Submodule closed subcategories of module categories

Dissertation zur Erlangung des Doktorgrades
der Fakultät für Mathematik
der Universität Bielefeld

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September 24, 2017

Acknowledgements

First and foremost, I am very grateful to my supervisor Henning Krause for his support, his advice and for suggesting the interesting topic of this thesis.

I would like to thank the BIREP group in general, but especially Julia Sauter, Andrew Hubery and Bill Crawley-Boevey for their discussions and feedback.

Furthermore, I want to express my gratitude to Hugh Thomas for his comments and suggestions.

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1 | Introduction

1.1 Motivation

We are interested in submodule closed subcategories of abelian length categories $\mathcal{A}$, that is, (full, additive) subcategories, which are closed under subobjects. If $\mathcal{A}$ is a module category, these subcategories are also called submodule closed.

While submodule closed subcategories have not yet been extensively studied, they are a very interesting topic with many connections to different parts of representation theory. For example, if A is a finite dimensional algebra, then every infinite submodule closed subcategory of $\mathsf{mod}\, A$ contains a minimal infinite submodule closed category, see [16].

Submodule closed subcategories can also be used to prove that there is a filtration of the Ziegler spectrum that is indexed by the Gabriel-Roiter filtration, see [12].

Furthermore, if A is a hereditary Artin algebra, then there is a natural bijection between the elements of the Weyl group of A and the full, additive cofinite submodule closed subcategories of $\mathsf{mod}\, A$. This has been proved in [14] for algebras over finite and algebraically closed fields and is proved in general in this thesis.

Another connection arises in the second main part of the thesis: If A is of colocal type, then the lattice formed by full, additive submodule closed subcategories of $\mathsf{mod}\, A$ is distributive. Algebras of colocal type have been studied repeatedly: for example, a first characterization dates back to H. Tachikawa in 1959, see [20]; two gaps in the proof were filled by T. Sumioka in 1984, see [19].

In this thesis, we give a new characterisation for algebras of colocal type, which is especially simple for algebras over an algebraically closed field.

For algebras of colocal type over such fields, we can completely describe the lattice $\mathsf{S}(\mathsf{mod}\, A)$. In fact, we get another connection to an important item in representation theory: Young diagrams.

1.2   Main results

In Chapter 3 we take a look at hereditary Artin algebras A. Let $\mathsf{mod}\,A$ be the category of finitely generated right modules over A.

A full, additive subcategory of $\mathsf{mod}\,A$ is called *cofinite* if it contains all but finitely many indecomposable modules in $\mathsf{mod}\,A$.

We get the following result:

Theorem 1.2.1. *Let A be a hereditary Artin algebra. Then there exists a natural bijection between the elements of the Weyl group of A and the full, additive cofinite submodule closed subcategories of $\mathsf{mod}\,A$.*

To prove this theorem, we first show the following result, which is important in its own right. We use the notation $X \mid Y$ if the module X is a direct summand of Y and $X \nmid Y$ if X is not a direct summand of Y:

Proposition 1.2.2. *Let A be a hereditary Artin algebra and $M \in \mathsf{mod}\,A$ indecomposable and preinjective. Suppose that $U \in \mathsf{mod}\,A$, and M is not a direct summand of U. There is a monomorphism $M \rightarrowtail U$ if and only if there is some $m \in \mathbb{N}$ with three sequences of modules*

$$(X_1, X_2, \ldots, X_m)$$
$$(X_1', X_2', \ldots, X_m')$$
$$(Y_1, Y_2, \ldots, Y_m)$$

that fulfil the following conditions:

(S1) There is an Auslander-Reiten sequence

$$0 \longrightarrow M \longrightarrow X_1 \oplus X_1' \longrightarrow Y_1 \longrightarrow 0 .$$

(S2) For all $1 \leq i < m$, there is some $\alpha_i \in \mathbb{N}$ so that $X_i^{\alpha_i}$ is a direct summand of $X_i \oplus X_i'$, but not of U.

(S3) For $1 \leq i < m$, there is an Auslander-Reiten sequence of the form

$$0 \longrightarrow X_i \longrightarrow Z_i \longrightarrow \tau^{-1}X_i \longrightarrow 0 . \qquad (1.1)$$

Let Y_i' be the maximal module that is both a direct summand of Y_i and Z_i. Write $Y_i = Y_i' \oplus Y_i''$ and $Z_i = Y_i' \oplus Z_i'$.

If $\tau^{-1}X_i \mid X_i'$, then let X_i'' be the module so that $X_i' = \tau^{-1}X_i \oplus X_i''$ and set $Y_i''' := 0$. Otherwise, set $X_i'' := X_i'$ and $Y_i''' := \tau^{-1}X_i$.

The following equations hold:

$$X_{i+1} \oplus X'_{i+1} = X''_i \oplus Z'_i$$
$$Y_{i+1} = Y''_i \oplus Y'''_i.$$

(S4) *If for any $1 \leq i \leq m$, the module $X_i \oplus X'_i$ has an injective direct summand I, then $I \mid U$.*

(S5) *$X_m \oplus X'_m$ is a direct summand of U.*

In Chapter 4, we proceed to consider a much broader case: abelian length categories, a generalization of module categories. On the other hand, the question that we answer is much less general: we ask, in which cases such a category is of colocal type (that is, every subobject of an indecomposable object is itself indecomposable). Partly, we can also answer the question, in which cases the lattice of full, additive subobject closed subcategories is distributive.

To state the answers, we first need some notation:

Definition 1.2.3. For all simple objects $S, T \in \mathcal{A}$ let

$$d^1_S(S, T) := \dim_{\mathsf{End}(S)^{op}} \mathsf{Ext}^1(S, T)$$

and

$$d^1_T(S, T) := \dim_{\mathsf{End}(T)} \mathsf{Ext}^1(S, T).$$

Then we can show the following. For simplicity, we are equating objects with isomorphism classes of objects:

Theorem 1.2.4. *The category $\mathcal{A}$ is of colocal type if and only if the following conditions hold:*

(C1) *For all simple objects $S \in \mathcal{A}$*

$$\sum_{T \; simple} d^1_T(S, T) \leq 1.$$

(C2) *For all simple objects $S \in \mathcal{A}$*

$$\sum_{T \; simple} d^1_T(T, S) \leq 2.$$

(C3) If there is a simple object S' with $\mathsf{Ext}^1(S, S') \neq 0$, let $\mathcal{T}$ be the set of simple objects T for which $d^1_T(T, S) \neq 0$ and there is an indecomposable object Z of length 3 with $\mathrm{top}\, Z \cong T$ and $\mathrm{soc}\, Z \cong S'$. Then

$$\sum_{T \in \mathcal{T}} d^1_T(T, S) \leq 1.$$

While the last condition is more complicated then the first two, there are several ways to state it. In particular, it is often equivalent to a condition on the 2-extensions between simple objects:

Proposition 1.2.5. *Suppose that (C1) holds for all simple objects in $\mathcal{A}$.*
For fixed simple objects S and S' with $\mathsf{Ext}^1(S, S') \neq 0$, the following classes of objects are the same:

(a) the class of simple objects T so that $d^1_T(T, S) \neq 0$ and there is some indecomposable object Z of length 3 with $\mathrm{soc}\, Z \cong S'$ and $\mathrm{top}\, Z \cong T$

(b) the class of simple objects T so that $d^1_T(T, S) \neq 0$ and there is some indecomposable object Z of length greater or equal 3 with $\mathrm{soc}\, Z \cong S'$ and $\mathrm{top}\, Z \cong T$

   If $\mathsf{Ext}(S', S') = 0$, then these classes are the same as

(c) the class of simple objects T so that $d^1_T(T, S) \neq 0$ and there is some indecomposable object Z with $\mathrm{soc}\, Z \cong S'$ and $\mathrm{top}\, Z \cong T$

If S' is not part of an oriented cycle in the Ext-quiver of $\mathcal{A}$, then this class is even the same as

(d) the class of simple objects T so that $d^1_T(T, S) \neq 0$ and $\mathsf{Ext}^2(T, S') = 0$

For all abelian length categories $\mathcal{A}$ of colocal type, the lattice $\mathsf{S}(\mathcal{A})$ is distributive.

Finally, we look at the categories of the form $\mathcal{A} := \mathsf{mod}\, A$ for some Artin algebra A over an algebraically closed field k. If $\mathcal{A}$ is of colocal type, then it is equivalent to $\mathsf{mod}\, kQ/I$ for an especially simple quiver Q with admissible ideal I and we can completely describe the lattice $\mathsf{S}(\mathcal{A})$ up to isomorphism:

For every vertex m in Q, consider the paths that end in m and do not contain any relation in I. Under these paths, either one or two are maximal.

If there is only one, then we denote its length with k_m and set $l_m := 0$.

If there are two maximal paths, we denote their lengths with k_m and l_m.

For a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$ of a natural number, the *Young diagram* of λ is an array of squares with n rows and exactly λ_i squares in the i-th row.

The partitions of natural numbers form a lattice Y, ordered by the inclusion order on the Young diagrams. It is called *Young's lattice.*

Denote by $Y^{m,n}$ that sublattice of Young's lattice that contains exactly the partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{m'})$ where $\lambda_i \leq n$ for all $1 \leq i \leq m'$ and $m' \leq m$.

Then we get the following lattice isomorphism:

Theorem 1.2.6. *Suppose* $\mathsf{mod}\, A \equiv \mathsf{mod}\, kQ/I$ *with quiver* $Q = (Q_0, Q_1)$ *and admissible ideal* I. *If* A *is of colocal type, then*

$$\mathsf{S}(\mathsf{mod}\, A) \cong \prod_{m \in Q_0} Y^{k_m+1, l_m+1}.$$

1.3 Outline

In the second chapter of this thesis, we collect definitions and results needed for the formulation and proofs of our results. We start with an overview of abelian categories, in particular module categories. Then we concentrate on Auslander-Reiten sequences, the main tool to prove the first theorem in this thesis. The next section introduces hereditary algebras and one of their most important examples: quiver algebras.

In the fourth section we define the Weyl group. In a slight deviation of the usual practice, we give (and work with) the definition of the Weyl group as a Coxeter group, thus not defining the reflections on A, except as generators of the Weyl group.

We conclude this chapter with a section about string algebras, which are the path algebras of quivers with relations that have a special form. Since the modules over theses algebras are well known, they are very useful for some of the proofs in the fourth chapter.

In the third chapter, we prove that there is a natural bijection between the elements of the Weyl group of A and the full, additive cofinite submodule closed subcategories of $\mathsf{mod}\, A$. Oppermann, Reiten and Thomas have shown this in [14] for algebraically closed fields and finite fields. While we use the same bijection, we give a different method of proof that does not depend on the field.

First, we define an order on the Weyl group and show some properties of this order. The next section is devoted to an algorithm which, given a preinjective module M, constructs all modules which contain M as a submodule.

In the third section, we show how the structure of the Weyl group is connected to the submodule relations between preinjective modules. In the next sections, after proving two auxiliary results, we show first that the

bijection is well defined and then that it is surjective. Since the injectivity is clear by definition, this concludes the proof. In the last section of this chapter, we draw some corollaries.

The fourth chapter is devoted to abelian length categories $\mathcal{A}$ where $\mathsf{S}(\mathcal{A})$ is distributive. In particular, we characterize abelian length categories of colocal type.

First, we show that the distributivity of the lattice is equivalent to a simple condition on submodule relations in $\mathcal{A}$. Then we introduce categories of colocal type and prove that $\mathsf{S}(\mathcal{A})$ is distributive if $\mathcal{A}$ is of colocal type.

In the third section, we prove that certain conditions on the Ext-quiver of $\mathcal{A}$ must be fulfilled if $\mathsf{S}(\mathcal{A})$ is distributive; stricter conditions have to be fulfilled if $\mathcal{A}$ is of colocal type.

In the next section, we collect some auxiliary lemmas about 2-extensions. We need these to show in the fifth section that several different formulations of a condition are equivalent under certain assumptions. Afterwards, we prove that abelian length categories of colocal type fulfil this condition.

In the sixth section, we show that the conditions formulated in the third and fifth section are even sufficient for $\mathcal{A}$ being of colocal type. This proof also draws on the auxiliary lemmas in the fourth section.

We complete this chapter with a description of the lattice $\mathsf{S}(\mathcal{A})$: First we show that it is a Cartesian product of certain sublattices. Then, we take a closer look at categories $\mathcal{A}$ of colocal type which are equivalent to $\mathsf{mod}\, kQ/I$ for some field k, quiver Q and admissible ideal I, and see that $\mathsf{S}(\mathcal{A})$ has an especially simple form.

2 | Abelian length categories and Artin algebras

We are interested in abelian length categories, in particular in the module categories of Artin algebras. For a more detailed introduction into abelian categories, see for example [8]; for an introduction into Artin algebras and Auslander-Reiten theory consult for example [2].

2.1 Abelian categories and module categories

We start by giving the definition of an additive category, which we need to define abelian categories:

Definition 2.1.1. A category $\mathcal{C}$ is called *additive* if

- For all objects $A, B \in \mathcal{C}$, the morphism set $\mathsf{Hom}(A, B)$ is an abelian group.

- For all objects $A, B, C \in \mathcal{C}$, the composition morphism

$$\mathsf{Hom}(A, B) \times \mathsf{Hom}(B, C) \to \mathsf{Hom}(A, C) \qquad (2.1)$$

  is bilinear over $\mathbb{Z}$.

- The category $\mathcal{C}$ has finite sums.

A stronger concept is the following, see [21], Definition 3.3.4:

Definition 2.1.2. An additive category $\mathcal{C}$ is called *abelian* if the following three conditions hold:

- Every morphism has a kernel and a cokernel.

- Every monomorphism is the kernel of a morphism.

- Every epimorphism is the cokernel of a morphism.

Abelian categories have pushouts and pullbacks:

Definition 2.1.3. Let V, X_1, X_2 be objects in an abelian category $\mathcal{A}$ and $f_1 : V \to X_1, f_2 : V \to X_2$ be morphisms. Then there exist an object Y and morphisms g_1, g_2 so that the following diagram is commutative

$$
\begin{array}{ccc}
V & \xrightarrow{f_1} & X_1 \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g_1} \\
X_2 & \xrightarrow{g_2} & Y
\end{array}
\tag{2.2}
$$

and for every object Z with morphisms $g_1' : X_1 \to Z$, $g_2' : X_2 \to Z$ which form a commutative diagram

$$
\begin{array}{ccc}
V & \xrightarrow{f_1} & X_1 \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g_1'} \\
X_2 & \xrightarrow{g_2'} & Z
\end{array} \quad ,
$$

there is a unique morphism $\phi : Y \to Z$ with $\phi g_1 = g_1'$ and $\phi g_2 = g_2'$.

If f_1 is a monomorphism, then g_2 is also a monomorphism.

The diagram (2.2) is called a *pushout*.

Dually, for two morphisms $g_1 : X_1 \to Y$, $g_2 : X_2 \to Y$, there exists a *pullback*, that is, a commutative diagram

$$
\begin{array}{ccc}
V & \xrightarrow{f_1} & X_1 \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g_1} \\
X_2 & \xrightarrow{g_2} & Y
\end{array} \quad ,
$$

so that for every object U with morphisms $f_1' : U \to X_1$, $f_2' : U \to X_2$, which form a commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{f_1'} & X_1 \\
\downarrow{\scriptstyle f_2'} & & \downarrow{\scriptstyle g_1} \\
X_2 & \xrightarrow{g_2} & Y
\end{array} \quad ,
$$

there is a unique morphism $\phi : U \to V$ with $f_1 \phi = f_1'$ and $f_2 \phi = f_2'$.

If g_1 is an epimorphism, then f_2 is also an epimorphism.

Pullbacks and pushouts are closely linked to short exact sequences:

Proposition 2.1.4. *Let $V, X_1, X_2, Y \in \mathsf{mod}\, A$ with morphisms $f_1 : V \to X_1$, $f_2 : V \to X_2$, $g_1 : X_1 \to Y$, $g_2 : X_2 \to Y$. Then the following statements are equivalent:*

1. *The diagram*

$$V \xrightarrow{f_1} X_1 \qquad\qquad (2.3)$$
$$\downarrow{\scriptstyle f_2} \qquad \downarrow{\scriptstyle g_1}$$
$$X_2 \xrightarrow{g_2} Y$$

   is a pushout and a pullback.

2. *The sequence*

$$0 \longrightarrow V \xrightarrow{\left[\begin{smallmatrix} -f_1 \\ f_2 \end{smallmatrix}\right]} X_1 \oplus X_2 \xrightarrow{[g_1\ g_2]} Y \longrightarrow 0$$

   is exact.

In fact, if $\begin{bmatrix} -f_1 \\ f_2 \end{bmatrix}$ is a monomorphism, then it suffices to demand that (2.3) is a pushout. If $\begin{bmatrix} g_1 & g_2 \end{bmatrix}$ is an epimorphism, then it suffices to demand that (2.3) is a pullback.

Pushouts and pullbacks have the following important property:

Proposition 2.1.5. *If*

$$V \xrightarrow{e_1} X_1$$
$$\downarrow{\scriptstyle e_2} \qquad \downarrow{\scriptstyle f_1}$$
$$X_2 \xrightarrow{f_2} Y_1$$

and

$$X_1 \xrightarrow{g} Y_2$$
$$\downarrow{\scriptstyle f_1} \qquad \downarrow{\scriptstyle h_2}$$
$$Y_1 \xrightarrow{h_1} Z$$

are both pullbacks (pushouts), then the square

$$V \xrightarrow{ge_1} Y_2$$
$$\downarrow{\scriptstyle e_2} \qquad \downarrow{\scriptstyle h_2}$$
$$X_2 \xrightarrow{h_1 f_2} Z$$

is itself a pullback (pushout).

Recall that an object is called *simple* if is does not contain proper, non-zero subobjects.

In this thesis, we consider *abelian length categories*, that is, abelian categories where every object is of finite length and the isomorphism classes of objects form a set (see for example [7], p. 81).

Definition 2.1.6. Let $\mathcal{A}$ be an abelian category. An object $X \in \mathcal{A}$ is said to be of *finite length* if there is a filtration $X = X_0 \supset X_1 \supset \cdots \supset X_n = 0$ so that X_{i-1}/X_i is simple for all $1 \leq i \leq n$. We call $l(X) := n$ the *length* of X.

Definition 2.1.7. An object of finite length is called *indecomposable* if it cannot be written as a direct sum of proper subobjects.

Let A be an Artin algebra. Then $\mathsf{mod}\,A$, the category of finitely generated right modules over A, is an abelian length category.

On the other hand, we have the following, see for example [21], Theorem 3.3.6:

Theorem 2.1.8. *Every abelian category is a full subcategory of a category* $A - Mod$ *for some algebra* A, *where* $A - Mod$ *denotes the left modules over* A.

For an object $X \in \mathcal{A}$ define the *Loewy length* of X to be the smallest $n \in \mathbb{N}$ with a filtration

$$X = X_0 \supset X_1 \supset \cdots \supset X_n = 0$$

so that X_{i-1}/X_i is semisimple (i.e. a direct summand of simple objects) for all $1 \leq i \leq n$.

Then the following holds, see [7], 8.2:

Theorem 2.1.9. *The abelian length category* $\mathcal{A}$ *is equivalent to the module category of an Artin ring if and only if*

1. $\mathcal{A}$ *has only finitely many simple objects.*

2. $\dim_{\mathsf{End}(T)}(S, T) < \infty$ *for all simple objects* $S, T \in \mathcal{A}$.

3. *The supremum of the Loewy lengths of the objects in* $\mathcal{A}$ *is finite.*

Denote by A^{op} the opposite algebra of A, that is, the algebra where left and right multiplication are exchanged (see for example [21], Definition 1.1.7). Then the left modules over A^{op} are just the right modules over A:

Proposition 2.1.10. *There is a duality* $D : \mathsf{mod}(A^{op}) \to \mathsf{mod}\,A$.

Thus, the following property of finitely generated modules generalizes to abelian length categories:

Definition 2.1.11. Let $\mathcal{A}$ be an abelian length category and X an object in $\mathcal{A}$. Then there are indecomposable objects $X_1, \ldots, X_n$ so that

$$X \cong \bigoplus_{i=1}^{n} X_i.$$

The objects $X_1, \ldots, X_n$ are unique up to order and isomorphism.

If the object Y is isomorphic to a direct summand of X, we will use the notation $Y \mid X$. Accordingly, we will write $Y \nmid X$ if Y is not isomorphic to a direct summand of X.

Now we define subcategories:

Definition 2.1.12. We call $\mathcal{C}'$ a *subcategory* of $\mathcal{C}$ if all objects and morphisms of $\mathcal{C}'$ are objects and morphisms in $\mathcal{C}$. Such a subcategory is called *full* if for all objects $A, B \in \mathcal{C}'$, the equality $\mathsf{Hom}_{\mathcal{C}}(A, B) = \mathsf{Hom}_{\mathcal{C}'}(A, B)$ holds.

We are interested in the following kind of subcategories:

Definition 2.1.13. A full, additive subcategory $\mathcal{C}$ of an abelian length category $\mathcal{A}$ is called *subobject closed* if for every object $X \in \mathcal{C}$ and all subobjects X' of X, we have $X' \in \mathcal{C}$.

Definition 2.1.14. A full, additive subcategory $\mathcal{C}$ of the module category $\mathsf{mod}\, A$ of an Artin algebra A is called *cofinite* if all indecomposable modules, except finitely many, lie in $\mathcal{C}$.

As in [3], Definition 2.6.1, for $n \in \mathbb{N}$, we can define an *n-fold extension* of an object X by an object X' as an exact sequence

$$0 \longrightarrow X' \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow X_0 \longrightarrow X \longrightarrow 0 \,.$$

A map between two extensions is a commutative diagram

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & X' & \longrightarrow & X_{n-1} & \longrightarrow & \ldots & \longrightarrow & X_0 & \longrightarrow & X & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & X' & \longrightarrow & X'_{n-1} & \longrightarrow & \ldots & \longrightarrow & X'_0 & \longrightarrow & X & \longrightarrow & 0
\end{array}
$$

By adding symmetry and transitivity, this can be completed to an equivalence relation and we can define $\mathsf{Ext}^n(X, X')$ to be the set of equivalence classes of n-fold extensions of X by X'.

These are groups; in particular, for $\mathsf{Ext}^1(X, X')$, the abelian group structure corresponds to the Baer sum, see [21], section 1.8.2:

For

$$\eta_1 : \quad 0 \longrightarrow X' \xrightarrow{\ f_1\ } X_0 \xrightarrow{\ g_1\ } X \longrightarrow 0$$

and

$$\eta_2 : \quad 0 \longrightarrow X' \xrightarrow{\ f_2\ } X_0' \xrightarrow{\ g_2\ } X \longrightarrow 0$$

there is an object Z_1 with a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X' \oplus X' & \longrightarrow & Z_1 & \longrightarrow & X & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow{\scriptstyle\left[\begin{smallmatrix} \mathrm{id} \\ \mathrm{id} \end{smallmatrix}\right]} & & \\
0 & \longrightarrow & X' \oplus X' & \xrightarrow{\left[\begin{smallmatrix} f_1 & 0 \\ 0 & f_2 \end{smallmatrix}\right]} & X_0 \oplus X_0' & \xrightarrow{\left[\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right]} & X \oplus X & \longrightarrow & 0
\end{array}
$$

and an object Z with a commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & X' \oplus X' & \longrightarrow & Z_1 & \longrightarrow X \longrightarrow 0 \ . \\
& & \downarrow{\scriptstyle[\,\mathrm{id}\ \mathrm{id}\,]} & & \downarrow & \| \\
0 & \longrightarrow & X' & \longrightarrow & Z & \longrightarrow X \longrightarrow 0
\end{array}
$$

The object Z_1 can be found by taking the pullback, while Z can be found via the pushout.

Then

$$0 \longrightarrow X' \longrightarrow Z \longrightarrow X \longrightarrow 0$$

is the exact sequence $\eta_1 + \eta_2$.

It is possible to view $\mathsf{Ext}^1(X, X')$ as a module over $\mathsf{End}(X', X')$ and over $\mathsf{End}(X, X)^{op}$.

2.2 Auslander-Reiten sequences

An important tool for our proofs are Auslander-Reiten sequences, which we introduce in this section.

Let A be an Artin algebra. First, we need the definition of certain special modules:

Definition 2.2.1. A module $P \in \mathsf{mod}\, A$ is *projective* if for any epimorphism $g : X \to Y$ with $X, Y \in \mathsf{mod}\, A$ and morphism $h : P \to Y$, there is a morphism $s : P \to X$ such that $gs = h$.

A module $I \in \mathsf{mod}\, A$ is *injective* if for any monomorphism $f : X \to Y$ with $X, Y \in \mathsf{mod}\, A$ and morphism $h : X \to I$, there is a morphism $s : Y \to I$ such that $sf = h$.

Now, we can define projective covers:

Definition 2.2.2. Let $X, Y, W \in \mathsf{mod}\,A$. An epimorphism $f : X \twoheadrightarrow Y$ is called an *essential epimorphism* if for all morphisms $g : W \to X$ the morphism $fg : W \to Y$ is an epimorphism if and only if g is an epimorphism.

Definition 2.2.3. Let $X \in \mathsf{mod}\,A$. A *projective cover* of X is an essential epimorphism $P \twoheadrightarrow X$ so that P is a projective module.

An analogous definition exists for injective modules:

Definition 2.2.4. For $X \subset Y \in \mathsf{mod}\,A$, we say that Y is an *essential extension* of X if $W \cap X \neq 0$ for all submodules W of Y. A monomorphism $X \to I$ is called an *injective envelope* of X if I is injective and an essential extension of B.

For simplicity, when referring to the modules P and I, we will also call them projective covers and injective envelopes, respectively.
We have the following:

Theorem 2.2.5. *Every module $X \in \mathsf{mod}\,A$ has projective covers and injective envelopes which are unique up to isomorphism.*

The indecomposable simple, projective and injective modules are connected:

Proposition 2.2.6. *Every Artin algebra A has a finite number n of non-isomorphic simple modules $S_1, \ldots, S_n$. Their projective covers $P_1, \ldots, P_n$ are a complete list of non-isomorphic indecomposable projective A-modules.*

Their injective envelopes $I_1, \ldots, I_n$ are a complete list of non-isomorphic indecomposable injective A-modules. For all $1 \leq i \leq n$, we have $\mathrm{soc}\, I_i = S_i$.

Using the duality $D : \mathsf{mod}(A^{op}) \to \mathsf{mod}\,A$, we can define the *Auslander-Reiten translation* τ:

Proposition 2.2.7. *There is a map* $\mathrm{Tr} : \mathsf{mod}\,A \to \mathsf{mod}(A^{op})$, *so that for* $\tau = D\,\mathrm{Tr}$ *the following holds:*

1. *$\tau\left(\bigoplus_{i=1}^{n} M_i\right) \cong \bigoplus_{i=1}^{n} \tau M_i$ for $M_1, \ldots, M_n \in \mathsf{mod}\,A$.*

2. *$\tau M = 0$ if and only if M is projective.*

3. *$\tau^{-1}M = 0$ if and only if M is injective.*

4. *τM is non-injective for all $M \in \mathsf{mod}\,A$.*

5. If no direct summand of M, N is projective, then $\tau M \cong \tau N$ if and only if $M \cong N$.

Next, we introduce *almost split morphisms*:

Definition 2.2.8. Let $X, Y \in \mathsf{mod}\, A$. A morphism $f : X \to Y$ is called a *split epimorphism* if the identity $1_Y : Y \to Y$ factors through f, that is, if there is some $g : Y \to X$ so that $1_Y = fg$.

The morphism f is called a *split monomorphism* if the identity $1_X : X \to X$ factors through f.

Definition 2.2.9. A morphism $f : X \to Y$ is called *right almost split* if the following conditions are fulfilled:

1. f is not a split epimorphism.

2. Any morphism $M \to Y$ with $M \in \mathsf{mod}\, A$, which is not a split epimorphism factors through f.

A morphism $f : X \to Y$ is called *left almost split* if the following conditions are fulfilled:

1. f is not a split monomorphism.

2. Any morphism $X \to M$ with $M \in \mathsf{mod}\, A$, which is not a split monomorphism factors through f.

Especially important are exact sequences where the morphisms are almost split:

Definition 2.2.10. Let $X, Y, Z \in \mathsf{mod}\, A$. Then the exact sequence

$$0 \longrightarrow X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \longrightarrow 0 \tag{2.4}$$

is called an *almost split sequence* or *Auslander-Reiten sequence* if f is left almost split and g is right almost split.

For every indecomposable, non-injective module X, there is an Auslander-Reiten sequence (2.4). The same holds for every indecomposable, non-projective module Z:

Theorem 2.2.11. *Let $X, Z \in \mathsf{mod}\, A$. Then the following assertions are equivalent:*

1. There is some module $Y \in \mathsf{mod}\, A$ with an Auslander-Reiten sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0 \, .$$

2. X is indecomposable, non-injective and $Z \cong \tau^{-1}X$.

3. Z is indecomposable, non-projective and $X \cong \tau Z$.

In fact, we have the following:

Theorem 2.2.12. *Auslander-Reiten sequences are unique up to isomorphism, that is, if*

$$0 \longrightarrow X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \longrightarrow 0$$

and

$$0 \longrightarrow X \xrightarrow{\ f'\ } Y' \xrightarrow{\ g'\ } Z \longrightarrow 0$$

are Auslander-Reiten sequences, then there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & X & \xrightarrow{\ f'\ } & Y' & \xrightarrow{\ g'\ } & Z & \longrightarrow & 0
\end{array}
$$

There is another important kind of morphisms:

Definition 2.2.13. A morphism $f : X \to Y$ is called *irreducible* if the following conditions are fulfilled:

1. f is neither a split epimorphism nor a split monomorphism.

2. If there are $t : X \to M$ and $s : M \to Y$ so that $f = st$, then either s is a split monomorphism or t is a split epimorphism.

An irreducible morphism is either a monomorphism or an epimorphism.

Irreducible morphisms are the components of almost split morphisms:

Theorem 2.2.14. *(a) Let X be an indecomposable, non-injective module and $f : X \to Y$ a morphism. Then f is irreducible if and only if there are some modules Y', Z, morphisms $f' : X \to Y'$, $g : Y \to Z$, $g' : Y' \to Z$ and an Auslander-Reiten sequence*

$$0 \longrightarrow X \xrightarrow{\ \left[\begin{smallmatrix} f \\ f' \end{smallmatrix}\right]\ } Y \oplus Y' \xrightarrow{\ [g\ g']\ } Z \longrightarrow 0 . \qquad (2.5)$$

(b) Dually, let Z be an indecomposable, non-projective module and $g : Y \to Z$ a morphism. Then g is irreducible if and only if there are some modules X, Y' and morphisms $f : X \to Y$, $f' : X \to Y'$, $g : Y \to Z$, $g' : Y' \to Z$ so that (2.5) is an Auslander-Reiten sequence.

2.3   Hereditary algebras and quivers

In this section, we take A to be a *hereditary* algebra, that is, an algebra, where all left ideals are projective. For such an algebra, every submodule of a projective module is itself projective. We collect some properties that we need in particular in Chapter 3.

Equivalently, we can give the following definition (see [15], Section 4.1):

Definition 2.3.1. An algebra A is *hereditary* if for all simple modules $S, T \in \mathsf{mod}\, A$, we have $\mathsf{Ext}^2(S, T) = 0$.

We are mainly working with the following kind of modules:

Definition 2.3.2. An indecomposable module $X \in \mathsf{mod}\, A$ is called preprojective if there is some non-zero projective module P and a non-negative integer n so that $\tau^n X = P$.

We set $\nu(X) := n$.

An arbitrary $M \in \mathsf{mod}\, A$ is called *preprojective* if its indecomposable direct summands are preprojective.

An indecomposable module $X \in \mathsf{mod}\, A$ is called preinjective if there is some non-zero injective module I and a non-negative integer n so that $\tau^{-n} X = I$.

We set $\mu(X) := n$.

An arbitrary $M \in \mathsf{mod}\, A$ is called *preinjective* if its indecomposable direct summands are preinjective.

Let $\mathcal{P}$ be the full subcategory of $\mathsf{mod}\, A$ that consists of the preprojective modules and $\mathcal{I}$ be the full subcategory of $\mathsf{mod}\, A$ that consists of the preinjective modules.

Proposition 2.3.3. *The duality* $D : \mathsf{mod}\, A \to \mathsf{mod}\, A^{op}$ *induces a duality between* $\mathcal{P}$ *and* $\mathcal{I}$.

So results for preprojective modules induce analogous results for preinjective modules and the other way around.

We need the following properties of preinjective modules:

Proposition 2.3.4. *Let M be in $\mathcal{P}$ or $\mathcal{I}$. If there are indecomposable modules $M = M_0, M_1, \ldots, M_{n-1}, M_n = M$ and non-zero morphisms $f_i : M_{i-1} \to M_i$ for $1 \leq i \leq n$, then f_i is an isomorphism for all $1 \leq i \leq n$.*

Lemma 2.3.5. *Let A be a hereditary Artin algebra and $X, Y \in \mathsf{mod}\, A$ be indecomposable. If there is an irreducible morphism $f : X \to Y$, then $X \in \mathcal{I}$ if and only if $Y \in \mathcal{I}$.*

Lemma 2.3.6. *Let A be a hereditary Artin algebra and $X, Y \in \mathcal{I}$ be inde-composable. If there is an irreducible morphism $f : X \to Y$, then*

1. *If X is injective, then Y is injective.*

2. *If X is not injective, then $0 \leq \mu(X) - 1 \leq \mu(Y) \leq \mu(X)$.*

Proposition 2.3.7. *Let $f : X \to Y$ be an irreducible morphism. Then the following holds:*

(a) If no direct summand of X is projective, then $\tau f : \tau X \to \tau Y$ is irre-ducible.

(b) If no direct summand of Y is injective, then $\tau^{-1}X \to \tau^{-1}Y$ is irreducible.

(c) The translation τ preserves monomorphisms.

Lemma 2.3.8. *Let $S_1, \ldots, S_n$ be a complete list of simple A-modules up to isomorphism. Then for all $1 \leq i, j \leq n$ either $\mathsf{Ext}^1(S_i, S_j) = 0$ or $\mathsf{Ext}^1(S_j, S_i) = 0$.*

Furthermore, $\mathsf{Ext}^1(S_i, S_j) \neq 0$ if and only if there is an irreducible mor-phism $I_j \to I_i$ for the injective envelopes I_i, I_j of S_i, S_j.

Lemma 2.3.9. *Let I_i, I_j be the injective envelopes of S_i, S_j with an irre-ducible morphism $I_j \twoheadrightarrow I_i$. Then the multiplicity of I_j in the Auslander-Reiten sequence that ends in I_i is $\dim_{\mathsf{End}_A(S_i)^{op}}(\mathsf{Ext}^1(S_i, S_j))$. On the other hand, the multiplicity of τI_i in the Auslander-Reiten sequence that ends in I_j is $\dim_{\mathsf{End}_A(S_j)}(\mathsf{Ext}^1(S_i, S_j))$.*

An important example for hereditary algebras are path algebras of quiv-ers:

Definition 2.3.10. A *quiver* $Q = (Q_0, Q_1)$ is an oriented graph where Q_0 is the set of vertices, while Q_1 is the set of arrows between vertices. If $\alpha : i \to j$ is an arrow, its *start point* is $s(\alpha) = i$ and its *end point* is $e(\alpha) = j$.

For all $i \in Q_0$, there is a *trivial path* e_i with $s(e_i) = e(e_i) = i$.

A *path* in this quiver is either a trivial path or a sequence $p = \alpha_n \ldots \alpha_1$ of arrows so that $e(\alpha_i) = s(\alpha_{i+1})$ for all $1 \leq i < n$. In this case, we define $s(p) = s(\alpha_1)$ and $e(p) = e(\alpha_n)$.

An *oriented cycle* is a non-trivial path so that $e(p) = s(p)$.

A quiver is called *finite* if Q_0 is a finite set.

It is natural to define the following algebra:

Definition 2.3.11. Let k be a field and kQ be the vector space with the paths of Q as a basis. Then there is an algebra structure on kQ induced by the concatenation of paths. That is, the product of $\alpha_n \ldots \alpha_1$ and $\alpha'_{n'} \ldots \alpha'_1$ is $\alpha_n \ldots \alpha_1 \alpha'_{n'} \ldots \alpha'_1$ if $s(\alpha_1) = e(\alpha'_{n'})$ and 0 otherwise.

Analogously, for every path p, we have $e_i p = p$ if $e(p) = i$ and 0 otherwise; $pe_i = p$ if $s(p) = i$ and 0 otherwise.

The module category of every Artin algebra is equivalent to the module category of a basic Artin algebra. So the following result in fact describes the module categories of all hereditary Artin algebras over algebraically closed fields:

Proposition 2.3.12. *Let k be a field and Q be a finite Quiver without oriented cycles. Then kQ is a hereditary Artin algebra.*

If k is algebraically closed, then for every basic hereditary Artin algebra A over k, there is some quiver Q so that A is isomorphic to kQ.

We can furthermore define representations over quivers:

Definition 2.3.13. Let $Q = (Q_0, Q_1)$ be a quiver and k a field. A *representation* (V, f) of Q over k is a set of vector spaces $\{V_i \mid i \in Q_0\}$ over k together with linear maps $f_\alpha : V_i \to V_j$ for each arrow $\alpha : i \to j$.

A morphism $h : (V, f) \to (V', f')$ between representations is a collection $\{h_i : V_i \to V'_i\}_{i \in Q_0}$ of linear maps so that the diagrams

$$
\begin{array}{ccc}
V_i & \xrightarrow{h_i} & V'_i \\
{\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle f'_\alpha} \\
V_j & \xrightarrow{h_j} & V'_j
\end{array}
$$

commute for all arrows $\alpha : i \to j$ in Q_1.

The composition of morphisms is of course induced by the composition of linear maps.

Then we get the following:

Proposition 2.3.14. *The category of representations of Q over k is equivalent to the category of finite dimensional modules over kQ.*

For every abelian length category $\mathcal{A}$, we can define its Ext-quiver: The vertices are given by a complete set of non-isomorphic simple objects in $\mathcal{A}$. If S and T are simple objects, there is an arrow $S \to T$ if $\mathsf{Ext}^1(S, T) \neq 0$.

If $\mathcal{A}$ is equivalent to the module category of a hereditary Artin algebra, then its Ext-quiver has no oriented cycle.

Furthermore, for every Artin algebra A, one can define its *Auslander-Reiten quiver*: Its vertices are the isomorphism classes of modules in mod A and there is an arrow between two isomorphism classes $[M]$ and $[N]$ if there is an irreducible morphism $M \to N$.

Attached to such an arrow is the label (a, b), where a is the maximal positive integer with an irreducible morphism $M^a \to N$, while b is the maximal positive integer with an irreducible morphism $M \to N^b$.

Let A be a hereditary algebra and $[M] \to [N]$ have the valuation (a, b). If $\tau^\alpha M$ and $\tau^\alpha N$ are non-zero for some integer α, then the valuation of $[\tau^\alpha M] \to [\tau^\alpha N]$ is also (a, b). If $\tau^{\beta+1}N$ and $\tau^\beta M$ are non-zero for some integer β, then $[\tau^{\beta+1}N] \to [\tau^\beta M]$ has the valuation (b, a).

The following holds:

Proposition 2.3.15. *Let A be an Artin algebra. If the Auslander-Reiten quiver of A has a finite component, then A is representation finite.*

Moreover, if A is hereditary and representation finite, then every module in mod A is both preinjective and preprojective and the number of components of the Auslander-Reiten quiver of A is the same as the number of blocks of A. So if A is indecomposable as an algebra, then the Auslander-Reiten quiver of A consists only of one component.

For simplicity, when drawing an Auslander-Reiten quiver, we will omit all labels (a, b) where $a = b = 1$ and use representatives of the isomorphism classes as vertices.

In the same vein, we will not always differentiate between modules (or objects) and isomorphism classes of modules (or objects) when the meaning is clear. For example, when we have a complete set $I_1, \ldots, I_n$ of non-isomorphic injective modules over a hereditary Artin algebra A, we will simply call $I_1, \ldots, I_n$ the injective modules over A and say that every indecomposable, preinjective module M is of the form $\tau^r I_i$ for some $1 \le i \le n$ and some $r \in \mathbb{N}$, when it actually is only isomorphic to such a module.

