} \longrightarrow \mathcal{L}$ be an R-linear functor with right adjoint G. Then the following statements hold:*

(1) *The adjunction isomorphism*

$$\operatorname{Hom}^*_{\mathcal{K}}(X, GY) \xrightarrow{\cong} \operatorname{Hom}^*_{\mathcal{L}}(FX, Y)$$

is R-linear.

(2) *The functor G is R-linear.*

Examples

We give some examples of tensor triangulated categories and its actions, that naturally appear in commutative algebra and representation theory.

Example 2.3.12. Let R be a commutative ring. Then the derived category $(\mathcal{D}(R), \otimes^{\mathbf{L}}_R, R)$ of $\operatorname{Mod} R$ is a tensor triangulated category with $\operatorname{End}^*_{\mathcal{D}(R)}(R) \cong R$, viewed as graded ring concentrated in degree 0. Letting $\mathcal{D}(R)$ acts on itself, we obtain an action of R on $\mathcal{D}(R)$.

Example 2.3.13. Let A be an algebra over a field $\Bbbk$ and $A^e = A \otimes A^{\mathrm{op}}$ be the enveloping algebra of A over $\Bbbk$. Then $(\mathcal{D}(A^e), \otimes^{\mathbf{L}}_A, A)$ is a tensor triangulated category with

$$\operatorname{End}^*_{\mathcal{D}(A^e)}(A) \cong \operatorname{Ext}^*_{A^e}(A, A) \cong \operatorname{HH}^*(A/\Bbbk),$$

which is the Hochschild cohomology of $\mathcal{D}(A)$. The action $\mathcal{D}(A^e) \times \mathcal{D}(A)\colon \xrightarrow{-\otimes^{\mathbf{L}}_A -} \mathcal{D}(A)$ induces an action of $\operatorname{HH}^*(A/\Bbbk)$ on $\mathcal{D}(A)$; see [BKSS15] and [Sol06] for details.

Example 2.3.14. Let $\Bbbk G$ be the group algebra of a finite group G over a field $\Bbbk$. The stable category $(\underline{\operatorname{Mod}}\, \Bbbk G, \otimes, \Bbbk)$ is a tensor triangulated category with

$$\underline{\operatorname{End}}^*_{\Bbbk G}(\Bbbk) \cong \widehat{\operatorname{Ext}}^*(\Bbbk, \Bbbk) \cong \widehat{\operatorname{H}}^*(G, \Bbbk),$$

which is the Tate cohomology ring of G. We refer to [Car96] for details.

Example 2.3.15. Again, let $\Bbbk G$ be as in Example 2.3.14. The homotopy category of injectives $(\mathcal{K}(\operatorname{Inj} \Bbbk G), \otimes, \mathbf{i}\Bbbk)$ is a tensor triangulated category with

$$\operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} \Bbbk G)}(\mathbf{i}\Bbbk) \cong \operatorname{Ext}^*_{\Bbbk G}(\mathbf{i}\Bbbk) \cong \operatorname{H}^*(G, \Bbbk),$$

which is the group cohomology ring of G. Letting $\mathcal{K}(\operatorname{Inj} \Bbbk G)$ acts on itself, we obtain an action of $\operatorname{H}^*(G, \Bbbk)$ on $\mathcal{K}(\operatorname{Inj} \Bbbk G)$. We refer to [BK08] for details.

Remark 2.3.16. The last two examples can be generalized to any finite dimensional Hopf algebra over a field $\Bbbk$. We will discuss this in Section 2.5.

2.4 Local cohomology and duality

We summarize the theory of local cohomology for triangulated categories in [BIK08, BIK11, BIK12] and local duality in [BIKP16].

Compactly generated triangulated categories

Local cohomology functors are defined for R-linear compactly generated triangulated category, where R is a graded-commutative noetherian ring. Therefore, we explain first the meaning of compact generation following [Nee96, Nee01].

Let $\mathcal{T}$ be a triangulated category admitting arbitrary set-indexed coproducts. A triangulated subcategory of $\mathcal{T}$ is called *localizing* if it is closed under taking coproducts, and called *thick* if closed under taking direct summands. We write $\mathrm{Loc}_{\mathcal{T}}(\mathcal{C})$ (resp., $\mathrm{Thick}_{\mathcal{T}}(\mathcal{C})$) for the smallest localizing (resp., thick) subcategory containing a given class of object $\mathcal{C} \subset \mathcal{T}$. Localizing subcategories in $\mathcal{T}$ are always thick.

An object $C \in \mathcal{T}$ is *compact* if the functor $\mathrm{Hom}_{\mathcal{T}}(C, -)$ commutes with all coproducts. The class of compact objects in $\mathcal{T}$ forms a thick subcategory of $\mathcal{T}$, which we denote by $\mathcal{T}^{\mathrm{c}}$. The category $\mathcal{T}$ is *compactly generated* if it is generated by a set of compact objects, that is there exists a set $\mathcal{G}$ of compact objects such that $\mathcal{T} = \mathrm{Loc}_{\mathcal{T}}(\mathcal{G})$. In this case, we have $\mathcal{T}^{\mathrm{c}} = \mathrm{Thick}(\mathcal{G})$.

Example 2.4.1. (1) Let A be a ring. A complex in $\mathcal{D}(A)$ is *perfect* if it is isomorphic, in $\mathcal{D}(A)$, to a bounded complex of finitely generated projective modules. The category $\mathcal{D}(A)$ is compactly generated by A, and the full subcategory $\mathcal{D}(A)^{\mathrm{op}}$ of compact objects is precisely the full subcategory $\mathcal{D}^{\mathrm{perf}}(A)$ of perfect complexes.

(2) Let A be a finite dimensional self-injective algebra. Then the stable category $\underline{\mathrm{Mod}}\, A$ is compactly generated and an object in $\underline{\mathrm{Mod}}\, A$ is compact if and only if it is stably-isomorphic to a finitely generated module

(3) Let A be a noetherian ring. The triangulated category $\mathcal{K}(\mathrm{Inj}\, A)$ is compactly generated and the canonical functor $\mathcal{K}(A) \longrightarrow \mathcal{D}(A)$ induces an equivalence $\mathcal{K}^{\mathrm{c}}(\mathrm{Inj}\, A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, A)$ (see [Kra05, Proposition 2.3]).

Compactly generated triangulated categories satisfies Brown representability theorem [Kel94, Nee96]. It is variation of a classical theorem of Brown [Bro62] from homotopy theory. Recall that a functor $\mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{A}$ from $\mathcal{T}$ to an abelian category $\mathcal{A}$ is said to be *cohomological* if it sends each exact triangle in $\mathcal{T}$ to an exact sequence in $\mathcal{A}$.

Theorem 2.4.2. *Let $\mathcal{T}$ be a compactly generated triangulated category. For a functor $H\colon \mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$ the following are equivalent.*

(1) *The functor H is cohomological and preserves set-indexed coproducts.*

(2) *There exists an object X in $\mathcal{T}$ such that $H \cong \mathrm{Hom}_{\mathcal{T}}(-, X)$.*

Here is one useful consequence of the Brown representability theorem. Recall that a left adjoint of a functor preserves all small colimits, in particular it preserves set-indexed coproducts.

Corollary 2.4.3. *Let $F\colon \mathcal{T} \longrightarrow \mathcal{U}$ be an exact functor between triangulated categories and suppose that $\mathcal{T}$ is compactly generated. Then F has a right adjoint if and only if it preserves set-indexed coproducts.*

Localization

An exact functor $L\colon \mathcal{T} \longrightarrow \mathcal{T}$ is called a *localization functor* if there exists natural transformation $\eta\colon \mathrm{Id}_{\mathcal{T}} \longrightarrow L$, called *adjunction* such that $L(\eta_X)\colon LX \longrightarrow L^2X$ is an isomorphism and $L(\eta_X) = \eta_{LX}$ for all objects $X \in \mathcal{T}$. Given such a localization functor L, the adjunction $\mathrm{Id}_{\mathcal{T}} \xrightarrow{\eta} L$ induces, for each object $X \in \mathcal{T}$, a natural *localization triangle*

$$\varGamma X \longrightarrow X \longrightarrow LX \longrightarrow \Sigma\varGamma X.$$

This gives rise to an exact functor $\varGamma\colon \mathcal{T} \longrightarrow \mathcal{T}$ and natural transformation $\varGamma \xrightarrow{\theta} \mathrm{Id}_{\mathcal{T}}$ satisfying $\varGamma(\theta_X)$ is an isomorphism and $\varGamma(\theta_X) = \theta_{\varGamma X}$ for all $X \in \mathcal{T}$. The functor $\varGamma$ is called the *colocalization functor* corresponding to L.

Example 2.4.4. Let A be a ring and $\mathcal{T} = \mathcal{K}(A)$. Then the K-injective resolution $\mathbf{i}\colon \mathcal{K}(A) \longrightarrow \mathcal{K}(A)$ is a localization functor with $\eta_X\colon X \longrightarrow \mathbf{i}X$ is the natural quasi-isomorphism. Dually, the K-projective resolution $\mathbf{p}\colon \mathcal{K}(A) \longrightarrow \mathcal{K}(A)$ is a colocalization functor with $\theta_X\colon \mathbf{p}X \longrightarrow X$ the natural map. But note that $\mathbf{i}$ is not the colocalization functor corresponding to $\mathbf{i}$, since there is no triangle

$$\mathbf{p}X \longrightarrow X \longrightarrow \mathbf{i}X \longrightarrow \Sigma\mathbf{p}X$$

in general.

Local cohomology and support

From now on, R denotes a graded-commutative noetherian ring. $\mathcal{T}$ a compactly generated R-linear triangulated category with arbitrary coproducts. We write $\operatorname{Spec} R$ for the set of homogeneous prime ideals of R. Given a homogeneous ideal $\mathfrak{a}$ in R, we set

$$\mathcal{V}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq \mathfrak{a}\}.$$

Such subsets are the closed sets in the *Zariski topology* on $\operatorname{Spec} R$.

A subset $\mathcal{V}$ of $\operatorname{Spec} R$ is said to be *specialization closed* if $\mathcal{V}(\mathfrak{p}) \subseteq \mathcal{V}$ for all $\mathfrak{p} \in \mathcal{V}$. Thus, specialization closed subsets are precisely the unions of closed subsets of $\operatorname{Spec} R$. For each specialization closed subset $\mathcal{V} \subseteq \operatorname{Spec} R$, there exists unique, up to isomorphism, a localization functor $L_{\mathcal{V}}\colon \mathcal{T} \longrightarrow \mathcal{T}$ with a property that $L_{\mathcal{V}}X = 0$ if and only if $\operatorname{Hom}^*_{\mathcal{T}}(C, X)_{\mathfrak{p}} = 0$ for all $C \in \mathcal{T}^{\mathrm{c}}$ and $\mathfrak{p} \in \operatorname{Spec} R\backslash\mathcal{V}$. This gives rise to a colocalization functor $\varGamma_{\mathcal{V}}\colon \mathcal{T} \longrightarrow \mathcal{T}$ corresponding to $L_{\mathcal{V}}$. We call $\varGamma_{\mathcal{V}}X$ the *local cohomology* of X supported on $\mathcal{V}$.

Fix a prime $\mathfrak{p}$ in $\operatorname{Spec} R$. We set

$$\mathcal{Z}(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} R \,|\, \mathfrak{q} \not\subseteq \mathfrak{p}\},$$

so $\mathcal{V}(\mathfrak{p})\backslash\mathcal{Z}(\mathfrak{p}) = \{\mathfrak{p}\}$. Note that $\mathcal{V}(\mathfrak{p})$ and $\mathcal{Z}(\mathfrak{p})$ are specialization closed. We define an exact functor $\varGamma_{\mathfrak{p}} \colon \mathcal{T} \longrightarrow \mathcal{T}$ by

$$\varGamma_{\mathfrak{p}} X = \varGamma_{\mathcal{V}(\mathfrak{p})} L_{\mathcal{Z}(\mathfrak{p})} X \quad \text{for each } X \in \mathcal{T},$$

and let $\varGamma_{\mathfrak{p}}\mathcal{T}$ denotes its essential image. An object X in $\mathcal{T}$ is in $\varGamma_{\mathfrak{p}}\mathcal{T}$ if and only if the R-module $\operatorname{Hom}^*_{\mathcal{T}}(C, X)$ is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion for every compact object C.

The example below justifies the language of local cohomology functor.

Example 2.4.5. Let R be a commutative noetherian ring. We view R as a graded ring concentrated in degree 0, when it acts on $\mathcal{D}(R)$. Thus, $\mathcal{D}(R)$ is an R-linear compactly generated. Let $\mathfrak{a}$ be an ideal of R. For each R-module M, consider the submodule

$$G_{\mathfrak{a}} M = \{m \in M \,|\, \mathfrak{a}^n m = 0 \quad \text{for some integer } n \geq 0\}.$$

The assignment $M \longmapsto G_{\mathfrak{a}} M$ is an additive, left-exact functor on the category of R-modules, called the $\mathfrak{a}$-*torsion functor*. We denote by $\mathbf{R}G_{\mathfrak{a}} \colon \mathcal{D}(R) \longrightarrow \mathcal{D}(R)$ the right derived functor of $F_{\mathfrak{a}}$. Then we have an isomorphism $\mathbf{R}G_{\mathfrak{a}} \cong \varGamma_{\mathcal{V}(\mathfrak{a})}$. [BIK08, Theorem 9.1].

The support of an object $X \in \mathcal{T}$ is by definition the set

$$\operatorname{supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \,|\, \varGamma_{\mathfrak{p}} X \neq 0\},$$

and we set

$$\operatorname{supp}_R \mathcal{T} = \bigcup_{X \in \mathcal{T}} \operatorname{supp}_R X.$$

We have the following formal properties of support.

Theorem 2.4.6 ([BIK08, Theorem 5.2])**.** *For each object X in $\mathcal{T}$, $\operatorname{supp}_R X = \emptyset$ if and only if $X = 0$.*

Theorem 2.4.7 ([BIK08, Theorem 5.6])**.** *Let $\mathcal{V}$ be a specialization closed subset of $\operatorname{Spec} R$. For each X in $\mathcal{T}$, the following equalities hold*

$$\operatorname{supp}_R \varGamma_{\mathcal{V}} X = \mathcal{V} \cap \operatorname{supp}_R X$$
$$\operatorname{supp}_R L_{\mathcal{V}} X = (\operatorname{Spec} R \backslash \mathcal{V}) \cap \operatorname{supp}_R X.$$

Corollary 2.4.8 ([BIK08, Corollary 5.9])**.** *Let $\mathfrak{p}$ be a point in $\operatorname{Spec} R$ and X a nonzero object in $\mathcal{T}$. Then $\varGamma_{\mathfrak{p}} X \cong X$ if and only if $\operatorname{supp}_R X = \{fp\}$.*

Corollary 2.4.9. *Let $\mathfrak{p}, \mathfrak{q}$ be primes in $\operatorname{Spec} R$. Then $\varGamma_{\mathfrak{p}} \circ \varGamma_{\mathfrak{q}}$ is naturally isomorphic to $\varGamma_{\mathfrak{p}}$ if $\mathfrak{p} = \mathfrak{q}$, and zero otherwise.*

Injective cohomology objects

Let R be a graded-commutative noetherian ring and $\mathcal{T}$ a compactly generated R-linear triangulated category. Given an object C in $\mathcal{T}^c$ and an injective R-module I. Consider the functor

$$\operatorname{Hom}_R(\operatorname{Hom}^*_{\mathcal{T}}(C,-), I)\colon \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{T}.$$

This functor is cohomological and preserve coproducts. Brown representability yields an object $T(C,I)$ in $\mathcal{T}$ such that

$$\operatorname{Hom}_R(\operatorname{Hom}^*_{\mathcal{T}}(C,-), I) \cong \operatorname{Hom}_{\mathcal{T}}(-, T(C,I)).$$

By Yoneda lemma, this yields a functor

$$T\colon \mathcal{T}^c \times \operatorname{Inj} R \longrightarrow \mathcal{T}.$$

For each $\mathfrak{p}$ in $\operatorname{Spec} R$, we write $I(\mathfrak{p})$ for the injective hull of $R/\mathfrak{p}$ and set

$$T_{\mathfrak{p}} := T(-, I(\mathfrak{p}))\,,$$

viewed as a functor $\mathcal{T}^c \to \mathcal{T}$.

The following proposition justifies the terminology.

Proposition 2.4.10. *Let $(\mathcal{T}, \otimes, \mathbb{1})$ be a tensor triangulated category and take $R = \operatorname{End}^*_{\mathcal{T}}(\mathbb{1})$. Suppose that $\mathcal{T}$ is compactly generated and R is noetherian. For all $E \in \operatorname{Inj} R$, we have an isomorphism of graded R-modules*

$$\operatorname{Hom}^*_R(\mathbb{1}, T(\mathbb{1}, E))) \cong E.$$

In particular $\operatorname{Hom}^*_R(\mathbb{1}, T(C,E))$ is injective.

Proof. This is a direct consequence of the defining property of the object $T(\mathbb{1}, E)$, namely we have the following isomorphisms of graded R-modules

$$\operatorname{Hom}^*_{\mathcal{T}}(\mathbb{1}, T(\mathbb{1}, E)) \cong \operatorname{Hom}^*_R(\operatorname{End}^*_{\mathcal{T}}(\mathbb{1}), E) = \operatorname{Hom}^*_R(R, E) \cong E.$$

□

We call a compactly generated R-linear triangulated category $\mathcal{T}$ is *noetherian* if, for any compact object C, the R-module $\operatorname{End}^*_{\mathcal{T}}(C)$ is finitely generated, and equivalently, for all compact objects C, D, the R-module $\operatorname{Hom}^*_{\mathcal{T}}(C,D)$ is finitely generated.

Proposition 2.4.11 ([BIK11, Proposition 5.4])**.** *If the R-linear category $\mathcal{T}$ is noetherian, then*

$$\operatorname{supp}_R T(C,I) = \operatorname{supp}_R C \cap \operatorname{supp}_R I$$

for all $C \in \mathcal{T}^c$ and $I \in \operatorname{Inj} R$. In particular, for each $\mathfrak{p} \in \operatorname{Spec} R$ the object $T_{\mathfrak{p}}(C)$ is in $\Gamma_{\mathfrak{p}}\mathcal{T}$ for each $C \in \mathcal{T}^c$.

The Gorenstein property

Let R be a graded-commutative noetherian ring and $\mathcal{T}$ a compactly generated R-linear triangulated category. We say that $\mathcal{T}$ is *Gorenstein* with respect to the action of R if there is an R-linear triangle equivalence $F\colon \mathcal{T}^{c} \xrightarrow{\simeq} \mathcal{T}^{c}$ and for every $\mathfrak{p} \in \operatorname{supp}_R(\mathcal{T})$ there is an integer $d(\mathfrak{p})$ and a natural isomorphism

$$\varGamma_{\mathfrak{p}} \circ F \cong \Sigma^{d(\mathfrak{p})} \circ T_{\mathfrak{p}}$$

of functors $\mathcal{T}^{c} \longrightarrow \mathcal{T}$. In this context F is called a *global Serre functor*, because localising at $\mathfrak{p}$ induces a Serre functor $\Sigma^{-d(\mathfrak{p})}F_{\mathfrak{p}}$ in the sense of Bondal and Kapranov [BK89].

