Noise-Induced Synchronization in Circulant Networks of Weakly Coupled Commensurate Oscillators

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To my mother Marianne.

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Mathematics is the music of reason.

James Joseph Sylvester

Abstract

In this thesis we investigate the exchange of energy and the evolution of phase differences in circulant networks of weakly noise-coupled commensurate oscillators. We introduce a generalized synchronization concept called eigenmode synchronization which beyond the classical notions of in-phase and anti-phase synchronization, also distinguishes between other phase-locking configurations corresponding to eigenmodes of the uncoupled system. We examine the interplay of deterministic and multiplicative-noise coupling and in particular verify that the latter can amplify some of the system's eigenmodes. Such an amplification is shown to induce an asymptotic eigenmode synchronization which even persists in the presence of an additive noise perturbation. Application of the Euler-Fermat theorem from number theory, finally allows us to relate a class of circulant noise-coupling topologies to their induced synchronization patterns. Specifically, we will identify critical numbers of oscillators at which these induced synchronization patterns change.

The synchronization results are obtained by studying a complex outer-product process which captures all of the uncoupled system's first integrals. In the weak-coupling limit, this process is shown to satisfy an averaging principle, i.e. after time-rescaling, it weakly converges towards an 'effective' limiting process governed by an averaged drift and diffusion term. This averaging result is proven by adaptation of an averaging principle based on the generalized convergence of Dirichlet forms. Application of the averaging theorem in particular allows us to identify a class of nonlinear perturbations of the drift term which yield a vanishing contribution to the evolution of the effective process.

Contents

1 Introduction

1.1. Synchronization of coupled oscillators

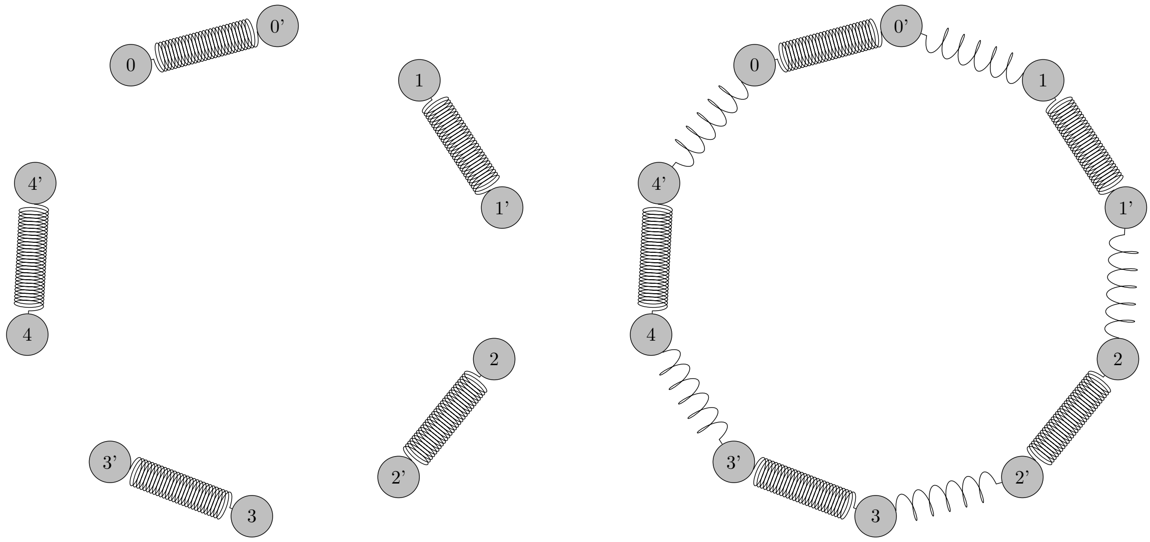
1.1.1. The notion of synchronization

Oscillating systems are an integral aspect of nature. They can be observed on all scales, from the microscopic realm of quantum mechanics, where even particle-like objects as molecules can exhibit a wave-like behavior on the quantum scale,[1] to the macroscopic scale of astrophysics with its rotating and revolving planets, stars and galaxies. At an intermediate scale, oscillations can, for instance, be observed in mechanical systems such as clocks, pendulums or musical instruments ([KPR01]); they occur in electrical circuits, for example in *resistor-inductor circuits* ([DQJQT+15]), and they are also an important part of many biological systems, for instance of the pacemaker cells that give rise to the rhythm of the human heartbeat ([MMJ87]). Both natural and man-made oscillatory systems are usually comprised of many oscillating elements (oscillators) which are interacting with one another ([KPR01]). As a simple example of such a system one can consider a collection of pendulums or spring oscillators, which are interacting with one another by means of a weak coupling, for instance realized by interconnecting springs. This is illustrated in Fig. 1.1 which depicts a *ring* of five spring oscillators. The oscillators consist of pairs of massive particles k and k' which are connected by a strong coupling, represented in the figure by 'tightly curled' springs (Fig. 1.1a). Each of these oscillators is weakly coupled to its two neighbors (*nearest-neighbor* coupling), illustrated in Fig. 1.1b by 'loosely curled' springs.

One can similarly consider a pair of metronomes "placed on a freely moving base" ([Pan02]) or a pair of pendulum clocks on a wooden structure.[2] Here, the coupling is provided by the moving base or the wooden structure respectively, which allow for a transfer of energy between the metronomes or clocks. In all of these examples, one can observe that the oscillators begin to align their movement until they eventually oscillate coherently - either in unison, i.e. with a vanishing phase difference ([PONA16]), or with opposed phases ([ADGK+08]), corresponding to a phase shift of $\pm\pi$. This is referred to as an *in-phase* and *anti-phase synchronization* of the oscillators, respectively.

[1] c.f. for instance [JMM+12] where a quantum interference of large organic molecules was demonstrated

[2] In [BSRW02] and [PONA16] the original experiment of Christiaan Huygens [Huy86] was recreated and analyzed.



(a) Five uncoupled harmonic oscillators

(b) Weak nearest-neighbor coupling of oscillators

Figure 1.1.: Ring of oscillators

Synchronization is a universal phenomenon which describes the emergence of order induced by the interactions within a system of coupled oscillators (c.f. [MMO04], Introduction). The core mechanism of synchronization is the “adjustment of rhythms due to an interaction” ([KPR01], Preface). A historical account of the research on synchronization can be found in the introductions of [KPR01] and [ADGK+08].

Synchronization not only occurs in the mechanical examples stated above, but is ubiquitous in physics, chemistry and biology (c.f. [KP09], Section 1). On an astronomical scale we can observe a synchronization of the moon’s spin with its orbiting time, which is why we always see the same side of the moon. Here, the coupling link is given by the gravitational pull of the tides (c.f. [Str12], Chapter 1). In the realm of biology, a prime example is given by the observation of thousands of fireflies, which in some South Asian forests can be observed to flash in unison (c.f. [ABV+05], Introduction). Without one firefly being a designated ‘conductor’, studies have shown the synchronized blinking to be the result of each firefly adjusting its rhythm to the one observed by the light signals of its *neighboring* fireflies (c.f. [Str03]). Similarly, pacemaker cells are coupled to their neighboring cells and can be observed to synchronize. Unlike a single pacemaker cell, this combined synchronized system is able to “generate an impulse with sufficiently high current” ([Osa17]) to cause the contractions of the heart (c.f. [MMJ87] and [Osa17]). A comprehensive review of research articles on applications in biology, medicine, chemistry, social sciences and engineering can be found in [DB14], Section 1.2.

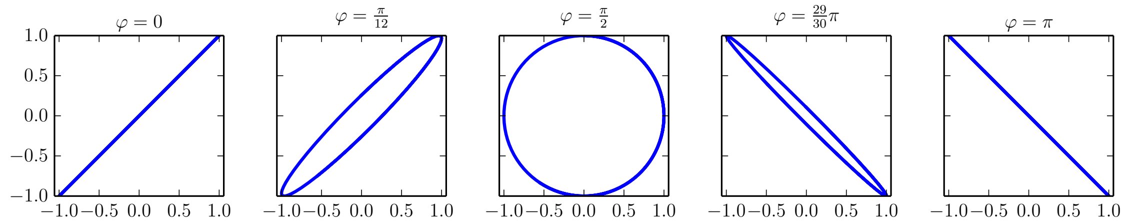


Figure 1.2.: Combined evolution of two phase-locked oscillators at phase difference φ

1.1.2. Characterization of synchronization

1.1.2.1. Phase synchronization and phase-locking

We can characterize the synchronization of a system of n oscillators in terms of their *phase differences*. As previously noted, asymptotically vanishing phase differences correspond to an oscillatory system where eventually all oscillators move in unison. This will be called (asymptotic) *in-phase synchronization*[3] or *complete phase synchronization*.[4] If the phase differences between all oscillators only approach constant but nonvanishing values, this is referred to as (asymptotic) *phase-locking*[5] or *complete frequency synchronization*.[6]

Synchronization and phase-locking of a two-oscillator system can be visualized by representing the combined evolution of the oscillators' elongations in a two-dimensional plot. If for instance we assume that both oscillators have the same frequency and are phase-locked at a phase difference of φ, we obtain a so-called *Lissajous figure*.[7] Fig. 1.2 illustrates the φ-dependence of this combined evolution. In the case of $\varphi = 0$ (*in-phase synchronization*) we obtain a degenerate Lissajous figure along the '$x = y$' line. On increasing the phase difference, this line turns into an ellipse, until for $\varphi = \pi$ (*anti-phase synchronization*) the Lissajous curve is again degenerate, running along the '$x = -y$' line.[8] Note that, for all values of φ, we obtain a closed Lissajous curve since the oscillators were assumed to evolve at the same frequency.[9]

1.1.2.2. Eigenmode synchronization

In a *two*-oscillator system we have distinguished between two particular modes of synchronization, namely in-phase and anti-phase synchronization, which were defined as phase-locking at a phase difference of $\varphi = 0$ or $\varphi = \pm\pi$, respectively. In the previous section we have already generalized the notion of in-phase synchronization to a larger number of oscillators by requiring that the phase differences between all oscillators (asymptotically) vanish. For anti-phase synchronization there is no such immediate generalization, since for $n > 2$ it is impossible for all $\binom{n}{2}$ oscillator pairs

[3] c.f. [Gre10], Chapter 7
[4] c.f. [CHY11], Definition 2.1
[5] c.f. [AR04], Introduction
[6] c.f. [CHY11], Definition 2.1
[7] c.f. [Arn89], Chapter 2, Section 5, Example 2
[8] c.f. [KPR01], Section 3.1.3
[9] A Lissjous curve is closed if and only if the frequency ratio is given by a rational number, i.e. if the oscillators have *commensurate* frequencies, c.f. [DI07], Fig. 1.

to achieve a phase difference of $\pm\pi$. In the special case of a 'ring-like' coupling (c.f. Fig. 1.1b) of an *even number* of oscillators, we can define a meaningful notion of anti-synchronization by requiring that *neighboring* oscillators[10] evolve with opposite phases.

As illustrated in this case of anti-synchronization, a definition of suitable synchronization modes in terms of oscillator phase differences is not always apparent and may depend on the coupling topology. However, all of the previous definitions can be captured and naturally generalized by looking at the *eigenmodes* of a coupled system of identical harmonic oscillators. Note in particular, that in the case of two oscillators (of frequency κ), the *eigenmodes*[11]

$$x_0(t) \coloneqq \mathrm{e}^{i\kappa t}\begin{pmatrix}1\\1\end{pmatrix}, \quad x_1(t) \coloneqq \mathrm{e}^{i\kappa t}\begin{pmatrix}1\\-1\end{pmatrix}, \tag{1.1}$$

correspond exactly to what we have called in-phase and anti-phase synchronization, i.e. to phase-locking at a phase difference of $\varphi = 0$ and $\varphi = \pm\pi$. The system can thus be said to synchronize, if it eventually evolves according to one its eigenmodes. Similarly, a ring of n *nearest-neighbor* coupled oscillators of frequency κ turns out to have n eigenmodes given by[12]

$$x^{(l)}(t) \coloneqq \mathrm{e}^{i\kappa t}\left(1, \mathfrak{u}^l, \mathfrak{u}^{2l}, \ldots, \mathfrak{u}^{(n-1)l}\right)^\top, \quad l \in \{0, ..., n-1\}, \tag{1.2}$$

where $\mathfrak{u} \coloneqq \exp\left(\frac{2\pi i}{n}\right)$ denotes the n-th unit root (c.f. Section 2.2.1). These eigenmodes correspond to eigenvectors of the discrete Fourier transform (c.f. Section 2.2) and are in some contexts referred to as *phonons*.[13] The zeroth eigenmode

$$x_0(t) = \mathrm{e}^{i\kappa t}\left(1, \ldots, 1\right)^\top \tag{1.3}$$

can be identified as in-phase synchronization of all oscillators and in the case of an *even* number of oscillators, the $\frac{n}{2}$'th eigenmode

$$x_{\frac{n}{2}}(t) = \mathrm{e}^{i\kappa t}\left(1, -1, \ldots, 1, -1\right)^\top \tag{1.4}$$

represents an anti-phase synchronization as described above. If we denote by $\phi_j^{(l)}(t)$ the phase of the j'th oscillator evolving according to the l'th eigenmode, i.e.

$$\phi_j^{(l)}(t) = \kappa\, t + \left(\frac{2\pi}{n}\right) l\, j, \tag{1.5}$$

we find that neighboring oscillators exhibit a *constant* phase difference of

$$\varphi^{(l)} \coloneqq \phi_{j+1}^{(l)}(t) - \phi_j^{(l)}(t) = \frac{2\pi l}{n}. \tag{1.6}$$

[10] where the *neighborhood* relation is defined along this ring; c.f. Section 3.2.1.2 for a formal definition
[11] c.f. [Gre10], Chapter 7, i.p. Fig. 7.2 and [Pes06], Chapter 3.2
[12] c.f. [Pes06], Section 3.2
[13] c.f. [MA94], Eq. (1.7)

In this way, all of the eigenmodes can be characterized by the constant phase difference, which is referred to as *phase cohesiveness*.[14] Fig. 1.3 illustrates these constant phase relations for the case of a four-oscillator system. Note that eigenmodes 1 and $n-1$ only differ by reversal of the orientation, i.e. they can be interpreted as an *eigenmode pair*, related through a reflection mapping. For a general number $n \geq 3$ of oscillators, one similarly finds that eigenmode k and $n-k$ coincide up to an orientation reversal. This 'eigenmode pairing' will play an important role in studying the evolution of the noise-coupled system, c.f. Section 6.2.2.

As in the two-oscillator case, we say that an n-oscillator system synchronizes if it 'eventually' evolves according to one of its eigenmodes, c.f. Definition 6.31 for a formal definition. We will refer to this generalized concept as *eigenmode synchronization*[15] or as synchronization towards a particular eigenmode. Note that this is a generalization of the classical notion of synchronization, which usually only encompasses the special cases of in-phase and anti-phase synchronization (c.f. Section 1.3).

1.2. Related work on symmetries and conserved quantities

The study of dynamical systems can be greatly facilitated by identifying its symmetries and the corresponding conserved quantities. In this section we will review some work on this subject which will prove relevant for this thesis.

1.2.1. Coupling topologies and their symmetries

A dynamical system $(x(t))_{t\geq 0}$ on a manifold M, specified by an ordinary differential equation (ODE) of the form

$$\dot{x}(t) = f(x), \tag{1.7}$$

is said to exhibit a Γ *symmetry*[16] *for a symmetry group* Γ *acting on the manifold* M, if[17]

$$f(\gamma x) = \gamma f(x), \quad \forall x \in M, \gamma \in \Gamma. \tag{1.8}$$

In the following, we recall results on the symmetries of a system of n identical oscillators. Such a system can exhibit both discrete symmetries, corresponding to an invariance under exchange or relabeling of the oscillators, and continuous symmetries, capturing for instance a rotational symmetry.

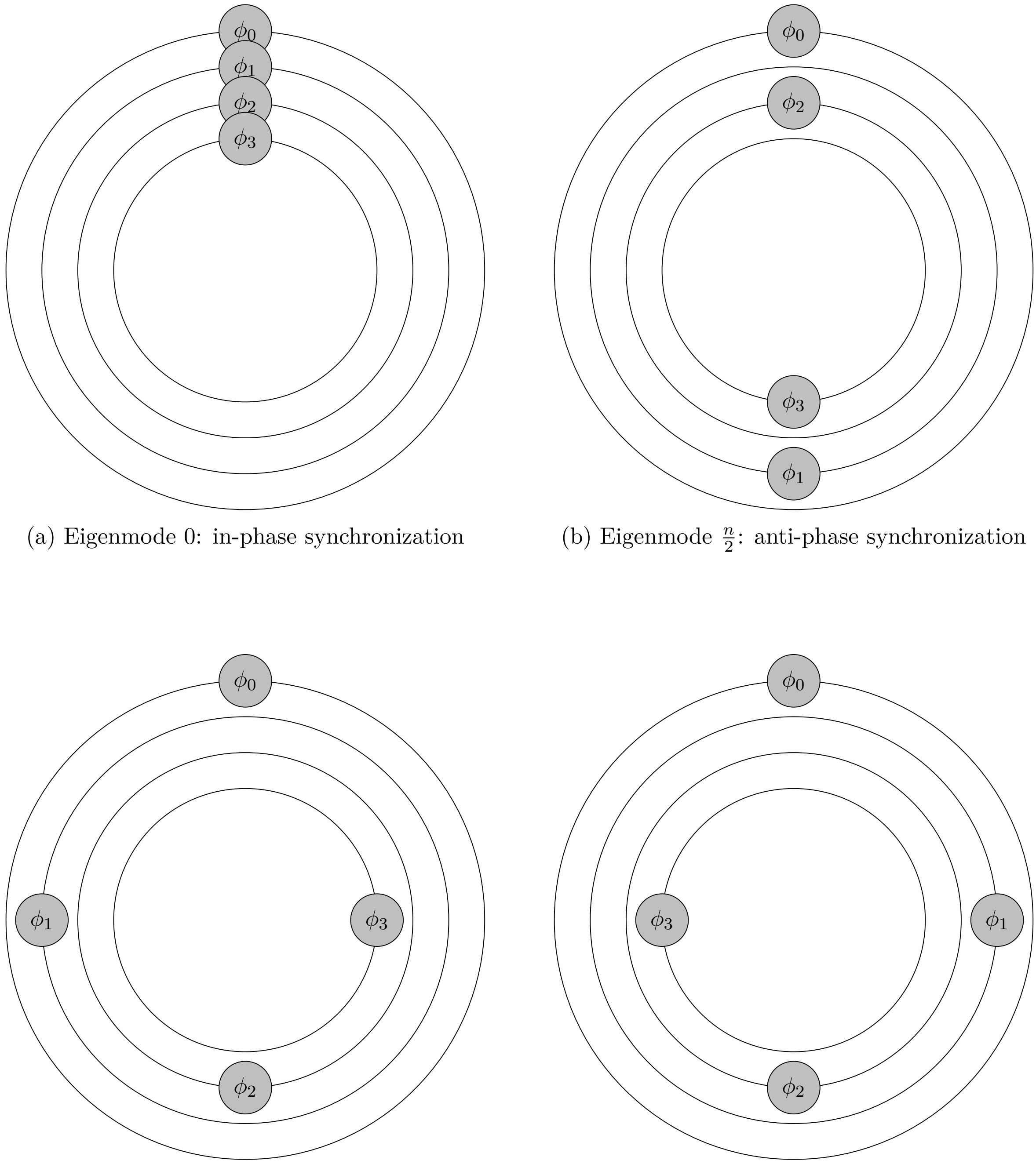
Discrete symmetries In [AKS90], examples for the discrete symmetry groups $S_n, \mathbb{Z}_n$ and D_n are given. Here, S_n denotes the *symmetric group* of all permutations, $\mathbb{Z}_n := \mathbb{Z}/(n\mathbb{Z})$ the *cyclic group*, generated by cyclic permutations and D_n the *dihedral group*, generated by cyclic permutations

[14] c.f. [DB14], Section 3.1 and Fig. 5, b)

[15] Note that this term does *not* refer to a synchronization of two eigenmodes, as for instance used in the theory of phonation, c.f. [Zha11]. In the context of this thesis it is rather used to refer to a synchronization procedure that (asymptotically) drives the system towards one of its eigenmodes.

[16] more precisely one says that the ODE is *equivariant under the action of* Γ

[17] [AS92], Definition 1.3



(a) Eigenmode 0: in-phase synchronization

(b) Eigenmode $\frac{n}{2}$: anti-phase synchronization

(c) Eigenmode 1: counter-clockwise 'wave'

(d) Eigenmode $n-1$: clockwise 'wave'

Figure 1.3.: Eigenmode phase-locking configurations for $n = 4$ oscillators

and a reflection mapping (c.f. [BFG07]). If oscillators are either uncoupled or if we have a global all-to-all coupling, then the discrete symmetries of this system are represented by the symmetric group S_n, since any exchange of oscillators leaves the system invariant. A *ring* of oscillators (*nearest-neighbor coupling*, c.f. Section 3.2.1.2) by contrast exhibits a $\mathbb{Z}_n$ symmetry if there is a preferred direction and a D_n symmetry if the coupling is symmetric, i.e. if the ring is invariant under a reflection mapping which inverts the order of the oscillators.

Continuous symmetries If the oscillator coupling only depends on the *differences* of the oscillator phases (as for instance in the *Kuramoto model*, c.f. Section 1.3.1 below), then the system is invariant under a *global phase shift*, which corresponds to a time-shift of the system ([AS92]). In this case we have an S^1 symmetry, where $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$.[18]

In [AW02], Section V, an n-dimensional *commensurate oscillator* is studied, which is defined as a system of n uncoupled one-dimensional harmonic oscillators, whose frequencies have a rational dependence. The Hamiltonian of this system can be written as[19]

$$H(q,p) = \frac{\omega}{2} \sum_{k=0}^{n-1} \frac{1}{m_k} (p_k^2 + q_k^2), \tag{1.9}$$

where (q_k, p_k) denotes the position and momentum of the k'th oscillator and $m_k \in \mathbb{N}$ characterizes the k'th oscillator's frequency compared the other rationally dependent frequencies. In the special case of $m_k = 1$ for all k, this reduces to the Hamiltonian of an *isotropic* n dimensional oscillator, i.e. to a system of n identical uncoupled harmonic oscillators. As remarked on in [AW02], in addition to the global phase-shift discussed above, this system is furthermore invariant under $SU(n)$ transformations, which we will discuss in more detail in Sections 3.1.2.1 and 3.1.2.2 below.

1.2.2. Conserved quantities

The *Noether theorem* implies that continuous symmetries can be related to conserved quantities of the system, c.f. Section 3.1.2.3. For a commensurate oscillator (c.f. Eq. (1.9)), explicit expressions for these conserved quantities are given in [AW02]. In the two-dimensional case they can be constructed as *sesquilinear* expressions[20] of the form

$$c_l := \frac{1}{2}\, x^\dagger \, \sigma_l \, x, \quad l = 0, ..., 3, \tag{1.10}$$

where $\sigma_0 = \mathbb{1}_{2\times 2}$ and σ_l, $l = 1, 2, 3$, are the *Pauli matrices*. The $\mathbb{C}^2$-valued vector x is defined by[21]

$$x_k := \sqrt{\frac{I_k}{m_k}}\, \mathrm{e}^{i\, m_k \varphi_k}, \quad k \in \{0, 1\}, \tag{1.11}$$

[18] c.f. [AS92], Section 1.2 and c.f. Section 3.1.2.2 in this thesis
[19] [AW02], Eq. (60)
[20] c.f. [AW02], Eq. (26), adapted notation
[21] c.f. [AW02], Eq. (20)

where I_k, φ_k are the *action-angle* variables parametrizing the evolution of the k'th oscillator, i.e.

$$\frac{1}{\sqrt{2}}(q_k + p_k) = \sqrt{I_k}\, \mathrm{e}^{i\,\varphi_k}, \quad k \in \{0,1\}\,. \tag{1.12}$$

For the two-dimensional *isotropic* oscillator we will characterize these conserved quantities I_k in terms of energy and phase differences between the oscillators and relate them to the notion of in-phase and anti-phase synchronization, c.f. Example 3.6. Similarly, Eqs. (69-71) from [AW02] provide sesquilinear expressions for the conserved quantities of an n-dimensional commensurate oscillator, which are constructed by mimicking quantum mechanical creation and annihilation operators. A discussion for the n-dimensional isotropic oscillator follows in Eq. (3.48), where we will identify these quantities with the components of a complex outer-product construction.

1.3. Related work on synchronization

We recall prominent oscillator- and phase-coupling models as well as established results on their synchronization behavior. In Section 1.6, we will subsequently discuss how the model studied in this thesis is related to the models presented in this section.

1.3.1. Kuramoto model

1.3.1.1. Standard model and generalizations

The archetypal model for studying synchronization effects is the so-called *Kuramoto model*.[22] It describes the phase evolution of a population of n (limit-cycle) oscillators, where the phases θ_i interact by means of a *sinusoidal coupling* of the form[23]

$$\dot{\theta}_i = \omega_i + \frac{K}{n}\sum_{j=0}^{n-1}\sin(\theta_j - \theta_i), \quad i \in \{0, ..., n-1\}\,. \tag{1.13}$$

Here the $\omega_i \in \mathbb{R}$ denote the oscillator frequencies, K the global coupling strength and n the number of oscillators. This model describes the *mean-field* case of an all-to-all coupling, i.e. we are considering a *complete coupling graph* (c.f. [RPJK16]). Synchronization is frequently measured in terms of a process called *order parameter*, which is defined as

$$r(t) := \left|\frac{1}{n}\sum_{j=0}^{n-1}\mathrm{e}^{i\,\theta_j(t)}\right| \in \{0,1\}\,. \tag{1.14}$$

This order parameter equals one, if and only if all phases are aligned, corresponding to in-phase synchronization. By contrast it vanishes if the phases are ‘evenly’ distributed along the circle, s.t. the center of mass of the points $\mathrm{e}^{i\,\theta_j}$ vanishes.

[22] A review of the history of the Kuramoto model is given in [Str00]. An overview of research on the Kuramoto model can be found in the reviews [ABV+05], [GCR14a] and [RPJK16].

[23] c.f. [Str00]

Remark 1.1 (Vanishing order parameter for eigenmode phase-locking)
Note that the order parameter in particular vanishes in the case of anti-phase synchronization. This applies more generally to any phase-locking state corresponding to an eigenmode $l > 0$, i.e. to the case of

$$\theta_j(t) = \phi_j^{(l)}(t) = \kappa\, t + \left(\frac{2\pi}{n}\right) l\, j, \tag{1.15}$$

as given by Eq. (1.5). For the order parameter we consequently find that

$$r(t) = \left| \frac{1}{n} \sum_{j=0}^{n-1} \mathrm{e}^{i\phi_j^{(l)}(t)} \right| = \left| \frac{1}{n} \sum_{j=0}^{n-1} \mathrm{e}^{i\left(\frac{2\pi l j}{n}\right)} \right| = 0, \quad \forall l \in \{1, ..., n-1\}\,, \tag{1.16}$$

where we have employed a 'geometric series'-argument.[a] This implies that the order parameter is not suitable for distinguishing between a random distribution of phases, which happen to cancel in the sense of Eq. (1.14), and an eigenmode phase-locking pattern, as for instance anti-phase synchronization.

[a] c.f. Eq. (2.64)

General coupling topologies The standard Kuramoto model can be generalized to the case of a more general coupling topology. In [ADGK+08], the authors for instance replace the global coupling constant K by "weighted interaction factors" K_{ij} and a *connectivity matrix*[24] C which yields a system of the form

$$\dot{\theta}_i = \omega_i + \sum_j K_{ij}\, C_{ij} \sin(\theta_j - \theta_i), \quad i \in \{0, ..., n-1\}\,. \tag{1.17}$$

The constants K_{ij} thus denote the coupling strength with which oscillator j influences oscillator i, which only takes effect if the entry C_{ij} connectivity matrix is non-vanishing, i.e. equal to one. Note that Eq. (1.17), just like the standard Kuramoto model, is given by a superposition of *pair-coupling terms* of the form $\sin(\theta_j - \theta_i)$. This is generalized in [JMB05], in which a system of the form

$$\dot{\theta} = \omega - \frac{K}{n}\, B\, \sin(B^\top \theta) \tag{1.18}$$

is considered[25] , where B is the "incidence matrix of the unweighted graph" and θ, ω denote the vectors of phases and frequencies, respectively. In components, this can be written as

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j,k} B_{ij} \sin(B_{kj}\, \theta_k), \quad i \in \{0, ..., n-1\}\,, \tag{1.19}$$

i.e. the argument of the sinusodial weight function is allowed to depend on a linear combination of all angles θ_k, rather than just on a difference of the form $\theta_j - \theta_i$. In [CM09], Eq. (1.18) is

[24] c.f. [ADGK+08], Section 3.1.2., where we have adapted the notation
[25] c.f. [JMB05] Eq. (4) and [CM08], Section 3

studied in the special case of a "ring-like coupling structure", i.e. in the case of *nearest-neighbor* coupling.

Inertial Kuramoto model with interaction frustration The Kuramoto model has also been generalized to include the effects of *inertia* by extending it to a *second-order system* of the form[26]

$$m_i\,\ddot{\theta}_i + d_i\dot{\theta}_i = \omega_i + \sum_{j=0}^{n-1} K_{ij}\sin(\theta_j - \theta_i), \quad i \in \{0, ..., n-1\}\,, \tag{1.20}$$

where m_i denotes the *mass* of the i'th oscillator, d_i represents the associated friction coefficient and $K_{ij} = K_{ji}$ the coupling strength between oscillators i and j. In [HKL14], the inertial model is specialized to the case of a global coupling topology, but it allows for an *interaction frustration* of the form

$$m_i\,\ddot{\theta}_i + \dot{\theta}_i = \omega_i + \frac{K}{n}\sum_{j=0}^{n-1}\sin(\theta_j - \theta_i + \alpha), \quad i \in \{0, ..., n-1\}\,, \tag{1.21}$$

where $\alpha \in \mathbb{R}$ is the so-called frustration parameter.

1.3.1.2. Synchronization results

Infinite dimensional system The Kuramoto model is frequently studied in the limit of a large number of oscillators, i.e. in the limit of $n \to \infty$, see for instance [SM91], [AS98], [ABS00], [OA08], [HS11b], [HS12] and [GPP12], [GCR14b]. Note that the scaling of the coupling constant as $\frac{K}{n}$ plays a crucial role in this limiting procedure. In the $n \to \infty$ limit, a *self-consistency condition*[27] for the order parameter and a *Fokker-Planck-type* equation[28] for the probability density of the oscillator distribution are studied.

The self-consistency approach is based on the assumption that the oscillators converge to a stationary distribution and ultimately exhibit a constant order parameter.[29] From the consistency condition one can derive a critical coupling strength K_c, which depends on the distribution of the frequencies ω_i. Below this critical value, the order paramter vanishes, and above, it continuously increases as[30]

$$r = \sqrt{1 - \frac{K_c}{K}}. \tag{1.22}$$

In the limit of $K \to \infty$, the order parameter thus approaches the value of $r = 1$, corresponding to in-phase synchronization.

[26] c.f. [CHY11] and i.p. [DB14], Section 1.1, Eq. (2), with slightly adapted notation
[27] c.f. [Str00] for an application of this condition
[28] c.f. for instance [AS98] and [ABS00]
[29] c.f. [Str00]; in [HS11a] a rigorous proof of the validity of this assumption can be found
[30] c.f. [Str00], Section 4

Finite dimensional system For the standard Kuramoto model, a general asymptotic phase-locking result is given in [HKR16]. Under the assumption of $\frac{1}{n}\sum_i \omega_i = 0$, $\frac{1}{n}\sum_i \theta_i(0) = 0$ and given a non-vanishing initial order parameter, the existence of a critical coupling strength K_c is proven. This critical coupling strength is characterized by the fact that for $K \geq K_c$, the system $\theta(t)$ always asymptotically approaches a phase-locking configuration θ^∞, i.e.

$$\lim_{t\to\infty} \|\theta(t) - \theta^\infty\|_\infty = 0. \tag{1.23}$$

In the setup of a more general coupling topology, as specified in Eq. (1.18), [JMB05] provides the following phase-locking result. If the oscillator frequencies ω_i are identical, then for all coupling topologies and any coupling strength $K > 0$, the system asymptotically approaches a phase-locking state ([JMB05], Theorem 1). If the oscillator frequencies do not coincide, phase-locking is achieved under the additional assumption of K surpassing a critical coupling constant, which is explicitly given in terms of the coupling topology and the vector of frequencies ([JMB05], Theorem 2).

In [CM09], Eq. (1.18) is studied in the special case of a *ring* of identical oscillators. It is shown that a ring of *five* or more oscillators does *not* allow for *in-phase* synchronization ([CM09], Theorem 6.1).

Synchronization results for the inertial Kuramoto model (with frustration) are provided in [CHY11] and [HKL14]. In the case of identical oscillators, the system exhibits a complete phase-frequency synchronization, i.e. in-phase synchronization[31], provided that the product mK of mass and coupling strength satisfies a certain bound and that the initial conditions are 'restricted to a half-circle'.[32] For non-identical oscillators, in-phase synchronization is not possible but the system achieves complete frequency synchronization, i.e. phase-locking, again assuming certain bounds on mK and the initial conditions.

1.3.2. Noise influence

In this section we discuss research on the influence of noise on the synchronization process.

1.3.2.1. Persistence of synchronization under additive noise

An additive-noise term can be considered as a random perturbation of the system, resulting for instance from a noisy environment. One might expect such an 'disordered' perturbation to be detrimental when it comes to a system achieving synchronization. Nevertheless, there are results showing that a synchronization behavior can persist under an additive-noise perturbation. In [CK05] for instance, two linearly coupled dynamical systems of the form

$$\mathrm{d}X(t) = f(X(t))\,\mathrm{d}t + K(Y(t) - X(t))\,\mathrm{d}t, \tag{1.24a}$$
$$\mathrm{d}Y(t) = g(Y(t))\,\mathrm{d}t + K(X(t) - Y(t))\,\mathrm{d}t, \tag{1.24b}$$

[31] recall Section 1.1.2.1

[32] In the case of a non-vanishing frustration α, one has to impose slightly stronger assumptions on the initial conditions, c.f. [HKL14]

are analyzed, where $X(t), Y(t)$ are $\mathbb{R}^n$-valued processes, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are required to satisfy a *one-sided dissipative Lipschitz condition* and $K > 0$ denotes the strength of the linear coupling between $X(t)$ and $Y(t)$. Both dynamical systems are subsequently perturbed by isotropic additive noise, i.e.

$$\mathrm{d}X(t) = f(X(t))\,\mathrm{d}t + K(Y(t) - X(t))\,\mathrm{d}t + \alpha\,\mathrm{d}B_1(t), \tag{1.25a}$$
$$\mathrm{d}Y(t) = g(Y(t))\,\mathrm{d}t + K(X(t) - Y(t))\,\mathrm{d}t + \beta\,\mathrm{d}B_2(t), \tag{1.25b}$$

where $B_1(t)$ and $B_2(t)$ are independent $\mathbb{R}^n$-valued Brownian motions. It is shown that the processes $X(t)$ and $Y(t)$ synchronize in the limit of $K \to \infty$, in spite of the additive-noise perturbation. More precisely, it is shown in [CK05] that for every $K > 0$, the system Eq. (1.25) admits a unique stationary solution $(\hat{X}^K(t), \hat{Y}^K(t))$, that pathwise converges on compact time intervals, i.e. for all $T_1 \leq T_2$ we have

$$\lim_{K\to\infty} \sup_{t\in[T_1,T_2]} \left| \begin{pmatrix} \hat{X}^K(t) \\ \hat{Y}^K(t) \end{pmatrix} - \begin{pmatrix} \hat{Z}(t) \\ \hat{Z}(t) \end{pmatrix} \right| = 0, \quad \mathbb{P}\text{-a.s.} \tag{1.26}$$

Here $\hat{Z}(t)$ is the stationary solution of

$$\mathrm{d}\hat{Z}(t) = \frac{1}{2}\left(f(\hat{Z}(t)) + g(\hat{Z}(t))\right)\mathrm{d}t + \frac{1}{2}\left(\alpha\,\mathrm{d}B_1(t) + \beta\,\mathrm{d}B_2(t)\right), \tag{1.27}$$

which can be interpreted as an average of the two SDEs given in Eq. (1.25).

1.3.2.2. Noise-assisted synchronization resulting from multiplicative noise

Roughly speaking, *multiplicative noise* denotes a white-noise perturbation with a strength linearly depending on the perturbed process, i.e. it can be interpreted as a feedback term. As will be illustrated in the examples below, multiplicative noise can have a 'favorable' influence on the system, such as the facilitation of synchronization or the enhancement of energy transport within a system.

Multiplicative-noise coupling Multiplicative noise arises, for instance, if the coupling strength of a deterministic coupling term is subjected to a white-noise perturbation, in which case it can be referred to as "*white-noise-based coupling*" (c.f. [LC06]) or simply as *multiplicative-noise coupling*, c.f. Section 3.3.1. In [LC06], an $\mathbb{R}^n$-valued *driving system* $x(t)$ is studied, which is determined by an ODE of the form[33]

$$\dot{x}(t) = A\,x(t) + f[x(t)]. \tag{1.28}$$

Here $A \in \mathbb{R}^{n\times n}$ is a constant matrix and $f : \mathbb{R}^n \to \mathbb{R}^n$ a nonlinear function, which satisfies a Lipschitz condition. An $\mathbb{R}^n$-valued *response system* $y(t)$ given by[34]

$$\dot{y}(t) = A\,y(t) + f[y(t)] + H[y(t) - x(t)]\,\dot{W}(t) \tag{1.29}$$

[33] [LC06], Eq. (1)
[34] [LC06], Eq. (3)

is coupled to the driving system $x(t)$ by means of a *white-noise-based coupling* $H[y(t)-x(t)]\,\dot{W}(t)$, where $(W(t))_{t\geq 0}$ is an m-dimensional Brownian motion and $H:\mathbb{R}^n\to\mathbb{R}^{n\times m}$ denotes the "*noise-coupling strength function*". By means of a *strong law of large numbers* argument,[35] this noise coupling is shown to induce asymptotic pathwise synchronization[36] of $x(t)$ and $y(t)$, provided certain assumptions on A, f and H are satisfied.[37] More precisely, it is shown that the deviation process $e(t) \coloneqq x(t)-y(t)$ satisfies[38]

$$\limsup_{t\to\infty}\frac{\log|e(t)|}{t}<0, \quad \mathbb{P}\text{-a.s.}, \tag{1.30}$$

which corresponds to the notion of *complete stochastic synchronization* given in [RS16].

All-to-all noise coupling In [XTX12], the theory of [LC06] is applied to a network of n nodes, which are *globally* (i.e. all-to-all) coupled by multiplicative noise, i.e.

$$\dot{x}_i(t) = f(x_i(t)) + (c + d\,\xi(t))\sum_{j=0}^{n-1} a_{ij}x_j, \quad i\in\{0,\ldots,n-1\}\,, \tag{1.31}$$

where $f:\mathbb{R}^n\to\mathbb{R}^n$ determines the dynamics of the uncoupled nodes and is required to satisfy a one-sided Lipschitz condition. The matrix a, defined by

$$a_{ij} = \begin{cases} -(n-1), & i=j,\\ 1, & i\neq j \end{cases} \tag{1.32}$$

specifies the global coupling topology and $\xi(t)=\dot{W}(t)$ denotes a one-dimensional Gaussian white noise. The parameters c and d finally represent the strength of the deterministic- and the noise coupling, respectively. The deviation process $e(t)$ is an $\mathbb{R}^n$-valued process defined by

$$e_i(t) \coloneqq x_i(t) - \hat{x}(t), \tag{1.33}$$

where $\hat{x}(t)$ is determined by an average of the node drift terms[39]

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = \frac{1}{n}\sum_{i=0}^{n-1} f(x_i(t)). \tag{1.34}$$

Provided that c, d and $f(x)$ satisfy certain assumptions given in [XTX12], it is shown that the system achieves complete stochastic synchronization, i.e. the deviation process satisfies Eq. (1.30).

Directed circular noise coupling In [XTS14a], the authors show that a similar result as in the previous paragraph can be obtained in the case of a *unidirectionally* coupled *ring* of three nodes,

[35] c.f. [LC06], Eq. (9)
[36] [LC06], Proposition I
[37] [LC06], Eq. (2), and assumptions (1),(2) of Proposition I
[38] c.f. [LC06], Eq. (12)
[39] c.f. Eq. (1.27), where the evolution of the limiting process is also determined by an averaged SDE

i.e. for a directed *nearest-neighbor coupling* of the form

$$\dot{x}_1(t) = f(x_1(t)) + (c + d\,\xi(t))(x_2(t) - x_1(t)), \tag{1.35a}$$
$$\dot{x}_2(t) = f(x_2(t)) + (c + d\,\xi(t))(x_3(t) - x_2(t)), \tag{1.35b}$$
$$\dot{x}_3(t) = f(x_3(t)) + (c + d\,\xi(t))(x_1(t) - x_3(t)). \tag{1.35c}$$

A generalization to a circular coupling of more than three nodes appears to be challenging, but numerical simulations seem to indicate that a synchronization is possible for a sufficiently strong noise coupling ([XTS14b]).

Noise-assisted energy transport An *experimental* result illustrating a further favorable effect of multiplicative-noise coupling, is presented in [DQJQT+15]. Here, a system of three identical capacitively coupled *RLC* oscillators is studied, i.e. oscillators built from resistor (R), inductor (L) and conductor (C). This system can be modeled by[40]

$$\frac{\mathrm{d}V_n}{\mathrm{d}t} = -\frac{1}{C}\left[i_n + \frac{V_n}{R} + \sum_{m\neq n}^{3} C_{nm}\left(\frac{\mathrm{d}V_n}{\mathrm{d}t} - \frac{\mathrm{d}V_m}{\mathrm{d}t}\right)\right], \tag{1.36}$$
$$\frac{\mathrm{d}i_n}{\mathrm{d}t} = \frac{V_n}{L}, \tag{1.37}$$

where V_n denotes the volatage, i_n the current of the n'th oscillator and the quantities C_{nm} represent the coupling strengths of the capacitive coupling links. One of these coupling links is subjected to a noise signal, i.e.[41]

$$C_{12} \rightsquigarrow C_{12}(t) \coloneqq C_{12}\left[1 + \varphi(t)\right], \tag{1.38}$$

where $\varphi(t)$ is a centered Gaussian process. In the experiment it has been observed that this noise perturbation significantly increases the efficiency of the energy transport between the oscillators.[42] This is referred to as a *noise-assisted energy transport* (c.f. [DQJQT+15]).

1.4. Related work on weakly coupled systems and averaging theory

1.4.1. Weak coupling

In this section we focus on a collection of oscillators, which are *weakly* coupled, i.e. where coupling terms are small compared to the drift terms determining the evolution of the individual oscillators. In [AS92], the authors study a system of identical, weakly coupled oscillators, with a phase evolution given by[43]

$$\dot{\theta}_i(t) = 1 + \varepsilon\, U_i(\theta(t)) + \mathcal{O}(\varepsilon^2), \quad i \in \{0, ..., n-1\}. \tag{1.39}$$

[40] [DQJQT+15], Eqs. (1),(2)
[41] [DQJQT+15], Eq. (3)
[42] More precisely, in the specific setup a relative enhancement of $22.5 \pm 3.6\%$ was observed.
[43] [AS92] with adapted notation

Such a system can be viewed as a “perturbation of an uncoupled system”.[44] In [AS92] it is approximated by an *averaged system* of the form

$$\dot{\theta}_i(t) = 1 + \varepsilon\, \hat{U}_i(\theta(t)), \quad i \in \{0, ..., n-1\}\,, \tag{1.40}$$

where $\hat{U}$ is a phase-averaged version of U given by

$$\hat{U}(t) \coloneqq \frac{1}{2\pi} \int_0^{2\pi} U(\theta(t) + t\,\mathbf{1})\, \mathrm{d}t, \quad \mathbf{1} \coloneqq (1, 1, ..., 1)^\top \in \mathbb{R}^n. \tag{1.41}$$

Averaging approximations such as in Eq. (1.40) can be rigorously phrased in terms of weak convergence results, which even extend to an SDE setup. These convergence results are given by an averaging theory to be discussed in the following Sections 1.4.2 and 1.4.3.

Dipole-dipole interaction A *quantum-mechanical dipole-dipole interaction* between n monomers is studied in [BE11]. The expansion coefficients $c_k(t)$ of the wavefunction $|\Psi(t)\rangle$ in terms of the excitation states $|\pi_k\rangle$, i.e.[45]

$$|\Psi(t)\rangle = \sum_k c_k(t)\, |\pi_k\rangle\,, \tag{1.42}$$

evolve according to[46]

$$i\,\dot{c}_k(t) = \omega_k\, c_k(t) + \sum_l \frac{V_{kl}}{\hbar}\, c_l(t). \tag{1.43}$$

The constants V_{kl} denote the expansion coefficients of a dipole-dipole interaction operator V, i.e.[47]

$$V_{kl} \coloneqq \langle \pi_k |\, V\, |\pi_l\rangle\,. \tag{1.44}$$

The *classical-mechanics* analogue of interacting oscillating dipoles can be modeled by

$$\dot{x}_k(t) = \frac{p_k}{M_k}, \tag{1.45a}$$

$$\dot{p}_k(t) = -M_k\, \omega_k^2\, x_k - \sum_l K_{kl}\, x_l, \tag{1.45b}$$

where $x_k(t)$ denotes the position and $p_k(t)$ the momentum of the k'th oscillator, while K plays the role of a coupling matrix. Representing the k'th oscillator by a complex-valued process[48]

$$\tilde{z}_k(t) \coloneqq \tilde{x}_k(t) + i\tilde{p}_k(t), \tag{1.46}$$

[44] [AS92], Section 1.1
[45] [BE11], Eq. (2)
[46] [BE11], Eq. (6) with a slight change in notation, setting $\omega_k \coloneqq \frac{\epsilon_k}{\hbar}$
[47] [BE11], Eq. (4)
[48] [BE11], Eq. (14)

where $\tilde{x}_k(t)$ and $\tilde{p}_k(t)$ denote rescaled versions of position and momentum of the k'th oscillator, we obtain[49]

$$i\dot{\tilde{z}}_k(t) = \omega_k\, \tilde{z}_k(t) + \sum_l \frac{2V_{kl}}{\hbar} \operatorname{Re}\left(\tilde{z}_l(t)\right). \tag{1.47}$$

Note that the quantum-mechanical evolution of Eq. (1.43) and the classical evolution of Eq. (1.47) only differ by a factor of two and by the *real part*. In the case of identical monomers (i.e. $\epsilon_k = \epsilon_l$) and of a weak coupling, more precisely if[50]

$$\frac{|V_{kl}|}{\hbar} \ll \omega_k, \quad \forall k, l, \tag{1.48a}$$

$$|\omega_k - \omega_l| \ll \omega_k, \quad \forall k, l, \tag{1.48b}$$

both solutions can be shown to be approximately equivalent.[51]

Coupled dipoles with random frequencies In [EB12] the setup of [BE11], i.p. Eq. (1.47), is modified by introducing a noise perturbation of the oscillation frequencies, i.e.

$$\omega_k \quad \rightsquigarrow \quad \omega_k(t) \coloneqq \omega_k + \sqrt{\gamma_k}\, \xi_k(t), \tag{1.49}$$

where $\xi(t)$ is an n-dimensional white noise and $\gamma_k > 0$ a constant factor representing the noise intensity. This perturbation induces an SDE of the form[52]

$$\mathrm{d}z_k(t) = \left(-i\omega_k\, z_n(t) - i\sum_l K_{kl}\operatorname{Re}\left(z_l(t)\right) - \frac{\gamma_k}{2} z_k(t)\right)\mathrm{d}t + \sqrt{\gamma_k} z_k(t)\,\mathrm{d}W_k(t). \tag{1.50}$$

In [EB12] this system is analyzed by studying the *expectation value* of the *complex outer product* $z(t)z^\dagger(t)$, i.e.[53]

$$p_{kl}(t) \coloneqq \mathbb{E}\left(z_k(t)\overline{z}_l(t)\right), \quad \forall k, l, \tag{1.51}$$

which gives rise to a linear matrix-valued ODE.[54] As in [BE11], it is shown that in the case of weak coupling and (nearly) identical frequencies (as specified in Eq. (1.48)), the classical system can be approximated by a quantum-mechanical version, i.e. a version without a 'restriction to the real part'.

[49] This rescaling and 'complexification' step will be illustrated in more detail in Section 3.1.1.
[50] [EB12], Eqs. (32),(33)
[51] We will observe a similar behavior for the weak-coupling limit of a system of identical oscillators, c.f. Section 1.6.2 and Remark 5.6.
[52] [EB12], Eq. (15)
[53] [EB12], Eqs. (14), (18) in adapted notation
[54] [EB12], Appendix

1.4.2. Averaging theory

The approximation of a weakly coupled system by an averaged version, as for instance the approximation of Eq. (1.39) by Eq. (1.40), has been studied in great detail. We recall a series of results by Freidlin, Wentzell and collaborators, which yield precise convergence results in the weak-coupling limit.

1.4.2.1. Hamiltonian as first integral of two-dimensional system

In this section we provide an outline of results on weakly perturbed Hamiltonian systems. In the unperturbed case, the Hamiltonian, i.e. the 'total energy' of the system, is a *conserved* quantity (*first integral*). In the perturbed case, however, the Hamiltonian is in general no longer conserved and its evolution is studied in the weak-coupling limit.

Non-degenerate noise In [DF98] a two-dimensional weakly perturbed system of the form[55]

$$\mathrm{d}X^\varepsilon(t) = \overline{\nabla} H(X^\varepsilon(t))\,\mathrm{d}t + \varepsilon\, U(X^\varepsilon(t))\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B(t), \quad X_0^\varepsilon = x \in \mathbb{R}^2, \tag{1.52}$$

is analyzed, where $B(t)$ denotes a two-dimensional Brownian motion and

$$\overline{\nabla} H(x) = (\partial_{x_2} H(x), -\partial_{x_1} H(x))^\top \tag{1.53}$$

is the so-called *skew-gradient* of the Hamiltonian $H(x)$. Note that $H : \mathbb{R}^2 \to \mathbb{R}$ is a *first integral* of the $\varepsilon = 0$ system, i.e.

$$\mathrm{d}H(X^0(t)) = \nabla H(X^0(t))\ \mathrm{d}X^0(t) = \nabla H(X^0(t))\,\overline{\nabla} H(X^0(t))\,\mathrm{d}t = 0, \tag{1.54}$$

which is a consequence of ∇ and $\overline{\nabla}$ being 'orthogonal'. A transformation to the "fast time"[56] $\frac{t}{\varepsilon}$ yields the time-rescaled process $Y^\varepsilon(t) := X^\varepsilon(\frac{t}{\varepsilon})$, whose evolution is determined by[57]

$$\mathrm{d}Y^\varepsilon(t) = \frac{1}{\varepsilon}\overline{\nabla} H(Y^\varepsilon(t))\,\mathrm{d}t + U(Y^\varepsilon(t))\,\mathrm{d}t + \mathrm{d}B(t), \quad Y^\varepsilon(0) = y \in \mathbb{R}^2. \tag{1.55}$$

Note that the only explicit ε-dependence of the right-hand side of Eq. (1.55) is given by the $\frac{1}{\varepsilon}\overline{\nabla} H(Y^\varepsilon(t))$ term. However, by a similiar argument as in Eq. (1.54) it follows that this part does *not* contribute to the evolution of the time-rescaled system's Hamiltonian $H(Y^\varepsilon(t))$. More precisely, one finds that[58]

$$\mathrm{d}H(Y^\varepsilon(t)) = \nabla H(Y^\varepsilon(t))\ [U(Y^\varepsilon(t)) + \mathrm{d}W(t)] + \frac{1}{2}\,\Delta H(Y^\varepsilon(t))\,\mathrm{d}t. \tag{1.56}$$

In order to study the weak-coupling limit ($\varepsilon \to 0$) of the process $H(Y^\varepsilon(t))$, the space $\mathbb{R}^2$ is mapped to a quotient space Γ (which in this case turns out to be a *graph*) by identifying all points within each connected component of the Hamiltonian's level sets. For each $H' \geq 0$, we can

[55] c.f. [DF98], Eq. (2) with slightly adapted notation
[56] [BG06], Section 1.2
[57] c.f. [DF98], Eq. (3)
[58] [DF98], Eq. (10)

decompose the level set $\{x \mid H(x) = H'\}$ into a finite number of *connected* components denoted by $C_i(H)$, i.e. we obtain a map

$$\pi : \mathbb{R}^2 \to \Gamma, x \to (i(x), H(x)), \tag{1.57}$$

which identifies each point with its representation on the graph Γ. Specifically, $i(x)$ is an index labeling the connected component which contains x, i.e.

$$x \in C_{i(x)}(H(x)). \tag{1.58}$$

The projection π provides a decomposition of the evolution of $Y^\varepsilon(t)$ into a "slow component"

$$\pi^\varepsilon(t) := \pi(Y^\varepsilon(t)) \in \Gamma \tag{1.59}$$

and a "fast" component, given by the evolution of $Y^\varepsilon(t)$ "along the level sets" of the Hamiltonian.[59] The averaging result in [DF98] is now formulated in terms of a weak convergence of Γ-valued processes in the limit of $\varepsilon \to 0$. More precisely, under certain assumptions on the Hamiltonian $H(x)$ and the drift term $U(x)$, it is shown that the "slow" process $\pi^\varepsilon(t)$ *weakly convergences* to a Markov process $\hat{\pi}(t)$, where $\hat{\pi}(t)$ is an 'averaged' or 'effective' process on the graph Γ, specified on the edges of Γ by a generator of the form[60]

$$\hat{L}_i f(i, H) = \frac{1}{2}\hat{A}_i(H)\, f''(i, H) + \hat{U}_i(H)\, f'(i, H). \tag{1.60}$$

Here, $\hat{U}$ and $\hat{A}$ denote an averaged drift term and diffusion matrix, defined by

$$\hat{U}_i(H) = \frac{\oint_{C_i(H)} \left(\nabla H(x) \cdot U(x) + \frac{1}{2}\Delta H(x)\right) \frac{\mathrm{d}l}{|\nabla H(x)|}}{\oint_{C_i(H)} \frac{\mathrm{d}l}{|\nabla H(x)|}}, \tag{1.61}$$

$$\hat{A}_i(H) = \frac{\oint_{C_i(H)} |\nabla H(x)|^2 \frac{\mathrm{d}l}{|\nabla H(x)|}}{\oint_{C_i(H)} \frac{\mathrm{d}l}{|\nabla H(x)|}}. \tag{1.62}$$

On the vertices of the graph Γ, the process is determined by certain *gluing conditions*, specified in [DF98], Eq. (8). The proof of this averaging result is based on a convergence of Laplace transforms, on a tightness result and on the uniqueness of the solution to the martingale problem associated to the generator $\hat{L}$.[61]

Degenerate noise In [FW98], an averaging theory analogous to [DF98] is developed for the special case of a perturbed oscillator[62]

$$\ddot{q}^\varepsilon(t) = -f(q(t)) + \sqrt{\varepsilon}\,\dot{B}(t), \tag{1.63}$$

[59] [FW04], Introduction
[60] c.f. [DF98], Eqs. (5)-(7)
[61] c.f. [FW12], Chapter 8, i.p. Lemma 3.1 (and its proof), as well as Lemma 3.2
[62] [FW98], Eq. (1.1), in slightly adapted notation

which can be rewritten as[63]

$$\mathrm{d}q^\varepsilon(t) = p^\varepsilon(t)\,\mathrm{d}t, \qquad q^\varepsilon(0) = x \in \mathbb{R}, \tag{1.64}$$
$$\mathrm{d}p^\varepsilon(t) = -f(q^\varepsilon(t))\,\mathrm{d}t + \sqrt{\varepsilon}\,dB(t), \qquad p^\varepsilon(0) = y \in \mathbb{R}. \tag{1.65}$$

Here, $B(t)$ denotes a *one*-dimensional Brownian motion, i.e. we are in the case of a *degenerate* noise perturbation, which only acts on one coordinate of the $\mathbb{R}^2$ phase space.

This setup is generalized in [BF00] to include a weak drift term, depending on both position and momentum, as well as a position-dependend dispersion matrix, i.e.[64]

$$\ddot{q}^\varepsilon(t) = -f(q^\varepsilon(t)) + \varepsilon\,U(\dot{q}^\varepsilon(t), q^\varepsilon(t)) + \sqrt{\varepsilon}\,\sigma(q^\varepsilon(t))\dot{W}(t), \tag{1.66}$$

which can similarly be transformed into a two-dimensional first order SDE. As in [DF98], the phase-space $\mathbb{R}^2$ is mapped to a graph Γ and a weak convergence of Γ-valued processes related to the Hamiltonian of the time-rescaled system is proven.[65]

1.4.2.2. Higher dimensional system

In the previous section, an $\mathbb{R}^2$-valued, weakly perturbed system (e.g. an oscillator) was investigated by studying the evolution of its Hamiltonian. This evolution was identified with a Γ-valued process, where Γ is a *graph*. In a higher dimensional case one can proceed similarly by identifying the first integrals of the *unperturbed* system and studying their evolution.

Multiple first integrals and open book space In [FW04], an N-dimensional, weakly coupled system of the form[66]

$$\mathrm{d}X^\varepsilon(t) = V(X^\varepsilon(t))\,\mathrm{d}t + \varepsilon\,U(X^\varepsilon(t))\,\mathrm{d}t + \sqrt{\varepsilon}\,\Sigma(X^\varepsilon(t))\,\mathrm{d}B(t), \tag{1.67}$$

is analyzed, where V is the drift term of the unperturbed system, while U and Σ denote the drift term and dispersion matrix of the weak perturbation. As usual, a time-rescaled version $Y^\varepsilon(t) := X^\varepsilon(\frac{t}{\varepsilon})$ of the perturbed process is examined, which evolves according to[67]

$$\mathrm{d}Y^\varepsilon(t) = \left[\frac{1}{\varepsilon}\,V(Y^\varepsilon(t)) + U(Y^\varepsilon(t))\right]\mathrm{d}t + \Sigma(Y^\varepsilon(t))\,\mathrm{d}B(t). \tag{1.68}$$

While in the previous section the first integral under consideration was given by the Hamiltonian, the setting is now generalized to a *vector*

$$\mathsf{P} : \mathbb{R}^N \to \mathbb{R}^m \tag{1.69}$$

[64] [BF00] Eq. (2.1) with slightly changed notation
[65] c.f. [BF00], Theorem 2.6, Theorem 2.7; more precisely a double limit of $\varepsilon \to 0$ followed by '$\sigma \to 0$' is considered
[66] c.f. [FW04], Eq. (1.2) in slightly adapted notation, setting $N := n + m$
[67] c.f. [FW04], Eq. (1.2); [FW12], Eq. (1.3)

of first integrals of the unperturbed system, i.e.

$$\mathsf{P}(X^0(t)) = \mathsf{P}(X^0(0)), \quad \forall t \geq 0, \tag{1.70}$$

which can equivalently be formulated as[68]

$$\mathrm{D}\,\mathsf{P}|_x\, V(x) = 0, \quad \forall x \in \mathbb{R}^N. \tag{1.71}$$

Identifying all connected components of the level sets of P yields a quotient space Γ, and a projection $\pi : \mathbb{R}^N \to \Gamma$ to this quotient space.

In [FW06] and [FW12], Chapter 9, the authors study the special case of n oscillators determined by[69]

$$\ddot{q}_i = -V_i'(q_i), \quad i \in \{0, ..., n-1\}, \tag{1.72}$$

giving rise to an $\mathbb{R}^{2n}$-valued process[70]

$$X(t) \coloneqq (p_0, ..., p_{n-1}, q_0, ..., q_{n-1})^\top (t). \tag{1.73}$$

After introduction of a weak perturbation, the evolution of $Y^\varepsilon(t) \coloneqq X^\varepsilon(\frac{t}{\varepsilon})$ is represented by an SDE of the form[71]

$$\mathrm{d}Y^\varepsilon(t) = \frac{1}{\varepsilon}\overline{\nabla} H(Y^\varepsilon(t))\,\mathrm{d}t + U(Y^\varepsilon(t))\,\mathrm{d}t + \Sigma(Y^\varepsilon(t))\,\mathrm{d}B(t), \tag{1.74}$$

where $\Sigma \in \mathbb{R}^{2n,2n}$ is a *constant* matrix and

$$\overline{\nabla} \coloneqq \left(\begin{array}{c|c} 0 & -\mathbb{1}_{n\times n} \\ \hline \mathbb{1}_{n\times n} & 0 \end{array}\right) \nabla \tag{1.75}$$

denotes the *skew gradient*. Here the combined Hamiltonian is given by $H(x) \coloneqq \sum_i H_i(q_i, p_i)$, where

$$H_i(q_i, p_i) \coloneqq \frac{p_i^2}{2} + V_i(q_i). \tag{1.76}$$

Eq. (1.74) thus corresponds to Eq. (1.68) in the special case of $N \coloneqq 2n$ and $V(x) \coloneqq \overline{\nabla} H(x)$. Employing the Hamiltonians of the individual oscillators as first integrals, i.e. setting

$$\mathsf{P}_i(X(t)) \coloneqq H_i(q_i(t), p_i(t)), \tag{1.77}$$

[68] c.f. [FW12], Section 9.1

[69] [FW06], Eq. (1.1)

[70] c.f. [FW06], Eq. (1.2), in a slightly reordered form with adapted notation

[71] [FW06], Eq. (1.4) stated as an SDE with a slightly changed notation, absorbing a factor of $\sqrt{\kappa}$ into the dispersion matrix σ

the quotient space Γ decomposes into a Cartesian product of the form

$$\Gamma = \Gamma_0 \times \Gamma_2 \times \dots \times \Gamma_{n-1}, \tag{1.78}$$

where Γ_i denotes the graph defined as the quotient space of the i'th oscillator with respect to the connected level sets of $H_i(q_i, p_i)$. A quotient space given as such a product of graphs is called an *open book space*.[72]

Assumptions and restrictions In order to obtain an averaging result for the projected process $\pi^\varepsilon(t) := \pi(Y^\varepsilon(t))$, Freidlin and Wentzell (c.f. [FW06] and correspondingly [FW12], Section 9.3) impose a series of restrictions on the weakly coupled system given in Eq. (1.74). We state those of the assumptions which will be violated by the noise-coupled oscillator system studied in this thesis and which are *not* required for the averaging result provided in Chapter 4.

Non-resonance assumption.[73] If $\omega_i(x_i)$ denotes the frequency of the i'th unperturbed oscillator starting in $x_i \in \mathbb{R}$, then the set of points $x \in \mathbb{R}^{2n}$, s.t. the frequencies $\omega_1(x_1), \dots, \omega_n(x_n)$ are *rationally dependent*, is required to have a zero Lebesgue measure. In particular, this does not allow for commensurate oscillators.

Bounded drift. The deterministic perturbation term $U(x)$ is required to be bounded.[74] Note that this in particular prohibits linear coupling terms.

Block-diagonal additive noise. The dispersion matrix Σ is required to be a constant matrix of block-diagonal form, "having 2×2 nonzero matrices $\sigma_i = (\sigma_{i;jk})^2_{j,k=1}$ on the diagonal, and 0 elsewhere".[75] This particularly excludes a multiplicative-noise coupling.

Averaging result on open book space Under certain assumptions (i.p. the ones specified in the previous paragraph), $\pi^\varepsilon(t)$ is shown to weakly converge[76] to a process $\widehat{\pi}(t)$ on the open book space Γ, where $\widehat{\pi}(t)$ evolves according to averaged versions of the drift term and the diffusion matrix that govern $\pi^\varepsilon(t)$, together with certain gluing conditions, c.f. [FW06], Section 4 and [FW12], Section 9.3 for more details.

1.4.3. Averaging theory via dirichlet forms

In [BR14] it is shown that averaging results similar to ones discussed in the previous section can be obtained by employing the theory of *Dirichlet forms*[77] and their generalized convergence. The work is in particular based on [MR92], [Hin98], [Sta99] and [Töl06].

In the main part of [BR14], an averaging theorem analogous to the Freidlin-Wentzell results of Section 1.4.2.1 is derived, i.e. a two dimensional, weakly perturbed Hamiltonian system is studied

[72] c.f. [FW12], Chapter 9, Section 1
[73] c.f. [FW06], Section 4 Condition ⋆ and [FW12], Section 9.3 Condition ⋆
[74] [FW06], Proposition 4.1
[75] [FW06], Section 1
[76] [FW12], Section 9.3, Theorem 3.2
[77] c.f. [MR92] for a comprehensive introduction to the theory of Dirichlet forms and [Kas15] for an overview on its history and some recent developements

in the weak-coupling limit. On the 'fast' time-scale, this system is determined by[78]

$$\mathrm{d}Y^\varepsilon(t) = \frac{1}{\varepsilon}\overline{\nabla}H(Y^\varepsilon(t))\,\mathrm{d}t + U(Y^\varepsilon(t))\,\mathrm{d}t + \sigma\,\mathrm{d}B(t), \tag{1.79}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is the Hamiltonian, $U : \mathbb{R} \to \mathbb{R}$ a drift term, "playing the role of a friction term"[79] and $\sigma \in \mathbb{R}$ a constant, c.f. Eq. (1.55). As in Section 1.4.2.1, the Hamiltonian plays the role of a first integral.

In the last section, this setup is generalized to an $\mathbb{R}^N$-valued system resembling Eq. (1.68), i.e. of the form[80]

$$\mathrm{d}Y^\varepsilon(t) = \frac{1}{\varepsilon}V(Y^\varepsilon(t))\,\mathrm{d}t + U(Y^\varepsilon(t))\,\mathrm{d}t + \Sigma(Y^\varepsilon(t))\,\mathrm{d}B(t), \tag{1.80}$$

where $V : \mathbb{R}^N \to \mathbb{R}^N$ is the drift term of the uncoupled system (generalizing the skew-product term), $U : \mathbb{R}^N \to \mathbb{R}^N$ the drift term of the deterministic coupling, $\Sigma : \mathbb{R}^N \to \mathbb{R}^{N,N'}$ the dispersion matrix and $B(t)$ an N'-dimensional Brownian motion. Similar to Eq. (1.69), an m-dimensional vector $\mathsf{P} : \mathbb{R}^N \to \mathbb{R}^m$ of *first integrals* of the uncoupled system is studied, i.e. a vector satisfying[81] $(\mathrm{D}\,\mathsf{P})V = 0$. As in the Freidlin-Wentzell setup, Γ denotes the quotient space induced by identifying all elements within a connected level set of P. The map $\pi : \mathbb{R}^N \to \Gamma$ analogously denotes the corresponding projection mapping to the quotient space Γ.

Assumptions and restrictions We state some of the central restrictions imposed by [BR14].[82]

Boundedness and Lipschitz continuity. The drift and dispersion terms V, U and Σ are required to both be bounded and Lipschitz continuous. As noted in the previous section, this in particular excludes linear coupling terms.

Uniform ellipticity. The diffusion matrix $\Sigma\Sigma^\top$ is required to be uniformly elliptic. This can be ensured by the presence of an isotropic additive-noise term, c.f. Lemma 4.6.

Supermedian measure. The existence of a "supermedian measure" μ on $\mathbb{R}^N$ is assumed. This measure is furthermore required to be preserved under the evolution generated by V.[83] Under certain assumptions, and in particular in the case studied in this thesis (c.f. Lemma 4.9), μ can be chosen as the Lebesgue measure on $\mathbb{R}^N$.

Dirichlet forms and Mosco convergence Unlike in the approach by Freidlin and Wentzell, the proof of the averaging result in [BR14] does not rely on martingale problems and the convergence of Laplace transforms but rather on a convergence of Dirichlet forms. For each ε, the process $Y^\varepsilon(t)$ can be associated with its infinitesimal generator[84] $\mathcal{A}_\varepsilon$, which in turn gives rise to a Dirichlet

[78] c.f. [BR14], Eq. (2) with slightly adapted notation
[79] [BR14], Section 1
[80] c.f. [BR14], Section 5 with adapted notation
[81] c.f. [BR14], Definition 4.29
[82] c.f. [BR14], Assumption 3
[83] c.f. [BR14], Eq. (132) and Remark after Assumption 3
[84] c.f. [BR14], Eq. (136)

form[85] E_ε defined on a subspace of $L^2(\mathbb{R}^N)$ by[86]

$$E_\varepsilon(f,g) \coloneqq -\int_{\mathbb{R}^N} (\mathcal{A}_\varepsilon f)(x) g(x) \mu(\mathrm{d}x). \tag{1.81}$$

The Dirichlet forms E_ε are shown to converge in a generalized sense (also called *Mosco convergence*)[87] to a the *projected Dirichlet form*[88] $\mathcal{E}$, defined on a subspace of $L^2_{\pi_*\mu}(\Gamma)$ by[89]

$$\mathcal{E}(f,g) \coloneqq E_\varepsilon(f\circ\pi, g\circ\pi), \tag{1.82}$$

a definition which turns out to be independent of the choice of $\varepsilon > 0$. The projected Dirichlet form $\mathcal{E}$ can be identified with a process[90] $\widehat{\pi}$ on Γ, which can be shown to evolve according to averaged versions[91] of the drift term and the diffusion matrix of $\pi(Y^\varepsilon)$. The Mosco convergence is shown to imply a convergence of the finite dimensional distributions,[92] which by means of a tightness result[93] is lifted to the process level, giving rise to the aspired weak convergence[94] of the processes $(\pi^\varepsilon(t))_{t\geq 0}$ to the limiting process $(\widehat{\pi}(t))_{t\geq 0}$.

Open steps While the two dimensional case is discussed in detail in [BR14], the generalization to the multidimensional case presented in the last section is only 'sketched' with respect to the identification of the limiting process.[95] For an application of the averaging theory to a specific multidimensional setup, one in particular needs to decompose the quotient space Γ into k-dimensional submanifolds and determine the k-dimensional Jacobian on these submanifolds in order to obtain a generalized *coarea formula*.[96] Such a formula is required for an explicit calculation of the generator associated to the limiting process.[97] In Chapter 4, i.p. Sections 4.4 and 4.5, these steps are performed for the case of a weakly coupled oscillator system.

1.5. Synchronization of noise-coupled Kuramoto oscillators

As a further motivation for studying noise-coupled oscillator systems, we present a new result on *in-phase* synchronization for a two-oscillator system, subject to a noise-perturbed Kuramoto coupling.

[85] c.f. [BR14], Proposition 6
[86] c.f. [BR14], Eq. (137)
[87] c.f. [BR14], Proposition 8
[88] c.f. [BR14], Proposition 7
[89] c.f. [BR14], Eq. (146)
[90] c.f. [BR14], Section 5.4
[91] c.f. [BR14], Eqs. (154), (156) and (157)
[92] c.f. [BR14], Proposition 4
[93] c.f. [BR14], Proposition 9, following from boundedness of drift dispersion terms
[94] c.f. [BR14], Theorem 6
[95] "In order to obtain a more intuitive representation of this process one should write the Dirichlet form as a scalar product in $L^2(\Gamma)$. In this last section, we would like to expose what one should expect and how computations could be made.", c.f. [BR14], Section 5.4
[96] c.f. [BR14], Section 5.4.2
[97] "For an explicit computation of the generator Eq. 159, one would need a lemma analogous to Lemma 5 to compute the derivatives of the averaging a_G and φ_G", c.f. [BR14], Section 5.4

Proposition 1.2 (Kuramoto model with noise-perturbed coupling strength) Consider a two-oscillator Kuramoto model with a randomly perturbed coupling strength, i.e.

$$\mathrm{d}\theta_i(t) = \frac{1}{2}\sum_{j=0}^{1} \sin(\theta_j(t) - \theta_i(t))(K\,\mathrm{d}t + \sigma\,\mathrm{d}B(t)), \quad i \in \{0,1\}\,. \tag{1.83}$$

The evolution of the distance $\eta(t) := \theta_1(t) - \theta_0(t)$ is consequently given by

$$\mathrm{d}\eta(t) = -K\sin(\eta(t))\,\mathrm{d}t - \sigma\sin(\eta(t))\,\mathrm{d}B(t). \tag{1.84}$$

If $K > \sigma^2/2$, then we have an *asymptotic in-phase synchronization*, i.e. an almost sure convergence of

$$\cos\eta(t) \to 1, \quad t \to \infty. \tag{1.85}$$

Proof. Itô's formula yields

$$\mathrm{d}\cos(\eta(t)) = \left(K\sin^2(\eta(t)) - \frac{\sigma^2}{2}\cos(\eta(t))\sin^2(\eta(t))\right)\mathrm{d}t + \sigma\sin^2(\eta(t))\,\mathrm{d}B(t), \tag{1.86}$$

and consequently, $Z(t) := \frac{1+\cos(\eta(t))}{2} \in [0,1]$ satisfies

$$\begin{aligned}\mathrm{d}Z(t) &= Z(t)(1-Z(t))\left[(2K + \sigma^2 - 2\sigma^2 Z(t))dt + 2\sigma\,\mathrm{d}B(t)\right]\\ &= f(Z(t))\left[(\alpha + 2\sigma^2 f'(Z(t)) + 2\sigma^2 Z(t))dt + 2\sigma\,\mathrm{d}B(t)\right],\end{aligned} \tag{1.87}$$

where $f(z) := z(1-z)$ and $\alpha := 2K - \sigma^2$. By a *comparison result* (c.f. [KS91], Proposition 5.2.18), $Z(t)$ is thus bounded below by the solution $(\tilde{Z}(t))_{t\geq 0}$ of

$$\mathrm{d}\tilde{Z}(t) = f(\tilde{Z}(t))\left[(\alpha + 2\sigma^2 f'(\tilde{Z}(t)))\,\mathrm{d}t + 2\sigma\,\mathrm{d}B(t)\right].$$

Since $h(z) := \log(z/(1-z))$ satisfies $h'(z) = 1/f(z)$, Itô's formula shows that $h(\tilde{Z}(t))$ is a *Brownian motion with drift*, i.e.

$$\mathrm{d}h(\tilde{Z}(t)) = \alpha\,\mathrm{d}t + 2\sigma\,\mathrm{d}B(t). \tag{1.88}$$

Solving for $\tilde{Z}(t)$, we obtain a bound on $\cos\eta(t)$ given by

$$1 \geq \cos\eta(t) = 2Z(t) - 1 \geq 2\tilde{Z}(t) - 1 = 2\left[1 + \frac{1-Z_0}{Z_0}e^{-(\alpha t + 2\sigma B(t))}\right]^{-1} - 1. \tag{1.89}$$

For $\alpha > 0$, i.e. $K > \sigma^2/2$, we therefore have almost sure convergence of

$$\alpha t + 2\sigma B(t) \to \infty, \quad t \to \infty, \tag{1.90}$$

and the result follows. □

1.6. Setup, objectives and methods

1.6.1. Setup

In this thesis we investigate a *circulant*[98] network of weakly noise-coupled commensurate oscillators. A formal definition of this system will be given in Chapter 3, c.f. in particular Section 3.4.

The class of circulant coupling topologies can roughly speaking be interpreted as a class of all 'rotionally invariant' coupling topologies for a *ring* of oscillators, c.f. Section 3.2.1.2. In particular, it encompasses the special cases of a *nearest-neighbor* and a global *all-to-all* coupling, which are usually studied (c.f. Section 1.3). Moreover, this class allows for a *directed* coupling, which generalizes the special cases of symmetric and unidirectional coupling (c.f. Remark 3.11).

We examine a circulant deterministic coupling (Section 3.2.1) combined with a circulant multiplicative-noise coupling (Section 3.3.1). The latter generalizes the nearest-neighbor and all-to-all versions of the white-noise-based coupling introduced in Section 1.3.2.2. In the spirit of the environmental noise discussed in Section 1.3.2.1, we also allow for an additive-noise perturbation (Section 3.3.3).

The circulant coupling topology of the deterministic coupling may differ from the one of the multiplicative-noise coupling, i.e. we study the interplay of a general combination of drift and diffusion (Definition 3.35). As in Sections 1.4.2 and 1.4.3, both terms are, however, assumed to be 'weak' compared to the evolution of the uncoupled system (c.f. Definition 3.35 and Section 3.5.1).

1.6.2. Objectives

Effects of circulant deterministic coupling We analyze effects of the previously outlined weak perturbations on a system of commensurate oscillators. We first study a general circulant and directed *deterministic* coupling and investigate its induced periodic exchange of energy between the oscillators, as well as the evolution of the oscillator phase differences (c.f. Section 6.1, i.p. Section 6.1.2).

Noise-induced synchronization In the spirit of Section 1.3.2.2, we subsequently examine the 'positive' influence of a circulant multiplicative-noise coupling by studying the emergence of *noise-induced synchronization* (Section 6.4.2). Here we employ the generalized notion of *eigenmode synchronization* introduced in Section 1.1.2.2, which represents a natural generalization of *in-phase* and *anti-phase* synchronization to a system of more than two oscillators (Definition 6.31). In particular, this allows for a much more detailed description of the system compared to just studying the *order parameter* of the oscillator phases, c.f. Eq. (1.14) and Remark 1.1. Accordingly, we also generalize the concept of *complete stochastic synchronization*:[99] Instead of requiring an (asymptotically) 'exponentially fast' *in-phase* synchronization, we consider a notion of stochastic synchronization that is defined as an 'exponentially fast' convergence to an eigenmode phase-locking configuration as a limiting state, c.f. Remark 6.34.

[98] c.f. Section 3.2.1.2
[99] c.f. [RS16] and Eq. (1.30)

Coupling topologies vs. synchronization patterns We relate a class of noise-coupling topologies to their induced synchronization patterns (Section 6.4.3). In particular, we examine whether or not a global all-to-all noise-coupling can induce an in-phase synchronization (Example 6.46). Analogously to the deterministic Kuramoto-model result of [CM09] (c.f. Section 1.3.1.2), we furthermore investigate if there is a critical number of oscillators above which a nearest-neighbor coupled ring of oscillators does not allow for in-phase synchronization, c.f. Theorem 6.45 and Table 6.1.

Persistence of synchronization We subsequently study the *persistence* of the observed noise-induced synchronization patterns[100] under an isotropic additive-noise perturbation (Section 6.5, i.p. Theorem 6.63). Note that such an investigation persues a similar goal as Section 1.3.2.1, where a *deterministic* coupling is shown to induce synchronization even in the presence of an environmental noise.

Averaging effects and nonlinear perturbations Finally, we also examine *averaging* effects resulting from the scale hierarchy between the evolution of the uncoupled oscillators and the weak perturbations. We follow the idea of [BE11] (c.f. Section 1.4.1) that a weak *deterministic* dipole-coupling (characterized by a 'real-part' operation in the equation, c.f. Eq. (1.47)) can be approximated by its *quantum-mechanical* analogue (characterized by the absence of such a 'real-part' operation, c.f. Eq. (1.43)). We show that in the weak-coupling limit, the circulant 'classical-mechanics' drift term (Eq. (3.161)) can be approximated by an 'effective' drift term (Eq. (5.25)), which in essence is related to the original drift term by removing the 'restriction to the real part', c.f. Remark 5.6. This approximation result is further generalized to a class of *nonlinear* drift terms, given as divergence free perturbations of linear drift terms. We show that for weak coupling strengths, these nonlinear drift terms limit can be effectively approximated by their linearizations (Definition 3.20 and Lemma 5.7). The notion of 'approximation' used in the previous statements is made rigorous by proving an averaging principle (Theorem 4.71) which yields a weak convergence of the involved processes.

1.6.3. Methods

Identification of symmetries and first integrals of uncoupled system We investigate the *uncoupled* system of oscillators and identify $U(n)$ as its group of continuous symmetries (Section 3.1.2.1). Applying results from [AW02] and in accordance with Noether's theorem, we relate the infinitesimal generators of the $U(n)$ symmetry transformations to a set of first integrals, i.e. conserved quantities, of the uncoupled system (Section 3.1.2). Subsequently, we relate these first integrals to the components of a *complex outer-product process* (Section 3.1.2.4).

Characterization of weak coupling We introduce weak coupling terms, as outlined in Section 1.6.1, and treat these as perturbations[101] which break some of the uncoupled system's symmetries. Investigating the evolution of the complex outer-product process (Section 3.5.2) allows us to determine the induced evolution of the first integrals corresponding to these broken

[100] induced by a *multiplicative*-noise coupling
[101] c.f. [AS92], Section 1.1

symmetries. This generalizes the approach proposed in [EB12], where the *expectation value* of the complex outer-product process is studied (Eq. (1.51)). As preparation for the application of an averaging principle, we perform a time-rescaling (Section 3.5.1), enabling us to study the outer-product process on a 'fast' time-scale.

Averaging result for complex outer product We prove an averaging principle which can be applied to the multidimensional case of a vector of first integrals given by the components of the complex outer product.[102] The results of [FW06] and [FW12] are *not* applicable, since the system of commensurate oscillators outlined in Section 1.6.1 violates the *non-resonance* condition, the assumption of a bounded drift and the restriction to a constant block-diagonal dispersion matrix, c.f. Section 1.4.2.2. Similarly, the averaging result of [BR14] cannot directly be applied, since drift and dispersion term violate the boundedness assumption, c.f. Section 1.4.3.[103]

Therefore we adapt the methods of [BR14] and prove an averaging result suitable for the oscillator system under consideration (Chapter 4). This particularly implies that we need to drop the boundedness assumption (c.f. Remark 4.3) and develop a new proof of the strong sector condition. Furthermore we provide an identification of the quotient space Γ (Sections 4.3.2 and 4.4) and of the limiting process $\widehat{\pi}$ on Γ (Sections 4.5.1 and 4.6).

Effective evolution of outer-product process We employ the *residue theorem* in order to evaluate the averaging integrals, which determine the drift and diffusion terms of the effective outer-product process (Chapter 5). We investigate the resulting effective SDE by means of the *Baker-Campbell-Hausdorff* formula (Section 6.1.1) and the complex Itô formula (Section 2.3.2). A *ratio-limit theorem* (c.f. [Löc15]) allows us to determine the *asymptotic* effective evolution (Section 6.3.2.2) which leads to a result on noise-induced synchronization (Section 6.4.2). By means of a time-change result, we can prove that the synchronization behavior persists in the presence of additive noise (Section 6.5, i.p. Proposition 6.53).

Classification of synchronization modes via Euler-Fermat A essential result from number theory, namely the *Euler-Fermat* theorem, allows us to classify the relation between coupling topology and its induced synchronization mode (c.f. Section 6.4.3, i.p. Proposition 6.43). This enables us to identify critical numbers of oscillators, at which this relation is altered (Lemma 6.44 and Theorem 6.45).

1.7. Outline and interdependence

1.7.1. Outline

We start with a technical chapter, where we provide basic notations and results needed throughout this thesis. This chapter is not intended to offer substantially new insights, but rather focuses on translating well-known, real-valued results like Itô's formula into a complex-valued notation,

[102] Note that a $\mathbb{C}^{n,n}$-valued outer-product matrix, such as the complex outer product, can of course be identified with a $\mathbb{C}^{n^2}$-valued vector, c.f. Definition 2.2.

[103] The nonlinear perturbations furthermore violate the global Lipschitz assumption.

which will prove more suitable for modeling a system of oscillators. Chapter 2 in particular aims at studying $\mathbb{C}^{n,n}$-valued stochastic processes, which will be of central interest to us throughout this thesis. For this purpose, we start with an identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and employ this identification in order to relate linear maps, complex product constructions and complex derivatives to their real-valued counterparts. Subsequently, we recall basic properties of the *discrete Fourier transform* (DFT), focusing on its relation to circulant matrices. We then introduce the complex Brownian motion and show its invariance under a DFT. Next we introduce complex-valued stochastic differential equations (SDEs) and provide a version of Itô's formula applicable to complex-valued stochastic processes. Finally, we study matrix-valued processes with a particular focus on the complex outer-product process.

In Chapter 3 we specify the main model of weakly coupled oscillators which we want to investigate in this thesis. We first introduce a system of uncoupled oscillators, study its symmetries and identify the conserved quantities related to these symmetries. We show that these conserved quantities correspond to the components of the systems complex outer product. In the following sections, we introduce *weak* coupling and perturbation terms, encompassing deterministic interactions as well as multiplicative, regularizing and additive-noise terms. Each of these terms is required to exhibit a circulant structure, which we employ to simplify the system's description by performing a discrete Fourier transform. Finally, we speed up the evolution by means of a time rescaling and study the complex outer-product process. This process turns out to be no longer constant, resulting from the fact that some of the uncoupled systems symmetries were broken by the weak interaction terms.

In Chapter 4 we derive an *averaging result* for the complex outer-product process of a weakly coupled oscillator system under some general assumptions on the coupling terms. This result will in particular be applicable to the system described in the previous section. The *weakness* of the coupling, represented by a parameter ε, implies a scale hierarchy between the fast evolution of the oscillators and the slow influence of the interactions between the oscillators. In the *scaling limit* of $\varepsilon \to 0$, the averaging result allows us to approximate the outer-product process by an *effective process*, which is governed by *averaged* drift and diffusion terms. We prove this averaging theorem by adapting the strategy of [BR14], i.e. by establishing a convergence of the associated Dirichlet forms. We first state the general setup and then show that the bilinear form related to the complex outer-product process is a Dirichlet form. In following sections we examine the equivalence relation induced by the complex outer-product mapping. In particular, we study the corresponding quotient space and the so-called *projected Dirichlet form* on this space. Finally, we prove that the Dirichlet forms related to the weakly coupled system converge to this projected Dirichlet form, which can be associated to an *effective* process. Verifying the tightness of the involved processes, subsequently allows us to infer a weak convergence of the associated stochastic processes.

Since the averaging result of Chapter 4 is applicable to the oscillator system of Chapter 3, we are interested in understanding the evolution of the *effective* limiting process. Chapter 5 is consequently devoted to explicitly determining the corresponding SDE. We therefore calculate the averaged drift term as well as the averaged diffusion matrices corresponding to multiplicative, regularizing and additive noise. The averaging calculations can be identified with line integrals that can be calculated by means of the residue theorem. After having identified the averaged *diffusion* matrices, we also determine corresponding *dispersion* matrices which give rise to these diffusion matrices.

In Chapter 6 we solve the effective SDE stated in the previous chapter. In the deterministic case, the Baker-Campbell-Hausdorff formula allows us to explicitly solve this matrix-valued equation. For the full stochastic system, we subsequently focus on the evolution of the diagonal elements of the outer-product process, which can be interpreted as the systems *eigenmode amplitudes*. We first study their evolution in the homogeneous case, i.e. in the absence of additive noise, where we obtain asymptotic *synchronization* results. In particular, we apply a number theoretic result in order to relate noise-coupling topologies to their induced synchronization states. The synchronization results are then shown to persist in the inhomogeneous case, provided that we have a sufficiently small additive-noise perturbation. Finally, we apply the averaging theorem in order to obtain a synchronization statement for the original, unaveraged system.

1.7.2. Interdependence of chapters

The chapters are designed in a modular fashion, i.e. subsequent chapters mainly refer to the final results of the previous chapters:

Chapter 2 provides basic technical tools needed in all of the following chapters. The complex outer-product process, which is the central object of interest, is introduced and motivated in Chapter 3. Its evolution equation can be found in Proposition 3.39. Chapter 4 subsequently develops an averaging theory which applies to this process and which is summarized in Theorem 4.71. This theorem characterizes a matrix-valued limiting process by means of an SDE, which is given in terms of certain averaging integrals. In Chapter 5 these integrals are evaluated and the resulting explicit SDE is stated in Theorem 5.26 and Corollary 5.28. Chapter 6 is finally devoted to solving this SDE and characterizing its solutions.

2 Complex structures

The description of a system of harmonic oscillators can be significantly simplified by adopting a complex-valued notation, c.f. Section 3.1.1. In essence, we identify the state space $\mathbb{R}^{2n}$ with its complex analogue $\mathbb{C}^n$. This identification will be given in Section 2.1, where we will focus on linear maps and their representation in terms of real- and complex-valued matrices. In Section 2.2 we will present a particular complex linear map, namely the *discrete Fourier transform* (DFT). This transformation proves to be useful in describing so-called *circulant* coupling structures in a ring of oscillators, c.f. Lemma 2.27 and Section 3.2.1.5.

Since we want to allow for *random* perturbations of the oscillator system, its evolution will be given by a $\mathbb{C}^n$-valued stochastic differential equation (SDE). For this reason, Section 2.3 will cover basic definitions and results concerning complex-valued SDEs. In particular, we will present a version of Itô's formula which can be applied to complex-valued processes.

In order to capture all of the uncoupled system's conserved quantities (Section 3.1.2), we will look at a *complex outer product* of the system's state vector with itself (Section 3.5.2). As preparation for studying its evolution in the coupled case, Section 2.4 employs the complex Itô formula in order to derive a matrix-valued SDE, describing the evolution of this complex *outer-product process*.

2.1. Complex linear transformations

2.1.1. Identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$

We obtain a pointwise identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by interpreting the first n components of a point in $\mathbb{R}^{2n}$ as the real part and its last n components as the imaginary part of the corresponding point in $\mathbb{C}^n$. Such an identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ allows us to obtain real-valued representations of complex linear transformations. A $\mathbb{C}^{n,n'}$-valued matrix can thus be identified with a corresponding $\mathbb{R}^{2n,2n'}$-valued matrix, which exhibits a certain block structure, c.f. Eq. (2.3).

Definition 2.1 (Identification of complex and real spaces)
For any $n \in \mathbb{N}$, we can isomorphically identify the space $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by the mapping[a]

$$\mathfrak{j}^{(n)} : \mathbb{C}^n \to \mathbb{R}^{2n},\ x \to \mathfrak{j}(x) := \begin{pmatrix} \mathrm{Re}\,(x) \\ \mathrm{Im}\,(x) \end{pmatrix}. \tag{2.1}$$

Conjugation with this mapping allows us to identify complex- with real-valued maps, i.e. for all $n, n' \in \mathbb{N}$ we define

$$\mathfrak{j}_{\#}^{(n',n)} : \mathrm{Maps}(\mathbb{C}^{n'}, \mathbb{C}^n) \to \mathrm{Maps}(\mathbb{R}^{2n'}, \mathbb{R}^{2n}), \quad f \to \mathfrak{j}_{\#}^{(n',n)}(f) := \mathfrak{j}^{(n)} \circ f \circ (\mathfrak{j}^{(n')})^{-1}. \tag{2.2}$$

In the special case of complex linear maps, this identification induces a representation of complex $n \times n'$ matrices in terms of real-valued $2n \times 2n'$ matrices, i.e.

$$\mathfrak{J}^{(n,n')} : \mathbb{C}^{n,n'} \hookrightarrow \mathbb{R}^{2n,2n'},\ \mathsf{M} \to \mathfrak{J}^{(n,n')}(\mathsf{M}) := \left(\begin{array}{c|c} \mathrm{Re}\,(\mathsf{M}) & -\mathrm{Im}\,(\mathsf{M}) \\ \hline \mathrm{Im}\,(\mathsf{M}) & \mathrm{Re}\,(\mathsf{M}) \end{array}\right). \tag{2.3}$$

Furthermore we introduce the related embedding

$$\breve{\mathfrak{J}}^{(n,n')} : \mathbb{C}^{n,n'} \hookrightarrow \mathbb{R}^{2n,2n'},\ \mathsf{M} \to \breve{\mathfrak{J}}^{(n,n')}(\mathsf{M}) := \left(\begin{array}{c|c} \mathrm{Re}\,(\mathsf{M}) & \mathrm{Im}\,(\mathsf{M}) \\ \hline \mathrm{Im}\,(\mathsf{M}) & -\mathrm{Re}\,(\mathsf{M}) \end{array}\right), \tag{2.4}$$

which will be useful for certain matrix decompositions (c.f. Eq. (2.9) and Lemma 2.6 below). A generalization of $\mathfrak{j}$ is given by

$$\mathfrak{J}_{1/2}^{(n,n')} : \mathbb{C}^{n,n'} \to \mathbb{R}^{2n,n'},\ \mathsf{M} \to \mathfrak{J}_{1/2}^{(n,n')}(\mathsf{M}) := \left(\begin{array}{c} \mathrm{Re}\,(\mathsf{M}) \\ \hline \mathrm{Im}\,(\mathsf{M}) \end{array}\right). \tag{2.5}$$

If no ambiguity arises, we will drop the superscripts denoting the dimensions involved.

[a] c.f. [DO10], Chapter 1

Note that $\mathfrak{J}_{1/2}$ corresponds to the '*left half*' of $\mathfrak{J}$ and is indeed a generalization of $\mathfrak{j}$, since for $n' = 1$ the maps $\mathfrak{j}$ and $\mathfrak{J}_{1/2}$ coincide. We will employ $\mathfrak{J}_{1/2}$ in order to represent a complex linear transformation of a *real-valued* vector, c.f. Eq. (2.10) below.

For future use, we will denote the canonical isomorphism that maps all n^2 components of an $n \times n$ matrix to an n^2-dimensional vector as follows.

$$\begin{array}{ccc} \mathrm{Lin}_{\mathbb{C}}(\mathbb{C}^{n'}, \mathbb{C}^{n}) & \xhookrightarrow{\mathfrak{j}_{\#}^{(n',n)}} & \mathrm{Lin}_{\mathbb{R}}(\mathbb{R}^{2n'}, \mathbb{R}^{2n}) \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{C}^{n,n'} & \xhookrightarrow{\mathfrak{J}^{(n,n')}} & \mathbb{R}^{2n,2n'} \end{array}$$

Figure 2.1.: Complex linear maps

Definition 2.2 (Matrix-vector identification)
Let for all $n, N \in \mathbb{N}$,

$$\widehat{\psi}_{\mathbb{C}}^{(n)} : \mathbb{C}^{n,n} \to \mathbb{C}^{n^2}, \quad \mathsf{M} \to (\mathsf{M}_{0,0}, \mathsf{M}_{0,1}, \ldots, \mathsf{M}_{0,n-1}, \mathsf{M}_{1,0}, \mathsf{M}_{1,1}, \ldots, \mathsf{M}_{n-1,n-1})^{\top}, \tag{2.6}$$
$$\widehat{\psi}_{\mathbb{R}}^{(N)} : \mathbb{R}^{N,N} \to \mathbb{R}^{N^2}, \quad \mathsf{M} \to (\mathsf{M}_{0,0}, \mathsf{M}_{0,1}, \ldots, \mathsf{M}_{0,N-1}, \mathsf{M}_{1,0}, \mathsf{M}_{1,1}, \ldots, \mathsf{M}_{N-1,N-1})^{\top}, \tag{2.7}$$

denote the canonical ismorphisms of $\mathbb{C}^{n,n}$ with $\mathbb{C}^{n^2}$ and of $\mathbb{R}^{N,N}$ with $\mathbb{R}^{N^2}$, respectively, which map all matrix elements to a single vector, in a column-wise manner. Again, we will drop the superscripts if there is no ambiguity.

Note that in Definition 2.2 and throughout this thesis, we apply the convention of labeling indices starting at *zero* instead of one. This convention will prove advantageous in the context of discrete Fourier transforms, c.f. Section 2.2. In Fig. 2.1 we identify complex and real linear maps with their matrix representations. The following lemma shows Fig. 2.1 represents a *commutative diagram*.

Lemma 2.3 (Complex linear maps)
Let $f \in \mathrm{Lin}_{\mathbb{C}}(\mathbb{C}^{n'}, \mathbb{C}^{n})$ and let $\mathsf{M}_f \in \mathbb{C}^{n,n'}$ denote its representing complex matrix. Then $\mathfrak{J}(\mathsf{M}_f) \in \mathbb{R}^{2n,2n'}$ coincides with the matrix representation of $\mathfrak{j}_{\#}^{(n',n)}(f)$, i.e. the diagram of Fig. 2.1 *commutes*.

For all $\mathsf{M} \in \mathbb{C}^{n,n'}$ and $v \in \mathbb{C}^{n'}$ we have

$$\mathfrak{J}\,(\mathsf{M})\ \mathfrak{j}(v) = \mathfrak{j}\,(\mathsf{M}\, v)\,, \tag{2.8}$$
$$\breve{\mathfrak{J}}\,(\mathsf{M})\ \mathfrak{j}(v) = \mathfrak{j}\,(\mathsf{M}\, \overline{v})\,. \tag{2.9}$$

A complex linear transformation of a *real-valued* vector can be represented in terms of $\mathfrak{J}_{1/2}$, i.e. for $\mathsf{M} \in \mathbb{C}^{n,n'}$ and $v \in \mathbb{R}^{n'}$ we have

$$\mathfrak{J}_{1/2}\,(\mathsf{M})\ v = \mathfrak{j}\,(\mathsf{M}\, v) \tag{2.10}$$

Moreover, in the case of $n = n'$, the map $\mathfrak{J}$ is an *injective homomorphism*, i.e. for $\mathsf{M}, \mathsf{M}' \in \mathrm{Lin}_{\mathbb{C}}(\mathbb{C}^{n}, \mathbb{C}^{n})$ we have $\mathfrak{J}(\mathsf{M}\mathsf{M}') = \mathfrak{J}(\mathsf{M})\mathfrak{J}(\mathsf{M}')$ and $\mathfrak{J}(\mathsf{M}) = 0$ implies $\mathsf{M} = 0$.

Proof. Let $f \in \mathrm{Lin}_{\mathbb{C}}(\mathbb{C}^{n'}, \mathbb{C}^{n})$ and let $\mathsf{M}_f \in \mathbb{C}^{n,n'}$ denote its representing complex matrix. We show that $\mathfrak{J}(\mathsf{M}_f) \in \mathbb{R}^{2n,2n'}$ coincides with the matrix representation of $\mathfrak{j} \circ f \circ \mathfrak{j}^{-1}$.[1] For all $x \in \mathbb{C}^{n'}$,

[1] For the sake of brevity, we have dropped the superscripts specifying the involved dimensions.

we have

$$\mathfrak{J}(\mathsf{M}_f)\,\mathfrak{j}(x) = \left(\begin{array}{c|c} \mathrm{Re}\,(\mathsf{M}_f) & -\mathrm{Im}\,(\mathsf{M}_f) \\ \hline \mathrm{Im}\,(\mathsf{M}_f) & \mathrm{Re}\,(\mathsf{M}_f) \end{array}\right) \begin{pmatrix} \mathrm{Re}(x) \\ \mathrm{Im}(x) \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{Re}\,(\mathsf{M}_f)\,\mathrm{Re}\,(x) - \mathrm{Im}\,(\mathsf{M}_f)\,\mathrm{Im}\,(x) \\ \mathrm{Im}\,(\mathsf{M}_f)\,\mathrm{Re}\,(x) + \mathrm{Re}\,(\mathsf{M}_f)\,\mathrm{Im}\,(x) \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{Re}\,(\mathsf{M}_f\,x) \\ \mathrm{Im}\,(\mathsf{M}_f\,x) \end{pmatrix} = \mathfrak{j}(\mathsf{M}_f\,x) = \mathfrak{j}\circ f(x), \tag{2.11}$$

which yields the first statement as well as Eq. (2.8). Similarly, we can obtain Eq. (2.9) by observing that for $\mathsf{M} \in \mathbb{C}^{n,n'}$ and $v \in \mathbb{C}^{n'}$ we find that

$$\breve{\mathfrak{J}}(\mathsf{M})\,\mathfrak{j}(x) = \left(\begin{array}{c|c} \mathrm{Re}\,(\mathsf{M}) & \mathrm{Im}\,(\mathsf{M}) \\ \hline \mathrm{Im}\,(\mathsf{M}) & -\mathrm{Re}\,(\mathsf{M}) \end{array}\right) \begin{pmatrix} \mathrm{Re}(x) \\ \mathrm{Im}(x) \end{pmatrix}$$
$$= \left(\begin{array}{c|c} \mathrm{Re}\,(\mathsf{M}) & -\mathrm{Im}\,(\mathsf{M}) \\ \hline \mathrm{Im}\,(\mathsf{M}) & \mathrm{Re}\,(\mathsf{M}) \end{array}\right) \begin{pmatrix} \mathrm{Re}(x) \\ -\mathrm{Im}(x) \end{pmatrix} = \mathfrak{J}(\mathsf{M})\mathfrak{j}(\overline{x}) = \mathfrak{j}(\mathsf{M}\,\overline{x}), \tag{2.12}$$

where in the last step we have applied Eq. (2.8). In the special of $v \in \mathbb{R}^{n'}$, we observe that

$$\mathfrak{J}_{1/2}\,(\mathsf{M})\;v = \begin{pmatrix} \mathrm{Re}\,(\mathsf{M}) \\ \mathrm{Im}\,(\mathsf{M}) \end{pmatrix} v = \left(\begin{array}{c|c} \mathrm{Re}\,(\mathsf{M}) & -\mathrm{Im}\,(\mathsf{M}) \\ \hline \mathrm{Im}\,(\mathsf{M}) & \mathrm{Re}\,(\mathsf{M}) \end{array}\right) \begin{pmatrix} v \\ 0 \end{pmatrix} = \mathfrak{J}(\mathsf{M})\mathfrak{j}(v) = \mathfrak{j}\,(\mathsf{M}\,v)\,. \tag{2.13}$$

The statement on $\mathfrak{J}$ being an injective homomorphism in the case of $n = n'$, similarly follows by direct calculation. □

Note however that $\mathfrak{J}$ is *not* surjective, since elements of $\mathfrak{J}(\mathbb{C}^{n,n'})$ can be characterized by the fact that they can be written in a particular block-diagonal form, i.e.

$$M \in \mathbb{C}^{n,n'} \quad \Leftrightarrow \quad M = A + iB,\ \text{for}\ A, B \in \mathbb{R}^{n,n'} \quad \Leftrightarrow \quad \mathfrak{J}(M) = \left(\begin{array}{c|c} A & -B \\ \hline B & A \end{array}\right). \tag{2.14}$$

Multiplying an element of $\mathbb{C}^n$ by the imaginary unit i can be interpreted as a complex linear map. This allows us to define an $\mathbb{R}^{2n,2n}$-representation of this multiplication as

$$I \coloneqq \mathfrak{J}(i\,\mathbb{1}_{n\times n}) = \left(\begin{array}{c|c} 0 & -\mathbb{1}_{n\times n} \\ \hline \mathbb{1}_{n\times n} & 0 \end{array}\right). \tag{2.15}$$

Employing this matrix I, we can obtain an alternative characterization of $\mathfrak{J}\left(\mathbb{C}^{n,n}\right)$ by means of a *commutator* relation.

Definition 2.4 (Commutator and anticommutator)
For quadratic matrices A, B the *commutator* of these matrices is given by

$$[A, B] \coloneqq A\,B - B\,A, \tag{2.16}$$

while the *anticommutator* is defined as

$$\{A, B\} \coloneqq A\,B + B\,A. \tag{2.17}$$

Note that the commutator of two matrices vanishes, if and only if they commute.

Lemma 2.5 (Characterization of $\mathfrak{J}\left(\mathbb{C}^{n,n}\right)$)
We can characterize $\mathfrak{J}\left(\mathbb{C}^{n,n}\right)$ as those $\mathbb{R}^{2n,2n}$ matrices which commute with I,[a] i.e.

$$\mathfrak{J}\left(\mathbb{C}^{n,n}\right) = \left\{M \in \mathbb{R}^{2n,2n} \mid [M, I] = 0\right\}. \tag{2.18}$$

[a]These matrices are called $\mathbb{C}$-linear, c.f. [Ubø87], Section 3.

Proof. A simple calculation yields that a matrix $M \in \mathrm{Lin}_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ commutes with I if and only if it can be written in the block-diagonal form of Eq. (2.14). □

A generic $\mathbb{R}^{2n,2n}$ matrix does *not* commute with I. However, it can be decomposed into a '$\mathbb{C}$-*linear*' part which commutes with I, allowing for a complex-valued representation via $\mathfrak{J}$, and a remaining '$\mathbb{C}$-*antilinear*' part (c.f. [Ubø87], Section 3), which allows for a representation in terms of $\breve{\mathfrak{J}}$.

Lemma 2.6 (Matrix decomposition with respect to linear complex structure)
A matrix $M \in \mathbb{R}^{2n,2n}$ can be decomposed as

$$M = M_+ + M_-, \tag{2.19}$$

where

$$M_+ \coloneqq \frac{M - IMI}{2} \tag{2.20}$$

commutes with I, i.e. $[M_+, I] = 0$ and

$$M_- \coloneqq \frac{M + IMI}{2} \tag{2.21}$$

anticommutes with I, i.e. $\{M_-, I\} = 0$. If $M \in \mathrm{Lin}_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is of the form

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \quad \text{for some } A, B, C, D \in \mathbb{R}^{n,n},$$

this decomposition can be written as

$$M = \frac{1}{2}\left(\begin{array}{c|c} A + D & -(C - B) \\ \hline C - B & A + D \end{array}\right) + \frac{1}{2}\left(\begin{array}{c|c} A - D & B + C \\ \hline B + C & D - A \end{array}\right), \tag{2.22}$$

i.e. we find that

$$M_+ = \frac{1}{2}\mathfrak{J}\left[(A + D) + i(C - B)\right], \tag{2.23a}$$

$$M_- = \frac{1}{2}\breve{\mathfrak{J}}\left[(A - D) + i(C + B)\right]. \tag{2.23b}$$

Proof. The assertions can be checked by direct calculation. For instance, we can make use of

$$I^2 = \mathfrak{J}(i\,\mathbb{1}_{n\times n})\,\mathfrak{J}(i\,\mathbb{1}_{n\times n}) = \mathfrak{J}(-\mathbb{1}_{n\times n}) = -\mathbb{1}_{2n\times 2n}, \tag{2.24}$$

in order to find that

$$M_+\,I = \frac{M - I\,M\,I}{2}\,I = \frac{(-I^2)M\,I + I\,M}{2} = I\,\frac{(-I)\,M\,I + M}{2} = I\,M_+, \tag{2.25}$$

which yields $[M_+, I] = 0$. ☐

In the following lemma we show that Hermitian conjugation of a $\mathbb{C}^{n,n}$ matrix translates to transposition of the corresponding $\mathbb{R}^{2n,2n}$ matrix. In particular, this implies that a *unitary* complex-valued matrix gives rise to an *orthogonal* real-valued matrix. This will be of importance when studying the discrete Fourier transformation (DFT) of a complex Brownian motion in Lemma 2.35.

Lemma 2.7 (Hermitian conjugation)
For all $\mathsf{M} \in \mathbb{C}^{n,n}$, we have $\mathfrak{J}(\mathsf{M}^\dagger) = \mathfrak{J}(\mathsf{M})^\top$. Consequently, we obtain the equivalencies

i) M is hermitian, if and only if $\mathfrak{J}(\mathsf{M})$ is symmetric,

ii) M is unitary, if and only if $\mathfrak{J}(\mathsf{M})$ is orthogonal.

Proof. For $\mathsf{M} \in \mathbb{C}^{n,n}$ we find that

$$\mathfrak{J}(\mathsf{M}^\dagger) = \left(\begin{array}{c|c} \mathrm{Re}\left(\mathsf{M}^\dagger\right) & -\mathrm{Im}\left(\mathsf{M}^\dagger\right) \\ \hline \mathrm{Im}\left(\mathsf{M}^\dagger\right) & \mathrm{Re}\left(\mathsf{M}^\dagger\right) \end{array}\right) = \left(\begin{array}{c|c} \mathrm{Re}\left(\mathsf{M}\right)^\top & \mathrm{Im}\left(\mathsf{M}\right)^\top \\ \hline -\mathrm{Im}\left(\mathsf{M}\right)^\top & \mathrm{Re}\left(\mathsf{M}\right)^\top \end{array}\right)$$

$$= \left(\begin{array}{c|c} \mathrm{Re}\left(\mathsf{M}\right) & -\mathrm{Im}\left(\mathsf{M}\right) \\ \hline \mathrm{Im}\left(\mathsf{M}\right) & \mathrm{Re}\left(\mathsf{M}\right) \end{array}\right)^\top = \mathfrak{J}(\mathsf{M})^\top.$$

The equivalencies now follow from the fact that $\mathfrak{J}$ is an injective homomorphism. □

2.1.2. Products

In this section we relate real- and complex-valued versions of several well-known product constructions as for instance the outer product. We start with the so-called *Frobenius product* (c.f. [BR14], Eq. (136)) of two real- or complex-valued square matrices of the same size, which is defined as follows.

Definition 2.8 (Frobenius product)
For $A, B \in \mathbb{K}^{n',n'}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we define the Frobenius product $A : B$ as

$$A : B := \mathrm{tr}(A^\dagger B) = \sum_{k,l=0}^{n'-1} \overline{A_{k,l}}\, B_{k,l}. \tag{2.26}$$

Note that, as in Definition 2.2, we start labeling indices at *zero*. The Frobenius product allows for a simple representations of the Itô correction term in Itô's formula and of the diffusion part of an infinitesimal generator, c.f. Theorem 2.36 and Corollary 2.37. If, in particular, we have a Frobenius product of two $\mathbb{R}^{2n,2n}$ matrices and want to adopt a complex-valued notation, we can first decompose these matrices according to Lemma 2.6 and then make use of the following lemma, which shows that mixed terms do not contribute.