2.4 The Weyl group as a Coxeter group

We define words following [13]:

Definition 2.4.1. Let S be a set. We call S an *alphabet* and its elements *letters*. A *word* over the alphabet S is a finite sequence

$$(s_1, s_2, \ldots, s_n), s_i \in S.$$

The product of two words is just the concatenation of the sequences. This product is associative and by identifying a letter $s \in S$ with the sequence (s), we can write the word $(s_1, s_2, \ldots, s_n)$ as the product $s_1 s_2 \ldots s_n$. The neutral element for this product is the empty word, which we accordingly denote as 1. Thus, the set of words over S together with the concatenation forms a *monoid S^**.

If $\underline{w} := s_1 s_2 \ldots s_n$ is a word over S, then $l(\underline{w}) := n$ is called the *length* of $\underline{w}$. Furthermore, a word of the form $\underline{v} = s_{i_1} s_{i_2} \ldots s_{i_m}$ with

$$1 \le i_1 < i_2 < \cdots < i_m \le n$$

and $m \le n$ is a *subword* of $\underline{w}$.

If $\underline{v} = s_1 s_2 \ldots s_m$ with $m \le n$, then we say that $\underline{v}$ is an *initial subword* of $\underline{w}$.

An introduction into Coxeter groups can be found in [4]. We only need the following properties:

Definition 2.4.2. Let S be a set and W a group generated by S. Then W is called a *Coxeter group* if all relations have the form $(ss')^{m(s,s')} = 1$ with $s, s' \in S$ so that

1. $m(s, s') = 1$ if and only if $s = s'$.

2. If $m(s, s')$ exists, then $m(s', s)$ also exists and $m(s, s') = m(s', s)$.

If there is no relation between s and s', then we write $m(s, s') = m(s', s) = \infty$.

We can describe the Coxeter group W through the monoid S^*:

Proposition 2.4.3. *Let S be a set and S^* the monoid of words over S. Let W be a Coxeter group generated by S with relations $(ss')^{m(s,s')} = 1$.*

Set $\equiv$ to be the equivalence relation on S^* which is generated by allowing the insertion or deletion of words of the form

$$(ss')^{m(s,s')} = \underbrace{ss'ss' \ldots ss'}_{2m(s,s') \ letters}$$

for all $m(s, s') < \infty$. Then $S^*/\equiv$ is isomorphic to W.

We will use the following notation:

Definition 2.4.4. Set $\{ss'\}^a := \underbrace{ss'ss' \ldots}_{a \ letters}$.

The next lemma makes it easier to work with the relations:

Lemma 2.4.5. *Let S, W and $\equiv$ be as in Proposition 2.4.3 and $s, s' \in S$. The equivalence of words $\{s's\}^a \equiv \{ss'\}^a$ holds if and only if $m(s, s')$ is a factor of a.*

Let $S_1, \ldots S_n$ with $n \in \mathbb{N}$ be a complete list of non-isomorphic simple modules of the Artin algebra A.

We can associate to A a Cartan matrix, as in [2], pp 69, 241 and 288:

Definition 2.4.6. To a hereditary Artin algebra A we associate the *Cartan matrix* $C = (c_{ij})_{nn}$ of the underlying graph of the quiver A^{op}.

That is, we set $c_{ii} = 2$. If $i \neq j$ and $\mathsf{Ext}^1(S_i, S_j) = \mathsf{Ext}^1(S_j, S_i) = 0$, then $c_{ij} = c_{ji} = 0$. Finally, if $\mathsf{Ext}^1(S_i, S_j) \neq 0$, set

$$c_{ij} = -\dim_{\mathsf{End}_A(S_i)^{op}} \mathsf{Ext}^1(S_i, S_j)$$

and

$$c_{ji} = -\dim_{\mathsf{End}_A(S_j)} \mathsf{Ext}^1(S_i, S_j).$$

A description of the Weyl group as a Coxeter group can be found in [11], Proposition 3.13:

Proposition 2.4.7. *The Weyl group associated to A with the Cartan matrix $(c_{ij})_{nn}$ is a Coxeter group generated by the reflections $s_1, s_2, \ldots, s_n$ with relations $s_i^2 = 1$ for all $1 \leq i \leq n$ and $(s_i s_j)^{m_{ij}} = 1$ for all $i \neq j$, where m_{ij} depends on $c_{ij} c_{ji}$ in the following way:*

$$\begin{array}{c|ccccc} c_{ij}c_{ji} & 0 & 1 & 2 & 3 & \geq 4 \\ \hline m_{ij} & 2 & 3 & 4 & 6 & \infty \end{array}.$$

We can write all relations as $(s_i s_j)^{m_{ij}}$ if we set $m_{ij} := 1$ for $i = j$.

Every element of the Weyl group is the equivalence class of several different words over the alphabet $S := \{s_1, s_2, \ldots, s_n\}$. To distinguish between the elements of the Weyl group and the words over S, we will always use underlined letters to denote words and normal letters for Weyl group elements.

Remark 2.4.8. For $A = kQ$ with a field k and a quiver Q without oriented cycles, the relations depend only on the edges in the underlying graph of Q, see e.g. [14], p. 570:

We have $m_{ij} = 2$ if there is no edge between the vertices i and j and $m_{ij} = 3$ if there is exactly one edge between i and j. If there are two or more edges between i and j, then $m_{ij} = \infty$.

Example 2.4.9. Let Q be the quiver

$$\begin{array}{ccc} 1 & \to & 3 \\ \downarrow & & \uparrow \\ 4 & \leftarrow & 2 \end{array}.$$

The Weyl group of $A = kQ$ is a Coxeter group with the relations $(s_i s_j)^{m_{ij}} = 1$ and the following values for m_{ij}:

	1	2	3	4
1	1	2	3	3
2	2	1	3	3
3	3	3	1	2
4	3	3	2	1

.

2.5 String Algebras

To define string algebras, we need quivers with relations:

Definition 2.5.1. Let k be a field and Q a quiver. A *relation* on Q over k is a k-linear combination $a_1 p_1 + \cdots + a_n p_n$ of paths $p_1, \ldots, p_n$ with $a_1, \ldots, a_n \in k$, $e(p_1) = \cdots = e(p_n)$ and $s(p_1) = \cdots = s(p_n)$. Let I be an ideal in kQ generated by a set of relations ρ. Then (Q, ρ) is called a *quiver with relations* and kQ/I is called the *path algebra* of (Q, ρ). Furthermore, we call I *admissible* if $J^t \subset I \subset J^2$, for some $t \in \mathbb{N}$, where J is the ideal generated by all arrows.

We can define the representations (V, f) of (Q, ρ) analogously to the representations of Q in Definition 2.3.13, but with the following additional condition:

For a path $p = \alpha_1 \alpha_2 \ldots \alpha_n$ where α_i is an arrow for $1 \leq i \leq n$, we set $f_p := f_{\alpha_1} f_{\alpha_2} \ldots f_{\alpha_n}$. Then for every relation $a_1 p_1 + \cdots + a_n p_n$ in ρ, we demand $a_1 f_{p_1} + \cdots + a_n f_{p_n} = 0$.

As before, $\mathsf{mod}\, kQ/I$ is equivalent to the category of representations of (Q, ρ).

Remark 2.5.2. Let Q be a finite quiver and k a field. If Q has no oriented cycles, then to every vertex i corresponds a simple module $S_i \in \mathsf{mod}\, kQ$. If the number of arrows $i \to j$ is denoted by n, then

$$\dim_{\mathsf{End}(S_i)^{op}} \mathsf{Ext}^1(S_i, S_j) = \dim_{\mathsf{End}(S_j)} \mathsf{Ext}^1(S_i, S_j) = n.$$

The same result holds for arbitrary finite quivers with an admissible ideal I and $\mathsf{mod}\, kQ/I$.

Furthermore, $\mathsf{Ext}^2(S_i, S_j) \neq 0$ if and only if there is a relation $a_1 p_1 + \cdots + a_n p_n$ in the set that generates I with $e(p_1) = i$ and $s(p_1) = j$.

Similarly to the hereditary case, the following theorem describes the module categories of all Artin algebras over algebraically closed fields:

Theorem 2.5.3. *Every basic, finite dimensional algebra over an algebraically closed field k is isomorphic to kQ/I for some quiver Q and admissible ideal I.*

A special kind of quiver algebras are string algebras as described in [5]:

Definition 2.5.4. Suppose that Q is a quiver and I an ideal in kQ which is generated by a set of zero relations.
   Then $A = kQ/I$ is a *string algebra* if and only if

1. Any vertex of Q is starting point of at most two arrows.

2. Any vertex of Q is end point of at most two arrows.

3. Given an arrow β, there is at most one arrow γ with $s(\beta) = e(\gamma)$ and $\beta\gamma \notin I$.

4. Given an arrow γ, there is at most one arrow β with $s(\beta) = e(\gamma)$ and $\beta\gamma \notin I$

5. Given an arrow β_1, there is some bound $n(\beta)$ such that any path $\beta_1\beta_2\ldots\beta_{n(\beta)}$ contains a subpath in I.

6. Given an arrow β, there is some bound $n'(\beta)$ such that any path $\beta_1\beta_2\ldots\beta_{n'(\beta)}$ with $\beta_{n'(\beta)} = \beta$ contains a subpath in I.

Definition 2.5.5. We can take the formal inverse β^{-1} of an arrow β by defining $e(\beta^{-1}) := s(\beta_n)$ and $s(\beta^{-1}) := e(\beta)$.
   A *string* is a word $w = \beta_1\beta_2\ldots\beta_n$ so that

- β_i is either an arrow or the inverse of an arrow for all $1 \leq i \leq n$

- $s(\beta_i) = e(\beta_{i+1})$ for all $1 \leq i \leq n$

- w does not contain a relation in I

The multiplication of strings is analogous to the multiplication of paths of a quiver.
   A *band* is a string $w = \beta_1\beta_2\ldots\beta_n$ such that every power of w is defined and does not contain a relation in I; furthermore w may not be a power of a string $w' \neq w$.

String algebras are especially useful, since their modules are well known, also from [5]:

Definition 2.5.6. Suppose that $w = \beta_1\beta_2\ldots\beta_n$ is a string. Set $u(i) = e(\beta_{i+1})$, for $0 \leq i < n$, and $u(n) = s(\beta_n)$. The *string module $M(w)$* is defined as the representation where for every $v \in Q_0$, the vector space $M(w)_v$ has as basis

$$\{z_i \mid u(i) = v\}$$

with $z_i \neq z_j$ for $i \neq j$. If β_i is an arrow, then $f_{\beta_i}(z_{i-1}) = z_i$, otherwise $f_{\beta_i^{-1}}(z_i) = z_{i-1}$. For all other arrows α, we have $f_\alpha = 0$.

Now suppose that w is even a band and $\phi : Z \to Z$ is an automorphism on a vector space over k. The *band module $M(w, \phi)$* is defined as the representation with

$$M(w, \phi)_v = \bigoplus_{e(\beta_{i+1})=v} Z_i$$

where $Z_i = Z$.

If β_1 is an arrow and $z \in Z_1$, then $f_{\beta_1}(z) = \phi(z_1) \in Z_0$. If β_1^{-1} is an arrow, then for $z \in Z_0$, $f_{\beta_1^{-1}}(z) = \phi^{-1}(z) \in Z_1$.

Let $2 \leq i \leq n$. If β_i is an arrow and $z \in Z_i$, then $f_{\beta_i}(z) = z \in Z_{i-1}$; if β_i^{-1} is an arrow and $z \in Z_{i-1}$, then $f_{\beta_i^{-1}}(z) = z \in Z_i$.

For all other arrows α, we have $f_\alpha = 0$.

Lemma 2.5.7. *Let $A = kQ$ be a string algebra with a string $w = \beta_1\beta_2\ldots\beta_n$.*

1. *All A-modules are isomorphic to a string module or a band module*

2. *Two string modules $M(w)$ and $M(w')$ are isomorphic if and only if $w = w'$ or $w' = w^{-1} := \beta_n^{-1}\beta_{n-1}^{-1}\ldots\beta_1^{-1}$.*

3. *Two band modules $M(w, \phi)$ and $M(w', \phi')$ are isomorphic if and only if ϕ and ϕ' are similar and w or w^{-1} is a cyclic permutation of w'.*

4. *No band module is isomorphic to a string module.*

In [6], p. 34 there is a result about morphisms between tree modules that reduces very nicely to monomorphisms between string modules:

Lemma 2.5.8. *$M(w)$ is a submodule of $M(w')$ if and only if there are arrows α, β and strings w_1, w_2 so that w' is of the form*

$$w_1\alpha^{-1}w\beta w_2$$

or

$$w\beta w_2$$

or

$$w_1\alpha^{-1}w.$$

3 | Submodule closed categories and the Weyl group

In this chapter, let A be a hereditary Artin algebra over an arbitrary field.

We aim to prove that there is a natural bijection between the Weyl group and the set of full additive cofinite submodule closed subcategories of the module category. Oppermann, Reiten and Thomas have shown this in [14] for algebraically closed fields and finite fields. While we use the same bijection, we will give a completely different method of proof that does not depend on the field.

First of all, we regard the Weyl group as a Coxeter group. This allows us to regard the Weyl group elements as equivalence classes of words. In Section 3.1, we define a total order on these words and call the smallest word of each equivalence class *leftmost*. Then we collect some results about this order.

We conclude Section 3.1 by stating the bijection, which is induced by a map between words of Weyl group elements and sets of preinjective modules. In Section 3.4 to Section 3.6, we will prove that a cofinite, full additive subcategory is submodule closed if and only if a leftmost word is mapped to its complement. Since we can assign a unique leftmost word to every element of the Weyl group, this gives a bijection between the full additive cofinite submodule closed subcategories and the Weyl group.

For this proof, we will use the results of Section 3.2, which is devoted to monomorphisms between preinjective modules. In particular, we give a way to construct all modules that contain a given preinjective module as a submodule. This allows us to draw some lemmas in Section 3.3 about the structure of full additive cofinite submodule closed subcategories and how they are related to the words of Weyl group elements.

In the sections 3.4 to 3.6, we use this to prove inductively that the proposed bijection exists. Finally, we conclude this chapter with some corollaries.

Note that in the following, a submodule closed subcategory will always

mean a full additive submodule closed subcategory of $\mathsf{mod}\,A$. Furthermore, we are equating modules with isomorphism classes of modules, since submodule closed subcategories are closed under isomorphisms.

3.1 Leftmost words

Let A be a hereditary Artin algebra and $\mathsf{mod}\,A$ the category of finitely presented modules over A. Furthermore, let $\mathcal{I}$ be the subcategory of $\mathsf{mod}\,A$ consisting of all preinjective modules.

We order the simple modules $S_1, \ldots, S_n$ of A with injective envelopes $I_1, \ldots, I_n$ in such a way that $\mathsf{Hom}(I_i, I_j) = 0$ if $i < j$. This is possible by Lemma 2.3.8.

Furthermore, let W be the Weyl group of A. Denote by

$$S := \{s_1, s_2, \ldots, s_n\}$$

the set of generators of W and by

$$(s_i s_j)^{m_{ij}} = 1$$

the defining relations of W.

Definition 3.1.1. Consider $\mathcal{N} = (\mathbb{N}_0 \times \{1, 2, \ldots, n\}, <)$, where $<$ is the lexicographic order: for pairs $(r, i), (r', j) \in \mathcal{N}$, we have $(r, i) < (r', j)$ if and only if one of the following holds:

1. $r < r'$

2. $r = r'$ and $i < j$.

Let $\underline{w} = s_{i_1} s_{i_2} \ldots s_{i_m}$ be a word over the alphabet S and $0 = r_1 \leq r_2 \leq \cdots \leq r_m \in \mathbb{N}_0$ the smallest non-negative integers so that

$$(r_1, i_1) < (r_2, i_2) < \cdots < (r_m, i_m)$$

is fulfilled. Then we define

$$\rho(\underline{w}) := (r_1, i_1)(r_2, i_2) \ldots (r_m, i_m).$$

Example 3.1.2. Consider the Weyl group of the quiver Q from Example 2.4.9. If we set $\underline{w} := s_2 s_3 s_1 s_3 s_4 s_1$ then

$$\rho(\underline{w}) = (0, 2)(0, 3)(1, 1)(1, 3)(1, 4)(2, 1).$$

Now we can define a total order $<_l$ on the words of W; this is again a lexicographic order:

Definition 3.1.3. Consider two words $\underline{w}, \underline{w}'$ with

$$\rho(\underline{w}) = (r_1, i_1)(r_2, i_2) \ldots (r_m, i_m)$$

and

$$\rho(\underline{w}') = (r_1', i_1')(r_2', i_2') \ldots (r_{m'}', i_{m'}').$$

We write $\underline{w} <_l \underline{w}'$ if one of the following holds:

1. $m < m'$

2. $m = m'$ and there is a $j \in \mathbb{N}$ so that

$$(r_1, i_1) = (r_1', i_1'), (r_2, i_2) = (r_2', i_2'), \ldots, (r_{j-1}, i_{j-1}) = (r_{j-1}', i_{j-1}')$$

and

$$(r_j, i_j) < (r_j', i_j').$$

Now we define the leftmost word; this definition can be found for example in [1], p. 411:

Definition 3.1.4. We call a word $\underline{w}$ for $w \in W$ *leftmost* if for every other word $\underline{w}'$ for w the inequality $\underline{w} <_l \underline{w}'$ holds.

Example 3.1.5. For the Weyl group from Example 2.4.9, the words

$$s_3 <_l s_2 s_3 <_l s_3 s_2 <_l s_2 s_3 s_2$$

are all leftmost words and

$$s_2 s_3 s_2 <_l s_3 s_2 s_3 <_l s_2 s_3 s_1 s_2 s_1.$$

are all words for the same element of the Weyl group.

Since $<_l$ is a total order, every element $w \in W$ has a unique leftmost word. Obviously, the leftmost word is *reduced*, that is, it has the smallest possible length for a word of w.

We follow with a Lemma about the order $<_l$ and the relations:

Lemma 3.1.6. *Suppose that* $\underline{w}_1 = \underline{u}\{s_i s_j\}^{m_{ij}}\underline{v}$ *for some reflections* s_i, s_j, $i \neq j$, *words* $\underline{u}$, $\underline{v}$. *Set*

$$\rho(\underline{w}_1) = \rho(\underline{u}) \underbrace{(p, i)(q, j)(p+1, i) \ldots}_{m_{ij} \ pairs} \rho_1$$

for some $p, q \in \mathbb{N}_0$ *and a sequence of pairs* ρ_1. *Set*

$$\underline{w}_2 = \underline{u}\{s_j s_i\}^{m_{ij}}\underline{v}.$$

Then $\underline{w}_2 <_l \underline{w}_1$ *if and only if both of the following conditions are fulfilled:*

1. $1 \leq q$.

2. Let (r, k) be a pair in $\rho(\underline{u})$. Then $(r, k) < (q - 1, j)$.

Proof. Let $\rho(\underline{w}_2) = \rho(\underline{u}) \underbrace{(q', j)(p', i) \dots}_{m_{ij} \text{ pairs}} \rho_2$ for some sequence of pairs ρ_2.

Suppose that $\underline{w}_2 <_l \underline{w}_1$. Then $(q', j) < (p, i)$ by Definition 3.1.3 and thus $q = q' + 1$. So the first condition is fulfilled.

Now consider a pair (r, k) in $\rho(\underline{u})$. Then $(r, k) < (q', j) = (q - 1, j)$ and the second condition is fulfilled.

On the other hand, suppose that the conditions 1 and 2 are fulfilled. Then q' is the smallest integer so that (q', j) is bigger than all (r, k) in $\rho(\underline{u})$. By the second condition, $(q', j) \leq (q - 1, j)$.

Furthermore, q is the smallest integer so that $(p, i) < (q, j)$. It follows that $(q - 1, j) < (p, i)$, since $i \neq j$.

Together, $(q', j) \leq (q - 1, j) < (p, i)$ and by Definition 3.1.3, we have $\underline{w}_2 <_l \underline{w}_1$. $\square$

The following lemmas are important for the induction with which we prove the main theorem of this chapter:

Lemma 3.1.7. *Let $\underline{x}, \underline{x}', \underline{y}$ be words and $s_i \neq s_j$ reflections. We suppose that the words $\underline{w} = \underline{x} s_i \underline{y}$ and $\underline{w}' = \underline{x}' s_j \underline{y}$ are equivalent, $\underline{x}$ is reduced and $\underline{w} <_l \underline{w}'$. Let $\underline{z}$ be the longest initial subword that $\underline{w}$ and $\underline{w}'$ share. If there is no $\underline{x}'' \equiv \underline{x}'$ that shares an initial subword with $\underline{w}$ which is longer than $\underline{z}$, then there are pairs $(r, h), (s, i), (t, j)$ and series of pairs $\rho_1, \rho_2, \rho_3, \rho_4$ so that $\rho(\underline{x}) = \rho_1 (r, h) \rho_2$*

$$\rho(\underline{w}) = \rho_1 (r, h) \rho_2 (s, i) \rho_3$$

and there is some word $\underline{w}'' \equiv \underline{w}$ with

$$\rho(\underline{w}'') = \rho_1 \rho_2 (s, i)(t, j) \rho_4. \tag{3.1}$$

Either $\rho_3 = \rho_4$, or a pair (q, g) is in ρ_4 if and only if $(q - 1, g)$ is in ρ_3.

If $\underline{v}$ is the initial subword of $\underline{w}$ with $\rho(\underline{v}) = \rho_1$, then no relation on reflections in $\underline{v}$ is needed to transform $\underline{w}$ into $\underline{w}''$.

Proof. We prove this inductively on the number of relations that are needed to transform $\underline{w}'$ into $\underline{w}$. Without loss of generality, we assume that m_{ij} is odd. If m_{ij} is even, we only need to relabel s_i and s_j in the arguments below.

If there is some word $\underline{x}_1$ so that

$$\underline{w} = \underline{x}_1 \{s_i s_j\}^{m_{ij}} \underline{y} <_l \underline{x}_1 \{s_j s_i\}^{m_{ij}} \underline{y} = \underline{w}' \tag{3.2}$$

then $\underline{w}'' := \underline{x}_1\{s_j s_i\}^{m_{ij}}\underline{y}$ fulfils the assertions by Lemma 3.1.6.

To conclude the basis of our induction, we note that for two words

$$\underline{x}_1\{s_j s_i\}^{m_{ij}}\underline{y} <_l \underline{x}_1\{s_i s_j\}^{m_{ij}}\underline{y},$$

weget a similar result: Then

$$\rho(\underline{x}_1\{s_j s_i\}^{m_{ij}}) = \rho_1 \rho_2'(s-1,i)(t-1,j)\rho_4',$$

where a pair $(q-1,g)$ is in ρ_2' if and only if (q,g) is in ρ_2'. Furthermore, either $\rho_4' = \rho_3$ or a pair $(q-1,g)$ is in ρ_4' if and only if (q,g) is in ρ_3.

Now suppose that $\underline{w} = \underline{u}\{s_l s_k\}^{m_{kl}}\underline{v}$ is not of the form in equation (3.2), but the assertion is true for $\underline{w}_1 = \underline{u}\{s_k s_l\}^{m_{kl}}\underline{v}$.

If $\underline{w}_1 <_l \underline{w}'$, then there is some word $\underline{w}_1''$ with

$$\rho(\underline{w}_1) = \rho_1'(r',h')\rho_2'(s,i)\rho_3'$$

and

$$\rho(\underline{w}_1'') = \rho_1' \rho_2'(s,i)(t,j)\rho_4'.$$

for some pair (r',h') and series of pairs $\rho_1', \rho_2', \rho_3', \rho_4'$. Either $\rho_3' = \rho_4'$, or a pair (q,g) is in ρ_4' if and only if $(q-1,g)$ is in ρ_3'.

There is some $\underline{u}'$ so that $\rho_1' = \rho(\underline{u}')$.

Furthermore, there is a pair (q,l) and a series of pairs ρ so that

$$\rho(\underline{w}) = \rho(\underline{u})(q,l)\rho.$$

If $\underline{u}' = \underline{u}$ or $\underline{u}' = \underline{u}\{s_l s_k\}^{m_{lk}-1}$, then we set $(r,h) := (q,l)$. Since $\underline{w} <_l \underline{w}'$, we can assume that $\underline{w} <_l \underline{w}_1$ and by Lemma 3.1.6, the assertion is true.

Since $\underline{x}$ is reduced, there is only one other case: the word $\underline{w}_1''$ has $\{s_k s_l\}^{m_{kl}}$ as a subword and the relation $\{s_k s_l\}^{m_{kl}} \equiv \{s_l s_k\}^{m_{kl}}$ gives a word $\underline{w}''$ that fulfils (3.1).

It remains to prove the assertion if $\underline{w}' <_l \underline{w}_1$.

Then we can inductively assume that there is some $\underline{w}_1''$ so that

$$\rho(\underline{w}_1'') = \rho_1 \rho_2'(s-1,i)(t-1,j)\rho_4',$$

where a pair $(q-1,g)$ is in ρ_2' if and only if (q,g) is in ρ_2'. Furthermore, either $\rho_4' = \rho_3$ or a pair $(q-1,g)$ is in ρ_4' if and only if (q,g) is in ρ_3.

But there is no $\underline{x}'' \equiv \underline{x}'$ that shares an initial subword with $\underline{w}$ which is longer than $\underline{z}$, the longest initial subword that $\underline{w}$ and $\underline{w}'$ share.

Since $\underline{w}_1''$ and $\underline{w}_1$ share the initial subword $\underline{u}'$ with $\rho(\underline{u}_1) = \rho_1$, this is only possible if $\underline{u}' = \underline{u}\{s_l s_k\}^{m_{lk}-1}$.

Again, we set $(r,h) := (q,l)$. Since $\underline{x}$ is leftmost , $\underline{w} <_l \underline{w}_1$ and by Lemma 3.1.6, the assertion is true. $\qquad \square$

Completely analogously, we can prove the following:

Lemma 3.1.8. *Let $\underline{x}, \underline{x}', \underline{y}$ be words and $s_i \neq s_j$ reflections. If the words $\underline{w} = \underline{x}s_i\underline{y}$ and $\underline{w}' = \underline{x}'s_j\underline{y}$ are equivalent, $\underline{x}$ is leftmost and $\underline{w}' <_l \underline{w}$, then there are pairs $(r, h), (s, i)$ and series of pairs $\rho_1, \rho_2, \rho_3, \rho_4$ so that $\rho(\underline{x}') = \rho_1\rho_2$*

$$\rho(\underline{w}) = \rho_1\rho_2(s, i)\rho_3$$

and there is some word $\underline{w}'' \equiv \underline{w}$ with

$$\rho(\underline{w}'') = \rho_1(r, h)\rho_2\rho_4$$

with $(r, h) \neq (s, i)$ or $\rho_2 \neq 0$.

Either $\rho_3 = \rho_4$, or a pair (q, g) is in ρ_4 if and only if $(q + 1, g)$ is in ρ_3.

If $\underline{v}$ is the initial subword of $\underline{w}$ with $\rho(\underline{v}) = \rho_1$, then no relation on reflections in $\underline{v}$ is needed to transform $\underline{w}$ into $\underline{w}''$.

We get the following corollary:

Corollary 3.1.9. *If $\underline{u}\{s_is_j\}^{m_{ij}-1}$ and $\underline{u}s_j$ are leftmost, then either $\underline{u}\{s_is_j\}^{m_{ij}}$ is leftmost or $\underline{u}s_j < \underline{u}s_i$.*

Proof. If we have $\underline{u}\{s_js_i\}^{m_{ij}} <_l \underline{u}\{s_is_j\}^{m_{ij}}$, then $\underline{u}s_j < \underline{u}s_i$.

Furthermore, if $\underline{u}\{s_is_j\}^{m_{ij}}$ is not leftmost, but $\underline{u}\{s_is_j\}^{m_{ij}-1}$ is, then we can write $\underline{w} = \underline{x}s_i\underline{y}$ and there is some $\underline{w}' = \underline{x}'s_j\underline{y}$ so that $\underline{w} \equiv \underline{w}'$, $\underline{x}$ is leftmost and $\underline{w}' <_l \underline{w}$.

By Lemma 3.1.8, there are some words $\underline{u}_1, \underline{u}_2$ and a reflection s_h so that

$$\underline{u} = \underline{u}_1\underline{u}_2$$

$$\underline{w}'' = \underline{u}_1 s_h \underline{u}_2\{s_is_j\}^{m_{ij}-1} \equiv \underline{w}$$

and $\underline{w}'' <_l \underline{w}$.

So $\underline{u}s_j \equiv \underline{u}_1 s_h \underline{u}_2$ and $\underline{u}_1 s_h \underline{u}_2 <_l \underline{u}s_j$.   □

Remark 3.1.10. Note that in Lemma 3.1.8, we do not actually need to assume that $\underline{x}$ is leftmost; it is sufficient that $\underline{x}$ is reduced and the following holds: let $\underline{x}'' \equiv \underline{x}$ so that $\underline{x}'' <_l \underline{x}$. Furthermore, assume that $\underline{x}_2$ is the maximal initial subword that $\underline{x}'$ and $\underline{x}$ share. Then there is some $\underline{w}' \equiv \underline{w}$ with $\underline{w}' <_l \underline{x}'s_i\underline{y}$.

So analogously to 3.1.9, we see: Suppose that there is some word $\underline{w}'$ so that $\underline{u}\{s_is_j\}^{m_{ij}} \equiv \underline{w}'$ and for all $\underline{u}' \equiv \underline{u}\{s_is_j\}^{m_{ij}-1}$, we have $\underline{w}' <_l \underline{u}'s_i$ if $m_{ij} = 3$ and $\underline{w}' <_l \underline{u}'s_j$ otherwise. Then $\underline{u}s_j$ is not leftmost.

Lemma 3.1.11. *Suppose that $\underline{w} \equiv \underline{u}\{s_is_j\}^{m_{ij}}$ with $m_{ij} \geq 3$ and $\underline{u}$ reduced. If there is some $i \neq j \neq k$ with $\underline{w} = \underline{u}'s_k\{s_is_j\}^{m_{ij}-m}$ for some even $m \geq 2$ or $\underline{w} = \underline{u}'s_k\{s_js_i\}^{m_{ij}-m}$ for some odd $m \geq 3$, then $m_{ik} = 2$ or $m_{jk} = 2$.*

Proof. This is a simple, inductive proof: without loss of generality, we can assume that $\underline{w} = \underline{u}_1 s_k \{s_i s_j\}^{m_{ij}-2}$ and $m_{jk} = 3$. Then there is some $\underline{u}_2$ so that

$$\underline{w} \equiv \underline{u}_2 s_k s_j s_k \{s_i s_j\}^{m_{ij}-2} \equiv \underline{u}_2 s_j s_k \{s_j s_i\}^{m_{ij}-1}.$$

So there is some $\underline{u}_3$ so that

$$\underline{u}_2 s_j s_k \equiv \underline{u}_3 \{s_i s_k\}^{m_{ik}}.$$

If $m_{ik} \geq 3$, then we have the same situation as before, only considering a shorter word. Since $\underline{w}$ is finite, we see with induction on the length of $\underline{w}$ that $m_{ik} = 2$ or $m_{jk} = 2$. □

Similarly, we can prove the following:

Lemma 3.1.12. *Suppose that $\underline{u} s_j$ is leftmost, but $\underline{u} s_j s_i$ is not leftmost for some word $\underline{u}$ and some reflections s_i, s_j with $m_{ij} \geq 4$.*
Then there are $s, t \in \mathbb{N}$ so that $\rho(\underline{u} s_j s_i) = \rho(\underline{u})(t, j)(s, i)$ and $\rho(\underline{u})$ contains the pair $(s-1, i)$. If $m_{ij} = 6$, then $\rho(\underline{u})$ additionally contains the pairs $(s-2, i)$ and $(t-1, j)$.

Proof. Suppose that the assertions are not fulfilled.
We can without loss of generality assume that there is some $\underline{u}'$ and reflections $s_{k_1}, \ldots, s_{k_m}, s_{l_1}, \ldots, s_{l_{m'}}$ so that

$$\underline{u} s_j s_i = \underline{u}' s_j s_{k_1} \ldots s_{k_m} s_i s_{l_1} \ldots s_{l_{m'}} \{s_j s_i\}^{m_{ij}-2} =: \underline{u}'' \{s_j s_i\}^{m_{ij}-2}$$

Then there are $s', t', t'', q_1, \ldots, q_m, r_1, \ldots, r_{m'} \in \mathbb{N}$ so that

$$\rho(\underline{u}'' s_j) = \rho(\underline{u}')(t', j)(q_1, k_1) \ldots (q_m, k_m)(s', i)(r_1, l_1) \ldots (r_{m'}, l_{m'})(t'', j).$$

If $(t''-1, j) < (s', i)$, then the assertions of the lemma are fulfilled. Otherwise, we get one of the following cases:

(a) There is some $1 \leq o \leq m'$ with $(t''-1, j) < (r_o, l_o)$ with $m_{l_o, j} \geq 3$.

(b) The words $s_i s_{l_1} \ldots s_{l_{m'}} s_j$ and $\underline{u} s_i$ are not leftmost, contrary to the assumptions.

So we can assume that the first case is fulfilled. Furthermore, without loss of generality, we can assume $o = m'$.
If $m_{il_1} = \cdots = m_{im'} = 2$, then

$$\underline{u}'' \equiv \underline{u}' s_j s_{k_1} \ldots s_{k_m} s_{l_1} \ldots s_{l_{m'}} s_i =: \underline{u}'''.$$

Either $\underline{u}''' <_l \underline{u}''$, contrary to the assumptions, or there is some $s'' > s'$ so that

$$\rho(\underline{u}''') = \rho(\underline{u}')(t', j)(q_1, k_1) \ldots (q_m, k_m)(r_1, l_1) \ldots (r_{m'}, l_{m'})(s'', i).$$

Since $\underline{u}s_j$ is leftmost, but $\underline{u}s_j s_i$ is not, there is some v_1 so that

$$\underline{u}s_j s_i \equiv \underline{v}_1 \{s_j s_i\}^{m_{ij}}. \tag{3.3}$$

Thus,

$$\underline{u}'' \equiv \underline{u}' \{s_j s_{l_{m'}}\}^{m_{jl_{m'}}} s_i$$

and we are in an analogous situation to before, only considering a shorter word. Since the length of $\underline{w}$ is finite, the assertions of the lemma are inductively true under these assumptions.

So we can assume without loss of generality that $m_{il_1} \geq 3$. Furthermore, we have $(s'+1, i) < (t''-1, j) < (r_{m'}, l_{m'})$. So there is at least one $1 \leq o \leq m'$ so that $m_{ol_{m'}} \geq 3$, since otherwise $s_j s_{k_1} \ldots s_{k_m} s_i s_{l_1}$ is equivalent to a smaller word (that begins with s_j), contrary to the assumption that $\underline{u}s_j$ is leftmost.

Then there is some $\underline{v}_2$ so that

$$\underline{u}' s_j s_{k_1} \ldots s_{k_m} s_i s_{l_1} \equiv \underline{v}_2 \{s_i s_{l_1}\}^{m_{il_1}}.$$

Because of (3.3), Lemma 3.1.7, Lemma 3.1.8 and $m_{il_1} \geq 3$, we get some $\underline{v}_3$ so that

$$\underline{u}'' \equiv \underline{v}_3 s_j s_i s_{l_1} \ldots s_{l_m} s_i.$$

Since $m_{jl_m} \geq 3$ and $m_{l_o l_m} \geq 3$, we are in the same situation as before, only considering a word of shorter length. Inductively, the proof is complete.

$\square$

Now we can define an assignment which maps the words of the Weyl group to the cofinite full additive subcategories of $\mathsf{mod}\, A$. We will show that this map yields a bijection between the Weyl group and the set of cofinite submodule closed subcategories.

Let $\tau = D\mathrm{Tr}$ be the Auslander-Reiten translation, see Proposition 2.2.7. By Definition 2.3.2, every indecomposable preinjective module is isomorphic to $\tau^r I_i$ for some $r \in \mathbb{N}$ and $1 \leq i \leq n$.

Definition 3.1.13. We can identify the pairs in $\mathcal{N}$ and the indecomposable preinjective modules by setting $(r, i) = \tau^r I_i$.

Not only does this give us a natural order on the preinjective modules, but this also yields an injective map from the words of the Weyl group to the cofinite full additive subcategories of $\mathsf{mod}\, A$: If

$$\rho(\underline{w}) = (r_1, i_1)(r_2, i_2) \ldots (r_m, i_m),$$

then $\underline{w} \mapsto \mathcal{C}_{\underline{w}}$, where $\mathcal{C}_{\underline{w}}$ is the full additive category with

$$\text{ind}\, \mathcal{C}_{\underline{w}} = \text{ind}\, A \setminus \{(r_1, i_1), (r_2, i_2), \ldots, (r_m, i_m)\}.$$

For a Weyl group element w with the leftmost word $\underline{w}$, define $\mathcal{C}_w := \mathcal{C}_{\underline{w}}$.

Example 3.1.14. Let A be as in Example 2.4.9 and $\underline{w} = s_1 s_2 s_3 s_2 s_4 s_1$. Then

$$\rho(\underline{w}) = (0,1)(0,2)(0,3)(1,2)(1,4)(2,1)$$

and

$$\text{ind}\, \mathcal{C}_{\underline{w}} = \text{ind}\, A \setminus \{I_1, I_2, I_3, \tau I_2, \tau I_4, \tau^2 I_1\}.$$

We will prove that the restriction of this map on the leftmost words is a bijection between those and the cofinite submodule closed subcategories.

Since every element of the Weyl group has a unique leftmost word, this gives a bijection between the elements of the Weyl group and the cofinite submodule closed subcategories.

The same bijection is used in [14].

3.2 Monomorphisms between preinjective modules

An observation makes the aim of the chapter much simpler to achieve: the cofinite submodule closed subcategories of the module category correspond naturally to the cofinite submodule closed subcategories of $\mathcal{I}$, the category of the preinjective modules.

Thus we devote this section to preinjective modules. In particular, we give a way to construct all modules U that contain a given preinjective, indecomposable module M as a submodule.

In Section 3.3 we will use this to show the connection to the Coxeter structure of the Weyl group. In Section 3.4 to 3.6, we will use this connection to prove that the bijection that we described exists.

Proposition 3.2.1. *There is a bijection between full additive cofinite submodule closed subcategories of* $\text{mod}\, A$ *and full additive cofinite submodule closed subcategories of* $\mathcal{I}$*. It maps the category* $\mathcal{C}$ *to the category* $\mathcal{C}' = \mathcal{C} \cap \mathcal{I}$*. Furthermore,*

$$\text{ind}\, A \setminus \mathcal{C} = \text{ind}\, \mathcal{I} \setminus \mathcal{C}.$$

Proof. This is completely analogous to [14], Proposition 2.2:

If A is representation finite, then $\mathsf{mod}\,A = \mathcal{I}$ and there is nothing to prove. Suppose that A is not representation finite. Since $\mathcal{C}$ is cofinite, there is some $r \in \mathbb{N}$, so that $\tau^r I_1, \tau^r I_2, \ldots, \tau^r I_n \in \mathcal{C}$. Now suppose that M is a preprojective or regular module. Then $\tau^{-r} M$ exists and has an injective envelope I. Since τ^r preserves monomorphisms by 2.3.7, $M \subseteq \tau^r I \in \mathcal{C}$. So $M \in \mathcal{C}$ and $\mathsf{ind}\,A \setminus \mathcal{C} = \mathsf{ind}\,\mathcal{I} \setminus \mathcal{C}$.

