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Pairwise Stable Networks in Homogeneous Societies

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Abstract

We study general properties of pairwise stable networks in homogeneous societies, i.e. when agents' utilities differ only with respect to their network position while their names do not matter. Rather than assuming a particular functional form of utility, we impose general link externality conditions on utility such as ordinal convexity and ordinal strategic complements. Depending on these rather weak notions of link externalities, we show that pairwise stable networks of various structure exist. For stronger versions of the convexity and strategic complements conditions, we are even able to characterize all pairwise stable networks: they are *nested split graphs* (NSG). We illustrate these results with many examples from the literature, including utility funtions that arise from games with strategic complements played on the network and utility funtions that depend on centrality measures such as Bonacich centrality.

Keywords: Network Formation, Noncooperative Games, Convexity, Strategic Complements
JEL-Classification: A14, C72, D85

1. Introduction

Starting with the seminal contribution of Jackson and Wolinsky (1996), a substantial literature has evolved modeling strategic network formation. Economic agents in these models have a preference ordering over the set of networks. Examples include firms' profit when forming R&D networks (Goyal and Joshi, 2003), countries' social welfare when forming trade agreements (Goyal and Joshi, 2006a), or agents' payoff from bargaining on a network (Gauer and Hellmann, 2017). Since the structure of interaction, i.e. the social network, affects economic outcomes, such as profits of firms, countries' social welfare, or bargaining outcomes, it is interesting to economists which kind of interaction structures emerge when links are formed strategically. The seminal concept of such equilibrium outcomes is the notion of *pairwise stability* (Jackson and Wolinsky, 1996). A central question is then under which conditions stable networks exist and which structure they have.

In this paper, we approach this question from a very general point. Rather than assuming

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a particular functional form of utility, we simply look at settings where each agent's utility depends only on her network position but not on her name. In other words, the utility function from the network is as general as possible with the restriction that all players are ex-ante homogeneous. We then show that ordinal link externality conditions on the utility function are sufficient for the existence of stable networks of particular architecture. These ordinal link externality conditions define solely the impact that new links have on incentives to form own links. In particular, we impose ordinal convexity, which is a single crossing property of marginal utility in own links, and ordinal strategic complements, i.e. a single crossing property of marginal utility in other agents' links.

We show that if one of these link externalities on marginal utility is positive then pairwise stable networks of certain structure exist. Which class of networks arise as stable depends on which externality property is satisfied. If strategic complements are satisfied, then a regularity assumpion bounding the externalities from own links guarantees that either the empty network or the complete network are always pairwise stable (Theorem 1). If on the other hand convexity is satisfied together with a regularity assumption, then there exists a dominant group network[1] which is pairwise stable (Theorem 3).

While these link externality properties guarantee existence, they are not sufficient to characterize classes of networks which contain all pairwise stable networks. To achieve that, we impose stronger assumptions expressing a general desire to be connected to prominent nodes which is commonly observed in network formation models (Goyal and Joshi, 2006b). We show that with these preference for prominence notions, all pairwise stable networks are contained in the class of nested split graphs (Theorem 2). Nested split graphs (Cvetković and Rowlinson, 1990) are networks where the set of neighbors of any two players can be ordered according to the set inclusion ordering. While the assumptions required for this characterization result are arguably stronger than the ordinal positive link externality conditions, we show that in many general environments the reverse is also true such that these assumptions are naturally satisfied when strategic complements and convexity are given (Proposition 1). As the society becomes more and more homogeneous, such that for linking decisions the network position of others do not matter, then the pairwise stable networks are only found in subclasses of the nested split graphs, the dominant group networks (Theorem 4).

We illustrate our general results with respect to several important applications. Among those is a model of network formation such that the utility of players is given by their Bonacich centrality (Bonacich, 1987). Such a utility function arises e.g. when individuals form costly links in the first stage and then engage in a second stage game of strategic complementaries between neighbors in the network. Indeed, Ballester et al. (2006) show that the unique pure strategy equilibrium of the second stage in such a game is determined by the Bonacich centrality. This measure of centrality counts the number of paths emanating from a given node which are discounted by the length of each path with a common discount factor. Utility functions given by Bonacich centrality give rise to the positive link externalities (Proposition 2) and, even more interestingly, for small discount factors, also our strong preference for prominence notions are satisfied (Proposition 3). Hence applying our general results to utility given by Bonacich centrality, we can conclude that either the empty network or the complete network are necessarily pairwise stable (for any discount factor), while all

[1]Dominant group networks are such that there is a completely connected subset of players while the remaining players have no links, see also Definition 8.

pairwise stable networks are of nested split structure if the discount factor is small enough. If, rather than sequentially, choice of links and efforts are made simultaneously, then, by adopting the framework of Hiller (2017), we show that a general class of games satisfy our positive link externalities (Proposition 4).

General properties of stable networks are of high interest for several reasons. Our results may help characterize stable networks for future (maybe very complex) models of network formation, and they provide reasoning why certain stability structures emerge in existing models of network formation: the driving force are the link externality conditions. That our results are applicable to so many settings is due to the generality of our approach and the fact that the assumption of a homogeneous society is not restrictive as almost all models of strategic network formation share this property (cf. e.g. several surveys and textbooks including Jackson, 2003, 2006; Goyal, 2005, 2007; Vega-Redondo, 2007; Jackson, 2008; Easly and Kleinberg, 2010; Hellmann and Staudigl, 2014).

Although the literature on strategic network formation is enormous, only few results concerning these general structural properties can be found. Exceptions are Jackson and Watts (2001) and Chakrabarti and Gilles (2007) who use the restrictive assumption of a potential function (Monderer and Shapley, 1996) to prove existence of stable networks, and Hellmann (2013a) who – similar to our approach – uses link externality conditions to show existence and uniqueness of stable networks. A recent paper, Bich and Morhaim (2017) shows existence of weighted pairwise stable networks using the ideas from Nash for the mixed extension for games. In light of their general approach, all these papers, however, are not able to show existence of pairwise stable networks of certain structure. We fill this gap with the help of the link externality conditions in a homogeneous society.

Assuming more structure on the functional form of utility, Goyal and Joshi (2006b) are also able to show existence of particular stable network structures such as regular networks, dominant group structures, and exclusive group structures[2] depending on *cardinal* link externalities.[3] They, however, assume a specific form of utility depending only on a particular network statistic, the vector of agents' degrees. We show that some of their results can be generalized such that they hold for arbitrary utility functions in a homogeneous society, such that the ordinal versions of the link externality conditions are sufficient, and such that some of their sufficient conditions are not required.[4] Thereby, our results are applicable to many examples of utility which are not captured in the framework of Goyal and Joshi (2006b), Jackson and Watts (2001) and Chakrabarti and Gilles (2007). In these examples, our results contribute substantially more than the more general setup in Hellmann (2013a). Among those is the afore mentioned utility function given by Bonacich centrality.

The rest of the paper is organized as follows. Section 2 defines the model and presents the important assumptions and definitions used throughout the paper. Sections 3 and 4 present the results ordered by the externalities that are respectively assumed. Section 5 concludes.

[2]Regular networks are such that all nodes have the same number of neighbors (degree), while we refer the reader to Goyal and Joshi (2006b) for a definition of exclusive group structures.

[3]Throughout this paper, we only need ordinal notions of link externalities defined via a single crossing property of marginal utility. Goyal and Joshi (2006b), instead use the stronger, cardinal versions.

[4]Note, however, that Goyal and Joshi (2006b) use a stronger stability condition, namely pairwise Nash stability. While their existence results are therefore stronger, our characterization results are stonger.

2. The model

Let $N = \{1, 2, ..., n\}$ be a finite set of agents with $n \geq 3$. Depending on the application these can be firms, countries, individuals, etc. These economic agents strategically form links and, thus, are henceforth called players. Throughout this paper we will assume network formation to be undirected. A connection or link between two players $i \in N$ and $j \in N$, $i \neq j$ will be denoted by $\{i, j\}$ which we abbreviate for simplicity by $ij = ji := \{i, j\}$. We then define the complete network by $g^N = \{ij \mid i, j \in N, i \neq j\}$ and we define the set of all networks by $G = \{g \mid g \subseteq g^N\}$.

We will further denote the set of links of some player i in a network g by $L_i(g) = \{ij \in g \mid j \in N\}$, and all other links $L_{-i}(g) := g - L_i(g)$, where $g - g' := g \setminus g'$ denotes the network obtained by deleting the set of links $g' \cap g$ from network g. Analogously, $g + g' := g \cup g'$. The set of player i's neighbors is given by $N_i(g) = \{j \in N \mid ij \in g\}$ and $\eta_i(g) = \#N_i(g)$ is called the degree of player i.

Players have preferences over networks. The profile of utility functions is denoted by $u(g) = (u_1(g), u_2(g), ..., u_n(g))$, where u_i is a mapping from G to $\mathbb{R}$ for all $i \in N$. The decision of adding or deleting links is based on the marginal utility of each link. We denote the marginal utility of deleting a set of links $l \subseteq g$ from g as $\Delta u_i(g, l) := u_i(g) - u_i(g - l)$, and similarly the marginal utility of adding a set of links $l \subseteq g^N - g$ to g as $\Delta u_i(g+l, l) = u_i(g+l) - u_i(g)$. Observe that in this definition, $u_i(g)$ may include any kind of disutilities arising in network g such as costs of link formation. In many examples from the literature linear costs of link formation are assumed, such that the utility function has the form $u_i(g) = v(g) - c\eta_i(g)$, where $c > 0$ is some constant. Altogether, we will call $\mathbb{G} = (N, G, u)$ a society.

2.1. Network Formation and Stability

The study of equilibrium/stability of networks has been a subject of interest in many models of network formation. Depending on the rules of network formation which are assumed in a given model, there are many definitions of equilibrium at hand. Here, we present only the well-known concept of pairwise stability introduced by Jackson and Wolinsky (1996).[5]

Definition 1 ***(Pairwise Stability)***:
A network g in a society $\mathbb{G} = (N, G, u)$ is pairwise stable (PS) if

$$\begin{array}{lll} (i) & \forall ij \in g: & \Delta u_i(g, ij) \geq 0 \;\; \textit{and} \;\; \Delta u_j(g, ij) \geq 0; \\ (ii) & \forall ij \notin g: & \Delta u_i(g+ij, ij) > 0 \;\Rightarrow\; \Delta u_j(g+ij, ij) < 0. \end{array}$$

This approach to stability defines desired properties directly on the set of networks. The implicit assumption of network formation underlying this approach is that players are in control of their links; any player can unilaterally delete a given link, but to form a link both involved players need to agree. The networks which satisfy property (i) of Definition 1 are called *link deletion proof* and the networks which satisfy (ii) are called *link addition proof.*

[5] A game theoretic foundation and a comparison of the several definitions of stability can be found in Bloch and Jackson (2006).

The intuition behind the definition of pairwise stability is that two players form a link if one is strictly better off and the other is not worse off when forming the link, while a link is deleted if one of the two involved players is strictly better off deleting the link. It should be noted that this definition of stability is rather a necessary condition of stability as it is fairly weak. It can be refined to account for multiple link deletion, called Pairwise Nash stability (Bloch and Jackson, 2006), to account for network formation with transfers, called Pairwise stability with transfers (Bloch and Jackson, 2007), and many more (see e.g. Jackson, 2008; Hellmann and Staudigl, 2014, for a further discussion on different approaches to stability).[6]

2.2. Homogeneity

The central assumption underlying this paper is homogeneity of the society. That is we assume all players to be ex-ante equal in order to assure that differences in utility solely depend on players' respective network positions but not on their name.We will establish this with the following anonymity condition on the utility profile.