Lemma 2.9 (Decomposition of Frobenius product)
For all $A, B \in \mathbb{R}^{2n,2n}$, we find that

$$A : B = A_+ : B_+ + A_- : B_-, \tag{2.27}$$

where $M_\pm := \frac{1}{2}(M \pm IM(-I))$, as defined in Lemma 2.6.

Proof. Employing the decompositions introduced in Lemma 2.6, we obtain

$$A : B = (A_+ + A_-) : (B_+ + B_-) = A_+ : B_+ + A_- : B_-, \tag{2.28}$$

since the mixed terms cancel, which can be seen as follows. We observe that $\breve{A} \coloneqq I\,A(-I)$ and $\breve{B} \coloneqq I\,B(-I)$ satisfy

$$\breve{A} : \breve{B} = \operatorname{tr}\left[(I\,A(-I))^\dagger I\,B(-I)\right] = \operatorname{tr}\left[I\,A^\dagger(-I)I\,B(-I)\right] = \operatorname{tr}\left[A^\dagger\,B\right] = A : B,$$

where we have made use of $I^\dagger = I^\top = (-I)$ and the cyclic property of the trace. Now the result follows, i.e.

$$\begin{aligned} A_+ : B_- + A_- : B_+ &= \frac{1}{4}\left[\left(A + \breve{A}\right) : \left(B - \breve{B}\right) + \left(A - \breve{A}\right) : \left(B + \breve{B}\right)\right] \\ &= \frac{1}{2}\left[A : B - \breve{A} : \breve{B}\right] = 0. \end{aligned}$$

□

We will come back to this result in the proof of Lemma 2.17, where we consider the special case of a Frobenius product with a *Hessian matrix*. Now we turn to the *scalar product* construction and observe that a complex scalar product[2] in $\mathbb{C}^n$ can be represented in terms of real scalar products in $\mathbb{R}^{2n}$.

Lemma 2.10 (Scalar product)
For any $x, y \in \mathbb{C}^n$, we have

$$x^\dagger y = X^\top Y + i\,X^\top(-I)Y, \tag{2.29}$$

which in particular implies that

$$X^\top Y = \operatorname{Re}\left(x^\dagger y\right), \tag{2.30}$$

where we have set $X \coloneqq \mathfrak{j}(x)$ and $Y \coloneqq \mathfrak{j}(y)$.

Proof. For $x, y \in \mathbb{C}^n$, we calculate

$$\begin{aligned} x^\dagger y &= (\operatorname{Re}(x) - i\operatorname{Im}(x))^\top(\operatorname{Re}(y) + i\operatorname{Im}(y)) \\ &= \left[(\operatorname{Re}(x))^\top\operatorname{Re}(y) + (\operatorname{Im}(x))^\top\operatorname{Im}(y)\right] + i\left[(\operatorname{Re}(x))^\top\operatorname{Im}(y) - (\operatorname{Im}(x))^\top\operatorname{Re}(y)\right] \\ &= (\operatorname{Re}(x), \operatorname{Im}(x))^\top\begin{pmatrix}\operatorname{Re}(y)\\ \operatorname{Im}(y)\end{pmatrix} + i\,(\operatorname{Re}(x), \operatorname{Im}(x))^\top\left(\begin{array}{c|c} 0 & \mathbb{1}_{n\times n} \\ \hline -\mathbb{1}_{n\times n} & 0\end{array}\right)\begin{pmatrix}\operatorname{Re}(y)\\ \operatorname{Im}(y)\end{pmatrix} \\ &= X^\top Y + i\,X^\top(-I)Y. \end{aligned}$$

□

Of special interest to us will be the *outer-product* construction, which takes two vectors as an input and gives rise to a matrix, containing all possible component-wise products of the vectors.

[2] more precisely, the *canonical Hermitian inner product*

Definition 2.11 (Outer product)
For $n \in \mathbb{N}$ and $x, y \in \mathbb{K}^n$, the outer product of x and y is given by $xy^\dagger$, i.e.

$$\left(xy^\dagger\right)_{k,l} = x_k\,\overline{y_l}, \quad \forall k,l \in \{0, ..., n-1\}\,. \tag{2.31}$$

In the case of $\mathbb{K} = \mathbb{C}$, we will call this a *complex* outer product and in the case of $\mathbb{K} = \mathbb{R}$ a *real* outer product. In the latter case, the hermitian conjugate can be replaced by a transposition.

The complex outer product will be at the center of our attention, since it can be used to capture the system's first integrals, c.f. Section 3.1.2 (i.p. Section 3.1.2.4) and Section 3.5.2. For these and subsequent chapters it will be important to have the following results at our disposal, which will allow us to go back and forth between complex- and real-valued representations. In a first step, we show that a complex outer product of two vectors $x, y \in \mathbb{C}^n$ can be expressed in terms of a real outer product of the corresponding vectors $X := \mathfrak{j}(x) \in \mathbb{R}^{2n}$ and $Y := \mathfrak{j}(y) \in \mathbb{R}^{2n}$.

Lemma 2.12 (Complex outer product)
For any $x, y \in \mathbb{C}^n$, the *complex outer product* $xy^\dagger$ has the real-valued representation

$$\mathfrak{J}(xy^\dagger) = XY^\top + I\,XY^\top(-I), \tag{2.32}$$

where we have set $X := \mathfrak{j}(x)$ and $Y := \mathfrak{j}(y)$.

Proof. Let $x, y, z \in \mathbb{C}^n$ and let $X, Y, Z \in \mathbb{R}^{2n}$ denote their real-valued representations. Employing Eq. (2.8), we observe that

$$\begin{aligned}
\mathfrak{J}(xy^\dagger)\,\mathfrak{j}(z) &= \mathfrak{j}\left(x\,(y^\dagger z)\right) \\
&= \mathfrak{J}\left((y^\dagger z)\,\mathbb{1}_{n\times n}\right)\,\mathfrak{j}(x) \\
&= \left[\mathrm{Re}(y^\dagger z)\,\mathfrak{J}(\mathbb{1}_{n\times n}) + \mathrm{Im}(y^\dagger z)\,\mathfrak{J}\,(i\,\mathbb{1}_{n\times n})\right]\mathfrak{j}(x) \\
&= \left[\mathrm{Re}(y^\dagger z)\,\mathbb{1}_{2n\times 2n} + \mathrm{Im}(y^\dagger z)\,I\right]\,\mathfrak{j}(x) \\
&= \left[(Y^\top Z)\,\mathbb{1}_{2n\times 2n} + (Y^\top(-I)Z)\,I\right]\,X \\
&= (Y^\top Z)\,X + (Y^\top(-I)Z)\,(I\,X) \\
&= XY^\top Z + I\,XY^\top(-I)Z \\
&= \left[XY^\top + I\,XY^\top(-I)\right]\,Z,
\end{aligned}$$

where in the fifth step we have employed the real-valued representation of the scalar product given by Lemma 2.10. Since $z \in \mathbb{C}^n$ (and thus $Z \in \mathbb{R}^{2n}$) was chosen arbitrarily, the result follows. ☐

In a second step, we want to look at the inverse direction, i.e. we represent a *real* outer product in terms of complex outer products. In fact, we will prove a slightly more general result on the decomposition of a 'generalized' outer product (c.f. Eq. (2.34)). This will be needed in the proof of Proposition 4.63 and directly yields the desired representation of the real outer product (c.f. Eq. (2.36)).

Lemma 2.13 (Generalized outer product)
For $\mathsf{M}, \widetilde{\mathsf{M}} \in \mathbb{C}^{n,n'}$, the generalized outer product $\mathfrak{J}_{1/2}(\mathsf{M})\, \mathfrak{J}_{1/2}\left(\widetilde{\mathsf{M}}\right)^\top$ can be decomposed as

$$\left(\mathfrak{J}_{1/2}(\mathsf{M})\, \mathfrak{J}_{1/2}\left(\widetilde{\mathsf{M}}\right)^\top\right)_+ = \frac{1}{2}\mathfrak{J}\left(\mathsf{M}\widetilde{\mathsf{M}}^\dagger\right), \tag{2.33a}$$

$$\left(\mathfrak{J}_{1/2}(\mathsf{M})\, \mathfrak{J}_{1/2}\left(\widetilde{\mathsf{M}}\right)^\top\right)_- = \frac{1}{2}\breve{\mathfrak{J}}\left(\mathsf{M}\widetilde{\mathsf{M}}^\top\right), \tag{2.33b}$$

i.e.

$$\mathfrak{J}_{1/2}(\mathsf{M})\, \mathfrak{J}_{1/2}\left(\widetilde{\mathsf{M}}\right)^\top \pm I\, \mathfrak{J}_{1/2}(\mathsf{M})\, \mathfrak{J}_{1/2}\left(\widetilde{\mathsf{M}}\right)^\top(-I) = \begin{cases}\mathfrak{J}\left(\mathsf{M}\widetilde{\mathsf{M}}^\dagger\right)\\ \breve{\mathfrak{J}}\left(\mathsf{M}\widetilde{\mathsf{M}}^\top\right)\end{cases}. \tag{2.34}$$

In particular, it follows that

$$\mathfrak{J}_{1/2}(\mathsf{M})\, \mathfrak{J}_{1/2}\left(\widetilde{\mathsf{M}}\right)^\top = \frac{1}{2}\left[\mathfrak{J}\left(\mathsf{M}\widetilde{\mathsf{M}}^\dagger\right) + \breve{\mathfrak{J}}\left(\mathsf{M}\widetilde{\mathsf{M}}^\top\right)\right]. \tag{2.35}$$

For the special case of $n' = 1$, i.e. $\mathsf{M} = x, \widetilde{\mathsf{M}} = y$, for some $x, y \in \mathbb{C}^n$, this yields

$$XY^\top = \frac{1}{2}\left(\mathfrak{J}(xy^\dagger) + \breve{\mathfrak{J}}(xy^\top)\right), \tag{2.36}$$

where we have set $X := \mathfrak{j}(x)$ and $Y := \mathfrak{j}(y)$.

Proof. The generalized outer product can be written as

$$\begin{aligned}\mathfrak{J}_{1/2}(\mathsf{M})\, \mathfrak{J}_{1/2}\left(\widetilde{\mathsf{M}}\right)^\top &= \left(\begin{array}{c}\mathrm{Re}(\mathsf{M})\\ \hline \mathrm{Im}(\mathsf{M})\end{array}\right)\left(\begin{array}{c|c}\mathrm{Re}\left(\widetilde{\mathsf{M}}\right)^\top & \mathrm{Im}\left(\widetilde{\mathsf{M}}\right)^\top\end{array}\right)\\ &= \left(\begin{array}{c|c}\mathrm{Re}(\mathsf{M})\,\mathrm{Re}\left(\widetilde{\mathsf{M}}\right)^\top & \mathrm{Re}(\mathsf{M})\,\mathrm{Im}\left(\widetilde{\mathsf{M}}\right)^\top\\ \hline \mathrm{Im}(\mathsf{M})\,\mathrm{Re}\left(\widetilde{\mathsf{M}}\right)^\top & \mathrm{Im}(\mathsf{M})\,\mathrm{Im}\left(\widetilde{\mathsf{M}}\right)^\top\end{array}\right) =: \left(\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right).\end{aligned}$$

Since

$$A \pm D = \mathrm{Re}(\mathsf{M})\,\mathrm{Re}\left(\widetilde{\mathsf{M}}\right)^\top \pm \mathrm{Im}(\mathsf{M})\,\mathrm{Im}\left(\widetilde{\mathsf{M}}\right)^\top = \begin{cases}\mathrm{Re}\left(\mathsf{M}\widetilde{\mathsf{M}}^\dagger\right)\\ \mathrm{Re}\left(\mathsf{M}\widetilde{\mathsf{M}}^\top\right)\end{cases}, \tag{2.37a}$$

and

$$C \mp B = \mathrm{Im}(\mathsf{M})\,\mathrm{Re}\left(\widetilde{\mathsf{M}}\right)^\top \mp \mathrm{Re}(\mathsf{M})\,\mathrm{Im}\left(\widetilde{\mathsf{M}}\right)^\top = \begin{cases}\mathrm{Im}\left(\mathsf{M}\widetilde{\mathsf{M}}^\dagger\right)\\ \mathrm{Im}\left(\mathsf{M}\widetilde{\mathsf{M}}^\top\right)\end{cases}, \tag{2.37b}$$

we can observe that Eq. (2.23) yields the aspired decompositions, i.e.

$$\left(\mathfrak{J}_{1/2}\left(\mathsf{M}\right)\mathfrak{J}_{1/2}\left(\widetilde{\mathsf{M}}\right)^{\top}\right)_{+} = \frac{1}{2}\mathfrak{J}\left[\left(A+D\right)+i\left(C-B\right)\right] = \frac{1}{2}\mathfrak{J}(\mathsf{M}\widetilde{\mathsf{M}}^{\dagger}), \tag{2.38a}$$

$$\left(\mathfrak{J}_{1/2}\left(\mathsf{M}\right)\mathfrak{J}_{1/2}\left(\widetilde{\mathsf{M}}\right)^{\top}\right)_{-} = \frac{1}{2}\breve{\mathfrak{J}}\left[\left(A-D\right)+i\left(C+B\right)\right] = \frac{1}{2}\breve{\mathfrak{J}}(\mathsf{M}\widetilde{\mathsf{M}}^{\top}). \tag{2.38b}$$

Recalling Eqs. (2.20) and (2.21), we find that Eq. (2.34) now follows directly from Eq. (2.33). Finally, Eq. (2.35) can be obtained by solving Eq. (2.34) for $\mathfrak{J}_{1/2}\left(\mathsf{M}\right)\mathfrak{J}_{1/2}\left(\widetilde{\mathsf{M}}\right)^{\top}$ and Eq. (2.36) is a consequence of $\mathfrak{J}_{1/2}(x) = \mathfrak{j}(x) = X$. □

2.1.3. Complex derivatives

We introduce notations for real and complex partial derivatives and present results which relate real- and complex versions of gradient and Hessian.

Definition 2.14 (Real and complex gradient and Hessian operators)
Let ∇ denote the *gradient* operator (or *Nabla* operator) in $\mathbb{R}^{2n}$, i.e. the vector of all partial derivatives ∇_k in $\mathbb{R}^{2n}$. In accordance with Definition 2.1, we denote the first n components of ∇ by $\nabla^{(\mathrm{Re})}$ and its last n components by $\nabla^{(\mathrm{Im})}$, i.e. we have

$$\nabla = \begin{pmatrix} \nabla^{(\mathrm{Re})} \\ \nabla^{(\mathrm{Im})} \end{pmatrix}. \tag{2.39}$$

The real *Hessian* operator is given by an outer product of the gradient operator with itself, i.e. given by $\nabla\nabla^{\top}$.

We define the *complex* gradient operators in $\mathbb{C}^n$ by (c.f. [Ran03])

$$\partial := \frac{1}{2}(\nabla^{(\mathrm{Re})} - i\,\nabla^{(\mathrm{Im})}), \tag{2.40a}$$

$$\overline{\partial} := \frac{1}{2}(\nabla^{(\mathrm{Re})} + i\,\nabla^{(\mathrm{Im})}). \tag{2.40b}$$

Similarly, the complex Hessian operator is defined as $\partial\partial^{\dagger}$.

Finally, we denote by $\overleftarrow{\nabla}$ and $\overleftarrow{\partial}$, the real- and complex-valued Nabla operators which are acting on functions to their *left*, instead of to their right.

Note that the symbol ∂ usually distinguishes *partial* derivatives from total derivatives. Since we will only be dealing with partial derivatives, we can employ $\partial = (\partial_0, \ldots, \partial_{n-1})^{\top}$ in order to denote *complex* partial derivatives, while *real* partial derivatives are represented by the symbol $\nabla = (\nabla_0, \ldots, \nabla_{2n-1})^{\top}$. A simple calculation allows us to recover the standard properties of complex partial derivatives.

Lemma 2.15 (Complex derivatives)
For all $x \in \mathbb{C}^n$ and $k, l \in \{0, ..., n-1\}$, we have

$$\partial_k x_l = \overline{\partial_k} \overline{x_l} = \delta_{k,l}, \tag{2.41}$$
$$\partial_k \overline{x_l} = \overline{\partial_k} x_l = 0. \tag{2.42}$$

Proof. For $x \in \mathbb{C}^n$ and $k, l \in \{0, ..., n-1\}$ a direct calculation yields

$$\partial_k x_l = \frac{\left(\nabla_k^{(\mathrm{Re})} - i\nabla_k^{(\mathrm{Im})}\right)(\mathrm{Re}(x_l) + i\,\mathrm{Im}(x_l))}{2} = \frac{\nabla_k^{(\mathrm{Re})}\,\mathrm{Re}(x_l) + (-i)\nabla_k^{(\mathrm{Im})}\,(i\,\mathrm{Im}(x_l))}{2} = \delta_{k,l},$$

and the other results follow similarly. □

We extend the definition of $\mathfrak{j}$ and $\mathfrak{J}$ to differential operators in a natural way by means of test functions, i.e. for all continuously differentiable functions $F : \mathbb{R}^{2n} \to \mathbb{R}$ we define

$$\mathfrak{j}(\partial)\, F := \mathfrak{j}\,(\partial f)\,, \tag{2.43}$$
$$\mathfrak{J}(\partial\partial^\dagger)\, F := \mathfrak{J}\left(\partial\partial^\dagger f\right), \tag{2.44}$$

where $f := F \circ \mathfrak{j} : \mathbb{C}^n \to \mathbb{R}$. This definition allows us to apply the decomposition results from the previous section to differential operators as well. In particular, we can represent the real gradient ∇ and Hessian $\nabla\nabla^\top$ in terms of their complex counterparts.

Lemma 2.16 (Complex representation of gradient and Hessian)
We have

$$\nabla = \mathfrak{j}\left(2\,\overline{\partial}\right), \tag{2.45}$$

as well as

$$\nabla\nabla^\top = 2\left[\mathfrak{J}(\overline{\partial}\,\overline{\partial}^\dagger) + \breve{\mathfrak{J}}(\overline{\partial}\,\overline{\partial}^\top)\right], \tag{2.46}$$

where

$$\left(\nabla\nabla^\top\right)_+ = 2\,\mathfrak{J}(\overline{\partial}\,\overline{\partial}^\dagger), \tag{2.47}$$
$$\left(\nabla\nabla^\top\right)_- = 2\,\breve{\mathfrak{J}}(\overline{\partial}\,\overline{\partial}^\top). \tag{2.48}$$

Proof. By definition of the complex derivatives we obtain

$$\mathfrak{j}\left(2\,\overline{\partial}\right) = \mathfrak{j}\left(\nabla^{(\mathrm{Re})} + i\,\nabla^{(\mathrm{Im})}\right) = \begin{pmatrix} \nabla^{(\mathrm{Re})} \\ \nabla^{(\mathrm{Im})} \end{pmatrix} = \nabla. \tag{2.49}$$

Combining the representation of Eq. (2.45) with the results of Lemma 2.13 yields

$$\nabla\nabla^\top = \mathfrak{j}\left(2\,\overline{\partial}\right)\left[\mathfrak{j}\left(2\,\overline{\partial}\right)\right]^\top = 2\left[\mathfrak{J}(\overline{\partial}\overline{\partial}^\dagger) + \breve{\mathfrak{J}}(\overline{\partial}\overline{\partial}^\top)\right], \tag{2.50}$$

where the first term corresponds to $\left(\nabla\nabla^\top\right)_+$ and the second one to $\left(\nabla\nabla^\top\right)_-$. □

Finally, we consider a Frobenius product of a generalized outer product with a Hessian operator and provide a complex-valued representation.

Lemma 2.17 (Frobenius product of Hessian and outer product)
For all $\mathsf{M}, \widetilde{\mathsf{M}} \in \mathbb{C}^{n,n'}$, we find that

$$\frac{1}{2}\mathfrak{J}_{1/2}(\mathsf{M})\,\mathfrak{J}_{1/2}\left(\widetilde{\mathsf{M}}\right)^\top : \nabla\nabla^\top = \operatorname{Re}\left(\mathsf{M}\widetilde{\mathsf{M}}^\dagger : \partial\partial^\dagger + \mathsf{M}\widetilde{\mathsf{M}}^\top : \partial\partial^\top\right). \tag{2.51}$$

In particular, $\mathsf{M} = \widetilde{\mathsf{M}}$ yields

$$\frac{1}{2}\mathfrak{J}_{1/2}(\mathsf{M})\,\mathfrak{J}_{1/2}(\mathsf{M})^\top : \nabla\nabla^\top = \mathsf{M}\mathsf{M}^\dagger : \partial\partial^\dagger + \frac{1}{2}\left(\mathsf{M}\mathsf{M}^\top : \partial\partial^\top + \overline{\mathsf{M}\mathsf{M}^\top} : \overline{\partial\partial^\top}\right). \tag{2.52}$$

Proof. We decompose the Frobenius product analogously to Lemma 2.9, evaluate the $\mathfrak{J}_{1/2}$-terms by employing Lemma 2.13 and rewrite the Hessian via Lemma 2.16, which gives us

$$\begin{aligned}
\mathfrak{J}_{1/2}(\mathsf{M})\,\mathfrak{J}_{1/2}\left(\widetilde{\mathsf{M}}\right)^\top : \nabla\nabla^\top &= \mathfrak{J}\left(\mathsf{M}\widetilde{\mathsf{M}}^\dagger\right) : \mathfrak{J}\left(\overline{\partial\partial^\dagger}\right) + \breve{\mathfrak{J}}\left(\mathsf{M}\widetilde{\mathsf{M}}^\top\right) : \breve{\mathfrak{J}}\left(\overline{\partial\partial^\top}\right) \\
&= 2\left[\operatorname{Re}\left(\mathsf{M}\widetilde{\mathsf{M}}^\dagger\right) : \operatorname{Re}\left(\partial\partial^\dagger\right) - \operatorname{Im}\left(\mathsf{M}\widetilde{\mathsf{M}}^\dagger\right) : \operatorname{Im}\left(\partial\partial^\dagger\right)\right] \\
&\quad + 2\left[\operatorname{Re}\left(\mathsf{M}\widetilde{\mathsf{M}}^\top\right) : \operatorname{Re}\left(\partial\partial^\top\right) - \operatorname{Im}\left(\mathsf{M}\widetilde{\mathsf{M}}^\top\right) : \operatorname{Im}\left(\partial\partial^\top\right)\right] \\
&= 2\operatorname{Re}\left(\mathsf{M}\widetilde{\mathsf{M}}^\dagger : \partial\partial^\dagger + \mathsf{M}\widetilde{\mathsf{M}}^\top : \partial\partial^\top\right).
\end{aligned}$$

If $\mathsf{M} = \widetilde{\mathsf{M}}$, then both $\mathsf{M}\mathsf{M}^\dagger$ as well as $\partial\partial^\dagger$ are hermitian, which is why

$$\overline{\mathsf{M}\mathsf{M}^\dagger} : \overline{\partial\partial^\dagger} = \left(\mathsf{M}\mathsf{M}^\dagger\right)^\top : \left(\partial\partial^\dagger\right)^\top = \mathsf{M}\mathsf{M}^\dagger : \partial\partial^\dagger. \tag{2.53}$$

□

2.1.4. Holomorphic functions and the residue theorem

We distinguish two notions of differentiability.[3] A function $f : \mathbb{C}^n \to \mathbb{C}^m$ is called (continuously) $\mathbb{R}$-differentiable,[4] if its representing function $F := \mathfrak{j} \circ f \circ \mathfrak{j}^{-1} : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ is (continuously) differentiable in the usual sense. A stronger notion is given by the concept of complex differentiability or the equivalent notion of *holomorphy* (c.f. [Sch05], Theorem 1.2.25).

Definition 2.18 (Holomorphic function)
Let $\mathcal{U} \subset \mathbb{C}^n$ be an open set. A continuously $\mathbb{R}$-differentiable function

$$f : \mathcal{U} \to \mathbb{C}^m, \tag{2.54}$$

[3] c.f. [FL11], Section I.5, which covers real- and complex differentiability in the one-dimensional case
[4] c.f. [Sch05], Section 1.2.3

is called *holomorphic* on the set $\mathcal{U}$, if it satisfies the *Cauchy-Riemann equation* on $\mathcal{U}$ in each variable, i.e. if[a]

$$\overline{\partial}_k f = 0, \text{ on } \mathcal{U}, \quad \forall k \in \{0, ..., n-1\}. \tag{2.55}$$

[a]c.f. [Ran03] and [Sch05], Theorem 1.2.25

We recall the notion of a *line integral*.

Definition 2.19 (Line integral[a])
"Let c be a smooth curve given by $(\mathsf{c}(t))_{a\leq t\leq b}$ and suppose that f is continuous at all the points $\mathsf{c}(t)$. Then, the integral of f along c is given by

$$\int_{\mathsf{c}} f(z)\,\mathrm{d}z = \int_a^b f(\mathsf{c}(t))\,\dot{\mathsf{c}}(t)\,\mathrm{d}t." \tag{2.56}$$

[a]We state Definition 4.3 of [BN10], making only slight changes to the notation.

For all $r > 0$, we denote the open disc of radius r with the origin removed by

$$K_{*,r} := \{z \in \mathbb{C} \mid 0 < |z| < r\}. \tag{2.57}$$

This is called a *deleted neighborhood*[5] *of* 0. If the integrand of a *closed* line integral is given by a *holomorphic* function, we can evaluate such an integral by means of the *residue theorem*.

Theorem 2.20 (Cauchy's residue theorem[a])
Let $h : \mathbb{C} \to \mathbb{C}$ be holomorphic in a simply connected domain $D \subset \mathbb{C}$, "except for isolated singularities at $z_1, z_2, ..., z_m$". Let c be a closed curve contained in D, surrounding these singularities without intersecting any of them. Then

$$\frac{1}{2\pi i}\int_{\mathsf{c}} h(z)\,\mathrm{d}z = \sum_{k=1}^{m} n(\mathsf{c}, z_k)\ \mathrm{Res}|_{z_k}(h), \tag{2.58}$$

where

$$n(\mathsf{c}, a) = \frac{1}{2\pi i}\int_{\mathsf{c}} \frac{\mathrm{d}z}{z-a}, \tag{2.59}$$

"is called the *winding number of* c *around* a"[b] and $\mathrm{Res}|_{z_0}(h)$ denotes the *residue of* h at z_0,[c] i.e. if $h(z) = \sum_{-\infty}^{\infty} a_k(z-z_0)^k$ in a deleted neighborhood of z_0, then $\mathrm{Res}|_{z_0}(h) := a_{-1}$.

[a]We state [BN10], Theorem 10.5, making slight changes to the notation and collecting the definitions of the involved quantities from the corresponding Chapter 10.
[b][BN10], Definition 10.2
[c]c.f. [BN10], Definition 10.1

For a vector-valued function $h : \mathbb{C} \to \mathbb{C}^m$, we extend the residue definition by setting $\mathrm{Res}|_{z_0}(h) := (\mathrm{Res}|_{z_0}(h_k))_{k\in\{0,...,m-1\}}$. We note that $h : \mathbb{C} \to \mathbb{C}^m$ is holomorphic, if all of its components h_k are holomorphic functions, c.f. Definition 2.18.

[5]c.f. [BN10], Section 9.1

2.2. Discrete Fourier transform

We introduce the notions of a *discrete Fourier transform* (DFT) and of a *circulant matrix*. The central result, Lemma 2.27, will relate both concepts by stating that circulant matrices are precisely the ones that are diagonalized by a DFT. Finally, we show that the DFT gives rise to a reflection mapping, which will later on prove useful in distinguishing directed and undirected coupling topologies, c.f. Assumption 3.10.

2.2.1. Definitions and basic properties

Let $\mathfrak{u} := \exp\left(\frac{2\pi i}{n}\right)$ denote the n-th unit root and define (see for instance [MP72])

$$Q := \frac{1}{\sqrt{n}}\left(\mathfrak{u}^{k\cdot l}\right)_{k,l=0,\ldots,n-1} = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & \mathfrak{u} & \mathfrak{u}^2 & \ldots & \mathfrak{u}^{n-2} & \mathfrak{u}^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \mathfrak{u}^{n-1} & \mathfrak{u}^{(n-1)\cdot 2} & \ldots & \mathfrak{u}^{(n-1)\cdot(n-2)} & \mathfrak{u}^{(n-1)\cdot(n-1)} \end{pmatrix}. \quad (2.60)$$

Example 2.21 (DFT matrix for $n = 2, 3, 4$)
For $n = 2, 3, 4$, we have

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (2.61)$$

$$Q = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2\pi i}{3}} & e^{\frac{4\pi i}{3}} \\ 1 & e^{\frac{4\pi i}{3}} & e^{\frac{8\pi i}{3}} \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{-1+i\sqrt{3}}{2} & \frac{-1-i\sqrt{3}}{2} \\ 1 & \frac{-1-i\sqrt{3}}{2} & \frac{-1+i\sqrt{3}}{2} \end{pmatrix}, \quad (2.62)$$

$$Q = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}. \quad (2.63)$$

Note that for all *even* n (and *only* for even n), the matrix Q contains a column of the form $\frac{1}{\sqrt{n}}(1, -1, \ldots)$, i.e. with alternating signs. As we will see later on, this vector corresponds to an *anti-synchronization* eigenmode, which can therefore only occur if n is an even number.

In the following lemma, we show that Q is a *unitary* matrix (c.f. [MP72]), i.e. its inverse is given by its Hermitian conjugate $Q^\dagger$.

Lemma 2.22 (Unitarity of Q)
Q is a unitary matrix, i.e. $QQ^\dagger = Q^\dagger Q = \mathbb{1}_{n\times n}$.

Proof. For all $i, j \in \{0, ..., n-1\}$, we have[6]

$$(QQ^\dagger)_{i,j} = \frac{1}{n}\sum_{k=0}^{n-1} \mathfrak{u}^{i\cdot k}\overline{\mathfrak{u}}^{k\cdot j} = \frac{1}{n}\sum_{k=0}^{n-1}(\mathfrak{u}^{i-j})^k = \delta_{i,j}, \tag{2.64}$$

where for the case of $i \neq j$, we have employed the summation formula of the geometric series, giving us

$$(QQ^\dagger)_{i,j} = \frac{1-(\mathfrak{u}^{i-j})^n}{1-\mathfrak{u}^{i-j}} = \frac{1-(\mathfrak{u}^n)^{i-j}}{1-\mathfrak{u}^{i-j}} = 0,$$

since $\mathfrak{u}^n = 1$. By an analogous calculation, the statement follows for $Q^\dagger Q$. □

This unitarity result finally allows us to define the DFT and its inverse.

Definition 2.23 (Discrete Fourier transform (DFT))
The complex linear mapping

$$\mathcal{F} : \mathbb{C}^n \to \mathbb{C}^n, \quad x \to \tilde{x} := \mathcal{F}(x) := Q^\dagger x \tag{2.65}$$

is called *discrete Fourier transform*. By virtue of Lemma 2.22, an inverse transformation exists and is given by

$$\mathcal{F}^{-1} : \mathbb{C}^n \to \mathbb{C}^n, \quad \tilde{x} \to x := \mathcal{F}^{-1}(\tilde{x}) := Q\tilde{x}. \tag{2.66}$$

Note that the assignment of $Q^\dagger$ to the DFT and Q to its inverse is purely conventional - we could have equally well switched their respective roles.

2.2.2. Convolution theorem

We introduce the notion of a *circulant matrix* and employ convolution formulas for the DFT in order to show that the DFT diagonalizes circulant matrices. We start by introducing a *cyclic index convention*, which will implicitly be assumed in the remainder of this thesis.

Definition 2.24 (Cyclic index convention)
Any given vector $a \in \mathbb{C}^n$ is extended to an n-periodic sequence by setting[a]

$$a_{k+z\cdot n} := a_k \text{ for all } k \in \{0, ..., n-1\}, z \in \mathbb{Z}. \tag{2.67}$$

[a] c.f. [MP72] and [Won11], p. 2

This convention amounts to mapping the index set of integers $\mathbb{Z}$ to the cyclic group $\mathbb{Z}_n := \mathbb{Z}/(n\mathbb{Z})$ and representing its elements by the set $\{0, ..., n-1\}$. Making use of this index convention, we can define a *circular convolution* of two vectors and the related notion of a *circulant matrix* as follows.

[6] c.f. [Sch10], Chapter 7.1

Definition 2.25 (Circular convolution and circulant matrices)
The *circulant convolution* of two vectors $a, b \in \mathbb{C}^n$ is defined as[a]

$$a \circledast b := \left(\sum_{l=0}^{n-1} a_{k-l}\, b_l \right)_{k=0,\dots,n-1} . \tag{2.68}$$

For a given vector $a \in \mathbb{C}^n$, we define a corresponding *circulant matrix*[b] as

$$A := \operatorname{cycl}(a) := (a_{k-l})_{k,l=0,\dots,n-1} . \tag{2.69}$$

[a] c.f. [Won11], Definition 4.1
[b] c.f. [Won11], Chapter 3

The notion of a circulant matrix allows us to rewrite a circular convolution as a matrix-vector multiplication[7], i.e. we have (in the notation of the previous definition)

$$a \circledast b = (\operatorname{cycl}(a))\, b = A\, b. \tag{2.70}$$

In the following theorem, we show that a DFT transforms a circular convolution into *pointwise multiplication*, which for $x, y \in \mathbb{C}^n$, we define as

$$(x \odot y)_k := x_k\, y_k, \quad \forall k \in \{0, \dots, n-1\} . \tag{2.71}$$

This is a special case of the so-called *Hadamard product*, c.f. [Mil07], Definition 1.1.

Theorem 2.26 (Convolution formulas)
For all $x, y \in \mathbb{C}^n$, we have[a]

$$\mathcal{F}(x \circledast y) = \sqrt{n}\, \mathcal{F}(x) \odot \mathcal{F}(y), \tag{2.72}$$

$$\mathcal{F}^{-1}(x \circledast y) = \sqrt{n}\, \mathcal{F}^{-1}(x) \odot \mathcal{F}^{-1}(y), \tag{2.73}$$

from which it follows that

$$\mathcal{F}^{-1}(x \odot y) = \frac{1}{\sqrt{n}}\, \mathcal{F}^{-1}(x) \circledast \mathcal{F}^{-1}(y), \tag{2.74}$$

$$\mathcal{F}(x \odot y) = \frac{1}{\sqrt{n}}\, \mathcal{F}(x) \circledast \mathcal{F}(y). \tag{2.75}$$

[a] c.f. for instance [Won11], Theorem 1.7 (where a slightly different normalization convention is used)

Proof. These statements follow by direct calculation, using similar techniques as in the proof of Lemma 2.22.

These convolution formulas allow us to characterize circulant matrices as those matrices which can be diagonalized by a DFT, c.f. [RE11], where it is shown that "a tensor of arbitrary order, which is circulant with respect to two particular modes, can be diagonalized in those modes by discrete Fourier transforms."

[7] c.f. [Won11], Proposition 4.3

Lemma 2.27 (Diagonal matrices in DFT eigenbasis)
A matrix $A \in \mathbb{C}^{n,n}$ is *diagonalized* by the DFT, i.e. $\tilde{A} := Q^\dagger A Q$ is a diagonal matrix, if and only if A is a *circulant* matrix, i.e. $A = \text{cycl}\,(a)$ for some $a \in \mathbb{C}^n$. More specifically, the DFT

$$\tilde{A} := Q^\dagger A Q = Q^\dagger \text{cycl}\,(a)\, Q, \tag{2.76}$$

of a circulant matrix $A = \text{cycl}\,(a)$, is given by

$$\tilde{A} = \text{diag}(\sqrt{n}\, Q^\dagger a). \tag{2.77}$$

Proof. A matrix $A \in \mathbb{C}^{n,n}$ is diagonalized by the DFT if and only if there is some $\lambda \in \mathbb{C}^n$, s.t.

$$Q^\dagger A\, Q = \text{diag}(\lambda),$$

i.e. if and only if for some $\lambda \in \mathbb{C}^n$ we have

$$\begin{aligned} A &= Q\,\text{diag}(\lambda) Q^\dagger = Q\,\text{diag}(\lambda) \sum_{k=0}^{n-1} \left(e_k \,(e_k)^\top \right) Q^\dagger = \sum_{k=0}^{n-1} Q\,(\lambda \odot e_k)\,(e_k)^\top\, Q^\dagger \\ &= \sum_{k=0}^{n-1} \mathcal{F}^{-1}(\lambda \odot e_k)\,(e_k)^\top\, Q^\dagger = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \mathcal{F}^{-1}(\lambda) \circledast \mathcal{F}^{-1}(e_k)\,(e_k)^\top\, Q^\dagger \\ &= \frac{1}{\sqrt{n}}\,\text{cycl}\,(Q\lambda)\, Q \sum_{k=0}^{n-1} \left(e_k\,(e_k)^\top \right) Q^\dagger = \text{cycl}\left(\frac{Q\lambda}{\sqrt{n}} \right), \end{aligned}$$

where in the second step, we have inserted a decomposition of the unit matrix, in the fifth step, we have employed the convolution identity (2.74) and in the sixth step, we have represented the circulant convolution in terms of a circulant matrix. Thus we have shown that if A is diagonalizable by the DFT with diagonal λ, then it can be represented as the circulant matrix generated by $\sqrt{n}^{-1} Q\lambda$.
Let on the other hand $A = \text{cycl}\,(a)$ for some $a \in \mathbb{C}^n$. Setting $\lambda := \sqrt{n}\, Q^\dagger a$ in the calculation above, we find that $\tilde{A} := Q^\dagger A Q = \text{diag}(\sqrt{n}\, Q^\dagger a)$, i.e. A is diagonalized by the DFT. □

In Section 3.2.1.2 we will argue that a circulant matrix is the natural choice for introducing a linear coupling in a ring of oscillators. Employing Lemma 2.27, we subsequently find that all of these coupling matrices can be simultaneously diagonalized by the DFT (Section 3.2.1.5).

2.2.3. Symmetries

Recalling that Definition 2.24 periodically extends the admissible set of indices for a $\mathbb{C}^n$-vector from $\{0, ..., n-1\}$ to the whole of $\mathbb{Z}$, we can introduce a *reflection mapping*[8] R, which reorders the components of a vector by performing a sign-change on the indices.

[8] c.f. [Hor10]

Definition 2.28 (Reflection und symmetry projection)
We define the so-called *reflection mapping* R as

$$R : \mathbb{C}^n \to \mathbb{C}^n,\ x \to Rx := (x_{n-k})_{k=0,\dots,n-1} = (x_{-k})_{k=0,\dots,n-1}\,, \tag{2.78}$$

i.e. we can represent this linear map by the $\mathbb{R}^{n,n}$-valued matrix

$$R := (\delta_{i,n-j})_{i,j}\,. \tag{2.79}$$

An element $x \in \mathbb{C}^n$ is called *even*, if $Rx = x$ and *odd*, if $Rx = -x$, c.f. [MP72]. Making use of R, we can define *projection mappings* $\mathcal{P}_\pm$ to the space of even and odd vectors (c.f. [Can11]) as

$$\mathcal{P}_+ := \frac{\mathbb{1}_{n\times n} + R}{2}, \quad \mathcal{P}_- := \frac{\mathbb{1}_{n\times n} - R}{2}. \tag{2.80}$$

Note that indeed we have $R\,(\mathcal{P}_+ x) = \mathcal{P}_+ x$ and $R\,(\mathcal{P}_- x) = -\mathcal{P}_- x$, while the projection properties (idempotency) $\mathcal{P}_+^2 = \mathcal{P}_+$ and $\mathcal{P}_-^2 = \mathcal{P}_-$ follow from the identity $R^2 = \mathbb{1}_{n\times n}$. The reflection mapping R can be expressed in terms of the DFT.

Lemma 2.29 (Reflection in terms of DFT)
Reflection mapping and DFT can be related by[a] $R = Q^2 = (Q^\dagger)^2$.

[a] c.f. [Hor10] and [Can11]

Proof. For $i, j \in \{0, \dots, n-1\}$, we proceed in a similar fashion as in the proof of Lemma 2.22, i.e.

$$(QQ)_{i,j} = \frac{1}{n}\sum_{k=0}^{n-1} \mathfrak{u}^{i\cdot k}\mathfrak{u}^{k\cdot j} = \frac{1}{n}\sum_{k=0}^{n-1} (\mathfrak{u}^{i+j})^k = \delta_{i,n-j} = R_{i,j},$$

where for the case of $i \neq n-j$ we again made use of the summation formula for the *geometric series*, giving us

$$(QQ)_{i,j} = \frac{1-(\mathfrak{u}^{i+j})^n}{1-\mathfrak{u}^{i+j}} = \frac{1-(\mathfrak{u}^n)^{i+j}}{1-\mathfrak{u}^{i+j}} = 0,$$

since $\mathfrak{u}^n = 1$. Since R is a real-valued matrix, the analogous statement for $(Q^\dagger)^2$ can be obtained by complex conjugation. □

In the following lemma we show that the parity of a vector, as defined in Definition 2.28, persists under the DFT. In the special case of a DFT of an $\mathbb{R}^n$-valued vector, we furthermore find that reflection mapping and complex conjugation coincide.

Lemma 2.30 (DFT of even and odd vectors[a])
i) If $x \in \mathbb{C}^n$ is even (odd), then $\tilde{x} := \mathcal{F}(x) = Q^\dagger x$ is even (odd) as well. More generally, we have $\mathcal{P}_\pm\,\tilde{x} = \widetilde{\mathcal{P}_\pm\, x}$, i.e. projection operators commute with the DFT.

ii) If $x \in \mathbb{R}^n$, then we obtain $\mathcal{P}_+ \tilde{x} = \mathrm{Re}(\tilde{x})$ as well as $\mathcal{P}_- \tilde{x} = i\,\mathrm{Im}(\tilde{x})$.

iii) If $x \in \mathbb{R}^n$ is even (odd), we find that $\tilde{x} \in \mathbb{R}^n$ ($\tilde{x} \in i\,\mathbb{R}^n$).

[a] The results can for instance be found in [Can11] and [Won11].

Proof. i) Let $x \in \mathbb{C}^n$ be even (odd), i.e. $Rx = \pm x$. Since $R = (Q^\dagger)^2$ commutes with $Q^\dagger$, we can conclude that $R\tilde{x} = RQ^\dagger x = Q^\dagger R x = \pm Q^\dagger x = \pm\tilde{x}$. The general statement follows by the same line of argument.

ii) Let $x \in \mathbb{R}^n$. Since Q and $Q^\dagger$ are symmetric matrices, we have $\overline{Q^\dagger} = Q$, from which it follows that[9] $\overline{\tilde{x}} = \overline{Q^\dagger x} = Qx = Q(Q^\dagger R Q^\dagger)x = R\tilde{x}$. Recalling the definition of the projection operators yields the result.

iii) Let $x \in \mathbb{R}^n$ be even (odd). According to part i), $\tilde{x}$ inherits this parity, which is why we can apply part ii) to find that the imaginary (real) part of $\tilde{x}$ vanishes. □

As noted before, these results will be used to distinguish directed and undirected coupling topologies, c.f. remark below Assumption 3.10.

Finally, we provide a technical result on the absolute value of weighted rows of the DFT matrix, which we will employ in the proof of Lemma 6.5.

Lemma 2.31 (Weighted DFT rows)
Let $k \in \{0, ..., n-1\}$ and denote by $q^{(k)}$ the k'th row of the DFT matrix $Q^\dagger$, i.e.

$$q^{(k)} := Q^\dagger_{k,\cdot} = (Q^\dagger_{k,j})_j = \frac{1}{\sqrt{n}}\,(\overline{\mathrm{u}}^{k\,j})_j. \tag{2.81}$$

Then for all $z \in \mathbb{C}$ it follows that

$$\left\|\mathrm{Re}\left(z\,q^{(k)}\right)\right\|^2 = \begin{cases} (\mathrm{Re}\,(z))^2, & k \in \{0, n/2\}, \\ \frac{1}{2}\,|z|^2, & k \in \{1, ..., n-1\} \setminus \{n/2\}, \end{cases} \tag{2.82}$$

$$\left\|\mathrm{Im}\left(z\,q^{(k)}\right)\right\|^2 = \begin{cases} (\mathrm{Im}\,(z))^2, & k \in \{0, n/2\}, \\ \frac{1}{2}\,|z|^2, & k \in \{1, ..., n-1\} \setminus \{n/2\}, \end{cases} \tag{2.83}$$

as well as

$$\left(\mathrm{Re}\left(q^{(k)}\right)\right)^\top \mathrm{Im}\left(q^{(k)}\right) = 0, \quad \forall k \in \{0, ..., n-1\}. \tag{2.84}$$

Proof. Let $k \in \{0, ..., n-1\}$ and $z \in \mathbb{C}$. We observe that

$$\mathrm{Re}\left(z\,q^{(k)}\right) = \mathrm{Re}\,(z)\;\mathrm{Re}\left(q^{(k)}\right) - \mathrm{Im}\,(z)\;\mathrm{Im}\left(q^{(k)}\right), \tag{2.85}$$

[9] c.f. [Won11], Proposition 1.25

which implies that

$$\left\|\mathrm{Re}\left(z\, q^{(k)}\right)\right\|^2 = (\mathrm{Re}\,(z))^2 \left\|\mathrm{Re}\left(q^{(k)}\right)\right\|^2 + (\mathrm{Im}\,(z))^2 \left\|\mathrm{Im}\left(q^{(k)}\right)\right\|^2 - 2\,\mathrm{Re}\,(z)\,\mathrm{Im}\,(z) \left(\mathrm{Re}\left(q^{(k)}\right)\right)^\top \mathrm{Im}\left(q^{(k)}\right). \tag{2.86}$$

In the case of $k \in \{0, n/2\}$, it follows that $q^{(k)} \in \mathbb{R}^n$, which implies that

$$\left\|\mathrm{Re}\left(q^{(k)}\right)\right\|^2 = 1, \quad \left\|\mathrm{Im}\left(q^{(k)}\right)\right\|^2 = 0, \quad \left(\mathrm{Re}\left(q^{(k)}\right)\right)^\top \mathrm{Im}\left(q^{(k)}\right) = 0. \tag{2.87}$$

For $k \in \{1, ..., n-1\} \setminus \{n/2\}$ on the other hand, Lemma 2.29 yields

$$\begin{aligned} 0 = R_{kk} = (Q^\dagger\, Q^\dagger)_{kk} &= \left(q^{(k)}\right)^\top q^{(k)} \\ &= \left(\mathrm{Re}\left(q^{(k)}\right)\right)^\top \mathrm{Re}\left(q^{(k)}\right) - \left(\mathrm{Im}\left(q^{(k)}\right)\right)^\top \mathrm{Im}\left(q^{(k)}\right) + 2i \left(\mathrm{Re}\left(q^{(k)}\right)\right)^\top \mathrm{Im}\left(q^{(k)}\right), \end{aligned} \tag{2.88}$$

which implies that

$$\left\|\mathrm{Re}\left(q^{(k)}\right)\right\|^2 = \frac{1}{2}, \quad \left\|\mathrm{Im}\left(q^{(k)}\right)\right\|^2 = \frac{1}{2}, \quad \left(\mathrm{Re}\left(q^{(k)}\right)\right)^\top \mathrm{Im}\left(q^{(k)}\right) = 0. \tag{2.89}$$

Inserting Eqs. (2.87) and (2.89) into Eq. (2.86) now yields Eq. (2.82). The analogous result on the imaginary part (c.f. Eq. (2.83)) can be verified similarly and Eq. (2.84) has already been shown in Eqs. (2.87) and (2.89). □

Note that Lemma 2.31 distinguishes between the index sets $\{0, n/2\}$ and $\{1, ..., n-1\} \setminus \{n/2\}$. This distinction which will prove to be crucial in studying the system's asymptotic synchronization behavior, c.f. Chapter 6, i.p. Section 6.2.2.

2.3. Complex-valued SDEs

This section aims at introducing complex-valued stochastic differential equations (SDEs) by decomposing the complex-valued objects into their well-known real-valued components. In a first step, we introduce $\mathbb{C}^n$-valued martingales, with a particular focus on the example of complex Brownian motion. Subsequently, we study the effect of a DFT on a complex-valued SDE and finally present a version of Itô's formula which is applicable to such a complex-valued setting. All stochastic integrals in this thesis will be interpreted in the sense of Itô.[10]

2.3.1. Complex martingales

For any two real-valued, square-integrable martingales $(X(t))_{t\geq 0}, (Y(t))_{t\geq 0}$, we denote the *covariation process*[11] by $(\langle X, Y\rangle\,(t))_{t\geq 0}$. This definition can be linearly extended to the case of

[10] c.f. [Øks98], Chapter 3.3 for a comparison of Itô and Stratonovich integrals; c.f. [PMH+13] for an experimental result on an electric curcuit, perturbed by multiplicative noise, which agrees with Itô's or Stratonovich's convention, depending on the parameters of system

[11] as defined in [KS91], Chapter 1, Definition 5.5; also called cross-variation process

complex-valued martingales.

Definition 2.32 (Covariation process for complex martingales)
Let $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ be $\mathbb{C}$-valued, square integrable martingales. We define their covariation process by means of the following decomposition into real and imaginary parts as

$$\begin{aligned}\langle X, Y\rangle (t) &:= \langle \mathrm{Re}(X), \mathrm{Re}(Y)\rangle (t) - \langle \mathrm{Im}(X), \mathrm{Im}(Y)\rangle (t) \\ &\quad + i\left[\langle \mathrm{Re}(X), \mathrm{Im}(Y)\rangle (t) + \langle \mathrm{Im}(X), \mathrm{Re}(Y)\rangle (t)\right],\end{aligned} \tag{2.90}$$

where the right-hand contains only well-defined covariation processes of real-valued martingales. Note that for vanishing imaginary parts, this definition reduces to the real-valued case.
Let now $(M(t))_{t\geq 0}$ and $(N(t))_{t\geq 0}$ be $\mathbb{C}^n$-valued, square integrable martingales. We define the following covariation processes as

$$\left\langle M, N^{\dagger}\right\rangle_{k,l}(t) := \left\langle M_k, \overline{N_l}\right\rangle(t), \qquad k,l \in \{0, ..., n-1\}, \tag{2.91a}$$

$$\left\langle M, N^{\top}\right\rangle_{k,l}(t) := \langle M_k, N_l\rangle(t), \qquad k,l \in \{0, ..., n-1\}. \tag{2.91b}$$

A $\mathbb{C}^n$-valued martingale $(M(t))_{t\geq 0}$ satisfying

$$\left\langle M, M^{\top}\right\rangle(t) = 0, \quad \forall t \geq 0, \tag{2.92}$$

is called a *conformal* martingale.[a]

[a] The definition of a conformal martingale and equivalent characterizations can be found in [Ubø87], Section 4, i.p. Theorem 1.

In the following, we will consider complex martingales whose real and imaginary parts are independent. A particularly important example of such a martingale is given by a so-called *complex Brownian motion*, which can be constructed by combining two independent $\mathbb{R}^n$-valued Brownian motions into a $\mathbb{C}^n$-valued process.

Definition 2.33 (Complex Brownian motion)
Let $(B(t))_{t\geq 0}$ and $(B'(t))_{t\geq 0}$ be independent $\mathbb{R}^n$-valued Brownian motions. Then

$$W(t) := B(t) + iB'(t), \quad t \geq 0, \tag{2.93}$$

is called a *complex Brownian motion.*[a]

[a] c.f. [Kat15], Eq. (1.54)

Complex Brownian motion turns out to be a conformal martingale.

Lemma 2.34 (Covariation of real and complex Brownian motion)
Let $(B(t))_{t\geq 0}$ and $(B'(t))_{t\geq 0}$ be independent $\mathbb{R}^n$-valued Brownian motions and let $(W(t))_{t\geq 0}$, defined by $W(t) := B(t) + iB'(t)$, denote the corresponding complex Brownian motion. In

the real-valued case we recover the well-known result of

$$\left\langle B, B^\top \right\rangle (t) \equiv \left\langle B, B^\dagger \right\rangle (t) = t\ \mathbb{1}_{n\times n}, \tag{2.94}$$

while for complex Brownian motion we find that (c.f. [Cas09])

$$\left\langle W, W^\dagger \right\rangle (t) = 2\,t\ \mathbb{1}_{n\times n}, \qquad \left\langle W, W^\top \right\rangle (t) = 0. \tag{2.95}$$

This implies that complex Brownian motion $(W(t))_{t\geq 0}$ is a conformal martingale.

Proof. For $k, l \in \{0, ..., n-1\}$ we find that

$$\left\langle W_k, \overline{W}_l \right\rangle (t) = \langle B_k, B_l\rangle (t) + \langle B'_k, B'_l\rangle (t) = 2\,\delta_{k,l}\,t, \tag{2.96}$$

while

$$\langle W_k, W_l\rangle (t) = \langle B_k, B_l\rangle (t) - \langle B'_k, B'_l\rangle (t) = 0. \tag{2.97}$$

□

The DFT of a complex Brownian motion again constitutes a complex Brownian motion.

Lemma 2.35 (DFT of complex Brownian motion)
Let $(W(t))_{t\geq 0}$ be a complex Brownian motion, i.e. $W(t) = B(t)+iB'(t)$ for some independent, $\mathbb{R}^n$-valued Brownian motions $(B(t))_{t\geq 0}$ and $(B'(t))_{t\geq 0}$. Then the DFT $(\widetilde{W}(t))_{t\geq 0}$ defined by $\widetilde{W}(t) := Q^\dagger W(t)$ again constitutes a complex Brownian motion, i.e. we have $\widetilde{W}(t) = \hat{B}(t) + i\hat{B}'(t)$ for some independent, $\mathbb{R}^n$-valued Brownian motions $(\hat{B}(t))_{t\geq 0}$ and $(\hat{B}'(t))_{t\geq 0}$.

Proof. We observe that

$$\begin{pmatrix} \hat{B}(t) \\ \hat{B}'(t) \end{pmatrix} := \mathfrak{j}(Q^\dagger W(t)) = \mathfrak{J}(Q^\dagger)\,\mathfrak{j}(W(t)) = \left(\begin{array}{c|c} \mathrm{Re}\left(Q^\dagger\right) & -\mathrm{Im}\left(Q^\dagger\right) \\ \hline \mathrm{Im}\left(Q^\dagger\right) & \mathrm{Re}\left(Q^\dagger\right) \end{array}\right) \begin{pmatrix} B(t) \\ B'(t) \end{pmatrix}.$$

Now $Q^\dagger$ is unitary, so we can make use of Lemma 2.7 to infer that $\mathfrak{J}(Q^\dagger)$ is orthogonal. The result thus follows from the well-known fact that Brownian motion starting at the origin is *rotationally invariant*.[12] □

The proof only relies on the unitarity of $Q^\dagger$ and thus actually yields the stronger result of a complex Brownian motion being invariant under unitary transformations. Note that $Q^\dagger$ is in general a *complex*-valued transformation matrix, i.e. for $n > 2$ we have $\mathrm{Re}\left(Q^\dagger\right) \neq Q^\dagger$. This is why, for $n > 2$, the processes $(\hat{B}(t))_{t\geq 0}$ and $(\widetilde{B}(t))_{t\geq 0}$ generally do *not* coincide:

$$\hat{B}(t) := \mathrm{Re}(\widetilde{W}(t)) = \mathrm{Re}\left(Q^\dagger B(t) + i\,Q^\dagger B'(t)\right) \neq Q^\dagger B(t) =: \widetilde{B}(t). \tag{2.98}$$

Similarly, $(\hat{B}'(t))_{t\geq 0}$ and $(\widetilde{B'}(t))_{t\geq 0}$ do *not* agree in general.

[12] c.f. [KS91], Chapter 3, Problem 3.18

2.3.2. Complex Itô formula

We adapt Itô's formula to a complex-valued setting, i.e. a setting where we can allow for a $\mathbb{C}^n$-valued stochastic process $(x(t))_{t\geq 0}$, a $\mathbb{C}^m$-valued martingale $(M(t))_{t\geq 0}$ and an $\mathbb{R}$-differentiable function $f : \mathbb{C}^n \to \mathbb{R}$. If we rewrite those processes and the function f in terms of their real-valued counterparts, c.f. Definition 2.1, we can of course simply apply the well-known real-valued version of Itô's formula. The aim of the following theorem is to rewrite the resulting SDE in a form that only contains the original complex-valued quantities and complex partial derivatives, without relying on any real-valued representations.

Theorem 2.36 (Complex Itô formula)
Let $f : \mathbb{C}^n \to \mathbb{C}$, s.t. $F : \mathbb{R}^{2n} \to \mathbb{R}^2$, $F := \mathfrak{j} \circ f \circ \mathfrak{j}^{-1}$ is twice continuously differentiable. Let $(x(t))_{t\geq 0}$ be the solution of an SDE of the form

$$\mathrm{d}x(t) = u(x(t))\,\mathrm{d}t + \sigma(x(t))\,\mathrm{d}M(t), \tag{2.99}$$

with drift term $u : \mathbb{C}^n \to \mathbb{C}^n$, dispersion matrix $\sigma : \mathbb{C}^n \to \mathbb{C}^{n,m}$ and $(M(t))_{t\geq 0}$ given by a $\mathbb{C}^m$-valued, square-integrable martingale. Then it follows that[a]

$$\begin{aligned}\mathrm{d}f(x(t)) = {} & (\partial^\top f)(x(t))\,\mathrm{d}x(t) + (\partial^\dagger f)(x(t))\,\mathrm{d}\overline{x}(t) \\ & + (\partial\partial^\dagger f)(x(t)) : \left(\sigma(x(t))\,\mathrm{d}\left\langle M, M^\dagger\right\rangle(t)\,\sigma^\dagger(x(t))\right) \\ & + \frac{1}{2}(\partial\partial^\top f)(x(t)) : \left(\sigma(x(t))\,\mathrm{d}\left\langle M, M^\top\right\rangle(t)\,\sigma^\top(x(t))\right) \\ & + \frac{1}{2}(\overline{\partial\partial^\top} f)(x(t)) : \left(\overline{\sigma(x(t))\,\mathrm{d}\langle M, M^\top\rangle(t)\,\sigma^\top(x(t))}\right).\end{aligned} \tag{2.100}$$

[a] A statement of an analogous formula can be found in [Ubø87], Section 2.

Note that we have employed a shorthand notation for the terms containing the covariation processes, i.e.

$$\left(\sigma(x(t))\,\mathrm{d}\left\langle M, M^\dagger\right\rangle(t)\sigma^\dagger(x(t))\right)_{i,j} := \sum_{k,l}[\sigma(x(t))]_{i,k}\,[\sigma^\dagger(x(t))]_{l,j}\,\mathrm{d}\left\langle M, M^\dagger\right\rangle_{k,l}(t), \tag{2.101}$$

and analogously for the other terms.

Proof. We prove the formula by transformation to an SDE in $\mathbb{R}^{2n}$, making use of the shorthands $X(t) := \mathfrak{j}(x(t)) \in \mathbb{R}^{2n}$, $U := \mathfrak{j} \circ u \circ \mathfrak{j}^{-1} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $\Sigma := \mathfrak{J} \circ \sigma \circ \mathfrak{j}^{-1} : \mathbb{R}^{2n} \to \mathbb{R}^{2n,2m}$ and $\mathcal{M} := \mathfrak{j}(M) \in \mathbb{R}^{2m}$. We only need to treat the special case of a *real*-valued functions $f : \mathbb{C}^n \to \mathbb{R}$. The general case then follows immediately by application of the result to both real and imaginary part of f. The real-valued Itô formula[13] yields

$$\mathrm{d}f(x(t)) = \mathrm{d}F(X(t)) = (\nabla F(X(t)))^\top\,\mathrm{d}X(t) + \frac{1}{2}\left(\nabla\nabla^\top F(X(t))\right) : \mathrm{d}\left\langle \Sigma\mathcal{M}, (\Sigma\mathcal{M})^\top\right\rangle(t),$$

[13] c.f. [KS91], Theorem 3.3.6

where we have omitted a notational reference to the $X(t)$-dependence of Σ. By Lemma 2.10, we can rewrite the first term as

$$\begin{aligned}((\nabla F)(X(t)))^\top \mathrm{d}X(t) &= \mathrm{Re}\left[\left(\mathfrak{j}^{-1}[(\nabla F)(X(t))]\right)^\dagger \mathrm{d}x(t)\right] \\ &= 2\,\mathrm{Re}\left[(\overline{\partial} f)(x(t))^\dagger \mathrm{d}x(t)\right] \\ &= 2\,\mathrm{Re}\left[(\partial^\top f(x(t)))\,\mathrm{d}x(t)\right] \\ &= (\partial^\top f)(x(t))\,\mathrm{d}x(t) + (\partial^\dagger f)(x(t))\,\mathrm{d}\overline{x}(t), \end{aligned} \tag{2.102}$$

where we have employed Lemma 2.10 and Eq. (2.45). Eq. (2.102) yields the complex representation of the drift and dispersion terms, confirming the first line of Eq. (2.100). For the Itô correction terms we observe that Eq. (2.11) provides the identification

$$\Sigma\mathcal{M} = \mathfrak{j}(\sigma\, M) = \mathfrak{J}_{1/2}(\sigma\, M), \tag{2.103}$$

where in the second step we have applied that $\mathfrak{J}_{1/2}$ is a generalization of the function $\mathfrak{j}$ to $\mathbb{C}^{n,n'}$ matrices, which in the special case of $n' = 1$ coincides with $\mathfrak{j}$. This allows us to employ Eq. (2.52) of Lemma 2.17, which yields

$$\begin{aligned}\frac{1}{2}\,(\nabla\nabla^\top F(X(t))) : \mathrm{d}\left\langle \Sigma\mathcal{M}, (\Sigma\mathcal{M})^\top\right\rangle(t) = (\partial\partial^\dagger f)(x(t)) : \mathrm{d}\left\langle (\sigma M), (\sigma M)^\dagger\right\rangle(t) \\ + \frac{1}{2}\,(\partial\partial^\top f)(x(t)) :\, \mathrm{d}\left\langle (\sigma M), (\sigma M)^\top\right\rangle(t) \\ + \frac{1}{2}\,(\overline{\partial\partial^\top} f)(x(t)) :\, \mathrm{d}\overline{\left\langle (\sigma M), (\sigma M)^\top\right\rangle}(t),\end{aligned}$$

again omitting the $x(t)$-dependence of the dispersion matrices. Recalling the convention of Eq. (2.101), the result follows. □

As a direct application, we can derive a representation of the generator of a complex process. Note that the diffusion terms in the generator depend on both $\sigma\sigma^\dagger$ and $\sigma\sigma^\top$.

Corollary 2.37 (Generator for complex processes)
Let $(x(t))_{t\geq 0}$ be the $\mathbb{C}^n$-valued solution of the complex SDE

$$\mathrm{d}x(t) = u(x(t))\,\mathrm{d}t + \sigma(x(t))\,\mathrm{d}M(t), \tag{2.104}$$

with drift term $u : \mathbb{C}^n \to \mathbb{C}^n$, dispersion matrix $\sigma : \mathbb{C}^n \to \mathbb{C}^{n,m}$ and a $\mathbb{C}^m$-valued martingale $(M(t))_{t\geq 0}$.[a]

Then the *infinitesimal generator*[b] of $(x(t))_{t\geq 0}$ is given by

$$\begin{aligned}\mathcal{A}f(x) = \left(u^\top(x)\partial + u^\dagger(x)\overline{\partial}\right) f(x) + \left(\sigma \frac{\mathrm{d}\left\langle M, M^\dagger\right\rangle}{\mathrm{d}t}\sigma^\dagger\right)(x) : (\partial\partial^\dagger) f(x) \\ + \frac{1}{2}\left(\sigma \frac{\mathrm{d}\left\langle M, M^\top\right\rangle}{\mathrm{d}t}\sigma^\top\right)(x) : (\partial\partial^\top) f(x) \\ + \frac{1}{2}\left(\overline{\sigma \frac{\mathrm{d}\left\langle M, M^\top\right\rangle}{\mathrm{d}t}\sigma^\top}\right)(x) : (\overline{\partial\partial^\top}) f(x),\end{aligned} \tag{2.105}$$

where $f : \mathbb{C}^n \to \mathbb{R}$, such that $F : \mathbb{R}^{2n} \to \mathbb{R}$, $F := f \circ \mathrm{j}^{-1}$ is twice continuously differentiable. In particular, if $(M(t))_{t\geq 0}$ is an $\mathbb{R}^n$-valued Brownian motion, the terms simplify to

$$\begin{aligned}\mathcal{A}f(x) = \left(u^\top(x)\partial + u^\dagger(x)\overline{\partial}\right) f(x) \\ + \left[(\sigma\sigma^\dagger)(x) : (\partial\partial^\dagger) + \frac{1}{2}\left((\sigma\sigma^\top)(x) : (\partial\partial^\top) + (\overline{\sigma\sigma^\top})(x) : (\overline{\partial\partial^\top})\right)\right] f(x).\end{aligned} \tag{2.106}$$

[a] We implicitly assume existence and uniqueness of a solution to Eq. (2.104).
[b] c.f. [Øks98], Definition 7.3.1 and Theorem 7.5.4

Proof. The generator can be determined as in the proofs of [Øks98], Lemma 7.3.2 and Theorem 7.3.3. It can be translated into a complex-valued representation by means of Theorem 2.36. In the special case of $(M(t))_{t\geq 0}$ being an $\mathbb{R}^n$-valued Brownian motion, we find that

$$\frac{\mathrm{d}\left\langle M, M^\dagger\right\rangle}{\mathrm{d}t} = \frac{\mathrm{d}\left\langle M, M^\top\right\rangle}{\mathrm{d}t} = \mathbb{1}_{n\times n},$$

according to Lemma 2.34. □

2.4. Matrix-valued SDEs

We employ the complex Itô formula in order to derive a matrix-valued SDE, describing the evolution of the complex outer-product process. Employing the matrix-vector identification of Definition 2.2, we can define a matrix-valued SDE by its corresponding vector-valued SDE.[14]

2.4.1. Evolution of the complex outer-product process

We denote by p the function that maps a $\mathbb{C}^n$ vector to its complex outer product.

[14] c.f. [Pfa12], i.p. Definitions 2.15, 3.11, 3.21 and 3.31

Definition 2.38 (Complex outer product)
The complex outer-product mapping is defined as

$$\mathsf{p} : \mathbb{C}^n \to \mathbb{C}^{n,n}, \quad x \to xx^\dagger. \tag{2.107}$$

Note that for any $x \in \mathbb{C}^n$, the corresponding matrix $\mathsf{p}(x)$ is hermitian. Given a stochastic $\mathbb{C}^n$-valued process $(x(t))_{t\geq 0}$, we now examine the evolution of its associated complex outer-product process[15] $(\mathsf{p}(x(t)))_{t\geq 0} = (x(t)x^\dagger(t))_{t\geq 0}$.

Theorem 2.39 (Evolution of outer-product process)
Let $(x(t))_{t\geq 0}$ be the solution of the SDE

$$\mathrm{d}x(t) = u(x(t))\,\mathrm{d}t + \sigma(x(t))\,\mathrm{d}M(t), \tag{2.108}$$

where $u : \mathbb{C}^n \to \mathbb{C}^n$, $\sigma : \mathbb{C}^n \to \mathbb{C}^{n,m}$ denote drift and dispersion terms and $(M(t))_{t\geq 0}$ is a $\mathbb{C}^m$-valued martingale.[a] Then the evolution of the outer-product process $\mathsf{p}(x_t)(t)$ is given by

$$\begin{aligned}\mathrm{d}\mathsf{p}(x(t)) = &\left[u(x(t))\,x(t)^\dagger + \text{h.c.}\right]\mathrm{d}t + \left[x(t)\,(\sigma(x(t))\,\mathrm{d}M(t))^\dagger + \text{h.c.}\right] \\ &+ \left(\sigma(x(t))\,\mathrm{d}\left\langle M, M^\dagger\right\rangle(t)\,\sigma^\dagger(x(t))\right),\end{aligned} \tag{2.109}$$

where we have employed the shorthand 'h.c.' to denote the *hermitian conjugate* of the term to its left, i.e. $[A + \text{h.c.}] := A + A^\dagger$ for any matrix $A \in \mathbb{C}^{n,n}$.

If the martingale $(M(t))_{t\geq 0}$ is given by $M(t) := \mathcal{Q}^\dagger\,\boldsymbol{B}(t)$ for some $\mathcal{Q} \in \mathbb{C}^{m,m}$ and an $\mathbb{R}^m$-valued Brownian motion $(\boldsymbol{B}(t))_{t\geq 0}$, then we obtain

$$\mathrm{d}\mathsf{p}(x(t)) = u_{\mathsf{p}}(x(t))\,\mathrm{d}t + \sigma_{\mathsf{p}}(x(t))\,\mathrm{d}\boldsymbol{B}(t), \tag{2.110}$$

where for any $x \in \mathbb{C}^n$, the matrix-valued drift term $u_{\mathsf{p}}(x) \in \mathbb{C}^{n,n}$ is given by

$$u_{\mathsf{p}}(x) := \left[u(x)\,x^\dagger + \text{h.c.}\right] + \sigma(x)(\mathcal{Q}^\dagger\mathcal{Q})\sigma^\dagger(x), \tag{2.111}$$

and the $\mathbb{C}^{n^2,m}$-valued dispersion matrix $\sigma_{\mathsf{p}}(x)$ is defined as

$$(\sigma_{\mathsf{p}})_{ij,r}(x) := x_i\,\overline{(\sigma(x)\mathcal{Q}^\dagger)}_{jr} + \overline{x}_j\,(\sigma(x)\mathcal{Q}^\dagger)_{ir}, \quad i,j \in \{0,...,n-1\}\,,\ r \in \{0,...,m-1\}\,, \tag{2.112}$$

where we have used a '*double index*' ij for labeling the n^2 rows of the matrix σ_{p}.

[a] Again, we implicitly assume existence and uniqueness of such a solution.

[15] c.f. [SC08], Section 5.8, where a *real-valued* outer-product process is studied

Proof. For $i, j \in \{0, ..., n-1\}$, application of the complex Itô formula of Theorem 2.36 to the complex-valued function $f(x) \coloneqq x_i \overline{x}_j$ yields[16]

$$\begin{aligned}
\mathrm{d}(xx^\dagger)_{i,j} &= \mathrm{d}f(x) \\
&= \sum_k \left((\overline{\partial}_k f)\, \mathrm{d}\overline{x_k} + (\partial_k f)\, \mathrm{d}x_k \right) + \sum_{k,l} (\partial_k \overline{\partial}_l f) \left(\sigma \,\mathrm{d} \left\langle M, M^\dagger \right\rangle \sigma^\dagger \right)_{k,l} \\
&\quad + \frac{1}{2} \sum_{k,l} \left\{ (\partial_k \partial_l f) \left(\sigma \,\mathrm{d} \left\langle M, M^\top \right\rangle \sigma^\top \right)_{k,l} + (\overline{\partial}_k \overline{\partial}_l f) \left(\overline{\sigma \,\mathrm{d} \langle M, M^\top \rangle \sigma^\top} \right)_{k,l} \right\} \\
&= x_i \,\mathrm{d}\overline{x}_j + \overline{x}_j \,\mathrm{d}x_i + \left(\sigma \,\mathrm{d} \left\langle M, M^\dagger \right\rangle \sigma^\dagger \right)_{i,j} \\
&= [x_i \,\overline{u}_j + \overline{x}_j u_i] \,\mathrm{d}t + \left[x_i \,\overline{(\sigma \,\mathrm{d}M)_j} + \overline{x}_j (\sigma \,\mathrm{d}M)_i \right] + \left(\sigma \,\mathrm{d} \left\langle M, M^\dagger \right\rangle \sigma^\dagger \right)_{i,j},
\end{aligned}$$

where we have made use of the identities

$$\begin{aligned}
\partial_k f(x) &= (\partial_k x_i) \overline{x}_j = \delta_{k,i}\, \overline{x}_j, \\
\overline{\partial}_k f(x) &= x_i (\overline{\partial}_k \overline{x}_j) = \delta_{k,j}\, x_i, \\
\partial_k \overline{\partial}_l f(x) &= (\partial_k x_i)(\overline{\partial}_l \overline{x}_j) = \delta_{k,i}\, \delta_{l,j}, \\
\partial_k \partial_l f(x) &= \overline{\partial}_k \overline{\partial}_l f(x) = 0.
\end{aligned}$$

Eq. (2.109) now follows. In the special case of M being given by $M(t) \coloneqq \mathcal{Q}^\dagger \boldsymbol{B}(t)$, for some $\mathcal{Q} \in \mathbb{C}^{m,m}$ and an $\mathbb{R}^m$-valued Brownian motion $(\boldsymbol{B}(t))_{t \geq 0}$, we can conclude that

$$\sigma \,\mathrm{d} \left\langle M, M^\dagger \right\rangle \sigma^\dagger = \sigma \mathcal{Q}^\dagger \,\mathrm{d} \left\langle \boldsymbol{B}, \boldsymbol{B}^\dagger \right\rangle \mathcal{Q} \sigma^\dagger = \sigma (\mathcal{Q}^\dagger \mathcal{Q}) \sigma^\dagger \,\mathrm{d}t,$$

where in the last step we have employed Lemma 2.34. This gives us the Itô correction term appearing in Eq. (2.111). We can furthermore verify that Eq. (2.112) specifies the dispersion matrix of the complex outer-product process, by rewriting the stochastic integral terms as

$$\begin{aligned}
x_i \,\overline{(\sigma \,\mathrm{d}M)_j} + \overline{x}_j (\sigma \,\mathrm{d}M)_i &= x_i \,\overline{[\sigma \mathcal{Q}^\dagger \,\mathrm{d}\boldsymbol{B}]_j} + \overline{x}_j [\sigma \mathcal{Q}^\dagger \,\mathrm{d}\boldsymbol{B}]_i \\
&= \sum_{r=0}^{m-1} x_i \,\overline{(\sigma \mathcal{Q}^\dagger)}_{jr} \,\mathrm{d}\overline{\boldsymbol{B}}_r + \overline{x}_j (\sigma \mathcal{Q}^\dagger)_{ir} \,\mathrm{d}\boldsymbol{B}_r \\
&= \sum_{r=0}^{m-1} \underbrace{[x_i \,\overline{(\sigma \mathcal{Q}^\dagger)}_{jr} + \overline{x}_j (\sigma \mathcal{Q}^\dagger)_{ir}]}_{=:(\sigma_{\mathsf{p}})_{ij,r}(x)} \,\mathrm{d}\boldsymbol{B}_r.
\end{aligned}$$

where in the last step we have used that $\boldsymbol{B}(t) \in \mathbb{R}^m$, which yields $\overline{\boldsymbol{B}}(t) = \boldsymbol{B}(t)$. □

In the following chapter we will apply this theorem to the special case of

$$\mathcal{Q} \coloneqq \mathbb{1}_{5\times5} \otimes Q = \mathrm{blockdiag}(Q, Q, Q, Q, Q) \in \mathbb{C}^{5n,5n}, \tag{2.113}$$

c.f. Eq. (3.175) and Proposition 3.39, where Q is the matrix corresponding to the inverse DFT.

[16]In the following calculations we shorten the notation by omitting all time dependencies as well as the x-dependencies of u and σ.

2.4.2. Diffusion matrix for evolution of outer-product process

We conclude this chapter by calculating the diffusion matrices $\sigma_{\mathsf{p}}\,\sigma_{\mathsf{p}}^{\dagger}$ and $\sigma_{\mathsf{p}}\,\sigma_{\mathsf{p}}^{\top}$ corresponding to the outer-product evolution studied in the previous section.

Lemma 2.40 (Diffusion parts)
Given the setting of the previous Theorem 2.39, we assume $\mathcal{Q}$ to be a symmetric, *unitary* matrix and define $\mathcal{R} := (\mathcal{Q}^{\dagger})^2$.[a] Then the *diffusion terms* of the the outer product evolution are for all $x \in \mathbb{C}^n$ given by

$$\left(\sigma_{\mathsf{p}}\,\sigma_{\mathsf{p}}^{\dagger}\right)_{ij,kl}(x) = (xx^{\dagger})_{ik}\,\overline{(\sigma\sigma^{\dagger})}_{jl}(x) + \overline{(xx^{\dagger})}_{jl}\,(\sigma\sigma^{\dagger})_{ik}(x) + (xx^{\top})_{il}\,\overline{(\sigma\mathcal{R}\sigma^{\top})}_{jk}(x) + \overline{(xx^{\top})}_{jk}\,(\sigma\mathcal{R}\sigma^{\top})_{il}(x), \tag{2.114}$$

$$\left(\sigma_{\mathsf{p}}\,\sigma_{\mathsf{p}}^{\top}\right)_{ij,kl}(x) = \left(\sigma_{\mathsf{p}}\,\sigma_{\mathsf{p}}^{\dagger}\right)_{ij,lk}(x), \tag{2.115}$$

for all indices $i, j, k, l \in \{0, ..., n-1\}$.

[a] In the following chapters, we will apply this lemma in the special case of $\mathcal{Q} := \mathbb{1}_{5\times 5} \otimes Q \in \mathbb{C}^{5n,5n}$ and $\mathcal{R} := \mathbb{1}_{5\times 5} \otimes R$ (c.f. Eq. (3.175) and Theorem 4.71, i.p. Eq. (4.234)). Here Q is the matrix corresponding to the inverse DFT and $R = Q^2$ denotes the reflection matrix.

Proof. We observe that

$$\begin{aligned}\left(\sigma_{\mathsf{p}}\,\sigma_{\mathsf{p}}^{\dagger}\right)_{ij,kl} &= \sum_{r=0}^{m-1}\left(x_i\,\overline{(\sigma\mathcal{Q}^{\dagger})}_{jr} + \overline{x}_j\,(\sigma\mathcal{Q}^{\dagger})_{ir}\right)\overline{\left(x_k\,\overline{(\sigma\mathcal{Q}^{\dagger})}_{lr} + \overline{x}_l\,(\sigma\mathcal{Q}^{\dagger})_{kr}\right)} \\ &= (xx^{\dagger})_{ik}\,\overline{(\sigma\mathcal{Q}^{\dagger}\mathcal{Q}\sigma^{\dagger})}_{jl} + \overline{(xx^{\dagger})}_{jl}\,(\sigma\mathcal{Q}^{\dagger}\mathcal{Q}\sigma^{\dagger})_{ik} \\ &\quad + (xx^{\top})_{il}\,\overline{(\sigma\mathcal{Q}^{\dagger}(\mathcal{Q}^{\dagger})^{\top}\sigma^{\top})}_{jk} + \overline{(xx^{\top})}_{jk}\,(\sigma\mathcal{Q}^{\dagger}(\mathcal{Q}^{\dagger})^{\top}\sigma^{\top})_{il} \\ &= (xx^{\dagger})_{ik}\,\overline{(\sigma\sigma^{\dagger})}_{jl} + \overline{(xx^{\dagger})}_{jl}\,(\sigma\sigma^{\dagger})_{ik} + (xx^{\top})_{il}\,\overline{(\sigma\mathcal{R}\sigma^{\top})}_{jk} + \overline{(xx^{\top})}_{jk}\,(\sigma\mathcal{R}\sigma^{\top})_{il},\end{aligned}$$

where in the last step we have made use of the symmetry and unitarity of $\mathcal{Q}$. Similarly, we conclude that

$$\begin{aligned}\left(\sigma_{\mathsf{p}}\,\sigma_{\mathsf{p}}^{\top}\right)_{ij,kl} &= \sum_{r=0}^{m-1}\left(x_i\,\overline{(\sigma\mathcal{Q}^{\dagger})}_{jr} + \overline{x}_j\,(\sigma\mathcal{Q}^{\dagger})_{ir}\right)\left(x_k\,\overline{(\sigma\mathcal{Q}^{\dagger})}_{lr} + \overline{x}_l\,(\sigma\mathcal{Q}^{\dagger})_{kr}\right) \\ &= (xx^{\top})_{ik}\,\overline{(\sigma\mathcal{Q}^{\dagger}(\mathcal{Q}^{\dagger})^{\top}\sigma^{\top})}_{jl} + \overline{(xx^{\top})}_{jl}\,(\sigma\mathcal{Q}^{\dagger}(\mathcal{Q}^{\dagger})^{\top}\sigma^{\top})_{ik} \\ &\quad + (xx^{\dagger})_{il}\,\overline{(\sigma\mathcal{Q}^{\dagger}\mathcal{Q}\sigma^{\dagger})}_{jk} + \overline{(xx^{\dagger})}_{jk}\,(\sigma\mathcal{Q}^{\dagger}\mathcal{Q}\sigma^{\dagger})_{il} \\ &= (xx^{\top})_{ik}\,\overline{(\sigma\mathcal{R}\sigma^{\top})}_{jl} + \overline{(xx^{\top})}_{jl}\,(\sigma\mathcal{R}\sigma^{\top})_{ik} + (xx^{\dagger})_{il}\,\overline{(\sigma\sigma^{\dagger})}_{jk} + \overline{(xx^{\dagger})}_{jk}\,(\sigma\sigma^{\dagger})_{il} \\ &= \left(\sigma_{\mathsf{p}}\,\sigma_{\mathsf{p}}^{\dagger}\right)_{ij,lk}.\end{aligned}$$

□

3 Weakly coupled oscillators

In this chapter we introduce a model describing a *ring* (or more precisely, a *circulant network*) of weakly coupled harmonic oscillators. In Section 3.1 we start by introducing n harmonic oscillators of identical frequency. We observe that this uncoupled system gives rives to a large class of symmetries, which can be captured by the *Lie group* $U(n)$.[1] By virtue of Noether's theorem, these symmetries can be related to conserved quantities of the uncoupled system, also called *first integrals*[2] of the system. These are of special interest to us for the following reason. Once we introduce a weak coupling between the oscillators (Section 3.2, ff.), such an interaction will break some of the system's symmetries and thus give rise to a slow evolution of the previously conserved quantities. For instance, we will find a periodic exchange of energy between oscillators, as well as a change of their relative oscillation phases, c.f. Section 6.1.2. This is why first integrals are suitable quantities for determining the impact of a weak coupling on the oscillator system. They will consequently be studied closely within this thesis (c.f. Section 3.5.2, ff.). In particular, these quantities will prove useful for determining the system's synchronization behavior (c.f. Section 6.2, ff.).

Following the introduction of the uncoupled system, we will subsequently augment our model with weak interactions and perturbations. In Section 3.2 we introduce a *deterministic* coupling between the oscillators, while Section 3.3 encompasses *stochastic* perturbations. Here we will allow for both multiplicative noise (Section 3.3.1) as well as additive noise (Section 3.3.3). For technical reasons we will furthermore introduce a regularizing noise (Section 3.3.2). Since we want to study a *ring* of oscillators, all of the aforementioned interactions will be subject to a symmetry assumption, which enforces a *circulant* structure, c.f. for instance Assumption 3.10 and Fig. 3.1.

While the various forms of weak perturbations are introduced separately, it is important to note that they are not considered as mutually exclusive cases but are to be understood as one combined and interacting system, which will be summarized in Section 3.4.

In Section 3.5 we employ the *separation of timescales*, which is a consequence of both deterministic coupling and noise perturbations being weak compared to the strong self-coupling of the oscillators. We perform a *time-rescaling* which accelerates the evolution of the system (Sec-

[1] c.f. [Böh11], Chapter 4, Example 8
[2] c.f. [Nol14], Section 2.4.4

tion 3.5.1). Hereafter, we restrict our attention to a study of the system's first integrals. As it turns out, all of these first integrals can be neatly captured by the *complex outer product* of the system's state vector with itself (Section 3.5.2). In the following chapters we will study this matrix-valued process, with a particular focus on the asymptotic long-time behavior of its diagonal elements. This will allow us to obtain results on the system's synchronization behavior.

3.1. Uncoupled commensurate oscillators

We investigate an uncoupled system of second order ordinary differential equations (ODEs), capturing the evolution of n identical harmonic oscillators. By employing a complex notation, we reduce the description to a system of complex-valued first order ODEs. A study of the system's symmetries finally yields a concise description of its conserved quantities.

3.1.1. Strong linear coupling of massive particles

Let $n \in \mathbb{N}$ denote the number of oscillators and let $j \in \{0, ..., n-1\}$. We can model each oscillator as a set of two massive particles, coupled by spring. More precisely, let $\theta_j(t)$ and $\theta'_j(t)$ denote the positions of two *particles*,[3] both of the same mass $\mathfrak{m}$. Introducing a deterministic linear *cross-coupling* of strength K, the system can be described by the set of second order ODEs[4]

$$\mathfrak{m}\,\ddot{\theta}_j(t) = \frac{\mathsf{K}}{2}\,\Big(\theta'_j(t) - \theta_j(t)\Big)\,, \tag{3.1a}$$

$$\mathfrak{m}\,\ddot{\theta}'_j(t) = \frac{\mathsf{K}}{2}\,\Big(\theta_j(t) - \theta'_j(t)\Big)\,. \tag{3.1b}$$

The distance $\check{\eta}_j(t) := \theta_j(t) - \theta'_j(t)$, i.e. the j'th oscillator's elongation, satisfies the *harmonic oscillator* equation

$$\mathfrak{m}\,\ddot{\check{\eta}}_j(t)_k = -\mathsf{K}\,\check{\eta}_j(t). \tag{3.2}$$

Denoting the momentum by $\check{p}_j(t) := \mathfrak{m}\,\dot{\check{\eta}}_j(t)$ and introducing the space-rescaled quantities $p_j(t) := \sqrt{\frac{1}{2\mathfrak{m}}}\,\check{p}_j(t)$ and $\eta_j(t) := \sqrt{\frac{\mathsf{K}}{2}}\,\check{\eta}_j(t)$, we find a reduction to the set of *first* order ODEs

$$\dot{\eta}_j(t) = \sqrt{\frac{\mathsf{K}}{2}}\,\frac{\check{p}_j(t)}{\mathfrak{m}} = \sqrt{\frac{\mathsf{K}}{\mathfrak{m}}}\,p_j(t), \tag{3.3a}$$

$$\dot{p}_j(t) = \sqrt{\frac{1}{2\mathfrak{m}}}\,\dot{\check{p}}_j(t) = -\frac{\mathsf{K}}{\sqrt{2\mathfrak{m}}}\,\check{\eta}_j(t) = -\sqrt{\frac{\mathsf{K}}{\mathfrak{m}}}\,\eta_j(t). \tag{3.3b}$$

[3] Note that the 'prime' in $\theta'_j(t)$ does *not* symbolize a derivative but rather indicates that $\theta'_j(t)$ is the location of the particle which will be strongly coupled to the one located at $\theta_j(t)$.

[4] $\dot{\theta}(t) := \frac{\mathrm{d}}{\mathrm{d}t}\theta(t)$ and $\ddot{\theta}(t) := (\frac{\mathrm{d}}{\mathrm{d}t})^2\,\theta(t)$ denote the first and second order time derivatives of a function $\theta(t)$.

Combining for each oscillator both rescaled momentum and distance into one complex-valued process[5]

$$x_j(t) := p_j(t) + i\,\eta_j(t) \tag{3.4}$$

and introducing the constant $\kappa := \sqrt{\frac{\mathsf{K}}{\mathsf{m}}}$, we can capture the real-valued pair of ODEs given by Eq. (3.3) in one complex-valued ODE of the form

$$\dot{x}_j(t) = \dot{p}_j(t) + i\,\dot{\eta}_j(t) = \kappa\,(-\eta_j(t) + i\,p_j(t)) = i\,\kappa\,x_j(t). \tag{3.5}$$

From this equation we can readily infer the solution

$$x_j(t) = \mathrm{e}^{i\,\kappa\,t}\,x_j(0). \tag{3.6}$$

We can represent the initial condition $x_j(0)$ in polar coordinates as

$$x_j(0) = |x_j(0)|\,\mathrm{e}^{i\phi_j}, \quad \phi_j \in [0, 2\pi), \tag{3.7}$$

where ϕ_j is called the initial *phase* of the j'th oscillator. This allows us to rewrite Eq. (3.6) as

$$x_j(t) = |x_j(0)|\,\mathrm{e}^{i(\phi_j+\kappa\,t)}\,. \tag{3.8}$$

The *phase* of the j'th oscillator at time t can thus be identified as $\phi_j + \kappa\,t$, where the term $\kappa\,t$ is independent of the index j, implying that all oscillators have the same frequency κ. In vector notation, the evolution of the system $x := (x_0, ..., x_{n-1}) \in \mathbb{C}^n$ is given by

$$x(t) = \mathrm{e}^{i\,\kappa\,t}\,x_0, \tag{3.9}$$

i.e. the system evolves by a *global phase-shift*. The combined system is called an *isotropic* n-dimensional oscillator, c.f. [AW02].

Remark 3.1 (Uncoupled system)
We will refer to the system introduced in this section as the *uncoupled* system, which references the absence of any interaction between the oscillators.

3.1.2. Symmetries and conserved quantities

3.1.2.1. Hamiltonian

The uncoupled oscillator system introduced in the previous section can be described by the *Hamiltonian*[6,7]

$$H(p,\eta) := \frac{\kappa}{2}\sum_{j=0}^{n-1}\left(p_j^2 + \eta_j^2\right) = \frac{\kappa}{2}\sum_{j=0}^{n-1}|x_j|^2 = \frac{\kappa}{2}\,x^\dagger x = \frac{\kappa}{2}\,\|x\|^2 \equiv H(x), \tag{3.10}$$

[5] c.f. [Kum76], Eq. (3) and [AW02], Eq. (6)
[6] [Nol14], Section 2.2.1 defines the concepts of a Hamiltonian.
[7] c.f. [AW02], Section V, for the Hamiltonian of an n-dimensional commensurate oscillator

which can be verified by observing that *Hamilton's equations*[8]

$$\dot{\eta}_j = \frac{\partial H}{\partial p_j} = \kappa\, p_j, \tag{3.11a}$$

$$\dot{p}_j = -\frac{\partial H}{\partial \eta_j} = -\kappa\, \eta_j, \tag{3.11b}$$

reproduce Eq. (3.3). We show that the Hamiltonian is invariant under unitary transformations, i.e. under any transformation of the form $x \to x' := Ux$, with $U \in U(n)$, where[9]

$$U(n) := \left\{ U \in \mathbb{C}^{n,n} \mid U^\dagger = U^{-1} \right\}. \tag{3.12}$$

Note in particular, that for $n = 1$ we obtain the set of all *phase-factors* $\mathrm{e}^{i\varphi}$, i.e.

$$U(1) = \left\{ u \in \mathbb{C} \mid \overline{u} = u^{-1} \right\} = \{ u \in \mathbb{C} \mid |u| = 1 \} = \left\{ \mathrm{e}^{i\varphi} \mid \varphi \in [0, 2\pi) \right\}. \tag{3.13}$$

The invariance of the Hamiltonian given in Eq. (3.10) directly follows by evaluating the Hamiltonian at a transformed point $x' := Ux$, i.e.

$$H(x') = \frac{\kappa}{2}\, (x')^\dagger\, x' = \frac{\kappa}{2}\, (Ux)^\dagger\, Ux = \frac{\kappa}{2}\, x^\dagger\, U^\dagger U x = \frac{\kappa}{2}\, x^\dagger x = H(x). \tag{3.14}$$

3.1.2.2. Symmetry group

In this section we aim for a representation of unitary matrices in terms of so-called *generators*. These generators represent 'infinitesimal versions' of the unitary transformations and will later on prove useful for determining the conserved quantity related to a given symmetry, c.f. Section 3.1.2.3. Let $U \in U(n)$ be a unitary matrix. Since

$$1 = \det(U\, U^\dagger) = \det(U)\, \det(U^\dagger) = \det(U)\overline{\det(U)} = |\det(U)|^2\,, \tag{3.15}$$

we conclude that the determinant of the matrix U has to be an element of $U(1)$, i.e.

$$\det U = \mathrm{e}^{i\varphi}, \text{ for some } \varphi \in [0, 2\pi). \tag{3.16}$$

The matrix U can thus be decomposed into a $U(1)$ *phase factor* $\mathrm{e}^{i\,\frac{\varphi}{n}}$ and a *special unitary matrix* $\tilde{U}$, i.e.

$$U = \mathrm{e}^{i\,\frac{\varphi}{n}}\, \tilde{U}, \text{ with } \varphi \in [0, 2\pi) \text{ and } \tilde{U} \in SU(n), \tag{3.17}$$

where[10]

$$SU(n) := \left\{ \tilde{U} \in \mathbb{C}^{n,n} \mid \tilde{U}^\dagger = \tilde{U}^{-1} \text{ and } \det(\tilde{U}) = 1 \right\}. \tag{3.18}$$

[8] c.f. [Nol14], Section 2.2.1, Eqs. (2.11) and (2.12)
[9] c.f. [Böh11], Tabelle 4.1
[10] c.f. [Böh11], Tabelle 4.1

We verify this decomposition by solving Eq. (3.17) for $\tilde{U}$, and calculating its determinant, i.e.

$$\det(\tilde{U}) = \det\left(\mathrm{e}^{-i\frac{\varphi}{n}}\, U\right) = \mathrm{e}^{-i\varphi} \det U = \mathrm{e}^{-i\varphi}\, \mathrm{e}^{i\varphi} = 1. \tag{3.19}$$

This decomposition is commonly denoted as $U(n) = U(1) \otimes SU(n)$. We have seen that all elements of $U(1)$ have an 'exponential' representation in terms of a phase φ. Similarly, $SU(n)$ matrices can be represented in terms of *generators*

$$\mathsf{T}^{(a)} \in \mathbb{C}^{n,n}, \quad a = 1..., n^2 - 1, \tag{3.20}$$

which are elements of the corresponding *Lie algebra*[11] $su(n)$, i.e. for any special unitary matrix $\tilde{U} \in SU(n)$, there are coefficients[12] $(\alpha_a)_{a=1,...,n^2-1} \in \mathbb{R}^{n^2-1}$, s.t.

$$\tilde{U} = \exp\left(i \sum_{a=1}^{n^2-1} \alpha_a\, \mathsf{T}^{(a)}\right). \tag{3.21}$$

Setting $\mathsf{T}^{(0)} := \frac{1}{2}\, \mathbb{1}_{n\times n}$, we can represent a general $U(n)$ matrix U as

$$U = \exp\left(i \sum_{a=0}^{n^2-1} \alpha_a\, \mathsf{T}^{(a)}\right) = \mathrm{e}^{i\frac{\alpha_0}{2}} \exp\left(i \sum_{a=1}^{n^2-1} \alpha_a\, \mathsf{T}^{(a)}\right), \tag{3.22}$$

for some $(\alpha_a)_{a=0,...,n^2-1} \in \mathbb{R}^{n^2}$. Recall that the uncoupled system of Eq. (3.9) evolves by a global phase shift, i.e. by a U(1) transformation. While the $su(n)$ generators are not uniquely determined, they do need to be *hermitian* and *traceless*, as we will see in the following remark.

Remark 3.2 (Generator properties[a])
For any $a \in \{1, ..., n^2 - 1\}$ and all $\alpha_a \in \mathbb{R}$, the matrix $\exp\left(i\, \alpha_a\, \mathsf{T}^{(a)}\right)$ is special unitary. Unitarity implies that

$$\mathbb{1}_{n\times n} = \exp\left(i\, \alpha_a\, \mathsf{T}^{(a)}\right) \exp\left(-i\, \alpha_a\, \mathsf{T}^{(a)\dagger}\right) = \exp\left(i\, \alpha_a\, (\mathsf{T}^{(a)} - \mathsf{T}^{(a)\dagger})\right), \quad \forall \alpha_a \in \mathbb{R}, \tag{3.23}$$

where have employed the identity $[\exp(A)]^\dagger = \exp(A^\dagger)$. Eq. (3.23) is only satisfied if we have $\mathsf{T}^{(a)} = \mathsf{T}^{(a)\dagger}$, i.e. if the generator $\mathsf{T}^{(a)}$ is *hermitian*. The unit value of the determinant on the other hand yields

$$1 = \det\left(\exp\left(i\, \alpha_a\, \mathsf{T}^{(a)}\right)\right) = \exp\left(i\, \alpha_a \operatorname{tr}\left(\mathsf{T}^{(a)}\right)\right), \quad \forall \alpha_a \in \mathbb{R}, \tag{3.24}$$

[11] c.f. [Gil08], Chapter 4 for a characterization of a Lie algebra as the structure emerging from the linearization of a Lie group "in the neighborhood of its identity"; c.f. [Gil08], Chapter 7 for a study of the exponentiation operation and the question of how to recover a Lie group from its Lie algebra; c.f. [EW06], Chapter 1 for an abstract definition of a Lie algebra as a vector space together with a particular bilinear map called *Lie bracket*

[12] c.f. [GSS07], Appendix A

where we have used that $\det[\exp(A)] = \exp(\mathrm{tr}(A))$. Eq. (3.24) implies that $\mathrm{tr}\left(\mathsf{T}^{(a)}\right) = 0$, i.e. the generator needs to be *traceless*.

[a] c.f. [Böh11], Eq. (6.27); note that $\mathbf{T}^{(a)}$ being hermitian implies that $i\,\mathbf{T}^{(a)}$ is antihermitian

We conclude this section by looking at the example of $n = 2$.

Example 3.3 (Pauli matrices)
The generators of $U(2)$ can be chosen as $\mathsf{T}^{(0)} := \frac{1}{2}\,\mathbb{1}_{2\times 2}$ and $\mathsf{T}^{(a)} := \frac{\sigma^{(a)}}{2}$, $a \in \{1,2,3\}$, where

$$\sigma^{(1)} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^{(2)} := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^{(3)} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.25}$$

are the well-known *Pauli* matrices.[a] The generators satisfy the commutation relations[b]

$$\left[\mathsf{T}^{(a)}, \mathsf{T}^{(b)}\right] = \sum_{c=1}^{3} i\,\epsilon_{a,b,c}\,\mathsf{T}^{(c)}, \quad \forall a,b \in \{1,2,3\}\,, \tag{3.26}$$

where $\epsilon_{a,b,c}$ is the *Levi-Civita symbol*, i.e.

$$\epsilon_{1,2,3} = \epsilon_{2,3,1} = \epsilon_{3,1,2} = 1, \tag{3.27a}$$
$$\epsilon_{3,2,1} = \epsilon_{2,1,3} = \epsilon_{1,3,2} = -1, \tag{3.27b}$$
$$\epsilon_{a,b,c} = 0, \qquad \text{otherwise.} \tag{3.27c}$$

[a] c.f. [Böh11], Eq. (2.43)
[b] c.f. [Geo99], Eq. (3.1)

3.1.2.3. Conserved quantities

In accordance with *Noether's theorem* there are conserved quantities of the Hamiltonian system, related to the symmetries of the system.[13] In this section we prove that for all $a \in \{0, ..., n^2 - 1\}$, the process[14]

$$c^{(a)}(t) := x^{\dagger}(t)\mathsf{T}^{(a)}x(t) \tag{3.28}$$

is a conserved quantity, i.e. a *first integral* related to a symmetry transformation of the form $U = \exp(i\,\alpha_a\,\mathsf{T}^{(a)})$. For this purpose we show that $(c^{(a)}(t))_{t\geq 0}$ is conserved under the flow generated by $\mathsf{T}^{(b)}$, if and only if the generators $\mathsf{T}^{(a)}$ and $\mathsf{T}^{(b)}$ commute.

Theorem 3.4 (Characterization of conserved quantities)
Let $a, b \in \{0, ..., n^2 - 1\}$. Let the evolution of the $(x(t))_{t\geq 0}$ be generated by $\mathsf{T}^{(b)}$, i.e.

$$\dot{x}(t) = i\,\alpha_b\,\mathsf{T}^{(b)}\,x(t), \quad \forall t \geq 0, \tag{3.29}$$

[13] c.f. [Der10], Chapter 7 and [DiB11], Proposition 11.1
[14] We generalize the sesquilinear expressions of [Kum76], Eq. (7) and [AW02], Eq. (26), which are introduced in the special case of $u(2)$.

for some $\alpha_b \neq 0$. Then

$$c^{(a)}(t) := x^{\dagger}(t)\mathsf{T}^{(a)}x(t) \tag{3.30}$$

defines a constant process for all choices of $x(0) \in \mathbb{C}^n$, if and only if

$$\left[\mathsf{T}^{(a)}, \mathsf{T}^{(b)}\right] = 0. \tag{3.31}$$

Proof. The solution of Eq. (3.29) is given by

$$x(t) = \exp\left(i\, t\, \alpha_b\, \mathsf{T}^{(b)}\right) x(0), \quad \forall t \geq 0, \tag{3.32}$$

where $x(0) \in \mathbb{C}^n$ denotes the initial condition. Now

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} c^{(a)}(t) &= x^{\dagger}(t)\mathsf{T}^{(a)}\dot{x}(t) + (\dot{x}(t))^{\dagger}\mathsf{T}^{(a)}x(t) \\ &= i\,\alpha_b\, x^{\dagger}(t)\, \mathsf{T}^{(a)}\mathsf{T}^{(b)}\, x(t) - i\alpha_b\, x^{\dagger}(t)\, \mathsf{T}^{(b)}\mathsf{T}^{(a)}\, x(t) \\ &= i\,\alpha_b\, x^{\dagger}(t) \left[\mathsf{T}^{(a)}, \mathsf{T}^{(b)}\right] x(t). \end{aligned} \tag{3.33}$$

If $[\mathsf{T}^{(a)}, \mathsf{T}^{(b)}] = 0$, then the right-hand side vanishes for all $t \geq 0$ and all choices of $x(0)$, i.e. $(c^{(a)}(t))_{t\geq 0}$ is constant. A vanishing commutator therefore turns out to be a *sufficient* condition for $(c^{(a)}(t))_{t\geq 0}$ to be conserved. Evaluating Eq. (3.33) at $t = 0$ on the other hand shows us that it is also a *necessary* condition to ensure that the right-hand side vanishes for any choice of $x(0)$. □

Returning to the uncoupled oscillator system of Eq. (3.9), we conclude that *all* processes $(c^{(a)}(t))_{t\geq 0}$ are conserved.

Corollary 3.5 (Conserved quantities of uncoupled oscillator system)
Let $(x(t))_{t\geq 0}$ denote the process of Eq. (3.9). Then the processes

$$(c^{(a)}(t))_{t\geq 0}, \quad a \in \left\{0, ..., n^2 - 1\right\}, \tag{3.34}$$

are *conserved*, i.e. $c^{(a)}(t) = c^{(a)}(0)$, for all $t \geq 0$.

Proof. Comparing Eqs. (3.9) and (3.32), this follows from Theorem 3.4 in the special case of $b = 0$ and $\alpha_b = 2\kappa$, since

$$\left[\mathsf{T}^{(a)}, \mathsf{T}^{(b)}\right] = \left[\mathsf{T}^{(a)}, \frac{1}{2}\mathbb{1}_{n\times n}\right] = 0, \quad \forall a = 0, ..., n^2 - 1. \tag{3.35}$$

□

Note that not all of these constants are *functionally independent*. By [AW02], an isotropic n-dimensional oscillator "possesses the maximal number of $(2n-1)$ functionally independent constants of motion". For instance, we could choose the oscillators' *energies* $\|x_k\|^2, k \in \{0, ..., n-1\}$, together with their $n-1$ independent *phase differences*. Once we introduce interactions between the oscillators, however, it will prove much easier to keep track of the sesquilinear forms of Eq. (3.28), rather than directly studying the evolution of the phase differences.

In the following example we illustrate the conserved quantities of Corollary 3.5 in the case of $n = 2$ and show how they are related to the oscillators' energies and phase differences.

Example 3.6 (Conserved quantities for two-oscillator system)
Choosing the $U(2)$ generators given in Example 3.3, we obtain[a]

$$c^{(0)}(t) = \frac{1}{2}\, x^\dagger(t) \mathbb{1}_{n\times n}\, x(t) = \frac{|x_0(t)|^2 + |x_1(t)|^2}{2} = \frac{\|x(t)\|^2}{2} \tag{3.36a}$$

$$c^{(1)}(t) = \frac{1}{2}\, x^\dagger(t)\, \sigma^{(1)} x(t) = \frac{\overline{x}_0(t)\, x_1(t) + \overline{x}_1(t)\, x_0(t)}{2} = \mathrm{Re}\left(\overline{x}_0(t)\, x_1(t)\right), \tag{3.36b}$$

$$c^{(2)}(t) = \frac{1}{2}\, x^\dagger(t) \sigma^{(2)}\, x(t) = (-i)\, \frac{\overline{x}_0(t)\, x_1(t) - \overline{x}_1(t)\, x_0(t)}{2} = \mathrm{Im}\left(\overline{x}_0(t)\, x_1(t)\right), \tag{3.36c}$$

$$c^{(3)}(t) = \frac{1}{2}\, x^\dagger(t) \sigma^{(3)}\, x(t) = \frac{\overline{x}_0(t)\, x_0(t) - \overline{x}_1(t)\, x_1(t)}{2} = \frac{|x_0(t)|^2 - |x_1(t)|^2}{2}. \tag{3.36d}$$

Note that $c^{(0)}$ is related to the *total energy* of the system, since by Eq. (3.10) we have

$$H(x(t)) = \kappa\, c^{(0)}(t). \tag{3.37}$$

Similarly, $c^{(3)}$ captures the *energy difference* between the two oscillators. The *phase difference*[b]

$$\varphi(t) \coloneqq (\phi_1 + \kappa\, t) - (\phi_0 + \kappa\, t) = \phi_1 - \phi_0 \tag{3.38}$$

turns out to be a constant and is represented by $c^{(1)}$ and $c^{(2)}$,[c] since

$$c^{(1)}(t) = |x_0(t)|\, |x_1(t)|\, \cos\left(\varphi(t)\right), \tag{3.39}$$

$$c^{(2)}(t) = |x_0(t)|\, |x_1(t)|\, \sin\left(\varphi(t)\right). \tag{3.40}$$

We conclude that the conserved quantities are related to energy and phase differences of the oscillators, as well as their total energy.[d] Observing that

$$\sqrt{\left|c^{(1)}(t)\right|^2 + \left|c^{(2)}(t)\right|^2 + \left|c^{(3)}(t)\right|^2} = c^{(0)}(t), \tag{3.41}$$

we find that the vector $\left(c^{(1)}(t), c^{(2)}(t), c^{(3)}(t)\right)^\top$ corresponds to a point on a sphere of radius $c^{(0)}$ in $\mathbb{R}^3$. If we represent the energy difference $c^{(3)}(t) \in [-c^{(0)}(t), c^{(0)}(t)]$ as

$$c^{(3)}(t) = c^{(0)}(t) \cos\left(\psi(t)\right), \quad \text{where } \psi(t) \in [0, \pi], \tag{3.42}$$

we obtain the spherical coordinate representation

$$\begin{pmatrix} c^{(1)} \\ c^{(2)} \\ c^{(3)} \end{pmatrix}(t) = c^{(0)}(t) \begin{pmatrix} \sin\left(\psi(t)\right) \cos\left(\varphi(t)\right) \\ \sin\left(\psi(t)\right) \sin\left(\varphi(t)\right) \\ \cos\left(\psi(t)\right) \end{pmatrix}, \tag{3.43}$$

where we have made use of

$$\begin{aligned}
|x_0(t)|\,|x_1(t)| &= \sqrt{\left(\frac{|x_0(t)|^2+|x_1(t)|^2}{2}\right)^2 - \left(\frac{|x_0(t)|^2-|x_1(t)|^2}{2}\right)^2} \\
&= \sqrt{\left|c^{(0)}(t)\right|^2 - \left|c^{(3)}(t)\right|^2} = c^{(0)}(t)\,\sqrt{1-|\cos(\psi(t))|^2} \\
&= c^{(0)}(t)\,\,|\sin(\psi(t))| = c^{(0)}(t)\,\sin(\psi(t))\,. \qquad (3.44)
\end{aligned}$$

In the last step we have applied $\sin(\psi(t)) \geq 0$, which holds since $\psi(t) \in [0, \pi]$ by definition. In Section 6.1.2 we will come back to Eq. (3.43) and employ this spherical representation in order to visualize the influence of a weak deterministic coupling on the system.

[a] c.f. [Kum76], Eqs. (8) and (9)
[b] Recall Eq. (3.7) and Eq. (3.8).
[c] Note that $c^{(2)}(t) = p_0(t)\,\eta_1(t) - p_1(t)\,\eta_0(t)$ can be interpreted as the conserved *angular momentum* of a two-dimensional isotropic oscillator, c.f. [DI07], Eq. (16b).
[d] c.f. Eq. (3.44) for a representation of $|x_0(t)|\,|x_1(t)|$ in terms of energy difference and total energy.