Thus the assignment $\mathcal{C} \mapsto \mathcal{C} \cap \mathcal{I}$ is a bijection between the full additive cofinite submodule closed subcategories of $\mathsf{mod}\,A$ and the full additive cofinite submodule closed subcategories of $\mathcal{I}$. $\qquad\square$

We start the construction of exact sequences with a lemma that holds for all Artin algebras:

Lemma 3.2.2. *Let A be an arbitrary Artin algebra and $M, X \in \mathsf{mod}\,A$ indecomposable. Let*

$$0 \longrightarrow M \xrightarrow{\left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right]} X \oplus X' \xrightarrow{\left[\begin{smallmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{smallmatrix}\right]} Y \oplus Y' \longrightarrow 0 \qquad (3.4)$$

be an exact sequence for some $X', Y, Y' \in \mathsf{mod}\,A$, so that there is some $Z \in \mathsf{mod}\,A$ and an AR-sequence

$$0 \longrightarrow X \xrightarrow{\left[\begin{smallmatrix} g_{11} \\ f_2' \end{smallmatrix}\right]} Y \oplus Z \xrightarrow{[g_1'\ g_2']} \tau^{-1}X \longrightarrow 0 \,. \qquad (3.5)$$

If for some $U \in \mathsf{mod}\,A$, a monomorphism $h : M \rightarrowtail U$ factors through $f = \left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right]$ and $X \nmid U$, then h also factors through $f'' = \left[\begin{smallmatrix} -f_2' f_1 \\ f_2 \end{smallmatrix}\right]$ and the following sequence is exact:

$$0 \longrightarrow M \xrightarrow{\left[\begin{smallmatrix} -f_2' f_1 \\ f_2 \end{smallmatrix}\right]} Z \oplus X' \xrightarrow{\left[\begin{smallmatrix} g_2' & g_1' g_{12} \\ 0 & g_{22} \end{smallmatrix}\right]} \tau^{-1}X \oplus Y' \longrightarrow 0 \,.$$

Proof. By (3.5), the sequence

$$0 \longrightarrow X \xrightarrow{\left[\begin{smallmatrix} g_{11} \\ f_2' \\ 0 \end{smallmatrix}\right]} Y \oplus Z \oplus Y' \xrightarrow{\left[\begin{smallmatrix} g_1' & g_2' & 0 \\ 0 & 0 & \mathrm{id}_{Y'} \end{smallmatrix}\right]} \tau^{-1}X \oplus Y' \longrightarrow 0$$

is also exact. By Proposition 2.1.4, the diagrams

$$
\begin{array}{ccc}
M & \xrightarrow{\ f_1\ } & X \\
{\scriptstyle -f_2}\big\downarrow & & \big\downarrow{\scriptstyle \left[\begin{smallmatrix} g_{11} \\ 0 \end{smallmatrix}\right]} \\
X' & \xrightarrow[\left[\begin{smallmatrix} g_{12} \\ g_{22} \end{smallmatrix}\right]]{} & Y \oplus Y'
\end{array}
\qquad (3.6)
$$

and

$$
\begin{array}{ccc}
X & \xrightarrow{\;-f_2'\;} & Z \\[2pt]
{\scriptstyle\left[\begin{smallmatrix} g_{11} \\ 0 \end{smallmatrix}\right]}\Big\downarrow & & \Big\downarrow{\scriptstyle\left[\begin{smallmatrix} g_2' \\ 0 \end{smallmatrix}\right]} \\[2pt]
Y \oplus Y' & \xrightarrow[\left[\begin{smallmatrix} g_1' & 0 \\ 0 & \mathrm{id}_{Y'} \end{smallmatrix}\right]]{} & \tau^{-1}X \oplus Y'
\end{array}
$$

are both pushouts and pullbacks. So the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\;-f_2'f_1\;} & Z \\[2pt]
{\scriptstyle -f_2}\Big\downarrow & & \Big\downarrow{\scriptstyle\left[\begin{smallmatrix} g_2' \\ 0 \end{smallmatrix}\right]} \\[2pt]
X' & \xrightarrow[\left[\begin{smallmatrix} g_1'g_{12} \\ g_{22} \end{smallmatrix}\right]]{} & \tau^{-1}X \oplus Y'
\end{array}
$$

is itself a pushout and a pullback by Proposition 2.1.5. Therefore, the sequence

$$
0 \longrightarrow M \xrightarrow{\left[\begin{smallmatrix} -f_2'f_1 \\ f_2 \end{smallmatrix}\right]} Z \oplus X' \xrightarrow{\left[\begin{smallmatrix} g_2' & g_1'g_{12} \\ 0 & g_{22} \end{smallmatrix}\right]} \tau^{-1}X \oplus Y' \longrightarrow 0
$$

is exact. It remains to show that $h : M \rightarrowtail U$ factors through $f'' = \left[\begin{smallmatrix} -f_2'f_1 \\ f_2 \end{smallmatrix}\right]$.

Since we have assumed that h factors through $f = \left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right]$, there is a morphism $s = [s_1 s_2] : X \oplus X' \to U$ so that

$$
h = \left[\begin{smallmatrix} s_1 & s_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right] = s_1 f_1 + s_2 f_2
$$

By the Definition 2.2.10, the morphism $s_1 : X \to U$ factors through $\left[\begin{smallmatrix} g_{11} \\ f_2' \end{smallmatrix}\right]$: there is a morphism $s' = [s_1' s_2'] : Y \oplus Z \to U$ so that

$$
s_1 = \left[\begin{smallmatrix} s_1' & s_2' \end{smallmatrix}\right] \left[\begin{smallmatrix} g_{11} \\ f_2' \end{smallmatrix}\right] = s_1' g_{11} + s_2' f_2'.
$$

So we get

$$
h = s_1' g_{11} f_1 + s_2' f_2' f_1 + s_2 f_2.
$$

Since (3.6) is commutative, we have

$$
h = -s_1' g_{12} f_2 + s_2' f_2' f_1 + s_2 f_2 = \left[\begin{smallmatrix} -s_2' & s_2 - s_1' g_{12} \end{smallmatrix}\right] \left[\begin{smallmatrix} -f_2'f_1 \\ f_2 \end{smallmatrix}\right] \tag{3.7}
$$

and h factors through f''. $\qquad\square$

We can even say more:

Lemma 3.2.3. *Let A be a hereditary Artin algebra, $M \in \mathcal{I}$ and $U \in \mathrm{mod}\,A$. Suppose that the sequences of modules*

$$(X_1, X_2, \ldots, X_m)$$
$$(X_1', X_2', \ldots, X_m')$$
$$(Y_1, Y_2, \ldots, Y_m)$$

fulfil the following conditions:

(S1) There is an Auslander-Reiten sequence

$$0 \longrightarrow M \longrightarrow X_1 \oplus X_1' \longrightarrow Y_1 \longrightarrow 0 .$$

(S2) For all $1 \le i < m$, there is some $\alpha_i \in \mathbb{N}$ so that $X_i^{\alpha_i} \mid X_i \oplus X_i'$, but $X_i^{\alpha_i} \nmid U$.

(S3) For $1 \le i < m$, there is an Auslander-Reiten sequence of the form

$$0 \longrightarrow X_i \longrightarrow Z_i \longrightarrow \tau^{-1} X_i \longrightarrow 0 .$$

Let Y_i' be the maximal module that is a direct summand of both Y_i and Z_i. Write $Y_i = Y_i' \oplus Y_i''$ and $Z_i = Y_i' \oplus Z_i'$.

If $\tau^{-1} X_i \mid X_i'$, then let X_i'' be the module so that $X_i' = \tau^{-1} X_i \oplus X_i''$ and set $Y_i''' := 0$. Otherwise, set $X_i'' := X_i'$ and $Y_i''' := \tau^{-1} X_i$.

The following equations hold:

$$X_{i+1} \oplus X_{i+1}' = X_i'' \oplus Z_i'$$
$$Y_{i+1} = Y_i'' \oplus Y_i'''.$$

Then for all $1 \le i \le m$ there is an exact sequence

$$0 \longrightarrow M \xrightarrow{\;f_i\;} X_i \oplus X_i' \xrightarrow{\;g_i\;} Y_i \longrightarrow 0 . \tag{3.8}$$

Furthermore, if a monomorphism $M \rightarrowtail U$ exists, then it factors through all f_i.

To prove Lemma 3.2.3, we need the following observation:

Remark 3.2.4. Suppose that

$$X_i \oplus X_i' = X_i \oplus X_{i+1} \oplus B_i$$

and that C_i is the maximal module that is both a direct summand of Y_i and Z_{i+1}. Furthermore, write $Y_i = C_i \oplus C'_i$ and $Z_{i+1} = C_i \oplus D_i$. Set $B_i = B'_i$ and $C''_i = \tau^{-1} X_i$ if $\tau^{-1} X_i \nmid B_i$ and $B_i = B'_i \oplus \tau^{-1} X_i$ and $C''_i = 0$ if $\tau^{-1} X_i \mid B_i$. Then the sequences of modules

$$(X_1, X_2 \ldots, X_{i-1}, X_{i+1}, X_i, X_{i+2}, X_{i+3}, \ldots, X_m)$$
$$(X'_1, X'_2 \ldots, X'_{i-1}, X_i \oplus B_i, B'_i \oplus D_i, X'_{i+2}, X'_{i+3}, \ldots, X'_m)$$
$$(Y_1, Y_2, \ldots, Y_{i-1}, Y_i, C'_i \oplus C''_i, Y_{i+2}, Y_{i+3}, \ldots, Y_m)$$

also fulfil the conditions (S1) - (S3).

Note that only the i-th and $(i+1)$-th elements of these sequences differ from the elements in the original sequences.

We can easily generalize this to the following: If $i < j_1 < j_2 < \cdots < j_l$, there is an irreducible morphism $X_{j_k} \rightarrowtail X_{j_{k+1}}$ for all $1 \le k \le l$ and $X_{j_1} \mid X_i$, then there are two sequences with X'_m and Y_m as their m-th elements that together with

$$(X_1, \ldots, X_{i-1}, X_{j_1}, X_{j_2}, \ldots, X_{j_l}, X_i, X_{i+1}, \ldots X_{j_1-1}, X_{j_1+1}, \ldots$$
$$\ldots, X_{j_2-1}, X_{j_2+1}, \ldots, X_{j_l-1}, X_{j_l+1}, \ldots, X_l) \tag{3.9}$$

fulfil (S1) - (S3).

Furthermore, there are sequences of modules that fulfil (S1) - (S3) with X_m, X'_m, Y_m as their m-th elements so that $X''_i = X'_i$ for all $1 \le i \le m$:

By Definition 3.1.13, if there is a morphism $X_i \to X_j$, then $X_j < X_i$. So we can use the above to get sequences that fulfils (S1) -(S3) with X_m, X'_m, Y_m as their m-th elements so that $X_1 \ge X_2 \ge \cdots \ge X_{m-1}$. Then $X > \tau^{-1} I_i$ for all $X \mid X'_1$ and $X \mid Z_j$ with $j \le i$ and thus $X''_i = X'_i$ for all $1 \le i < m$.

Proof of Lemma 3.2.3. We prove the lemma inductively. By Remark 3.2.4, it is sufficient to prove the assertion for all sequences so that $X''_i = X'_i$ for all $1 \le i < m$.

For these sequences, we additionally show the following: If there is an indecomposable direct summand X of $X_m \oplus X'_m$ and τX_i of Y_m so that an irreducible morphism $X \to \tau X_i$ exists, then one of the following holds:

(a) There is a direct summand $X' \cong X$ of $X_m \oplus X'_m$ so that the component $X' \to Y_m$ of g_m is irreducible and $g_m(X') \subseteq \tau^{-1} X_i$

(b) Either $X \cong X_j$ for some $i < j < m$ or X is isomorphic to a direct summand of Y'_i.

If $m = 1$, the assertion is obvious by definition of the Auslander-Reiten sequence.

Now suppose that it holds for all series of modules of length $m \in \mathbb{N}$ or smaller. We want to show that it also holds for sequences of length $m+1$ by applying Lemma 3.2.2.

To do this, we need to prove that there is an exact sequence of the form

$$0 \longrightarrow M \longrightarrow X_m \oplus X_m' \xrightarrow{\left[\begin{smallmatrix} g_{m1} & g_{m2} \\ 0 & g_{m3} \end{smallmatrix}\right]} Y_m' \oplus Y_m'' \longrightarrow 0. \qquad (3.10)$$

so that g_{m1} is irreducible.

Suppose that Y_m' has some direct summands $Y_{m1}', Y_{m2}', \ldots, Y_{mk}'$ and g_m has a component

$$diag(g_{11}, g_{22}, \ldots, g_{kk}) : X_m^k \to \bigoplus_{l=1}^{k} Y_{ml}'$$

where $g_{11}, g_{22}, \ldots, g_{kk}$ are irreducible and $diag(g_{11}, g_{22}, \ldots, g_{kk})$ is the diagonal matrix with entries $g_{11}, g_{22}, \ldots, g_{kk}$. Then there is a copy of X_m on which this restricts to

$$\begin{bmatrix} g_{11} \\ g_{22} \\ \vdots \\ g_{kk} \end{bmatrix} : X_m \to \bigoplus_{l=1}^{k} Y_{ml}',$$

an irreducible morphism.

By condition (S3) and since $Y_1'' = 0$, every indecomposable direct summand of Y_m has the form $\tau^{-1}X_i$ for some $1 \leq i < m$.

If for all $\tau^{-1}X_i \mid Y_m'$, there is some copy X of X_m so that the component $X \to \tau^{-1}Y_i$ of g_m is irreducible and $g_m(X) \subseteq \tau^{-1}X_i$, then the above and the induction hypothesis mean that we can apply Lemma 3.2.2.

Suppose that there is some $\tau^{-1}X_i \mid Y_m'$, so that the above is not the case.

Since $Y_m' \mid Z_m$, there is an irreducible morphism between X_m and $\tau^{-1}X_i$. By the inductive hypothesis, one of the following holds:

(a) $X_m \cong X_j$ for some $i < j \leq m$

(b) $X_m \mid Y_i'$.

We show that there are sequences

$$(X_1^{(1)}, X_2^{(1)}, \ldots, X_m^{(1)})$$
$$(X_1'^{(1)}, X_2'^{(1)}, \ldots, X_m'^{(1)}) \qquad (3.11)$$
$$(Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_m^{(1)})$$

that fulfil (S1)-(S3) and have X_m, X'_m, Y_m as their m-th elements so that $\tau^{-1}X_i = \tau^{-1}X^{(1)}_{i'}$, but $X_m \not\cong X^{(1)}_j$ for all $i' < j \le m$ and $X_m \nmid Y'^{(1)}_{i'}$.

Furthermore, we want to show that $X''^{(1)}_l = X'^{(1)}_l$ for all $1 \le l < m$. Since $X_m = X^{(1)}_m$, $X'_m = X'^{(1)}_m$, $Y_m = Y^{(1)}_m$, this is already clear for $i = m$.

Obviously, we have $X_m \mid X_m \oplus X'_m$, so either $X_m \mid X'_1$ or $X_m \mid Z'_k$ for some $1 \le k \le m$, $k \ne i$.

In the first case, (b) is not possible, since Y_i and $X_i \oplus X'_i$ do not share direct summands. In case (a), Remark 3.2.4 yields a sequence $(X^{(2)}_1, \ldots, X^{(2)}_m)$, where X_j comes before X_i.

In the second case, we can get a new sequence where X_k comes before X_i, since $Z'_i \mid X_m \oplus X'_m$ (otherwise, $\tau^{-1}X_i$ would not be a direct summand of Y_m). In case (b), this sequence is already the one we need; in case (a), we can again get a another sequence by Remark 3.2.4 where X_j comes before X_i.

If we call this new exact sequence $(X^{(2)}_1, \ldots, X^{(2)}_m)$, then it is clear by (3.9) that $X''^{(2)}_l = X'^{(2)}_l$ holds for all $1 \le l < m$.

Since there are only finitely many j with $i' < j \le m$, we get sequences of the form (3.11) after finitely many steps.

The inductive assumption gives us an exact sequence

$$0 \longrightarrow M \xrightarrow{f'_m} X_m \oplus X'_m \xrightarrow{g'_m} Y_m \longrightarrow 0 \qquad (3.12)$$

where the component $X_m \to Y_m$ of g'_m is irreducible and $g'_m(X_m) \subseteq \tau^{-1}X_i$.

If $\tau^{-1}X_i = Y_m$, then it is sufficient to look at the sequence (3.12) instead of

$$0 \longrightarrow M \xrightarrow{f_m} X_m \oplus X'_m \xrightarrow{g_m} Y_m \longrightarrow 0 . \qquad (3.13)$$

If there is some $\tau^{-1}X_k$ so that $\tau^{-1}X_i \oplus \tau^{-1}X_k \mid Y_m$, then we can assume that g_m induces an indecomposable morphism $X_m \to \tau^{-1}X_k$ and $g_m(X_m) \subset \tau^{-1}X_k$.

So together (3.13) and (3.12) give a new exact sequence

$$0 \longrightarrow M \xrightarrow{f''_m} X_m \oplus X'_m \xrightarrow{g''_m} Y_m \longrightarrow 0 ,$$

where the induced morphisms $X_m \to \tau^{-1}X_i$ and $X_m \to \tau X_k$ of g''_m are irreducible and $g_m(X_m) \subset \tau^{-1}X_i \oplus \tau^{-1}X_k$.

Inductively, there is an exact sequence of the form (3.10), where g_{m1} is irreducible and we can use Lemma 3.2.2 to get an exact sequence

$$0 \longrightarrow M \xrightarrow{f_{m+1}} X_{m+1} \oplus X'_{m+1} \xrightarrow{g_{m+1}} Y_{m+1} \longrightarrow 0 .$$

If there is a monomorphism $M \rightarrowtail U$, then it factors through f_{m+1}.

This gives us not only the assertion of the lemma but also the additional assumptions we have made:

Let

$$g_m = \begin{bmatrix} g'_{m1} & g'_{m2} \end{bmatrix} : Y'_m \oplus Z'_m \to \tau^{-1}X_m$$

be the epimorphism of the AR-sequence. By (3.10) and Lemma 3.2.2, we get

$$g_{m+1} = \begin{bmatrix} g'_{m2} & g'_{m1}g_{m2} \\ 0 & g_{m3} \end{bmatrix} : Z'_m \oplus X'_m \to \tau^{-1}X_m \oplus Y''_m.$$

Let X be a direct summand of $Z'_m \oplus X'_m$ and $\tau^{-1}X_i$ a direct summand of $\tau^{-1}X_m \oplus Y''_m$ so that there is an irreducible morphism $X \to \tau^{-1}X_i$.

If $i = m$, then X is a direct summand of Z_m. If it is also a direct summand of Z'_m, then $g_{m+1}(X) = \begin{bmatrix} g'_{m2} \\ 0 \end{bmatrix}(X)$ and (a) holds. Otherwise, X is a direct summand of Y'_m.

If $i \neq m$ and X is a direct summand of X'_m, then either (b) holds or

$$\begin{bmatrix} g_{m2} \\ g_{m3} \end{bmatrix}(X'_m) \subset \tau^{-1}X_i.$$

Thus $g_{m2}(X'_m) = 0$ and $g_{m+1}(X) = \begin{bmatrix} 0 \\ g_{m3} \end{bmatrix}(X)$. So (a) holds.

Finally, suppose that $i \neq m$ and X is not a direct summand of X'_m. Because of the irreducible morphism between X and $\tau^{-1}X_i$, the former is a direct summand of Z_i. By (S3), either it is a direct summand of Y'_i or of X_j for some $i < j < m$. □

A perhaps simpler way to interpret the sequences of modules used in the lemma above is the following:

Remark 3.2.5. Suppose that X_i, X'_i, Y_i are the i-th elements of sequences that fulfil (S1) - (S3). Then $X_{i+1} \oplus X'_{i+1}$ are defined by taking the exact sequence

$$0 \longrightarrow M \longrightarrow X_i \oplus X'_i \longrightarrow Y_i \longrightarrow 0 .$$

and the Auslander-Reiten sequence

$$0 \longrightarrow X_i \longrightarrow Z_i \longrightarrow \tau^{-1}X_i \longrightarrow 0 .$$

We can add these sequences together and get

$$0 \longrightarrow M \oplus X_i \longrightarrow X_i \oplus X'_i \oplus Z_i \longrightarrow Y_i \oplus \tau^{-1}X_i \longrightarrow 0 .$$

Then X_i is the maximal module that is a direct summand of both the first and the second term: we still get an exact sequence if we delete it in both terms:

$$0 \longrightarrow M \longrightarrow X'_i \oplus Z_i \longrightarrow Y_i \oplus \tau^{-1} X_i \longrightarrow 0 \ .$$

The same holds for G_i, the maximal module that is both a direct summand of the middle term and the last term. Deleting this in both terms gives us an exact sequence

$$0 \longrightarrow M \longrightarrow X_{i+1} \oplus X'_{i+1} \longrightarrow Y_{i+1} \longrightarrow 0 \ .$$

These modules have some interesting properties:

Corollary 3.2.6. *If there is a monomorphism $h : M \rightarrowtail U$, then for sequences of modules*

$$(X_1, X_2, \ldots, X_m)$$
$$(X'_1, X'_2, \ldots, X'_m)$$
$$(Y_1, Y_2, \ldots, Y_m),$$

which fulfil (S1) - (S3), there is a monomorphism $X_i \oplus X'_i \rightarrowtail Y_i \oplus U$ for all $1 \le i < m$.

Thus, every injective direct summand of $X_i \oplus X'_i$ is a direct summand of U.

Proof. By Lemma 3.2.3, there is an exact sequence

$$0 \longrightarrow M \xrightarrow{\ f_i\ } X_i \oplus X'_i \xrightarrow{\ g_i\ } Y_i \longrightarrow 0$$

for $1 \le i \le m$ so that h factors through f_i. Thus, there is some morphism h_i with $h = h_i f_i$. So h_i is a monomorphism on $\mathsf{Im}\, f_i$. Since $\mathsf{Ker}\, g_i = \mathsf{Im}\, f_i$, the morphism

$$\begin{bmatrix} g_i \\ h_i \end{bmatrix} : X_i \oplus X'_i \rightarrowtail Y_i \oplus U$$

is a monomorphism.

So every injective direct summand I of $X_i \oplus X'_i$ is a direct summand of $Y_i \oplus U$. Since $X_i \oplus X'_i$ and Y_i do not share any direct summands, I is even a direct summand of U. $\qquad\qquad\square$

We can use the following lemma to show that there is an algorithm that, for given indecomposable, preinjective module M constructs all U with $M \rightarrowtail U$.

Lemma 3.2.7. *Suppose there is an irreducible morphism between $(s,i) = \tau^s I_i$ and $(t,j) = \tau^t I_j$. Then either $s = t$ and $i > j$ or $s = t+1$ and $i < j$.*

Proof. By 2.3.6,

$$s - 1 \leq t \leq s.$$

Furthermore, we have ordered the injective modules so that $\mathsf{Hom}(I_i, I_j) = 0$ if $i < j$.

By 2.3.7, we get $\mathsf{Hom}(\tau^s I_i, \tau^s I_j) = 0$ if $i < j$. By 2.2.14, there is an irreducible morphism $\tau^s I_i \to \tau^{s-1} I_j$ if and only if there is an irreducible morphism $\tau^s I_j \to \tau^s I_i$.

So if $s = t$, then $i > j$ and if $s = t+1$ then $i < j$. $\qquad\square$

Proposition 3.2.8. *Let A be a hereditary Artin algebra with $M \in \mathsf{mod}\, A$ indecomposable and preinjective. Let $U \in \mathsf{mod}\, A$, so that M is not a direct summand of U. There is a monomorphism $M \rightarrowtail U$ if and only if for some $m \in \mathbb{N}$ there are three sequences of modules*

$$(X_1, X_2, \ldots, X_m)$$
$$(X_1', X_2', \ldots, X_m')$$
$$(Y_1, Y_2, \ldots, Y_m)$$

that fulfil the conditions (S1) - (S3) and furthermore

(S4) If for some $1 \leq i \leq m$ the module $X_i \oplus X_i'$ has an injective direct summand I, then $I \mid U$.

(S5) $X_m \oplus X_m'$ is a direct summand of U.

Proof. To prove this, we use Lemma 3.2.3: since the sequences fulfil (S1)-(S3), there are exact sequences of the form

$$0 \longrightarrow M \xrightarrow{f_i} X_i \oplus X_i' \xrightarrow{g_i} Y_i \longrightarrow 0$$

for all $1 \leq i \leq m$. If a monomorphism $M \rightarrowtail U$ exists, it factors through f_i for all $1 \leq i \leq m$.

Thus one direction is obvious: if such sequences of modules exist, $f_m : M \rightarrowtail U$ is a monomorphism.

On the other hand, suppose that no series of modules fulfil (S1) - (S5).

If M is injective, then it cannot be a submodule of U. Otherwise, there are series of modules that fulfil (S1) - (S3), since there is an AR-sequence that starts in M and we can set $m = 1$.

If (S4) is not fulfilled, then M cannot be a submodule of U by Corollary 3.2.6.

Otherwise, there is some non-injective X_{m+1} and some $\alpha_{m+1} \in \mathbb{N}$ so that $X_{m+1}^{\alpha_{i+1}} \mid X_m \oplus X_m'$, but $X_{m+1}^{\alpha_{i+1}} \nmid U$. So we can extend the sequences of modules to

$$(X_1, X_2, \ldots X_m, X_{m+1})$$
$$(X_1', X_2', \ldots X_m', X_{m+1}')$$
$$(Y_1, Y_2, \ldots Y_m, Y_{m+1})$$

so that these series fulfil (S1) - (S3). If these sequences fulfil (S4), we can extend them again to sequences of length $m + 2$.

We have $M = (r, i)$ for some $r \in \mathbb{N}$ and $1 \leq i \leq n$. Every indecomposable direct summand of $X_1 \oplus X_1'$ is of the form $(r', j) < (r, i)$ for some $r' \in \mathbb{N}_0$ and $1 \leq j \leq n$. Furthermore, if $X_1 = (r', j)$, then every direct summand of Z_1' is of the form $(r'', k) < (r', j)$, and analogously for $X_2, X_3, \ldots$.

So after finitely many steps, either we find sequences that do not fulfil (S4), or there is some m' so that every direct summand of $X_m \oplus X_m'$ is injective. If (S4) is still fulfilled, then (S5) is also fulfilled, a contradiction to our assumption. □

The proof of Proposition 3.2.8 shows the following:

Corollary 3.2.9. *Let M and U be preinjective modules over A. If $M \subset U$, then all sequences of modules that fulfil (S1) - (S3) can be extended to sequences of modules that fulfil (S1) - (S5).*

If $M \not\subset U$, then all sequences of modules that fulfil (S1) - (S3) can be extended to sequences that fulfil (S1) - (S3) so that $X_m \oplus X_m'$ has an injective direct summand that is not a direct summand of U.

Remark 3.2.10. By Corollary 3.2.9, we can use the proposition as an algorithm that finds out for given indecomposable, preinjective M and modules U, if there is a monomorphism $M \rightarrowtail U$. Alternatively, we can use it to construct all U with $M \subset U$.

Note that it is very simple to generalize this for arbitrary preinjective M:

Corollary 3.2.11. *Let M be a preinjective module so that $M = \bigoplus_{i=1}^{m} M_i$ with M_i indecomposable. Let U be some module in $\mathsf{mod}\, A$. Denote the middle term of the Auslander-Reiten sequence that starts in M_i by N_i.*

Furthermore, order $M_1, \ldots, M_m$ so that there is some $0 \leq k \leq m$ with $M_i \mid U$ if and only if $i \leq k$.

Suppose that the sequences of modules

$$(X_1, X_2, \ldots, X_m)$$
$$(X_1', X_2', \ldots, X_m')$$
$$(Y_1, Y_2, \ldots, Y_m)$$

fulfil (S2), (S3) and

(S'1) We have

$$X_1 \oplus X_1' = \bigoplus_{i=1}^{k} M_i \oplus \bigoplus_{i=k+1}^{m} N_i$$

and

$$Y_1 = \bigoplus_{i=k+1}^{m} \tau^{-1} M_i.$$

Then for all $1 \leq i \leq m$, there is an exact sequence

$$0 \longrightarrow M \overset{f_i}{\longrightarrow} X_i \oplus X_i' \overset{g_i}{\longrightarrow} Y_i \longrightarrow 0 .$$

There is a monomorphism $M \rightarrowtail U$ if and only if there is some $m' > m$ and modules $X_{m+1}, \ldots X_{m'}$, $X_{m+1}', \ldots X_{m'}'$, $Y_{m+1}, \ldots Y_{m'}$ so that the sequences

$$(X_1, X_2, \ldots, X_{m'})$$
$$(X_1', X_2', \ldots, X_{m'}')$$
$$(Y_1, Y_2, \ldots, Y_{m'})$$

fulfil (S'1) and (S2) - (S5).

Furthermore, if a monomorphism $M \rightarrowtail U$ exists, then it factors through all f_i.

Example 3.2.12. Take A as in Example 2.4.9. A part of the preinjective component of the AR-quiver of A is:

Suppose that we want to know whether $M = \tau I_3$ is a submodule of, say, $U = I_2 \oplus I_3 \oplus I_4$.

Then by (S1), $X_1 \oplus X_1' = \tau I_1 \oplus \tau I_2$ and $Y_1 = I_3$. Since neither τI_1 nor τI_2 is a direct summand of U, we arbitrarily set $X_1 := \tau I_1$.

The AR-sequence

$$0 \longrightarrow \tau I_1 \longrightarrow I_3 \oplus I_4 \longrightarrow I_1 \longrightarrow 0$$

and (S3) show that $X_2 \oplus X_2' = \tau I_2 \oplus I_4$ and $Y_2 = I_1$. Since I_4 is a direct summand of U, we set $X_2 := \tau I_2$ to fulfil (S2). Using the AR-sequence

$$0 \longrightarrow \tau I_2 \longrightarrow I_3 \oplus I_4 \longrightarrow I_2 \longrightarrow 0$$

we get $X_3 \oplus X_3' = I_3 \oplus I_4^2$ and $Y_3 = I_1 \oplus I_2$. Since I_4^2 is injective, but not a direct summand of U, the condition (S4) is not fulfilled and there is no monomorphism between M and U.

We have one more lemma:

Lemma 3.2.13. *Let M be an indecomposable, preinjective module and $U \in$* mod A. *If the sequences*

$$(X_1, X_2, \ldots, X_m)$$
$$(X_1', X_2', \ldots, X_m')$$
$$(Y_1, Y_2, \ldots, Y_m)$$

(3.14)

fulfil (S1) - (S3), then for every $1 \leq i \leq m$, there is an exact sequence

$$0 \longrightarrow X_i \oplus X_i' \longrightarrow Y_i \oplus X_m \oplus X_m' \longrightarrow Y_m \longrightarrow 0 .$$

(3.15)

Furthermore, if there is an exact sequence

$$0 \longrightarrow X_i \oplus X_i' \longrightarrow Y_i \oplus U \longrightarrow Z \longrightarrow 0$$

(3.16)

then there is also an exact sequence

$$0 \longrightarrow M_0 \longrightarrow U \longrightarrow Z \longrightarrow 0 .$$

(3.17)

Proof. Let Y be the maximal module so that $Y \mid Y_i$ and $Y \mid Y_m$. Furthermore, suppose that $Y_i = Y \oplus Y'$ and $Y_m = Y \oplus Y''$.

We use Corollary 3.2.11 on $X_i \oplus X_i'$ and $Y \oplus U$. Take $i < j_1 < j_2 \cdots < j_l$ so that X_{j_k} are those modules in the sequence $(X_{i+1}, \ldots X_m)$ which are already a direct summand of $X_i \oplus X_i'$. Then

$$(X_{m+1}, \ldots, X_{j_1-1}, X_{j_1+1}, \ldots, X_{j_2-1}, X_{j_2+1}, \ldots, X_{j_l-1}, X_{j_l+1}, \ldots, X_m)$$

is part of a triple of sequences that fulfil (S'1) and (S2) - (S5) with respect to $X_i \oplus X_i'$ and $Y' \oplus U$.

So the same construction that yields an exact sequence

$$0 \longrightarrow M \longrightarrow X_m \oplus X_m' \longrightarrow Y_m \longrightarrow 0 ,$$

also gives an exact sequence

$$0 \longrightarrow X_i \oplus X_i' \longrightarrow Y' \oplus X_m \oplus X_m' \longrightarrow Y'' \longrightarrow 0 ,$$

when used on $X_i \oplus X_i'$ and $Y' \oplus U$ instead of M and U. Adding Y to both the middle and the last term gives (3.15).

The exact sequence (3.16) is given by a sequence of modules that fulfil (S'1) and (S2) - (S5). Together with the sequences (3.14), this yields the exact sequence 3.17. □

3.3   Preinjective modules and the Weyl group

In this section we connect our results about preinjective modules with the relations of the Weyl group.

First we give a Lemma that shows the connection between the AR-sequences and the relations:

Lemma 3.3.1. *Fix two integers $1 \leq i, j \leq n$. Set*

$$\alpha := \max\{v \mid \text{there are } s, t \text{ with an irreducible morphism } (s, i) \to (t, j)^v\}$$
$$\beta := \max\{v \mid \text{there are } s, t \text{ with an irreducible morphism } (t, j) \to (s, i)^v\}$$

Let $(s_i s_j)^{m_{ij}}$ be the defining relation of the Weyl group as in Lemma 2.4.7. Then the value of m_{ij} depends on $\alpha\beta$ in the following way:

$\alpha\beta$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞.

Proof. From 2.3.7, we know that α and β do not depend on s. Let $(c_{ij})_{nn}$ be the Cartan matrix. By Lemma 2.3.9, either $\alpha = c_{ij}$ and $\beta = c_{ji}$ or $\beta = c_{ij}$ and $\alpha = c_{ji}$. Lemma 2.4.7 gives the stated values for m_{ij}.   □

Now we define a recursion that plays a fundamental role in the proof of the bijection:

Definition 3.3.2. For given $\alpha, \beta \in \mathbb{N}$ define a recursion formula by

$$E(0) = 1$$
$$E(1) = \alpha$$
$$E(2m) = \max(\beta E(2m - 1) - E(2m - 2), 0)$$
$$E(2m + 1) = \max(\alpha E(2m) - E(2m - 1), 0)$$

for all $m \in \mathbb{N}$.

This recursion is directly linked to the Weyl group:

Lemma 3.3.3. *Let α, β be as in Lemma 3.3.1. Then*

$$E(m) = 0 \iff m \geq m_{ij} - 1.$$

Proof. If $\alpha\beta < 4$, then we get the following values for $m \leq 6$:

$\alpha\beta$	m_{ij}	$E(0)$	$E(1)$	$E(2)$	$E(3)$	$E(4)$	$E(5)$	$E(6)$
0	2	1	0	0	0	0	0	0
1	3	1	1	0	0	0	0	0
2	4	1	α	1	0	0	0	0
3	6	1	α	2	α	1	0	0

$$(3.18)$$

Obviously, $E(m) = 0$ if $\alpha\beta < 4$ and $m > 6$.

If $\alpha\beta \geq 4$, then $m_{ij} = \infty$ by Lemma 3.3.1 and we need to show that $E(m) \neq 0$ for all $m \in \mathbb{N}$.

Since $E(2) = \alpha\beta - 1 > E(0) > 0$, we get inductively for $m > 1$:

$$\begin{aligned}
E(2m) &= \beta E(2m-1) - E(2m-2) \\
&= (\alpha\beta - 1)E(2m-2) - \beta E(2m-3) \\
&= (\alpha\beta - 1)E(2m-2) - E(2m-2) - E(2m-4) \\
&> (\alpha\beta - 3)E(2m-2) \\
&\geq E(2m-2).
\end{aligned}$$

The proof that $E(2m+1) > E(2m-1) > 0$ is completely analogous. $\qquad\square$

Next, we need some notation:

Definition 3.3.4. Fix $s \in \mathbb{N}_0$ and $1 \leq i \neq j \leq n$ and let $M_0 := \tau^s I_i$. If $s \geq 1$ or $j < i$, let t be the integer with $(s-1, i) < (t, j) < (s, i)$. Denote $M_1 := \tau^t I_j$, $M_2 := \tau^{s-1} I_i$, $M_3 := \tau^{t-1} I_j$, ...

The following lemma is a key part in the proof that there is a bijection between cofinite, submodule closed subcategories and the elements of the Weyl group:

Lemma 3.3.5. *Let U be a module so that $M_k \nmid U$ for all $M_k \neq 0$ with $0 \leq k \leq m_{ij} - 1$. Furthermore, let α, β be as in Lemma 3.3.1. Then for all $m \geq 1$ with $M_{m+1} \neq 0$ and $E(m-1) > 0$, there are series of modules that fulfil (S1) - (S3) and yield exact sequences*

$$0 \longrightarrow M_0 \xrightarrow{f_m} M_m^{E(m)} \oplus U_m \longrightarrow M_{m+1}^{E(m-1)} \longrightarrow 0 \qquad (3.19)$$

so that no $M_k \nmid U_m$ for all $M_k \neq 0$ with $0 \leq k \leq m_{ij} - 1$.

If there is a monomorphism $M_0 \rightarrowtail U$, then it factors through f_m for all m.

Proof. If M_0 is injective, there is nothing to show. So we can assume that an AR-sequence starts in M_0.

Let α, β be as in Lemma 3.3.1. Then there are modules M', N' so that

$$0 \longrightarrow M_0 \longrightarrow M_1^\alpha \oplus N' \longrightarrow M_2 \longrightarrow 0 \qquad (3.20)$$

and

$$0 \longrightarrow M_1 \longrightarrow M_2^\beta \oplus M' \longrightarrow M_3 \longrightarrow 0$$

are AR-sequences. Note that by 2.3.7, for all non-injective M_{2m-1}, $m \in \mathbb{N}_0$, there are AR-sequences of the form

$$0 \longrightarrow M_{2m-1} \longrightarrow M_{2m}^\beta \oplus \tau^m M' \longrightarrow M_{2m+1} \longrightarrow 0 . \qquad (3.21)$$

For all non-injective M_{2m}, $m \in \mathbb{N}$ they are of the form

$$0 \longrightarrow M_{2m} \longrightarrow M_{2m+1}^\alpha \oplus \tau^m N' \longrightarrow M_{2m+2} \longrightarrow 0 .$$

If we set $U_1 := N'$, the AR-sequence that starts in M_0 is the exact sequence

$$0 \longrightarrow M_0 \longrightarrow M_1^{E(1)} \oplus U_1 \longrightarrow M_2^{E(0)} \longrightarrow 0 .$$

If M_1 is injective, then the proof is complete. So we can assume that an AR-sequence starts in M_1 and use Lemma 3.2.3.

Since $M_1 \nmid U$, we set

$$X_1 := X_2 := X_3 := \cdots := X_{E(1)} := M_1,$$

Then we get sequences that fulfil conditions (S1) - (S3) by setting

$$X_1' := M_1^{E(1)-1} \oplus N',$$
$$X_2' := M_1^{E(1)-2} \oplus M_2^{\beta-1} \oplus N' \oplus M',$$
$$\cdots$$
$$X_{E(1)}' := M_2^{(E(1)-1)\beta-1} \oplus N' \oplus (M')^{E(1)-1}$$

and

$$Y_1 := M_2,$$
$$Y_2 := M_3,$$
$$Y_3 := M_3^2,$$
$$\cdots$$
$$Y_{E(1)} := M_3^{E(1)-1}$$

We get

$$X_{E(1)+1} \oplus X'_{E(1)+1} = M_2^{E(1)\beta-1} \oplus N' \oplus (M')^{E(1)}$$

and $Y_{E(1)+1} = M_3^{E(1)}$. Thus by Lemma 3.2.3 there is some f_1 so that the following sequence is exact:

$$0 \longrightarrow M_0 \xrightarrow{f_1} M_2^{E(1)\beta-1} \oplus N' \oplus (M')^{E(1)} \longrightarrow M_3^{E(1)} \longrightarrow 0 .$$

If there is a monomorphism $M \rightarrowtail U$, then it factors through f_1.

Since $U_2 := N' \oplus (M')^{E(1)}$, we can write the exact sequence as

$$0 \longrightarrow M_0 \xrightarrow{f_1} M_2^{E(2)} \oplus U_2 \longrightarrow M_3^{E(1)} \longrightarrow 0 .$$

We show the rest inductively: Suppose that

$$0 \longrightarrow M_0 \longrightarrow M_{2m-1}^{E(2m-1)} \oplus U_{2m-1} \longrightarrow M_{2m}^{E(2m-2)} \longrightarrow 0$$

is an exact sequence and $E(2m-1) \neq 0$. Furthermore, suppose that this exact sequence is yielded by sequences of modules of the length $m'-1$. Then

$$X_{m'} := M_{2m-1}, X'_{m'+1} := M_{2m-1}^{E(2m-1)-1} \oplus U_{2m-1}, Y_{m'} := M_{2m}^{E(2m-2)}$$

are elements of sequences that fulfil the condition (S1) - (S3) of Lemma 3.2.3.

If M_{2m-1} is injective, then $M_{2m+1} = \tau M_{2m-1} = 0$ and there is nothing to prove. If M_{2m-1} is not injective, then the AR-sequence (3.21) exists. As above, we set

$$X_{m'+1} := \cdots := X_{m'+E(2m-1)-1} := M_{2m-1}.$$

This determines $X'_{m'+1}, \ldots, X'_{m'+E(2m-1)-1}$ and $Y_{m'+1}, \ldots, Y_{m'+E(2m-1)-1}$ completely.