Definition 2 ***(Anonymity)***:
Let $g_\pi := \{\pi(i)\pi(j) \mid ij \in g\}$ *be the network obtained from a network* g *by some permutation of players* $\pi \colon N \to N$. *A profile of utility functions is anonymous if*

$$u_i(g) = u_{\pi(i)}(g_\pi). \tag{2.1}$$

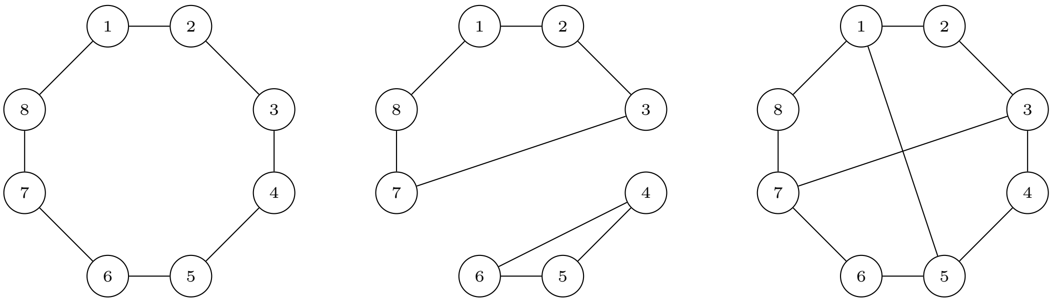
A society $\mathbb{G}$ with a profile of utility functions satisfying anonymity will be called homogeneous. As noted above, players in a homogeneous society are anonymous in the sense that players in symmetric network positions receive the same utility. The notion of symmetric position in a network, implied by Definition 2, is such that two players $i, j \in N$, $i \neq j$ are *symmetric* in a network $g \in G$ if there exists a permutation of the set of players $\pi : N \to N$ such that $\pi(i) = j$ and $g_\pi = g$. This is most trivially satisfied if two players $i, j \in N$, $i \neq j$ share the same neighbors (disregarding a possible common link), i.e. $N_i(L_{-j}(g)) = N_j(L_{-i}(g))$. On the other hand, having the same degree is a necessary condition for two players to be in a symmetric position.

Consequently, a network $g \in G$ is called a *symmetric network* if all players are in a symmetric position.[7] Hence, a necessary condition for g to be symmetric is that it is regular, i.e. that all players have the same degree. However, this condition is not sufficient (see Figure 1). Some examples of symmetric positions in a network and symmetric networks are given in Figure 1.

Moreover, with the notion of homogeneous society, it is easy to see that *symmetric links*

[6]Some results presented here generalize to the stronger concept of pairwise Nash stability, also known as pairwise equilibria. Pairwise Nash stable networks are immune against deletion of any subsets of own links. Specifically, it is known that ordinal concavity of the utility function (see Definition 3) implies that all pairwise stable networks are also pairwise Nash stable (Calvó-Armengol and Ilkiliç, 2009; Hellmann, 2013a). Any result in this paper that does not require convexity, hence, also holds for pairwise Nash stability under the additional assumption of concavity. Further, the results of this paper which hold for all pairwise stable networks, trivially also extend to pairwise Nash stability.

[7]The graph theoretic equivalent to symmetric networks we consider here are not symmetric, but vertex-transitive graphs. In this setup, we need symmetry of the players, that is symmetry of vertices whereas symmetry in graph theory would also demand edges to be symmetric. For details see e.g. Biggs (1994).



(a) Symmetric network (b) Non-symmetric network (c) Non-symmetric network

Figure 1: Networks (a) and (b) are regular, but only (a) is symmetric. In network (b), two players of different components are not in symmetric positions. In network (c), players 1, 3, 5, and 7, respectively players $2, 4, 6$, and 8 are in symmetric positions, while the network is obviously not.

provide the same marginal utility. For this, however, a symmetry on links has to be imposed. To simplify things, note that for two players whose neighborhood coincide (disregarding a mutual connection), links from both players to any third player are symmetric which implies (ii) and (iii) of Lemma 1.

Lemma 1.
Let some profile of utility functions u satisfy anonymity. Then the following statements are true:

$$(i) \quad u_i(g) = u_j(g), \text{ if } i \text{ and } j \text{ are symmetric},$$

$$(ii) \quad \Delta u_i(g+ik, ik) = \Delta u_j(g+jk, jk) \quad \forall k \in N \backslash N_i(g), \quad \text{if } N_i(L_{-j}(g)) = N_j(L_{-i}(g)),$$

$$(iii) \quad \Delta u_k(g+ik, ik) = \Delta u_k(g+jk, jk) \quad \forall k \in N \backslash N_i(g), \quad \text{if } N_i(L_{-j}(g)) = N_j(L_{-i}(g)).$$

The proof of Lemma 1 as well as all following results can be found in the appendix. To illustrate Lemma 1, note that Figure 1(a) is a symmetric network. Hence by (i), all players $i \in N = \{1, 2, \dots, 8\}$ receive the same utility. However, for no two players $i, j \in N$ in this network we have $N_i(L_{-j}(g)) = N_j(L_{-i}(g))$. This means that (ii) and (iii) of Lemma 1 do not apply here. Indeed, incentives to create links may differ considerably. If, for instance, utility is distance based (consider e.g. the symmetric connections model in Jackson and Wolinsky, 1996), then for Player 1 the link to Player 5 might be a lot more desirable then the link to Player 3. Instead, consider the non-symmetric network Figure 1(b). There, players 4, 5, and 6, are not only in symmetric positions and, thereby, receive the same utility by Lemma 1(i), but they also share the same neighbors, i.e. $N_i(L_{-j}(g)) = N_j(L_{-i}(g))$ holds for $i, j \in \{4, 5, 6\}$, $i \neq j$. Thus, by Lemma 1(ii), they receive the same marginal utility from the connection to any player from the other component $k \in N \setminus \{4, 5, 6\}$. Vice versa, all other players $k \in N \setminus \{4, 5, 6\}$ have equal incentives to connect to players $i, j \in \{4, 5, 6\}$ by Lemma 1(iii). Note that both of these properties do not depend on which functional form of utility we apply, but only on the anonymity condition. As an example, consider again the symmetric connections model where we have $\Delta u_i(g + ik, ik) = \delta + 2\delta^2 + 2\delta^3 - c$ and $\Delta u_k(g + ik, ik) = \delta + 2\delta^2 - c$ for all $i \in \{4, 5, 6\}$ and $k \in N \setminus \{4, 5, 6\}$ for g as in Figure 1b.

2.3. Non-Existence of Pairwise Stable Networks in Homogeneous Societies

Even if the society is homogeneous, pairwise stable networks may fail to exist. The following example proves this.

Example 1. Let $N = 5$ and utility for all $i \in N$ be such that

$$u_i(g) = \begin{cases} \eta_i(g) & \text{if } g \in \{\{ij\}, \{ij, ik\}, \{ij, jk\}, \{ij, kl, lm\}\} \text{ for } i \neq j \neq k \neq l \neq m, \\ -\eta_i(g) & \text{else} \end{cases}$$

It is straightforward to see that this utility function is well defined and satisfies anonymity since names of players do not matter. To see that there does not exist a pairwise stable network, consider Figure 2 where up to any permutation of players the only networks where a subset of players receives positive utility are shown. First, the empty network is not pairwise stable since any two players have an incentive to form a link receiving utility of 1 each. Thus, there exists an improvement path from the empty network to e.g. g_1. None of the networks which yield positive utility for a subset of players is pairwise stable either, since an improvement cycle as the one displayed in Figure 2 exists, where the arrows indicate an improvement, i.e. two players adding a link which is mutually beneficial or a player deleting a link which yields negative marginal utility. In any network not of the form displayed in Figure 2 (up to permutation), players receive $u_i(g) = -\eta_i(g)$ and hence have incentives to delete links until one of the networks in the cycle is reached. Hence there does not exist a pairwise stable network although the utility function is anonymous.[8]

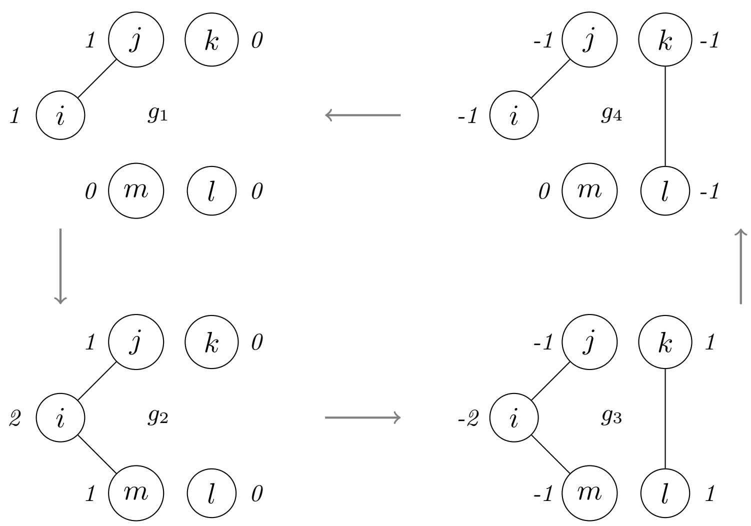


Figure 2: The only possible network structures which may yield positive utility for some players (up to permutation) in Example 1 form a cycle with improvement paths indicated by gray arrows.

2.4. Link externalities

Additionally to existence problems, it is impossible to say anything about stability of particular network structures without any assumptions on the utility function. In the literature

[8]It is easy to check that at least 5 players are necessary to get non existence of a pairwise stable network in a homogeneous society.

on network formation, however, many utility functions admit certain link externality conditions. By link externalities we mean conditions on how marginal utility is affected when links are added to or deleted from a network. Hence, without losing much of the generality of our approach, we will examine whether stable networks of certain structure exist if various combinations of link externalities in the context of homogeneous societies are satisfied. We will consider the weakest version of link externalities in the literature, namely the ordinal versions presented in Hellmann (2013a).[9] For the sake of convenience, in the rest of the paper we will speak about convexity, concavity, strategic complements and strategic substitutes, keeping in mind that what is used are the respective ordinal formulations of Definition 3.

Definition 3 ***(Ordinal link externalities)***:
A utility function u_i satisfies ordinal convexity (concavity) *in own links if for all $g \in G$, $l_i \subseteq L_i(g^N - g)$ and $ij \notin g + l_i$ it holds that*

$$\Delta u_i(g + ij, ij) \geq 0 \;\Rightarrow\; (\Leftarrow)\; \Delta u_i(g + l_i + ij, ij) \geq 0. \tag{2.2}$$

A utility function u_i satisfies ordinal strategic complements (substitutes) *if for all $g \in G$, $l_{-i} \subseteq L_{-i}(g^N - g)$ and $ij \in L_i(g^N - g)$ it holds that*

$$\Delta u_i(g + ij, ij) \geq 0 \;\Rightarrow\; (\Leftarrow)\; \Delta u_i(g + l_{-i} + ij, ij) \geq 0. \tag{2.3}$$

The ordinal versions of link externality conditions are given by single crossing properties of marginal utility with respect to additional own or other players links. If the utility function of a player is such that once a given link yields positive marginal utility, the marginal utility of this link always stays positive when this player adds some other links, then convexity is satisfied. In this case we also speak of positive externalities from own links. If we replace positive by negative, the utility function satisfies concavity. In the same sense, strategic complements capture positive externalities of other players links, while strategic substitutes capture negative externalities of other players' links for the incentive to form own links.

3. Strategic Complements

To gain some insights into the structure of pairwise stable networks, we assume in this section that the profile of utility functions satisfies the ordinal notion of strategic complements. Hence, links become more valuable when links between *other players* are added. When, moreover, externalities from *own* links can be bounded from below such that some weak link monotonicity conditions are satisfied, existence of a pairwise stable network is guarenteed and some additional structural properties concerning the empty and the complete network are implied (Section 3.1). Using more restrictive versions of these link externalities such that for marginal utility only link positions in terms of degree matter, we are able to show that all pairwise stable networks are contained in the set of nested split graphs (Section 3.2). We also provide applications for our results in Section 3.3.

[9]Ordinal link externalities as first defined by Hellmann (2013a) are implied by the more commonly used but stronger cardinal link externalities (see e.g. Bloch and Jackson, 2006, 2007; Goyal and Joshi, 2006b), as well as, by several related concepts such as α-submodularity (Calvó-Armengol and Ilkiliç, 2009).