3.1.2.4. Complex outer product

Finally, we show that all of the conserved quantities $c^{(a)}(t)$, $a \in \{0, ..., n^2-1\}$, can be captured by the complex outer product $(xx^\dagger)(t)$. For $k, l \in \{0, ..., n-1\}$, we denote by $\mathsf{E}^{(k,l)} \in \mathbb{C}^{n,n}$ the matrix which has a 1 at position (k, l) and zeros everywhere else, i.e.

$$\mathsf{E}^{(k,l)}_{k',l'} := \delta_{k,k'}\delta_{l,l'}\,, \quad \forall k', l' \in \{0, ..., n-1\}\,. \qquad (3.45)$$

A possible choice of $SU(n)$ generators is given by the set[15]

$$\left\{\mathsf{T}^{(1;k,l)}, \mathsf{T}^{(2;k,l)} \mid k, l \in \{0, ..., n-1\}\,,\, k < l\right\} \cup \left\{\mathsf{T}^{(3;k)} \mid k \in \{0, ..., n-2\}\right\}, \qquad (3.46)$$

where

$$\begin{aligned}
\mathsf{T}^{(1;k,l)} &:= \frac{1}{2}\left(\mathsf{E}^{(k,l)} + \mathsf{E}^{(l,k)}\right), &(3.47a)\\
\mathsf{T}^{(2;k,l)} &:= \frac{(-i)}{2}\left(\mathsf{E}^{(k,l)} - \mathsf{E}^{(l,k)}\right), &(3.47b)\\
\mathsf{T}^{(3;k)} &:= \frac{1}{2}\left(\mathsf{E}^{(k,k)} - \mathsf{E}^{(k+1,k+1)}\right), &(3.47c)
\end{aligned}$$

are higher dimensional generalizations[16] of the Pauli matrices (up to a prefactor). Note that there are indeed $2\,\frac{n\,(n-1)}{2} + (n-1) = n^2 - 1$ generators in the set specified by Eq. (3.46). According to the previous section, these generators give rise to the set

$$\left\{c^{(1;k,l)}(t), c^{(2;k,l)}(t) \mid k, l \in \{0, ..., n-1\}\,,\, k < l\right\} \cup \left\{c^{(3;k)}(t) \mid k \in \{0, ..., n-2\}\right\}, \qquad (3.48)$$

[15] c.f. [AW02], Section V.B
[16] For $n = 2$ the indices are fixed to $k = 0$, $l = 1$ and Eq. (3.47) reproduces the $SU(2)$ generators of Example 3.3.

of conserved quantities, where[17]

$$
\begin{aligned}
c^{(1;k,l)}(t) &:= x^\dagger(t)\, \mathsf{T}^{(1;k,l)}\, x(t), && \text{(3.49a)}\\
c^{(2;k,l)}(t) &:= x^\dagger(t)\, \mathsf{T}^{(2;k,l)}\, x(t), && \text{(3.49b)}\\
c^{(3;k)}(t) &:= x^\dagger(t)\, \mathsf{T}^{(3;k)}\, x(t). && \text{(3.49c)}
\end{aligned}
$$

We can combine each pair of real-valued constant processes $c^{(1;k,l)}(t)$ and $c^{(2;k,l)}(t)$ into a single complex quantity, which can be identified with an off-diagonal element of the *complex outer product*:

$$
\begin{aligned}
c^{(1;k,l)}(t) - i\, c^{(2;k,l)}(t) &= x^\dagger(t) \left(\mathsf{T}^{(1;k,l)} - i\, \mathsf{T}^{(2;k,l)}\right) x(t)\\
&= x^\dagger(t)\, \mathsf{E}^{(l,k)}\, x(t)\\
&= \overline{x_l(t)}\, x_k(t) = (xx^\dagger)_{k,l}(t). \qquad \text{(3.50)}
\end{aligned}
$$

Similarly, the processes $c^{(3;k)}(t)$ can be expressed in terms of diagonal elements of the complex outer product, i.e.

$$
\begin{aligned}
c^{(3;k)}(t) = x^\dagger(t)\mathsf{T}^{(3;k)}\, x(t) &= \frac{1}{2}\left(x^\dagger(t)\mathsf{E}^{(k,k)}\, x(t) - x^\dagger(t)\mathsf{E}^{(k+1,k+1)}\, x(t)\right)\\
&= \frac{1}{2}\left(|x_k(t)|^2 - |x_{k+1}(t)|^2\right)\\
&= \frac{1}{2}\left((xx^\dagger)_{k,k}(t) - (xx^\dagger)_{k+1,k+1}(t)\right). \qquad \text{(3.51)}
\end{aligned}
$$

These terms can be interpreted as the *energy differences* between ‘neighboring’ oscillators k and $k+1$. Finally, the conserved quantity $c^{(0)}(t)$ can be related to the trace of the outer product, i.e.

$$
c^{(0)}(t) = x^\dagger(t)\, \mathsf{T}^{(0)}\, x(t) = \frac{1}{2}\, x^\dagger(t)\, \mathbb{1}_{n\times n}\, x(t) = \frac{1}{2}\, \|x(t)\|^2 = \frac{1}{2}\,\mathrm{tr}\left((xx^\dagger)(t)\right), \qquad \text{(3.52)}
$$

and interpreted as the *total energy* of the system. We illustrate this identification of the system’s outer product with its conserved quantities in the two-oscillator case.

Example 3.7 (Outer product for two-oscillator system)
We continue with Example 3.6 and observe that the complex outer product $(xx^\dagger)(t)$ can be written as

$$
(xx^\dagger)(t) = \begin{pmatrix} c^{(0)} + c^{(3)} & c^{(1)} - i\, c^{(2)} \\ c^{(1)} + i\, c^{(2)} & c^{(0)} - c^{(3)} \end{pmatrix}(t). \qquad \text{(3.53)}
$$

This follows from Eqs. (3.50) to (3.52) and can also be directly inferred from Eq. (3.36).

[17] c.f. [AW02], Eqs. (69)-(71)

3.2. Weak deterministic drift

We introduce a weak *deterministic* coupling, i.e. a drift term which gives rise to an interaction between the oscillators. In Section 3.2.1 we consider *linear* interactions given by a drift term $u_{\text{lin}}(x)$ and in Section 3.2.2 we will also allow for certain divergence-free forms of *nonlinear* coupling, represented by a drift term $u_{\text{nl}}(x)$. The combined coupling will then be given by the sum of both contributions, i.e.

$$u(x) \coloneqq u_{\text{lin}}(x) + u_{\text{nl}}(x), \quad \forall x \in \mathbb{C}^n. \tag{3.54}$$

3.2.1. Circulant linear coupling

The linear coupling term may depend on the space-variables $\eta_k(t)$ (*space-coupling*) as well as the moment-variables $p_k(t)$ (*momentum-coupling*). We motivate the structure of the linear interaction terms, focusing on the example of space-coupling (3.2.1.1). In particular, we introduce the aforementioned circularity assumption (3.2.1.2). Subsequently, we transfer the results to the case of momentum-coupling (Section 3.2.1.3) and state the general combined case (Section 3.2.1.4). Finally, we apply a *discrete Fourier transform* (DFT), which will allow us to diagonalize the circulant coupling matrices (Section 3.2.1.5).

3.2.1.1. Linear space-coupling

Gradient flow arising from harmonic two-oscillator coupling potentials Recall that $\check{p}_k(t), \check{\eta}_k(t)$ denote momentum and elongation of the k'th unrescaled oscillator in the absence of any interaction, c.f. Section 3.1.1. Similarly, let $\check{p}^\varepsilon_k(t), \check{\eta}^\varepsilon_k(t)$ now denote momentum and elongation of the k'th unrescaled oscillator in the presence of a weak interaction of strength ε. We first consider the special case of a *gradient-type* coupling, i.e. we examine a perturbed system of the form

$$\mathrm{d}\check{p}^\varepsilon_k(t) = -\mathsf{K}\,\check{\eta}^\varepsilon_k(t)\,\mathrm{d}t - \varepsilon\,\mathsf{K}\left(\frac{\partial V}{\partial \check{\eta}_k}\right)(\check{\eta}^\varepsilon(t))\,\mathrm{d}t, \qquad k \in \{0, ..., n-1\}, \tag{3.55a}$$

$$\mathrm{d}\check{\eta}^\varepsilon_k(t) = \frac{\check{p}^\varepsilon_k(t)}{\mathfrak{m}}\,\mathrm{d}t, \qquad k \in \{0, ..., n-1\}. \tag{3.55b}$$

We assume that the coupling potential V is composed of harmonic *two-oscillator interactions*

$$V(\check{\eta}_0, ..., \check{\eta}_{n-1}) \coloneqq \frac{1}{2} \sum_{\substack{i,j=0,\\ i<j}}^{n-1} A_{ij}(\check{\eta}_i - \check{\eta}_j)^2, \tag{3.56}$$

where the elements $A_{ij} \in \mathbb{R}$ are the components of a *pair-coupling matrix* A. The corresponding negative gradient is given by

$$-\left(\frac{\partial V}{\partial \check{\eta}_k}\right)(\check{\eta}^\varepsilon(t)) = \sum_{\substack{i=0,\\ i<k}}^{n-1} A_{ik}(\check{\eta}^\varepsilon_i(t) - \check{\eta}^\varepsilon_k(t)) - \sum_{\substack{j=0,\\ k<j}}^{n-1} A_{kj}(\check{\eta}^\varepsilon_k(t) - \check{\eta}^\varepsilon_j(t)). \tag{3.57}$$

Employing the notation $A_{kj} := A_{jk}$, for $k > j$, as well as $A_{kk} := 0$, this can be written as

$$-\left(\frac{\partial V}{\partial \check{\eta}_k}\right)(\check{\eta}^\varepsilon(t)) = -\sum_{j=0}^{n-1} A_{kj}(\check{\eta}_k^\varepsilon(t) - \check{\eta}_j^\varepsilon(t)). \tag{3.58}$$

The potential therefore gives rise to a weakly coupled system of the form

$$\mathrm{d}\check{p}_k^\varepsilon(t) = -\mathsf{K}\check{\eta}_k^\varepsilon(t)\,\mathrm{d}t + \varepsilon\,\mathsf{K}\sum_{j=0}^{n-1} A_{kj}(\check{\eta}_j^\varepsilon(t) - \check{\eta}_k^\varepsilon(t))\,\mathrm{d}t, \qquad k \in \{0, ..., n-1\}, \tag{3.59a}$$

$$\mathrm{d}\check{\eta}_k^\varepsilon(t) = \check{p}_k^\varepsilon(t)\,\mathrm{d}t, \qquad k \in \{0, ..., n-1\}. \tag{3.59b}$$

Analogously to Section 3.1.1, we define the space-rescaled quantities $p_k^\varepsilon(t) := \sqrt{\frac{1}{2\mathsf{m}}}\,\check{p}_k^\varepsilon(t)$ and $\eta_k^\varepsilon(t) := \sqrt{\frac{\mathsf{K}}{2}}\,\check{\eta}_k^\varepsilon(t)$, which satisfy

$$\mathrm{d}p_k^\varepsilon(t) = -\kappa\,\eta_k^\varepsilon(t)\,\mathrm{d}t + \varepsilon\,\kappa\sum_{j=0}^{n-1} A_{kj}(\eta_j^\varepsilon(t) - \eta_k^\varepsilon(t))\,\mathrm{d}t, \qquad k \in \{0, ..., n-1\}, \tag{3.60a}$$

$$\mathrm{d}\eta_k^\varepsilon(t) = \kappa\,p_k^\varepsilon(t)\,\mathrm{d}t, \qquad k \in \{0, ..., n-1\}. \tag{3.60b}$$

Introducing the complex notation $x_k^\varepsilon(t) := p_k^\varepsilon(t) + i\,\eta_k^\varepsilon(t)$,[18] we conclude that

$$\mathrm{d}x_k^\varepsilon(t) = i\,\kappa\,x_k^\varepsilon(t)\,\mathrm{d}t + \varepsilon\,\kappa\sum_{j=0}^{n-1} A_{kj}\left[\mathrm{Im}(x_j^\varepsilon(t)) - \mathrm{Im}(x_k^\varepsilon(t))\right]\mathrm{d}t, \quad k \in \{0, ..., n-1\}. \tag{3.61}$$

Defining the linear drift term $u_{\mathrm{lin}} : \mathbb{C}^n \to \mathbb{C}^n$ by

$$(u_{\mathrm{lin}})_k(x) := \sum_{j=0}^{n-1} A_{kj}(\mathrm{Im}(x_j) - \mathrm{Im}(x_k)), \quad k \in \{0, ..., n-1\}, \tag{3.62}$$

we can represent Eq. (3.61) by the $\mathbb{C}^n$-valued ODE

$$\mathrm{d}x^\varepsilon(t) = i\,\kappa\,x^\varepsilon(t)\,\mathrm{d}t + \varepsilon\,\kappa\,u_{\mathrm{lin}}(x^\varepsilon(t))\,\mathrm{d}t. \tag{3.63}$$

Remark 3.8
Note that the constant κ is just an overall factor on the right-hand side, which can be absorbed in a time-change, c.f. Section 3.5 below.

Introducing the row-sums $\alpha_k := \sum_{j=0}^{n-1} A_{kj}$, we can rewrite the drift term as

$$u_{\mathrm{lin}}(x) = (A - \mathrm{diag}\,(\alpha))\,\mathrm{Im}(x) = L\,\mathrm{Im}(x), \tag{3.64}$$

[18] c.f. Eq. (3.4)

where

$$L := A - \text{diag}(\alpha) \tag{3.65}$$

is the so-called *Laplacian matrix* or *graph Laplacian*[19] of the coupling graph, corresponding to the pair-coupling matrix A.

Directed coupling and self-coupling We now generalize the setup to a setting beyond a gradient flow. First of all, we allow for a *directed* coupling, i.e. for an asymmetric pair-coupling matrix $A_{ik} \neq A_{ki}$. As we will see later on (c.f. Proposition 6.1), such an asymmetry can induce an amplification (or damping) of some of the system's eigenmodes. We will illustrate two directed-coupling examples in Fig. 3.1. We further generalize by no longer requiring that L exhibits the structure of a Laplacian matrix. This allows us to also include *self-coupling effects*, e.g. $L \propto \mathbb{1}_{n\times n}$.

From now we will refer to L as the *(space-)coupling matrix* and study a drift term of the form $u_{\text{lin}}(x) = L\,\text{Im}(x)$.

3.2.1.2. Circulant coupling topologies

We define the notion of a *coupling distance* between oscillators. This allows us to classify which oscillators are *nearest neighbors*, *next-to-nearest neighbors* and so forth. Since we want to model a *ring* of oscillators, such a neighborhood relation has to be defined in a cyclic way w.r.t. to their labels.

Definition 3.9 (Cyclic coupling distance)
The *directed coupling distance* between two oscillators is defined as

$$d : \mathbb{Z}^2 \to \mathbb{Z}_n \cong \{0, ..., n-1\}, (k,l) \to d(k,l) := (k-l) \bmod n, \tag{3.66}$$

where for any $m \in \mathbb{Z}$, we denote by $(m \bmod n)$ the unique value $m' \in \{0, ..., n-1\}$ which satisfies $m = z \cdot n + m'$ for some $z \in \mathbb{Z}$.

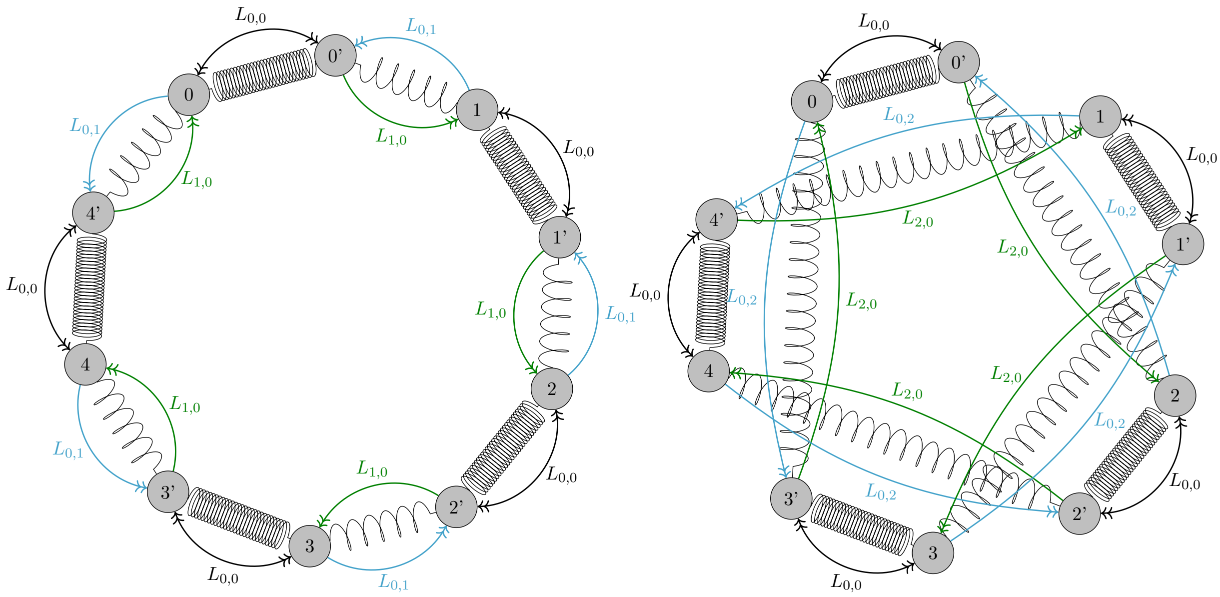
The *symmetric coupling distance* between two oscillators is given by

$$d_s : \mathbb{Z}^2 \to \mathbb{Z}_n \cong \{0, ..., n-1\}, (k,l) \to d_s(k,l) := \min\{d(k,l), d(l,k)\}. \tag{3.67}$$

Oscillators k and l are called *nearest neighbors*, if $d_s(k,l) = 1$ and *next-to-nearest neighbors* if $d_s(k,l) = 2$.

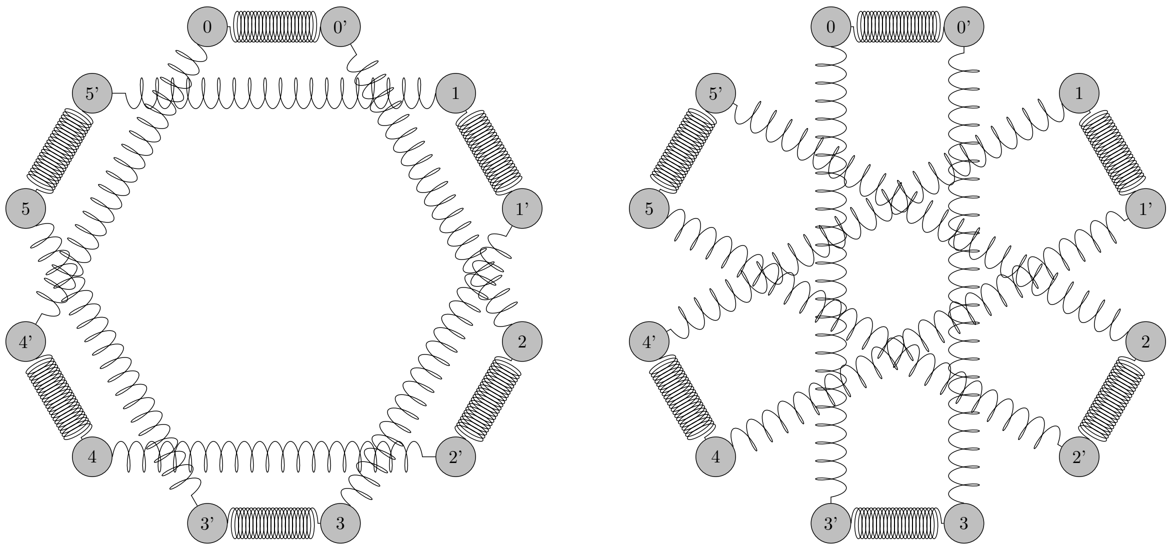
Note that Definition 3.9 indeed defines a *cyclic* neighborhood relation, in the sense that the oscillators $n-1$ and 0 are nearest neighbors (c.f. also Fig. 3.1a). This notion of a coupling distance allows us to characterize those coupling topologies which are 'rotationally invariant'. These so-called *circulant* coupling structures can be characterized by the fact that the coupling strength between any two oscillators only depends on the *cyclic coupling distance* d between these oscillators.

[19] The concept of a graph Laplacian is introduced in [New10], Chapter 6.13. Note that the sign in our definition differs from the one in the literature.



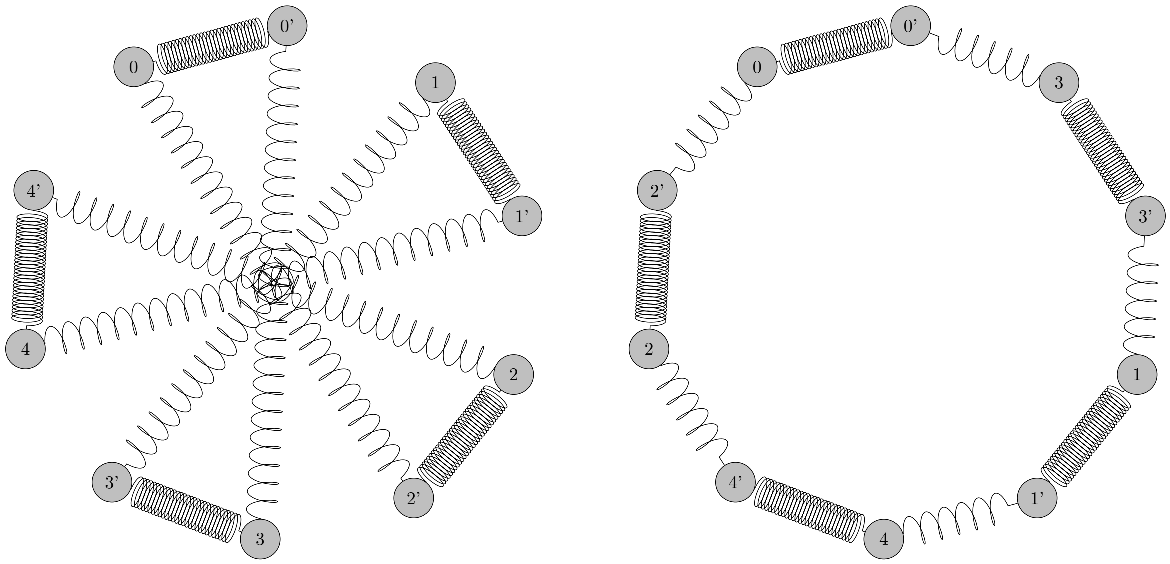
(a) Nearest-neighbor (distance-1) coupling (b) Next-to-nearest-neighbor (distance-2) coupling

Figure 3.1.: Circulant network of oscillators



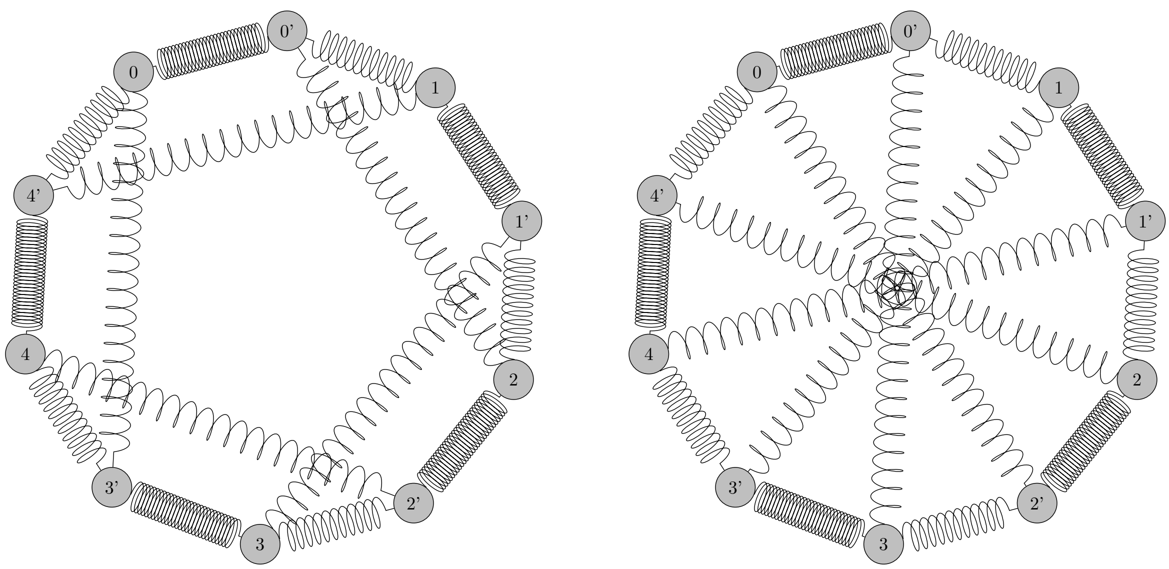
(a) Distance-2 coupling (b) Distance-3 coupling

Figure 3.2.: Distance-l coupling of n oscillators, where $\gcd(l, n) > 1$.



(a) Distance-3 coupling (b) Distance-3 coupling of *reordered* oscillators

Figure 3.3.: Reordering of oscillators



(a) Combined distance-1 and distance-2 coupling (b) Combined distance-1 and distance-3 coupling

Figure 3.4.: Superposition of distance-k coupling topologies

This requirement is illustrated in Fig. 3.1, where on the left (Fig. 3.1a), a typical example of *nearest-neighbor coupling* is given, which we can identify as a *distance-1-coupling*, since oscillators are coupled, if and only if their symmetric coupling distance is equal to one. Fig. 3.1b, by comparison, illustrates the case of *next-to-nearest neighbor coupling*, which by the same line of argument can be identified as a *distance-2 coupling*. In the plots we can distinguish three kinds of coupling links. There is *self-coupling*, represented by black arrows and labeled with elements of the diagonal of L, here for example by $L_{0,0}$. This is of course a symmetric coupling. Coupling links between *distinct* oscillators, however, are allowed to be *directed*. The strength $L_{0,1}$ (depicted in blue) with which oscillator 1 influences oscillator 0 for example, does *not* need to coincide with $L_{1,0}$ (depicted in green), which describes the reverse direction.

Now the circulant symmetry is reflected by the fact, that for instance all the distance-0-links are of the same strength $L_{0,0}$, which implies that all elements of the diagonal of L are required to be identical. Similarly, all distance-1 links (Fig. 3.1a) have to exhibit an identical coupling structure and the same is required for the distance-2 links (Fig. 3.1b).

In order to formalize this symmetry requirement, we recall that a *circulant matrix* (c.f. Eq. (2.69)) can be characterized by the fact that its elements only depend on the difference between row and column index. This is exactly the requirement we want to impose on our coupling matrix L, which leads us to the assumption of L being a *circulant* matrix, generated by some vector $\mathfrak{l} \in \mathbb{R}^n$.

Assumption 3.10 (Circulant space-coupling matrix)
We assume that the coupling matrix L is *circulant*, i.e. that there exists some $\mathfrak{l} \in \mathbb{R}^n$, s.t.

$$L = \operatorname{cycl}(\mathfrak{l}) = (\mathfrak{l}_{k-l})_{k,l=0,\dots,n-1}\,. \tag{3.68}$$

We will refer to $\mathfrak{l}$ as the *space-coupling vector*.

The *coupling matrix* L can be associated with a directed graph by choosing $\{0, ..., n-1\}$ as the set of vertices and interpreting the elements of L as the edge weights.[20] We will use the terms *coupling structure* or *coupling topology* to refer to this coupling graph. A coupling matrix L and its associated graph are called *even* (*odd*), if the coupling vector $\mathfrak{l}$ is *even* (*odd*).[21]

Remark 3.11 (Directed coupling)
An *even* coupling will also be referred to as *symmetric* or *undirected*. A general coupling, which is not symmetric (i.e. $\mathcal{P}_{-}\mathfrak{l} \neq 0$) will be called *directed*.

Note that by Eq. (2.70), we find that Assumption 3.10 allows us to rewrite the space-coupling drift term as a circular convolution, i.e.

$$u_{\text{lin}}(x) := L\,\text{Im}(x) = \mathfrak{l} \circledast \text{Im}(x). \tag{3.69}$$

Circulant coupling matrices in particular encompass a *unidirectional distance-l* coupling (c.f. Eq. (1.35)), where $l \in \{0, ..., n-1\}$. In this case, oscillators are coupled if and only if their coupling distance d is equal to l, i.e.

$$L_{ij} \propto \delta_{i-j,l}. \tag{3.70}$$

[20] An introduction to networks, graphs and their matrix representations can be found in [New10], Chapter 6.
[21] as specified in Definition 2.28

Recalling Eq. (3.68), this can be achieved by choosing a coupling vector proportional to the l'th unit vector e_l, i.e.

$$\mathfrak{l} \propto e_l := (\delta_{i,l})_{i\in\{0,\dots,n-1\}}. \tag{3.71}$$

By linearity, any superposition of circulant coupling structures again constitutes a circulant coupling and can be represented by the sum of the respective coupling vectors. An *even* (*odd*) distance-l coupling for instance can be obtained as the sum (difference) of a unidirectional distance-l and a unidirectional distance-$(n-l)$ coupling.

Example 3.12 (Directed and undirected circulant coupling topologies)
Examples of *even* (*symmetric*) circulant coupling topologies:

Distance-0 (self coupling)	$\mathfrak{l} = (1, 0 \cdots 0)^\top,$	(3.72)
Distance-1 (nearest-neighbor coupling)	$\mathfrak{l} = (0, 1, 0 \cdots 0, 1)^\top,$	(3.73)
Distance-2 (next-to-nearest-neighbor coupling)	$\mathfrak{l} = (0, 0, 1, 0 \cdots 0, 1, 0)^\top,$	(3.74)
Distance-l ($l \in \left\{0, \dots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$)	$\mathfrak{l} = (\delta_{k,l} + \delta_{k,n-l})_{k\in\{0,\dots,n-1\}},$	(3.75)
All-to-all coupling	$\mathfrak{l} = (0, 1 \cdots 1)^\top.$	(3.76)

Examples of *odd* circulant coupling topologies:

Distance-1	$\mathfrak{l} = (0, 1, 0 \cdots 0, -1)^\top,$	(3.77)
Distance-l ($l \in \left\{0, \dots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$)	$\mathfrak{l} = (\delta_{k,l} - \delta_{k,n-l})_{k\in\{0,\dots,n-1\}}.$	(3.78)

If we look at a distance-l coupling of n oscillators, where l and n are *not* coprime, i.e. $\gcd(l, n) > 1$, then the system can be decomposed into $\gcd(l, n)$ independent subsystems of $\frac{n}{\gcd(l,n)}$ oscillators each. This follows from the theory of subgroups of the cyclic group $\mathbb{Z}_n$, c.f. [Nor12], Lemma 2.7. An illustration of this decoupling is given in Fig. 3.2, where (keeping in mind Assumption 3.10) we have dropped the coupling-strength labels. If, on the other hand, l and n *are* coprime, then we can transform a distance-l coupling into a nearest-neighbor coupling by a relabeling of the oscillators, c.f. Fig. 3.3. However, for a general superposition of different distance-l coupling topologies, c.f. Fig. 3.4, there is no such relabeling which would reduce the system to a nearest-neighbor coupling.

Finally, we note that the circularity assumption on the coupling matrix L is in particular compatible with the special case of $L = (A - \operatorname{diag}(\alpha))$ arising as the Laplacian matrix of some pair coupling matrix A, provided that A itself is circulant.

Remark 3.13 (Laplacian matrix in the case of a circulant pair-coupling matrix)
Let $A \in \mathbb{R}^{n,n}$ be a circulant matrix, i.e. $A = \operatorname{cycl}(\mathfrak{a})$, for some $\mathfrak{a} \in \mathbb{R}^n$. Then the corresponding Laplacian matrix $L := (A - \operatorname{diag}(\alpha))$, with $\alpha_k = \sum_{j=0}^{n-1} A_{kj}$, is also a circulant matrix,

and the drift term (3.64) can be represented as

$$u_{\text{lin}}(x) = (\operatorname{cycl}(\mathfrak{a}) - \alpha_0 \, \mathbb{1}_{n\times n}) \operatorname{Im}(x) = \mathfrak{a} \circledast \operatorname{Im}(x) - \alpha_0 \operatorname{Im}(x) \tag{3.79}$$

Proof. We observe that $\alpha_k = \sum_{j=0}^{n-1} \mathfrak{a}_{k-j} = \sum_{j=0}^{n-1} \mathfrak{a}_j = \alpha_0$ for all $k = 0, ..., n-1$, where we have employed the cyclic index convention introduced in Definition 2.24. Therefore, $\operatorname{diag}(\alpha)$ satisfies the circularity assumption and L, as the sum of two circulant matrices, as well. The representation of $u(x)$ in terms of a circular convolution again follows from Eq. (2.70). □

3.2.1.3. Linear momentum-coupling

Similar to the previously introduced drift terms, which linearly depend on the space variables $\check{\eta}_k$, we now introduce analogous perturbation terms, linearly depending on the momenta $\check{p}_k$. After rescaling, such a *momentum-coupling* is of the form

$$\mathrm{d}p_k^\varepsilon(t) = -\kappa\, \eta_k^\varepsilon(t)\,\mathrm{d}t + \varepsilon\,\kappa \sum_{j=0}^{n-1} L'_{kj}\, p_j^\varepsilon(t)\,\mathrm{d}t, \qquad k \in \{0, ..., n-1\}, \tag{3.80a}$$

$$\mathrm{d}\eta_k^\varepsilon(t) = \quad \kappa\, p_k^\varepsilon(t)\,\mathrm{d}t, \qquad k \in \{0, ..., n-1\}, \tag{3.80b}$$

which in complex vector notation can be written as

$$\mathrm{d}x^\varepsilon(t) = i\,\kappa\, x^\varepsilon(t)\,\mathrm{d}t + \varepsilon\,\kappa\, L'\, p^\varepsilon(t)\,\mathrm{d}t. \tag{3.81}$$

As above, we impose a rotational symmetry by requiring that L' is a circulant matrix.

Assumption 3.14 (Circulant momentum-coupling matrix)
We assume that the momentum-coupling matrix L' is circulant, i.e. that there exists some $\mathfrak{l}' \in \mathbb{R}^n$, called *momentum-coupling vector*, s.t. $L' = \operatorname{cycl}(\mathfrak{l}')$.

By Eq. (2.70) we conclude that the momentum-coupling drift term can be written as

$$u'_{\text{lin}}(x) := L' \operatorname{Im}(x) = \mathfrak{l}' \circledast \operatorname{Re}(x). \tag{3.82}$$

The examples are similar to the ones in previous section. Note in particular, that a momentum self-coupling, corresponding to $\mathfrak{l}' = \pm(1, 0, ..., 0)^\top$, can be interpreted as a global, homogeneous *amplification*- or *friction* term, depending on the sign.

3.2.1.4. Combined space- and momentum-coupling

We now allow for a combination of both space- and momentum-coupling.

Definition 3.15 (Combined circulant drift term)
From now on we will denote by $u_{\mathrm{lin}}(x)$ a general combination of space- and momentum-coupling, i.e.

$$u_{\mathrm{lin}}(x) := L' \operatorname{Re}(x) + L \operatorname{Im}(x) = \mathfrak{l}' \circledast \operatorname{Re}(x) + \mathfrak{l} \circledast \operatorname{Im}(x), \tag{3.83}$$

where $L' = \operatorname{cycl}(\mathfrak{l}')$ and $L = \operatorname{cycl}(\mathfrak{l})$ are the circulant coupling matrices generated by the coupling vectors $\mathfrak{l}', \mathfrak{l} \in \mathbb{R}^n$.[a]

[a] Recall Assumptions 3.10 and 3.14.

Employing a complex notation, we can obtain a unified description.

Definition 3.16 (Complex deterministic-coupling vector)
We combine the space- and momentum-coupling vectors into one complex coupling vector

$$\lambda := \mathfrak{l}' - i\,\mathfrak{l} \in \mathbb{C}^n \tag{3.84}$$

and denote the corresponding circulant coupling matrix by

$$\Lambda := \operatorname{cycl}(\lambda)\,. \tag{3.85}$$

By linearity, this implies that $\Lambda = L' - iL$, i.e. $L' = \operatorname{Re}(\Lambda)$ and $L = -\operatorname{Im}(\Lambda)$. Eq. (3.83) can thus be written as

$$u_{\mathrm{lin}}(x) = \operatorname{Re}(\Lambda)\operatorname{Re}(x) - \operatorname{Im}(\Lambda)\operatorname{Im}(x) = \operatorname{Re}(\Lambda x) = \operatorname{Re}(\lambda \circledast x). \tag{3.86}$$

The weakly coupled system can consequently be described by

$$\mathrm{d}x^\varepsilon(t) = i\,\kappa\, x^\varepsilon(t)\,\mathrm{d}t + \varepsilon\,\kappa\, u_{\mathrm{lin}}(x^\varepsilon(t))\,\mathrm{d}t = i\,\kappa\, x^\varepsilon(t)\,\mathrm{d}t + \varepsilon\,\kappa \operatorname{Re}(\Lambda\, x^\varepsilon(t))\,\mathrm{d}t, \tag{3.87}$$

or equivalently by

$$\mathrm{d}x^\varepsilon(t) = i\,\kappa\, x^\varepsilon(t)\,\mathrm{d}t + \varepsilon\,\kappa \operatorname{Re}(\lambda \circledast x^\varepsilon(t))\,\mathrm{d}t. \tag{3.88}$$

We continue with Example 3.6 and show that a nearest-neighbor coupling of two oscillators can be related to the first Pauli matrix.

Example 3.17 (Deterministic coupling of two oscillators)
In the case of two oscillators, a *distance*-1 coupling is the only non-trivial coupling topology, apart from self-coupling effects. For a *symmetric* distance-1 coupling, the complex coupling vector is given by (c.f. Example 3.12)

$$\lambda = \begin{pmatrix} 0 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathfrak{l}_1' - i\,\mathfrak{l}_1 \end{pmatrix}, \tag{3.89}$$

where $\mathfrak{l}'_1$ describes the momentum-coupling strength, while $\mathfrak{l}_1$ represents the strength of the space-coupling. According to Definition 3.16, the coupling matrix is thus given by

$$\Lambda = \mathrm{cycl}\,(\lambda) = \begin{pmatrix} 0 & \lambda_1 \\ \lambda_1 & 0 \end{pmatrix} = \lambda_1\, \sigma^{(1)}, \tag{3.90}$$

where $\sigma^{(1)}$ is the first Pauli matrix, as defined in Example 3.3. Following Eq. (3.87), the weakly coupled system evolves according to

$$\begin{aligned} \mathrm{d}x^\varepsilon(t) &= \begin{pmatrix} i\,\kappa & \frac{\varepsilon\,\kappa}{2}\,\lambda_1 \\ \frac{\varepsilon\,\kappa}{2}\,\lambda_1 & i\,\kappa \end{pmatrix} x^\varepsilon(t)\ \mathrm{d}t + \begin{pmatrix} 0 & \frac{\varepsilon\,\kappa}{2}\,\overline{\lambda_1} \\ \frac{\varepsilon\,\kappa}{2}\,\overline{\lambda_1} & 0 \end{pmatrix} \overline{x^\varepsilon}(t)\ \mathrm{d}t \\ &= \left(i\,2\kappa\,\mathsf{T}^{(0)} + \varepsilon\,\kappa\,\lambda_1\,\mathsf{T}^{(1)}\right) x^\varepsilon(t)\ \mathrm{d}t + \varepsilon\,\kappa\,\overline{\lambda_1}\,\mathsf{T}^{(1)}\,\overline{x^\varepsilon}(t)\ \mathrm{d}t, \end{aligned} \tag{3.91}$$

where $\mathsf{T}^{(0)} := \frac{1}{2}\,\mathbb{1}_{n\times n}$ and $\mathsf{T}^{(1)} := \frac{1}{2}\,\sigma^{(1)}$, as given by Example 3.3. The coupled system is no longer $U(2)$ invariant, which is why the quantities $c^a(t)$ of Example 3.6 cannot all be expected to remain constant. In Remark 5.6 we will show that in a *scaling limit* (specified in Chapter 4), the second term does *not* contribute to the evolution of the quantities $c^{(a)}(t)$. This will allow us to show that $c^{(0)}$ and $c^{(1)}$ remain constant while $c^{(2)}(t)$ and $c^{(3)}(t)$ oscillate, corresponding to a periodic exchange of energy between the two oscillators, c.f. Section 6.1.2.

3.2.1.5. Transformation to coupling eigenbasis via DFT

As shown in Lemma 2.27, circulant matrices are precisely those matrices which can be diagonalized by a DFT. This is why we will now study the transformed process $\tilde{x}^\varepsilon(t) := Q^\dagger x^\varepsilon(t)$, allowing us to significantly simplify the evolution equations.

Lemma 3.18 (DFT of drift terms)
The evolution of $\tilde{x}^\varepsilon(t) := Q^\dagger x^\varepsilon(t)$, with $x^\varepsilon(t)$ solving Eq. (3.87), is given by

$$\mathrm{d}\tilde{x}^\varepsilon(t) = i\,\kappa\,\tilde{x}^\varepsilon(t)\,\mathrm{d}t + \varepsilon\,\kappa\,\widetilde{u_{\mathrm{lin}}}(\tilde{x}^\varepsilon(t))\,\mathrm{d}t, \tag{3.92}$$

where

$$\widetilde{u_{\mathrm{lin}}}(\tilde{x}) := \sqrt{n}\left(\tilde{\mathfrak{l}}' \odot \widetilde{\mathrm{Re}(Q\tilde{x})} + \tilde{\mathfrak{l}} \odot \widetilde{\mathrm{Im}(Q\tilde{x})}\right) = \frac{\sqrt{n}}{2}\left(\mathrm{diag}(\widetilde{\lambda})\tilde{x} + R\,\overline{\mathrm{diag}(\widetilde{\lambda})\tilde{x}}\right). \tag{3.93}$$

The transformed coupling vectors $\tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}'$ inherit the parity of the coupling vectors $\mathfrak{l}, \mathfrak{l}' \in \mathbb{R}^n$.

Proof. Let $\tilde{x} \in \mathbb{C}^n$ and set $x := Q\tilde{x}$. By Eq. (3.83) we have

$$u_{\mathrm{lin}}(x) = \mathfrak{l}' \circledast \mathrm{Re}(x) + \mathfrak{l} \circledast \mathrm{Im}(x). \tag{3.94}$$

This can be transformed by employing the convolution formula of Theorem 2.26, according to which circular convolutions are translated into pointwise multiplications, confirming the first representation of $\widetilde{u_{\mathrm{lin}}}$ stated in Eq. (3.93). wrepresentation we recall Eq. (3.86), which implies that

$$\widetilde{u_{\text{lin}}}(\tilde{x}) = Q^{\dagger} u_{\text{lin}}(x) = \frac{1}{2} Q^{\dagger} \left(\Lambda x + \overline{\Lambda x} \right) = \frac{1}{2} \left[(Q^{\dagger} \Lambda Q)(Q^{\dagger} x) + (Q^{\dagger})^2 \overline{(Q^{\dagger} \Lambda Q)(Q^{\dagger} x)} \right]$$
$$= \frac{1}{2} \left[\widetilde{\Lambda} \tilde{x} + R \, \overline{\widetilde{\Lambda} \tilde{x}} \right], \tag{3.95}$$

where the transformed coupling matrix $\widetilde{\Lambda} := Q^{\dagger} \Lambda Q$ is given by (recall Lemma 2.27)

$$\widetilde{\Lambda} = Q^{\dagger} \operatorname{cycl}(\lambda) \, Q = \sqrt{n} \operatorname{diag}(\widetilde{\lambda}). \tag{3.96}$$

The statement on the Fourier-transformed vectors $\tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}'$, finally follows from Lemma 2.30. □

Continuing with Example 3.12, we calculate the transformed coupling vectors.

Lemma 3.19 (Directed and undirected circulant coupling topologies)
The DFT's of the coupling vectors introduced in Example 3.12, are given as follows.

Even (symmetric) circulant coupling topologies:

Distance-0 $$\tilde{\mathfrak{l}} = \frac{1}{\sqrt{n}} (1, 1, \ldots, 1)^{\top}, \tag{3.97}$$

Distance-1 $$\tilde{\mathfrak{l}} = \frac{2}{\sqrt{n}} \left(1, \cos\left(\frac{2\pi}{n}\right), \cos\left(\frac{4\pi}{n}\right), \ldots, \cos\left(\frac{2\pi\,(n-1)}{n}\right) \right)^{\top}, \tag{3.98}$$

Distance-2 $$\tilde{\mathfrak{l}} = \frac{2}{\sqrt{n}} \left(1, \cos\left(\frac{4\pi}{n}\right), \cos\left(\frac{8\pi}{n}\right), \ldots, \cos\left(\frac{4\pi\,(n-1)}{n}\right) \right)^{\top}, \tag{3.99}$$

Distance-l $$\tilde{\mathfrak{l}} = \left(\frac{2}{\sqrt{n}} \cos\left(\frac{2\pi\,k\,l}{n}\right) \right)_{k \in \{0,\ldots,n-1\}}, \tag{3.100}$$

All-to-all $$\tilde{\mathfrak{l}} = \frac{1}{\sqrt{n}} (n-1, -1, \ldots, -1)^{\top}. \tag{3.101}$$

Directed circulant coupling topologies

Distance-1 $$\tilde{\mathfrak{l}} = \frac{(-2i)}{\sqrt{n}} \left(1, \sin\left(\frac{2\pi}{n}\right), \sin\left(\frac{4\pi}{n}\right), \ldots, \sin\left(\frac{2\pi\,(n-1)}{n}\right) \right)^{\top}, \tag{3.102}$$

Distance-k $$\tilde{\mathfrak{l}} = \left(\frac{(-2\,i)}{\sqrt{n}} \sin\left(\frac{2\pi\,k\,l}{n}\right) \right)_{k \in \{0,\ldots,n-1\}}. \tag{3.103}$$

Proof. Calculating the DFT of *even* (*odd*) distance-l coupling, we find that

$$\tilde{\mathfrak{l}}_k = \sum_{k'=0}^{n-1} Q^{\dagger}_{k,k'} \left(\delta_{k',l} \pm \delta_{k',(n-l)} \right) = Q^{\dagger}_{k,l} \pm Q^{\dagger}_{k,(n-l)} = Q^{\dagger}_{k,l} \pm \overline{Q^{\dagger}_{k,l}}$$
$$= \begin{cases} \frac{2}{\sqrt{n}} \cos\left(\frac{2\pi\,k\,l}{n}\right), & \textit{even} \text{ distance-}l \text{ coupling}, \\ \frac{(-2\,i)}{\sqrt{n}} \sin\left(\frac{2\pi\,k\,l}{n}\right), & \textit{odd} \text{ distance-}l \text{ coupling}, \end{cases} \tag{3.104}$$

where in the second to last step we have made use of[22]

$$Q^{\dagger}_{k,(n-l)} = \frac{1}{\sqrt{n}}\,\overline{\mathfrak{u}}^{k\,(n-l)} = \frac{1}{\sqrt{n}}\,\mathfrak{u}^{k\,l} = \frac{1}{\sqrt{n}}\,\overline{\overline{\mathfrak{u}}^{k\,l}} = \overline{Q^{\dagger}_{k,l}}. \tag{3.105}$$

For the case of all-to-all coupling we observe that

$$\tilde{\mathfrak{l}}_k = \sum_{k'=1}^{n-1} Q^{\dagger}_{k,k'} = \frac{1}{\sqrt{n}}\sum_{k'=1}^{n-1}\overline{\mathfrak{u}}^{k\,k'} = \frac{\left[\sum_{k'=0}^{n-1}\overline{\mathfrak{u}}^{k\,k'}\right]-1}{\sqrt{n}} = \frac{[n\,\delta_{k,0}]-1}{\sqrt{n}}, \tag{3.106}$$

which follows by a similar argument as in the proof of Lemma 2.22. □

Symmetric distance-l coupling will be studied extensively in Section 6.4.3.

3.2.2. Nonlinear pertubations

We introduce a class of *nonlinear* perturbations u_{nl} of the drift term, which will prove to have a vanishing impact on the *averaged system*, c.f. Theorem 4.71 and Lemma 5.7.

Definition 3.20 (Nonlinear coupling)
Let $\mathsf{A} : \mathbb{C}^n \to \mathbb{C}^{n,n}$ denote an antihermitian, *nonlinear coupling matrix* of the form

$$\mathsf{A}_{k,l}(x) = \begin{cases} h_{k,l}(\overline{x}_k, x_l), & k < l, \\ 0, & k = l, \\ -\overline{\mathsf{A}}_{l,k}(x), & k > l, \end{cases} \tag{3.107}$$

where the functions $h_{k,l} : \mathbb{C}^2 \to \mathbb{C}$ are *holomorphic* and antisymmetric, i.e.

$$h_{k,l}(-z_1, -z_2) = -h_{k,l}(z_1, z_2), \tag{3.108}$$

s.t. $\overline{h_{k,l}(z)} = h_{k,l}(\overline{z})$ for all $z \in \mathbb{C}^2$.[a] Let the nonlinear drift term $u_{\mathrm{nl}} : \mathbb{C}^n \to \mathbb{C}^n$ be given by

$$u_{\mathrm{nl}}(x) \coloneqq \mathsf{A}(x)\,x \tag{3.109}$$

and let $U_{\mathrm{nl}} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ denote the real-valued representation of u_{nl}, i.e.

$$U_{\mathrm{nl}}(X) \coloneqq \mathsf{j} \circ u_{\mathrm{nl}} \circ \mathsf{j}^{-1}(X). \tag{3.110}$$

[a] This is satisfied if the holomorphic function $h_{k,l}$ has *real-valued* expansion coefficients.

Note that A can be interpreted as a nonlinear perturbation of the coupling matrices studied in the previous section. We show that U_{nl} is a *divergence free* drift term, which satisfies a *weak coercivity*[23] condition.

[22] Recall that $\mathfrak{u} \coloneqq \exp\left(\frac{2\pi i}{n}\right)$ denotes the n'th unit root and that Q is a symmetric matrix defined by Eq. (2.60).
[23] c.f. [LR15], Eq. (3.4)

Lemma 3.21 (Properties of nonlinear drift)
The real-valued representation U_{nl} of the nonlinear drift term u_{nl} satisfies

$$X^\top U_{\mathrm{nl}}(X) = 0, \quad \forall\, X \in \mathbb{R}^{2n}, \quad \text{(weak coercivity)}, \tag{3.111}$$

as well as

$$\nabla^\top U_{\mathrm{nl}}(X) = 0, \quad \forall\, X \in \mathbb{R}^{2n}, \quad \text{(divergence free)}. \tag{3.112}$$

Proof. Let $x \in \mathbb{C}^n$ and let $X := \mathrm{j}(x)$ denote its real-valued representation. Note that, by definition, $\mathsf{A}(x)$ is an antihermitian matrix, i.e.

$$\mathsf{A}^\dagger(x) = -\mathsf{A}(x). \tag{3.113}$$

This implies that $x^\dagger \mathsf{A}(x)\, x$ is imaginary:

$$\overline{(x^\dagger \mathsf{A}(x)\, x)} = \left(x^\dagger \mathsf{A}(x)\, x\right)^\dagger = x^\dagger \mathsf{A}^\dagger(x)\, x = -x^\dagger \mathsf{A}(x)\, x. \tag{3.114}$$

Application of Lemma 2.10 therefore yields

$$X^\top U_{\mathrm{nl}}(X) = \mathrm{Re}\left(x^\dagger u_{\mathrm{nl}}(x)\right) = \mathrm{Re}\left(x^\dagger \mathsf{A}(x)\, x\right) = 0. \tag{3.115}$$

By Eq. (2.45), Lemma 2.10 and the product rule we furthermore obtain that

$$\nabla^\top U_{\mathrm{nl}}(X) = 2\,\mathrm{Re}\left(\partial^\top u_{\mathrm{nl}}(x)\right) = 2\,\mathrm{Re}\left[\left(\partial^\top \mathsf{A}(x)\right) x + \mathrm{tr}\left(\mathsf{A}(x)\right)\right] = 2\,\mathrm{Re}\left[\left(\partial^\top \mathsf{A}(x)\right) x\right], \tag{3.116}$$

since

$$\sum_{k,l} \mathsf{A}_{k,l}(x)\, \partial_k x_l = \sum_{k,l} \mathsf{A}_{k,l}(x)\, \delta_{k,l} = \sum_k \mathsf{A}_{k,k}(x) = \mathrm{tr}\left(\mathsf{A}(x)\right) = 0. \tag{3.117}$$

Moreover, by Eq. (3.107) we find that

$$(\partial^\top \mathsf{A}(x))_l = \sum_{k<l} \partial_k \mathsf{A}_{k,l}(x) - \sum_{k>l} \partial_k \overline{\mathsf{A}}_{l,k}(x) = \sum_{k<l} \partial_k h_{k,l}(\overline{x}_k, x_l) - \sum_{k>l} \partial_k \underbrace{\overline{h_{l,k}(\overline{x}_l, x_k)}}_{=h_{l,k}(x_l, \overline{x_k})} = 0, \tag{3.118}$$

since $h_{k,l}$ is holomorphic and $\partial_k \overline{x}_k = \partial_k x_l = 0$ for $k \neq l$. Thus we conclude that $\nabla^\top U_{\mathrm{nl}} = 0$. □

In particular, Definition 3.20 encompasses nonlinear pair-coupling matrices of the following form.

Remark 3.22 (Nonlinear pair-coupling)
Let

$$h : \mathbb{C} \to \mathbb{C}, \quad z \to h(z) := \sum_{r=0}^{\infty} \alpha^{(r)} z^r \tag{3.119}$$

denote a holomorphic function with *real-valued* expansion coefficients $\alpha^{(r)} \in \mathbb{R}$, satisfying

$$h(-z) = -h(z), \tag{3.120}$$

i.e. $\alpha^{(r)} = 0$, if r even. Let furthermore $C \in \mathbb{C}^{n,n}$ denote a hermitian matrix. Then

$$\mathsf{A}_{k,l}(x) := C_{k,l}\, h(\overline{x}_k - x_l), \quad \forall k, l \in \{0, ..., n-1\} \tag{3.121}$$

defines a nonlinear coupling matrix of the form specified in Definition 3.20.

Proof. For all $x \in \mathbb{C}^n$, the matrix $\mathsf{A}(x)$ is antihermitian, since

$$\mathsf{A}^\dagger_{k,l}(x) = \overline{\mathsf{A}}_{l,k}(x) = \overline{C_{l,k}}\, \overline{h(\overline{x}_l - x_k)} = C^\dagger_{k,l}\, h(x_l - \overline{x}_k) = -C_{k,l}\, h(\overline{x}_k - x_l) = -\mathsf{A}_{k,l}, \tag{3.122}$$

where in the third step we have employed the holomorphy of h together with the fact that its expansion coeffients are real-valued, which implies that $\overline{h(z)} = h(\overline{z})$. In the fourth step we have made use of C being hermitian and h being antisymmetric. Moreover, $\mathsf{A}(x)$ can be written as in Eq. (3.107) if we set

$$h_{k,l}(z_1, z_2) := C_{k,l}\, h(z_1 - z_2), \tag{3.123}$$

which defines holomorphic and antisymmetric functions $h_{k,l} : \mathbb{C}^2 \to \mathbb{C}$, as required. □

Note that C plays the role of a constant coupling matrix (similiar to the previous section), which is now weighted by a nonlinear coupling-function h. One can think of the coupling function h as given by the unique analytic extension of a real analytic and antisymmetric function $f : \mathbb{R} \to \mathbb{R}$, such as for instance $f(x) = \sin(x)$.[24]

Example 3.23 (Complex Kuramoto coupling)
If in the setup of Remark 3.22 we choose $h(x) := \sin(x)$, we obtain a complex Kuramoto-like coupling matrix of the form

$$\mathsf{A}_{k,l}(x) := C_{k,l}\, \sin(\overline{x}_k - x_l), \tag{3.124}$$

which gives rise to a nonlinear drift term u_{nl} of the form

$$(u_{\mathrm{nl}})_k(x) = \sum_{l=0}^{n-1} C_{k,l} \sin(\overline{x}_k - x_l)\, x_l. \tag{3.125}$$

[24]This follows from the *identity theorem*, c.f. [Jän13], Section 3.5, Theorem 12.

3.3. Weak stochastic perturbations

3.3.1. Multiplicative noise

We define a randomly perturbed analogue to the linear circulant coupling of Section 3.2.1. More precisely, we introduce a linear *multiplicative-noise* term. Following the lines of the previous section, we first of all define a complex circulant-coupling matrix (c.f. Definition 3.16), which in this context will be called $\mathcal{V}$.

Definition 3.24 (Multiplicative-noise coupling vector)
Let $\mathfrak{n}', \mathfrak{n} \in \mathbb{R}^n$ and define

$$\nu := \mathfrak{n}' - i\,\mathfrak{n} \in \mathbb{C}^n, \tag{3.126}$$

where $\mathfrak{n}'$ and $\mathfrak{n}$ play the role of a momentum- and space-coupling vector, respectively. The complex coupling vector ν encompasses both contributions and gives rise to the circulant coupling matrix

$$\mathcal{V} := \operatorname{cycl}(\nu)\,. \tag{3.127}$$

These noise-coupling vectors determine the multiplicative-noise dispersion matrix $\sigma_{\text{mult}}(x)$. We will specify this matrix in Definition 3.25 after giving a brief informal derivation of how this term can arise as a *white-noise perturbation* of a circulant deterministic coupling.

Informal motivation of noise-coupling structure

Note that the term

$$\operatorname{Re}(\mathcal{V}\,x) = \operatorname{Re}(\nu \circledast x) \tag{3.128}$$

has the exact same form as the circulant drift term of Eq. (3.86). We now construct the noise-coupling term as a white-noise perturbation of this deterministic drift term, i.e. we look at a circular convolution of the drift term with an n-dimensional white-noise vector. A circulant drift corresponds to an ODE of the form

$$\dot{x} = \operatorname{Re}(\nu \circledast x) \tag{3.129}$$

and circular convolution of such a circulant drift term with an n-dimensional white-noise perturbation ξ_{mult}, can thus (heuristically) be written as

$$\dot{x} = \operatorname{Re}(\nu \circledast x) \circledast \xi_{\text{mult}}.$$

If we represent this expression in terms of an SDE, we obtain

$$\mathrm{d}x = \operatorname{Re}(\nu \circledast x) \circledast \mathrm{d}B_{\text{mult}} = \operatorname{cycl}\left(\operatorname{Re}(\nu \circledast x)\right) \mathrm{d}B_{\text{mult}}, \tag{3.130}$$

where B_{mult} is an n dimensional Brownian motion.

Definition of noise-coupling term

Formalizing the results of the previous paragraph, we introduce a *linear multiplicative noise* with a *circulant dispersion matrix* as specified in the following definition.

Definition 3.25 (Multiplicative-noise dispersion matrix)
Let the multiplicative-noise dispersion matrix $\sigma_{\text{mult}} : \mathbb{C}^n \to \mathbb{C}^{n,n}$ be defined by

$$\sigma_{\text{mult}}(x) := \operatorname{cycl}\left(\mathfrak{n}' \circledast \operatorname{Re}(x) + \mathfrak{n} \circledast \operatorname{Im}(x)\right) = \operatorname{Re}\left(\operatorname{cycl}\left(\nu \circledast x\right)\right). \tag{3.131}$$

The contribution of the multiplicative noise is given by the term

$$\sqrt{\varepsilon\,\kappa}\;\sigma_{\text{mult}}(x^\varepsilon(t))\,\mathrm{d}B_{\text{mult}}(t) = \sqrt{\varepsilon\,\kappa}\;\operatorname{Re}\left(\operatorname{cycl}\left(\nu \circledast x^\varepsilon(t)\right)\right)\mathrm{d}B_{\text{mult}}(t), \tag{3.132}$$

where B_{mult} is an n-dimensional, real-valued Brownian motion. Note that the scaling factor of $\sqrt{\varepsilon\,\kappa}$ captures the relative 'weakness' of the noise perturbation. This factor will be eliminated in the time-rescaling step of Section 3.5.

It will prove useful to decompose the components $\nu_k \in \mathbb{C}$ of the coupling vector $\nu \in \mathbb{C}^n$ into absolute value and phase-factor.

Definition 3.26 (Coupling angles)
For every $k \in \{0, ..., n-1\}$ we introduce the polar coordinate representation

$$\nu_k = |\nu_k|\; \mathrm{e}^{i\,\gamma_k}, \qquad \text{where } \gamma_k \in [0, 2\pi). \tag{3.133}$$

The phases γ_k will be called *coupling angles*.

The coupling angles describe the relation between space- and momentum-parts of the multiplicative-noise coupling. As illustrated in Fig. 3.5, we distinguish the following cases (w.r.t. the k'th oscillator):

Pure momentum coupling	$\mathfrak{n}_k = 0,$	$\nu_k = \mathfrak{n}'_k \in \mathbb{R},$	$\cos(2\gamma_k) = 1,$
Momentum dominated coupling	$\lvert\mathfrak{n}'_k\rvert > \lvert\mathfrak{n}_k\rvert,$	$\nu_k = \mathfrak{n}'_k - i\,\mathfrak{n}_k,$	$\cos(2\gamma_k) \in (0, 1),$
Equal coupling	$\lvert\mathfrak{n}'_k\rvert = \lvert\mathfrak{n}'_k\rvert,$	$\nu_k = \mathfrak{n}'_k(1 \mp i),$	$\cos(2\gamma_k) = 0,$
Space dominated coupling	$\lvert\mathfrak{n}'_k\rvert < \lvert\mathfrak{n}_k\rvert,$	$\nu_k = \mathfrak{n}'_k - i\,\mathfrak{n}_k,$	$\cos(2\gamma_k) \in (-1, 0),$
Pure space coupling	$\mathfrak{n}'_k = 0,$	$\nu_k = -i\,\mathfrak{n}_k \in i\,\mathbb{R},$	$\cos(2\gamma_k) = -1,$

We thus find that $\cos(2\gamma_k)$ is positive (depicted in blue in Fig. 3.5), if the momentum coupling $\mathfrak{n}'_k$ dominates in *absolute* value compared to the space coupling $\mathfrak{n}'_k$. This distinction will prove useful in analyzing the behavior of the averaged system, c.f. Lemma 6.27 and Remark 6.29.

In the context of multiplicative-noise coupling, we will restrict our attention to undirected (i.e. symmetric) coupling links, which is why we impose the following restriction.

Assumption 3.27 (Symmetric noise-coupling)
In the following we assume that ν is *even*, i.e. $R\,\nu = \nu$ (c.f. Definition 2.28).

As in the case of deterministic coupling, the SDE can be simplified by performing a DFT.

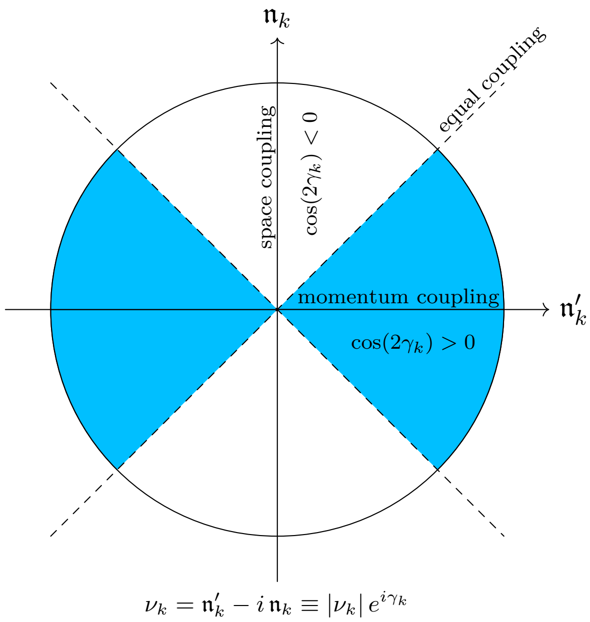


Figure 3.5.: Noise-coupling angle γ_k

Lemma 3.28 (DFT of multiplicative-noise term)
The transformed multiplicative-noise term is given by

$$\sqrt{\varepsilon\,\kappa}\,\widetilde{\sigma_{\text{mult}}}(\tilde{x}^{\varepsilon}(t))\,\mathrm{d}\widetilde{B_{\text{mult}}}(t), \tag{3.134}$$

where

$$\begin{aligned}\widetilde{\sigma_{\text{mult}}}(\tilde{x}) &\coloneqq n\ \text{diag}\left(\widetilde{\mathfrak{n}'}\odot \widetilde{\text{Re}(Q\tilde{x})} + \tilde{\mathfrak{n}}\odot\widetilde{\text{Im}(Q\tilde{x})}\right)\\ &= \frac{n}{2}\ \text{diag}\left(\text{diag}(\tilde{\nu})\tilde{x} + R\,\overline{\text{diag}(\tilde{\nu})\tilde{x}}\right),\end{aligned} \tag{3.135}$$

$$\widetilde{B_{\text{mult}}}(t) \coloneqq Q^{\dagger}B_{\text{mult}}(t). \tag{3.136}$$

Proof. The DFT of the multiplicative-noise term of Eq. (3.132) is given by

$$Q^{\dagger}\left[\sqrt{\varepsilon\,\kappa}\ \sigma_{\text{mult}}(x^{\varepsilon}(t))\,\mathrm{d}B_{\text{mult}}(t)\right] = \sqrt{\varepsilon\,\kappa}\ Q^{\dagger}\sigma_{\text{mult}}(x^{\varepsilon}(t))Q\,\mathrm{d}Q^{\dagger}B_{\text{mult}}(t). \tag{3.137}$$

We can calculate the DFT of the circulant matrix $\sigma_{\text{mult}}(x)$ by employing Eq. (2.77) from Lemma 2.27, which for all $x\in\mathbb{C}^n$ yields

$$\begin{aligned}Q^{\dagger}\sigma_{\text{mult}}(x)\,Q &= Q^{\dagger}\,\text{cycl}\left(\mathfrak{n}'\circledast\text{Re}(x) + \mathfrak{n}\circledast\text{Im}(x)\right)Q\\ &= \sqrt{n}\,\text{diag}\left(Q^{\dagger}\left(\mathfrak{n}'\circledast\text{Re}(x)+\mathfrak{n}\circledast\text{Im}(x)\right)\right)\\ &= n\,\text{diag}\left(\widetilde{\mathfrak{n}'}\odot\widetilde{\text{Re}(x)} + \tilde{\mathfrak{n}}\odot\widetilde{\text{Im}(x)}\right).\end{aligned} \tag{3.138}$$

In the last step we have applied the convolution formula (2.72) from Theorem 2.26, which allows us to replace the circular convolutions by pointwise multiplications of the transformed coupling vectors. The representation of $\widetilde{\sigma_{\text{mult}}}$ in terms of $\widetilde{\nu}$ follows along the lines of the proof of Lemma 3.18. □

Note that from Assumption 3.27 and Lemma 2.30 it follows that $\widetilde{\nu}$ inherits the parity of ν, i.e.

$$\widetilde{\nu}_k = \widetilde{\nu}_{n-k}, \quad \forall k \in \{0, ..., n-1\}\,. \tag{3.139}$$

Analogously to Definition 3.26, we introduce a polar decomposition of the transformed noise-coupling vector $\widetilde{\nu}$, i.e. we decompose each component $\widetilde{\nu}_k$ into its absolute value and its phase in the complex plane.

Definition 3.29 (Transformed noise-coupling vector and coupling angles)
For every $k \in \{0, ..., n-1\}$, we introduce the polar-coordinate representation

$$\widetilde{\nu}_k = |\widetilde{\nu}_k|\; \mathrm{e}^{i\,\tilde{\gamma}_k}, \tag{3.140}$$

where, as in Definition 3.26, we refer to the values $\tilde{\gamma}_k$ as *coupling angles* (of the transformed system).

Note that the angles $\tilde{\gamma}_k$ are *not* given by the DFT of the vector $(\gamma_k)_k$ of phases, but are defined as the phases of the transformed vector $\widetilde{\nu}$. In the following lemma we will show that for distance-l coupling, all coupling angles $\tilde{\gamma}_k$ coincide with γ_l, up to a possible shift of $\pm\pi$.

Lemma 3.30 (Distance-l coupling angles of transformed system)
Let $\nu = \nu_l\,(\delta_{k,l} + \delta_{k,n-l})_{k\in\{0,...,n-1\}}$, i.e. we assume a symmetric distance-l coupling (c.f. Example 3.12) with coupling strength $\nu_l = |\nu_l|\; \mathrm{e}^{i\,\gamma_l}$. Then for all $k \in \{0, ..., n-1\}$, s.t. $\widetilde{\nu}_k \neq 0$, it follows that

$$\cos(2\tilde{\gamma}_k) = \cos(2\gamma_l). \tag{3.141}$$

Proof. By Eq. (3.100) we conclude that

$$\widetilde{\nu}_k = \nu_l\,\frac{2}{\sqrt{n}}\,\cos\left(\frac{2\pi\,k\,l}{n}\right), \tag{3.142}$$

i.e. up to a real-valued factor, all components $\widetilde{\nu}_k$ coincide with ν_l. If this factor is positive, then the phases of $\widetilde{\nu}_k$ and ν_l agree, and if it is negative, they agree up to a $\pm\pi$ shift, i.e.

$$\tilde{\gamma}_k = \begin{cases} \gamma_l, & \cos\left(\frac{2\pi\,k\,l}{n}\right) > 0, \\ \gamma_l \pm \pi, & \cos\left(\frac{2\pi\,k\,l}{n}\right) < 0, \end{cases} \tag{3.143}$$

where we have omitted the case of $\cos\left(\frac{2\pi\,k\,l}{n}\right) = 0$, in which the phase of $\widetilde{\nu}_k = 0$ is not well defined. The result now follows by observing that

$$\cos\left(2(\gamma_l \pm \pi)\right) = \cos(2\gamma_l). \tag{3.144}$$

□

3.3.2. Regularizing noise

For technical reasons, c.f. Remark 4.7 and Lemma 4.23, we introduce a *regularizing*-noise term. It is comprised of two independent versions of multiplicative noise similar to Definition 3.25, only without a restriction of 'taking the real part'.

Definition 3.31 (Regularizing noise)
Let the *regularizing-noise* dispersion matrix $\boldsymbol{\sigma}_{\text{reg}} : \mathbb{C}^n \to \mathbb{C}^{n,2n}$ be given by

$$\boldsymbol{\sigma}_{\text{reg}}(x) \coloneqq \Big(\sigma_{\text{reg}}(x) \mid \sigma'_{\text{reg}}(x)\Big) \coloneqq \frac{\sqrt{n}}{2}\,\sigma_r\left(\text{cycl}\,(x) \mid i\,\text{cycl}\,(Rx)\right), \tag{3.145}$$

where $\sigma_r > 0$.

The parameter σ_r will be chosen as a small value, c.f. Remark 6.36 for a discussion of the influence of regularizing noise on the system's asymptotic synchronization behavior. The contribution of the regularizing noise is given by the term

$$\sqrt{\varepsilon\,\kappa}\;\boldsymbol{\sigma}_{\text{reg}}(x^\varepsilon(t))\,\mathrm{d}\boldsymbol{B}_{\text{reg}}(t) = \sqrt{\varepsilon\,\kappa}\;\frac{\sqrt{n}}{2}\,\sigma_r\,\Big(\text{cycl}\,(x)\,\mathrm{d}B_{\text{reg}}(t) + i\,\text{cycl}\,(Rx)\,\mathrm{d}B'_{\text{reg}}(t)\Big)\,, \tag{3.146}$$

where $\boldsymbol{B}_{\text{reg}} = ((B_{\text{reg}})^\top, (B'_{\text{reg}})^\top)^\top$ and $B_{\text{reg}}, B'_{\text{reg}}$ are independent, n-dimensional, real-valued Brownian motions, which are independent of B_{mult}. Lemma 5.15 will illustrate why this choice of regularizing noise is particularly suitable, namely, unlike its multiplicative-noise counterpart, it is invariant under a certain averaging procedure.

As in the previous sections, we can perform a DFT in order to diagonalize the circulant matrices.

Lemma 3.32 (DFT of regularizing noise)
The transformed regularizing-noise term is given by

$$\sqrt{\varepsilon\,\kappa}\,\widetilde{\boldsymbol{\sigma}_{\text{reg}}}(\tilde{x}^\varepsilon(t))\,\mathrm{d}\widetilde{\boldsymbol{B}_{\text{reg}}}(t), \tag{3.147}$$

where

$$\widetilde{\boldsymbol{\sigma}_{\text{reg}}}(\tilde{x}) \coloneqq \frac{n}{2}\,\sigma_r\,\left(\text{diag}(\tilde{x}) \mid i\,\text{diag}(R\tilde{x})\right), \tag{3.148}$$

$$\widetilde{\boldsymbol{B}_{\text{reg}}}(t) \coloneqq \left(\Big(Q^\dagger B_{\text{reg}}(t)\Big)^\top, \Big(Q^\dagger B'_{\text{reg}}(t)\Big)^\top\right)^\top. \tag{3.149}$$

Proof. Omitting the scalar prefactor of $\sqrt{\varepsilon\,\kappa}$, the DFT of the regularizing-noise term of Eq. (3.146) is given by

$$\begin{aligned} &Q^\dagger\left[\Big(\sigma_{\text{reg}}(x^\varepsilon(t)) \mid \sigma'_{\text{reg}}(x^\varepsilon(t))\Big)\,\mathrm{d}\begin{pmatrix} B_{\text{reg}} \\ B'_{\text{reg}} \end{pmatrix}(t)\right] \\ &\quad= \Big(Q^\dagger\sigma_{\text{reg}}(x^\varepsilon(t))Q \mid Q^\dagger\sigma'_{\text{reg}}(x^\varepsilon(t))Q\Big)\,\mathrm{d}\begin{pmatrix} Q^\dagger B_{\text{reg}} \\ Q^\dagger B'_{\text{reg}} \end{pmatrix}(t). \end{aligned} \tag{3.150}$$

The transformation of the circulant matrices σ_{reg} and σ'_{reg} under the discrete Fourier transformation is given by Lemma 2.27, which yields

$$Q^\dagger \sigma_{\text{reg}}(x) Q = \frac{\sqrt{n}}{2}\, \sigma_r\, Q^\dagger \operatorname{cycl}(x)\, Q = \frac{n}{2}\, \sigma_r\, \operatorname{diag}(Q^\dagger x), \quad \forall x \in \mathbb{C}^n, \tag{3.151a}$$

$$Q^\dagger \sigma'_{\text{reg}}(x) Q = i\, \frac{\sqrt{n}}{2}\, \sigma_r\, Q^\dagger \operatorname{cycl}(Rx)\, Q = i\, \frac{n}{2}\, \sigma_r\, \operatorname{diag}(R\, Q^\dagger x), \quad \forall x \in \mathbb{C}^n, \tag{3.151b}$$

where in the last step we have made use of the fact that R and $Q^\dagger$ commute. □

3.3.3. Additive noise

The noise terms introduced so far are all (linearly) dependent on the system's state vector and consequently might become small, or vanish altogether. By contrast, we now also want to allow for an isotropic *additive* noise, thereby introducing a global random perturbation which does *not* depend on the state of the oscillator system.

Definition 3.33 (Additive noise)
Let the additive-noise dispersion matrix $\boldsymbol{\sigma}_{\text{add}} \in \mathbb{C}^{n,2n}$ be defined as

$$\boldsymbol{\sigma}_{\text{add}} := \left(\sigma_{\text{add}} \mid \sigma'_{\text{add}}\right) := \frac{\sigma_0}{\sqrt{2}} \left(\mathbb{1}_{n\times n} \mid i\, \mathbb{1}_{n\times n}\right), \tag{3.152}$$

where $\sigma_0 > 0$.

The additive-noise contribution is given by

$$\begin{aligned} \sqrt{\varepsilon\,\kappa}\, \boldsymbol{\sigma}_{\text{add}}\, \mathrm{d}\boldsymbol{B}_{\text{add}}(t) &= \sqrt{\varepsilon\,\kappa}\, \left(\sigma_{\text{add}}\, \mathrm{d}B_{\text{add}}(t) + \sigma'_{\text{add}}\, \mathrm{d}B'_{\text{add}}(t)\right) \\ &= \sqrt{\varepsilon\,\kappa}\, \frac{\sigma_0}{\sqrt{2}}\, \mathrm{d}\left(B_{\text{add}}(t) + iB'_{\text{add}}(t)\right), \end{aligned} \tag{3.153}$$

where $\boldsymbol{B}_{\text{add}} = ((B_{\text{add}})^\top, (B'_{\text{add}})^\top)^\top$ and $B_{\text{add}}, B'_{\text{add}}$ are independent, n-dimensional and real-valued Brownian motions, which are independent of B_{mult} and $\boldsymbol{B}_{\text{reg}}$. Eq. (3.153) thus corresponds to a perturbation induced by the *complex* Brownian motion $B_{\text{add}}(t) + iB'_{\text{add}}(t)$, i.e. both the real part and the imaginary part of the oscillator system are subject to an additive-noise perturbation. In the following chapter we will show that such a complex additive noise ensures a *uniform ellipticity* of the combined diffusion matrix, c.f. Lemma 4.6.

We observe that *isotropic* additive noise is invariant under the discrete Fourier transform.

Lemma 3.34 (DFT of additive noise)
The transformed additive-noise term is given by

$$\sqrt{\varepsilon\,\kappa}\, \widetilde{\boldsymbol{\sigma}_{\text{add}}}\, \mathrm{d}\widetilde{\boldsymbol{B}_{\text{add}}}(t) = \sqrt{\varepsilon\,\kappa}\, \frac{\sigma_0}{\sqrt{2}}\, \mathrm{d}(\widetilde{B_{\text{add}}} + i\, \widetilde{B'_{\text{add}}})(t), \tag{3.154}$$

where

$$\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}} := \left(\widetilde{\sigma_{\mathrm{add}}} \mid \widetilde{\sigma'_{\mathrm{add}}}\right) = \frac{\sigma_0}{\sqrt{2}} \left(\mathbb{1}_{n\times n} \mid i\, \mathbb{1}_{n\times n}\right) = \boldsymbol{\sigma}_{\mathrm{add}}, \tag{3.155}$$

$$\widetilde{\boldsymbol{B}_{\mathrm{add}}}(t) := ((\widetilde{B_{\mathrm{add}}})^\top, (\widetilde{B'_{\mathrm{add}}})^\top)^\top := ((Q^\dagger B_{\mathrm{add}})^\top, (Q^\dagger B'_{\mathrm{add}})^\top)^\top. \tag{3.156}$$

Proof. We find that

$$Q^\dagger \boldsymbol{\sigma}_{\mathrm{add}}\, \mathrm{d}\boldsymbol{B}_{\mathrm{add}}(t) = Q^\dagger \sigma_{\mathrm{add}}\, \mathrm{d}B_{\mathrm{add}}(t) + Q^\dagger \sigma'_{\mathrm{add}}\, \mathrm{d}B'_{\mathrm{add}}(t). \tag{3.157}$$

The first term, for instance, can be rewritten as

$$Q^\dagger \sigma_{\mathrm{add}}\, \mathrm{d}B_{\mathrm{add}}(t) = \left(Q^\dagger \sigma_{\mathrm{add}} Q\right) \mathrm{d} \left(Q^\dagger B_{\mathrm{add}}\right)(t), \tag{3.158}$$

where

$$\widetilde{\sigma_{\mathrm{add}}} := Q^\dagger \sigma_{\mathrm{add}} Q = Q^\dagger \left(\sigma_0\, \mathbb{1}_{n\times n}\right) Q = \sigma_0\, \mathbb{1}_{n\times n}, \tag{3.159}$$

since $\sigma_0 > 0$ is a scalar value and Q a unitary matrix. The second terms follows analogously. □

Note that $(\widetilde{B_{\mathrm{add}}} + i\, \widetilde{B'_{\mathrm{add}}})(t)$ is the DFT of a complex Brownian motion, which by Lemma 2.35 again constitutes a complex Brownian motion.

3.4. Combined stochastic differential equation

We summarize the results of the previous sections and present the combined system, both before and after the DFT. Furthermore we prove existence and uniqueness of a strong solution to these combined SDE's.

3.4.1. Original system

The system of weakly coupled and perturbed oscillators is described by the following $\mathbb{C}^n$-valued SDE.

Definition 3.35 (SDE of weakly coupled oscillators)
For $\varepsilon > 0$ we consider the SDE

$$\mathrm{d}x^\varepsilon(t) = i\, \kappa\, x^\varepsilon(t)\, \mathrm{d}t + \varepsilon\, \kappa\, u(x^\varepsilon(t))\, \mathrm{d}t + \sqrt{\varepsilon\, \kappa}\, \boldsymbol{\sigma}(x^\varepsilon(t))\, \mathrm{d}\boldsymbol{B}(t), \tag{3.160}$$

where $u(x) = u_{\mathrm{lin}}(x) + u_{\mathrm{nl}}(x)$ is composed of a linear drift term

$$u_{\mathrm{lin}}(x) := \mathrm{Re}(\lambda \circledast x), \quad \lambda \in \mathbb{C}^n, \tag{3.161}$$

and a nonlinear perturbation

$$u_{\mathrm{nl}}(x) = \mathsf{A}(x)\, x. \tag{3.162}$$

Here $\mathsf{A}(x)$ is an antihermitian, nonlinear coupling matrix as specified in Definition 3.20. The dispersion matrix $\boldsymbol{\sigma}(x)$ is given by

$$\boldsymbol{\sigma}(x) \coloneqq (\sigma_{\text{mult}}(x) \mid \boldsymbol{\sigma}_{\text{reg}}(x) \mid \boldsymbol{\sigma}_{\text{add}}), \tag{3.163}$$

where the multiplicative-noise matrix $\sigma_{\text{mult}}(x)$ is defined as

$$\sigma_{\text{mult}}(x) \coloneqq \operatorname{Re}(\operatorname{cycl}(\nu \circledast x)), \quad \nu \in \mathbb{C}^n, \text{ s.t. } \mathcal{P}_{-}\nu = 0, \tag{3.164}$$

the regularizing-noise matrix $\boldsymbol{\sigma}_{\text{reg}}(x)$ is given by

$$\boldsymbol{\sigma}_{\text{reg}}(x) \coloneqq \frac{\sqrt{n}}{2}\,\sigma_r\,(\operatorname{cycl}(x) \mid i\operatorname{cycl}(Rx)), \quad \sigma_r > 0, \tag{3.165}$$

and the isotropic additive noise is represented by the constant dispersion matrix

$$\boldsymbol{\sigma}_{\text{add}} \coloneqq \frac{\sigma_0}{\sqrt{2}}(\mathbb{1}_{n\times n} \mid i\,\mathbb{1}_{n\times n}). \tag{3.166}$$

Finally, $(\boldsymbol{B}(t))_{t\geq 0}$ denotes a $5n$-dimensional real-valued Brownian motion, which we decompose as

$$\boldsymbol{B} = \left((B_{\text{mult}})^\top, (B_{\text{reg}})^\top, \left(B'_{\text{reg}}\right)^\top, (B_{\text{add}})^\top, (B'_{\text{add}})^\top\right)^\top, \tag{3.167}$$

in terms of the independent, n-dimensional and real-valued Brownian motions B_{mult}, B_{reg}, B'_{reg}, B_{add} and B'_{add}.

3.4.2. Existence and uniqueness

3.4.2.1. Strong uniqueness

The drift and dispersion terms of the SDE specified in Definition 3.35 are clearly *locally Lipschitz-continuous*, since $u_{\text{lin}}(x)$ and $\boldsymbol{\sigma}(x)$ are *linear* maps, while the nonlinear term $u_{\text{nl}}(x)$ is continuously $\mathbb{R}$-differentiable (recall that the nonlinear coupling matrix was defined in terms of holomorphic functions). Thus, *strong uniqueness*[25] follows from [KS91], Theorem 5.2.5.

3.4.2.2. Existence of a strong solution

Given a (not necessarily deterministic) initial condition $x^\varepsilon(0)$, the local Lipschitz condition is also sufficient to ensure the *existence* of a *strong solution*[26] up to an *explosion-time*[27]; c.f. [BS96], Section III.4.17 or [LW14], Theorem 1.1 for an even more general condition. Note that some authors require a strong solution to be non-exploding, i.e. to be globally defined for all times $t \geq 0$, c.f. for instance [KS91], Definition 2.1.

[25] c.f. [KS91], Definition 5.2.3
[26] c.f. [BS96], Section III.4.15
[27] c.f. [KS91] for

3.4.2.3. Non-explosion

In the absence of a nonlinear drift term, i.e. if $u_{\mathrm{nl}} = 0$, Eq. (3.160) is a *linear* stochastic differential equation, which clearly satisfies both *global Lipschitz* and *linear growth conditions*[28]. Thus, by [KS91], Theorem 5.2.9, we find that for any given initial condition there is a *strong solution* which does *not* explode. For the general case, we first note that a *nonlinear* holomorphic function cannot be globally Lipschitz continuous. This is a consequence of Liouville's theorem, as outlined in the following remark.

Remark 3.36 (Lipschitz continuity enforces linearity)
If a function is both globally holomorphic (i.e. an *entire* function) and globally Lipschitz continuous, then it is an affine linear function.

Proof. Global Lipschitz continuity implies that all partial derivatives need to be bounded. Since these partial derivatives are holomorphic functions themselves, it follows by *Liouville's Theorem*[29] that they are constant. □

A less restrictive but still sufficient non-explosion condition is given in [LR15], Theorem 3.1.1, Eq. (3.4).[30] In particular, we find that a *weak coercivity*[31] of the drift, i.e.

$$X^\top U_{\mathrm{nl}}(X) \leq 0, \quad \forall\, X \in \mathbb{R}^{2n}, \tag{3.168}$$

suffices to ensure the existence of a global solution. By Eq. (3.111) we know that Eq. (3.111) is satisfied.

3.4.3. Transformed system

As we have seen, the circularity assumptions on the coupling matrices allow us to simplify the description of the system by performing a DFT.

Lemma 3.37 (Combined system in DFT eigenbasis)
In the DFT eigenbasis, the system (3.160) can be written as

$$\mathrm{d}\tilde{x}^\varepsilon(t) = i\,\kappa\,\tilde{x}^\varepsilon(t)\,\mathrm{d}t + \varepsilon\,\kappa\,\tilde{u}(\tilde{x}^\varepsilon(t))\,\mathrm{d}t + \sqrt{\varepsilon\,\kappa}\,\widetilde{\boldsymbol{\sigma}}(\tilde{x}^\varepsilon(t))\,\mathrm{d}\widetilde{\boldsymbol{B}}(t), \tag{3.169}$$

where the drift-term $\tilde{u}(\tilde{x})$ is given by

$$\tilde{u}(\tilde{x}) := \frac{\sqrt{n}}{2}\left(\mathrm{diag}(\widetilde{\lambda})\tilde{x} + R\,\overline{\mathrm{diag}(\widetilde{\lambda})\tilde{x}}\right) + \widetilde{u_{\mathrm{nl}}}(\tilde{x}), \tag{3.170}$$

and the dispersion matrix $\widetilde{\boldsymbol{\sigma}}(\tilde{x})$ is defined as

$$\widetilde{\boldsymbol{\sigma}}(\tilde{x}) := \left(\widetilde{\sigma_{\mathrm{mult}}}(\tilde{x}) \mid \widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}(\tilde{x}) \mid \widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}\right). \tag{3.171}$$

[28] as for instance given in [KS91], Theorem 5.2.5, Eqs. (2.12) and (2.13)
[29] c.f. [Bor13], Korollar 14.2
[30] c.f. also [LW14], Theorem 1.4.
[31] c.f. [LR15], Eq. (3.4)

The transformed dispersion matrices are given by

$$\widetilde{\sigma_{\text{mult}}}(\tilde{x}) = \frac{n}{2}\,\operatorname{diag}\left(\operatorname{diag}(\widetilde{\nu})\tilde{x} + R\,\overline{\operatorname{diag}(\widetilde{\nu})\tilde{x}}\right), \tag{3.172}$$

$$\widetilde{\boldsymbol{\sigma}_{\text{reg}}}(\tilde{x}) := \frac{n}{2}\,\sigma_r\,\left(\operatorname{diag}(\tilde{x}) \mid i\operatorname{diag}(R\tilde{x})\right), \tag{3.173}$$

$$\widetilde{\boldsymbol{\sigma}_{\text{add}}} = \frac{\sigma_0}{\sqrt{2}}\left(\mathbb{1}_{n\times n} \mid i\,\mathbb{1}_{n\times n}\right), \tag{3.174}$$

and $\widetilde{\boldsymbol{B}}$ denotes a transformed Brownian defined as

$$\widetilde{\boldsymbol{B}} := \mathcal{Q}_5^\dagger \boldsymbol{B}, \text{ where } \mathcal{Q}_5 := \mathbb{1}_{5\times 5}\otimes Q = \operatorname{blockdiag}(Q,Q,Q,Q,Q) \in \mathbb{C}^{5n,5n}, \tag{3.175}$$

and where '⊗' denotes the *Kronecker product.*[a]

[a] c.f. [Pol14], Section 5

Proof. This follows from Lemmas 3.18, 3.28, 3.32 and 3.34. □

Note that we have not given an explicit representation of the DFT $\widetilde{u_{\text{nl}}} := Q^\dagger u_{\text{nl}}$ of the nonlinear perturbation. This is because this term will prove to have a vanishing influence on the *averaged* system, c.f. Lemma 5.7, which is why we don't need to simplify it any further. For future reference, we furthermore observe that (employing Lemma 2.34)

$$\left\langle \widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{B}}^\dagger \right\rangle(t) = \mathcal{Q}_5^\dagger \left\langle \boldsymbol{B}, \boldsymbol{B}^\dagger \right\rangle(t)\mathcal{Q}_5 = t\,\mathbb{1}_{5n\times 5n}, \tag{3.176a}$$

while

$$\left\langle \widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{B}}^\top \right\rangle(t) = \mathcal{Q}_5^\dagger \left\langle \boldsymbol{B}, \boldsymbol{B}^\top \right\rangle(t)\mathcal{Q}_5^\dagger = t\,\mathcal{R}_5, \tag{3.176b}$$

where we have set

$$\mathcal{R}_5 := (\mathcal{Q}_5^\dagger)^2 = \mathbb{1}_{5\times 5}\otimes R = \operatorname{blockdiag}(R,R,R,R,R). \tag{3.177}$$

Since Eq. (3.169) is a *linear* transformation of Eq. (3.160), the existence, uniqueness and non-explosion results of the previous section still apply.

3.5. Time rescaling and evolution of outer-product process

Recall that the small parameter ε models the fact that all perturbations, both deterministic and stochastic, are weak compared to the strong self-coupling of the oscillators, i.e. ε represents a *scale hierarchy* of the system. Ultimately, we want to exploit this scale hierarchy, which by means of a stochastic averaging result (Theorem 4.71) will allow us to obtain an effective description of the system (Chapter 6). As a preparation for this averaging step we need to perform two kinds of transformations on the system. First we will perform a *time rescaling* by means of which we speed up the system's evolution by a factor of ε. Subsequently, we study the *complex outer product* of the rescaled system vector.

3.5.1. Time rescaling

We look at the system on the *fast* timescale $\frac{t}{\varepsilon}$, i.e. we study the process $\tilde{y}^{\varepsilon}(t) \coloneqq \tilde{x}^{\varepsilon}(\frac{t}{\kappa\,\varepsilon})$.

Lemma 3.38 (Time-rescaled system in DFT eigenbasis)
The evolution of the time-rescaled system $\tilde{y}^{\varepsilon}(t) \coloneqq \tilde{x}^{\varepsilon}(\frac{t}{\kappa\,\varepsilon})$ is given by

$$\mathrm{d}\tilde{y}^{\varepsilon}(t) = i\left(\frac{1}{\varepsilon}\right)\tilde{y}^{\varepsilon}(t)\,\mathrm{d}t + \tilde{u}(\tilde{y}^{\varepsilon}(t))\,\mathrm{d}t + \tilde{\boldsymbol{\sigma}}(\tilde{y}^{\varepsilon}(t))\,\mathrm{d}\widetilde{\boldsymbol{B}}(t). \tag{3.178}$$

Its generator is given by

$$\begin{aligned}\mathcal{A}_{\varepsilon} f(\tilde{y}) \coloneqq{}& i\left(\frac{1}{\varepsilon}\right)\left(\tilde{y}^{\top}\,\partial + \tilde{y}^{\dagger}\overline{\partial}\right) f(\tilde{y}) + \left(\tilde{u}^{\top}(\tilde{y})\,\partial + \tilde{u}^{\dagger}(\tilde{y})\,\overline{\partial}\right) f(\tilde{y}) \\ &+ \left[(\tilde{\sigma}\tilde{\sigma}^{\dagger})(\tilde{y}) : (\partial\partial^{\dagger}) + \frac{1}{2}\left((\tilde{\boldsymbol{\sigma}}\,\mathcal{R}_5\,\tilde{\boldsymbol{\sigma}}^{\top})(\tilde{y}) : (\partial\partial^{\top}) + (\overline{\tilde{\boldsymbol{\sigma}}\,\mathcal{R}_5\,\tilde{\boldsymbol{\sigma}}^{\top}})(\tilde{y}) : (\overline{\partial\partial^{\top}})\right)\right] f(\tilde{y}).\end{aligned} \tag{3.179}$$

Proof. The first result follows directly by applying a *time-change* result for stochastic integrals.[32] In essence, we make use of the basic scaling behavior of Brownian motion, i.e. of $\sqrt{\varepsilon}\left(\boldsymbol{B}_{\frac{t}{\varepsilon}}\right)_{t\geq 0}$ again being a $5n$-dimensional Brownian motion.[33] For the representation of the generator we employ Corollary 2.37 and Eq. (3.176). □

Note that the perturbation terms no longer have a prefactor of ε, i.e. they now operate on a scale of 'order one' and don't vanish in the scaling limit of $\varepsilon \to 0$. The price we pay for this, is that the global oscillator evolution now operates at the fast rate of $\frac{1}{\varepsilon}$, which would diverge in the above scaling limit. As we will see in the following section, however, this term does *not* contribute to the evolution of the complex outer product.

3.5.2. Complex outer-product process

As noted in Section 3.1.2, the complex outer product captures the *first integrals*, i.e. the conserved quantities of the uncoupled system. For this reason, it does not come as a surprise that the global phase evolution (i.e. the evolution characterizing the uncoupled system) has no influence on the evolution of the complex outer product.

Proposition 3.39 (Evolution of complex outer-product process)
The evolution of the complex outer product of the time-rescaled system is given by

$$\begin{aligned}\mathrm{d}(\tilde{y}^{\varepsilon}(\tilde{y}^{\varepsilon})^{\dagger})(t) ={}& \left[\tilde{u}(\tilde{y}^{\varepsilon}(t))\,(\tilde{y}^{\varepsilon}(t))^{\dagger} + \text{h.c.}\right]\mathrm{d}t + \left[\tilde{y}^{\varepsilon}(t)\left(\tilde{\boldsymbol{\sigma}}(\tilde{y}^{\varepsilon}(t))\,\mathrm{d}\widetilde{\boldsymbol{B}}(t)\right)^{\dagger} + \text{h.c.}\right] \\ &+ \tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}^{\dagger}(\tilde{y}^{\varepsilon}(t))\,\mathrm{d}t,\end{aligned} \tag{3.180}$$

[32] c.f. [KS91], Proposition 5.4.8
[33] c.f. [KS91], Lemma 2.9.4

i.e. the complex outer-product process $\mathsf{p}(\tilde{y}^\varepsilon(t)) = (\tilde{y}^\varepsilon(\tilde{y}^\varepsilon)^\dagger)(t)$ satisfies

$$\mathrm{d}\mathsf{p}(\tilde{y}^\varepsilon(t)) = u_{\mathsf{p}}(\tilde{y}^\varepsilon(t))\,\mathrm{d}t + \sigma_{\mathsf{p}}(\tilde{y}^\varepsilon(t))\,\mathrm{d}\boldsymbol{B}(t), \tag{3.181}$$

where

$$u_{\mathsf{p}}(\tilde{y}) := \tilde{u}(\tilde{y})\,\tilde{y}^\dagger + \tilde{y}\,(\tilde{u}(\tilde{y}))^\dagger + \tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}^\dagger(\tilde{y}), \tag{3.182}$$

$$(\sigma_{\mathsf{p}})_{ij,r}(\tilde{y}) := \tilde{y}_i\,\overline{(\tilde{\boldsymbol{\sigma}}(\tilde{y})\mathcal{Q}_5^\dagger)_{jr}} + \overline{\tilde{y}}_j\,(\tilde{\boldsymbol{\sigma}}(\tilde{y})\mathcal{Q}_5^\dagger)_{ir}, \quad i,j \in \{0,...,n-1\}\,,\ r \in \{0,...,5n-1\}\,. \tag{3.183}$$

Proof. We apply Theorem 2.39 to the setting of Eq. (3.178), i.e. in particular we set $m = 5n$, choose $\boldsymbol{B}(t)$ as defined in Definition 3.35 and $\mathcal{Q} := \mathcal{Q}_5$ as given in Lemma 3.37. Thus we find that $\widetilde{\boldsymbol{B}}(t)$ can be identified with the martingale $M(t) := \mathcal{Q}^\dagger\,\boldsymbol{B}(t)$ of Theorem 2.39. Plugging in the vector-valued drift term $i\left(\frac{1}{\varepsilon}\right)\tilde{y}^\varepsilon(t) + \tilde{u}(\tilde{y}^\varepsilon(t))$ of Eq. (3.178), we find that according to Eq. (2.111) the matrix-valued drift term $u_{\mathsf{p}}(\tilde{y})$ is given by

$$\begin{aligned}
u_{\mathsf{p}}(\tilde{y}) &= \left[\left(i\left(\frac{1}{\varepsilon}\right)\tilde{y} + \tilde{u}(\tilde{y})\right)\tilde{y}^\dagger + \text{h.c.}\right] + \left(\tilde{\boldsymbol{\sigma}}(\mathcal{Q}_5^\dagger\mathcal{Q}_5)\tilde{\boldsymbol{\sigma}}^\dagger\right)(\tilde{y}) \\
&= \left[\left(i\left(\frac{1}{\varepsilon}\right)\tilde{y} + \tilde{u}(\tilde{y})\right)\tilde{y}^\dagger + \tilde{y}\left((-i)\left(\frac{1}{\varepsilon}\right)\tilde{y}^\dagger + \tilde{u}^\dagger(\tilde{y})\right)\right] + \left(\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}^\dagger\right)(\tilde{y}) \\
&= \left[\tilde{u}(\tilde{y})\tilde{y}^\dagger + \text{h.c.}\right] + \left(\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}^\dagger\right)(\tilde{y}),
\end{aligned}$$

where in the second to last step we have made use of $\mathcal{Q}_5$ being unitary. In a similar way, the structure of σ_{p} follows from Eq. (2.112). □

Note in particular, that the ε-dependent parts have indeed canceled out. Furthermore note that Eq. (3.181) is written in terms of a *real*-valued Brownian motion $(\boldsymbol{B}(t))_{t\geq 0}$, while Eq. (3.180) was specified in terms of the DFT $(\widetilde{\boldsymbol{B}}(t))_{t\geq 0}$ of this Brownian motion. This is due to the fact that we have 'absorbed' the transformation matrix $\mathcal{Q}_5^\dagger$ into the definition of σ_{p}.

4 Averaging theory

In this chapter we develop an averaging theory allowing us to find an approximation of a *weakly* coupled system of oscillators by an *averaged* version of this system. More precisely, we prove an averaging result, Theorem 4.71, which applies to the evolution of the outer-product process $\mathsf{p}^\varepsilon(t) := \mathsf{p}(\tilde{y}^\varepsilon(t)) = \tilde{y}^\varepsilon(t)\tilde{y}^\varepsilon(t)^\dagger$, as given by Proposition 3.39. The theorem shows that, in the scaling limit of $\varepsilon \to 0$, the process $(\mathsf{p}^\varepsilon(t))_{t\geq 0}$ converges in law to an *effective* process $(\widehat{\mathsf{p}}(t))_{t\geq 0}$ whose evolution is governed by an *averaged* drift and diffusion term.

In order to obtain this result, we closely follow [BR14]. In this publication, the authors show that an averaging principle can essentially be obtained by relating the involved stochastic processes to *Dirichlet forms* and proving that these Dirichlet forms converge in a generalized sense. The main part of this publication examines a two-dimensional system and studies the averaged evolution of *one* first integral, which in this case is given by the Hamiltonian. The final chapter 5 of [BR14] is of particular interest to us, since it provides a sketch of how the method can be extended to higher dimensional cases. Our main strategy will be to adapt these results. In particular, it will prove necessary to match the complex-valued description of the previous chapter to the real-valued setup in [BR14], c.f. Definition 4.4. Moreover, some of the underlying assumptions of [BR14] are violated in the coupled oscillator case (Remark 4.3), which is why we will provide new arguments for these steps.

4.1. Setup and assumptions

4.1.1. Assumptions

We state those of the central assumptions on drift- and dispersion terms in [BR14], which will turn out to be compatible with our system.

Assumption 4.1 (Drift terms, dispersion matrix and SDE setup[a])
Given $N, N' \in \mathbb{N}$, let

$$V, U : \mathbb{R}^N \to \mathbb{R}^N \quad \text{and} \quad \Sigma : \mathbb{R}^N \to \mathbb{R}^{N,N'} \tag{4.1}$$

be continuously differentiable functions s.t. there is a strongly unique and non-exploding strong solution to the SDE (4.6), with initial conditions as specified in Definition 4.2 below. We assume that the dispersion matrix Σ is chosen s.t. the diffusion matrix

$$A \coloneqq \Sigma\Sigma^\top : \mathbb{R}^N \to \mathbb{R}^{N,N}, \tag{4.2}$$

is twice continuously differentiable and *uniformly elliptic*,[b] i.e. there exists a $c > 0$ s.t.

$$\xi^\top A(X)\,\xi \geq c\,\|\xi\|^2, \quad \forall X, \xi \in \mathbb{R}^N. \tag{4.3}$$

Furthermore, we assume that the (in-)equalities

$$\nabla^\top V(X) = 0, \quad \forall X \in \mathbb{R}^N, \tag{4.4}$$

$$-\nabla^\top U(X) + \frac{1}{2}\nabla\nabla^\top : A(X) \leq 0, \quad \forall X \in \mathbb{R}^N. \tag{4.5}$$

are satisfied.

[a] c.f. [BR14], Assumption 3

[b] As it turns out, an even stronger condition will be fulfilled, c.f. Lemma 4.6 below.

Making use of the functions specified in Assumption 4.1, we can present the real-valued SDE setup from [BR14].

Definition 4.2 (SDE and infinitesimal generator[a])
Let $(\boldsymbol{B}(t))_{t\geq 0}$ be an $\mathbb{R}^{N'}$-valued Brownian motion and let, for all $\varepsilon > 0$, $(Y^\varepsilon(t))_{t\geq 0}$ denote the unique strong solution of the SDE

$$\mathrm{d}Y^\varepsilon(t) = \frac{1}{\varepsilon}V(Y^\varepsilon(t))\,\mathrm{d}t + U(Y^\varepsilon(t))\,\mathrm{d}t + \Sigma(Y^\varepsilon(t))\,\mathrm{d}\boldsymbol{B}(t), \tag{4.6}$$

with initial distribution μ_ε given by $\mathrm{d}\mu_\varepsilon = \zeta_\varepsilon\,\mathrm{d}\mu$, where μ denotes the Lebesgue measure on $\mathbb{R}^N$ and ζ_ε an $L^2(\mathbb{R}^N)$ function, s.t.[b]

$$\sup_{\varepsilon>0}\mathbb{E}\left(\|Y_0^\varepsilon\|^2\right) < \infty. \tag{4.7}$$

For any $\varepsilon > 0$, we denote by $\mathcal{A}_\varepsilon$ the *infinitesimal generator* corresponding to $(Y^\varepsilon(t))_{t\geq 0}$, i.e.

$$\mathcal{A}_\varepsilon f \coloneqq \frac{1}{\varepsilon}V^\top\nabla f + U^\top\nabla f + \frac{1}{2}A : \nabla\nabla^\top f, \quad \forall f \in C_c^2(\mathbb{R}^N). \tag{4.8}$$

[a] c.f. [BR14], Section 5

[b] This assumption is not explicitly stated in [BR14] but will be required in order to prove tightness - c.f. Lemma 4.68, below.

As noted before, there are some differences to the setup in [BR14], which require us to diverge at some points from the approach taken therein.

$$\begin{array}{ccc} \mathbb{R}^N & \xrightarrow{V,U} & \mathbb{R}^N \\ \mathrm{j}\big\uparrow & & \big\uparrow\mathrm{j} \\ \mathbb{C}^n & \xrightarrow{i,\tilde{u}} & \mathbb{C}^n \end{array}$$

(a) Drift terms

$$\begin{array}{ccc} \mathbb{R}^N & \xrightarrow{\Sigma} & \mathbb{R}^{N,5n} \\ \mathrm{j}\big\uparrow & & \big\uparrow\mathfrak{J}_{1/2} \\ \mathbb{C}^n & \xrightarrow{\widetilde{\boldsymbol{\sigma}}\mathcal{Q}_5^\dagger} & \mathbb{C}^{n,5n} \end{array}$$

(b) Dispersion matrix

Figure 4.1.: Real vs complex representation of drift terms and dispersion matrix

Remark 4.3 (Generalizations)
Note that [BR14], Assumption 3, requires V, U and Σ to be *Lipschitz continuous* and the authors make use of this assumption to infer both pathwise uniqueness and strong existence of a solution to Eq. (4.6). Since Lipschitz continuity is only a sufficient but not a necessary condition, we have replaced it by the weaker assumption of strong existence and uniqueness, which will allow us to consider certain nonlinear holomorphic drift terms, c.f. Section 3.4.2, i.p. Remark 3.36.

Moreover, note that Assumption 3 of [BR14] imposes the stronger assumption of V, U and Σ being *bounded*, which in conjunction with the uniform ellipticity requirement of Eq. (4.3) yields an equivalence of the norm induced by Eq. (4.61) with the $H^1(\mathbb{R}^N)$ norm, c.f. [BR14], Proposition 6. The boundedness assumption of [BR14] however is violated in the case studied in this thesis, since the multiplicative noise for instance is modeled by a linear, and thus *unbounded*, dispersion term. For this reason, we need to adapt all arguments which rely on the boundedness assumption, c.f. in particular Lemma 4.23, Remark 4.26 and Lemma 4.68.

We show that we can identify SDE (4.6) as a real-valued representation of the time-rescaled oscillator system given in Eq. (3.169). For this purpose, we provide explicit choices of the functions V, U and Σ, which allow us to model the oscillator system of the previous section (c.f. Remark 4.5) and which satisfy the requirements of Assumption 4.1 (c.f. Lemma 4.9).

Definition 4.4 (Matching of averaging setup to coupled oscillator case)
Let $N := 2n$, $N' := 5n$, and define (c.f. Fig. 4.1)

$$V : \mathbb{R}^N \to \mathbb{R}^N, \quad X \to V(X) := \mathfrak{J}(i)\, X, \tag{4.9}$$

$$U : \mathbb{R}^N \to \mathbb{R}^N, \quad X \to U(X) := \mathrm{j} \circ \tilde{u} \circ \mathrm{j}^{-1}(X), \tag{4.10}$$

$$\Sigma : \mathbb{R}^N \to \mathbb{R}^{N,N'}, \quad X \to \Sigma(X) := \mathfrak{J}_{1/2}\left(\widetilde{\boldsymbol{\sigma}}(\mathrm{j}^{-1}(X))\mathcal{Q}_5^\dagger\right), \tag{4.11}$$

where $\tilde{u}$ and $\widetilde{\boldsymbol{\sigma}}$ are defined as in Lemma 3.37.[a] Let furthermore $\boldsymbol{B}$ denote the $5n$-dimensional, real-valued Brownian motion as specified in Eq. (3.167).

[a] Recall the identification mappings j, $\mathfrak{J}$ and $\mathfrak{J}_{1/2}$ introduced in Definition 2.1.

In the following remark we note that the 'matching choices' of Definition 4.4 indeed reproduce the Fourier transformed oscillator system introduced in the previous chapter.

Remark 4.5 (Matching of SDEs)
Definition 4.4 allows for an identification of the process $(Y^{\varepsilon}(t))_{t\geq 0}$ defined in Eq. (4.6) with the process $(\tilde{y}^{\varepsilon}(t))_{t\geq 0}$ given by Lemma 3.38:

Inserting the drift and dispersion terms of Definition 4.4 into Eq. (4.6), we are in the situation of

$$\mathrm{d}Y^{\varepsilon}(t) = \frac{1}{\varepsilon}\,\mathfrak{J}(i)\,Y^{\varepsilon}(t)\,\mathrm{d}t + \mathsf{j}\circ\tilde{u}\circ\mathsf{j}^{-1}(Y^{\varepsilon}(t))\,\mathrm{d}t + \mathfrak{J}_{1/2}\left(\tilde{\boldsymbol{\sigma}}(\mathsf{j}^{-1}(Y^{\varepsilon}(t)))\mathcal{Q}_5^{\dagger}\right)\mathrm{d}\boldsymbol{B}(t), \tag{4.12}$$

which by Lemma 2.3 is nothing but the real-valued representation of the complex-valued SDE of Lemma 3.38, i.e.

$$\mathrm{d}\tilde{y}^{\varepsilon}(t) = \left(\frac{1}{\varepsilon}\right)\, i\,\tilde{y}^{\varepsilon}(t)\,\mathrm{d}t + \,\tilde{u}(\tilde{y}^{\varepsilon}(t))\,\mathrm{d}t + \tilde{\boldsymbol{\sigma}}(\tilde{y}^{\varepsilon}(t))\,\mathrm{d}\widetilde{\boldsymbol{B}}(t). \tag{4.13}$$

We can therefore identify both processes via $Y^{\varepsilon}\equiv\mathsf{j}(\tilde{y}^{\varepsilon})$.