Since $E(2m) = \beta E(2m-1) - E(2m-2)$, we get

$$X_{m'+E(2m-1)} \oplus X'_{m'+E(2m-1)} = M_{2m}^{E(2m)} \oplus U_{2m-1} \oplus (\tau^{m-1} M')^{E(2m-1)}$$
$$Y_{m'+E(2m-1)} = M_{2m+1}^{E(2m-1)}$$

Together with $U_{2m} := U_{2m-1} \oplus (\tau^{m-1} M')^{E(2m-1)}$, this yields an exact sequence

$$0 \longrightarrow M_0 \xrightarrow{f_{2m}} M_{2m}^{E(2m)} \oplus U_{2m} \longrightarrow M_{2m+1}^{E(2m-1)} \longrightarrow 0$$

for some f_{2m}. By Lemma 3.2.3, $M \rightarrowtail U$ factors through f_{2m}.

Analogously, we can construct

$$0 \longrightarrow M_0 \xrightarrow{f_{2m+1}} M_{2m+1}^{E(2m+1)} \oplus U_{2m+1} \longrightarrow M_{2m+2}^{E(2m)} \longrightarrow 0$$

if $E(2m) \neq 0$ and $M_{2m+2} \neq 0$. □

Corollary 3.3.6. *Let U_m be as in Lemma 3.3.5. If $(r,l) \mid U_m$, then $M_0 > (r,l) > M_{m+1}$.*

If $M_0 > (r,l) > M_1$, then $(r,l) \mid U_m$ if and only if $m_{il} \geq 3$.

If $M_1 > (r,l) > M_m$, then $(r,l) \mid U_m$ if and only if $m_{il} + m_{jl} \geq 5$.

If $M_m > (r,l) > M_{m+1}$ and m is even, then $(r,l) \mid U_m$ if and only if $m_{il} \geq 3$. If m is odd, then $(r,l) \mid U_m$ if and only if $m_{jl} \geq 3$.

Proof. This is obvious from the proof of Lemma 3.3.5. □

Remark 3.3.7. Note that $m_{ij} = m_{ji}$. If we fix s, t as in Definition 3.3.4, we can set $M_0' := \tau^t I_j$, $M_1' := \tau^{s-1} I_i$, $M_2' := \tau^{t-1} I_j$, $M_3' := \tau^{s-2} I_i$, ... and

$$E'(0) = 1$$
$$E'(1) = \beta$$
$$E'(2m) = \alpha E(2m - 1) - E(2m - 2)$$
$$E'(2m + 1) = \beta E(2m) - E(2m - 1).$$

With this definition, we get analogous results to 3.3.3, 3.3.5 and 3.3.6.

3.4 Preliminaries for the main theorem

This section collects some preliminaries which are necessary to prove that there is a bijection between the Weyl group and the cofinite, submodule closed subcategories: First, we show that every cofinite submodule closed subcategory is of the form $\mathcal{C}_{\underline{w}}$ for some word $\underline{w}$.

Then we will prove an auxiliary result that will make the inductions in the next section possible.

Lemma 3.4.1. *If a cofinite, full additive subcategory $\mathcal{C}$ of $\mathsf{mod}\, A$ is submodule closed, then there is a word $\underline{w}$ over $S = \{s_1, s_2, \ldots s_n\}$ with $\mathcal{C} = \mathcal{C}_{\underline{w}}$.*

Proof. By Lemma 3.2.1,

$$\mathsf{ind}\, A \setminus \mathcal{C} = \mathsf{ind}\, \mathcal{I} \setminus \mathcal{C} =: \{(r_1, i_1), (r_2, i_2), \ldots (i_m, r_m)\}.$$

for some $m \in \mathbb{N}$ and modules $(r_1, i_1) < (r_2, i_2) < \cdots < (r_m, i_m)$.

Suppose that for all words $\underline{w}$ over S

$$\rho(\underline{w}) \neq (r_1, i_1)(r_2, i_2) \ldots (r_m, i_m).$$

By Definition 3.1.1, either $r_1 > 0$ or there is some $1 \leq j \leq m - 1$ so that

$$(r_j, i_j) < (r_{j+1}, i_{j+1} - 1).$$

In the first case, $\mathcal{C}$ contains the middle term of the AR-sequence that starts in (r_1, i_1) by Lemma 3.2.7. In the second case, $\mathcal{C}$ contains the middle term of the AR-sequence that starts in (r_{j+1}, i_{j+1}). In both cases, $\mathcal{C}$ is not submodule closed.

So by Definition 3.1.13, there is some $\underline{w}$ with

$$\rho(\underline{w}) = (r_1, i_1)(r_2, i_2) \dots (r_m, i_m)$$

and $\mathcal{C} = \mathcal{C}_{\underline{w}}$. □

Recall that $\mathcal{C}_w = \mathcal{C}_{\underline{w}}$, where $\underline{w}$ is the leftmost word for w. So we need to prove that the word $\underline{w}$ in Lemma 3.4.1 is leftmost. Furthermore, we need the other direction, namely, that $\mathcal{C}_{\underline{w}}$ is submodule closed if $\underline{w}$ is leftmost.

We will use the following lemma for the proofs of both directions:

Lemma 3.4.2. *Suppose that the words $\underline{w}$ and $\underline{w}''$ are equivalent and there are pairs $(r, h), (s, i), (t+1, j)$ and series of pairs ρ_1, ρ_2, ρ_4 so that*

$$\rho(\underline{w}) = \rho_1(r, h)\rho_2(s, i)\rho_3,$$

$$\rho(\underline{w}'') = \rho_1\rho_2(s, i)(t+1, j)\rho_4$$

and either $\rho_3 = \rho_4$, or a pair (q, g) is in ρ_4 if and only if $(q-1, g)$ is in ρ_3.

Furthermore, suppose that the word $\underline{x}$ with $\rho(\underline{x}) = \rho_1(r, h)\rho_2$ is reduced and $m_{ij} \geq 3$.

If $M_0 = (s, i), M_1 = (t, j), M_2, \dots, M_{m_{ij}-3} \notin \mathcal{C}_{\underline{w}}$, then there are sequences of modules as in Lemma 3.2.3 (used on M_0 and any $U \in \mathcal{C}_{\underline{w}}$) that yield some $U', Y \in \mathcal{I}$ so that

$$0 \longrightarrow M_0 \longrightarrow M_{m_{ij}-2} \oplus U' \longrightarrow Y \longrightarrow 0 \tag{3.22}$$

is an exact sequence and either $Y \in \mathcal{C}_{\underline{w}}$ or both $Y = (r, h)^{E(m_{ij}-3)}$ and $U' \in \mathcal{C}_{\underline{w}}$ hold.

Proof. We show this by induction on the number m of Coxeter relations needed to transform $\underline{w}$ into $\underline{w}''$.

Furthermore, we show that a few additional assertions hold, which we need for the inductive proof:

(A1) Let (r', h') be a module so that $(r', h')^\gamma \mid Y$. If $Z \in \mathcal{C}_{\underline{w}}$ is a direct summand of the middle term of the AR-sequence that ends in (r', h'), then $Z^\gamma \mid U'$.

(A2) Let V be the maximal direct summand of U', so that for every $(q, g) \mid V$ there is some $(r', h') \mid Y$ with an indecomposable morphism $(q, g) \rightarrow (r', h')$. Furthermore, let (q', g') be the biggest indecomposable direct summand of V and (r'', h'') the smallest indecomposable direct summand of Y.

If there is some $o \in \mathbb{N}_0$ so that $Y, \tau^{-1}Y, \ldots, \tau^{-o}Y \in \mathcal{C}_{\underline{w}}$ and $\tau^{-o-1}Y \notin \mathcal{C}_{\underline{w}}$, then one of the following holds:

(a) $\tau^{-o-1}Y = (r, h)^{E(m_{ij}-3)}$ and
$$\tau^{-1}V, \tau^{-2}V, \ldots, \tau^{-o-1}V \in \mathcal{C}_{\underline{w}}.$$

(b) Let V' be the maximal direct summand of V so that for every $(q, g) \mid V'$ there is some $0 \leq k \leq o + 1$ so that $(q - k, g) \notin \mathcal{C}_{\underline{w}}$. Then there is some module Y' with an exact sequence

$$0 \longrightarrow \tau^{-o-1}V' \longrightarrow \tau^{-o-1}Y \oplus U'' \longrightarrow Y' \longrightarrow 0 . \qquad (3.23)$$

Either $Y' \in \mathcal{C}_{\underline{w}}$ or both $Y' = (r, h)^{E(m_{ij}-3)}$ and $U'' \in \mathcal{C}'$ hold, where
$$\text{ind}\,\mathcal{C}' = \text{ind}\,\mathcal{C}_{\underline{w}} \setminus \mathcal{M}$$
with

$$\mathcal{M} = \left\{ M \in \text{ind}\,\mathcal{I} \mid \exists 0 \leq k \leq o : \left\{ \begin{array}{l} \tau^{k+1}M \notin \mathcal{C}_{\underline{w}} \\ (r'', h'') < \tau^{k+1}M \\ \tau^{k+1}M < (q'-1, g') \end{array} \right\} \right\}$$

For all $U \in \mathcal{C}'$, there are sequences of modules that fulfil (S'1), (S2) and (S3) with respect to $\tau^{-o-1}V'$ and U and yield (3.23) as in Corollary 3.2.11.

Furthermore, (A1) and (A2) still hold if we exchange U', Y and $\mathcal{C}_{\underline{w}}$ for U'', Y' and $\mathcal{C}'$ respectively.

If there are some reflections s_k, s_l and words $\underline{u}, \underline{v}$ so that

$$w = \underline{u}\{s_k s_l\}^{m_{kl}}\underline{v} \quad \text{and} \quad w'' = \underline{u}\{s_l s_k\}^{m_{kl}}\underline{v},$$

then this is the result of Lemma 3.3.5.

Now suppose that $w = \underline{u}\{s_k s_l\}^{m_{kl}}\underline{v}$ and the lemma, (A1) and (A2) are proved for the word $\underline{w}_1 = \underline{u}\{s_l s_k\}^{m_{kl}}\underline{v}$. Furthermore, assume that the transformation of $\underline{w}$ into $\underline{w}_1$ is the first step in the transformation of $\underline{w}$ into $\underline{w}''$.

Either there are modules $U'_{(1)}, Y_{(1)}$ so that the exact sequence given by the inductive assumptions is

$$0 \longrightarrow M_0 \longrightarrow M_{m_{ij}-2} \oplus U'_{(1)} \longrightarrow Y_{(1)} \longrightarrow 0 , \qquad (3.24)$$

or we can write the exact sequence given by the inductive assumption as a τ-translate or a τ^{-1}-translate of (3.24).

Let $\underline{w}_1 \equiv \underline{w}''_1$ with

$$\rho(\underline{w}_1) = \rho'_1(r_1, h_1)\rho'_2(s', i)\rho'_3$$

and

$$\rho(\underline{w}''_1) = \rho'_1\rho'_2(s', i)(t' + 1, j)\rho'_4.$$

so that either $\rho'_3 = \rho'_4$, or a pair (q, g) is in ρ'_4 if and only if $(q - 1, g)$ is in ρ'_3.

Furthermore, we can assume without loss of generality that m_{kl} is even. Then there are $q_1, q_2 \in \mathbb{N}_0$ and a series of pairs ρ'' so that

$$\rho(\underline{w}) = \rho(\underline{u})(q_1 - \frac{m_{kl}}{2} + 1, k)(q_2 - \frac{m_{kl}}{2} + 1, l)\ldots(q_1, k)(q_2, l)\rho''.$$

We can assume that (r, h) is in the series of pairs $\rho(\underline{u})(q_1 - \frac{m_{kl}}{2} + 1, k)(q_2 - \frac{m_{kl}}{2} + 1, l)\ldots(q_1, k)(q_2, l)$, since otherwise there is nothing to show. Analogously, we assume that the pair (r_1, h_1) is in the series of pairs $\rho(\underline{u}\{s_l s_k\}^{m_{kl}})$. Furthermore, we can assume $\underline{w}''_1 \neq \underline{w}''$ and $M_0 > (q_2, l)$: Let $\underline{x}$ be the word with $\rho(\underline{x}) = \rho_1(r, h)\rho_2(s, i)$ and $\underline{x}'$ be the word so that $\underline{w} = \underline{x}\underline{x}'$. Then there is some $\underline{x}''$ so that $\underline{w}'' = \underline{x}''\underline{x}'$.

Analogously to Lemma 3.3.5, if $m_{lk} \geq 3$, then there is some $X \in \mathcal{C}_{\underline{w}}$ so that

$$0 \longrightarrow (q_2, l) \longrightarrow (q_1, k) \oplus X \longrightarrow (q_1 - \frac{m_{kl}}{2} + 1, k) \qquad (3.25)$$

is an exact sequence.

We have two different cases to consider:

First, assume that $\underline{u}$ is also an initial subword of $\underline{w}''$. Then $\underline{u}\{s_l s_k\}^{m_{kl}-1}$ is an initial subword of $\underline{w}''_1$, since $\underline{x}$ with $\rho(\underline{x}) = \rho_1(r, h)\rho_2$ is reduced. Furthermore, $\underline{w} <_l \underline{w}_1$ and $(q_1 - \frac{m_{kl}}{2} + 1, k) = (r, h)$.

If $(r_1, h_1) \in \mathcal{C}_{\underline{w}}$, then we can set $Y := Y_{(1)}$ and we have $(q_1, k) = (r_1 - 1, h_1)$.

(A1) holds by the inductive assumption, (A2) holds by (3.25), (A1) and Lemma 3.2.13.

On the other hand, if $(r_1, h_1) \notin \mathcal{C}_{\underline{w}}$, then $(q_1, k) = (r_1, h_1)$. Furthermore, if we have $Y_{(1)} = (r_1, h_1)^{E(m_{ij}-3)}$, then (A1), (3.25) and Lemma 3.2.13 give

an exact sequence of the form (3.22) with $Y = (r, h)^{E(m_{ij}-3)}$. (A1) holds obviously.

If $Y_{(1)} \neq (r_1, h_1)^{E(m_{ij}-3)}$, then we set $Y := Y_{(1)}$ and we only need to prove that (A2) holds. Analogously to above, this is the result of Lemma 3.2.13 and (3.25).

It remains to prove the assumption in the case that $\underline{u}$ is not an initial subword of $\underline{w}''$.

If $\underline{w} <_l \underline{w}_1$, then $\rho(w_1)$ contains the series of pairs

$$\rho(\underline{u})(q_2 - \frac{m_{kl}}{2} + 1, l) \ldots (q_1, k)(q_2, l)(q_1 + 1, k)$$

and the exact sequence given by the induction is either (3.24) or the τ-translate of (3.24). We can assume without loss of generality, that some indecomposable direct summand of $Y_{(1)}$ or $\tau Y_{(1)}$ respectively is smaller than $(q_1 + 1, k)$. Otherwise, the arguments below hold analogously for an exact sequence given by (A2).

By Proposition 3.2.8, the exact sequence yielded by the inductive assumption is given by sequences of modules

$$(X_1, X_2, \ldots, X_m)$$
$$(X_1', X_2', \ldots, X_m')$$
$$(Y_1, Y_2, \ldots, Y_m)$$

that fulfil (S1) - (S3). By Remark 3.2.4, we can assume that $X_\gamma'' = X_\gamma'$ for all X_γ'.

In the following we begin with the case where the exact sequence given by the inductive assumption is (3.24).

Since $M_0 > (q_1 + 1, k)$ and by Lemma 3.2.3, these series of modules yield an exact sequence

$$0 \longrightarrow M_0 \longrightarrow X_\gamma \oplus X_\gamma' \longrightarrow Y_\gamma \longrightarrow 0 \qquad (3.26)$$

so that for every $(r_1', h_1') \mid Y_\gamma$ the inequality $(q_1, k) < (r_1', h_1')$ holds. By the inductive assumption, we can even assume that $(q_1, k) < (r_1', h_1') \leq (q_1+1, k)$ for all $(r_1', h_1') \notin \mathcal{C}_{\underline{w}_1}$. So either $(r_1', h_1') = (q_2, l)$ or $(r_1', h_1') = (q_1 + 1, k)$

Furthermore, there is an irreducible morphism $X_\gamma \to (r_1', h_1')$ and $(q_1 + 1, k) \leq X_\gamma \notin \mathcal{C}_{\underline{w}}$. Analogously to Lemma 3.3.5, if $m_{lk} \geq 3$, then there is some $X \in \mathcal{C}_{\underline{w}}$ so that

$$0 \longrightarrow (q_1 + 1, k) \longrightarrow (q_2, l) \oplus X \longrightarrow (q_1 - \frac{m_{ij}}{2} + 1, k) \qquad (3.27)$$

with $X \in \mathcal{C}_{\underline{w}_1}$. Together with the τ-translate of the exact sequence (3.25), this shows that $X_\gamma \neq (q_2+1, l)$ and $X_\gamma \neq (q_1+1, k)$: Otherwise, by Lemma 3.2.13, we would get an exact sequence where either $(q_1 - \frac{m_{kl}}{2} + 1, k)$ or $(q_2 - \frac{m_{kl}}{2} + 1, l)$ is a direct summand of the last term, but every direct summand of the middle term is in $\mathcal{C}_{\underline{w}}$. This is a contradiction to the inductive assumption.

Since $(q_1 + 2, k) > X_\gamma > (q_1 + 1, k)$, we have $\tau^{-1} X_\gamma \in \mathcal{C}_{\underline{w}_1}$. Inductively, $Y_{(1)} \in \mathcal{C}_{\underline{w}}$ and $(q_1, k) < (r_1', h_1')$ for every $(r_1', h_1') \mid Y_{(1)}$.

If $(q_1 + 1, k) \mid Y_\gamma$ for any exact sequence of the form (3.26), then there is such a sequence so that $Y_\gamma \in \mathcal{C}_{\underline{w}}$, but $Y_\gamma \notin \mathcal{C}_{\underline{w}_1}$.

Otherwise, the sequence (3.24) is already of the form (3.22).

In the latter case, it is easily seen that this sequence fulfils (A1) and (A2): the former holds by the inductive assumption. Define V, V', $\mathcal{C}'$ as in (A2) and let $V_{(1)}, V_{(1)}'$ be the corresponding modules, $\mathcal{C}_{(1)}'$ the corresponding category for the sequence (3.24). Then $V = V_{(1)}$, $V' = V_{(1)}'$ and $\mathcal{C}' = \mathcal{C}_{(1)}'$. Thus, assertion (2) also holds.

So assume that there is some γ with $(q_1 + 1, k) \mid Y_\gamma \in \mathcal{C}_{\underline{w}}$. This sequence is of the form (3.22) and (A1) holds. Let $\alpha \in \mathbb{N}$ be the maximal exponent so that $(q_1 + 1, k)^\alpha \mid Y_\gamma$

By construction, we can write $X_\gamma \oplus X_\gamma' = B_1 \oplus B_1' \oplus M_{m_{ij}-2}$ so that there is an exact sequence

$$0 \longrightarrow B_1 \longrightarrow C_1 \oplus (q_1 + 1, k)^\alpha \longrightarrow \tau^{-1} B_1 \ .$$

By Remark 3.2.4 and Lemma 3.2.13, $U_{(1)}' = B_1' \oplus C_1$. If $Y_\gamma = D \oplus (q_1+1, k)^\alpha$, then $Y_{(1)} = D \oplus \tau^{-1} B_1$.

So we can write $V_{(1)}' = B_1'' \oplus C_1''$ with $B_1'' \mid B_1'$ and $C_1'' \mid C_1$. We get $V' = B_1'' \oplus B_1$. By Proposition 3.2.8, Lemma 3.2.13 and the inductive assumption, assertion (2) is fulfilled.

If we still have $\underline{w} <_l \underline{w}_1$, but the exact sequence given by the inductive assumption is the τ-translate of (3.24), then analogously we have $\tau Y_{(1)} \in \mathcal{C}_{\underline{w}_1}$ and $(q_1, k) < (r_1', h_1')$ for all direct summands (r_1', h_1') of $\tau Y_{(1)}$. Suppose that $\tau Y_{(1)}, Y_{(1)}, \ldots, \tau^{-o+1} Y_{(1)} \in \mathcal{C}_{\underline{w}}$ and $\tau^{-o} Y \notin \mathcal{C}_{\underline{w}}$. If $o > 0$, then (3.24) is of the form (3.22). By the inductive hypothesis, (A2) is fulfilled.

If $o = 0$, then we use assertion (A2) of the inductive hypothesis: by Lemma 3.2.13, there is an exact sequence of the form (3.22) that fulfils (A1) and (A2).

It only remains to prove the case $\underline{w}_1 <_l \underline{w}$. Then $\rho(w_1)$ contains the series of pairs

$$\rho(\underline{u})(q_2 - \frac{m_{kl}}{2}, l)(q_1 - \frac{m_{kl}}{2} + 1, k) \ldots (q_2 - 1, l)(q_1, k).$$

The exact sequence given by the inductive assumption is either (3.24) or the τ^{-1}-translate. In the first case, we show (analogously to above) that $Y_{(1)} \in \mathcal{C}_{\underline{w}_1}$ and $(q_1, k) < (r'_1, h'_1)$ for all $(r'_1, h'_1) \mid Y_{(1)}$.

If $(q_2, l) \nmid Y_{(1)}$, then the sequence (3.24) is already of the form (3.22) and the assertions hold.

Otherwise, $\tau^{-1} Y_{(1)} \notin \mathcal{C}_{\underline{w}_1}$. Analogously to before, $(q_1, k) \nmid \tau^{-1} X'_{(1)}$ by Lemma 3.3.5 and (A2): If $(q_1, k) \mid \tau^{-1} X'_{(1)}$ we use Lemma 3.2.13 and get an exact sequence

$$0 \longrightarrow \tau^{-o-1} V'_{(1)} \longrightarrow \tau^{-o-1} Y_{(1)} \oplus U''_{(1)} \longrightarrow Y'_{(1)} \longrightarrow 0 \ .$$

where $(q_1 - \frac{m_{kl}}{2}, k) \mid Y'_{(1)}$ or $(q_2 - \frac{m_{kl}}{2}, l) \mid Y'_{(1)}$ but every direct summand of $U''_{(1)}$ is in $\mathcal{C}'_{(1)}$. This is a contradiction to the inductive assumption.

For every module B with an irreducible morphism $B \to (q_2, l)$, we have $(q_2 - 1, l) < \tau^{-1} B < (q_2, l)$. If $B \mid X'_{(1)}$, then $\tau^{-1} B \in \mathcal{C}_{\underline{w}}$: all modules between (q_1, k) and (q_2, l) are in $\mathcal{C}_{\underline{w}_1}$, since the exact sequence given by the inductive assumption is (3.24).

So $B \notin \mathcal{C}_{\underline{w}_1}$ by the definition of $X'_{(1)}$. Thus, we also have $B \notin \mathcal{C}_{\underline{w}}$.

Let α be the maximal integer so that $(q_2, l)^\alpha \mid Y_{(1)}$.

Analogously to the arguments above, we can define $B_1 \mid X'_{(1)}$ so that there is an exact sequence

$$0 \longrightarrow B_1 \longrightarrow C_1 \oplus (q_1 + 1, k)^\alpha \longrightarrow \tau^{-1} B_1 \ .$$

As before, the assertion (A2) of the inductive hypothesis and Lemma 3.2.13 show that there is an exact sequence of the form (3.22) and that (A1) and (A2) are fulfilled.

If the exact sequence yielded by the inductive hypothesis is the τ^{-1}-translate of (3.24), then similar to the arguments above we get an exact sequence (3.22) so that $Y \in \mathcal{C}_{\underline{w}}$ and $(q_1, k) < (r', h')$ for every direct summand (r', h') of Y. This exact sequence fulfils (A1) and (A2). □

3.5 The first direction

In this section we show inductively that for every $w \in W$, the category $\mathcal{C}_w$ is submodule closed. Afterwards, it only remains to show that every cofinite, submodule closed category is of the form $\mathcal{C}_w$.

We begin with the basis of the induction:

Lemma 3.5.1. *Let $m_{ij} < \infty$ and $U_1, \ldots, U_{m_{ij}-1}$ be as in Lemma 3.3.5. If $U_1, \ldots, U_{m_{ij}-1} \in \mathcal{C}_{\underline{w}}$ and $M_0 \notin \mathcal{C}_{\underline{w}}$, then $\mathcal{C}_{\underline{w}}$ is not submodule closed and $\underline{w}$ is not leftmost.*

Proof. Since $M_0 \subseteq U_{m_{ij}-1}$, the category $\mathcal{C}_{\underline{w}}$ is not submodule closed. Let $\underline{w}$ be a word for the element $w \in W$. By Definition 3.3.4, $M_0 = (s, i)$ and $M_1 = (t, j)$.

Suppose that $M_1 \in \mathcal{C}_{\underline{w}}$. Let $\underline{u}$ be the initial subword of $\underline{w}$ that is defined through the inequality $(r, k) \leq (s - 1, i)$ for every pair (r, k) in $\rho(\underline{u})$.

Then there are reflections $s_{k_1}, s_{k_2}, \ldots, s_{k_m}$ and a word $\underline{v}$ so that

$$\underline{w} = \underline{u} s_{k_1} s_{k_2} \ldots, s_{k_m} s_i \underline{v}.$$

Since $U_1 \in \mathcal{C}_{\underline{w}}$, we have

$$m_{k_1 i} = m_{k_2 i} = \cdots = m_{k_m i} = 2$$

by Lemmas 3.3.1 and 3.2.7. So

$$\underline{w}' = \underline{u} s_i s_{k_1} s_{k_2} \ldots s_{k_m} \underline{v}$$

is equivalent to $\underline{w}$ and thus a word for w. Since $(r, k) \leq (s - 1, i)$ for all reflections (r, k) in $\rho(\underline{u})$, we see that either $\underline{w}' <_l \underline{w}$, or $\underline{w}$ is not reduced.

Clearly, the same argument holds if $M_m \in \mathcal{C}_{\underline{w}}$ for some $1 \leq m \leq m_{ij} - 1$.

It remains to prove that $\mathcal{C}_{\underline{w}}$ is not submodule closed if $M_0, \ldots, M_{m_{ij}-1} \notin \mathcal{C}_{\underline{w}}$. Without loss of generality, we can assume that $M_{m_{ij}-1} = (p, i)$ for some $p \in \mathbb{N}$ and $M_{m_{ij}-2} = (q, j)$ for some $q \in \mathbb{N}$. Suppose that (r, k) is a pair in $\underline{w}$. If $(q - 1, j) < (r, k) < (t, j)$ then we use that $U_1, U_2, \ldots U_{m_{ij}-1} \in \mathcal{C}_{\underline{w}}$ and get $m_{jk} = 2$ by 3.3.6. If $(p, i) < (r, k) < (s, i)$ then $m_{ik} = 2$.

Let $\underline{u}'$ be the initial subword of $\underline{w}$ that is defined through the inequality $(r, k) \leq (q - 1, j)$ for every pair (r, k) in $\rho(\underline{u})$.

Then there are reflections $s_{k_1}, \ldots s_{k_m}$ with

$$m_{k_1 j} = m_{k_2 j} = \cdots = m_{k_m j} = 2$$

so that $\underline{w} \equiv \underline{w}'$ for

$$\underline{w}' = \underline{u}' s_{k_1} \ldots s_{k_m} \{s_i s_j\}^{m_{ij}} \underline{v}'. \tag{3.28}$$

So $\underline{w}$ is also equivalent to

$$\underline{w}'' = \underline{u}' s_j s_{k_1} \ldots s_{k_m} \{s_i s_j\}^{m_{ij}-1} \underline{v}' \tag{3.29}$$

and either $\underline{w}'' <_l \underline{w}$ or $\underline{w}$ is not reduced. $\qquad \square$

This proof even shows the following:

Corollary 3.5.2. *If $m_{ij} < \infty$, $M_0, M_1, \ldots, M_{m_{ij}-1} \notin \mathcal{C}_{\underline{w}}$ and $\underline{w}$ is reduced, then there is a word $\underline{w}' \equiv \underline{w}$ with $\underline{w}' <_l \underline{w}$, pairs $(r, h) = M_{m_{ij}-1}$, $(r', h') \neq (r, h)$ and series of pairs ρ_1, ρ_2, ρ_3 so that*

$$\rho(\underline{w}') = \rho_1(r, h)\rho_2 \quad and \quad \rho(\underline{w}) = \rho_1(r', h')\rho_3.$$

For the inductive step, we still need some lemmas:

Lemma 3.5.3. *Suppose that for some $\underline{w}$, we have $M_0 \notin \mathcal{C}_{\underline{w}}$ and M_0 is a submodule of $U \in \mathcal{C}$. Let $U_{m_{ij}-1}$ be as in Lemma 3.3.5 with modules $(r_k, l_k) \notin \mathcal{C}_{\underline{w}}$ for $1 \leq k \leq a$ so that $\bigoplus_{k=1}^{a}(r_k, l_k) \mid U_{m_{ij}-1}$.*
Let

$$
\begin{aligned}
&(X_1, X_2, \ldots, X_m) \\
&(X_1', X_2', \ldots, X_m') \\
&(Y_1, Y_2, \ldots, Y_m)
\end{aligned}
\tag{3.30}
$$

be the sequences of modules that yield the exact sequences

$$\eta_k : \quad 0 \longrightarrow M_0 \xrightarrow{f_k} M_k^{E(k)} \oplus U_k \longrightarrow M_{k+1}^{E(k-1)} \longrightarrow 0$$

for all $1 \leq k \leq m_{ij} - 1$.
Then one of the following holds:

(a) *There is some $1 \leq m' \leq m_{ij}-1$ and some $U' \in \mathcal{C}_{\underline{w}}$ with a monomorphism $M_{m'} \rightarrowtail U'$*

(b) *If $(X_1, \ldots X_m, X_{m+1}, \ldots X_{m'})$ is part of a triple of sequences that fulfils (S1) - (S5), there is some $1 \leq k \leq m'$ so that $M_{m_{ij}} \mid X_k \oplus X_k'$. Furthermore, for $l \in \mathbb{N}$, $1 \leq k \leq a$ and $M_{m_{ij}-1} < (r_k - l, l_k)$, we have $(r_k - l, l_k) \notin \mathcal{C}_{\underline{w}}$.*

(c) *$a = 1$, $m_{il_1} + m_{jl_1} = 5$ and there is no indecomposable morphism $M_{m''} \to (r_1, l_1)$ for $m'' < m_{ij} - 3$.*

(d) *$a = 1$ and $(r_1, l_1) < M_{m_{ij}-1}$.*

If $(r_k - 1, l_k) \in \mathcal{C}_{\underline{w}}$ for some $1 \leq k \leq a$, then (a) holds.

Proof. By Corollary 3.3.6, for all $1 \leq k \leq a$, there are some $X \in \mathcal{I}$, $\beta_k \in \mathbb{N}$ and $2 \leq m_k \leq m_{ij}$ so that the AR-sequence that starts in (r_k, l_k) is of the form

$$0 \longrightarrow (r_k, l_k) \longrightarrow M_{m_k}^{\beta_k} \oplus X \longrightarrow (r_k - 1, l_k) \longrightarrow 0 .$$

First, suppose that $a > 1$. By Lemma 3.3.3 and Lemma 3.3.5, there is an exact sequence

$$0 \longrightarrow M_0 \longrightarrow U_{m_{ij}-1} \longrightarrow M_{m_{ij}} \longrightarrow 0 \ .$$

and we can assume that $X_m \oplus X'_m = U_{m_{ij}-1}$ and $Y_m = M_{m_{ij}}$. By Corollary 3.2.9, we can set $X_m := (r_1, l_1)$ and $X_{m+1} := (r_2, l_2)$. If $M_{m_1} = M_{m_2} = M_{m_{ij}}$, then $M_{m_{ij}} \mid X_{m+2} \oplus X'_{m+2}$ by (S3). Thus (b) is fulfilled.

Otherwise, set $\gamma = m + m_{ij} + 2 - m_1$, $\delta = \gamma + m_{ij} - m_2$

$$X_{m+3} := M_{m_1}, X_{m+4} := M_{m_1+1}, \ldots, X_\gamma := M_{m_{ij}-1}$$

and

$$X_{\gamma+1} := M_{m_2}, X_{\gamma+2} := M_{m_2+1}, \ldots X_\delta := M_{m_{ij}-1}.$$

Then $M_{m_{ij}} \mid X_{\delta+1} \oplus X'_{\delta+1}$ and (b) holds.

On the other hand, suppose that there is an indecomposable morphism $M_{m''} \to (r_1, l_1)$ for some $m'' < m_{ij} - 3$. By Corollary 3.3.6, we have $(r_1-1, l_1) \mid U_{m_{ij}-1}$. By the construction of $\eta_{m''+1}$ from $\eta_{m''+1}$, there must be a module $U'_{m''+1}$ so that we can write $\eta_{m''+1}$ as the following:

$$0 \longrightarrow M_0 \longrightarrow M_{m''+1}^{E(m''+1)} \oplus (r_1, l_1)^{E(m'')} \oplus U'_{m''+1} \longrightarrow M_{m''+2}^{E(m'')} \ .$$

So there is a module U' with an exact sequence

$$0 \longrightarrow M_0 \longrightarrow M_{m''+1}^{E(m''+1)} \oplus U' \longrightarrow (r_1 - 1, l_1)^{E(m''+1)} \longrightarrow 0 \ .$$

By Corollary 3.2.6 and the monomorphism $M_0 \rightarrowtail U$, we get a monomorphism

$$M_{m_{ij}-1} \rightarrowtail U \oplus (r_1 - 1, l_1).$$

So either (a) is fulfilled or $(r - 1, l_1) \notin \mathcal{C}_{\underline{w}}$. In this case, $a > 1$ and (b) is fulfilled.

Finally, suppose that $m_{il_1} + m_{jl_1} > 5$ and $a = 1$. By Lemma 3.3.1, $\beta_1 > 1$ if either $m_{il_1} \geq 4$ or $m_{il_2} \geq 4$. We can set $X_{m+1} := (r_1, l_1)$ and $X_{m+2} := X_{m+3} := M_{m_1}$. Analogously to the case $a > 1$, this yields some $m' \in \mathbb{N}$ so that $M_{m_{ij}} \mid X_{m'} \oplus X'_{m'}$.

If $m_{il_1} = m_{jl_1} = 3$, then either (d) holds, or we can write the AR-sequence that starts in (r_1, l_1) as

$$0 \longrightarrow (r_1, l_1) \longrightarrow M_{m_1} \oplus M_{m_1+1} \oplus X' \longrightarrow (r_1 - 1, l_1) \longrightarrow 0 \qquad (3.31)$$

for some $X' \in \mathcal{I}$ and some $1 \leq m_1 \leq m_{ij} - 1$. Since $X_{m+1} := (r_1, l_1)$, $X_{m+2} := M_{m_1}$ and $X_{m+3} = M_{m_1+1}$, we see that (b) holds. $\qquad\square$

We can generalize this lemma in the following way, which we will need for an induction:

Remark 3.5.4. Let us assume that the assumptions of Lemma 3.5.3 hold, except that M_0 is not a submodule of some $U \in \mathcal{C}_{\underline{w}}$ but of $X \oplus U$ for some indecomposable $X \notin \mathcal{C}_{\underline{w}}$. If $X = M_k$ for some $1 \leq k \leq m_{ij} - 1$, we alter the assumptions on the sequences (3.30) accordingly so that they fulfil (S1) - (S3) with respect to M_0 and $X \oplus U$.

Furthermore, we assume that $X^2 \mid X_k \oplus X'_k$ for some $1 \leq k \leq m$: then for every sequence $(X_1, \ldots, X_m, X_{m+1}, \ldots, X_{m'})$ which is part of a triple of sequences that fulfils (S1) - (S5), we have some $1 \leq k \leq m'$ so that $X_k = X$.

So every argument in the proof of 3.5.3 still holds and we still get that one of the conditions (a) - (d) must be fulfilled.

If we take a look at case (b) of 3.5.3 and suppose that $a = 1$, then by Lemma 3.2.13, there is a monomorphism $(r_1, l_1) \rightarrowtail M_{m_{ij}} \oplus U$. So there are sequences of modules

$$(^1X_1, {}^1X_2, \ldots, {}^1X_o)$$
$$(^1X'_1, {}^1X'_2, \ldots, {}^1X'_o)$$
$$(^1Y_1, {}^1Y_2, \ldots, {}^1Y_o)$$

which fulfil (S1) - (S5) with respect to (r_1, l_1) and $M_{m_{ij}} \oplus U$. Since $M_{m_{ij}} \mid X_k \oplus X'_k$ for some $1 \leq k \leq m'$, there must be some k' so that $M^2_{m_{ij}} \mid^1 X_{k'} \oplus X'_{k'}$.

We still need a result about the case $a > 1$:

Lemma 3.5.5. *Suppose that for some $\underline{w}$, we have $M_0 \notin \mathcal{C}_{\underline{w}}$ and M_0 is a submodule of $U \in \mathcal{C}$. Let $U_{m_{ij}-1}$ be as in Lemma 3.3.5 with modules $(r_k, l_k) \notin \mathcal{C}_{\underline{w}}$ for $1 \leq k \leq a$ so that $\bigoplus_{k=1}^a (r_k, l_k) \mid U_{m_{ij}-1}$.*

Furthermore, suppose that for all $1 \leq k \leq a$, there is an irreducible morphism $M_m \to (r_k, l_k)$ for some $0 \leq m < m_{ij} - 2$.

If $a > 1$, then one of the following holds:

(a) There is some $N < M$, $N \notin \mathcal{C}_{\underline{w}}$ that is a submodule of some $U' \in \mathcal{C}_{\underline{w}}$.

(b) We have $a = 2$, $m_{l_1 l_2} = 2$, $m_{il_k} = 3$, $m_{jl_k} = 2$ for $1 \leq k \leq 2$ and $m_{ij} = 3$.

(c) We have $a = 2$, $l_1 = l_2$, $m_{il_1} = 3$ and $m_{ij} = 3$.

In case (b),
$$M_{m_{ij}}, M_{m_{ij}+1}, M_{m_{ij}+2}, M_{m_{ij}+3} \notin \mathcal{C}_{\underline{w}}.$$
and
$$(r_1 - 1, l_1), (r_2 - 1, l_2), (r_1 - 2, l_1), (r_2 - 2, l_2) \notin \mathcal{C}_{\underline{w}}.$$

Proof. The proof is analogous to that of Lemma 3.3.5. First note that if $m_{ij} = 3$, we get $m_{il_k} \geq 3$ for $1 \leq k \leq a$, since there is an irreducible morphism $M_m \rightarrow (r_k, l_k)$ for some $0 \leq m < m_{ij} - 2$.

If $m_{l_1 l_2} \neq 2$, we can exchange j and l_2 in the calculations below.

Define α_{ki}, β_{ki}, α_{kj}, β_{kj} analogous to α and β with i, l_k and j, l_k instead of i, j. If $m_{l_1 l_2} = 2$, but (b) is not fulfilled, then

$$\alpha\beta + \sum_{k=1}^{a} (\alpha_{ki} + \alpha_{kj})(\beta_{ki} + \beta_{kj}) \geq 4.$$

By Corollary 3.2.11, we can do completely analogous calculations to the case $\alpha\beta = 4$. In these calculations, we construct exact sequences which contain modules of the form M_o, (o', l_1) and (o'', l_2) for some $o, o', o'' \in \mathbb{N}$.

It remains to show that (a) is fulfilled if any of these modules is in $\mathcal{C}_{\underline{w}}$.

By Lemma 3.5.3, if $(r_k - 1, l_k) \in \mathcal{C}_{\underline{w}}$ for some $1 \leq k \leq a$, then (a) holds. The rest follows inductively with the same argument as in 3.5.3. $\square$

Now we can show the following, which is the last lemma that we need to prove the first direction of the main theorem:

Lemma 3.5.6. *Let $\underline{w}$ be a word so that $(s, i) = M_0, M_1, \ldots, M_{m_{ij}-1} \notin \mathcal{C}_{\underline{w}}$. Then there are words $\underline{u}, \underline{v}$ so that $\underline{w} = \underline{u} s_i \underline{v}$ and there is some ρ with $\rho(\underline{w}) = \rho(\underline{u})(s, i)\rho$. Suppose that there is some $U \in \mathcal{C}_{\underline{w}}$ with a monomorphism $M_0 \rightarrowtail U$ and for every $X < M_0$ with some $U' \in \mathcal{C}_{\underline{w}}$ and a monomorphism $X \rightarrowtail U'$, we have $X \in \mathcal{C}_{\underline{w}}$.*

Then there exists some $\underline{u}'$ so that

$$\underline{w} \equiv \underline{u}' \{s_i s_j\}^{m_{ij}} \underline{v}. \tag{3.32}$$

Proof. By Lemma 3.3.3 and 3.3.5, Proposition 3.2.8 yields an exact sequence

$$0 \longrightarrow M_0 \longrightarrow U_{m_{ij}-1} \longrightarrow M_{m_{ij}} \longrightarrow 0 . \tag{3.33}$$

If $(r, l) \in \mathcal{C}_{\underline{w}}$ for all $(r, l) \mid U_{m_{ij}-1}$ with $(r, l) > M_{m_{ij}-1}$, then (3.33) is obvious by Corollary 3.3.6.