Example 2.4.12. Let A be a commutative noetherian ring and $\mathcal{D}(A)$ the derived category of A-modules. This is an A-linear compactly generated tensor triangulated category. Recall that the ring A is *Gorenstein* if for each $\mathfrak{p} \in \operatorname{Spec} A$ the injective dimension of $A_{\mathfrak{p}}$, as a module over itself, is finite. By Grothendieck's local duality theorem [BH93, Section 3.5], this is equivalent to an isomorphism of $A_{\mathfrak{p}}$-modules

$$\varGamma_{\mathfrak{p}} A \cong \Sigma^{-\dim A_{\mathfrak{p}}} I(\mathfrak{p})\,.$$

Thus $\mathcal{D}(A)$ is Gorenstein with dualising object A and $d(\mathfrak{p}) = -\dim A_{\mathfrak{p}}$.

Local Serre duality

Here we discuss the notion of local Serre duality for an essentially small R-linear triangulated category following [BIKP16].

Let $\mathcal{C}$ be an essentially small R-linear triangulated category. Fix $\mathfrak{p} \in \operatorname{Spec} R$ and let $\mathcal{C}_{\mathfrak{p}}$ denote the triangulated category that is obtained from $\mathcal{C}$ by keeping the objects of $\mathcal{C}$ and setting

$$\operatorname{Hom}^*_{\mathcal{C}_{\mathfrak{p}}}(X,Y) := \operatorname{Hom}^*_{C}(X,Y)_{\mathfrak{p}}.$$

Then $\mathcal{C}_{\mathfrak{p}}$ is an $R_{\mathfrak{p}}$-linear triangulated category and localising the morphisms induces an exact functor $\mathcal{C} \longrightarrow \mathcal{C}_{\mathfrak{p}}$.

Let $\gamma_{\mathfrak{p}}\mathcal{C}$ be the full subcategory of $\mathfrak{p}$-torsion objects in $\mathcal{C}_{\mathfrak{p}}$, namely

$$\gamma_{\mathfrak{p}}\mathcal{C} := \{X \in \mathcal{C}_{\mathfrak{p}} \mid \operatorname{End}^*_{\mathcal{C}_{\mathfrak{p}}}(X) \text{ is } \mathfrak{p}\text{-torsion}\}.$$

This is a thick subcategory of $\mathcal{C}_{\mathfrak{p}}$.

Remark 2.4.13. Let $\mathcal{T}$ be a compactly generated R-linear triangulated category. Set $\mathcal{C} := \mathcal{T}^c$ and fix $\mathfrak{p} \in \operatorname{Spec} R$. The triangulated categories $L_{\mathcal{Z}(\mathfrak{p})}\mathcal{T}$ and $\varGamma_{\mathfrak{p}}\mathcal{T}$ are compactly generated. The left adjoint of the inclusion $L_{\mathcal{Z}(\mathfrak{p})}\mathcal{T} \hookrightarrow \mathcal{T}$ induces (up to direct summands) a triangle equivalence $\mathcal{C}_{\mathfrak{p}} \xrightarrow{\simeq} (L_{\mathcal{Z}(\mathfrak{p})}\mathcal{T})^c$ and restricts to a triangle equivalence (also up to direct summands)

$$\gamma_{\mathfrak{p}}\mathcal{C} \xrightarrow{\sim} (\varGamma_{\mathfrak{p}}\mathcal{T})^c\,.$$

This follows from the fact that the localisation functor $\mathcal{T} \longrightarrow L_{\mathcal{Z}(\mathfrak{p})}\mathcal{T}$ preserves compactness and that for compact objects X, Y in $\mathcal{T}$

$$\operatorname{Hom}^*_{\mathcal{T}}(X,Y)_{\mathfrak{p}} \xrightarrow{\sim} \operatorname{Hom}^*_{L_{\mathcal{Z}(\mathfrak{p})}\mathcal{T}}(X_{\mathfrak{p}}, Y_{\mathfrak{p}}) .$$

For details we refer to [BIK15].

Let R be a graded commutative ring that is *local*; thus there is a unique homogeneous maximal ideal, say $\mathfrak{m}$. We call an R-linear triangle equivalence $F\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ a *Serre functor* if for all objects X, Y in $\mathcal{C}$ there is an natural isomorphism

$$\operatorname{Hom}_R(\operatorname{Hom}^*_{\mathcal{C}}(X,Y), I(\mathfrak{m})) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(Y, FX) . \tag{2.4.1}$$

The situation when R is a field was the one considered in [BK89].

For an arbitrary graded commutative ring R, we say that an R-linear triangulated category $\mathcal{C}$ satisfies *local Serre duality* if there exists an R-linear triangle equivalence $F\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ such that for every $\mathfrak{p} \in \operatorname{Spec} R$ and some integer $d(\mathfrak{p})$ the induced functor $\Sigma^{-d(\mathfrak{p})}F_{\mathfrak{p}}\colon \gamma_{\mathfrak{p}}\mathcal{C} \xrightarrow{\sim} \gamma_{\mathfrak{p}}\mathcal{C}$ is a Serre functor for the $R_{\mathfrak{p}}$-linear category $\gamma_{\mathfrak{p}}\mathcal{C}$. Thus for all objects X, Y in $\gamma_{\mathfrak{p}}\mathcal{C}$ there is a natural isomorphism

$$\operatorname{Hom}_R(\operatorname{Hom}^*_{\mathcal{C}_{\mathfrak{p}}}(X,Y), I(\mathfrak{p})) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}_{\mathfrak{p}}}(Y, \Sigma^{-d(\mathfrak{p})}F_{\mathfrak{p}}X) .$$

The Gorenstein property of an R-linear compactly generated triangulated category implies that the full subcategory of compact objects has local Serre duality.

Proposition 2.4.14 ([BIKP16, Proposition 7.3])**.** *Let R be a graded commutative noetherian ring and $\mathcal{T}$ a compactly generated R-linear triangulated category. Suppose that $\mathcal{T}$ is Gorenstein, with global Serre functor F and shifts $\{d(\mathfrak{p})\}$. Then for each $\mathfrak{p} \in \operatorname{supp}_R(\mathcal{T})$, object $X \in (\varGamma_{\mathfrak{p}}\mathcal{T})^c$ and $Y \in L_{\mathcal{Z}(\mathfrak{p})}\mathcal{T}_{\mathfrak{p}}$ there is a natural isomorphism*

$$\operatorname{Hom}_R(\operatorname{Hom}^*_{\mathcal{T}}(X,Y), I(\mathfrak{p})) \cong \operatorname{Hom}_{\mathcal{T}}(Y, \Sigma^{-d(\mathfrak{p})}F_{\mathfrak{p}}(X)) .$$

Corollary 2.4.15 ([BIKP16, Corollary 7.4])**.** *Let R be a graded commutative noetherian ring and $\mathcal{T}$ a compactly generated R-linear triangulated category. If $\mathcal{T}$ is Gorenstein, then $\mathcal{T}^{\mathrm{c}}$ satisfies local Serre duality.*

Example 2.4.16. Using the notation of Example 2.4.12, when A is a (commutative noetherian) Gorenstein ring, local Serre duality reads: For each $\mathfrak{p} \in \operatorname{Spec} A$ and $n \in \mathbb{Z}$ there are natural isomorphisms

$$\operatorname{Hom}_{A_{\mathfrak{p}}}(\operatorname{Ext}^n_{A_{\mathfrak{p}}}(X,Y), I(\mathfrak{p})) \cong \operatorname{Ext}^{n+\dim A_{\mathfrak{p}}}_{A_{\mathfrak{p}}}(Y,X)$$

where X is a perfect complexes of $A_{\mathfrak{p}}$-modules with finite length cohomology, and Y is a complex of $A_{\mathfrak{p}}$-modules.

2.5 Hopf algebras

Let us fix a field $\Bbbk$ and let $\otimes$ denote tensor product over $\Bbbk$. We denote by τ the natural isomorphism $\tau_{M,N}\colon M \otimes N \longrightarrow N \otimes M$, $m \otimes n \longmapsto n \otimes m$ for $M, N \in \operatorname{Mod}\Bbbk$.

Definition of Hopf algebras

Definition 2.5.1. A Hopf algebra over a field $\Bbbk$ is an algebra A, together with $\Bbbk$-linear map $\Delta\colon A \longrightarrow A \otimes A$, $\varepsilon\colon A \longrightarrow \Bbbk$, and $S\colon A \longrightarrow A$, called comultiplication, counit, and antipode, satisfying the following conditions:

(a) The triple (A, Δ, ε) is a coalgebra over $\Bbbk$, namely that the diagrams

$$\begin{array}{ccc} A & \xrightarrow{\ \Delta\ } & A \otimes A \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes A} \\ A \otimes A & \xrightarrow{\ A\otimes\Delta\ } & A \otimes A \otimes A, \end{array} \qquad \begin{array}{ccccc} \Bbbk \otimes A & \xleftarrow{\ \cong\ } & A & \xrightarrow{\ \cong\ } & A \otimes \Bbbk \\ & {\scriptstyle\varepsilon\otimes A}\nwarrow & {\scriptstyle\Delta}\downarrow & \nearrow{\scriptstyle A\otimes\varepsilon} & \\ & & A \otimes A & & \end{array}$$

are commutative.

(b) The map Δ and ε are algebra homomorphisms.

(c) The map S makes the following diagram

$$\begin{array}{ccccc} A \otimes A & \xleftarrow{\ \Delta\ } & A & \xrightarrow{\ \Delta\ } & A \otimes A \\ & & \downarrow{\scriptstyle\varepsilon} & & \\ {\scriptstyle A\otimes S}\downarrow & & \Bbbk & & \downarrow{\scriptstyle S\otimes A} \\ & & \downarrow{\scriptstyle u} & & \\ A \otimes A & \xrightarrow{\ m\ } & A & \xleftarrow[\ m\ ]{} & A \otimes A \end{array}$$

commutative, where $u\colon \Bbbk \longrightarrow A$ is the unit map and $m\colon A \otimes A \longrightarrow A$ is the multiplication map. A Hopf algebra A is called *cocommutative* if the triangle

$$\begin{array}{ccc} & A & \\ {\scriptstyle\Delta}\swarrow & & \searrow{\scriptstyle\Delta} \\ A \otimes A & \xrightarrow{\ \tau\ } & A \otimes A \end{array}$$

is commutative.

The following notation for the comultiplication, called the *Sweedler notation*, will be very useful. For an element a in a Hopf algebra A, the element $\Delta(a)$ in $A \otimes A$ will be abbreviated to

$$\Delta(a) = \sum a_1 \otimes a_2.$$

For example, a Hopf algebra A is cocommutative if $\sum a_1 \otimes a_2 = \sum a_2 \otimes a_1$ for all a in A.

Example 2.5.2. Let G be a finite group. Then the group algebra $A = \Bbbk G$ is a Hopf algebra with

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}$$

for all $g \in G$. It is clear that $\Bbbk G$ is cocommutative.

Example 2.5.3. Suppose that the characteristic of the field $\Bbbk$ is a prime number p. Then the $\Bbbk$-algebra

$$A = k[X_1, \dots, X_n]/(X_1^p, \dots, X_n^p)$$

is a Hopf algebra with

$$\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i, \quad \varepsilon(X_i) = 0, \quad S(X_i) = -X_i$$

for $i = 1, \dots, n$. This Hopf algebra is also cocommutative.

Basic properties

We collect some basic properties of a Hopf algebra and refer to [SY11] for details.

Proposition 2.5.4. *Let A be a Hopf algebra with antipode S. Then S is an algebra homomorphism from A to A^{op}, that is $S(ab) = S(b)S(a)$ for all $a, b \in A$.*

Remark 2.5.5. As a consequence of the above proposition, we have a homomorphism $\delta \colon A \xrightarrow{\Delta} A \otimes A \xrightarrow{A \otimes S} A \otimes A^{\mathrm{op}} = A^e$ of $\Bbbk$-algebras. This induce a restriction functor $\operatorname{Mod} A^e \longrightarrow \operatorname{Mod} A$ which will be useful in Section 4.4.

Example 2.5.6. For $A = \Bbbk G$ for a finite group G, the homomorphism δ is given by $\delta(g) = g \otimes g^{-1}$. Thus, for a $(\Bbbk G)^e$-module M, the restriction along δ gives a $\Bbbk G$-module M with left action given by $g \cdot m = (g \otimes g^{-1}) \cdot m$.

Now recall that a $\Bbbk$-algebra A is Frobenius if $\operatorname{Hom}_{\Bbbk}(A, \Bbbk)$ is isomorphic to A as left and right A-module (but not necessarily isomorphic as A^e-module). It is called *symmetric* if A and $\operatorname{Hom}_{\Bbbk}(A, \Bbbk)$ are isomorphic as A^e-module.

Theorem 2.5.7 ([Par71])**.** *A finite dimensional Hopf algebra over $\Bbbk$ is Frobenius. In particular, it is self-injective.*

Example 2.5.8. The group algebra $\Bbbk G$ for a finite group G is symmetric. There is an isomorphism $\varphi \colon \Bbbk G \xrightarrow{\cong} \operatorname{Hom}_{\Bbbk}(\Bbbk G, \Bbbk)$, given by

$$\varphi(g)(h) = \begin{cases} 1, & \text{if } g = h \\ 0, & \text{otherwise} \end{cases}$$

for all $g, h \in G$. See [SY11, Section VI.5]

The tensor structure

Throughout this subsection, let us fix a Hopf algebra $(A, \Delta, \varepsilon, S)$. The module category $\operatorname{Mod} A$ has a tensor structure which we describe below. For $M, N \in \operatorname{Mod} A$, the $\Bbbk$-module $M \otimes N$ has a natural structure of an $(A \otimes A)$-module with multiplication $(a \otimes b)(m \otimes n) = am \otimes bn$ for $a, b \in A$, $m \in M$, and $n \in N$. We make it into an A-module by restriction along the comultiplication Δ. Using Sweedler notation, the A-module structure is given by

$$a(m \otimes n) = \sum a_1 n \otimes a_2 n.$$

The 1-dimensional $\Bbbk$-module $\Bbbk$ is an A-module via ε. The triple $(\operatorname{Mod} A, \otimes, \Bbbk)$ forms a tensor category, where the coherence axioms is the same as those in $(\operatorname{Mod} \Bbbk, \otimes, \Bbbk)$. For example, the natural isomorphism $(M \otimes N) \otimes L \xrightarrow{\cong} M \otimes (N \otimes L)$ of $\Bbbk$-modules is also an A-linear isomorphism, for all $M, N, L \in \operatorname{Mod} A$. Moreover, if A is cocommutative, then the tensor structure is symmetric. Again, the $\Bbbk$-linear isomorphism $\tau_{M,N}$ gives a symmetry $M \otimes N \xrightarrow{\cong} N \otimes M$ as A-modules.

For $M, N \in \operatorname{Mod} A$, the $\Bbbk$-module $\operatorname{Hom}_{\Bbbk}(M, N)$ has a natural structure of A^e-module with multiplication

$$((a \otimes b)f)(m) = af(bm), \quad m \in M$$

for $a, b \in A$, and $f \in \operatorname{Hom}_{\Bbbk}(M, N)$. By restriction along the $\Bbbk$-algebra homomorphism $A \xrightarrow{\Delta} A \otimes A \xrightarrow{A \otimes S} A^e$, the A^e-module $\operatorname{Hom}_{\Bbbk}(M, N)$ becomes an A-module. Using Sweedler notation, the A-module structure is given by

$$(af)(m) = \sum a_1 f(S(a_2)m), \quad m \in M$$

for $a \in A$ and $f \in \operatorname{Hom}_{\Bbbk}(M, N)$.

Theorem 2.5.9. *For any A-modules X, M, N, there is an isomorphism*

$$\operatorname{Hom}_A(M \otimes X, Y) \cong \operatorname{Hom}_A(M, \operatorname{Hom}_{\Bbbk}(X, N))$$

of $\Bbbk$-vector spaces which is natural in X, M, N. In particular, $- \otimes X$ is a left adjoint to $\operatorname{Hom}_{\Bbbk}(X, -)$.

Corollary 2.5.10. *Let P, M be A-modules with P projective. Then $P \otimes M$ is a projective A-module.*

The homotopy category of Hopf algebras

Now we discuss the homotopy category of Hopf algebras. Again, let us fix a Hopf algebra $(A, \Delta, \varepsilon, S)$. The homotopy category $\mathcal{K}(A)$ have a tensor triangulated structure $(\otimes, \Bbbk)$ where the tensor product of complexes is defined using (direct sum) totalization of the tensor product in $\operatorname{Mod} A$. More precisely, given complexes X, Y in $\mathcal{K}(A)$, the complex $X \otimes Y$ have component

$$(X \otimes Y)^n = \bigoplus_{i \in \mathbb{Z}} X^i \otimes Y^{n-i}$$

and differential

$$d_{X \otimes Y}(x \otimes y) = d_X(x) \otimes y + (-1)^i x \otimes d_Y(y)$$

for element $x \in X^i$ and $y \in Y^j$. If A is cocommutative, the tensor structure is again symmetric.

Now we consider the homotopy category $\mathcal{K}(\operatorname{Inj} A)$ of injective A-modules. In this case, we assume additionally that A is finite dimensional. Thus, A is self-injective by Theorem 2.5.7. In particular, we have $\operatorname{Proj} A = \operatorname{Inj} A$ and thus $\mathcal{K}(\operatorname{Inj} A)$ is closed under $\otimes$ by Corollary 2.5.10. It turns out that $\mathcal{K}(\operatorname{Inj} A)$ becomes a tensor triangulated category with this tensor product. The unit is given by the injective resolution $\mathbf{i}\Bbbk$ of $\Bbbk$ as an A-module (see Theorem 4.4.8).

Considering the action of the tensor triangulated category $(\mathcal{K}(\operatorname{Inj} A), \otimes, \mathbf{i}\Bbbk)$ on itself, we obtain a ring action of the graded endomorphism ring $\operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A)}(\mathbf{i}\Bbbk)$ on $\mathcal{K}(\operatorname{Inj} A)$. This ring is the cohomology ring of A, which we discuss next.

Cohomology of Hopf algebras

Let A be a Hopf algebra over $\Bbbk$. Recall that $\Bbbk$ becomes an A-module via the counit $A \longrightarrow \Bbbk$. The *cohomology* of A with coefficients in an A-module M is defined by

$$\mathrm{H}^n(A, M) = \operatorname{Ext}^n_A(\Bbbk, M).$$

The cohomology ring of A is the graded ring

$$\mathrm{H}^*(A, \Bbbk) = \operatorname{Ext}^*_A(\Bbbk, \Bbbk)$$

with multiplication given by Yoneda product. This is a graded-commutative ring, for example, since it is isomorphic to the graded endomorphism ring of the tensor unit $\mathbf{i}\Bbbk$ of the tensor triangulated category $(\mathcal{K}(\operatorname{Inj} A), \otimes, \mathbf{i}\Bbbk)$ by Theorem 2.3.4.