Recall that the *diffusion matrix* was defined as $A := \Sigma\Sigma^{\top}$. This implies that for $X\in\mathbb{R}^N$ and $x := \mathsf{j}^{-1}(X)$ we have

$$A(X) = \mathfrak{J}_{1/2}\left(\tilde{\boldsymbol{\sigma}}(x)\mathcal{Q}_5^{\dagger}\right)\left(\mathfrak{J}_{1/2}\left(\tilde{\boldsymbol{\sigma}}(x)\mathcal{Q}_5^{\dagger}\right)\right)^{\top} = \frac{1}{2}\left[\mathfrak{J}\left(\tilde{\boldsymbol{\sigma}}(x)\tilde{\boldsymbol{\sigma}}^{\dagger}(x)\right) + \breve{\mathfrak{J}}\left(\tilde{\boldsymbol{\sigma}}(x)\mathcal{R}_5\tilde{\boldsymbol{\sigma}}^{\top}(x)\right)\right], \tag{4.14}$$

where we have employed Eq. (2.35) of Lemma 2.13 in order to evaluate the $\mathfrak{J}_{1/2}$-terms.

We derive a bound on this diffusion matrix which in particular yields a *uniform ellipticity* result. In fact, we provide an even stronger bound that will prove crucial for obtaining the so-called weak sector condition, c.f. Lemma 4.23 below.

Lemma 4.6 (Uniform ellipticity)
For all $X,\xi\in\mathbb{R}^N$, the diffusion matrix $A(X)$ satisfies the uniform ellipticity condition

$$\xi^{\top}A(X)\,\xi \geq \|\mathfrak{D}(X)\,\xi\|^2 \geq \frac{\sigma_0^2}{2}\,\|\xi\|^2\,, \tag{4.15}$$

where

$$\mathfrak{D}(X) = \mathfrak{J}\left[\mathrm{diag}\left(\mathfrak{d}\left(\mathsf{j}^{-1}(X)\right)\right)\right], \tag{4.16}$$

$$\mathfrak{d}_k(x) = \sqrt{\frac{\sigma_0^2}{2} + \frac{n^2}{8}\,\sigma_r^2\,\left(|x_k|^2 + |x_{n-k}|^2\right)}. \tag{4.17}$$

Proof. Let $X,\xi\in\mathbb{R}^N$ and let $x := \mathsf{j}^{-1}(X)$ denote the complex-valued representation of X. The decomposition of the dispersion matrix (c.f. Eq. (3.171))

$$\tilde{\boldsymbol{\sigma}}(\tilde{x}) := (\widetilde{\sigma_{\mathrm{mult}}}(\tilde{x}) \mid \widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}(\tilde{x}) \mid \widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}) \tag{4.18}$$

allows us to decompose the diffusion matrix as

$$\begin{aligned}A(X) = \mathfrak{J}_{1/2}\left(\widetilde{\sigma_{\mathrm{mult}}}(x)Q^\dagger\right)\left(\mathfrak{J}_{1/2}\left(\widetilde{\sigma_{\mathrm{mult}}}(x)Q^\dagger\right)\right)^\top \\ + \mathfrak{J}_{1/2}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}(x)\mathcal{Q}_2^\dagger\right)\left(\mathfrak{J}_{1/2}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}(x)\mathcal{Q}_2^\dagger\right)\right)^\top \\ + \mathfrak{J}_{1/2}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}\mathcal{Q}_2^\dagger\right)\left(\mathfrak{J}_{1/2}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}\mathcal{Q}_2^\dagger\right)\right)^\top .\end{aligned} \tag{4.19}$$

By Eq. (2.35) we observe that

$$\begin{aligned}\mathfrak{J}_{1/2}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}(x)\mathcal{Q}_2^\dagger\right)\left(\mathfrak{J}_{1/2}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}(x)\mathcal{Q}_2^\dagger\right)\right)^\top &= \frac{1}{2}\left[\mathfrak{J}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}(x)\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}^\dagger(x)\right) + \breve{\mathfrak{J}}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}(x)\mathcal{R}_2\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}^\top(x)\right)\right] \\ &= \frac{1}{2}\mathfrak{J}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}(x)\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}^\dagger(x)\right),\end{aligned} \tag{4.20a}$$

as well as

$$\begin{aligned}\mathfrak{J}_{1/2}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}\mathcal{Q}_2^\dagger\right)\left(\mathfrak{J}_{1/2}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}\mathcal{Q}_2^\dagger\right)\right)^\top &= \frac{1}{2}\left[\mathfrak{J}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}^\dagger\right) + \breve{\mathfrak{J}}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}\mathcal{R}_2\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}^\top\right)\right] \\ &= \frac{1}{2}\mathfrak{J}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}^\dagger\right),\end{aligned} \tag{4.20b}$$

since (c.f. Lemmas 5.15 and 5.18 for an explicit calculation)

$$\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}(x)\mathcal{R}_2\,\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}^\top(x) = 0, \tag{4.21a}$$

$$\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}\mathcal{R}_2\,\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}^\top = 0. \tag{4.21b}$$

Combining Eqs. (4.19) and (4.20), we can conclude that

$$\begin{aligned}\xi^\top A(X)\,\xi &\geq \frac{1}{2}\xi^\top\,\mathfrak{J}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}^\dagger(x) + \widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}^\dagger\right)\xi \\ &= \xi^\top\,\mathfrak{D}^\top(X)\mathfrak{D}(X)\,\xi \equiv \|\mathfrak{D}(X)\,\xi\|^2 .\end{aligned} \tag{4.22}$$

where the inequality follows from

$$\xi^\top\,\mathfrak{J}_{1/2}\left(\widetilde{\sigma_{\mathrm{mult}}}(x)Q^\dagger\right)\left(\mathfrak{J}_{1/2}\left(\widetilde{\sigma_{\mathrm{mult}}}(x)Q^\dagger\right)\right)^\top\xi = \left\|\left(\mathfrak{J}_{1/2}\left(\widetilde{\sigma_{\mathrm{mult}}}(x)Q^\dagger\right)\right)^\top\xi\right\|^2 \geq 0. \tag{4.23}$$

In the second step of Eq. (4.22) we have made use of Eqs. (3.148) and (3.155), which yield

$$\begin{aligned}\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}^\dagger(x) &= \left(\frac{n}{2}\,\sigma_r\right)^2 (\mathrm{diag}(x)\mid i\,\mathrm{diag}(Rx))\begin{pmatrix}\mathrm{diag}(\overline{x}) \\ (-i)\mathrm{diag}(\overline{Rx})\end{pmatrix} \\ &= \frac{n^2}{4}\,\sigma_r^2\,\mathrm{diag}\left(\left(|x_k|^2 + |x_{n-k}|^2\right)_k\right),\end{aligned} \tag{4.24}$$

$$\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}^\dagger = \frac{\sigma_0^2}{2}\,(\mathbb{1}_{n\times n}\mid i\,\mathbb{1}_{n\times n})\begin{pmatrix}\mathbb{1}_{n\times n} \\ -i\,\mathbb{1}_{n\times n}\end{pmatrix} = \sigma_0^2\,\mathbb{1}_{n\times n}, \tag{4.25}$$

and thus imply that

$$\mathfrak{J}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}}^{\dagger}+\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}^{\dagger}(x)\right)=\mathfrak{J}\left(\sigma_0^2\,\mathbb{1}_{n\times n}+\frac{n^2}{4}\,\sigma_r^2\,\mathrm{diag}\left(\left(|x_k|^2+|x_{n-k}|^2\right)_k\right)\right)$$
$$=2\,\mathfrak{D}^{\top}(X)\mathfrak{D}(X). \tag{4.26}$$

□

Lemma 4.6 shows that the additive noise yields a uniform ellipticity bound on the diffusion matrix, which is further strengthened by the regularizing noise. In the following Remark 4.7, we will comment on why this cannot, to the same extend, be achieved by the multiplicative-noise term.

Remark 4.7 (Multiplicative versus regularizing noise)
In Eq. (4.22) we have used a crude lower bound on the multiplicative-noise diffusion matrix, estimating (c.f. Eq. (4.23))

$$\xi^{\top}\,\mathfrak{J}_{1/2}\left(\widetilde{\sigma_{\mathrm{mult}}}(x)Q^{\dagger}\right)\left(\mathfrak{J}_{1/2}\left(\widetilde{\sigma_{\mathrm{mult}}}(x)Q^{\dagger}\right)\right)^{\top}\xi\geq 0. \tag{4.27}$$

Unlike for the regularizing noise, one can observe that $\widetilde{\sigma_{\mathrm{mult}}}R\,\widetilde{\sigma_{\mathrm{mult}}}^{\top}\neq 0$, since for all $i\in\{0,...,n-1\}$ we have (c.f. Lemma 5.10)

$$\left(\widetilde{\sigma_{\mathrm{mult}}}R\,(\widetilde{\sigma_{\mathrm{mult}}})^{\top}\right)_{i,n-i}(x)=\frac{n^2}{4}\big[\,|\widetilde{\nu}_i|^2\,|x_i|^2+|\widetilde{\nu}_{n-i}|^2\,|x_{n-i}|^2$$
$$+\left(\widetilde{\nu}_i\widetilde{\nu}_{n-i}x_ix_{n-i}+\overline{\widetilde{\nu}_i\widetilde{\nu}_{n-i}x_ix_{n-i}}\right)\big]. \tag{4.28}$$

This implies that we cannot neglect the second term in the representation (c.f. Eq. (4.20))

$$\mathfrak{J}_{1/2}\left(\widetilde{\sigma_{\mathrm{mult}}}(x)Q^{\dagger}\right)\left(\mathfrak{J}_{1/2}\left(\widetilde{\sigma_{\mathrm{mult}}}(x)Q^{\dagger}\right)\right)^{\top}=\frac{1}{2}\,\mathfrak{J}\left(\widetilde{\sigma_{\mathrm{mult}}}(x)\widetilde{\sigma_{\mathrm{mult}}}^{\dagger}(x)\right)$$
$$+\frac{1}{2}\,\check{\mathfrak{J}}\left(\widetilde{\sigma_{\mathrm{mult}}}(x)R\,\widetilde{\sigma_{\mathrm{mult}}}^{\top}(x)\right). \tag{4.29}$$

Moreover, this second term is generally not positive definite (c.f. Eq. (2.4)) and therefore does not allow for an estimate along the lines of Eq. (4.27).

Investigating the *first* term, we observe further differences to the case of regularizing noise. We can make use of Eq. (3.135) to observe that $\widetilde{\sigma_{\mathrm{mult}}}\widetilde{\sigma_{\mathrm{mult}}}^{\dagger}(x)$ is a diagonal matrix with diagonal elements given by

$$\left(\widetilde{\sigma_{\mathrm{mult}}}\widetilde{\sigma_{\mathrm{mult}}}^{\dagger}\right)_{k,k}(x)=\left(\frac{n}{2}\right)^2\left(\mathrm{diag}(\widetilde{\nu})x+R\,\overline{\mathrm{diag}(\widetilde{\nu})x}\right)_k\overline{\left(\mathrm{diag}(\widetilde{\nu})\tilde{x}+R\,\overline{\mathrm{diag}(\widetilde{\nu})x}\right)_k}$$
$$=\frac{n^2}{4}\left[|\widetilde{\nu}_k|^2\,|x_k|^2+|\widetilde{\nu}_{n-k}|^2\,|x_{n-k}|^2\right]$$
$$+\frac{n^2}{4}\left[\left(\widetilde{\nu}_k\widetilde{\nu}_{n-k}x_kx_{n-k}+\overline{\widetilde{\nu}_k\widetilde{\nu}_{n-k}x_kx_{n-k}}\right)\right]. \tag{4.30}$$

Note however, that unlike Eq. (4.24), this can generally *not* be bounded below by a term proportional to $\left(|x_k|^2 + |x_{n-k}|^2\right)$. Considering for instance the case of $\widetilde{\nu}_k = \widetilde{\nu}_{n-k} \in \mathbb{R}^+$, we find that Eq. (4.30) simplifies to

$$\left(\widetilde{\sigma_{\text{mult}}}\widetilde{\sigma_{\text{mult}}}^\dagger(x)\right)_{k,k} = \frac{n^2}{4}\,|\widetilde{\nu}_k|^2\,\left[|x_k|^2 + |x_{n-k}|^2 + 2\,\mathrm{Re}\left(x_k x_{n-k}\right)\right], \tag{4.31}$$

which vanishes for all elements $x \in \mathbb{C}^n$ satisfying $x_k = -x_{n-k}$. Recalling Definition 3.25, this observation does not come as a surprise, since

$$\sigma_{\text{mult}}(x) = \mathrm{Re}\left(\mathrm{cycl}\left(\nu \circledast x\right)\right), \tag{4.32}$$

vanishes if $\nu \circledast x \in i\,\mathbb{R}^n$. By contrast, there is no real part appearing in the regularizing-noise diffusion matrix, which allows for an estimate as outlined in Eq. (4.24). This will be of crucial importance in proving the so-called *sector condition* later on, c.f. Lemma 4.23.

In order to satisfy Eq. (4.5), we need to impose the following global growth condition, which, as we shall see later on, does not impose any relevant restriction to our results, c.f. Assumption 6.54.

Assumption 4.8 (Global growth of the system)
In the following we assume that

$$\mathfrak{l}_0' > 2n\sum_{k=0}^{n-1}\left(|\widetilde{\nu}_k|^2 + \sigma_r\right). \tag{4.33}$$

We are now in a position to verify that the choices of V, U and Σ meet the requirements of Assumption 4.1.

Lemma 4.9 (Setup satisfies averaging assumptions)
Given Assumption 4.8, the functions of Definition 4.4 satisfy the conditions of Assumption 4.1.

Proof. Existence and uniqueness of solution to SDE: In Remark 4.5 we have seen that the drift and dispersion terms of Definition 4.4 give rise to a real-valued representation of the oscillator system from the previous section. By Section 3.4.2, this system satisfies strong uniqueness and has a strong, non-exploding solution for any given initial condition.

Uniform ellipticity was proven in Lemma 4.6 above.

For any $X \in \mathbb{R}^N$, we observe that

$$\nabla^\top V(X) = \nabla^\top \mathfrak{J}(i)\,X = \mathrm{tr}(\mathfrak{J}(i)) = 0, \tag{4.34}$$

which follows from the general identity

$$\nabla^\top M\,X = \sum_{k,l} M_{k,l}\nabla_k X_l = \sum_{k,l} M_{k,l}\,\delta_{k,l} = \sum_k M_{k,k} = \mathrm{tr}(M). \tag{4.35}$$

Employing the results of Lemma 4.10 below, we furthermore find that for any $X \in \mathbb{R}^N$

$$(-\nabla^\top U + \frac{1}{2}\nabla\nabla^\top : A)(X) \leq -n\,\mathfrak{l}_0' + 2n^2 \sum_{k=0}^{n-1}\left(|\widetilde{\nu}_k|^2 + \sigma_r^2\right) =: -c \tag{4.36}$$

$$< 0, \tag{4.37}$$

where in the last step we have applied Assumption 4.8. □

Lemma 4.10
For all $X \in \mathbb{R}^N$, the following identities hold:

i) $\nabla^\top U(X) = \mathrm{Re}\,(\mathrm{tr}(\Lambda(X))) = n\,\mathfrak{l}_0'$

ii) $\frac{1}{2}\,(\nabla\nabla^\top : A)(X) \leq 2n^2\,\sum_{k=0}^{n-1}\left(|\widetilde{\nu}_k|^2 + \sigma_r^2\right)$.

Proof. Let $X \in \mathbb{R}^n$ and set $x := \mathsf{j}^{-1}(X) \in \mathbb{C}^n$.

i) By Eq. (3.112) we know that the nonlinear perturbation of the drift term is divergence free. This carries over to the Fourier transformed system, i.e.

$$\nabla^\top(\mathsf{j} \circ \widetilde{u_{\mathrm{nl}}} \circ \mathsf{j}^{-1}(X)) = 0. \tag{4.38}$$

It thus remains for us to calculate the divergence of the linear drift term. Making use of Eq. (2.30) and Lemma 2.16 we obtain the complex representation

$$-\nabla^\top(\mathsf{j} \circ \widetilde{u_{\mathrm{lin}}} \circ \mathsf{j}^{-1}(X)) = -\left[\mathsf{j}(2\overline{\partial})\right]^\top \mathsf{j}\,(\widetilde{u_{\mathrm{lin}}}(x)) = -\mathrm{Re}\left[2\,\partial^\top \widetilde{u_{\mathrm{lin}}}(x)\right]. \tag{4.39}$$

We recall that by Eq. (3.93) we have

$$\widetilde{u_{\mathrm{lin}}}(x) = \frac{\sqrt{n}}{2}\left(\mathrm{diag}(\widetilde{\lambda})\,x + R\,\overline{\mathrm{diag}(\widetilde{\lambda})\,\overline{x}}\right), \tag{4.40}$$

which together with Lemma 2.15 and Eqs. (3.96), (4.35) and (4.38) implies that

$$2\,\partial^\top \tilde{u}(x) = \mathrm{tr}\left[\sqrt{n}\,\mathrm{diag}(\widetilde{\lambda})\right] = \mathrm{tr}(\widetilde{\Lambda}) = \mathrm{tr}(Q^\dagger \Lambda Q) = \mathrm{tr}(\Lambda) = \mathrm{tr}(\mathrm{cycl}\,(\lambda)) = n\,\lambda_0 \tag{4.41}$$

and allows us to conclude that

$$-\nabla^\top(\mathsf{j} \circ \tilde{u} \circ \mathsf{j}^{-1}(X)) = -\mathrm{Re}(n\,\lambda_0) = -n\,\mathfrak{l}_0'. \tag{4.42}$$

ii) We recall that by Eq. (2.52) we know that[1]

$$\frac{1}{2}\,\nabla\nabla^\top : \mathfrak{J}_{1/2}\,(\mathsf{M})\,\mathfrak{J}_{1/2}\,(\mathsf{M})^\top = \partial\partial^\dagger : \mathsf{M}\mathsf{M}^\dagger + \mathrm{Re}\left(\partial\partial^\top : \mathsf{M}\mathsf{M}^\top\right), \tag{4.43}$$

[1] Note that we employ a modified version with interchanged factors, which can be derived analogously to Eq. (2.52).

which in the special case of $\mathsf{M} = \widetilde{\boldsymbol{\sigma}}(x)\mathcal{Q}_5^\dagger$ yields (c.f. Eq. (4.14))

$$\frac{1}{2}(\nabla\nabla^\top : A)(X) = \partial\partial^\dagger : \widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\sigma}}^\dagger(x) + \operatorname{Re}\left(\partial\partial^\top : \widetilde{\boldsymbol{\sigma}}\mathcal{R}_5\widetilde{\boldsymbol{\sigma}}^\top(x)\right). \tag{4.44}$$

We recall that by Eqs. (4.24) and (4.30) we have

$$\begin{aligned}
\widetilde{\sigma_{\text{mult}}}\widetilde{\sigma_{\text{mult}}}^\dagger(x) &= \frac{n^2}{4}\operatorname{diag}\left(\left(|\widetilde{\nu}_k|^2\,|x_k|^2 + |\widetilde{\nu}_{n-k}|^2\,|x_{n-k}|^2 + 2\operatorname{Re}(\widetilde{\nu}_k\widetilde{\nu}_{n-k}x_k x_{n-k})\right)_k\right),\\
\widetilde{\boldsymbol{\sigma}_{\text{reg}}}\widetilde{\boldsymbol{\sigma}_{\text{reg}}}^\dagger(x) &= \frac{n^2}{4}\,\sigma_r^2\operatorname{diag}\left(\left(|x_k|^2 + |x_{n-k}|^2\right)_k\right),
\end{aligned}$$

and note that from Eq. (4.21) it follows that

$$\widetilde{\boldsymbol{\sigma}}\mathcal{R}_5\widetilde{\boldsymbol{\sigma}}^\top(x) = \widetilde{\sigma_{\text{mult}}}R\left(\widetilde{\sigma_{\text{mult}}}\right)^\top(x), \tag{4.45}$$

where the only non-vanishing elements of $\widetilde{\sigma_{\text{mult}}}R\left(\widetilde{\sigma_{\text{mult}}}\right)^\top$ are of the form[2]

$$\begin{aligned}
\left(\widetilde{\sigma_{\text{mult}}}R\left(\widetilde{\sigma_{\text{mult}}}\right)^\top\right)_{k,n-k}(x) = \frac{n^2}{4}\Big[&|\widetilde{\nu}_k|^2\,|x_k|^2 + |\widetilde{\nu}_{n-k}|^2\,|x_{n-k}|^2\\
&+\left(\widetilde{\nu}_k\widetilde{\nu}_{n-k}x_k x_{n-k} + \overline{\widetilde{\nu}_k\widetilde{\nu}_{n-k}x_k x_{n-k}}\right)\Big].
\end{aligned} \tag{4.46}$$

These results allow us to conclude that

$$\begin{aligned}
\frac{1}{2}(\nabla\nabla^\top : A)(X) &= \partial\partial^\dagger : \widetilde{\sigma_{\text{mult}}}\widetilde{\sigma_{\text{mult}}}^\dagger(x) + \partial\partial^\dagger : \widetilde{\boldsymbol{\sigma}_{\text{reg}}}\widetilde{\boldsymbol{\sigma}_{\text{reg}}}^\dagger(x)\\
&\quad + \operatorname{Re}\left[\partial\partial^\top : \widetilde{\sigma_{\text{mult}}}R\left(\widetilde{\sigma_{\text{mult}}}\right)^\top(x)\right]\\
&= \frac{n^2}{2}\sum_{k=0}^{n-1}\partial_k\overline{\partial}_k\left\{\left(|\widetilde{\nu}_k|^2+\sigma_r^2\right)|x_k|^2 + \left(|\widetilde{\nu}_{n-k}|^2+\sigma_r^2\right)|x_{n-k}|^2\right\}\\
&\quad + \frac{n^2}{2}\sum_{k=0}^{n-1}\operatorname{Re}\left[\partial_k\partial_{n-k}\left(\widetilde{\nu}_k\widetilde{\nu}_{n-k}x_k x_{n-k} + \overline{\widetilde{\nu}_k\widetilde{\nu}_{n-k}x_k x_{n-k}}\right)\right]\\
&= \frac{n^2}{2}\sum_{k=0}^{n-1}\left[\left(|\widetilde{\nu}_k|^2+\sigma_r^2\right)\left(1+\mathbb{1}_{k\in\{0,n/2\}}\right) + \operatorname{Re}\left(\widetilde{\nu}_k\,\widetilde{\nu}_{n-k}\right)\right]\\
&\leq 2\,n^2\sum_{k=0}^{n-1}\left(|\widetilde{\nu}_k|^2+\sigma_r^2\right),
\end{aligned} \tag{4.47}$$

where we have employed $\widetilde{\nu}_k = \widetilde{\nu}_{n-k}$ and have applied Young's inequality. □

[2] c.f. proof of Lemma 5.10

4.2. Dirichlet forms

In the following let $\mathcal{H}$ denote a real *Hilbert space* with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Of particular interest to us will be the real Hilbert space $L^2(\mathbb{R}^N)$ with scalar product $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^N)}$. In this section as well as in Sections 4.5 to 4.8 we restrict ourselves to the case of a *linear* drift term, i.e. $u = u_{\text{lin}}$. The nonlinear term u_{nl} can be interpreted as a *divergence free perturbation* (Eq. (3.112)) and can analogously be dealt with by employing the framework of *generalized Dirichlet forms*, c.f. [Sta99]. In Lemma 5.7 it can be shown to yield a vanishing contribution to the evolution of the averaged system.

4.2.1. Generators and corresponding semigroups

We state results from [MR92] allowing us to infer that the process $(Y^\varepsilon(t))_{t\geq 0}$ gives rise to a *strongly continuous contraction semigroup*.[3]

Theorem 4.11 (Strongly continuous contraction semigroup)
For every fixed $\varepsilon > 0$, the family $(T^\varepsilon(t))_{t>0}$ of linear operators on $L^2(\mathbb{R}^N)$, defined by

$$T^\varepsilon(t)f(X) \coloneqq \mathbb{E}^X(f(Y^\varepsilon(t))), \quad \forall f \in L^2(\mathbb{R}^N),\ X \in \mathbb{R}^N,\ t > 0, \tag{4.48}$$

is a strongly continuous contraction semigroup $(T^\varepsilon(t))_{t>0}$ on $L^2(\mathbb{R}^N)$.

Proof. We first note that $(Y^\varepsilon(t))_{t\geq 0}$ is a *right process*,[4] since it is a *continuous* process which satisfies the *strong Markov property*.[5] Assumptions (4.4) and (4.5) imply that for every positive $f \in C_c^2(\mathbb{R}^N)$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \mathcal{A}_\varepsilon f(X)\,\mu(\mathrm{d}X) &= \int_{\mathbb{R}^N} \frac{1}{\varepsilon} V^\top \nabla f + U^\top \nabla f + \frac{1}{2} A : \nabla\nabla^\top f\,\mu(\mathrm{d}X) \\ &= \int_{\mathbb{R}^N} \left[-\frac{1}{\varepsilon}\nabla^\top V - \nabla^\top U + \frac{1}{2}\nabla\nabla^\top : A\right] f\,\mu(\mathrm{d}X) \\ &\leq 0.\end{aligned}$$

From this observation we can conclude that μ is a *supermedian measure*[6] for the transition semigroup of $(Y^\varepsilon(t))_{t\geq 0}$. Now the result follows from [MR92]: Lemma IV.2.1, Proposition IV.2.3 and the comment on page 98 ensure that the assumptions of Proposition II.4.3. are met, which yields the result. □

[3]This notion is introduced in [MR92], Definition I.1.6. Note that Diagram 2 on page 27 of [MR92] provides a comprehensive summary of how such a continuous contraction semigroup can be related to an associated densely defined operator, to a strongly continuous contraction resolvent and to a coercive closed form.

[4]c.f. [MR92], Definition IV.1.8.

[5]The continuity assumption is part of the definition of a strong solution, c.f. [KS91], Definition 5.2.1. The strong Markov property follows from [KS91], Theorem 5.4.20, which is applicable since the drift and dispersion terms are *bounded on compact subsets* and Eq. (4.6) is *well posed* by Assumption 4.1.

[6]c.f. [MR92], Section II.4 a) and [BR14], Eq. (29)

4.2.2. Bilinear forms

Following [MR92], we recall the notion of a *coercive closed form* and show that it applies to our setup. From now on, we denote by $D \subset \mathcal{H}$ a linear subspace of $\mathcal{H}$ and by $\mathcal{E} : D \times D \to \mathbb{R}$ a bilinear map.[7] Of particular interest to us are bilinear forms induced by the *infinitesimal generators* $\mathcal{A}_\varepsilon$.

Definition 4.12 (Bilinear form corresponding to infinitesimal generator)
For any $\varepsilon > 0$, we define the bilinear form $E_\varepsilon : C_c^2(\mathbb{R}^N) \times C_c^2(\mathbb{R}^N) \to \mathbb{R}$ as[a]

$$\begin{aligned} E_\varepsilon(f, g) &\coloneqq \langle -\mathcal{A}_\varepsilon f, g \rangle_{L^2(\mathbb{R}^N)} \\ &\equiv - \int_{\mathbb{R}^N} \left[\frac{1}{\varepsilon}(V^\top \nabla f) + (U^\top \nabla f) + \frac{1}{2} A : \nabla\nabla^\top f \right] g \,\mathrm{d}\mu, \quad \forall f, g \in C_c^2(\mathbb{R}^N). \end{aligned}$$

[a] c.f. Eq. (137) in [BR14]

Theorem 4.11 implies that E_ε is positive definite.

Lemma 4.13 (Positive definiteness)
$(E_\varepsilon, C_c^2(\mathbb{R}^N))$ is a positive definite bilinear form, i.e.

$$E_\varepsilon(f, f) \geq 0, \quad \forall f \in C_c^2(\mathbb{R}^N). \tag{4.49}$$

Proof. This follows from Theorem 4.11 and [MR92], Remark I.1.13. □

4.2.2.1. Symmetry decomposition

As it turns out, the bilinear form E_ε, as defined above, is generally *not* symmetric. It can, however, be decomposed into a symmetric and an antisymmetric part, c.f. Lemma 4.17 below.

Definition 4.14 (Decomposition into symmetric and antisymmetric part[a])
For a general bilinear form $\mathcal{E} : D \times D \to \mathbb{R}$, we denote by $\tilde{\mathcal{E}}$ its *symmetric part* and by $\check{\mathcal{E}}$ its *antisymmetric part*, i.e.

$$\tilde{\mathcal{E}}(f, g) \coloneqq \frac{\mathcal{E}(f, g) + \mathcal{E}(g, f)}{2}, \quad \forall f, g \in D, \tag{4.50}$$

$$\check{\mathcal{E}}(f, g) \coloneqq \frac{\mathcal{E}(f, g) - \mathcal{E}(g, f)}{2}, \quad \forall f, g \in D. \tag{4.51}$$

[a] c.f. Section I.2 in [MR92], i.p. Eq. (2.1)

For the decomposition of E_ε, the following vector fields will prove useful.

[7] c.f. Section I.2 of [MR92]

Definition 4.15 (Vector fields)
Let $\Phi, \Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ be defined by (c.f. [BR14], Eqs. (134) and (135))

$$\Phi := U - \frac{1}{2}\left(\nabla^\top A\right)^\top, \tag{4.52}$$

$$\Phi_\varepsilon := \frac{1}{\varepsilon} V + \left[U - \frac{1}{2}\left(\nabla^\top A\right)^\top\right]. \tag{4.53}$$

Inserting the choices for U and A specified in Definition 4.4, we can obtain a complex-valued representation of Φ as well as an upper bound.

Lemma 4.16 (Vector fields of oscillator system)
The complex valued version $\mathrm{j}^{-1} \circ \Phi \circ \mathrm{j} : \mathbb{C}^n \to \mathbb{C}^n$ of the vector field $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$\begin{aligned}\left(\mathrm{j}^{-1} \circ \Phi \circ \mathrm{j}(x)\right)_k = \tilde{u}_k(x) &- \frac{n^2}{8}\left(\left[|\widetilde{\nu}_k|^2 + \sigma_r^2\right] x_k + \overline{\widetilde{\nu}}_k \overline{\widetilde{\nu}}_{n-k} \overline{x}_{n-k}\right)\left(1 + \mathbb{1}_{\{0,n/2\}}(k)\right) \\ &- \frac{n^2}{8}\left(|\widetilde{\nu}_{n-k}|^2\, \overline{x_{n-k}} + \widetilde{\nu}_k \widetilde{\nu}_{n-k}\, x_k\right)\left(1 + \mathbb{1}_{\{0,n/2\}}(k)\right).\end{aligned}$$

In the linear case of $u = u_{\mathrm{lin}}$, we conclude that there is a constant $c_k^\Phi > 0$, providing us with an upper bound of the form

$$\left|\left(\mathrm{j}^{-1} \circ \Phi \circ \mathrm{j}(x)\right)_k\right|^2 \le c_k^\Phi \left(|x_k|^2 + |x_{n-k}|^2\right), \quad \forall x \in \mathbb{C}^n. \tag{4.54}$$

Proof. Let $X \in \mathbb{R}^n$ and define $x := \mathrm{j}^{-1}(X)$. We recall that

$$A(X) = \mathfrak{J}_{1/2}\left(\widetilde{\boldsymbol{\sigma}}(x)\mathcal{Q}_5^\dagger\right)\left(\mathfrak{J}_{1/2}\left(\widetilde{\boldsymbol{\sigma}}(x)\mathcal{Q}_5^\dagger\right)\right)^\top = \frac{1}{2}\left[\mathfrak{J}\left(\widetilde{\boldsymbol{\sigma}}(x)\widetilde{\boldsymbol{\sigma}}^\dagger(x)\right) + \breve{\mathfrak{J}}\left(\widetilde{\boldsymbol{\sigma}}(x)\mathcal{R}_5\widetilde{\boldsymbol{\sigma}}^\top(x)\right)\right] \tag{4.55}$$

and employ Eqs. (2.8), (2.9), (4.45) and (4.46) in order to obtain that

$$\begin{aligned}\Phi(X) &= U(X) - \frac{1}{2}\left(\nabla^\top (A(X)\right)^\top \\ &= \mathrm{j}(\tilde{u}(x)) - \frac{1}{4}\left(\nabla^\top \mathfrak{J}\left(\widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\sigma}}^\dagger(x)\right)\right)^\top - \frac{1}{4}\left(\nabla^\top \breve{\mathfrak{J}}\left(\widetilde{\boldsymbol{\sigma}}\mathcal{R}_5\widetilde{\boldsymbol{\sigma}}^\top(x)\right)\right)^\top \\ &= \mathrm{j}(\tilde{u}(x)) - \frac{1}{2}\left(\left[\mathrm{j}(\overline{\partial})\right]^\top \mathfrak{J}\left(\widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\sigma}}^\dagger(x)\right)\right)^\top - \frac{1}{2}\left(\left[\mathrm{j}(\overline{\partial})\right]^\top \breve{\mathfrak{J}}\left(\widetilde{\boldsymbol{\sigma}}\mathcal{R}_5\widetilde{\boldsymbol{\sigma}}^\top(x)\right)\right)^\top \\ &= \mathrm{j}(\tilde{u}(x)) - \frac{1}{2}\left(\mathfrak{J}\left(\widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\sigma}}^\dagger(x)\right)\right)^\top \mathrm{j}\left(\overleftarrow{\overline{\partial}}\right) - \frac{1}{2}\left(\breve{\mathfrak{J}}\left(\widetilde{\boldsymbol{\sigma}}\mathcal{R}_5\widetilde{\boldsymbol{\sigma}}^\top(x)\right)\right)^\top \mathrm{j}\left(\overleftarrow{\overline{\partial}}\right) \\ &= \mathrm{j}(\tilde{u}(x)) - \frac{1}{2}\mathrm{j}\left(\widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\sigma}}^\dagger(x)\overleftarrow{\overline{\partial}}\right) - \frac{1}{2}\mathrm{j}\left(\widetilde{\boldsymbol{\sigma}}\mathcal{R}_5\widetilde{\boldsymbol{\sigma}}^\top(x)\overleftarrow{\partial}\right) \\ &= \mathrm{j}(\tilde{u}(x)) - \frac{n^2}{8}\mathrm{j}\left[\left(\left([|\widetilde{\nu}_k|^2 + \sigma_r^2]x_k + \overline{\widetilde{\nu}}_k\overline{\widetilde{\nu}}_{n-k}\overline{x}_{n-k}\right)\right)\left(1 + \mathbb{1}_{\{0,n/2\}}(k)\right)\right] \\ &\qquad - \frac{n^2}{8}\mathrm{j}\left[\left(|\widetilde{\nu}_{n-k}|^2\,\overline{x_{n-k}} + \widetilde{\nu}_k\widetilde{\nu}_{n-k}\,x_k\right)_k\right]\left(1 + \mathbb{1}_{\{0,n/2\}}(k)\right),\end{aligned} \tag{4.56}$$

where in the last step we have applied

$$\begin{aligned}\overline{\partial}_k(\widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\sigma}}^\dagger)_{k,k} &= \frac{n^2}{4}\,\overline{\partial}_k\biggl(\Bigl[|\widetilde{\nu}_k|^2+\sigma_r^2\Bigr]|x_k|^2+\Bigl[|\widetilde{\nu}_{n-k}|^2+\sigma_r^2\Bigr]|x_{n-k}|^2\\&\qquad\qquad+\widetilde{\nu}_k\widetilde{\nu}_{n-k}x_kx_{n-k}+\overline{\widetilde{\nu}}_k\overline{\widetilde{\nu}}_{n-k}\overline{x}_k\overline{x}_{n-k}\biggr)\\&=\frac{n^2}{4}\left(\Bigl[|\widetilde{\nu}_k|^2+\sigma_r^2\Bigr]x_k+\overline{\widetilde{\nu}}_k\overline{\widetilde{\nu}}_{n-k}\overline{x}_{n-k}\right)\left(1+\mathbb{1}_{\{0,n/2\}}(k)\right),\end{aligned}\tag{4.57}$$

as well as

$$\begin{aligned}\partial_{n-k}\left(\widetilde{\boldsymbol{\sigma}}\mathcal{R}_5\widetilde{\boldsymbol{\sigma}}^\top\right)_{k,n-k}(x) &= \partial_{n-k}\widetilde{\sigma_{\text{mult}}}R\left(\widetilde{\sigma_{\text{mult}}}\right)^\top(x)\\&=\partial_{n-k}\left(\widetilde{\sigma_{\text{mult}}}R\left(\widetilde{\sigma_{\text{mult}}}\right)^\top\right)_{k,n-k}(x)\\&=\frac{n^2}{4}\,\partial_{n-k}\biggl[|\widetilde{\nu}_k|^2|x_k|^2+|\widetilde{\nu}_{n-k}|^2|x_{n-k}|^2\\&\qquad+\left(\widetilde{\nu}_k\widetilde{\nu}_{n-k}x_kx_{n-k}+\overline{\widetilde{\nu}_k\widetilde{\nu}_{n-k}x_kx_{n-k}}\right)\biggr]\\&=\frac{n^2}{4}\left(|\widetilde{\nu}_{n-k}|^2\,\overline{x_{n-k}}+\widetilde{\nu}_k\widetilde{\nu}_{n-k}\,x_k\right)\left(1+\mathbb{1}_{\{0,n/2\}}(k)\right).\end{aligned}\tag{4.58}$$

By Eq. (3.93) we have

$$\widetilde{u_{\text{lin}}}(x)=\frac{\sqrt{n}}{2}\left(\operatorname{diag}(\widetilde{\lambda})x+R\,\overline{\operatorname{diag}(\widetilde{\lambda})x}\right),\tag{4.59}$$

which implies that

$$|(\widetilde{u_{\text{lin}}})_k(x)|^2=\frac{n}{4}\left|\widetilde{\lambda}_kx_k+\widetilde{\lambda}_{n-k}x_{n-k}\right|^2\le\frac{n}{2}\left(\left|\widetilde{\lambda}_k\right|^2+\left|\widetilde{\lambda}_{n-k}\right|^2\right)\left(|x_k|^2+|x_{n-k}|^2\right),\tag{4.60}$$

where in the last step we have employed *Young's inequality*. Together with Eq. (4.56) this yields the aspired upper bound. □

Lemma 4.17 (Decomposition)
The decomposition $E_\varepsilon=\tilde{E}_\varepsilon+\check{E}_\varepsilon$ of the bilinear form E_ε into a symmetric part $\tilde{E}_\varepsilon$ and an antisymmetric part $\check{E}_\varepsilon$ yields

$$\tilde{E}_\varepsilon(f,g)=\frac{1}{2}\int_{\mathbb{R}^N}\Bigl[(\nabla f)^\top A\,\nabla g+(\nabla^\top\Phi)\,f\,g\Bigr]\,\mathrm{d}\mu,\tag{4.61}$$

$$\check{E}_\varepsilon(f,g)=\frac{1}{2}\int_{\mathbb{R}^N}\Phi_\varepsilon^\top(f\,\nabla g-g\,\nabla f)\,\mathrm{d}\mu.\tag{4.62}$$

Proof. Employing Eq. (4.4), this follows as in the proof of [BR14], Proposition 6.

The following observation will prove to be essential.

Remark 4.18 (Dependence on scaling parameter)
The symmetric part $\tilde{E}_\varepsilon$ does *not* depend on the parameter ε.

As we will see in Lemma 4.51 below, for a certain class of functions, the antisymmetric part $\check{E}_\varepsilon$ does not depend on ε as well. For the symmetric part we have the following estimate.

Remark 4.19 (Estimate on symmetric part)
Note that by Eqs. (4.34) and (4.36) we know that

$$\begin{aligned}\operatorname{div}(\Phi_\varepsilon) &= \frac{1}{\varepsilon}\nabla^\top V + \nabla^\top U - \frac{1}{2}\nabla\nabla^\top : A \\ &= \nabla^\top U - \frac{1}{2}\nabla\nabla^\top : A \\ &\geq n\,\mathfrak{l}_0' - 2n^2\sum_{k=0}^{n-1}\left(|\tilde{\nu}_k|^2 + \sigma_r^2\right) \equiv c > 0,\end{aligned} \tag{4.63}$$

which is why, for all $f \in C_c^2(\mathbb{R}^N)$, we can conclude that

$$\begin{aligned}E_\varepsilon(f,f) = \tilde{E}_\varepsilon(f,f) &= \frac{1}{2}\int_{\mathbb{R}^N}\left[(\nabla f)^\top A\,\nabla f + (\nabla^\top\Phi)\,f^2\right]\mathrm{d}\mu \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N}(\nabla f)^\top A\,\nabla f\,\mathrm{d}\mu + \frac{c}{2}\int_{\mathbb{R}^N} f^2\,\mathrm{d}\mu.\end{aligned} \tag{4.64}$$

4.2.2.2. Closed forms

In this section we show that E_ε can be extended to a *coercive closed form*, c.f. Theorem 4.25. For this purpose we will introduce the following definitions.

Definition 4.20 (Norms induced by positive definite bilinear form)
For $\alpha > 0$ we set[a]

$$\mathcal{E}^\alpha(f,g) := \mathcal{E}(f,g) + \alpha\,\langle f,g\rangle_{\mathcal{H}}\,, \quad \forall f,g \in D, \tag{4.65}$$

$$\tilde{\mathcal{E}}^\alpha(f,g) := \tilde{\mathcal{E}}(f,g) + \alpha\,\langle f,g\rangle_{\mathcal{H}}\,, \quad \forall f,g \in D. \tag{4.66}$$

If $(\mathcal{E}, D)$ is *positive definite*, we can define *equivalent norms* $\|\cdot\|_{\tilde{\mathcal{E}}^\alpha}$ by setting[b]

$$\|f\|_{\tilde{\mathcal{E}}^\alpha} := \left(\tilde{\mathcal{E}}^\alpha(f,f)\right)^{1/2}, \quad \forall f \in D. \tag{4.67}$$

[a] c.f. Section I.2 in [MR92], i.p. Eq. (2.2) and [BR14], Eqs. (21), (22)
[b] c.f. Exercise I.2.1 in [MR92]

Next, we define the notion of a symmetric closed form.

Definition 4.21 (Symmetric closed forms[a])
A pair $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is called *symmetric closed form* (on $\mathcal{H}$) if the following conditions are fulfilled:

i) $\mathcal{D}(\mathcal{E})$ is a dense linear subspace of $\mathcal{H}$,
ii) $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$ is a positive definite bilinear form,
iii) $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is *symmetric* (i.e. $\mathcal{E} = \tilde{\mathcal{E}}$),
iv) $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is *closed* on $\mathcal{H}$ (i.e. $\mathcal{D}(\mathcal{E})$ is complete w.r.t. the norm $\|\cdot\|_{\tilde{\mathcal{E}}^1}$).

[a] c.f. Definitions I.2.3 of [MR92]

A *non*-symmetric Dirichlet form can be controlled by its symmetric part, if it satisfies the following so-called *weak sector condition*.

Definition 4.22 (Sector conditions[a])
$(\mathcal{E}, D)$ is said to satisfy the *weak sector condition* if it is positive definite and if there is a so-called *continuity constant* $K > 0$, s.t.

$$\left|\mathcal{E}^1(f,g)\right| \leq K \left(\mathcal{E}^1(f,f)\right)^{1/2} \left(\mathcal{E}^1(g,g)\right)^{1/2}, \quad \forall f,g \in D. \tag{4.68}$$

$(\mathcal{E}, D)$ is said to satisfy the *strong sector condition* if it is positive definite and if there is a constant $K > 0$, s.t.

$$|\mathcal{E}(f,g)| \leq K \left(\mathcal{E}(f,f)\right)^{1/2} \left(\mathcal{E}(g,g)\right)^{1/2}, \quad \forall f,g \in D. \tag{4.69}$$

A positive linear operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ on $L^2(\mathbb{R}^N)$ satisfies the *strong sector condition* if there is a constant $K' > 0$, s.t.

$$\left|\langle \mathcal{A}f, g\rangle_{L^2(\mathbb{R}^N)}\right| \leq K' \left(\langle \mathcal{A}f, f\rangle_{L^2(\mathbb{R}^N)}\right)^{1/2} \left(\langle \mathcal{A}g, g\rangle_{L^2(\mathbb{R}^N)}\right)^{1/2}, \quad \forall f,g \in \mathcal{D}(\mathcal{A}). \tag{4.70}$$

[a] c.f. Section I.2 in [MR92], i.p. Eqs. (2.3), (2.4) and (2.5)

The following statement on the strong sector condition of the bilinear form $(E_\varepsilon, C_c^2(\mathbb{R}^N))$ plays an essential role in the construction of a Dirichlet form from the stochastic process $(Y^\varepsilon(t))_{t\geq 0}$. Results from [MR92] and [RS95] will allow us to construct a proof of this important property.

Lemma 4.23 ($(E_\varepsilon, C_c^2(\mathbb{R}^N))$ and $(1-\mathcal{A}_\varepsilon)$ satisfy the strong sector condition)
For every $\varepsilon > 0$, the bilinear form $(E_\varepsilon, C_c^2(\mathbb{R}^N))$ of Definition 4.12 and the operator $(1-\mathcal{A}_\varepsilon)$ satisfy the strong sector condition,[a] i.e. there are $K_\varepsilon, K'_\varepsilon > 0$, s.t. for all $f,g \in C_c^2(\mathbb{R}^N)$:

$$|E_\varepsilon(f,g)| \leq K_\varepsilon \left(E_\varepsilon(f,f)\right)^{1/2} \left(E_\varepsilon(g,g)\right)^{1/2}, \tag{4.71}$$

$$\left|\langle(1-\mathcal{A}_\varepsilon)f, g\rangle_{L^2(\mathbb{R}^N)}\right| \leq K'_\varepsilon \left(\langle(1-\mathcal{A}_\varepsilon)f, f\rangle_{L^2(\mathbb{R}^N)}\right)^{1/2} \left(\langle(1-\mathcal{A}_\varepsilon)g, g\rangle_{L^2(\mathbb{R}^N)}\right)^{1/2}. \tag{4.72}$$

[a] Recall this section's assumption of the drift term being linear. The nonlinear perturbation violates the sector condition, which is why in this case an extended framework of generalized Dirichlet forms is required.

Proof. Let $\varepsilon > 0$ be fixed. In a first step we estimate the *antisymmetric* part[8] and show that there is a constant $K_\varepsilon > 0$, s.t.

$$\left|\check{E}_\varepsilon(f,g)\right| \le K_\varepsilon \left(E_\varepsilon(f,f)\right)^{1/2} \left(E_\varepsilon(g,g)\right)^{1/2}, \quad \forall f,g \in C_c^2(\mathbb{R}^N). \tag{4.73}$$

We plug in Eq. (4.62) for the left-hand side and find[9]

$$\left|\check{E}_\varepsilon(f,g)\right| \le \frac{1}{2}\int_{\mathbb{R}^N} \left|f\,(\nabla g)^\top \Phi_\varepsilon\right| \mathrm{d}\mu + \frac{1}{2}\int_{\mathbb{R}^N} \left|g\,(\nabla f)^\top \Phi_\varepsilon\right| \mathrm{d}\mu.$$

We only need to estimate the first term, since a similiar bound for the second term can subsequently be obtained by interchanging the roles of f and g. For the first term, application of Hölder's inequality yields

$$\begin{aligned}\int_{\mathbb{R}^N} \left|(\nabla g)^\top \Phi_\varepsilon\, f\right| \mathrm{d}\mu &= \int_{\mathbb{R}^N} \left|\left((\nabla g)^\top \mathfrak{D}\right)\left(\mathfrak{D}^{-1}\Phi_\varepsilon\, f\right)\right| \mathrm{d}\mu \\ &\le \left(\int_{\mathbb{R}^N} \|\mathfrak{D}\,\nabla g\|^2\, \mathrm{d}\mu\right)^{1/2} \left(\int_{\mathbb{R}^N} \left\|\mathfrak{D}^{-1}\Phi_\varepsilon\right\|^2 f^2\, \mathrm{d}\mu\right)^{1/2}.\end{aligned} \tag{4.74}$$

Note that we have made use of the fact that for every $X \in \mathbb{R}^N$ the matrix $\mathfrak{D}(X)$, as defined in Eq. (4.16) of Lemma 4.6, is invertible. By the uniform ellipticity result of Lemma 4.6 and the estimate on the symmetric part given in Eq. (4.64), we know that[10]

$$\int_{\mathbb{R}^N} \|\mathfrak{D}\,\nabla g\|^2\, \mathrm{d}\mu \le \int_{\mathbb{R}^N} (\nabla g)^\top A\,\nabla g\, \mathrm{d}\mu \le 2\,E_\varepsilon(g,g). \tag{4.75}$$

From the upper bound on Φ given in Eq. (4.54) we conclude that for all $X \in \mathbb{R}^N$,

$$\begin{aligned}|(\Phi_\varepsilon(X))_k|^2 &\le \left|\frac{1}{\varepsilon}\,V_k\right|^2 + |(\Phi(X))_k|^2 \\ &\le \frac{1}{\varepsilon^2}\,|x_k|^2 + c_k^\Phi\left(|x_k|^2 + |x_{n-k}|^2\right),\end{aligned} \tag{4.76}$$

where we have set $x := \mathfrak{j}^{-1}(X)$. This implies a boundedness of the integrand in the second factor of Eq. (4.74), i.e.

$$c_\varepsilon := \sup_{X\in\mathbb{R}^N} \left\|\mathfrak{D}^{-1}(X)\Phi_\varepsilon(X)\right\|^2 = \sum_{k=0}^{n-1} \sup_{x\in\mathbb{C}^n} \frac{|[\Phi_\varepsilon(\mathfrak{j}(x))]_k|^2}{\frac{\sigma_0^2}{2} + \frac{n^2}{8}\,\sigma_r^2\left(|x_k|^2 + |x_{n-k}|^2\right)} < \infty. \tag{4.77}$$

Note that the presence of regularizing noise ($\sigma_r \neq 0$) plays a crucial role in this step. Eq. (4.77) now implies that

$$\int_{\mathbb{R}^N} \left\|\mathfrak{D}^{-1}\Phi_\varepsilon\right\|^2 f^2\, \mathrm{d}\mu \le c_\varepsilon \int_{\mathbb{R}^N} f^2\, \mathrm{d}\mu \le \frac{2c_\varepsilon}{c}\, E_\varepsilon(f,f), \tag{4.78}$$

[8][MR92] Exercise I.2.1(iv) shows that this is the relevant part for proving the sector condition.
[9]We adapt the proof of [RS95] Theorem 1.2, c.f. in particular Eq. (1.37).
[10]c.f. [RS95], Eq. (1.26)

where in the last step we have employed Eq. (4.64). Combining the previous results, we obtain Eq. (4.73).

An estimate on the *symmetric* part $\tilde{E}_\varepsilon$ can be obtained by application of the Cauchy-Schwarz inequality, which yields

$$\left|\tilde{E}_\varepsilon(f,g)\right| \leq \left(E_\varepsilon(f,f)\right)^{1/2} \left(E_\varepsilon(g,g)\right)^{1/2}, \tag{4.79}$$

and Eq. (4.71) follows.

Finally note that Eq. (4.72) follows from the Cauchy-Schwarz inequality, Eq. (4.64) and Eq. (4.71):

$$\begin{aligned}
\left|\langle(1-\mathcal{A}_\varepsilon)f,g\rangle_{L^2(\mathbb{R}^N)}\right| &\leq \left|\langle -\mathcal{A}_\varepsilon f,g\rangle_{L^2(\mathbb{R}^N)}\right| + \left(\langle f,f\rangle_{L^2(\mathbb{R}^N)}\right)^{1/2} \left(\langle g,g\rangle_{L^2(\mathbb{R}^N)}\right)^{1/2} \\
&\leq K_\varepsilon \left(E_\varepsilon(f,f)\right)^{1/2} \left(E_\varepsilon(g,g)\right)^{1/2} + \frac{2}{c}\left(E_\varepsilon(f,f)\right)^{1/2} \left(E_\varepsilon(g,g)\right)^{1/2} \\
&\leq \left(K_\varepsilon + \frac{2}{c}\right) \left(\langle -\mathcal{A}_\varepsilon f,f\rangle_{L^2(\mathbb{R}^N)}\right)^{1/2} \left(\langle -\mathcal{A}_\varepsilon g,g\rangle_{L^2(\mathbb{R}^N)}\right)^{1/2} \\
&\leq \left(K_\varepsilon + \frac{2}{c}\right) \left(\langle (1-\mathcal{A}_\varepsilon) f,f\rangle_{L^2(\mathbb{R}^N)}\right)^{1/2} \left(\langle (1-\mathcal{A}_\varepsilon) g,g\rangle_{L^2(\mathbb{R}^N)}\right)^{1/2}. \quad \square
\end{aligned}$$

The weak sector condition allows us to extend the notion of Definition 4.21 to the case of non-symmetric forms.

Definition 4.24 (Coercive closed forms[a])
A pair $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is called *coercive closed form* (on $\mathcal{H}$) if the following conditions are fulfilled:

i) $\mathcal{D}(\mathcal{E})$ is a dense linear subspace of $\mathcal{H}$,

ii) $\mathcal{E}:\mathcal{D}(\mathcal{E})\times\mathcal{D}(\mathcal{E})\to\mathbb{R}$ is a bilinear form,

iii) $(\tilde{\mathcal{E}},\mathcal{D}(\mathcal{E}))$ is a symmetric closed form on $\mathcal{H}$,

iv) $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ satisfies the weak sector condition (4.68).

[a] c.f. Definition I.2.4 of [MR92]

The following theorem allows us to extend the (non-symmetric) bilinear form E_ε to a coercive closed form on $L^2(\mathbb{R}^N)$.

Theorem 4.25 (Coercive closed form corresponding to stochastic process[a])
Let $\mathcal{D}(E_\varepsilon)$ be defined as the completion of $C_c^2(\mathbb{R}^N)$ w.r.t. $\|\cdot\|_{\tilde{E}_\varepsilon^1}$. Furthermore, denote the unique bilinear extension of E_ε to $\mathcal{D}(E_\varepsilon)$, which is continuous w.r.t. $\|\cdot\|_{\tilde{E}_\varepsilon^1}$, again by E_ε.
Then $\mathcal{D}(E_\varepsilon)$ is independent of ε and $(E_\varepsilon,\mathcal{D}(E_\varepsilon))$ is a coercive closed form on $L^2(\mathbb{R}^N)$, s.t.

$$E_\varepsilon(f,g) = \langle -\mathcal{A}_\varepsilon f,g\rangle_{L^2(\mathbb{R}^N)}, \quad \forall f\in\mathcal{D}(\mathcal{A}_\varepsilon), g\in\mathcal{D}(E_\varepsilon). \tag{4.80}$$

[a] c.f. [MR92], Theorem I.2.15

Proof. By Remark 4.18, $\tilde{E}_\varepsilon$ is independent of ε, which implies that $\mathcal{D}(E_\varepsilon)$ is independent of ε as well. According to Theorem 4.11, the transition semigroup $(T^\varepsilon(t))_{t>0}$, associated to the infinitesimal generator $\mathcal{A}_\varepsilon$, is a strongly continuous contraction semigroup $(T^\varepsilon(t))_{t>0}$ on $L^2(\mathbb{R}^N)$ and by Lemma 4.23, $(1-\mathcal{A}_\varepsilon)$ satisfies the strong sector condition for operators. The result follows now from [MR92], Lemma I.2.14 and Theorem I.2.15. □

Remark 4.26 (Domain $\mathcal{D}(E_\varepsilon)$)
In [BR14], the domain $\mathcal{D}(E_\varepsilon)$ coincides with $H^1(\mathbb{R}^N)$, c.f. Proposition 6, which is a consequence of drift and dispersion terms being bounded. This does not, however, hold in our setting, since we do not impose a boundedness restriction.

4.2.2.3. Dirichlet forms

A coercive closed form satisfying certain *contraction properties*[11] is called a *Dirichlet form*.

Definition 4.27 (Dirichlet form)
A coercive closed form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\mathbb{R}^N)$ is called a *Dirichlet form*[a] if for all $f \in \mathcal{D}(\mathcal{E})$, it satisfies the so-called *contraction properties*

$$f^+ \wedge 1 \in \mathcal{D}(\mathcal{E}), \tag{4.81a}$$

$$\mathcal{E}\left(f + f^+ \wedge 1, f - f^+ \wedge 1\right) \geq 0, \tag{4.81b}$$

$$\mathcal{E}\left(f - f^+ \wedge 1, f + f^+ \wedge 1\right) \geq 0. \tag{4.81c}$$

A Dirichlet $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ form is said to be *regular* if $C_c^2(\mathbb{R}^N) \cap \mathcal{D}(\mathcal{E})$ is dense in $\mathcal{D}(\mathcal{E})$ w.r.t. $\|\cdot\|_{\tilde{\mathcal{E}}^\alpha}$ and in $C_c(\mathbb{R}^N)$ w.r.t. to the uniform norm $\|\cdot\|_\infty$.

It is said to satisfy the *local property*[b] if for any $f, g \in \mathcal{D}(\mathcal{E})$ with $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$, it follows that $\mathcal{E}(f, g) = 0$.

[a]c.f. [MR92], Definition I.4.5
[b][BR14], p.1038 and [MR92], Section IV.4.a, p.118

In Theorem 4.25 we have seen that E_ε is a coercive closed form on $L^2(\mathbb{R}^N)$. We can strengthen this result by showing that E_ε actually constitutes a Dirichlet form.

Lemma 4.28 (Dirichlet form corresponding to stochastic process)
For any $\varepsilon > 0$, the coercive closed form E_ε is a *Dirichlet form*, which is *regular* and satisfies the *local property*.[a]

[a]This statement corresponds to [BR14], Proposition 6.

Proof. We follow the proof of [BR14], Proposition 6.

Dirichlet form: The contraction property of E_ε follows from [MR92], Section II.2.d. (p. 48), i.p.

[11]c.f. [BR14], Eq. (2.14)

Eq. (2.18), which is applicable due to the assumptions we have stated in Eqs. (4.4) and (4.5).
Regularity: $\mathcal{D}(E_\varepsilon)$ was defined in Theorem 4.25 as the completion of $C_c^2(\mathbb{R}^N)$ w.r.t. $\|\cdot\|_{\tilde{E}^1_\varepsilon}$, which is why $C_c^2(\mathbb{R}^N) \cap \mathcal{D}(E_\varepsilon) = C_c^2(\mathbb{R}^N)$ is dense in $\mathcal{D}(E_\varepsilon)$ w.r.t. $\|\cdot\|_{\tilde{E}^1_\varepsilon}$.[12] Moreover, $C_c^2(\mathbb{R}^N)$ is dense in $C_c(\mathbb{R}^N)$ w.r.t. $\|\cdot\|_\infty$.
Local property: If $f, g \in \mathcal{D}(E_\varepsilon)$ with $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$, then $E_\varepsilon(f,g) = 0$ is apparent from Definition 4.12. □

Both regularity and local property will play a crucial role in proving certain desirable properties of the limiting process, c.f. Proposition 4.58. The local property for instance will allow us to obtain a continuity result for the sample paths.

4.3. First integrals and induced projection

We develop a formalism to study the evolution of conserved quantities, or so-called *first integrals*.[13] Namely, we employ the system's first integrals to define an equivalence relation and subsequently analyze the resulting quotient space.

4.3.1. First integrals

We characterize functions which are conserved under the fast evolution generated by $\frac{1}{\varepsilon} V$.

Definition 4.29 (First integrals[a])
Let $\mathsf{P} : \mathbb{R}^N \to \mathbb{R}^m$ be a continuously differentiable function with *compact level sets*, s.t.

$$\mathrm{D}\,\mathsf{P}(X)\, V(X) = 0, \quad \forall X \in \mathbb{R}^N, \tag{4.82}$$

where

$$\mathrm{D}\,\mathsf{P}(X) \coloneqq \begin{pmatrix} \nabla_0 \mathsf{P}_0 & \cdots & \nabla_{N-1}\mathsf{P}_0 \\ \vdots & & \vdots \\ \nabla_0 \mathsf{P}_{m-1} & \cdots & \nabla_{N-1}\mathsf{P}_{m-1} \end{pmatrix}(X) \;\in \mathbb{R}^{m,N} \tag{4.83}$$

denotes the corresponding Jacobian matrix. The components of P are called *first integrals* of the flow generated by V.

[a] c.f. Assumption 3 of [BR14].

Note that Eq. (4.82) implies that all components of P, i.e. the first integrals, are constant along the flow induced by V. As we have seen, the first integrals of the uncoupled oscillator system correspond to the conserved Noether quantities of the $U(n)$ symmetry. All of these first integrals can be captured by the *complex outer product*, c.f. Section 3.1.2. This is why from now on we will choose P to be the real-valued representation of the complex outer product.

[12] Recall that the norms $\{\|\cdot\|_{\tilde{E}^\alpha_\varepsilon} \mid \alpha > 0\}$ are equivalent, as was noted in Definition 4.20.
[13] c.f. introduction to Chapter 3

Definition 4.30 (Outer product components as first integrals)
Let $m = N^2 \equiv 4n^2$ and define

$$\mathsf{P} : \mathbb{R}^N \to \mathbb{R}^{N,N} \cong \mathbb{R}^m, \quad X \to XX^\top + I\,XX^\top(-I) \equiv \mathfrak{J}\left(\mathsf{j}^{-1}(X)\,\mathsf{j}^{-1}(X)^\dagger\right), \tag{4.84}$$

where we have made use of Lemma 2.12 in order to obtain the equivalent representation. The corresponding complex-valued map is given by

$$\mathsf{p} : \mathbb{C}^n \to \mathbb{C}^{n,n} \cong \mathbb{C}^{n^2}, \quad x \to \mathsf{p}(x) := \mathfrak{J}^{-1} \circ \mathsf{P} \circ \mathsf{j}(x) = xx^\dagger. \tag{4.85}$$

In order to verify Eq. (4.82), we need to calculate the Jacobian of the outer-product map. For future use, c.f. Proposition 4.63, we furthermore determine the Frobenius product of a general $\mathbb{R}^{N,N}$ matrix with the Hessian matrix of P.

Lemma 4.31 (Jacobian matrix and Hessian of outer product map)
For all $X, Y \in \mathbb{R}^N$, we find that

$$\mathrm{D}\,\mathsf{P}|_X\, Y = \mathfrak{J}\left(\mathsf{j}^{-1}(X)\mathsf{j}^{-1}(Y)^\dagger + \mathsf{j}^{-1}(Y)\mathsf{j}^{-1}(X)^\dagger\right), \tag{4.86}$$

i.e. $\mathrm{D}\,\mathsf{P}|_X\, Y$ has a complex linear structure and for all $x, y \in \mathbb{C}^n$, we obtain

$$\mathfrak{J}^{-1}\left[\mathrm{D}\,\mathsf{P}|_{\mathsf{j}(x)}\ \mathsf{j}(y)\right] = xy^\dagger + \text{h.c.} \tag{4.87}$$

Let $X \in \mathbb{R}^N$ and $A \in \mathbb{R}^{N,N}$ be a positive semidefinite matrix. Then the Frobenius product of A with the Hessian of the complex outer product is given by

$$\frac{1}{2} A : \nabla\nabla^\top \mathsf{P}\Big|_X = A^{(+)} + I\,A^{(+)}\,(-I), \tag{4.88}$$

where $A^{(+)} := \frac{A+A^\top}{2}$ denotes the symmetric part of A.

Proof. Let $X, Y \in \mathbb{R}^N$ and denote by $x := \mathsf{j}^{-1}(X), y := \mathsf{j}^{-1}(Y)$ their complex-valued representations. For $k \in \{0, ..., m-1\}$ we note that

$$(\mathrm{D}\,\mathsf{P}|_X\, Y)_k = \sum_{l=0}^{N-1} Y_l (\nabla_l \mathsf{P}_k)|_X = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathsf{P}_k(X + t\,Y),$$

and conclude that

$$\begin{aligned}
\mathrm{D}\,\mathsf{P}|_X\, Y &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathsf{P}(X + t\,Y) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathfrak{J}\left(\mathsf{p}(x + t\,y)\right) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathfrak{J}\left[(x + t\,y)(x + t\,y)^\dagger\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \left[\mathfrak{J}\left(xx^\dagger\right) + t\,\mathfrak{J}\left(xy^\dagger + yx^\dagger\right) + t^2\,\mathfrak{J}\left(yy^\dagger\right)\right] \\
&= \mathfrak{J}\left(xy^\dagger + yx^\dagger\right),
\end{aligned}$$

which yields Eq. (4.87).

Let $X \in \mathbb{R}^N$ and $A \in \mathbb{R}^{N,N}$ be a positive semidefinite matrix. Denote by $A^{(+)}$ the symmetric part of A and by $A^{(+)} = \Sigma\Sigma^\top$ a decomposition[14] of $A^{(+)}$. Let furthermore $V^{(r)}$ denote the r'th column of Σ, i.e. $V_k^{(r)} := \Sigma_{k,r}$, for all $k \in \{0, ..., N-1\}$, $r \in \{0, ..., N'-1\}$. By a similar calculation as before we obtain

$$\begin{aligned} \frac{1}{2} A : \nabla\nabla^\top \mathsf{P}\Big|_X &= \frac{1}{2} \sum_{k,l} A_{k,l} \nabla_k \nabla_l \mathsf{P}|_X = \frac{1}{2} \sum_{k,l} A_{k,l}^{(+)} \nabla_k \nabla_l \mathsf{P}\Big|_X \\ &= \frac{1}{2} \sum_{k,l,r} \Sigma_{k,r} \Sigma_{l,r} \nabla_k \nabla_l \mathsf{P}|_X = \frac{1}{2} \sum_{k,l,r} [V_k^{(r)} \nabla_k] \, [V_l^{(r)} \nabla_l] \, \mathsf{P}\Big|_X \\ &= \frac{1}{2} \sum_r \frac{\mathrm{d}^2}{(\mathrm{d}t)^2}\bigg|_{t=0} \mathsf{P}(X + t\, V^{(r)}) = \mathfrak{J}\left(\sum_r v^{(r)} {v^{(r)}}^\dagger\right) \\ &= \sum_r V^{(r)} {V^{(r)}}^\top + I\, V^{(r)} {V^{(r)}}^\top (-I) = A^{(+)} + I\, A^{(+)}\, (-I). \qquad \square \end{aligned}$$

The previous lemma now puts us in a position to verify that P is indeed a first integral.

Lemma 4.32 (First integrals satisfy averaging assumptions)
The outer-product map P of Definition 4.30 satisfies the assumptions of Definition 4.29, i.e. it has compact level sets and satisfies

$$\mathrm{D}\,\mathsf{P}|_X \; V(X) = 0, \quad \forall X \in \mathbb{R}^N. \tag{4.89}$$

Proof. Compact level sets: This will be proven in Lemma 4.35 below.

Invariance under V: For $X \in \mathbb{R}^N$ we define the shorthands $x := \mathrm{j}^{-1}(X), v := \mathrm{j}^{-1}(V(X))$ and apply Lemma 4.31, i.e.

$$\mathrm{D}\,\mathsf{P}|_X \; V(X) = \mathfrak{J}\left(xv^\dagger + vx^\dagger\right) = 0,$$

since $v = i\,x$ and therefore $xv^\dagger + vx^\dagger = -i\,xx^\dagger + i\,xx^\dagger = 0$. $\square$

4.3.2. Projection induced by first integrals

We want to study the evolution of the complex outer product P and are consequently no longer interested in distinguishing between different system states which yield the same value of P. More precisely, we identify all points within a connected component of a P level set.

Definition 4.33 (Equivalence classes and projection induced by first integrals)
We introduce an equivalence relation $\sim_\mathsf{P}$ on $\mathbb{R}^N$, defined by

$$X \sim_\mathsf{P} Y \quad :\Leftrightarrow \quad X, Y \in \mathbb{R}^N \text{ lie in the same } \textit{connected component} \text{ of a } \textit{level set} \text{ of } \mathsf{P}. \tag{4.90}$$

[14] e.g. a Cholesky decomposition

We denote the equivalence class of an element $X \in \mathbb{R}^N$ by $[X]$, the corresponding quotient space by

$$\Gamma := (\mathbb{R}^N / \sim_{\mathsf{P}}) \equiv \left\{ [X] \mid X \in \mathbb{R}^N \right\}, \tag{4.91}$$

and the projection to this space by

$$\pi : \mathbb{R}^N \to \Gamma, \quad X \to [X]\,. \tag{4.92}$$

For any function $f : \Gamma \to \mathbb{R}$, we define the *lift*[a]

$$\underline{f} : \mathbb{R}^N \to \mathbb{R} : \quad \underline{f}(X) := f \circ \pi(X) = f([X]), \tag{4.93}$$

i.e. the lift is given by the *pullback*[b]

$$\pi_* : \mathbb{R}^\Gamma \to \mathbb{R}^{\mathbb{R}^N}, \quad f \to \pi_*(f) := \underline{f} := f \circ \pi. \tag{4.94}$$

[a] c.f. [BR14], p. 1058
[b] c.f. [BR14], Section 3.1, Eq. (44)

In the special case of P having *connected* level sets, the description of the quotient space simplifies, since it can be isomorphically identified with the image of P.

Lemma 4.34 (Special case of connected level sets)
If all level sets of P are connected, then there is an *isomorphism* ψ which identifies the image of P with the quotient space Γ, i.e.

$$\psi : \mathsf{P}(\mathbb{R}^N) \to \Gamma, \quad P \to \psi(P) := \pi(X) = [X]\,, \quad \text{for } X \in \mathsf{P}^{-1}(P), \tag{4.95}$$

i.e. we have $\psi \circ \mathsf{P} = \pi$, which allows us rewrite the lift $\underline{f}$ of a function $f : \Gamma \to \mathbb{R}$ as

$$\underline{f} := f \circ \pi = f \circ \psi \circ \mathsf{P}. \tag{4.96}$$

Proof. The function ψ is *well-defined*, i.e. independent of the choice of $X \in \mathsf{P}^{-1}(P)$ since for $X, Y \in \mathsf{P}^{-1}(P)$ we know that X and Y are in the same level set of P, which by assumption is connected. By definition, it now follows that X and Y are equivalent.
Injectivity: Assume that $\psi(P) = \psi(P')$ for some $P, P' \in \mathsf{P}(\mathbb{R}^N)$. Taking $X \in \mathsf{P}^{-1}(P)$ and $Y \in \mathsf{P}^{-1}(P')$, the definition of ψ yields $\pi(X) = \pi(Y)$, i.e. X and Y are equivalent. In particular, this implies that $P = \mathsf{P}(X) = \mathsf{P}(Y) = P'$.
Surjectivity: By definition, any element of Γ can be represented as $[X]$ for some $X \in \mathbb{R}^N$. Setting $P := \mathsf{P}(X)$ we find $\psi(P) = \pi(X) = [X]$. □

The relations between the maps we have just introduced can be neatly summarized by the commutative diagram of Fig. 4.2, where we note that the isomorphism ψ generally only exists in the case of connected level sets.

Now we proceed to identify the equivalence relation induced by our choice of P as the complex outer-product mapping. In particular, we show that the complex outer product has connected components. For this purpose, we return to a complex-valued description.

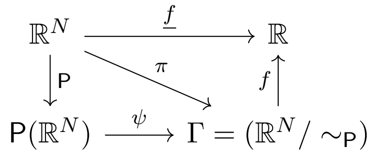


Figure 4.2.: Quotient space and lifts

Lemma 4.35 (Complex outer-product equivalence)
The equivalence relation induced by the complex-valued representation p of the first integrals P (c.f. Definition 4.30) is given by

$$\forall\, x, y \in \mathbb{C}^n : \quad x \sim_{\mathsf{p}} y \quad \Leftrightarrow \quad x = \mathrm{e}^{i\,\Delta\phi}\, y, \text{ for some } \Delta\phi \in [0, 2\pi) \tag{4.97a}$$

$$\Leftrightarrow \quad x = z\, y, \quad \text{for some } z \in S^1. \tag{4.97b}$$

The equivalence class of an element $x \in \mathbb{C}^n$ will be denoted by $[x]_{\mathbb{C}}$ and the induced quotient space $\Gamma_{\mathbb{C}} \coloneqq \mathbb{C}^n / \sim_{\mathsf{p}}$ is given by

$$\Gamma_{\mathbb{C}} = \mathbb{C}^n / S^1. \tag{4.98}$$

All level sets of Γ are *compact* and *connected*, which is why there is an isomorphism

$$\psi_{\mathbb{C}} : \mathsf{p}(\mathbb{C}^n) \to \Gamma_{\mathbb{C}}, \quad \hat{p} \to \psi_{\mathbb{C}}(\hat{p}) \coloneqq [x]_{\mathbb{C}}\,, \quad \forall x \in \mathsf{p}^{-1}(\hat{p}). \tag{4.99}$$

This isomorphism allows us to identify the first-integrals map p with its corresponding projection mapping $\pi_{\mathbb{C}}$ to the quotient space $\Gamma_{\mathbb{C}}$, i.e.

$$\pi_{\mathbb{C}} = \psi_{\mathbb{C}} \circ \mathsf{p} : \mathbb{C}^n \to \Gamma_{\mathbb{C}}, \quad x \to \pi_{\mathbb{C}}(x) = \psi_{\mathbb{C}} \circ \mathsf{p}(x). \tag{4.100}$$

Proof. Equivalence relation: Let $x, y \in \mathbb{C}^n$. For all $k \in \{0, ..., n-1\}$ with $|x_k| \neq 0$ ($|y_k| \neq 0$ respectively), we employ the polar decompositions

$$x_k \equiv |x_k|\, \mathrm{e}^{i\,\phi_k^x}, \quad y_k \equiv |y_k|\, \mathrm{e}^{i\,\phi_k^y}, \quad \text{where} \quad \phi_k^x, \phi_k^y \in [0, 2\pi). \tag{4.101}$$

We obtain the following characterization of the equivalence relation induced by p:

$$\begin{aligned}
x \sim_{\mathsf{p}} y :\Leftrightarrow\ & x x^\dagger = y y^\dagger \\
\Leftrightarrow\ & x_k\, \overline{x_l} = y_k\, \overline{y_l}, \quad \forall k, l \in \{0, ..., n-1\} \\
\Leftrightarrow\ & |x_k|\, |x_l|\, \mathrm{e}^{i\,(\phi_k^x - \phi_l^x)} = |y_k|\, |y_l|\, \mathrm{e}^{i\,(\phi_k^y - \phi_l^y)}, \quad \forall k, l \in \{0, ..., n-1\} \\
\Leftrightarrow\ & \quad |x_k| = |y_k|\,, \quad \forall k \in \{0, ..., n-1\} \\
& \wedge\ \phi_k^x - \phi_l^x \equiv \phi_k^y - \phi_l^y \bmod 2\pi, \quad \forall k, l \in \{0, ..., n-1\}\,, \text{ s.t. } |x_k|\,|x_l| \neq 0 \\
\Leftrightarrow\ & \quad |x_k| = |y_k|\,, \quad \forall k \in \{0, ..., n-1\} \\
& \wedge\ \phi_k^x - \phi_k^y \equiv \phi_l^x - \phi_l^y \bmod 2\pi, \quad \forall k, l \in \{0, ..., n-1\}\,, \text{ s.t. } |x_k|\,|x_l| \neq 0,
\end{aligned}$$

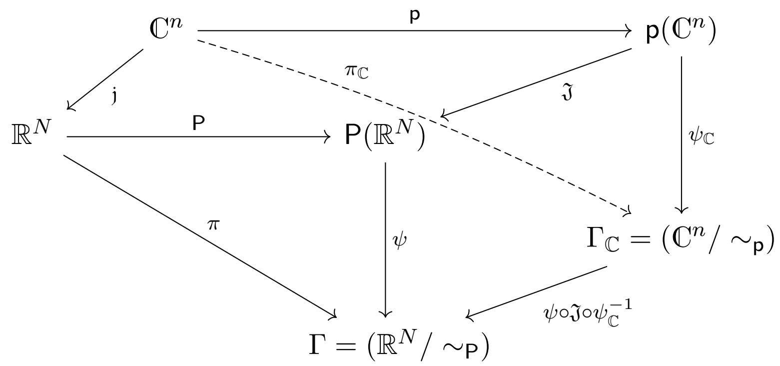


Figure 4.3.: Real- and complex-valued quotient space

i.e. we can conclude that

$$\begin{aligned} x \sim_\mathsf{p} y \Leftrightarrow x = \mathrm{e}^{i\Delta\phi}\, y, &\quad \text{for some } \Delta\phi \in [0, 2\pi) \\ \Leftrightarrow x = z\, y, &\quad \text{for some } z \in S^1. \end{aligned}$$

Quotient space: From the characterization of the equivalence relation it follows that

$$\Gamma_\mathbb{C} \coloneqq \mathbb{C}^n / \sim_\mathsf{p} = \mathbb{C}^n / S^1. \tag{4.102}$$

Level sets: All level sets are of the form

$$\mathsf{p}^{-1}(xx^\dagger) = \left\{ x\, z \mid z \in S^1 \right\}, \quad \forall x \in \mathbb{C}^n. \tag{4.103}$$

These level sets are clearly both compact and connected.
Isomorphism: This follows by the same arguments as given in the proof of Lemma 4.34. □

In the commutative diagram of Fig. 4.3 we have summarized the relations between real- and complex valued descriptions of the quotient space under consideration. Points in $\mathbb{C}^n$ and $\mathbb{R}^N = \mathbb{R}^{2n}$ respectively are identified by means of j, while matrices in $\mathsf{p}(\mathbb{C}^n)$ and $\mathsf{P}(\mathbb{R}^N)$ are related by the map $\mathfrak{J}$, c.f. Definition 2.1. The quotient spaces $\Gamma_\mathbb{C}$ and Γ can therefore be identified by employing the isomorphisms $\psi_\mathbb{C}$ and ψ, i.e. we find that $\Gamma = \psi \circ \mathfrak{J} \circ \psi_\mathbb{C}^{-1}(\Gamma_\mathbb{C})$.

4.4. Decomposition of quotient space and coarea formula

The aim of this section is to decompose an integral over $\mathbb{R}^N$ into integrals over level sets of P, c.f. Lemma 4.48. In particular we will identify certain sets of Lebesgue measure zero, which can be neglected in the context of such integrations, c.f. Lemma 4.37.

4.4.1. Decomposition of quotient space

We analyze the quotient space Γ by decomposing[15] it according to the *rank* of the Jacobian $\mathrm{D}\,\mathsf{P}$. As will be shown in Lemma 4.38, this decomposition yields the following connected submanifolds of $\mathbb{R}^{N,N}$.

Definition 4.36 (Decomposition of the quotient space Γ)
We decompose Γ into subsets of the form

$$I_\Gamma^l := \left\{[X] \mid X \in \mathbb{R}^N, \text{ s.t. } \left|\left\{i \in \{0, ..., n-1\} \mid (\mathfrak{j}^{-1}(X))_i = 0\right\}\right| = l\right\}, \tag{4.104}$$

where $l \in \{0, ..., n\}$, i.e. I_Γ^l collects all equivalence classes of points whose complex representation has l components which are equal to zero. Since any element of $\mathbb{C}^n$ clearly has a number of zero components lying in $\{0, ..., n\}$, the elements in Eq. (4.104) form a partition of the quotient space, i.e. $\Gamma = \bigcup_{l=0}^n I_\Gamma^l$. The isomorphism ψ carries this decomposition over to a partition of $\mathsf{P}(\mathbb{R}^N)$, given by subsets of the form

$$I_\mathsf{P}^l := \psi^{-1}(I_\Gamma^l), \quad l \in \{0, ..., n\}. \tag{4.105}$$

The corresponding inverse images

$$R^l := \pi^{-1}(I_\Gamma^l) = \mathsf{P}^{-1}(I_\mathsf{P}^l), \quad l \in \{0, ..., n\}, \tag{4.106}$$

yield a decomposition of $\mathbb{R}^N = \bigcup_{l=0}^n R^l$.

Note that for $l = n$ we only capture the origin, i.e. $I_\Gamma^n = \{[0]\}$, while for $l = 0$ we find that

$$I_\Gamma^0 = \left\{[X] \mid X \in \mathbb{R}^N, \text{ s.t. } (\mathfrak{j}^{-1}(X))_i \neq 0, \forall i \in \{0, ..., n-1\}\right\} \tag{4.107}$$

collects all non-degenerate points.

Lemma 4.37 (Lebesgue measure of decomposing sets)
Note that

$$\mu\left(\bigcup_{l=1}^n R^l\right) = 0, \tag{4.108}$$

i.e. μ-almost all points $X \in \mathbb{R}^N$ lie in R^0.

[15] We apply the strategy outlined in [BR14], Section 5.4.1.

Proof. We observe that[16]

$$\begin{aligned}\mu_{\mathbb{R}^N}\left(\bigcup_{l=1}^{n} R^l\right) &= \mu_{\mathbb{R}^N}\left(\left\{X\in\mathbb{R}^N \mid \left|\left\{i\in\{0,...,n-1\} \mid (\mathfrak{j}^{-1}(X))_i=0\right\}\right|\geq 1\right\}\right)\\ &= \mu_{\mathbb{C}^n}\left(\{x\in\mathbb{C}^n \mid |\{i\in\{0,...,n-1\} \mid x_i=0\}|\geq 1\}\right)\\ &= 0.\end{aligned}$$

□

Lemma 4.38 (Rank of Jacobian matrix)
For $l\in\{0,...,n-1\}$, the rank of $\mathrm{D}\,\mathsf{P}$ restricted to R^l is equal to $N-2l-1=2(n-l)-1$.

Proof. We first consider the case of $l=0$ and show that the rank of $\mathrm{D}\,\mathsf{P}$ on R^0 is constant and given by $N-1$. Let $X\in R^0$, set $x:=\mathfrak{j}^{-1}(X)\in\mathbb{C}^n$ and choose $X^{(1)},...,X^{(N-2)}\in\mathbb{R}^N$, s.t.

$$\left\{X, IX, X^{(1)},...,X^{(N-2)}\right\}$$

constitutes a basis of $\mathbb{R}^N$, where $I:=\mathfrak{J}(i)$, as in Eq. (2.15). Such a choice of basis is possible since $X\neq 0$, which ensures that X and IX are $\mathbb{R}$-linearly independent.

Setting $x^{(k)}:=\mathfrak{j}^{-1}(X^{(k)})\in\mathbb{C}^n$, for all $k=1,...,N-2$, we notice that according to Eq. (4.87) of Lemma 4.31 we have

$$\begin{aligned}&\left\{\mathfrak{J}^{-1}\left[\mathrm{D}\,\mathsf{P}|_X\ X\right],\mathfrak{J}^{-1}\left[\mathrm{D}\,\mathsf{P}|_X\ X^{(1)}\right],...,\mathfrak{J}^{-1}\left[\mathrm{D}\,\mathsf{P}|_X\ X^{(N-2)}\right]\right\}\\ &\qquad=\left\{xx^\dagger+\text{h.c.}, x^{(1)}x^\dagger+\text{h.c.},...,x^{(N-2)}x^\dagger+\text{h.c.}\right\},\end{aligned}$$

while $\mathfrak{J}^{-1}\left[\mathrm{D}\,\mathsf{P}|_X\ IX\right]=ixx^\dagger+\text{h.c.}=0$. Thus it suffices to show the $\mathbb{R}$-linear independence of these elements. Let $\lambda,\lambda_1,...,\lambda_{N-2}\in\mathbb{R}$ be s.t.

$$\left(\lambda\, xx^\dagger+\sum_{k=1}^{N-2}\lambda_k\, x^{(k)}x^\dagger\right)+\text{h.c.}=0. \tag{4.109}$$

Setting $y:=\lambda\, x+\sum_{k=1}^{N-2}\lambda_k\, x^{(k)}$, we observe that

$$\begin{aligned} & xy^\dagger+yx^\dagger=0 && \text{(4.110)}\\ \Leftrightarrow\quad & x_i\overline{y}_j+\overline{x}_j y_i=0, \qquad \forall i,j\in\{0,...,n-1\}\\ \Leftrightarrow\quad & y_i=-\frac{x_i}{\overline{x}_j}\,\overline{y}_j, \qquad \forall i,j\in\{0,...,n-1\}, && \text{(4.111)}\end{aligned}$$

where we have used that $x_i\neq 0$, for all $i\in\{0,...,n-1\}$, since $X\in R^0$. Solving Eq. (4.111) for $\overline{y}_j$ and exchanging the index i for an index k we get

$$\overline{y}_j=-\frac{\overline{x}_j}{x_k}\,y_k,\quad \forall j,k\in\{0,...,n-1\}. \tag{4.112}$$

[16] Recall that μ denotes the Lebesgue measure on $\mathbb{R}^N$ or $\mathbb{C}^n$ respectively. In this proof we will explicitly distinguish between both cases by means of a subscript.

Combining Eqs. (4.111) and (4.112), we find that

$$y_i = \frac{y_k}{x_k}\, x_i, \quad \forall i, k \in \{0, ..., n-1\}\,. \tag{4.113}$$

Since $x_i \neq 0$, the ratio $\frac{y_k}{x_k}$ is equal to a constant c which is independent of l:

$$\frac{y_k}{x_k} = c, \quad \forall k \in \{0, ..., n-1\}\,, \tag{4.114}$$

i.e. we can infer that $y = c\,x$. Recalling Eq. (4.110) we find

$$0 = yx^\dagger + xy^\dagger = (c + \bar{c})\, xx^\dagger, \tag{4.115}$$

which yields $c = i\,|c|$, i.e. $y = i\,|c|\,x$. Since we have defined y as an expansion in terms of all the basis elements *except* ix, we can conclude that $y = 0$ and thus $\lambda, \lambda_1, ..., \lambda_{N-2} = 0$. The cases of $l > 0$ follow by the same line of argument. □

4.4.2. Coarea formula

We follow [BR14], Section 5.4.2 in order to derive a *coarea formula* for integration w.r.t. to level sets of the complex outer product. This formula will play a central role in identifying the averaged drift- and diffusion terms later on. For the coarea formula we will require the notion of a *k-dimensional Jacobian* which in turn relies on the concept of k-dimensional *parallelepipeds.*

Definition 4.39 (Parallelepiped)
For $k \in \{1, ..., N-1\}$ and linearly independent vectors $X^{(1)}, ..., X^{(k)} \in \mathbb{R}^N$, we define the *k-dimensional parallelepiped* spanned by these vectors as the set

$$\varnothing := \left\langle X^{(1)}, ..., X^{(k)} \right\rangle := \left\{ \lambda_1 X^{(1)} + ... + \lambda_k X^{(k)} \mid \lambda_i \in [0,1],\, \forall i \in \{1, ..., k\} \right\} \tag{4.116}$$

and represent it by the matrix

$$\widehat{\varnothing} := \left(X^{(1)} \mid ... \mid X^{(k)} \right) \in \mathbb{R}^{N,k}. \tag{4.117}$$

The volume of a parallelepiped can be neatly expressed in terms of its representing matrix.

Lemma 4.40 (Volume of parallelepiped)
Let $k \in \{1, ..., N-1\}$, $\varnothing$ be a k-dimensional parallelepiped and $\widehat{\varnothing}$ its representing matrix. Then the k-dimensional volume of $\varnothing$ is given by

$$\mathcal{H}_k(\varnothing) = \sqrt{\det(\widehat{\varnothing}\widehat{\varnothing}^\top)}, \tag{4.118}$$

where $\mathcal{H}_k$ denotes the k-dimensional *Hausdorff measure.*[a]

[a] c.f. [BR14], Section 5.4.2

Proof. This follows for instance from [Pen07], Theorem 7. □

Now we are in a position to define a k-dimensional Jacobian, which unlike the Jacobian matrix is scalar valued, i.e. given by a real-valued number.

Definition 4.41 (k-dimensional Jacobian)
For $k \in \{1, ..., N-1\}$, the *k-dimensional Jacobian* $\mathrm{D}_k\,\mathsf{P}$ of the map P in a point $X \in \mathbb{R}^N$ is defined as[a]

$$\mathrm{D}_k\,\mathsf{P}(X) := \sup\left\{\frac{\mathcal{H}_k(\mathrm{D}\,\mathsf{P}|_X\,\varnothing)}{\mathcal{H}_k(\varnothing)} \mid \varnothing \text{ is a } k\text{-dimensional parallelepiped in } \mathbb{R}^N\right\}. \tag{4.119}$$

[a] c.f. [BR14], Eq. (148)

We state the general form of the coarea formula, as given in [BR14].

Lemma 4.42 (Coarea formula)
For $l \in \{0, ..., n-1\}$ and an integrable function $f : \mathbb{R}^N \to \mathbb{R}$, we have the integral formula

$$\int_{R^l} f(X)\,\mathrm{d}X = \int_{I^l_\Gamma}\left[\int_{\pi^{-1}(\gamma)} \frac{f(Y)}{\mathrm{D}_{N-2l-1}\,\mathsf{P}(Y)}\,\mathrm{d}\mathcal{H}_{2l+1}(Y)\right]\mathrm{d}\mathcal{H}_{N-2l-1}(\gamma). \tag{4.120}$$

Proof. This formula follows from Barret 2014, Section 5.4.2, Eq. (149), if we recall that by Lemma 4.38 the rank of $\mathrm{D}\,\mathsf{P}$ restricted to R^l is equal to $N-2l-1$.

Next we exploit the specific structure of P. We show that the k-dimensional Jacobian is constant on level sets of P (Lemma 4.45), which will allow us to simplify the coarea formula (Lemma 4.48). In a first step, we introduce the notion of a complex rotation of a parallelepiped, which is defined by means of multiplying all spanning vectors with a global phase-factor.

Definition 4.43 (Complex rotation of parallelepiped)
Let $\varphi \in [0, 2\pi)$. For any $k \in \{1, ..., N-1\}$ and any k-dimensional parallelepiped

$$\varnothing = \left\langle X^{(1)}, ..., X^{(k)}\right\rangle, \tag{4.121}$$

spanned by $\mathbb{R}$-linearly independent vectors $X^{(1)}, ..., X^{(k)} \in \mathbb{R}^N$, with complex representations $x^{(i)} := \mathrm{j}^{-1}(X^{(i)})$, we define the 'rotated' parallelepiped $\varnothing_\varphi$ as

$$\varnothing_\varphi := \left\langle \mathrm{j}(\mathrm{e}^{i\varphi}\,x^{(1)}), ..., \mathrm{j}(\mathrm{e}^{i\varphi}\,x^{(k)})\right\rangle = \left\langle \mathfrak{J}(\mathrm{e}^{i\varphi}\,\mathbb{1}_{n\times n})X^{(1)}, ..., \mathfrak{J}(\mathrm{e}^{i\varphi}\,\mathbb{1}_{n\times n})X^{(k)}\right\rangle. \tag{4.122}$$

Denoting by $\widehat{\varnothing}, \widehat{\varnothing}_\varphi$ the representing matrices of the parallelepipeds $\varnothing, \varnothing_\varphi$, we find that

$$\widehat{\varnothing}_\varphi = \left(\mathfrak{J}(\mathrm{e}^{i\varphi}\,\mathbb{1}_{n\times n})X^{(1)} \mid ... \mid \mathfrak{J}(\mathrm{e}^{i\varphi}\,\mathbb{1}_{n\times n})X^{(k)}\right) = \mathfrak{J}(\mathrm{e}^{i\varphi}\,\mathbb{1}_{n\times n})\,\widehat{\varnothing}. \tag{4.123}$$

We show that such a rotation does not affect the volume of the parallelepiped.

Lemma 4.44 (Volume of parallelepiped invariant under complex rotation)
Let $\varphi \in [0, 2\pi)$, $k \in \{1, ..., N-1\}$ and $\varnothing$ be a k-dimensional parallelepiped. Then we have

$$\mathcal{H}_k(\varnothing_\varphi) = \mathcal{H}_k(\varnothing). \tag{4.124}$$

Proof. Application of Lemma 2.7 yields

$$\mathfrak{J}(\mathrm{e}^{i\varphi}\, \mathbb{1}_{n\times n})^\top \mathfrak{J}(\mathrm{e}^{i\varphi}\, \mathbb{1}_{n\times n}) = \mathfrak{J}(\mathrm{e}^{-i\varphi}\, \mathbb{1}_{n\times n})\mathfrak{J}(\mathrm{e}^{i\varphi}\, \mathbb{1}_{n\times n}) = \mathfrak{J}(\mathbb{1}_{n\times n}) = \mathbb{1}_{N\times N}. \tag{4.125}$$

Denoting by $\widehat{\varnothing}, \widehat{\varnothing}_\varphi$ the representing matrices of the parallelepipeds $\varnothing, \varnothing_\varphi$, we conclude that

$$\mathcal{H}_k(\varnothing_\varphi) = \sqrt{\det(\widehat{\varnothing}_\varphi^\top \widehat{\varnothing}_\varphi)} = \sqrt{\det\left(\widehat{\varnothing}^\top \mathfrak{J}(\mathrm{e}^{i\varphi})^\top \mathfrak{J}(\mathrm{e}^{i\varphi})\widehat{\varnothing}\right)} = \sqrt{\det(\widehat{\varnothing}^\top \widehat{\varnothing})} = \mathcal{H}_k(\varnothing), \tag{4.126}$$

where in the first and last step we have used the volume formula from Lemma 4.40. □

The previous lemma allows us to prove that the value of a k-dimensional Jacobian is indeed constant on level sets of the complex outer product.

Lemma 4.45 (k-dimensional Jacobian constant on P level sets)
For all $k \in \{1, ..., N-1\}$, the k-dimensional Jacobian $\mathrm{D}_k\,\mathsf{P}$ is constant on P level sets, i.e. for all $X, Y \in \mathbb{R}^N$ we have

$$X \sim_{\mathsf{P}} Y \quad \Rightarrow \quad \mathrm{D}_k\,\mathsf{P}(X) = \mathrm{D}_k\,\mathsf{P}(Y). \tag{4.127}$$

Proof. Let $X, Y \in \mathbb{R}^N$, s.t. $X \sim_{\mathsf{P}} Y$, i.e. setting $x := \mathsf{j}^{-1}(X)$ and $y := \mathsf{j}^{-1}(Y)$ we know that (recall Lemma 4.35)

$$x = \mathrm{e}^{-i\,\Delta\phi}\, y, \text{ for some } \Delta\phi \in [0, 2\pi). \tag{4.128}$$

Note that if

$$\varnothing = \left\langle X^{(1)}, ..., X^{(k)} \right\rangle, \tag{4.129}$$

is a k-dimensional parallelepiped, spanned by the $\mathbb{R}$-linearly independent vectors $X^{(1)}, ..., X^{(k)}$, with complex representation $x^{(i)} := \mathsf{j}^{-1}(X^{(i)})$ for all $i = 1, ..., k$, then by Lemma 4.31 it follows that

$$\begin{aligned} \mathrm{D}\,\mathsf{P}|_X \varnothing &= \left\langle \mathfrak{J}(x^{(1)}x^\dagger + \text{h.c.}), ..., \mathfrak{J}(x^{(k)}x^\dagger + \text{h.c.}) \right\rangle \\ &= \left\langle \mathfrak{J}(\mathrm{e}^{i\,\Delta\phi}\, x^{(1)}y^\dagger + \text{h.c.}), ..., \mathfrak{J}(\mathrm{e}^{i\,\Delta\phi}\, x^{(k)}y^\dagger + \text{h.c.}) \right\rangle \\ &= \mathrm{D}\,\mathsf{P}|_Y \varnothing_{\Delta\phi}. \end{aligned} \tag{4.130}$$

Note that any k-dim. parallelepiped $\varparallelogram$ can be represented as $\varparallelogram = \varparallelogram'_{\Delta\phi}$ with $\varparallelogram' := \varparallelogram_{-\Delta\phi}$. Therefore we find that

$$\left\{\varparallelogram \mid \varparallelogram \ k\text{-dim. parallelepiped in } \mathbb{R}^N\right\} = \left\{\varparallelogram'_{\Delta\phi} \mid \varparallelogram' \ k\text{-dim. parallelepiped in } \mathbb{R}^N\right\}. \tag{4.131}$$

Equations (4.130), (4.131) and Lemma 4.44 now yield the aspired result:

$$\begin{aligned} \mathrm{D}_k \mathsf{P}(X) &:= \sup\left\{\frac{\mathcal{H}_k(\mathrm{D}\,\mathsf{P}|_X \varparallelogram)}{\mathcal{H}_k(\varparallelogram)} \mid \varparallelogram \ k\text{-dimensional parallelepiped in } \mathbb{R}^N\right\} \\ &= \sup\left\{\frac{\mathcal{H}_k(\mathrm{D}\,\mathsf{P}|_Y \varparallelogram_{\Delta\phi})}{\mathcal{H}_k(\varparallelogram)} \mid \varparallelogram \ k\text{-dimensional parallelepiped in } \mathbb{R}^N\right\} \\ &= \sup\left\{\frac{\mathcal{H}_k(\mathrm{D}\,\mathsf{P}|_Y \varparallelogram')}{\mathcal{H}_k(\widehat{\varparallelogram'}_{-\Delta\phi})} \mid \varparallelogram' \ k\text{-dimensional parallelepiped in } \mathbb{R}^N\right\} \\ &= \sup\left\{\frac{\mathcal{H}_k(\mathrm{D}\,\mathsf{P}|_Y \varparallelogram')}{\mathcal{H}_k(\varparallelogram')} \mid \varparallelogram' \ k\text{-dimensional parallelepiped in } \mathbb{R}^N\right\} \\ &= \mathrm{D}_k \mathsf{P}(Y). \end{aligned}$$

□

The previous result enables us to write the k-dimensional Jacobian as the lift of a function $\mathrm{D}_k^\Gamma \mathsf{P}$ which is defined on Γ.

Definition 4.46 (k-dimensional Jacobian on Γ)
For $k \in \{1, ..., N-1\}$, we define

$$\mathrm{D}_k^\Gamma \mathsf{P} : \Gamma \to \mathbb{R}, \quad \gamma \to \mathrm{D}_k^\Gamma \mathsf{P}(\gamma) := \mathrm{D}_k \mathsf{P}(X), \qquad \text{where } X \in \pi^{-1}(\gamma). \tag{4.132}$$

According to Lemma 4.45, the right-hand side does not depend on the choice of $X \in \pi^{-1}(\gamma)$, i.e. $\mathrm{D}_k^\Gamma \mathsf{P}$ is well-defined. The k-dimensional Jacobian $\mathrm{D}_k^\Gamma \mathsf{P}$ allows us to identify the density of the image measure

$$\pi_*\mu := \mu\pi^{-1}, \tag{4.133}$$

i.e. the density of the Lebesgue measure's push forward under the projection mapping π.

Lemma 4.47 (Pushforward measure on Γ)
The image measure $\pi_*\mu$ is given by

$$\mathrm{d}\pi_*\mu(\gamma) = \frac{\mathcal{H}_1\left(\pi^{-1}(\gamma)\right)}{\mathrm{D}_{N-1}^\Gamma \mathsf{P}(\gamma)}\,\mathrm{d}\mathcal{H}_{N-1}(\gamma), \quad \forall \gamma \in \Gamma, \tag{4.134}$$

i.e. for all measurable $A \subset \Gamma$, we have

$$\pi_*\mu(A) = \int_A \frac{\mathcal{H}_1\left(\pi^{-1}(\gamma)\right)}{\mathrm{D}_{N-1}^\Gamma \mathsf{P}(\gamma)}\,\mathrm{d}\mathcal{H}_{N-1}(\gamma). \tag{4.135}$$

Proof. Let $A \subset \Gamma$ be measurable. Lemma 4.37 allows us to reduce any $\mathbb{R}^N$ integral to an integral over R^0, i.e.

$$\pi_*\mu(A) = \mu\pi^{-1}(A) = \int_{\mathbb{R}^N} \mathbb{1}_A \circ \pi(X)\,\mathrm{d}X = \int_{R^0} \mathbb{1}_A \circ \pi(X)\,\mathrm{d}X. \tag{4.136}$$

Since $R^0 = \pi^{-1}(I_\Gamma^0)$, we conclude that

$$\pi_*\mu(A) = \pi_*\mu(A \cap I_\Gamma^0). \tag{4.137}$$

This observation allows us to apply the coarea formula Lemma 4.42 for $l = 0$, i.e.

$$\begin{aligned}
\pi_*\mu(A) &= \int_{I_\Gamma^0} \left[\int_{\pi^{-1}(\gamma)} \frac{\mathbb{1}_A \circ \pi(Y)}{\mathrm{D}_{N-1}\,\mathsf{P}(Y)}\,\mathrm{d}\mathcal{H}_1(Y)\right] \mathrm{d}\mathcal{H}_{N-1}(\gamma) \\
&= \int_{I_\Gamma^0} \left[\int_{\pi^{-1}(\gamma)} \mathrm{d}\mathcal{H}_1(Y)\right] \frac{\mathbb{1}_A(\gamma)}{\mathrm{D}^\Gamma_{N-1}\,\mathsf{P}(\gamma)}\,\mathrm{d}\mathcal{H}_{N-1}(\gamma) \\
&= \int_{I_\Gamma^0 \cap A} \frac{\mathcal{H}_1\left(\pi^{-1}(\gamma)\right)}{\mathrm{D}^\Gamma_{N-1}\,\mathsf{P}(\gamma)}\,\mathrm{d}\mathcal{H}_{N-1}(\gamma),
\end{aligned}$$

where in the second to last step we have made use of Lemma 4.45 and Definition 4.46. Now the result follows by recalling Eq. (4.137). □

Finally, we can now decompose Lebesgue integrals over $\mathbb{R}^N$ into 'averaging' integrations over each fixed, non-degenerate level set γ of π and an integration over all these level sets $\gamma \in I_\Gamma^0$, weighted by the pushforward measure.

Lemma 4.48 (Outer product version of coarea formula)
For any integrable function $f : \mathbb{R}^N \to \mathbb{R}$, the following integral formula holds:

$$\int_{\mathbb{R}^N} f(X)\,\mathrm{d}X = \int_{I_\Gamma^0} \left[\frac{1}{\mathcal{H}_1\left(\pi^{-1}(\gamma)\right)} \int_{\pi^{-1}(\gamma)} f(Y)\,\mathrm{d}\mathcal{H}_1(Y)\right] \mathrm{d}\pi_*\mu(\gamma). \tag{4.138}$$

Proof. Lemma 4.37 allows to apply the coarea formula of Lemma 4.42 with $l = 0$, which yields

$$\begin{aligned}
\int_{\mathbb{R}^N} f(X)\,\mathrm{d}X &= \int_{I_\Gamma^0} \left[\int_{\pi^{-1}(\gamma)} \frac{f(Y)}{\mathrm{D}_{N-1}\,\mathsf{P}(Y)}\,\mathrm{d}\mathcal{H}_1(Y)\right] \mathrm{d}\mathcal{H}_{N-1}(\gamma) \\
&= \int_{I_\Gamma^0} \left[\frac{1}{\mathcal{H}_1\left(\pi^{-1}(\gamma)\right)} \int_{\pi^{-1}(\gamma)} f(Y)\,\mathrm{d}\mathcal{H}_1(Y)\right] \frac{\mathcal{H}_1\left(\pi^{-1}(\gamma)\right)}{\mathrm{D}^\Gamma_{N-1}\,\mathsf{P}(\gamma)}\,\mathrm{d}\mathcal{H}_{N-1}(\gamma) \\
&= \int_{I_\Gamma^0} \left[\frac{1}{\mathcal{H}_1\left(\pi^{-1}(\gamma)\right)} \int_{\pi^{-1}(\gamma)} f(Y)\,\mathrm{d}\mathcal{H}_1(Y)\right] \mathrm{d}\pi_*\mu(\gamma),
\end{aligned}$$

where in the second step we have employed Lemma 4.45 and Definition 4.46 in order to pull the $(N-1)$-dimensional Jacobian out of the inner integral. In the last step, application of Lemma 4.47 has allowed us to identify the pushforward measure $\pi_*\mu$. □

4.5. Projected Dirichlet form

In the previous two sections we have defined and analyzed the quotient space induced by the projection π arising from the complex outer-product map. In particular, we have seen in Eq. (4.94) that this projection induces the pullback mapping

$$\pi_* : \mathbb{R}^\Gamma \to \mathbb{R}^{\mathbb{R}^N}, \quad f \to \underline{f} := f \circ \pi, \tag{4.139}$$

where $\underline{f}$ is called the *lift* of f. We note that lifted functions are invariant under the evolution induced by V.

Lemma 4.49 (Lifted functions are constant w.r.t. fast evolution)
We observe that[a]

$$V^\top \nabla \underline{f} = 0, \quad \forall f \in C^1(\Gamma). \tag{4.140}$$

[a]c.f. [BR14], Lemma 6

Proof. The proof of [BR14], Lemma 6 directly applies to our case as well.

Employing the pullback π_*, we can define an L^2-Hilbert space on Γ.

Definition 4.50 (Pullback spaces)
We set[a]

$$L^2_{\pi_*\mu}(\Gamma) := \pi_*^{-1}(L^2(\mathbb{R}^N)), \tag{4.141}$$

with a scalar product given by

$$\langle f, g \rangle_{L^2_{\pi_*\mu}(\Gamma)} := \left\langle \underline{f}, \underline{g} \right\rangle_{L^2(\mathbb{R}^N)}. \tag{4.142}$$

[a]c.f. [BR14], Section 3.1, Eqs. (45) and (46)

Similarly, we can define a pullback of the Dirichlet form[17] E_ε, which turns out to be independent of the scaling parameter ε.

Lemma 4.51 (Projected bilinear form on Γ)
Employing the pullback mapping π_*, we define the domain[a]

$$\mathcal{D}(\mathcal{E}) := \pi_*^{-1}\left(\mathcal{D}(E_\varepsilon)\right), \tag{4.143}$$

where we note that by Theorem 4.25, this definition is independent of the choice of $\varepsilon > 0$.

[17]As noted in the introduction of Section 4.2, we restrict the presentation to the case of a linear drift term in this section. For this reason, E_ε constitutes a Dirichlet form, while in the presence of a nonlinear perturbation one would need to employ the framework of *generalized* Dirichlet forms.

The restriction of E_ε to $\mathcal{D}(\mathcal{E})$ does not depend on ε and will be denoted by $\mathcal{E}$, i.e.

$$\begin{aligned}\mathcal{E}(f,g) &:= E_\varepsilon(\underline{f},\underline{g})\\ &= -\int_{\mathbb{R}^N}\left[U^\top\nabla\underline{f} + \frac{1}{2}A:\nabla\nabla^\top\underline{f}\right]\underline{g}\,\mathrm{d}\mu\\ &= \frac{1}{2}\int_{\mathbb{R}^N}\left[(\nabla\underline{f})^\top A\,\nabla\underline{g} + (\nabla^\top\Phi)\,\underline{f}\,\underline{g} + \Phi^\top(\underline{f}\,\nabla\underline{g} - \underline{g}\,\nabla\underline{f})\right]\mathrm{d}\mu. \end{aligned} \tag{4.144}$$

[a] This definition corresponds to the one in c.f. [BR14], Eq. (47), where the special case of $\mathcal{D}(E_\varepsilon) = H^1(\mathbb{R}^N)$ is employed.

Proof. This representation is similar to the one given in Eq. (146) of [BR14] and directly follows from Lemmas 4.17 and 4.49. □

We will refer to $\mathcal{E}$ as the *projected Dirichlet form*. In Proposition 4.53 we will prove that this bilinear form indeed defines a Dirichlet form.

Lemma 4.52 (Pullback Hilbert spaces[a])
$(L^2_{\pi_*\mu}(\Gamma), \langle\cdot,\cdot\rangle_{L^2_{\pi_*\mu}(\Gamma)})$ and $(\mathcal{D}(\mathcal{E}), \tilde{\mathcal{E}}^1(\cdot,\cdot))$ are *Hilbert spaces* and $\mathcal{D}(\mathcal{E})$ is a *dense* linear subspace of $L^2_{\pi_*\mu}(\Gamma)$.

[a] This Lemma is an adaptation of [BR14], Lemma 4.

Proof. The results follow completely analogous to the proof of [BR14], Lemma 4, making use of $\mathcal{D}(E_\varepsilon)$ being a dense subspace of $L^2(\mathbb{R}^N)$. □

In Remark 4.26 we have noted that some of the results in [BR14] are relying on the fact that the domain $\mathcal{D}(E_\varepsilon)$ coincides with $H^1(\mathbb{R}^N)$, which does *not* hold in our case. Similarly, we can observe that $\mathcal{D}(\mathcal{E})$ does not coincide with

$$H^1_{\pi_*\mu}(\Gamma) := \pi_*^{-1}(H^1(\mathbb{R}^N)). \tag{4.145}$$

Lemma 4.52 enables us to prove that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ inherits the property of being a regular Dirichlet form.

Proposition 4.53 (Projected Dirichlet form on Γ)
$(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form on $L^2_{\pi_*\mu}(\Gamma)$, which is regular and satisfies the local property.[a]

[a] Recall that we have assumed a linear drift term in this section.

Proof. This statement corresponds to Proposition 7 of [BR14], which generalizes Theorem 1 of the same source. We follow the proof of Theorem 1, where we need to make some modifications however, since $\mathcal{D}(E_\varepsilon) \neq H^1(\mathbb{R}^N)$ in our case. We need to verify the conditions of Definitions 4.24 and 4.27:

Lemma 4.52 states that $\mathcal{D}(\mathcal{E}))$ is a *dense* linear subspace of $L^2_{\pi_*\mu}(\Gamma)$ and that $(\tilde{\mathcal{E}}, \mathcal{D}(\mathcal{E}))$ is *closed* on $L^2_{\pi_*\mu}(\Gamma)$.