3.1. Link Monotonicity

When the incentives to form links are increasing in both own and other players' links, then the utility function is supermodular. However, we cannot use results from game theory, as the structure of pairwise stable networks is different from Nash equilibria.[10] We show here that we do not need to assume positive link externalities from both own and other players' links to arrive at an analogous result. Instead, these are relaxed in two ways: first, strategic complements only need to hold in ordinal terms, and second, externalities from own links may not even satisfy the single crossing property, but instead shall not be "too negative". To account for the latter, we introduce a quite general link monotonicity condition.

Definition 4 ***(κ-Link Monotonicity)***:
*A utility function u_i satisfies κ-*link monotonicity *if for all $g \in G$, for all $j, k \in N \setminus \{i\}$ with $ij, ik \notin g$, all $l_{-i} \in L_{-i}(g)$ with $|l_{-i}| = \kappa$:*

$$\Delta u_i(g + ij, ij) < (\leq)\ 0 \quad \Rightarrow \quad \Delta u_i(g + ik + l_{-i} + ij, ij) < (\leq)\ 0, \tag{3.1}$$

If some player i's utility function satisfies κ-link monotonicity and if i has an incentive to add some link to a network, then i still wants to add the link even after i added some other link and other players added exactly κ links. In the context of strategic complements which imply that i keeps the desire to add a link after other players added links anyway, Definition 4 puts an additional restriction on the externalities from own links which cannot be too negative as to dominate the (positive) externalities from κ links of other players. Hence, in the case of strategic complements, κ-link monotocity implies κ'-link monotonicity for all $\kappa' > \kappa$. Thus, the larger κ, the weaker is the restriction on the externalities from own links in the presence of strategic complements. In particular, κ link monotonicity is weaker than the convexity assumption which (together with strategic complements) requires κ-Link Monotonicity to hold for all $\kappa \geq 0$.

In the following we derive boudaries on κ that guarantee stability of the empty or the complete network.

Theorem 1.
Suppose the profile of utility functions u satisfies strategic complements, anonymity and $\hat{\kappa}(n)$-link monotonicity for $\hat{\kappa}(n) := \lfloor n/2 \rfloor - 1$. If the empty network is not pairwise stable, then the complete network is pairwise stable, and vice versa.
If, more restrictively, u satisfies $\bar{\kappa}(n)$-link monotonicity for $\bar{\kappa}(n) := \left\lfloor \sqrt{2(n-1)(n-2)} \right\rfloor - (n-1)$, then if the empty network is not pairwise stable, the complete network is uniquely pairwise stable, and vice versa.

The result consists of two parts. First, if externalities from a single own link do not dominate the externalities from $\lfloor n/2 \rfloor - 1$ other players' links, then either the empty or the complete network is guaranteed to be pairwise stable. If we restrict the externalities from own links

[10]The non-cooperative game underlying network formation is due to Myerson (1991), where the intentions to form links are announced. Nash equilibria of this game are immune to multiple link deletion and do not consider link addition.

further such that $\bar{\kappa}(n)$-link monotonicity is satisfied, then the structure of pairwise stable networks is very reminiscent of the structure of Nash equilibria in a supermodular game: if multiple networks are pairwise stable, then there always exists a smallest and a largest stable network in the sense of the set inclusion ordering, namely the empty and the complete network. To the contrary, if one of these networks (empty network or complete network) fails to be pairwise stable, then the other network must be uniquely pairwise stable, i.e. the least and maximal stable network coincide. For this result, however, we do not require supermodularity of the utility function, as both strategic complements and convexity have been relaxed.

Let us elaborate on the interpretation of link monotonicity and the odd looking condition on $\kappa(n)$ in combination with strategic complements which basically implies that the externalities from own links cannot be "too negative". Since the larger κ, the weaker is the restriction on the externalities from own links in the presence of strategic complements and $\bar{\kappa}(n), \hat{\kappa}(n) \to \infty$ for $n \to \infty$, the restriction on the externalities from own links gets smaller for larger societies. We can conclude that in large homogeneous societies ($n \to \infty$), (strict cardinal) strategic complements alone are sufficient for the result.

For small n, instead, $\hat{\kappa}(n)$ and $\bar{\kappa}(n)$-link monotonicity become more restrictive. If e.g. $3 \leq n \leq 5$, then 0-link monotonicity is required for the second part of Theorem 1. Note that 0-link monotonicity requires the externalities from own links to satisfy a single crossing property and is, therefore, equivalent to convexity in own links. As a direct consequence of Theorem 1, we therefore get the same result in case of ordinal positive externalities since convexity and strategic complements imply κ-link monotonicity for $\kappa = 0$. No additional restrictions on n are hence required.

Corollary 1.
Suppose the profile of utility functions u satisfies the strategic complements property, convexity in own links and anonymity. If the empty network is not pairwise stable, then the complete network is uniquely pairwise stable, and vice versa.

Thus, in this section we provide strong existence results, as strategic complements together with a regularity assumptions that the externalities from own links cannot be "too negative" imply that the pairwise stable networks have an interesting structure: one out of a set of two networks (empty or complete) network is always pairwise stable, while if there exist multiple stable networks, then both must be pairwise stable.

3.2. Prominence-based Utility Functions

Although it is possible to gain some insights into the structure of pairwise stable networks in a homogeneous society when ordinal link externalities are not too negative, these assumptions are not sufficient to characterize all pairwise stable networks. In particular, it would be interesting to examine which stable structures emerge when the least and maximal stable network do not coincide, such that multiple stable networks exist. However, in the general framework that we imposed so far, there is little hope to say more about the structure of pairwise stable networks without putting stronger assumptions on the utility function.

We therefore focus attention on the relative sizes of the link externalities. By that we mean the following. Consider a player $i \in N$ and a network $g \in G$ and suppose there are two

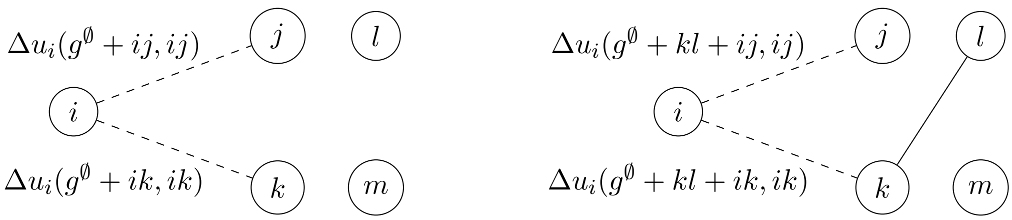


Figure 3: The marginal utilities of the links ij and ik in the networks $g^{\emptyset}$ and $g^{\emptyset}+kl$.

players $k,l \in N$ such that $kl, ik \notin g$ form the link kl (see Figure3). Then by strategic complements, if $\Delta u_i(g+ik,ik) \geq 0$ then $\Delta u_i(g+ik+kl,ik) \geq 0$ and if $\Delta u_i(g+ij,ij) \geq 0$ then $\Delta u_i(g+ij+kl,ij) \geq 0$ for some $j \neq k,l$, $ij \notin g$. However, strategic complements do not specify on which links the effect of other players' links is stronger (and this cannot be captured by the cardinal notion of either). In other words, does the addition of the link kl increase the incentive for player $i \notin \{k,l\}$ more to link to k (resp. l) than to $j \notin \{k,l\}$, or vice versa?

In this setting it would be quite natural to say that the externality of the link kl on the incentive for player i to form a link to k is larger than the externality of the link kl on the incentive for player i to form a link to j. Coupled with the cardinal notion of strategic complements which means that the externalities are both positive, this implies that the addition of the link kl increases the marginal utility of the link ik more than the link ij.

First, we only apply this idea to players $k,l \in N$ who are in completely symmetric positions such that $N_j(L_{-k}(g)) = N_k(L_{-j}(g))$. Thus, for any $i \in N$ by Lemma 1, $\Delta u_i(g+ij,ij) = \Delta u_i(g+ik,ik)$ and hence by above reasoning, $\Delta u_i(g'+ij,ij) \leq \Delta u_i(g'+ik,ik)$ if $N_j(g'_{-k}) \subseteq N_k(g'_{-j})$. Using only the ordinal version we receive the property of Weak Preference for Prominence in Definition 5. The stronger notion of Strong Preference for Prominence in Definition 5, goes beyond that by applying the logic also to players with different degrees.

Definition 5 ***(Weak and Strong Preference for Prominence)***:
A utility function u_i satisfies weak preference for prominence (WPP) *if for all $g \in G$, whenever there exist $j,k \in N \backslash N_i(g)$ such that $N_j(L_{-k}(g)) \subseteq N_k(L_{-j}(g))$ it holds that*

$$\Delta u_i(g+ij,ij) \geq (>)0 \quad \Rightarrow \quad \Delta u_i(g+ik,ik) \geq (>)0, \tag{3.2}$$

A utility profile u_i satisfies strong preference for prominence (SPP) *if for all $g \in G$, $\eta(g) \in \{0,..,n-1\}^n$ such that $\eta_j(g) \leq \eta_k(g)$ it holds that*

$$\Delta u_i(g,ij) \geq 0 \;\Rightarrow\; \Delta u_i(g+ik,ik) > 0. \tag{3.3}$$

The notions of weak and strong preference of prominence, as the names suggest, have a quite intuitive interpretation, expressing a preference for nodes with many neighbors. WPP is an extremely weak notion of preference for prominence. It simply requires that if a link to a node is desirable then a link should be also desirable to a more prominent node where the prominence relation is a partial ordering given by the set inclusion ordering. When we consider, instead, a prominence relation such that a node is more prominent if and only if it has more neighbors, we receive a complete ordering on the set of nodes making the notion of preference for prominence more demanding and therefore defined as strong preference for prominence (SPP).

Although the latter notion of SPP seems demanding at first sight, it may be very naturally satisfied in societies where strategic complements are given. We elaborate this in Section 3.3.1 for the framework of Goyal and Joshi (2006b) where SPP is implied by strategic complements as the externalities from other players' links act homogeneoulsy on marginal utility since the Goyal and Joshi utility functions depend on fewer network statistics (see Proposition 1). Hence, assuming SPP instead of strategic complements can also be seen as strengthening the homogeneity assumption when strategic complements are satisfied.

In a similar way as SPP represents strategic complements in a more homogeneous society, we can also consider externalities from own links. We call this stronger notion anonymous convexity.

Definition 6 ***(Anonymous Convexity)***:
A utility profile u satisfies anonymous convexity (AC) *if for all $g \in G$, for all $i, j, k \in N$, and for all $\eta(g) \in \{0, .., n-1\}^n$ such that $\eta_i(g) \leq \eta_j(g)$ it holds that*

$$\Delta u_i(g, ik) \geq 0 \;\Rightarrow\; \Delta u_j(g + jk, jk) \geq 0. \tag{3.4}$$

Similar to above, anonymous convexity (AC) implicitly assumes a higher degree of homogeneity compared to convexity: if a player i likes the connection to k then any player with more links also has an incentive to connect to k. In a more homogeneous society where players with same degree have the same incentives, this formulation reflects the idea of ordinal convexity since once the marginal utility of a link is positive, it stays positive if own links are added. Hence AC translates the convexity notion to other players. We show in Section 3.3.1 that AC is very naturally implied by convexity in homogeneous societies by the example of the Goyal and Joshi utility functions (Proposition 1).

Recall that we aim at characterizing a class of networks which incorporates all pairwise stable networks. The set of networks that we will need is given by the following definition.

Definition 7 ***(Nested Split Graphs)***:
A network $g \in G$ is a nested split graph *(NSG) if for all players $i, j, k \in N$ such that*

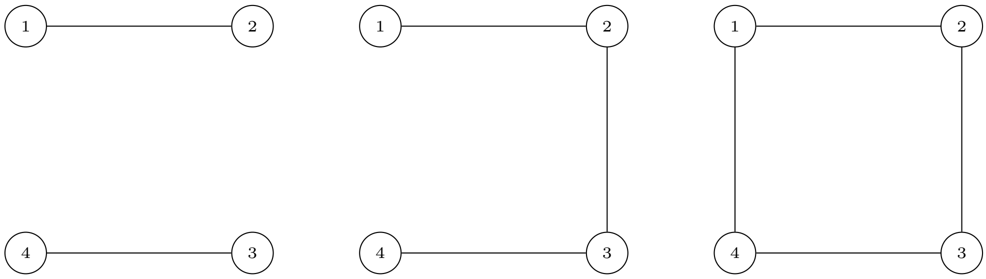
$$\eta_i(g) \geq \eta_j(g) \geq \eta_k(g),$$

we have that if $ik \in g$ then also $ij \in g$ and if $jk \in g$ then also $ik \in g$.