If A is finite dimensional and cocommutative, then the cohomology ring $\mathrm{H}^*(A, \Bbbk)$ is noetherian. This is a theorem of Friedlander and Suslin which generalizes the theorems of Golod ([Gol59]), Venkov ([Ven59]), and Evens ([Eve61]) in the case of group algebra.

Theorem 2.5.11 ([FS97]). *Let A be a finite dimensional cocommutative Hopf algebra over a field $\Bbbk$. The cohomology ring $\mathrm{H}^*(A, \Bbbk)$ is finitely generated as a $\Bbbk$-algebra, and the graded $\mathrm{H}^*(A, \Bbbk)$-module $\operatorname{Ext}^*_A(M, N)$ is finitely generated for all finitely generated A-modules M, N,*

Chapter 3

Idempotents in tensor triangulated categories

Idempotents in tensor triangulated categories have been studied for various purposes. Rickard constructed in [Ric97] idempotent modules in the stable module category $\underline{\mathrm{Mod}}\,\Bbbk G$ of the group algebra $\Bbbk G$ for some finite group G. These modules are used to define support varieties for infinitely generated $\Bbbk G$-modules in [BCR96] and to classify thick tensor ideal of $\underline{\mathrm{mod}}\,\Bbbk G$ in [BCR97]. In [BF11] Balmer and Favi generalized Rickard's construction in the context of tensor triangular geometry and connected the generalized idempotents to the so called telescope conjecture. In the general case of non-symmetric tensor category, idempotents are considered in [BD14] in the study of character sheaves on unipotent groups, and recently in [Hog17] to generalize the notions of cohomology and Tate cohomology in tensor triangulated categories.

One basic fact that is used repeatedly in the above examples is that an idempotent e in a tensor category $\mathcal{T}$ induce a full subcategory, denoted by $e\mathcal{T}e$ (see Definition 3.1.1), which has a natural structure of tensor category with unit e. Moreover, if $\mathcal{T}$ is tensor triangulated then $e\mathcal{T}e$ is also tensor triangulated. Our aim in this chapter is to relate the abstract theory of idempotents with actions of tensor triangulated categories. We show that an idempotent in a tensor triangulated category $\mathcal{T}$ acting on a triangulated category $\mathcal{K}$ induce an action of $e\mathcal{T}e$ on a full subactegory of $\mathcal{K}$, denoted by $e\mathcal{K}$. We will apply this idea in the following chapter to obtain various actions of Hochschild cohomology ring.

3.1 Idempotents

We begin with the definition of an idempotent in a tensor category following [BD14]. Note that the definition below only depends on the tensor structure.

Definition 3.1.1. Let $(\mathcal{M}, \otimes, \mathbb{1})$ be a tensor category and e an object in $\mathcal{M}$.

(a) The object e is said to be a *weak idempotent* if $e \otimes e \cong e$. If e is a weak idempotent, then let $e\mathcal{M}, \mathcal{M}e, e\mathcal{M}e$ denote the full subcategory

$$\begin{aligned} e\mathcal{M} &= \{X \in \mathcal{M} \,|\, X \cong e \otimes Y,\, Y \in \mathcal{M}\} \\ \mathcal{M}e &= \{X \in \mathcal{M} \,|\, X \cong Y \otimes e,\, Y \in \mathcal{M}\} \\ e\mathcal{M}e &= \{X \in \mathcal{M} \,|\, X \cong e \otimes Y \otimes e,\, Y \in \mathcal{M}\}. \end{aligned}$$

The subcategory $e\mathcal{M}e$ is called the *corner subcategory* defined by e.

(b) A morphism $\mathbb{1} \xrightarrow{\eta} e$ is said to be a *unital idempotent arrow* if both morphisms

$$\mathbb{1} \otimes e \xrightarrow{\eta \otimes e} e \otimes e \quad \text{and} \quad e \otimes \mathbb{1} \xrightarrow{e \otimes \eta} e \otimes e$$

are isomorphisms. Dually, a morphism $e \xrightarrow{\varepsilon} \mathbb{1}$ is said to be *counital idempotent arrow* if both morphisms

$$e \otimes e \xrightarrow{\varepsilon \otimes e} \mathbb{1} \otimes e \quad \text{and} \quad e \otimes e \xrightarrow{e \otimes \varepsilon} e \otimes \mathbb{1}$$

are isomorphisms.

(c) The object e is said to be a *unital idempotent* if there exists a unital idempotent arrow $\mathbb{1} \longrightarrow e$. It is said to be a *counital idempotent* if there exists a counital idempotent arrow $e \longrightarrow \mathbb{1}$. We say that e is an *idempotent* to mean that e is either a unital or counital idempotent.

Remark 3.1.2. It is not difficult to see that, for weak idempotent e, the full subcategories $e\mathcal{M}$, $\mathcal{M}e$, and $e\mathcal{M}e$ have the following alternative description:

$$\begin{aligned} e\mathcal{M} &= \{X \in \mathcal{M} \,|\, X \cong e \otimes X\} \\ \mathcal{M}e &= \{X \in \mathcal{M} \,|\, X \cong X \otimes e\} \\ e\mathcal{M}e &= \{X \in \mathcal{M} \,|\, X \cong e \otimes X \otimes e\}. \end{aligned}$$

We summarize some results in [BD14] that we will need. First let us fix a tensor category $(\mathcal{M}, \otimes, \mathbb{1})$.

Lemma 3.1.3 ([BD14, Lemma 3.14])**.** *Let e be an idempotent in $\mathcal{M}$. Then*

$$e\mathcal{M}e = e\mathcal{M} \cap \mathcal{M}e.$$

Lemma 3.1.4 ([BD14, Lemma 3.15])**.** *Let $\eta\colon \mathbb{1} \longrightarrow e$ be a unital idempotent arrow. An object $X \in \mathcal{M}$ belongs to $e\mathcal{M}$ (resp., $\mathcal{M}e$) if and only if the morphism $\eta \otimes X\colon \mathbb{1} \otimes X \longrightarrow e \otimes X$ (resp., $X \otimes \eta\colon X \otimes \mathbb{1} \longrightarrow X \otimes e$) is an isomorphism.*

Remark 3.1.5. The lemma above also holds for counital idempotent arrow with an analogous proof.

Lemma 3.1.6 ([BD14, Lemma 3.18])**.** *If e is an idempotent in $\mathcal{M}$, then the corner subcategory $e\mathcal{M}e$ is a tensor category with tensor product $\otimes$ and unit e.*

Remark 3.1.7. We describe the coherence isomorphisms in $(e\mathcal{M}e, \otimes, e)$. The associativity isomorphism $(X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$ is inherited from $(\mathcal{M}, \otimes, \mathbb{1})$. The left unit isomorphism $e \otimes X \xrightarrow{\cong} X$ is obtained by composing $e \otimes X \xrightarrow{\cong} \mathbb{1} \otimes X$ with the left unit isomorphism $\mathbb{1} \otimes X \xrightarrow{\cong} X$ in $\mathcal{M}$. The right unit isomorphism $X \otimes e \xrightarrow{\cong} X$ is obtained in a similar way.

Remark 3.1.8. If $e \in \mathcal{M}$ is a weak idempotent, then the corner subcategory $e\mathcal{M}e$ is closed under $\otimes$, but may fail to be monoidal. An example is given in [BD14, Remark 3.19].

We add a simple lemma to obtain an idempotent inside a corner subcategory.

Lemma 3.1.9. *Let $\eta\colon \mathbb{1} \longrightarrow e$ and $\varepsilon\colon f \longrightarrow \mathbb{1}$ be a unital and counital idempotent arrows in $\mathcal{M}$. If f belongs to $e\mathcal{M}e$ then the composite $\eta\varepsilon\colon f \longrightarrow e$ is a counital idempotent arrow in $e\mathcal{M}e$ and $f(e\mathcal{M}e)f = f\mathcal{M}f$.*

Proof. Since f is a counital idempotent in $\mathcal{M}$, $\varepsilon \otimes f$ and $f \otimes \varepsilon$ are isomorphisms. If $f \in e\mathcal{M}e$ then the maps $\eta \otimes f$ and $f \otimes \eta$ are isomorphisms by Lemma 3.1.3 and Lemma 3.1.4. Thus $(\eta\varepsilon) \otimes f$ and $f \otimes (\eta\varepsilon)$ are isomorphisms. The second part follows from the fact that $f \otimes e \cong f \cong e \otimes f$ since f is in $e\mathcal{M}e$. □

Now we focus on idempotents in tensor triangulated categories. Henceforth, we fix a tensor triangulated category $(\mathcal{T}, \otimes, \mathbb{1})$. Recall that the tensor product $\otimes$ need not be symmetric, but we assume that it is exact in both variables.

In tensor triangulated categories, unital and counital idempotent come in pairs.

Proposition 3.1.10 ([BF11, Proposition 3.1])**.** *For a triangle*

$$f \xrightarrow{\varepsilon} \mathbb{1} \xrightarrow{\eta} e \longrightarrow \Sigma f$$

in $\mathcal{T}$, the following statements are equivalent:

(i) *The morphism ε is a counital idempotent arrow.*

(ii) *The morphism η is a unital idempotent arrow.*

(iii) $e \otimes f = f \otimes e = 0$.

In that case,

$$e\mathcal{T} = \mathrm{Ker}(f \otimes -), \quad \mathcal{T}e = \mathrm{Ker}(- \otimes f)$$

$$f\mathcal{T} = \mathrm{Ker}(e \otimes -), \quad \mathcal{T}f = \mathrm{Ker}(- \otimes e).$$

Remark 3.1.11. Although [BF11] only deals with symmetric tensor products, the proof there works in the general case. The main point is that the tensor product $\otimes$ is exact in both variables; compare with [Hog17, Definition 4.2].

Definition 3.1.12. A triangle

$$f \xrightarrow{\varepsilon} \mathbb{1} \xrightarrow{\eta} e \longrightarrow \Sigma f$$

satisfying the equivalent condition of Proposition 3.1.10 is called an *idempotent triangle*. We also say that e is the complement of f, and vice versa.

Corollary 3.1.13. *Let e be an idempotent in $\mathcal{T}$. Then $e\mathcal{T}$, $\mathcal{T}e$, and $e\mathcal{T}e$ are thick subcategory of $\mathcal{T}$. If $\otimes$ preserve coproducts, then they are localizing.*

Proof. This is clear since $e\mathcal{T}$ and $\mathcal{T}e$ are the kernels of the exact functors $f \otimes -$ and $- \otimes f$, respectively, where f is the complement of e. ☐

Theorem 3.1.14. *Let e be an idempotent in $\mathcal{T}$. Then the corner subcategory $(e\mathcal{T}e, \otimes, e)$ is tensor triangulated.*

Proof. By Lemma 3.1.6, $e\mathcal{T}e$ is a tensor category, and by Corollary 3.1.13, $e\mathcal{T}e$ is a triangulated subcategory of $\mathcal{T}$. It remains to show that both structures are compatible. To do this, the only axioms that we need to show is that the following diagrams are commutative

$$\begin{array}{ccc} e \otimes \Sigma X & \xrightarrow{\cong} & \Sigma X \\ {\scriptstyle\cong}\downarrow & & \| \\ \Sigma(e \otimes X) & \xrightarrow{\cong} & \Sigma X \end{array} \qquad \begin{array}{ccc} \Sigma X \otimes e & \xrightarrow{\cong} & \Sigma X \\ {\scriptstyle\cong}\downarrow & & \| \\ \Sigma(X \otimes e) & \xrightarrow{\cong} & \Sigma X \end{array}$$

for all $X \in e\mathcal{T}e$, where $\cong$ denotes various coherence axioms in $e\mathcal{T}e$. The commutativity of the left square follows from the following commutative diagram

$$\begin{array}{ccccc} e \otimes \Sigma X & \xrightarrow{\cong} & \mathbb{1} \otimes \Sigma X & \xrightarrow{\cong} & \Sigma X \\ {\scriptstyle\cong}\downarrow & & {\scriptstyle\cong}\downarrow & & \| \\ \Sigma(e \otimes X) & \xrightarrow{\cong} & \Sigma(\mathbb{1} \otimes X) & \xrightarrow{\cong} & \Sigma X. \end{array}$$

and similarly for the right one. ☐

Corollary 3.1.15. *Let e be a unital or counital idempotent in $\mathcal{T}$. Then the graded endomorphism ring*

$$\operatorname{End}^*_{\mathcal{T}}(e) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{T}}(e, \Sigma^n e)$$

of e is graded commutative.

Proof. This follows from Theorem 2.3.4 and Theorem 3.1.14; see also [Hog17, Theorem 4.21]. ☐

3.2 Actions induced by idempotents

Now we extend the theory of idempotent to deal with actions of tensor triangulated categories. Throughout this section let $*\colon \mathcal{T}\times\mathcal{K}\longrightarrow\mathcal{K}$ be an action of $(\mathcal{T},\otimes,\mathbb{1})$ on a triangulated category $\mathcal{K}$. As usual, we suppress all the coherence isomorphisms.

Let e be an idempotent. We denote by $e\mathcal{K}$ the full subcategory

$$e\mathcal{K}=\{M\in\mathcal{K}\,|\,M\cong e*N,\ N\in\mathcal{K}\}.$$

We shall show that the action of $\mathcal{T}$ on $\mathcal{K}$ together with e induce an action of the corner subcategory $e\mathcal{T}e$ on $e\mathcal{K}$.

Lemma 3.2.1. *Let*

$$f\longrightarrow\mathbb{1}\longrightarrow e\longrightarrow\Sigma f$$

be an idempotent triangle in $\mathcal{T}$. For an object M in $\mathcal{K}$, the following are equivalent:

(i) *The induced map $\mathbb{1}*M\longrightarrow e*M$ (resp., $f*M\longrightarrow\mathbb{1}*M$) is an isomorphism.*

(ii) *$M\cong e*M$ (resp., $M\cong f*M$).*

(iii) *$M\cong e*N$ (resp., $M\cong f*N$) for some $N\in\mathcal{K}$.*

(iv) *$f*M=0$ (resp., $e*M=0$).*

In particular, $e\mathcal{K}=\mathrm{Ker}(f*-)$ and $f\mathcal{K}=\mathrm{Ker}(e*-)$.

Proof. It is clear that (i) implies (ii) and (ii) implies (iii). Assuming (iii), we have

$$f*M\cong f*(e*N)\cong(f\otimes e)*N=0$$

since $f\otimes e=0$. Assuming (iv), since $-*M$ is exact, we have from the triangle

$$f*M\longrightarrow\mathbb{1}*M\longrightarrow e*M\longrightarrow\Sigma f*M$$

that the induced map $\mathbb{1}*M\longrightarrow e*M$ is an isomorphism. □

Remark 3.2.2. The subcategory $e\mathcal{K}$ and $f\mathcal{K}$ are thick since it is the kernel of the exact functor $f*-$ and $e*-$, respectively. They are localizing if $-*-$ is exact in the second variable.

Theorem 3.2.3. *Let e be an idempotent in $\mathcal{T}$. The action of $\mathcal{T}$ on $\mathcal{K}$ induces an action of $e\mathcal{T}e$ on $e\mathcal{K}$ simply by restriction. More precisely, the bifunctor $*$ restricts to a bifunctor $*\colon e\mathcal{T}e\times e\mathcal{K}\longrightarrow e\mathcal{K}$ which defines an action of $e\mathcal{T}e$ on $e\mathcal{K}$.*

Proof. It is easy to see that the bifunctor $*$ restricts to a bifunctor on the induced subcategories. Indeed, if X is in $e\mathcal{T}e$ and M is in $e\mathcal{K}$, then we have $e \cong e \otimes X \otimes e$ and $M \cong e * M$. Hence,

$$X * M \cong (e \otimes X \otimes e) * (e * M) \cong e * (X * M)$$

by associativity and $e \otimes e \cong e$. Thus $X * M$ belongs to $e\mathcal{K}$.

To see that the restricted bifunctor defines an action, we need to specify the coherence isomorphisms for the action. These coherence isomorphisms are the same as those for the action of $\mathcal{T}$ on $\mathcal{K}$, except only for the left unit isomorphism, which is given by the composition of the natural isomorphisms

$$e * M \xrightarrow{\cong} \mathbb{1} * M \xrightarrow{\cong} M.$$

We need to check that the following diagrams are commutative for all $X \in e\mathcal{T}e$ and $M \in e\mathcal{M}$:

$$\begin{array}{ccc} (X \otimes e) * M & \xrightarrow{\cong} & X * (e * M) \\ & {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\cong} & \\ & X * M, & \end{array}$$

$$\begin{array}{ccc} (e \otimes X) * M & \xrightarrow{\cong} & e * (X * M) \\ & {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\cong} & \\ & X * M, & \end{array}$$

$$\begin{array}{ccc} e * \Sigma M & \xrightarrow{\cong} & \Sigma M \\ \downarrow{\scriptstyle\cong} & & \| \\ \Sigma(e * M) & \xrightarrow{\cong} & \Sigma M \end{array}$$

The proof that the bottom square is commutative is similar to the one in Theorem 3.1.14. The commutativity of the top triangle follows from the following commutative diagrams

$$\begin{array}{ccc} (X \otimes e) * M & \xrightarrow{\cong} & X * (e * M) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ (X \otimes \mathbb{1}) * M & \xrightarrow{\cong} & X * (\mathbb{1} * M) \\ & {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\cong} & \\ & X * M. & \end{array}$$

The commutativity of the middle one is similar. □

Next we discuss the induced ring actions. The following is a direct corollary of Theorem 3.2.3 and Proposition 2.3.9.

Corollary 3.2.4. *Let e be an idempotent in $\mathcal{T}$. There category $e\mathcal{K}$ is* $\operatorname{End}_{\mathcal{T}}(e)$*-linear where the ring action is given by*

$$- * M\colon \operatorname{End}^*_{\mathcal{T}}(e) \longrightarrow \operatorname{End}^*_{e\mathcal{K}}(M)$$

for all $M \in e\mathcal{K}$.

Thus, if e is an idempotent in $\mathcal{T}$ and M is an object in $e\mathcal{K}$ we have two graded ring homomorphisms, namely $\operatorname{End}^*_{\mathcal{T}}(\mathbb{1}) \longrightarrow \operatorname{End}^*_{e\mathcal{K}}(M)$ and $\operatorname{End}^*_{\mathcal{T}}(e) \longrightarrow \operatorname{End}^*_{e\mathcal{K}}(M)$ from the action of $\operatorname{End}^*_{\mathcal{T}}(\mathbb{1})$ and $\operatorname{End}^*_{\mathcal{T}}(e)$, respectively. On the other hand, letting $\mathcal{T}$ acts on itself, we get a ring homomorphism $\operatorname{End}^*_{\mathcal{T}}(\mathbb{1}) \xrightarrow{-\otimes e} \operatorname{End}^*_{\mathcal{T}}(e)$. The following proposition relates these three maps.