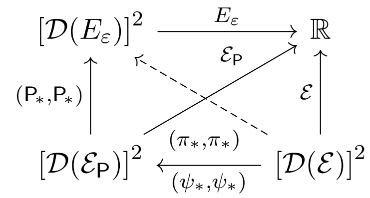


Figure 4.4.: Projected Dirichlet Form

The *positive definiteness* of $\tilde{\mathcal{E}}$ is directly inherited from $\tilde{E}_\varepsilon$; $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfies the *strong sector condition* as a consequence of $(E_\varepsilon, \mathcal{D}(E_\varepsilon))$ fulfilling it and the contraction properties (4.81) and the local property are inherited as well (c.f. Eqs. (61) and (62) in [BR14]).

The *regularity* of the Dirichlet form follows as in [BR14], Eq. (63), with $H^1(\mu)$ replaced by $\mathcal{D}(E_\varepsilon)$. □

In the next lemma, we employ the fact that Γ and $\mathsf{P}(\mathbb{R}^N)$ are isomorphic, which allows us to define a corresponding projected Dirichlet form $\mathcal{E}_\mathsf{P}$ on $\mathsf{P}(\mathbb{R}^N)$.

Lemma 4.54 (Projected Dirichlet form on $\mathsf{P}(\mathbb{R}^N)$)
The bilinear form

$$\mathcal{E}_\mathsf{P}(f, g) := \mathcal{E}(f \circ \psi^{-1}, g \circ \psi^{-1}) \tag{4.146}$$

with domain

$$\mathcal{D}(\mathcal{E}_\mathsf{P}) := \mathsf{P}^{-1}\left(\mathcal{D}(E_\varepsilon)\right), \tag{4.147}$$

defines a Dirichlet form on

$$L^2_{\mathsf{P}_*\mu}(\mathsf{P}(\mathbb{R}^N)) := \mathsf{P}_*^{-1}(L^2(\mathbb{R}^N)), \tag{4.148}$$

where

$$\mathsf{P}_* : \mathbb{R}^{\mathsf{P}(\mathbb{R}^N)} \to \mathbb{R}^{\mathbb{R}^N}, \quad f \to \underline{f} := f \circ \mathsf{P}, \tag{4.149}$$

is a pullback similar to π_*. $(\mathcal{E}_\mathsf{P}, \mathcal{D}(\mathcal{E}_\mathsf{P}))$ constitutes a regular Dirichlet form, which satisfies the local property.

Proof. Since the map $\psi : \mathsf{P}(\mathbb{R}^N) \to \Gamma$ is an ismorphism (c.f. Lemma 4.34), this follows directly from Proposition 4.53. □

The relation between $\mathcal{E}$ and $\mathcal{E}_\mathsf{P}$ is illustrated by the commutative diagram of Fig. 4.4.

4.5.1. Identification of Dirchlet form

We have seen that the projected bilinear form $\mathcal{E}$ of Lemma 4.51 is Dirichlet form. The aim of this section is to rewrite Eq. (4.144) in terms of quantities defined on the quotient space Γ.

Roughly speaking, we want to 'integrate out' the S^1 degree of freedom which is captured by the equivalence relation of Definition 4.33 and Lemma 4.35. This will be possible since $\mathcal{E}$ is defined in terms of lifted functions which are invariant under such a global phase transformation.

For $f \in C^1(\mathsf{p}(\mathbb{C}^n))$, we define $\partial_{\mathsf{p}} f \in \mathbb{C}^{n,n} \cong \mathbb{C}^{n^2}$ as the complex gradient of $f(\hat{p})$ w.r.t. $\hat{p}$.[18] Analogously, for $f \in C^1(\mathsf{P}(\mathbb{R}^N))$, we define the real-valued representation of the complex gradient as[19]

$$\nabla_{\mathsf{P}} f := \mathfrak{J}(\overline{\partial_{\mathsf{p}}}(f \circ \mathfrak{J})) \in \mathbb{R}^{N,N} \cong \mathbb{R}^{N^2}. \tag{4.150}$$

We note that all $\mathbb{R}^N$-gradients ∇ which appear in Eq. (4.144), are acting on *lifted* functions, which allows us to rewrite these derivatives in terms of ∇_{P}.

Lemma 4.55 (Gradient and Hessian of a lifted function)
For $f \in C^2(\Gamma)$, the gradient of the lifted function $\underline{f}$ can be written as

$$\nabla \underline{f}\Big|_X = (\mathrm{D}\,\mathsf{P})^\top\Big|_X \,\nabla_{\mathsf{P}}(f \circ \psi)|_{\mathsf{P}(X)}\,, \quad \forall X \in R^0, \tag{4.151}$$

and its Hessian is given by

$$\begin{aligned}(\nabla\nabla^\top \underline{f})_{k,l}\Big|_X &= \Big[(\mathrm{D}\,\mathsf{P})^\top\Big|_X \,\nabla_{\mathsf{P}}\nabla_{\mathsf{P}}^\top(f \circ \psi)\Big|_{\mathsf{P}(X)}\,(\mathrm{D}\,\mathsf{P})|_X\Big]_{k,l} \\ &\quad + \Big[(\nabla\nabla^\top)_{k,l}\mathsf{P}\Big|_X\Big]^\top \nabla_{\mathsf{P}}(f \circ \psi)|_{\mathsf{P}(X)}\,, \qquad \forall k,l \in \{0, \dots, N-1\}\,. \end{aligned} \tag{4.152}$$

For any matrix $A \in \mathbb{R}^{N,N}$, we find that

$$\begin{aligned} A : (\nabla\nabla^\top \underline{f})\Big|_X &= \Big[(A : \nabla\nabla^\top)\mathsf{P}\Big|_X\Big]^\top \nabla_{\mathsf{P}}(f \circ \psi)|_{\mathsf{P}(X)} \\ &\quad + \Big[(\mathrm{D}\,\mathsf{P})|_X\, A\, (\mathrm{D}\,\mathsf{P})^\top\Big|_X\Big] : \nabla_{\mathsf{P}}\nabla_{\mathsf{P}}^\top(f \circ \psi)\Big|_{\mathsf{P}(X)}\,, \quad \forall X \in R^0. \end{aligned} \tag{4.153}$$

Proof. Let $f \in C^2(\Gamma)$. For all $k \in \{0, \dots, N-1\}$ and $X \in R^0$, an application of Lemma 4.34 and the chain-rule yield

$$\begin{aligned} \nabla_k \underline{f}|_X &= \frac{\mathrm{d}(f \circ \pi)}{\mathrm{d}X_k}\Big|_X = \frac{\mathrm{d}((f \circ \psi) \circ \mathsf{P})}{\mathrm{d}X_k}\Big|_X \\ &= \sum_{k'=0}^{m-1} \frac{\mathrm{d}(f \circ \psi)}{\mathrm{d}P_{k'}}\Big|_{\mathsf{P}(X)} \frac{\mathrm{d}P_{k'}}{\mathrm{d}X_k}\Big|_X \\ &= \Big[(\mathrm{D}\,\mathsf{P})^\top\Big|_X \,\nabla_{\mathsf{P}}(f \circ \psi)|_{\mathsf{P}(X)}\Big]_k . \end{aligned}$$

[18] "Note that this derivative has value in the respective tangent space of the submanifold", c.f. [BR14], Section 5.4.3.

[19] Compared to Eq. (2.45) we have omitted the factor of two, in order to compensate for the degeneracy of ∇_{P} arising from the structure of the map $\mathfrak{J}$, c.f. Eq. (4.184) below.

Similarly, we find that for all $k, l \in \{0, ..., N-1\}$,

$$\begin{aligned}
\nabla_k \nabla_l \underline{f}\Big|_X &= \nabla_l \left(\sum_{k'=0}^{m-1} \frac{\mathrm{d}(f \circ \psi)}{\mathrm{d}P_{k'}}\bigg|_{\mathsf{P}(X)} \frac{\mathrm{d}P_{k'}}{\mathrm{d}X_k}\bigg|_X \right) \\
&= \sum_{k',l'=0}^{m-1} \frac{\mathrm{d}^2(f \circ \psi)}{\mathrm{d}P_{k'}\,\mathrm{d}P_{l'}}\bigg|_{\mathsf{P}(X)} \frac{\mathrm{d}P_{k'}}{\mathrm{d}X_k}\bigg|_X \frac{\mathrm{d}P_{l'}}{\mathrm{d}X_l}\bigg|_X + \sum_{k'=0}^{m-1} \frac{\mathrm{d}(f \circ \psi)}{\mathrm{d}P_{k'}}\bigg|_{\mathsf{P}(X)} \frac{\mathrm{d}^2 P_{k'}}{\mathrm{d}X_k\,\mathrm{d}X_l}\bigg|_X \\
&= \left[(\mathrm{D}\,\mathsf{P})^\top\Big|_X \nabla_{\mathsf{P}} \nabla_{\mathsf{P}}^\top (f \circ \psi)\Big|_{\mathsf{P}(X)} (\mathrm{D}\,\mathsf{P})|_X \right]_{k,l} + \left[(\nabla\nabla^\top)_{k,l} \mathsf{P}\Big|_X \right]^\top \nabla_{\mathsf{P}} (f \circ \psi)|_{\mathsf{P}(X)} .
\end{aligned}$$

For all $A \in \mathbb{R}^{N,N}$ we can therefore conclude that

$$\begin{aligned}
&A : \nabla\nabla \underline{f}\Big|_X \\
&= \sum_{k',l',k,l=0}^{m-1} \frac{\mathrm{d}P_{k'}}{\mathrm{d}X_k}\bigg|_X A_{k,l} \frac{\mathrm{d}P_{l'}}{\mathrm{d}X_l}\bigg|_X \frac{\mathrm{d}^2(f \circ \psi)}{\mathrm{d}P_{k'}\,\mathrm{d}P_{l'}}\bigg|_{\mathsf{P}(X)} + \sum_{k',k,l=0}^{m-1} A_{k,l} \frac{\mathrm{d}^2 P_{k'}}{\mathrm{d}X_k\,\mathrm{d}X_l}\bigg|_X \frac{\mathrm{d}(f \circ \psi)}{\mathrm{d}P_{k'}}\bigg|_{\mathsf{P}(X)} \\
&= \left[(\mathrm{D}\,\mathsf{P})|_X\, A\, (\mathrm{D}\,\mathsf{P})^\top\Big|_X \right] : \nabla_{\mathsf{P}} \nabla_{\mathsf{P}}^\top (f \circ \psi)\Big|_{\mathsf{P}(X)} + \left[(A : \nabla\nabla^\top)\mathsf{P}\Big|_X \right]^\top \nabla_{\mathsf{P}} (f \circ \psi)|_{\mathsf{P}(X)} . \qquad \square
\end{aligned}$$

We define an $\mathbb{R}^m$-valued drift term U_{P} and an $\mathbb{R}^{m,m}$-valued diffusion matrix A_{P}, capturing the evolution of the outer product P. Subsequently we introduce 'averaged' versions $\widehat{U}_{\mathsf{P}}$ and $\widehat{A}_\Gamma$ by integrating these terms over given P level sets. Note that we average the diffusion and not the dispersion matrix.

Definition 4.56 (Averaged drift term and diffusion matrix)
Let the drift term U_{P}, dispersion matrix Σ_{P} and diffusion matrix A_{P} be given by[a,b]

$$U_{\mathsf{P}} : \mathbb{R}^N \to \mathbb{R}^m, \quad Y \to U_{\mathsf{P}}(Y) := (\mathrm{D}\,\mathsf{P})|_Y\, U(Y) + \frac{1}{2} \left(A(Y) : \nabla\nabla^\top \right) \mathsf{P}\Big|_Y , \tag{4.154}$$

$$\Sigma_{\mathsf{P}} : \mathbb{R}^N \to \mathbb{R}^{m,N'}, \quad Y \to \Sigma_{\mathsf{P}}(Y) := (\mathrm{D}\,\mathsf{P})|_Y\, \Sigma(Y), \tag{4.155}$$

$$A_{\mathsf{P}} : \mathbb{R}^N \to \mathbb{R}^{m,m}, \quad Y \to A_{\mathsf{P}}(Y) := (\Sigma_{\mathsf{P}} \Sigma_{\mathsf{P}}^\top)(Y) = (\mathrm{D}\,\mathsf{P})|_Y\, A(Y)\, (\mathrm{D}\,\mathsf{P})^\top\Big|_Y . \tag{4.156}$$

We define an averaged drift term $\widehat{U}_{\mathsf{P}}$ and an averaged diffusion matrix $\widehat{A}_{\mathsf{P}}$ as

$$\widehat{U}_{\mathsf{P}} : \mathsf{P}(\mathbb{R}^N) \to \mathbb{R}^m, \quad \hat{P} \to \widehat{U}_{\mathsf{P}}(\hat{P}) := \frac{1}{\mathcal{H}_1\left(\mathsf{P}^{-1}(\hat{P})\right)} \int_{\mathsf{P}^{-1}(\hat{P})} U_{\mathsf{P}}(Y)\,\mathrm{d}\mathcal{H}_1(Y), \tag{4.157}$$

$$\widehat{A}_{\mathsf{P}} : \mathsf{P}(\mathbb{R}^N) \to \mathbb{R}^{m,m}, \quad \hat{P} \to \widehat{A}_{\mathsf{P}}(\hat{P}) := \frac{1}{\mathcal{H}_1\left(\mathsf{P}^{-1}(\hat{P})\right)} \int_{\mathsf{P}^{-1}(\hat{P})} A_{\mathsf{P}}(Y)\,\mathrm{d}\mathcal{H}_1(Y). \tag{4.158}$$

Application of the isomorphism ψ yields the corresponding terms defined on Γ, i.e.

$$\widehat{U}_\Gamma : \Gamma \to \mathbb{R}^m, \quad \gamma \to \widehat{U}_\Gamma(\gamma) := \frac{1}{\mathcal{H}_1(\pi^{-1}(\gamma))} \int_{\pi^{-1}(\gamma)} U_{\mathsf{P}}(Y)\, \mathrm{d}\mathcal{H}_1(Y), \tag{4.159}$$

$$\widehat{A}_\Gamma : \Gamma \to \mathbb{R}^{m,m}, \quad \gamma \to \widehat{A}_\Gamma(\gamma) := \frac{1}{\mathcal{H}_1(\pi^{-1}(\gamma))} \int_{\pi^{-1}(\gamma)} A_{\mathsf{P}}(Y)\, \mathrm{d}\mathcal{H}_1(Y). \tag{4.160}$$

[a] The following definitions roughly follow [BR14], Section 5.4.3.

[b] To understand the dimensions involved, recall that $N' := 5n$ captures the dimension of the real-valued Brownian motion $(\boldsymbol{B}(t))_{t\geq 0}$ under consideration (Definition 4.4), while for any $X \in \mathbb{R}^N$ we obtain an $N \times N$ matrix $\mathsf{P}(X)$, which we can identify with a vector in $\mathbb{R}^m$, where $m := N^2$ (Definition 4.30). In this regard, an $\mathbb{R}^m$-valued drift term U_{P} can also be viewed as $N \times N$-*matrix*-valued drift term.

Employing the coarea formula (Lemma 4.48) and the results on derivatives of lifted functions (Lemma 4.55), we are now in a position to represent the projected Dirichlet form $\mathcal{E}$ in terms of the averaged drift and diffusion.

Proposition 4.57 (Representation of projected Dirichlet form)
The projected Dirichlet form of Lemma 4.51 and Proposition 4.53 is, for $f, g \in \mathcal{D}(\mathcal{E})$, given by[a]

$$\mathcal{E}(f,g) = -\int_{I_\Gamma^0} \Big[\widehat{U}_\Gamma(\gamma)^\top \; \nabla_{\mathsf{P}}(f \circ \psi)|_{\psi^{-1}(\gamma)} + \frac{1}{2}\, \widehat{A}_\Gamma(\gamma) : \nabla_{\mathsf{P}}\nabla_{\mathsf{P}}^\top (f\circ\psi)\Big|_{\psi^{-1}(\gamma)} \Big] g(\gamma)\, \mathrm{d}\pi_*\mu(\gamma). \tag{4.161}$$

Analogously, for $f, g \in \mathcal{D}(\mathcal{E}_{\mathsf{P}})$, we find that

$$\mathcal{E}_{\mathsf{P}}(f,g) = -\int_{I_{\mathsf{P}}^0} \Big[\widehat{U}_{\mathsf{P}}(\hat{P})^\top \; \nabla_{\mathsf{P}} f|_{\hat{P}} + \frac{1}{2}\, \widehat{A}_{\mathsf{P}}(\hat{P}) : \nabla_{\mathsf{P}}\nabla_{\mathsf{P}}^\top f\Big|_{\hat{P}} \Big] g(\hat{P})\, \mathrm{d}\mathsf{P}_*\mu(\hat{P}). \tag{4.162}$$

[a] c.f. [BR14], Eq. (158), where a slightly different representation was chosen, which is more tuned towards distinguishing symmetric and antisymmetric parts of the Dirichlet form

Proof. Let $f, g \in \mathcal{D}(\mathcal{E})$. By Lemmas 4.37 and 4.51 we have

$$\mathcal{E}(f,g) := E_\varepsilon(\underline{f}, \underline{g}) = -\int_{R^0} \Big[(U^\top(X)\nabla \underline{f}(X)) + \frac{1}{2} A(X) : \nabla\nabla^\top \underline{f}(X) \Big] \underline{g}(X)\, \mathrm{d}\mu(X). \tag{4.163}$$

Employing Lemma 4.55 to rewrite the gradient and Hessian of $\underline{f}$, we find

$$\begin{aligned} \mathcal{E}(f,g) = &-\int_{R^0} \Big[U^\top(X)\, (\mathrm{D}\,\mathsf{P})^\top\Big|_X \; \nabla_{\mathsf{P}}(f\circ\psi)|_{\mathsf{P}(X)} \Big]\, \underline{g}(X)\; \mathrm{d}X \\ &- \int_{R^0} \Big[\frac{1}{2} \Big(A(X) : \nabla\nabla^\top \Big) \mathsf{P}\Big|_X \Big]^\top \nabla_{\mathsf{P}}(f\circ\psi)|_{\mathsf{P}(X)}\, \underline{g}(X)\; \mathrm{d}X \\ &- \frac{1}{2} \int_{R^0} \Big[(\mathrm{D}\,\mathsf{P})|_X\, A(X)\, (\mathrm{D}\,\mathsf{P})^\top\Big|_X \Big] : \nabla_{\mathsf{P}}\nabla_{\mathsf{P}}^\top (f\circ\psi)\Big|_{\mathsf{P}(X)} \underline{g}(X)\; \mathrm{d}X. \end{aligned}$$

Applying the coarea formula given in Lemma 4.48 and 'pulling out' all factors which are constant on P level sets, we obtain the result

$$\mathcal{E}(f,g) = -\int_{I_\Gamma^0} \left[\widehat{U}_\Gamma(\gamma)^\top \; \nabla_{\mathsf{P}}(f\circ\psi)|_{\psi^{-1}(\gamma)} + \frac{1}{2}\widehat{A}_\Gamma(\gamma) : \nabla_{\mathsf{P}}\nabla_{\mathsf{P}}^\top(f\circ\psi)\Big|_{\psi^{-1}(\gamma)} \right] g(\gamma)\,\mathrm{d}\pi_*\mu(\gamma),$$

where $\widehat{U}_\Gamma$ and $\widehat{A}_\Gamma$ were given in Definition 4.56. Application of the isomorphism ψ yields Eq. (4.162). □

4.6. Identification of process associated to projected Dirichlet form

We identify the infinitesimal generator $\mathcal{A}$ associated to the Dirichlet form $\mathcal{E}$ (Proposition 4.59). This generator uniquely determines the distribution of its associated process which we will call $(\widehat{\pi}(t))_{t\geq 0}$.

4.6.1. Process associated with projected Dirichlet form

We observe that one can *associate* a process to the projected Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Proposition 4.58 (Process associated to projected Dirichlet form[a])
There is a weakly unique *continuous Hunt process* $(\widehat{\pi}(t))_{t\geq 0}$, *properly associated*[b] to the projected Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

[a] The possibility of an association of a continuous Hunt process and some of the references used in the following proof are stated in [BR14], Remark below Theorem 1.
[b] as defined in [MR92], Definition IV.2.5

Proof. By Proposition 4.53 we know that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a regular Dirichlet form. This allows us to conclude that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is *quasi-regular*[20] as well, c.f. [MR92], Section IV 4.a). Quasi-regularity in turn implies the existence of a *μ-tight special standard process*,[21] which is *properly associated* with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, c.f. [MR92] Theorem IV.3.5. By definition of *association*,[22] the distribution of such a process is uniquely determined by the Dirichlet form. Moreover, quasi-regularity implies that the associated process is a *Hunt process*.[23] This follows from [MR92], Proposition V.2.12, Theorem V.2.13 and Remark V.2.14. Since $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ has the local property (Proposition 4.53), the continuity of $\widehat{\pi}$ is as consequence of [MR92], Theorem V.1.5. □

In the following section we will illustrate how the law of $\widehat{\pi}$ is determined by its associated Dirichlet form.

[20] as defined in [MR92], Definition IV.3.1
[21] as defined in [MR92], Definition IV.1.13
[22] c.f. [MR92], Definition IV.2.5
[23] as defined in [MR92], Definition IV.1.13

4.6.2. Infinitesimal generator related to projected Dirichlet form

We relate the projected Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ to an operator $\mathcal{A}$, which can be identified as the *infinitesimal generator* of the process $\widehat{\pi}$ introduced in the previous section. This will allows us to (uniquely) determine the law of $\widehat{\pi}$.

Proposition 4.59 (Generator associated to projected Dirichlet form on Γ[a])
Let the domain of the infinitesimal generator $\mathcal{A}$ be given by

$$\mathcal{D}(\mathcal{A}) := \left\{ f \in \mathcal{D}(\mathcal{E}) \mid g \to \mathcal{E}(f,g) \text{ is continuous w.r.t. } \|\cdot\|_{L^2_{\pi_*\mu}(\Gamma)} \text{ on } \mathcal{D}(\mathcal{E}) \right\}. \tag{4.164}$$

Let the infinitesimal operator be defined by

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \to L^2_{\pi_*\mu}(\Gamma), \quad f \to \mathcal{A}f := \breve{f}, \tag{4.165}$$

where $\breve{f}$, for given $f \in \mathcal{D}(\mathcal{A})$, is the *unique* element of $L^2_{\pi_*\mu}(\Gamma)$ s.t.

$$\mathcal{E}(f,g) = -\left\langle \breve{f}, g \right\rangle_{L^2_{\pi_*\mu}(\Gamma)} \equiv -\int_\Gamma \breve{f}\, g \,\mathrm{d}\pi_*\mu, \quad \forall g \in \mathcal{D}(\mathcal{E}). \tag{4.166}$$

Then $\mathcal{A}$ is the infinitesimal generator of the process $\widehat{\pi}$, introduced in Proposition 4.58, and $1 - \mathcal{A}$ satisfies the strong sector condition.

[a] c.f. [BR14], Eq. (80) and [MR92], Proposition I 2.16

Proof. Proposition I.2.16 of [MR92] is applicable, since by Proposition 4.53, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form and thus in particular a coercive closed form. It implies that $\mathcal{A}$, as defined above, is the generator of the strongly continuous contraction semigroup $(T(t))_{t\geq 0}$, associated to the projected Dirichlet form $\mathcal{E}$, which in turn is associated to the process $\widehat{\pi}$, c.f. Proposition 4.58 and [MR92], Definition IV.2.5. □

The isomorphism ψ of Lemma 4.34 allows us to transfer these results to the Dirichlet form $\mathcal{E}_{\mathsf{P}}$, to which we can now associate an infinitesimal generator $\mathcal{A}_{\mathsf{P}}$.

Corollary 4.60 (Generator associated to projected Dirichlet form on $\mathsf{P}(\mathbb{R}^N)$)
Let the domain of the infinitesimal generator $\mathcal{A}_{\mathsf{P}}$ be given by

$$\mathcal{D}(\mathcal{A}_{\mathsf{P}}) := \psi_* \left(\mathcal{D}(\mathcal{A}) \right) \tag{4.167}$$

and let the infinitesimal operator be defined by

$$\mathcal{A}_{\mathsf{P}} : \mathcal{D}(\mathcal{A}_{\mathsf{P}}) \to L^2_{\mathsf{P}_*\mu}(\mathsf{P}(\mathbb{R}^N)), \quad f \to \mathcal{A}_{\mathsf{P}} f := (\mathcal{A}(f \circ \psi^{-1})) \circ \psi, \tag{4.168}$$

i.e. $\mathcal{A}_{\mathsf{P}} f$ is, for given $f \in \mathcal{D}(\mathcal{A}_{\mathsf{P}})$, determined by the unique element of $L^2_{\mathsf{P}_*\mu}(\mathsf{P})$, s.t.

$$\mathcal{E}_{\mathsf{P}}(f,g) = -\left\langle \mathcal{A}_{\mathsf{P}} f, g \right\rangle_{L^2_{\mathsf{P}_*\mu}(\mathsf{P}(\mathbb{R}^N))} \equiv -\int_{\mathsf{P}(\mathbb{R}^N)} \breve{f}\, g \,\mathrm{d}\mathsf{P}_*\mu, \quad \forall g \in \mathcal{D}(\mathcal{A}_{\mathsf{P}}). \tag{4.169}$$

Then $\mathcal{A}_{\mathsf{P}}$ is the infinitesimal generator of the process $\widehat{\mathsf{P}}(t) := \psi^{-1}(\widehat{\pi}(t))$ and associated to the Dirichlet form $\mathcal{E}_{\mathsf{P}}$.

Proposition 4.57 allows us to express $\mathcal{A}_{\mathsf{P}}$ in terms of averaged drift and diffusion.

Lemma 4.61 (Identification of infinitesimal generator $\mathcal{A}_{\mathsf{P}}$)
The infinitesimal generator $\mathcal{A}_{\mathsf{P}}$, as introduced in Corollary 4.60, is given by

$$\mathcal{A}_{\mathsf{P}} f := \widehat{U}_{\mathsf{P}}^{\top} \nabla_{\mathsf{P}} f + \frac{1}{2} \widehat{A}_{\mathsf{P}} : \nabla_{\mathsf{P}} \nabla_{\mathsf{P}}^{\top} f, \quad \forall f \in \mathcal{D}(\mathcal{A}_{\mathsf{P}}). \tag{4.170}$$

Proof. The identification of the infinitesimal generator follows directly from its definition and Eq. (4.162). □

In the remaining part of this section we introduce complex-valued representations of the averaged drift and diffusion terms.

Definition 4.62 (Complex averaged drift term and diffusion matrix)
We define an averaged complex-valued drift term $\widehat{u}_{\mathsf{p}}$ as

$$\begin{aligned} &\widehat{u}_{\mathsf{p}} : \mathsf{p}(\mathbb{C}^n) \to \mathbb{C}^{n,n} \cong \mathbb{C}^{n^2}, \\ &\qquad \hat{p} \to \widehat{u}_{\mathsf{p}}(\hat{p}) := \frac{1}{\mathcal{H}_1(\mathsf{p}^{-1}(\hat{p}))} \oint_{\mathsf{p}^{-1}(\hat{p})} \Big(y\, \tilde{u}(y)^{\dagger} + \text{h.c.} \Big) + (\widetilde{\boldsymbol{\sigma}} \widetilde{\boldsymbol{\sigma}}^{\dagger})(y)\, \mathrm{d}\mathcal{H}_1(y), \end{aligned} \tag{4.171}$$

and averaged diffusion matrices $\widehat{a}_{\mathsf{p}}, \widehat{a}'_{\mathsf{p}}$ by

$$\widehat{a}_{\mathsf{p}}, \widehat{a}'_{\mathsf{p}} : \mathsf{p}(\mathbb{C}^n) \to \mathbb{C}^{n^2,n^2},$$

$$\hat{p} \to \widehat{a}_{\mathsf{p}}(\hat{p}) := \frac{1}{\mathcal{H}_1(\mathsf{p}^{-1}(\hat{p}))} \oint_{\mathsf{p}^{-1}(\hat{p})} (\sigma_{\mathsf{p}}\, \sigma_{\mathsf{p}}^{\dagger})(y)\ \mathrm{d}\mathcal{H}_1(y), \tag{4.172a}$$

$$\hat{p} \to \widehat{a}'_{\mathsf{p}}(\hat{p}) := \frac{1}{\mathcal{H}_1(\mathsf{p}^{-1}(\hat{p}))} \oint_{\mathsf{p}^{-1}(\hat{p})} (\sigma_{\mathsf{p}}\, \sigma_{\mathsf{p}}^{\top})(y)\ \mathrm{d}\mathcal{H}_1(y), \tag{4.172b}$$

where σ_{p} is given by Eq. (3.183).

We note that $\sigma_{\mathsf{p}} : \mathbb{C}^n \to \mathbb{C}^{n^2,N'}$ can be written as

$$\sigma_{\mathsf{p}}(y) = \left(\left(\widetilde{\boldsymbol{\sigma}}(y) \mathcal{Q}_5^{\dagger}\right)^{(0)} y^{\dagger} + \text{h.c.} \mid \ldots \mid \left(\widetilde{\boldsymbol{\sigma}}(y) \mathcal{Q}_5^{\dagger}\right)^{(N'-1)} y^{\dagger} + \text{h.c.} \right), \quad \forall y \in \mathbb{C}^n, \tag{4.173}$$

where for all $r \in \{0, \ldots, N'-1\}$, we denote by $\left(\widetilde{\boldsymbol{\sigma}}(y) \mathcal{Q}_5^{\dagger}\right)^{(r)}$ the r'th column of the matrix $\widetilde{\boldsymbol{\sigma}}(y) \mathcal{Q}_5^{\dagger}$. This can be seen as follows. According to Eq. (3.183), $(\sigma_{\mathsf{p}})_{ij,r}$ is for $y \in \mathbb{C}^n$ and all indices $i, j \in \{0, \ldots, n-1\}$, $r \in \{0, \ldots, N'-1\}$ given by

$$(\sigma_{\mathsf{p}})_{ij,r}(y) := y_i\, \overline{(\widetilde{\boldsymbol{\sigma}}(y) \mathcal{Q}_5^{\dagger})_{jr}} + \overline{y}_j\, (\widetilde{\boldsymbol{\sigma}}(y) \mathcal{Q}_5^{\dagger})_{ir}. \tag{4.174}$$

A reordering allows us to rewrite this in terms of the column vectors $\left(\widetilde{\boldsymbol{\sigma}}(y)\mathcal{Q}_5^\dagger\right)^{(r)}$ as

$$(\sigma_{\mathsf{p}})_{ij,r}(y) = (\widetilde{\boldsymbol{\sigma}}(y)\mathcal{Q}_5^\dagger)_{ir}\,\overline{y}_j + y_i\,\overline{(\widetilde{\boldsymbol{\sigma}}(y)\mathcal{Q}_5^\dagger)_{jr}} = \left(\left(\widetilde{\boldsymbol{\sigma}}(y)\mathcal{Q}_5^\dagger\right)^{(r)} y^\dagger + y\left(\left(\widetilde{\boldsymbol{\sigma}}(y)\mathcal{Q}_5^\dagger\right)^{(r)}\right)^\dagger\right)_{ij},$$

which yields Eq. (4.173). From Eq. (2.115) of Lemma 2.40 we can thus conclude that for all $i,j,k,l \in \{0,...,n-1\}$ we have

$$(\sigma_{\mathsf{p}}\,\sigma_{\mathsf{p}}^\dagger)_{ij,kl}(y) = (\sigma_{\mathsf{p}}\,\sigma_{\mathsf{p}}^\top)_{ij,lk}(y), \quad \forall y \in \mathbb{C}^n, \tag{4.175}$$

and consequently

$$(\widehat{a}_{\mathsf{p}})_{ij,kl}\,(\hat{p}) = \left(\widehat{a}'_{\mathsf{p}}\right)_{ij,lk}(\hat{p}), \quad \forall \hat{p} \in \mathsf{p}(\mathbb{C}^n). \tag{4.176}$$

The following proposition allows us to represent the elements of the infinitesimal generator $\mathcal{A}_{\mathsf{P}}$ in terms of $\widehat{u}_{\mathsf{p}}$, $\widehat{a}_{\mathsf{p}}$, $\widehat{a}'_{\mathsf{p}}$ and ∂_{p}.

Proposition 4.63 (Complex representation of drift and diffusion matrix)
Let $\hat{P} \in \mathsf{P}(\mathbb{R}^N)$ and denote by $\hat{p} := \mathfrak{J}^{-1}(\hat{P}) \in \mathsf{p}(\mathbb{C}^n)$ its complex-valued representation. Then for all $f \in C^2(\mathsf{P}(\mathbb{R}^N))$, it follows that

$$\widehat{U}_{\mathsf{P}}(\hat{P})^\top\;\nabla_{\mathsf{P}} f|_{\hat{P}} = \left(\widehat{u}_{\mathsf{p}}^\top(\hat{p})\,\partial_{\mathsf{p}} + \widehat{u}_{\mathsf{p}}^\dagger(\hat{p})\,\overline{\partial_{\mathsf{p}}}\right)(f\circ\mathfrak{J})\Big|_{\hat{p}}, \tag{4.177}$$

as well as

$$\frac{1}{2}\,\widehat{A}_{\mathsf{P}}(\hat{P}) : \nabla_{\mathsf{P}}\nabla_{\mathsf{P}}^\top f\Big|_{\hat{P}} = \left[\widehat{a}_{\mathsf{p}}(\hat{p}) : \partial_{\mathsf{p}}\partial_{\mathsf{p}}^\dagger + \frac{1}{2}\left(\widehat{a}'_{\mathsf{p}}(\hat{p}) : \partial_{\mathsf{p}}\partial_{\mathsf{p}}^\top + \overline{\widehat{a}'_{\mathsf{p}}}(\hat{p}) : \overline{\partial_{\mathsf{p}}\partial_{\mathsf{p}}^\top}\right)\right](f\circ\mathfrak{J})|_{\hat{P}}. \tag{4.178}$$

Proof. Averaged drift term:
Let $\hat{P} \in \mathsf{P}(\mathbb{R}^N)$ and $Y \in \mathsf{P}^{-1}(\hat{P})$. Denote their complex representations by $y := \mathsf{j}^{-1}(Y)$ and $\hat{p} := \mathsf{p}(y) = yy^\dagger$. Recall that in Definition 4.56 we have set

$$U_{\mathsf{P}}(Y) := (\mathrm{D}\,\mathsf{P})|_Y\,U(Y) + \frac{1}{2}\left(A(Y) : \nabla\nabla^\top\right)\mathsf{P}\Big|_Y. \tag{4.179}$$

Application of Lemma 4.31 allows us to evaluate the Jacobian and the Hessian of the outer product map, which appear in the definition of U_{P}, i.e.

$$(\mathrm{D}\,\mathsf{P})|_Y\,U(Y) = \mathfrak{J}\left(\mathsf{j}^{-1}(Y)\left(\mathsf{j}^{-1}(U(Y))\right)^\dagger + \text{h.c.}\right) = \mathfrak{J}\left(y\,\tilde{u}(y)^\dagger + \text{h.c.}\right), \tag{4.180}$$

$$\frac{1}{2}\left(A(Y) : \nabla\nabla^\top\right)\mathsf{P}\Big|_Y = A^{(+)}(Y) + I\,A^{(+)}(Y)\,(-I) = A(Y) + I\,A(Y)\,(-I), \tag{4.181}$$

since $A(Y)$ is symmetric. Recalling that by Eq. (4.14) we have

$$A(Y) = \mathfrak{J}_{1/2}\left(\widetilde{\boldsymbol{\sigma}}(y)\mathcal{Q}_5^\dagger\right)\left(\mathfrak{J}_{1/2}\left(\widetilde{\boldsymbol{\sigma}}(y)\mathcal{Q}_5^\dagger\right)\right)^\top, \tag{4.182}$$

we conclude that (c.f. Lemmas 2.6 and 2.13)

$$A(Y) + I\, A(Y)\,(-I) = 2\,(A(Y))_+ = \mathfrak{J}\left((\widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\sigma}}^\dagger)(y)\right). \tag{4.183}$$

Thus we find that (recall Eq. (4.157) and Definition 4.62)

$$\begin{aligned}\widehat{U}_{\mathsf{P}}(\hat{P}) &:= \frac{1}{\mathcal{H}_1\left(\mathsf{P}^{-1}(\hat{P})\right)} \int_{\mathsf{P}^{-1}(\hat{P})} U_{\mathsf{P}}(Y)\,\mathrm{d}\mathcal{H}_1(Y) \\ &= \frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} \mathfrak{J}\left(y\,\tilde{u}(y)^\dagger + \text{h.c.}\right) + \mathfrak{J}\left((\widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\sigma}}^\dagger)(y)\right)\mathrm{d}\mathcal{H}_1(y) = \mathfrak{J}(\widehat{u}_{\mathsf{p}}(\hat{p})).\end{aligned}$$

We observe that for all $A, B \in \mathbb{C}^{n,n}$ we have

$$\mathfrak{J}(A) : \mathfrak{J}(B) = 2\,(\mathrm{Re}\,(A) : \mathrm{Re}\,(B) + \mathrm{Im}\,(A) : \mathrm{Im}\,(B)) = 2\,\mathrm{Re}\left(\overline{A} : B\right). \tag{4.184}$$

Identifying the $\mathbb{C}^{n,n}$ matrices with vectors in $\mathbb{C}^m = \mathbb{C}^{n^2}$, this can be written as

$$A^\top B = 2\,\mathrm{Re}\left(A^\dagger\, B\right) = A^\top\,\overline{B} + A^\dagger\, B. \tag{4.185}$$

Recalling Eq. (4.150), we thus find that

$$\widehat{U}_{\mathsf{P}}(\hat{P})^\top\ \nabla_{\mathsf{P}} f|_{\hat{P}} = \left(\widehat{u}_{\mathsf{p}}^\top(\hat{p})\,\partial_{\mathsf{p}} + \widehat{u}_{\mathsf{p}}^\dagger(\hat{p})\,\overline{\partial_{\mathsf{p}}}\right)(f \circ \mathfrak{J})\Big|_{\hat{P}}. \tag{4.186}$$

Averaged diffusion matrix:
Recall that by Definition 4.56 and Lemma 4.31 we can represent the r'th column of Σ_{P} as

$$\begin{aligned}\Sigma_{\mathsf{P}}^{(r)}(Y) &= \mathrm{D}\,\mathsf{P}|_Y\,\Sigma^{(r)}(Y) = \mathfrak{J}\left(\mathsf{j}^{-1}(\Sigma^{(r)}(Y))\left(\mathsf{j}^{-1}(Y)\right)^\dagger + \text{h.c.}\right) \\ &= \mathfrak{J}\left(\left(\widetilde{\boldsymbol{\sigma}}(y)\mathcal{Q}_5^\dagger\right)^{(r)} y^\dagger + \text{h.c.}\right) = \mathfrak{J}\left(\sigma_{\mathsf{p}}{}^{(r)}(y)\right),\end{aligned} \tag{4.187}$$

where in the third step we have applied

$$\mathsf{j}^{-1}(\Sigma^{(r)}(Y)) = \left(\widetilde{\boldsymbol{\sigma}}(y)\mathcal{Q}_5^\dagger\right)^{(r)}, \tag{4.188}$$

which follows from Eq. (4.11), and in the last step we have employed the representation of σ_{p} given in Eq. (4.173). Application of Eqs. (4.150) and (4.185) therefore yields

$$\left(\Sigma_{\mathsf{P}}^{(r)}\right)^\top\ \nabla_{\mathsf{P}} = \left(\sigma_{\mathsf{p}}{}^{(r)}\right)^\top\,\partial_{\mathsf{p}} + \left(\sigma_{\mathsf{p}}{}^{(r)}\right)^\dagger\,\overline{\partial_{\mathsf{p}}}, \tag{4.189}$$

which by $A_{\mathsf{P}} = \Sigma_{\mathsf{P}}\Sigma_{\mathsf{P}}^\top$ implies that

$$\frac{1}{2}\,A_{\mathsf{P}} : \nabla_{\mathsf{P}}\nabla_{\mathsf{P}}^\top = \sigma_{\mathsf{p}}\sigma_{\mathsf{p}}^\dagger : \partial_{\mathsf{p}}\partial_{\mathsf{p}}^\dagger + \frac{1}{2}\left(\sigma_{\mathsf{p}}\sigma_{\mathsf{p}}^\top : \partial_{\mathsf{p}}\partial_{\mathsf{p}}^\top + \overline{\sigma_{\mathsf{p}}\sigma_{\mathsf{p}}^\top} : \overline{\partial_{\mathsf{p}}\partial_{\mathsf{p}}^\top}\right). \tag{4.190}$$

Performing the averaging integration and recalling Eq. (4.172), now yields the result. □

4.7. Generalized convergence of Dirichlet forms and associated weak convergence

We show that the Dirichlet forms E_ε converge in a generalized sense towards the projected Dirichlet form $\mathcal{E}$ (Proposition 4.66). Subsequently we show that this gives rise to a weak convergence of the processes $(\pi(Y^\varepsilon(t)))_{t\geq 0}$ to the process $(\widehat{\pi}(t))_{t\geq 0}$ (Theorem 4.69). In order to simplify notation, we introduce the following shorthands for the projected processes:

$$\pi^\varepsilon(t) := \pi(Y^\varepsilon(t)), \tag{4.191a}$$

$$\mathsf{P}^\varepsilon(t) := \mathsf{P}(Y^\varepsilon(t)) = Y^\varepsilon(t)\,(Y^\varepsilon(t))^\top + I\,Y^\varepsilon(t)\,(Y^\varepsilon(t))^\top, \tag{4.191b}$$

$$\mathsf{p}^\varepsilon(t) := \mathsf{p}(\tilde{y}^\varepsilon(t)) = \tilde{y}^\varepsilon(t)(\tilde{y}^\varepsilon(t))^\dagger. \tag{4.191c}$$

We define the concept of convergence of $L^2(\mathbb{R}^N)$-functions towards projected functions in $L^2_{\pi_*\mu}(\Gamma)$.

Definition 4.64 (Weak and strong convergence towards a projected function[a])
A sequence $(f_k)_k \subset L^2(\mathbb{R}^N)$ is said to *weakly converge* to a function $f \in L^2_{\pi_*\mu}(\Gamma)$, if

$$\sup_k \langle f_k, f_k\rangle_{L^2(\mathbb{R}^N)} < \infty, \tag{4.192a}$$

$$\langle f_k, g\circ\pi\rangle_{L^2(\mathbb{R}^N)} \to \langle f\circ\pi, g\circ\pi\rangle_{L^2(\mathbb{R}^N)} \equiv \langle f, g\rangle_{L^2_{\pi_*\mu}(\Gamma)}, \quad \forall g \in L^2_{\pi_*\mu}(\Gamma). \tag{4.192b}$$

The sequence $(f_k)_k \subset L^2(\mathbb{R}^N)$ *strongly* converges to $f \in L^2_{\pi_*\mu}(\Gamma)$, if

$$\|f_k - f\circ\pi\|_{L^2(\mathbb{R}^N)} \to 0, \quad k\to\infty. \tag{4.193}$$

[a] c.f. [BR14], Definition 2; [Töl06], Definitions 2.4, 2.5

Now we can introduce the notion of *generalized convergence* of Dirichlet forms.

Definition 4.65 (Generalized convergence[a])
For a sequence $(\varepsilon_k)_k$ converging to zero, E_{ε_k} is said to *converge to $\mathcal{E}$ in the generalized sense*,[b] if the following conditions hold:

i) Let $(f_k)_k \subset L^2(\mathbb{R}^N)$ be sequence, which *weakly convergences* to some $f \in L^2_{\pi_*\mu}(\Gamma)$ (as specified in Definition 4.64), and which satisfies

$$\sup_k \tilde{E}^1_{\varepsilon_k}(f_k, f_k) < \infty, \tag{4.194}$$

where we have set $E_\varepsilon(f,f) = \infty$, for $f \notin \mathcal{D}(E_\varepsilon)$. Then it follows that $f \in \mathcal{D}(\mathcal{E})$.

ii) For any sequence $k_l \nearrow \infty$, any sequence $(f_l)_l \subset L^2(\mathbb{R}^N)$ *weakly* converging to some $f \in \mathcal{D}(\mathcal{E})$, s.t.

$$\sup_l \tilde{E}^1_{\varepsilon_{k_l}}(f_l, f_l) < \infty, \tag{4.195}$$

and for all $g \in \mathcal{D}(\mathcal{E})$, there exists a sequence $(g_l)_l \subset L^2(\mathbb{R}^N)$, which *strongly* converges to g, s.t.

$$\liminf_{l} E_{\varepsilon_{k_l}}(f_l, g_l) \leq \mathcal{E}(f, g). \tag{4.196}$$

[a] This definition follows from [BR14], Definition 3, which in turn is based on [Hin98], Section 3 and [Töl06], Definitions 2.40, 2.43. Note that the functions $\mathbf{\Phi^k}$ and $\mathbf{\Theta^k}$, appearing in the definitions of [Hin98] and [Töl06] respectively, are equivalent to the norms $\|\cdot\|_{\tilde{E}^1_{\varepsilon_k}}$, as is shown in [Töl06], Lemma 2.39(iv).

[b] c.f. [Töl06], Definition 2.43; in [BR14], this type of convergence is referred to as *Mosco-convergence*.

This notion of generalized convergence is realized by the sequence $(E_\varepsilon)_{\varepsilon>0}$ of Dirichlet forms, which in the scaling limit of $\varepsilon \to 0$, converges to the projected Dirichlet form $\mathcal{E}$.

Proposition 4.66 (Generalized convergence to projected Dirichlet form)
E_{ε_k} converges in the generalized sense to $\mathcal{E}$, as $\varepsilon_k \to 0$.[a]

[a] c.f. [BR14], Proposition 8.

Proof. We verify the conditions of Definition 4.65 by adapting the proof of [BR14], Theorem 2.

i) Let $(f_k)_k \subset L^2(\mathbb{R}^N)$ be a sequence, which *weakly convergences* to some $f \in L^2_{\pi_*\mu}(\Gamma)$ and which satisfies Eq. (4.194). Since by Remark 4.18, the norms $\|\cdot\|_{\tilde{E}^1_{\varepsilon_k}}$ do not depend on ε_k, Eq. (4.194) implies that $(f_k)_k$ is a bounded sequence in the complete space $(\mathcal{D}(E_\varepsilon), \|\cdot\|_{\tilde{E}^1_\varepsilon})$, which again is independent of the choice of $\varepsilon > 0$. By the *Banach-Alaoglu theorem*,[24] there exists a subsequence f_{k_l}, which *weakly* converges in $(\mathcal{D}(E_\varepsilon), \|\cdot\|_{\tilde{E}^1_\varepsilon})$ to some element $f' \in \mathcal{D}(E_\varepsilon)$. In particular, this implies a weak convergence in $L^2(\mathbb{R}^N)$, i.e.

$$\langle f_{k_l}, g' \rangle_{L^2(\mathbb{R}^N)} \to \langle f', g' \rangle_{L^2(\mathbb{R}^N)}, \quad \forall g' \in L^2(\mathbb{R}^N), \tag{4.197}$$

which, by restricting to the smaller set $\pi_*(L^2_{\pi_*\mu}(\Gamma)) \subset L^2(\mathbb{R}^N)$ of test functions, yields

$$\langle f_{k_l}, g \circ \pi \rangle_{L^2(\mathbb{R}^N)} \to \langle f', g \circ \pi \rangle_{L^2(\mathbb{R}^N)}, \quad \forall g \in L^2_{\pi_*\mu}(\Gamma). \tag{4.198}$$

On the other hand, the subsequence f_{k_l} by assumption also weakly converges to some $f \in L^2_{\pi_*\mu}(\Gamma)$ in the sense specified in Definition 4.64, i.e.

$$\langle f_{k_l}, g \circ \pi \rangle_{L^2(\mathbb{R}^N)} \to \langle f \circ \pi, g \circ \pi \rangle_{L^2(\mathbb{R}^N)}, \quad \forall g \in L^2_{\pi_*\mu}(\Gamma). \tag{4.199}$$

Comparing Eqs. (4.198) and (4.199) and recalling that $f' \in \mathcal{D}(E_\varepsilon)$, yields $f \circ \pi \in \mathcal{D}(E_\varepsilon)$, i.e. $f \in \pi_*^{-1}(\mathcal{D}(E_\varepsilon)) \equiv \mathcal{D}(\mathcal{E})$, c.f. Lemma 4.51.

ii) Let $f, g \in \mathcal{D}(\mathcal{E})$, $k_l \nearrow \infty$ and let $(f_l)_l \subset L^2(\mathbb{R}^N)$ be *weakly* converging to $f \in \mathcal{D}(\mathcal{E})$, s.t. Eq. (4.195) holds. Let $g_l := g \circ \pi$ for all $l \in \mathbb{N}$, which obviously yields the required strong convergence of $(g_l)_l$ to g (c.f. Eq. (4.193)). Thus we are left with proving that this choice of $(g_l)_l$ satisfies Eq. (4.196), which now can be rewritten as

$$\liminf_{l\to\infty} E_{\varepsilon_{k_l}}(f_l, g \circ \pi) \leq \mathcal{E}(f, g) \tag{4.200}$$

[24] c.f. [Con90], Chapter V, Theorem 3.1

Condition (4.195) implies that the sequence $(f_l)_l$ is bounded in $(\mathcal{D}(E_\varepsilon), \|\cdot\|_{\tilde{E}_\varepsilon^1})$, which, again by the *Banach-Alaoglu theorem*, yields the existence of a subsequence $(f_{l_m})_m$, weakly converging to some f', i.e.

$$\tilde{E}_\varepsilon^1(f_{l_m}, g') \to \tilde{E}_\varepsilon^1(f', g'), \quad \forall g' \in \mathcal{D}(E_\varepsilon). \tag{4.201}$$

In particular, this weak convergence holds in $L^2(\mathbb{R}^N)$, i.e.

$$\langle f_{l_m}, g' \rangle_{L^2(\mathbb{R}^N)} \to \langle f', g' \rangle_{L^2(\mathbb{R}^N)}, \quad \forall g' \in L^2(\mathbb{R}^N). \tag{4.202}$$

As in the proof of **i)**, we recall that this subsequence also weakly converges to $f \in \mathcal{D}(\mathcal{E})$, which allows us to identify f' with $f \circ \pi$. Thus we can conclude that

$$\tilde{E}_\varepsilon(f_{l_m}, g') \to \tilde{E}_\varepsilon(f \circ \pi, g'), \quad \forall g' \in \mathcal{D}(E_\varepsilon), \tag{4.203}$$

which finally puts us in a position to verify Eq. (4.200):

$$\begin{aligned}
\liminf_{l\to\infty} E_{\varepsilon_{k_l}}(f_l, g\circ\pi) &= \liminf_{l\to\infty} \tilde{E}_{\varepsilon_{k_l}}(f_l, g\circ\pi) + \liminf_{l\to\infty} \check{E}_{\varepsilon_{k_l}}(f_l, g\circ\pi) \\
&\le \liminf_{m\to\infty} \tilde{E}_{\varepsilon_{k_{l_m}}}(f_{l_m}, g\circ\pi) + \liminf_{m\to\infty} \check{E}_{\varepsilon_{k_{l_m}}}(f_{l_m}, g\circ\pi) \\
&= \tilde{E}_\varepsilon(f\circ\pi, g\circ\pi) + \check{E}_\varepsilon(f\circ\pi, g\circ\pi) \\
&= E_\varepsilon(f\circ\pi, g\circ\pi) \\
&= \mathcal{E}(f, g).
\end{aligned}$$

Here we have made use of Eq. (4.203) with $g' := g \circ \pi \in \mathcal{D}(E_\varepsilon)$ to obtain convergence for the symmetric part. In order to obtain the convergence of the asymmetric part we have applied

$$\begin{aligned}
\check{E}_{\varepsilon_{k_{l_m}}}(f_{l_m}, g\circ\pi) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(\frac{1}{\varepsilon_{k_{l_m}}} V^\top + \Phi^\top\right)(f_{l_m}\,\nabla(g\circ\pi) - g\circ\pi\,\nabla f_{l_m})\,\mathrm{d}\mu \\
&= \frac{1}{2}\left\langle f_{l_m}, \Phi^\top\nabla(g\circ\pi)\right\rangle_{L^2(\mathbb{R}^N)} + \frac{1}{2}\left\langle f_{l_m}, \nabla^\top(\Phi\, g\circ\pi)\right\rangle_{L^2(\mathbb{R}^N)} \\
&= \left\langle f_{l_m}, \Phi^\top\nabla(g\circ\pi)\right\rangle_{L^2(\mathbb{R}^N)} + \frac{1}{2}\left\langle f_{l_m}, (\nabla^\top\Phi)\, g\circ\pi\right\rangle_{L^2(\mathbb{R}^N)},
\end{aligned} \tag{4.204}$$

where the ε-dependent term vanishes, since $V^\top\,\nabla(g\circ\pi) = 0$ according to Lemma 4.49 (lifted functions are constant w.r.t. fast evolution) and

$$\int_{\mathbb{R}^N} \nabla^\top\,(V\,g\circ\pi)\,f_{l_m}\;\mathrm{d}\mu = \int_{\mathbb{R}^N}\left(V^\top\,\nabla(g\circ\pi) + \left(\nabla^\top V\right) g\circ\pi\right)\,f_{l_m}\;\mathrm{d}\mu = 0,$$

making use of $\nabla^\top V = 0$, which follows from Eq. (4.34). Convergence of $\check{E}_{\varepsilon_{k_{l_m}}}(f_{l_m}, g\circ\pi)$ now follows by application of Eq. (4.202) to Eq. (4.204), since $\Phi^\top\nabla(g\circ\pi)$ and $(\nabla^\top\Phi)\,g\circ\pi$ are elements of $L^2(\mathbb{R}^N)$:[25]

$(\nabla^\top\Phi)\,g\circ\pi$ lies in $L^2(\mathbb{R}^N)$, since $\nabla^\top\Phi$ is constant, which follows from Remark 4.19 and the

[25]In the proof of [BR14], Theorem 2, the argument given at this point is based on the boundedness of Φ, which does not hold in our case.

proof of Lemma 4.10 (note that this remains valid in the presence of a nonlinear *divergence free* perturbation of the drift term). For the term $\Phi^\top \nabla(g \circ \pi)$ we recall that by Eqs. (4.74), (4.75) and (4.77) we have

$$\begin{aligned}\int_{\mathbb{R}^N} \left|\Phi^\top \nabla(g\circ\pi)\right|^2 \mathrm{d}\mu &= \int_{\mathbb{R}^N} \left|\Phi^\top \mathfrak{D}^{-1}\,\mathfrak{D}\,\nabla(g\circ\pi)\right|^2 \mathrm{d}\mu \\ &\leq \int_{\mathbb{R}^N} \left\|\mathfrak{D}^{-1}\Phi\right\|^2 \|\mathfrak{D}\,\nabla(g\circ\pi)\|^2 \,\mathrm{d}\mu \\ &\leq \sup_{X\in\mathbb{R}^N} \left\|\mathfrak{D}^{-1}(X)\Phi(X)\right\|^2 \int_{\mathbb{R}^N} \|\mathfrak{D}\,\nabla(g\circ\pi)\|^2 \,\mathrm{d}\mu \\ &\leq c\,E_\varepsilon\left((g\circ\pi),(g\circ\pi)\right) \equiv c\,\mathcal{E}\left(g,g\right) < \infty,\end{aligned}$$

where c is a finite, positive constant, which unlike in Eq. (4.77) does not depend on ε, since we are only requiring a bound on Φ and not on Φ_ε. □

Generalized convergence can be shown to imply convergence of the *finite dimensional* distributions of the associated processes.

Proposition 4.67 (Convergence of finite dimensional distributions[a])
Assume that the initial distributions $\mu_\varepsilon := \mathbb{P}(Y_0^\varepsilon)^{-1}$ are absolutely continuous w.r.t. to μ and that their densities $(\zeta_\varepsilon)_{\varepsilon>0}$ weakly converge in $L^2(\mathbb{R}^N)$ to $\zeta\circ\pi$, where $\zeta \in L^2_{\pi_*\mu}(\Gamma)$, s.t. $\mathrm{d}\widehat{\mu} = \zeta\circ\pi\,\mathrm{d}\mu$ defines a probability measure $\widehat{\mu}$.[b] Then, in the limit of $\varepsilon\to 0$, the *finite dimensional distributions* of $(\pi^\varepsilon)_{\varepsilon>0}$ weakly converge to the finite dimensional distributions of the process $\widehat{\pi}$, with initial distribution $\pi_*\widehat{\mu}$ given by $\mathrm{d}(\pi_*\widehat{\mu}) = \zeta\,\mathrm{d}(\pi_*\mu)$.

[a] c.f. [BR14], Proposition 4
[b] Note that the assumptions imply that $\mu_{\varepsilon_k} \Rightarrow \widehat{\mu}$.

Proof. Let $\varepsilon_k > 0$, s.t. $\varepsilon_k \to 0$ and let $(T^{(k)}(t))_{t\geq 0}, (T(t))_{t\geq 0}$, be the continuous contraction semigroups associated to E_{ε_k} and $\mathcal{E}$. According to [BR14], Theorem 3; [Hin98], Theorem 3.5 and [Töl06], Theorem 2.53, the following two notions of convergence are now equivalent:

i) E_{ε_k} converges to $\mathcal{E}$ in the generalized sense.

ii) $T^{(k)}(t)$ strongly converges to $T(t)$, for all $t \geq 0$.

Since Proposition 4.66 ensures i), we can now employ the equivalent condition ii) and the Markov property of the processes $(\pi^{\varepsilon_k})_k$ in order to prove the convergence of the finite dimensional distributions. This is done in the proof of [BR14], Proposition 4 and can be directly applied to this case as well. □

In order to extend the convergence of finite dimensional distributions to the process level, we need a *tightness* result for the involved distributions. This is established in the following Lemma 4.68. Its statement corresponds to [BR14], Proposition 9; the proof given in [BR14], however, does *not* apply since we do not have bounded drift and diffusion terms. For this reason we provide a new proof, suitable to our setup.

Lemma 4.68 (Tightness)
The distributions of the families of processes $(\pi^\varepsilon)_{\varepsilon>0}$ and $(\mathsf{P}^\varepsilon)_{\varepsilon>0}$ are *tight.*[a]

[a] c.f. [BR14], Proposition 9

Proof. Since there is an isomorphism between the families $(\pi^\varepsilon)_{\varepsilon>0}$ and $(\mathsf{P}^\varepsilon)_{\varepsilon>0}$ (c.f. Lemma 4.34), it suffices to prove the tightness of the latter, or of its complex representation $(\mathsf{p}^\varepsilon)_{\varepsilon>0}$. By [KS91], Problem 2.4.11, this can be achieved by showing that for all $k, l \in \{0, ..., n-1\}$, we have

i) $\sup_{\varepsilon>0} \mathbb{E}\left(\left|\mathsf{p}^\varepsilon_{k,l}(0)\right|\right) < \infty$,

ii) $\sup_{\varepsilon>0} \mathbb{E}\left(\left|\mathsf{p}^\varepsilon_{k,l}(t) - \mathsf{p}^\varepsilon_{k,l}(s)\right|^\alpha\right) \le C_T\, |t-s|^{1+\beta}, \quad \forall T > 0$ and $0 \le s, t \le T$,

for some positive constants α, β and C_T (depending on $T > 0$).

Let $k, l \in \{0, ..., n-1\}$ and $\alpha > 2$.

Now **i)** follows from Eq. (4.7) and the *Cauchy-Schwarz* inequality:

$$\sup_{\varepsilon>0} \mathbb{E}\left(\left|\mathsf{p}^\varepsilon_{k,l}(0)\right|\right) \le \sup_{\varepsilon>0} \sqrt{\mathbb{E}\left(|\tilde{y}^\varepsilon_k(0)|^2\right)}\,\sqrt{\mathbb{E}\left(|\tilde{y}^\varepsilon_l(0)|^2\right)} \le \sup_{\varepsilon>0} \mathbb{E}\left(\|Y^\varepsilon(0)\|^2\right) < \infty. \tag{4.205}$$

In order to prove **ii)**, we first observe that $\alpha > 1$ and *Jensen's inequality* yield

$$|a+b|^\alpha \le 2^\alpha \left(\frac{|a|+|b|}{2}\right)^\alpha \le 2^{\alpha-1}\left(|a|^\alpha + |b|^\alpha\right), \quad \forall\, a, b \in \mathbb{R}, \tag{4.206}$$

which by Eq. (3.181) implies that

$$\begin{aligned}
\mathbb{E}\left(\left|\mathsf{p}^\varepsilon_{k,l}(t) - \mathsf{p}^\varepsilon_{k,l}(s)\right|^\alpha\right) &= \mathbb{E}\left(\left|\int_s^t (u_{\mathsf{p}})_{kl}(\tilde{y}^\varepsilon(\tau))\,\mathrm{d}\tau + \int_s^t \left(\sigma_{\mathsf{p}}(\tilde{y}^\varepsilon(\tau))\,\mathrm{d}\boldsymbol{B}(\tau)\right)_{kl}\right|^\alpha\right) \\
&\le 2^{\alpha-1}\,\mathbb{E}\left(\left|\int_s^t (u_{\mathsf{p}})_{kl}(\tilde{y}^\varepsilon(\tau))\,\mathrm{d}\tau\right|^\alpha\right) \\
&\quad + 2^{\alpha-1}\,\mathbb{E}\left(\left|\left(\int_s^t \sigma_{\mathsf{p}}(\tilde{y}^\varepsilon(\tau))\,\mathrm{d}\boldsymbol{B}(\tau)\right)_{kl}\right|^\alpha\right).
\end{aligned} \tag{4.207}$$

Estimate of first term:

We employ *Jensen's inequality* and *Fubini's theorem* which yield

$$\begin{aligned}
\mathbb{E}\left(\left|\int_s^t (u_{\mathsf{p}})_{kl}(\tilde{y}^\varepsilon(\tau))\,\mathrm{d}\tau\right|^\alpha\right) &= (t-s)^\alpha\,\mathbb{E}\left(\left|\frac{1}{t-s}\int_s^t (u_{\mathsf{p}})_{kl}(\tilde{y}^\varepsilon(\tau))\,\mathrm{d}\tau\right|^\alpha\right) \\
&\le (t-s)^{\alpha-1}\mathbb{E}\left(\int_s^t |(u_{\mathsf{p}})_{kl}(\tilde{y}^\varepsilon(\tau))|^\alpha\,\mathrm{d}\tau\right) \\
&\le (t-s)^{\alpha-1}\int_s^t \mathbb{E}\left(|(u_{\mathsf{p}})_{kl}(\tilde{y}^\varepsilon(\tau))|^\alpha\right)\mathrm{d}\tau.
\end{aligned} \tag{4.208}$$

Note that from Eq. (3.93) we obtain the estimate

$$|(\widetilde{u_{\mathrm{lin}}})_k(\tilde{y})| = \frac{\sqrt{n}}{2}\left|\tilde{\lambda}_k\,\tilde{y}_k + \overline{\tilde{\lambda}_{n-k}}\,\overline{\tilde{y}_{n-k}}\right| \le \frac{\sqrt{n}}{2}\left(\left|\tilde{\lambda}_k\right| + \left|\tilde{\lambda}_{n-k}\right|\right)\sqrt{|\tilde{y}_k|^2 + |\tilde{y}_{n-k}|^2}.$$

Similarly, Eq. (3.171) yields

$$\left|(\widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\sigma}}^\dagger)_{kl}(\tilde{y})\right| = \left|\left(\widetilde{\sigma_{\text{mult}}}\widetilde{\sigma_{\text{mult}}}^\dagger + \widetilde{\boldsymbol{\sigma}_{\text{reg}}}\widetilde{\boldsymbol{\sigma}_{\text{reg}}}^\dagger + \widetilde{\boldsymbol{\sigma}_{\text{add}}}\widetilde{\boldsymbol{\sigma}_{\text{add}}}^\dagger\right)_{kl}(\tilde{y})\right|,$$

where by Eqs. (4.24), (4.25) and (4.30) we have

$$\begin{aligned}
\widetilde{\sigma_{\text{mult}}}\widetilde{\sigma_{\text{mult}}}^\dagger(\tilde{y}) &= \frac{n^2}{4}\operatorname{diag}\left(\left(|\widetilde{\nu}_k|^2\,|\tilde{y}_k|^2 + |\widetilde{\nu}_{n-k}|^2\,|\tilde{y}_{n-k}|^2 + 2\operatorname{Re}(\widetilde{\nu}_k\widetilde{\nu}_{n-k}\tilde{y}_k\tilde{y}_{n-k})\right)_k\right),\\
\widetilde{\boldsymbol{\sigma}_{\text{reg}}}\widetilde{\boldsymbol{\sigma}_{\text{reg}}}^\dagger(\tilde{y}) &= \frac{n^2}{4}\,\sigma_r^2\operatorname{diag}\left(\left(|\tilde{y}_k|^2 + |\tilde{y}_{n-k}|^2\right)_k\right),\\
\widetilde{\boldsymbol{\sigma}_{\text{add}}}\widetilde{\boldsymbol{\sigma}_{\text{add}}}^\dagger &= \sigma_0^2\,\mathbb{1}_{n\times n}.
\end{aligned}$$

Recalling that $\widetilde{\nu}_k = \widetilde{\nu}_{n-k}$ (Assumption 3.27), we can estimate

$$\left|(\widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\sigma}}^\dagger)_{kl}(\tilde{y})\right| \le \left[n^2\left(|\widetilde{\nu}_k|^2 + |\widetilde{\nu}_{n-k}|^2 + \sigma_r^2\right)\left(|\tilde{y}_k|^2 + |\tilde{y}_{n-k}|^2\right) + \sigma_0^2\right]\delta_{k,l}. \tag{4.209}$$

Now by Eq. (3.182), we can conclude that[26]

$$\begin{aligned}
|(u_{\mathsf{p}})_{kl}(\tilde{y})| &= \left|\left[\tilde{u}_k(\tilde{y})\,\overline{\tilde{y}_l} + \overline{\tilde{u}_l(\tilde{y})}\,\tilde{y}_k\right] + (\widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\sigma}}^\dagger)_{kl}(\tilde{y})\right|\\
&\le \frac{\sqrt{n}}{2}\Bigg[\left(\left|\widetilde{\lambda}_k\right| + \left|\widetilde{\lambda}_{n-k}\right|\right)\sqrt{|\tilde{y}_k|^2 + |\tilde{y}_{n-k}|^2}\;|\tilde{y}_l|\\
&\qquad + \left(\left|\widetilde{\lambda}_l\right| + \left|\widetilde{\lambda}_{n-l}\right|\right)\sqrt{|\tilde{y}_l|^2 + |\tilde{y}_{n-l}|^2}\;|\tilde{y}_k|\Bigg]\\
&\quad + \left[n^2\left(|\widetilde{\nu}_k|^2 + |\widetilde{\nu}_{n-k}|^2 + \sigma_r^2\right)\left(|\tilde{y}_k|^2 + |\tilde{y}_{n-k}|^2\right) + \sigma_0^2\right]\delta_{k,l}\\
&\le c\sqrt{c + |\tilde{y}_k|^2 + |\tilde{y}_{n-k}|^2}\sqrt{c + |\tilde{y}_l|^2 + |\tilde{y}_{n-l}|^2},
\end{aligned}\tag{4.210}$$

for a suitably chosen constant $c > 1$, which only depends on $\widetilde{\lambda}, \widetilde{\nu}, \sigma_r$ and σ_0. This estimate allows us to infer that

$$\begin{aligned}
\mathbb{E}\left(|(u_{\mathsf{p}})_{kl}(\tilde{y})|^\alpha\right) &\le c^\alpha\,\mathbb{E}\left(\left(c + |\tilde{y}_k|^2 + |\tilde{y}_{n-k}|^2\right)^{\alpha/2}\left(c + |\tilde{y}_l|^2 + |\tilde{y}_{n-l}|^2\right)^{\alpha/2}\right)\\
&\le c^\alpha\sqrt{\mathbb{E}\left[\left(c + |\tilde{y}_k|^2 + |\tilde{y}_{n-k}|^2\right)^\alpha\right]}\sqrt{\mathbb{E}\left[\left(c + |\tilde{y}_l|^2 + |\tilde{y}_{n-l}|^2\right)^\alpha\right]}\\
&\le c^\alpha\,2^{\alpha-1}\sqrt{c^\alpha + \mathbb{E}\left[\left(|\tilde{y}_k|^2 + |\tilde{y}_{n-k}|^2\right)^\alpha\right]}\sqrt{c^\alpha + \mathbb{E}\left[\left(|\tilde{y}_l|^2 + |\tilde{y}_{n-l}|^2\right)^\alpha\right]},
\end{aligned}\tag{4.211}$$

where we have applied the inequalities of *Cauchy-Schwarz* and *Jensen*. For further estimates we introduce the notation (c.f. also Eq. (6.307) below)

$$\rho_k^\varepsilon(t) := |\tilde{y}_k^\varepsilon(t)|^2, \tag{4.212}$$

$$\rho_k^{+,\varepsilon}(t) := |\tilde{y}_k^\varepsilon(t)|^2 + |\tilde{y}_{n-k}^\varepsilon(t)|^2, \tag{4.213}$$

[26] Recall that we have restricted the presentation in this section to the case of a linear drift term.

which together with Eq. (4.211) allows us to rewrite Eq. (4.208) as

$$\begin{aligned} &\mathbb{E}\left(\left|\int_s^t (u_{\mathsf{p}})_{kl}(\tilde{y}^\varepsilon(\tau))\,\mathrm{d}\tau\right|^\alpha\right) \\ &\quad\leq (t-s)^{\alpha-1}\, c^\alpha\, 2^{\alpha-1} \int_s^t \sqrt{c^\alpha + \mathbb{E}\left[\left(\rho_k^{+,\varepsilon}(\tau)\right)^\alpha\right]}\sqrt{c^\alpha + \mathbb{E}\left[\left(\rho_l^{+,\varepsilon}(\tau)\right)^\alpha\right]}\,\mathrm{d}\tau. \end{aligned} \tag{4.214}$$

As a final step we show that

$$\sup_{0\leq\tau\leq T} \mathbb{E}\left[\left(\rho_k^{+,\varepsilon}(\tau)\right)^\alpha\right] \leq C_T, \tag{4.215}$$

where $C_T > 0$ is a constant independent of ε. Together with Eq. (4.214) this will allow us to obtain the following estimate which is in compliance with the structure required in **ii)**:

$$\mathbb{E}\left(\left|\int_s^t (u_{\mathsf{p}})_{kl}(\tilde{y}^\varepsilon(\tau))\,\mathrm{d}\tau\right|^\alpha\right) \leq (C_T+\alpha)\, c^\alpha\, 2^{\alpha-1}\,(t-s)^\alpha, \quad \forall 0\leq s,t\leq T, \quad \forall\varepsilon>0. \tag{4.216}$$

We will prove Eq. (4.215) by means of a *Gronwall-type estimate*. First observe that by Itô's formula we have (note that $\rho_k^{+,\varepsilon}(t)$ is a *real*-valued process)

$$\mathrm{d}\left(\rho_k^{+,\varepsilon}(\tau)\right)^\alpha = \alpha\left(\rho_k^{+,\varepsilon}(\tau)\right)^{\alpha-1}\mathrm{d}\rho_k^{+,\varepsilon}(\tau) + \frac{\alpha(\alpha-1)}{2}\left(\rho_k^{+,\varepsilon}(\tau)\right)^{\alpha-2}\mathrm{d}\left\langle\rho_k^{+,\varepsilon}\right\rangle(\tau), \tag{4.217}$$

where by Eq. (3.181),

$$\begin{aligned} \mathrm{d}\rho_k^{+,\varepsilon}(\tau) &= \mathrm{d}\mathsf{p}_{kk}^\varepsilon(\tau) + \mathrm{d}\mathsf{p}_{n-k,n-k}^\varepsilon(\tau) \\ &= \left((u_{\mathsf{p}})_{kk}(\tilde{y}^\varepsilon(\tau)) + (u_{\mathsf{p}})_{n-k,n-k}(\tilde{y}^\varepsilon(\tau))\right)\mathrm{d}\tau \\ &\quad + (\sigma_{\mathsf{p}}(\tilde{y}^\varepsilon(\tau))\,\mathrm{d}\boldsymbol{B}(\tau))_{kk} + (\sigma_{\mathsf{p}}(\tilde{y}^\varepsilon(\tau))\,\mathrm{d}\boldsymbol{B}(\tau))_{n-k,n-k}\,. \end{aligned} \tag{4.218}$$

By [KS91], Problem 1.5.7 and *Young's inequality* we can conclude that

$$\begin{aligned} \mathrm{d}\left\langle\rho_k^{+,\varepsilon}\right\rangle(\tau) &\leq 2\left(\mathrm{d}\left\langle\rho_k^\varepsilon\right\rangle(\tau) + \mathrm{d}\left\langle\rho_{n-k}^\varepsilon\right\rangle(\tau)\right) \\ &= 2\left[\left(\sigma_{\mathsf{p}}(\tilde{y}^\varepsilon(\tau))\sigma_{\mathsf{p}}^\top(\tilde{y}^\varepsilon(\tau))\right)_{k,k;k,k} + \left(\sigma_{\mathsf{p}}(\tilde{y}^\varepsilon(\tau))\sigma_{\mathsf{p}}^\top(\tilde{y}^\varepsilon(\tau))\right)_{n-k,n-k;n-k,n-k}\right]\mathrm{d}\tau \\ &= 4\left[\rho_k^\varepsilon(\tau)(\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}^\dagger)_{kk}(\tilde{y}^\varepsilon(\tau)) + \mathrm{Re}\left([\tilde{y}_k^\varepsilon(\tau)]^2\,\overline{\left(\tilde{\boldsymbol{\sigma}}\mathcal{R}_5\tilde{\boldsymbol{\sigma}}^\top\right)}_{kk}(\tilde{y}^\varepsilon(\tau))\right)\right]\mathrm{d}\tau \\ &\quad + 4\left[\rho_{n-k}^\varepsilon(\tau)(\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}^\dagger)_{n-k,n-k}(\tilde{y}^\varepsilon(\tau))\right. \\ &\qquad\left. + \mathrm{Re}\left([\tilde{y}_{n-k}^\varepsilon(\tau)]^2\,\overline{\left(\tilde{\boldsymbol{\sigma}}\mathcal{R}_5\tilde{\boldsymbol{\sigma}}^\top\right)}_{n-k,n-k}(\tilde{y}^\varepsilon(\tau))\right)\right]\mathrm{d}\tau \\ &\leq 8\,\rho_k^{+,\varepsilon}(\tau)\left((\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}^\dagger)_{kk}(\tilde{y}^\varepsilon(\tau)) + (\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}^\dagger)_{n-k,n-k}(\tilde{y}^\varepsilon(\tau))\right)\mathrm{d}\tau \\ &\leq 16\,\rho_k^{+,\varepsilon}(\tau)\left[n^2\left(|\tilde{\nu}_k|^2 + |\tilde{\nu}_{n-k}|^2 + \sigma_r^2\right)\rho_k^{+,\varepsilon}(\tau) + \sigma_0^2\right]\mathrm{d}\tau, \end{aligned} \tag{4.219}$$

where we have employed Lemma 2.40 and Eq. (4.209).

Moreover, we have applied Eqs. (4.45) and (4.46), which yield

$$\begin{aligned}\mathrm{Re}\left([\tilde y_k^\varepsilon(\tau)]^2\ \overline{\left(\tilde{\boldsymbol\sigma}\mathcal{R}_5\tilde{\boldsymbol\sigma}^\top\right)}_{kk}(\tilde y^\varepsilon(\tau))\right) &= \mathrm{Re}\left([\tilde y_k^\varepsilon(\tau)]^2\ \overline{\left(\widetilde{\sigma_{\mathrm{mult}}}R\left(\widetilde{\sigma_{\mathrm{mult}}}\right)^\top\right)}_{kk}(\tilde y^\varepsilon(\tau))\right)\\ &= \mathrm{Re}\left([\tilde y_k^\varepsilon(\tau)]^2\ \overline{(\widetilde{\sigma_{\mathrm{mult}}}\widetilde{\sigma_{\mathrm{mult}}}^\top)}_{kk}(\tilde y^\varepsilon(\tau))\right)\delta_{k,n-k}\\ &\le \mathrm{Re}\left([\tilde y_k^\varepsilon(\tau)]^2\ \overline{(\widetilde{\sigma_{\mathrm{mult}}}\widetilde{\sigma_{\mathrm{mult}}}^\dagger)}_{kk}(\tilde y^\varepsilon(\tau))\right)\delta_{k,n-k}\\ &\le |\tilde y_k^\varepsilon(\tau)|^2\ (\tilde{\boldsymbol\sigma}\tilde{\boldsymbol\sigma}^\dagger)_{kk}(\tilde y^\varepsilon(\tau)).\end{aligned}$$

Now Eqs. (4.210) and (4.217) to (4.219) imply that

$$\begin{aligned}&\mathbb{E}\left[\left(\rho_k^{+,\varepsilon}(t)\right)^\alpha\right]\\ &\quad\le \mathbb{E}\left[\left(\rho_k^{+,\varepsilon}(0)\right)^\alpha\right] + 2\alpha\, c\,\mathbb{E}\left[\int_0^t \left(\rho_k^{+,\varepsilon}(\tau)\right)^{\alpha-1}\left(\rho_k^{+,\varepsilon}(\tau)+c\right)\mathrm{d}\tau\right]\\ &\qquad + 8\,\alpha(\alpha-1)\,\mathbb{E}\left[\int_0^t \left(\rho_k^{+,\varepsilon}(\tau)\right)^{\alpha-1}\left(n^2\left(|\tilde\nu_k|^2+|\tilde\nu_{n-k}|^2+\sigma_r^2\right)\rho_k^{+,\varepsilon}(\tau)+\sigma_0^2\right)\mathrm{d}\tau\right]\\ &\quad\le \mathbb{E}\left[\left(\rho_k^{+,\varepsilon}(0)\right)^\alpha\right] + 2\alpha\, c\,(c+1)\int_0^t \mathbb{E}\left[\left(\rho_k^{+,\varepsilon}(\tau)\right)^\alpha\right]\mathrm{d}\tau + 2\alpha\, c^2\, t\\ &\qquad + 8\,\alpha(\alpha-1)\,\left(n^2\left(|\tilde\nu_k|^2+|\tilde\nu_{n-k}|^2+\sigma_r^2\right)+\sigma_0^2\right)\int_0^t \mathbb{E}\left[\left(\rho_k^{+,\varepsilon}(\tau)\right)^\alpha\right]\mathrm{d}\tau + \sigma_0^2\, t\\ &\quad\le C(1+t) + C'\left(\int_0^t \mathbb{E}\left[\left(\rho_k^{+,\varepsilon}(\tau)\right)^\alpha\right]\mathrm{d}\tau\right),\end{aligned}\tag{4.220}$$

for suitable constants $0 < C, C' < \infty$, which by Eq. (4.205) can be chosen independently of ε. Note that we have used *Fubini's theorem* as well as the estimate

$$\rho^{\alpha-1}\,(a\,\rho + c) \le (a+c)\,\rho^\alpha + c,\tag{4.221}$$

which holds for all $\rho, a, c > 0$. Now by *Gronwall's inequality*, it follows that

$$\mathbb{E}\left[\left(\rho_k^{+,\varepsilon}(t)\right)^\alpha\right] \le C(1+t)\,\mathrm{e}^{C'\,t} \le C(1+T)\,\mathrm{e}^{C'\,T} =: C_T,\quad \forall 0\le t\le T,\tag{4.222}$$

which concludes the proof of Eq. (4.215) and thus validates Eq. (4.216).

Estimate of second term:

An upper bound for the second term of Eq. (4.207) can be obtained by using the *Burkholder-Davis-Gundy inequality*[27]

$$\begin{aligned}\mathbb{E}\left(\left|\left(\int_s^t \sigma_{\mathsf{p}}(\tilde y^\varepsilon(\tau))\,\mathrm{d}\boldsymbol{B}(\tau)\right)_{kl}\right|^\alpha\right) &\le c_\alpha\,\mathbb{E}\left(\left[\int_s^t \left(\sigma_{\mathsf{p}}(\tilde y^\varepsilon(\tau))\,{\sigma_{\mathsf{p}}}^\dagger(\tilde y^\varepsilon(\tau))\right)_{kl,kl}\mathrm{d}\tau\right]^{\alpha/2}\right)\\ &\le (t-s)^{\alpha/2-1}\,c_\alpha\int_s^t \mathbb{E}\left(\left|\left(\sigma_{\mathsf{p}}(\tilde y^\varepsilon(\tau))\,{\sigma_{\mathsf{p}}}^\dagger(\tilde y^\varepsilon(\tau))\right)_{kl,kl}\right|^{\alpha/2}\right)\mathrm{d}\tau,\end{aligned}\tag{4.223}$$

[27] c.f. [KS91], Theorem 3.3.28

where $c_\alpha > 0$ is a constant which only depends on α. In the second step we have, as before, applied *Jensen's inequality* and *Fubini's theorem*. Notice that by Lemma 2.40 and our previous estimate (Eq. (4.209)) we have

$$\begin{aligned}\left|\left(\sigma_{\mathsf{p}}(\tilde{y}^\varepsilon(\tau))\,{\sigma_{\mathsf{p}}}^\dagger(\tilde{y}^\varepsilon(\tau))\right)_{kl,kl}\right| &\le \rho_k^\varepsilon(\tau)\,(\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}^\dagger)_{ll}(\tilde{y}^\varepsilon(\tau)) + \rho_l^\varepsilon(\tau)\,(\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}^\dagger)_{kk}(\tilde{y}^\varepsilon(\tau))\\ &\quad + |\tilde{y}_k^\varepsilon(\tau)|\,|\tilde{y}_l^\varepsilon(\tau)|\,|\tilde{\boldsymbol{\sigma}}_{kk}(\tilde{y}^\varepsilon(\tau))|\,|\tilde{\boldsymbol{\sigma}}_{n-l,n-l}(\tilde{y}^\varepsilon(\tau))|\\ &\quad + |\tilde{y}_l^\varepsilon(\tau)|\,|\tilde{y}_k^\varepsilon(\tau)|\,|\tilde{\boldsymbol{\sigma}}_{ll}(\tilde{y}^\varepsilon(\tau))|\,|\tilde{\boldsymbol{\sigma}}_{n-k,n-k}(\tilde{y}^\varepsilon(\tau))|\\ &\le c\,\left(\rho_k^{+,\varepsilon}(\tau)+1\right)\,\left(\rho_l^{+,\varepsilon}(\tau)+1\right), \end{aligned} \tag{4.224}$$

for a suitable, ε-independent constant $c > 0$. Now by the *Cauchy-Schwarz inequality* and *Jensen's inequality* it follows that

$$\begin{aligned}\mathbb{E}\left(\left|\left(\sigma_{\mathsf{p}}(\tilde{y}^\varepsilon(\tau))\,{\sigma_{\mathsf{p}}}^\dagger(\tilde{y}^\varepsilon(\tau))\right)_{kl,kl}\right|^{\alpha/2}\right) &\le c^{\alpha/2}\sqrt{\mathbb{E}\left(\left|\rho_k^{+,\varepsilon}(\tau)+1\right|^\alpha\right)}\sqrt{\mathbb{E}\left(\left|\rho_l^{+,\varepsilon}(\tau)+1\right|^\alpha\right)}\\ &\le c^{\alpha/2}\,2^{\alpha-1}\sqrt{\mathbb{E}\left(\left|\rho_k^{+,\varepsilon}(\tau)\right|^\alpha\right)+1}\sqrt{\mathbb{E}\left(\left|\rho_l^{+,\varepsilon}(\tau)\right|^\alpha\right)+1}. \end{aligned} \tag{4.225}$$

By Eq. (4.215), the right-hand side can be bounded uniformly in $0 \le \tau \le T$ and $\varepsilon > 0$ by some constant C_T, which is why Eq. (4.223) yields

$$\mathbb{E}\left(\left|\int_s^t \left(\sigma_{\mathsf{p}}(\tilde{y}^\varepsilon(\tau))\,\mathrm{d}\boldsymbol{B}(\tau)\right)_{kl}\right|^\alpha\right) \le (t-s)^{\alpha/2}\,c_\alpha\,c^{\alpha/2}\,2^{\alpha-1}\,(C_T+1), \quad \forall\, 0 \le s,t, \le T, \quad \forall \varepsilon > 0. \tag{4.226}$$

Together with Eq. (4.216) this yields **ii)** and concludes the proof. □

Application of this tightness result allows us to extend Proposition 4.67.

Theorem 4.69 (Weak convergence of π process[a])
Assume that the initial distributions $\mu_\varepsilon = \mathbb{P}(Y_0^\varepsilon)^{-1}$ are absolutely continuous w.r.t. to μ and that their densities $(\zeta_\varepsilon)_{\varepsilon>0}$ weakly converge in $L^2(\mathbb{R}^N)$ to $\zeta \circ \pi$, where $\zeta \in L^2_{\pi_*\mu}(\Gamma)$, s.t. $\mathrm{d}\widehat{\mu} = \zeta \circ \pi\,\mathrm{d}\mu$ defines a probability measure $\widehat{\mu}$.

Then, in the limit of $\varepsilon \to 0$, the processes $(\pi^\varepsilon)_{\varepsilon>0}$ weakly converge to the process $\widehat{\pi}$, with initial distribution $\pi_*\widehat{\mu}$ given by $\mathrm{d}(\pi_*\widehat{\mu}) = \zeta\,\mathrm{d}(\pi_*\mu)$.

[a] c.f. [BR14], Theorem 6

Proof. As noted in the proof of [BR14], Theorem 6, this statement is a direct consequence of the convergence of the finite dimensional distributions (Proposition 4.67) and the tightness result (Lemma 4.68), which extends the convergence to the process level (c.f. [KS91], Theorem 2.4.15). Note that the $L^2(\mathbb{R}^N)$ convergence of the densities $(\zeta_\varepsilon)_{\varepsilon>0}$ in particular ensures that the condition of '$\sup_{\varepsilon>0}\mathbb{E}\left(\|Y_0^\varepsilon\|^2\right) < \infty$' (c.f. Eq. (4.7)) is satisfied. □

Since all level sets of P are connected (c.f. Lemmas 4.34 and 4.35), this convergence result translates to a convergence of the first integrals.

Corollary 4.70 (Weak convergence of P process)
Under the assumptions of Theorem 4.69, the processes $(\mathsf{P}^{\varepsilon_k})_k$ weakly converge to the process $\widehat{\mathsf{P}} := \psi^{-1}(\widehat{\pi})$ with initial distribution $\mathsf{P}_*\widehat{\mu}$.

Proof. This follows directly from Theorem 4.69 and the isomorphism ψ, which was specified in Lemma 4.34.

4.8. Averaging Theorem

Combining the previous steps we obtain our main result.

Theorem 4.71 (Averaging theorem)
Let $(\tilde{y}^\varepsilon(t))_{t\geq 0}$ be the time-rescaled process specified in Lemma 3.38, i.e. determined by

$$\mathrm{d}\tilde{y}^\varepsilon(t) = i\left(\frac{1}{\varepsilon}\right)\tilde{y}^\varepsilon(t)\,\mathrm{d}t + \tilde{u}(\tilde{y}^\varepsilon(t))\,\mathrm{d}t + \tilde{\boldsymbol{\sigma}}(\tilde{y}^\varepsilon(t))\,\mathrm{d}\widetilde{\boldsymbol{B}}(t). \tag{4.227}$$

Assume that the initial distributions $\mathbb{P}(\tilde{y}^\varepsilon(0))^{-1}$ are absolutely continuous w.r.t. to the Lebesgue measure μ on $\mathbb{C}^n$. Furthermore assume that, in the limit of $\varepsilon \to 0$, the densities $\zeta_\varepsilon := \frac{\mathrm{d}\mathbb{P}(\tilde{y}^\varepsilon(0))^{-1}}{\mathrm{d}\mu}$ weakly converge in $L^2(\mathbb{C}^n)$ to $\zeta \circ \mathsf{p}$, where $\zeta \in L^2_{\mathsf{p}_*\mu}(\mathsf{p}(\mathbb{C}^n))$, s.t. $\mathrm{d}\widehat{\mu} = \zeta \circ \mathsf{p}\,\mathrm{d}\mu$ defines a probability measure $\widehat{\mu}$.[a]

In the limit of $\varepsilon \to 0$, the processes $(\mathsf{p}^\varepsilon(t))_{t\geq 0}$ defined by $\mathsf{p}^\varepsilon(t) := \mathsf{p}(\tilde{y}^\varepsilon(t)) = \tilde{y}^\varepsilon(t)(\tilde{y}^\varepsilon(t))^\dagger$, weakly converge to a process $(\widehat{\mathsf{p}}(t))_{t\geq 0}$, i.e.

$$(\mathsf{p}^\varepsilon(t))_{t\geq 0} \Rightarrow (\widehat{\mathsf{p}}(t))_{t\geq 0}, \quad \varepsilon \to 0, \tag{4.228}$$

The limiting process $(\widehat{\mathsf{p}}(t))_{t\geq 0}$ has an initial distribution $\mathsf{p}_*\widehat{\mu}$ given by $\mathrm{d}(\mathsf{p}_*\widehat{\mu}) = \zeta\,\mathrm{d}(\pi_*\mu)$ and its law is specified by the generator

$$\begin{aligned}\mathcal{A}f(\hat{p}) = &\left(\widehat{u}_{\mathsf{p}}^\top(\hat{p})\,\partial_{\mathsf{p}} + \widehat{u}_{\mathsf{p}}^\dagger(\hat{p})\,\overline{\partial_{\mathsf{p}}}\right) f(\hat{p})\\ &+ \left[\widehat{a}_{\mathsf{p}}(\hat{p}) : (\partial_{\mathsf{p}}\partial_{\mathsf{p}}^\dagger) + \frac{1}{2}\left(\widehat{a}'_{\mathsf{p}}(\hat{p}) : (\partial_{\mathsf{p}}\partial_{\mathsf{p}}^\top) + \overline{\widehat{a}'_{\mathsf{p}}}(\hat{p}) : (\overline{\partial_{\mathsf{p}}\partial_{\mathsf{p}}^\top})\right)\right] f(\hat{p}),\end{aligned} \tag{4.229}$$

i.e. its evolution is governed by an *averaged* drift term $\widehat{u}_{\mathsf{p}}$ of the form

$$\widehat{u}_{\mathsf{p}}(\hat{p}) := \widehat{u}_{\mathsf{p},\det}(\hat{p}) + \widehat{u}_{\mathsf{p},\text{Itô}}(\hat{p}), \tag{4.230}$$

where

$$\widehat{u}_{\mathsf{p},\det}(\hat{p}) := \frac{1}{\mathcal{H}_1(\mathsf{p}^{-1}(\hat{p}))}\oint_{\mathsf{p}^{-1}(\hat{p})}\left(\tilde{u}(\tilde{y})\tilde{y}^\dagger + \text{h.c.}\right)\mathrm{d}\mathcal{H}_1(\tilde{y}), \tag{4.231}$$

$$\widehat{u}_{\mathsf{p},\text{Itô}}(\hat{p}) := \frac{1}{\mathcal{H}_1(\mathsf{p}^{-1}(\hat{p}))}\oint_{\mathsf{p}^{-1}(\hat{p})}\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}^\dagger(\tilde{y})\,\mathrm{d}\mathcal{H}_1(\tilde{y}), \tag{4.232}$$

and by averaged diffusion terms $\widehat{a}_{\mathsf{p}}, \widehat{a}'_{\mathsf{p}}$ defined as

$$(\widehat{a}_{\mathsf{p}})_{ij,kl}(\hat{p}) := \frac{1}{\mathcal{H}_1(\mathsf{p}^{-1}(\hat{p}))} \oint_{\mathsf{p}^{-1}(\hat{p})} (\sigma_{\mathsf{p}}\, \sigma_{\mathsf{p}}^{\dagger})_{ij,kl}(\tilde{y})\, \mathrm{d}\mathcal{H}_1(\tilde{y}) \tag{4.233}$$

$$\begin{aligned} &= \frac{\hat{p}_{ik}}{\mathcal{H}_1(\mathsf{p}^{-1}(\hat{p}))} \oint_{\mathsf{p}^{-1}(\hat{p})} \left(\overline{\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}^{\dagger}}(\tilde{y})\right)_{jl} \mathrm{d}\mathcal{H}_1(\tilde{y}) \\ &+ \frac{\overline{\hat{p}}_{jl}}{\mathcal{H}_1(\mathsf{p}^{-1}(\hat{p}))} \oint_{\mathsf{p}^{-1}(\hat{p})} \left(\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}^{\dagger}(\tilde{y})\right)_{ik} \mathrm{d}\mathcal{H}_1(\tilde{y}) \\ &+ \frac{1}{\mathcal{H}_1(\mathsf{p}^{-1}(\hat{p}))} \oint_{\mathsf{p}^{-1}(\hat{p})} (\tilde{y}\tilde{y}^{\top})_{il} \left(\overline{\tilde{\boldsymbol{\sigma}}\mathcal{R}\tilde{\boldsymbol{\sigma}}^{\top}(\tilde{y})}\right)_{jk} \mathrm{d}\mathcal{H}_1(\tilde{y}) \\ &+ \frac{1}{\mathcal{H}_1(\mathsf{p}^{-1}(\hat{p}))} \oint_{\mathsf{p}^{-1}(\hat{p})} \overline{(\tilde{y}\tilde{y}^{\top})}_{jk} \left(\tilde{\boldsymbol{\sigma}}\mathcal{R}\tilde{\boldsymbol{\sigma}}^{\top}(\tilde{y})\right)_{il} \mathrm{d}\mathcal{H}_1(\tilde{y}), \end{aligned} \tag{4.234}$$

$$\left(\widehat{a}'_{\mathsf{p}}\right)_{ij,kl}(\hat{p}) := (\widehat{a}_{\mathsf{p}})_{ij,lk}(\hat{p}), \tag{4.235}$$

where $\mathcal{R} := \mathbb{1}_{5\times 5} \otimes R = \mathrm{blockdiag}(R, R, R, R, R) \in \mathbb{C}^{5n,5n}$.

[a]Note that we have redefined the densities ζ_ε, ζ from Theorem 4.69 in terms of the complex outer product p instead of the projection π. This is an equivalent choice, since p and π can be identified, c.f. Lemmas 4.34 and 4.35 and Fig. 4.3.

Proof. Remark 4.5 and Lemma 4.9 imply that Theorem 4.69 can be applied to the family of processes $\mathsf{p}^\varepsilon(t) := \mathsf{p}(\tilde{y}^\varepsilon(t)) = \tilde{y}^\varepsilon(t)(\tilde{y}^\varepsilon(t))^\dagger$. Application of Theorem 4.69 yields Eq. (4.228) since by Lemma 4.35 we know that π and p are isomorphic. The generator describing the limiting process $(\widehat{\mathsf{p}}(t))_{t\geq 0}$ is given by Lemma 4.61 and Proposition 4.63, which yield Eqs. (4.229) to (4.233). Equations (4.234) and (4.235) follow from Lemma 2.40 in the special case of $\mathcal{Q} := \mathbb{1}_{5\times 5} \otimes Q$ and $\mathcal{R} := \mathbb{1}_{5\times 5} \otimes R$, where Q is the matrix corresponding to the inverse DFT and $R = Q^2$ denotes the reflection matrix. Comparison of Eq. (4.229) with the results of Corollary 2.37, finally allows us to identify $\widehat{u}_{\mathsf{p}}$ and $\widehat{a}_{\mathsf{p}}, \widehat{a}'_{\mathsf{p}}$ as the drift and diffusion terms governing the process $(\widehat{\mathsf{p}}(t))_{t\geq 0}$. □

We conclude this chapter by noting that a decomposition of the averaged diffusion matrix $\widehat{a}_{\mathsf{p}}$ allows us to separately calculate the averaging contributions of multiplicative, regularizing and additive noise.

Lemma 4.72 (Decomposition of diffusion matrix)
We can decompose the averaged diffusion matrix into a multiplicative-, regularizing- and an additive-noise part, i.e.

$$\widehat{a}_{\mathsf{p}} = \widehat{a}_{\mathsf{p},\mathrm{mult}} + \widehat{a}_{\mathsf{p},\mathrm{reg}} + \widehat{a}_{\mathsf{p},\mathrm{add}}, \tag{4.236}$$

where for all "type ∈ {mult, reg, add}" we define

$$
\begin{aligned}
\left(\widehat{a}_{\mathsf{p},\text{type}}\right)_{ij,kl}(\hat{p}) \coloneqq\ & \frac{\hat{p}_{ik}}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} \left(\overline{\widetilde{\sigma_{\text{type}}}\widetilde{\sigma_{\text{type}}}^{\dagger}}(\tilde{y})\right)_{jl} \mathrm{d}\mathcal{H}_1(\tilde{y}) \\
&+ \frac{\overline{\hat{p}}_{jl}}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} \left(\widetilde{\sigma_{\text{type}}}\widetilde{\sigma_{\text{type}}}^{\dagger}(\tilde{y})\right)_{ik} \mathrm{d}\mathcal{H}_1(\tilde{y}) \\
&+ \frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} (\tilde{y}\tilde{y}^{\top})_{il} \left(\overline{\widetilde{\sigma_{\text{type}}}\mathcal{R}_m\, \widetilde{\sigma_{\text{type}}}^{\top}(\tilde{y})}\right)_{jk} \mathrm{d}\mathcal{H}_1(\tilde{y}) \\
&+ \frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} \overline{(\tilde{y}\tilde{y}^{\top})}_{jk} \left(\widetilde{\sigma_{\text{type}}}\mathcal{R}_m\, \widetilde{\sigma_{\text{type}}}^{\top}(\tilde{y})\right)_{il} \mathrm{d}\mathcal{H}_1(\tilde{y}),
\end{aligned} \tag{4.237}
$$

where $m = 1$ for "*type* = *mult*" and $m = 2$ otherwise. Here $\mathcal{R}_2 \coloneqq \text{blockdiag}(R, R)$, while $\mathcal{R}_1 \coloneqq R$ (c.f. comment after Lemma 3.37).

Proof. Since $\widetilde{\boldsymbol{\sigma}} = (\widetilde{\sigma_{\text{mult}}} \mid \widetilde{\boldsymbol{\sigma}_{\text{reg}}} \mid \widetilde{\boldsymbol{\sigma}_{\text{add}}})$, the diffusion matrices which need to be averaged decompose as follows:

$$
\widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\sigma}}^{\dagger}(\tilde{y}) = \widetilde{\sigma_{\text{mult}}}(\widetilde{\sigma_{\text{mult}}})^{\dagger}(\tilde{y}) + \widetilde{\boldsymbol{\sigma}_{\text{reg}}}\widetilde{\boldsymbol{\sigma}_{\text{reg}}}^{\dagger}(\tilde{y}) + \widetilde{\boldsymbol{\sigma}_{\text{add}}}(\widetilde{\boldsymbol{\sigma}_{\text{add}}})^{\dagger},
$$

and similarly we have:

$$
\widetilde{\boldsymbol{\sigma}}\mathcal{R}_5\, \widetilde{\boldsymbol{\sigma}}^{\top}(\tilde{y}) = \widetilde{\sigma_{\text{mult}}} R \left(\widetilde{\sigma_{\text{mult}}}\right)^{\top}(\tilde{y}) + \widetilde{\boldsymbol{\sigma}_{\text{reg}}}\mathcal{R}_2 \left(\widetilde{\boldsymbol{\sigma}_{\text{reg}}}\right)^{\top}(\tilde{y}) + \widetilde{\boldsymbol{\sigma}_{\text{add}}}\mathcal{R}_2 \left(\widetilde{\boldsymbol{\sigma}_{\text{add}}}\right)^{\top}. \qquad \square
$$

5 Averaged system

In this chapter we explicit determine the averaged drift term $\widehat{u}_{\mathsf{p}}$ and the averaged diffusion matrix $\widehat{a}_{\mathsf{p}}$, which govern the evolution of the limiting process $(\mathsf{p}^{\varepsilon}(t))_{t\geq 0}$, c.f. Theorem 4.71. For this purpose, we evaluate the averaging integrals by application of the residue theorem. We furthermore provide a representation of the averaged diffusion matrix $\widehat{a}_{\mathsf{p}}$ in terms of an 'effective' dispersion matrix $\widehat{\sigma_{\mathsf{p}}}$, i.e. $\widehat{a}_{\mathsf{p}} = \widehat{\sigma_{\mathsf{p}}}\,\widehat{\sigma_{\mathsf{p}}}^{\dagger}$, c.f. Proposition 5.24.

5.1. Averaging via residue theorem

For $y_* \in \mathbb{C}^n$ and $\hat{p} := \mathsf{p}(y_*)$, we observe that the level set $\mathsf{p}^{-1}(\hat{p})$ of the complex outer product can be represented as

$$\mathsf{p}^{-1}(\hat{p}) = \left\{ z\, y_* \mid z \in S^1 \right\}, \tag{5.1}$$

which is why for any continuous function $f : \mathbb{C}^n \to \mathbb{C}$, we can parametrize the averaging integral over this level set as

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} f(y)\, \mathrm{d}\mathcal{H}_1(y) = \frac{1}{2\pi} \int_0^{2\pi} f\left(\mathrm{e}^{it}\, y_*\right) \mathrm{d}t. \tag{5.2}$$

Application of the residue theorem (c.f. Theorem 2.20) allows us to evaluate integrals of this form.

Proposition 5.1 (Averaging integration via residue theorem)
Let $f : \mathbb{C}^n \to \mathbb{C}^m$ be continuous, $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$ and $y_* \in \mathsf{p}^{-1}(\hat{p})$. If there is a function

$$h_{y_*} : \mathbb{C} \to \mathbb{C}^m, \tag{5.3}$$

satisfying

$$h_{y_*}(z) = f\left(z\, y_*\right), \quad \forall z \in S^1, \tag{5.4}$$

and which is holomorphic in $K_{*,1+\epsilon}$ for some $\epsilon > 0$ (c.f. Eq. (2.57)), then it follows that

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} f(y)\, \mathrm{d}\mathcal{H}_1(y) = \mathrm{Res}|_{z=0} \left(\frac{h_{y_*}(z)}{z}\right). \tag{5.5}$$

For the case of $n = m$, we in particular find that

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} f(y)\, y^\dagger\, \mathrm{d}\mathcal{H}_1(y) = \mathrm{Res}|_{z=0} \left(\frac{h_{y_*}(z)}{z^2}\right) y_*^\dagger. \tag{5.6}$$

Proof. Employing Eq. (5.2), we can, for all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$ and all $y_* \in \mathsf{p}^{-1}(\hat{p})$, rewrite the averaging integration in terms of an integral in the complex plane. In the general case we find

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} f(y)\, \mathrm{d}\mathcal{H}_1(y) = \frac{1}{2\pi} \int_0^{2\pi} f(y_*\, \mathrm{e}^{it})\, \mathrm{d}t = \frac{1}{2\pi i} \oint_{S^1} \frac{f\left(z\, y_*\right)}{z}\, \mathrm{d}z,$$

where in the second to last step we have set $z(t) := \mathrm{e}^{it}$ and made use of $\mathrm{d}z(t) = iz(t)\, \mathrm{d}t$. Employing Eq. (5.4) allows us to replace $f\left(z\, y_*\right)$ in the integrand by $h_{y_*}(z)$. Since h_{y_*} was assumed to be holomorphic in a neighborhood of the unit disc (with exception of the origin), we can apply Theorem 2.20 and obtain the desired result. For the special case of $n = m$, we similarly observe that

$$\begin{aligned}\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} f(y)\, y^\dagger\, \mathrm{d}\mathcal{H}_1(y) &= \left(\frac{1}{2\pi} \int_0^{2\pi} f(y_*\, \mathrm{e}^{it})\, \mathrm{e}^{-it}\, \mathrm{d}t\right) y_*^\dagger \\ &= \left(\frac{1}{2\pi i} \oint_{S^1} \frac{h_{y_*}(z)}{z^2}\, \mathrm{d}z\right) y_*^\dagger,\end{aligned}$$

which by virtue of Theorem 2.20 concludes the proof. □

If f is a *holomorphic* function, the averaging integration turns out to have a *linearizing* effect.

Proposition 5.2 (Averaging of holomorphic functions)
Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a *holomorphic* function with an expansion at $y = 0$ of the form

$$f_k(y) = \sum_{l_0,\ldots,l_{n-1}=0}^{\infty} \alpha_k^{(l_0,\ldots,l_{n-1})}\, y_0^{l_0} \cdots y_{n-1}^{l_{n-1}}, \quad \forall y \in \mathbb{C}^n, \quad \forall k \in \{0, ..., n-1\}. \tag{5.7}$$

Then for all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$ it follows that

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} f(y)\, y^\dagger\, \mathrm{d}\mathcal{H}_1(y) = \mathsf{L}\, \hat{p}, \tag{5.8}$$

where the components of the matrix L are defined as

$$\mathsf{L}_{ij} = \alpha_i^{(0,\ldots,l_j=1,\ldots,0)}, \tag{5.9}$$

i.e. $y \to \mathsf{L}\, y$ is the linearization of f.
More specifically, we find that both real- and imaginary part of f contribute equally to this

result, i.e.

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} \mathrm{Re}\,(f)\,(y)\,y^\dagger\;\mathrm{d}\mathcal{H}_1(y) = \frac{1}{2}\,\mathsf{L}\,\hat{p}, \tag{5.10a}$$

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} i\,\mathrm{Im}\,(f)\,(y)\,y^\dagger\;\mathrm{d}\mathcal{H}_1(y) = \frac{1}{2}\,\mathsf{L}\,\hat{p}. \tag{5.10b}$$

Proof. Let $y_* \in \mathsf{p}^{-1}(\hat{p})$ and notice that for all $k \in \{0, ..., n-1\}$, the map

$$z \to (h_{y_*})_k(z) := f_k(z\,y_*) = \sum_{l_0,\ldots,l_{n-1}=0}^{\infty} \alpha_k^{(l_0,\ldots,l_{n-1})}\,(y_*)_0^{l_0} \cdots (y_*)_{n-1}^{l_{n-1}}\; z^{l_0+\ldots+l_{n-1}}, \tag{5.11}$$

is holomorphic in all of $\mathbb{C}$ (i.e. $(h_{y_*})_k$ is an *entire* function). Moreover, we find that

$$\begin{aligned}
\mathrm{Res}|_{z=0}\left(\frac{(h_{y_*})_k(z)}{z^2}\right) &= \sum_{l_0+\ldots+l_{n-1}=1} \alpha_k^{(l_0,\ldots,l_{n-1})}\,(y_*)_0^{l_0} \cdots (y_*)_{n-1}^{l_{n-1}} \\
&= \alpha_k^{(1,0,\ldots,0)}(y_*)_0 + \alpha_k^{(0,1,\ldots,0)}(y_*)_1 + \ldots + \alpha_k^{(0,0,\ldots,1)}(y_*)_{n-1} \\
&= \sum_{l=0}^{n-1} \mathsf{L}_{k,l}\,(y_*)_l = (\mathsf{L}\,y_*)_k\,.
\end{aligned}$$

Thus, Proposition 5.1 yields

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} f(y)\,y^\dagger\;\mathrm{d}\mathcal{H}_1(y) = \mathrm{Res}|_{z=0}\left(\frac{h_{y_*}(z)}{z^2}\right) y_*^\dagger = \mathsf{L}\,y_* y_*^\dagger = \mathsf{L}\,\hat{p}. \tag{5.12}$$

Since

$$\overline{z} = \frac{1}{z}, \quad \forall z \in S^1, \tag{5.13}$$

we conclude that for all $k \in \{0, ..., n-1\}$, the map

$$(\hat{h}_{y_*})_k(z) := \sum_{l_0,\ldots,l_{n-1}=0}^{\infty} \overline{\alpha}_k^{(l_0,\ldots,l_{n-1})}\,(\overline{y_*})_0^{l_0} \cdots (\overline{y_*})_{n-1}^{l_{n-1}}\; \frac{1}{z^{l_0+\ldots+l_{n-1}}}, \quad \forall z \in \mathbb{C}_*, \tag{5.14}$$

is holomorphic on $\mathbb{C}_*$. Furthermore, it coincides on S^1 with $\overline{f_k}(z\,y_*)$, since $z^{-1} = \overline{z}$ for all $z \in S^1$. Now Eq. (5.14) implies that

$$\mathrm{Res}|_{z=0}\left(\frac{\hat{h}_{y_*}}{z^2}\right) = 0, \tag{5.15}$$

which is why Proposition 5.1 yields

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} \overline{f}(y)\,y^\dagger\;\mathrm{d}\mathcal{H}_1(y) = \mathrm{Res}|_{z=0}\left(\frac{\hat{h}_{y_*}(z)}{z^2}\right) y_*^\dagger = 0, \tag{5.16}$$

and Eq. (5.10) follows. □

We state two immediate consequences of the previous proposition.