In a nested split graph the neighborhood structure of all players is nested in the sense that for any two players $i, j \in N$ the set of their neighbors can be ordered according to the set inclusion order, i.e. $N_i(L_{-j}(g)) \subseteq N_j(L_{-i}(g))$ or $N_i(L_{-j}(g)) \supseteq N_j(L_{-i}(g))$. Our Definition 7 can be straightforwardly seen to be equivalent to the ones in Cvetković and Rowlinson (1990), Mahadev and Peled (1995), and Simić et al. (2006). In particular, a network is NSG if and only if it does not contain a path (P_4), a cycle (C_4) or two connected pairs ($K_{2,2}$) when restricted to any 4 players (see Figure 4).[11] Moreover, nested split graphs maximize the largest eigenvalue of networks that contain the same number of links.[12]

[11] The subgraph of some nodes $I \subset N$ from network g is the network $g_I \subset g$, such that $g_I = \{ij \mid i, j \in I, ij \in g\}$.

[12] For a further elaboration on nested split graphs see König et al. (2014).



(a) $K_{2,2}$: Two complete components of size 2

(b) P_4: Path of length 4

(c) C_4: Cycle of size 4

Figure 4: A network is a nested split graph if it does not contain a set of four players who form one of the subgraphs $K_{2,2}, P_4$, and C_4.

More importantly for our purposes, the set of nested split graphs contains all pairwise stable networks when the profile of utility functions satisfies SPP and AC.

Theorem 2.
Suppose a profile of utility functions satisfies SPP and AC. Then any pairwise stable network is a nested split graph.

Although the utility function is not specified in our framework, we learn a lot about the structure of pairwise stable networks when SPP and AC are satisfied: any two players' neighborhoods can be ordered with respect to the set inclusion order. This reduces the set of possible candidates for PS networks considerably as the set of NSG's only make up a very small fraction of the set of all possible networks G.

Further, note that for this result anonymity is not explicitly required. Instead a different kind of homogeneity is implicitly captured by the assumptions AC and SPP. These require the externalities from own links (AC) and other players' links (SPP) to act homogeneously across all players on the incentives to form links. However, the anonymity property itself is not necessary for the utility function to satisfy AC and SPP.

3.3. Applications

The assumptions in previous results may seem demanding at first sight, in particular for Theorem 2. In this section we want to show that there exists a lot of models in the literature on network formation that are captured by our approach.

3.3.1. Playing the Field and Local Spillovers

In Goyal and Joshi (2006b), two utility functions with a particular structure –called playing the field and local spillovers– are studied with respect to existence of stable networks. A utility function is of *playing the field* type if benefits can be written as a function $f : \{0, 1, \dots, n-1\} \times \{0, 1, \dots, (n-1)^2\} \to \mathbb{R}$ of own degree $\eta_i(g)$ and the number of links of other players' links $\eta_{-i}(g) := 2|L_{-i}(g)| = \sum_{j \neq i} \eta_j(g) - \eta_i(g)$ net of per unit link formation costs $c \in \mathbb{R}_+$ such that,

$$u_i^{PF}(g) = f(\eta_i(g), \eta_{-i}(g)) - c\eta_i(g). \tag{3.5}$$

A utility function is of *local spillover* type if there exists functions $f_1, f_2, f_3 : \{0, ..., n-1\} \to \mathbb{R}$ such that with these functions applied to own degree, neighbors' degrees and non-neighbors' degrees, respectively, utility can be expressed as the sum of these functions net of costs,

$$u_i^{LS}(g) := f_1(\eta_i(g)) + \sum_{j \in N_i} f_2(\eta_j(g)) + \sum_{k \notin N_i \cup \{i\}} f_3(\eta_k(g)) - c\eta_i(g). \quad (3.6)$$

Both of these utility functions reduce the network to only one characteristic: the vector of degrees $\eta(g) = (\eta_1(g), \ldots, \eta_n(g))$. To establish existence of stable networks, Goyal and Joshi (2006b) additionally assume various combinations of cardinal notions of link externalities.[13] The more restrictive assumptions of SPP and AC are implied by the definitions of strategic complements and convexity in the above utility functions which is formally stated in the following result.

Proposition 1.
If u^{PF} satisfies convexity and strategic complements, then u^{PF} satisfies AC and SPP.
If u^{LS} satisfies convexity and strategic complements, then u^{LS} satisfies AC and SPP.

If strategic complements and convexity are satisfied, Goyal and Joshi (2006b) show that in Playing the Field (3.5) all pairwise Nash stable (PNS)[14] networks are of dominant group architecture[15] and in Local Spillovers (3.6) all PNS networks are of interlinked star architecture.[16] Both types of network structures –even though distinct– belong to the larger set of nested split graphs which is confirmed by Theorem 2 since SPP and AC are satisfied for playing the field and local spillover utility functions if strategic complements and convexity hold by Proposition 1. Thus, although different functional forms of utility may imply very different stable networks we are able to show that in very homogeneous societies, the convexity and strategic complements properties are the driving force for the emergence of nested split graphs which particularly contain dominant group architectures and interlinked stars. Since PNS networks are also PS, our characterization result is more general not only with respect to the functional form of utility, but also with respect to the stability notion used.

A result that either the empty or complete network is always pairwise stable cannot be found in Goyal and Joshi (2006b) although their utility functions and the link externality conditions are far less general. The reason is to be found in stability notions used. If we additionally assume (ordinal) concavity in own links then we get by Bloch and Jackson (2006) that both equilibrium concepts PS and PNS coincide. Applying Bloch and Jackson's result

[13] Note, if u^{PF} is (cardinally) convex, then $f(\eta_i + 1, \eta_{-i}) - f(\eta_i, \eta_{-i})$ is increasing in η_i, while $f(\eta_i + 1, \eta_{-i}) - f(\eta_i, \eta_{-i})$ is increasing in η_{-i} if (cardinal) strategic complements are satisfied. Similarly, if u^{LS} is (cardinally) convex, then $f_1(\eta_i)$ is increasing in η_i, while $f_2(\eta_j + 1) - f_3(\eta_j)$ is increasing in η_j for all $j \neq i$ if strategic complements are satisfied. It is straightforward to see that our assumption of homogeneity is straightforwardly satisfied and our ordinal link externalities are implied by their cardinal notions.

[14] A pairwise Nash stable network is link addition proof such that (ii) of Definiton 1 is satisfied and additionally for all $l_i \in L_i(g)$ we have $\Delta u_i(g, l_i) \geq 0$ for all $i \in N$.

[15] A network g is of dominant group architecture if one group of players is completely connected while the remaining players stay isolated, see Definition 8.

[16] A network g is of interlinked star architecture if there exists $M \subset N$ such that $i \in M$, $i \neq j \to ij \in g$ and $i, j \in N \setminus M$ $i \neq j \to ij \notin g$. In other words, one group of players is completely connected while the remaining players connect have links to all players in the completely connected group but do not connect among themselves.

on equivalence of PNS and PS, we would require (ordinal) concavity to hold in order to state an existence result for PNS as a Corollary of Theorem 1. Hence the externalities from own links need to be negative, but at the same time cannot be too negative to satisfy $\kappa-$link monotonicity for Theorem 1 to hold. These results can hence be seen as complementing and extending Goyal and Joshi's results to PS (instead of PNS) in a more general environment.

3.3.2. Bonacich Centrality

With our general approach we are able to study interesting utility functions which do not fall into the class of playing the field or local spillover games in Goyal and Joshi (2006b). One such example where more than the degree distribution matters for utility is given by the important class of utility functions which depend on players' Bonacich centrality.

Bonacich (1987) introduced a parametric family of centrality measures in order to formulate the intuitive idea that the centrality of a single node in a network should depend on the centrality of its neighbors. This self-referential definition of centrality leads to an eigenvector-based measure, which can be defined as follows: Let $\mathbf{A}(g)$ be the $n \times n$ adjacency matrix of a given network g and $\mathbf{1}$ be the $n \times 1$ vector with all entries equal to 1.[17] Noting that $(\mathbf{A}(g))^k\mathbf{1}$ counts the total number of walks of length k and letting $\delta > 0$ be a given parameter, discounting for walk length and chosen in such a way that $[\mathbf{I}-\delta\mathbf{A}(g)]^{-1}$ exists,[18] the centrality index proposed by Bonacich (1987) is then given by,

$$\mathbf{b}(\delta, g) = \sum_{n=0}^{\infty} \delta^n \mathbf{A}^n \mathbf{1} = [I - \delta\mathbf{A}]^{-1}\mathbf{1}. \tag{3.7}$$

This centrality measure is actually a Nash equilibrium of an interesting class of non-cooperative games: Suppose there are N agents who are involved in a team production problem (for an in-depth introduction of this game, see Ballester et al., 2006). Each player chooses a non-negative quantity $x_i \geq 0$, interpreted as efforts invested in the team production. Efforts are costly, and the level of effort invested by the other players affects the utility of player i. To capture these effects, player i's payoff from an effort profile $\mathbf{x} = (x_i, \mathbf{x}_{-i})$ is given by

$$\pi^{BC}(g, x_i, \mathbf{x}_{-i}) = x_i - \frac{1}{2}x_i^2 + \delta \sum_{j \in N_i} x_i x_j. \tag{3.8}$$

Ballester et al. (2006) show that this game has a unique Nash equilibrium $\mathbf{x}^* = \mathbf{b}(\delta, g)$. Given network g, and discount factor $\delta \in \mathbb{R}$, so that (3.7) is well defined, the equilibrium payoff of player i can be computed as[19]

$$\pi_i^{BC}(g, \mathbf{x}^*) = \frac{1}{2} b_i(\delta, g)^2. \tag{3.9}$$

There are many other examples of games where equilibrium is given by a function of the Bonacich centrality. Among those are models of production economy (Acemoglu et al., 2012),

[17]The adjacency matrix $\mathbf{A}(g)$ of a network g is a matrix with entries $a_{ij}(g) = 1$ if $ij \in g$ and $a_{ij}(g) = 0$ otherwise. Note that $\mathbf{A}$ is necessarily symmetric as we consider undirected network formation.

[18]The necessary condition for this to be the case is that $0 < \delta < \lambda_1(\mathbf{A})^{-1}$, where $\lambda_1(\mathbf{A})$ is the eigenvalue of $\mathbf{A}$ having largest modulus.

[19]To see this, note that $b_i(g, \delta) = 1 + \delta \sum_{j \in N_i} b_j(g, \delta)$.

R&D cooperation (König, 2012), local public goods (Allouch, 2012; Bramoullé et al., 2014), and trade (Bosker and Westbrock, 2014).

In a stage game where players can first form the network prior to engaging in such a game, they anticipate equilibrium payoffs as a function of Bonacich centrality $f\left(b_i(\delta, g)\right)$ in the second stage. In the Ballester et al. game, we have $f(x) = \frac{1}{2}x^2$ by (3.9), but more generally we will assume f to be *increasing* and *convex* for the results in this section. Assuming cost of link formation to be linear in the number of links, we then arrive at a general class of utility functions,

$$u_i^{BC}(g) = f\left(b_i(\delta, g)\right) - \eta_i c. \tag{3.10}$$

When considering link formation with the utility function $u_i^{BC}(g)$ as the objective, we have to make sure that $b_i(g, \delta)$ is well defined for any network. Since the largest eigenvalue $\lambda_1(g)$ is maximized for the complete network g^N, and we need $\delta < \frac{1}{\lambda_1(g)}$ for $b_i(g, \delta)$ to exist, we assume $0 < \delta < \frac{1}{\lambda_1(g^N)} = \frac{1}{n-1}$ in order to define a consistent model of network formation.