Proposition 3.2.5. *Let e be an idempotent in $\mathcal{T}$. For each object M in $e\mathcal{K}$, the following triangle*

$$\begin{array}{ccc} \operatorname{End}^*_{\mathcal{T}}(\mathbb{1}) & \xrightarrow{\quad -*M \quad} & \operatorname{End}^*_{e\mathcal{K}}(M) \\ & {\scriptstyle -\otimes e}\searrow \quad \nearrow{\scriptstyle -*M} & \\ & \operatorname{End}^*_{\mathcal{T}}(e). & \end{array}$$

is commutaive.

Proof. Consider the following diagram of exact functors

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\quad -*_M \quad} & \mathcal{K} \\ & {\scriptstyle -\otimes e}\searrow \quad \nearrow{\scriptstyle -*_M} & \\ & \mathcal{T}. & \end{array}$$

The diagram is commutative by associativity and isomorphism $e * M \cong M$:

$$(X \otimes e) * M \cong X * (e * M) \cong X * M,$$

which is natural in $X \in \mathcal{T}$. The proposition then follows by evaluation at $\mathbb{1}$. □

We give a sufficient condition to ensure that the map $\operatorname{End}^*_{\mathcal{T}}(\mathbb{1}) \xrightarrow{-\otimes f} \operatorname{End}^*_{\mathcal{T}}(f)$ is bijective for a counital idempotent f in $\mathcal{T}$.

Proposition 3.2.6. *Let*

$$f \xrightarrow{\varepsilon} \mathbb{1} \xrightarrow{\eta} e \longrightarrow \Sigma f$$

be an idempotent triangle. Then the following triangle

$$\begin{array}{ccc} \mathrm{End}^*_{\mathcal{T}}(\mathbb{1}) & \xrightarrow{-\otimes f} & \mathrm{End}^*_{\mathcal{T}}(f) \\ & {\scriptstyle -\circ\varepsilon} \searrow \quad \swarrow {\scriptstyle \varepsilon\circ -} & \\ & \mathrm{Hom}^*_{\mathcal{T}}(f, \mathbb{1}) & \end{array}$$

is commutative. In particular, if $\mathrm{Hom}^*_{\mathcal{T}}(f, e) = 0 = \mathrm{Hom}^*_{\mathcal{T}}(e, \mathbb{1})$*, the map* $\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}) \longrightarrow \mathrm{End}^*_{\mathcal{T}}(f)$ *is bijective.*

Proof. First note that, $\mathrm{Hom}^*_{\mathcal{T}}(f, \mathbb{1})$ is an $(\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}), \mathrm{End}^*_{\mathcal{T}}(f))$-bimodule via composition such that

$$\Sigma^{|\alpha|} r \circ \alpha = (-1)^{|r||\alpha|} \Sigma^{|r|} \alpha \circ (r \otimes f)$$

for each homogeneous $r \in \mathrm{End}^*_{\mathcal{T}}(\mathbb{1})$ and $\alpha \in \mathrm{Hom}^*_{\mathcal{T}}(f, \mathbb{1})$. In particular, since $\varepsilon \in \mathrm{Hom}^*_{\mathcal{T}}(f, \mathbb{1})$ is of degree zero, we have

$$r \circ \varepsilon = \Sigma^{|r|} \varepsilon \circ (r \otimes f)$$

for each homogeneous element $r \in \mathrm{End}^*_{\mathcal{T}}(\mathbb{1})$. This shows that the triangle commute.

Moreover, if $\mathrm{Hom}^*_{\mathcal{T}}(f, e) = 0$ then the map $\mathrm{End}^*_{\mathcal{T}}(f) \longrightarrow \mathrm{Hom}^*_{\mathcal{T}}(f, \mathbb{1})$ is bijective, and if $\mathrm{Hom}^*_{\mathcal{T}}(e, \mathbb{1}) = 0$ then $\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}) \longrightarrow \mathrm{Hom}^*_{\mathcal{T}}(f, \mathbb{1})$ is bijective. The last statement follows from these by using the commutative diagram. □

3.3 Functors induced by idempotents

In the last section of this chapter we discuss functors induced by idempotents. Throughout we fix tensor triangulated categories $(\mathcal{T}, \otimes_{\mathcal{T}}, \mathbb{1}_{\mathcal{T}})$ and $(\mathcal{U}, \otimes_{\mathcal{U}}, \mathbb{1}_{\mathcal{U}})$. Let $F\colon \mathcal{T} \longrightarrow \mathcal{U}$ be an exact functor together with an isomorphism $\phi_0\colon \mathbb{1}_{\mathcal{U}} \longrightarrow F(\mathbb{1}_{\mathcal{T}})$ and a natural isomorphism

$$\phi_{X,Y}\colon FX \otimes_{\mathcal{U}} FY \longrightarrow F(X \otimes_{\mathcal{T}} Y)$$

for $X, Y \in \mathcal{T}$. Thus, we have commutative square

$$\begin{array}{ccc} \mathcal{T} \times \mathcal{T} & \xrightarrow{-\otimes_{\mathcal{T}}-} & \mathcal{T} \\ {\scriptstyle F\times F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{U} \times \mathcal{U} & \xrightarrow{-\otimes_{\mathcal{U}}-} & \mathcal{U}. \end{array}$$

Lemma 3.3.1. *Let* (F, ϕ, ϕ_0) *be as above. Then* F *preserve idempotents and idempotent triangles, and restricts to a functor between the corresponding corner subcategories. More precisely, for an idempotent* $e \in \mathcal{T}$*, its image* $e' = F(e)$ *is an idempotent in* $\mathcal{U}$ *and* F *restricts to a functor* $e\mathcal{T}e \longrightarrow e'\mathcal{U}e'$*.*

Proof. We take care of the case of unital idempotents first. Let e be a unital idempotent with idempotent arrow $\eta\colon \mathbb{1} \longrightarrow e$. Consider the composite

$$\mathbb{1}_{\mathcal{U}} \xrightarrow{\phi_0} F(\mathbb{1}_{\mathcal{T}}) \xrightarrow{F(\eta)} F(e).$$

We claim that it is an idempotent arrow in $\mathcal{U}$. In fact, the commutative diagram

$$\begin{array}{ccccc}
\mathbb{1}_{\mathcal{U}} \otimes e & \xrightarrow[\cong]{\phi_0 \otimes F(e)} & F(\mathbb{1}_{\mathcal{T}}) \otimes F(e) & \xrightarrow{F(\eta)\otimes F(e)} & F(e) \otimes F(e) \\
 & & {\scriptstyle \phi_{\mathbb{1}_{\mathcal{T}},e}} \Big\downarrow {\scriptstyle \cong} & & {\scriptstyle \phi_{e,e}} \Big\downarrow {\scriptstyle \cong} \\
 & & F(\mathbb{1}_{\mathcal{T}} \otimes e) & \xrightarrow[\cong]{F(\eta \otimes e)} & F(e \otimes e)
\end{array}$$

implies that the map $(F(\eta) \circ \phi_0) \otimes F(e)$ is an isomorphism. Similarly, the map $F(e) \otimes (F(\eta) \circ \phi_0)$ is also an isomorphism, and the claim follows.

If f is a counital idempotent given by idempotent arrow $\varepsilon\colon f \longrightarrow \mathbb{1}$, then the composite

$$F(f) \xrightarrow{F(\varepsilon)} F(\mathbb{1}_{\mathcal{T}}) \xrightarrow{\phi_0^{-1}} \mathbb{1}_{\mathcal{U}}$$

is an idempotent arrow in $\mathcal{U}$. The proof is similar as the unital case.

Now let

$$f \xrightarrow{\varepsilon} \mathbb{1} \xrightarrow{\eta} e \longrightarrow \Sigma f$$

be an idempotent triangle in $\mathcal{T}$. From the commutative diagram

$$\begin{array}{ccccccc}
F(f) & \xrightarrow{F(\varepsilon)} & F(\mathbb{1}_{\mathcal{T}}) & \xrightarrow{F(\eta)} & F(e) & \longrightarrow & \Sigma F(f) \\
\Big\| & & {\scriptstyle \cong} \Big\downarrow {\scriptstyle \phi_0} & & \Big\| & & \Big\| \\
F(f) & \longrightarrow & \mathbb{1}_{\mathcal{U}} & \longrightarrow & F(e) & \longrightarrow & \Sigma F(f)
\end{array}$$

we conclude that the bottom row is a triangle since the top row is. Thus the bottom row is an idempotent triangle in $\mathcal{U}$.

The last part follows from the isomorphism $F(e \otimes X \otimes e) \cong F(e) \otimes FX \otimes F(e)$ for $X \in \mathcal{T}$, which is obtained by applying ϕ. $\square$

Remark 3.3.2. The idempotent e and the triple (F, ϕ, ϕ_0) as above induce another triple (F, ϕ, ϕ_0'), where F is the restricted functor $e\mathcal{T}e \longrightarrow e'\mathcal{U}e'$, ϕ is the restricted natural isomorphism

$$FX \otimes_{\mathcal{U}} FY \xrightarrow{\cong} F(X \otimes_{\mathcal{T}} Y)$$

for $X, Y \in e\mathcal{T}e$ and ϕ_0' is the identity on $F(e) = e'$.

Now we discuss functors which 'compatible' with actions. Suppose that $*\colon \mathcal{T} \times \mathcal{K} \longrightarrow \mathcal{K}$ and $\star\colon \mathcal{U} \times \mathcal{L} \longrightarrow \mathcal{L}$ are actions of $\mathcal{T}$ and $\mathcal{U}$ on triangulated categories $\mathcal{K}$ and $\mathcal{L}$. Let

(F, ϕ, ϕ_0) be as before and let $G\colon \mathcal{K} \longrightarrow \mathcal{L}$ be an exact functor together with natural isomorphism

$$\psi_{X,M}\colon FX \star GM \longrightarrow G(X * M).$$

Thus, the square

$$\begin{array}{ccc} \mathcal{T} \times \mathcal{K} & \xrightarrow{-*-} & \mathcal{K} \\ {\scriptstyle F\times G}\downarrow & & \downarrow{\scriptstyle G} \\ \mathcal{U} \times \mathcal{L} & \xrightarrow{-\star-} & \mathcal{L}. \end{array}$$

is commutative.

Lemma 3.3.3. *Let $(F, \phi, \phi_0, G, \psi)$ be as above. Then we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{End}^*_{\mathcal{T}}(\mathbb{1}_{\mathcal{T}}) & \xrightarrow{-*M} & \mathrm{End}^*_{\mathcal{K}}(M) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ \mathrm{End}^*_{\mathcal{U}}(\mathbb{1}_{\mathcal{U}}) & \xrightarrow{-\star GM} & \mathrm{End}^*_{\mathcal{L}}(GM). \end{array}$$

Proof. The natural isomorphism ψ implies that the square

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{-*M} & \mathcal{K} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ \mathcal{U} & \xrightarrow{-\star GM} & \mathcal{L} \end{array}$$

is commutative for $M \in \mathcal{K}$. The above square induces the required commutative square by evaluation at $\mathbb{1}_{\mathcal{T}}$. □

Similar to Lemma 3.3.1, the functor G restrict to a functor between subcategories of defined by idempotents.

Lemma 3.3.4. *Let $(F, \phi, \phi_0, G, \psi)$ be as above and e be an idempotent in $\mathcal{T}$, and let $e' = F(e)$. The functor G restrict to a functor $e\mathcal{K} \longrightarrow e'\mathcal{L}$.*

Proof. For $M \in \mathcal{K}$, we have $G(e * M) \cong e' * GM \in e'\mathcal{L}$ by applying ψ. □

Remark 3.3.5. In the special case that $\mathcal{K} = \mathcal{L}$ and $G = \mathrm{Id}_{\mathcal{K}}$, the restricted functor $G\colon e\mathcal{K} \longrightarrow e'\mathcal{L}$ is the identity on objects and morphisms. In fact, it is really the identity functor on $e\mathcal{K}$. Indeed, we have equality $e\mathcal{K} = e'\mathcal{L}$ since we have isomorphism $\psi_{e,M}\colon e' \star M \xrightarrow{\cong} e * M$ for all $M \in \mathcal{K}$.

Combining everything, we obtain the following theorem.

Theorem 3.3.6. *Let* $(F, \phi, \phi_0, G, \psi)$ *be as above and* e *be an idempotent in* $\mathcal{T}$*, and let* $e' = F(e)$*. Then the following diagram*

$$\begin{array}{ccc} \mathrm{End}^*_{\mathcal{T}}(e) & \xrightarrow{-*M} & \mathrm{End}^*_{e\mathcal{K}}(M) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ \mathrm{End}^*_{\mathcal{U}}(e') & \xrightarrow{-\star GM} & \mathrm{End}^*_{e'\mathcal{L}}(GM) \end{array}$$

is commutative for all $M \in e\mathcal{K}$.

Proof. The point is that the quintuple $(F, \phi, \phi_0, G, \psi)$ restricts to another quintuple $(F, \phi, \phi'_0, G, \psi)$ on the full subcategories defined by e and e', namely $F\colon e\mathcal{T}e \longrightarrow e'\mathcal{U}e'$, $\phi_0\colon e' \xrightarrow{=} F(e)$,

$$\phi_{X,Y}\colon FX \otimes_{\mathcal{U}} FY \longrightarrow F(X \otimes_{\mathcal{T}} Y)$$

for all $X, Y \in e\mathcal{T}e$, and $G\colon e\mathcal{K} \longrightarrow e'\mathcal{L}$,

$$\psi_{X,M}\colon FX \star GM \longrightarrow G(X * M),$$

for all $X \in e\mathcal{T}e$, $M \in e\mathcal{K}$. Now apply lemma 3.3.3. □

Chapter 4

Actions of Hochschild cohomology

Actions of Hochschild cohomology were used by Snashall and Solberg in [SS04] to study and develop the theory of support varieties for finitely generated modules over an arbitrary finite dimensional algebra. This is as an analogue of the support varieties for group algebra which is defined using group cohomology [Car81, Car83, Qui71] , and the support varieties for complete intersection ring defined using Shukla cohomology [Avr89, AB00]. The action of the Hochschild cohomology can be extended to the derived category to define support varieties for complexes [BKSS15, Sol06].

For finite dimensional self-injective algebras, we define an action of Hochschild cohomology on the homotopy category of injectives, and we show that this extends the previous action on the derived category. For finite dimensional Hopf algebras, we relate this with the canonical action of the ordinary cohomology ring.

In fact, the action of the Hochschild cohomology that we define is induced by an action of a tensor triangulated category in which the graded endomorphism of the tensor unit is isomorphic to the Hochschild cohomology. We will apply the idempotent theory that we developed in Chapter 3 to obtain such actions of tensor triangulated categories Similarly, in the Hopf algebra case, the relation between the action of the Hochschild cohomology and the ordinary cohomology comes from a functor compatible with the actions of tensor triangulated categories.

4.1 Hochschild cohomology

Let A be an algebra over a field $\Bbbk$. The *n-th Hochschild cohomology* of A with coefficient in an A^e-module M is defined to be

$$\mathrm{HH}^n(A/\Bbbk; M) = \mathrm{Ext}^n_{A^e}(A, M).$$

Thus we have an isomorphism

$$\mathrm{HH}^n(A/\Bbbk; M) \cong \mathrm{Hom}_{\mathcal{D}(A^e)}(A, \Sigma^n M)$$

for A^e-module M, and we view it as a complex in the usual way. The graded ring

$$\mathrm{HH}^*(A/\Bbbk; A) = \bigoplus_{i\in\mathbb{Z}} \mathrm{H}^*(A/\Bbbk; A)$$

together with multiplication given by Yoneda product will be denoted by $\mathrm{HH}^*(A/\Bbbk)$.

Theorem 4.1.1 ([Ger63])**.** *The graded ring* $\mathrm{HH}^*(A/\Bbbk)$ *is graded-commutative.*

Thus, when $\mathrm{HH}^*(A/\Bbbk)$ is noetherian and acts on a compactly generated triangulated category $\mathcal{K}$, we can apply Benson-Iyengar-Krause support theory to study $\mathcal{K}$. This is the case for finite dimensional cocommutative Hopf algebra A (see Theorem 5.1.3). Alternatively, one can use a graded ring homomorphism $H \longrightarrow \mathrm{HH}^*(A/\Bbbk)$, where H is a graded-commutative noetherian ring.

In the following sections, we describe actions of $\mathrm{HH}^*(A/\Bbbk)$ on various compactly generated triangulated categories.

4.2 Action on the derived category

In this section, we describe the action of Hochschild cohomology on the homotopy category of K-projective complexes over arbitrary $\Bbbk$-algebra, and relate it with the action on the derived category.

Let us fix an algebra A over a field $\Bbbk$ and let $A^e = A \otimes A^{\mathrm{op}}$ be the enveloping algebra of A over $\Bbbk$. Let $\mathbf{p}A$ denotes the projective resolution of A as an A^e-module. Recall that it is a complex of projective A^e-modules which is K-projective in $\mathcal{K}(A^e)$ together with a quasi-isomorphism $\mathbf{p}A \xrightarrow{\varepsilon} A$.

Proposition 4.2.1. *Let B be a $\Bbbk$-algebra and $X \in \mathcal{K}(A \otimes B^{\mathrm{op}})$ be such that X is K-projective over B^{op}. Then the natural map*

$$\mathbf{p}A \otimes_A X \xrightarrow{\varepsilon\otimes_A X} A \otimes_A X \cong X$$

is a K-projective resolution of X in $\mathcal{K}(A \otimes B^{\mathrm{op}})$. Similarly, for $Y \in \mathcal{K}(B \otimes A^{\mathrm{op}})$ such that Y is K-projective over B, the natural map

$$Y \otimes_A \mathbf{p}A \xrightarrow{Y\otimes_A \varepsilon} Y \otimes_A A \cong Y$$

is a K-projective resolution of Y in $\mathcal{K}(B \otimes A^{\mathrm{op}})$.

Proof. We only prove the first statement. Complete the natural map $\mathbf{p}A \xrightarrow{\varepsilon} A$ into a triangle

$$\mathbf{p}A \xrightarrow{\varepsilon} A \longrightarrow Z \longrightarrow \Sigma\mathbf{p}A$$

in $\mathcal{K}(A^e)$ with Z acyclic. Since A is projective as an A^{op}-modules, Z is contractible as a complex of A^{op}-modules. Thus the complex $Z \otimes_A X$ is acyclic and the induced map $\mathbf{p}A \otimes_A X \longrightarrow A \otimes_A X \cong X$ is a quasi-isomorphisms.