Otherwise, we get $M_{m_{ij}} \notin \mathcal{C}_{\underline{w}}$, since there is a monomorphism $U_{m_{ij}-1} \rightarrowtail M_{m_{ij}} \oplus U$ by Lemma 3.2.6.

There are direct summands $(r_1, l_1), \ldots, (r_a, l_a) \notin \mathcal{C}_{\underline{w}}$ of $U_{m_{ij}-1}$ so that $M_{m_{ij}-1} < (r_k, l_k)$ for all $1 \leq k \leq a$.

We can assume without loss of generality that m_{ij} is odd; otherwise we only need to exchange s_j and s_i in the arguments below.

If for all $1 \leq k \leq a$, there is no morphism $M_m \to (r_k, l_k)$ for any $0 \leq m < m_{ij} - 2$, then $m_{l_k i} = 2$ and for some word $\underline{x}$ we have

$$\underline{w} = \underline{x} s_i s_{l_1} \ldots s_{l_a} \{s_j s_i\}^{m_{ij}-1} \equiv \underline{x} s_{l_1} \ldots s_{l_a} \{s_i s_j\}^{m_{ij}-1}.$$

So we can assume that for some (r_k, l_k), there is a morphism $M_m \to (r_k, l_k)$ for some $0 \leq m < m_{ij} - 2$.

By Lemma 3.5.3, one of the following cases hold:

(a) $a = 1$, $m_{il_1} + m_{jl_1} = 5$ and there is no indecomposable morphism $M_{m''} \to (r_1, l_1)$ for $m'' < m_{ij} - 3$.

(b) There is some m' so that $M_{m_{ij}} \mid X_{m'} \oplus X'_{m'}$

In case (a) there are words $\underline{u}'', \underline{v}$, so that either

$$\underline{w} = \underline{u}'' s_i s_{l_1} \{s_j s_i\}^{m_{ij}-1} \underline{v} \tag{3.34}$$

or both $m_{jl_1} = 2$ and

$$\underline{w} = \underline{u}'' s_i s_j s_{l_1} \{s_i s_j\}^{m_{ij}-2} \underline{v}. \tag{3.35}$$

Then there is some word $\underline{u}_1$ so that $\underline{w} \equiv \underline{u}_1 s_{l_1} \{s_i s_j\}^{m_{ij}} \underline{v}$: If $m_{il_1} = 2$, there is nothing to show. If $m_{jl_1} = 2$, then we use that $M_{m_{ij}}$ is of the form (q, j) for some q and there is a monomorphism $U_{m_{ij}-1} \rightarrowtail M_{m_{ij}} \oplus U$: Let $\underline{u}_2$ be the subword of $\underline{u}_1$ that does not contain the reflection that corresponds to (q, j). By the inductive assumption, $\underline{u}_2 s_i s_{l_1}$ is equivalent to a word $\underline{u}_3 \{s_i s_{l_1}\}^{m_{il_1}}$.

Now we go back to looking at $\underline{u}_1$, not $\underline{u}_2$. Since $m_{il_1} = 3$ and m_{jl_1}, there is some word $\underline{u}_4$ with $\underline{u}_1 s_i s_{l_1} \equiv \underline{u}_4 \{s_i s_{l_1}\}^{m_{il_1}}$ and thus

$$\underline{w} \equiv \underline{u}'' s_{l_1} \{s_i s_j\}^{m_{ij}} \underline{v}$$

for some word $\underline{u}''$.

On the other hand, suppose that

$$M_{m_{ij}} \mid X_{m'} \oplus X'_{m'}. \tag{3.36}$$

Since we have $\bigoplus_{k=1}^{a} (r_k, l_k) \mid U_{m_{ij}}$, there is a monomorphism

$$\bigoplus_{k=1}^{a} (r_k, l_k) \rightarrowtail U \oplus M_{m_{ij}}. \tag{3.37}$$

Assume that $(r_1, l_1) \leq (r_2, l_2) \leq \cdots \leq (r_a, l_a)$.

Let the AR-sequence that starts in (r_a, l_a) be

$$0 \longrightarrow (r_a, l_a) \longrightarrow M_{m_1} \oplus Z_1 \longrightarrow (r_a - 1, l_1) \longrightarrow 0 .$$

If $a = 1$, then by Lemma 3.5.3, $\underline{w}$ must have the form (3.34) or (3.35). The only difference to case (a) is that $m_{il_1} + m_{jl_2} > 5$.

We can use case (a) as the basis of an induction: Instead of the modules $M_0, M_1, (r_1, l_1)$, we take $(r_1, l_1), M_{m_1}, M_{m_1+1}$ and use the same arguments as before. Since $M_{m_{ij}} \mid X_{m'} \oplus X'_{m'}$, we can use Remark 3.5.4 and either we get the analogue to case (a) above or the analogue to the case (b). In the first case, we get $m_{il_1} + m_{ij} = 5$ or $m_{jl_1} + m_{ij} = 5$ and there is some $\underline{u}'_1$ so that $\underline{w}$ is equivalent to a word with the subword $\{s_{l_1} s_i\}^{m_{il_1}}$ if m_1 is even and $\{s_{l_1} s_j\}^{m_{jl_1}}$ is odd. Thus we also get $\underline{w} \equiv \underline{u}' \{s_i s_j\}^{m_{ij}} \underline{v}$ for some words $\underline{u}', \underline{v}$. In the case (b), we continue this inductively.

After finitely many steps we get

$$\underline{w} \equiv \underline{u}' \{s_i s_j\}^{m_{ij}} \underline{v}.$$

If $a \neq 1$, then $a = 2$ by Lemma 3.5.5. If $l_1 = l_2$, then $l_1 = l_2$, $m_{il_1} = 3$ and $m_{ij} = 3$. We can exchange j and l_1 to get the case $a = 1$. Otherwise, $m_{l_1 l_2} = 2$, $m_{il_k} = 3$, $m_{jl_k} = 2$ for $1 \leq k \leq 2$ and $m_{ij} = 3$. Furthermore,

$$M_{m_{ij}}, M_{m_{ij}+1}, M_{m_{ij}+2}, M_{m_{ij}+3} \notin \mathcal{C}_{\underline{w}}.$$

and

$$(r_1 - 1, l_1), (r_2 - 2, l_2), (r_1 - 2, l_1), (r_2 - 2, l_2) \notin \mathcal{C}_{\underline{w}}.$$

Analogously to before, we see inductively that $\underline{w}$ is equivalent to a word with the subword

$$s_{l_1} s_{l_2} s_j s_i s_{l_1} s_{l_2} s_j s_i s_{l_1} s_{l_2} s_j s_i.$$

(For the purpose of this induction, we can treat the word above completely analogously to a word of the form $\{s_i s_j\}^{m_{ij}}$ with $m_{ij} = 6$. As in the proof of 3.5.5, all calculations are the same by Corollary 3.2.11.)

We have the following equivalences, where bold reflections denote those which differ from the reflections in the word above:

$$s_{l_1} s_{l_2} s_j s_i s_{l_1} s_{l_2} s_j s_i s_{l_1} s_{l_2} s_j s_i$$

$$\equiv s_{l_1} s_{l_2} s_j s_i \mathbf{s_j s_{l_1} s_{l_2}} s_i \mathbf{s_{l_2} s_{l_1}} s_j s_i$$

$$\equiv s_{l_1} s_{l_2} \mathbf{s_i s_j s_i} s_{l_1} \mathbf{s_i s_{l_2} s_i} s_{l_1} s_j s_i$$

$$\equiv s_{l_1} s_{l_2} s_i s_j \mathbf{s_{l_1} s_i s_{l_1}} s_{l_2} s_i s_{l_1} s_j s_i$$

$$\equiv s_{l_1} s_{l_2} s_i s_j s_{l_1} s_i \mathbf{s_{l_2} s_{l_1}} s_i s_{l_1} s_j s_i$$

$$\equiv s_{l_1} s_{l_2} s_i s_j s_{l_1} s_i s_{l_2} \mathbf{s_{l_1} s_i} s_j s_i.$$

So $\underline{w}$ is equivalent to a word with the subword $s_i s_j s_i$ and the assertion is true. $\qquad\square$

Finally, we can prove the first direction of our main result:

Lemma 3.5.7. *If $\underline{w}$ is a leftmost word, then $\mathcal{C}_{\underline{w}}$ is submodule closed.*

Proof. Suppose that $\mathcal{C}_{\underline{w}}$ is not submodule closed. Then there is some $M_0 \in \operatorname{ind}\mathcal{I}\backslash\mathcal{C}_{\underline{w}}$ and some $U \in \mathcal{C}_{\underline{w}}$ with a monomorphism $M_0 \rightarrowtail U$. Furthermore, we can assume that for every $X < M_0$ with some $U' \in \mathcal{C}_{\underline{w}}$ and a monomorphism $X \rightarrowtail U'$, we have $X \in \mathcal{C}_{\underline{w}}$.

We use induction on the length m of the sequences of modules in Proposition 3.2.8 applied on M_0 and U. If $m = 1$, then $\underline{w}$ is not leftmost by Lemma 3.5.1.

Now suppose that $\underline{w}$ is not leftmost if the sequences have the length m or smaller. We prove that this is also the case if they have length $m + 1$:

We can assume without loss of generality that $M_1 \notin \mathcal{C}_{\underline{w}}$, since $m + 1 > 1$. On the other hand, by Lemma 3.3.5, the sequences of modules induce an exact sequence

$$0 \longrightarrow M_0 \longrightarrow M_1^{E(1)} \oplus U_1 \longrightarrow M_2 \longrightarrow 0 \ .$$

So by Corollary 3.2.6, there is a monomorphism $M_1^{E(1)} \rightarrowtail M_2 \oplus U$. Since $M_1 < M_0$, our assumptions yield $M_2 \notin \mathcal{C}_{\underline{w}}$. By the same argument, we get $M_3, M_4, \ldots, M_{m_{ij}-1} \notin \mathcal{C}_{\underline{w}}$ and by (S4) this means $m_{ij} < \infty$.

By Lemma 3.5.6, if we choose $\underline{u}$ and $\underline{v}$ so that $\underline{w} = \underline{u}s_i\underline{v}$ and there is some ρ so that $\rho(\underline{w}) = \rho(\underline{u})(s,i)\rho$, then $\underline{w} \equiv \underline{u}'\{s_is_j\}^{m_{ij}}\underline{v}$ for some word $\underline{u}'$.

We still need to show that $\underline{u}'\{s_is_j\}^{m_{ij}}\underline{v}$ is equivalent to a word which is smaller than $\underline{w}$.

To do this, we use Lemma 3.1.7. Either there is nothing to show, or there are $\rho_1, \ldots, \rho_4$ and a pair (r, h) so that

$$\rho(\underline{w}) = \rho_1(r,h)\rho_2(s,i)\rho_3 \tag{3.38}$$

and there is some $\underline{w}'' \equiv \underline{w}$ with

$$\rho(\underline{w}'') = \rho_1\rho_2(s,i)(t+1,j)\rho_3. \tag{3.39}$$

We can assume that the word $\underline{x}$ with $\rho(\underline{x}) = \rho_1(r,h)\rho_2$ is reduced, because otherwise there is nothing to prove. So by Lemma 3.4.2, there are some sequences of modules as in Lemma 3.2.3 that yield some $U' \in \mathcal{I}$ and an exact sequence

$$0 \longrightarrow M_0 \longrightarrow M_{m_{ij}-2} \oplus U' \longrightarrow Y \longrightarrow 0$$

with either $Y \in \mathcal{C}_{\underline{w}}$ or $Y = (r, h)^{E(m_{ij}-3)}$.

By Corollary 3.2.6, there is a monomorphism

$$M_{m_{ij}-2} \rightarrowtail U \oplus Y \in \mathcal{C}_{\underline{w}''}.$$

By (3.38), (3.39) and the induction hypothesis, $\underline{w}''$ is not leftmost.

So there is some $\underline{w}_3 \equiv \underline{w}''$ with $\underline{w}_3 <_l \underline{w}''$. We still need to show that $\underline{w}_3 <_l \underline{w}$.

If $\underline{w}''$ is not reduced, this is obvious. If $Y \in \mathcal{C}_{\underline{w}}$, we can use the inductive assumption.

So suppose that $\underline{w}''$ is reduced and $Y = (r, h)^{E(m_{ij}-3)}$. We denote the sequences of modules that fulfil (S1) - (S5) with respect to M_0 and U by

$$(X_1, X_2, \ldots, X_m)$$
$$(X_1', X_2', \ldots, X_m')$$
$$(Y_1, Y_2, \ldots, Y_m).$$

There are sequences of modules

$$({}^1X_1, {}^1X_2, \ldots, {}^1X_{m'})$$
$$({}^1X_1', {}^1X_2', \ldots, {}^1X_{m'}')$$
$$({}^1Y_1, {}^1Y_2, \ldots, {}^1Y_{m'})$$

that fulfil (S1) - (S5) with respect to $M_{m_{ij}-2}$ and $U \oplus (r, h)^{E(m_{ij}-3)}$. Let (r', h') be the smallest indecomposable direct summand of ${}^1Y_{m'}$. Then $(r', h') < (r, h)$.

If there is a pair $(r'', h'') \neq (r', h')$ and series of pairs $\rho_1', \rho_2', \rho_3'$ so that we can write

$$\rho(\underline{w}_3) = \rho_1'(r', h')\rho_2' \quad \text{and} \quad \rho(\underline{w}) = \rho_1'(r'', h'')\rho_3',$$

then $\underline{w}_3 <_l \underline{w}''$ implies $\underline{w}_3 <_l \underline{w}$.

A simple induction on m' shows that this is indeed the case: If $m' = 1$, then $(r, h) = M_{m_{ij}-1}$, $(r', h') = M_{m_{ij}}$ and the assertion is true by Corollary 3.5.2.

By 3.2.13, the smallest direct summand of ${}^1Y_{m'}$ is also the smallest direct summand of Y_m and the inductive step is obvious.

So $\underline{w}$ is not leftmost and the proof is complete. $\qquad\square$

3.6 The other direction

In this section we finally conclude the proof that the map between words and full additive cofinite subcategories of $\mathsf{mod}\, A$ introduced in Definition

3.1.13 gives rise to a bijection between the leftmost words and the cofinite submodule closed subcategories. Since every element of the Weyl group has a unique leftmost element, this gives a bijection between the Weyl group elements and the cofinite, submodule closed subcategories.

Again, we start with the basis of an induction:

Lemma 3.6.1. *(a) Let $\underline{w} := \underline{u}\{s_i s_j\}^{m_{ij}}\underline{v}$. If $\mathcal{C}_{\underline{w}}$ is submodule closed, then*

$$\underline{w} <_l \underline{u}\{s_j s_i\}^{m_{ij}}\underline{v}.$$

(b) The category $\mathcal{C}_{\underline{u}s_i s_i \underline{v}}$ is not submodule closed.

(c) Let $\underline{w}' := \underline{u}\{s_j s_i\}^{m_{ij}+1}\underline{v}$. Then $\mathcal{C}_{\underline{w}'}$ is not submodule closed.

Proof. We prove (a) by contraposition. By Definition 3.1.13, $\mathsf{ind}\,\mathcal{I} \setminus \mathcal{C}_{\underline{w}}$ consists of the modules which correspond to the reflections in $\underline{w}$.

Assume that

$$\underline{u}\{s_j s_i\}^{m_{ij}}\underline{v} <_l \underline{u}\{s_i s_j\}^{m_{ij}}\underline{v} = \underline{w}$$

and

$$\rho(\underline{w}) = \rho(\underline{u})\underbrace{(p,i)(q,j)(p+1,i)\ldots}_{m_{ij}\text{ pairs}}\rho_1$$

for a sequence of pairs ρ_1.

By Lemma 3.1.6, the module $(q-1,j)$ exists and by Definition 3.1.13, $\mathcal{C}_{\underline{w}}$ contains all indecomposable, preinjective modules M with $(q-1,j) < M < (p,i)$ or $(p,i) < M < (q,j)$.

First, suppose that $m_{ij} = 2$. In this case, $\mathcal{C}_{\underline{w}}$ contains the middle term of the AR-sequence that starts in (q,j) by Lemmas 3.2.7 and 3.3.1. Since $(q,j) \notin \mathcal{C}_{\underline{w}}$, the subcategory is not submodule closed.

Now let $m_{ij} \geq 3$. In 3.3.4, we defined $M_0 := (s,i)$, $M_1 := (t,j), \ldots$ for some arbitrary, fixed s, t. By Remark 3.3.7, we can assume without loss of generality that m_{ij} is odd and we can choose s, t so that $M_{m_{ij}-1} = (p,i)$.

Then $M_{m_{ij}} = (q-1,j) \neq 0$ and by Lemma 3.3.3, $E(m_{ij} - 2) \neq 0$. By Lemma 3.3.5, there is an exact sequence

$$0 \longrightarrow M_0 \longrightarrow (M_{m_{ij}-1})^{E(m_{ij}-1)} \oplus U_{m_{ij}-1} \longrightarrow M_{m_{ij}}^{E(m_{ij}-2)} \longrightarrow 0$$

so that no $M_0, M_1, \ldots, M_{m_{ij}-1}$ is a direct summand of $U_{m_{ij}-1}$. By Lemma 3.3.3 we have $E(m_{ij} - 1) = 0$, so there is a monomorphism

$$M_0 \rightarrowtail U_{m_{ij}-1}.$$

It remains to show that $U_{m_{ij}-1} \in \mathcal{C}_{\underline{w}_2}$ by Corollary 3.3.6: If X be a direct summand of $U_{m_{ij}-1}$, then $M_{m_{ij}} < X < M_0$ and thus $X \in \mathcal{C}_{\underline{w}}$.

By Lemma 3.2.7, part (b) is obvious.

The proof of (c) is completely analogous to the proof of (a). □

Finally, we are prepared to prove that Definition 3.1.13 gives a bijection. Recall that $\mathcal{C}_w = \mathcal{C}_{\underline{w}}$ if $\underline{w}$ is the leftmost word for w:

Theorem 3.6.2. *The map $w \mapsto \mathcal{C}_w$ is a bijection between the elements of the Weyl group of A and the cofinite submodule closed subcategories of* mod A.

Proof. The map is well defined by Lemma 3.5.7 and obviously injective. It remains to prove that it is surjective, i.e. that for all cofinite submodule closed subcategories $\mathcal{C}$ of mod A, there is a $w \in W$, so that $\mathcal{C} = \mathcal{C}_w$.

We already know that $\mathcal{C} = \mathcal{C}_{\underline{w}}$ for some word from Lemma 3.4.1, so we only need to show that $\underline{w}$ is leftmost.

Assume that the word $\underline{w}$ for the element $w \in W$ is not leftmost. We show that $\mathcal{C}_{\underline{w}}$ is not submodule closed by induction on the number of Coxeter relations that are needed to transform $\underline{w}$ into a smaller word.

If only one relation is needed, then the theorem is the result of Lemma 3.6.1. Now suppose that the assertion is true if we need m or less relations and that we need $m + 1$ relations to transform $\underline{w}$ into a smaller word.

Then there are some $1 \leq i, j \leq n$ and some words $\underline{x}$, $\underline{x}'$, $\underline{y}$ so that

$$\underline{w} = \underline{x}s_i\underline{y} \equiv \underline{x}'s_j\underline{y} = \underline{w}' \tag{3.40}$$

and

$$\underline{w}' <_l \underline{w}$$

with $i \neq j$.

Thus, there are some words $\underline{w}'', \underline{x}''$ so that

$$\underline{w} \equiv \underline{w}'' = \underline{x}''\{s_i s_j\}^{m_{ij}}\underline{y}.$$

Because of the inductive assumption, we can suppose that $\underline{w} \leq_l \underline{w}''$ and that $\underline{x}$ is leftmost. Obviously, we can choose $\underline{x}''$ to be leftmost.

Let the reflection s_i in (3.40) correspond to M_0. We can assume that $m_{ij} \geq 3$, since there is nothing to show if the middle term of the Auslander-Reiten sequence that starts in M_0 is contained in $\mathcal{C}_{\underline{w}}$. We can also assume that there is some word $\underline{x}''$ so that $\underline{x} = \underline{x}''s_j$: If there are $s_{k_1}, \ldots s_{k_m}$ with $m_{k_1,i} = \cdots = m_{k_m,i}$ and $\underline{x} = \underline{x}''s_j s_{k_1} \ldots s_{k_m}$, then $\underline{x}s_i \equiv \underline{x}''s_j s_i s_{k_1} \ldots s_{k_m}$ and if there is some $U \in \mathcal{C}_{\underline{x}''s_j s_i}$ with a monomorphism $M_0 \rightarrowtail U$, then $U \in \mathcal{C}_{\underline{w}}$.

Without loss of generality, we can assume that m_{ij} is odd; otherwise we relabel i and j and get the same arguments by Remark 3.3.7.

By Lemma 3.1.12, we can suppose that $M_0, M_1, \ldots, M_{m_{ij}-3}, M_{m_{ij}-2} \in \mathcal{C}_{\underline{w}}$. We show that there is some $U \in \mathcal{C}_{\underline{w}}$ with a monomorphism $M_0 \rightarrowtail U$.

Let $(q,j) := M_{m_{ij}-2}$. We use Lemma 3.1.7: Because $m > 1$, there is some $\underline{w}_3 \equiv \underline{w}$ with series of pairs $\rho_1, \ldots, \rho_4$ and a pair (r,h) so that

$$\rho(\underline{w}) = \rho_1(r,h)\rho_2(q,j)\rho_3(s,i)\rho_4 \tag{3.41}$$

and

$$\rho(\underline{w}_3) = \rho_1\rho_2(q,j)\rho_3(s,i)(t+1,j)\rho_4. \tag{3.42}$$

By Lemma 3.4.2, if $m_{ij} \geq 3$, then there is an exact sequence

$$0 \longrightarrow M_0 \longrightarrow M_{m_{ij}-2} \oplus U' \longrightarrow Y \longrightarrow 0$$

so that either $Y \in \mathcal{C}_{\underline{w}}$ or both $Y = (r,h)^{E(m_{ij}-3)}$ and $U' \in \mathcal{C}_{\underline{w}}$ hold.

We want to show that there is some $U'' \in \mathcal{C}_{\underline{w}}$ and a monomorphism

$$M_{m_{ij}-2} = (q,j) \rightarrowtail U'' \oplus Y. \tag{3.43}$$

We prove this inductively: First, note that the word $\underline{x}''s_j$ is not leftmost Lemma 3.1.9 and 3.1.10.

If $\underline{w} = \underline{w}''$, then $(r,h) = M_{m_{ij}-1}$. So by the inductive hypothesis, there is a monomorphism $(q,j) \rightarrowtail U'' \oplus (r,h)^\gamma$.

Since $E(1) = E(m_{ij}-3)$ by table (3.18), we see that $Y = (r,h)^{E(m_{ij}-3)}$ and $\gamma = E(m_{ij}-3)$.

The inductive step is completely analogous to the one in Lemma 3.4.2.

By our assumptions, $\underline{x}$ is leftmost and thus $Y \notin \mathcal{C}_{\underline{w}}$ by Lemma 3.5.7. So $U' \in \mathcal{C}_{\underline{w}}$.

By Lemma 3.2.13, there is a monomorphism $M_0 \rightarrowtail U' \oplus U'' \in \mathcal{C}_{\underline{w}}$ and $\mathcal{C}_{\underline{w}}$ is not submodule closed. $\qquad\square$

3.7   Some consequences

We conclude the chapter with a generalization and a corollary:

As in [14], Section 8 we can extend the notion of leftmost words:

Definition 3.7.1. Define infinite words analogously to words, only as infinite instead of finite sequences. We say that an (infinite) word is *leftmost* if any initial subword of finite length is leftmost.

Analogously to [14], Theorem 8.1, we get the following:

Theorem 3.7.2. *There is a bijection between the (finite and infinite) leftmost words over $S = \{s_1, s_2, \ldots s_n\}$ and the submodule closed subcategories of $\mathcal{I}$, the preinjective component of $\mathsf{mod}\, A$.*

Proof. This is completely analogous to [14], 8.1:

Let $\mathcal{C}$ be a submodule closed subcategory of $\mathcal{I}$. Since $\mathcal{I}$ contains at most a countable number of indecomposable modules, we can set

$$\mathsf{ind}\, \mathcal{I} \setminus \mathcal{C} =: \{(r_1, i_1), (r_2, i_2), \ldots\}$$

and $(r_1, i_1) < (r_2, i_2) < \ldots$. Then the subcategory $\mathcal{C}_m$ with

$$\mathsf{ind}\, \mathcal{I} \setminus \mathcal{C}_m = \{(r_1, i_1), (r_2, i_2), \ldots, (r_m, i_m)\}$$

is submodule closed for all $m \in \mathbb{N}$. By Lemma 3.2.1 and Theorem 3.6.2, the words $\underline{w}_m$ with

$$\rho(\underline{w}_m) = (r_1, i_1)(r_2, i_2) \ldots (r_m, i_m)$$

are leftmost for all $m \in \mathbb{N}$. By Definition 3.7.1, the (infinite) word $\underline{w}$ with

$$\rho(\underline{w}) = (r_1, i_1)(r_2, i_2) \ldots$$

is leftmost and $\mathcal{C} = \mathcal{C}_{\underline{w}}$.

On the other hand assume that the (infinite) word $\underline{w}$ with

$$\rho(\underline{w}) = (r_1, i_1)(r_2, i_2) \ldots$$

is leftmost. Then the words with

$$\rho(\underline{w}_m) = (r_1, i_1)(r_2, i_2) \ldots (r_m, i_m)$$

are leftmost for all $m \in \mathbb{N}$. By 3.6.2, the categories $\mathcal{C}_{\underline{w}_m}$ are submodule closed. Thus $\mathcal{C}_{\underline{w}}$ is also submodule closed: if there was a module $M \notin \mathcal{C}_{\underline{w}}$ and some module $U \in \mathcal{C}_{\underline{w}}$ with a monomorphism $M \rightarrowtail U$, then $U \in \mathcal{C}_{\underline{w}_m}$ for all $m \in \mathbb{N}$ and there is some $m \in \mathbb{N}$ so that $M \notin \mathcal{C}_{\underline{w}_m}$, since M is finitely generated. $\square$

We can draw a further corollary. Let A' be a hereditary and let the module category $\mathsf{mod}\, A'$ be equivalent to the subcategory of $\mathsf{mod}\, A$ with the simple modules S_j, $j \in J$ for some $J \subseteq \{1, 2, \ldots n\}$. Let $\mathcal{I}_{A'}$ be the subcategory of $\mathsf{mod}\, A'$ consisting of all preinjective modules.

Corollary 3.7.3. *There is a bijection between the submodule closed subcategories of $\mathcal{I}_{A'}$ and the submodule closed subcategories of $\mathcal{I}$ which contain all $\tau^r I_i$ with $r \in \mathbb{N}_0$ and $i \in \mathbb{N} \setminus J$.*

Proof. The words in the Weyl group of A' are exactly the words in the Weyl group of A which only consist of reflections s_j with $j \in J$. $\square$

4 | Distributive lattices of subobject closed subcategories

In this chapter we consider abelian length categories, a generalization of module categories over Artin algebras. For a category $\mathcal{A}$ we are interested in the lattice $\mathsf{S}(\mathcal{A})$ of full additive subobject closed subcategories of $\mathcal{A}$. In particular, we are trying to find cases in which the lattice $\mathsf{S}(\mathcal{A})$ is distributive.

In Section 4.1, we show that the distributivity of $\mathsf{S}(\mathcal{A})$ is equivalent to a condition on the submodule relations in $\mathcal{A}$, which is much easier to work with. We can show in the next section that the following is an even stronger property: every subobject of an indecomposable object in $\mathcal{A}$ is itself indecomposable. Such categories are said to be of colocal type.

We characterize these categories in Section 4.3 to 4.6: First, we show that two conditions on the Ext-quiver hold. Weaker conditions hold if $\mathsf{S}(\mathcal{A})$ is distributive. In Section 4.4, we collect some auxiliary lemmas that are mainly concerned with 2-extensions.

These results are needed in Section 4.5, in which we give different formulations and a proof of the third condition that abelian length categories of colocal type fulfil. Again, we see that a weaker condition is fulfilled if $\mathsf{S}(\mathcal{A})$ is distributive.

In Section 4.6, we use the results of Section 4.4 to prove that every abelian length category which fulfils the three conditions is of colocal type.

Returning to the lattice $\mathsf{S}(\mathcal{A})$, we prove in the next section that it is the Cartesian product of certain sublattices.

Finally, in the last section, we assume that $\mathcal{A} \equiv \mathsf{mod}\, kQ/I$ for some field k, quiver Q with an admissible ideal I. In this case, A is of colocal type if and only if A is a string algebra and no vertex in Q is starting point of more than one arrow. For these algebras, we get a complete, explicit description of the lattice $\mathsf{S}(\mathsf{mod}\, A)$.

Note that in this chapter, we are equating objects with isomorphism classes of objects. In particular, all sums over simple objects are actually sums over isomorphism classes of simple objects.

4.1   Conditions on indecomposable objects

Let $\mathcal{A}$ be an abelian length category. The main result of this section is a characterisation of abelian length categories with distributive lattices $\mathsf{S}(\mathcal{A})$ in terms of the subobject relations between the objects of $\mathcal{A}$.

   We start with the definition of a distributive lattice, as given for example in [17], p. 69:

Definition 4.1.1. A lattice L is called *distributive* if

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$$

for all $a, b, c \in L$.

   Now let $\mathsf{S}(\mathcal{A})$ be the set of full additive subobject closed subcategories as in [12]. It is partially ordered by inclusion and a complete lattice.

   The join $a \vee b$ for two categories $a, b \in \mathsf{S}(\mathcal{A})$ is the smallest full additive subobject closed subcategory, which contains both a and b. The meet $a \wedge b$ is of course the largest category in $\mathsf{S}(\mathcal{A})$ that is contained in both a and b.

   The meet coincides with the intersection $a \cap b$: all subobjects of direct sums of objects in $a \cap b$ are again objects in $a \cap b$, since a and b are subobject closed. The join consists of all subobjects of direct sums of objects in a and b.

   Every category in $\mathsf{S}(\mathcal{A})$ is completely determined by the isomorphism classes of indecomposable objects it contains.

   For a class $\mathcal{X}$ of objects let $\mathsf{sub}\,\mathcal{X}$ be the category that consists of all subobjects of direct sums of objects in $\mathcal{X}$. This is the smallest category in $\mathsf{S}(\mathcal{A})$ that contains $\mathcal{X}$. Set $\mathsf{sub}\,X := \mathsf{sub}\{X\}$.

   In the following case, $\mathsf{S}(\mathcal{A})$ is not distributive:

Lemma 4.1.2. *If there exists an indecomposable object $X \in \mathcal{A}$, and objects $Y_1, Y_2 \in \mathcal{A}$ so that $X \in \mathsf{sub}\,Y_1 \vee \mathsf{sub}\,Y_2$ but $X \notin \mathsf{sub}\,Y_i$ for all $1 \leq i \leq 2$, then*

$$(\mathsf{sub}\,Y_1 \vee \mathsf{sub}\,Y_2) \wedge \mathsf{sub}\,X \neq (\mathsf{sub}\,Y_1 \wedge \mathsf{sub}\,X) \vee (\mathsf{sub}\,Y_2 \wedge \mathsf{sub}\,X).$$

Proof. By the assumption

$$X \in \mathsf{sub}\,Y_1 \vee \mathsf{sub}\,Y_2$$

and by definition

$$X \in \mathsf{sub}\,X,$$

so
$$X \in (\mathsf{sub}\, Y_1 \vee \mathsf{sub}\, Y_2) \wedge \mathsf{sub}\, X.$$

But
$$X \notin (\mathsf{sub}\, Y_1 \wedge \mathsf{sub}\, X) \vee (\mathsf{sub}\, Y_2 \wedge \mathsf{sub}\, X),$$

since otherwise there were some objects

$$X_i \in \mathsf{sub}\, Y_i \wedge \mathsf{sub}\, X$$

for $1 \le i \le 2$ with a monomorphism

$$f : X \rightarrowtail X_1 \oplus X_2.$$

But since $X_1 \oplus X_2 \in \mathsf{sub}\, X$, there is some $\alpha \in \mathbb{N}$ and a monomorphism g

$$g : X_1 \oplus X_2 \rightarrowtail X^\alpha.$$

Since gf is a monomorphism, its image is isomorphic to X and g induces a morphism $g' : X_1 \oplus X_2 \to X$ so that $g'f$ is an isomorphism on X. Thus, f splits.

Because X is indecomposable, there is an $i \in I$, so that X is a direct summand of X_i. Since $X_i \in \mathsf{sub}\, Y_i$, this means that $X \in \mathsf{sub}\, Y_i$, contrary to the assumption.

So

$$(\mathsf{sub}\, Y_1 \vee \mathsf{sub}\, Y_2) \wedge \mathsf{sub}\, X \ne (\mathsf{sub}\, Y_1 \wedge \mathsf{sub}\, X) \vee (\mathsf{sub}\, Y_2 \wedge \mathsf{sub}\, X)$$

and the proof is complete. $\square$

In fact, we get the following equivalence:

Proposition 4.1.3. *The following statements are equivalent:*

1. *The lattice $\mathsf{S}(\mathcal{A})$ is distributive*

2. *If $X \in \mathcal{A}$ is indecomposable and there are objects $Y_1, Y_2 \in \mathcal{A}$, so that $X \in \mathsf{sub}\, Y_1 \vee \mathsf{sub}\, Y_2$ then $X \in \mathsf{sub}\, Y_i$ for some $1 \le i \le 2$.*

3. *For all index sets I and categories $a_i \in \mathsf{S}(\mathcal{A})$, $i \in I$ we have*

$$\mathsf{ind}(\bigvee_{i \in I} a_i) = \bigcup_{i \in I} \mathsf{ind}\, a_i.$$

Proof. $(1) \Rightarrow (2)$ is clear from the Lemma 4.1.2.

$(2) \Rightarrow (3)$: The direction

$$\mathsf{ind}(\bigvee_{i \in I} a_i) \supseteq \bigcup_{i \in I} \mathsf{ind}\, a_i$$

is clear. For the other direction, we look at an indecomposable object

$$X \in \bigvee_{i \in I} a_i.$$

There are objects $A_i \in a_i$ with a monomorphism

$$X \rightarrowtail \bigoplus_{i \in I} A_i$$

and thus

$$X \in \bigvee_{i \in I} \mathsf{sub}\, A_i.$$

The object $\bigoplus_{i \in I} A_i$ must be of finite length; thus $A_i = 0$ for all except finitely many $i \in I$. By (2) and an induction, we get

$$X \in \mathsf{sub}\, A_i$$

for at least one $i \in I$ and thus

$$X \in a_i.$$

So

$$X \in \bigcup_{i \in I} \mathsf{ind}\, a_i$$

and

$$\mathsf{ind}(\bigvee_{i \in I} a_i) = \bigcup_{i \in I} \mathsf{ind}\, a_i.$$

$(3) \Rightarrow (1)$: Let $a, b, c \in \mathsf{S}(\mathcal{A})$. Then

$$\begin{aligned}
\mathsf{ind}((a \vee b) \wedge c) &= (\mathsf{ind}\, a \cup \mathsf{ind}\, b) \cap \mathsf{ind}\, c \\
&= (\mathsf{ind}\, a \cap \mathsf{ind}\, c) \cup (\mathsf{ind}\, a \cap \mathsf{ind}\, c) \\
&= \mathsf{ind}((a \wedge c) \vee (a \wedge c)).
\end{aligned}$$

Since a, b, c are completely determined by their indecomposable objects,

$$(a \vee b) \wedge c = (a \wedge b) \vee (a \wedge c)$$

and $\mathsf{S}(\mathcal{A})$ is distributive. $\qquad\square$

We can generalize the notion of a distributive lattice as e.g. in [9], p. 1227:

Definition 4.1.4. A complete lattice Λ is a *frame* if for all index sets I and elements a, b_i with $i \in I$ the equation

$$a \wedge \left(\bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \wedge b_i)$$

holds.

Obviously, every frame is also distributive. But in general, not every distributive lattice is a frame. An exception are lattices of subobject closed categories:

Corollary 4.1.5. *The lattice* $\mathsf{S}(\mathcal{A})$ *is distributive if and only if it is a frame.*

Proof. This follows from part (3) of Proposition 4.1.3. $\square$

4.2 Categories of colocal type

For the following categories, $\mathsf{S}(\mathcal{A})$ is always distributive:

Definition 4.2.1. We call a category $\mathcal{A}$ *of colocal type* if any subobject of an indecomposable object is itself indecomposable. If there is some Artin algebra A so that $\mathcal{A} = \mathsf{mod}\, A$, then we also say that A is of colocal type.

To show this, we need the following lemma:

Lemma 4.2.2. *(a) If there are objects V_1, V_2, X with a monomorphism*

$$f = \begin{bmatrix} f_1 & f_2 \end{bmatrix} : V_1 \oplus V_2 \rightarrowtail X,$$

then there is also a monomorphism

$$X \rightarrowtail \mathsf{Coker}\, f_1 \oplus \mathsf{Coker}\, f_2.$$

(b) If there are objects X, Y_1, Y_2 with a monomorphism

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : X \rightarrowtail Y_1 \oplus Y_2,$$

then there is also a monomorphism

$$\mathsf{Ker}\, f_1 \oplus \mathsf{Ker}\, f_2 \rightarrowtail X.$$

Proof. First, we prove that (a) holds: There is an exact sequence

$$0 \longrightarrow V_1 \oplus V_2 \xrightarrow{\left[\begin{smallmatrix} f_1 & 0 \\ 0 & f_2 \end{smallmatrix}\right]} X \oplus X \xrightarrow{g} \mathsf{Coker}\, f_1 \oplus \mathsf{Coker}\, f_2 \longrightarrow 0$$

for some morphism g with $\mathsf{Ker}\, g \cong V_1 \oplus V_2$. So g induces a monomorphism

$$X \rightarrowtail \mathsf{Coker}\, f_1 \oplus \mathsf{Coker}\, f_2.$$

The proof of (b) is similar: We get a morphism f' with an exact diagram

So

$$0 \longrightarrow \mathsf{Ker}\, f_1 \oplus \mathsf{Ker}\, f_2 \longrightarrow X \longrightarrow \mathsf{Coker}\, f \longrightarrow 0$$

is an exact sequence and there is a monomorphism $\mathsf{Ker}\, f_1 \oplus \mathsf{Ker}\, f_2 \rightarrowtail X$. $\quad\square$

Proposition 4.2.3. *If $\mathcal{A}$ is of colocal type, then $\mathsf{S}(\mathcal{A})$ is distributive. Furthermore, for all objects $X, Y_1, Y_1 \in \mathcal{A}$ with a monomorphism $X \rightarrowtail Y_1 \oplus Y_2$, there is a monomorphism $X \rightarrowtail Y_1$ or $X \rightarrowtail Y_2$.*

Proof. Assume that every subobject of an indecomposable object is indecomposable. Further suppose that there are objects X, Y_1, Y_2, so that X is a subobject of $Y_1 \oplus Y_2$. Then there is a monomorphism

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : X \rightarrowtail Y_1 \oplus Y_2$$

and by Lemma 4.2.2, we get a monomorphism $\mathsf{Ker}\, f_1 \oplus \mathsf{Ker}\, f_2 \rightarrowtail X$.

Since $\mathcal{A}$ is of colocal type, either $\mathsf{Ker}\, f_1 = 0$ or $\mathsf{Ker}\, f_2 = 0$. Thus there is a monomorphism $X \rightarrowtail Y_1$ or $X \rightarrowtail Y_2$. By Proposition 4.1.3, the lattice $\mathsf{S}(\mathcal{A})$ is distributive. $\quad\square$

Lemma 4.2.4. *If $\mathcal{A}$ is not of colocal type, then there are objects V_1, V_2, non-simple objects Y_1, Y_2, an indecomposable object X and a simple object S with exact sequences*

$$0 \longrightarrow V_1 \oplus V_2 \longrightarrow X \longrightarrow S \longrightarrow 0 , \qquad (4.1)$$

$$0 \longrightarrow X \longrightarrow Y_1 \oplus Y_2 \longrightarrow S \longrightarrow 0 . \qquad (4.2)$$

For such objects, the following sequences are exact

$$0 \longrightarrow V_j \longrightarrow Y_i \longrightarrow S \longrightarrow 0 \qquad (4.3)$$

for $i, j = 1, 2$ and $i \neq j$.