The profile of utility functions u^{BC} obviously satisfies anonymity. Moreover, the following results states that u^{BC} also satisfies positive link externalities, i.e. convexity and strategic complements.

Proposition 2.
If f is increasing and convex, then u_i^{BC} as defined by (3.10) *satisfies stratgic complements and convexity.*

The result is intuitive since more own or other players' links increase the number of paths that an additional link creates. A convex transformation does not change this fact and since linking costs are linear, marginal utility is increasing in own and other players' links.

Thus, we can apply Corollary 1, to conclude that either the empty network or the complete network is uniquely pairwise stable, or both are pairwise stable when utility net of costs is given by a convex function of the Bonacich centrality. It is worth noting that to our best knowledge, there is so far only one result from the literature that can be applied to shed some light into the structure of pairwise stable networks when individuals form links according to u_i^{BC}. From Hellmann (2013a) it is known that a pairwise stable network exists. Other models are not applicable, since u_i^{BC} does not fall in the category of playing the field and local spillover games of Goyal and Joshi (2006b), and does not allow for a network potential (cf. Jackson and Watts, 2001; Chakrabarti and Gilles, 2007). We go beyond showing existence since Corollary 1 is applicable.

Further by restricting to low discount factors, we show in Proposition 3 that u^{BC} satisfies SPP and AC and therefore all pairwise stable networks are of nested split architecture.

Proposition 3.
If f is increasing and convex and $\delta < \frac{1}{(n-1)^2}$, then u_i^{BC} as defined by (3.10) *satisfies SPP and AC.*

Although the utility function given by the Bonacich centrality seems to be quite a complex object since it considers the infinite discounted sum of all possible paths in the networks, it is possible to characterize the set of pairwise stable networks at least for low enough discount factors. This is due to the fact that u^{BC} satisfies SPP for these low discount factors since the benefits from second order connections (degree of neighbors) dominate any benefits from higher order connections which is shown in the proof of Proposition 3. Hence, although our results hold for general utility functions, they are still applicable to interesting classes of utility functions and help characterize the structure of PS networks, even where no results are available so far.

3.3.3. Simultaneous Choice of Links and Efforts under Strategic Complementarities

In Section 3.3.2, we presented a two stage game where the network is formed prior to action choice in a game between neighbors in the network. Suppose, instead, that action choice and link formation are done simultaneously. Such a framework is employed in two recent papers by Baetz (2015) and Hiller (2017).[20] The assumption of simultaneous choices of network and actions simplifies analysis a lot compared to a two stage game. The reason is that when network formation takes place before action choice as in the previous section, then the effects of forming links on the equilibrium of the second stage have to be taken into account.[21] For instance, the resulting utility function from the second stage equilibrium outcome of the Ballester et al. (2006) game is a function of the Bonacich centrality and is quite a complex object. We needed additional assumptions on δ in Proposition 3 to characterize all pairwise stable networks. Instead for equilibria in games of simultaneous choice of links and efforts, only single player deviations have to be considered taking other players' equilibrium effort choices as given.

Both frameworks of Baetz (2015) and Hiller (2017) are almost identical differing only in the curvature assumption on the value function and the type of network formation (directed vs undirected). We discus here briefly the model due to Hiller (2017). Adapting Hiller's notation and setup to our framework and letting $x_i \in \mathbb{R}$, utility is given by

$$u_i(g, \mathbf{x}) = \pi(x_i, \sum_{k \in N_i(g)} x_k) - \eta_i c \tag{3.11}$$

such that $\partial\pi(x,y)/\partial y, \partial^2\pi(x,y)/(\partial x \partial y) > 0$ and $c > 0$. Hiller further assumes that for all $i \in N$ best reply effort choices satisfy $\bar{x}_i(g, \mathbf{x}_{-i}) = \bar{x}(\sum_{k \in N_i(g)} x_k)$ with $\bar{x}(0) > 0$, $0 < \lim_{y \to \infty} \bar{x}'(y) < 1/(n-1)$ and either $\bar{x}''(y) < 0$ or $\bar{x}''(y) = 0$ for all $y \in \mathbb{R}$. Moreover, gross payoffs π evaluated at best reply can be written as $\pi(\bar{x}_i(g, \mathbf{x}_{-i}), \sum_{k \in N_i(g)} x_k) = v(\sum_{k \in N_i(g)} x_k)$ with $v(0) \geq 0$, $v' > 0$, and $v'' \geq 0$. One example, where all these assumptions are satisfied, is given by the Ballester et al. utility function, see (3.8).

Given these assumptions, Hiller finds that for each network $g \in G$ there exists a unique Nash equilibirum of effort choices (see Hiller (2017), Proposition 1) denoted by $\mathbf{x}^*(g)$. To account for pairwise nature of network formation also deviations by two players are allowed for equilibrium considerations. Thus, when players $i, j \in N$ connect in network g, we denote the vector of deviation effort levels by $\mathbf{x}^{ij}(g+ij)$ with entries $x_k^{ij}(g+ij) = x_k^*(g)$ for $k \neq i, j$,

[20] For a general treatment of games under strategic complementarities on a fixed network, see also Belhaj et al. (2014).

[21] See also Baetz (2015), where the author states: "Unfortunately, such a [two stage] model is a lot less tractable: to solve it via backward induction, one would need to characterize the equilibrium activity on any exogenous network – an as yet unsolved problem for an arbitrary concave best response function."

and $x_k^{ij}(g+ij) = \bar{x}\left(g+ij, \mathbf{x}_{-k}^{ij}(g+ij)\right)$ for $k = i, j$. Marginal utility of such a deviation can then be defined by

$$\Delta^d u_i(g+ij, ij) := u_i(g+ij, \mathbf{x}^{ij}(g+ij)) - u_i(g, \mathbf{x}^*(g)) \tag{3.12}$$

Similarly, when player i deletes links $l_i \subseteq L_i(g)$ denote the vector of deviation effort choice levels by $\mathbf{x}^i(g-l_i)$ with entries $x_k^i(g-l_i) = x_k^*(g)$ for all $k \neq i$ and $x_i^i(g-l_i) = \bar{x}\left(g-l_i, \mathbf{x}_{-i}^*(g)\right)$. Marginal utility of such a deviation is hence given by

$$\Delta^d u_i(g, l_i) := u_i(g, \mathbf{x}^*(g)) - u_i(g-l_i, \mathbf{x}^i(g-l_i)) \tag{3.13}$$

Assuming that the unique equilibrium effort levels are obtained, we can then define a network $g^* \in G$ to be pairwise stable if for all $ij \in g^*$: $\Delta^d u_i(g^*, ij) \geq 0$ and for all $ij \notin g^*$: $\Delta^d u_i(g^*+ij, ij) > 0 \Rightarrow \Delta^d u_i(g^*+ij, ij) < 0$.[22] In other words a network-efforts pair $(g^*, \mathbf{x}^*)$ is an equilibrium if no two players can profitably deviate by forming a link in g^* and no single player can benefit by deleting a link in g^* while the unique equilibrium in efforts $\mathbf{x}^* = \mathbf{x}^*(g)$ obtains.

Considering the so defined marginal utility of deviations, we find that strategic complements and convexity are satisfied under the assumptions imposed by Hiller.

Proposition 4.
In the simultaneous move game of links and efforts given by Hiller (2017), marginal utility of deviations satisfies strategic complements and convexity in own links.

Thus, we can immediately apply Theorem 1 to conclude that either the empty network of the complete network are uniquely pairwise stable or both are pairwise stable. Not surprisingly, Hiller (2017) finds the same result in his paper (see Hiller, 2017, Proposition 2). Hiller (2017) continues to show that all pairwise Nash stable networks are nested split graphs. Although SPP and AC are not necessarily satisfied it is possible to show that quite similar properties obtain yielding increasing marginal utility with respect to the effort exerted by the players (instead of increasing marginal utility with respect to the degree) which is the driving force for Hiller's result.

4. Convexity

We finally want to study the structure of pairwise stable networks in homogeneous societies when strategic complements are not necessarily satisfied. To obtain results we will assume that at least the externalities from own links satisfy a single crossing property such that the utility function is convex in own links. Recall that ordinal convexity as given in Definition 3 orders the externalities from own links on marginal utility in a way that, once positive, it will stay positive whenever own links are added to the network. In presence of this form of complementarity between own links, the intuition is that players that already have links are likely to strive for more. Notice, however, that due to ambiguous marginal effects of other links, cycling behavior may still arise such that pairwise stable networks may fail to exist.

[22] In Hiller (2017), a PNE is a strategy profile $(g^*, \mathbf{x}^*)$ which is a NE and satisfies for all $ij \notin g$: $\Delta^d u_i(g^* + ij, ij) > 0 \Rightarrow \Delta^d u_i(g^* + ij, ij) < 0$. The difference to our notion is analogous to that between pairwise Nash stability and pairwise stability as multiple link deletion is allowed as a deviation.

4.1. Weak Preference for Prominence

To the contrary, we show in the following that with the additional assumption of WPP as in Definition 5 stable networks will exist. To show existence we define the following class of networks.

Definition 8 ***(Dominant Group Networks)***:
A network $g \in G$ *is a* dominant group network *if there exists* $E \subset N$ *such that* $ij \in g \Leftrightarrow i, j \in E$, $i \neq j$.

In other words, a network is of dominant group architecture if a subset of players E is completely connected, while the remaining players stay isolated. Now, in a homogeneous society if for some $E \subset N$ a dominant group network is pairwise stable, then any dominant group network of same size is pairwise stable for all $\tilde{E} \subset N$ with $|\tilde{E}| = |E|$. Since, therefore, stable dominant group networks are completely characterized by the size of their dominant group in a homogeneous society, we also write g_m^{dg} to denote dominant group networks of size m with $1 \leq m \leq n$. For $m = n$, g_m^{dg} is the complete network, while for $m = 1$, g_m^{dg} is the empty network. The following result shows that there exists $m \in \{1, \ldots, n\}$ such that all dominant group networks g_m^{dg} are pairwise stable, if WPP and convexity are satisfied in a homogeneous society.

Theorem 3.
Suppose the profile of utility functions satisfies convexity, anonymity, and WPP. Then, there exists a pairwise stable network of dominant group architecture g_m^{dg}*, for some* $1 \leq m \leq n$.

The intuition for Theorem 3 is as follows. First, as marginal utility satisfies convexity, players' incentive to form a link is not destroyed by additional own links. Second, players tend to connect to others that already have more links, due to WPP. Both effects together point to networks where players either have many or no links. In Theorem 3, we then naturally find existence of a stable networks in the extreme case, namely one completely connected subgroup and one subgroup of isolated players.

Let us emphasize, again, that WPP is a very weak assumption. Recall that the only restriction imposed by the WPP assumption is that the desire to form links stays positive when connecting to more prominent nodes where prominence is meant with respect to the set inclusion order of the neighborhoods. As the set inclusion order is only a partial order, the assumption is not binding in networks where no two players' neighborhood structures can be ordered. Further, it is very naturally satisfied in many utility functions where players have a desire to be central in the network. As an example, consider some self-referential definition of centrality where a player is central if her neighbors are central. Then, clearly, the connection to a player j such that any of j's neighbor is also a neighbor of some other player k increases centrality by a smaller amount as the connection to k. As an example for a utility function depending on a self-referential definition of centrality, u^{BC} given by (3.10), obviously satisfies WPP.

4.2. Independence of Network Position

WPP and convexity imply the existence of pairwise stable networks in a homogeneous society as shown in Section 4.1. However, to characterize all pairwise stable networks, these conditions are not sufficient. The main reason is that WPP is too weak to really exclude other network structures from being pairwise stable. Instead, consider the following stronger condition.