It remains to prove that $\mathbf{p}A \otimes_A X$ is K-projective. Since X is K-projective over B^{op}, the functor $\mathrm{Hom}_{B^{\mathrm{op}}}(X, -)$ preserves acyclic complexes. Thus for any acyclic complex W in $\mathcal{K}(A \otimes B^{\mathrm{op}})$ we have

$$\mathrm{Hom}_{\mathcal{K}(A \otimes B^{\mathrm{op}})}(\mathbf{p}A \otimes_A X, W) \cong \mathrm{Hom}_{\mathcal{K}(A^e)}(\mathbf{p}A, \mathrm{Hom}_{B^{\mathrm{op}}}(X, W)) = 0$$

since $\mathbf{p}A$ is K-projective and $\mathrm{Hom}_{B^{\mathrm{op}}}(X, W)$ is acyclic. Thus, $\mathbf{p}A \otimes_A X$ is K-projective over $A \otimes B^{\mathrm{op}}$, as required. □

We consider $\mathcal{K}(A^e)$ as a tensor triangulated category with tensor product $\otimes_A$ and unit A. The fact, that $\mathbf{p}A \xrightarrow{\varepsilon} A$ is a counital idempotent arrow in $\mathcal{K}(A^e)$, is well-known; see, for example, [Hog17, Example 1.5]. We determine the corresponding corner subcategory.

Proposition 4.2.2. *The natural map $\mathbf{p}A \xrightarrow{\varepsilon} A$ is a counital idempotent arrow in $\mathcal{K}(A^e)$, and the corner subcategory $\mathbf{p}A \cdot \mathcal{K}(A^e) \cdot \mathbf{p}A$ defined by $\mathbf{p}A$ is the full subcategory $\mathcal{K}_{\mathrm{proj}}(A^e) \subset \mathcal{K}(A^e)$ of K-projective complexes. In particular, $(\mathcal{K}_{\mathrm{proj}}(A^e), \otimes_A, \mathbf{p}A)$ is a tensor triangulated category.*

Proof. Let X be a K-projective complex over A^e. Hence, it is K-projective over A^{op}, since for all acyclic complex W in $\mathcal{K}(A^{\mathrm{op}})$ we have

$$\mathrm{Hom}_{\mathcal{K}(A^{\mathrm{op}})}(X, W) \cong \mathrm{Hom}_{\mathcal{K}(A^e)}(X, \mathrm{Hom}_{\Bbbk}(A, W)) = 0.$$

We can use Proposition 4.2.1 to obtain that the map $\mathbf{p}A \otimes_A X \xrightarrow{\varepsilon \otimes_A X} A \otimes_A X$ is a K-projective resolution of X in $\mathcal{K}(A^e)$. But X itself is K-projective, hence $\varepsilon \otimes_A X$ is an isomorphism. Similarly, $X \otimes \mathbf{p}A \xrightarrow{X \otimes_A \varepsilon} X \otimes A$ is also an isomorphism. In particular $\mathbf{p}A$ is a counital idempotent (since $\mathbf{p}A$ is K-projective) and all K-projective complexes over A^e lie in the corner subcategory $\mathbf{p}A \cdot \mathcal{K}(A^e) \cdot \mathbf{p}A$ (by Lemma 3.1.3 and Lemma 3.1.4).

Now we prove that each complex in the corner subcategory is K-projective. We apply Proposition 4.2.1 twice. Let Y be a complex in $\mathcal{K}(A^e)$. Since Y is clearly K-projective over $\Bbbk$, $\mathbf{p}A \otimes_A Y$ is K-projective over A, and hence $\mathbf{p}A \otimes_A Y \otimes_A \mathbf{p}A$ is K-projective over A^e. Thus $\mathbf{p}A \cdot \mathcal{K}(A^e) \cdot \mathbf{p}A$ is contained in $\mathcal{K}_{\mathrm{proj}}(A^e)$, as claimed. □

Next we consider the action $\mathcal{K}(A^e) \times \mathcal{K}(A) \xrightarrow{- \otimes_A -} \mathcal{K}(A)$ given by tensor product of complexes over A. The idempotent $\mathbf{p}A$ induces an action of $\mathbf{p}A \cdot \mathcal{K}(A^e) \cdot \mathbf{p}A = \mathcal{K}_{\mathrm{proj}}(A^e)$ on $\mathbf{p}A \cdot \mathcal{K}(A)$. We determine the latter category.

Proposition 4.2.3. *The full subcategory* $\mathbf{p}A \cdot \mathcal{K}(A)$ *is the full subcategory* $\mathcal{K}_{\mathrm{proj}}(A)$ *in* $\mathcal{K}(A)$ *of K-projective complexes. In particular, there is an action of* $\mathcal{K}_{\mathrm{proj}}(A^e)$ *on* $\mathcal{K}_{\mathrm{proj}}(A)$ *given by* $\otimes_A$.

Proof. The proof is basically the same as that of Proposition 4.2.2. For any complex M in $\mathcal{K}(A)$, the natural map $\mathbf{p}A \otimes_A M \longrightarrow A \otimes M \cong M$ is a K-projective resolution of M by Proposition 4.2.1. In particular, $\mathbf{p}A \otimes_A M$ is K-projective, and if M is K-projective, then the map $\mathbf{p}A \otimes_A M \longrightarrow A \otimes_A M$ is an isomorphism. This proves that $\mathbf{p}A \cdot \mathcal{K}(A) = \mathcal{K}_{\mathrm{proj}}(A)$ by Lemma 3.2.1. □

As a direct corollary, we have an action of the Hochschild cohomology on the homotopy category of K-projective complexes.

Corollary 4.2.4. *The Hochschild cohomology ring* $\mathrm{HH}^*(A/\Bbbk)$ *acts on the full subcategory* $\mathcal{K}_{\mathrm{proj}}(A)$ *of K-projective complexes over* A*, where the ring action is given by*

$$\mathrm{HH}^*(A/\Bbbk) \cong \mathrm{End}^*_{\mathcal{K}_{\mathrm{proj}}(A^e)}(\mathbf{p}A) \xrightarrow{-\otimes_A M} \mathrm{End}^*_{\mathcal{K}_{\mathrm{proj}}(A)}(M)$$

for each M *in* $\mathcal{K}_{\mathrm{proj}}(A)$.

We explain the relation with the action of the Hochschild cohomology on $\mathcal{D}(A)$ defined in [BKSS15, Sol06]. The composite

$$F\colon \mathcal{K}_{\mathrm{proj}}(A^e) \xhookrightarrow{\mathrm{inc}} \mathcal{K}(A^e) \xrightarrow{\mathrm{can}} \mathcal{D}(A^e)$$

is an equivalence of triangulated categories. In fact, it is also a tensor triangulated functor with isomorphism $\mathbf{p}A \xrightarrow{\cong} A$ and natural isomorphisms

$$FX \otimes^{\mathbf{L}}_A FY \xrightarrow{\cong} F(X \otimes_A Y)$$

for $X, Y \in \mathcal{K}_{\mathrm{proj}}(A^e)$. Similarly, the composite

$$G\colon \mathcal{K}_{\mathrm{proj}}(A) \xhookrightarrow{\mathrm{inc}} \mathcal{K}(A) \xrightarrow{\mathrm{can}} \mathcal{D}(A)$$

is an equivalence of categories, which is compatible with the actions of $\mathcal{K}_{\mathrm{proj}}(A^e)$ on $\mathcal{K}_{\mathrm{proj}}(A)$ and $\mathcal{D}(A^e)$ on $\mathcal{D}(A)$. In particular, the following diagram

$$\begin{array}{ccc}
\mathcal{K}_{\mathrm{proj}}(A^e) \times \mathcal{K}_{\mathrm{proj}}(A) & \xrightarrow{-\otimes_A -} & \mathcal{K}_{\mathrm{proj}}(A) \\
{\scriptstyle F\times G}\downarrow\cong & & {\scriptstyle G}\downarrow\cong \\
\mathcal{D}(A^e) \times \mathcal{D}(A) & \xrightarrow{-\otimes^{\mathbf{L}}_A -} & \mathcal{D}(A)
\end{array}$$

is commutative. This induces a commutative diagram

$$\begin{array}{ccccc}
\mathrm{HH}^*(A/\Bbbk) & \xrightarrow{\cong} & \mathrm{End}^*_{\mathcal{K}_{\mathrm{proj}}(A^e)}(\mathbf{p}A) & \xrightarrow{-\otimes_A M} & \mathrm{End}^*_{\mathcal{K}_{\mathrm{proj}}(A)}(M) \\
 & \searrow^{\cong} & \downarrow\cong & & {\scriptstyle G}\downarrow\cong \\
 & & \mathrm{End}^*_{\mathcal{D}(A^e)}(A) & \xrightarrow{-\otimes^{\mathbf{L}}_A GM} & \mathrm{End}^*_{\mathcal{D}(A)}(GM)
\end{array}$$

For each $M \in \mathcal{K}_{\mathrm{proj}}(A)$. In other words, the equivalence $G\colon \mathcal{K}_{\mathrm{proj}}(A) \xrightarrow{\cong} \mathcal{D}(A)$ is $\mathrm{HH}^*(A/\Bbbk)$-linear.

4.3 Action on the homotopy category of injectives

In this section, we will define an action of the Hochschild cohomology $\mathrm{HH}^*(A/\Bbbk)$ on the homotopy category $\mathcal{K}(\mathrm{Inj}\, A)$ of injectives over finite dimensional self-injective $\Bbbk$-algebra A. The new action extends the action on $\mathcal{K}_{\mathrm{proj}}(A) \simeq \mathcal{D}(A)$ explained in the previous section, in the sense that the embedding $\mathcal{D}(A) \xhookrightarrow{\mathrm{inc}} \mathcal{K}(\mathrm{Inj}\, A)$ defined by the left adjoint of the natural functor

$$\mathcal{K}(\mathrm{Inj}\, A) \xhookrightarrow{\mathrm{inc}} \mathcal{K}(A) \xrightarrow{\mathrm{can}} \mathcal{D}(A)$$

is $\mathrm{HH}^*(A/\Bbbk)$-linear.

Throughout this section let A be a finite dimensional self-injective $\Bbbk$-algebra. Thus the full subcategory $\mathrm{Proj}\, A$ of projective A-modules coincides with the full subcategory $\mathrm{Inj}\, A$ of injective A-modules. The enveloping algebra A^e over $\Bbbk$ is also finite dimensional and self-injective (see, for example, [SY11, Proposition IV.11.5]), and hence $\mathrm{Proj}\, A^e = \mathrm{Inj}\, A^e$.

We will prove that the injective resolution $A \xrightarrow{\eta} \mathbf{i}A$ is a unital idempotent arrow in $\mathcal{K}(A^e)$ and compute the corresponding corner subcategory.

Lemma 4.3.1. *Let B be a $\Bbbk$-algebra. Let P be a projective A^e-module and X a $(A \otimes B^{\mathrm{op}})$-module, which is projective as B^{op}-module. Then the $(A \otimes B^{\mathrm{op}})$-module $P \otimes_A X$ is projective. Similarly, if Y is a $(B \otimes A^{\mathrm{op}})$-module which is projective as B-module, then $Y \otimes_A P$ is projective as $(B \otimes A^{\mathrm{op}})$-module.*

Proof. We only prove the first statement. The functor $\mathrm{Hom}_{A\otimes B^{\mathrm{op}}}(P \otimes_A X, -)$ is exact since it is naturally isomorphic to $\mathrm{Hom}_{A^e}(P, \mathrm{Hom}_{B^{\mathrm{op}}}(X, -))$, which is exact by assumption. Thus $P \otimes_A X$ is projective. □

Lemma 4.3.2. *Let X, Y be A^e-modules which are projective and injective, respectively. Then the A^e-modules $\mathrm{Hom}_A(X, Y)$ and $\mathrm{Hom}_{A^{\mathrm{op}}}(X, Y)$ are both injective.*

Proof. Note that X is projective as left and right A-module by Lemma 4.3.1 (take $B = \Bbbk$, $X = Y = A$). The functor $\mathrm{Hom}_{A^e}(-, \mathrm{Hom}_A(X, Y))$ is exact since it is naturally isomorphic to $\mathrm{Hom}_{A^e}(X \otimes_A -, Y)$, which is exact. Thus $\mathrm{Hom}_A(X, Y)$ is injective. □

Lemma 4.3.3. *Let X be a complex in $\mathcal{K}(\mathrm{Inj}\, A^e)$. The natural map $A \xrightarrow{\eta} \mathbf{i}A$ induces isomorphisms*

$$A \otimes_A X \xrightarrow{\eta \otimes_A X} \mathbf{i}A \otimes_A X \quad \textit{and} \quad X \otimes_A A \xrightarrow{X \otimes_A \eta} X \otimes_A \mathbf{i}A.$$

Proof. First, observe that $A \otimes_A X$ and $\mathbf{i}A \otimes_A X$ belong to $\mathcal{K}(\operatorname{Inj} A^e)$, as tensor product of two injective A^e-modules is injective by Lemma 4.3.1 (since projectives and injectives coincide). For each $Y \in \mathcal{K}(\operatorname{Inj} A^e)$, the complex $\operatorname{Hom}_{A^{\mathrm{op}}}(X, Y)$ also lies in $\mathcal{K}(\operatorname{Inj} A^e)$ by Lemma 4.3.2. We consider the commutative diagram

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathcal{K}(A^e)}(\mathbf{i}A \otimes_A X, Y) & \xrightarrow{\cong} & \operatorname{Hom}_{\mathcal{K}(A^e)}(\mathbf{i}A, \operatorname{Hom}_{A^{\mathrm{op}}}(X, Y)) \\
\downarrow & & \downarrow \\
\operatorname{Hom}_{\mathcal{K}(A^e}(A \otimes_A X, Y) & \xrightarrow{\cong} & \operatorname{Hom}_{\mathcal{K}(A^e)}(A, \operatorname{Hom}_{A^{\mathrm{op}}}(X, Y)).
\end{array}$$

The right vertical map is a bijection by [Kra05, Lemma 2.1]. It follows that the left vertical map is also a bijection. By Yoneda lemma, $A \otimes_A X \longrightarrow \mathbf{i}A \otimes_A X$ is also an isomorphism. The proof of the second isomorphism is similar. □

Proposition 4.3.4. *The natural map $A \xrightarrow{\eta} \mathbf{i}A$ is a unital idempotent arrow in $\mathcal{K}(A^e)$ and the corner subcategory $\mathcal{T} = \mathbf{i}A \cdot \mathcal{K}(A^e) \cdot \mathbf{i}A$ is the replete closure $\mathcal{K}(\operatorname{Inj} A^e)$, in the sense that $\mathcal{T}$ contains $\mathcal{K}(\operatorname{Inj} A^e)$ and the canonical inclusion $\mathcal{K}(\operatorname{Inj} A^e) \xhookrightarrow{\text{inc}} \mathcal{T}$ is an equivalence. Moreover, $\mathcal{K}(\operatorname{Inj} A^e)$ is a tensor triangulated subcategory of $\mathcal{T}$.*

Proof. It is immediate from Lemma 4.3.3 that $A \xrightarrow{\eta} \mathbf{i}A$ is a unital idempotent arrow (since $\mathbf{i}A$ is a complex of injectives) and that $\mathcal{K}(\operatorname{Inj} A^e)$ is contained in $\mathcal{T}$ (by Lemma 3.1.3 and Lemma 3.1.4). Now take $X \in \mathcal{T}$, hence $X \cong \mathbf{i}A \otimes_A X \otimes_A \mathbf{i}A$. The latter is a complex of injectives by Lemma 4.3.1 (since injectives and projectives coincide). Thus, $\mathcal{K}(\operatorname{Inj} A^e) \xhookrightarrow{\text{inc}} \mathcal{T}$ is dense, and hence an equivalence.

The last statements follows from the fact that $\mathcal{K}(\operatorname{Inj} A^e) = \mathcal{K}(\operatorname{Proj} A^e)$ is closed under tensor products (by Lemma 4.3.1), and contains the unit $\mathbf{i}A$. □

Next we consider the usual action $\mathcal{K}(A^e) \times \mathcal{K}(A) \xrightarrow{-\otimes_A -} \mathcal{K}(A)$.

Lemma 4.3.5. *Let M be a complex in $\mathcal{K}(\operatorname{Inj} A)$. The natural map $A \xrightarrow{\eta} \mathbf{i}A$ induces isomorphism $A \otimes_A M \xrightarrow{\eta \otimes_A M} \mathbf{i}A \otimes_A M$.*

Proof. The proof is similar to Lemma 4.3.3. □

Proposition 4.3.6. *The subcategory $\mathbf{i}A \cdot \mathcal{K}(A)$ defined by $\mathbf{i}A$ is the replete closure of $\mathcal{K}(\operatorname{Inj} A)$.*

Proof. The proof is similar to Proposition 4.3.4, but we use Lemma 4.3.5 instead of Lemma 4.3.3. □

Combining everything, we arrive at the main theorem of this section.

Theorem 4.3.7. *Let A be a finite dimensional self-injective algebra over $\mathcal{K}$. Then the tensor triangulated category $\mathcal{K}(\operatorname{Inj} A^e)$ acts on $\mathcal{K}(\operatorname{Inj} A)$ via tensor product of complexes over A. In particular, we get an action of $\operatorname{HH}^*(A/\Bbbk)$ on $\mathcal{K}(\operatorname{Inj} A)$ given by*

$$\operatorname{HH}^*(A/\Bbbk) \cong \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A^e)}(\mathbf{i}A) \xrightarrow{-\otimes_A M} \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A)}(M)$$

for each $M \in \mathcal{K}(\operatorname{Inj} A)$.

Let us explain that the defined action of $\operatorname{HH}^*(A/\Bbbk)$ on $\mathcal{K}(\operatorname{Inj} A)$ extends the action on $\mathcal{D}(A)$. For simplicity, we identify $\mathbf{i}A \cdot \mathcal{K}(A^e) \cdot \mathbf{i}A$ with $\mathcal{K}(\operatorname{Inj} A)$ (see Proposition 4.3.4). First, since $\mathbf{p}A \longrightarrow A$ and $A \longrightarrow \mathbf{i}A$ are idempotent arrows in $\mathcal{K}(A^e)$, and $\mathbf{p}A$ lies in $\mathcal{K}(\operatorname{Inj} A)$, the composite $\mathbf{p}A \longrightarrow \mathbf{i}A$ is an idempotent arrow in $\mathcal{K}(\operatorname{Inj} A)$ by Lemma 3.1.9. Proposition 3.2.5 yields the following commutative diagram

$$\begin{array}{ccccc}
\operatorname{HH}^*(A/\Bbbk) & \xrightarrow{\cong} & \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A^e)}(\mathbf{i}A) & \xrightarrow{-\otimes_A M} & \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A)}(M) \\
 & & \downarrow{\scriptstyle -\otimes_A \mathbf{p}A} & & \| \\
\operatorname{HH}^*(A/\Bbbk) & \xrightarrow{\cong} & \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A^e)}(\mathbf{p}A) & \xrightarrow{-\otimes_A M} & \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A)}(M)
\end{array}$$

for all $M \in \mathcal{K}_{\text{proj}}(A)$. The vertical arrow $-\otimes_A \mathbf{p}A$ is an isomorphism by Proposition 3.2.6. Identifying $\mathcal{D}(A)$ as $\mathcal{K}_{\text{proj}}(A)$ via the left adjoint of the natural functor $\mathcal{K}(\operatorname{Inj} A) \longrightarrow \mathcal{D}(A)$, we may interpret the commutative diagram above as saying that the embedding $\mathcal{D}(A) \simeq \mathcal{K}_{\text{proj}}(A) \xhookrightarrow{\text{inc}} \mathcal{K}(\operatorname{Inj} A)$ is a $\operatorname{HH}^*(A/\Bbbk)$-linear triangulated functor.