Proof. Suppose that X is an indecomposable object and there is a monomorphism

$$f = \begin{bmatrix} f_1 & f_2 \end{bmatrix} : V_1 \oplus V_2 \rightarrowtail X$$

with $V_1 \neq 0 \neq V_2$.

Let S be a simple factor module of $\mathsf{Coker}\, f$. Then there is some V with $V_1 \oplus V_2 \subseteq V$ and an exact sequence

$$0 \longrightarrow V \longrightarrow X \longrightarrow S \longrightarrow 0 .$$

If V is indecomposable, then it is of smaller length than X and we can regard V instead of X. So we can assume that $\mathsf{Coker}\, f = S$ and $V_1 \oplus V_2 = V$. By Lemma 4.2.2, there is a monomorphism

$$g : X \rightarrowtail \mathsf{Coker}\, f_1 \oplus \mathsf{Coker}\, f_2.$$

The following diagram is exact for all $i, j \in \{1, 2\}$, $i \neq j$, since all columns and the first and second row are exact:

By this, we know that the following diagram is exact, since all columns and the first and second row are exact:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_1 \oplus V_2 & \longrightarrow & X & \longrightarrow & S & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g} & & \downarrow & & \\
0 & \longrightarrow & V_1 \oplus V_2 & \longrightarrow & \mathsf{Coker}\, f_1 \oplus \mathsf{Coker}\, f_2 & \longrightarrow & S^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathsf{Coker}\, g & \longrightarrow & S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

So $\mathsf{Coker}\, g = S$. With $Y_1 = \mathsf{Coker}\, f_1$ and $Y_2 = \mathsf{Coker}\, f_2$, we get the exact sequences (4.1) - (4.3). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

4.3   Conditions on the Ext-quiver

Using pullbacks and pushouts, we show in this section that every abelian length category $\mathcal{A}$ of colocal type has to fulfil the conditions (C1) and (C2) in Theorem 1.2.4. Weaker conditions hold if $\mathsf{S}(\mathcal{A})$ is distributive.

Recall that we defined for simple objects $S, T \in \mathcal{A}$

$$
d_S^1(S, T) := \dim_{\mathsf{End}(S)^{op}} \mathsf{Ext}^1(S, T)
$$

and

$$
d_T^1(S, T) := \dim_{\mathsf{End}(T)} \mathsf{Ext}^1(S, T).
$$

We begin with an auxiliary lemma:

Lemma 4.3.1. *Let $\mathcal{A}$ be an abelian length category and S, T_1, T_2 simple objects. Then the following holds:*

(a) Let there be indecomposable objects X_1, X_2 with exact sequences

$$
0 \longrightarrow T_1 \xrightarrow{\ f_1\ } X_1 \xrightarrow{\ g_1\ } S \longrightarrow 0
$$

and

$$
0 \longrightarrow T_2 \xrightarrow{\ f_2\ } X_2 \xrightarrow{\ g_2\ } S \longrightarrow 0.
$$

If $T_1 \cong T_2$, *we suppose that these exact sequences are linearly independent over* $\mathsf{End}(T_1)$.

Then the object Y *in the following pullback is indecomposable:*

$$
\begin{array}{ccc}
Y & \longrightarrow & X_1 \\
\downarrow & & \downarrow {\scriptstyle g_1} \\
X_2 & \xrightarrow[g_2]{} & S
\end{array}
\qquad .
\tag{4.4}
$$

Furthermore, there is an exact sequence

$$
0 \longrightarrow T_1 \oplus T_2 \longrightarrow Y \longrightarrow S \longrightarrow 0 \ .
$$

(b) *Let there be indecomposable objects* X_1, X_2 *with exact sequences*

$$
0 \longrightarrow S \xrightarrow{f_1} X_1 \xrightarrow{g_1} T_1 \longrightarrow 0
$$

and

$$
0 \longrightarrow S \xrightarrow{f_2} X_2 \xrightarrow{g_2} T_2 \longrightarrow 0.
$$

If $T_1 \cong T_2$, *we suppose that these exact sequences are linearly independent over* $\mathsf{End}(T_1)^{op}$.

Then the object Y *in the following pushout is indecomposable:*

$$
\begin{array}{ccc}
S & \xrightarrow{f_1} & X_1 \\
{\scriptstyle f_2} \downarrow & & \downarrow \\
X_2 & \longrightarrow & Y
\end{array}
\qquad .
$$

Furthermore, there is an exact sequence

$$
0 \longrightarrow S \longrightarrow Y \longrightarrow T_1 \oplus T_2 \longrightarrow 0 \ .
$$

Proof. We only prove (a), since (b) is completely analogous. In this case, the

following diagram is exact by 2.1.4:

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
T_1 \oplus T_2 & & Y & & S \\
\| & & \downarrow & & \downarrow \\
0 \longrightarrow T_1 \oplus T_2 \longrightarrow X_1 \oplus X_2 \longrightarrow S^2 \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow 0 \longrightarrow S ===== S \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

and we get some morphisms $f_1' : T_1 \to Y$ and $f_2' : T_2 \to Y$ so that

$$
0 \longrightarrow T_1 \oplus T_2 \xrightarrow{[f_1'\ f_2']} Y \longrightarrow S \longrightarrow 0
$$

is an an exact sequence.

Suppose that Y was decomposable. Since X_1 and X_2 are indecomposable, S cannot be a direct summand of Y.

So we can assume that $Y = T_1 \oplus X$ for some object X of length 2. Since (4.4) is commutative, this yields a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f_1'} & X_1 \\
\downarrow{\scriptstyle f_2'} & & \downarrow{\scriptstyle g_1} \\
X_2 & \xrightarrow{g_2} & S
\end{array}
$$

with $\mathsf{Im}(g_1 f_1') = S = \mathsf{Im}(g_2 f_2')$. Furthermore, $\mathsf{Im}(f_1') \ncong S$, since otherwise $\mathsf{Im}(f_1') \rightarrowtail X_1 \twoheadrightarrow S$ splits. Taking the length of X_1 into consideration, it is obvious that $\mathsf{Im}(f_1') = X_1$ is the only remaining possibility and f_1' is an isomorphism. Analogously, $\mathsf{Im}(f_2') = X_2$.

Thus, X is isomorphic to both X_1 and X_2, $T_1 \cong T_2$ and for $1 \le i \le 2$, there are commutative diagrams

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T_1 & \longrightarrow & X & \xrightarrow{g_i f_i'} & S & \longrightarrow & 0 \ . \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle f_i'} & & \| & & \\
0 & \longrightarrow & T_i & \longrightarrow & X_i & \xrightarrow{g_i} & S & \longrightarrow & 0
\end{array}
$$

Since $g_1 f_1' = g_2 f_2'$, this is a contradiction to the assumption that η_1 and η_2 are linearly independent over $\mathsf{End}(T_1)$.

So Y must be indecomposable. $\qquad\square$

Now we can prove that condition (C1) holds if $\mathcal{A}$ is of colocal type:

Lemma 4.3.2. *Let $\mathcal{A}$ be an abelian length category. The following is true for all simple objects $S \in \mathcal{A}$:*

(a) If $\mathcal{A}$ is of colocal type, then

$$\sum_{T \ simple} d_T^1(S,T) \leq 1.$$

(b) If $\mathsf{S}(\mathcal{A})$ is distributive, then there is at most one T with $d_T^1(S,T) \neq 0$.

Proof. We begin with the proof of (a). If

$$\sum_{T \ simple} d_T^1(S,T) \geq 2,$$

then there are simple objects T_1, T_2 and indecomposable objects X_1, X_2 with exact sequences

$$\eta_1 : \ 0 \longrightarrow T_1 \xrightarrow{\ f_1\ } X_1 \xrightarrow{\ g_1\ } S \longrightarrow 0$$

and

$$\eta_2 : \ 0 \longrightarrow T_2 \xrightarrow{\ f_2\ } X_2 \xrightarrow{\ g_2\ } S \longrightarrow 0$$

which are linearly independent over $\mathsf{End}(T_1)$ if $T_1 \cong T_2$.

By Lemma 4.3.1, the object Y in the following pullback is indecomposable:

$$
\begin{array}{ccc}
Y & \longrightarrow & X_1 \\
\downarrow & & \downarrow{\scriptstyle g_1} \\
X_2 & \xrightarrow{\ g_2\ } & S
\end{array}
$$

and has $T_1 \oplus T_2$ as a subobject.

So $\mathcal{A}$ is not of colocal type.

To prove (b), we suppose that $T_1 \not\cong T_2$. Then $X_1 \not\cong X_2$ and for every epimorphism $h_i : Y \twoheadrightarrow X_i$, we have $\mathsf{Ker}\, h_i \cong T_j$ with $1 \leq i \neq j \leq 2$. If $h_i' : Y \to X_i$ is not an epimorphism, then $\mathsf{Im}(h_i') \cong T_i$ and thus $T_j \mid \mathrm{soc}(\ker h_i')$. But T_j^2 is not a subobject of Y and by Lemma 4.2.2, there is no monomorphism $Y \rightarrowtail X_i^n$ for any $1 \leq i \leq 2$ and $n \in \mathbb{N}$. So $\mathsf{S}(\mathcal{A})$ is not distributive by Proposition 4.1.3. $\qquad\square$

To show that (C2) holds if $\mathcal{A}$ is of colocal type, we use the following auxiliary lemma:

Lemma 4.3.3. *Let $\mathcal{A}$ be an abelian category and $S, T_1, \ldots, T_n$ simple objects in $\mathcal{A}$ so that there are exact sequences with indecomposable middle terms*

$$\eta_i : \; 0 \longrightarrow S \xrightarrow{\;f_i\;} X_i \longrightarrow T_i \longrightarrow 0$$

for $1 \leq i \leq n$. Furthermore, suppose that $2 \leq n$. Let $f := \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix}$ and suppose that there is a isomorphism $g : \bigoplus_{i=1}^{n-1} X_i \to \bigoplus_{i=1}^{n-1} X_i$ and a monomorphism $g_n : X_n \rightarrowtail \bigoplus_{i=1}^{n-1} X_i$ so that the following diagram is commutative

$$\begin{array}{ccc}
S & \xrightarrow{\;\;f\;\;} & \bigoplus_{i=1}^{n-1} X_i \\
{\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle g} \\
X_n & \xrightarrow{\;g_n\;} & \bigoplus_{i=1}^{n-1} X_i
\end{array} \tag{4.5}$$

or an isomorphism $g_n' : X_n \to X_n$ and an epimorphism $g' : \bigoplus_{i=1}^{n-1} X_i \twoheadrightarrow X_n$ so that the following diagram is commutative

$$\begin{array}{ccc}
S & \xrightarrow{\;\;f\;\;} & \bigoplus_{i=1}^{n-1} X_i \\
{\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle g'} \\
X_n & \xrightarrow{\;g_n'\;} & X_n
\end{array} \;. \tag{4.6}$$

Then there are $1 \leq i_1, \ldots, i_m \leq n-1$ so that $T_{i_1} \cong T_n, \ldots, T_{i_m} \cong T_n$ and η_n is linearly dependent of $\eta_{i_1}, \ldots, \eta_{i_m}$ over T_n.

Proof. We only prove this for (4.5), since the proof for (4.6) is analogous.

Since S^2 is not a subobject of X_n, the image of the concatenation

$$\phi : S^{n-1} \rightarrowtail \bigoplus_{i=1}^{n-1} X_i \twoheadrightarrow \mathsf{Coker}\, g^{-1} g_n$$

is S^{n-2}. Set

$$F := \begin{bmatrix} f_1 & 0 & 0 & \ldots & 0 \\ 0 & f_2 & 0 & \ldots & 0 \\ & & \ddots & & \\ 0 & 0 & \ldots & 0 & f_{n-1} \end{bmatrix} \;.$$

Then the following diagram is exact, because all rows and the first and second column are exact:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S & \xrightarrow{\,f_n\,} & X_n & \longrightarrow & T_n & \longrightarrow & 0 \\
& & {\scriptstyle [\mathrm{id}\ \ldots\ \mathrm{id}]}\downarrow & & {\scriptstyle g^{-1}g_n}\downarrow & & \downarrow & & \\
0 & \longrightarrow & S^{n-1} & \xrightarrow{\,F\,} & \bigoplus_{i=1}^{n-1} X_i & \longrightarrow & \bigoplus_{i=1}^{n-1} T_i & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^{n-2} & \xrightarrow{\,\phi\,} & \mathsf{Coker}\, g^{-1}g_n & \longrightarrow & \mathsf{Coker}\,\phi & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Thus, there are $1 \leq i_1, \ldots, i_m \leq n-1$ so that $T_{i_1} \cong T_n, \ldots, T_{i_m} \cong T_n$ and η_n is linearly dependent of $\eta_{i_1}, \ldots, \eta_{i_m}$ over T_n. $\qquad\square$

Now we prove in particular that (C2) holds if $\mathcal{A}$ is of colocal type:

Lemma 4.3.4. *Let $\mathcal{A}$ be an abelian length category. The following holds for all simple objects $S \in \mathcal{A}$:*

(a) If $\mathcal{A}$ is of colocal type, then

$$
\sum_{T \ simple} d_T^1(T, S) \leq 2. \tag{4.7}
$$

(b) If $\mathsf{S}(\mathcal{A})$ is distributive, then there are at most two non-isomorphic simple modules T with $d_T^1(T, S) \neq 0$.

Proof. We start with the proof of (a). If

$$
\sum_{T \ simple} d_T^1(T, S) \geq 3,
$$

then we have three exact sequences

$$
\eta_i : \ 0 \longrightarrow S \xrightarrow{\,f_i\,} X_i \longrightarrow T_i \longrightarrow 0
$$

for some indecomposable objects $X_i \in \mathcal{A}$ and simple T_i with $i \in 1, 2, 3$. If $T_i \cong T_j$ for some $i \neq j$, then η_i and η_j are linearly independent over $\mathsf{End}(T_i)^{op}$. If $T_1 \cong T_2 \cong T_3$, then over $\mathsf{End}(T_i)^{op}$ none of the exact sequences is a linear combination of the other two.

There is a pushout

$$
\begin{array}{ccc}
S & \xrightarrow{\left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right]} & X_1 \oplus X_2 \\
{\scriptstyle f_3}\downarrow & & \downarrow{\scriptstyle [g_1\ g_2]} \\
X_3 & \xrightarrow{g_3} & Y
\end{array} \quad .
$$

We show that Y is indecomposable: By 2.1.3, $\begin{bmatrix} g_1 & g_2 \end{bmatrix}$ and g_3 are monomorphisms.

Furthermore, $l(X_i) = 2$, $l(X_1 \oplus X_2) = 4$ and by 2.1.4, $l(Y) = 5$.

Suppose that $Y = Y_1 \oplus Y_2$ for some indecomposable Y_1 and write

$$
g_i = \begin{bmatrix} g_{1i} \\ g_{2i} \end{bmatrix}
$$

with $g_{ji} : X_i \to Y_j$ for all $1 \leq i \leq 3$ and $1 \leq j \leq 2$.

Without loss of generality, we can assume that $g_{13}f_3 \neq 0$ and $g_{11}f_1 \neq 0$. Then g_{11} and g_{13} are both monomorphisms; otherwise their image was isomorphic to S and they would split. By Lemma 4.3.3, none of them is an isomorphism and thus $l(Y_1) \geq 3$. If $g_{23}f_3 \neq 0$, we analogously get $l(Y_2) \geq 3$, a contradiction to $l(Y) = 5$.

So we can assume that $g_{23}f_3 = 0$. If $g_{22}f_2 \neq 0$, then $g_{21}f_1 + g_{22}f_2 = 0$. By Lemma 4.3.3, we get again $l(Y_2) \geq 3$ and thus $l(Y_1 \oplus Y_2) = 6$, a contradiction to $l(Y) = 5$.

So g_{11}, g_{12}, g_{13} are all monomorphisms and $g_{23}f_3 = g_{22}f_2 = g_{12}f_1 = 0$. Since $\begin{bmatrix} g_1 & g_2 \end{bmatrix}$ is a monomorphism by 2.1.3, we get

$$
S^2 \rightarrowtail \mathsf{Im} \begin{bmatrix} g_{11} & g_{12} \end{bmatrix} \rightarrowtail Y_1. \tag{4.8}
$$

Thus $\mathsf{Ker} \begin{bmatrix} g_{11} & g_{12} \end{bmatrix} = 0$ and this morphism is a monomorphism. We get $4 = l(X_1 \oplus X_2) < Y_1$ and $Y_1 \cong Y$.

By (4.8), $\mathcal{A}$ is not of colocal type.

To prove (b), we can assume that the objects T_1, T_2 and T_3 are pairwise non-isomorphic.

Since $\begin{bmatrix} g_1 & g_2 \end{bmatrix} : X_1 \oplus X_2 \rightarrowtail Y$ is a monomorphism, there is also a monomorphism $Y \rightarrowtail \mathsf{Coker}\, g_1 \oplus \mathsf{Coker}\, g_2$ by Lemma 4.2.2.

For $1 \leq i \neq j \neq k \leq 3$, the following diagrams are exact, because all rows and the first and second column are exact:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X_k \oplus X_j & = & X_k \oplus X_j & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
0 \longrightarrow S & \longrightarrow & X_1 \oplus X_2 \oplus X_3 & \longrightarrow & Y & \longrightarrow & 0 & & \\
\| & & \downarrow & & \downarrow & & & & \\
0 \longrightarrow S & \longrightarrow & X_i & \longrightarrow & T_i & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

and

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow S & \longrightarrow & X_k \oplus X_j & \longrightarrow & \mathsf{Coker}\, g_i & \longrightarrow 0 \\
\downarrow & & \downarrow & & \| & & \\
0 \longrightarrow X_i & \xrightarrow{g_i} & Y & \longrightarrow & \mathsf{Coker}\, g_i & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow T_i & = & T_i & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}
$$

By Lemma 4.3.1, $\mathsf{Coker}\, g_i$ is indecomposable and there is an exact sequence

$$0 \longrightarrow S \longrightarrow \mathsf{Coker}\, g_i \longrightarrow T_j \oplus T_k \longrightarrow 0 \, .$$

In particular, the socle of $\mathsf{Coker}\, g_i$ is S and T_i is not a subobject of any indecomposable object $Z \not\cong \mathsf{Coker}\, g_i$ with an epimorphism $(\mathsf{Coker}\, g_i)^n \twoheadrightarrow Z$.

If there was any $n \in \mathbb{N}$ with a monomorphism $\phi : Y \rightarrowtail (\mathsf{Coker}\, g_i)^n$, then

there was some Z' so that the following diagram was exact:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^2 & \longrightarrow & Y & \longrightarrow & T_1 \oplus T_2 \oplus T_3 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \phi} & & \downarrow & & \\
0 & \longrightarrow & S^2 & \longrightarrow & (\mathsf{Coker}\, g_i)^n & \longrightarrow & Z' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{Coker}\, \phi & = & \mathsf{Coker}\, \phi & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

But for the smallest n with such a monomorphism, the component $Y \to$ $\mathsf{Coker}\, g_i$ of ϕ is non-zero for every copy of $\mathsf{Coker}\, g_i$, so no direct summand of Z' is isomorphic to $\mathsf{Coker}\, g_i$, a contradiction.

By Proposition 4.1.3, $\mathsf{S}(\mathcal{A})$ is not distributive. $\qquad\square$

The following lemma is especially important if $\mathcal{A} = \mathsf{mod}\, A$ for some Artin algebra A over an algebraically closed field k, since in this case, $\mathsf{End}(S) \cong \mathsf{End}(T)$ for all simple modules $S, T \in \mathsf{mod}\, A$, since $\mathsf{mod}\, A$ is equivalent to the module category of a quiver algebra, see 2.5.3.

Lemma 4.3.5. *Let* $\mathsf{S}(\mathcal{A})$ *be distributive. Then for all simple modules* S, T *with* $\mathsf{End}(S) \cong \mathsf{End}(T)$*, we have*

$$d_S^1(S, T) = 1 = d_T^1(S, T). \tag{4.9}$$

Proof. Suppose that 4.9 is not fulfilled. Then

$$2 \leq d_S^1(S, T) = d_T^1(S, T) = \dim_{\mathsf{End}(S)^{op}} \mathsf{Ext}^1(S, T) = \dim_{\mathsf{End}(T)} \mathsf{Ext}^1(S, T).$$

There are $\eta_1', \eta_2' \in \mathsf{Ext}^1(S, T)$ which are linearly independent over $\mathsf{End}(S)^{op}$ and $\eta_3', \eta_4' \in \mathsf{Ext}^1(S, T)$ which are linearly independent over $\mathsf{End}(T)$. We can choose η_1, η_2 out of these so that they are linearly independent over both $\mathsf{End}(S)^{op}$ and $\mathsf{End}(T)$: If η_1', η_2' are linearly dependent over $\mathsf{End}(T)$, then at least one of η_3', η_4' is linearly independent of both of them over $\mathsf{End}(T)$. We can assume this to be η_3'. Over $\mathsf{End}(S)^{op}$, η_3' is linearly independent of at least one of η_1' and η_2'.

There are indecomposable objects X_1, X_2 so that we can write η_1 and η_2 as

$$\eta_i : \ 0 \longrightarrow T \xrightarrow{\ f_i\ } X_i \xrightarrow{\ g_i\ } S \longrightarrow 0$$

with $1 \leq i \leq 2$.

First, assume that $X_1 \cong X_2$.

If $g_1 f_2 \neq 0$, then $\operatorname{Im} g_1 f_2 \cong S \cong T$, since S and T are simple. Because $\operatorname{End}(S)$ is a division ring, the identity factors through $g_1 f_2$. So these morphisms split, contrary to the assumption that $X_1 \cong X_2$ is indecomposable. Analogously, we get $g_2 f_1 = 0$.

So $\operatorname{Im} f_1 = \operatorname{Im} f_2 = \operatorname{Ker} g_1 = \operatorname{Ker} g_2$. Thus there are isomorphisms $\phi : T \to T$ and $\psi : S \to S$ so that $f_2 = f_1 \phi$ and $g_2 = \psi g_1$.

But $\operatorname{End}(S) \cong \operatorname{End}(T)$, so we can also regard ψ as an isomorphism over T and ϕ as an isomorphism over S. Over $\operatorname{End}(S)^{op}$, we get $\eta_2 = \eta_1 \phi \psi^{-1}$ and over $\operatorname{End}(T)$, we have $\eta_2 = \phi^{-1} \psi \eta_1$, contrary to the assumption that η_1 and η_2 are linearly independent over these division rings.

So $X_1 \not\cong X_2$. By Lemma 4.3.1, there is a pullback

$$
\begin{array}{ccc}
Y & \xrightarrow{\ f\ } & X_1 \\
\downarrow{\scriptstyle g} & & \downarrow \\
X_2 & \longrightarrow & S
\end{array}
$$

so that Y is indecomposable. Furthermore,

$$T \oplus T \cong \operatorname{Ker} f \oplus \operatorname{Ker} g \subset Y.$$

Suppose that for $1 \leq i \leq 2$, there is some monomorphism $h_i : T \rightarrowtail Y$ so that $X_i \cong \operatorname{Coker} h_i$. Then

$$\begin{bmatrix} h_1 & h_2 \end{bmatrix} : T^2 \rightarrowtail Y$$

is a monomorphism.

Since there is only one way to embed $T \oplus T$ into Y, h_1 and h_2 form a basis of $\operatorname{Hom}(T, Y)$ over $\operatorname{End}(T)$.

If there are objects X_1', X_2' which are middle terms of exact sequences η_1', η_2' in $\operatorname{Ext}^1(S, T)$ and a monomorphism

$$Y \rightarrowtail X_1' \oplus X_2',$$

then there is a monomorphism

$$\begin{bmatrix} h_1' & h_2' \end{bmatrix} : T^2 \rightarrowtail Y$$

so that $X_1' = \operatorname{Coker} h_1'$ and $X_2' = \operatorname{Coker} h_2'$, since $l(Y) = 3$, $l(X_1') = l(X_2') = 2$ and $\operatorname{soc} Y = T^2$.

There are some $\alpha, \beta, \in \mathsf{End}(T)$ so that $h_1' = h_1\alpha + h_2\beta$. Without loss of generality, we can assume that $\beta \neq 0$.

Since $\begin{bmatrix} h_1' & h_2' \end{bmatrix}$ is a monomorphism, h_1' and h_2' must be orthogonal to each other: $h_2' = (h_2\alpha - h_1\beta)\gamma$ for some $\gamma \in \mathsf{End}(T)$. Thus, there are exact sequences with h_1' and h_2' as monomorphisms that are linearly independent of each other over both $\mathsf{End}(S)^{op}$ and $\mathsf{End}(T)$. So $X_1' \not\cong X_2'$ and in particular, $Y \notin \mathsf{sub}\, X_1$ and $Y \notin \mathsf{sub}\, X_2$.

By Proposition 4.1.3, the proof is complete. $\qquad\qquad\qquad\qquad \square$

4.4   2-Extensions

In this section, we collect some results about exact sequences in $\mathsf{Ext}^2(S, T)$, where S and T are simple objects. We will need these results in the later sections.

Lemma 4.4.1. *Suppose that S', T are simple objects. If*

$$\eta: \quad 0 \longrightarrow S' \longrightarrow M \longrightarrow N \longrightarrow T \longrightarrow 0$$

is an exact sequence in $\mathsf{Ext}^2(T, S')$, then there is some exact sequence

$$\eta': \quad 0 \longrightarrow S' \longrightarrow M' \longrightarrow N' \longrightarrow T \longrightarrow 0$$

with a map $\eta' \to \eta$ in $\mathsf{Ext}^2(S', T)$ and $\mathrm{top}\, N' \cong T$. The induced morphisms of objects $M' \rightarrowtail M$ and $N' \rightarrowtail N$ are monomorphisms. Furthermore, there is an exact sequence

$$\eta'': \quad 0 \longrightarrow S' \longrightarrow M'' \longrightarrow N'' \longrightarrow T \longrightarrow 0$$

with a map $\eta \to \eta''$ in $\mathsf{Ext}^2(S', T)$ and $\mathrm{soc}\, M'' \cong S'$. The induced morphisms of objects $M \twoheadrightarrow M''$ and $N \twoheadrightarrow N''$ are epimorphisms.

Proof. We only prove the first part, since the second one is completely analogous.

If $\mathrm{top}\, N = T \oplus T'$ for some non-zero object T', let $h: N \twoheadrightarrow T'$ be an epimorphism. The concatenation $\mathsf{Ker}\, h \rightarrowtail N \twoheadrightarrow T$ is non-zero, so $T \mid \mathrm{top}(\mathsf{Ker}\, h)$. If $\mathrm{top}(\mathsf{Ker}\, h) \neq T$, we repeat the construction. Since the object T has finite length, we get an object N' with $\mathrm{top}\, N' = T$ after finitely many steps.

Let g be the epimorphism of the exact sequence. Then there are morphisms f, g' with an exact diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{Ker}\, g' & \longrightarrow & N' & \overset{g'}{\longrightarrow} & T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle f} & & \| & & \\
0 & \longrightarrow & \mathsf{Ker}\, g & \longrightarrow & N & \overset{g}{\longrightarrow} & T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{Coker}\, f & =\!\!= & \mathsf{Coker}\, f & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\qquad (4.10)
$$

Furthermore, there is a concatenation of epimorphisms $M \twoheadrightarrow \mathsf{Ker}\, g \twoheadrightarrow \mathsf{Coker}\, f$ and thus some object M' with an exact diagram

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S' & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & T & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \| & & \\
0 & \longrightarrow & S' & \longrightarrow & M & \longrightarrow & N & \longrightarrow & T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{Coker}\, f & =\!\!= & \mathsf{Coker}\, f & \longrightarrow & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & 0 & & 0 & & & & & &
\end{array}
\qquad (4.11)
$$

$\square$

Note that the exact sequences η' and η'' are not unique: in the construction of N', we have used an arbitrary epimorphism $h : N \twoheadrightarrow T'$, where $\mathrm{top}\, N = T \oplus T'$. If $T^2 \mid \mathrm{top}\, N$, there can be such epimorphisms with non-isomorphic kernels.

But let m be the maximal integer so that $T^m \mid \mathrm{top}\, N$. Then there is some T'' so that $\mathrm{top}\, N = T^m \oplus T''$. If there are two epimorphisms $h_1 : N \twoheadrightarrow T''$ and $h_2 : N \twoheadrightarrow T''$, then there is some isomorphism $\chi : T'' \to T''$ so that $h_1 = \chi h_2$. So $\mathsf{Ker}\, h_1 \cong \mathsf{Ker}\, h_2$.

Furthermore, the following diagram is exact, since all columns and the second and third row are exact:

$$
\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & & (4.12)\\
 & \downarrow & & \downarrow & & \downarrow & & & \\
0 \longrightarrow & \mathsf{Ker}\,h & \longrightarrow & \mathsf{Ker}\,h_1 & \longrightarrow & S^{m-1} & \longrightarrow & 0 & \\
 & \| & & \downarrow & & \downarrow & & & \\
0 \longrightarrow & \mathsf{Ker}\,h & \longrightarrow & N & \overset{h}{\longrightarrow} & T' & \longrightarrow & 0 & \\
 & \downarrow & & \downarrow{\scriptstyle h_1} & & \downarrow & & & \\
0 \longrightarrow & T'' & = & T'' & & & \longrightarrow & 0 & \\
 & & & \downarrow & & \downarrow & & & \\
 & & & 0 & & 0 & & &
\end{array}
$$

Note that neither η' nor η'' in the lemma above are necessarily unique: If $T^2 \mid$ top N or $S^2 \mid$ soc M respectively, then we can choose different epimorphisms $N \twoheadrightarrow T$ to construct N' or monomorphisms $S \rightarrowtail N$ to construct M''.

We still need two auxiliary lemmas:

Lemma 4.4.2. *Let $\mathcal{A}$ be an abelian length category. Let there be simple objects $S_1, \ldots, S_n \in \mathcal{A}$ with $\mathsf{Ext}^1(S_i, S_{i+1}) \neq 0$ for all $1 \leq i < n$ and for all simple $S \in \mathcal{A}$*

$$
\sum_{T \text{ simple}} d_T^1(S, T) \leq 1.
$$

Suppose that there is some indecomposable object Z with soc $Z = S_n$ and $S_1 \mid$ top Z or alternatively $S_n \mid$ soc Z and $S_1 =$ top Z.

Then for all $1 \leq i \leq j \leq n$, there is some indecomposable X_{ij} with soc $X_{ij} = S_j$ and top $X_{ij} = S_i$ so that for all $1 \leq i \leq n$, there is a chain of monomorphisms

$$
X_{ii} \rightarrowtail X_{i-1,i} \rightarrowtail \cdots \rightarrowtail X_{1,i}
$$

and a chain of epimorphisms

$$
X_{in} \twoheadrightarrow X_{i,n-1} \twoheadrightarrow \cdots \twoheadrightarrow X_{ii}.
$$

If soc Z is simple, we can choose these objects so that there is a monomorphism $X_{1,n} \rightarrowtail Z$. If top Z is simple, we can choose these objects so that there is an epimorphism $Z \twoheadrightarrow X_{1,n}$.

Proof. We show this inductively. If $n = 1$, this is clear. So suppose that the assertion is proved for $n - 1$.

We can assume that $\operatorname{soc} Z$ is simple, since the proof is completely analogous if $\operatorname{top} Z$ is simple. Let

$$S_n = Z_m \rightarrowtail Z_{m-1} \rightarrowtail \cdots \rightarrowtail Z_0 = Z$$

be a filtration of Z so that Z_i/Z_{i+1} is simple. Then $\operatorname{soc} Z_i$ is simple and Z_i is indecomposable for all $0 \le i \le m$. Furthermore, for every simple T with a morphism $f : Z \twoheadrightarrow T$, we can choose Z_1 to be $\operatorname{Ker} f$. If $T \oplus S_1 \mid \operatorname{top} Z$, then the concatenation of morphisms $Z_1 \rightarrowtail Z \twoheadrightarrow S_1$ is non-zero and thus $S_1 \mid \operatorname{top} Z_1$.

So there is some object Z_i with $\operatorname{soc} Z_i = S_n$ and $\operatorname{top} Z_i = S_1$.

Note that every submodule and every factor module of Z_i is indecomposable.

Then $Z_i/Z_{i+1} = S_1$. There is some simple object T_1 and an indecomposable object X so that the following diagram is exact:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_{i+2} & \longrightarrow & Z_{i+1} & \longrightarrow & T_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_{i+2} & \longrightarrow & Z_i & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_1 & =\!=\!= & S_1 & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

So $\operatorname{Ext}^1(S_1, T) \ne 0$. Analogously, there is some $i_2 > i$ with $T_2 \mid \operatorname{top} Z_{i_2}$ and $\operatorname{Ext}^1(T, T_2) \ne 0$. Inductively, we get some $k \in \mathbb{N}$ with

$$T_3 \mid \operatorname{top} Z_{i_3}, \ldots, T_k \mid \operatorname{top} Z_{i_k} \quad \text{and} \quad T_{k+1} = S_n$$

and $\operatorname{Ext}(T_j, T_{j+1}) \ne 0$. Since the only object T_k with $\operatorname{Ext}^1(T_k, S_n) \ne 0$, is S_{n-1}, inductively the sequence $(T_1, \ldots, T_{k+1})$ must either be of the form

$$(S_1, \ldots, S_{n-1}, S_n)$$

or (if $\operatorname{Ext}^1(S_n, S_n) \ne 0$) of the form

$$(S_1, \ldots, S_{n-1}, S_n, S_n, \ldots, S_n)$$

or (if $S_1 \to S_2 \cdots \to S_n \to S_1$ is an oriented cycle in the Ext-quiver of $\mathcal{A}$) of the form

$$(S_1, \ldots, S_{n-1}, S_n, S_1, \ldots, S_{n-1}, S_n, \ldots, S_1, \ldots, S_{n-1}, S_n).$$

Thus $\operatorname{soc} Z_{i+1} = S_n$ and $S_2 \mid \operatorname{top} Z_{i+1}$. By the inductive assumption, the objects X_{j_1, j_2} and the chains of monomorphisms and epimorphisms exist for all $2 \leq j_1 \leq j_2 \leq n$. We can choose them so that there is a chain of monomorphisms $X_{2,n} \rightarrowtail Z_{i+1} \rightarrowtail Z_i =: X_{1,n}$.

  We construct the remaining objects $X_{1,i}$ inductively for all $i = n - 1, \ldots i = 1$: We choose them to be the objects that make the middle column in the following diagram exact. Then all columns and the first and second row of the diagram are exact; thus the last row is also exact:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{Ker} f_2 & =\!=\!= & \operatorname{Ker} f_2 & \longrightarrow & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_{2,i+1} & \xrightarrow{f_1} & X_{1,i+1} & \longrightarrow & \operatorname{Coker} f_1 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f_2} & & \downarrow & & \| & & \\
0 & \longrightarrow & X_{2,i} & \longrightarrow & X_{1,i} & \longrightarrow & \operatorname{Coker} f_1 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

These objects fulfil all assertions made in the lemma and the proof is complete.   □

  From the proof above, we can see:

Corollary 4.4.3. *Let $\mathcal{A}$ be an abelian length category with*

$$\sum_{T\ simple} d_T^1(S, T) \leq 1$$

for all simple objects $S \in \mathcal{A}$.

  Let Z be an object. If $\operatorname{top} Z = T$ is simple and there is an object Z' with an exact sequence

$$0 \longrightarrow Z' \longrightarrow Z \longrightarrow T \longrightarrow 0 \,,$$

then for every simple $T' \mid \operatorname{top} Z'$, we have $\operatorname{Ext}^1(T, T') \neq 0$.

Now we can prove the following:

Lemma 4.4.4. *Let $\mathcal{A}$ be an abelian length category. Suppose that there are simple objects $S_1, \ldots, S_n \in \mathcal{A}$ with $\mathsf{Ext}^1(S_i, S_{i+1}) \neq 0$ for all $1 \leq i < n$ and for all simple $S \in \mathcal{A}$ we have*

$$\sum_{T \ simple} d_T^1(S, T) \leq 1. \tag{4.13}$$

Then for every indecomposable object Z with $\mathrm{top}\, Z = S_1$ and $\mathrm{soc}\, Z = \bigoplus_{i=1}^{n} S_i^{m_i}$ we have $\bigoplus_{i=1}^{n} m_i = 1$.

Proof. Suppose that a Z exists with $\bigoplus_{i=1}^{n} m_i > 1$. By Corollary 4.4.3, there is some $n' \geq n$, so that we can define

$$S_{n+1} = S_1, \ldots, S_{2n} = S_n, S_{2n+1} = S_1, \ldots, S_{n'} = S_n$$

and there is a filtration

$$S_n^{m_n} = Z_{n'} \rightarrowtail Z_{n'-1} \rightarrowtail \cdots \rightarrowtail Z_0 = Z$$

so that $Z_{i-1}/Z_i = S_i^{m_i'}$ for all $0 < i \leq n'$ and some $m_i' \in \mathbb{N}$. We prove the assertion by induction on n'.

Suppose that $n' = 1$. Then there is some Y with

$$0 \longrightarrow S \longrightarrow Z \longrightarrow Y \longrightarrow 0 \, .$$

The object Y is indecomposable, since $\mathrm{top}\, Y \cong \mathrm{top}\, Z \cong T$. Furthermore, $\mathrm{soc}\, Y = S^{m_n - 1}$. Inductively, we can assume that $m_n = 2$. Then $\mathrm{soc}\, Z \cong S^2$ and the cokernel of $S \rightarrowtail Z$ is some X that is an indecomposable middle term of an exact sequence in $\mathsf{Ext}^1(S_1, S_2)$. By 2.1.4 and Lemma 4.2.2, there are such middle terms X_1, X_2 so that the following is both a pushout and a pullback:

$$\begin{array}{ccc} Z & \xrightarrow{f_1} & X_1 \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g_1} \\ X_2 & \xrightarrow{g_2} & T \end{array} \quad . \tag{4.14}$$

By (4.13), X_1 and X_2 are the middle terms of exact sequences which are linearly dependent over T. But then $X_1 \cong X_2$ and there is a commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{e_1} & X_1 \\ \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle g_1} \\ X_2 & \xrightarrow{g_2} & T \end{array} \quad , $$

a contradiction to (4.14): If there was any ϕ so that $e_1 = f_1\phi$, then f_1 would split.

Now suppose that the assertion is proved for $n' - 1 \geq 2$. By Corollary 4.4.3, top $Z_1 = S_2^{m_2'}$ and thus $m_2' > 1$. There is some Z' which makes the following diagram exact:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_2 & = & Z_2 & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_1 & \longrightarrow & Z & \longrightarrow & S_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & S_2^{m_2} & \longrightarrow & Z' & \longrightarrow & S_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Since top Z is simple, top $Z' = $ top $Z = S_1$, the object Z' is indecomposable and the proof is complete. $\qquad\square$

We can say even more:

Lemma 4.4.5. *Let $\mathcal{A}$ be an abelian length category. Suppose that there are simple objects $S_1, \ldots, S_n \in \mathcal{A}$ with $\mathsf{Ext}^1(S_i, S_{i+1}) \neq 0$ for all $1 \leq i < n$ and for all simple $S \in \mathcal{A}$*

$$
\sum_{T \ simple} d_T^1(S, T) \leq 1. \tag{4.15}
$$

Then the following holds:

(a) *If the objects X_{ij} in Lemma 4.4.2 exist, then we can choose them so that $l(X_{ij}) = j - i + 1$ for all $1 \leq i \leq j \leq n$.*

(b) *For every object Z of length $n' > n$ with top $Z = S_1$ and soc $Z = S_n$, we can define*

$$
S_{n+1} = S_1, \ldots, S_{2n} = S_n, S_{2n+1} = S_1, \ldots, S_{n'} = S_n \tag{4.16}
$$

so that $Z \cong X_{1,n'}$.