Corollary 5.3 (Averaging of conjugate of holomorphic function)
Let $f : \mathbb{C}^n \to \mathbb{C}^n$ and assume that its complex conjugate $\overline{f}$ is an *entire* function (i.e. holomorphic on all of $\mathbb{C}^n$). Then for all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$ it follows that

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} f(y)\, y^\dagger \, \mathrm{d}\mathcal{H}_1(y) = 0. \tag{5.17}$$

Proof. This follows along the same lines as in the proof of Proposition 5.2, c.f. Eq. (5.16). □

Corollary 5.4 (Averaging of complex linear function)
Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a complex linear mapping, i.e. there is some matrix $K \in \mathbb{C}^{n,n}$, s.t.

$$f(y) = K\, y, \quad \forall y \in \mathbb{C}^n. \tag{5.18}$$

Then for all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$, we find that

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} \mathrm{Re}\left(f(y)\right)\, y^\dagger \, \mathrm{d}\mathcal{H}_1(y) = \frac{1}{2} K\, \hat{p}, \tag{5.19a}$$

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} i\,\mathrm{Im}\left(f(y)\right)\, y^\dagger \, \mathrm{d}\mathcal{H}_1(y) = \frac{1}{2} K\, \hat{p}. \tag{5.19b}$$

Proof. K coincides with the linear expansion matrix L defined in Eq. (5.9), so the results follow from Proposition 5.2. □

5.2. Averaged drift

The results of the previous section in particular allow us to calculate the averaged *deterministic* drift term $\widehat{u}_{\mathsf{p},\mathrm{det}}$. The averaged contribution $\widehat{u}_{\mathsf{p},\mathrm{Itô}}$ from the Itô correction will be determined in Lemma 5.25.

Lemma 5.5 (Averaged linear drift)
For all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$, the averaged contribution from the *linear* drift term is given by

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} \left(u_{\mathrm{lin}}(y)\, y^\dagger + \mathrm{h.c.}\right) \mathrm{d}\mathcal{H}_1(y) = \frac{\sqrt{n}}{2} \left[\mathrm{diag}\left(\widetilde{\lambda}\right)\, \hat{p} + \hat{p}\, \overline{\mathrm{diag}\left(\widetilde{\lambda}\right)}\right] \tag{5.20}$$

Proof. By Eq. (3.93) we have

$$\tilde{u}(\tilde{y}) = \frac{\sqrt{n}}{2} \left(\mathrm{diag}(\widetilde{\lambda})\tilde{y} + R\,\overline{\mathrm{diag}(\widetilde{\lambda})\tilde{y}}\right). \tag{5.21}$$

Application of Corollary 5.3 yields that the second summand of Eq. (5.21) does not contribute to the integral of Eq. (5.20) and we can make use of Corollary 5.4 to find that

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} \left(u_{\mathrm{lin}}(y)\, y^{\dagger} + \mathrm{h.c.}\right) \,\mathrm{d}\mathcal{H}_1(y) = \frac{\sqrt{n}}{2} \operatorname{diag}(\widetilde{\lambda})\,\hat{p} + \mathrm{h.c.} \tag{5.22}$$

Since $\hat{p}$ is hermitian, the result follows. □

We remark that the 'complex conjugated parts' of the linear drift term $u_{\mathrm{lin}}(\tilde{y})$ have no impact on the effective outer-product drift term $\widehat{u}_{\mathsf{p},\mathrm{det}}$.

Remark 5.6 (Complex conjugated terms do not contribute to averaged drift)
In the proof of Lemma 5.5 we have seen that the second summand of the drift term

$$\widetilde{u_{\mathrm{lin}}}(\tilde{y}) = \frac{\sqrt{n}}{2} \left(\operatorname{diag}(\widetilde{\lambda})\tilde{y} + R\,\overline{\operatorname{diag}(\widetilde{\lambda})\tilde{y}}\right) \tag{5.23}$$

does *not* contribute to the averaged drift term $\widehat{u}_{\mathsf{p},\mathrm{det}}$ of Eq. (5.20). Recalling Eq. (3.95), we observe that this term is the DFT of $\frac{1}{2}\,\overline{\Lambda y}$. This implies that, starting with the drift term

$$u_{\mathrm{lin}}(y) = \mathrm{Re}\left(\Lambda y\right) = \frac{1}{2}\left(\Lambda y + \overline{\Lambda y}\right), \tag{5.24}$$

only the first term contributes to the averaged evolution. Thus we could replace $u_{\mathrm{lin}}(y)$ by an 'effective' drift term of the form

$$u_{\mathrm{eff}}(y) := \frac{1}{2}\Lambda y, \tag{5.25}$$

without affecting the averaged evolution of the system.

A nonlinear drift term, of the form specified in Definition 3.20, can also be shown to yield a vanishing contribution to the averaged evolution.

Lemma 5.7 (Averaging of nonlinear perturbation)
The nonlinear coupling term u_{nl} introduced in Definition 3.20 does not contribute to the evolution of the averaged system, i.e.

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} u_{\mathrm{nl}}(y)\, y^{\dagger} \,\mathrm{d}\mathcal{H}_1(y) = 0. \tag{5.26}$$

Proof. For $k \in \{0, ..., n-1\}$ and $y \in \mathbb{C}^n$ we observe that

$$\begin{aligned}
(u_{\mathrm{nl}})_k(y) &= \sum_{r>k} \mathsf{A}_{k,r}(y)\, y_r + \sum_{r<k} \mathsf{A}_{k,r}(y)\, y_r \\
&= \sum_{r>k} h_{k,r}(\overline{y}_k, y_r)\, y_r - \sum_{r<k} \overline{h_{r,k}(\overline{y}_r, y_k)}\, y_r \\
&= \sum_{r>k} h_{k,r}(\overline{y}_k, y_r)\, y_r - \sum_{r<k} h_{r,k}(y_r, \overline{y}_k)\, y_r,
\end{aligned} \tag{5.27}$$

where in the second step we have employed Eq. (3.107). In the final step we have applied the identity $\overline{h_{r,k}(z)} = h_{r,k}(\overline{z})$, which holds by assumption (c.f. Definition 3.20), to the case of $z = (\overline{y}_r, y_k) \in \mathbb{C}^2$. Let $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$ and $y_* \in \mathsf{p}^{-1}(\hat{p})$. Defining

$$h_{y_*} : \mathbb{C} \to \mathbb{C}^n \tag{5.28}$$

by

$$(h_{y_*})_k\,(z) := \sum_{r>k} h_{k,r}\left(\frac{(\overline{y_*})_k}{z}, z\,(y_*)_r\right) z\,(y_*)_r - \sum_{r<k} h_{r,k}\left(z\,(y_*)_r, \frac{(\overline{y_*})_k}{z}\right) z\,(y_*)_r, \tag{5.29}$$

we obtain a function, which is holomorphic in $K_{*,1+\epsilon}$ for $\epsilon > 0$, and which satisfies

$$h_{y_*}(z) = u_{\mathrm{nl}}\,(z\,y_*)\,, \quad \forall z \in S^1. \tag{5.30}$$

This allows us to apply Eq. (5.6), from which we obtain that

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} u_{\mathrm{nl}}(y)\,y^\dagger\,\mathrm{d}\mathcal{H}_1(y) = \mathrm{Res}|_{z=0}\left(\frac{h_{y_*}(z)}{z^2}\right) y_*^\dagger. \tag{5.31}$$

Note that the holomorphy of the function $h_{k,r}$ implies that we have an analytic expansion of the form

$$h_{k,r}(z_0, z_1) = \sum_{l_0,l_1=0}^{\infty} \alpha_{k,r}^{(l_0,l_1)}\, z_0^{l_0}\, z_1^{l_1}, \quad \forall\,(z_0, z_1) \in \mathbb{C}^2. \tag{5.32}$$

Due to the antisymmetry (Eq. (3.108)), only *odd* combinations of z_0 and z_1 are admissable, i.e.

$$\alpha_{k,r}^{(l_0,l_1)} \neq 0 \quad \Rightarrow \quad l_0 - l_1 \in 2\mathbb{Z} + 1. \tag{5.33}$$

This implies that all terms in the expansion of $h_{k,r}((\overline{y_*})_k/z, z\,(y_*)_r)$ contain a factor of z^l, where l is an *odd* number, i.e. $l \in 2\mathbb{Z}+1$. Consequently, $h_{k,r}((\overline{y_*})_k/z, z\,(y_*)_r)\,z\,(y_*)_r$ contains only *even* powers of z and by a similar argument, the same is true for the second term of Eq. (5.29). Recalling the definition of the residue, we consequently find that

$$\mathrm{Res}|_{z=0}\left(\frac{h_{y_*}(z)}{z^2}\right) = 0. \tag{5.34}$$

□

We conclude this section by combining the averaging results of the linear drift term and of its nonlinear perturbation.

Lemma 5.8 (Averaged drift)
The averaged drift term $\widehat{u}_{\mathsf{p},\mathrm{det}}$ for the *deterministic* outer-product evolution, as defined in Eq. (4.231), is given by

$$\widehat{u}_{\mathsf{p},\mathrm{det}}(\hat{p}) = \frac{\sqrt{n}}{2}\left[\mathrm{diag}\left(\tilde{\lambda}\right)\,\hat{p} + \hat{p}\,\overline{\mathrm{diag}\left(\tilde{\lambda}\right)}\right]. \tag{5.35}$$

Proof. According to Eq. (4.231), $\widehat{u}_{\mathsf{p},\mathrm{det}}$ is given by

$$\widehat{u}_{\mathsf{p},\mathrm{det}}(\hat{p}) \coloneqq \frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} \left(\tilde{u}(\tilde{y})\tilde{y}^{\dagger} + \mathrm{h.c.}\right) \mathrm{d}\mathcal{H}_1(\tilde{y}), \quad \forall \hat{p} \in \mathsf{p}(\mathbb{C}^n), \tag{5.36}$$

where $u(\tilde{y}) = u_{\mathrm{lin}}(\tilde{y}) + u_{\mathrm{nl}}(\tilde{y})$, c.f. Eq. (3.54). Now the result follows from Lemmas 5.5 and 5.7.□

5.3. Averaged multiplicative noise

In this section we will first calculate the averaged multiplicative-noise diffusion matrix $\widehat{a}_{\mathsf{p},\mathrm{mult}}$ and subsequently construct a decomposition in terms of a dispersion matrix $\widehat{\sigma}_{\mathsf{p},\mathrm{mult}}$.

5.3.1. Calculation of averaged diffusion matrix

As in the deterministic case, we employ the residue theorem in order to evaluate the averaging integrals. We separately calculate the averaged versions of $\widetilde{\sigma_{\mathrm{mult}}}\widetilde{\sigma_{\mathrm{mult}}}^{\dagger}$ and $\widetilde{\sigma_{\mathrm{mult}}}\, R\, \widetilde{\sigma_{\mathrm{mult}}}^{\top}$, which appear in the definition of $\widehat{a}_{\mathsf{p},\mathrm{mult}}$, c.f. Eq. (4.237).

Lemma 5.9 (Averaging of $\widetilde{\sigma_{\mathrm{mult}}}\widetilde{\sigma_{\mathrm{mult}}}^{\dagger}$)
For all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$, we find that

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} \left(\widetilde{\sigma_{\mathrm{mult}}}\left(\widetilde{\sigma_{\mathrm{mult}}}\right)^{\dagger}\right)(\tilde{y})\, \mathrm{d}\mathcal{H}_1(\tilde{y}) = \frac{n^2}{4} \mathrm{diag}\left(\left(|\tilde{\nu}_k|^2(\hat{p}_{kk} + \hat{p}_{n-k,n-k})\right)_k\right). \tag{5.37}$$

Proof. Recall that by Eq. (3.135) we have

$$\widetilde{\sigma_{\mathrm{mult}}}(\tilde{y}) = \frac{n}{2}\, \mathrm{diag}\left(\mathrm{diag}(\tilde{\nu})\tilde{y} + R\, \overline{\mathrm{diag}(\tilde{\nu})\tilde{y}}\right), \quad \forall \tilde{y} \in \mathbb{C}^n, \tag{5.38}$$

which is why we can restrict all following calculations to the diagonal. Let $k \in \{0, ..., n-1\}$, $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$ and $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$. For all $\tilde{y} \in \mathbb{C}^n$, we set

$$f(\tilde{y}) \coloneqq \left|(\widetilde{\sigma_{\mathrm{mult}}})_{k,k}\right|^2(\tilde{y}) = \frac{n^2}{4}\Big[|\tilde{\nu}_k|^2 |\tilde{y}_k|^2 + |\tilde{\nu}_{n-k}|^2 |\tilde{y}_{n-k}|^2 \\ + \left(\tilde{\nu}_k \tilde{\nu}_{n-k} \tilde{y}_k \tilde{y}_{n-k} + \overline{\tilde{\nu}_k \tilde{\nu}_{n-k} \tilde{y}_k \tilde{y}_{n-k}}\right)\Big]. \tag{5.39}$$

Note that $h_{\tilde{y}_*}$, defined by

$$h_{\tilde{y}_*}(z) \coloneqq \frac{n^2}{4}\Big[|\tilde{\nu}_k|^2 |(\tilde{y}_*)_k|^2 + |\tilde{\nu}_{n-k}|^2 |(\tilde{y}_*)_{n-k}|^2 \\ + \left(\tilde{\nu}_k \tilde{\nu}_{n-k} (\tilde{y}_*)_k (\tilde{y}_*)_{n-k}\, z^2 + \overline{\tilde{\nu}_k \tilde{\nu}_{n-k} (\tilde{y}_*)_k (\tilde{y}_*)_{n-k}}\, \frac{1}{z^2}\right)\Big], \tag{5.40}$$

coincides for all $z \in S^1$ with $f(z\,\tilde{y}_*)$. Moreover, $h_{\tilde{y}_*}$ is holomorphic in $K_{*,1+\epsilon}$ for all $\epsilon > 0$. Proposition 5.1 therefore yields

$$\begin{aligned}\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} \left(\widetilde{\sigma_{\text{mult}}}\,(\widetilde{\sigma_{\text{mult}}})^\dagger\right)_{k,k}(\tilde{y})\,\mathrm{d}\mathcal{H}_1(\tilde{y}) &= \operatorname{Res}|_{z=0}\left(\frac{h_{y_*}(z)}{z}\right) \\ &= \frac{n^2}{4}\left[|\tilde{\nu}_k|^2\,|(\tilde{y}_*)_k|^2 + |\tilde{\nu}_{n-k}|^2\,|(\tilde{y}_*)_{n-k}|^2\right] \\ &= \frac{n^2}{4}\left[|\tilde{\nu}_k|^2\,\hat{p}_{kk} + |\tilde{\nu}_{n-k}|^2\,\hat{p}_{n-k,n-k}\right],\end{aligned}$$

where in the last step we have employed $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$. Since we have $\tilde{\nu}_k = \tilde{\nu}_{n-k}$,[1] the result follows. □

Similarly, we can determine the averaged version of $\widetilde{\sigma_{\text{mult}}}R\,(\widetilde{\sigma_{\text{mult}}})^\top$.

Lemma 5.10 (Averaging of $\widetilde{\sigma_{\text{mult}}}R\left(\widetilde{\sigma_{\text{mult}}}\right)^\top$)
For all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$ and all $i, j, k, l \in \{0, ..., n-1\}$, we find that

$$\begin{aligned}\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} \overline{(\tilde{y}\tilde{y}^\top)}_{jk}\left(\widetilde{\sigma_{\text{mult}}}R\,\widetilde{\sigma_{\text{mult}}}^\top(\tilde{y})\right)_{il}\mathrm{d}\mathcal{H}_1(\tilde{y}) \\ = \frac{n^2}{4}\left(\operatorname{diag}(\tilde{\nu}\odot\hat{p}_{\cdot,j})\,R\operatorname{diag}(\tilde{\nu}\odot\hat{p}_{\cdot,k})\right)_{il}.\end{aligned} \tag{5.41}$$

This implies that, for all $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$, we have

$$\frac{1}{2\pi i}\oint_{S^1}\frac{1}{z^3}\left(\widetilde{\sigma_{\text{mult}}}R\,\widetilde{\sigma_{\text{mult}}}^\top(z\,y_*)\right)\mathrm{d}z = \frac{n^2}{4}\operatorname{diag}(\tilde{\nu}\odot\tilde{y}_*)\,R\operatorname{diag}(\tilde{\nu}\odot\tilde{y}_*). \tag{5.42}$$

Proof. Recall that $\widetilde{\sigma_{\text{mult}}}$ is given by

$$\widetilde{\sigma_{\text{mult}}}(\tilde{y}) = \frac{n}{2}\operatorname{diag}\left(\operatorname{diag}(\tilde{\nu})\tilde{y} + R\,\overline{\operatorname{diag}(\tilde{\nu})\tilde{y}}\right), \quad \forall \tilde{y} \in \mathbb{C}^n, \tag{5.43}$$

i.e. it is a *diagonal* matrix. This allows us to conclude that $\left(\widetilde{\sigma_{\text{mult}}}R\,(\widetilde{\sigma_{\text{mult}}})^\top\right)_{i,l}$, like the reflection matrix R itself, vanishes if $l \neq (n-i)$. Thus we have

$$\left(\widetilde{\sigma_{\text{mult}}}R\,(\widetilde{\sigma_{\text{mult}}})^\top\right)_{i,l} = (\widetilde{\sigma_{\text{mult}}})_{i,i}(\widetilde{\sigma_{\text{mult}}})_{l,l}\,R_{i,l} = (\widetilde{\sigma_{\text{mult}}})_{i,i}(\widetilde{\sigma_{\text{mult}}})_{l,l}\,\delta_{l,n-i}, \tag{5.44}$$

i.e.

$$\begin{aligned}\left(\widetilde{\sigma_{\text{mult}}}R\,(\widetilde{\sigma_{\text{mult}}})^\top\right)_{i,n-i}(\tilde{y}) &= (\widetilde{\sigma_{\text{mult}}})_{i,i}(\widetilde{\sigma_{\text{mult}}})_{n-i,n-i}(\tilde{y}) = \left|(\widetilde{\sigma_{\text{mult}}})_{i,i}\right|^2(\tilde{y}) \\ &= \frac{n^2}{4}\Big[\,|\tilde{\nu}_i|^2\,|\tilde{y}_i|^2 + |\tilde{\nu}_{n-i}|^2\,|\tilde{y}_{n-i}|^2 \\ &\qquad + \left(\tilde{\nu}_i\tilde{\nu}_{n-i}\tilde{y}_i\tilde{y}_{n-i} + \overline{\tilde{\nu}_i\tilde{\nu}_{n-i}\tilde{y}_i\tilde{y}_{n-i}}\right)\Big],\end{aligned}$$

[1] c.f. remark below Lemma 3.28

where we have employed $(\widetilde{\sigma_{\mathrm{mult}}})_{n-i,n-i} = \overline{(\widetilde{\sigma_{\mathrm{mult}}})_{i,i}}$ (c.f. Eq. (5.43)) and afterwards have applied the result from Eq. (5.39). Now let $i, j, k \in \{0, ..., n-1\}$, $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$ and $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$. For all $\tilde{y} \in \mathbb{C}^n$, we define

$$\begin{aligned} f(\tilde{y}) &:= \overline{(\tilde{y}\tilde{y}^\top)}_{jk} \left(\widetilde{\sigma_{\mathrm{mult}}} R \left(\widetilde{\sigma_{\mathrm{mult}}}\right)^\top (\tilde{y})\right)_{i,n-i} \\ &= \frac{n^2}{4} \overline{(\tilde{y}\tilde{y}^\top)}_{jk} \left[|\widetilde{\nu}_i|^2 |\tilde{y}_i|^2 + |\widetilde{\nu}_{n-i}|^2 |\tilde{y}_{n-i}|^2 + \left(\widetilde{\nu}_i \widetilde{\nu}_{n-i} \tilde{y}_i \tilde{y}_{n-i} + \overline{\widetilde{\nu}_i \widetilde{\nu}_{n-i} \tilde{y}_i \tilde{y}_{n-i}} \right) \right]. \end{aligned}$$

Note that $h_{\tilde{y}_*}$ defined by

$$\begin{aligned} h_{\tilde{y}_*}(z) &:= \frac{n^2}{4} \overline{(\tilde{y}_* \tilde{y}_*^\top)}_{jk} \frac{1}{z^2} \left(|\widetilde{\nu}_i|^2 |(\tilde{y}_*)_i|^2 + |\widetilde{\nu}_{n-i}|^2 |(\tilde{y}_*)_{n-i}|^2 \right) \\ &\quad + \frac{n^2}{4} \overline{(\tilde{y}_* \tilde{y}_*^\top)}_{jk} \frac{1}{z^2} \left(\widetilde{\nu}_i \widetilde{\nu}_{n-i} (\tilde{y}_*)_i (\tilde{y}_*)_{n-i} \, z^2 + \overline{\widetilde{\nu}_i \widetilde{\nu}_{n-i} (\tilde{y}_*)_i (\tilde{y}_*)_{n-i}} \, \frac{1}{z^2} \right) \end{aligned}$$

coincides for $z \in S^1$ with $f(z\tilde{y}_*)$ and is holomorphic in $K_{*,1+\epsilon}$, for all $\epsilon > 0$. Therefore Proposition 5.1 is applicable and yields

$$\begin{aligned} \frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} & \overline{(\tilde{y}\tilde{y}^\top)}_{jk} \left(\widetilde{\sigma_{\mathrm{mult}}} R \left(\widetilde{\sigma_{\mathrm{mult}}}\right)^\top\right)_{i,n-i} (\tilde{y}) \, \mathrm{d}\mathcal{H}_1(\tilde{y}) \\ &= \operatorname{Res}|_{z=0} \left(\frac{h_{y_*}(z)}{z} \right) \\ &= \frac{n^2}{4} \overline{(\tilde{y}_* \tilde{y}_*^\top)}_{jk} \widetilde{\nu}_i \widetilde{\nu}_{n-i} \, (\tilde{y}_*)_i (\tilde{y}_*)_{n-i} \\ &= \frac{n^2}{4} \widetilde{\nu}_i \, (\tilde{y}_*)_i \overline{(\tilde{y}_*)_j} \; \widetilde{\nu}_{n-i} \, (\tilde{y}_*)_{n-i} \overline{(\tilde{y}_*)_k} \\ &= \frac{n^2}{4} \widetilde{\nu}_i \, \hat{p}_{i,j} \; \widetilde{\nu}_{n-i} \, \hat{p}_{n-i,k}, \end{aligned} \tag{5.45}$$

which yields Eq. (5.41). Finally, we note that Eq. (5.42) follows by observing that (c.f. proof of Proposition 5.1)

$$\begin{aligned} \frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} & \overline{(\tilde{y}\tilde{y}^\top)}_{jk} \left(\widetilde{\sigma_{\mathrm{mult}}} R \left(\widetilde{\sigma_{\mathrm{mult}}}\right)^\top\right)_{i,n-i} (\tilde{y}) \, \mathrm{d}\mathcal{H}_1(\tilde{y}) \\ &= \frac{1}{2\pi i} \oint_{S^1} \overline{(z\tilde{y}_*)_j \, (z\tilde{y}_*)_k} \left(\widetilde{\sigma_{\mathrm{mult}}} R \, \widetilde{\sigma_{\mathrm{mult}}}^\top\right)_{i,n-i} (z \, y_*) \, \frac{\mathrm{d}z}{z} \\ &= \overline{(\tilde{y}_*)_j \, (\tilde{y}_*)_k} \left[\frac{1}{2\pi i} \oint_{S^1} \frac{1}{z^3} \left(\widetilde{\sigma_{\mathrm{mult}}} R \, \widetilde{\sigma_{\mathrm{mult}}}^\top\right)_{i,n-i} (z \, y_*) \, \mathrm{d}z \right], \end{aligned} \tag{5.46}$$

which in conjunction with Eq. (5.45) yields the aspired result. □

We summarize the previous results by stating the averaged diffusion matrix.

Lemma 5.11 (Averaged multiplicative-noise diffusion matrix)
The averaged multiplicative-noise diffusion matrix is, for all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$, given by

$$\begin{aligned}(\widehat{a}_{\mathsf{p},\mathrm{mult}})_{ij,kl}(\hat{p}) = {}& \frac{n^2}{4}\hat{p}_{ik}\,|\widetilde{\nu}_j|^2\,\overline{(\hat{p}_{jj} + \hat{p}_{n-j,n-j})}\,\delta_{jl} \\ &+ \frac{n^2}{4}\,\overline{\hat{p}}_{jl}\,|\widetilde{\nu}_i|^2\,(\hat{p}_{ii} + \hat{p}_{n-i,n-i})\,\delta_{ik} \\ &+ \frac{n^2}{4}\overline{(\mathrm{diag}(\widetilde{\nu}\odot\hat{p}_{\cdot,i})\,R\,\mathrm{diag}(\widetilde{\nu}\odot\hat{p}_{\cdot,l}))_{jk}} \\ &+ \frac{n^2}{4}\,(\mathrm{diag}(\widetilde{\nu}\odot\hat{p}_{\cdot,j})\,R\,\mathrm{diag}(\widetilde{\nu}\odot\hat{p}_{\cdot,k}))_{il}\,, \quad \forall i,j,k,l \in \{0,\ldots,n-1\}\,. \end{aligned} \tag{5.47}$$

Proof. This result follows by inserting the calculated averages of Lemma 5.9 and Lemma 5.10 into Eq. (4.237). □

5.3.2. Representation of averaged diffusion matrix via effective dispersion matrix

The aim of this section is to find a decomposition of the averaged multiplicative-noise diffusion matrix $\widehat{a}_{\mathsf{p},\mathrm{mult}}$ in terms of a dispersion term $\widehat{\sigma}_{\mathsf{p},\mathrm{mult}}$, i.e.

$$\widehat{a}_{\mathsf{p},\mathrm{mult}} = \widehat{\sigma}_{\mathsf{p},\mathrm{mult}}\,\widehat{\sigma}^{\dagger}_{\mathsf{p},\mathrm{mult}}. \tag{5.48}$$

For this purpose it proves to be useful to find an *effective* dispersion matrix

$$\widehat{\boldsymbol{\sigma}}_{\mathrm{mult}} = (\widehat{\sigma}_{\mathrm{mult}} \mid \widehat{\sigma'}_{\mathrm{mult}}) \in \mathbb{C}^{n,2n}, \tag{5.49}$$

s.t. $\widehat{a}_{\mathsf{p},\mathrm{mult}}$ can be represented as

$$\begin{aligned}(\widehat{a}_{\mathsf{p},\mathrm{mult}})_{ij,kl}(\hat{p}) = {}& \hat{p}_{ik}\,\overline{(\widehat{\boldsymbol{\sigma}}_{\mathrm{mult}}(\tilde{y}_*)\widehat{\boldsymbol{\sigma}}^{\dagger}_{\mathrm{mult}}(\tilde{y}_*))_{jl}} + \overline{\hat{p}}_{jl}\,(\widehat{\boldsymbol{\sigma}}_{\mathrm{mult}}(\tilde{y}_*)\widehat{\boldsymbol{\sigma}}^{\dagger}_{\mathrm{mult}}(\tilde{y}))_{ik} \\ &+ (\tilde{y}_*\tilde{y}_*^{\top})_{il}\,\overline{(\widehat{\boldsymbol{\sigma}}_{\mathrm{mult}}(\tilde{y}_*)\,\mathcal{R}_2\,\widehat{\boldsymbol{\sigma}}^{\top}_{\mathrm{mult}}(\tilde{y}_*))_{jk}} \\ &+ \overline{(\tilde{y}_*\tilde{y}_*^{\top})_{jk}}\,(\widehat{\boldsymbol{\sigma}}_{\mathrm{mult}}(\tilde{y}_*)\,\mathcal{R}_2\,\widehat{\boldsymbol{\sigma}}^{\top}_{\mathrm{mult}}(\tilde{y}_*))_{il}, \quad \forall \hat{p} \in \mathsf{p}(\mathbb{C}^n), \end{aligned} \tag{5.50}$$

where $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$. Such a representation in terms of $\widehat{\boldsymbol{\sigma}}_{\mathrm{mult}}$ will allow us to define a matrix $\widehat{\sigma}_{\mathsf{p},\mathrm{mult}}$, yielding the desired decomposition, c.f. Lemma 5.14 below.

Remark 5.12 (Effective system interpretation)
Note that an oscillator system perturbed by $\widehat{\boldsymbol{\sigma}}_{\mathrm{mult}}$ instead of $\widetilde{\sigma_{\mathrm{mult}}}$ would give rise to an outer-product evolution whose diffusion matrix coincides with the one in Eq. (5.50), c.f. Theorem 2.39, Lemma 2.40 and Proposition 3.39. This is the reason why we can informally interpret $\widehat{\boldsymbol{\sigma}}_{\mathrm{mult}}$ as a dispersion matrix which describes an averaged or *effective* evolution of the *vector*-valued oscillator system in the averaging limit. Similarly, the matrix $\widehat{\sigma}_{\mathsf{p},\mathrm{mult}}$ describes the effective evolution of the *matrix*-valued outer product and will also be called *effective* dispersion matrix. In the context of dispersion matrices we prefer the term *effective* to the

term *averaged*, since 'averaged dispersion' suggests that a dispersion matrix was averaged, when in fact the diffusion matrix is subject to the averaging procedure.

Note that for this representation to hold and in particular to be well-defined, the right-hand side of Eq. (5.50) has to be independent of the particular choice of $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$. Comparing Eq. (5.50) with the definition of $\hat{a}_{\mathsf{p}}$ given in Eq. (4.237), we find the following two conditions which need to be satisfied for any choice of $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$:

$$\widehat{\boldsymbol{\sigma}}_{\text{mult}}(\tilde{y}_*)\,\widehat{\boldsymbol{\sigma}}^{\dagger}_{\text{mult}}(\tilde{y}_*) \stackrel{!}{=} \frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} \left(\widetilde{\sigma_{\text{mult}}\sigma_{\text{mult}}}^{\dagger}(\tilde{y})\right) \mathrm{d}\mathcal{H}_1(\tilde{y}), \tag{5.51a}$$

$$\widehat{\boldsymbol{\sigma}}_{\text{mult}}(\tilde{y}_*)\,\mathcal{R}_2\,\widehat{\boldsymbol{\sigma}}^{\top}_{\text{mult}}(\tilde{y}_*) \stackrel{!}{=} \frac{1}{2\pi i} \oint_{S^1} \frac{1}{z^3} \left(\widetilde{\sigma_{\text{mult}} R\,\sigma_{\text{mult}}}^{\top}(z\,y_*)\right) \mathrm{d}z. \tag{5.51b}$$

Recalling the definitions of $\widetilde{\boldsymbol{\sigma}_{\text{mult}}}$ and $\mathcal{R}_2 = \text{blockdiag}(R, R)$, as well as the previous results of Lemmas 5.9 and 5.10 on the averaging integrals, we can rewrite these conditions as

$$\widehat{\sigma}_{\text{mult}}(\tilde{y}_*)\,\widehat{\sigma}^{\dagger}_{\text{mult}}(\tilde{y}_*) + \widehat{\sigma'}_{\text{mult}}(\tilde{y}_*)\,\widehat{\sigma'}^{\dagger}_{\text{mult}}(\tilde{y}_*) \stackrel{!}{=} \frac{n^2}{4}\,\text{diag}\left(\left(|\tilde{\nu}_k|^2(\hat{p}_{kk} + \hat{p}_{n-k,n-k})\right)_k\right), \tag{5.52a}$$

$$\widehat{\sigma}_{\text{mult}}(\tilde{y}_*)\,R\,\widehat{\sigma}^{\top}_{\text{mult}}(\tilde{y}_*) + \widehat{\sigma'}_{\text{mult}}(\tilde{y}_*)\,R\,\widehat{\sigma'}^{\top}_{\text{mult}}(\tilde{y}_*) \stackrel{!}{=} \frac{n^2}{4}\,\text{diag}(\tilde{\nu} \odot \tilde{y}_*)\,R\,\text{diag}(\tilde{\nu} \odot \tilde{y}_*). \tag{5.52b}$$

The following lemma provides a solution to this representation problem.

Lemma 5.13 (Effective multiplicative noise)
A solution to the representation problem (5.52) is given by

$$\widehat{\sigma}_{\text{mult}}(\tilde{y}) \coloneqq \frac{n}{2}\,\text{diag}\,(\tilde{\nu} \odot \tilde{y})\,\mathrm{e}^{i\,\frac{\pi}{6}}, \tag{5.53a}$$

$$\widehat{\sigma'}_{\text{mult}}(\tilde{y}) \coloneqq \frac{n}{2}\,\text{diag}\,(\tilde{\nu} \odot (R\,\tilde{y}))\,\mathrm{e}^{-i\,\frac{\pi}{6}}. \tag{5.53b}$$

Proof. As an ansatz for solving Eq. (5.52), we choose a form similiar to the transformed regularizing noise (c.f. Eq. (3.173)), i.e.

$$\widehat{\sigma}_{\text{mult}}(\tilde{y}) \coloneqq \frac{n}{2}\,\text{diag}\,(\tilde{\nu} \odot \tilde{y})\,\mathrm{e}^{i\,\alpha}, \tag{5.54a}$$

$$\widehat{\sigma'}_{\text{mult}}(\tilde{y}) \coloneqq \frac{n}{2}\,\text{diag}\,(\tilde{\nu} \odot (R\,\tilde{y}))\,\mathrm{e}^{i\,\beta}, \tag{5.54b}$$

where α, β are angles to be chosen appropriately. We observe that, independent of α and β, we have

$$\begin{aligned}
&\widehat{\sigma}_{\text{mult}}(\tilde{y})\,\widehat{\sigma}^{\dagger}_{\text{mult}}(\tilde{y}) + \widehat{\sigma'}_{\text{mult}}(\tilde{y})\,\widehat{\sigma'}^{\dagger}_{\text{mult}}(\tilde{y}) \\
&\quad = \frac{n^2}{4}\left[\text{diag}\,(\tilde{\nu} \odot \tilde{y})\,\overline{\text{diag}\,(\tilde{\nu} \odot \tilde{y})} + \text{diag}\,(\tilde{\nu} \odot (R\tilde{y}))\,\overline{\text{diag}\,(\tilde{\nu} \odot (R\tilde{y}))}\right] \\
&\quad = \frac{n^2}{4}\,\text{diag}\left(\left(|\tilde{\nu}_k|^2\,(|\tilde{y}_k|^2 + |\tilde{y}_{n-k}|^2)\right)_k\right),
\end{aligned} \tag{5.55}$$

which for $\tilde{y} = \tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$ yields Eq. (5.52a). Similarly, we find that

$$\begin{aligned}
&\widehat{\sigma}_{\text{mult}}(\tilde{y})\, R\, \widehat{\sigma}^{\top}_{\text{mult}}(\tilde{y}) + \widehat{\sigma'}_{\text{mult}}(\tilde{y})\, R\, \widehat{\sigma'}^{\top}_{\text{mult}}(\tilde{y}) \\
&\quad = \frac{n^2}{4}\left[\operatorname{diag}(\widetilde{\nu}\odot\tilde{y})\, R \operatorname{diag}(\widetilde{\nu}\odot\tilde{y})\, \mathrm{e}^{2i\alpha} + \operatorname{diag}(\widetilde{\nu}\odot(R\tilde{y}))\, R \operatorname{diag}(\widetilde{\nu}\odot(R\tilde{y}))\, \mathrm{e}^{2i\beta}\right] \\
&\quad = \frac{n^2}{4}\left[\operatorname{diag}(\widetilde{\nu}\odot\tilde{y})\, R \operatorname{diag}(\widetilde{\nu}\odot\tilde{y})\right]\left(\mathrm{e}^{2i\alpha} + \mathrm{e}^{2i\beta}\right),
\end{aligned} \tag{5.56}$$

where we have employed that

$$\operatorname{diag}(\widetilde{\nu}\odot(R\tilde{y}))\, R \operatorname{diag}(\widetilde{\nu}\odot(R\tilde{y})) = \operatorname{diag}(\widetilde{\nu}\odot\tilde{y})\, R \operatorname{diag}(\widetilde{\nu}\odot\tilde{y}), \tag{5.57}$$

which can be seen as follows. We note that by definition of the reflection matrix (c.f. Definition 2.28), it suffices to verify Eq. (5.57) for matrix elements of the form $(\cdot)_{k,n-k}$, since all other components vanish on both sides. For $k \in \{0, ..., n-1\}$, we find that

$$\begin{aligned}
\left[\operatorname{diag}(\widetilde{\nu}\odot(R\tilde{y}))\, R \operatorname{diag}(\widetilde{\nu}\odot(R\tilde{y}))\right]_{k,n-k} &= \widetilde{\nu}_k\, (R\tilde{y})_k\; \widetilde{\nu}_{n-k}\, (R\tilde{y})_{n-k} \\
&= \widetilde{\nu}_k\, \widetilde{\nu}_{n-k}\, \tilde{y}_k\, \tilde{y}_{n-k} \\
&= \left[\operatorname{diag}(\widetilde{\nu}\odot\tilde{y})\, R \operatorname{diag}(\widetilde{\nu}\odot\tilde{y})\right]_{k,n-k}.
\end{aligned} \tag{5.58}$$

Now Eq. (5.56) yields Eq. (5.52b), provided that

$$\mathrm{e}^{2i\alpha} + \mathrm{e}^{2i\beta} = 1. \tag{5.59}$$

This condition is satisfied if we choose $\alpha = \frac{\pi}{6}$ and $\beta = -\frac{\pi}{6}$, since in this case

$$\mathrm{e}^{2i\alpha} + \mathrm{e}^{2i\beta} = \mathrm{e}^{2i\alpha} + \mathrm{e}^{-2i\alpha} = 2\cos(2\alpha) = 2\cos\left(\frac{\pi}{3}\right) = 1. \tag{5.60}$$

□

The effective dispersion matrix $\widehat{\boldsymbol{\sigma}}_{\text{mult}} = (\widehat{\sigma}_{\text{mult}} \mid \widehat{\sigma'}_{\text{mult}}) \in \mathbb{C}^{2n,n}$, as given by the previous lemma, allows us to obtain the aspired decomposition of $\widehat{a}_{\mathsf{p},\text{mult}}$.

Lemma 5.14 (Effective multiplicative-noise dispersion matrix)
The dispersion matrix $\widehat{\sigma}_{\mathsf{p},\text{mult}}$, defined for all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$ and an arbitrary choice of $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$ by

$$(\widehat{\sigma}_{\mathsf{p},\text{mult}})_{ij,r}(\hat{p}) := (\tilde{y}_*)_i \left(\overline{\widehat{\boldsymbol{\sigma}}_{\text{mult}}(\tilde{y}_*)\mathcal{Q}_2^\dagger}\right)_{jr} + \overline{(\tilde{y}_*)_j}\left(\widehat{\boldsymbol{\sigma}}_{\text{mult}}(\tilde{y}_*)\mathcal{Q}_2^\dagger\right)_{ir}, \quad i,j \in \{0, ..., n-1\},\ r \in \{0, ..., 2n-1\}, \tag{5.61}$$

provides a decomposition of the averaged diffusion matrix, i.e.

$$\widehat{a}_{\mathsf{p},\text{mult}}(\hat{p}) = \widehat{\sigma}_{\mathsf{p},\text{mult}}(\hat{p})\, \widehat{\sigma}^\dagger_{\mathsf{p},\text{mult}}(\hat{p}). \tag{5.62}$$

Proof. First of all note that the right-hand side of Eq. (5.61) is independent of the choice of $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$, i.e. $\widehat{\sigma}_{\mathsf{p},\text{mult}}$ is well-defined. For all $i, j, r \in \{0, ..., n-1\}$ and all $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$ for instance we observe that

$$(\tilde{y}_*)_i \left(\overline{\widehat{\boldsymbol{\sigma}}_{\text{mult}}(\tilde{y}_*)}\right)_{jr} = (\tilde{y}_*)_i \left(\overline{\widehat{\sigma}_{\text{mult}}(\tilde{y}_*)}\right)_{jr} = \frac{n}{2}\,\overline{\widetilde{\nu}}_j\,\hat{p}_{ij}\,\delta_{jr}\,\mathrm{e}^{-i\frac{\pi}{6}}\,. \tag{5.63}$$

The statement now follows from Theorem 2.39, Lemma 2.40, and Eqs. (4.237) and (5.51). □

Note that such a decomposition of a diffusion matrix is of course *not* unique. The degrees of freedom we have in choosing a respresenting dispersion matrix, however, do not affect the law of the resulting process. Consider for instance a decomposition of a positive definite matrix $a \in \mathbb{C}^{n,n}$ as $a = \sigma\sigma^\dagger$, where $\sigma \in \mathbb{C}^{n,n}$. Then for any unitary matrix $U \in U(n)$, the matrix $\sigma_U := \sigma U$ also yields a decomposition of a, i.e.

$$\sigma_U \sigma_U^\dagger = \sigma U U^\dagger \sigma^\dagger = \sigma\sigma^\dagger = a. \tag{5.64}$$

Such a unitary transformation can however be 'absorbed' in the definition of the complex Brownian motion, since for any $\mathbb{C}^n$-valued complex Brownian motion $(W(t))_{t\geq 0}$, the transformed process $(U\,W(t))_{t\geq 0}$ is a complex Brownian motion as well, c.f. Lemma 2.35 and its subsequent remark.

5.4. Averaged regularizing noise

We show that the regularizing noise is *invariant* under the averaging procedure.

5.4.1. Invariance of regularizing noise

Lemma 5.15 (Averaging of $\widetilde{\boldsymbol{\sigma}_{\text{reg}}}\widetilde{\boldsymbol{\sigma}_{\text{reg}}}^\dagger$ and $\widetilde{\boldsymbol{\sigma}_{\text{reg}}}\mathcal{R}_2\left(\widetilde{\boldsymbol{\sigma}_{\text{reg}}}\right)^\top$)
For all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$ and all $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$, we find that the terms

$$\left(\widetilde{\boldsymbol{\sigma}_{\text{reg}}}(\widetilde{\boldsymbol{\sigma}_{\text{reg}}})^\dagger\right)(\tilde{y}_*) = \frac{n^2}{4}\,\sigma_r^2\,\operatorname{diag}\left((\hat{p}_{kk} + \hat{p}_{n-k,n-k})_k\right), \tag{5.65}$$

$$\widetilde{\boldsymbol{\sigma}_{\text{reg}}}\mathcal{R}_2\left(\widetilde{\boldsymbol{\sigma}_{\text{reg}}}\right)^\top(\tilde{y}_*) = 0, \tag{5.66}$$

are constant on level sets of p and are thus invariant under the averaging procedure of Eq. (4.237), i.e.

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)}\oint_{\mathsf{p}^{-1}(\hat{p})}\left(\widetilde{\boldsymbol{\sigma}_{\text{reg}}}(\widetilde{\boldsymbol{\sigma}_{\text{reg}}})^\dagger\right)(\tilde{y})\,\mathrm{d}\mathcal{H}_1(\tilde{y}) = \widetilde{\boldsymbol{\sigma}_{\text{reg}}}(\widetilde{\boldsymbol{\sigma}_{\text{reg}}})^\dagger(\tilde{y}_*), \tag{5.67}$$

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)}\oint_{\mathsf{p}^{-1}(\hat{p})}\overline{(\tilde{y}\tilde{y}^\top)}\left(\widetilde{\boldsymbol{\sigma}_{\text{reg}}}\mathcal{R}_2\left(\widetilde{\boldsymbol{\sigma}_{\text{reg}}}\right)^\top\right)(\tilde{y})\,\mathrm{d}\mathcal{H}_1(\tilde{y}) = \overline{(\tilde{y}_*\tilde{y}_*^\top)}\left(\widetilde{\boldsymbol{\sigma}_{\text{reg}}}\mathcal{R}_2\left(\widetilde{\boldsymbol{\sigma}_{\text{reg}}}\right)^\top\right)(\tilde{y}_*). \tag{5.68}$$

Proof. By Eq. (3.148) it follows that

$$\begin{aligned}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}})^{\dagger}\right)(\tilde{y}) &= \frac{n^2}{4}\,\sigma_r^2\left[\operatorname{diag}(\tilde{y})\,\overline{\operatorname{diag}(\tilde{y})} + i\operatorname{diag}(R\tilde{y})\,(-i)\overline{\operatorname{diag}(R\tilde{y})}\right]\\ &= \frac{n^2}{4}\,\sigma_r^2\operatorname{diag}\left(\left(|\tilde{y}_k|^2 + |\tilde{y}_{n-k}|^2\right)_k\right)\\ &= \frac{n^2}{4}\,\sigma_r^2\operatorname{diag}\left((\hat{p}_{kk} + \hat{p}_{n-k,n-k})_k\right), \quad \forall \tilde{y} \in \mathsf{p}^{-1}(\hat{p}).\end{aligned}$$

Similarly, we find that

$$\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}\mathcal{R}_2\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}\right)^{\top}(\tilde{y}) = \frac{n^2}{4}\,\sigma_r^2\,\left[\operatorname{diag}(\tilde{y})\,R\operatorname{diag}(\tilde{y}) - \operatorname{diag}(R\tilde{y})\,R\operatorname{diag}(R\tilde{y})\right] = 0, \quad \forall \tilde{y} \in \mathbb{C}^n,$$

where the last step follows along the lines of Eq. (5.57). In both cases, we have an integrand which is *constant* on any level set of the form $\mathsf{p}^{-1}(\hat{p})$ and is thus invariant under the averaging procedure of Eq. (4.237). □

The invariance of the diffusion terms allows us to state the regularizing-noise contribution to the diffusion matrix $\widehat{a}_{\mathsf{p}}$.

Lemma 5.16 (Averaged regularizing-noise diffusion matrix)
The averaged regularizing-noise diffusion matrix for all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$ is given by

$$\left(\widehat{a}_{\mathsf{p},\mathrm{reg}}\right)_{ij,kl}(\hat{p}) = \hat{p}_{ik}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}})^{\dagger}\right)_{jl} + \overline{\hat{p}}_{jl}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}})^{\dagger}\right)_{ik}, \quad \forall i,j,k,l \in \{0,\ldots,n-1\}. \tag{5.69}$$

Proof. This follows directly from Lemma 5.15 and Eq. (4.237). □

5.4.2. Representation of averaged diffusion matrix via effective dispersion matrix

Analogous to Section 5.3.2, we want to represent $\widehat{a}_{\mathsf{p},\mathrm{reg}}$ in terms of an *effective* dispersion matrix $\widehat{\boldsymbol{\sigma}}_{\mathrm{reg}} := (\widehat{\sigma}_{\mathrm{reg}} \mid \widehat{\sigma'}_{\mathrm{reg}})$, which is determined by the conditions (c.f. Eq. (5.51))

$$\widehat{\boldsymbol{\sigma}}_{\mathrm{reg}}\,\widehat{\boldsymbol{\sigma}}^{\dagger}_{\mathrm{reg}}(\tilde{y}_*) \stackrel{!}{=} \frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)}\oint_{\mathsf{p}^{-1}}\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}})^{\dagger}(\tilde{y})\,\mathrm{d}\mathcal{H}_1(\tilde{y}), \tag{5.70a}$$

$$\widehat{\boldsymbol{\sigma}}_{\mathrm{reg}}\,\mathcal{R}_2\,\widehat{\boldsymbol{\sigma}}^{\top}_{\mathrm{reg}}(\tilde{y}_*) \stackrel{!}{=} \frac{1}{2\pi i}\oint_{S^1}\frac{1}{z^3}\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}\mathcal{R}_2\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}\right)^{\top}(z\,y_*)\right)\mathrm{d}z, \tag{5.70b}$$

for all $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$ and all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$. By the *invariance* results of Lemma 5.15, these conditions simplify to

$$\widehat{\boldsymbol{\sigma}}_{\mathrm{reg}}\,\widehat{\boldsymbol{\sigma}}^{\dagger}_{\mathrm{reg}}(\tilde{y}_*) \stackrel{!}{=} \widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}})^{\dagger}(\tilde{y}_*) = \frac{n^2}{4}\,\sigma_r^2\operatorname{diag}\left((\hat{p}_{kk} + \hat{p}_{n-k,n-k})_k\right), \tag{5.71a}$$

$$\widehat{\boldsymbol{\sigma}}_{\mathrm{reg}}\,\mathcal{R}_2\,\widehat{\boldsymbol{\sigma}}^{\top}_{\mathrm{reg}}(\tilde{y}_*) \stackrel{!}{=} \widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}\mathcal{R}_2\left(\widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}\right)^{\top}(\tilde{y}_*) = 0, \tag{5.71b}$$

and are thus satisfied by the trivial choice of

$$\widehat{\boldsymbol{\sigma}}_{\mathrm{reg}}(\tilde{y}) := (\widehat{\sigma}_{\mathrm{reg}}(\tilde{y}) \mid \widehat{\sigma'}_{\mathrm{reg}}(\tilde{y})) := \frac{n}{2}\,\sigma_r\,\left(\mathrm{diag}(\tilde{y}) \mid i\,\mathrm{diag}(\tilde{y})\right) \equiv \widetilde{\boldsymbol{\sigma}_{\mathrm{reg}}}(\tilde{y}). \tag{5.72}$$

The matrix $\widehat{\boldsymbol{\sigma}}_{\mathrm{reg}}$ now enables us to define a dispersion matrix $\widehat{\sigma}_{\mathsf{p},\mathrm{reg}}$, giving rise to the diffusion matrix $\widehat{a}_{\mathsf{p},\mathrm{reg}}$.

Lemma 5.17 (Effective regularizing-noise dispersion matrix)
The dispersion matrix $\widehat{\sigma}_{\mathsf{p},\mathrm{reg}}$, defined for all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$ and any choice of $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$ by

$$\begin{aligned}(\widehat{\sigma}_{\mathsf{p},\mathrm{reg}})_{ij,r}(\hat{p}) := (\tilde{y}_*)_i \left(\overline{\widehat{\boldsymbol{\sigma}}_{\mathrm{reg}}(\tilde{y}_*)\mathcal{Q}_2^\dagger}\right)_{jr} + \overline{(\tilde{y}_*)_j}\left(\widehat{\boldsymbol{\sigma}}_{\mathrm{reg}}(\tilde{y}_*)\mathcal{Q}_2^\dagger\right)_{ir}, \\ i,j \in \{0,\dots,n-1\}\,,\ r \in \{0,\dots,2n-1\}\,,\end{aligned} \tag{5.73}$$

provides a decomposition of the averaged additive-noise diffusion matrix, i.e.

$$\widehat{a}_{\mathsf{p},\mathrm{reg}}(\hat{p}) = \widehat{\sigma}_{\mathsf{p},\mathrm{reg}}(\hat{p})\,\widehat{\sigma}^\dagger_{\mathsf{p},\mathrm{reg}}(\hat{p}). \tag{5.74}$$

Proof. Note that, as in Lemma 5.14, the right-hand side of Eq. (5.61) is independent of the choice of $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$, i.e. $\widehat{\sigma}_{\mathsf{p},\mathrm{reg}}$ is well-defined. The statement now follows from Theorem 2.39, Lemma 2.40, and Eqs. (4.237) and (5.70). □

5.5. Averaged additive noise

As in the case of regularizing noise, we observe that the additive noise is *invariant* under the averaging procedure.

5.5.1. Invariance of additive noise

The averaged additive-noise diffusion matrices exhibit the following invariant structure.

Lemma 5.18 (Averaging of $\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}\boldsymbol{\sigma}_{\mathrm{add}}}^\dagger$ and $\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}\mathcal{R}_2}\,(\boldsymbol{\sigma}_{\mathrm{add}})^\top$)
The terms

$$\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}(\boldsymbol{\sigma}_{\mathrm{add}})^\dagger} = \sigma_0^2\,\mathbb{1}_{n\times n}, \tag{5.75}$$

$$\widetilde{\boldsymbol{\sigma}_{\mathrm{add}}\mathcal{R}_2\,(\boldsymbol{\sigma}_{\mathrm{add}})^\top} = 0, \tag{5.76}$$

are constant on level sets of p and are thus invariant under the averaging procedure of Eq. (4.237), i.e. for all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$ and all $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$, we find that

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} \left(\widetilde{\boldsymbol{\sigma}_{\text{add}}}(\widetilde{\boldsymbol{\sigma}_{\text{add}}})^\dagger\right) \mathrm{d}\mathcal{H}_1(\tilde{y}) = \widetilde{\boldsymbol{\sigma}_{\text{add}}}(\widetilde{\boldsymbol{\sigma}_{\text{add}}})^\dagger, \tag{5.77}$$

$$\frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}(\hat{p})} \overline{(\tilde{y}\tilde{y}^\top)} \left(\widetilde{\boldsymbol{\sigma}_{\text{add}}}\mathcal{R}_2\left(\widetilde{\boldsymbol{\sigma}_{\text{add}}}\right)^\top\right)(\tilde{y})\, \mathrm{d}\mathcal{H}_1(\tilde{y}) = \overline{(\tilde{y}_*\tilde{y}_*^\top)} \left(\widetilde{\boldsymbol{\sigma}_{\text{add}}}\mathcal{R}_2\left(\widetilde{\boldsymbol{\sigma}_{\text{add}}}\right)^\top\right). \tag{5.78}$$

Proof. Inserting $\widetilde{\boldsymbol{\sigma}_{\text{add}}}$, as defined in Eq. (3.155), we observe that

$$\widetilde{\boldsymbol{\sigma}_{\text{add}}}(\widetilde{\boldsymbol{\sigma}_{\text{add}}})^\dagger = \frac{\sigma_0^2}{2}\left(\mathbb{1}_{n\times n} \mid i\,\mathbb{1}_{n\times n}\right)\begin{pmatrix} \mathbb{1}_{n\times n} \\ -i\,\mathbb{1}_{n\times n}\end{pmatrix} = \sigma_0^2\,\mathbb{1}_{n\times n} \tag{5.79}$$

and

$$\widetilde{\boldsymbol{\sigma}_{\text{add}}}\mathcal{R}_2\left(\widetilde{\boldsymbol{\sigma}_{\text{add}}}\right)^\top = \frac{\sigma_0^2}{2}\left(\mathbb{1}_{n\times n} \mid i\,\mathbb{1}_{n\times n}\right)\left(\begin{array}{c|c} R & 0 \\ \hline 0 & R\end{array}\right)\begin{pmatrix} \mathbb{1}_{n\times n} \\ i\,\mathbb{1}_{n\times n}\end{pmatrix} = \frac{\sigma_0^2}{2}(R-R) = 0. \tag{5.80}$$

□

Consequently, we obtain the following representation of the averaged additive-noise diffusion matrix.

Lemma 5.19 (Averaged diffusion matrix)
The averaged additive-noise diffusion matrix is for all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$ given by

$$\begin{aligned}(\widehat{a}_{\mathsf{p},\text{add}})_{ij,kl}(\hat{p}) &= \hat{p}_{ik}\left(\overline{\widetilde{\boldsymbol{\sigma}_{\text{add}}}(\widetilde{\boldsymbol{\sigma}_{\text{add}}})^\dagger}\right)_{jl} + \overline{\hat{p}}_{jl}\left(\widetilde{\boldsymbol{\sigma}_{\text{add}}}(\widetilde{\boldsymbol{\sigma}_{\text{add}}})^\dagger\right)_{ik} \\ &= \sigma_0^2\left(\hat{p}_{ik}\,\delta_{jl} + \overline{\hat{p}}_{jl}\,\delta_{ik}\right), \quad \forall i,j,k,l \in \{0,\ldots,n-1\}.\end{aligned} \tag{5.81}$$

Proof. Inserting our averaging results from Lemma 5.18 into the definition given in Eq. (4.237), we obtain the result. □

5.5.2. Representation of averaged diffusion matrix via effective dispersion matrix

As in the previous cases, we aim for a representation of the averaged additive-noise diffusion matrix $\widehat{a}_{\mathsf{p},\text{add}}$ in terms of an *effective* dispersion matrix $\widehat{\boldsymbol{\sigma}}_{\text{add}}$. This matrix is determined by the conditions (c.f. Eqs. (5.51) and (5.70))

$$\widehat{\boldsymbol{\sigma}}_{\text{add}}\,\widehat{\boldsymbol{\sigma}}_{\text{add}}^\dagger \stackrel{!}{=} \frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)} \oint_{\mathsf{p}^{-1}} \widetilde{\boldsymbol{\sigma}_{\text{add}}}(\widetilde{\boldsymbol{\sigma}_{\text{add}}})^\dagger\, \mathrm{d}\mathcal{H}_1(\tilde{y}), \tag{5.82a}$$

$$\widehat{\boldsymbol{\sigma}}_{\text{add}}\,\mathcal{R}_2\,\widehat{\boldsymbol{\sigma}}_{\text{add}}^\top(\tilde{y}_*) \stackrel{!}{=} \frac{1}{2\pi i}\oint_{S^1} \frac{1}{z^3}\left(\widetilde{\boldsymbol{\sigma}_{\text{add}}}\mathcal{R}_2\left(\widetilde{\boldsymbol{\sigma}_{\text{add}}}\right)^\top\right)\mathrm{d}z, \tag{5.82b}$$

for all $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$ and all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$. The invariance result of Lemma 5.18 simplifies these conditions to

$$\widehat{\boldsymbol{\sigma}}_{\text{add}}\, \widehat{\boldsymbol{\sigma}}_{\text{add}}^{\dagger} \overset{!}{=} \widetilde{\boldsymbol{\sigma}_{\text{add}}} (\widetilde{\boldsymbol{\sigma}_{\text{add}}})^{\dagger} = \sigma_0^2\, \mathbb{1}_{n\times n}, \tag{5.83a}$$

$$\widehat{\boldsymbol{\sigma}}_{\text{add}}\, \mathcal{R}_2\, \widehat{\boldsymbol{\sigma}}_{\text{add}}^{\top} \overset{!}{=} \widehat{\boldsymbol{\sigma}}_{\text{add}}\, \mathcal{R}_2\, \widehat{\boldsymbol{\sigma}}_{\text{add}}^{\top} = 0, \tag{5.83b}$$

allowing us to choose $\widehat{\boldsymbol{\sigma}}_{\text{add}}$ as (c.f. Eq. (5.72))

$$\widehat{\boldsymbol{\sigma}}_{\text{add}} \coloneqq (\widehat{\sigma}_{\text{add}} \mid \widehat{\sigma}_{\text{add}}) \coloneqq \frac{\sigma_0}{\sqrt{2}}\, (\mathbb{1}_{n\times n} \mid i\, \mathbb{1}_{n\times n}) \equiv \widetilde{\boldsymbol{\sigma}_{\text{add}}}. \tag{5.84}$$

Employing $\widehat{\boldsymbol{\sigma}}_{\text{add}}$ allows us to obtain a decomposition of $\widehat{a}_{\mathsf{p},\text{add}}$ in terms of an effective dispersion matrix $\widehat{\sigma}_{\mathsf{p},\text{add}}$. However, we need to deviate slightly from the approach taken in the previous cases, as is illustrated in the following remark.

Remark 5.20 (An ill-defined approach)
Following the steps of the previous sections, we could attempt to define $\widehat{\sigma}_{\mathsf{p},\text{add}}$ as

$$(\widehat{\sigma}_{\mathsf{p},\text{add}})_{ij,r}(\hat{p}) \coloneqq (\tilde{y}_*)_i \left(\overline{\widehat{\boldsymbol{\sigma}}_{\text{add}}\, \mathcal{Q}_2^{\dagger}}\right)_{jr} + \overline{(\tilde{y}_*)_j} \left(\widehat{\boldsymbol{\sigma}}_{\text{add}}\, \mathcal{Q}_2^{\dagger}\right)_{ir}, \tag{5.85}$$

where $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$. This would indeed yield a decomposition of the form

$$\widehat{a}_{\mathsf{p},\text{add}}(\hat{p}) = \widehat{\sigma}_{\mathsf{p},\text{add}}(\hat{p})\, \widehat{\sigma}_{\mathsf{p},\text{add}}^{\dagger}(\hat{p}).$$

However, unlike in Lemmas 5.14 and 5.17, Eq. (5.85) does *not* yield a well-defined dispersion matrix, since the right-hand side of Eq. (5.85) depends on the specific choice of $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$.

We can obtain a well-defined expression by decomposing the outer product $\hat{p}$ not in terms of $\tilde{y}_*$ but in terms of a vector $\beta(\hat{p})$, given by the following lemma.

Lemma 5.21 (Decomposition of complex outer product)
Let $\beta : \mathsf{p}(\mathbb{C}^n) \to \mathbb{C}^n$ be defined by

$$\beta_i(\hat{p}) \coloneqq \begin{cases} \frac{\hat{p}_{i0}}{\sqrt{\hat{p}_{00}}}, & \text{if } \hat{p}_{00} \neq 0, \\ \frac{\hat{p}_{i1}}{\sqrt{\hat{p}_{11}}}, & \text{if } \hat{p}_{00} = 0,\ \hat{p}_{11} \neq 0, \\ \frac{\hat{p}_{i2}}{\sqrt{\hat{p}_{22}}}, & \text{if } \hat{p}_{00} = \hat{p}_{11} = 0,\ \hat{p}_{22} \neq 0, \\ \vdots & \vdots \\ \frac{\hat{p}_{i,n-1}}{\sqrt{\hat{p}_{n-1,n-1}}}, & \text{if } \hat{p}_{00} = \hat{p}_{11} = \cdots = \hat{p}_{n-2,n-2} = 0,\ \hat{p}_{n-1,n-1} \neq 0, \\ 0, & \text{if } \hat{p} = 0, \end{cases} \tag{5.86}$$

for all $i \in \{0, \dots, n-1\}$ and all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$. Then it follows that

$$\beta_i(\hat{p})\, \overline{\beta_k}(\hat{p}) = \hat{p}_{ik}, \quad \forall i, k \in \{0, \dots, n-1\}, \forall \hat{p} \in \mathsf{p}(\mathbb{C}^n). \tag{5.87}$$

Proof. Let $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$. If $\hat{p} = 0$ then Eq. (5.87) is trivially satisfied. Assume now that $\hat{p} \neq 0$ and let $j \in \{0, ..., n-1\}$, s.t. $\hat{p}_{00} = ... = \hat{p}_{j-1,j-1} = 0$ and $\hat{p}_{jj} \neq 0$. In this case $\beta(\hat{p})$ is given by

$$\beta_i(\hat{p}) = \frac{\hat{p}_{ij}}{\sqrt{\hat{p}_{jj}}}, \quad \forall i \in \{0, ..., n-1\}\,. \tag{5.88}$$

It consequently follows that

$$\beta_i(\hat{p})\,\overline{\beta_k}(\hat{p}) = \frac{\hat{p}_{ij}}{\sqrt{\hat{p}_{jj}}}\,\frac{\overline{\hat{p}_{kj}}}{\sqrt{\hat{p}_{jj}}} = \frac{\hat{p}_{ij}\,\hat{p}_{jk}}{\hat{p}_{jj}} = \frac{(\tilde{y}_*)_i\,\left(\overline{\tilde{y}_*}\right)_j\,(\tilde{y}_*)_j\,\left(\overline{\tilde{y}_*}\right)_k}{|(\tilde{y}_*)_j|^2} = \hat{p}_{ik}, \quad \forall \tilde{y}_* \in \mathsf{p}^{-1}(\hat{p}). \tag{5.89}$$

□

Lemma 5.21 enables us to obtain a well-defined decomposition of $\widehat{\sigma}_{\mathsf{p},\text{add}}$.

Lemma 5.22 (Effective additive-noise dispersion matrix)
The dispersion matrix $\widehat{\sigma}_{\mathsf{p},\text{add}}$, defined by

$$(\widehat{\sigma}_{\mathsf{p},\text{add}})_{ij,r}(\hat{p}) := \beta_i(\hat{p})\left(\overline{\widehat{\boldsymbol{\sigma}}_{\text{add}}\,\mathcal{Q}_2^\dagger}\right)_{jr} + \overline{\beta_j(\hat{p})}\left(\widehat{\boldsymbol{\sigma}}_{\text{add}}\,\mathcal{Q}_2^\dagger\right)_{ir}, \qquad i,j \in \{0, ..., n-1\}\,,\; r \in \{0, ..., 2n-1\}\,, \tag{5.90}$$

provides a decomposition of the averaged additive-noise diffusion matrix, i.e.

$$\widehat{a}_{\mathsf{p},\text{add}}(\hat{p}) = \widehat{\sigma}_{\mathsf{p},\text{add}}(\hat{p})\,\widehat{\sigma}^\dagger_{\mathsf{p},\text{add}}(\hat{p}). \tag{5.91}$$

Proof. Let $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$ and $i,j,k,l \in \{0, ..., n-1\}$. We introduce the shorthand $\mathcal{M} := \widehat{\boldsymbol{\sigma}}_{\text{add}}\mathcal{Q}_2^\dagger$, recall that (c.f. Eq. (5.83))

$$\mathcal{M}\mathcal{M}^\dagger = \widehat{\boldsymbol{\sigma}}_{\text{add}}\widehat{\boldsymbol{\sigma}}^\dagger_{\text{add}} = \sigma_0^2\,\mathbb{1}_{n\times n}, \tag{5.92}$$

$$\mathcal{M}\mathcal{M}^\top = \widehat{\boldsymbol{\sigma}}_{\text{add}}\mathcal{R}_2\widehat{\boldsymbol{\sigma}}^\top_{\text{add}} = 0, \tag{5.93}$$

and verify Eq. (5.91) by calculation:

$$\begin{aligned}\left(\widehat{\sigma}_{\mathsf{p},\text{add}}(\hat{p})\,\widehat{\sigma}^\dagger_{\mathsf{p},\text{add}}(\hat{p})\right)_{ij,kl} &= \sum_r \left(\beta_i(\hat{p})\,\overline{\mathcal{M}}_{jr} + \overline{\beta_j(\hat{p})}\,\mathcal{M}_{ir}\right)\left(\overline{\beta_k(\hat{p})}\,\mathcal{M}_{lr} + \beta_l(\hat{p})\,\overline{\mathcal{M}_{kr}}\right)\\ &= \beta_i(\hat{p})\overline{\beta_k(\hat{p})}\,\left(\mathcal{M}\mathcal{M}^\dagger\right)_{lj} + \beta_l(\hat{p})\overline{\beta_j(\hat{p})}\,\left(\mathcal{M}\mathcal{M}^\dagger\right)_{ik}\\ &= \sigma_0^2\,(\hat{p}_{ik}\,\delta_{jl} + \hat{p}_{lj}\,\delta_{ik}) = \sigma_0^2\,\left(\hat{p}_{ik}\,\delta_{jl} + \overline{\hat{p}}_{jl}\,\delta_{ik}\right)\\ &= (\widehat{a}_{\mathsf{p},\text{add}})_{ij,kl}\,(\hat{p}),\end{aligned}$$

where in third step we have employed Lemma 5.21 and in the last step we have identified $\widehat{a}_{\mathsf{p},\text{add}}$ by means of Lemma 5.19. □

We have seen that a complex Brownian motion is invariant under the averaging procedure, c.f. Eq. (5.84). By contrast, a standard (real-valued) Brownian motion is *not* invariant, as we will observe in the following remark.

Remark 5.23 (Averaging influence on real-valued Brownian motion)
Note that a *real-valued* additive noise, corresponding to a dispersion matrix of the form

$$\sigma_{\text{add}} = \sigma_0 \, \mathbb{1}_{n\times n} \tag{5.94}$$

would give rise to the same conditions as stated in Eq. (5.83), i.e.

$$\widehat{\boldsymbol{\sigma}}_{\text{add}} \, \widehat{\boldsymbol{\sigma}}_{\text{add}}^{\dagger} \overset{!}{=} \sigma_{\text{add}}\sigma_{\text{add}}{}^{\dagger} = \sigma_0^2 \, \mathbb{1}_{n\times n}, \tag{5.95a}$$
$$\widehat{\boldsymbol{\sigma}}_{\text{add}} \, \mathcal{R}_2 \, \widehat{\boldsymbol{\sigma}}_{\text{add}}^{\top} = 0. \tag{5.95b}$$

In this case, the trivial choice of $\widehat{\boldsymbol{\sigma}}_{\text{add}} \coloneqq \sigma_{\text{add}}$ does *not* yield a solution to Eq. (5.95), since

$$\sigma_{\text{add}} R \, \sigma_{\text{add}}^{\top} = \sigma^2 \, \mathbb{1}_{n\times n} \neq 0. \tag{5.96}$$

Instead, the solution to Eq. (5.95) is again (as in Eq. (5.84)) given by

$$\widehat{\boldsymbol{\sigma}}_{\text{add}} \coloneqq \frac{\sigma_0}{\sqrt{2}} \left(\mathbb{1}_{n\times n} \,|\, i \, \mathbb{1}_{n\times n}\right) \equiv \widetilde{\boldsymbol{\sigma}_{\text{add}}}. \tag{5.97}$$

This implies that an additive noise perturbation corresponding to a *real-valued* Brownian motion effectively acts on the averaged system in the same way as a *complex* Brownian motion.

5.6. Combined noise and Itô correction

Collecting the results from the previous sections, we are in a position to obtain a decomposition of the averaged diffusion matrix

$$\widehat{a}_{\mathsf{p}} = \widehat{a}_{\mathsf{p},\text{mult}} + \widehat{a}_{\mathsf{p},\text{reg}} + \widehat{a}_{\mathsf{p},\text{add}}, \tag{5.98}$$

as defined by Eq. (4.236).

Proposition 5.24 (Averaged dispersion matrix)
The *effective dispersion matrix* $\widehat{\sigma_{\mathsf{p}}}$, defined, for all $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$, by

$$\widehat{\sigma_{\mathsf{p}}} : \mathsf{p}(\mathbb{C}^n) \to \mathbb{C}^{n^2,6n}, \quad \hat{p} \to \widehat{\sigma_{\mathsf{p}}}(\hat{p}) \coloneqq \left(\widehat{\sigma}_{\mathsf{p},\text{mult}}(\hat{p}) \,|\, \widehat{\sigma}_{\mathsf{p},\text{reg}}(\hat{p}) \,|\, \widehat{\sigma}_{\mathsf{p},\text{add}}(\hat{p})\right), \tag{5.99}$$

provides a decomposition of the averaged diffusion matrix $\widehat{a}_{\mathsf{p}}$, i.e.

$$\widehat{a}_{\mathsf{p}}(\hat{p}) = \widehat{\sigma_{\mathsf{p}}}(\hat{p}) \, \widehat{\sigma_{\mathsf{p}}}^{\dagger}(\hat{p}), \quad \forall \hat{p} \in \mathsf{p}(\mathbb{C}^n). \tag{5.100}$$

Proof. This follows from Lemmas 4.72, 5.14, 5.17 and 5.22. □

The diffusion averages from the previous sections furthermore allow us to determine the averaged Itô correction $\widehat{u}_{\mathsf{p},\text{Itô}}$.

Lemma 5.25 (Averaged Itô correction)
The averaged Itô correction is given by

$$\widehat{u}_{\mathsf{p},\text{Itô}}\left(\hat{p}\right) = \frac{n^2}{4}\operatorname{diag}\left(\left(\left[|\widetilde{\nu}_k|^2 + \sigma_r^2\right]\left(\hat{p}_{kk} + \hat{p}_{n-k,n-k}\right)\right)_k\right) + |\sigma_0|^2\, \mathbb{1}_{n\times n}. \tag{5.101}$$

Proof. Recalling Eq. (4.232), we observe that

$$\begin{aligned}\widehat{u}_{\mathsf{p},\text{Itô}}\left(\hat{p}\right) &:= \frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)}\oint_{\mathsf{p}^{-1}(\hat{p})} \widetilde{\boldsymbol{\sigma}\boldsymbol{\sigma}^\dagger}(\tilde{y})\,\mathrm{d}\mathcal{H}_1(\tilde{y}) \\ &= \frac{1}{\mathcal{H}_1\left(\mathsf{p}^{-1}(\hat{p})\right)}\oint_{\mathsf{p}^{-1}(\hat{p})} \widetilde{\sigma_{\text{mult}}(\sigma_{\text{mult}})^\dagger}(\tilde{y}) + \widetilde{\boldsymbol{\sigma}_{\text{reg}}(\boldsymbol{\sigma}_{\text{reg}})^\dagger}(\tilde{y}) + \widetilde{\boldsymbol{\sigma}_{\text{add}}(\boldsymbol{\sigma}_{\text{add}})^\dagger}\,\mathrm{d}\mathcal{H}_1(\tilde{y}) \\ &= \frac{n^2}{4}\operatorname{diag}\left(\left(\left[|\widetilde{\nu}_k|^2 + \sigma_r^2\right]\left(\hat{p}_{kk} + \hat{p}_{n-k,n-k}\right)\right)_k\right) + |\sigma_0|^2\, \mathbb{1}_{n\times n},\end{aligned} \tag{5.102}$$

where in the last step we have inserted the averaged diffusion matrices, which where calculated in Lemmas 5.9, 5.15 and 5.18. □

5.7. Combined system

We conclude this chapter by providing an explicit matrix-valued SDE, which describes the evolution of the effective outer-product process.

Theorem 5.26 (Combined system)
The evolution of the effective outer-product process is given by

$$\mathrm{d}\hat{p}(t) = \widehat{u}_{\mathsf{p}}(\hat{p})\,\mathrm{d}t + \widehat{\sigma_{\mathsf{p}}}(\hat{p})\,\mathrm{d}\boldsymbol{B}(t), \tag{5.103}$$

where the $\mathbb{C}^{n^2} \cong \mathbb{C}^{n,n}$-valued drift term $\widehat{u}_{\mathsf{p}}(\hat{p}) = \widehat{u}_{\mathsf{p},\text{det}}(\hat{p}) + \widehat{u}_{\mathsf{p},\text{Itô}}\left(\hat{p}\right)$ is composed of a deterministic-coupling part

$$\widehat{u}_{\mathsf{p},\text{det}}(\hat{p}) = \frac{\sqrt{n}}{2}\left[\operatorname{diag}\left(\widetilde{\lambda}\right)\hat{p} + \hat{p}\,\overline{\operatorname{diag}\left(\widetilde{\lambda}\right)}\right], \tag{5.104}$$

and an Itô correction of the form

$$\widehat{u}_{\mathsf{p},\text{Itô}}\left(\hat{p}\right) := \frac{n^2}{4}\operatorname{diag}\left(\left(\left[|\widetilde{\nu}_k|^2 + \sigma_r^2\right]\left(\hat{p}_{kk} + \hat{p}_{n-k,n-k}\right)\right)_k\right) + |\sigma_0|^2\, \mathbb{1}_{n\times n}. \tag{5.105}$$

The $\mathbb{C}^{n^2,6n}$-valued dispersion matrix $\widehat{\sigma_{\mathsf{p}}}(\hat{p})$ is given by

$$\widehat{\sigma_{\mathsf{p}}}(\hat{p}) := \left(\widehat{\sigma}_{\mathsf{p},\text{mult}}(\hat{p}) \mid \widehat{\sigma}_{\mathsf{p},\text{reg}}(\hat{p}) \mid \widehat{\sigma}_{\mathsf{p},\text{add}}(\hat{p})\right). \tag{5.106}$$

where

$$(\widehat{\sigma}_{\mathsf{p},\text{mult}})_{ij,r}(\hat{p}) \coloneqq (\tilde{y}_*)_i \left(\overline{\widehat{\boldsymbol{\sigma}}_{\text{mult}}(\tilde{y}_*)\mathcal{Q}_2^\dagger}\right)_{jr} + \overline{(\tilde{y}_*)_j} \left(\widehat{\boldsymbol{\sigma}}_{\text{mult}}(\tilde{y}_*)\mathcal{Q}_2^\dagger\right)_{ir}, \tag{5.107a}$$

$$(\widehat{\sigma}_{\mathsf{p},\text{reg}})_{ij,r}(\hat{p}) \coloneqq (\tilde{y}_*)_i \left(\overline{\widehat{\boldsymbol{\sigma}}_{\text{reg}}(\tilde{y}_*)\mathcal{Q}_2^\dagger}\right)_{jr} + \overline{(\tilde{y}_*)_j} \left(\widehat{\boldsymbol{\sigma}}_{\text{reg}}(\tilde{y}_*)\mathcal{Q}_2^\dagger\right)_{ir}, \tag{5.107b}$$

$$(\widehat{\sigma}_{\mathsf{p},\text{add}})_{ij,r}(\hat{p}) \coloneqq \beta_i(\hat{p}) \left(\overline{\widehat{\boldsymbol{\sigma}}_{\text{add}}\mathcal{Q}_2^\dagger}\right)_{jr} + \overline{\beta_j(\hat{p})} \left(\widehat{\boldsymbol{\sigma}}_{\text{add}}\mathcal{Q}_2^\dagger\right)_{ir}, \tag{5.107c}$$

for all $i, j \in \{0, ..., n-1\}$, $r \in \{0, ..., 2n-1\}$, $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$ and all $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$. The $\mathbb{C}^n$-valued function $\beta(\hat{p})$ is defined as in Eq. (5.86) and the $\mathbb{C}^{n,2n}$-valued dispersion matrices on the right-hand side of Eq. (5.107) are given by

$$\widehat{\boldsymbol{\sigma}}_{\text{mult}}(\tilde{y}) \coloneqq \frac{n}{2} \left(\operatorname{diag}(\tilde{\nu} \odot \tilde{y})\, \mathrm{e}^{i\,\frac{\pi}{6}} \mid \operatorname{diag}(\tilde{\nu} \odot (R\,\tilde{y}))\, \mathrm{e}^{-i\,\frac{\pi}{6}}\right), \tag{5.108a}$$

$$\widehat{\boldsymbol{\sigma}}_{\text{reg}}(\tilde{y}) \coloneqq \frac{n}{2}\, \sigma_r \left(\operatorname{diag}(\tilde{y}) \mid i \operatorname{diag}(R\tilde{y})\right), \tag{5.108b}$$

$$\widehat{\boldsymbol{\sigma}}_{\text{add}} \coloneqq \frac{\sigma_0}{\sqrt{2}} \left(\mathbb{1}_{n\times n} \mid i\, \mathbb{1}_{n\times n}\right). \tag{5.108c}$$

$(\boldsymbol{B}(t))_{t\geq 0}$ finally denotes a $6n$-dimensional, real-valued Brownian motion.

Proof. We recall that by Theorem 4.71, the evolution of the effective outer-product process $\hat{p}(t)$ is governed by an effective drift term of the form $\widehat{u}_{\mathsf{p}} = \widehat{u}_{\mathsf{p},\text{det}} + \widehat{u}_{\mathsf{p},\text{Itô}}$ (defined in Eq. (4.230)) and a diffusion matrix $\widehat{a}_{\mathsf{p}}$ given by Eq. (4.234). In Lemmas 5.8 and 5.25 we have explicitly calculated the drift terms $\widehat{u}_{\mathsf{p},\text{det}}$ and $\widehat{u}_{\mathsf{p},\text{Itô}}$ and in Proposition 5.24 we have provided a decomposition of the *diffusion* matrix $\widehat{a}_{\mathsf{p}}$ into an averaged *dispersion* matrix $\widehat{\sigma_{\mathsf{p}}}$ of the form

$$\widehat{\sigma_{\mathsf{p}}}(\hat{p}) \coloneqq \left(\widehat{\sigma}_{\mathsf{p},\text{mult}}(\hat{p}) \mid \widehat{\sigma}_{\mathsf{p},\text{reg}}(\hat{p}) \mid \widehat{\sigma}_{\mathsf{p},\text{add}}(\hat{p})\right), \tag{5.109}$$

where $\widehat{\sigma}_{\mathsf{p},\text{mult}}, \widehat{\sigma}_{\mathsf{p},\text{reg}}$ and $\widehat{\sigma}_{\mathsf{p},\text{add}}$ were defined in Eqs. (5.61), (5.73) and (5.90). As commented on in the proofs of Lemmas 5.14 and 5.17, the matrices of Eq. (5.107) are *well-defined*, i.e. they do *not* depend on the choice of $\tilde{y}_* \in \mathsf{p}^{-1}(\hat{p})$ and can be expressed entirely in terms of $\hat{p}$. The evolution of $\hat{p}(t)$ is thus determined by an SDE of the form

$$\mathrm{d}\hat{p}(t) = \widehat{u}_{\mathsf{p}}(\hat{p}(t))\,\mathrm{d}t + \widehat{\sigma_{\mathsf{p}}}(\hat{p}(t))\,\mathrm{d}\boldsymbol{B}(t),$$

where $(\boldsymbol{B}(t))_{t\geq 0}$, denotes a $6n$-dimensional, real-valued Brownian motion, and the result follows. □

The so-called *Yamada-Watanabe condition* allows us to prove strong existence and uniqueness for the SDE given in the previous theorem.

Lemma 5.27 (Existence and uniqueness for averaged outer-product process)
The stochastic differential equation given by Eq. (5.103) has a strong, non-exploding solution, which is strongly unique.

Proof. The *existence* of a weak non-exploding solution to Eq. (5.103) follows from the previous chapter, i.p. from Proposition 4.58 and the averaging result of Theorem 4.71. Despite the fact that Eq. (5.103) does *not* satisfy a local Lipschitz condition,[2] we still have *strong uniqueness* because the *Yamada-Watanabe condition* is fulfilled, c.f. [AS13], Theorem 2.1 and Corollary 2.19, which are based on [YW71], Theorem 1. Weak existence and strong uniqueness together imply the existence of a strong (non-exploding) solution, c.f. [Che02], Proposition 1.6 and Diagram 1.□

We insert the representations of drift and dispersion terms into Eq. (5.103), in order to obtain an explicit SDE for the effective outer-product process.

Corollary 5.28 (Explicit representation of effective SDE)
The evolution of the effective outer-product process $\hat{p}(t)$ is determined by the SDE

$$\begin{aligned} \mathrm{d}\hat{p}(t) = &\frac{\sqrt{n}}{2}\left(\operatorname{diag}\left(\widetilde{\lambda}\right)\,\hat{p}(t) \,+\, \hat{p}(t)\,\overline{\operatorname{diag}\left(\widetilde{\lambda}\right)}\right)\mathrm{d}t \\ &+\left(\frac{n^2}{4}\operatorname{diag}\left(\left(\left[|\widetilde{\nu}_k|^2+\sigma_r^2\right]\left(\hat{p}_{kk}(t)+\hat{p}_{n-k,n-k}(t)\right)\right)_k\right)+|\sigma_0|^2\,\mathbb{1}_{n\times n}\right)\mathrm{d}t \\ &+\frac{n}{2}\left[\tilde{y}(t)\left\{\operatorname{diag}\left(\widetilde{\nu}\odot\tilde{y}(t)\right)\mathrm{e}^{i\,\frac{\pi}{6}}\;\mathrm{d}\widetilde{B_{\mathrm{mult}}}(t)\right\}^\dagger+\mathrm{h.c.}\right] \\ &+\frac{n}{2}\left[\tilde{y}(t)\left\{\operatorname{diag}\left(\widetilde{\nu}\odot R\,\tilde{y}(t)\right)\mathrm{e}^{-i\,\frac{\pi}{6}}\;\mathrm{d}\widetilde{B'_{\mathrm{mult}}}(t)\right\}^\dagger+\mathrm{h.c.}\right] \\ &+\frac{n}{2}\,\sigma_r\left[\tilde{y}(t)\left\{\operatorname{diag}\left(\tilde{y}(t)\right)\,\mathrm{d}\widetilde{B_{\mathrm{reg}}}(t)+i\operatorname{diag}\left(R\,\tilde{y}(t)\right)\,\mathrm{d}\widetilde{B'_{\mathrm{reg}}}(t)\right\}^\dagger+\mathrm{h.c.}\right] \\ &+\frac{\sigma_0}{\sqrt{2}}\left[\beta(\hat{p}(t))\left(\mathrm{d}(\widetilde{B_{\mathrm{add}}}(t)+i\,\widetilde{B'_{\mathrm{add}}}(t))\right)^\dagger+\mathrm{h.c.}\right], \end{aligned} \tag{5.110}$$

where $\tilde{y}(t) \in \mathsf{p}^{-1}(\hat{p}(t))$ and $B_{\mathrm{mult}}, B'_{\mathrm{mult}}, B_{\mathrm{reg}}, B'_{\mathrm{reg}}, B_{\mathrm{add}}$, and B'_{add} are independent, n-dimensional and real-valued Brownian motions.

Note that the right-hand side of Eq. (5.110) is well-defined, i.e. independent of the choice of $\tilde{y}(t) \in \mathsf{p}^{-1}(\hat{p}(t))$. This follows from the fact that it can be expressed entirely in terms of $\hat{p}(t)$ (c.f. proof of Theorem 5.26).

Proof. Following Theorem 5.26, $\hat{p}(t)$ is determined by an SDE of the form

$$\begin{aligned} \mathrm{d}\hat{p}(t) &= \widehat{u}_{\mathsf{p}}(\hat{p}(t))\,\mathrm{d}t+\widehat{\sigma_{\mathsf{p}}}(\hat{p}(t))\,\mathrm{d}\boldsymbol{B}(t), \\ &= \left(\widehat{u}_{\mathsf{p},\mathrm{det}}(\hat{p}(t))+\widehat{u}_{\mathsf{p},\mathrm{Itô}}\,(\hat{p}(t))\right)\mathrm{d}t \\ &\quad+\widehat{\sigma}_{\mathsf{p},\mathrm{mult}}(\hat{p}(t))\,\mathrm{d}\boldsymbol{B}_{\mathrm{mult}}(t)+\widehat{\sigma}_{\mathsf{p},\mathrm{reg}}(\hat{p}(t))\,\mathrm{d}\boldsymbol{B}_{\mathrm{reg}}(t)+\widehat{\sigma}_{\mathsf{p},\mathrm{add}}(\hat{p}(t))\,\mathrm{d}\boldsymbol{B}_{\mathrm{add}}(t), \end{aligned}$$

where we have represented the $6n$-dimensional Brownian motion

$$\boldsymbol{B}(t)=\left((\boldsymbol{B}_{\mathrm{mult}})^\top(t),(\boldsymbol{B}_{\mathrm{reg}})^\top(t),(\boldsymbol{B}_{\mathrm{add}})^\top(t)\right)^\top \tag{5.111}$$

[2] Local Lipschitz continuity is violated by the additive-noise dispersion term.

in terms of the $2n$-dimensional Brownian motions

$$\boldsymbol{B}_{\text{mult}} = \left((B_{\text{mult}})^{\top}, (B'_{\text{mult}})^{\top} \right)^{\top}, \tag{5.112a}$$

$$\boldsymbol{B}_{\text{reg}} = \left((B_{\text{reg}})^{\top}, \left(B'_{\text{reg}}\right)^{\top} \right)^{\top}, \tag{5.112b}$$

$$\boldsymbol{B}_{\text{add}} = \left((B_{\text{add}})^{\top}, (B'_{\text{add}})^{\top} \right)^{\top}. \tag{5.112c}$$

Employing the representations of $\widehat{\sigma}_{\mathsf{p},\text{mult}}$, $\widehat{\sigma}_{\mathsf{p},\text{reg}}$ and $\widehat{\sigma}_{\mathsf{p},\text{add}}$ given in Eq. (5.107), we thus find that

$$\begin{aligned} \mathrm{d}\hat{p}(t) = {} & [\widehat{u}_{\mathsf{p},\text{det}}(\hat{p}(t)) + \widehat{u}_{\mathsf{p},\text{Itô}}\,(\hat{p}(t))]\,\mathrm{d}t \\ & + \left[\tilde{y}(t) \left\{ \widehat{\sigma}_{\text{mult}}(\tilde{y}(t))\,\mathrm{d}\widetilde{B_{\text{mult}}}(t) + \widehat{\sigma'}_{\text{mult}}(\tilde{y}(t))\,\mathrm{d}\widetilde{B'_{\text{mult}}}(t) \right\}^{\dagger} + \text{h.c.} \right] \\ & + \left[\tilde{y}(t) \left\{ \widehat{\sigma}_{\text{reg}}(\tilde{y}(t))\,\mathrm{d}\widetilde{B_{\text{reg}}}(t) + \widehat{\sigma'}_{\text{reg}}(\tilde{y}(t))\,\mathrm{d}\widetilde{B'_{\text{reg}}}(t) \right\}^{\dagger} + \text{h.c.} \right] \\ & + \left[\left(\beta_i(\hat{p}(t)) \left(\overline{\widehat{\sigma}_{\text{add}}\, Q^{\dagger}\,\mathrm{d}B_{\text{add}}(t) + \widehat{\sigma'}_{\text{add}}\, Q^{\dagger}\,\mathrm{d}B'_{\text{add}}(t)} \right)_j \right)_{ij} + \text{h.c.} \right], \end{aligned}$$

where $\tilde{y}(t) \in \mathsf{p}^{-1}(\hat{p}(t))$. The result now follows by inserting the representations of $\widehat{u}_{\mathsf{p},\text{det}}$ and $\widehat{u}_{\mathsf{p},\text{Itô}}$ given in Eqs. (5.104) and (5.105) as well as the definitions of $\widehat{\boldsymbol{\sigma}}_{\text{mult}}$, $\widehat{\boldsymbol{\sigma}}_{\text{reg}}$ and $\widehat{\boldsymbol{\sigma}}_{\text{add}}$, provided by Eq. (5.108). □

6 Effective evolution

In this chapter we study the evolution of the averaged system as given by Theorem 5.26, which we will call *effective evolution*.[1] The *deterministic*, i.e. only drift-coupled case of this system will be investigated in Section 6.1. Subsequently, we study the 'full' randomly perturbed system, with a special focus on the *diagonal* elements of the outer-product process, which we interpret as *eigenmode amplitudes*. We derive evolution equations for these diagonal elements (Section 6.2) and first solve these equations in the *homogeneous case*, i.e. in the absence of additive noise (c.f. Section 6.4). In particular, we derive synchronization results for the averaged system, which in Section 6.5 will be shown to persist in the general *inhomogeneous* case, i.e. in the presence of additive noise.

6.1. Deterministic evolution

We examine the system of Theorem 5.26 in the *deterministic* case, i.e. setting

$$\widetilde{\nu} = \sigma_r = \sigma_0 = 0. \tag{6.1}$$

Under this assumption, Eq. (5.110) reduces to a matrix-valued ordinary differential equation (ODE).

6.1.1. Solution to ODE

The ODE which describes the deterministic system can directly be solved by employing the *Baker–Campbell–Hausdorff* formula.

Proposition 6.1 (Effective drift evolution)
For a given starting point $\hat{p}(0) \in \mathbb{C}^{n,n}$, the (unique) solution to the matrix-valued ODE

$$\mathrm{d}\hat{p}(t) = \widehat{u}_{\mathsf{p},\mathrm{det}}(\hat{p}(t))\,\mathrm{d}t \equiv \frac{\sqrt{n}}{2}\left(\mathrm{diag}\left(\widetilde{\lambda}\right)\,\hat{p}(t)\, + \hat{p}(t)\,\overline{\mathrm{diag}\left(\widetilde{\lambda}\right)}\right)\mathrm{d}t \tag{6.2}$$

[1] c.f. Remark 5.12

is given by

$$\hat{p}(t) = \exp\left(\frac{\sqrt{n}}{2}\,\mathrm{diag}(\widetilde{\lambda})\,t\right)\,\hat{p}(0)\,\exp\left(\frac{\sqrt{n}}{2}\,\overline{\mathrm{diag}(\widetilde{\lambda})\,t}\right), \tag{6.3}$$

i.e. for $k, l \in \{0, ..., n-1\}$, we have

$$\hat{p}_{k,l}(t) = \exp\left(\frac{\sqrt{n}}{2}\left(\mathrm{Re}(\widetilde{\lambda}_k) + \mathrm{Re}\left(\widetilde{\lambda}_l\right)\right)\,t\right)\,\exp\left(\frac{\sqrt{n}}{2}\,i\left(\mathrm{Im}(\widetilde{\lambda}_k) - \mathrm{Im}\left(\widetilde{\lambda}_l\right)\right)\,t\right)\,\hat{p}_{k,l}(0), \tag{6.4}$$

where

$$\mathrm{Re}(\widetilde{\lambda}) = \mathcal{P}_+\,\tilde{\mathfrak{l}}' + (-i)\mathcal{P}_-\,\tilde{\mathfrak{l}} \quad \text{and} \quad \mathrm{Im}(\widetilde{\lambda}) = (-i)\mathcal{P}_-\,\tilde{\mathfrak{l}}' - \mathcal{P}_+\,\tilde{\mathfrak{l}}. \tag{6.5}$$

In particular, the evolution of the diagonal elements $\hat{\rho}_k(t) := \hat{p}_{k,k}(t)$ is governed by the vector-valued ODE

$$\mathrm{d}\hat{\rho}(t) = \sqrt{n}\,\mathrm{diag}\left(\mathrm{Re}(\widetilde{\lambda})\right)\,\hat{\rho}(t)\,\mathrm{d}t \tag{6.6}$$

which gives rise to the solution

$$\hat{\rho}(t) = \exp\left(\sqrt{n}\,\mathrm{diag}\left(\mathrm{Re}(\widetilde{\lambda})\right)\,t\right)\,\hat{\rho}(0) = \exp\left(\sqrt{n}\,\mathrm{diag}\left(\mathcal{P}_+\tilde{\mathfrak{l}}' + (-i)\mathcal{P}_-\tilde{\mathfrak{l}}\right)\,t\right)\,\hat{\rho}(0). \tag{6.7}$$

Proof. Using product rule and differentiability of the matrix exponentials, we can directly verify that Eq. (6.3) solves Eq. (6.2) (c.f. Remark 6.2 for a motivation of Eq. (6.3)). This is the *unique* solution, since Eq. (6.2) is a *linear* ODE, which in particular implies that the drift term is globally Lipschitz continuous. Eq. (6.4) follows by employing that the matrix exponential of a diagonal matrix is again a diagonal matrix and Eq. (6.5) can be verified by decomposing $\widetilde{\lambda} := \tilde{\mathfrak{l}}' - i\,\tilde{\mathfrak{l}}$ into its real- and imaginary part:

$$\mathrm{Re}(\widetilde{\lambda}) = \mathrm{Re}(\tilde{\mathfrak{l}}') + \mathrm{Im}(\tilde{\mathfrak{l}}) = \mathcal{P}_+\,\tilde{\mathfrak{l}}' - i\,\mathcal{P}_-\,\tilde{\mathfrak{l}}, \tag{6.8a}$$
$$\mathrm{Im}(\widetilde{\lambda}) = \mathrm{Im}(\tilde{\mathfrak{l}}') - \mathrm{Re}(\tilde{\mathfrak{l}}) = -i\,\mathcal{P}_-\,\tilde{\mathfrak{l}}' - \mathcal{P}_+\,\tilde{\mathfrak{l}}, \tag{6.8b}$$

where we have applied Lemma 2.30 ii). Eq. (6.7) now directly follows by setting $k = l$ in Eq. (6.4) and by employing Eq. (6.5). □

The ansatz of Eq. (6.3) can be motivated as follows.

Remark 6.2 (Solution via generalized Baker–Campbell–Hausdorff formula)
Setting $\widetilde{A} := \frac{\sqrt{n}}{2}\,\mathrm{diag}\left(\mathrm{Re}(\widetilde{\lambda})\right)$ and $\widetilde{B} := \frac{\sqrt{n}}{2}\,\mathrm{diag}\left(\mathrm{Im}(\widetilde{\lambda})\right)$ we can rewrite Eq. (6.2) as

$$\mathrm{d}\hat{p}(t) = \left(\left\{\widetilde{A}, \hat{p}(t)\right\} + i\,\left[\widetilde{B}, \hat{p}(t)\right]\right)\mathrm{d}t, \tag{6.9}$$

where $[A, B] := AB - BA$ denotes the *commutator* and $\{A, B\} := AB + BA$ the *anticommutator* of two quadratic matrices A, B, c.f. Definition 2.4. From the *Baker–Campbell–Hausdorff* formula (cf. [Pai13]) we infer that

$$e^{i\,\widetilde{B}\,t}\,\hat{p}(0)\,e^{-i\widetilde{B}\,t} = \hat{p}(0) + i\,t\,\left[\widetilde{B},\hat{p}(0)\right] + \frac{(i\,t)^2}{2!}\left[\widetilde{B},\left[\widetilde{B},\hat{p}(0)\right]\right] + \ldots \tag{6.10}$$

Differentiating both sides w.r.t. time t yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left(e^{i\,\widetilde{B}\,t}\,\hat{p}(0)\,e^{-i\widetilde{B}\,t}\right)\\ &= i\left[\widetilde{B},\hat{p}(0)\right] + i\,(i\,t)\left[\widetilde{B},\left[\widetilde{B},\hat{p}(0)\right]\right] + i\,\frac{(i\,t)^2}{2!}\left[\widetilde{B},\left[\widetilde{B},\left[\widetilde{B},\hat{p}(0)\right]\right]\right] + \ldots\\ &= i\left[\widetilde{B},\left(e^{i\,\widetilde{B}\,t}\,\hat{p}(0)\,e^{-i\widetilde{B}\,t}\right)\right].\end{aligned} \tag{6.11}$$

Thus we have shown that in the case of $\widetilde{A}=0$, the solution to Eq. (6.9) is given by

$$\hat{p}(t) = e^{i\,\widetilde{B}\,t}\,\hat{p}(0)\,e^{-i\widetilde{B}\,t}. \tag{6.12}$$

In the general case we similarly find[a]

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{(\widetilde{A}+i\,\widetilde{B})\,t}\,\hat{p}(0)\,e^{(\widetilde{A}-i\,\widetilde{B})\,t}\right) &= \left\{\widetilde{A},\left(e^{(\widetilde{A}+i\,\widetilde{B})\,t}\,\hat{p}(0)\,e^{(\widetilde{A}-i\,\widetilde{B})\,t}\right)\right\}\\ &\quad + i\left[\widetilde{B},\left(e^{(\widetilde{A}+i\,\widetilde{B})\,t}\,\hat{p}(0)\,e^{(\widetilde{A}-i\,\widetilde{B})\,t}\right)\right],\end{aligned}$$

which shows that

$$\hat{p}(t) = e^{(\widetilde{A}+i\,\widetilde{B})\,t}\,\hat{p}(0)\,e^{(\widetilde{A}-i\,\widetilde{B})\,t} = \exp\left(\frac{\sqrt{n}}{2}\,\mathrm{diag}(\widetilde{\lambda})\,t\right)\,\hat{p}(0)\,\exp\left(\frac{\sqrt{n}}{2}\,\overline{\mathrm{diag}(\widetilde{\lambda})}\,t\right) \tag{6.13}$$

solves Eq. (6.9) and thus Eq. (6.2).

[a] employing results from [Pai13] and [MP10]

The diagonal elements

$$\hat{p}_k(t) \coloneqq \hat{p}_{k,k}(t) \equiv |\tilde{y}_k(t)|^2\,, \quad \forall \tilde{y}(t) \in \mathsf{p}^{-1}(\hat{p}(t)), \tag{6.14}$$

are of particular interest to us since they correspond to the squared absolute values (or '*amplitudes*') of the system's eigenmodes, c.f. Definition 6.3 below. By Eqs. (6.5) and (6.7) we observe that the evolution of these eigenmode amplitudes is only affected by the *real part* of $\widetilde{\lambda}$, i.e. by *even* momentum-coupling $\mathcal{P}_+\tilde{\mathfrak{l}}'$ and *odd* space-coupling $\mathcal{P}_-\tilde{\mathfrak{l}}$. The *imaginary part* of $\widetilde{\lambda}$ on the other hand induces oscillations in the off-diagonal elements of $\hat{p}$, c.f. Eq. (6.4). As we will illustrate in the following Section 6.1.2, these oscillations can be interpreted as a periodic exchange of energy between the oscillators.

6.1.2. Energy- and phase evolution of two-oscillator system

6.1.2.1. Generator approach

We continue with Example 3.17 and study how the conserved quantities $\mathsf{T}^{(a)}$, $a \in \{0,1,2,3\}$, of the uncoupled system, evolve in the presence of a weak deterministic coupling. Here we focus on the case of *space*-coupling, i.e. we choose $\lambda = -i\,\mathfrak{l}$. Instead of directly employing Proposition 6.1 (which we will be will be focusing on in the next paragraph), we first return to the original system

$$\mathrm{d}x^{\varepsilon}(t) = \left(i\,2\kappa\,\mathsf{T}^{(0)} - i\,\varepsilon\,\kappa\,\mathfrak{l}_1\,\mathsf{T}^{(1)}\right) x^{\varepsilon}(t)\;\mathrm{d}t + i\,\varepsilon\,\kappa\,\mathfrak{l}_1\,\mathsf{T}^{(1)}\,\overline{x^{\varepsilon}}(t)\;\mathrm{d}t \tag{6.15}$$

of Eq. (3.91), which implies that the *time-rescaled* system $y^{\varepsilon}(t) \coloneqq x^{\varepsilon}(\frac{t}{\kappa\,\varepsilon})$ evolves according to

$$\mathrm{d}y^{\varepsilon}(t) = \left(\frac{2i}{\varepsilon}\,\mathsf{T}^{(0)} - i\,\mathfrak{l}_1\,\mathsf{T}^{(1)}\right) y^{\varepsilon}(t)\;\mathrm{d}t + i\,\mathfrak{l}_1\,\mathsf{T}^{(1)}\,\overline{y^{\varepsilon}}(t)\;\mathrm{d}t. \tag{6.16}$$

By Remark 5.6, we know that the last term of Eq. (6.16) does *not* contribute to the averaged system, i.e. it suffices to study the 'effective' ODE

$$\mathrm{d}y^{\varepsilon}(t) = i\,\left(\frac{2}{\varepsilon}\,\mathsf{T}^{(0)} - \mathfrak{l}_1\,\mathsf{T}^{(1)}\right) y^{\varepsilon}(t)\;\mathrm{d}t. \tag{6.17}$$

Analogously to Eq. (3.30) we define[2,3]

$$c^{(a)}(t) \coloneqq (y^{\varepsilon})^{\dagger}(t)\mathsf{T}^{(a)}y^{\varepsilon}(t), \quad a \in \{0,1,2,3\}\,. \tag{6.18}$$

From Eq. (3.33) in the proof of Theorem 3.4 it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,c^{(a)}(t) = i\,(y^{\varepsilon})^{\dagger}(t)\,\left[\mathsf{T}^{(a)}, \frac{2}{\varepsilon}\,\mathsf{T}^{(0)} - \,\mathfrak{l}_1\,\mathsf{T}^{(1)}\right]\,y^{\varepsilon}(t) = (-i)\,\mathfrak{l}_1\,(y^{\varepsilon})^{\dagger}(t)\,\left[\mathsf{T}^{(a)}, \mathsf{T}^{(1)}\right]\,y^{\varepsilon}(t), \tag{6.19}$$

where in the second step we have made use of $\mathsf{T}^{(0)} = \frac{1}{2}\mathbb{1}_{n\times n}$ commuting with every generator $\mathsf{T}^{(a)}$, i.e. the evolution of $c^{(a)}(t)$ turns out to be independent of ε. In particular, we find that the total energy $c^{(0)}$ remains constant since $\mathsf{T}^{(0)}$ commutes with $\mathsf{T}^{(1)}$. For $a \in \{1,2,3\}$, we recall the commutation relations given in Example 3.3, to find that[4]

$$\frac{\mathrm{d}}{\mathrm{d}t}\,c^{(a)}(t) = \mathfrak{l}_1\,\sum_{d=1}^{3}\,\epsilon_{a,1,d}\,(y^{\varepsilon})^{\dagger}(t)\mathsf{T}^{(d)}y^{\varepsilon}(t) = \mathfrak{l}_1\,\sum_{d=1}^{3}\,\epsilon_{a,1,d}\,c^{(d)}(t), \tag{6.20}$$

[2] These processes will turn out to be independent of the specific choice of ε, which is why we have omitted an explicit reference to ε in the notation.

[3] c.f. [Kum76], Eq. (7)

[4] c.f. [Kum76], Eq. (10)

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} c^{(1)} \\ c^{(2)} \\ c^{(3)} \end{pmatrix}(t) = \mathfrak{l}_1 \begin{pmatrix} 0 \\ -c^{(3)} \\ c^{(2)} \end{pmatrix}(t), \tag{6.21}$$

since $\epsilon_{2,1,3} = -1$ and $\epsilon_{3,1,2} = 1$, while all other $\epsilon_{a,1,c}$ terms vanish. We conclude that both $c^{(2)}(t)$ and $c^{(3)}(t)$ satisfy an harmonic oscillator equation:

$$\frac{\mathrm{d}^2}{(\mathrm{d}t)^2}\, c^{(2)}(t) = -(\mathfrak{l}_1)^2\, c^{(2)}(t), \tag{6.22a}$$

$$\frac{\mathrm{d}^2}{(\mathrm{d}t)^2}\, c^{(3)}(t) = -(\mathfrak{l}_1)^2\, c^{(3)}(t). \tag{6.22b}$$

Recalling the interpretations of $c^{(3)}$ as energy difference and of $c^{(1)}, c^{(2)}$ in terms of the phase difference (c.f. Example 3.6), we conclude that the space-coupling induces a *periodic exchange of energy* between the oscillators, while at the same time the phase difference varies periodically. More precisely, by Eq. (3.41) we know that the vector $(c^{(1)}(t), c^{(2)}(t), c^{(3)}(t))^\top$ is restricted to a sphere of radius $c^{(0)}(t) = c^{(0)}(0)$, while Eq. (6.21) implies that $c^{(1)}(t) = c^{(1)}(0)$ also remains constant. The evolution of the vector is thus restricted to a circle given by the intersection of the $\left\{c \in \mathbb{R}^3 \mid c_1 = c^{(1)}(0)\right\}$-plane with the sphere of radius $c^{(0)}(0)$. This is illustrated in Fig. 6.1, which shows the vector field on the 'energy sphere', highlighting the rotation at constant $c^{(1)}$ value. The coloring indicates the speed of evolution, i.e. the absolute value of the vector field.[5] The red-colored area in front surrounds the fixed point $c^{(1)} = c^{(0)}$, whose counterpart $c^{(1)} = -c^{(0)}$ lies on the not visible backwards facing part of the sphere. Note that these fixed-points correspond to *in-phase synchronization* and *anti-phase* synchronization, which can be seen as follows. At the fixed-points we have $c^{(1)} = \pm c^{(0)}$, which by the spherical coordinates representation of Eq. (3.43) implies that $\cos(\varphi) = \pm 1$, i.e. $\varphi \in \{0, \pi\}$. Since φ denotes the phase difference between the oscillators, $\varphi = 0$ corresponds to *in-phase* synchronization while $\varphi = \pi$ represents *anti-phase* synchronization.

6.1.2.2. Evolution of outer product

Now we study the evolution of the effective two-oscillator system by means of the outer-product approach, i.e. by employing Proposition 6.1. Analogously to Example 3.7, we find that

$$y^\varepsilon(t)(y^\varepsilon(t))^\dagger = \begin{pmatrix} c^{(0)} + c^{(3)} & c^{(1)} - i\,c^{(2)} \\ c^{(1)} + i\,c^{(2)} & c^{(0)} - c^{(3)} \end{pmatrix}(t), \tag{6.23}$$

where the quantities $c^{(a)}$ are defined as in Eq. (6.18).