4.4 Action on the homotopy category of injectives of Hopf algebras

Let us fix a Hopf algebra A over $\Bbbk$ with multiplication Δ, counit ε, and antipode S. Recall that Δ is a $\Bbbk$-algebra homomorphism from A to $A \otimes A$ and S is a $\Bbbk$-algebra homomorphism from A to A^{op}. Therefore, we have an algebra map $\delta\colon A \longrightarrow A^e$ obtained by composition $A \xrightarrow{\Delta} A \otimes A \xrightarrow{A \otimes S} A^e$. Using Sweedler notation, δ is defined by $\delta(a) = \sum a_1 \otimes S(a_2)$. Let $(-)^{\text{ad}}$ denotes the induced restriction functor $\operatorname{Mod} A^e \longrightarrow \operatorname{Mod} A$. Thus, the A-module structure of M^{ad} (for $M \in \operatorname{Mod} A^e$) is given by

$$am = \sum (a_1 \otimes S(a_2))m = \sum a_1 m S(a_2)$$

for all $a \in A$ and $m \in M$.

Now we consider the left adjoint of the restriction functor $(-)^{\text{ad}}$. This functor sends an A-module M to an A^e-module $A^e \otimes_A M$ where A^e is viewed as an (A^e, A)-bimodule with

$$(a \otimes b)(a' \otimes b')c = (a \otimes b)(a' \otimes b')\delta(c) = aa'c_1 \otimes S(c_2)b'b.$$

This left adjoint has another description which we explain (see Lemma 4.4.1). Let X be an A-module. Then $X \otimes A$ has a structure of an A-module induced by Δ (from the left A-module structure of A). It also has an A^{op}-module action induced by the right action of A on itself. In short, $X \otimes A$ is an A^e-module, with action determined by

$$(a \otimes b)(x \otimes c) = \sum a_1 x \otimes a_2 cb.$$

The assignment $X \longmapsto X \otimes A$ defines an exact functor $\operatorname{Mod} A \longrightarrow \operatorname{Mod} A^e$ that we denote by Φ.

Lemma 4.4.1. *There is a natural isomorphism of A^e-modules $A^e \otimes_A X \cong \Phi(X)$ for $X \in \operatorname{Mod} A$. In particular Φ is a left adjoint of $(-)^{\mathrm{ad}}$.*

Proof. We use Eilenberg-Watts theorem [Eil60, Wat60]. Note that Φ is right exact and preserves direct sums. By Eilenberg-Watts theorem, Φ is naturally isomorphic to $\Phi(A) \otimes_A -$, where $\Phi(A)$ is viewed as an (A^e, A)-bimodule, where the right action of A is obtained from the right multiplication of A on itself. Thus, it suffices to show that $\Phi(A)$ is isomorphic to A^e as (A^e, A)-bimodule. Such an isomorphism can be given by $A^e \longrightarrow \Phi(A)$, $a \otimes b \longmapsto \sum a_1 \otimes a_2 b$; see [CI17, Construction 2.5]. □

Remark 4.4.2. By construction, the natural isomorphism $A^e \otimes_A X \xrightarrow{\cong} \Phi(X)$ is given by $a \otimes b \otimes x \longmapsto \sum a_1 x \otimes a_2 b$, with inverse $x \otimes b \longmapsto 1 \otimes b \otimes x$.

Since the functors Φ and $(-)^{\mathrm{ad}}$ are exact, we have the following formal corollary.

Corollary 4.4.3. *The functor Φ preserves projective modules and $(-)^{\mathrm{ad}}$ preserves injective modules.*

Proof. This is a formal consequence of the fact that a left (resp. right) adjoint of an exact functor preserves projective (resp. injective) modules. For $P \in \operatorname{Mod} A$ projective, there is a natural isomorphism

$$\operatorname{Hom}_{A^e}(\Phi(P), -) \cong \operatorname{Hom}_A(P, (-)^{\mathrm{ad}}),$$

and the right hand side is exact. Thus $\Phi(P)$ is a projective A^e-module. Similarly, for $Q \in \operatorname{Mod} A^e$ injective, the A-module Q^{ad} is injective since $\operatorname{Hom}_A(-, Q^{\mathrm{ad}})$ is naturally isomorphic to an exact functor $\operatorname{Hom}_{A^e}(\Phi(-), Q)$. □

Lemma 4.4.4. *For each A-modules X, Y, M, there is an isomorphism of A-modules*

$$\psi_{X,M} \colon \Phi(X) \otimes_A M \longrightarrow X \otimes M,$$
$$x \otimes a \otimes m \longmapsto x \otimes am,$$

and an isomorphism of A^e-modules

$$\phi_{X,Y} \colon \Phi(X) \otimes_A \Phi(Y) \longrightarrow \Phi(X \otimes Y),$$
$$x \otimes a \otimes y \otimes b \longmapsto x \otimes ay \otimes b,$$

which are natural in X, M and X, Y, respectively.

Proof. The first isomorphism is a part of the proof of Lemma 13 in [PW09].

We show the second one. By forgetting the right action of A on $\Phi(Y)$, we get an isomorphism of A-modules

$$\Phi(X) \otimes_A \Phi(Y) \cong X \otimes \Phi(Y) \cong \Phi(X \otimes Y).$$

The first isomorphism follows from the first part and the second one by associativity $X \otimes (Y \otimes A) \cong (X \otimes Y) \otimes A$. It easy to see that this isomorphism is also compatible with the right action of A, since $(x \otimes a \otimes y \otimes b)c = x \otimes a \otimes y \otimes bc$ in $\Phi(X) \otimes_A \Phi(Y)$ and $(x \otimes ay \otimes b)c = x \otimes ay \otimes bc$ in $\Phi(X \otimes Y)$. □

Lemma 4.4.5. *The canonical* $\Bbbk$*-linear isomorphism* $A \cong \Bbbk \otimes A$ *induce an* A^e*-linear isomorphism* $\phi_0 \colon A \xrightarrow{\cong} \Phi(\Bbbk)$.

Proof. This is straighforward; see [CI17, Construction 2.5] □

Passing to the homotopy category, the functor $\Phi\colon \operatorname{Mod} A \longrightarrow \operatorname{Mod} A^e$ induces a triangulated functor $\mathcal{K}(A) \longrightarrow \mathcal{K}(A^e)$, which we also denote by Φ. We have an isomorphism $\phi_0 \colon A \longrightarrow \Phi(\Bbbk)$ from Lemma 4.4.5 and a natural isomorphism in $\mathcal{K}(A^e)$

$$\begin{aligned} \phi_{X,Y} \colon \Phi(X) \otimes_A \Phi(Y) &\longrightarrow \Phi(X \otimes Y), \\ x \otimes a \otimes y \otimes b &\longmapsto x \otimes ay \otimes b, \end{aligned}$$

for $X, Y \in \mathcal{K}(A)$ induced by ϕ in Lemma 4.4.4.

As usual, we regard $\mathcal{K}(A)$ and $\mathcal{K}(A^e)$ as tensor triangulated category with tensor product $\otimes$ and $\otimes_A$, and tensor unit $\Bbbk$ and A, respectively.

Proposition 4.4.6. *The triple* (Φ, ϕ, ϕ_0) *is a tensor triangulated functor from* $\mathcal{K}(A)$ *to* $\mathcal{K}(A^e)$.

Proof. Since the tensor product of complexes are defined using totalization of the tensor product of modules, it suffices to show that the tensor product of modules are compatible with the coherence isomorphisms, namely that the diagrams

$$\begin{array}{ccc} (\Phi(X) \otimes_A \Phi(Y)) \otimes_A \Phi(Z) & \xrightarrow{\cong} & \Phi(X) \otimes_A (\Phi(Y) \otimes_A \Phi(Z)) \\ \downarrow{\scriptstyle \phi_{X,Y} \otimes_A \Phi(Z)} & & \downarrow{\scriptstyle \Phi(X) \otimes_A \phi_{Y,Z}} \\ \Phi(X \otimes Y) \otimes_A \Phi(Z) & & \Phi(X) \otimes_A \Phi(Y \otimes Z) \\ \downarrow{\scriptstyle \phi_{X \otimes Y, Z}} & & \downarrow{\scriptstyle \phi_{X, Y \otimes Z}} \\ \Phi((X \otimes Y) \otimes Z) & \xrightarrow{\cong} & \Phi(X \otimes (Y \otimes Z)). \end{array}$$

and

$$\begin{array}{ccc} \Phi(\Bbbk)\otimes_A \Phi(X) & \xrightarrow{\phi_{\Bbbk,X}} & \Phi(\Bbbk\otimes X) \\ {\scriptstyle \phi_0\otimes_A\Phi(X)}\downarrow & & \downarrow{\scriptstyle \cong} \\ A\otimes_A \Phi(X) & \xrightarrow{\cong} & \Phi(X) \end{array} \qquad \begin{array}{ccc} \Phi(X)\otimes_A \Phi(\Bbbk) & \xrightarrow{\phi_{X,\Bbbk}} & \Phi(X\otimes \Bbbk) \\ {\scriptstyle \Phi(X)\otimes_A\phi_0}\downarrow & & \downarrow{\scriptstyle \cong} \\ \Phi(X)\otimes_A A & \xrightarrow{\cong} & \Phi(X) \end{array}$$

are commutative for $X, Y, Z \in \operatorname{Mod} A$. These are straightforward as we only need to check the commutativity on the generators. For example, $(\Phi(X)\otimes_A \Phi(Y))\otimes_A \Phi(Z)$ is generated by elements of the form $x\otimes 1\otimes y\otimes 1\otimes z\otimes 1$ with $x\in X$, $y\in Y$ and $z\in Z$. The images of this element along two different ways of the top hexagon are equal, namely $x\otimes y\otimes z\otimes 1$. Thus, the top diagram is commutative. □

Next we consider the actions $\mathcal{K}(A)\times\mathcal{K}(A)\xrightarrow{-\otimes-}\mathcal{K}(A)$ and $\mathcal{K}(A^e)\times\mathcal{K}(A)\xrightarrow{-\otimes_A-}\mathcal{K}(A)$. Note that we have a natural isomorphism in $\mathcal{K}(A)$

$$\begin{aligned} \psi_{X,M}\colon \Phi(X)\otimes_A M &\longrightarrow X\otimes M, \\ x\otimes a\otimes m &\longmapsto x\otimes am, \end{aligned}$$

for $X, M\in\mathcal{K}(A)$ induced by ψ in Lemma 4.4.4.

Proposition 4.4.7. *The quintuple* $(\Phi,\phi,\phi_0,\mathrm{Id}_{\mathcal{K}(A)},\psi)$ *is compatible with the action of* $\mathcal{K}(A)$ *and* $\mathcal{K}(A^e)$ *on* $\mathcal{K}(A)$.

Proof. Similar as Proposition 4.4.6, it suffices to show that the diagrams

$$\begin{array}{ccc} (\Phi(X)\otimes_A\Phi(Y))\otimes_A M & \xrightarrow{\cong} & \Phi(X)\otimes_A(\Phi(Y)\otimes_A M) \\ \downarrow{\scriptstyle \phi_{X,Y}\otimes_A M} & & \downarrow{\scriptstyle \Phi(X)\otimes_A\psi_{Y,M}} \\ \Phi(X\otimes Y)\otimes_A M & & \Phi(X)\otimes_A(Y\otimes M) \\ \downarrow{\scriptstyle \psi_{X\otimes Y,M}} & & \downarrow{\scriptstyle \psi_{X,Y\otimes M}} \\ (X\otimes Y)\otimes M & \xrightarrow{\cong} & X\otimes(Y\otimes M). \end{array}$$

and

$$\begin{array}{ccc} \Phi(\Bbbk)\otimes_A M & \xrightarrow{\psi_{\Bbbk,M}} & \Bbbk\otimes X \\ {\scriptstyle \phi_0\otimes_A M}\downarrow & & \downarrow{\scriptstyle\cong} \\ A\otimes_A M & \xrightarrow{\cong} & M \end{array}$$

are commutative for $X, Y, M\in\operatorname{Mod} A$. These are also a straightforward checking on the generators. □

From here on, we assume additionally that A is finite dimensional over $\Bbbk$. Finite dimensional Hopf algebras are Frobenius [Par71]. In particular, they are self-injective, Thus the result from the previous section applies, namely there is an action of the Hochschild cohomology on the homotopy category of injectives. We will relate it with the action of the ordinary cohomology ring. In fact, our method yields an alternative proof of the action of the Hochschild cohomology.

Proposition 4.4.8 ([BK08, Proposition 5.3])**.** *The injective resolution $\Bbbk \xrightarrow{\eta} \mathbf{i}\Bbbk$ of the trivial module $\Bbbk$ is a unital idempotent arrow $\mathcal{K}(A)$, and the following statements hold.*

(1) *Both the subcategories $\mathbf{i}\Bbbk \cdot \mathcal{K}(A) \cdot \mathbf{i}\Bbbk$ and $\mathbf{i}\Bbbk \cdot \mathcal{K}(A)$ coincide with the replete closure of $\mathcal{K}(\operatorname{Inj} A)$.*

(2) *The action of $\mathcal{K}(A)$ on itself given by tensor product over $\Bbbk$ restricts to an action of $\mathcal{K}(\operatorname{Inj} A)$ on itself.*

(3) *There is a ring action of $\mathrm{H}^*(A, \Bbbk)$ on $\mathcal{K}(\operatorname{Inj} A)$ given by*

$$\mathrm{H}^*(A, \Bbbk) \cong \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A)}(\mathbf{i}\mathcal{K}) \xrightarrow{-\otimes M} \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A)}(M).$$

Remark 4.4.9. Actually, [BK08] only treat the cocommutative Hopf algebras case. In this case, the tensor product $\otimes$ on $\mathcal{K}(A)$ is symmetric. But the proof works also in the general case. The main ingredients are:

(1) For $X \in \mathcal{K}(\operatorname{Inj} A)$, the complexes $\mathbf{i}\Bbbk \otimes X$ and $X \otimes \mathbf{i}\Bbbk$ belong to $\mathcal{K}(\operatorname{Inj} A)$.

(2) For $X \in \mathcal{K}(\operatorname{Inj} A)$, the functor $X \otimes -$ and $- \otimes X$ have right adjoints which sends complexes in $\mathcal{K}(\operatorname{Inj} A)$ to complexes in $\mathcal{K}(\operatorname{Inj} A)$.

(3) Lemma 2.1 in [Kra05] and Yoneda lemma.

We determine the image $\Phi(\mathbf{i}\Bbbk)$.

Lemma 4.4.10. *Let $\eta \colon \Bbbk \longrightarrow \mathbf{i}\Bbbk$ be the injective resolution of $\Bbbk$. The composite $A \xrightarrow{\phi_0} \Phi(\Bbbk) \xrightarrow{\Phi(\eta)} \Phi(\mathbf{i}\Bbbk)$ is an injective resolution of A in $\mathcal{K}(A^e)$.*

Proof. Complete η into a triangle

$$Z \longrightarrow \Bbbk \xrightarrow{\eta} \mathbf{i}\Bbbk \longrightarrow \Sigma Z$$

in $\mathcal{K}(A)$ with Z acyclic. Since Φ preserves acyclic complexes, $\Phi(Z)$ is acyclic, and hence $\Phi(\eta)$ is a quasi-isomorphism. By Lemma 4.4.3, Φ preserves injective modules as functor $\operatorname{Mod} A \longrightarrow \operatorname{Mod} A^e$. Thus, $\Phi(\mathbf{i}\Bbbk)$ is a bounded below complex of injective A^e-modules, and hence it is K-injective. The conclusion follows since ϕ_0 is an isomorphism. □

Using our machinery in Section 3.3 with input quintuple $(\Phi, \phi, \phi_0, \mathrm{Id}_{\mathcal{K}(A)}, \psi)$ and idempotent $\Bbbk \xrightarrow{\eta} \mathbf{i}\Bbbk$, we get the following theorem, which relates the actions of the ordinary cohomology ring and the Hochschild cohomology ring.

Theorem 4.4.11. *The canonical ring action of the ordinary cohomology* $\mathrm{H}^*(A, \Bbbk)$ *on* $\mathcal{K}(\mathrm{Inj}\, A)$ *factors through the action of the Hochschild cohomology* $\mathrm{HH}^*(A/\Bbbk)$*. More precisely, for each complex* M *in* $\mathcal{K}(\mathrm{Inj}\, A)$*, we have a commutative diagram*

$$\begin{array}{ccccc} \mathrm{H}^*(A, \Bbbk) & \xrightarrow{\cong} & \mathrm{End}^*_{\mathcal{K}(A)}(\mathbf{i}\Bbbk) & \xrightarrow{-\otimes M} & \mathrm{End}^*_{\mathcal{K}(\mathrm{Inj}\, A)}(M) \\ & & \Big\downarrow{\scriptstyle \Phi} & & \Big\| \\ \mathrm{HH}^*(A/\Bbbk) & \xrightarrow{\cong} & \mathrm{End}^*_{\mathcal{K}(A^e)}(\Phi(\mathbf{i}\Bbbk)) & \xrightarrow{-\otimes_A M} & \mathrm{End}^*_{\mathcal{K}(\mathrm{Inj}\, A)}(M). \end{array}$$

As a direct consequence of the last proposition, we have the following corollary first proved in [GK93]. We denote by $\alpha \colon \mathrm{H}^*(A, \Bbbk) \longrightarrow \mathrm{HH}^*(A/\Bbbk)$ the unique map such that

$$\begin{array}{ccc} \mathrm{H}^*(A, \Bbbk) & \xrightarrow{\cong} & \mathrm{End}^*_{\mathcal{K}(A)}(\mathbf{i}\Bbbk) \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \Phi} \\ \mathrm{HH}^*(A/\Bbbk) & \xrightarrow{\cong} & \mathrm{End}^*_{\mathcal{K}(A^e)}(\Phi(\mathbf{i}\Bbbk)). \end{array}$$

Corollary 4.4.12. *The map* $\alpha \colon \mathrm{H}^*(A, \Bbbk) \longrightarrow \mathrm{HH}^*(A/\Bbbk)$ *splits. In particular,* $\mathrm{H}^*(A, \Bbbk)$ *is a direct summand of* $\mathrm{HH}^*(A/\Bbbk)$ *as a module over* $\mathrm{H}^*(A, \Bbbk)$*.*

Proof. Take $X = \mathbf{i}\Bbbk$ in Proposition 4.4.11. □

Remark 4.4.13. The above corollary holds in the general case of (not necessarily finite dimensional) Hopf algebra. To show this, one uses the projective resolution $\mathbf{p}\Bbbk \longrightarrow \Bbbk$, instead of injective resolution of $\Bbbk$. The projective resolution is always a idempotent in $\mathcal{K}(A^e)$, while the fact that the injective resolution is an idempotent need finite dimensional assumption.