Proof. We prove (a) first: From the construction of X_{ij} it is obvious that $l(X_{ij}) \geq j - i + 1$. If $i = j$, we can choose $X_{ij} = S_i$ with length 1. Suppose that $l(X_{ij}) = j - i + 1$ for all $j - i \leq m$. Then we can also find such X_{ij} for $j - i = m + 1$:

If $l(X_{ij}) > j - i + 1$, then the cokernel Y of $S_j \rightarrowtail X_{ij}$ is indecomposable and not isomorphic to $X_{i,j-1}$, but there is an epimorphism $Y \twoheadrightarrow X_{i,j-1}$. By (4.15) and Lemma 4.4.4, the socle of Y is simple; by Corollary 4.4.3, $\operatorname{soc} Y = S_{j-1}$.

We can repeat this construction and since S_j is the only simple object with $\mathsf{Ext}^1(S_{j-1}, S_j) \neq 0$, we get an indecomposable object Y' with an exact sequence

$$0 \longrightarrow S_j \longrightarrow Y' \longrightarrow X_{i,j-1} \longrightarrow 0 .$$

Then $l(Y) = j - i + 1$. Set $X'_{i',j'} := Y'$ for $i' = i$ and $j' = j$. Furthermore, $X'_{i',j'} := X_{i',j'}$ if $i \leq i' \leq j' \leq j - 1$. Then we can construct $X'_{i',j}$ inductively, analogous to the construction in 4.4.2. These objects fulfil $\operatorname{soc} X'_{ij} = S_j$ and $\operatorname{top} X'_{ij} = S_i$ and for all $1 \leq i \leq n$, there is a chain of monomorphisms

$$X'_{ii} \rightarrowtail X'_{i-1,i} \rightarrowtail \cdots \rightarrowtail X'_{1,i}$$

and a chain of epimorphisms

$$X'_{in} \twoheadrightarrow X'_{i,n-1} \twoheadrightarrow \cdots \twoheadrightarrow X'_{ii}.$$

To prove (b), we note that by Lemma 4.4.2, there is an epimorphism $Z \twoheadrightarrow X_{1,n}$ and by (a) we can assume that $l(X_{1,n}) = n$. So we can use an analogous construction as in (a) to get a filtration

$$X_{1,n} \rightarrowtail Z_{n+1} \rightarrowtail \cdots \rightarrowtail Z_{n'} = Z$$

where Z_i/Z_{i-1} is simple. Denoting $S_i := Z_i/Z_{i-1}$ for $n < i \leq n'$, we see that this Definition fulfils (4.16) by Corollary 4.4.3 and (4.15). $\square$

Now we can show the following:

Lemma 4.4.6. *For all simple objects $S \in \mathcal{A}$ let*

$$\sum_{T \ simple} d^1_T(S, T) \leq 1. \tag{4.17}$$

Suppose that there are fixed simple objects S', S, T and indecomposable objects X, Z so that $\mathsf{Ext}^1(T, S) \neq 0$ and the following sequences are exact:

$$0 \longrightarrow S' \longrightarrow X \longrightarrow S \longrightarrow 0$$

and

$$0 \longrightarrow X \longrightarrow Z \longrightarrow T \longrightarrow 0 . \tag{4.18}$$

Then the following holds:

(a) If for all $S' \cong S_1, S_2, \ldots, S_n \cong T$ with $\mathsf{Ext}^1(S_i, S_{i+1}) \neq 0$ for $1 \leq i < n$, so that $X_{1,n-1}$ and $X_{2,n}$ as in Lemma 4.4.5 exist, we have $n = 3$ and $S_2 \cong S$, then $\mathsf{Ext}^2(S', T) = 0$.

(b) All exact sequences of the form (4.18) are equivalent over $\mathsf{End}(X)$, while all exact sequences of the form

$$0 \longrightarrow S' \longrightarrow Z \longrightarrow X' \longrightarrow 0 \qquad\qquad (4.19)$$

with indecomposable X' are equivalent over $\mathsf{End}(S')$. In particular, all objects Z so that (4.18) exists are isomorphic.

Proof. First, we prove (a): Suppose that

$$\eta : \; 0 \longrightarrow S' \longrightarrow M \longrightarrow N \longrightarrow T \longrightarrow 0$$

is an exact sequence in $\mathsf{Ext}^2(S', T)$ that does not split. By Lemma 4.4.1, there is an exact sequence

$$\eta' : \; 0 \longrightarrow S' \xrightarrow{\; b \;} M' \longrightarrow N' \longrightarrow T \longrightarrow 0$$

with top $N' = T$ and $\eta' \to \eta$. Thus η and η' are equivalent. Again by Lemma 4.4.1, we can assume that $\mathrm{soc}\, M' \cong S'$. There is an epimorphism $M' \twoheadrightarrow \mathsf{Coker}\, b$ and a monomorphism $\mathsf{Coker}\, b \rightarrowtail N'$. If $\mathsf{Coker}\, b = 0$, then η' splits and there is nothing to show.

Otherwise, there are some objects R and R', so that the following diagram is exact, since all columns and the first and second row are exact:

Thus, top $M' \cong$ top $\mathsf{Coker}\, b$ and analogously, $\mathrm{soc}\, \mathsf{Coker}\, b = \mathrm{soc}\, N'$. By Corollary 4.4.3, there are some $S' \cong S_1, S_2, \ldots, S_n \cong T$ with $\mathsf{Ext}^1(S_i, S_{i+1}) \neq 0$ for

$1 \leq i < n$. Furthermore, $S_{n-1} \mid \text{top } M'$ and $S_1 \mid \text{soc } N'$. By the assumption, $\mathsf{Coker}\, b \cong \text{top } M' \cong \text{soc } N' \cong S^m$ for some $m \in \mathbb{N}$.

By Lemma 4.4.4, we get $m = 1$ and thus $M' \cong X$, where X is the middle term of an exact sequence in $\mathsf{Ext}(S, S')$ and $N' \cong X'$, where X' is the middle term of an exact sequence in $\mathsf{Ext}(T, S)$.

It remains to show that all exact sequences of the form

$$\eta_1 : \; 0 \longrightarrow S' \xrightarrow{\;f\;} X \xrightarrow{\;g'g\;} X' \xrightarrow{\;h\;} T \longrightarrow 0$$

with $g : X \twoheadrightarrow S$, $g' : S \rightarrowtail X'$ are equivalent to the split exact sequence

$$0 \longrightarrow S' \longrightarrow S' \longrightarrow T \longrightarrow T \longrightarrow 0 \,.$$

We show that for all monomorphisms $c : S' \rightarrowtail Z$ and $c' : X \rightarrowtail Z$, there is some monomorphism b and some epimorphism d with an exact sequence

$$\eta_2 : \; 0 \longrightarrow S' \xrightarrow{\left[\begin{smallmatrix} \mathrm{id} \\ b \end{smallmatrix}\right]} S' \oplus X \xrightarrow{[\,c\; c'\,]} Z \xrightarrow{\;d\;} T \longrightarrow 0$$

and a morphism of exact sequences $\eta_2 \to \eta_1$. Since there is obviously a morphism from η_2 into the split exact sequence, this proves that $\mathsf{Ext}^2(S', T) = 0$.

By Lemma 4.4.2, there is an epimorphism $\phi : Z \twoheadrightarrow X'$. Obviously, $\mathsf{Im}(\phi c') \cong S$. There is an isomorphism ψ so that the following diagram is commutative:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{Im}(\phi c') & \lhook\joinrel\longrightarrow & X' & \xrightarrow{\;h\;} & T & \longrightarrow & 0 \,. \\
& & \downarrow{\scriptstyle \psi} & & \| & & \| & & \\
0 & \longrightarrow & S & \xrightarrow{\;g'\;} & X' & \xrightarrow{\;h\;} & T & \longrightarrow & 0
\end{array}
$$

Since (4.17) holds, there are isomorphisms ψ', ψ'' and a monomorphism f' so that the following diagram is commutative

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & S' & \xrightarrow{\;b\;} & X & \xrightarrow{\;\phi c'\;} & \mathsf{Im}(\phi c') & \longrightarrow & 0 \,. \\
& & \downarrow{\scriptstyle \psi''} & & \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & S' & \xrightarrow{\;f\;} & X & \xrightarrow{\;g\;} & S & \longrightarrow & 0
\end{array}
$$

We can set $d := h\phi$ and get the commutative diagram

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & S' & \xrightarrow{\left[\begin{smallmatrix} \mathrm{id} \\ b \end{smallmatrix}\right]} & S' \oplus X & \xrightarrow{[\,c\; c'\,]} & Z & \xrightarrow{\;d\;} & T & \longrightarrow & 0 \,. \\
& & \downarrow{\scriptstyle \psi''} & & \downarrow{\scriptstyle [0\;\psi']} & & \downarrow{\scriptstyle \phi} & & \| & & \\
0 & \longrightarrow & S' & \xrightarrow{\;f\;} & X & \xrightarrow{\;g'g\;} & X' & \xrightarrow{\;h\;} & T & \longrightarrow & 0
\end{array}
\qquad (4.20)
$$

So $\mathsf{Ext}^2(S', T) = 0$.

Part (b) is simple to show: By (4.17), every isomorphism on T induces an isomorphism on X' and thus on S. On the other hand, every isomorphism on S induces an isomorphism on X. So every isomorphism on T induces an isomorphism on X and all exact sequences of the form (4.18) must be equivalent over $\mathsf{End}(X)$.

Analogously, all exact sequences of the form (4.19) are equivalent over S'.
$\square$

It is possible to generalize this to the following:

Lemma 4.4.7. *For all simple objects $S \in \mathcal{A}$ let*

$$\sum_{T \ simple} d_T^1(S, T) \leq 1. \tag{4.21}$$

Then the following holds if there are some fixed simple objects S, T with an object Z so that $\mathrm{soc}(Z) = S$ and $\mathrm{top}(Z) = T$:

(a) *If there are unique objects $S' \cong S_0, S_1, \ldots, S_n \cong T$ with $\mathsf{Ext}^1(S_i, S_{i+1}) \neq 0$ for $0 \leq i < n$, so that $X_{0,n-1}$ and $X_{1,n}$ as in Lemma 4.4.5 exist, then $\mathsf{Ext}^2(S, T) = 0$.*

(b) *If $T \cong S_1, S_2, \ldots, S_n \cong S$ and the objects X_{ij} are defined as in 4.4.5, then all exact sequences of the form*

$$0 \longrightarrow S \longrightarrow X_{1,n} \longrightarrow X_{2,n} \longrightarrow 0$$

are equivalent over $\mathsf{End}(S)$ and all exact sequences of the form

$$0 \longrightarrow X_{2,n} \longrightarrow X_{1,n} \longrightarrow T \longrightarrow 0$$

are equivalent over $\mathsf{End}(X_{2,n})$; in particular all objects of the form $X_{1,n}$ are isomorphic.

Proof. By (4.21), there are up to isomorphism uniquely determined simple objects $T \cong S_1, S_2, \ldots, S_n \cong S$ with $\mathsf{Ext}(S_i, S_{i+1}) \neq 0$. Furthermore, by Lemma 4.4.2, there are $X_{1,n-1}$ and $X_{2,n}$ so that the following sequence is exact:

$$\eta: \ 0 \longrightarrow S \longrightarrow X_{1,n-1} \longrightarrow X_{2,n} \longrightarrow T \longrightarrow 0 \ .$$

We prove the lemma with induction on n. For $n = 2$, the lemma is obvious; for $n = 3$ it is the result of Lemma 4.4.6. Suppose that the assertion holds for $n - 1$.

By Lemma 4.4.2, there is a monomorphism $X_{1,n-1} \rightarrowtail X_{1,n}$ and an epimorphism $X_{1,n} \twoheadrightarrow X_{2,n}$.

Now we can show both (a) and (b) completely analogously to 4.4.6. $\qquad\square$

Now we show the other direction of 4.4.6:

Lemma 4.4.8. *Suppose that for all simple objects $S \in \mathcal{A}$*

$$\sum_{T \ simple} d_T^1(S,T) \leq 1.$$

Let $S, S', T \in \mathcal{A}$ be fixed simple objects with $\mathsf{Ext}^1(S, S') \neq 0 \neq \mathsf{Ext}^1(T, S)$ so that for all $T \cong S_1, S_2, \ldots, S_n \cong S'$ with $\mathsf{Ext}^1(S_i, S_{i+1}) \neq 0$ for $1 \leq i < n$, we have $n = 3$ and $S_2 \cong S$.

If $\mathsf{Ext}^2(T, S') = 0$, then there is some indecomposable object Z of length 3 with $\operatorname{soc} Z = S'$ and $\operatorname{top} Z = T$.

Proof. By the assumptions, there are indecomposable objects X, X' with exact sequences

$$0 \longrightarrow S' \xrightarrow{f} X \xrightarrow{g} S \longrightarrow 0 \qquad\qquad (4.22)$$

$$0 \longrightarrow S \xrightarrow{g'} X' \xrightarrow{h} T \longrightarrow 0 \ . \qquad\qquad (4.23)$$

So the following sequence is also exact:

$$\eta_1 : \ 0 \longrightarrow S' \xrightarrow{f} X \xrightarrow{g'g} X' \xrightarrow{h} T \longrightarrow 0 \ .$$

If $\mathsf{Ext}^2(S', T) = 0$, then η_1 is equivalent to the split exact sequence

$$0 \longrightarrow S' \longrightarrow S' \longrightarrow T \longrightarrow T \longrightarrow 0 \ .$$

So there are maps of exact sequences $\eta_1 \to \eta_2 \leftarrow \eta_3 \to \cdots \leftarrow \eta_m$ or $\eta_1 \leftarrow \eta_2 \to \eta_3 \leftarrow \cdots \to \eta_m$ so that η_m or η_{m+1} with $\eta_m \to \eta_{m+1}$ splits for some $m \in \mathbb{N}$.

Let η be an exact sequence with $\eta \to \eta_1$. If we start with an exact sequence η with $\eta_1 \to \eta$, the proof is analogous.

Then there are objects M, N, morphisms c, d, e, ϕ_1, ψ_1, and a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & S' & \xrightarrow{c} & M & \xrightarrow{d} & N & \xrightarrow{e} & T & \longrightarrow 0 \ . \\
& & \| & & \downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\psi_1} & & \| & \\
0 & \longrightarrow & S' & \xrightarrow{f} & X & \xrightarrow{g'g} & X' & \xrightarrow{h} & T & \longrightarrow 0
\end{array}
$$

By Lemma 4.4.1, there is some exact sequence

$$\eta' : \; 0 \longrightarrow S' \longrightarrow M' \longrightarrow N' \longrightarrow T \longrightarrow 0$$

with a map $\eta' \to \eta$ in $\mathsf{Ext}^2(S', T)$ and top $N' \cong T$. Since $\eta' \to \eta \to \eta_1$, we can assume that $\eta = \eta'$ and thus top $N = T$. Then η'' as defined in Lemma 4.4.1 fulfils soc $M'' = S'$ and top $N'' = T$.

We can assume that some

$$\eta'' : \; 0 \longrightarrow S' \xrightarrow{f''} M'' \xrightarrow{g''} N'' \xrightarrow{h''} T \longrightarrow 0$$

exists, which is not isomorphic to η_1.

Let $\phi_2 : M \to M''$ and $\psi_2 : N \to N''$ be the morphisms that are induced by $\eta \to \eta''$. The morphisms ψ_1 and ψ_2 are epimorphisms: since top $X' = T = $ top N'' and $h\psi_1 = h''\psi_2 = e \neq 0$, the cokernels of these morphisms are zero. Thus, ϕ_1 and ϕ_2 are also epimorphisms, since

$$g'g\phi_1 = \psi_1 d \neq 0 \neq \psi_2 d = g''\phi_2$$

and

$$\phi_1 c = f \neq 0 \neq f'' = \phi_2 c.$$

Similarly to the arguments in Lemma 4.4.1, $\mathsf{Ker}\,\phi$ is a subobject of $\mathsf{Im}\,d$ and thus there is a monomorphism $\chi : \mathsf{Ker}\,\phi \rightarrowtail N$. Obviously, there is also a monomorphism $\chi_i : \mathsf{Ker}\,\phi \rightarrowtail \mathsf{Ker}\,\phi_i \cong \mathsf{Ker}\,\psi_i$ for $1 \leq i \leq 2$. The following diagram is exact, since all rows and the first and second column are exact:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{Ker}\,\phi & \xrightarrow{\chi_i} & \mathsf{Ker}\,\psi_i & \longrightarrow & \mathsf{Coker}\,\chi_i & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{Ker}\,\phi & \xrightarrow{\chi} & N & \longrightarrow & \mathsf{Coker}\,\chi & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{Im}\,\psi_i & = & \mathsf{Im}\,\psi_i & \longrightarrow & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

So there is an exact sequence

$$\theta : \; 0 \longrightarrow S' \longrightarrow \mathsf{Im}\,\phi \longrightarrow \mathsf{Coker}\,\chi \longrightarrow T \longrightarrow 0$$

with maps $\theta \to \eta_1$ and $\theta \to \eta''$. Thus, we can assume that $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$ is a monomorphism.

If $\operatorname{soc} M = S'$, then ϕ_1 is not only an epimorphism, but also monomorphism and thus an isomorphism. Since η is an exact sequence, ϕ_2 is also an isomorphism. But the same holds for ϕ_2 and ψ_2, so $\eta'' \cong \eta_1$, contradictory to our assumptions.

We get $S'^2 \mid \operatorname{soc} M$ and thus $S' \mid \operatorname{soc} N$.

By the assumption, we have $n = 3$ and $S_2 \cong S$ for all $S' \cong S_1, S_2, \ldots, S_n \cong T$ with $\operatorname{Ext}^1(S_i, S_{i+1}) \neq 0$ for $1 \leq i < n$. Furthermore, $d_S(S, S') = 1 = d_T(T, S)$.

By Corollary 4.4.3 and Lemma 4.4.4, we see that either $N'' \cong X'$ or $N'' \cong T$, since all exact sequences in $\operatorname{Ext}^1(T, S)$ are equivalent over T. In the first case, length considerations yield $M'' \cong X$. Since the morphisms of η_1 are arbitrarily chosen, we can assume that the second case holds. Then η'' splits and $M'' \cong S'$.

So the epimorphism ψ_2 splits and we get $M = X \oplus S'$. We get $c = \begin{bmatrix} f \\ h \end{bmatrix}$ and its cokernel is isomorphic to X. If we set $N =: Z$, then there are short exact sequences

$$0 \longrightarrow X \longrightarrow Z \longrightarrow T \longrightarrow 0$$

and

$$0 \longrightarrow S' \longrightarrow Z \longrightarrow X' \longrightarrow 0$$

and Z fulfils the assertions. $\qquad\square$

4.5 The third condition

In this section, we prove that (C3) holds if $\mathcal{A}$ is of colocal type.

First note that the following equivalence holds:

Proposition 4.5.1. *Suppose that for all simple objects $S \in \mathcal{A}$*

$$\sum_{T \text{ simple}} d_T^1(S, T) \leq 1.$$

For fixed simple objects S and S' with $\operatorname{Ext}^1(S, S') \neq 0$, the following classes of objects are the same:

(a) the class of simple objects T so that $d_T(T, S) \neq 0$ and there is some indecomposable object Z of length 3 with $\operatorname{soc} Z \cong S'$ and $\operatorname{top} Z \cong T$

(b) the class of simple objects T so that $d_T(T, S) \neq 0$ and there is some indecomposable object Z of length greater or equal 3 with $\operatorname{soc} Z \cong S'$ and $\operatorname{top} Z \cong T$

   If $\mathsf{Ext}(S', S') = 0$, then these classes are the same as

(c) the class of simple objects T so that $d_T(T, S) \neq 0$ and there is some indecomposable object Z with $\operatorname{soc} Z \cong S'$ and $\operatorname{top} Z \cong T$

If S' is not part of an oriented cycle in the Ext-quiver of $\mathcal{A}$, then this class is even the same as

(d) the class of simple objects T so that $d_T(T, S) \neq 0$ and $\mathsf{Ext}^2(T, S') = 0$

Proof. The equivalence of (a) and (b) is just Lemma 4.4.2 and Lemma 4.4.5. Part (c) is obvious, since under these assumptions $S \not\cong S'$, $\mathsf{Ext}(S, S) = 0$, $T \not\cong S$ and $\mathsf{Ext}^1(T, S') = 0$. Part (d) is proved in Lemma 4.4.6 (a) and Lemma 4.4.8: If there is a cycle in the Ext-quiver of $\mathcal{A}$, then it is oriented. So if S' is not part of an oriented cycle in the Ext-quiver of $\mathcal{A}$, then for all $T \cong S_1, S_2, \ldots, S_n \cong S'$ with $\mathsf{Ext}^1(S_i, S_{i+1}) \neq 0$ for $1 \leq i < n$, we have $n = 3$ and $S_2 \cong S$. $\qquad\square$

   We still need one lemma before we can prove that (C3) holds if $\mathcal{A}$ is of colocal type:

Lemma 4.5.2. *Let $\mathcal{A}$ be an abelian length category. Suppose that there are simple objects S, S' with $\mathsf{Ext}^1(S, S') \neq 0$ and for $1 \leq i \leq 2$ there are objects T_i, X_i and Z_i with exact sequences*

$$0 \longrightarrow S \xrightarrow{\ g_i\ } X_i \xrightarrow{\ h_i\ } T_i \longrightarrow 0 , \qquad\qquad (4.24)$$

which are linearly independent over $\mathsf{End}(T_1)^{op}$ if $T_1 \cong T_2$ and

$$0 \longrightarrow S' \longrightarrow Z_i \longrightarrow X_i \longrightarrow 0 .$$

If $T_1 \cong T_2$, we furthermore assume that $d_{S'}(S, S') = 1$.
   Then there is a pushout

$$
\begin{array}{ccc}
S' & \longrightarrow & Z_1 \\
\downarrow & & \downarrow \\
Z_2 & \longrightarrow & Y
\end{array}
$$

so that Y is indecomposable.

Proof. If $Z_1 \not\cong Z_2$, we can choose the monomorphisms $f_1 : S' \rightarrowtail Z_1$ and $f_2 : S' \rightarrowtail Z_2$ arbitrarily.

Otherwise, we denote the kernel of $Z_1 \twoheadrightarrow T_1$ with X_3. Since $l(Z_1) = 3$, there is an exact sequence

$$0 \longrightarrow S' \overset{f}{\longrightarrow} X_3 \overset{g}{\longrightarrow} S \longrightarrow 0 \ .$$

We have $\operatorname{Im} gg_i \cong S \cong \operatorname{soc} X_i$. The image of the concatenation $\phi_i : X_3 \rightarrowtail Z_i \twoheadrightarrow X_i$ is also $\operatorname{soc} X_i$ and because of $d_{S'}(S, S') = 1$, there is some isomorphism ψ_i so that $\phi_i \psi_i = gg_i$.

So we can choose f_1' and f_2' so that the following diagrams are exact for $1 \le i \le 2$:

$$
\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & S' & = & S' & \longrightarrow & 0 & \longrightarrow \\
& \downarrow {\scriptstyle f} & & \downarrow {\scriptstyle f f_i'} & & \downarrow & \\
0 \longrightarrow & X_3 & \overset{f_i'}{\longrightarrow} & Z_i & \longrightarrow & T_i & \longrightarrow 0 \\
& \downarrow {\scriptstyle g} & & \downarrow & & \downarrow & \\
0 \longrightarrow & S & \overset{g_i}{\longrightarrow} & X_i & \overset{h_i}{\longrightarrow} & T_i & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}
\qquad (4.25)
$$

We set $f_1 = f_1'f$ and $f_2 = f_2'f$ and form the pushout:

$$
\begin{array}{ccc}
S' & \overset{f_1}{\longrightarrow} & Z_1 \\
{\scriptstyle f_2} \downarrow & & \downarrow {\scriptstyle g_1} \\
Z_2 & \underset{g_2}{\longrightarrow} & Y
\end{array}
\ .
$$

Now, we show that Y is indecomposable:

Since $\operatorname{soc} Z_1 = \operatorname{soc} Z_2 = S'$, there is some indecomposable direct summand Y' of Y with monomorphisms $Z_1 \rightarrowtail Y'$ and $Z_2 \rightarrowtail Y'$. The following diagram

is exact for $1 \leq i \neq j \leq 2$, since its rows and the first two columns are exact:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & (4.26) \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Z_i & =\!=\!= & Z_i & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
0 \longrightarrow S' & \longrightarrow & Z_1 \oplus Z_2 & \longrightarrow & Y & \longrightarrow & 0 & & \\
\| & & \downarrow & & \downarrow & & & & \\
0 \longrightarrow S' & \longrightarrow & Z_j & \longrightarrow & X_j & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Note that $l(Y) = 5$ and $l(Y') \geq l(Z_1) = 3$. Thus, the cokernel of $\phi_i : Z_i \rightarrowtail Y'$ cannot be simple; otherwise the cokernel of $Z_i \rightarrowtail Y$ would be semisimple. So either $Y \cong Y_1$ or $Y_1 \cong Z_1$. But the last case would give us a commutative diagram

$$
\begin{array}{ccc}
S' & \xrightarrow{f_1} & Z_1 \\
f_2 \downarrow & & \downarrow \cong \\
Z_2 & \xrightarrow[\cong]{} & Y_1
\end{array}
$$

So $T_1 \cong T_2$ and with (4.25), this contradicts the assumption that the exact sequences (4.24) are linearly independent over $\mathsf{End}(T_i)^{op}$.

So Y is indecomposable. $\qquad\square$

The next Lemma shows in particular that (C3) is fulfilled if $\mathcal{A}$ is colocal.

Lemma 4.5.3. *Let $\mathcal{A}$ be an abelian length category. The following holds for all simple objects $S \in \mathcal{A}$:*

(a) *Let $\mathcal{A}$ be of colocal type. If there is a simple object S' with $\mathsf{Ext}(S, S') \neq 0$ and a set $\mathcal{T}$ of simple objects T so that $d_T(T, S) \neq 0$ and there is an indecomposable object Z of length 3 with $\mathrm{top}\, Z \cong T$ and $\mathrm{soc}\, Z \cong S'$, then*

$$
\sum_{T \in \mathcal{T}} d_T(T, S) \leq 1.
$$

(b) *Let $\mathsf{S}(\mathcal{A})$ be distributive. Assume that there is a simple object $S' \not\cong S$ with $\mathsf{Ext}(S, S') \neq 0$ and a set $\mathcal{T}$ of simple objects T so that $d_T(T, S) \neq 0$*

and there is an indecomposable object Z of length 3 with $\operatorname{top} Z \cong T$ *and* $\operatorname{soc} Z \cong S'$. *If* $d_S(S, S') \geq d_{S'}(S, S')$ *or there are two non-isomorphic objects in* $\mathcal{T}$, *then*

$$\sum_{T \in \mathcal{T}} d_T(T, S) \leq 1.$$

Proof. We show (a) first: Since $\mathcal{A}$ is colocal,

$$d_S(S, S') \geq d_{S'}(S, S') = 1$$

by Lemma 4.3.2. By Lemma 4.5.2, there are some $T_1, T_2 \in \mathcal{T}$, an indecomposable middle term X_3 of an exact sequence in $\mathsf{Ext}^1(S, S')$, indecomposable objects Z_1, Z_2 with

$$0 \longrightarrow X_3 \longrightarrow Z_i \longrightarrow T_i \longrightarrow 0$$

and an indecomposable object Y so that the following is a pushout:

$$
\begin{array}{ccc}
S' & \longrightarrow & Z_1 \\
\downarrow & & \downarrow \\
Z_2 & \longrightarrow & Y
\end{array}
.
$$

In the following, we use the same notation as in Lemma 4.5.2: in its proof, we either have chosen or can choose f, f_1, f_2 so that the upper left square of the following diagram is commutative. Thus, the diagram is commutative and exact, since its columns and the first two rows are exact:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & S' & = & S' & \longrightarrow & 0 & & \\
 & & {\scriptstyle [f\ f]}\downarrow & & {\scriptstyle [f_1\ f_2]}\downarrow & & \downarrow & & \\
0 & \longrightarrow & X_3 \oplus X_3 & \longrightarrow & Z_1 \oplus Z_2 & \longrightarrow & T_1 \oplus T_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & S \oplus X_3 & \longrightarrow & Y & \longrightarrow & T_1 \oplus T_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
. \qquad (4.27)
$$

So $\operatorname{soc} Y = S \oplus S'$ and $\mathcal{A}$ is not of colocal type.

To show (b), we assume that $S \not\cong S'$. Let $k : S \rightarrowtail Y$ and $l : S' \rightarrowtail Y$ be induced by the last row of the diagram (4.27). Then there is a monomorphism $Y \rightarrowtail \mathsf{Coker}\, k \oplus \mathsf{Coker}\, l$ by Lemma 4.2.2.

For $1 \le i \ne j \le 2$, we get the following exact diagram, since all columns and the first and second row are exact:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & S & =\!=\!= & S & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow{\scriptstyle k} & & \downarrow & & \\
0 & \longrightarrow & S \oplus Z_i & \longrightarrow & Y & \longrightarrow & T_j & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Z_i & \longrightarrow & \mathsf{Coker}\, k & \longrightarrow & T_j & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\qquad . \qquad (4.28)
$$

By Lemma 4.2.2, there is an indecomposable direct summand of $\mathsf{Coker}\, k$ of which Z_1 is a subobject. The last row of the diagram above cannot split, since Y is indecomposable. So $\mathsf{Coker}\, k$ is also indecomposable and has the socle S'. Thus, $\mathrm{soc}\, Y \cong S' \oplus S$ is not a subobject of any direct sum of copies of $\mathsf{Coker}\, k$ and neither is Y. Analogously, the next diagram is exact for all $1 \le i \ne j \le 2$:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & S' & =\!=\!= & S' & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_i & \longrightarrow & Z_1 \oplus Z_2 & \longrightarrow & Z_j & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_i & \longrightarrow & Y & \longrightarrow & X_j & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\qquad .
$$

And finally, the following diagram is exact, because all columns and the

second and third row are exact:

Thus, we have $\mathsf{Coker}\, l \cong X_1 \oplus X_2$, soc $\mathsf{Coker}\, l \cong S^2$ and Y is not a subobject of any direct sum of copies of $\mathsf{Coker}\, l$.

□

4.6 An Equivalence Theorem

We begin with the following special case, from which the general statement follows:

Lemma 4.6.1. *Let A be a hereditary Artin algebra and $\mathcal{A} \equiv \mathsf{mod}\, A$. Assume that $\mathcal{A}$ fulfils the following conditions:*

(C1) For all simple objects $S \in \mathcal{A}$

$$\sum_{T \ simple} d_T^1(S, T) \leq 1.$$

(C2) For all simple objects $S \in \mathcal{A}$

$$\sum_{T \ simple} d_T^1(T, S) \leq 2.$$

(C'3) If there is a simple object $S \in \mathcal{A}$ with

$$\sum_{T \ simple} d_T^1(T, S) = 2,$$

then $\mathsf{Ext}^1(S, S') = 0$ for all simple $S' \in \mathcal{A}$.

Then the following holds:

(a) A is of colocal type.

(b) A is representation finite.

Proof. We can assume without loss of generality that A is indecomposable as an algebra. To show both (a) and (b), we construct the preinjective component of the Auslander-Reiten quiver of A. We will see that this component is finite and thus the complete Auslander-Reiten quiver of A by Proposition 2.3.15.

The Ext-quiver of A does not contain oriented cycles, since A is hereditary. By (C1), the Ext-quiver does not contain other cycles, either. So there are two possible cases:

(a) We have $d_T^1(S,T) \leq 1$ and $d_S^1(S,T) \leq 1$ for all simple modules $S, T \in \mathsf{mod}\, A$. Then we can order the simple modules $S_1, S_2, \ldots, S_n$ of $\mathsf{mod}\, A$ so that for some $1 \leq l \leq n$ we have $\mathsf{Ext}^1(S_i, S_{i+1}) \neq 0$ for all $1 \leq i < l$ and $\mathsf{Ext}^1(S_{i+1}, S_i) \neq 0$ for all $l \leq i < n$. By (C1) - (C2) and (C'3), $\mathsf{Ext}^1(S, T) = 0$ for all other simple modules S, T.

(b) We can denote the simple modules by $S_1, \ldots, S_n$ so that

$$d_{S_i}^1(S_i, S_{i+1}) = 1 = d_{S_{i+1}}^1(S_i, S_{i+1})$$

for all $1 \leq i \leq n-2$, $d_{S_n}^1(S_{n-1}, S_n) = 1$, $d_{S_{n-1}}^1(S_{n-1}, S_n) = 2$ and $\mathsf{Ext}^1(S, T) = 0$ for all other simple modules S, T.

We will only prove the assertion in case (a). In case (b), we can construct the Auslander-Reiten quiver completely analogously to the first case and thus show the assertion completely analogously. The result of this construction is the following Auslander-Reiten quiver:
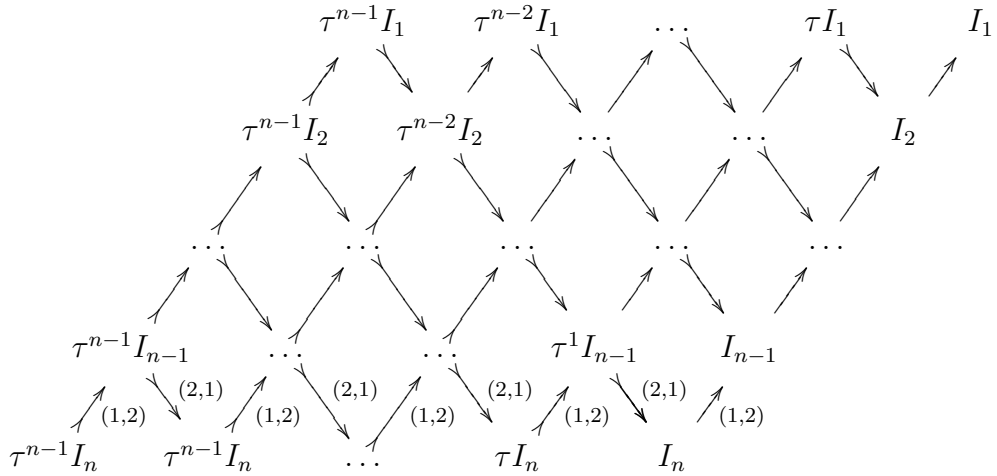
The arrows of the form $\rightarrowtail$ denote monomorphisms.

Now suppose that (a) holds. Additionally, we assume without loss of generality that $l > n - l$. Then the following is part of the preinjective component of the Auslander-Reiten quiver of A by 2.3.7 and 2.3.9:
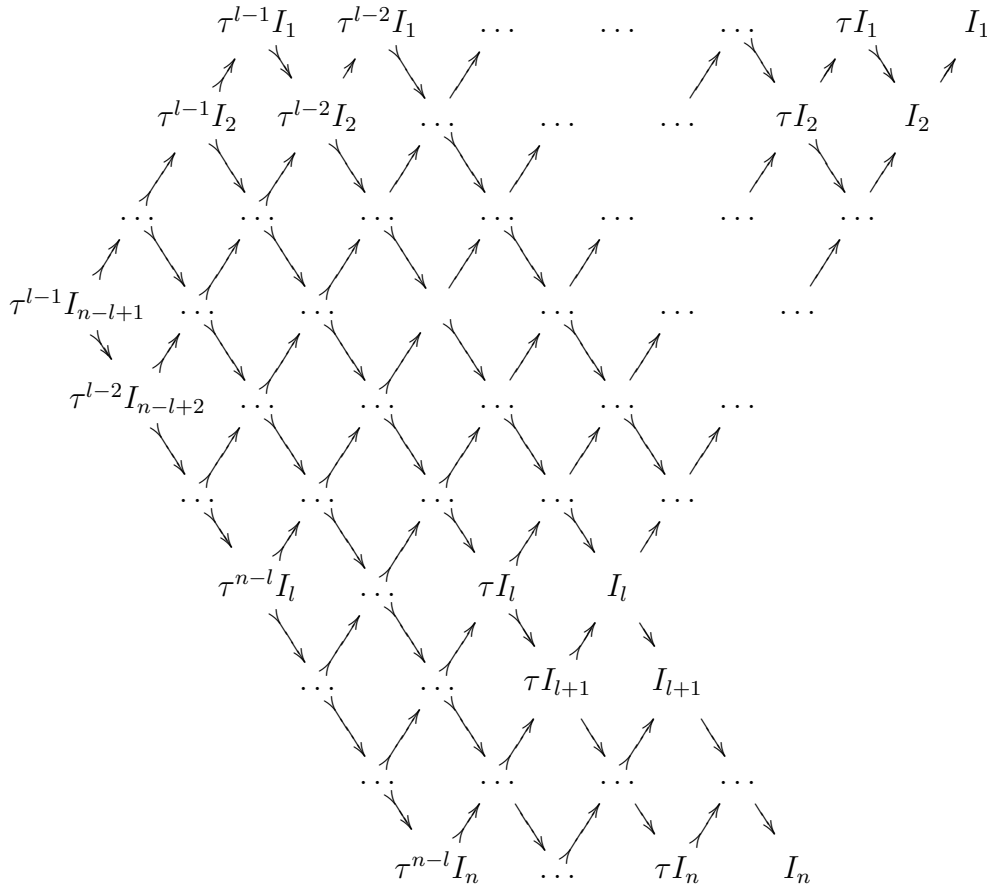
If the arrows of the form $\rightarrowtail$ correspond to monomorphisms, it is obvious that the above is a complete component of the Auslander-Reiten quiver of A and thus the complete Auslander-Reiten quiver: We have $\tau^l I_1 = 0$ and $\tau^{n-l+1} I_n = 0$, since no Auslander-Reiten sequence can start in these modules. Inductively,

$$\tau^l I_2 = \cdots = \tau^l I_{n-l+1} = 0$$

and

$$\tau^{n-l+1} I_{n-1} = \cdots = \tau^{n-l+1} I_l = \tau^{n-l+2} I_{l-1} = \cdots = \tau^{l-1} I_{n-l+2} = 0.$$

To show that the arrows of the form $\rightarrowtail$ denote indeed monomorphisms, we use Lemma 3.2.13: There are Auslander-Reiten sequences

$$0 \longrightarrow \tau^i I_1 \longrightarrow \tau^{i-1} I_2 \longrightarrow \tau^{i-1} I_1 \longrightarrow 0$$

for $\tau^i I_1 \neq 0$ and

$$0 \longrightarrow \tau^i I_j \longrightarrow \tau^{i-1} I_{j+1} \oplus \tau^i I_{j-1} \longrightarrow \tau^{i-1} I_j \longrightarrow 0$$

for all $1 < j \leq l - 1$ and i with $\tau^i I_j \neq 0$. If there is an exact sequence

$$0 \longrightarrow \tau^i I_{j-1} \longrightarrow \tau^{i-1} I_j \longrightarrow \tau^{i-1} I_1 \longrightarrow 0$$

then there is also an exact sequence

$$0 \longrightarrow \tau^i I_j \longrightarrow \tau^{i-1} I_{j+1} \longrightarrow \tau^{i-1} I_1 \longrightarrow 0 \ . \qquad (4.29)$$

It possible to show that $\tau^i I_{j-1}, \tau^i I_{j-2}, \ldots, \tau^i I_1$ are non-zero and thus, we get indeed the exact sequences above. But this is not necessary for our proof, since we are only interested in the monomorphisms of the exact sequences and if $\tau^i I_{j-1} = 0$, there is obviously a monomorphism $\tau^i I_j \rightarrowtail \tau^{i-1} I_{j+1}$, just with a different cokernel.

Analogously, we can assume in the following that all modules in the exact sequences below are non-zero.

There are Auslander-Reiten sequences

$$0 \longrightarrow \tau^i I_n \longrightarrow \tau^{i-1} I_{n-1} \longrightarrow \tau^{i-1} I_n \longrightarrow 0$$

for $\tau^i I_n \neq 0$ and

$$0 \longrightarrow \tau^i I_j \longrightarrow \tau^{i-1} I_{j-1} \oplus \tau^i I_{j+1} \longrightarrow \tau^{i-1} I_j \longrightarrow 0$$

for all $l < j \leq n - 1$ with $\tau^i I_j \neq 0$. We get an exact sequence

$$0 \longrightarrow \tau^i I_j \longrightarrow \tau^{i-1} I_{j-1} \longrightarrow \tau^{i-1} I_n \longrightarrow 0 \qquad (4.30)$$

for $m < j \leq n$ and i with $\tau^i I_j \neq 0$.

The remaining Auslander-Reiten sequences are

$$0 \longrightarrow \tau^i I_l \longrightarrow \tau^i I_{l-1} \oplus \tau^i I_{l+1} \longrightarrow \tau^{i-1} I_l \longrightarrow 0$$

for $\tau^i I_l \neq 0$.