Definition 9:
A utility function u_i satisfies independence of the network position of other players (INP) *if for all $g \in G$, whenever there exist $j \in N \backslash N_i(g)$ such that $\Delta u_i(g + ij, ij) \geq 0$ then $\Delta u_i(g + ik, ik) > 0$ for all $k \in N \backslash N_i(g)$.*

INP is a quite strong assumption. If player i's utility function satisfies INP, then the network position of players with whom i can form a link cannot play a big role. In particular, it must be that if i wants to connect to *some* player, then i wants to connect to *any* player. In this sense the marginal utility is independent (in an ordinal sense) of the network position of other players. Clearly, if a utility function satisfies INP, then it also satisfies WPP (and even stronger: SPP, see Definiton 5).

Now, in combination with the convexity assumption in this section, INP has also strong implications. When the network position of other players does not matter for the willingness to form links, the convexity assumption then implies that a player either wants to form no links or all possible links. Straightforwardly, we then get that only dominant group networks can be pairwise stable. Further, existence is still guaranteed since INP implies WPP and hence Theorem 3 still applies.

Theorem 4.
Suppose a profile of utility functions satisfies convexity and INP. Then, any pairwise stable network is of dominant group architecture.

We may compare this result with Theorem 2. There, we used SPP and AC to show that any pairwise stable network is a nested split graph. Note that dominant group networks are in fact nested split graphs of special structure. Thus, characterizing pairwise stable networks by the dominant group architecture is a stronger result than characterizing them by a nested split graph. The conditions required cannot be compared in the same way. While it is clear that INP implies SPP, it is the other way around with convexity coupled with anonymity and AC.

While the conditions used in Theorem 4 may seem demanding, these only have to hold in ordinal terms. We also show that there exist some applications where these are satisfied.

4.3. Applications

Consider, again, the Playing the Field utility function from Goyal and Joshi (2006b) defined in Section 3.3.1 by (3.5). If convexity is satisfied, Goyal and Joshi (2006b) show that all pairwise Nash stable network networks belong to the class of dominant group networks. For such a network to exist, however, either strategic complements or strategic substitutes are required.

When computing marginal utility in the Playing the Field utility function we get from (3.5),

$$\Delta u_i^{PF}(g) = f(\eta_i + 1, \eta_{-i}) - f(\eta_i, \eta_{-i}) - c \tag{4.1}$$

Recall that u^{PF} satisfies (strict) convexity if the right-hand side of (4.1) is (strictly) increasing in the first argument. Since the right-hand side of (4.1) is independent of other players' network position, INP and, hence, WPP are straightforwardly satisfied with convexity. Hence, with Theorems 3 and 4, we complement and extend the results from Goyal and Joshi (2006b) in the following way: we show in Theorem 3 that only WPP and convexity are sufficient for the existence of a pairwise stable (dominant group) network. For this result we do not require a restriction on the externalities from other players' links, we do not assume a particular functional form of utility, and we do not rely on the cardinal notions. Note, however, that existence of a pairwise Nash stable network is also not guaranteed by our results. Theorem 4, moreover, is a true generalization of their characterization result as we show that all pairwise stable networks are characterized by the dominant group structure. This includes, trivially also the pairwise Nash stable networks. The characterization result hence not only holds for utility functions which are of playing the field type and satisfy convexity, but instead for all utility functions as long as the ordinal notions of convexity and INP are satisfied.

One classical example in the literature, where the assumptions of convexity and INP are satisfied, is a Cournot oligopoly where firms can form bilateral collaboration links lowering marginal costs before competing in quantities (Goyal and Joshi, 2003, 2006b; Dawid and Hellmann, 2014). In these models, equilibrium quantities are given by

$$q_i(g) = \frac{(a - \gamma_0) + (n-1)\gamma\eta_i(g) - \gamma \sum_{j \neq i} \eta_j(L_{-i}(g))}{n+1}, \quad i \in N.$$

With Cournot profits given by $\pi_i(g) = q_i^2(g)$, this results in marginal profit of an additional link $ij \notin g$ being equal to

$$\Delta\pi_i(g + ij, ij) = \frac{\gamma(n-1)}{(n+1)^2} \Big[2(\alpha - \gamma_0) + \gamma(n-1) + 2\gamma n \eta_i(g) - 2\gamma \sum_{j \neq i} \eta_j(g)\Big]^2 - c.$$

Clearly, as $\eta_{-i} = \sum_{j \neq i} \eta_j(g) - \eta_i$, marginal utility is then just a function of own and other players' number of links and, hence, the associated utility function is of playing the field type. In particular, WPP, INP and anonymity are satisfied. Moreover, Dawid and Hellmann (2014) show that (cardinal) convexity is satisfied and also conclude that all pairwise stable networks are of dominant group structure. Theorem 4 could have worked as a shortcut for this result.

Finally, for completeness note that in terms of utility functions satisfying the local spillover property defined by (3.6), Theorems 3 and 4 do not deliver more than what we already found in Section 3.3.1. The reason is that marginal utility of local spillover utility functions can be written as

$$\Delta u_i^{LS}(g + ij, ij) = f_1(\eta_i + 1) - f_1(\eta_i) + f_2(\eta_j + 1) - f_3(\eta_j) - c.$$

Thus, utility functions of local spillovers type satisfy WPP only if $f_2(\eta_j + 1) - f_3(\eta_j)$ as a function of η_j crosses the point $f_1(\eta_i + 1) - f_1(\eta_i) - c$ just once for all $1 \leq \eta_i \leq n$. This is due to the fact that for every $1 \leq \eta_i \leq n$ and for every $1 \leq \eta_j \leq \eta_k \leq n$, we can always find a network such that $N_j \setminus \{k\} \subset N_k \setminus \{j\}$. But note that this defines strategic complements for local spillover utility functions which implies also that SPP is satisfied by Proposition 1. Hence, even though much weaker for general utility functions, WPP coincides with SPP for utility functions satisfying the local spillover property. Since convexity of u^{LS} implies AC we can then directly apply Theorem 2 delivering stronger results than Theorem 3. Requiring INP for u^{LS} is too strong as it implies that either all $f_2(\eta_j + 1) - f_3(\eta_j)$ as a function of η_j lie above the point $f_1(\eta_i + 1) - f_1(\eta_i) - c$ or all lie below $f_1(\eta_i + 1) - f_1(\eta_i) - c$ for all η_i. Expressed in cardinal terms, $f_2(\eta_j + 1) - f_3(\eta_j)$ must be a constant, implying u^{LS} is of Playing the Field type which is constant in the second argument.

5. Conclusion

We have shown that in a very general environment of network formation, it is possible to derive results on the structural properties of pairwise stable networks by exploiting the ordinal link externality conditions in a homogeneous society. While almost all models in the literature (that we can think of) share the homogeneity assumption, we have shown that the link externality conditions are also quite often satisfied. This paper hence contributes to a better understanding what the driving force for the structure of pairwise stable networks in those model are: the link externality conditions. The results in this paper may, moreover, be used to characterize pairwise stable networks in future even very complex models of network formation that satisfy the link externality conditions (which e.g. arise from multistage games).

For the results on existence of pairwise stable networks, we do not rely on the assumptions of a potential function which is very restrictive or on the assumption of supermodularity of the utility function. Instead, supermodularity can be weakened such that the externalities from own links only satisfy a boundary condition while the externalities from other players' links only need to satisfy a single crossing property to have the structure of pairwise stable networks like the structure of Nash equilibria in supermodular games.

We have thereby improved on the literature that also use the link externality conditions to derive structural properties of stable networks. Compared to Hellmann (2013a), we are able to show existence of pairwise stable networks of specific structures like the empty and the complete network or the dominant group structure. The only additional assumption made is that of a homogeneous society while some other assumptions are relaxed (like the externality conditions of either own or other players links). On the other hand we have generalized some of the results in Goyal and Joshi (2006b): they hold for arbitrary functional forms of utility, they require only ordinal versions of externalities and some assumptions are not even needed.

While the present work exhibits a focus on positive link externalities it would be interesting for future research to show similar results in case of negative link externalities. Our conjecture for the case of both concavity and strategic substitutes however is that existence of pairwise stable networks is not always guaranteed. Second, a full characterization of pairwise stable networks if utility profiles are functions of Bonacich centrality still remains an open question. While we provide a first contribution to this goal, proving existence of a pairwise stable network for any discount factor and characterizing stable networks for low discount factors, it still remains a challenge to characterize stable networks for the rest of the set of admissible discount factors.

6. Appendix

Proof of Lemma 1. Let the profile of utility functions u satisfy anonymity.

(i). Suppose that $i, j \in N$ are symmetric such that there exists a permutation π with $\pi(i) = j$ and $g_\pi = g$. Then by anonymity, we get

$$u_i(g) = u_{\pi(i)}(g_\pi) = u_j(g).$$

(ii). Now let $i, j \in N$ such that $N_i(L_{-j}(g)) = N_j(L_{-i}(g))$. Define π_{ij} as the permutation where players i and j switch positions, that is

$$\pi_{ij} \colon N \to N,\ \pi_{ij}(k) = k\ \ \forall k \in N \setminus \{i, j\},\ \pi_{ij}(i) = j.$$

Then since $N_i(L_{-j}(g)) = N_j(L_{-i}(g))$ we have $g_{\pi_{ij}} = g$. Take now any $k \in N \backslash \{i, j\}$ and define $\tilde{g} = g + ik$. Anonymity then yields

$$u_i(g + ik) = u_i(\tilde{g}) = u_{\pi_{ij}(i)}(\tilde{g}_{\pi_{ij}}) = u_j(g + jk).$$

Then it directly follows that

$$\Delta u_i(g + ik, ik) = u_i(g + ik) - u_i(g) = u_j(g + jk) - u_j(g) = \Delta u_j(g + jk, jk).$$

(iii). By the same arguments as in (ii) we get

$$u_k(g + ik) = u_k(\tilde{g}) = u_{\pi_{ij}(k)}(\tilde{g}_{\pi_{ij}}) = u_k(g + jk).$$

and consequently

$$\Delta u_k(g + ik, ik) = u_k(g + ik) - u_k(g) = u_k(g + jk) - u_k(g) = \Delta u_j(g + jk, jk).$$

□

Proof of Theorem 1. Suppose the empty network $g^\emptyset$ is not pairwise stable (otherwise there is nothing to show). Then by Definition 1, there exists $i, j \in N$ such that $0 < \Delta u_i(g^\emptyset + ij, ij)$. Moreover, suppose that the profile of utility functions u satisfies anonymity. Then by Lemma 1, $0 < \Delta u_i(g^\emptyset + ij, ij)$ for all $i, j \in N$.