Chapter 5

Local duality

Let A be a finite dimensional cocommutative Hopf algebra. In this chapter, we will prove our main result, namely if A is symmetric, then the category $\mathcal{K}(\operatorname{Inj} A)$, viewed as a $\mathrm{HH}^*(A/\Bbbk)$-linear triangulated category, is Gorenstein. Then we discuss some formal consequences of this fact.

We recall first the following setup from Section 4.4 that we will use throughout the chapter. There is a tensor triangulated functor (Φ, ϕ, ϕ_0) from $(\mathcal{K}(A), \otimes, \Bbbk)$ to $(\mathcal{K}(A^e), \otimes_A, A)$ by Proposition 4.4.6. The functor Φ is a left adjoint of the functor $(-)^{\mathrm{ad}}$, by Lemma 4.4.1, and both functors restrict to an adjoint pair of functors between $\mathcal{K}(\operatorname{Inj} A)$ and $\mathcal{K}(\operatorname{Inj} A^e)$ by Corollary 4.4.3.

Now, for simplicity, we use the identifications

$$\mathrm{H}^*(A, \Bbbk) = \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A)}(\mathbf{i}\Bbbk) \quad \text{and} \quad \mathrm{HH}^*(A/\Bbbk) = \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A^e)}(\mathbf{i}A)$$

There is a ring map $\alpha\colon \mathrm{H}^*(A, \Bbbk) \longrightarrow \mathrm{HH}^*(A/\Bbbk)$, which is given by applying the functor $\Phi\colon \mathcal{K}(\operatorname{Inj} A) \longrightarrow \mathcal{K}(\operatorname{Inj} A^e)$. Moreover, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^*(A, \Bbbk) & \xrightarrow{-\otimes M} & \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A)}(M) \\ \alpha \big\downarrow & & \big\| \\ \mathrm{HH}^*(A/\Bbbk) & \xrightarrow{-\otimes_A M} & \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A)}(M) \end{array}$$

for all $M \in \mathcal{K}(\operatorname{Inj} A)$.

5.1 Finite generation of Hochschild cohomology

Let A be a finite dimensional cocommutative Hopf algebra over a field $\Bbbk$. Here we will prove that the Hochschild cohomology $\mathrm{HH}^*(A/\Bbbk)$ is finitely generated as a $\Bbbk$-algebra. In fact, we shall prove that the map $\alpha\colon \mathrm{H}^*(A, \Bbbk) \longrightarrow \mathrm{HH}^*(A/\Bbbk)$ is a finite map, that is $\mathrm{HH}^*(A/\Bbbk)$ is finitely generated as a $\mathrm{H}^*(A, \Bbbk)$-modules by restriction. The finite generation of $\mathrm{HH}^*(A/\Bbbk)$ follows immediately from this and Theorem 2.5.11 of Friedlander and Suslis.

Lemma 5.1.1. *The diagram*

$$\begin{array}{ccc} \mathrm{H}^*(A,\Bbbk) & \xrightarrow{-\otimes X} & \mathrm{End}^*_{\mathcal{K}(\mathrm{Inj}\,A)}(X) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\Phi} \\ \mathrm{HH}^*(A/\Bbbk) & \xrightarrow{-\otimes_A \Phi(M)} & \mathrm{End}^*_{\mathcal{K}(\mathrm{Inj}\,A^e)}(\Phi(X)). \end{array}$$

is commutative for all $X \in \mathcal{K}(\mathrm{Inj}\,A)$.

Proof. We let the tensor triangulated categories $\mathcal{K}(\mathrm{Inj}\,A)$ and $\mathcal{K}(\mathrm{Inj}\,A^e)$ act on themselves. The lemma follows from Lemma 3.3.3 by using the quintuple $(\Phi, \phi, \phi_0, \Phi, \phi)$ as input. □

Proposition 5.1.2. *Let X and Y be complexes in $\mathcal{K}(\mathrm{Inj}\,A)$ and $\mathcal{K}(\mathrm{Inj}\,A^e)$, respectively. There is a $\mathrm{H}^*(A,\Bbbk)$-linear isomorphism*

$$\mathrm{Hom}^*_{\mathcal{K}(\mathrm{Inj}\,A)}(X, Y^{\mathrm{ad}}) \cong \mathrm{Hom}^*_{\mathcal{K}(\mathrm{Inj}\,A^e)}(\Phi(X), Y),$$

and the functor $(-)^{\mathrm{ad}}\colon \mathcal{K}(\mathrm{Inj}\,A^e) \longrightarrow \mathcal{K}(\mathrm{Inj}\,A)$ is $\mathrm{H}^*(A,\Bbbk)$-linear. In particular, for A-module M, there is a $\mathrm{H}^*(A,\Bbbk)$-linear isomorphism

$$\mathrm{H}^*(A, M^{\mathrm{ad}}) \cong \mathrm{HH}^*(A/\Bbbk; M).$$

Proof. By restriction along α, the $\mathrm{HH}^*(A/\Bbbk)$-linear category $\mathcal{K}(\mathrm{Inj}\,A^e)$ can be viewed as a $\mathrm{H}^*(A,\Bbbk)$-linear category. Lemma 5.1.1 then says that the functor Φ is $\mathrm{H}^*(A,\Bbbk)$-linear. Thus, the first part follows directly from Lemma 2.3.11. The second part is application of the first by taking $X = \mathbf{i}\Bbbk$ and $Y = M$. □

Theorem 5.1.3. *The map $\alpha\colon\ \mathrm{H}^*(A,\Bbbk) \longrightarrow \mathrm{HH}^*(A/\Bbbk)$ is finite. In particular, $\mathrm{HH}^*(A/\Bbbk)$ is finitely generated as $\Bbbk$-algebra.*

Proof. Taking $M = A$ in Proposition 5.1.2, we obtain $\mathrm{H}^*(A,\Bbbk)$-linear isomorphisms

$$\mathrm{HH}^*(A/\Bbbk) \cong \mathrm{H}^*(A, A^{\mathrm{ad}}) \cong \mathrm{Ext}^*_A(\Bbbk, A^{\mathrm{ad}}).$$

The conclusion follows by Theorem 2.5.11, since $\Bbbk$ and A^{ad} are finitely generated as A-module. □

5.2 Change of ring actions

Let A be a finite dimensional Hopf algebra. In this section we explain some formal consequences of the fact, that the canonical action of the cohomology ring $\mathrm{H}^*(A,\Bbbk)$ on $\mathcal{K}(\mathrm{Inj}\,A)$ is obtained by a change of rings from the action of $\mathrm{HH}^*(A/\Bbbk)$ via the finite map α.

Local cohomology functors and support

The following are two applications of [BIK12, §7] in our setting. We denote by α^* the continous map $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ defined by $\alpha^*(\mathfrak{q}) = \alpha^{-1}(\mathfrak{q})$.

The first proposition relates local cohomology functors defined using the actions of $\mathrm{H}^*(A, \Bbbk)$ and $\mathrm{HH}^*(A/\Bbbk)$.

Proposition 5.2.1. *Let $\mathfrak{p} \in \operatorname{Spec} \mathrm{H}^*(A, \Bbbk)$ and $\mathcal{V} = (\alpha^*)^{-1}\{\mathfrak{p}\}$. Then there is an isomorphisms*

$$\varGamma_{\mathfrak{p}} \cong \coprod_{\mathfrak{q}} \varGamma_{\mathfrak{q}}$$

of functors $\mathcal{K}(\operatorname{Inj} A) \longrightarrow \mathcal{K}(\operatorname{Inj} A)$.

Proof. This is a direct application of [BIK12, Corollary 7.10]. □

The next proposition says that the support theory defined using $\mathrm{H}^*(A, \Bbbk)$ is 'contained' in the support theory defined using $\mathrm{HH}^*(A/\Bbbk)$.

Proposition 5.2.2. *For $X \in \mathcal{K}(\operatorname{Inj} A)$ there is an equality*

$$\operatorname{supp}_{\mathrm{H}^*(A,\Bbbk)} X = \alpha^*(\operatorname{supp}_{\mathrm{HH}^*(A/\Bbbk)})X.$$

Proof. This is a direct application of [BIK12, Corollary 7.8]. □

Injective cohomology functors

Let us start with the general setup. Let R, S be graded commutative noetherian rings and $\mathcal{T}, \mathcal{U}$ compactly generated R-linear and S-linear triangulated categories, respectively. Let $F \colon \mathcal{T} \longrightarrow \mathcal{U}$ be an exact functor and $\alpha \colon R \longrightarrow S$ be a finite map of graded rings such that the following diagram

$$\begin{array}{ccc} R & \longrightarrow & \operatorname{End}^*_{\mathcal{T}}(X) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle F} \\ S & \longrightarrow & \operatorname{End}^*_{\mathcal{U}}(FX) \end{array}$$

is commutative for all X in $\mathcal{T}$, where the horizontal arrows are from the actions of R and S. We also assume that F has a right adjoint G, and G has a right adjoint H. Since $\mathcal{T}$ and $\mathcal{U}$ are compactly generated the assumption is implies that G preserves coproducts, and F preserves compact objects [Nee96]. This setup applies to our situation by taking $R = \mathrm{H}^*(A, \Bbbk)$, $S = \mathrm{HH}^*(A/\Bbbk)$, $\mathcal{T} = \mathcal{U} = \mathcal{K}(\operatorname{Inj} A)$ and $F = G = H = \mathrm{Id}_{\mathcal{K}(\operatorname{Inj} A)}$.

Recall from Section 2.4 that we have injective cohomology functors $\mathcal{T}^c \times \operatorname{Inj} R \longrightarrow \mathcal{T}$ and $\mathcal{U}^c \times \operatorname{Inj} S \longrightarrow \mathcal{U}$ constructed using Brown representability. Both functors will be denoted by the same notation T; but this should not cause any confusion. On the other hand, the map $\alpha\colon R \longrightarrow S$ induces a functor

$$\mathrm{Hom}^*_R(S,-)\colon \operatorname{Inj} R \longrightarrow \operatorname{Inj} S.$$

The following proposition tells us that we have a commutative square

$$\begin{array}{ccc} \mathcal{T}^c \times \operatorname{Inj} R & \xrightarrow{T} & \mathcal{T} \\ {\scriptstyle F\times \mathrm{Hom}^*_R(S,-)}\downarrow & & \downarrow{\scriptstyle H} \\ \mathcal{U}^c \times \operatorname{Inj} S & \xrightarrow{T} & \mathcal{U} \end{array}$$

(see [BK02, Proposition 7.1] for the group algebra case).

Proposition 5.2.3. *There is an isomorphism*

$$H(T(C,I)) \cong T(FC, \mathrm{Hom}^*_R(S,I)),$$

which natural in $C \in \mathcal{T}^c$ and $I \in \operatorname{Inj} R$.

Proof. We have for each Y in $\mathcal{U}$ the following chain of natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{U}}(Y, T(FC, \mathrm{Hom}^*_R(S,I))) &\cong \mathrm{Hom}_S(\mathrm{Hom}^*_{\mathcal{U}}(FC,Y), \mathrm{Hom}^*_R(S,I)) \\ &\cong \mathrm{Hom}_R(\mathrm{Hom}^*_{\mathcal{U}}(FC,Y), I) \\ &\cong \mathrm{Hom}_R(\mathrm{Hom}^*_{\mathcal{T}}(C,GY), I) \\ &\cong \mathrm{Hom}_{\mathcal{T}}(GY, T(C,I)) \\ &\cong \mathrm{Hom}_{\mathcal{U}}(Y, H(T(C,I))). \end{aligned}$$

The first and the fourth are by definition. The second, the third and the last are by adjunctions. The conclusion follows by Yoneda Lemma. □

Now let $\operatorname{Spec} R$ and $\operatorname{Spec} S$ be the sets of homogeneous prime ideals in R and S, respectively. The map α induces a continuous map $\alpha^*\colon \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$, defined by $\alpha^*(\mathfrak{q}) = \alpha^{-1}(\mathfrak{q})$. For each prime $\mathfrak{p}$ in $\operatorname{Spec} R$, we write $I(\mathfrak{p})$ for the injective hull of $R/\mathfrak{p}$ as graded R-module and set

$$T_{\mathfrak{p}} := T(-, I(\mathfrak{p}))\colon \mathcal{T}^c \longrightarrow \mathcal{T}.$$

Similarly for each prime $\mathfrak{q}$ in $\operatorname{Spec} S$, we set

$$T_{\mathfrak{q}} := T(-, I(\mathfrak{q}))\colon \mathcal{U}^c \longrightarrow \mathcal{U}.$$

The assumption that α is finite implies the following proposition.

Proposition 5.2.4. *Let $\mathfrak{p}$ be a point in $\operatorname{Spec} R$ and let $\mathcal{V} = (\alpha^*)^{-1}\{\mathfrak{p}\} \subseteq \operatorname{Spec} S$. Then there is an isomorphism*

$$H \circ T_{\mathfrak{p}} \cong \coprod_{\mathfrak{q}\in\mathcal{V}} (T_{\mathfrak{q}} \circ F),$$

of functors from $\mathcal{T}^c$ to $\mathcal{U}$.

Proof. Since α is finite, the set $\mathcal{V}$ is finite and there is an isomorphism

$$\operatorname{Hom}^*_R(S, I(\mathfrak{p})) \cong \bigoplus_{\mathfrak{q}\in\mathcal{V}} I(\mathfrak{q})$$

of graded S-modules (see [Rah09]). Using this isomorphism and Proposition 5.2.3, we obtain natural isomorphisms

$$H(T_{\mathfrak{p}}(C)) \cong T(FC, \operatorname{Hom}^*_R(S, I)) \cong \coprod_{\mathfrak{q}\in\mathcal{V}} T_{\mathfrak{q}}(FC),$$

as required. □

We record the following special case of our situation.

Proposition 5.2.5. *Let $\mathcal{T} = \mathcal{U} = \mathcal{K}(\operatorname{Inj} A)$, $F = G = H = \operatorname{Id}_{\mathcal{K}(\operatorname{Inj} A)}$, $R = \operatorname{H}^*(A, \Bbbk)$ and $S = \operatorname{HH}^*(A/\Bbbk)$. There is a natural isomorphism*

$$T_{\mathfrak{p}} \cong \coprod_{\mathfrak{q}\in\mathcal{V}} T_{\mathfrak{q}}$$

for each prime $\mathfrak{p}$ in $\operatorname{Spec} \operatorname{H}^*(A, \Bbbk)$, where $\mathcal{V} = (\alpha^*)^{-1}\{\mathfrak{p}\} \subset \operatorname{Spec} \operatorname{HH}^*(A/\Bbbk)$.

5.3 The Gorenstein property of $\mathcal{K}(\operatorname{Inj} A)$

Let A be a finite dimensional cocommutative Hopf algebra over a field $\Bbbk$. We recall the following result of Benson, Iyengar, Krause, and Pevtsova.

Theorem 5.3.1 ([BIKP16, Theorem 6.4])**.** *The category $\mathcal{K}(\operatorname{Inj} A)$, viewed as an $\operatorname{H}^*(A, \Bbbk)$-linear triangulated category, is Gorenstein with the global Serre functor $\nu \colon \mathcal{K}^c(\operatorname{Inj} A) \longrightarrow \mathcal{K}^c(\operatorname{Inj} A)$ induced by the Nakayama functor $D(A) \otimes_A -$ and $d(\mathfrak{p}) = \dim \operatorname{H}^*(A, \Bbbk)/\mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec} \operatorname{H}^*(A, \Bbbk)$. More precisely, ν is a $\operatorname{H}^*(A, \Bbbk)$-linear triangle equivalence and there is an isomorphism*

$$\varGamma_{\mathfrak{p}} \circ \nu \cong \Sigma^{d(\mathfrak{p})} \circ T_{\mathfrak{p}}$$

of functors $\mathcal{K}^c(\operatorname{Inj} A) \longrightarrow \mathcal{K}(\operatorname{Inj} A)$ for every $\mathfrak{p}$ in $\operatorname{Spec} \operatorname{H}^*(A, \Bbbk)$.

We prove our main theorem using a 'transfer argument' via the finite map α. As in Theorem 5.3.1, we denote by ν the functor $\mathcal{K}^c(\operatorname{Inj} A) \longrightarrow \mathcal{K}^c(\operatorname{Inj} A)$ induced by the Nakayama functor $DA \otimes_A -$.

Proposition 5.3.2. *For every* $\mathfrak{q}$ *in* $\operatorname{Spec}\mathrm{HH}^*(A/\Bbbk)$ *there is an isomorphism*

$$\mathit{\Gamma}_{\mathfrak{q}} \circ \nu \cong \Sigma^{d(\mathfrak{q})} \circ T_{\mathfrak{q}}$$

of functors $\mathcal{K}^c(\operatorname{Inj} A) \to \mathcal{K}(\operatorname{Inj} A)$, *where* $d(\mathfrak{q}) = \dim \mathrm{HH}^*(A/\Bbbk)/\mathfrak{q}$ *for* $\mathfrak{q} \in \operatorname{Spec}\mathrm{HH}^*(A/\Bbbk)$.

Proof. Fix a prime $\mathfrak{q}$ in $\operatorname{Spec}\mathrm{HH}^*(A/\Bbbk)$ and set $\mathfrak{p} = \alpha^*(\mathfrak{q})$ in $\operatorname{Spec}\mathrm{H}^*(A, \Bbbk)$. By Theorem 5.3.1 we have

$$\mathit{\Gamma}_{\mathfrak{p}} \circ \nu \cong \Sigma^{d(\mathfrak{p})} \circ T_{\mathfrak{p}}, \tag{5.3.1}$$

where $d(\mathfrak{p}) = \dim \mathrm{H}^*(A, \Bbbk)/\mathfrak{p}$. Now let $\mathcal{V} = (\alpha^*)^{-1}\{\mathfrak{p}\}$. Clearly $\mathfrak{q}$ is in $\mathcal{V}$. Combining Proposition 5.2.1 and Proposition 5.2.5 with (5.3.1) gives us

$$\coprod_{\mathfrak{r}\in\mathcal{V}} (\mathit{\Gamma}_{\mathfrak{r}} \circ \nu) \cong \coprod_{\mathfrak{r}\in\mathcal{V}} (\Sigma^{d(\mathfrak{p})} \circ T_{\mathfrak{r}}). \tag{5.3.2}$$

Using Corollary 2.4.9, $\mathit{\Gamma}_{\mathfrak{q}}\mathit{\Gamma}_{\mathfrak{r}} \cong \mathit{\Gamma}_{\mathfrak{q}}$ if $\mathfrak{r} = \mathfrak{q}$, and $\mathit{\Gamma}_{\mathfrak{q}}\mathit{\Gamma}_{\mathfrak{r}} = 0$ otherwise. In particular, we have

$$\mathit{\Gamma}_{\mathfrak{q}} \left(\coprod_{\mathfrak{r}\in\mathcal{V}} (\mathit{\Gamma}_{\mathfrak{r}} \circ \nu) \right) \cong \mathit{\Gamma}_{\mathfrak{q}} \circ \nu.$$

Similarly, since $\mathcal{K}(\operatorname{Inj} A)$ is noetherian as $\mathrm{HH}^*(A/\Bbbk)$-linear triangulated category by Theorem 2.5.11, the object $T_{\mathfrak{r}}(C)$ belongs to $\mathit{\Gamma}_{\mathfrak{r}}\mathcal{K}(\operatorname{Inj} A)$ by Proposition 2.4.11. Therefore we have

$$\mathit{\Gamma}_{\mathfrak{q}} \left(\coprod_{\mathfrak{r}\in\mathcal{V}} (\Sigma^{d(\mathfrak{p})} \circ T_{\mathfrak{r}}) \right) \cong \Sigma^{d(\mathfrak{p})} \circ T_{\mathfrak{q}}.$$

Thus, applying $\mathit{\Gamma}_{\mathfrak{q}}$ to (5.3.2) gives us

$$\mathit{\Gamma}_{\mathfrak{q}} \circ \nu \cong \Sigma^{d(\mathfrak{p})} \circ T_{\mathfrak{q}}.$$

Since α is finite and $\mathfrak{p} = \alpha^*(\mathfrak{q})$, we have $d(\mathfrak{p}) = d(\mathfrak{q})$, and we are done. □

If A is symmetric, that is if DA is isomorphic to A as an A^e module, then the functor ν is isomorphic to the identity functor. In particular, it is $\mathrm{HH}^*(A/\Bbbk)$-linear. In this case, we have the following theorem.