[5] increasing from red (vanishing speed) to violate (fast evolution)

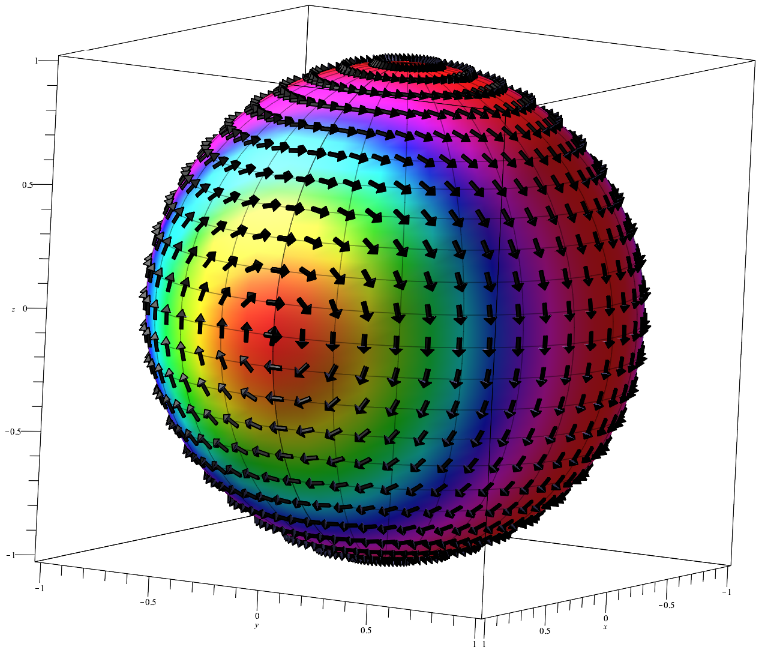


Figure 6.1.: Phase-/energy oscillation of weakly coupled two-oscillator system

Consequently, the outer-product process of the transformed system is given by

$$\begin{aligned}\tilde{y}^{\varepsilon}(t)(\tilde{y}^{\varepsilon}(t))^{\dagger} &= Q^{\dagger}\begin{pmatrix} c^{(0)}+c^{(3)} & c^{(1)}-i\,c^{(2)} \\ c^{(1)}+i\,c^{(2)} & c^{(0)}-c^{(3)}\end{pmatrix}(t)\,Q, \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1\end{pmatrix}\begin{pmatrix} c^{(0)}+c^{(3)} & c^{(1)}-i\,c^{(2)} \\ c^{(1)}+i\,c^{(2)} & c^{(0)}-c^{(3)}\end{pmatrix}(t)\,\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1\end{pmatrix} \\ &= \begin{pmatrix} c^{(0)}+c^{(1)} & c^{(3)}+i\,c^{(2)} \\ c^{(3)}-i\,c^{(2)} & c^{(0)}-c^{(1)}\end{pmatrix}(t). \end{aligned} \tag{6.24}$$

By Eq. (6.4), we conclude that (in the $\varepsilon \to 0$ limit, i.e. for the *averaged* system) we have

$$c^{(0)}(t)+c^{(1)}(t) = \exp\left(\sqrt{2}\,\mathrm{Re}(\tilde{\lambda}_0)\,t\right)\,\left(c^{(0)}(0)+c^{(1)}(0)\right), \tag{6.25a}$$

$$c^{(0)}(t)-c^{(1)}(t) = \exp\left(\sqrt{2}\,\mathrm{Re}(\tilde{\lambda}_1)\,t\right)\,\left(c^{(0)}(0)-c^{(1)}(0)\right), \tag{6.25b}$$

$$c^{(3)}(t)\pm i\,c^{(2)}(t) = \exp\left(\frac{1}{\sqrt{2}}\left(\mathrm{Re}(\tilde{\lambda}_0+\tilde{\lambda}_1)\pm i\,\mathrm{Im}(\tilde{\lambda}_0-\tilde{\lambda}_1)\right)\,t\right)\,\left(c^{(3)}(0)\pm i\,c^{(2)}(0)\right). \tag{6.25c}$$

Note that a circulant coupling of a two-oscillator system necessarily is *symmetric*, since $R = Q^2 = \mathbb{1}_{2\times 2}$ implies $\mathcal{P}_- = 0$. Recalling Eq. (6.5), we thus observe that

$$\mathrm{Re}(\tilde{\lambda}) = \mathcal{P}_+\,\tilde{\mathfrak{l}}' = \tilde{\mathfrak{l}}' \quad \text{and} \quad \mathrm{Im}(\tilde{\lambda}) = -\mathcal{P}_+\,\tilde{\mathfrak{l}} = -\tilde{\mathfrak{l}}. \tag{6.26}$$

This allows us to rewrite Eq. (6.25) as

$$c^{(0)}(t) + c^{(1)}(t) = \exp\left(\sqrt{2}\,\tilde{\mathfrak{l}}'_0\, t\right)\ \left(c^{(0)}(0) + c^{(1)}(0)\right), \tag{6.27a}$$

$$c^{(0)}(t) - c^{(1)}(t) = \exp\left(\sqrt{2}\,\tilde{\mathfrak{l}}'_1\, t\right)\ \left(c^{(0)}(0) - c^{(1)}(0)\right), \tag{6.27b}$$

$$\begin{aligned} c^{(3)}(t) \pm i\, c^{(2)}(t) &= \exp\left(\frac{\left(\tilde{\mathfrak{l}}'_0 + \tilde{\mathfrak{l}}'_1\right) \mp i\, \left(\tilde{\mathfrak{l}}_0 - \tilde{\mathfrak{l}}_1\right)}{\sqrt{2}}\, t\right)\ \left(c^{(3)}(0) \pm i\, c^{(2)}(0)\right) \\ &= \exp\left(\left(\mathfrak{l}'_0 \mp i\, \mathfrak{l}_1\right)\, t\right)\ \left(c^{(3)}(0) \pm i\, c^{(2)}(0)\right), \end{aligned} \tag{6.27c}$$

where in the last step we have made use of

$$\frac{\tilde{\mathfrak{l}}'_0 + \tilde{\mathfrak{l}}'_1}{\sqrt{2}} = (Q\,\tilde{\mathfrak{l}}')_0 = \mathfrak{l}'_0, \tag{6.28}$$

$$\frac{\tilde{\mathfrak{l}}_0 - \tilde{\mathfrak{l}}_1}{\sqrt{2}} = (Q\,\tilde{\mathfrak{l}})_1 = \mathfrak{l}_1. \tag{6.29}$$

Considering specifically the case of *space-coupling* as in the previous section, i.e. $\mathfrak{l}' = 0$ and thus $\lambda = -i\,\mathfrak{l}$, we find that

$$c^{(0)}(t) + c^{(1)}(t) = \left(c^{(0)}(0) + c^{(1)}(0)\right), \tag{6.30a}$$

$$c^{(0)}(t) - c^{(1)}(t) = \left(c^{(0)}(0) - c^{(1)}(0)\right), \tag{6.30b}$$

$$c^{(3)}(t) \pm i\, c^{(2)}(t) = \exp\left(\mp i\, \mathfrak{l}_1\, t\right)\ \left(c^{(3)}(0) \pm i\, c^{(2)}(0)\right). \tag{6.30c}$$

Thus we recover the result of both $c^{(0)}$ and $c^{(1)}$ being constant. Moreover, $c^{(2)}(t)$ and $c^{(3)}(t)$, as given by Eq. (6.30c), solve Eq. (6.22). As noted in the previous section, their evolution can be interpreted as an oscillation of energy- and phase difference.

6.2. Evolution equations for eigenmode amplitudes

We return to the full system given by Theorem 5.26 and study the evolution of the *diagonal elements* of the averaged outer-product process.

6.2.1. Eigenmode amplitudes

Definition 6.3 (Eigenmode amplitudes and total energy)
For each $k \in \{0, ..., n-1\}$, we define the process $\hat{\rho}_k(t)$, called *eigenmode amplitude*, as

$$\hat{\rho}_k(t) \coloneqq \hat{\rho}_{k,k}(t) \equiv \left|\tilde{y}_k(t)\right|^2, \quad \forall \tilde{y}(t) \in \mathsf{p}^{-1}(\hat{\rho}(t)). \tag{6.31}$$

The *total energy* $E(t)$ of the averaged system at time t is given by

$$E(t) := \operatorname{tr}(\hat{\rho}(t)) = \sum_{k=0}^{n-1} \hat{\rho}_k(t) \equiv \sum_{k=0}^{n-1} |\tilde{y}_k(t)|^2 , \quad \forall \tilde{y}(t) \in \mathsf{p}^{-1}(\hat{\rho}(t)). \tag{6.32}$$

The name '*eigenmode amplitude*' takes into account that $\tilde{y}(t) \in \mathsf{p}^{-1}(\hat{\rho}(t))$ can be interpreted as a representation of the averaged system in the Fourier eigenbasis, i.e. its components correspond to the system's eigenmodes and $|\tilde{y}_k(t)|^2$ represents the squared norm of the k'th eigenmode. We note that the *deterministic* evolution of these eigenmode amplitudes has already been studied in the previous section.

Remark 6.4 (Deterministic evolution of eigenmode amplitudes)
Recall that by Proposition 6.1,

$$\hat{\rho}(t) = \exp\left(\sqrt{n}\operatorname{diag}\left(\operatorname{Re}(\widetilde{\lambda})\right)\, t\right)\, \hat{\rho}(0) = \exp\left(\sqrt{n}\operatorname{diag}\left(\mathcal{P}_+\tilde{\mathfrak{l}}' + (-i)\mathcal{P}_-\tilde{\mathfrak{l}}\right)\, t\right)\, \hat{\rho}(0), \tag{6.33}$$

i.e. we find that $\hat{\rho}_k(t)$ exhibits an exponential growth or decay, depending on the sign of the *growth rate* $(\mathcal{P}_+\tilde{\mathfrak{l}}')_k - i\,(\mathcal{P}_-\tilde{\mathfrak{l}})_k \in \mathbb{R}$. Here, $\mathcal{P}_+\tilde{\mathfrak{l}}' = \widetilde{\mathcal{P}_+\mathfrak{l}'}$ is the DFT of the symmetric *momentum* coupling, i.e. this term captures the influence of friction acting on the system, while the factor $\mathcal{P}_-\tilde{\mathfrak{l}} = \widetilde{\mathcal{P}_-\mathfrak{l}}$ reflects the effect of *directed* space-coupling. Note, by contrast, that neither the anti-symmetric component of the momentum-coupling nor the symmetric part of the space-coupling contribute to the evolution of the eigenmode amplitudes.

The general evolution of the eigenmode amplitudes is much more involved. It proves to be useful to distinguish the cases of $k \in \{0, n/2\}$, the 'single-case', and $k \in \{1, ..., n-1\} \setminus \{n/2\}$, the 'pair-case'. These names derive from the fact that in the former case we have $k = n - k$, i.e. $\{k, n-k\}$ is a set with a *single* element, while in the latter, we find $k \neq n - k$ and $\{k, n-k\}$ consequently contains a *pair* of distinct indices. This distinction will turn out to be important, as will become apparent in the following lemma.

Lemma 6.5 (Effective evolution of eigenmode-amplitudes)
For $k \in \{0, n/2\}$, the effective evolution of the eigenmode-amplitudes $\hat{\rho}_k(t)$ is given by

$$\begin{aligned}
\mathrm{d}\hat{\rho}_k(t) = {} & \hat{\rho}_k(t)\, n \sqrt{\frac{|\widetilde{\nu}_k|^2 + 2\,(\operatorname{Re}(\widetilde{\nu}_k))^2}{2} + \sigma_r^2}\; \mathrm{d}(\breve{B}_{\text{hom}})_k \\
& + \hat{\rho}_k(t) \left(\frac{n^2}{2}\left[|\widetilde{\nu}_k|^2 + \sigma_r^2\right] + \sqrt{n}\operatorname{Re}(\widetilde{\lambda}_k)\right) \mathrm{d}t \\
& + \sigma_0\sqrt{2\,\hat{\rho}_k(t)}\; \mathrm{d}(\hat{B}_{\text{add}})_k + |\sigma_0|^2\; \mathrm{d}t,
\end{aligned} \tag{6.34}$$

and for $k \in \{1, ..., n-1\} \setminus \{n/2\}$, it is determined by

$$\begin{aligned} \mathrm{d}\hat{\rho}_k(t) = &\frac{n}{\sqrt{2}} \sqrt{|\widetilde{\nu}_k|^2 + \sigma_r^2} \sqrt{\hat{\rho}_k(t)} \sqrt{\hat{\rho}_k(t) + \hat{\rho}_{n-k}(t)}\, \mathrm{d}(\breve{B}_{\text{hom}})_k \\ &+ \left(\frac{n^2}{4} \left[|\widetilde{\nu}_k|^2 + \sigma_r^2 \right] (\hat{\rho}_k(t) + \hat{\rho}_{n-k}(t)) \right) \mathrm{d}t + \sqrt{n}\, \mathrm{Re}(\widetilde{\lambda}_k) \hat{\rho}_k(t)\, \mathrm{d}t \\ &+ \sigma_0 \sqrt{2\, \hat{\rho}_k(t)}\, \mathrm{d}(\hat{B}_{\text{add}})_k + |\sigma_0|^2\, \mathrm{d}t. \end{aligned} \tag{6.35}$$

Here, $\hat{B}_{\text{add}}$ is an n-dimensional, real-valued Brownian motion defined in Eq. (6.40). For all $k \in \{0, ..., n-1\}$, the process $(\breve{B}_{\text{hom}})_k$ is a real-valued Brownian motion given by Eqs. (6.45)-(6.47), however, $\breve{B}_{\text{hom}}$ is *not* an n-dimensional Brownian motion, since some of its components are not independent, c.f. Lemma 6.8.

Proof. Recall that by Eq. (5.110) of Theorem 5.26, the evolution of the effective system is given by[6]

$$\begin{aligned} \mathrm{d}\hat{p}(t) = &\frac{\sqrt{n}}{2} \left(\mathrm{diag}\left(\widetilde{\lambda}\right) \hat{p}(t) + \hat{p}(t)\, \overline{\mathrm{diag}\left(\widetilde{\lambda}\right)} \right) \mathrm{d}t \\ &+ \left(\frac{n^2}{4} \mathrm{diag} \left(\left(\left[|\widetilde{\nu}_k|^2 + \sigma_r^2 \right] [\hat{p}_{kk}(t) + \hat{p}_{n-k,n-k}(t)] \right)_k \right) + |\sigma_0|^2\, \mathbb{1}_{n\times n} \right) \mathrm{d}t \\ &+ \frac{n}{2} \left[\tilde{y}(t) \left\{ \mathrm{diag}\left(\widetilde{\nu} \odot \tilde{y}(t)\right) \mathrm{e}^{i\frac{\pi}{6}}\, \mathrm{d}\widetilde{B_{\text{mult}}} \right\}^\dagger + \text{h.c.} \right] \\ &+ \frac{n}{2} \left[\tilde{y}(t) \left\{ \mathrm{diag}\left(\widetilde{\nu} \odot R\,\tilde{y}(t)\right) \mathrm{e}^{-i\frac{\pi}{6}}\, \mathrm{d}\widetilde{B'_{\text{mult}}} \right\}^\dagger + \text{h.c.} \right] \\ &+ \frac{n}{2}\sigma_r \left[\tilde{y}(t) \left\{ \mathrm{diag}\left(\tilde{y}(t)\right) \mathrm{d}\widetilde{B_{\text{reg}}} + i\, \mathrm{diag}\left(R\,\tilde{y}(t)\right) \mathrm{d}\widetilde{B'_{\text{reg}}} \right\}^\dagger + \text{h.c.} \right] \\ &+ \frac{\sigma_0}{\sqrt{2}} \left[\beta(\hat{p}(t)) \left(\mathrm{d}(\widetilde{B_{\text{add}}} + i\,\widetilde{B'_{\text{add}}}) \right)^\dagger + \text{h.c.} \right], \end{aligned}$$

where $\tilde{y}(t) \in \mathsf{p}^{-1}(\hat{p}(t))$. In particular, the evolution of the *diagonal* elements $\hat{\rho}_k(t) := \hat{p}_{k,k}(t)$ is governed by

$$\begin{aligned} \mathrm{d}\hat{\rho}_k(t) = &n\, \mathrm{Re} \left[\widetilde{\nu}_k\, \hat{\rho}_k(t)\, e^{i\frac{\pi}{6}}\, \mathrm{d}(\widetilde{B_{\text{mult}}})_k + \widetilde{\nu}_k\, \hat{p}_{n-k,k}(t)\; e^{-i\frac{\pi}{6}}\, \mathrm{d}(\widetilde{B'_{\text{mult}}})_k \right] \\ &+ n\,\sigma_r\, \mathrm{Re} \left(\hat{\rho}_k(t)\, \mathrm{d}(\widetilde{B_{\text{reg}}})_k + i\, \hat{p}_{n-k,k}(t)\; \mathrm{d}(\widetilde{B'_{\text{reg}}})_k \right) \\ &+ \sigma_0 \sqrt{2}\, \mathrm{Re} \left(\overline{\beta_k}(\hat{p}(t))\, \mathrm{d}(\widetilde{W_{\text{add}}})_k \right) \\ &+ \left(\frac{n^2}{4} \left[|\widetilde{\nu}_k|^2 + \sigma_r^2 \right] (\hat{\rho}_k(t) + \hat{\rho}_{n-k}(t)) + |\sigma_0|^2 \right) \mathrm{d}t + \sqrt{n}\, \mathrm{Re}(\widetilde{\lambda}_k) \hat{\rho}_k(t)\, \mathrm{d}t, \end{aligned} \tag{6.36}$$

where $\widetilde{W_{\text{add}}} := \widetilde{B_{\text{add}}} + i\,\widetilde{B'_{\text{add}}}$ is the DFT of the complex Brownian motion $B_{\text{add}} + iB'_{\text{add}}$, which by Lemma 2.35 again constitutes a complex Brownian motion. Since such a complex Brownian

[6]Throughout this proof we will shorten the notation by omitting the time dependence of the appearing Brownian motions.

motion is invariant under a general unitary transformation, we can furthermore 'absorb' the phase of $\beta_k(t)$ into the definition of the complex Brownian motion, i.e.

$$\overline{\beta_k}(\hat{p}(t))\,\mathrm{d}(\widetilde{W}_{\mathrm{add}})_k = |\beta_k(\hat{p}(t))|\,\mathrm{d}\left[\mathrm{e}^{-i\,\varphi_k(t))}\,\mathrm{d}(\widetilde{W}_{\mathrm{add}})_k\right] = \sqrt{\hat{\rho}_k(t)}\;\mathrm{d}(\widetilde{\widetilde{W}}_{\mathrm{add}})_k, \tag{6.37}$$

where we have employed a polar-coordinates decomposition of the form[7]

$$\beta_k(t) = |\beta_k(t)|\,\mathrm{e}^{i\varphi_k(t)} = \sqrt{\hat{\rho}_k(t)}\,\mathrm{e}^{i\varphi_k(t)}, \tag{6.38}$$

and where

$$\mathrm{d}(\widetilde{\widetilde{W}}_{\mathrm{add}})_k \coloneqq \mathrm{e}^{-i\,\varphi_k(t))}\;\mathrm{d}(\widetilde{W}_{\mathrm{add}})_k \tag{6.39}$$

defines a complex Brownian motion $((\widetilde{\widetilde{W}}_{\mathrm{add}})_k(t))_{t\geq 0}$. We denote its real part by

$$(\hat{B}_{\mathrm{add}})_k(t) \coloneqq \mathrm{Re}\left((\widetilde{\widetilde{W}}_{\mathrm{add}})_k\right)(t), \tag{6.40}$$

which thus defines a real-valued Brownian motion. As in Lemma 2.31, we denote by $q^{(k)}$ the k'th row of the DFT matrix $Q^\dagger$ (c.f. Eq. (2.81)). This vector allows us to rewrite Eq. (6.36) in terms of real-valued Brownian motions, i.e.

$$\begin{aligned}\mathrm{d}\hat{\rho}_k(t) = {}& n\ \mathrm{Re}\left[\widetilde{\nu}_k\,\hat{\rho}_k(t)\,e^{i\,\frac{\pi}{6}}\,q^{(k)}\right]\mathrm{d}B_{\mathrm{mult}} + n\,\mathrm{Re}\left[\widetilde{\nu}_k\,\hat{p}_{n-k,k}(t)\ e^{-i\,\frac{\pi}{6}}\,q^{(k)}\right]\mathrm{d}B'_{\mathrm{mult}}\\ &+ n\,\sigma_r\ \mathrm{Re}\left[\hat{\rho}_k(t)q^{(k)}\right]\mathrm{d}B_{\mathrm{reg}} + n\,\sigma_r\ \mathrm{Re}\left[i\,\hat{p}_{n-k,k}(t)\,q^{(k)}\right]\mathrm{d}B'_{\mathrm{reg}}\\ &+ \sigma_0\sqrt{2\,\hat{\rho}_k(t)}\ \mathrm{d}(\hat{B}_{\mathrm{add}})_k\\ &+ \left(\frac{n^2}{4}\ \left[|\widetilde{\nu}_k|^2+\sigma_r^2\right](\hat{\rho}_k(t)+\hat{\rho}_{n-k}(t)) + |\sigma_0|^2\right)\mathrm{d}t + \sqrt{n}\,\mathrm{Re}(\widetilde{\lambda}_k)\hat{\rho}_k(t)\,\mathrm{d}t.\end{aligned} \tag{6.41}$$

We introduce the $\mathbb{R}^n$-valued shorthands

$$a_k(t) \coloneqq n\,\mathrm{Re}\left[\widetilde{\nu}_k\,\hat{\rho}_k(t)\,e^{i\,\frac{\pi}{6}}\,q^{(k)}\right], \tag{6.42a}$$

$$a'_k(t) \coloneqq n\,\mathrm{Re}\left[\widetilde{\nu}_k\,\hat{p}_{n-k,k}(t)\ e^{-i\,\frac{\pi}{6}}\,q^{(k)}\right], \tag{6.42b}$$

$$b_k(t) \coloneqq n\,\sigma_r\ \mathrm{Re}\left[\hat{\rho}_k(t)q^{(k)}\right], \tag{6.42c}$$

$$b'_k(t) \coloneqq n\,\sigma_r\ \mathrm{Re}\left[i\,\hat{p}_{n-k,k}(t)\,q^{(k)}\right], \tag{6.42d}$$

in terms of which Eq. (6.41) can be written as

$$\begin{aligned}\mathrm{d}\hat{\rho}_k(t) = {}& \left[a_k(t)\,\mathrm{d}B_{\mathrm{mult}} + a'_k(t)\,\mathrm{d}B'_{\mathrm{mult}} + b_k(t)\,\mathrm{d}B_{\mathrm{reg}} + b'_k(t)\,\mathrm{d}B'_{\mathrm{reg}}\right] + \sigma_0\sqrt{2\,\hat{\rho}_k(t)}\ \mathrm{d}(\hat{B}_{\mathrm{add}})_k\\ &+ \left(\frac{n^2}{4}\ \left[|\widetilde{\nu}_k|^2+\sigma_r^2\right](\hat{\rho}_k(t)+\hat{\rho}_{n-k}(t)) + |\sigma_0|^2\right)\mathrm{d}t + \sqrt{n}\,\mathrm{Re}(\widetilde{\lambda}_k)\hat{\rho}_k(t)\,\mathrm{d}t.\end{aligned} \tag{6.43}$$

[7] Recall that by definition of $\beta_k(t)$, c.f. Eq. (5.86), it follows that $|\beta_k(t)| = \sqrt{\hat{p}_{kk}(t)} = \sqrt{\hat{\rho}_k(t)}$.

Employing the independence of B_{mult}, B'_{mult}, B_{reg} and B'_{reg}, this can be further simplified to

$$\begin{aligned} \mathrm{d}\hat{\rho}_k(t) &= \sqrt{\|a_k(t)\|^2 + \|a'_k(t)\|^2 + \|b_k(t)\|^2 + \|b'_k(t)\|^2}\, \mathrm{d}(\breve{B}_{\text{hom}})_k + \sigma_0 \sqrt{2\,\hat{\rho}_k(t)}\, \mathrm{d}(\hat{B}_{\text{add}})_k \\ &\quad + \left(\frac{n^2}{4}\left[|\widetilde{\nu}_k|^2 + \sigma_r^2\right](\hat{\rho}_k(t) + \hat{\rho}_{n-k}(t)) + |\sigma_0|^2\right)\mathrm{d}t + \sqrt{n}\,\mathrm{Re}(\widetilde{\lambda}_k)\hat{\rho}_k(t)\,\mathrm{d}t. \end{aligned} \tag{6.44}$$

where for each $k \in \{0, ..., n-1\}$,

$$\mathrm{d}(\breve{B}_{\text{hom}})_k := \frac{a_k(t)\,\mathrm{d}B_{\text{mult}} + a'_k(t)\,\mathrm{d}B'_{\text{mult}} + b_k(t)\,\mathrm{d}B_{\text{reg}} + b'_k(t)\,\mathrm{d}B'_{\text{reg}}}{\sqrt{\|a_k(t)\|^2 + \|a'_k(t)\|^2 + \|b_k(t)\|^2 + \|b'_k(t)\|^2}} \tag{6.45}$$

defines a real-valued Brownian motion, since it satisfies the conditions of *Lévy's characterization*.[8] Note however, that $\breve{B}_{\text{hom}}$ is *not* an n-dimensional Brownian motion, since some of its components are not independent, c.f. Lemma 6.8.

For $k \in \{0, n/2\}$, we observe that $k = n - k \pmod n$, i.e. $\hat{p}_{n-k,k}(t) = \hat{\rho}_k(t)$. Employing Lemma 2.31, we conclude that for $k \in \{0, n/2\}$,

$$\begin{aligned} \|a_k(t)\|^2 + \|a'_k(t)\|^2 &= n^2\left[\mathrm{Re}\left(\widetilde{\nu}_k\,\hat{\rho}_k(t)\,e^{i\frac{\pi}{6}}\right)\right]^2 + n^2\left[\mathrm{Re}\left(\widetilde{\nu}_k\,\hat{\rho}_k(t)\;e^{-i\frac{\pi}{6}}\right)\right]^2 \\ &= \frac{n^2}{2}\left(|\widetilde{\nu}_k|^2 + 2\,(\mathrm{Re}\,(\widetilde{\nu}_k))^2\right)\,(\hat{\rho}_k(t))^2\,, \end{aligned} \tag{6.46a}$$

$$\|b_k(t)\|^2 + \|b'_k(t)\|^2 = n^2\sigma_r^2\,[\mathrm{Re}\,(\hat{\rho}_k(t))]^2 + n^2\sigma_r^2\,[\mathrm{Re}\,(i\hat{\rho}_k(t))]^2 = n^2\sigma_r^2\;(\hat{\rho}_k(t))^2\,, \tag{6.46b}$$

where we have applied the identity

$$\left(\mathrm{Re}(z\;e^{i\frac{\pi}{6}})\right)^2 + \left(\mathrm{Re}(z\;e^{-i\frac{\pi}{6}})\right)^2 = \frac{|z|^2 + 2(\mathrm{Re}(z))^2}{2}, \quad \forall z \in \mathbb{C}.$$

For $k \in \{1, ..., n-1\} \setminus \{n/2\}$, on the other hand, we obtain (again by Lemma 2.31)

$$\|a_k(t)\|^2 + \|a'_k(t)\|^2 = n^2\,\frac{|\widetilde{\nu}_k|^2}{2}\left[(\hat{\rho}_k(t))^2 + \hat{\rho}_k(t)\,\hat{\rho}_{n-k}(t)\right], \tag{6.47a}$$

$$\|b_k(t)\|^2 + \|b'_k(t)\|^2 = n^2\,\frac{\sigma_r^2}{2}\left[(\hat{\rho}_k(t))^2 + \hat{\rho}_k(t)\,\hat{\rho}_{n-k}(t)\right], \tag{6.47b}$$

where we have employed that $|\hat{p}_{n-k,k}(t)|^2 = \hat{\rho}_k(t)\,\hat{\rho}_{n-k}(t)$. For the denominator of Eq. (6.45) we consequently obtain

$$\sqrt{\|a_k(t)\|^2 + \|a'_k(t)\|^2 + \|b_k(t)\|^2 + \|b'_k(t)\|^2} = \frac{n}{\sqrt{2}}\sqrt{|\widetilde{\nu}_k|^2 + \sigma_r^2}\sqrt{\hat{\rho}_k(t)}\sqrt{\hat{\rho}_k(t) + \hat{\rho}_{n-k}(t)}. \tag{6.48}$$

Now the result follows from Eq. (6.44), together with Eqs. (6.46) and (6.47) □

[8] c.f. [KS91], Theorem 3.3.16

Note that Eq. (6.34) and Eq. (6.35) are obtained by restricting Eq. (5.110) to its diagonal elements. For this reason, strong existence and uniqueness for these equations are ensured by Lemma 5.27. Furthermore observe that for $k \in \{1, ..., n-1\} \setminus \{n/2\}$, the evolution of the k'th eigenmode amplitude is coupled to the amplitude of the $(n-k)$'th eigenmode, while the equations for $k \in \{0, n/2\}$ are both uncoupled. Therefore, the SDE decouples into an evolution of $\lfloor n/2 \rfloor + 1$ pairs of eigenmodes.

6.2.2. Pairing of eigenmodes

We focus on the more involved case of $k \in \{1, ..., n-1\} \setminus \{n/2\}$ and study the evolution of the corresponding eigenmode pair. For these eigenmode pairs it proves useful to perform the following 'change of coordinates'.

Definition 6.6 (Ampitude sum, difference and ratio)
For $k \in \{1, ..., n-1\} \setminus \{n/2\}$, we introduce a notation for the sum and difference of the amplitude pairs, as well as for their ratio, i.e.

$$\hat{\rho}_k^{\pm}(t) := [\hat{\rho}_k(t) \pm \hat{\rho}_{n-k}(t)] , \tag{6.49}$$

$$\hat{r}_k(t) := \frac{\hat{\rho}_k^-(t)}{\hat{\rho}_k^+(t)} \in [-1, 1] . \tag{6.50}$$

For $k \in \{0, n/2\}$, we simply define

$$\hat{\rho}_k^+(t) := \hat{\rho}_k(t), \text{ as well as } \hat{\rho}_k^-(t) := 0. \tag{6.51}$$

For future use (c.f. Lemma 6.8 below), we define the factors

$$\mathsf{c}_k := \frac{|\widetilde{\nu}_k|^2}{|\widetilde{\nu}_k|^2 + \sigma_r^2} \in [0, 1], \quad k \in \{1, ..., n-1\} \setminus \{n/2\} , \tag{6.52}$$

which capture the relative strength of multiplicative and regularizing noise. They converge to one in the limit of a vanishing regularizing noise, i.e. $\mathsf{c}_k \to 1$ as $\sigma_r \to 0$. The following lemma establishes the evolution equations for the processes $\hat{\rho}_k^+(t)$ and $\hat{\rho}_k^-(t)$ introduced in Definition 6.6.

Lemma 6.7 (Evolution of amplitude sum and difference)
For $k \in \{0, n/2\}$, we recall that $\hat{\rho}_k^+(t) \equiv \hat{\rho}_k(t)$, which according to Lemma 6.5 satisfies

$$\begin{aligned} \mathrm{d}\hat{\rho}_k^+(t) = {} & \hat{\rho}_k^+(t)\, n \sqrt{\frac{|\widetilde{\nu}_k|^2 + 2\,\mathrm{Re}(\widetilde{\nu}_k)^2}{2} + \sigma_r^2} \;\mathrm{d}(\breve{B}_{\mathrm{hom}})_k \\ & + \hat{\rho}_k^+(t) \left(\frac{n^2}{2} \left[|\widetilde{\nu}_k|^2 + \sigma_r^2 \right] + \sqrt{n}\, \mathrm{Re}(\widetilde{\lambda}_k) \right) \mathrm{d}t \\ & + \sigma_0 \sqrt{2\, \hat{\rho}_k^+(t)}\; \mathrm{d}(\hat{B}_{\mathrm{add}})_k + |\sigma_0|^2 \;\mathrm{d}t. \end{aligned} \tag{6.53}$$

For $k \in \{1, ..., n-1\} \setminus \{n/2\}$, the evolution of $\hat{\rho}_k^+(t)$ and $\hat{\rho}_k^-(t)$ is determined by

$$\begin{aligned}
\mathrm{d}\hat{\rho}_k^+(t) = {} & \left(\frac{n}{2}\sqrt{|\widetilde{\nu}_k|^2+\sigma_r^2}\right)\hat{\rho}_k^+(t)\sqrt{2+(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\;\mathrm{d}(\breve{B}_{\mathrm{hom}}^+)_k \\
& + \left(\frac{n^2}{4}\left[|\widetilde{\nu}_k|^2+\sigma_r^2\right]\right)2\,\hat{\rho}_k^+(t)\,\mathrm{d}t \\
& + \sqrt{n}\,\hat{\rho}_k^+(t)\left[(\mathcal{P}_+\tilde{\mathfrak{l}}')_k - i\,(\mathcal{P}_-\tilde{\mathfrak{l}})_k\hat{r}_k(t)\right]\mathrm{d}t \\
& + \sigma_0\sqrt{2\,\hat{\rho}_k^+(t)}\;\mathrm{d}(\breve{B}_{\mathrm{add}}^+)_k + 2\,|\sigma_0|^2\;\mathrm{d}t, \qquad (6.54a)
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}\hat{\rho}_k^-(t) = {} & \left(\frac{n}{2}\sqrt{|\widetilde{\nu}_k|^2+\sigma_r^2}\right)\hat{\rho}_k^+(t)\sqrt{2-(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\;\mathrm{d}(\breve{B}_{\mathrm{hom}}^-)_k \\
& + \sqrt{n}\,\hat{\rho}_k^+(t)\left[-i\,(\mathcal{P}_-\tilde{\mathfrak{l}})_k + (\mathcal{P}_+\tilde{\mathfrak{l}}')_k\,\hat{r}_k(t)\right]\mathrm{d}t \\
& + \sigma_0\sqrt{2\,\hat{\rho}_k^+(t)}\;\mathrm{d}(\breve{B}_{\mathrm{add}}^-)_k, \qquad (6.54b)
\end{aligned}$$

where $\breve{B}_{\mathrm{hom}}^{\pm}$ and $\breve{B}_{\mathrm{add}}^{\pm}$ are real-valued Brownian motions, defined by

$$\mathrm{d}(\breve{B}_{\mathrm{hom}}^{\pm})_k \coloneqq \frac{\sqrt{\hat{\rho}_k(t)}\;\mathrm{d}(\breve{B}_{\mathrm{hom}})_k \pm \sqrt{\hat{\rho}_{n-k}(t)}\;\mathrm{d}(\breve{B}_{\mathrm{hom}})_{n-k}}{\sqrt{\hat{\rho}_k^+(t)}\sqrt{1\pm\frac{1}{2}\,(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}, \qquad (6.55)$$

$$\mathrm{d}(\breve{B}_{\mathrm{add}}^{\pm})_k \coloneqq \frac{\sqrt{\hat{\rho}_k(t)}\;\mathrm{d}(\hat{B}_{\mathrm{add}})_k \pm \sqrt{\hat{\rho}_{n-k}(t)}\;\mathrm{d}(\hat{B}_{\mathrm{add}})_{n-k}}{\sqrt{\hat{\rho}_k^+(t)}}. \qquad (6.56)$$

Proof. Let $k \in \{1, ..., n-1\} \setminus \{n/2\}$. We rewrite Eq. (6.35) of Lemma 6.5 as:

$$\begin{aligned}
\mathrm{d}\hat{\rho}_k^{\pm}(t) = {} & \left(\frac{n}{2}\sqrt{|\widetilde{\nu}_k|^2+\sigma_r^2}\right)\sqrt{2}\sqrt{\hat{\rho}_k^+(t)}\left(\sqrt{\hat{\rho}_k(t)}\;\mathrm{d}(\breve{B}_{\mathrm{hom}})_k \pm \sqrt{\hat{\rho}_{n-k}(t)}\;\mathrm{d}(\breve{B}_{\mathrm{hom}})_{n-k}\right) \\
& + \left(\frac{n^2}{4}\left[|\widetilde{\nu}_k|^2+\sigma_r^2\right]\right)\left[\hat{\rho}_k^+(t) \pm \hat{\rho}_k^+(t)\right]\mathrm{d}t \\
& + \sqrt{n}\,\underbrace{\left[\mathrm{Re}(\widetilde{\lambda}_k)\hat{\rho}_k(t) \pm \mathrm{Re}(\widetilde{\lambda}_{n-k})\hat{\rho}_{n-k}(t)\right]}_{(\mathcal{P}_{\pm}\mathrm{Re}(\widetilde{\lambda}))_k\,\hat{\rho}_k^+(t)+(\mathcal{P}_{\mp}\mathrm{Re}(\widetilde{\lambda}))_k\,\hat{\rho}_k^-(t)}\mathrm{d}t \\
& + \sigma_0\left(\sqrt{2\,\hat{\rho}_k(t)}\;\mathrm{d}(\hat{B}_{\mathrm{add}})_k \pm \sqrt{2\,\hat{\rho}_{n-k}(t)}\;\mathrm{d}(\hat{B}_{\mathrm{add}})_{n-k}\right) + \left(|\sigma_0|^2 \pm |\sigma_0|^2\right)\mathrm{d}t, \qquad (6.57)
\end{aligned}$$

where we have applied the identity

$$a\,b \pm d\,c = \frac{a \pm d}{2}\,(b+c) + \frac{a \mp d}{2}\,(b-c),$$

with $a = \mathrm{Re}(\widetilde{\lambda}_k)$, $b = \hat{\rho}_k(t)$, $c = \mathrm{Re}(\widetilde{\lambda}_{n-k})$ and $d = \hat{\rho}_{n-k}(t)$. Recall that Eq. (6.8) provides the decomposition $\mathrm{Re}(\widetilde{\lambda}) = \mathcal{P}_+\,\tilde{\mathfrak{l}}' - i\,\mathcal{P}_-\,\tilde{\mathfrak{l}}$, which thus allows us to conclude that $\mathcal{P}_+\mathrm{Re}(\widetilde{\lambda}) = \mathcal{P}_+\,\tilde{\mathfrak{l}}'$ and $\mathcal{P}_-\mathrm{Re}(\widetilde{\lambda}) = -i\,\mathcal{P}_-\,\tilde{\mathfrak{l}}$.

In Lemma 6.8 below we will show that for $k \in \{1, ..., n-1\} \setminus \{n/2\}$ we have

$$\mathrm{d}\left\langle (\breve{B}_{\mathrm{hom}})_k, (\breve{B}_{\mathrm{hom}})_{n-k} \right\rangle (t) = \frac{1}{2} \sqrt{1 - [\hat{r}_k(t)]^2}\, \mathsf{c}_k \cos(2\tilde{\gamma}_k)\, \mathrm{d}t. \tag{6.58}$$

Employing Lévy's characterization of Brownian motion, we thus find that

$$\mathrm{d}(\breve{B}^{\pm}_{\mathrm{hom}})_k := \frac{\sqrt{\hat{\rho}_k(t)}\ \mathrm{d}(\breve{B}_{\mathrm{hom}})_k \pm \sqrt{\hat{\rho}_{n-k}(t)}\ \mathrm{d}(\breve{B}_{\mathrm{hom}})_{n-k}}{\sqrt{\hat{\rho}^+_k(t)}\ \sqrt{1 \pm \frac{1}{2}\ (1 - [\hat{r}_k(t)]^2)\ \mathsf{c}_k \cos(2\tilde{\gamma}_k)}} \tag{6.59}$$

defines, for each $k \in \{1, ..., n-1\} \setminus \{n/2\}$, standard Brownian motions $(\breve{B}^{+}_{\mathrm{hom}})_k$ and $(\breve{B}^{-}_{\mathrm{hom}})_k$, since

$$\mathrm{d}\left\langle \int_0^{\cdot} \sqrt{\hat{\rho}_k(s)}\ \mathrm{d}(\breve{B}_{\mathrm{mult}})_k \pm \int_0^{\cdot} \sqrt{\hat{\rho}_{n-k}(s)}\ \mathrm{d}(\breve{B}_{\mathrm{mult}})_{n-k} \right\rangle (t) \tag{6.60}$$

$$= \underbrace{(\hat{\rho}_k(t) + \hat{\rho}_{n-k}(t))}_{\hat{\rho}^+_k(t)}\,\mathrm{d}t \pm \underbrace{2\sqrt{\hat{\rho}_k(t)\hat{\rho}_{n-k}(t)}}_{\hat{\rho}^+_k(t)\,\sqrt{1-[\hat{r}_k(t)]^2}}\ \underbrace{\mathrm{d}\left\langle (\breve{B}_{\mathrm{mult}})_k, (\breve{B}_{\mathrm{mult}})_k \right\rangle (t)}_{\frac{1}{2}\sqrt{1-[\hat{r}_k(t)]^2}\,\mathsf{c}_k \cos(2\tilde{\gamma}_k)\,\mathrm{d}t}$$

$$= \hat{\rho}^+_k(t) \left(1 \pm \frac{1}{2} \left(1 - [\hat{r}_k(t)]^2\right) \mathsf{c}_k \cos(2\tilde{\gamma}_k)\right) \mathrm{d}t. \tag{6.61}$$

Similarly, we find that

$$\mathrm{d}\left\langle \int_0^{\cdot} \sqrt{\hat{\rho}_k(s)}\ \mathrm{d}(\hat{B}_{\mathrm{add}})_k \pm \int_0^{\cdot} \sqrt{\hat{\rho}_{n-k}(s)}\ \mathrm{d}(\hat{B}_{\mathrm{add}})_{n-k} \right\rangle (t) = \hat{\rho}^+_k(t)\,\mathrm{d}t, \tag{6.62}$$

which is why $(\breve{B}^{+}_{\mathrm{add}})_k$ and $(\breve{B}^{-}_{\mathrm{add}})_k$ defined by

$$\mathrm{d}(\breve{B}^{\pm}_{\mathrm{add}})_k := \frac{\sqrt{\hat{\rho}_k(t)}\ \mathrm{d}(\hat{B}_{\mathrm{add}})_k \pm \sqrt{\hat{\rho}_{n-k}(t)}\ \mathrm{d}(\hat{B}_{\mathrm{add}})_{n-k}}{\sqrt{\hat{\rho}^+_k(t)}} \tag{6.63}$$

are Brownian motions and the result follows. □

In the proof of this previous lemma we have employed the following result on the covariation process $\left\langle (\breve{B}_{\mathrm{hom}})_k, (\breve{B}_{\mathrm{hom}})_{n-k} \right\rangle (t)$, which proves to be nontrivial, since the Brownian motions are not independent.

Lemma 6.8 (Covariation process)
For $k \in \{1, ..., n-1\} \setminus \{n/2\}$, the covariation process of $(\breve{B}_{\mathrm{hom}})_k$ and $(\breve{B}_{\mathrm{hom}})_{n-k}$ is given by

$$\mathrm{d}\left\langle (\breve{B}_{\mathrm{hom}})_k, (\breve{B}_{\mathrm{hom}})_{n-k} \right\rangle (t) = \frac{1}{2} \sqrt{1 - [\hat{r}_k(t)]^2}\, \mathsf{c}_k \cos(2\tilde{\gamma}_k)\, \mathrm{d}t. \tag{6.64}$$

Proof. Recall that $(\breve{B}_{\mathrm{hom}})_k$ is given by (c.f. Eq. (6.45))

$$\mathrm{d}(\breve{B}_{\mathrm{hom}})_k := \frac{a_k(t)\,\mathrm{d}B_{\mathrm{mult}} + a'_k(t)\,\mathrm{d}B'_{\mathrm{mult}} + b_k(t)\,\mathrm{d}B_{\mathrm{reg}} + b'_k(t)\,\mathrm{d}B'_{\mathrm{reg}}}{\sqrt{\|a_k(t)\|^2 + \|a'_k(t)\|^2 + \|b_k(t)\|^2 + \|b'_k(t)\|^2}}, \tag{6.65}$$

where the ($\mathbb{R}^n$-valued) shorthands $a_k(t)$, $a'_k(t)$, $b_k(t)$ and $b'_k(t)$ were defined in Eq. (6.42). Furthermore recall that for $k \in \{1, ..., n-1\} \setminus \{n/2\}$, we know that (c.f. Eq. (6.48))

$$\begin{aligned} c_k(t) &:= \sqrt{\|a_k(t)\|^2 + \|a'_k(t)\|^2 + \|b_k(t)\|^2 + \|b'_k(t)\|^2} \\ &= \frac{n}{\sqrt{2}} \sqrt{|\widetilde{\nu}_k|^2 + \sigma_r^2} \sqrt{\hat{\rho}_k(t)} \sqrt{\hat{\rho}_k(t) + \hat{\rho}_{n-k}(t)}. \end{aligned} \tag{6.66}$$

Employing $\widetilde{\nu}_k = \widetilde{\nu}_{n-k}$ (c.f. remark below Lemma 3.28), this yields

$$c_k(t)\, c_{n-k}(t) = \frac{n^2}{2} \left(|\widetilde{\nu}_k|^2 + \sigma_r^2\right) \sqrt{\hat{\rho}_k(t)\, \hat{\rho}_{n-k}(t)}\, \left(\hat{\rho}_k(t) + \hat{\rho}_{n-k}(t)\right). \tag{6.67}$$

Since B_{mult}, B'_{mult}, B_{reg} and B'_{reg} are *independent* n-dimensional Brownian motions, we find that

$$\begin{aligned} \mathrm{d}\left\langle (\breve{B}_{\text{hom}})_k, (\breve{B}_{\text{hom}})_{n-k} \right\rangle (t) = &\frac{[a_k(t)]^\top a_{n-k}(t) + [a'_k(t)]^\top a'_{n-k}(t)}{c_k(t)\, c_{n-k}(t)}\, \mathrm{d}t \\ &+ \frac{[b_k(t)]^\top b_{n-k}(t) + [b'_k(t)]^\top b'_{n-k}(t)}{c_k(t)\, c_{n-k}(t)}\, \mathrm{d}t, \end{aligned} \tag{6.68}$$

since for instance

$$\begin{aligned} \mathrm{d}\left\langle \int^\cdot a_k(t)\, \mathrm{d}B_{\text{mult}}, \int^\cdot a_{n-k}(t)\, \mathrm{d}B_{\text{mult}} \right\rangle (t) &= [a_k(t)]^\top\, \mathrm{d}\left\langle B_{\text{mult}}, B_{\text{mult}} \right\rangle (t)\, a_{n-k}(t) \\ &= [a_k(t)]^\top\, \mathbb{1}_{n\times n}\, a_{n-k}(t)\ \mathrm{d}t \\ &= [a_k(t)]^\top\, a_{n-k}(t)\ \mathrm{d}t. \end{aligned} \tag{6.69}$$

Employing Lemma 2.31 and Eq. (6.42), we can calculate the terms in the numerator of Eq. (6.68). For the first term we obtain

$$\begin{aligned} [a_k(t)]^\top a_{n-k}(t) &= \left[n \operatorname{Re}\left(\widetilde{\nu}_k\, \hat{\rho}_k(t)\, e^{i\frac{\pi}{6}}\, q^{(k)}\right)\right]^\top n \operatorname{Re}\left(\widetilde{\nu}_{n-k}\, \hat{\rho}_{n-k}(t)\, e^{i\frac{\pi}{6}}\, q^{(n-k)}\right) \\ &= n^2\, \hat{\rho}_k(t)\, \hat{\rho}_{n-k}(t) \left[\operatorname{Re}\left(\widetilde{\nu}_k\, e^{i\frac{\pi}{6}}\, q^{(k)}\right)\right]^\top \operatorname{Re}\left(\widetilde{\nu}_k\, e^{i\frac{\pi}{6}}\, \overline{q^{(k)}}\right) \\ &= n^2\, \hat{\rho}_k(t)\, \hat{\rho}_{n-k}(t) \left(\operatorname{Re}\left(\widetilde{\nu}_k\, e^{i\frac{\pi}{6}}\right)\right)^2 \left\|\operatorname{Re}\left(q^{(k)}\right)\right\|^2 \\ &\quad - n^2\, \hat{\rho}_k(t)\, \hat{\rho}_{n-k}(t) \left(\operatorname{Im}\left(\widetilde{\nu}_k\, e^{i\frac{\pi}{6}}\right)\right)^2 \left\|\operatorname{Im}\left(q^{(k)}\right)\right\|^2 \\ &= \frac{n^2}{2} \hat{\rho}_k(t)\, \hat{\rho}_{n-k}(t) \left[\left(\operatorname{Re}\left(\widetilde{\nu}_k\, e^{i\frac{\pi}{6}}\right)\right)^2 - \left(\operatorname{Im}\left(\widetilde{\nu}_k\, e^{i\frac{\pi}{6}}\right)\right)^2\right], \end{aligned} \tag{6.70}$$

where in the third step we have made use of the decomposition

$$\begin{aligned} \left[\operatorname{Re}\left(\widetilde{\nu}_k\, e^{i\frac{\pi}{6}}\, q^{(k)}\right)\right]^\top \operatorname{Re}\left(\widetilde{\nu}_k\, e^{i\frac{\pi}{6}}\, \overline{q^{(k)}}\right) = &\left[\operatorname{Re}\left(\widetilde{\nu}_k\, e^{i\frac{\pi}{6}}\right) \operatorname{Re}\left(q^{(k)}\right) - \operatorname{Im}\left(\widetilde{\nu}_k\, e^{i\frac{\pi}{6}}\right) \operatorname{Im}\left(q^{(k)}\right)\right]^\top \\ &\cdot \left[\operatorname{Re}\left(\widetilde{\nu}_k\, e^{i\frac{\pi}{6}}\right) \operatorname{Re}\left(q^{(k)}\right) + \operatorname{Im}\left(\widetilde{\nu}_k\, e^{i\frac{\pi}{6}}\right) \operatorname{Im}\left(q^{(k)}\right)\right], \end{aligned}$$

noting that the mixed terms vanish by virtue of Eq. (2.84). Similarly, the second term yields

$$
\begin{aligned}
\left[a'_k(t)\right]^\top a'_{n-k}(t) &= \left[n\,\mathrm{Re}\left(\widetilde{\nu}_k\,\hat{p}_{n-k,k}(t)\,e^{-i\,\frac{\pi}{6}}\,q^{(k)}\right)\right]^\top n\,\mathrm{Re}\left(\widetilde{\nu}_{n-k}\,\hat{p}_{k,n-k}(t)\,e^{-i\,\frac{\pi}{6}}\,q^{(n-k)}\right) \\
&= n^2\left[\mathrm{Re}\left(\widetilde{\nu}_k\,e^{-i\,\frac{\pi}{6}}\,\hat{p}_{n-k,k}(t)\,q^{(k)}\right)\right]^\top \mathrm{Re}\left(\widetilde{\nu}_k\,e^{-i\,\frac{\pi}{6}}\,\overline{\hat{p}_{n-k,k}}(t)\,\overline{q^{(k)}}\right) \\
&= n^2\left(\mathrm{Re}\left(\widetilde{\nu}_k\,e^{-i\,\frac{\pi}{6}}\right)\right)^2\left\|\mathrm{Re}\left(\hat{p}_{n-k,k}(t)\,q^{(k)}\right)\right\|^2 \\
&\quad - n^2\left(\mathrm{Im}\left(\widetilde{\nu}_k\,e^{-i\,\frac{\pi}{6}}\right)\right)^2\left\|\mathrm{Im}\left(\hat{p}_{n-k,k}(t)\,q^{(k)}\right)\right\|^2 \\
&= \frac{n^2}{2}\hat{\rho}_k(t)\,\hat{\rho}_{n-k}(t)\left[\mathrm{Re}\left(\widetilde{\nu}_k\,e^{-i\,\frac{\pi}{6}}\right)^2 - \mathrm{Im}\left(\widetilde{\nu}_k\,e^{-i\,\frac{\pi}{6}}\right)^2\right].
\end{aligned}
\tag{6.71}
$$

Recalling the coupling-angle representation $\widetilde{\nu}_k = |\widetilde{\nu}_k|\;\mathrm{e}^{i\,\tilde{\gamma}_k}$ of Definition 3.29, we can represent the sum of Eqs. (6.70) and (6.71) as

$$
\left[a_k(t)\right]^\top a_{n-k}(t) + \left[a'_k(t)\right]^\top a'_{n-k}(t) = \frac{n^2}{2}\hat{\rho}_k(t)\,\hat{\rho}_{n-k}(t)\;|\widetilde{\nu}_k|^2\,\cos(2\tilde{\gamma}_k),
\tag{6.72}
$$

where we have employed the identity

$$
\cos^2\left(\tilde{\gamma}_k + \frac{\pi}{6}\right) - \sin^2\left(\tilde{\gamma}_k + \frac{\pi}{6}\right) + \cos^2\left(\tilde{\gamma}_k - \frac{\pi}{6}\right) - \sin^2\left(\tilde{\gamma}_k - \frac{\pi}{6}\right) = \cos(2\tilde{\gamma}_k).
\tag{6.73}
$$

Analogous calculations allow us to determine the third and fourth term in the numerator of Eq. (6.68):

$$
\begin{aligned}
\left[b_k(t)\right]^\top b_{n-k}(t) &= n^2\,\sigma_r^2\left[\mathrm{Re}\left(\hat{\rho}_k(t)q^{(k)}\right)\right]^\top \mathrm{Re}\left(\hat{\rho}_{n-k}(t)q^{(n-k)}\right) \\
&= n^2\,\sigma_r^2\,\hat{\rho}_k(t)\,\hat{\rho}_{n-k}(t)\;\left\|\mathrm{Re}\left(q^{(k)}\right)\right\|^2 \\
&= \frac{n^2}{2}\,\sigma_r^2\,\hat{\rho}_k(t)\,\hat{\rho}_{n-k}(t),
\end{aligned}
\tag{6.74a}
$$

$$
\begin{aligned}
\left[b'_k(t)\right]^\top b'_{n-k}(t) &= n^2\,\sigma_r^2\left[\mathrm{Re}\left(i\,\hat{p}_{n-k,k}(t)\,q^{(k)}\right)\right]^\top \mathrm{Re}\left(i\,\hat{p}_{k,n-k}(t)\,q^{(n-k)}\right) \\
&= n^2\,\sigma_r^2\left[\mathrm{Im}\left(\hat{p}_{n-k,k}(t)\,q^{(k)}\right)\right]^\top \mathrm{Im}\left(\overline{\hat{p}_{n-k,k}}(t)\,\overline{q^{(k)}}\right) \\
&= -n^2\,\sigma_r^2\left\|\mathrm{Im}\left(\hat{p}_{n-k,k}(t)\,q^{(k)}\right)\right\|^2 \\
&= -\frac{n^2}{2}\,\sigma_r^2\,\hat{\rho}_k(t)\,\hat{\rho}_{n-k}(t),
\end{aligned}
\tag{6.74b}
$$

which can be seen to cancel, i.e.

$$
\left[b_k(t)\right]^\top b_{n-k}(t) + \left[b'_k(t)\right]^\top b'_{n-k}(t) = 0.
\tag{6.75}
$$

Inserting the results of Eqs. (6.67), (6.72) and (6.75) into Eq. (6.68) we conclude that

$$
\begin{aligned}
\mathrm{d}\left\langle ((\breve{B}_{\mathrm{hom}}))_k, ((\breve{B}_{\mathrm{hom}}))_{n-k}\right\rangle(t) &= \frac{\frac{n^2}{2}\,|\tilde{\nu}_k|^2\,\hat{\rho}_k(t)\,\hat{\rho}_{n-k}(t)\cos(2\tilde{\gamma}_k)}{\frac{n^2}{2}\left(|\tilde{\nu}_k|^2+\sigma_r^2\right)\sqrt{\hat{\rho}_k(t)\,\hat{\rho}_{n-k}(t)}\,\left(\hat{\rho}_k(t)+\hat{\rho}_{n-k}(t)\right)}\,\mathrm{d}t \\
&= \frac{|\tilde{\nu}_k|^2}{\left(|\tilde{\nu}_k|^2+\sigma_r^2\right)}\,\cos(2\tilde{\gamma}_k)\,\frac{\sqrt{\hat{\rho}_k(t)\,\hat{\rho}_{n-k}(t)}}{\hat{\rho}_k(t)+\hat{\rho}_{n-k}(t)}\,\mathrm{d}t \\
&= \mathsf{c}_k\,\cos(2\tilde{\gamma}_k)\,\frac{1}{2}\sqrt{1-[\hat{r}_k(t)]^2}\,\mathrm{d}t. \qquad \square
\end{aligned}
$$

6.3. Amplitude ratio for eigenmode pair

A description of $\hat{\rho}_k^+(t)$ by means of Eq. (6.54a) hinges on an understanding of the ratio process $\hat{r}_k(t)$, which is why this section is devoted to the study of $\hat{r}_k(t)$. Recall that this process is only needed in the case of $k \in \{1, \ldots, n-1\} \setminus \{n/2\}$, to which we will restrict ourselves in this section.

6.3.1. SDE for amplitude ratio

Itô's formula and Lemma 6.7 allow us to derive an evolution equation for $\hat{r}_k(t)$.

Lemma 6.9 (Evolution of amplitude ratio for paired amplitudes)
For $k \in \{1, \ldots, n-1\} \setminus \{n/2\}$, we find that

$$
\begin{aligned}
\mathrm{d}\hat{r}_k(t) = {} & \left(\frac{n}{2}\sqrt{|\tilde{\nu}_k|^2+\sigma_r^2}\right)\sqrt{2-(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\,\sqrt{1-[\hat{r}_k(t)]^2}\,\mathrm{d}(\breve{B}^r_{\mathrm{hom}})_k \\
& - \left(\frac{n^2}{4}\left[|\tilde{\nu}_k|^2+\sigma_r^2\right]\right)\left[2-\left(1-[\hat{r}_k(t)]^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right]\hat{r}_k(t)\,\mathrm{d}t \\
& + \sqrt{n}\,(-i)(\mathcal{P}_{-}\tilde{\mathfrak{l}})_k\left(1-[\hat{r}_k(t)]^2\right)\mathrm{d}t \\
& + \sigma_0\sqrt{\frac{2}{\hat{\rho}_k^+(t)}}\sqrt{1-[\hat{r}_k(t)]^2}\,\mathrm{d}(\breve{B}^r_{\mathrm{add}})_k - |\sigma_0|^2\,\frac{2}{\hat{\rho}_k^+(t)}\,\hat{r}_k(t)\,\mathrm{d}t,
\end{aligned} \tag{6.76}
$$

where $(\breve{B}^r_{\mathrm{hom}})_k$ and $(\breve{B}^r_{\mathrm{add}})_k$, defined by

$$
\begin{aligned}
\mathrm{d}(\breve{B}^r_{\mathrm{hom}})_k &:= \frac{\sqrt{2-(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}{\sqrt{2-(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\,\sqrt{1-[\hat{r}_k(t)]^2}}\,\mathrm{d}(\breve{B}^-_{\mathrm{hom}})_k \\
&\quad - \hat{r}_k(t)\,\frac{\sqrt{2+(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}{\sqrt{2-(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\,\sqrt{1-[\hat{r}_k(t)]^2}}\,\mathrm{d}(\breve{B}^+_{\mathrm{hom}})_k,
\end{aligned} \tag{6.77}
$$

$$
\mathrm{d}(\breve{B}^r_{\mathrm{add}})_k := \frac{\mathrm{d}(\breve{B}^-_{\mathrm{add}})_k - \hat{r}_k(t)\,\mathrm{d}(\breve{B}^+_{\mathrm{add}})_k}{\sqrt{1-[\hat{r}_k(t)]^2}} \tag{6.78}
$$

are real-valued Brownian motions.

Proof. Application of Itô's formula yields

$$\begin{aligned}
\mathrm{d}\hat{r}_k(t) = & \left(\frac{1}{\hat{\rho}_k^+(t)}\right) \mathrm{d}\hat{\rho}_k^-(t) - \left(\frac{\hat{\rho}_k^-(t)}{[\hat{\rho}_k^+(t)]^2}\right) \mathrm{d}\hat{\rho}_k^+(t) \\
& + \frac{1}{2}\left[2\left(\frac{\hat{\rho}_k^-(t)}{[\hat{\rho}_k^+(t)]^3}\right) \mathrm{d}\left\langle \hat{\rho}_k^+\right\rangle(t) + 2\left(-\frac{1}{[\hat{\rho}_k^+(t)]^2}\right) \mathrm{d}\left\langle \hat{\rho}_k^-, \hat{\rho}_k^+\right\rangle(t)\right]. \qquad (6.79)
\end{aligned}$$

In order to evaluate the Itô correction terms, we determine the relevant covariation processes. We observe that

$$\begin{aligned}
& \mathrm{d}\left\langle (\breve{B}^+_{\mathrm{hom}})_k, (\breve{B}^-_{\mathrm{hom}})_k\right\rangle(t) \\
&= \frac{\mathrm{d}\left\langle \int_0^\cdot \sqrt{\hat{\rho}_k(s)}\,\mathrm{d}(\breve{B}_{\mathrm{hom}})_k\right\rangle(t) - \mathrm{d}\left\langle \int_0^\cdot \sqrt{\hat{\rho}_{n-k}(s)}\,\mathrm{d}(\breve{B}_{\mathrm{hom}})_{n-k}\right\rangle(t)}{\hat{\rho}_k^+(t)\sqrt{\left(1+\frac{1}{2}\,(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right)}\sqrt{\left(1-\frac{1}{2}\,(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right)}} \\
&= \frac{\hat{\rho}_k(t) - \hat{\rho}_{n-k}(t)}{\hat{\rho}_k^+(t)\sqrt{\left(1+\frac{1}{2}\,(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right)}\sqrt{\left(1-\frac{1}{2}\,(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right)}}\,\mathrm{d}t \\
&= \frac{2\,\hat{r}_k(t)}{\sqrt{(2+(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k))}\sqrt{(2-(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k))}}\,\mathrm{d}t, \qquad (6.80)
\end{aligned}$$

where in the first step the mixed terms involving $\mathrm{d}\left\langle (\breve{B}_{\mathrm{hom}})_k, (\breve{B}_{\mathrm{hom}})_{n-k}\right\rangle(t)$ cancel. In a similar way we calculate[9]

$$\begin{aligned}
& \mathrm{d}\left\langle (\breve{B}^+_{\mathrm{add}})_k, (\breve{B}^-_{\mathrm{add}})_k\right\rangle \\
&= \frac{\mathrm{d}\left\langle \int^\cdot \sqrt{\hat{\rho}_k}\,\mathrm{d}(\hat{B}_{\mathrm{add}})_k + \int^\cdot \sqrt{\hat{\rho}_{n-k}}\,\mathrm{d}(\hat{B}_{\mathrm{add}})_{n-k}, \int^\cdot \sqrt{\hat{\rho}_k}\,\mathrm{d}(\hat{B}_{\mathrm{add}})_k - \int^\cdot \sqrt{\hat{\rho}_{n-k}}\,\mathrm{d}(\hat{B}_{\mathrm{add}})_{n-k}\right\rangle}{\hat{\rho}_k^+} \\
&= \frac{\hat{\rho}_k - \hat{\rho}_{n-k}}{\hat{\rho}_k^+}\,\mathrm{d}t = \hat{r}_k\,\mathrm{d}t. \qquad (6.81)
\end{aligned}$$

Employing these results, we can rewrite Eq. (6.79) as

$$\begin{aligned}
\mathrm{d}\hat{r}_k(t) = & \left(\frac{n}{2}\sqrt{|\tilde{\nu}_k|^2+\sigma_r^2}\right)\sqrt{2-(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\;\mathrm{d}(\breve{B}^-_{\mathrm{hom}})_k \\
& - \left(\frac{n}{2}\sqrt{|\tilde{\nu}_k|^2+\sigma_r^2}\right)\hat{r}_k(t)\sqrt{2+(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\;\mathrm{d}(\breve{B}^+_{\mathrm{hom}})_k \\
& - \left(\frac{n^2}{4}\left[|\tilde{\nu}_k|^2+\sigma_r^2\right]\right)\hat{r}_k(t)\left[2-\left(1-[\hat{r}_k(t)]^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right]\mathrm{d}t \\
& + \sqrt{n}\left[(-i)(\mathcal{P}_-\tilde{\mathfrak{l}})_k + (\mathcal{P}_+\tilde{\mathfrak{l}}')_k\,\hat{r}_k(t)\right]\mathrm{d}t - \sqrt{n}\,\hat{r}_k(t)\left[(\mathcal{P}_+\tilde{\mathfrak{l}}')_k + (-i)(\mathcal{P}_-\tilde{\mathfrak{l}})_k\hat{r}_k(t)\right]\mathrm{d}t \\
& + \sigma_0\sqrt{\frac{2}{\hat{\rho}_k^+(t)}}\,\mathrm{d}(\breve{B}^-_{\mathrm{add}})_k - \hat{r}_k(t)\,\sigma_0\sqrt{\frac{2}{\hat{\rho}_k^+(t)}}\,\mathrm{d}(\breve{B}^+_{\mathrm{add}})_k - \frac{\hat{r}_k(t)}{\hat{\rho}_k^+(t)}\,2\,|\sigma_0|^2\,\mathrm{d}t,
\end{aligned}$$

[9] We drop the time dependence in order to shorten the notation.

i.e.

$$\begin{aligned} \mathrm{d}\hat{r}_k(t) = &\left(\frac{n}{2}\sqrt{|\tilde{\nu}_k|^2+\sigma_r^2}\right)\sqrt{2-(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\,\sqrt{1-[\hat{r}_k(t)]^2}\,\mathrm{d}(\breve{B}^r_{\mathrm{hom}})_k \\ &-\left(\frac{n^2}{4}\,\left[|\tilde{\nu}_k|^2+\sigma_r^2\right]\right)\hat{r}_k(t)\left[2-\left(1-[\hat{r}_k(t)]^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right]\mathrm{d}t \\ &+\sqrt{n}\,(-i)(\mathcal{P}_-\tilde{\mathfrak{l}})_k\left(1-[\hat{r}_k(t)]^2\right)\mathrm{d}t \\ &+\sigma_0\sqrt{\frac{2}{\hat{\rho}^+_k(t)}}\,\sqrt{1-[\hat{r}_k(t)]^2}\,\mathrm{d}(\breve{B}^r_{\mathrm{add}})_k-|\sigma_0|^2\,\frac{2}{\hat{\rho}^+_k(t)}\,\hat{r}_k(t)\,\mathrm{d}t, \end{aligned}$$

where we have introduced

$$\mathrm{d}(\breve{B}^r_{\mathrm{hom}})_k \coloneqq \frac{\mathrm{d}(\breve{B}^-_{\mathrm{hom}})_k}{\sqrt{1-[\hat{r}_k(t)]^2}}-\frac{\hat{r}_k(t)\,\sqrt{2+(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\;\mathrm{d}(\breve{B}^+_{\mathrm{hom}})_k}{\sqrt{1-[\hat{r}_k(t)]^2}\,\sqrt{2-(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}, \tag{6.82}$$

$$\mathrm{d}(\breve{B}^r_{\mathrm{add}})_k \coloneqq \frac{\mathrm{d}(\breve{B}^-_{\mathrm{add}})_k-\hat{r}_k(t)\;\mathrm{d}(\breve{B}^+_{\mathrm{add}})_k}{\sqrt{1-[\hat{r}_k(t)]^2}}, \tag{6.83}$$

which, for each $k\in\{1,...,n-1\}\setminus\{n/2\}$, defines real-valued Brownian motions $(\breve{B}^r_{\mathrm{hom}})_k$ and $(\breve{B}^r_{\mathrm{add}})_k$, since

$$\begin{aligned} &\mathrm{d}\Big\langle\int_0^{\cdot}\sqrt{2-(1-[\hat{r}_k(s)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\;\mathrm{d}(\breve{B}^-_{\mathrm{hom}})_k \\ &\qquad-\int_0^{\cdot}\hat{r}_k(s)\,\sqrt{2+(1-[\hat{r}_k(s)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\;\mathrm{d}(\breve{B}^+_{\mathrm{hom}})_k\Big\rangle(t) \\ &=\left[2-\left(1-[\hat{r}_k(t)]^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right]\mathrm{d}t+[\hat{r}_k(t)]^2\left(2+\left(1-[\hat{r}_k(t)]^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right)\mathrm{d}t \\ &\quad-2\,\hat{r}_k(t)\sqrt{2-(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\,\cdot \\ &\qquad\cdot\,\sqrt{2+(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\,\mathrm{d}\left\langle(\breve{B}^-_{\mathrm{hom}})_k,(\breve{B}^+_{\mathrm{hom}})_k\right\rangle(t) \\ &=\left[2-\left(1-[\hat{r}_k(t)]^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right]\left(1-[\hat{r}_k(t)]^2\right)\mathrm{d}t, \end{aligned}$$

where we have applied Eq. (6.80), which yields

$$\sqrt{2-(1-[\hat{r}_k]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\,\sqrt{2+(1-[\hat{r}_k]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\,\mathrm{d}\left\langle(\breve{B}^-_{\mathrm{hom}})_k,(\breve{B}^+_{\mathrm{hom}})_k\right\rangle=2\,\hat{r}_k\,\mathrm{d}t.$$

Similarly, we obtain

$$\begin{aligned} \mathrm{d}\left\langle\breve{B}^-_{\mathrm{add}}-\int_0^{\cdot}\hat{r}_k(s)\;\mathrm{d}(\breve{B}^+_{\mathrm{add}})_k\right\rangle(t) &=\left[1+[\hat{r}_k(t)]^2\right]\mathrm{d}t-2\hat{r}_k(t)\,\underbrace{\mathrm{d}\left\langle(\breve{B}^-_{\mathrm{add}})_k,(\breve{B}^+_{\mathrm{add}})_k\right\rangle(t)}_{=\hat{r}_k(t)\,\mathrm{d}t} \\ &=\left[1-[\hat{r}_k(t)]^2\right]\mathrm{d}t. \end{aligned}$$ □

In the *homogeneous* case ($\sigma_0=0$), we find that Eq. (6.76) reduces to a *one-dimensional* SDE, i.e. its right-hand side can be expressed entirely in terms of $\hat{r}_k(t)$ itself. This SDE will be analyzed in the following section.

6.3.2. Asymptotic evolution of ratio process in homogeneous case

We study the *asymptotic* evolution of $\hat{r}_k(t)$ in the case without additive noise (i.e. $\sigma_0 = 0$).

Corollary 6.10 (Evolution of amplitude ratio in homogeneous case)
For $k \in \{1, ..., n-1\} \setminus \{n/2\}$ and $\sigma_0 = 0$, the evolution of $\hat{r}_k(t)$ is given by

$$\begin{aligned} \mathrm{d}\hat{r}_k(t) = &\left(\frac{n}{2}\sqrt{|\tilde{\nu}_k|^2 + \sigma_r^2}\right)\sqrt{2 - (1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\,\sqrt{1-[\hat{r}_k(t)]^2}\,\mathrm{d}(\breve{B}^r_{\mathrm{hom}})_k \\ &-\left(\frac{n^2}{4}\left[|\tilde{\nu}_k|^2 + \sigma_r^2\right]\right)\left[2 - \left(1-[\hat{r}_k(t)]^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right]\hat{r}_k(t)\,\mathrm{d}t \\ &+\sqrt{n}\,(-i)(\mathcal{P}_-\tilde{\mathfrak{l}})_k\left(1-[\hat{r}_k(t)]^2\right)\mathrm{d}t, \end{aligned} \tag{6.84}$$

where $\breve{B}^r_{\mathrm{hom}}$, defined in Eq. (6.77), is a real-valued Brownian motion.

The *existence* of a strong solution to Eq. (6.84) follows by construction from the existence of a strong solution to Eq. (6.54) and, as in the proof of Lemma 5.27, strong *uniqueness* can be obtained from the *Yamada-Watanabe condition*. Note that $\hat{r}_k(0) = \pm 1$ yields

$$\mathrm{d}\hat{r}_k(0) = \mp 2\left(\frac{n^2}{4}\left[|\tilde{\nu}_k|^2 + \sigma_r^2\right]\right)\mathrm{d}t, \tag{6.85}$$

i.e. the process immediately enters the domain $(-1, 1)$. Moreover, we will show that the boundary points $\{\pm 1\}$ are inaccessible from within this domain, c.f. Lemma 6.15. It therefore suffices to study the evolution of $\hat{r}_k(t)$ for initial conditions restricted to the interval $(-1, 1)$.

6.3.2.1. Scale function and speed measure

In order to study the behavior of $\hat{r}_k(t)$ for some fixed $k \in \{1, ..., n-1\} \setminus \{n/2\}$, we employ the theory of [KS91], Chapter 5.5., i.p. results of the subsections "B. The Method of Removal of Drift" and "C. Feller's Test for Explosions".

Scale function Let $k \in \{1, ..., n-1\} \setminus \{n/2\}$. We calculate the *scale function* $\mathsf{s}_k(r)$ corresponding to the process $\hat{r}_k(t) \in [-1, 1]$. Note that according to Eq. (6.84), the drift term is given by

$$b_k(r) := -\left(\frac{n^2}{4}\left[|\tilde{\nu}_k|^2 + \sigma_r^2\right]\right)\left[2 - \left(1-r^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right] r + \sqrt{n}\,(-i)(\mathcal{P}_-\tilde{\mathfrak{l}})_k\left(1-r^2\right), \tag{6.86}$$

while the diffusion term is defined as

$$a_k(r) := \left(\frac{n^2}{4}\left[|\tilde{\nu}_k|^2 + \sigma_r^2\right]\right)\left[2 - \left(1-r^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right]\left(1-r^2\right). \tag{6.87}$$

We observe that the theory of [KS91], Section 5.5. is applicable.

Remark 6.11 (Nondegeneracy and integrability)
On the interval $I \coloneqq (-1,+1)$, the conditions *(ND')* and *(LI')* of [KS91], Section 5.5.C are fulfilled, since $a_k(r) > 0$ for $r \in I$ and

$$\int_{r-\epsilon}^{r+\epsilon} \frac{1+|b_k(r')|}{a_k(r')}\,\mathrm{d}r' \leq \frac{1+3\left(\frac{n^2}{4}\,[|\tilde{\nu}_k|^2+\sigma_r^2]\right)+\sqrt{n}\,\left|(-i)(\mathcal{P}_-\tilde{\mathfrak{l}})_k\right|}{\left(\frac{n^2}{4}\,[|\tilde{\nu}_k|^2+\sigma_r^2]\right)} \int_{r-\epsilon}^{r+\epsilon} \frac{1}{(1-(r')^2)}\,\mathrm{d}r'$$
$$\leq \text{const}\,\left(\tanh^{-1}(r+\epsilon)-\tanh^{-1}(r-\epsilon)\right) < \infty,$$

for all $r \in I$ and ϵ sufficiently small, s.t. $|r \pm \epsilon| < 1$.

The derivative of the scale function $\mathsf{s}_k(t)$ is therefore given by[10]

$$\begin{aligned}\frac{\mathrm{d}\mathsf{s}_k}{\mathrm{d}r}(r) &= \exp\left[\int_0^r (-2)\frac{b_k(r')}{a_k(r')}\,\mathrm{d}r'\right]\\ &= \exp\left[\int_0^r \frac{2r'}{1-(r')^2} - \frac{\sqrt{n}\,(-i)\,(\mathcal{P}_-\tilde{\mathfrak{l}})_k}{\frac{n^2}{4}\,[|\tilde{\nu}_k|^2+\sigma_r^2]}\,\frac{2}{2-(1-(r')^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\,\mathrm{d}r'\right]\\ &= \frac{1}{1-r^2}\,\exp\left[-\mathsf{f}_k\,\mathsf{t}(r,\tilde{\gamma}_k)\right],\end{aligned} \tag{6.88}$$

where f_k and $\mathsf{t}(r,\tilde{\gamma}_k)$ are given by the following definition.

Definition 6.12 (Asymmetry factor and arctan-function)
We define the 'asymmetry' factor

$$\mathsf{f}_k \coloneqq \frac{\sqrt{n}\,(-i)\,(\mathcal{P}_-\tilde{\mathfrak{l}})_k}{\frac{n^2}{4}\,[|\tilde{\nu}_k|^2+\sigma_r^2]}, \tag{6.89}$$

as well as the function

$$\begin{aligned}\mathsf{t}(r,\tilde{\gamma}_k) &\coloneqq \int_0^r \frac{2}{2-(1-(r')^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\,\mathrm{d}r'\\ &= \begin{cases} \frac{2}{\sqrt{|\mathsf{c}_k\cos(2\tilde{\gamma}_k)|\,(2-\mathsf{c}_k\cos(2\tilde{\gamma}_k))}}\tan^{-1}\left(r\sqrt{\frac{|\mathsf{c}_k\cos(2\tilde{\gamma}_k)|}{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}\right), & \text{if } \cos(2\tilde{\gamma}_k) > 0,\\ r, & \text{if } \cos(2\tilde{\gamma}_k) = 0.\\ \frac{2}{\sqrt{|\mathsf{c}_k\cos(2\tilde{\gamma}_k)|\,(2-\mathsf{c}_k\cos(2\tilde{\gamma}_k))}}\tanh^{-1}\left(r\sqrt{\frac{|\mathsf{c}_k\cos(2\tilde{\gamma}_k)|}{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}\right), & \text{if } \cos(2\tilde{\gamma}_k) < 0.\end{cases}\end{aligned} \tag{6.90}$$

Note that all constants f_k vanish in the case of symmetric space coupling, i.e. if $\mathcal{P}_-\tilde{\mathfrak{l}} = 0$, which is why this factor can be interpreted as an 'asymmetry' factor. The mapping $r \to \mathsf{t}(r,\tilde{\gamma}_k)$ is strictly

[10] c.f. [KS91], Section 5.5.C, Eq. (5.42)

increasing for all $\tilde{\gamma}_k$, which yields

$$\sup_{r\in[-1,+1]} |\mathsf{t}(r,\tilde{\gamma}_k)| \leq \max\left\{1, \frac{2\,\tanh^{-1}\left(\sqrt{\frac{|\mathsf{c}_k\cos(2\tilde{\gamma}_k)|}{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}\right)}{\sqrt{|\mathsf{c}_k\cos(2\tilde{\gamma}_k)|\,(2-\mathsf{c}_k\cos(2\tilde{\gamma}_k))}}\right\} < \infty, \tag{6.91}$$

where we made use of $\sqrt{\frac{|\mathsf{c}_k\cos(2\tilde{\gamma}_k)|}{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}} < 1$ and of $|\tan^{-1}(x)| \leq |\tanh^{-1}(x)|$, for all $x \in [-1,+1]$. The scale function $\mathsf{s}_k(r)$ can now by obtained by integrating Eq. (6.88), i.e.

$$\mathsf{s}_k(r) = \int_0^r \frac{\exp\left[-\mathsf{f}_k\,\mathsf{t}(r',\tilde{\gamma}_k)\right]}{1-(r')^2}\,\mathrm{d}r'. \tag{6.92}$$

If $\mathsf{f}_k = 0$, i.e. in the case $(\mathcal{P}_-\mathfrak{l})_k = 0$ of *symmetric* space-coupling, we find that

$$\mathsf{s}_k(r) = \tanh^{-1}(r), \tag{6.93}$$

which is independent of the chosen eigenmode k and independent of all coupling parameters.

Speed measure We calculate the *speed measure*,[11] whose density is given by

$$m_k(r) = \frac{2}{\frac{\mathrm{d}\mathsf{s}_k}{\mathrm{d}r}(r)a_k(r)} = \left(\frac{n^2}{4}\left[|\widetilde{\nu}_k|^2+\sigma_r^2\right]\right)^{-1}\frac{2\,\exp\left[\mathsf{f}_k\,\mathsf{t}(r,\tilde{\gamma}_k)\right]}{2-(1-r^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}. \tag{6.94}$$

In order to normalize the speed measure, we calculate the normalization constant $\int_{-1}^1 m_k(r')\,\mathrm{d}r'$. If $\mathsf{f}_k \neq 0$, we observe that

$$\begin{aligned}\int_{-1}^1 m_k(r')\,\mathrm{d}r' &= \left(\frac{n^2}{4}\left[|\widetilde{\nu}_k|^2+\sigma_r^2\right]\right)^{-1}(\mathsf{f}_k)^{-1}\left(\exp\left[\mathsf{f}_k\,\mathsf{t}(1,\tilde{\gamma}_k)\right]-\exp\left[\mathsf{f}_k\,\mathsf{t}(-1,\tilde{\gamma}_k)\right]\right)\\ &= 2(\sqrt{n}\,(-i)\,(\mathcal{P}_-\tilde{\mathfrak{l}})_k)^{-1}\,\sinh\left[\mathsf{f}_k\,\mathsf{t}(1,\tilde{\gamma}_k)\right],\end{aligned} \tag{6.95}$$

where in the first step we have made use of the identity $\int \mathsf{t}'(r)\exp(a\,\mathsf{t}(r))\,\mathrm{d}r = a^{-1}\exp(a\,\mathsf{t}(r))$ with $a = \mathsf{f}_k$ and in the second step we have applied the fact that $\mathsf{t}(-1,\tilde{\gamma}_k) = -\mathsf{t}(1,\tilde{\gamma}_k)$. Note that Eq. (6.95) is well-defined since $\mathsf{f}_k \neq 0$ and thus $(\mathcal{P}_-\tilde{\mathfrak{l}})_k \neq 0$.

In the symmetric case of $\mathsf{f}_k = 0$, we similarly obtain

$$\begin{aligned}\int_{-1}^1 m_k(r')\,\mathrm{d}r' &= \left(\frac{n^2}{4}\left[|\widetilde{\nu}_k|^2+\sigma_r^2\right]\right)^{-1}\int_{-1}^1 \frac{2}{2-(1-[r']^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\,\mathrm{d}r'\\ &= \left(\frac{n^2}{4}\left[|\widetilde{\nu}_k|^2+\sigma_r^2\right]\right)^{-1}\left[\mathsf{t}(1,\tilde{\gamma}_k)-\mathsf{t}(-1,\tilde{\gamma}_k)\right]\\ &= 2\left(\frac{n^2}{4}\left[|\widetilde{\nu}_k|^2+\sigma_r^2\right]\right)^{-1}\mathsf{t}(1,\tilde{\gamma}_k).\end{aligned} \tag{6.96}$$

[11] c.f. [KS91], Section 5.5.C.

The normalized density $\breve{m}_k(r) := \frac{m_k(r)}{\int_{-1}^{1} m_k(r')\,\mathrm{d}r'}$ of the speed measure is therefore given by

$$\breve{m}_k(r) = \frac{1}{2-(1-r^2)\mathsf{c}_k\cos(2\tilde{\gamma}_k))}\begin{cases}\frac{\mathsf{f}_k}{\sinh[\mathsf{f}_k\,\mathsf{t}(1,\tilde{\gamma}_k)]}\exp\left[\mathsf{f}_k\,\mathsf{t}(r,\tilde{\gamma}_k)\right], & \text{if } \mathsf{f}_k \neq 0,\\ (\mathsf{t}(1,\tilde{\gamma}_k))^{-1}, & \text{if } \mathsf{f}_k = 0.\end{cases} \tag{6.97}$$

6.3.2.2. Asymptotic behavior

We show that $\hat{r}_k(t)$ is *Harris recurrent* (Definition 6.16), which will allow us to employ a *ratio-limit theorem* (Theorem 6.18) in order to determine the long-time behavior of this process (Corollary 6.19).

Accessibility of boundary points

Lemma 6.13 (Scale function at boundary points)
For the boundary points $\{-1,+1\}$ of the interval $(-1,+1)$, the limit behavior of the scale function is given by

$$\lim_{r\searrow -1}\mathsf{s}_k(r) = -\infty, \tag{6.98}$$

$$\lim_{r\nearrow 1}\mathsf{s}_k(r) = +\infty, \tag{6.99}$$

i.e. $\mathsf{s}_k(r)$ is a so-called *space transformation on* $\mathbb{R}$.[a]

[a] c.f. [HL03], p. 29 and [Löc15], Proposition 3.1

Proof. We estimate:

$$|\mathsf{s}_k(r)| \geq \exp\left[-|\mathsf{f}_k|\sup_{r'\in[-1,+1]}\mathsf{t}(r',\tilde{\gamma}_k)\right]\underbrace{\left|\int_0^r \frac{1}{1-(r')^2}\,\mathrm{d}r'\right|}_{=\tanh^{-1}(r)}$$

$$\geq \exp\left[-|\mathsf{f}_k|\max\left\{1,\frac{2\tanh^{-1}\left(\sqrt{\frac{|\mathsf{c}_k\cos(2\tilde{\gamma}_k)|}{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}\right)}{\sqrt{|\mathsf{c}_k\cos(2\tilde{\gamma}_k)|\,(2-\mathsf{c}_k\cos(2\tilde{\gamma}_k))}}\right\}\right]|\tanh^{-1}(r)|, \tag{6.100}$$

where we have employed Eq. (6.91). Now the result follows from

$$\lim_{r\to\pm 1}\tanh^{-1}(r) = \pm\infty, \tag{6.101}$$

and the monotonicity of the scale function. □

We define the *exit time* of the process $\hat{r}_k(t)$ from the interval $(-1,1)$.[12]

[12] Recall that we have assumed that $\hat{r}_k(0) \in (-1,1)$, c.f. remark below Corollary 6.10.

Definition 6.14 (Exit time[a])
Let $l_m \searrow -1$ and $r_m \nearrow +1$ be strictly monotone sequences and define

$$S_k^{(m)} := \inf\{t \geq 0 \mid \hat{r}_k(t) \notin (l_m, r_m)\}, \quad m \in \mathbb{N}. \tag{6.102}$$

The pathwise limit of the monotone sequence $(S_k^{(m)})_m$

$$S_k := \lim_{m\to\infty} S_k^{(m)} \in [0, \infty], \tag{6.103}$$

is called *exit time* from the domain $I = (-1, +1)$.

[a] defined as in [KS91], Section 5.5., Definition 5.20 and Eq. (5.57)

Assuming that $\hat{r}_k(0) \in I$, we know that $\mathbb{P}(S_k > 0) = 1$.[13] Employing Lemma 6.13, we can strengthen this statement significantly.

Lemma 6.15 (Inaccessibility of boundary)
The boundary points $\{-1, +1\}$ of the interval $I = (-1, +1)$ are *inaccessible*, i.e.

$$\mathbb{P}(S_k = \infty) = 1, \tag{6.104}$$

provided that $\hat{r}_k(0) \in I$.

Proof. This follows from [KS91], Proposition 5.5.22, a). □

Harris recurrence and ergodic theorem Following [HL03] and [Löc15], we introduce the notion of a *Harris recurrent* process.

Definition 6.16 (Harris recurrence[a])
Let $(X(t))_{t\geq 0}$ be a right-continuous strong Markov process which takes values in a Polish space $(E, \mathcal{E})$ and has càdlàg paths, starting $\mathbb{P}_x$-a.s. in $X(0) = x \in E$. Then X is called *Harris recurrent* if there is a "σ-finite measure m on $(E, \mathcal{E})$, s.t."

$$m(A) > 0 \Rightarrow \forall x \in E: \quad \mathbb{P}_x\left(\int_0^\infty \mathbb{1}_A(X_s)\,\mathrm{d}s = \infty\right) = 1. \tag{6.105}$$

[a] c.f. [HL03], Section 1, Definition 1.1

The ratio process can be shown to be both *recurrent* and *Harris recurrent*.

Lemma 6.17 (Recurrence)
The process $\hat{r}_k(t)$ satisfies

$$\mathbb{P}\left\{\sup_{0\leq t<\infty} \hat{r}_k(t) = +1\right\} = \mathbb{P}\left\{\inf_{0\leq t<\infty} \hat{r}_k(t) = -1\right\} = 1, \tag{6.106}$$

[13] c.f. [KS91], comment in Section 5.5. below Eq. (5.20)

which implies that $\hat{r}_k(t)$ is *recurrent*, i.e. for every $r' \in (-1, +1)$ we have:

$$\mathbb{P}\left\{\hat{r}_k(t) = r' \text{ for some } 0 \leq t < \infty\right\} = 1. \tag{6.107}$$

Moreover, the process $\hat{r}_k(t)$ is also *Harris recurrent*.

Proof. Since the scale function is a space transformation on $\mathbb{R}$ (Lemma 6.13), equations (6.106) and (6.107) follow from [KS91], Proposition 5.5.22. Employing [HL03], Example 3.5 b), it follows that $\hat{r}_k(t)$ is Harris recurrent as well. □

According to [Löc15], Proposition 3.2, it follows that $\hat{r}_k(t)$ has an *invariant measure* whose density is (up to multiplication by a constant) given by the one of the speed measure. The density of the invariant *probability* measure corresponding to $\hat{r}_k(t)$, is therefore given by the density $\breve{m}(r)$ of the *normalized* speed measure.[14] It plays a key role in the following theorem, which allows us to represent long-time averages of $\hat{r}_k(t))$ in terms of integrals with respect to $\breve{m}$.

Theorem 6.18 (Ergodic theorem for the ratio process)
For any measurable, positive function $f : [-1, 1] \to \mathbb{R}$ and any starting point $r(0) \in (-1, 1)$, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(\hat{r}_k(s))\, \mathrm{d}s = \int_{-1}^1 f(r')\, \breve{m}(r')\, \mathrm{d}r', \quad \mathbb{P}^{r(0)}\text{-a.s.}, \tag{6.108}$$

i.e. the time-average along a given path converges towards the space-average with respect to the invariant probability distribution. Moreover, this result also holds in expectation:

$$\lim_{t\to\infty} \frac{1}{t}\, \mathbb{E}^{r(0)} \left(\int_0^t f(\hat{r}_k(s))\, \mathrm{d}s\right) = \int_{-1}^1 f(r')\, \breve{m}(r')\, \mathrm{d}r'. \tag{6.109}$$

Proof. These results follow as a special case of the so-called *ratio-limit theorem*, c.f. [Löc15], Theorem 1.6 and [HL03], Theorem 1.7, which both refer to [ADR69]. This theorem is applicable since $\hat{r}_k(t)$ is Harris recurrent, as was shown in Lemma 6.17. □

Note that the asymptotic time-averages do not dependent on the initial condition. Eq. (6.108) enables us to determine the long-time averages of $\hat{r}_k(t))$ and of $[\hat{r}_k(t))]^2$, which will be needed in the evaluation of $\hat{\rho}_k^+(t)$, c.f. Proposition 6.20.

Corollary 6.19 (Time averages of $\hat{r}_k(t)$ and $[\hat{r}_k(t)]^2$)
For any $r(0) \in (-1, 1)$, the processes $(\frac{1}{t}\int_0^t \hat{r}_k(s)\, \mathrm{d}s)_{t\geq 0}$ and $(\frac{1}{t}\int_0^t [\hat{r}_k(s)]^2\, \mathrm{d}s)_{t\geq 0}$ convergence $\mathbb{P}^{r(0)}$-a.s., i.e. the pathwise limits

$$\psi_k^{(1)} := \lim_{t\to\infty} \frac{1}{t} \int_0^t \hat{r}_k(s)\, \mathrm{d}s, \tag{6.110a}$$

$$\psi_k^{(2)} := \lim_{t\to\infty} \frac{1}{t} \int_0^t [\hat{r}_k(s)]^2\, \mathrm{d}s, \tag{6.110b}$$

[14] c.f. also [HL03], Eq. (3.8')

$\mathbb{P}^{r(0)}$-a.s. exist. In the symmetric case of $\mathsf{f}_k = 0$, these integrals yield

$$\psi_k^{(1)} = 0,$$

$$\psi_k^{(2)} = \begin{cases} \left[\sqrt{\frac{\mathsf{c}_k\cos(2\tilde{\gamma}_k)}{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}\tan^{-1}\left(\sqrt{\frac{\mathsf{c}_k\cos(2\tilde{\gamma}_k)}{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}\right)\right]^{-1} - \frac{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}{\mathsf{c}_k\cos(2\tilde{\gamma}_k)}, & \text{if } \cos(2\tilde{\gamma}_k) > 0, \\ \frac{1}{3}, & \text{if } \cos(2\tilde{\gamma}_k) = 0, \\ -\left[\sqrt{\frac{|\mathsf{c}_k\cos(2\tilde{\gamma}_k)|}{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}\tanh^{-1}\left(\sqrt{\frac{|\mathsf{c}_k\cos(2\tilde{\gamma}_k)|}{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}\right)\right]^{-1} + \frac{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}{|\mathsf{c}_k\cos(2\tilde{\gamma}_k)|}, & \text{if } \cos(2\tilde{\gamma}_k) < 0, \end{cases}$$

$$= \begin{cases} \frac{2}{\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\left(\frac{1}{\mathsf{t}(1,\tilde{\gamma}_k)} - 1\right) + 1, & \text{if } \cos(2\tilde{\gamma}_k) \neq 0, \\ \frac{1}{3}, & \text{if } \cos(2\tilde{\gamma}_k) = 0. \end{cases}$$

Proof. The existence of the limits is a direct consequence of Theorem 6.18. We calculate the speed measure integrals, assuming that $\mathsf{f}_k = 0$. In this case, the speed measure has a symmetric density, i.e. $\breve{m}(r') = \breve{m}(-r')$, which is why $\psi_k^{(1)}$ vanishes:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \hat{r}_k(s)\,\mathrm{d}s = \int_{-1}^{1} r'\,\breve{m}(r')\,\mathrm{d}r' = 0.$$

In the calculation of $\psi_k^{(2)}$ we distinguish three cases. For $\cos(2\tilde{\gamma}_k) > 0$, we $\mathbb{P}^{r(0)}$-a.s. obtain

$$\begin{aligned} \lim_{t\to\infty}\frac{1}{t}\int_0^t [\hat{r}_k(s)]^2\,\mathrm{d}s &= \int_{-1}^{1}(r')^2\,\breve{m}(r')\,\mathrm{d}r' \\ &= \frac{2}{\mathsf{t}(1,\tilde{\gamma}_k)}\left(\frac{1-\sqrt{\frac{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}{\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}\tan^{-1}\left(\sqrt{\frac{\mathsf{c}_k\cos(2\tilde{\gamma}_k)}{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}\right)}{\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\right) \\ &= \sqrt{\frac{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}{\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}\left(\frac{1}{\tan^{-1}\left(\sqrt{\frac{\mathsf{c}_k\cos(2\tilde{\gamma}_k)}{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}\right)} - \sqrt{\frac{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}{\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}\right) \\ &= \left[\sqrt{\frac{\mathsf{c}_k\cos(2\tilde{\gamma}_k)}{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}\tan^{-1}\left(\sqrt{\frac{\mathsf{c}_k\cos(2\tilde{\gamma}_k)}{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}}\right)\right]^{-1} - \frac{2-\mathsf{c}_k\cos(2\tilde{\gamma}_k)}{\mathsf{c}_k\cos(2\tilde{\gamma}_k)}, \end{aligned}$$

where we have employed Eq. (6.97). For $\cos(2\tilde{\gamma}_k) = 0$ and $\cos(2\tilde{\gamma}_k) < 0$, similar arguments allow us to obtain the stated pathwise limits. □

Note that in the symmetric case of $\mathsf{f}_k = 0$, the value $\psi_k^{(2)}$ has a continuous, π-periodic dependence on the noise-coupling angle $\tilde{\gamma}_k$. In Remark 6.29, we will discuss the coupling-angle dependence of $\psi_k^{(2)}$ in greater detail. More precisely, we will study the term $1 - \frac{1}{2}(1-\psi_k^{(2)})\mathsf{c}_k\cos(2\tilde{\gamma}_k)$, which is visualized in Fig. 6.2.

6.4. Evolution of homogeneous system

We study the asymptotic evolution of the eigenmode amplitudes in the *homogeneous* case of $\sigma_0 = 0$, i.e. in the absence of additive noise. The influence of the *inhomogeneous* additive-noise term will be discussed in the following section.

6.4.1. Asymptotic evolution of eigenmode amplitudes

Setting $\sigma_0 = 0$ in Eq. (6.34), we find that for $k \in \{0, n/2\}$, the evolution of $\hat{\rho}_k(t)$ is given by

$$\mathrm{d}\hat{\rho}_k(t) = \hat{\rho}_k(t) \left(n \sqrt{\frac{|\widetilde{\nu}_k|^2 + 2\,(\mathrm{Re}(\widetilde{\nu}_k))^2}{2} + \sigma_r^2}\; \mathrm{d}(\breve{B}_{\mathrm{hom}})_k + \frac{n^2}{2}\,\left[|\widetilde{\nu}_k|^2 + \sigma_r^2\right] \mathrm{d}t + \sqrt{n}\,\mathrm{Re}(\widetilde{\lambda}_k)\,\mathrm{d}t \right). \tag{6.111}$$

Recall that $\mathrm{Re}(\widetilde{\lambda}) = \mathrm{Re}(\widetilde{\mathfrak{l}}') + \mathrm{Im}(\widetilde{\mathfrak{l}}) = \mathcal{P}_+\,\widetilde{\mathfrak{l}}' - i\,\mathcal{P}_-\,\widetilde{\mathfrak{l}}$, by Eq. (6.8). Since for $k \in \{0, n/2\}$, we have $k = n - k$, it follows that $(\mathcal{P}_+\,\widetilde{\mathfrak{l}}')_k = \widetilde{\mathfrak{l}}'_k$ and $(\mathcal{P}_-\,\widetilde{\mathfrak{l}})_k = 0$, which allows us to simplify the SDE to

$$\mathrm{d}\hat{\rho}_k(t) = \hat{\rho}_k(t) \left(n \sqrt{\frac{|\widetilde{\nu}_k|^2 + 2\,(\mathrm{Re}(\widetilde{\nu}_k))^2}{2} + \sigma_r^2}\; \mathrm{d}(\breve{B}_{\mathrm{hom}})_k + \frac{n^2}{2}\,\left[|\widetilde{\nu}_k|^2 + \sigma_r^2\right] \mathrm{d}t + \sqrt{n}\,\widetilde{\mathfrak{l}}'_k\,\mathrm{d}t \right). \tag{6.112}$$

For $k \in \{1, ..., n-1\} \setminus \{n/2\}$ on the other hand we recall Eq. (6.54), which in the homogeneous case of $\sigma_0 = 0$ reduces to

$$\begin{aligned}
\mathrm{d}\hat{\rho}^+_k(t) = {} & \left(\frac{n}{2}\sqrt{|\widetilde{\nu}_k|^2 + \sigma_r^2}\right) \hat{\rho}^+_k(t)\,\sqrt{2 + (1 - [\hat{r}_k(t)]^2)\,\mathsf{c}_k \cos(2\tilde{\gamma}_k)}\;\mathrm{d}(\breve{B}^+_{\mathrm{hom}})_k \\
& + \left(\frac{n^2}{4}\,\left[|\widetilde{\nu}_k|^2 + \sigma_r^2\right]\right) 2\,\hat{\rho}^+_k(t)\,\mathrm{d}t \\
& + \sqrt{n}\,\hat{\rho}^+_k(t)\left[(\mathcal{P}_+\widetilde{\mathfrak{l}}')_k + (-i)\,(\mathcal{P}_-\widetilde{\mathfrak{l}})_k\,\hat{r}_k(t)\right] \mathrm{d}t, && (6.113) \\
\mathrm{d}\hat{\rho}^-_k(t) = {} & \left(\frac{n}{2}\sqrt{|\widetilde{\nu}_k|^2 + \sigma_r^2}\right) \hat{\rho}^+_k(t)\,\sqrt{2 - (1 - [\hat{r}_k(t)]^2)\,\mathsf{c}_k \cos(2\tilde{\gamma}_k)}\;\mathrm{d}(\breve{B}^-_{\mathrm{hom}})_k \\
& + \sqrt{n}\,\hat{\rho}^+_k(t)\left[-i\,(\mathcal{P}_-\widetilde{\mathfrak{l}})_k + (\mathcal{P}_+\widetilde{\mathfrak{l}}')_k\,\hat{r}_k(t)\right] \mathrm{d}t. && (6.114)
\end{aligned}$$

Itô's formula enables us to solve these SDEs. In the case of $k \in \{0, n/2\}$, we obtain an explicit solution, while for $k \in \{1, ..., n-1\} \setminus \{n/2\}$, the solution is given in terms of the k'th ratio process $\hat{r}_k(t)$.

Proposition 6.20 (Exponential solution)
For $k \in \{0, n/2\}$, the solution to Eq. (6.112) is given by

$$\hat{\rho}_k(t) = \exp\left(\breve{M}^+_k(t)\right) \exp\left(\left[\frac{n^2}{4}\,|\widetilde{\nu}_k|^2\,(-\cos(2\tilde{\gamma}_k)) + \sqrt{n}\,\widetilde{\mathfrak{l}}'_k\right] t\right) \hat{\rho}_k(0), \tag{6.115}$$

where the martingale $\breve{M}_k^+(t)$ is a rescaled Brownian motion defined by

$$\breve{M}_k^+(t) := n\sqrt{\frac{|\widetilde{\nu}_k|^2 + 2\,(\mathrm{Re}(\widetilde{\nu}_k))^2}{2} + \sigma_r^2}\,(\breve{B}_{\mathrm{hom}})_k(t). \tag{6.116}$$

For $k \in \{1, ..., n-1\} \setminus \{n/2\}$, the solution to Eq. (6.113) can be written as

$$\hat{\rho}_k^+(t) = \exp\left(\breve{M}_k^+(t)\right)\,\exp\left(\frac{n^2}{4}\,\left[|\widetilde{\nu}_k|^2 + \sigma_r^2\right]\left[1 - \frac{1}{2}\left(1 - \frac{1}{t}\int_0^t [\hat{r}_k(s)]^2\,\mathrm{d}s\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right]t\right)$$
$$\cdot\,\exp\left(\sqrt{n}\,\left[(\mathcal{P}_+\tilde{\mathfrak{l}}')_k + (-i)\,(\mathcal{P}_-\tilde{\mathfrak{l}})_k\,\frac{1}{t}\int_0^t \hat{r}_k(s)\,\mathrm{d}s\right]t\right)\hat{\rho}_k^+(0), \tag{6.117}$$

where $\breve{M}_k^+(t)$ is a martingale, given in terms of a stochastic integral as

$$\breve{M}_k^+(t) := \left(\frac{n}{2}\sqrt{|\widetilde{\nu}_k|^2 + \sigma_r^2}\right)\int_0^t \sqrt{2 + (1 - [\hat{r}_k(s)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\,\mathrm{d}(\breve{B}_{\mathrm{hom}}^+)_k. \tag{6.118}$$

Proof. Given $k \in \{0, n/2\}$, we solve Eq. (6.112) by applying Itô's formula to the logarithm, i.e.

$$\begin{aligned}
\mathrm{d}\ln(\hat{\rho}_k(t)) &= \frac{\mathrm{d}\hat{\rho}_k(t)}{\hat{\rho}_k(t)} + \frac{1}{2}\left(-\frac{1}{[\hat{\rho}_k(t)]^2}\,\mathrm{d}\left\langle\hat{\rho}_k\right\rangle(t)\right)\\
&= n\sqrt{\frac{|\widetilde{\nu}_k|^2 + 2\,(\mathrm{Re}(\widetilde{\nu}_k))^2}{2} + \sigma_r^2}\;\mathrm{d}(\breve{B}_{\mathrm{hom}})_k\\
&\quad + \left(\frac{n^2}{2}\left[|\widetilde{\nu}_k|^2 + \sigma_r^2\right] - \frac{n^2}{2}\left[\frac{|\widetilde{\nu}_k|^2 + 2\,\mathrm{Re}(\widetilde{\nu}_k)^2}{2} + \sigma_r^2\right]\right)\mathrm{d}t + \sqrt{n}\,\tilde{\mathfrak{l}}_k'\,\mathrm{d}t\\
&= n\sqrt{\frac{|\widetilde{\nu}_k|^2 + 2\,(\mathrm{Re}(\widetilde{\nu}_k))^2}{2} + \sigma_r^2}\;\mathrm{d}(\breve{B}_{\mathrm{hom}})_k\\
&\quad + \left(\frac{n^2}{4}\right)\left[|\widetilde{\nu}_k|^2 - 2\,\mathrm{Re}(\widetilde{\nu}_k)^2\right]\mathrm{d}t + \sqrt{n}\,\tilde{\mathfrak{l}}_k'\,\mathrm{d}t,
\end{aligned}$$

where $|\widetilde{\nu}_k|^2 - 2\,\mathrm{Re}(\widetilde{\nu}_k)^2 = \mathrm{Im}(\widetilde{\nu}_k)^2 - \mathrm{Re}(\widetilde{\nu}_k)^2 = |\widetilde{\nu}_k|^2\,\left(-\cos(2\tilde{\gamma}_k)\right)$. Integration and exponentiation now yields Eq. (6.115). Similarly, for $k \in \{1, ..., n-1\} \setminus \{n/2\}$, we solve Eq. (6.113), at least in terms of the process $\hat{r}_k(t)$. By Itô's formula we find that

$$\begin{aligned}
\mathrm{d}\ln(\hat{\rho}_k^+(t)) &= \frac{\mathrm{d}\hat{\rho}_k^+(t)}{\hat{\rho}_k^+(t)} + \frac{1}{2}\left(-\frac{1}{[\hat{\rho}_k^+(t)]^2}\,\mathrm{d}\left\langle\hat{\rho}_k^+\right\rangle(t)\right)\\
&= \left(\frac{n}{2}\sqrt{|\widetilde{\nu}_k|^2 + \sigma_r^2}\right)\sqrt{2 + (1 - [\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\;\mathrm{d}(\breve{B}_{\mathrm{hom}}^+)_k\\
&\quad + \left(\frac{n^2}{4}\,\left[|\widetilde{\nu}_k|^2 + \sigma_r^2\right]\right)\left[2 - \frac{1}{2}\left(2 + \left(1 - [\hat{r}_k(t)]^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right)\right]\mathrm{d}t\\
&\quad + \sqrt{n}\,\left[(\mathcal{P}_+\tilde{\mathfrak{l}}')_k + (-i)\,(\mathcal{P}_-\tilde{\mathfrak{l}})_k\,\hat{r}_k(t)\right]\mathrm{d}t,
\end{aligned}$$

i.e.

$$\begin{aligned} \mathrm{d}\ln(\hat{\rho}_k^+(t)) = & \left(\frac{n}{2}\sqrt{|\widetilde{\nu}_k|^2+\sigma_r^2}\right)\sqrt{2+(1-[\hat{r}_k(t)]^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\;\mathrm{d}(\breve{B}^+_{\mathrm{hom}})_k \\ & + \left(\frac{n^2}{4}\left[|\widetilde{\nu}_k|^2+\sigma_r^2\right]\right)\left[1-\frac{1}{2}\left(1-[\hat{r}_k(t)]^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right]\mathrm{d}t \\ & + \sqrt{n}\,\left[(\mathcal{P}_+\tilde{\mathfrak{l}}')_k + (-i)\,(\mathcal{P}_-\tilde{\mathfrak{l}})_k\,\hat{r}_k(t)\right]\mathrm{d}t, \end{aligned}$$

which by integration and exponentiation yields Eq. (6.117). □

6.4.1.1. Martingale evolution

An estimate on the *running maximum* of Brownian motion allows us to prove that the martingales $\breve{M}_k^+(t)$ of the previous proposition do not contribute to the *asymptotic growth rate*[15] of $\hat{\rho}_k^+(t)$.

Lemma 6.21 (Martingale evolution)
For any $k \in \{0, ..., n-1\}$, we have $\lim_{t\to\infty} \frac{\breve{M}_k^+(t)}{t} = 0$, $\mathbb{P}$-a.s.

Proof. First, we examine the case of $k \in \{1, ..., n-1\} \setminus \{n/2\}$. Employing a *time-change theorem for martingales*,[16] we find that there is a Brownian motion $\mathring{B}_k^+(t)$, s.t.

$$\breve{M}_k^+(t) = \mathring{B}_k^+\left(\left\langle \breve{M}^+\right\rangle(t)\right) = \mathring{B}_k^+\left(\frac{n^2}{4}\left[|\widetilde{\nu}_k|^2+\sigma_r^2\right]\int_0^t \left[2+\left(1-[\hat{r}_k(s)]^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right]\mathrm{d}s\right).$$

The asymptotic growth of the so-called *running maximum* $\sup_{s\le t}\left|\mathring{B}_k^+(s)\right|$ of the Brownian motion $(\mathring{B}_k^+(t))_{t\ge 0}$ is given by[17]

$$\limsup_{t\to\infty}\frac{\sup_{s\le t}\left|\mathring{B}_k^+(s)\right|}{\sqrt{2t\ln\ln t}} = \frac{\pi}{8}, \quad \mathbb{P}\text{-a.s.} \tag{6.119}$$

Moreover, we observe that $\left\langle \breve{M}_k^+\right\rangle(t) \le \frac{3n^2}{4}\left[|\widetilde{\nu}_k|^2+\sigma_r^2\right] t$, which together with Eq. (6.119) implies that

$$\begin{aligned} \limsup_{t\to\infty}\frac{\left|\breve{M}_k^+(t)\right|}{t} &= \limsup_{t\to\infty}\frac{\left|\mathring{B}_k^+\left(\left\langle \breve{M}^+\right\rangle(t)\right)\right|}{t} \\ &\le \limsup_{t\to\infty}\frac{\sup_{0\le s\le \frac{3n^2}{4}\left[|\widetilde{\nu}_k|^2+\sigma_r^2\right]t}\left|\mathring{B}_k^+(s)\right|}{t} \\ &\le \frac{3n^2}{4}\left[|\widetilde{\nu}_k|^2+\sigma_r^2\right]\limsup_{t\to\infty}\frac{\sup_{0\le s\le t}\left|\mathring{B}_k^+(s)\right|}{t} = 0. \end{aligned}$$

[15] c.f. Lemma 6.23 below
[16] c.f. [KS91], Section 3.4., Theorem 4.6
[17] [SP12], Theorem 11.7

In the simpler case of $k \in \{0, n/2\}$, we have $\breve{M}_k^+(t) = n\sqrt{\frac{|\widetilde{\nu}_k|^2 + 2\left(\mathrm{Re}(\widetilde{\nu}_k)\right)^2}{2} + \sigma_r^2}\,(\breve{B}_{\mathrm{hom}})_k(t)$ and a similar argument as before yields the desired result. □

6.4.1.2. Asymptotic growth

Definition 6.22 (Asymptotic equivalence and order)
Two real-valued, strictly positive stochastic processes $(X(t))_{t\geq 0}, (Y(t))_{t\geq 0}$ will be called *asymptotically equivalent*, denoted as

$$(X(t))_{t\geq 0} \sim (Y(t))_{t\geq 0}, \tag{6.120}$$

or as a shorthand $X(t) \sim Y(t)$, if

$$\ln(X(t)) - \ln(Y(t)) = o(t), \quad \mathbb{P}\text{-a.s.} \tag{6.121}$$

i.e., if for all $\varepsilon > 0$,

$$\lim_{t\to\infty}\left(\frac{X(t)}{Y(t)}\right)e^{-\varepsilon t} = \lim_{t\to\infty}\left(\frac{Y(t)}{X(t)}\right)e^{-\varepsilon t} = 0, \quad \mathbb{P}\text{-a.s.} \tag{6.122}$$

This allows us to define an *asymptotic order*. Two strictly positive stochastic processes $(X(t))_{t\geq 0}, (Y(t))_{t\geq 0}$ are called *asymptotically ordered*, i.e. $(X(t))_{t\geq 0}$ is *asymptotically bounded* by $(Y(t))_{t\geq 0}$, denoted as

$$(X(t))_{t\geq 0} \lesssim (Y(t))_{t\geq 0}, \tag{6.123}$$

or as a shorthand $X(t) \lesssim Y(t)$, if

$$\left(\frac{Y(t)}{X(t)}\right)_{t\geq 0} \sim (Z(t))_{t\geq 0}, \tag{6.124}$$

for some strictly positive stochastic process $(Z(t))_{t\geq 0}$, satisfying $\liminf_{t\to\infty} Z(t) \geq 1$.