Together with (4.29) and (4.30), we get exact sequences

$$0 \longrightarrow \tau^i I_l \longrightarrow \tau^i I_{l+1} \longrightarrow \tau^{i-1} I_1 \longrightarrow 0$$

and

$$0 \longrightarrow \tau^i I_l \longrightarrow \tau^i I_{l-1} \longrightarrow \tau^{i-1} I_n \longrightarrow 0 \ .$$

Inductively, we get exact sequences

$$0 \longrightarrow \tau^i I_j \longrightarrow \tau^i I_{j-1} \longrightarrow \tau^{i-l+j-1} I_n \longrightarrow 0 \qquad (4.31)$$

for all $\tau^i I_j \neq 0$ with $1 < j \leq l$ and $l - j < i$. Analogously,

$$0 \longrightarrow \tau^i I_j \longrightarrow \tau^i I_{j+1} \longrightarrow \tau^{i+l-j-1} I_1 \longrightarrow 0 \qquad (4.32)$$

is exact for all $\tau^i I_j \neq 0$ with $l \leq j \leq n$ and $j - l \leq i$.

We can find the injective envelope (see 2.2.4) of every indecomposable module M by taking the product of irreducible monomorphisms.

So M has an indecomposable injective envelope I with simple socle (see 2.2.6) and $\operatorname{soc} M = \operatorname{soc} I$. Thus every submodule of M is indecomposable and A is of colocal type. $\qquad \square$

Now we can prove Theorem 1.2.4:

Theorem 4.6.2. *The category $\mathcal{A}$ is of colocal type if and only if the following conditions hold:*

(C1) For all simple objects $S \in \mathcal{A}$

$$\sum_{T \; simple} d^1_T(S, T) \leq 1.$$

(C2) For all simple objects $S \in \mathcal{A}$

$$\sum_{T \; simple} d^1_T(T, S) \leq 2.$$

(C3) If there is a simple object S' with $\mathsf{Ext}^1(S, S') \neq 0$, let $\mathcal{T}$ be the class of simple objects T for which $d^1_T(T, S) \neq 0$ and there is an indecomposable object Z of length 3 with $\operatorname{top} Z \cong T$ and $\operatorname{soc} Z \cong S'$. Then

$$\sum_{T \in \mathcal{T}} d^1_T(T, S) \leq 1.$$

Proof. If $\mathcal{A}$ is of colocal type, then condition (C1) holds by Lemma 4.3.2, condition (C2) holds by Lemma 4.3.4 and condition (C3) by Lemma 4.5.3.

For the other direction, we use Lemma 4.2.4: If $\mathcal{A}$ is not of colocal type, there are objects X, Y_1, Y_2, S so that X is indecomposable, Y_1, Y_2 are not simple and there is an exact sequence

$$0 \longrightarrow X \longrightarrow Y_1 \oplus Y_2 \xrightarrow{[g_1 \; g_2]} S \longrightarrow 0 \; . \qquad (4.33)$$

If $Y \mid Y_1 \oplus Y_2$ is indecomposable, we can assume that $\operatorname{soc} Y$ is simple.

On the other hand, $S \mid \operatorname{top} Y$, since $Y \nmid X$.

First, we assume that for every $T \mid \operatorname{soc}(Y_1 \oplus Y_2)$ there are up to isomorphism unique objects $S \cong S_1, S_2, \ldots, S_n \cong T$ with $\mathsf{Ext}^1(S_i, S_{i+1}) \neq 0$ for $1 \leq i < n$ so that $X_{1,n-1}$ and $X_{2,n}$ as in Lemma 4.4.5 exist. By 4.4.2 and Lemma 4.4.7, these are objects in a full abelian subcategory $\mathcal{A}'$ of $\mathcal{A}$ so that $\mathsf{Ext}^2(S,T) = 0$ for all simple $S, T \in \mathcal{A}'$.

By 2.3.1, we can be embed $\mathcal{A}'$ into the module category of a hereditary Artin algebra A. With Lemma 4.6.1, at least one of the conditions (C1), (C2) or (C'3) is not fulfilled by $\mathsf{mod}\, A$. For a hereditary Artin algebra, the condition (C'3) is equivalent to (C3) by 4.5.1 and thus $\mathcal{A}'$ and $\mathcal{A}$ do not fulfil (C1) - (C3).

Now suppose that for some $T \mid \operatorname{soc}(Y_1 \oplus Y_2)$, there are non-isomorphic objects $S \cong S_1, S_2, \ldots, S_n \cong T$ and $S \cong S'_1, S'_2, \ldots, S'_{n'} \cong T$ so that we have $\mathsf{Ext}^1(S_i, S_{i+1}) \neq 0$ for $1 \leq i < n$, $\mathsf{Ext}^1(S'_i, S'_{i+1}) \neq 0$ for $1 \leq i < n'$ and there are indecomposable objects $X_{1,n-1}$, $X_{2,n}$ as in Lemma 4.4.5 and $X'_{1,n-1}$ with $\operatorname{soc} X'_{1,n-1} \cong S'_{n-1}$, $\operatorname{top} X'_{1,n-1} \cong S'_1$ and $X'_{2,n}$ with $\operatorname{soc} X'_{2,n} \cong S'_n$, $\operatorname{top} X'_{2,n} \cong S'_2$.

If we assume that $n' \leq n$, then $S_1 \cong S'_1, \ldots, S_{n'} \cong S'_{n'}$ by (C1), so $n' < n$. By Lemma 4.4.2 and Lemma 4.4.5, there are objects $X_{i,i+2}$ of length 3 with $\operatorname{soc} X_{i,i+2} = S_{i+2}$ and $\operatorname{top} X_{ij} = S_i$. So by condition (C3), we get

$$d_{S_i}(S_i, S_{i+1}) = 1$$

for all $1 \leq i \leq n$. Analogously, every $T' \mid \operatorname{soc}(Y_1 \oplus Y_2)$ must be of the form S_i for some $1 \leq i \leq n$.

So we can choose T so that for all indecomposable objects $Y \mid Y_1 \oplus Y_2$ there is some $1 \leq i \leq n$ with $\operatorname{soc} Y = S_i$. By (C1), there are some $m_i \in \mathbb{N}$ for $1 \leq i \leq n$ so that $\operatorname{top} Y = \bigoplus_{i=1}^n S_i^{m_i}$. Analogously to Lemma 4.4.4, $d_{S_i}(S_i, S_{i+1}) = 1$ means $\operatorname{top} Y = S = S_1$.

By Lemma 4.4.7, all objects with socle S_i and top S_1 are isomorphic to $X_{1,i}$ and all exact sequences of the form

$$0 \longrightarrow X_{2,i} \longrightarrow X_{1,i} \longrightarrow S_1 \longrightarrow 0$$

are equivalent over $\mathsf{End}(X_{2,i})$. So for all epimorphisms $g : X_{1,i} \twoheadrightarrow S_1$ and $g' : X_{1,i} \twoheadrightarrow S_1$, there is some isomorphism χ on $X_{1,i}$ so that $g' = \chi g$.

Since $S = S_1$ and $T \mid \operatorname{soc} Y_1 \oplus Y_2$, we get $X_{1,n} \mid Y_1 \oplus Y_2$ and can assume that $Y_1 = Y'_1 \oplus X_{1,n}$. By Lemma 4.4.2, there is an epimorphism $X_{1,n} \twoheadrightarrow X_{i,n}$.

There are morphisms f_1, f_2 so that the following is a commutative dia-

gram:

$$
\begin{array}{ccc}
X_{1,n} & \xrightarrow{\left[\begin{smallmatrix}\mathrm{id}\\f_1\end{smallmatrix}\right]} & X_{1,n} \oplus Y_1' \;. \\
{\scriptstyle f_2}\big\downarrow & & \big\downarrow{\scriptstyle g_1} \\
Y_2 & \xrightarrow{\;\;g_2\;\;} & T
\end{array}
$$

But by (4.33), there is a pullback

$$
\begin{array}{ccc}
X & \longrightarrow & Y_1 \quad. \\
\big\downarrow & & \big\downarrow{\scriptstyle g_1} \\
Y_2 & \xrightarrow{\;\;g_2\;\;} & T
\end{array}
$$

Thus, X cannot be indecomposable, a contradiction to the assumption and the proof is complete. $\qquad\square$

We can draw the following corollary:

Corollary 4.6.3. *If A is a colocal Artin algebra, then A is of finite representation type.*

Proof. Since A is an Artin algebra, it has finitely many non-isomorphic simple modules $S_1, S_2, \ldots, S_n$. The proof of Theorem 4.6.2 shows that every module in $\mathsf{mod}\,A$ can be either be regarded as a module in $\mathsf{mod}\,A'$, where A' is a hereditary subalgebra of A with simple modules $S_{i_1}, S_{i_2}, \ldots, S_{i_m}$ for some $m \leq n$. Or there is a path $S_1' \to S_2' \to \cdots \to S_{n'}'$ in the Ext-quiver of A which is part of an oriented cycle so that the module is of the form $X_{1,n'}$ and of length n'. Since A is an Artin algebra, there are only finitely many objects of the latter form.

By Lemma 4.6.1, every hereditary subalgebra of A is representation finite. Because there are only finitely many possibilities for $i_1, \ldots, i_m$, the algebra A is also representation finite. $\qquad\square$

4.7 The lattice $\mathsf{S}(\mathcal{A})$

We show in this section that the lattice $\mathsf{S}(\mathcal{A})$ is in fact the Cartesian product of certain sublattices.

For Artin algebras A over algebraically closed fields, we will use this in the next section.

We begin with the following lemma:

Lemma 4.7.1. *Suppose X is an indecomposable object and there is an index set I so that $X \subseteq \bigoplus_{i \in I} Y_i$. Set*

$$I' = \{i \in I \mid \text{there is a simple object } S \text{ with } S \subseteq X \text{ and } S \subseteq X_i\}.$$

Then $X \subseteq \bigoplus_{i \in I'} X_i$.

Proof. There is a morphism

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : X \rightarrowtail \bigoplus_{i \in I'} Y_i \oplus \bigoplus_{i \in I \setminus I'} Y_i$$

with

$$f_1 : X \to \bigoplus_{i \in I'} Y_i \quad \text{and} \quad f_2 : X \to \bigoplus_{i \in I \setminus I'} Y_i.$$

Furthermore, $\mathsf{Ker}(f_1) \oplus \mathsf{Ker}(f_2) \subseteq X$ and $\mathsf{Ker}(f_1) \subseteq \bigoplus_{i \in I \setminus I'} Y_i$. So there is no simple $S \subset \mathsf{Ker}(f_1)$ and thus $\mathsf{Ker}(f_1) = 0$, which implies that f_1 is a monomorphism and $X \subseteq \bigoplus_{i \in I'} Y_i$. $\square$

To simplify the notation, we define:

Definition 4.7.2. For a class $\mathcal{M}$ of indecomposable objects in $\mathcal{A}$ let

$$\mathsf{S}(\mathcal{M}) := \mathsf{S}(\mathsf{add}\,\mathcal{M}).$$

Under certain assumptions, $\mathsf{S}(\mathcal{M})$ is a sublattice of $\mathsf{S}(\mathcal{A})$.

Lemma 4.7.3. *Let $\mathcal{M}$ be a class of indecomposable objects in $\mathcal{A}$. If*

$$\mathsf{ind}\,\mathsf{sub}\,\mathcal{M} = \mathcal{M}, \tag{4.34}$$

then $\mathsf{S}(\mathcal{M})$ is a sublattice of $\mathsf{S}(\mathcal{A})$.

Proof. We need to show that for $C, C' \in \mathsf{S}(\mathcal{M})$, the join and the meet are again in $\mathsf{S}(\mathcal{M})$. The first direction is obvious: $C \wedge C' = C \cap C'$ and thus

$$\mathsf{ind}\,(C \wedge C') = \mathsf{ind}\,C \cap \mathsf{ind}\,C' \subseteq \mathcal{M},$$

since $\mathsf{ind}\,C, \mathsf{ind}\,C' \subseteq \mathcal{M}$. So $C \wedge C' \in \mathsf{S}(\mathcal{M})$.

On the other hand, the join $C \vee C'$ consists of all subobjects of direct sums of objects in C and C'. Thus, if $M \in \mathsf{ind}(C \vee C')$ then $M \in \mathsf{sub}\,\mathcal{M}$. By (4.34), $M \in \mathcal{M}$. So $C \vee C' \in \mathsf{S}(\mathcal{M})$ and $\mathsf{S}(\mathcal{M})$ is a sublattice of $\mathsf{S}(\mathcal{A})$. $\square$

We get the following homomorphism between lattices:

Lemma 4.7.4. *Let $\mathcal{A}$ be an abelian length category. Suppose that there is an index set I, and classes of indecomposable objects $\mathcal{M}_i$, $i \in I$ exist, so that*

(M1) $\bigcup_{i \in I} \mathcal{M}_i = \mathsf{ind}\, A$

(M2) $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ *for all $i, j \in I$ with $i \neq j$*

(M3) $\mathsf{ind}\,\mathsf{sub}\,\mathcal{M}_i = \mathcal{M}_i$ *for all $i \in I$.*

Denote $\mathcal{M} = \{\mathcal{M}_i \mid i \in I\}$. Then

$$f_{\mathcal{M}} : \mathsf{S}(\mathsf{mod}\ A) \to \prod_{i \in I} \mathsf{S}(\mathcal{M}_i)$$

$$C \to \prod_{i \in I} C_i$$

where C_i is given by

$$\mathsf{ind}\, C_i = \mathsf{ind}\, C \cap \mathcal{M}_i$$

is a lattice homomorphism.

Proof. By Lemma 4.7.3 and (M3), $\mathsf{S}(\mathcal{M}_i)$ is a lattice for every $i \in I$ and the Cartesian product exists. We have to show that $f_{\mathcal{M}}$ preserves meets and joins. Take $C, C' \in \mathsf{S}(\mathcal{M}_i)$. Then $f_{\mathcal{M}}$ preserves meets, since

$$\mathsf{ind}(C \wedge C') = \mathsf{ind}\, C \cap \mathsf{ind}\, C'$$

and

$$\mathsf{ind}(C \wedge C')_i = \mathsf{ind}(C \wedge C') \cap \mathcal{M}_i = (\mathsf{ind}\, C \cap \mathsf{ind}\, C') \cap \mathcal{M}_i = \mathsf{ind}\, C_i \cap \mathsf{ind}\, C'_i.$$

Thus $(C \wedge C')_i = C_i \wedge C'_i$ and

$$f_{\mathcal{M}}(C \wedge C') = \prod_{i \in I} (C \wedge C')_i = \prod_{i \in I} (C_i \wedge C'_i) = \prod_{i \in I} C_i \wedge \prod_{i \in I} C'_i = f_{\mathcal{M}}(C) \wedge f_{\mathcal{M}}(C').$$

The function also preserves joins: For some object M, we have $M \in \mathsf{ind}(C \vee C')_i$ if and only if $M \in \mathcal{M}_i$ and there are objects $x_1, \ldots, x_c \in \mathsf{ind}\, C$ and $x'_1, \ldots, x'_{c'} \in \mathsf{ind}\, C'$ for some $c, c' \in \mathbb{N}$ so that

$$M \subseteq \bigoplus_{k=1}^{c} x_k \oplus \bigoplus_{k=1}^{c'} x'_k.$$

Suppose that there is some $x_k \notin \mathcal{M}_i$ for some $1 \leq k \leq c$. By (M1), there is some $j \neq i$ with $x_k \in \mathcal{M}_j$. For every simple object $S \subseteq x_k$, we get $S \in \mathcal{M}_j$

with (M3) and thus $S \notin \mathcal{M}_i$ by (M2). So S is not a subobject of M and by Lemma 4.7.1, we get

$$M \subseteq \bigoplus_{\substack{k=1 \\ k \neq l}}^{c} x_k \oplus \bigoplus_{k=1}^{c'} x_k'.$$

So we can assume that $x_1, \ldots x_c, x_1', \ldots x_{c'}' \in \mathcal{M}_i$ and thus $M \in \mathsf{ind}(C_i \vee C_i')$.

Since $C_i \vee C_i' \in \mathsf{S}(\mathcal{M}_i)$ and $C_i \vee C_i' \leq C \vee C'$, the other direction is obvious. We get $\mathsf{ind}(C_i \vee C_i') = \mathsf{ind}(C \vee C')_i$ and $C_i \vee C_i' = (C \vee C')_i$. So

$$f_{\mathcal{M}}(C \vee C') = \prod_{i \in I}(C \vee C')_i = \prod_{i \in I}(C_i \vee C_i') = \prod_{i \in I} C_i \vee \prod_{i \in I} C_i' = f_{\mathcal{M}}(C) \vee f_{\mathcal{M}}(C')$$

and $f_{\mathcal{M}}$ is a lattice homomorphism. $\qquad \square$

Even better, $f_{\mathcal{M}}$ is an isomorphism:

Proposition 4.7.5. *Let $\mathcal{A}$ be an abelian length category and $\mathcal{M} = \{\mathcal{M}_i \mid i \in I\}$ be a family of classes of indecomposable objects that fulfil (M1) - (M3). Then $f_{\mathcal{M}}$ as defined in Lemma 4.7.4 is a lattice isomorphism between $\mathsf{S}(\mathcal{A})$ and $\prod_{i \in I} \mathsf{S}(\mathcal{M}_i)$.*

Proof. By Lemma 4.7.4, $f_{\mathcal{M}}$ is a homomorphism between lattices. To show that $f_{\mathcal{M}}$ is an isomorphism, we need to prove that f is injective and surjective.

Suppose that $f_{\mathcal{M}}(C) = f_{\mathcal{M}}(C')$ for some $C, C' \in \mathsf{S}(\mathsf{mod}\ A)$. Then

$$\prod_{i \in I} C_i = \prod_{i \in I} C_i'$$

and by (M2), we have $C_i = C_i'$ for all $i \in I$. This means

$$\mathsf{ind}\, C \cap \mathcal{M}_i = \mathsf{ind}\, C' \cap \mathcal{M}_i$$

for all $i \in I$. By (M1), $\mathsf{ind}\, C = \mathsf{ind}\, C'$ and $f_{\mathcal{M}}$ is injective.

Now take

$$\prod_{i \in I} C_i \in \prod_{i \in I} \mathsf{S}(\mathcal{M}_i).$$

Since all C_i are subobject closed subcategories of $\mathcal{A}$, we have $C_i \in \mathsf{S}(\mathcal{A})$ for all $i \in I$. We will show that

$$f_{\mathcal{M}}(\bigvee_{i \in I} C_i) = \prod_{i \in I} C_i.$$

It is obvious that $C_j \subseteq \left(\bigvee_{i \in I} C_i\right)_j$ for all $j \in I$ which implies

$$\prod_{i \in I} C_i \subseteq f_{\mathcal{M}}(\bigvee_{i \in I} C_i).$$

For the other direction, we need to show that $\left(\bigvee_{i\in I} C_i\right)_j \subseteq C_j$ for all $j \in I$, which is equivalent to

$$\left(\operatorname{ind} \bigvee_{i\in I} C_i\right) \cap \mathcal{M}_j \subseteq \operatorname{ind} C_j. \tag{4.35}$$

Suppose that $M \in \left(\operatorname{ind} \bigvee_{i\in I} C_i\right) \cap \mathcal{M}_j$. Then there are objects $N_i \in C_i$, so that

$$M \subseteq \bigoplus_{i\in I} N_i.$$

Set

$$I' = \{i \in I \mid \text{there is a simple object } S \text{ with } S \subseteq M \text{ and } S \subseteq N_i\}.$$

By Lemma 4.7.1,

$$M \subseteq \bigoplus_{i\in I'} N_i.$$

By (M3), we have $S \in \mathcal{M}_j$ for all simple modules $S \subseteq M \in \mathcal{M}_j$. On the other hand, if $S \subseteq N_i \in \mathcal{M}_i$, then $S \in \mathcal{M}_i$ and by (M2) we get $I' = \{j\}$. So (4.35) holds, $f_{\mathcal{M}}$ is surjective and thus a lattice isomorphism. $\qquad\square$

4.8 The structure of the lattice

Let $\operatorname{mod} A \equiv \operatorname{mod} kQ/I$ for some quiver Q and some admissible ideal I. This is always the case if A is an algebra over a closed field k. If A is of colocal type, then the lattice $\mathsf{S}(\operatorname{mod} A)$ is relatively simple and can be described completely.

By Theorem 4.6.2 we get the following:

Proposition 4.8.1. *Let A be an Artin algebra and $\operatorname{mod} A \equiv \operatorname{mod} kQ/I$ for some quiver Q and an admissible ideal I.*

(a) The algebra A is of colocal type if and only if $\mathsf{S}(\operatorname{mod} A)$ is distributive and for every subquiver of Q of the form

$$1 \xrightarrow{\beta} 2 \,\circlearrowright\, \alpha$$

$\alpha\beta \in I$ or $\alpha^2 \in I$.

(b) The algebra A is of colocal type if and only if it is a string algebra and no vertex in Q is starting point of more than one arrow.

Proof. First, we prove (a): By Lemma 4.2.3, $\mathsf{S}(\mathsf{mod}\,A)$ is distributive if A is of colocal type.

On the other hand, suppose that $\mathsf{S}(\mathsf{mod}\,A)$ is distributive and fulfils the condition above. We have $d_S(S,T) = d_T(S,T)$ for all simple modules $S, T \in \mathsf{mod}\,A$ by 2.5.2. Since $\mathsf{mod}\,kQ/I$ is equivalent to the category of representations of Q with the relations that generate I, we have $\mathsf{End}(S) \cong \mathsf{End}(T)$. So

$$d_S(S,T) = d_T(S,T) \leq 1$$

for all S, T by Lemma 4.3.5. Thus Lemma 4.3.2 (b), 4.3.4 (b) and 4.5.3 (b) show that $\mathsf{mod}\,A$ fulfils (C1) - (C3). By Theorem 4.6.2, A is of colocal type.

To show (b), suppose that A fulfils (C1) - (C3). By 2.5.2, this is equivalent to the following:

1. No vertex in Q is starting point of more than one arrow.

2. No vertex in Q is end point of more than two arrows.

3. Given an arrow β, there is at most one arrow γ with $s(\beta) = e(\gamma)$ and $\beta\gamma \notin I$.

Since A is an Artin algebra, the quiver Q must be finite.

Comparing this to Definition 2.5.4, it only remains to show that I is an ideal generated by zero relations. If Q does not contain oriented cycles, then for any given vertices i and j, there is at most one path ρ with $s(\rho) = i$ and $e(\rho) = j$.

In fact, any relation which is not a zero relation is of the following form, where ρ is an oriented cycle, ρ' is a subpath of ρ with $s(\rho') = e(\rho) = s(\rho)$, $a_1, \ldots a_n \in k$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n \in \mathbb{N}$:

$$a_1\rho'\rho^{\alpha_1} + a_2\rho'\rho^{\alpha_2} + \cdots + a_n\rho'\rho^{\alpha_n} = 0. \tag{4.36}$$

Now, we use that I is admissible: there must be some $t \in \mathbb{N}$, so that $\rho^t = 0$. So for every representation V of Q, there is some m so that

$$0 = \mathsf{Im}\, f_{\rho^m} \subsetneq \mathsf{Im}\, f_{\rho^{m-1}} \subsetneq \cdots \subsetneq \mathsf{Im}\, f_\rho,$$

where f_{ρ^i} denotes the map of the representation defined by the path ρ^i. We get $\rho^{\alpha_1} = \cdots = \rho^{\alpha_n} = 0$, since otherwise

$$\mathsf{Im}(f_{\rho'\rho^{\alpha_2}} + \cdots + f_{\rho'\rho^{\alpha_n}}) \subseteq \mathsf{Im}\, f_{\rho'\rho^{\alpha_2}} \subsetneq \mathsf{Im}\, f_{\rho'\rho^{\alpha_1}},$$

a contradiction to (4.36). $\qquad\square$

Furthermore, we get some useful properties:

Lemma 4.8.2. *If $A = kQ/I$ for some quiver $Q = (Q_0, Q_1)$ with admissible ideal I and A is of colocal type, then the following holds:*

(a) If Q contains a cycle, this cycle is oriented.

(b) At most two paths are maximal under all paths without relations that end in i.

(c) Every module in $\mathsf{ind}\, A$ is a string module.

(d) Every string is of the form

$$w = \alpha_{l_1}^{-1}\alpha_{l_1-1}^{-1} \ldots \alpha_1^{-1}\beta_1\beta_2 \ldots \beta_{l_2},$$

for some $l_1, l_2 \in \mathbb{N}_0$, and arrows $\alpha_1, \ldots, \alpha_{l_1}, \beta_1, \ldots, \beta_{l_2}$ or of the form e_m for some vertex m.

(e) We have $M(w') \subseteq M(w)$ if and only if there are $1 \le j_1 \le l_1$ and $1 \le j_2 \le l_2$ so that

$$w = \alpha_{j_1}^{-1}\alpha_{j_1-1}^{-1} \ldots \alpha_1^{-1}\beta_1\beta_2 \ldots \beta_{j_2},$$

or $w = e_m$ with $m = e(\alpha_1) = e(\beta_1)$.

Proof. (a) Every non-oriented cycle contains a vertex which is starting point of two arrows.

(b)Since A is a string algebra, there are at most two arrows which end in i by Definition 2.5.4 (2). By 2.5.4 (3), each of those arrows is part of only one maximal path that ends in i.

(c) From definition 2.5.5, it is obvious that every band corresponds to a cycle without relations. Since I is an admissible ideal, every oriented cycle of Q contains a relation in I. By (1), $A = kQ/I$ has no band modules and $\mathsf{ind}\, A$ consists only of string modules.

(d) There are no arrows α, β with $e(\beta^{-1}) = s(\beta) = s(\alpha)$.

(e) This follows from Lemma 2.5.8. $\square$

We use Proposition 4.7.5 to simplify the problem of describing $\mathsf{S}(\mathsf{mod}\, A)$ and start with the definition of a suitable family $\mathcal{M}$:

By Lemma 4.8.2 (2), there are at most two maximal paths without relation that end in a vertex $m \in Q_0$:

Definition 4.8.3. Suppose that there is at most one arrow α with $e(\alpha) = m$. Then there is only one path that is maximal under the paths without relation that ends in m. We denote its length with k_m and set $l_m := 0$.

If there are two arrows that end in m, there are two maximal paths. We denote their lengths with k_m and l_m.

Definition 4.8.4. Let $A = kQ/I$ for some quiver $Q = (Q_0, Q_1)$, $m \in Q_0$, and $M_m := M(w_m)$ be the module with

$$w_m = \alpha_{k_m}^{-1} \alpha_{l_1-1}^{-1} \ldots \alpha_1^{-1} \beta_1 \beta_2 \ldots \beta_{l_m}$$

so that $\alpha_{k_m} \alpha_1$ and $\beta_{l_m} \ldots \beta_1$ are the maximal paths that end in m. By Lemma 2.5.7 (2) and Lemma 4.8.2 (2), this module is well defined.

Furthermore, we define

$$\mathcal{M}_m := \{M \in \text{mod } A \mid M \subset M_m\}.$$

Lemma 4.8.5. *If A is of colocal type, then*

$$\mathsf{S}(\text{mod } A) \cong \prod_{m \in Q_0} \mathsf{S}(\mathcal{M}_m).$$

Proof. We need to prove that the sets $\mathcal{M}_m$, $m \in Q_0$ fulfil the conditions of Proposition 4.7.5:

(M1) is fulfilled by Lemma 4.8.2 (3), (4) and (5); (M2) and (M3) are fulfilled by Lemma 4.8.2 (5). $\qquad\square$

The lattices $\mathsf{S}(\mathcal{M}_m)$ for $m \in Q_0$ have a very simple description: they are all sublattices of Young's lattice, which is defined in [18], p. 58 and Example 3.4.4(b):

Definition 4.8.6. Take a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$ of a natural number, ordered so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The *Young diagram* of λ is an array of squares with n rows and exactly λ_i squares in the i-th row.

These partitions form a lattice Y, ordered by the inclusion order on the Young diagrams. It is called *Young's lattice*.
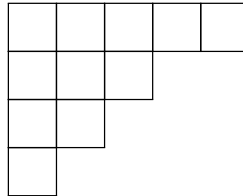
Let $\lambda' := (\lambda'_1, \lambda'_2, \lambda'_3, \ldots, \lambda'_n)$, suppose that $n \leq n'$ and set $\lambda_i := 0$ for $i > n$. Then

$$\lambda \wedge \lambda' = (\min(\lambda_1, \lambda'_1), \ldots \min(\lambda_n, \lambda'_n)))$$

and

$$\lambda \vee \lambda' = (\max(\lambda_1, \lambda'_1), \ldots \max(\lambda_{n'}, \lambda'_{n'}))).$$
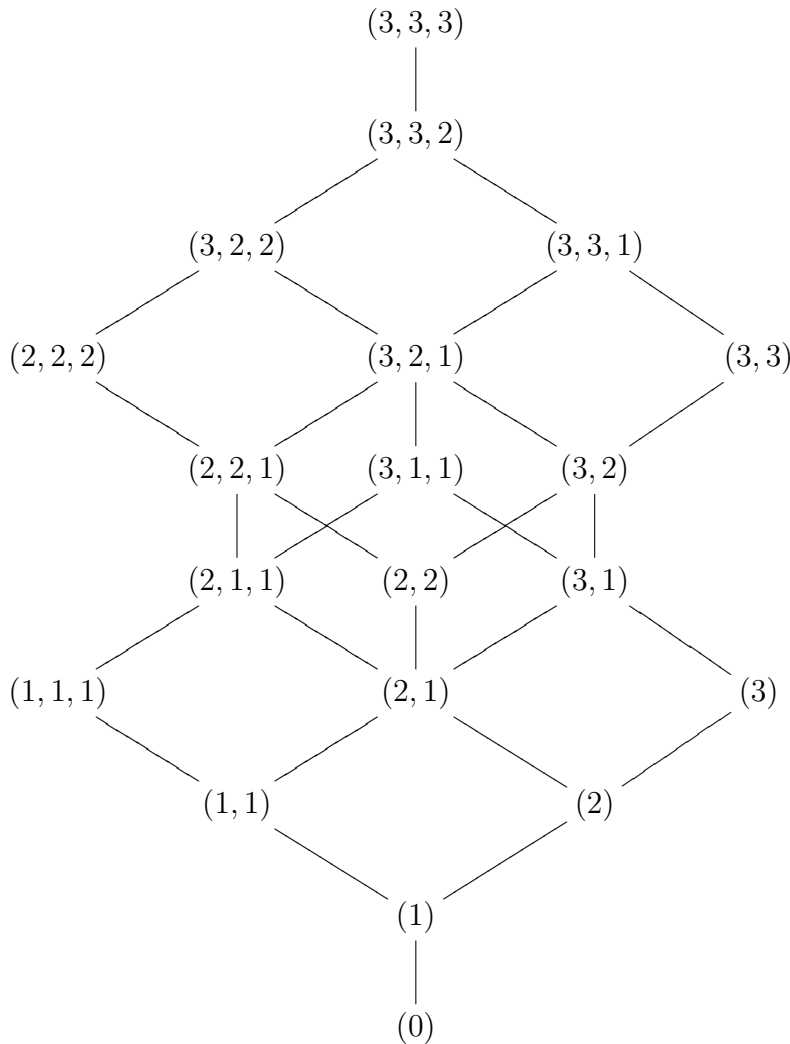
Example 4.8.7. The Young diagram of the partition $(5, 3, 2, 1)$ has the following form:

We will need the following lattices to describe $\mathsf{S}(\mathcal{M}_m)$ for $m \in Q_0$:

Definition 4.8.8. Denote by $Y^{m,n}$ that sublattice of Young's lattice that contains exactly the partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{m'})$ where $m' \leq m$ and $\lambda_i \leq n$ for all $1 \leq i \leq m'$. Equivalently, we can define $Y^{m,n}$ as the lattice given by all Young diagrams with at most m rows and at most n columns.

Example 4.8.9. The Hasse diagram of the lattice $Y^{3,3}$ is

$$
\begin{array}{c}
(3,3,3) \\
| \\
(3,3,2) \\
(3,2,2) \qquad (3,3,1) \\
(2,2,2) \qquad (3,2,1) \qquad (3,3) \\
(2,2,1) \quad (3,1,1) \quad (3,2) \\
(2,1,1) \quad (2,2) \quad (3,1) \\
(1,1,1) \qquad (2,1) \qquad (3) \\
(1,1) \qquad (2) \\
(1) \\
| \\
(0)
\end{array}
$$

Remark 4.8.10. Note that for $m, n \in \mathbb{N}$, we have $Y^{m,n} \cong Y^{n,m}$ and $Y^{1,n} \cong (\{0, 1 \ldots, n\}, <) \cong Y^{n,1}$.

Now, we can completely describe the distributive lattices $\mathsf{S}(\mathrm{mod}\ A)$:

Theorem 4.8.11. *Suppose $A = kQ/I$ with quiver $Q = (Q_0, Q_1)$ and admissible ideal I. If A is of colocal type, then*

$$\mathsf{S}(\mathrm{mod}\ A) \cong \prod_{m \in Q_0} Y^{k_m+1, l_m+1}.$$

Proof. By Lemma 4.8.5,

$$\mathsf{S}(\mathrm{mod}\ A) \cong \prod_{m \in Q_0} \mathsf{S}(\mathcal{M}_m).$$

If only one path $\alpha_1 \alpha_2 \ldots \alpha_{k_m}$ ends in m, it is obvious from Lemma 4.8.2 (5) that we can order the modules in $\mathcal{M}_m$ the following way:

$$M(e_m) \subseteq M(\alpha_1^{-1}) \subseteq M(\alpha_2^{-1}\alpha_1^{-1}) \subseteq \cdots \subseteq M(\alpha_{k_m}^{-1}\alpha_{k_m-1}^{-1} \ldots \alpha_1^{-1})$$

Thus

$$\mathsf{S}(\mathcal{M}_m) \cong (\{0, \ldots, k_m + 1\}, <) = Y^{k_m+1, 1}.$$

If there are two paths without relations $\alpha_1 \alpha_2 \ldots \alpha_{l_2}$ and $\beta_1 \beta_2 \ldots \beta_{l_2}$ are maximal under those that end in m, then by 4.8.2 (5) all modules in $\mathcal{M}_m$ are of the form $M(e_m) =: M(w_{0,0})$ or $M(w_{ij})$ with

$$w_{ij} = \alpha_i^{-1} \alpha_{i-1}^{-1} \ldots \alpha_1^{-1} \beta_1 \beta_2 \ldots \beta_j,$$

with $0 \leq i \leq k_m$, $0 \leq j_2 \leq l_m$ and at least one of them non-zero. Furthermore, $M(e_m) \subseteq M(w_{ij})$ and $M(w_{ij}) \subseteq M(i'j')$ if and only if $i \leq i'$ and $j \leq j'$.

For a submodule closed subcategory $C \in \mathcal{M}_m$, there is some $0 \leq c \leq k_m \in \mathbb{N}$ with

$$l_m \geq j_0 \geq j_1 \geq \cdots \geq j_c \geq 0$$

so that

$$\mathsf{ind}\ C = \{M(w_{ij}) \in \mathsf{ind}\ A \mid \text{there is some } 0 \leq h \leq c \text{ with } i \leq h, j \leq j_h\}.$$

We define

$$\lambda_C := (j_0 + 1, j_1 + 1, \ldots j_c + 1).$$

Then

$$f : \mathsf{S}(\mathcal{M}_m) \to Y^{k_m+1, l_m+1}, C \to \lambda_C$$

is obviously injective and surjective. We need to prove that f is a lattice homomorphism, that is, that it preserves joins and meets: Since $\mathsf{S}(\mathcal{M}_m)$ is distributive, for any two categories $C_1, C_2 \in \mathsf{S}(\mathcal{M}_m)$,

$$\mathsf{ind}(C_1 \wedge C_2) = \mathsf{ind}\ C_1 \cap \mathsf{ind}\ C_2$$

and

$$\mathsf{ind}(C_1 \vee C_2) = \mathsf{ind}\, C_1 \cup \mathsf{ind}\, C_2$$

by Proposition 4.1.3. From the definition of the joins and meets in Y^{k_m+1,l_m+1}, it is clear that f preserves them. $\square$

5 | Conclusion and outlook

The first result of this thesis is that for all hereditary Artin algebras A, there exists a natural bijection between the elements of the Weyl group associated to A and the cofinite submodule closed subcategories in $\mathsf{mod}\,A$. So we have shown that a result for algebraically closed fields holds for arbitrary fields.

Next, we get a new characterization of algebras and abelian length categories of colocal type with three conditions that are simple to check, especially if we work over an algebraically closed field, where the module category of an Artin algebra is equivalent to that of a quiver with relations.

In this case, we can completely describe the lattice $\mathsf{S}(\mathsf{mod}\,A)$ for all A of colocal type. It is the Cartesian product of sublattices of Young's lattice.

While these connections to several topics show the importance of subobject closed subcategories, there are still a myriad of questions that can be asked about them. The most obvious question with regard to the results above is the following: how does a characterization of arbitrary abelian length categories with distributive lattice $\mathsf{S}(\mathcal{A})$ looks like?

But we can also ask other questions: Is there a description for the finite submodule closed categories of a hereditary Artin algebra that is similar to the description of the cofinite ones? Is it possible to give a simple description for the lattices of submodule closed categories of all string algebras A, not only those with distributive lattice $\mathsf{S}(\mathsf{mod}\,A)$? Can we use submodule relations to characterize string algebras?

Glossary

Notation	Description	Page List
$\mid$	we write $X \mid Y$ if X is a direct summand of Y	15
$\nmid$	we write $X \nmid Y$ if X is not a direct summand of Y	15
$\{ss'\}^a$	a shorter notation for the word $\underbrace{ss'ss'\dots}_{a \text{ letters}}$	24
(r,i)	an alternative notation for the module $\tau^r I_i$	36
$<_l$	a total order defined on the words over the alphabet $\{s_1, \dots, s_n\}$	31
A^{op}	the opposite algebra of A	14
(C1) - (C3)	conditions on a triple of sequences of modules	7
$\mathcal{C}_{\underline{w}}$	the category defined by the word $\underline{w}$	37
$\mathcal{C}_w$	the category defined by the Weyl group element w	37
$d^1_S(S,T)$	a shorter notation for $\dim_{\mathsf{End}(S)^{op}} \mathsf{Ext}^1(S,T)$	7
$d^1_T(S,T)$	a shorter notation for $\dim_{\mathsf{End}(T)} \mathsf{Ext}^1(S,T)$	7
$E(m)$	a recursion formula	50
$E'(m)$	a recursion formula that is similar to $E(m)$	54
$\mathsf{Ext}^n(X,X')$	set of equivalence classes of n-fold extensions of X by X'	15
k_m	for a vertex m in a string algebra, this is the length of a maximal path without relations that ends in m	123
l_m	for a vertex m in a string algebra, $l_m = 0$ if at most one arrow has m as end point; otherwise this is the length of the maximal path without relations that ends in m and does not define k_m	123
$l(X)$	the length of the object X	14

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Notation	**Description**	**Page List**
m_{ij}	the defining relations of the Weyl group are $(s_i s_j)^{m_{ij}} = 1$	25
M_k	a certain preinjective module	51
M'_k	a preinjective module that is defined similarly to M_k	54
$\mathsf{mod}\, A$	the category of finitely generated right modules over A	14
$M(w)$	the module defined by the string w	27
$M(w, \phi)$	the module defined by the band w and the linear map ϕ	28
Q_0	the set of vertices of the quiver Q	21
Q_1	the set of arrows of the quiver Q	21
$\rho(\underline{w})$	a series of pairs defined by the word $\underline{w}$	30
(S1) - (S3)	conditions on a triple of sequences of modules	40
(S4) - (S5)	conditions on a triple of sequences of modules	46
(S'1)	condition on a triple of sequences of modules	48
$\mathsf{S}(\mathcal{A})$	the lattice of full additive subobject closed subcategories of $\mathcal{A}$	75
$\mathsf{S}(\mathcal{M})$	for a class of indecomposable objects $\mathcal{M}$, this is a shorter notation for $\mathsf{S}(\mathsf{add}\,\mathcal{M})$	118
$\mathsf{sub}\,\mathcal{X}$	the category that consists of all subobjects of direct sums of objects in the class $\mathcal{X}$	76
$\mathsf{sub}\,X$	for an object X, this is a shorter notation for $\mathsf{sub}\{X\}$	76
τ	the Auslander-Reiten translation	17
$Y^{m,n}$	the lattice that contains the partitions whose Young diagrams have $\leq m$ rows and $\leq n$ columns	125

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