Theorem 5.3.3. *Let* A *be a finite dimensional symmetric cocommutative Hopf algebra over a field* $\Bbbk$. *The category* $\mathcal{K}(\operatorname{Inj} A)$ *viewed as an* $\mathrm{HH}^*(A/\Bbbk)$*-linear triangulated category is Gorenstein, where the global Serre functor is just the identity functor on* $\mathcal{K}^c(\operatorname{Inj} A)$ *and* $d(\mathfrak{q}) = \dim \mathrm{HH}^*(A/\Bbbk)/\mathfrak{q}$ *for each* $\mathfrak{q} \in \operatorname{Spec}\mathrm{HH}^*(A/\Bbbk)$.

Recall that there are embeddings of $\underline{\operatorname{Mod}}\, A$ and $\mathcal{D}(A)$ inside $\mathcal{K}(\operatorname{Inj} A)$ by Theorem 2.2.1. This embedding is clearly $\mathrm{HH}^*(A/\Bbbk)$-linear if we regard $\underline{\operatorname{Mod}}\, A$ and $\mathcal{D}(A)$ as $\mathrm{HH}^*(A/\Bbbk)$-linear by restriction of the action on $\mathcal{K}(\operatorname{Inj} A)$. The following corollary is a straightforward consequences of Theorem 5.3.3

Corollary 5.3.4. *Let* A *be a finite dimensional symmetric cocommutative Hopf algebra over a field* $\Bbbk$. *As an* $\mathrm{HH}^*(A/\Bbbk)$*-linear triangulated category,* $\underline{\operatorname{Mod}}\, A$ *and* $\mathcal{D}(A)$ *are Gorenstein.*

5.4 Local Serre duality for $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)$

There are some formal consequences of the Gorenstein property for triangulated categories, namely local Serre duality and Auslander-Reiten triangles (see Section 2.4). In particular, this applies to our result in the previous section.

Let A be a finite dimensional symmetric cocommutative Hopf algebra, for instance $A = \Bbbk G$ for a finte group G. For simplicity, we write $\mathcal{K}$ for $\mathcal{K}(\operatorname{Inj} A)$. Thus $\mathcal{K}$ is Gorenstein as a $\mathrm{HH}^*(A/\Bbbk)$-linear triangulated category by Theorem 5.3.3. For each prime $\mathfrak{q} \in \operatorname{Spec} \mathrm{HH}^*(A/\Bbbk)$, we have a localizing subcategory $\varGamma_{\mathfrak{q}}\mathcal{K} \subset \mathcal{K}$ formed by the $\mathfrak{q}$-local and $\mathfrak{q}$-torsion objects. This category is compactly generated, and we denote the full subcategory of compact objects by $(\varGamma_{\mathfrak{q}}\mathcal{K})^{\mathrm{c}}$.

Proposition 5.4.1. *For each* $\mathfrak{q} \in \operatorname{Spec} \mathrm{HH}^*(A/\Bbbk)$*, object* $X \in (\varGamma_{\mathfrak{q}}\mathcal{K})^{\mathrm{c}}$*, and* $Y \in L_{\mathcal{Z}(\mathfrak{q})}\mathcal{K}$ *there is a natural isomorphism*

$$\operatorname{Hom}_{\mathrm{HH}^*(A/\Bbbk)}(\operatorname{Hom}^*_{\mathcal{K}}(X, Y), I(\mathfrak{q})) \cong \operatorname{Hom}_{\mathcal{K}}(Y, \Sigma^{-d(\mathfrak{q})} X),$$

where $d(\mathfrak{q}) = \dim \mathrm{HH}^*(A/\Bbbk)/\mathfrak{q}$.

Proof. Apply Proposition 2.4.14. □

The full subcategory of compact object $\mathcal{K}^{\mathrm{c}}$ in $\mathcal{K}$ may be identified with the bounded derived category $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)$ of A-modules (see Section 2.2). We view $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)$ as a $\mathrm{HH}^*(A/\Bbbk)$-linear triangulated category via restriction using this identification.

Corollary 5.4.2. *As* $\mathrm{HH}^*(A/\Bbbk)$*-linear triangulated category,* $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)$ *satisfies local Serre duality.*

Proof. Apply Corollary 2.4.15. □

Now we discuss the existence of Auslander-Reiten triangles. These were introduced by Happel for derived categories of finite dimensional algebras [Hap88]. Let $\mathcal{C}$ be a *Krull-Schmidt category*, that is, each object decomposes into a finite direct sum of objects with local endomorphism rings. An exact triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

in $\mathcal{C}$ is an *AR-triangle* if

(1) any morphism $X \longrightarrow X'$ that is not a split monomorphism factors through α;

(2) any morphism $Z' \longrightarrow Z$ that is not a split epimorphism factors through β;

(3) $\gamma \neq 0$.

We say that $\mathcal{C}$ *has AR-triangles* if for every indecomposable object X there are AR-triangles

$$V \longrightarrow W \longrightarrow X \longrightarrow \Sigma V \quad \text{and} \quad X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X.$$

In [Hap91] Happel established the connection between AR-triangles with the Gorenstein property, while Reiten and Van den Bergh [RVdB02] discovered the connection between AR-triangles and the existence of a Serre functor. The next corollary is the analogue of a result in [RVdB02, Section I.2] for triangulated categories that are Hom-finite over a field.

Corollary 5.4.3. *The full subcategory* $(\varGamma_{\mathfrak{q}}\mathcal{K})^{\mathrm{c}}$ *has AR-triangles for* $\mathfrak{q} \in \operatorname{Spec} \mathrm{HH}^*(A/\Bbbk)$*.*

Proof. Apply [BIKP16, Corollary 7.7]. □

Chapter 6

Conclusion and Outlook

In this thesis we have discussed various actions of Hocshschild cohomology and the Gorenstein property with respect to such actions. But some of our main results require quite strict assumptions to hold. In this chapter, we review those results again and discuss some possible generalizations or some related open problems.

Actions of $\mathrm{HH}^*(A/\Bbbk)$ on $\mathcal{K}(\operatorname{Inj} A)$

The first one concerns actions of Hochschild cohomology. In Section 4.2, we explain that there is an action of $\mathrm{HH}^*(A/\Bbbk)$ on the derived category $\mathcal{D}(A)$ for arbitrary $\Bbbk$-algebra A. When A is finite dimensional, $\mathcal{D}(A)$ may be embedded into $\mathcal{K}(\operatorname{Inj} A)$. The category $\mathcal{K}(\operatorname{Inj} A)$ is an important category in the representation theory of finite dimensional algebra. Thus, it is natural to ask whether the action of $\mathrm{HH}^*(A/\Bbbk)$ on $\mathcal{D}(A)$ can be extended to $\mathcal{K}(\operatorname{Inj} A)$. We gave an affirmative answer for this question for A self-injective in Section 4.3

We attempted to answer the general question for arbitrary finite dimensional algebra A. As usual, we consider the tensor triangulated category $(\mathcal{K}(A^e), \otimes_A, A)$. Now the full subcategory $\mathcal{K}(\operatorname{Inj} A^e)$ is not closed under $\otimes_A$ in general, but there is an equivalence $\mathcal{K}(\operatorname{Inj} A^e) \simeq \mathcal{K}(\operatorname{Proj} A^e)$ and $\mathcal{K}(\operatorname{Proj} A^e)$ is closed under $\otimes_A$. Similarly, there are also equivalence $\mathcal{K}(\operatorname{Inj} A) \simeq \mathcal{K}(\operatorname{Proj} A)$ and the action $\mathcal{K}(A^e) \times \mathcal{K}(A) \xrightarrow{-\otimes_A -} \mathcal{K}(A)$ restrict to a bifunctor $\mathcal{K}(\operatorname{Proj} A^e) \times \mathcal{K}(\operatorname{Proj} A) \xrightarrow{-\otimes_A -} \mathcal{K}(\operatorname{Proj} A)$. We do not call the second bifunctor action because we do not know whether $(\mathcal{K}(\operatorname{Proj} A^e), \otimes_A)$ has a tensor unit. This leads us to the following questions:

- Q1. Is there an idempotent U in $\mathcal{K}(A^e)$ such that $U \cdot \mathcal{K}(A^e) \cdot U = \mathcal{K}(\operatorname{Proj} A)$ and $U \cdot \mathcal{K}(A) = \mathcal{K}(\operatorname{Proj} A)$?
- Q2. Compute the graded endomorphism ring $\operatorname{End}^*_{\mathcal{K}(\operatorname{Proj} A^e)}(U)$.

One consequence of an affirmative answer to the first question is that U will be a tensor unit of $\mathcal{K}(\operatorname{Proj} A^e)$. A work of Jørgensen give us a natural candidate for the idempotent U. Let $B = A^e$. We consider a pair of functors

$$\mathcal{K}(\operatorname{Proj} B) \underset{\operatorname{Hom}_{B^{\mathrm{op}}}(-,B)}{\overset{\operatorname{Hom}_B(-,B)}{\rightleftarrows}} \mathcal{K}(\operatorname{Proj} B^{\mathrm{op}}).$$

For simplicity, we denote both functors by $(-)^*$. We take as U the complex $(\mathbf{p}A^*)^*$ in $\mathcal{K}(\operatorname{Proj} B)$, where $\mathbf{p}A^*$ is the projective resolution of A^* in $\mathcal{K}(\operatorname{Proj} B^{\mathrm{op}})$. There is a natural map $A \longrightarrow U$ given by the composition

$$A \longrightarrow A^{**} \longrightarrow (\mathbf{p}A^*)^* = U.$$

We are able to prove that U satisfies the requirements of Q1 under an assumption that $\operatorname{Hom}_{\Bbbk}(A_A, A_A)$ and $\operatorname{Hom}_{\Bbbk}({}_AA, {}_AA)$ are projective as A^e-module. Note that when A is self-injective, then this assumption is satisfied and $A \longrightarrow U$ is just an injective resolution of A. We do not know whether this is true without that assumption. We also do not know the answer of Q2 for this choice of U.

The Gorenstein property of $\mathcal{K}(\operatorname{Inj} A)$

Let A be a finite dimensional cocommutative Hopf algebra. In [BIKP16], it is showed that $\mathcal{K}(\operatorname{Inj} A)$ is Gorenstein with respect to the canonical action of $\mathrm{H}^*(A, \Bbbk)$. One of our main result that we proved in Section 5.3 is that $\mathcal{K}(\operatorname{Inj} A)$ is also Gorenstein with respect to an action of $\mathrm{HH}^*(A/\Bbbk)$, but we require an additional assumption that A is symmetric, namely $DA = \operatorname{Hom}_{\Bbbk}(A, \Bbbk)$ is isomorphic to A as A^e-modules. We need this assumption to ensure that the global Serre functor $\nu\colon \mathcal{K}^{\mathrm{c}}(\operatorname{Inj} A) \longrightarrow \mathcal{K}^{\mathrm{c}}(\operatorname{Inj} A)$ induced by the Nakayama functor $DA \otimes_A -$ is $\mathrm{HH}^*(A/\Bbbk)$-linear. The $\mathrm{HH}^*(A/\Bbbk)$-linearity is needed to obtain local Serre duality for the full subcategory $\mathcal{K}^{\mathrm{c}}(\operatorname{Inj} A) \simeq \mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)$ of compact objects. Thus, our question is:

Q3. Is the functor $DA \otimes_A -\colon \mathcal{K}^{\mathrm{c}}(\operatorname{Inj} A) \longrightarrow \mathcal{K}^{\mathrm{c}}(\operatorname{Inj} A)$ a $\mathrm{HH}^*(A/\Bbbk)$-linear functor for arbitrary finite dimensional cocommutative algebra A?

The obstruction to our attempt of proving the Q3 is that $\otimes_A$ is in general not symmetric. For example, we do not know whether there is an isomorphism $DA \otimes_A - \cong - \otimes_A DA$ of functors $\mathcal{K}(\operatorname{Inj} A) \longrightarrow \mathcal{K}(\operatorname{Inj} A)$. In fact, such a natural isomorphism is suffice to give an affirmative answer to Q3 using a standard argument below: If $DA \otimes_A - \cong - \otimes_A DA$, then the diagram

$$\begin{array}{ccc}
\mathcal{K}(\operatorname{Inj} A^e) & \xrightarrow{-\otimes_A M} & \mathcal{K}(\operatorname{Inj} A) \\
& {\scriptstyle -\otimes_A (DA \otimes_A M)} \searrow & \downarrow {\scriptstyle DA \otimes_A -} \\
& & \mathcal{K}(\operatorname{Inj} A)
\end{array}$$

is commutative for any $M \in \mathcal{K}(\operatorname{Inj} A)$. By evaluation at $\mathbf{i}A$, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{HH}^*(A/\Bbbk) & \xrightarrow{-\otimes_A M} & \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A)}(M) \\ & {\scriptstyle -\otimes_A (DA \otimes_A M)} \searrow & \downarrow {\scriptstyle DA\otimes_A -} \\ & & \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A)}(M) \end{array}$$

for all $M \in \mathcal{K}(\operatorname{Inj} A)$. In other words, $DA \otimes_A -$ is a $\mathrm{HH}^*(A/\Bbbk)$-linear functor.

Transfer of the Gorenstein property

Our method to prove that $\mathcal{K}(\operatorname{Inj} A)$ is Gorenstein as $\mathrm{HH}^*(A/\Bbbk)$-linear category is easily generalized to the following situation. Suppose that R, S are graded commutative noetherian ring acting on a compactly generated triangulated category $\mathcal{K}$ such that $\mathcal{K}$ is noetherian as R-linear and S-linear triangulated category Suppose also that there is a finite map $\alpha\colon R \longrightarrow S$ such that the action of R on $\mathcal{K}$ factors through the action of S, namely we have a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & \operatorname{End}^*_{\mathcal{K}}(M) \\ \downarrow {\scriptstyle \alpha} & & \Vert \\ S & \longrightarrow & \operatorname{End}^*_{\mathcal{K}}(M) \end{array}$$

for all $M \in \mathcal{K}$, where the horizontal arrows are the actions of R and S respectively. If $\mathcal{K}$ is Gorenstein as R-linear category, then $\mathcal{K}$ is also Gorenstein as S-linear category. Our question is whether the other direction is also true:

Q4. Let R, S, and $\mathcal{K}$ be as above. Is it true that if $\mathcal{K}$ is Gorenstein with respect to an action of S, then $\mathcal{K}$ is Gorenstein with respect to an action of R?

Support theory for $\mathcal{K}(\operatorname{Inj} A)$

This is one aspect that we have not touched at all. Let A be a finite dimensional cocommutative Hopf algebra. Thus $\mathrm{HH}^*(A/\Bbbk)$ is finitely generated and its action on $\mathcal{K}(\operatorname{Inj} A)$ gives us local cohomology functor and support on $\mathcal{K}(\operatorname{Inj} A)$ using Benson-Iyengar-Krause machinery. For instance, we have for each prime $\mathfrak{q} \in \operatorname{Spec} \mathrm{HH}^*(A/\Bbbk)$ a local cohomology functor $\varGamma_{\mathfrak{q}}\colon \mathcal{K}(\operatorname{Inj} A) \longrightarrow \mathcal{K}(\operatorname{Inj} A)$. The essential image of this functor, namely $\varGamma_{\mathfrak{q}}\mathcal{K}(\operatorname{Inj} A)$ is a localizing subcategory of $\mathcal{K}(\operatorname{Inj} A)$. We remind the reader that this means that $\varGamma_{\mathfrak{q}}\mathcal{K}(\operatorname{Inj} A)$ is a full triangulated categories of $\mathcal{K}(\operatorname{Inj} A)$ closed under arbitrary set-indexed direct sums.

In some situation, $\Gamma_{\mathfrak{q}}\mathcal{K}(\operatorname{Inj} A)$ is a 'minimal' compactly generated triangulated category, in the sense that any full localizing subcategory of $\Gamma_{\mathfrak{q}}\mathcal{K}(\operatorname{Inj} A)$ is either 0 or itself. This happens for example when $A = \Bbbk G$ for a finite p-group G. Thus, we ask the following problems.

Q5. Is it true that $\Gamma_{\mathfrak{q}}\mathcal{K}(\operatorname{Inj} A)$ is minimal for all $q \in \operatorname{Spec} \operatorname{HH}^*(A/\Bbbk)$. If not, for which A is this true?

Minimal compactly generated triangulated categories are important, because they are the 'building blocks', in some sense, of compactly generated triangulated categories.

Another question is the following.

Q6. 'Characterize' $\Gamma_{\mathfrak{q}}\mathcal{K}(\operatorname{Inj} A)$ among all localizing subcategory of $\mathcal{K}(\operatorname{Inj} A)$.

We explain what we might expect using an analogy. Consider the canonical action of $\operatorname{H}^*(A, \Bbbk)$ on $\mathcal{K}(\operatorname{Inj} A)$. Thus, for each prime $\mathfrak{p} \in \operatorname{Spec} \operatorname{H}^*(A, \Bbbk)$, we have localizaing subcategory $\Gamma_{\mathfrak{p}}\mathcal{K}(\operatorname{Inj} A)$. It turns out that this subcategory is not just localizing, but also a 'tensor ideal', namely if X belongs to $\Gamma_{\mathfrak{p}}\mathcal{K}(\operatorname{Inj} A)$, then so does $X \otimes Y$ (and $Y \otimes X$). In fact, the subcategory of this form is precisely the minimal tensor ideal of $\Gamma_{\mathfrak{p}}\mathcal{K}(\operatorname{Inj} A)$. On the other hand, for $\mathfrak{q} \in \operatorname{Spec} \operatorname{HH}^*(A/\Bbbk)$, the subcategory $\Gamma_{\mathfrak{q}}\mathcal{K}(\operatorname{Inj} A)$ might not be a tensor ideal in general.

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