Further, suppose the profile of utility functions u satisfies strategic complements and κ-link monotonicity for some $\kappa \in \mathbb{N}$. Let $g \in G \setminus \{g^\emptyset, g^N\}$ and suppose there exist a player $i \in N$ with $\eta_i(g) \leq \kappa\eta_{-i}(g)$. We first show that (if the empty network is not stable) such a player has an incentive to add any link $ij \notin g$. If $\eta_i(g) = 0$, then by strategic complements $0 < \Delta u_i(g^\emptyset + ij, ij) \Rightarrow 0 < u_i\left(g^\emptyset + L_{-i}(g) + ij, ij\right)$. Otherwise if $\eta_i(g) > 0$, we can label the set of i's links by $L_i(g) = \{ij_1, ..., ij_{\eta_i(g)}\}$. Since $\kappa \in \mathbb{N}$ and $\eta_i(g) \leq \kappa\eta_{-i}(g)$, we can then partition the set $L_{-i}(g)$ into $\eta_i(g)$ disjoint subsets $l_1^{-i}, l_2^{-i}, \ldots l_{\eta_i(g)}^{-i} \subset L_{-i}(g)$ such that $l_0^{-i} \dot\cup l_1^{-i} \dot\cup \ldots \dot\cup l_{\eta_i(g)}^{-i} = L_{-i}(g)$, and $|l_k^{-i}| \geq \kappa$ for all $k \in \{1, 2, \ldots, \eta_i(g)\}$. Since for all $g' \in G$, $ij, ik \notin g'$ and $l'_{-i} \in L_{-i}(g^N - g')$ such that $|l'_{-i}| \geq \kappa$ we can find a partition $l^\kappa_{-i} \dot\cup l^+_{-i} = l_{-i}$ with $l'_{-i} = \kappa$, we get by applying strategic complements and κ-link monotonicity,

$$\begin{aligned} 0 < \Delta u_i(g' + ij, ij) &\Rightarrow 0 < u_i(g' + l^+_{-i} + ij, ij) \\ &\Rightarrow 0 < u_i(g' + l^+_{-i} + ik + l^\kappa_{-i} + ij, ij) = u_i(g' + l_{-i} + ik + ij, ij) \end{aligned}$$

We conclude using induction over $1 \leq k \leq \eta_i(g)$,

$$
\begin{aligned}
0 < \Delta u_i(g^\emptyset + ij, ij) &\Rightarrow 0 < u_i(g^\emptyset + ij_1 + l_1^{-i} + ij, ij) \\
&\Rightarrow 0 < u_i(g^\emptyset + ij_1 + ij_2 + l_1^{-i} + l_2^{-i} + ij, ij) \\
&\Rightarrow \ldots \Rightarrow 0 < u_i \left(g^\emptyset + \cup_{k=1}^{\eta_i(g)} \left(ij_k + l_k^{-i} \right) + ij, ij \right)
\end{aligned}
$$

Thus if the empty network is not PS, then for any $g \in G$ any player $i \in N$ with $\eta_i(g) \leq \kappa \eta_{-i}(g)$ has an incentive to add any link and (analogously) has no incentive to delete a link. This implies that for a network $g \in G \setminus \{g^\emptyset, g^N\}$ to be pairwise stable, the set of players $E^\kappa(g) := \{i \in N | \eta_i(g) \leq \kappa \eta_{-i}(g)\}$ has to be completely connected.

To show the first part of the Theorem, let $\hat{\kappa}$ link monotonicity be satisfied with $\hat{\kappa} = \lfloor n/2 \rfloor - 1$. Note that in the complete network g^N, we have $\eta_i = n-1$ and $\eta_{-i} = (n-1)(n-2)/2 - (n-1) = (n-2)/2$ implying $\eta_i \leq \hat{\kappa} \eta_{-i}$ for all $i \in N$. Thus by above, if the empty network is not PS, then $\Delta u_i(g^N, ij) > 0$ for all $i, j \in N$ and, thus, the complete network is PS. Analogously, the empty network is PS if the complete network is not PS, implying the first part of the statement.

For the second part, let $\bar{\kappa} := \left\lfloor \sqrt{2(n-1)(n-2)} \right\rfloor - (n-1)$. To show that no other network than the complete network can be PS if the empty network is not PS, we now show that $E := E^{\bar{\kappa}}(g)$ cannot be completely connected for all $g \in G \setminus \{g^\emptyset, g^N\}$. Suppose to the contrary that it is and take $i \in \arg\min_{j \in E^C} \{\eta_j\}$ where $E^C := N \setminus E$ denotes the complement of E. Since $L_{-i} = g - L_i(g)$, we get $\frac{\eta_{-i}}{\eta_i} = \frac{|g| - \eta_i}{\eta_i}$. Note that since E is completely connected we have $|g| = \frac{|E|(|E|-1)}{2} + \sum_{j \in E^C} \eta_j - \left| \{ij \in g | i, j \in E^C\} \right|$. Thus,

$$
\frac{\eta_{-i}}{\eta_i} = \frac{1}{\eta_i} \left(\frac{|E|(|E|-1)}{2} + \sum_{j \in E^C} \eta_j - \left| \{ij \in g | i, j \in E^C\} \right| - \eta_i \right) \tag{6.1}
$$

For a fixed degree distribution $(\eta_j)_{j \in E^C}$ the right hand side of (6.1) is clearly minimal if $\left| \{ij \in g | i, j \in E^C\} \right|$ is maximal, meaning that it is minimal if there do not exist links between E and E^C, such that the minimum is achieved at $\left| \{ij \in g | i, j \in E^C\} \right| = \frac{1}{2} \sum_{j \in E^C} \eta_j$. Hence,

$$
\frac{\eta_{-i}}{\eta_i} \geq \frac{1}{\eta_i} \left(\frac{|E|(|E|-1)}{2} + \frac{1}{2} \sum_{j \in E^C} \eta_j - \eta_i \right)
$$

By construction, $\eta_i \leq \eta_j$ for all $j \in E^C$. Thus,

$$
\frac{\eta_{-i}}{\eta_i} \geq \frac{1}{\eta_i} \left(\frac{|E|(|E|-1)}{2} + \frac{1}{2} \left(\left| E^C \right| - 2 \right) \eta_i \right) \tag{6.2}
$$

Since the right hand side of (6.2) is decreasing in η_i, choosing η_i maximial (such that the maximum is achieved when E^C is completely connected, i.e. $\eta_i = \left| E^C \right| - 1$) obtains,

$$
\frac{\eta_{-i}}{\eta_i} \geq \frac{1}{2\left(E^C - 1\right)} \left(|E|(|E|-1) + \left(\left| E^C \right| - 2 \right) \left(\left| E^C \right| - 1 \right) \right) \tag{6.3}
$$

Now, setting $k := \left|E^C\right| - 1$ and, hence, $n - k - 1 = |E|$, we can find a lower bound for the right hand side of (6.3) by minimizing with respect to $k \in \mathbb{R}$. Solving the optimization problem

$$\min_{k\in\mathbb{R}} \tfrac{1}{2k}\left((n-k-1)(n-k-2)+k(k-1)\right)$$

we get $k^* := \sqrt{\frac{(n-1)(n-2)}{2}}$ as a global minimum which implies,

$$\begin{aligned}\frac{\eta_{-i}}{\eta_i} &\geq \tfrac{1}{2k^*}\left((n-k^*-1)(n-k^*-2)+k^*(k^*-1)\right) = \sqrt{2(n-1)(n-2)} - (n-1)\\ &\geq \left\lfloor\sqrt{2(n-1)(n-2)}\right\rfloor - (n-1) = \bar{\kappa}.\end{aligned}$$

Which contradicts our assumption that $i \in E^C$. Thus either E cannot be completely connected which means there exists two players $i, j \in E$ which have a strict incentive to form a link contradicting pairwise stability, or E^C is the empty set which contradicts that we chose $g \neq g^N$. Thus, no network other than the complete network can be pairwise stable. Stability of the complete network is easily checked as

$$\frac{\eta_{-i}}{\eta_i} = \frac{(n-1)(n-2)}{n-1} = n-2 \overset{n\geq 3}{\geq} \left\lfloor\sqrt{2(n-1)(n-2)}\right\rfloor - (n-1) = \bar{\kappa}$$

implying that all players have an incentive to keep their links in the complete network by $\bar{\kappa}$-link monotonicity.[23] We have hence shown that if the empty network is not PS then the complete network is uniquely PS. Completely analogous arguments constitute the reverse implication. □

Proof of Corollary 1. First note that by convexity and strategic complements,

$$\Delta u_i(g+ij,ij)(\geq) > 0 \;\Rightarrow\; \Delta u_i(g+ik+ij,ij)(\geq) > 0 \;\Rightarrow\; \Delta u_i(g+l_{-i}+ik+ij,ij)(\geq) > 0$$

for all $l_{-i} \in L_{-i}(g^N - g)$ with $0 \leq |l_i|$, i.e. 0-link monotonicity is satisfied. The statement is then directly implied by Theorem 1 □

Proof of Theorem 2. Suppose to the contrary that there exists a pairwise stable network which is not a nested split graph. Then by definition there exists a set of three distinct players i, j, k, such that $\eta_i(g) \geq \eta_j(g) \geq \eta_k(g)$, and either $ik \in g$ while $ij \notin g$ or $jk \in g$ while $ik \notin g$.

Suppose first $ik \in g, ij \notin g$. Since g is assumed to be stable, we have $\Delta u_i(g, ik) \geq 0$ and $\Delta u_k(g, ik) \geq 0$. Then however

$$\Delta u_i(g,ik) \geq 0 \;\Rightarrow\; \Delta u_i(g+ij,ij) > 0,$$

following by SPP, and further

$$\Delta u_k(g,ik) \geq 0 \;\Rightarrow\; \Delta u_j(g+ij,ij) \geq 0,$$

following by AC. Thus i and j would want to add a link to g, contradicting pairwise stability.

[23] For $n = 2$ the statement of the proposition is trivially satisfied without any additional assumptions.

If on the other hand $jk \in g, ik \notin g$ we can argue similarly

$$\Delta u_k(g, jk) \geq 0 \;\Rightarrow\; \Delta u_k(g + ik, ik) > 0,$$

by SPP, and

$$\Delta u_j(g, jk) \geq 0 \Rightarrow \Delta u_i(g + ik, ik) \geq 0,$$

by anonymous convexity. Again, i and k would want to add a link, so that g cannot be stable. □

Proof of Proposition 1. (i) Let $i, j \in N$ such that $\eta_i \leq \eta_j$ implying $\eta_{-i} \geq \eta_{-j}$. Letting $k \in N$, $k \neq i, j$, we get by convexity and strategic complements,

$$\begin{aligned}\Delta u_j^{PF}(g + jk, jk) &= f(\eta_j + 1, \eta_{-j}) - f(\eta_j, \eta_{-j}) \\ &\geq f(\eta_i, \eta_{-j}) - f(\eta_i - 1, \eta_{-j}) \\ &\geq f(\eta_i, \eta_{-i}) - f(\eta_i - 1, \eta_{-i}) \\ &= \Delta u_i^{PF}(g, ik),\end{aligned}$$

which implies AC according to Definition 6. Further, since degree of neighbors do not matter,

$$\Delta u_k^{PF}(g + ik, ik) = f(\eta_k + 1, \eta_{-k}) - f(\eta_k, \eta_{-k}) = \Delta u_k^{PF}(g + jk, jk),$$

which implies SPP according to Definition 5.

(ii) Let $i, j \in N$ such that $\eta_i \leq \eta_j$. Let $k \in N$, $k \neq i, j$. We have by convexity and strategic complements,

$$\begin{aligned}\Delta u_j^{LS}(g + jk, jk) &= f_1(\eta_j + 1) - f_1(\eta_j) + f_2(\eta_k + 1) - f_3(\eta_k) \\ &\geq f_1(\eta_i) - f_1(\eta_i - 1) + f_2(\eta_k) - f_3(\eta_k - 1) \\ &= \Delta u_i^{PF}(g, ik),\end{aligned}$$

which implies AC according to Definition 6. Finally, because of strategic complements alone,

$$\begin{aligned}\Delta u_k^{LS}(g + jk, jk) &= f_1(\eta_k + 1) - f_1(\eta_k) + f_2(\eta_j + 1) - f_3(\eta_j) \\ &\geq f_1(\eta_k + 1) - f_1(\eta_k) + f_2(\eta_i + 1) - f_3(\eta_i) \\ &= \Delta u_k^{PF}(g, ik),\end{aligned}$$

implying SPP according to Definition 5.

□

Proof of Proposition 2. First, we show (more generally) by induction over $k \in \mathbb{N}$ that $(\mathbf{A}' + \mathbf{B})^k - (\mathbf{A}')^k \geq (\mathbf{A} + \mathbf{B})^k - (\mathbf{A})^k$ if for all nonnegative $n \times n$ matrixes $\mathbf{A}, \mathbf{A}', \mathbf{B}$ with $\mathbf{A} \leq \mathbf{A}'$, where for matrices $\mathbf{A}$ and $\mathbf{B}$ we write $\mathbf{A} \leq \mathbf{B}$, if and only if the entries satisfy $a_{ij} \geq b_{ij}$ for all $i, j \in N$.

For $k = 1$, we have the assertion satisfied with equality,

$$\left(\mathbf{A}' + \mathbf{B}\right)^1 - \left(\mathbf{A}'\right)^1 = \mathbf{B} = (\mathbf{A} + \mathbf{B})^1 - (\mathbf{A})^1$$

Now suppose that the assertion holds for some $k \in \mathbb{N}$. Then,

$$\begin{aligned}
&\left(\mathbf{A}' + \mathbf{B}\right)^k - \left(\mathbf{A}'\right)^k \geq (\mathbf{A} + \mathbf{B})^k - (\mathbf{A})^k \\
\Rightarrow\ &(\mathbf{A}' + \mathbf{B}) \left(\left(\mathbf{A}' + \mathbf{B}\right)^k - \left(\mathbf{A}'\right)^k\right) \geq (\mathbf{A} + \mathbf{B}) \left((\mathbf{A} + \mathbf{B})^k - (\mathbf{A})^k\right) \\
\Leftrightarrow\ &\left(\mathbf{A}' + \mathbf{B}\right)^{k+1} - \left(\mathbf{A}'\right)^{k+1} - \mathbf{B}\left(\mathbf{A}'\right)^k \geq (\mathbf{A} + \mathbf{B})^{k+1} - (\mathbf{A})^{k+1} - \mathbf{B}\,(\mathbf{A})^k \\
\Rightarrow\ &\left(\mathbf{A}' + \mathbf{B}\right)^{k+1} - \left(\mathbf{A}'\right)^{k+1} \geq (\mathbf{A} + \mathbf{B})^{k+1} - (\mathbf{A})^{k+1}
\end{aligned}$$

where we repeatedly used that $\mathbf{A}' \geq \mathbf{A}$. Thus for $g, g' \in G$, with $g \subset g'$ and $ij \notin g'$ we can set $\mathbf{A} := \mathbf{A}(g)$, $\mathbf{A}' := \mathbf{A}(g')$ and $\mathbf{B} := \mathbf{A}(ij)$ implying $\mathbf{A} \leq \mathbf{A}'$ and we obtain

$$b(g' + ij, \delta) - b(g', \delta) = \sum_{k=0}^{n} \left(\mathbf{A}' + \mathbf{B}\right)^k - \left(\mathbf{A}'\right)^k \geq \sum_{k=0}^{n} (\mathbf{A} + \mathbf{B})^k - (\mathbf{A})^k = b(g + ij, \delta) - b(g, \delta).$$

Since f is an increasing and convex function and $b_i(g'+ij, \delta) \geq b_i(g', \delta) \geq 0$ and $b_i(g+ij, \delta) \geq b_i(g, \delta) \geq 0$, we then have,

$$\begin{aligned}
u_i^{BC}(g' + ij) - u_i^{BC}(g') &= f(b_i(g' + ij, \delta)) - f(b_i(g', \delta)) - c \\
&\geq f(b_i(g + ij, \delta)) - f(b_i(g, \delta)) - c \\
&= u_i^{BC}(g + ij) - u_i^{BC}(g),
\end{aligned}$$

Letting g' and g being such that $g' - g \subset L_{-i}(g^N - g)$ we obtain the (cardinal) strategic complements property and letting g' and g being such that $g' - g \subset L_i(g^N - g)$ we obtain (cardinal) convexity. □

Proof of Proposition 3. Remember that

$$u_i^{BC} = f(b_i(g)) - \eta_i(g)c = f(\mathbf{e}_i'(\sum_{t=0}^{\infty} \delta^t \mathbf{A}^t)\mathbf{1}) - \eta_i(g)c,$$

with $\mathbf{A}$ being the adjacency matrix of network g and $\mathbf{e}_i$ the i-th unit vector. Take some players $i, j, k \in N$ and a network g such that $ij \in g$, $ik \notin g$ and $\eta_j(g) \leq \eta_k(g)$. We get,

$$\begin{aligned}
\Delta u_i^{BC}(g + ik, ik) =& f(b_i(g + ik)) - f(b_i(g)) - c \\
=& f(b_i(g) + \delta + \delta^2(\eta_k(g) + 1) + \mathbf{e}_i'(\sum_{t=3}^{\infty} \delta^t((\mathbf{A} + \mathbf{A}(ik))^t - \mathbf{A}^t)\mathbf{1}) \\
&- f(b_i(g)) - c, \\
\geq& f(b_i(g) + \delta + \delta^2(\eta_k(g) + 1)) - f(b_i(g)) - c,
\end{aligned}$$

We can find an upper bound for the marginal utility of deleting j by considering utility of the complete network from order 3 on,[24]

$$\begin{aligned}\Delta u_i^{BC}(g,ij) \leq & f(b_i(g)+\delta+\delta^2(\eta_j(g))+\sum_{t=3}^{\infty}\delta^t\eta_j(g)(n-1)^{t-2})-f(b_i(g))-c,\\ =& f(b_i(g)+\delta+\delta^2\eta_j(g)+\delta^2\eta_j(g)(\sum_{t=0}^{\infty}\delta^t(n-1)^t-1))-f(b_i(g))-c\\ =& f(b_i(g)+\delta+\tfrac{\delta^2\eta_j(g)}{1-\delta(n-1)})-f(b_i(g)-c.\end{aligned}$$

Now, from $0<\delta<\frac{1}{(n-1)^2}$, we get $\frac{\eta_j(g)}{1-\delta(n-1)}<\frac{\eta_j(g)}{1-\frac{1}{(n-1)}}=\eta_j+\frac{\eta_j}{n-2}$. Since k is not connected to i in g, we have $\eta_j(g)\leq\eta_k(g)\leq n-2$. Thus, $\eta_j+\frac{\eta_j}{n-2}\leq\eta_k+\frac{n-2}{n-2}=\eta_k+1$, implying

$$f(b_i(g)+\delta+\delta^2\tfrac{\eta_j(g)}{1-\delta(n-1)})-f(b_i(g))-c\leq f(b_i(g)+\delta+\delta^2(\eta_k(g)+1))-f(b_i(g))-c,$$

since f is an increasing function. We conclude that for $0<\delta<\frac{1}{(n-1)^2}$ the following holds

$$\eta_j(g)\leq\eta_k(g)\ \Rightarrow\ \Delta u_i^{BC}(g,ij)\leq\Delta u_i^{BC}(g+ik,ik).$$

implying that u^{BC} satisfies SPP.

Letting $\eta_i(g)\leq\eta_j(g)$ and $\Delta u_i^{BC}(g,ik)\geq 0$, we get for $0<\delta<\frac{1}{(n-1)^2}$ the same bounds on third order terms, implying analogously,

$$\begin{aligned}0\leq\Delta u_i^{BC}(g,ik)&\leq\delta+\delta^2\eta_k(g)+\sum_{t=3}^{\infty}\delta^t\eta_k(g)(n-1)^{t-2})-c\\ &<\delta+\delta^2(\eta_k(g)+1)-c\leq\Delta u_j^{BC}(g+jk,jk),\end{aligned}$$

thus u^{BC} also satisfies AC. □

Proof of Proposition 4. First note that by Hiller (2017) Proposition 1, $\mathbf{x}^*(g)\leq\mathbf{x}^*(g')$ for all $g\subseteq g'$. We get $x_i^{ij}(g+ij)$ and $x_j^{ij}(g+ij)$ as a solution to the system of two equations $x_i^{ij}(g+ij)=\bar{x}\left(g+ij,(x_j^{ij}(g+ij),\mathbf{x}^*_{-ij}(g))\right)$ and $x_j^{ij}(g+ij)=\bar{x}\left(g+ij,(x_i^{ij}(g+ij),\mathbf{x}^*_{-ij}(g))\right)$ where $\mathbf{x}^*_{-ij}(g)$ is the vector obtained by deleting the entries i and j from $\mathbf{x}^*$. Now, since the best reply function $\bar{x}$ is strictly increasing and for all $g\subseteq g'$ and $\mathbf{x}^*_{-ij}(g)\leq\mathbf{x}^*_{-ij}(g')$, we also get $x_j^{ij}(g+ij)<x_j^{ij}(g'+ij)$ for all $g\subseteq g'$. Since the value function v is increasing and convex we then get for $g\subseteq g'$,

$$\begin{aligned}\Delta^d u_i(g'+ij,ij)&=u_i(g'+ij,\mathbf{x}^{ij}(g'+ij))-u_i(g,\mathbf{x}^*(g'))\\ &=v(\sum_{k\in N_i(g')}x_k^*(g'+ij)+x_j^{ij}(g'+ij))-v(\sum_{k\in N_i(g')}x_k^*(g'))\\ &\underset{v''\geq 0}{\geq}v(\sum_{k\in N_i(g)}x_k^*(g+ij)+x_j^{ij}(g'+ij))-v(\sum_{k\in N_i(g)}x_k^*(g))\\ &\underset{v'>0}{\geq}v(\sum_{k\in N_i(g)}x_k^*(g+ij)+x_j^{ij}(g+ij))-v(\sum_{k\in N_i(g)}x_k^*(g))\\ &=\Delta^d u_i(g+ij,ij).\end{aligned}$$

[24]Notice that the approximations used are quite rough. For example, instead of using the empty network as a lower bound approximation, one could instead use the star network of $\eta_k(g)+1$ players.

Since $g, g' \in G$ with $g \subseteq g'$ were chosen arbitrarily, we obtain the strategic complements property by restricting to g, g' with $g' - g \subseteq L_{-i}(g')$ and we obtain the convexity property by restricting to g, g' with $g' - g \subseteq L_i(g')$. □

Proof of Theorem 3. Consider a dominant group network g_m^{dg} and denote the set of completely connected players by E (of size $|E| = m$).Suppose that g_m^{dg} is not deletion proof. Then, there exists a player $i \in E$ such that $\Delta u_i(g_m^{dg}, ij) < 0$. By anonymity, $\Delta u_i(g_m^{dg}, ij) < 0$ implies $\Delta u_i(g_m^{dg}, ik) < 0$ for all $k \in E \setminus \{i\}$ since $N_i(g_m^{dg}) \setminus \{k\} = N_k(g_m^{dg}) \setminus \{i\}$, see Lemma 1.

Let $g_{m-1}^{dg} := g_m^{dg} - L_i(g_m^{dg})$ be the network obtained after deleting all of player i's links in g_m^{dg} which is again a dominant group network with dominant group $E \setminus \{i\}$. We then get by convexity that

$$0 > \Delta u_i(g_m^{dg} - ik + ik, ik) \quad \Rightarrow \quad 0 > \Delta u_i(g_m^{dg} - L_i(g_m^{dg}) + ik, ik) = \Delta u_i(g_{m-1}^{dg} + ik, ik)$$

for all $k \in E \setminus \{i\}$. By WPP, we then get $\Delta u_i(g_{m-1}^{dg} + ij, ij) < 0$ for all $j \in E^C$ since $N_j(g_{m-1}^{dg}) \subset N_k(g_{m-1}^{dg})$, $k \in E \setminus \{i\}$. Applying anonymity, we then get for any $j \in E^C$, $\Delta u_j(g_{m-1}^{dg} + jl, jl) < 0$ for all $l \in N \setminus \{j\}$ since $N_i(g_{m-1}^{dg}) = N_j(g_{m-1}^{dg}) = \emptyset$. Thus, the isolated players have no incentive to form a link, implying that the network g_{m-1}^{dg} is addition proof.

We have, hence, shown that if g_m^{dg} is not deletion proof, then g_{m-1}^{dg} is addition proof. Now, since the complete network ($m = n$) is trivially addition proof and the empty network ($m = 1$) is trivially deletion proof, there must exists a $1 \leq m \leq n$ such that g_m^{dg} is addition proof and deletion proof and, hence, pairwise stable.

□

Proof of Theorem 4. Suppose a profile of utility functions satisfies convexity and INP. Consider a network $g \in G$ which is not of dominant group structure meaning that there exists two (non-isolated) players $i, j \in N$ with $N_i(g), N_j(g) \neq \emptyset$ and such that $ij \notin g$. Suppose to the contrary that g is PS. For $k \in N_i(g) \neq \emptyset$ we then get $0 \leq \Delta u_i(g, ik) = \Delta u_i(g - ik + ik, ik)$. Together with INP, this implies $0 < \Delta u_i(g - ik + ij, ij)$ and, hence, by convexity $0 < \Delta u_i(g + ij, ij)$. Analogously we get $0 < \Delta u_i(g + ij, ij)$, contradicting that g is PS. □

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