Note that, as suggested by the name, asymptotic equivalence indeed defines an equivalence relation. We now provide a sufficient condition for a stochastic process to exhibit an asymptotic exponential growth.

Lemma 6.23 (Asymptotic exponential growth)
Let $(X(t))_{t\geq 0}$ be given by $X(t) = \exp(Z(t))$, where $(Z(t))_{t\geq 0}$ is a real-valued stochastic process satisfying $\lim_{t\to\infty} \frac{Z(t)}{t} = c$, $\mathbb{P}$-a.s., for some $c \in \mathbb{R}$. Then

$$(X(t))_{t\geq 0} \sim (\mathrm{e}^{ct})_{t\geq 0} \tag{6.125}$$

and c will be called the *asymptotic growth rate* of $(X(t))_{t\geq 0}$.

Proof. For all $\varepsilon > 0$, we observe that

$$\lim_{t\to\infty}\left(\frac{X(t)}{\mathrm{e}^{c\,t}}\right)e^{-\varepsilon t} = \lim_{t\to\infty}\exp\left(\left[\frac{Z(t)}{t} - c - \varepsilon\right]t\right) = 0, \quad \mathbb{P}\text{-a.s.},$$
$$\lim_{t\to\infty}\left(\frac{\mathrm{e}^{c\,t}}{X(t)}\right)e^{-\varepsilon t} = \lim_{t\to\infty}\exp\left(\left[-\frac{Z(t)}{t} + c - \varepsilon\right]t\right) = 0, \quad \mathbb{P}\text{-a.s.}$$

The asymptotic equivalence now follows from Definition 6.22. □

The processes $\hat{\rho}_k^+(t)$ can be shown to exhibit such an asymptotic exponential growth and we can identify their growth rates.

Theorem 6.24 (Asymptotic growth of $\hat{\rho}_k^+(t)$)
The asymptotic growth of $\hat{\rho}_k^+(t)$ is given by

$$\left(\hat{\rho}_k^+(t)\right)_{t\geq 0} \sim \left(\exp\left(\xi_k\,t\right)\right)_{t\geq 0}, \tag{6.126}$$

where the *growth rate* ξ_k is for $k \in \{0, n/2\}$ given by

$$\xi_k := \frac{n^2}{4}\left|\tilde{\nu}_k\right|^2\left(-\cos(2\tilde{\gamma}_k)\right) + \sqrt{n}\,\tilde{\mathfrak{l}}'_k, \tag{6.127}$$

and in the case of $k \in \{1, ..., n-1\} \setminus \{n/2\}$ by

$$\begin{aligned}\xi_k := &\frac{n^2}{4}\left[|\tilde{\nu}_k|^2 + \sigma_r^2\right]\left[1 - \frac{1}{2}\left(1 - \psi_k^{(2)}\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right]\\ &+ \sqrt{n}\left[(\mathcal{P}_+\tilde{\mathfrak{l}}')_k + (-i)\,(\mathcal{P}_-\tilde{\mathfrak{l}})_k\,\psi_k^{(1)}\right].\end{aligned} \tag{6.128}$$

Proof. The evolution of $(\hat{\rho}_k^+(t))_{t\geq 0}$ is given by Proposition 6.20 (recall that $\hat{\rho}_k^+(t) = \hat{\rho}_k(t)$ in the case of $k \in \{0, n/2\}$). In order to determine its asymptotic behavior, we first note that by Lemma 6.21 and Lemma 6.23 we can conclude that

$$\left(\exp\left(\breve{M}_k^+(t)\right)\right)_{t\geq 0} \sim 1, \qquad \text{for all } k \in \{0, ..., n-1\}. \tag{6.129}$$

Similarly, Corollary 6.19 in combination with Lemma 6.23 yields

$$\left(\exp\left(\int_0^t \hat{r}_k(s)\,\mathrm{d}s\right)\right)_{t\geq 0} \sim \left(\exp\left(\psi_k^{(1)}\,t\right)\right)_{t\geq 0}, \quad \forall k \in \{0, ..., n-1\}, \tag{6.130}$$
$$\left(\exp\left(\int_0^t [\hat{r}_k(s)]^2\,\mathrm{d}s\right)\right)_{t\geq 0} \sim \left(\exp\left(\psi_k^{(2)}\,t\right)\right)_{t\geq 0}, \quad \forall\,k \in \{0, ..., n-1\}, \tag{6.131}$$

and the result follows. □

Remark 6.25 (Vector of growth rates is even)
Note that the vector of growth rates is *even*, i.e. $\xi_k = \xi_{n-k}$, for all $k \in \{0, ..., n-1\}$. This is in agreement with $\hat{\rho}^+_k(t) = \hat{\rho}^+_{n-k}(t)$.

If there is a relative order of the growth rates, e.g. $\xi_k < \xi_l$ for some $k, l \in \{0, ..., \lfloor \frac{n}{2} \rfloor\}$, we can conclude that $\hat{\rho}^+_k(t)$ can asymptotically be neglected compared to $\hat{\rho}^+_l(t)$.

Corollary 6.26 (Eigenmode comparison)
Let $k, l \in \{0, ..., \lfloor \frac{n}{2} \rfloor\}$, s.t. $\xi_k < \xi_l$. Then it follows that

$$\lim_{t\to\infty} \frac{\hat{\rho}^+_k(t)}{\hat{\rho}^+_l(t)} = 0, \quad \mathbb{P}\text{-a.s.,} \tag{6.132}$$

i.e. eigenmode pairs with a larger growth rate asymptotically dominate.
If there is a unique largest growth rate, i.e. some $l \in \{0, ..., \lfloor \frac{n}{2} \rfloor\}$, s.t. $\xi_k < \xi_l$ for all $k \in \{0, ..., \lfloor \frac{n}{2} \rfloor\}$ with $k \neq l$, then its corresponding eigenmode pair dominates all other eigenmode pairs, i.e.

$$\lim_{t\to\infty} \frac{\sum_{k=0,\, k\notin\{l,n-l\}}^{n-1} \hat{\rho}_k(t)}{\hat{\rho}^+_l(t)} = \lim_{t\to\infty} \frac{\sum_{k=0,\, k\neq l}^{\lfloor n/2 \rfloor} \hat{\rho}^+_k(t)}{\hat{\rho}^+_l(t)} = 0, \quad \mathbb{P}\text{-a.s.} \tag{6.133}$$

If $\mathfrak{I} \subset \{0, ..., \lfloor \frac{n}{2} \rfloor\}$ denotes the index set of all largest growth rates, i.e.

$$\xi_l = \xi_{l'} > \xi_k, \quad \forall l, l' \in \mathfrak{I},\ \forall k \in \left\{0, ..., \left\lfloor \frac{n}{2} \right\rfloor\right\} \setminus \mathfrak{I}, \tag{6.134}$$

then we correspondingly find that for all $l \in \mathfrak{I}$,

$$\lim_{t\to\infty} \frac{\sum_{k\in\{0,...,\lfloor \frac{n}{2} \rfloor\}\setminus\mathfrak{I}} \hat{\rho}^+_k(t)}{\hat{\rho}^+_l(t)} = 0, \quad \mathbb{P}\text{-a.s.} \tag{6.135}$$

Proof. Assuming $\xi_k < \xi_l$, Theorem 6.24 yields

$$\lim_{t\to\infty} \frac{\hat{\rho}^+_k(t)}{\hat{\rho}^+_l(t)} = \lim_{t\to\infty} \exp\left(\ln\left(\hat{\rho}^+_k(t)\right) - \ln\left(\hat{\rho}^+_l(t)\right)\right) = \lim_{t\to\infty} \exp\left((\xi_k - \xi_l)\, t + o(t)\right) = 0,\ \mathbb{P}\text{-a.s.,}$$

and the other results follow similarly. □

Note that $\hat{\rho}^+_l(t)$ *exponentially* dominates $\hat{\rho}^+_k(t)$ under the assumption of $\xi_k < \xi_l$, i.e. Eq. (6.132) for instance can be strengthened to[18]

$$\limsup_{t\to\infty} \frac{1}{t} \log \left| \frac{\hat{\rho}^+_k(t)}{\hat{\rho}^+_l(t)} \right| < 0, \quad \mathbb{P}\text{-a.s.} \tag{6.136}$$

[18] c.f. [RS16], Definition 2

Employing the representations of $\psi_k^{(1)}$ and $\psi_k^{(2)}$ as given by Corollary 6.19, we can obtain explicit expressions for the growth rates ξ_k.

Lemma 6.27 (Asymptotic growth factors in symmetric case)
In the symmetric case of $\mathsf{f}_k = 0$, i.e. in the case of symmetric deterministic space-coupling $(\mathcal{P}_- \tilde{\mathfrak{l}})_k = 0$, we obtain a growth rate, given for $k \in \{0, n/2\}$ by

$$\xi_k := \frac{n^2}{4} |\tilde{\nu}_k|^2 (-\cos(2\tilde{\gamma}_k)) + \sqrt{n}\, \tilde{\mathfrak{l}}'_k, \tag{6.137}$$

while for $k \in \{1, ..., n-1\} \setminus \{n/2\}$ we have

$$\xi_k := \begin{cases} \frac{n^2}{4} \left[|\tilde{\nu}_k|^2 + \sigma_r^2\right] \dfrac{\mathsf{c}_k \cos(2\tilde{\gamma}_k) \sqrt{\frac{2-\mathsf{c}_k \cos(2\tilde{\gamma}_k)}{\mathsf{c}_k \cos(2\tilde{\gamma}_k)}}}{2 \tan^{-1}\left(\sqrt{\frac{\mathsf{c}_k \cos(2\tilde{\gamma}_k)}{2-\mathsf{c}_k \cos(2\tilde{\gamma}_k)}}\right)} + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k, & \cos(2\tilde{\gamma}_k) > 0, \\ \frac{n^2}{4} \left[|\tilde{\nu}_k|^2 + \sigma_r^2\right] + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k, & \cos(2\tilde{\gamma}_k) = 0, \\ \frac{n^2}{4} \left[|\tilde{\nu}_k|^2 + \sigma_r^2\right] \dfrac{|\mathsf{c}_k \cos(2\tilde{\gamma}_k)| \sqrt{\frac{2-\mathsf{c}_k \cos(2\tilde{\gamma}_k)}{|\mathsf{c}_k \cos(2\tilde{\gamma}_k)|}}}{2 \tanh^{-1}\left(\sqrt{\frac{|\mathsf{c}_k \cos(2\tilde{\gamma}_k)|}{2-\mathsf{c}_k \cos(2\tilde{\gamma}_k)}}\right)} + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k, & \cos(2\tilde{\gamma}_k) < 0, \end{cases} \tag{6.138}$$

where $\mathcal{P}_+ \tilde{\mathfrak{l}}' = \mathcal{P}_+ \mathrm{Re}(\tilde{\lambda}) = \frac{\mathrm{Re}(\tilde{\lambda}_k) + \mathrm{Re}(\tilde{\lambda}_{n-k})}{2}$.

In the particular case of *momentum-like* noise coupling, i.e. $\tilde{\nu} = \tilde{\mathfrak{n}}'$, we find

$$\xi_k = \begin{cases} -\frac{n^2}{4} |\tilde{\mathfrak{n}}'_k|^2 + \sqrt{n}\, \tilde{\mathfrak{l}}'_k, & k \in \{0, n/2\}, \\ \frac{n^2}{4} \left[|\tilde{\mathfrak{n}}'_k|^2 + \sigma_r^2\right] \dfrac{\mathsf{c}_k \sqrt{\frac{2-\mathsf{c}_k}{\mathsf{c}_k}}}{2 \tan^{-1}\left(\sqrt{\frac{\mathsf{c}_k}{2-\mathsf{c}_k}}\right)} + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k, & k \in \{1, ..., n-1\} \setminus \{n/2\}, \end{cases} \tag{6.139}$$

while for *space-like* noise coupling, i.e. $\tilde{\nu} = -i\, \tilde{\mathfrak{n}}$, we obtain

$$\xi_k = \begin{cases} \frac{n^2}{4} |\tilde{\mathfrak{n}}_k|^2 + \sqrt{n}\, \tilde{\mathfrak{l}}'_k, & k \in \{0, n/2\}, \\ \frac{n^2}{4} \left[|\tilde{\mathfrak{n}}_k|^2 + \sigma_r^2\right] \dfrac{\mathsf{c}_k \sqrt{\frac{2+\mathsf{c}_k}{\mathsf{c}_k}}}{2 \tanh^{-1}\left(\sqrt{\frac{\mathsf{c}_k}{2+\mathsf{c}_k}}\right)} + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k, & k \in \{1, ..., n-1\} \setminus \{n/2\}. \end{cases} \tag{6.140}$$

Proof. For $k \in \{0, n/2\}$, the results follow directly from Eq. (6.127). In the following, we therefore only need to examine the case of $k \in \{1, ..., n-1\} \setminus \{n/2\}$. In Corollary 6.19 we have seen that under the assumption of $(\mathcal{P}_- \tilde{\mathfrak{l}})_k = 0$, we have explicit representations for $\psi_k^{(1)}$ and $\psi_k^{(2)}$, which we now insert into the growth-rate formulas given in Theorem 6.24. Introducing the shorthand

$$\mathsf{s}_k := \sqrt{\frac{|\mathsf{c}_k \cos(2\tilde{\gamma}_k)|}{2 - \mathsf{c}_k \cos(2\tilde{\gamma}_k)}}, \tag{6.141}$$

we can represent the $[\hat{r}_k(t)]^2$ time-average by

$$\psi_k^{(2)} = \begin{cases} \frac{1}{\mathsf{s}_k \tan^{-1}(\mathsf{s}_k)} - \frac{2-\mathsf{c}_k \cos(2\tilde{\gamma}_k)}{\mathsf{c}_k \cos(2\tilde{\gamma}_k)}, & \cos(2\tilde{\gamma}_k) > 0, \\ -\frac{1}{2\,\mathsf{s}_k \tanh^{-1}(\mathsf{s}_k)} - \frac{2-\mathsf{c}_k \cos(2\tilde{\gamma}_k)}{\mathsf{c}_k \cos(2\tilde{\gamma}_k)}, & \cos(2\tilde{\gamma}_k) < 0, \end{cases} \tag{6.142}$$

where we have omitted the $\cos(2\tilde{\gamma}_k) = 0$ case, in which $\psi_k^{(2)}$ does not contribute to ξ_k. We employ Theorem 6.24 in order to conclude that

$$\begin{aligned} \xi_k &= \frac{n^2}{4} \left[|\tilde{\nu}_k|^2 + \sigma_r^2\right] \left(1 - \frac{1}{2}\left(1 - \psi_k^{(2)}\right) \mathsf{c}_k \cos(2\tilde{\gamma}_k)\right) + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k \\ &= \begin{cases} \frac{n^2}{4} \left[|\tilde{\nu}_k|^2 + \sigma_r^2\right] \frac{\mathsf{c}_k \cos(2\tilde{\gamma}_k)}{2\,\mathsf{s}_k \tan^{-1}(\mathsf{s}_k)} + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k, & \cos(2\tilde{\gamma}_k) > 0, \\ \frac{n^2}{4} \left[|\tilde{\nu}_k|^2 + \sigma_r^2\right] + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k, & \cos(2\tilde{\gamma}_k) = 0, \\ \frac{n^2}{4} \left[|\tilde{\nu}_k|^2 + \sigma_r^2\right] \frac{\mathsf{c}_k |\cos(2\tilde{\gamma}_k)|}{2\,\mathsf{s}_k \tanh^{-1}(\mathsf{s}_k)} + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k, & \cos(2\tilde{\gamma}_k) < 0. \end{cases} \end{aligned}$$

Note that in the *momentum-like noise coupling* case of $\tilde{\nu}_k = \tilde{\mathfrak{n}}'_k$, the noise-coupling angle is given by $\tilde{\gamma}_k \in \{0, \pi\}$. This follows from the assumption of ν being even, which by Lemma 2.30 implies that $\mathrm{Re}(\widetilde{\mathfrak{n}'}) = \tilde{\mathfrak{n}}'$. Thus we have $\cos(2\tilde{\gamma}_k) = 1 > 0$, as well as $\mathsf{s}_k = \sqrt{\frac{\mathsf{c}_k}{2-\mathsf{c}_k}}$ and find that

$$\xi_k = \frac{n^2}{4} \left[|\tilde{\mathfrak{n}}'_k|^2 + \sigma_r^2\right] \frac{\mathsf{c}_k \sqrt{\frac{2-\mathsf{c}_k}{\mathsf{c}_k}}}{2 \tan^{-1}\left(\sqrt{\frac{\mathsf{c}_k}{2-\mathsf{c}_k}}\right)} + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k. \tag{6.143}$$

In the *space-like noise-coupling* case of $\tilde{\nu}_k = -i\, \tilde{\mathfrak{n}}_k$, Lemma 2.30 implies that $\tilde{\nu}_k \in i\mathbb{R}$, i.e. $|\tilde{\gamma}_k| = \frac{\pi}{2}$. Therefore we have $\cos(2\tilde{\gamma}_k) = -1 < 0$ and $\mathsf{s}_k = \sqrt{\frac{\mathsf{c}_k}{2+\mathsf{c}_k}}$, which yields

$$\xi_k = \frac{n^2}{4} \left[|\tilde{\mathfrak{n}}_k|^2 + \sigma_r^2\right] \frac{\mathsf{c}_k \sqrt{\frac{2+\mathsf{c}_k}{\mathsf{c}_k}}}{2 \tanh^{-1}\left(\sqrt{\frac{\mathsf{c}_k}{2+\mathsf{c}_k}}\right)} + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k. \tag{6.144}$$

□

Note that for $k \in \{0, n/2\}$, the growth rates do *not* depend on the regularizing noise. For $k \in \{1, ..., n-1\} \setminus \{n/2\}$, they have a continuous dependence on the regularizing-noise intensity σ_r, c.f. Eq. (6.52). Since we will choose σ_r as a small parameter, it will prove useful to determine the $\sigma_r \to 0$ limit of the growth rates.

Lemma 6.28 (Asymptotic growth factors in symmetric case, $\sigma_r \to 0$ limit)
In the symmetric case of $\mathsf{f}_k = 0$, the $\sigma_r \to 0$ limit of the growth rates ξ_k is for $k \in \{0, n/2\}$ given by

$$\xi_k := \frac{n^2}{4} |\widetilde{\nu}_k|^2 \left(-\cos(2\tilde{\gamma}_k)\right) + \sqrt{n}\, \tilde{\mathfrak{l}}'_k, \tag{6.145}$$

while for $k \in \{1, ..., n-1\} \setminus \{n/2\}$ we obtain

$$\xi_k := \begin{cases} \frac{n^2}{4} |\widetilde{\nu}_k|^2 \frac{\cos(2\tilde{\gamma}_k) \sqrt{\frac{2-\cos(2\tilde{\gamma}_k)}{\cos(2\tilde{\gamma}_k)}}}{2 \tan^{-1}\left(\sqrt{\frac{\cos(2\tilde{\gamma}_k)}{2-\cos(2\tilde{\gamma}_k)}}\right)} + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k, & \cos(2\tilde{\gamma}_k) > 0, \\ \frac{n^2}{4} |\widetilde{\nu}_k|^2 + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k, & \cos(2\tilde{\gamma}_k) = 0, \\ \frac{n^2}{4} |\widetilde{\nu}_k|^2 \frac{|\cos(2\tilde{\gamma}_k)| \sqrt{\frac{2-\cos(2\tilde{\gamma}_k)}{|\cos(2\tilde{\gamma}_k)|}}}{2 \tanh^{-1}\left(\sqrt{\frac{|\cos(2\tilde{\gamma}_k)|}{2-\cos(2\tilde{\gamma}_k)}}\right)} + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k, & \cos(2\tilde{\gamma}_k) < 0. \end{cases} \tag{6.146}$$

If the case of *momentum-like* noise coupling, i.e. $\widetilde{\nu} = \widetilde{\mathfrak{n}'}$, we find

$$\xi_k = \begin{cases} -\frac{n^2}{4} |\tilde{\mathfrak{n}}'_k|^2 + \sqrt{n}\, \tilde{\mathfrak{l}}'_k, & k \in \{0, n/2\}, \\ \frac{n^2}{4} |\tilde{\mathfrak{n}}'_k|^2 \frac{2}{\pi} + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k, & k \in \{1, ..., n-1\} \setminus \{n/2\}. \end{cases} \tag{6.147}$$

For *space-like* noise coupling, i.e. $\widetilde{\nu} = -i\, \tilde{\mathfrak{n}}$, we obtain

$$\xi_k = \begin{cases} \frac{n^2}{4} |\tilde{\mathfrak{n}}_k|^2 + \sqrt{n}\, \tilde{\mathfrak{l}}'_k, & k \in \{0, n/2\}, \\ \frac{n^2}{4} |\tilde{\mathfrak{n}}_k|^2 \frac{\sqrt{3}}{2 \tanh^{-1}\left(\frac{1}{\sqrt{3}}\right)} + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k, & k \in \{1, ..., n-1\} \setminus \{n/2\}. \end{cases} \tag{6.148}$$

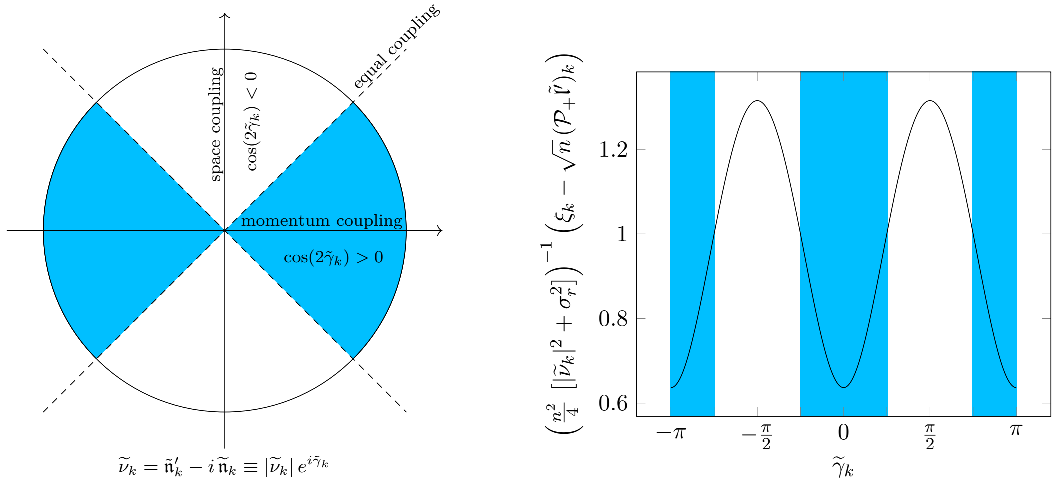
Proof. The results directly follow from Lemma 6.27 and the observation that $\mathsf{c}_k \to 1$ in the limit of $\sigma_r \to 0$, c.f. Eq. (6.52). Moreover, we have applied the identity $\tan^{-1}(1) = \frac{\pi}{4}$ which implies that

$$\frac{1}{2 \tan^{-1}(1)} = \frac{2}{\pi}. \tag{6.149}$$

□

We discuss the influence of the noise-coupling angle, i.e. the relation between space-coupling and momentum-coupling, on the growth rate.

Remark 6.29 (Growth rate dependence on noise-coupling angle)
Note that for $k \in \{0, n/2\}$, the sign of the multiplicative-noise contribution to the growth rate (Eq. (6.145)) depends on the factor of $(-\cos(2\tilde{\gamma}_k))$. As visualized in Fig. 6.2a, we consequently obtain a negative contribution for *momentum-dominated*[a] coupling (blue seg-



(a) Noise-coupling angle $\tilde{\gamma}_k$

(b) Multiplicative-noise contribution to ξ_k, for $k \in \{1, ..., n-1\} \setminus \{n/2\}$

Figure 6.2.: Influence of noise-coupling angle on growth rate

ments) and a positive contribution for *space-dominated* coupling (white segments). For $k \in \{1, ..., n-1\} \setminus \{n/2\}$, we obtain a more involved $\tilde{\gamma}_k$ dependence (c.f. Eq. (6.146)), which is plotted in Fig. 6.2b. As in the $\{0, n/2\}$ case, we can observe that the growth rate is largest if space coupling dominates (white segments) and smallest for dominating momentum coupling (blue segments). Unlike in the $\{0, n/2\}$ case, we now always get a *positive* noise contribution to the growth rate.

[a] Recall that the notion of *momentum-dominated* coupling was introduced in terms of the coupling angles γ_k, c.f. Definition 3.26 and Fig. 3.5. In the case of distance-l coupling however we have seen that the coupling angles $\tilde{\gamma}_k$ of Definition 3.29 can be related to γ_l, c.f. Lemma 3.30.

In Section 6.4.3 we will study the relation between distance-l coupling topologies and their induced largest growth rates.

6.4.2. Pathwise synchronization of energy-normalized effective system

In order to define a suitable notion of synchronization, we examine an *energy-normalized* version of the vector $\hat{\rho}(t)$.

Definition 6.30 (Energy normalization)
For each $l \in \{0, ..., n-1\}$, we define

$$\mathsf{e}_l(t) := \frac{\hat{\rho}_l(t)}{E(t)} = \frac{\hat{\rho}_l(t)}{\sum_{k=0}^{n-1} \hat{\rho}_k(t)} = \frac{\hat{\rho}_l(t)}{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \hat{\rho}_k^+(t)} \in [0, 1], \tag{6.150}$$

i.e. $\mathsf{e}(t) \in \mathcal{S}$, where $\mathcal{S} := \left\{x \in \mathbb{R}^n \mid \sum_{k=0}^{n-1} x_k = 1\right\}$ denotes the standard $(n-1)$-simplex. Analogously to Definition 6.6, we define

$$\mathsf{e}_l^+(t) := \begin{cases} \mathsf{e}_l(t), & \text{if } l \in \{0, n/2\}\,, \\ \mathsf{e}_l(t) + \mathsf{e}_{n-l}(t), & \text{if } l \in \{1, ..., n-1\} \setminus \{n/2\}\,. \end{cases} \tag{6.151}$$

We define the notion of *eigenmode synchronization* in terms of the normalized vector $\mathsf{e}(t)$.

Definition 6.31 (Eigenmode synchronization)
The effective system is said to *pathwise synchronize towards the l'th eigenmode* (for some $l \in \{0, ..., n-1\}$), if

$$\lim_{t\to\infty} \mathsf{e}(t) = e_l, \quad \mathbb{P}\text{-a.s.}, \tag{6.152}$$

where $e_l \in \mathbb{R}^n$ denotes the l'th unit vector. Synchronization towards the 0'th eigenmode will be referred to as *in-phase synchronization*, while synchronization towards the $(n/2)$'th eigenmode (possible only if n is even) will be called *anti-phase synchronization*.

More generally, let $C \subset \mathcal{S}$ be a *convex* subset. The effective system is said to *(pathwise) synchronize towards C*, if

$$\lim_{t\to\infty} \operatorname{dist}(\mathsf{e}(t), C) = 0, \quad \mathbb{P}\text{-a.s.} \tag{6.153}$$

The convex sets relevant to our case are the ones spanned by *pairs* of eigenmodes.

Definition 6.32 (Convex subsets spanned by eigenmode pairs)
For all $l \in \{0, ..., \lfloor \frac{n}{2} \rfloor\}$, we define C_l as the convex subset of $\mathcal{S}$, spanned by the l'th unit-vector pair e_l, e_{n-l}, i.e.

$$C_l := \{\lambda e_l + (1-\lambda) e_{n-l} \mid \lambda \in [0,1]\}\,. \tag{6.154}$$

For an index set $\mathfrak{I} \subset \{0, ..., \lfloor \frac{n}{2} \rfloor\}$, we similarly define $C_{\mathfrak{I}}$ as the convex subset of $\mathcal{S}$, spanned by the unit-vector pairs indexed in $\mathfrak{I}$, i.e.

$$C_{\mathfrak{I}} := \left\{ \sum_{l\in\mathfrak{I}} \lambda_l e_l + \lambda_{n-l} e_{n-l} \mid \sum_{l\in\mathfrak{I}} \lambda_l + \lambda_{n-l} \in [0,1] \right\}. \tag{6.155}$$

Furthermore let $\mathfrak{I}_0 := \{0, n/2\} \cap \mathbb{N}$, i.e.

$$C_{\mathfrak{I}_0} := \begin{cases} C_0, & \text{if } n \text{ odd}, \\ C_{\{0,n/2\}}, & \text{if } n \text{ even}. \end{cases} \tag{6.156}$$

The eigenmode-comparison result of Corollary 6.26 can now be interpreted as a synchronization result for the effective system.

Theorem 6.33 (Pathwise synchronization of effective system)
Assume that there is a *unique* largest growth rate, i.e. an index $l \in \{0, ..., \lfloor \frac{n}{2} \rfloor\}$, s.t. $\xi_k < \xi_l$, for all $k \in \{0, ..., n-1\}$ with $k \neq l$. Then the effective system pathwise synchronizes towards the convex subset C_l, i.e.

$$\lim_{t\to\infty} \operatorname{dist}(\mathsf{e}(t), C_l) = 0, \quad \mathbb{P}\text{-a.s.} \tag{6.157}$$

If the largest growth rate is given by ξ_0, the effective system pathwise synchronizes towards the 0'th eigenmode (*in-phase synchronization*).

If n is even and $\xi_{n/2}$ constitutes the largest growth rate, then the effective system pathwise synchronizes towards the $(n/2)$'th eigenmode (*anti-phase synchronization*).

In a more general setting where there might be multiple largest growth rates, we denote by $\mathfrak{I} \subset \{0, ..., \lfloor \frac{n}{2} \rfloor\}$ the index set of all largest growth rates, i.e.

$$\xi_l = \xi_{l'} > \xi_k, \quad \forall l, l' \in \mathfrak{I},\ \forall k \in \left\{0, ..., \left\lfloor \frac{n}{2} \right\rfloor\right\} \setminus \mathfrak{I}. \tag{6.158}$$

Then the effective system pathwise synchronizes towards the convex subset $C_{\mathfrak{I}}$, i.e.

$$\lim_{t\to\infty} \operatorname{dist}(\mathsf{e}(t), C_{\mathfrak{I}}) = 0, \quad \mathbb{P}\text{-a.s.} \tag{6.159}$$

Proof. Note that according to Corollary 6.26 we have

$$\lim_{t\to\infty} \frac{\sum_{k=0}^{n-1} \hat{\rho}_k(t)}{\hat{\rho}_l^+(t)} = 1 + \lim_{t\to\infty} \frac{\sum_{k=0, k\notin\{l, n-l\}}^{n-1} \hat{\rho}_k(t)}{\hat{\rho}_l^+(t)} = 1, \quad \mathbb{P}\text{-a.s.,} \tag{6.160}$$

which by inversion yields

$$\lim_{t\to\infty} \mathsf{e}_l^+(t) = \lim_{t\to\infty} \frac{\hat{\rho}_l^+(t)}{\sum_{k=0}^{n-1} \hat{\rho}_k(t)} = 1, \quad \mathbb{P}\text{-a.s.} \tag{6.161}$$

For all $k \neq l$, we can thus conclude that

$$\lim_{t\to\infty} \mathsf{e}_k^+(t) = 0, \quad \mathbb{P}\text{-a.s.,} \tag{6.162}$$

and Eq. (6.157) follows.

In the general setting of $\mathfrak{I}$ denoting the index set of all largest growth rates, Corollary 6.26 similarly yields

$$\lim_{t\to\infty} \frac{\sum_{k=0}^{n-1} \hat{\rho}_k(t)}{\sum_{l\in\mathfrak{I}} \hat{\rho}_l^+(t)} = 1 + \lim_{t\to\infty} \frac{\sum_{k\in\{0,...,\lfloor \frac{n}{2} \rfloor\}\setminus\mathfrak{I}} \hat{\rho}_k(t)}{\sum_{l\in\mathfrak{I}} \hat{\rho}_l^+(t)} = 1, \quad \mathbb{P}\text{-a.s.} \tag{6.163}$$

and thus equivalently (for the inverse process)

$$\lim_{t\to\infty} \sum_{l\in\mathfrak{I}} \mathsf{e}_l^+(t) = \lim_{t\to\infty} \frac{\sum_{l\in\mathfrak{I}} \hat{\rho}_l^+(t)}{\sum_{k=0}^{n-1} \hat{\rho}_k(t)} = 1, \quad \mathbb{P}\text{-a.s.} \tag{6.164}$$

□

The results of Theorem 6.33 can be strengthened by restating them in terms of a stronger notion of synchronization.

Remark 6.34 (Stochastic synchronization)
Note that by Eq. (6.136), synchronization is achieved 'exponentially fast'. Theorem 6.33 thus remains valid w.r.t. a stricter notion of synchronization, i.e. if we replace Eq. (6.153) by

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(\operatorname{dist}\left(\mathsf{e}(t), C\right)\right) < 0, \quad \mathbb{P}\text{-a.s.}, \tag{6.165}$$

which yields a definition of synchronization similar to the notion of "stochastic synchronization", as specified in [RS16], Definition 2.

6.4.3. Coupling topologies and induced synchronization modes

In the previous section we have seen that the asymptotic synchronization behavior depends on the order of the growth rates. In this section we will examine the relationship between coupling topologies and the growth rates they give rise to. In particular, we will relate *distance-l* noise-coupling to its resulting synchronization modes, c.f. Theorem 6.45. In this context we will show that there are *critical numbers of oscillators* from which this relationship changes.

We assume that we are in the *symmetric* case of $(\mathcal{P}_-\,\tilde{\mathfrak{l}})_k = 0$, which allows us employ the explicit growth-rate representations provided by Lemma 6.27. Moreover, we note that $(\mathcal{P}_-\,\tilde{\mathfrak{l}})_k = 0$ implies a 'decoupling' of the contributions from deterministic- and noise-parts. This is because the mixed term $(-i)\,(\mathcal{P}_-\tilde{\mathfrak{l}})_k\,\frac{1}{t}\int_0^t \hat{r}_k(s)\,\mathrm{d}s$ in Eq. (6.117) vanishes. The effects of the deterministic coupling have already been studied (c.f. Remark 6.4), so we can now focus on the *noise-coupling* effects.

In the following, the number $l \in \{0, ..., n-1\}$ represents the *distance-l* coupling topology and will be thought of as fixed. The index $k \in \{0, ..., n-1\}$, by comparison, will be used to distinguish the different growth rates ξ_k.

Lemma 6.35 (Asymptotic growth factors for distance-l coupling)
Let $l \in \{0, ..., n-1\}$. In the case of *momentum-like* distance-l noise coupling, the growth rates are given by

$$\xi_k = \begin{cases} -n + \sqrt{n}\,\tilde{\mathfrak{l}}'_k, & k \in \{0, n/2\}\,, \\ \left[n\,\cos^2\left(\frac{2\pi\,k\,l}{n}\right) + \frac{n^2}{4}\,\sigma_r^2\right] \dfrac{\mathsf{c}_k\sqrt{\frac{2-\mathsf{c}_k}{\mathsf{c}_k}}}{2\,\tan^{-1}\left(\sqrt{\frac{\mathsf{c}_k}{2-\mathsf{c}_k}}\right)} + \sqrt{n}\,(\mathcal{P}_+\tilde{\mathfrak{l}}')_k, & k \in \{1, ..., n-1\} \setminus \{n/2\}\,, \end{cases} \tag{6.166}$$

which in the limit of $\sigma_r \to 0$ simplifies to

$$\xi_k = \begin{cases} -n + \sqrt{n}\,\tilde{\mathfrak{l}}'_k, & k \in \{0, n/2\}\,, \\ n\,\cos^2\left(\frac{2\pi\,k\,l}{n}\right)\frac{2}{\pi} + \sqrt{n}\,(\mathcal{P}_+\tilde{\mathfrak{l}}')_k, & k \in \{1, ..., n-1\} \setminus \{n/2\}\,. \end{cases} \tag{6.167}$$

For *space-like* distance-l noise coupling we find that

$$\xi_k = \begin{cases} n + \sqrt{n}\,\tilde{\mathfrak{l}}'_k, & k \in \{0, n/2\}\,, \\ \left[n\,\cos^2\left(\frac{2\pi\,k\,l}{n}\right) + \frac{n^2}{4}\,\sigma_r^2\right] \frac{\mathsf{c}_k\sqrt{\frac{2+\mathsf{c}_k}{\mathsf{c}_k}}}{2\,\tanh^{-1}\left(\sqrt{\frac{\mathsf{c}_k}{2+\mathsf{c}_k}}\right)} + \sqrt{n}\,(\mathcal{P}_+\tilde{\mathfrak{l}}')_k, & k \in \{1, ..., n-1\} \setminus \{n/2\}\,, \end{cases} \tag{6.168}$$

which for $\sigma_r \to 0$ reduces to

$$\xi_k = \begin{cases} n + \sqrt{n}\,\tilde{\mathfrak{l}}'_k, & k \in \{0, n/2\}\,, \\ n\,\cos^2\left(\frac{2\pi\,k\,l}{n}\right) \frac{\sqrt{3}}{2\,\tanh^{-1}\left(\frac{1}{\sqrt{3}}\right)} + \sqrt{n}\,(\mathcal{P}_+\tilde{\mathfrak{l}}')_k, & k \in \{1, ..., n-1\} \setminus \{n/2\}\,. \end{cases} \tag{6.169}$$

Proof. We only discuss the case of *momentum-like* coupling; the case of *space-like* coupling case follows analogously. Let $\mathfrak{n}'$ be given as a distance-l coupling vector, i.e. (recall Example 3.12)

$$\mathfrak{n}'_k := \delta_{k,l} + \delta_{k,(n-l)}, \quad \forall k \in \{0, ..., n-1\}\,. \tag{6.170}$$

Here we have set the noise-coupling scale to 'one', i.e. we did not write $c\,(\delta_{k,l} + \delta_{k,(n-l)})$ for some coupling constant $c > 0$. This choice can be made without loss of generality, since in regard to the synchronization results, such a constant would only be relevant in comparison to the deterministic coupling. We can thus implicitly account for it by changing $(\mathcal{P}_+\tilde{\mathfrak{l}}')$. By Eq. (3.100) we know that

$$\tilde{\mathfrak{n}}'_k = \frac{2}{\sqrt{n}}\cos\left(\frac{2\pi\,k\,l}{n}\right). \tag{6.171}$$

Now the result follows directly from Lemma 6.27 and the fact that for $k \in \{0, n/2\}$ we have $\cos^2\left(\frac{2\pi\,k\,l}{n}\right) = 1$. For the limiting cases recall that $\mathsf{c}_k \to 1$ as $\sigma_r \to 0$ and refer to Lemma 6.28.□

Remark 6.36 (Choice of regularization parameter)
As already noted (c.f. remark below Lemma 6.27), the growth rates continuously depend on σ_r, which is a regularization parameter that can be chosen very small. Any strict ordering of growth rates in the case of $\sigma_r = 0$ will therefore persist in the $\sigma_r > 0$ case, provided σ_r is chosen small enough. In the following, we will compare the $\sigma_r = 0$ growth rates and implicitly assume that σ_r is chosen suitably small, so as not to interfere with any strict order. The regularizing noise *can* however make a difference in the case of *degeneracies*. If for instance $\sigma_r = 0$ yields $\xi_0 = \xi_k$ for some $k \in \{1, ..., n-1\} \setminus \{n/2\}$, then we find that $\xi_0 < \xi_k$ for any choice of $\sigma_r > 0$. This is due to the fact that the regularizing noise only affects the 'pair growth-rates', i.e. ξ_k for $k \in \{1, ..., n-1\} \setminus \{n/2\}$.

In a first step we look at a noise coupling of *two* oscillators.

Example 6.37 (Synchronization for $n = 2$)
For $n = 2$ we are exclusively in the case of $k \in \{0, n/2\} = \{0, 1\}$ and all vectors are automatically even, since $R = Q^2 = \mathbb{1}_{2\times 2}$, which implies that $\mathcal{P}_- = 0$. According to Lemma 6.27, the $\sigma_r \to 0$ limit of the growth rates is thus given by

$$\xi_k := |\widetilde{\nu}_k|^2 \left(-\cos(2\tilde{\gamma}_k)\right) + \sqrt{2}\,\tilde{\mathfrak{l}}'_k, \quad k \in \{0, 1\}\,. \tag{6.172}$$

Note that

$$\widetilde{\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \nu_0 \\ \nu_1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \nu_0 + \nu_1 \\ \nu_0 - \nu_1 \end{pmatrix}, \quad \tilde{\mathfrak{l}}' = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathfrak{l}'_0 + \mathfrak{l}'_1 \\ \mathfrak{l}'_0 - \mathfrak{l}'_1 \end{pmatrix}. \tag{6.173}$$

For distance-0 noise coupling, i.e. $\nu = (\nu_0, 0)^\top$, we have

$$\cos(2\tilde{\gamma}_0) = \cos(2\tilde{\gamma}_1) = \cos(2\gamma_0), \tag{6.174}$$

c.f. Lemma 3.30, and conclude that

$$\widetilde{\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} \nu_0 \\ \nu_0 \end{pmatrix} \quad \Rightarrow \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\,|\nu_0|^2 \left(-\cos(2\gamma_0)\right) + \sqrt{2}\,\tilde{\mathfrak{l}}'_0 \\ \frac{1}{2}\,|\nu_0|^2 \left(-\cos(2\gamma_0)\right) + \sqrt{2}\,\tilde{\mathfrak{l}}'_1 \end{pmatrix}, \tag{6.175}$$

while for distance-1 noise coupling, i.e. $\nu = (0, \nu_1)^\top$, we have

$$\cos(2\tilde{\gamma}_0) = \cos(2\tilde{\gamma}_1) = \cos(2\gamma_1), \tag{6.176}$$

and obtain

$$\widetilde{\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} \nu_1 \\ -\nu_1 \end{pmatrix} \quad \Rightarrow \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\,|\nu_1|^2 \left(-\cos(2\gamma_1)\right) + \sqrt{2}\,\tilde{\mathfrak{l}}'_0 \\ \frac{1}{2}\,|\nu_1|^2 \left(-\cos(2(\gamma_1 + \pi))\right) + \sqrt{2}\,\tilde{\mathfrak{l}}'_1 \end{pmatrix}. \tag{6.177}$$

We can therefore conclude that in both coupling cases, the noise does not distinguish between the eigenmodes, i.e. it gives rise to the same contribution to the growth rates ξ_0 and ξ_1. In these cases the synchronization pattern is therefore determined by the deterministic coupling, c.f. Remark 6.4.

This changes, if we look at superpositions of self- and distance-1 noise coupling. In the case of $\nu_0 = \nu_1$, for instance, we obtain

$$\widetilde{\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2\nu_0 \\ 0 \end{pmatrix} \quad \Rightarrow \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\,|\nu_0|^2 \left(-\cos(2\gamma_0)\right) + \sqrt{2}\,\tilde{\mathfrak{l}}'_0 \\ 0 + \sqrt{2}\,\tilde{\mathfrak{l}}'_1 \end{pmatrix}, \tag{6.178}$$

i.e. for momentum-dominated noise coupling we get a negative noise contribution to ξ_0, while a space-dominated noise coupling yields a positive one. If we now furthermore assume that we have a deterministic self-coupling, i.e. $\mathfrak{l}'_1 = 0$ and thus $\tilde{\mathfrak{l}}'_0 = \tilde{\mathfrak{l}}'_1$, we find that *momentum-dominated* noise coupling (c.f. Fig. 6.2) yields anti-phase synchronization, while *space-dominated* noise coupling gives rise to in-phase synchronization.

In the following, we can assume that $n > 2$. We compare the growth rates given by Eqs. (6.167) and (6.169) and determine the *maximal* ones.

Remark 6.38 (Momentum-like noise coupling)
In the *momentum*-like noise coupling case of Eq. (6.167), we obtain a negative noise contribution of $(-n)$ to the growth rates of $k \in \{0, n/2\}$, while all other growth rates obtain a positive noise contribution given by $n\cos^2\left(\frac{2\pi\,k\,l}{n}\right)\frac{2}{\pi}$. This is why for momentum-like noise coupling, the eigenmode pair $0, n/2$ never dominates, i.e. we are left with finding all indices $k \in \{1, ..., n-1\} \setminus \{n/2\}$ which maximize the factor $\cos^2\left(\frac{2\pi\,k\,l}{n}\right)$, c.f. Definition 6.41 and Proposition 6.43 below.

For *space*-like noise coupling, however, all growth rates obtain a positive noise contribution and the growth rates of the eigenmodes $0, \frac{n}{2}$ *can* be dominant. This is the case if and only if (c.f. Eq. (6.169))

$$\cos^2\left(\frac{2\pi\,k\,l}{n}\right)\frac{\sqrt{3}}{2\,\tanh^{-1}\left(\frac{1}{\sqrt{3}}\right)} < 1, \quad \forall\, k \in \{1, ..., n-1\} \setminus \{n/2\}\,. \tag{6.179}$$

This condition crucially depends on n and we will observe in Lemma 6.44 below that there are critical numbers of oscillators, which determine whether or not Eq. (6.179) can be fulfilled. We start with a monotonicity observation.

Lemma 6.39 (Monotonicity of $\cos^2(\frac{2\pi}{n})$)
The sequence $(\cos^2\left(\frac{2\pi}{n}\right))_{n\geq 4}$ is strictly increasing in n.

Proof. Note that $n \geq 4$ implies $\frac{2\pi}{n} \in [0, \frac{\pi}{2}]$. On this interval, the function

$$x \to \cos^2(x) = \frac{1+\cos(2x)}{2} \tag{6.180}$$

is strictly decreasing, which is why composition with the descreasing sequence $\frac{2\pi}{n}$ yields a strictly increasing sequence $(\cos^2\left(\frac{2\pi}{n}\right))_{n\geq 4}$. □

In the next step we examine the maximization of a cosine factor of the form $\cos^2(\frac{2\pi}{n}\,m)$.

Lemma 6.40 (Maximization of $\cos^2(\frac{2\pi}{n}\,m)$ factor)
Let $n > 2$. The set

$$\mathfrak{I}^{(n)} := \left\{m^* \in \{1, ..., n-1\} \setminus \{n/2\} \mid \cos^2\left(\frac{2\pi}{n}\,m^*\right) = \max_{m\in\{1,...,n-1\}\setminus\{n/2\}} \cos^2\left(\frac{2\pi}{n}\,m\right)\right\}$$

of all indices $m^* \in \{1, ..., n-1\} \setminus \{n/2\}$ which maximize the factor $\cos^2\left(\frac{2\pi\,m}{n}\right)$, is given by

$$\mathfrak{I}^{(n)} = \begin{cases} \{\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil\}, & \text{if } n \text{ odd,} \\ \{1, \frac{n}{2}-1, \frac{n}{2}+1, n-1\}, & \text{if } n \text{ even.} \end{cases} \tag{6.181}$$

At these indices we attain a maximal value of

$$\cos^2\left(\frac{2\pi}{n}\,m^*\right) = \begin{cases} \cos^2\left(\frac{\pi}{n}\right), & \text{if } n \text{ odd,} \\ \cos^2\left(\frac{2\pi}{n}\right), & \text{if } n \text{ even.} \end{cases} \tag{6.182}$$

Proof. We split the index set $\{1, ..., n-1\} \setminus \{n/2\}$ into four subsets, maximize $\cos^2(\frac{2\pi}{n}\,m)$ over each of these subsets and finally compare the obtained 'local' maximal values in order to obtain the global maximum.

We note that, similar to the previous proof, $1 \le m \le \frac{n}{4}$ implies $\frac{2\pi\,m}{n} \in \left[0, \frac{\pi}{2}\right]$. On this interval, $x \to \cos^2 x$ is strictly descreasing, which is why $m = 1$ maximizes the factor $\cos^2(\frac{2\pi}{n}\,m)$ in this regime.

For $\frac{n}{4} < m < \frac{n}{2}$, we similarly observe that $\frac{2\pi\,m}{n} \in \left[\frac{\pi}{2}, \pi\right]$. On this interval, $x \to \cos^2 x$ is strictly increasing and

$$m = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor, & \text{if } n \text{ odd,} \\ \frac{n}{2} - 1, & \text{if } n \text{ even,} \end{cases} \tag{6.183}$$

thus maximizes the cosine factor on the index subset under consideration.

By symmetry considerations, we find that for $\frac{n}{2} < m \le \frac{3n}{4}$, the cosine factor is maximized at

$$m = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & \text{if } n \text{ odd,} \\ \frac{n}{2} + 1, & \text{if } n \text{ even,} \end{cases} \tag{6.184}$$

while for $\frac{3n}{4} < m < n$, we obtain a maximal value at $m = n - 1$.

For n *even*, all 'local' maxima yield the same value of

$$\cos^2\left(\frac{2\pi}{n}\right) = \cos^2\left(\frac{2\pi}{n}\left(\frac{n}{2} \pm 1\right)\right) = \cos^2\left(\frac{2\pi}{n}(n-1)\right), \tag{6.185}$$

while for n *odd*, we note that $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$ and $\left\lceil \frac{n}{2} \right\rceil = \frac{n+1}{2}$ yield

$$\cos^2\left(\frac{2\pi}{n}\left\lfloor \frac{n}{2} \right\rfloor\right) = \cos^2\left(\frac{2\pi}{n}\left\lceil \frac{n}{2} \right\rceil\right) = \cos^2\left(\frac{\pi\,(n \pm 1)}{n}\right) = \cos^2\left(\frac{\pi}{n}\right). \tag{6.186}$$

By Lemma 6.39, we know that

$$\cos^2\left(\frac{\pi}{n}\right) = \cos^2\left(\frac{2\pi}{2n}\right) > \cos^2\left(\frac{2\pi}{n}\right), \quad \forall n \ge 4, \tag{6.187}$$

which implies that

$$\cos^2\left(\frac{2\pi}{n}\left\lfloor \frac{n}{2} \right\rfloor\right) = \cos^2\left(\frac{2\pi}{n}\left\lceil \frac{n}{2} \right\rceil\right) > \cos^2\left(\frac{2\pi}{n}\right) = \cos^2\left(\frac{2\pi}{n}(n-1)\right), \quad \forall n \ge 4. \tag{6.188}$$

For $n = 3$, we remark that $\left\lfloor \frac{n}{2} \right\rfloor = 1$ and $\left\lceil \frac{n}{2} \right\rceil = 2 = n-1$, i.e. $\{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil\} = \{1, \left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil, n-1\}$.□

Note that in Lemma 6.40 we have substituted the product '$k\,l$' appearing in Eq. (6.179), by a number $m \in \{1, ..., n-1\} \setminus \{n/2\}$, with respect to which we have maximized the cosine-squared factor. Now we return to the original factor of $\cos^2\left(\frac{2\pi\,k\,l}{n}\right)$ and determine the indices $k \in \{0, ..., n-1\}$ which maximize the term for given $l \in \{0, ..., n-1\}$.

Definition 6.41 (Maximization of $\cos^2(\frac{2\pi\,k\,l}{n})$ factor)
For $l \in \{1, ..., n-1\}$ we denote by $\mathfrak{I}^{(l,n)}$ the set of all indices $k^* \in \{1, ..., n-1\} \setminus \{n/2\}$, which maximize the factor $\cos^2\left(\frac{2\pi\,k\,l}{n}\right)$, i.e.

$$\mathfrak{I}^{(l,n)} := \left\{ k^* \in \{1, ..., n-1\} \setminus \{n/2\} \mid \cos^2\left(\frac{2\pi\,k^*\,l}{n}\right) = \max_{k \in \{1,...,n-1\} \setminus \{n/2\}} \cos^2\left(\frac{2\pi\,k\,l}{n}\right) \right\}. \tag{6.189}$$

We can explicitly determine the set $\mathfrak{I}^{(l,n)}$ by employing the *Euler-Fermat* theorem from number theory.

Theorem 6.42 (Euler-Fermat)
If l and n are *coprime*, it follows that

$$l^{\phi(n)} \equiv 1 \bmod n, \tag{6.190}$$

where $\phi(n)$ denotes *Euler's totient function*, which is defined as

$$\phi(n) := |\{k \in \{1, ..., n-1\} \mid k, n \text{ coprime}\}|. \tag{6.191}$$

This theorem allows us to determine the maximizing indices k^*. As it turns out, we have to distinguish several cases, depending on the value of the greatest common divisor $\gcd(l, n)$ of the coupling length l and the number of oscillators n.

Proposition 6.43 (Identification of $\mathfrak{I}^{(l,n)}$)
Let $n > 2$ and $l \in \{1, ..., n-1\}$. If l and n are *coprime*, or if $l' := \frac{l}{2}$, $n' := \frac{n}{2}$ are coprime integers, where $\frac{n}{2}$ is *odd*, we find that

$$\mathfrak{I}^{(l,n)} = \begin{cases} \left\{(k^* \lfloor \frac{n}{2} \rfloor \bmod n), (k^* \lceil \frac{n}{2} \rceil \bmod n)\right\}, & \text{if } n \text{ odd}, \\ \left\{k^*, n-k^*, \frac{n}{2}+k^*, \frac{n}{2}-k^*\right\}, & \text{if } n \text{ even}, \end{cases} \tag{6.192}$$

where

$$k^* := \begin{cases} \min\left\{(l^{\phi(n)-1} \bmod n), n - (l^{\phi(n)-1} \bmod n)\right\}, & \text{if } \gcd(l,n) = 1, \\ \min\left\{\left((l')^{\phi(n')-1} \bmod n'\right), n' - \left((l')^{\phi(n')-1} \bmod n'\right)\right\}, & \text{if } \gcd(l,n) = 2, \end{cases} \tag{6.193}$$

with an attained maximum value of

$$\cos^2\left(\frac{2\pi\, k^*\, l}{n}\right) = \begin{cases} \cos^2\left(\frac{\pi}{n}\right), & \text{if } \gcd(l,n)=1,\ n \text{ odd},\\ \cos^2\left(\frac{2\pi}{n}\right), & \text{if } \gcd(l,n)=1,\ n \text{ even},\\ \cos^2\left(\frac{4\pi}{n}\right), & \text{if } \gcd(l,n)=2,\ \frac{n}{2} \text{ odd}. \end{cases} \tag{6.194}$$

If $\gcd(l,n)=2$ and $\frac{n}{2}$ is *even*, or if $\gcd(l,n)>2$, we obtain

$$\mathfrak{J}^{(l,n)} = k^*\mathbb{N} \cap (\{1,...,n-1\} \setminus \{n/2\}), \tag{6.195}$$

where

$$k^* := \begin{cases} \frac{n}{\gcd(l,n)}, & \text{if } \frac{n}{\gcd(l,n)} \text{ odd},\\ \frac{n}{2\gcd(l,n)}, & \text{if } \frac{n}{\gcd(l,n)} \text{ even}, \end{cases} \tag{6.196}$$

giving rise to a maximum of

$$\cos^2\left(\frac{2\pi\, c\, k^*\, l}{n}\right) = 1, \quad \forall c \in \mathbb{N}. \tag{6.197}$$

Proof. For the purpose of maximizing $\cos^2\left(\frac{2\pi\, k\, l}{n}\right)$, we only have to look at the product $k\,l$ up to a multiple of n, i.e. we can restrict our calculations to $\mathbb{Z}/(n\mathbb{Z}) \cong \{0,...,n-1\}$. Recall that for any $m \in \mathbb{Z}$, we denote by $m \bmod n$ the unique value $m' \in \{0,...,n-1\}$ which satisfies $m = s\cdot n + m'$ for some $s \in \mathbb{Z}$. We distinguish between the cases of $\gcd(l,n)=1$, $\gcd(l,n)=2$ and $\gcd(l,n)\geq 2$.

i) If l and n are *coprime*, i.e. if $\gcd(l,n)=1$, we can apply Theorem 6.42 to obtain

$$l^{\phi(n)} \equiv 1 \bmod n. \tag{6.198}$$

We conclude that

$$1 \equiv l^{\phi(n)-1}\, l \equiv (l^{\phi(n)-1} \bmod n)\, l \mod n, \tag{6.199}$$

i.e. defining $k^\star := (l^{\phi(n)-1} \bmod n)$, we have $k^\star\, l \equiv 1 \bmod n$ and consequently observe that $(n-k^\star)\, l \equiv -1 \bmod n$. Defining $k^* := \min\{k^\star, n-k^\star\}$, we can furthermore enforce that $k^* \leq \frac{n}{2}$, while still knowing that $k^*\, l \equiv \pm 1 \bmod n$. This is sufficient for Eq. (6.194) to hold in the case of an even n, i.e.

$$\cos^2\left(\frac{2\pi\, k^*\, l}{n}\right) = \cos^2\left(\pm\frac{2\pi}{n}\right) = \cos^2\left(\frac{2\pi}{n}\right). \tag{6.200}$$

By Lemma 6.40 this corresponds to the maximal value which the cosine-squared factor can attain in the case of n being even.[19] For n being odd, we similarly observe that

$$\left(k^* \left\lfloor\frac{n}{2}\right\rfloor \bmod n\right) l \equiv k^*\, l \left\lfloor\frac{n}{2}\right\rfloor \equiv \pm\left\lfloor\frac{n}{2}\right\rfloor \bmod n, \tag{6.201}$$

[19] Note that $(k\,l \bmod n) \in \{1,...,n-1\} \setminus \{n/2\}$, since $(k\,l \bmod n) \in \{0, n/2\}$ would contradict the assumption of $\gcd(l,n)=1$.

which yields

$$\cos^2\left(\frac{2\pi\ (k^* \lfloor\frac{n}{2}\rfloor \bmod n)\ l}{n}\right) = \cos^2\left(\pm\frac{2\pi}{n}\left\lfloor\frac{n}{2}\right\rfloor\right) = \cos^2\left(\frac{\pi}{n}\right), \tag{6.202}$$

realizing the maximal value possible in the case of an odd number of oscillators.

We still need to show that these choices are *admissible*, i.e. $\mathfrak{I}^{(l,n)} \subset \{1, ..., n-1\} \setminus \{n/2\}$. We first prove that $k^* \in \{1, ..., n-1\} \setminus \{n/2\}$. By definition of the modulo operation we have $k^* \in \{0, ..., n-1\}$. Moreover, $k^* = 0$ can be excluded since in this case $(l^{\phi(n)-1} \bmod n) = 0$ would contradict Eq. (6.199). Thus we are left with showing that $k^* \neq n/2$. Assume that $k^* = n/2$, which in particular implies that n is even. Now $k^*\, l \equiv \pm 1 \bmod n$ yields $\frac{n}{2}\, l = N\, n \pm 1$ for some $N \in \mathbb{N}$, i.e. $l = 2\, N \pm \frac{2}{n}$, which implies $n = 2$, since the right-hand side has to be an integer. This however contradicts our assumption of $n > 2$. Recall that by definition of k^* as the minimum of $k^\star$ and $n - k^\star$, we have ensured that $k^* < n/2$, which is why (in the case of n being *even*) $n-1, \frac{n}{2} \pm k^* \in \{1, ..., n-1\} \setminus \{n/2\}$ as well. Similarly, in the case of n odd, it follows that $\left(k^* \lfloor\frac{n}{2}\rfloor \bmod n\right), \left(k^* \lceil\frac{n}{2}\rceil \bmod n\right) \in \{1, ..., n-1\} \setminus \{n/2\}$.

Finally we show that there are *no other values* $k \in \{1, ..., n-1\} \setminus \{n/2\}$, which realize the respective maximum. Assume that n is even (the case of n odd follows similarly) and that there is a $k \in (\{1, ..., n-1\} \setminus \{n/2\}) \setminus \{k^*, n-k^*, \frac{n}{2}+k^*, \frac{n}{2}-k^*\}$, s.t.

$$\cos^2\left(\frac{2\pi\, k\, l}{n}\right) = \cos^2\left(\frac{2\pi}{n}\right), \quad \text{i.e.} \quad (k\, l \bmod n) \in \left\{1, \frac{n}{2} \pm 1, n-1\right\}. \tag{6.203}$$

If $(k\, l \bmod n) = 1$, then Eq. (6.199) yields $(k - k^\star)\, l \equiv 0 \bmod n$, which implies $(k - k^\star) \equiv 0 \bmod n$, because l and n were assumed to be coprime. Since both k and $k^\star$ are elements of $\{0, ..., n-1\}$, they have to coincide, i.e. $k = k^\star \in \{k^*, n-k^*\}$ which contradicts our initial assumption. Similar arguments show that the other options of $(k\, l \bmod n) = \frac{n}{2} \pm 1$ or $(k\, l \bmod n) = n-1$ also yield contradictions.

ii) Let now $\gcd(l, n) = 2$ and assume that $\frac{n}{2}$ is an *odd* number. In this case $l' := \frac{l}{2}$ and $n' := \frac{n}{2}$ are coprime integers, to which we can apply all of the results from the first case, i.e.

$$k^* := \min\left\{\left((l')^{\phi(n')-1} \bmod n'\right), n' - \left((l')^{\phi(n')-1} \bmod n'\right)\right\} \in \{1, ..., n'-1\} \tag{6.204}$$

defines an admissable index k^*, satsifying $k^*\, l' \equiv \pm 1 \bmod n'$, i.e. $k^*\, l' = Nn' \pm 1$ and thus $k^*\, l = Nn \pm 2$. Consequently we find that

$$\cos^2\left(\frac{2\pi\, k^*\, l}{n}\right) = \cos^2\left(\frac{4\pi}{n}\right), \tag{6.205}$$

i.e. Eq. (6.194) is satisfied. Note that a larger value is not possible since $(k^*\, l \bmod n) = \pm\frac{n}{2}$ would imply that

$$k^*\, l = N\, n \pm \frac{n}{2}, \quad \text{for some } N \in \mathbb{Z}, \tag{6.206}$$

which however contradicts $\gcd(l,n) = 2$, according to which l and n are even numbers. Similarly, $k^* l \bmod n \equiv \pm 1$ would imply that

$$k^* l = N n \pm 1, \qquad \text{for some } N \in \mathbb{Z}, \tag{6.207}$$

again yielding a contradiction to both l and n being even numbers.

iii) Finally, if $\gcd(l,n) = 2$ and $\frac{n}{2}$ is *even* or if $\gcd(l,n) > 2$, we set

$$k^* := \begin{cases} \frac{n}{\gcd(l,n)}, & \text{if } \frac{n}{\gcd(l,n)} \text{ odd}, \\ \frac{n}{2\gcd(l,n)}, & \text{if } \frac{n}{\gcd(l,n)} \text{ even}, \end{cases} \tag{6.208}$$

and observe that for all $c \in \mathbb{N}$,

$$\cos^2\left(\frac{2\pi\, c\, k^*\, l}{n}\right) = \begin{cases} \cos^2\left(2\pi\, c\, \frac{l}{\gcd(l,n)}\right) = 1, & \text{if } \frac{n}{\gcd(l,n)} \text{ odd}, \\ \cos^2\left(\pi\, c\, \frac{l}{\gcd(l,n)}\right) = 1, & \text{if } \frac{n}{\gcd(l,n)} \text{ even}. \end{cases} \tag{6.209}$$

□

Comparing Eq. (6.179) and Eq. (6.194), allows us to classify critical numbers of oscillators.

Lemma 6.44 (Critical number of oscilllators for space-like noise coupling)
We observe that

$$\cos^2\left(\frac{\pi}{n}\right) \frac{\sqrt{3}}{2\tanh^{-1}\left(\frac{1}{\sqrt{3}}\right)} \begin{cases} < 1, & \text{if } 2 \le n < 7, \\ > 1, & \text{if } n \ge 7, \end{cases} \tag{6.210a}$$

and

$$\cos^2\left(\frac{2\pi}{n}\right) \frac{\sqrt{3}}{2\tanh^{-1}\left(\frac{1}{\sqrt{3}}\right)} \begin{cases} > 1, & \text{if } n = 2, \\ < 1, & \text{if } 2 < n < 13, \\ > 1, & \text{if } n \ge 13, \end{cases} \tag{6.210b}$$

as well as

$$\cos^2\left(\frac{4\pi}{n}\right) \frac{\sqrt{3}}{2\tanh^{-1}\left(\frac{1}{\sqrt{3}}\right)} \begin{cases} > 1, & \text{if } n = 2, \\ < 1, & \text{if } n = 3, \\ > 1, & \text{if } n = 4, \\ < 1, & \text{if } 4 < n < 26, \\ > 1, & \text{if } n \ge 26. \end{cases} \tag{6.210c}$$

Proof. We present a proof of Eq. (6.210b), the other statements follow analogously. The $n = 2$ case follows from $\cos^2(2\pi) = 1$ and

$$\frac{\sqrt{3}}{2\tanh^{-1}\left(\frac{1}{\sqrt{3}}\right)} \approx 1.315 > 1, \tag{6.211}$$

while for $n = 3$, we have $\cos^2(\frac{2\pi}{3}) = \frac{1}{4}$ and can conclude that

$$\frac{1}{4}\,\frac{\sqrt{3}}{2\,\tanh^{-1}\left(\frac{1}{\sqrt{3}}\right)} \approx 0.329 < 1. \tag{6.212}$$

For $n \geq 4$ finally, we can make use of the monotonicity (Lemma 6.39) and note that

$$\cos^2\left(\frac{2\pi}{12}\right)\frac{\sqrt{3}}{2\,\tanh^{-1}\left(\frac{1}{\sqrt{3}}\right)} \approx 0.986 < 1, \tag{6.213}$$

while

$$\cos^2\left(\frac{2\pi}{13}\right)\frac{\sqrt{3}}{2\,\tanh^{-1}\left(\frac{1}{\sqrt{3}}\right)} \approx 1.031 > 1, \tag{6.214}$$

which concludes the proof. □

Combining the previous results, we can finally give a complete characterization of the resulting synchronization states induced by a given distance-l noise coupling.

Theorem 6.45 (Synchronization for distance-l noise coupling)
Let $n > 2$ and $(\mathcal{P}_+\tilde{\mathfrak{l}}')_k = (\mathcal{P}_+\tilde{\mathfrak{l}}')_{k'}$, for all $k, k' \in \{0, ..., n-1\}$. Then the effective system *pathwise synchronizes* towards $C_{\mathfrak{J}}$, where for *momentum-like* distance-l coupling, the set $\mathfrak{J}$ is given by

$$\mathfrak{J} = \mathfrak{J}^{(l,n)}, \tag{6.215}$$

while for *space-like* distance-l coupling we have

$$\mathfrak{J} = \begin{cases} \mathfrak{J}_0, & \text{if } n \in \{3,5\} \cup \{4,6,...,12\},\ \gcd(l,n) = 1;\ \text{or}\ \{6,10,...,22\},\ \gcd(l,n) = 2, \\ \mathfrak{J}^{(l,n)}, & \text{otherwise,} \end{cases}$$

where $\mathfrak{J}_0 \coloneqq \{0, n/2\} \cap \mathbb{N}$.

Proof. According to Theorem 6.33, the effective system pathwise synchronizes towards $C_{\mathfrak{J}}$, where $\mathfrak{J}$ is the set of all indices corresponding to the largest growth rates. These growth rates are given in Eqs. (6.167) and (6.169) of Lemma 6.35. Since by assumption $(\mathcal{P}_+\tilde{\mathfrak{l}}')_k = (\mathcal{P}_+\tilde{\mathfrak{l}}')_{k'}$, for all $k, k' \in \{0, ..., n-1\}$, the largest growth rates are determined by the magnitude of $\cos^2\left(\frac{2\pi\,k\,l}{n}\right)$.

In the case of *momentum-like* distance-l coupling we have seen that that the $\mathfrak{J}_0$-eigenmodes (i.e. in-phase and anti-phase synchronization) never dominate, c.f. Remark 6.38. This implies that $\mathfrak{J} = \mathfrak{J}^{(l,n)}$, as stated in Eq. (6.215).

In the case of *space-like* distance-l coupling we have $\mathfrak{J} = \mathfrak{J}_0$, if and only if (c.f. Eq. (6.179))

$$\cos^2\left(\frac{2\pi\,k\,l}{n}\right)\frac{\sqrt{3}}{2\,\tanh^{-1}\left(\frac{1}{\sqrt{3}}\right)} < 1. \tag{6.216}$$

By Eqs. (6.194) and (6.210), this inequality is satisfied if and only if one of the following conditions is fulfilled[20]

i) $\gcd(n,l) = 1$, n odd and $2 < n < 7$,

ii) $\gcd(n,l) = 1$, n even and $2 < n < 13$,

iii) $\gcd(n,l) = 2$, $\frac{n}{2}$ odd and $4 < n < 26$.

Otherwise we have $\mathfrak{I} = \mathfrak{I}^{(l,n)}$, as in the momentum-like noise-coupling case. □

Lemma 6.44 and Theorem 6.45 imply that in the case of space-like distance-l coupling there are three *critical numbers* $n = 7$, $n = 13$ and $n = 23$, at which the synchronization behavior changes: Under the assumption of n being odd, we obtain in-phase synchronization, i.e. $\mathfrak{I} = \{0\}$, if and only if $n < 7$. If n is even and $\gcd(n,l) = 1$, then we obtain $\mathfrak{I} = \mathfrak{I}_0$ (i.e. a superposition of in-phase and anti-phase synchronization), if and only if $n < 13$. Similarly, if $\gcd(n,l) = 2$ (which in particular implies that n is even), $\mathfrak{I} = \mathfrak{I}_0$ holds, if and only if $\frac{n}{2}$ is odd and $n < 26$. The largest number which satisfies this condition is $n = 22$, which is why we can identify $n = 23$ as a critical number, allowing us to conclude that space-like distance-l coupling *cannot* induce an in-phase synchronization of a population of 23 or more oscillators.

These observations are illustrated in Table 6.1, which depicts the index set $\mathfrak{I} \cap \{0, 1, ..., \frac{n}{2}\}$ in its relation to the coupling distance l and the number of oscillators n. Note that due to the reflection symmetry ($k \in \mathfrak{I} \Rightarrow n - k \in \mathfrak{I}$), this restriction to $\{0, 1, ..., \frac{n}{2}\}$ can be applied without loss of generality. Cells which correspond to pairs (l, n) giving rise to $\mathfrak{I} = \mathfrak{I}_0$ are highlighted in green hues, while those giving rise to $\mathfrak{I} = \mathfrak{I}^{(l,n)}$ are marked with yellow hues. The color saturation on the other hand reflects the value of $\gcd(l, n)$.

We conclude this section by investigating the case of a global all-to-all coupling, which unlike distance-l coupling can be shown to induce in-phase synchronization even for a large number of oscillators.

Example 6.46 (All-to-all space-like noise coupling)
Let $\nu := -i\,\mathfrak{n}$, where $\mathfrak{n} = (0, 1, ..., 1)^\top$. We recall Eq. (3.101) of Lemma 3.19, which yields

$$\tilde{\mathfrak{n}} = \frac{1}{\sqrt{n}}(n-1, -1, ..., -1)^\top. \tag{6.217}$$

Applying Eq. (6.148), we conclude that

$$\xi_k = \frac{n}{4} \cdot \begin{cases} (n-1)^2, & \text{if } k = 0, \\ 1, & \text{if } k = n/2, \\ \frac{\sqrt{3}}{2\tanh^{-1}\left(\sqrt{\frac{1}{3}}\right)}, & \text{if } k \in \{1, ..., n-1\} \setminus \{n/2\}, \end{cases}$$

i.e. for all $n > 2$, the growth rate ξ_0 dominates and all-to-all coupling thus results in an *in-phase synchronization* of the system.

[20] Recall the underlying assumption of $n > 2$.

$l \setminus n$	2	3	4	5	6	7	8
1	$\{0,1\}$	$\{0\}$	$\{0,2\}$	$\{0\}$	$\{0,3\}$	$\{3\}$	$\{0,4\}$
2			$\{1\}$	$\{0\}$	$\{0,3\}$	$\{2\}$	$\{2\}$
3					$\{1,2\}$	$\{1\}$	$\{0,4\}$
4							$\{1,2,3\}$

$l \setminus n$	9	10	11	12	13	14	15
1	$\{4\}$	$\{0,5\}$	$\{5\}$	$\{0,6\}$	$\{6\}$	$\{1,6\}$	$\{7\}$
2	$\{2\}$	$\{0,5\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,7\}$	$\{4\}$
3	$\{3\}$	$\{0,5\}$	$\{2\}$	$\{2,4\}$	$\{2\}$	$\{2,5\}$	$\{5\}$
4	$\{1\}$	$\{0,5\}$	$\{4\}$	$\{3\}$	$\{5\}$	$\{0,7\}$	$\{2\}$
5		$\{1,2,3,4\}$	$\{1\}$	$\{0,6\}$	$\{4\}$	$\{3,4\}$	$\{3,6\}$
6				$\{1,2,...,5\}$	$\{1\}$	$\{0,7\}$	$\{5\}$
7						$\{1,2,...,7\}$	$\{1\}$

$l \setminus n$	16	17	18	19	20	21	22
1	$\{1,7\}$	$\{8\}$	$\{1,8\}$	$\{9\}$	$\{1,9\}$	$\{10\}$	$\{1,10\}$
2	$\{4\}$	$\{4\}$	$\{0,9\}$	$\{5\}$	$\{5\}$	$\{5\}$	$\{0,11\}$
3	$\{3,5\}$	$\{3\}$	$\{3,6\}$	$\{3\}$	$\{3,7\}$	$\{7\}$	$\{4,7\}$
4	$\{2,4,6\}$	$\{2\}$	$\{0,9\}$	$\{7\}$	$\{5\}$	$\{8\}$	$\{0,11\}$
5	$\{3,5\}$	$\{5\}$	$\{2,7\}$	$\{2\}$	$\{2,4,6,8\}$	$\{2\}$	$\{2,9\}$
6	$\{4\}$	$\{7\}$	$\{3,6\}$	$\{8\}$	$\{5\}$	$\{7\}$	$\{0,11\}$
7	$\{1,7\}$	$\{6\}$	$\{4,5\}$	$\{4\}$	$\{3,7\}$	$\{3,6,9\}$	$\{3,8\}$
8	$\{1,2,...,7\}$	$\{1\}$	$\{0,9\}$	$\{6\}$	$\{5,10\}$	$\{4\}$	$\{0,11\}$
9			$\{1,2,...,8\}$	$\{1\}$	$\{9,11\}$	$\{7\}$	$\{5,6\}$
10					$\{1,2,...,9\}$	$\{1\}$	$\{0,11\}$
11							$\{1,2,...,10\}$

$l \setminus n$	23	24	25	26	27
1	$\{11\}$	$\{1,11\}$	$\{12\}$	$\{1,12\}$	$\{13\}$
2	$\{6\}$	$\{6\}$	$\{6\}$	$\{1,12\}$	$\{7\}$
3	$\{4\}$	$\{4,8\}$	$\{4\}$	$\{4,9\}$	$\{9\}$
4	$\{3\}$	$\{3,6,9\}$	$\{3\}$	$\{6,7\}$	$\{10\}$
5	$\{7\}$	$\{5,7\}$	$\{5,10\}$	$\{5,8\}$	$\{8\}$
6	$\{2\}$	$\{2,4,8,10\}$	$\{2\}$	$\{4,9\}$	$\{9\}$
7	$\{5\}$	$\{5,7\}$	$\{9\}$	$\{2,11\}$	$\{2\}$
8	$\{10\}$	$\{3,6,9\}$	$\{11\}$	$\{3,10\}$	$\{5\}$
9	$\{9\}$	$\{4,8\}$	$\{7\}$	$\{3,10\}$	$\{3,6,9,12\}$
10	$\{8\}$	$\{6\}$	$\{5,10\}$	$\{5,8\}$	$\{4\}$
11	$\{1\}$	$\{1,11\}$	$\{8\}$	$\{6,7\}$	$\{11\}$
12		$\{1,2,...,10\}$	$\{1\}$	$\{2,11\}$	$\{9\}$
13				$\{1,2,...,12\}$	$\{1\}$

$\mathfrak{I} = \mathfrak{I}_0$
$\gcd(l,n) = 1$
$\gcd(l,n) = 2$, $n/2$ odd

$\mathfrak{I} = \mathfrak{I}^{(l,n)}$
$\gcd(l,n) = 1$
$\gcd(l,n) = 2$, $n/2$ odd
$\gcd(l,n) = 2$, $n/2$ even
$\gcd(l,n) > 2$

Table 6.1.: Space-like distance-l noise coupling, giving rise to synchronization towards $C_{\mathfrak{I}}$.

6.5. Evolution of inhomogeneous system

In this section we study the 'full' inhomogeneous system, i.e. we allow for additive noise and study its influence on the results from the previous section.

6.5.1. Evolution of eigenmode amplitudes and the ratio process

We distinguish between the inhomogeneous and homogeneous case by introducing a σ_0 superscript. In the inhomogeneous case, the processes $\hat{\rho}_k^{+}(t)$ and $\hat{r}_k(t)$ are now written as $\hat{\rho}_k^{(+,\sigma_0)}(t)$ and $\hat{r}_k^{(\sigma_0)}(t)$, while the solutions of the homogeneous system will be denoted by $\hat{\rho}_k^{(+,0)}(t)$ and $\hat{r}_k^{(0)}(t)$.

Definition 6.47 (Inhomogenous equation)
For $k \in \{1, ..., n-1\} \setminus \{n/2\}$, we denote by $\hat{\rho}_k^{(+,\sigma_0)}(t)$ the solution to the inhomogeneous SDE of Eq. (6.54a), i.e.

$$\begin{aligned} \mathrm{d}\hat{\rho}_k^{(+,\sigma_0)}(t) = {}& \hat{\rho}_k^{(+,\sigma_0)}(t) \left(\frac{n}{2}\sqrt{|\tilde{\nu}_k|^2 + \sigma_r^2}\right) \sqrt{2 + \left(1 - [\hat{r}_k^{(\sigma_0)}(t)]^2\right) \mathsf{c}_k \cos(2\tilde{\gamma}_k)}\; \mathrm{d}(\breve{B}^+_{\mathrm{hom}})_k \\ & + \hat{\rho}_k^{(+,\sigma_0)}(t) \left[\left(\frac{n^2}{4}\left[|\tilde{\nu}_k|^2 + \sigma_r^2\right]\right) 2 + \sqrt{n}\left[(\mathcal{P}_+\tilde{\mathfrak{l}}')_k - i\,(\mathcal{P}_-\tilde{\mathfrak{l}})_k\,\hat{r}_k^{(\sigma_0)}(t)\right]\right] \mathrm{d}t \\ & + \sigma_0 \sqrt{2\,\hat{\rho}_k^{(+,\sigma_0)}(t)}\; \mathrm{d}(\breve{B}^+_{\mathrm{add}})_k + 2\,|\sigma_0|^2\; \mathrm{d}t, \end{aligned} \tag{6.218}$$

while $\hat{r}_k^{(\sigma_0)}(t)$ denotes the process of Lemma 6.9, i.e. the solution of

$$\begin{aligned} \mathrm{d}\hat{r}_k^{(\sigma_0)}(t) = {}& \left(\frac{n}{2}\sqrt{|\tilde{\nu}_k|^2 + \sigma_r^2}\right) \sqrt{2 - \left(1 - [\hat{r}_k^{(\sigma_0)}(t)]^2\right) \mathsf{c}_k \cos(2\tilde{\gamma}_k)}\, \sqrt{1 - [\hat{r}_k^{(\sigma_0)}(t)]^2}\; \mathrm{d}(\breve{B}^r_{\mathrm{hom}})_k \\ & - \left(\frac{n^2}{4}\left[|\tilde{\nu}_k|^2 + \sigma_r^2\right]\right) \left[2 - \left(1 - [\hat{r}_k^{(\sigma_0)}(t)]^2\right) \mathsf{c}_k \cos(2\tilde{\gamma}_k)\right] \hat{r}_k^{(\sigma_0)}(t)\, \mathrm{d}t \\ & + \sqrt{n}\,(-i)(\mathcal{P}_-\tilde{\mathfrak{l}})_k \left(1 - [\hat{r}_k^{(\sigma_0)}(t)]^2\right) \mathrm{d}t \\ & + \sigma_0 \sqrt{\frac{2}{\hat{\rho}_k^{(+,\sigma_0)}(t)}}\, \sqrt{1 - [\hat{r}_k^{(\sigma_0)}(t)]^2}\, \mathrm{d}(\breve{B}^r_{\mathrm{add}})_k - |\sigma_0|^2 \, \frac{2}{\hat{\rho}_k^{(+,\sigma_0)}(t)}\, \hat{r}_k^{(\sigma_0)}(t)\, \mathrm{d}t. \end{aligned} \tag{6.219}$$

For $k \in \{0, n/2\}$, we define $\hat{\rho}_k^{(+,\sigma_0)}(t) := \rho_k(t)$ similarly to Lemma 6.7, i.e. as the solution of

$$\begin{aligned} \mathrm{d}\hat{\rho}_k^{(+,\sigma_0)}(t) = {}& \hat{\rho}_k^{(+,\sigma_0)}(t)\, n \sqrt{\frac{|\tilde{\nu}_k|^2 + 2\,(\mathrm{Re}(\tilde{\nu}_k))^2}{2} + \sigma_r^2}\, \mathrm{d}(\breve{B}_{\mathrm{hom}})_k \\ & + \hat{\rho}_k^{(+,\sigma_0)}(t) \left[\frac{n^2}{2}\left[|\tilde{\nu}_k|^2 + \sigma_r^2\right] + \sqrt{n}\,\tilde{\mathfrak{l}}'_k\right] \mathrm{d}t \\ & + \sigma_0 \sqrt{2\,\hat{\rho}_k^{(+,\sigma_0)}(t)}\; \mathrm{d}(\hat{B}_{\mathrm{add}})_k + |\sigma_0|^2\; \mathrm{d}t. \end{aligned} \tag{6.220}$$

It will prove useful to introduce a process $\hat{\rho}_k^{(+,\mathrm{h})}(t)$ which plays an intermediate role between $\hat{\rho}_k^{(+,\sigma_0)}(t)$ and $\hat{\rho}_k^{(+,0)}(t)$.

Definition 6.48 (Homogeneous equation depending on full ratio process)
For $k \in \{1, ..., n-1\} \setminus \{n/2\}$, we define $\hat{\rho}_k^{(+,\mathrm{h})}(t)$ as the solution to the homogeneous SDE

$$\begin{aligned} \mathrm{d}\hat{\rho}_k^{(+,\mathrm{h})}(t) &= \hat{\rho}_k^{(+,\mathrm{h})}(t) \left(\frac{n}{2} \sqrt{|\tilde{\nu}_k|^2 + \sigma_r^2} \right) \sqrt{2 + \left(1 - [\hat{r}_k^{(\sigma_0)}(t)]^2\right) \mathsf{c}_k \cos(2\tilde{\gamma}_k)}\ \mathrm{d}(\breve{B}_{\mathrm{hom}}^+)_k \\ &\quad + \hat{\rho}_k^{(+,\mathrm{h})}(t) \left[\left(\frac{n^2}{4} \left[|\tilde{\nu}_k|^2 + \sigma_r^2 \right] \right) 2 + \sqrt{n} \left[(\mathcal{P}_+ \tilde{\mathfrak{l}}')_k - i\, (\mathcal{P}_- \tilde{\mathfrak{l}})_k\, \hat{r}_k^{(\sigma_0)}(t) \right] \right] \mathrm{d}t. \end{aligned} \tag{6.221}$$

For $k \in \{0, n/2\}$, there is no $\hat{r}_k^{(\sigma_0)}(t)$ dependence and we can define $\hat{\rho}_k^{(+,\mathrm{h})}(t) := \hat{\rho}_k^{(+,0)}(t)$, i.e. $\hat{\rho}_k^{(+,\mathrm{h})}(t)$ is given as the solution of

$$\begin{aligned} \mathrm{d}\hat{\rho}_k^{(+,\mathrm{h})}(t) &= \hat{\rho}_k^{(+,\mathrm{h})}(t)\, n \sqrt{\frac{|\tilde{\nu}_k|^2 + 2\, (\mathrm{Re}(\tilde{\nu}_k))^2}{2} + \sigma_r^2}\ \mathrm{d}(\breve{B}_{\mathrm{hom}})_k \\ &\quad + \hat{\rho}_k^{(+,\mathrm{h})}(t) \left[\frac{n^2}{2} \left[|\tilde{\nu}_k|^2 + \sigma_r^2 \right] + \sqrt{n}\, \tilde{\mathfrak{l}}'_k \right] \mathrm{d}t. \end{aligned} \tag{6.222}$$

The initial conditions are in both cases assumed to coincide with the inhomogeneous ones, i.e. $\hat{\rho}_k^{(+,\mathrm{h})}(0) = \hat{\rho}_k^{(+,\sigma_0)}(0)$.

Note that $\hat{\rho}_k^{(+,\mathrm{h})}(t)$ is defined as a solution to Eq. (6.218) without the additive-noise terms of the last line, i.e. the only element which distinguishes $\hat{\rho}_k^{(+,\mathrm{h})}(t)$ from $\hat{\rho}_k^{(+,0)}(t)$ is the ratio process, which in the former case is given by $\hat{r}_k^{(\sigma_0)}(t)$ (i.e. depending on the additive noise) and in the latter case given by $\hat{r}_k^{(0)}(t)$. The solutions to Eqs. (6.221) and (6.222) can be obtained in the same way as in the proof of Proposition 6.20, the only distinction lying in the now σ_0-dependent ratio process $\hat{r}_k^{(\sigma_0)}(t)$, which in this context can be treated as a given process.

Lemma 6.49 (Exponential solution of homogeneous process)
For $k \in \{0, n/2\}$, we recover Eqs. (6.115) and (6.116), i.e.

$$\hat{\rho}_k^{(+,\mathrm{h})}(t) = \exp\left(\breve{M}_k^+(t)\right) \exp\left(\left[\frac{n^2}{4}\ |\tilde{\nu}_k|^2 \left(-\cos(2\tilde{\gamma}_k)\right) + \sqrt{n}\, \tilde{\mathfrak{l}}'_k \right] t \right) \hat{\rho}_k^{(+,\mathrm{h})}(0), \tag{6.223}$$

where the martingale $\breve{M}_k^+(t)$ is given by a rescaled Brownian motion of the form

$$\breve{M}_k^+(t) := n \sqrt{\frac{|\tilde{\nu}_k|^2 + 2\, (\mathrm{Re}(\tilde{\nu}_k))^2}{2} + \sigma_r^2}\, ((\breve{B}_{\mathrm{hom}})_k(t). \tag{6.224}$$

For $k \in \{1, ..., n-1\} \setminus \{n/2\}$, on the other hand, we have

$$\begin{aligned}\hat{\rho}_k^{(+,\mathrm{h})}(t) &= \exp\left(\breve{M}_k^+(t)\right) \\ &\quad \cdot \exp\left(\frac{n^2}{4}\left[|\tilde{\nu}_k|^2 + \sigma_r^2\right]\left[1 - \frac{1}{2}\left(1 - \frac{1}{t}\int_0^t [\hat{r}_k^{(\sigma_0)}(s)]^2\,\mathrm{d}s\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right]t\right) \\ &\quad \cdot \exp\left(\sqrt{n}\left[(\mathcal{P}_+\tilde{\mathfrak{l}}')_k + (-i)\,(\mathcal{P}_-\tilde{\mathfrak{l}})_k\,\frac{1}{t}\int_0^t \hat{r}_k^{(\sigma_0)}(s)\,\mathrm{d}s\right]t\right)\hat{\rho}_k^{(+,\mathrm{h})}(0), \end{aligned} \tag{6.225}$$

where the martingale $\breve{M}_k^+(t)$ is defined by

$$\breve{M}_k^+(t) := \left(\frac{n}{2}\sqrt{|\tilde{\nu}_k|^2 + \sigma_r^2}\right)\int_0^t \sqrt{2 + \left(1 - [\hat{r}_k^{(\sigma_0)}(s)]^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\,\mathrm{d}(\breve{B}_{\mathrm{hom}}^+)_k. \tag{6.226}$$

Proof. This follows analogously to the proof of Proposition 6.20 by application of Itô's formula to the logarithm of the process $\hat{\rho}_k^{(+,\mathrm{h})}(t)$, followed by integration and exponentiation. □

We want to determine the asymptotic growth of $\hat{\rho}_k^{(+,\mathrm{h})}(t)$, which will subsequently allow us to determine the growth of $\hat{\rho}_k^{(+,\sigma_0)}(t)$ (c.f. Theorem 6.61). For $k \in \{0, n/2\}$, the asymptotic evolution of $\hat{\rho}_k^{(+,\mathrm{h})}(t)$ can be recovered from the previous section, while for $k \in \{1, ..., n-1\} \setminus \{n/2\}$, we need to take into account the additive-noise dependence of the ratio process.

Remark 6.50 (Asymptotic evolution of $\hat{\rho}_k^{(+,\mathrm{h})}(t)$)
We observe that the statement on a sublinear growth of the martingales, i.e.

$$\lim_{t\to\infty}\frac{\breve{M}_k^+(t)}{t} = 0, \quad \forall k \in \{0, ..., n-1\}, \tag{6.227}$$

as given by Lemma 6.21, remains valid in the inhomogeneous case. This is because the proof only relied on the boundedness of $\hat{r}_k^{(0)}(t)$, which of course holds for $\hat{r}_k^{(\sigma_0)}(t)$ as well.

Since for $k \in \{0, n/2\}$ there is no dependence on the ratio process, we can recover the asymptotic behavior of Eq. (6.126), i.e.

$$\left(\hat{\rho}_k^{(+,\mathrm{h})}(t)\right)_{t\geq 0} \sim \left(\exp\left(\xi_k\,t\right)\right)_{t\geq 0}, \tag{6.228}$$

where the growth rate ξ_k is given by Eq. (6.127).

However, determining the asymptotic growth of $\hat{\rho}_k^{(+,\mathrm{h})}(t)$ in the case of $k \in \{1, ..., n-1\} \setminus \{n/2\}$ is more involved, since we have to estimate the time averages of $\hat{r}_k^{(\sigma_0)}(t)$ and $[\hat{r}_k^{(\sigma_0)}(t)]^2$. For small values of σ_0, we will show in Lemma 6.60 that these time averages can be approximated by the ones given in Corollary 6.19 for the case of $\sigma_0 = 0$. In Theorem 6.61, this approximation will be applied, yielding an estimate on the asymptotic growth of $\hat{\rho}_k^{(+,\mathrm{h})}(t)$.

In the next step, we relate the full solution $\hat{\rho}_k^{(+,\sigma_0)}(t)$ of the inhomogeneous equation to the homogeneous solution $\hat{\rho}_k^{(+,\mathrm{h})}(t)$. As will be shown below, they are related by a time-changed *squared Bessel process*.

Definition 6.51 (Squared Bessel process[a])
For a given one-dimensional, real-valued Brownian motion $(B(t))_{t\geq 0}$, the unique strong solution of the equation

$$\beta^{(x)}(t) = x + d \cdot t + 2\int_0^t \sqrt{\beta^{(x)}(s)}\,\mathrm{d}B_s, \tag{6.229}$$

is called a *squared Bessel process with dimension* $d \in \mathbb{N}$, starting at x.

[a] c.f. [JYC09], Section 6.1.2

In future estimates we will employ the following scaling behavior of squared Bessel processes.

Lemma 6.52 (Scaling behavior of squared Bessel processes[a])
If $(\beta^{(x)}(t))_{t\geq 0}$ is a squared Bessel process with dimension d starting at x, then it follows that for any $c > 0$, the rescaled process $\left(\frac{1}{c}\beta^{(x)}_{c\,t}\right)_{t\geq 0}$ is again a squared Bessel process with dimension d, starting at $\frac{x}{c}$.

[a] c.f. [JYC09], Proposition 6.1.4.1

Applying Itô's formula to the ratio of $\hat{\rho}_k^{(+,\sigma_0)}(t)$ and $\hat{\rho}_k^{(+,\mathrm{h})}(t)$ will allow us to obtain the aspired decomposition.

Proposition 6.53 (Representation of inhomogeneous SDE)
We find that

$$\hat{\rho}_k^{(+,\sigma_0)}(t) = \hat{\rho}_k^{(+,\mathrm{h})}(t)\,\beta_k^{(1)}(\tau_k(t)), \tag{6.230}$$

where $\beta_k^{(1)}(t)$ is a squared Bessel process starting at $x = 1$, with dimension d_k, where

$$d_k := \begin{cases} 2, & \text{if } k \in \{0, n/2\}\,, \\ 4, & \text{if } k \in \{1, ..., n-1\} \setminus \{n/2\}\,. \end{cases} \tag{6.231}$$

Moreover, the time-change $\tau_k(t)$ is defined by

$$\tau_k(t) := \frac{\sigma_0^2}{2}\int_0^t \frac{1}{\hat{\rho}_k^{(+,\mathrm{h})}(s)}\,\mathrm{d}s. \tag{6.232}$$

Proof. By virtue of Itô's formula, we find that for all $k \in \{0, ..., n-1\}$,

$$
\begin{aligned}
\mathrm{d}\left(\frac{\hat{\rho}_k^{(+,\sigma_0)}(t)}{\hat{\rho}_k^{(+,\mathrm{h})}(t)}\right) &= \left(\frac{1}{\hat{\rho}_k^{(+,\mathrm{h})}(t)}\right)\mathrm{d}\hat{\rho}_k^{(+,\sigma_0)}(t) - \left(\frac{\hat{\rho}_k^{(+,\sigma_0)}(t)}{[\hat{\rho}_k^{(+,\mathrm{h})}(t)]^2}\right)\mathrm{d}\hat{\rho}_k^{(+,\mathrm{h})}(t) \\
&\quad + \frac{1}{2}\left[2\left(\frac{\hat{\rho}_k^{(+,\sigma_0)}(t)}{[\hat{\rho}_k^{(+,\mathrm{h})}(t)]^3}\right)\mathrm{d}\left\langle \hat{\rho}_k^{(+,\mathrm{h})}\right\rangle(t)\right. \\
&\quad \left. + 2\left(-\frac{1}{[\hat{\rho}_k^{(+,\mathrm{h})}(t)]^2}\right)\mathrm{d}\left\langle \hat{\rho}_k^{(+,\sigma_0)}, \hat{\rho}_k^{(+,\mathrm{h})}\right\rangle(t)\right] \\
&= \left(\frac{1}{\hat{\rho}_k^{(+,\mathrm{h})}(t)}\right)\left[\sigma_0\sqrt{2\,\hat{\rho}_k^{(+,\sigma_0)}(t)}\;\mathrm{d}(\breve{B}^+_{\mathrm{add}})_k + \frac{d_k}{2}\,|\sigma_0|^2\,\mathrm{d}t\right] \\
&= \sqrt{2}\sqrt{\frac{\hat{\rho}_k^{(+,\sigma_0)}(t)}{\hat{\rho}_k^{(+,\mathrm{h})}(t)}}\,\frac{\sigma_0}{\sqrt{\hat{\rho}_k^{(+,\mathrm{h})}(t)}}\,\mathrm{d}(\breve{B}^+_{\mathrm{add}})_k + \frac{d_k}{2}\,\frac{|\sigma_0|^2}{\hat{\rho}_k^{(+,\mathrm{h})}(t)}\,\mathrm{d}t,
\end{aligned}
\tag{6.233}
$$

where in the second step we have made use of the fact, that Definition 6.47 and Definition 6.48 yield a cancellation of the homogeneous parts, as well as a cancellation of the Itô corrections. The latter, for instance, is a consequence of

$$
\left(\frac{\hat{\rho}_k^{(+,\sigma_0)}(t)}{[\hat{\rho}_k^{(+,\mathrm{h})}(t)]^3}\right)\mathrm{d}\left\langle \hat{\rho}_k^{(+,\mathrm{h})}\right\rangle(t) - \left(\frac{1}{[\hat{\rho}_k^{(+,\mathrm{h})}(t)]^2}\right)\mathrm{d}\left\langle \hat{\rho}_k^{(+,\sigma_0)}, \hat{\rho}_k^{(+,\mathrm{h})}\right\rangle(t) = 0.
$$

We define the strictly increasing (and thus invertible) time-change

$$
t \to \tau_k(t) := \frac{\sigma_0^2}{2}\int_0^t \frac{1}{\hat{\rho}_k^{(+,\mathrm{h})}(s)}\,\mathrm{d}s, \tag{6.234}
$$

and refer to its inverse by $\tau_k^{-1}(t)$. The time-changed version of the process in Eq. (6.233) is denoted by

$$
\beta_k^{(1)}(t) := \frac{\hat{\rho}_k^{(+,\sigma_0)}(\tau_k^{-1}(t))}{\hat{\rho}_k^{(+,\mathrm{h})}(\tau_k^{-1}(t))}. \tag{6.235}
$$

Employing a *time-change result for stochastic integrals*,[21] we conclude that

$$
\mathrm{d}\beta_k^{(1)}(t) = 2\sqrt{\beta_k^{(1)}(t)}\;\mathrm{d}B_k(t) + d_k\;\mathrm{d}t, \tag{6.236}
$$

where $B_k(t)$ is a real-valued Brownian motion[22] and $\beta_k^{(1)}(0) = 1$, since $\hat{\rho}_k^{(+,\sigma_0)}(0) = \hat{\rho}_k^{(+,\mathrm{h})}(0)$ by assumption. Definition 6.51 now allows us to identify $\beta_k^{(1)}(t)$ as a squared Bessel process with dimension d_k, starting at $x = 1$. As a final step, we can solve Eq. (6.235) for $\hat{\rho}_k^{(+,\sigma_0)}(t)$ and obtain the result. □

[21] c.f. [KS91], Section 3.4.A, in particular Theorem 3.4.6, Problem 3.4.7 and Proposition 3.4.8

[22] c.f. [KS91], Theorem 5.4.6, Eq. (4.16)

In Lemma 6.55 we will obtain an upper bound on the time-change clock $\tau_k(t)$ of Eq. (6.232), which will allow us to control the time-changed squared Bessel process $\beta_k^{(1)}(t)$ in the decomposition provided by Eq. (6.230). For this purpose we need some assumptions on the *global* growth of the system, which we can impose without loss of generality since they do not affect our synchronization results.

Assumption 6.54 (Global growth of the system - revisited)
Recall that in Assumption 4.8 we have imposed the lower bound

$$\mathfrak{l}_0' \geq 2n \sum_{k=0}^{n-1} \left(|\tilde{\nu}_k|^2 + \sigma_r \right). \tag{6.237}$$

In addition, we now assume that

$$\sqrt{n}\,\tilde{\mathfrak{l}}_k' > \frac{n^2}{4} |\tilde{\nu}_k|^2 \, |\cos(2\tilde{\gamma}_k)| \,, \qquad \text{for } k \in \{0, n/2\} \,, \tag{6.238a}$$

$$(\mathcal{P}_+ \tilde{\mathfrak{l}}')_k > \left| (\mathcal{P}_- \tilde{\mathfrak{l}})_k \right| , \qquad \text{for } k \in \{1, ..., n-1\} \setminus \{n/2\} \,. \tag{6.238b}$$

Without loss of generality, this can also be achieved by choosing a suitably large global amplification factor $\mathfrak{l}_0'$. Since

$$Q^\dagger \begin{pmatrix} \mathfrak{l}_0' \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \frac{\mathfrak{l}_0'}{\sqrt{n}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{6.239}$$

we find that $\mathfrak{l}_0'$ contributes equally to all $(\mathcal{P}_+\tilde{\mathfrak{l}}')_k$, without altering $(\mathcal{P}_-\tilde{\mathfrak{l}}')_k$. For this reason, $\mathfrak{l}_0'$ and thus all assumptions on $\mathfrak{l}_0'$, have no influence on the evolution of the energy normalized process $(\mathsf{e}(t))_{t\geq 0}$ of Definition 6.30, since any common summand to all growth rates simply cancels out. In particular, we find that the synchronization results are unaffected by the choice of $\mathfrak{l}_0'$.

These growth assumptions allow us to obtain *pathwise* upper bounds on the clocks $\tau_k(t)$.

Lemma 6.55 (Time-change bound)
Let $k \in \{0, ..., n-1\}$. Given Assumption 6.54, there are $\mathbb{R}^+$-valued random variables C_k, such that $\mathbb{P}$-a.s. we have an upper bound of the form

$$\int_0^t \frac{1}{\hat{\rho}_k^{(+,\mathrm{h})}(s)} \, \mathrm{d}s \leq C_k < \infty, \quad \forall t \geq 0. \tag{6.240}$$

For $\mathbb{P}$-a.a. paths we can thus bound the clocks $(\tau_k(t))_{t\geq 0}$ uniformly in time, i.e.

$$\tau_k(t) \leq \frac{\sigma_0^2}{2} C_k, \quad \forall t \geq 0. \tag{6.241}$$

Proof. Recall that by Lemma 6.49 we know that for $k \in \{0, n/2\}$,

$$\hat{\rho}_k^{(+,\mathrm{h})}(t) = \exp\left(\breve{M}_k^+(t)\right) \exp\left(\left[\frac{n^2}{4}\,|\widetilde{\nu}_k|^2\,(-\cos(2\tilde{\gamma}_k)) + \sqrt{n}\,\tilde{\mathfrak{l}}'_k\right] t\right) \hat{\rho}_k^{(+,\mathrm{h})}(0), \tag{6.242}$$

while for $k \in \{1, ..., n-1\} \setminus \{n/2\}$ we have

$$\begin{aligned}
\hat{\rho}_k^{(+,\mathrm{h})}(t) &= \exp\left(\breve{M}_k^+(t)\right) \\
&\quad \cdot \exp\left(\frac{n^2}{4}\,\left[|\widetilde{\nu}_k|^2 + \sigma_r^2\right]\left[1 - \frac{1}{2}\left(1 - \frac{1}{t}\int_0^t [\hat{r}_k^{(\sigma_0)}(s)]^2\,\mathrm{d}s\right)\mathsf{c}_k \cos(2\tilde{\gamma}_k)\right] t\right) \\
&\quad \cdot \exp\left(\sqrt{n}\,\left[(\mathcal{P}_+\tilde{\mathfrak{l}}')_k + (-i)\,(\mathcal{P}_-\tilde{\mathfrak{l}})_k\,\frac{1}{t}\int_0^t \hat{r}_k^{(\sigma_0)}(s)\,\mathrm{d}s\right] t\right) \hat{\rho}_k^{(+,\mathrm{h})}(0) \\
&\geq \exp\left(\breve{M}_k^+(t)\right) \exp\left(\frac{n^2}{8}\,\left[|\widetilde{\nu}_k|^2 + \sigma_r^2\right] t\right) \\
&\quad \cdot \exp\left(\sqrt{n}\,\left[(\mathcal{P}_+\tilde{\mathfrak{l}}')_k - \left|(-i)\,(\mathcal{P}_-\tilde{\mathfrak{l}})_k\right|\right] t\right) \hat{\rho}_k^{(+,\mathrm{h})}(0).
\end{aligned} \tag{6.243}$$

As was noted in Remark 6.50, even in the presence of additive noise we still obtain

$$\lim_{t\to\infty} \frac{\breve{M}_k^+(t)}{t} = 0, \quad \mathbb{P}\text{-a.s.}, \quad \forall k \in \{0, ..., n-1\}. \tag{6.244}$$

This allows us to conclude that $\hat{\rho}_k^{(+,\mathrm{h})}(t)$ can be *asymptotically bounded below*,[23] i.e.

$$\hat{\rho}_k^{(+,\mathrm{h})}(t) \gtrsim \exp(c_k\,t), \tag{6.245}$$

where

$$c_k := \begin{cases} \frac{n^2}{4}\,|\widetilde{\nu}_k|^2\,(-\cos(2\tilde{\gamma}_k)) + \sqrt{n}\,\tilde{\mathfrak{l}}'_k, & \text{if } k \in \{0, n/2\}, \\ \frac{n^2}{8}\,\left[|\widetilde{\nu}_k|^2 + \sigma_r^2\right] + \sqrt{n}\,\left[(\mathcal{P}_+\tilde{\mathfrak{l}}')_k - \left|(-i)\,(\mathcal{P}_-\tilde{\mathfrak{l}})_k\right|\right], & \text{if } k \in \{1, ..., n-1\} \setminus \{n/2\}. \end{cases}$$

Note that Assumption 6.54 ensures that $c_k > 0$ for all $k \in \{0, ..., n-1\}$. The asymptotic bound (6.245) now allows us to conclude that for all $\omega \in \Omega$ there is a positive random variable $c'_k = c'_k(\omega) > 0$, s.t.

$$\hat{\rho}_k^{(+,\mathrm{h})}(t) \geq c'_k \exp\left(\frac{c_k}{2}\,t\right) > 0. \tag{6.246}$$

More precisely, application of the logarithm to Eqs. (6.242) and (6.243) yields

$$\begin{aligned}
\log(\hat{\rho}_k^{(+,\mathrm{h})}(t)) \geq \breve{M}_k^+(t) + c_k\,t &\geq \begin{cases} \inf_{0\leq t\leq t_*} \breve{M}_k^+(t) + c_k\,t, & \text{if } t \leq t_*, \\ \left(c_k - \left|\frac{\breve{M}_k^+(t)}{t}\right|\right) t, & \text{if } t > t_* \end{cases} \\
&\geq -\sup_{0\leq t\leq t_*} \left|\breve{M}_k^+(t)\right| + \frac{c_k}{2}\,t,
\end{aligned}$$

[23] Recall that the notion of an asymptotic order was introduced in Definition 6.22.

where for given $\omega \in \Omega$, we have chosen $t_* = t_*(\omega)$, s.t. $\left|\frac{\check{M}_k^+(t)}{t}\right| \leq \frac{c_k}{2}$ for all $t \geq t_*$.[24] Setting

$$c_k' := \exp\left(- \sup_{0\leq t\leq t_*} \left|\check{M}_k^+(t)\right|\right) > 0, \tag{6.247}$$

therefore yields Eq. (6.246). This lower bound in turn implies that $\mathbb{P}$-a.s. we have

$$\int_0^t \frac{1}{\hat{\rho}_k^{(+,\mathrm{h})}(s)} \,\mathrm{d}s \leq \frac{1}{c_k'} \int_0^t \exp\left(-\frac{c_k}{2}\, s\right) \mathrm{d}s \leq \frac{2}{c_k'\, c_k} =: C_k, \quad \forall t > 0,$$

which concludes the proof. □

These pathwise bounds on the clocks $\tau_k(t)$ give rise to pathwise upper and lower bounds on the time-changed processes $\beta_k^{(1)}(\tau_k(t))$.

Lemma 6.56 (Bounds on time-changed squared Bessel process)
For all $k \in \{0, ..., n-1\}$, there are random variables $\check{\beta}_k, \hat{\beta}_k$, s.t. $\mathbb{P}$-a.s. we have

$$0 < \check{\beta}_k \leq \beta_k^{(1)}(\tau_k(t)) \leq \hat{\beta}_k < \infty, \quad \forall t \geq 0. \tag{6.248}$$

Proof. From Lemma 6.55 we can conclude that

$$\hat{\beta}_k := \sup_{t\geq 0} \beta_k^{(1)}(\tau_k(t)) \leq \sup_{0\leq t\leq \frac{\sigma_0^2}{2} C_k} \beta_k^{(1)}(t) < \infty, \tag{6.249}$$

since squared Bessel processes are non-exploding. Similarly, we find that

$$\check{\beta}_k := \inf_{t\geq 0} \beta_k^{(1)}(\tau_k(t)) \geq \inf_{0\leq t\leq \frac{\sigma_0^2}{2} C_k} \beta_k^{(1)}(t) > 0, \tag{6.250}$$

where in the last step we have made use of the *nonattainability of the origin*

$$\mathbb{P}\left(\beta_k^{(1)}(t) > 0, \forall t \geq 0\right) = 1, \tag{6.251}$$

which holds for all squared Bessel processes of dimension at least two, starting at a point $x > 0$, c.f. [KS91], Proposition 3.3.22. □

6.5.2. Asymptotic evolution

We can employ the bounds on $\beta_k^{(1)}(\tau_k(t))$ in order to prove the asymptotic equivalence of $\hat{\rho}_k^{(+,\sigma_0)}(t)$ and $\hat{\rho}_k^{(+,\mathrm{h})}(t)$.

[24] This is possible by virtue of Eq. (6.244).

Lemma 6.57 (Equivalence of homogeneous and inhomogeneous process)
For all $k \in \{0, ..., n-1\}$, we have

$$(\hat{\rho}_k^{(+,\sigma_0)}(t))_{t\geq 0} \sim (\hat{\rho}_k^{(+,\mathrm{h})}(t))_{t\geq 0}. \tag{6.252}$$

Proof. Proposition 6.53 and Lemma 6.56 together yield the $\mathbb{P}$-a.s. bounds

$$\hat{\rho}_k^{(+,\mathrm{h})}(t)\,\check{\beta}_k \leq \hat{\rho}_k^{(+,\sigma_0)}(t) \leq \hat{\rho}_k^{(+,\mathrm{h})}(t)\,\hat{\beta}_k, \quad \forall t \geq 0, \tag{6.253}$$

i.e.

$$\log(\hat{\rho}_k^{(+,\mathrm{h})}(t)) + \log(\check{\beta}_k) \leq \log(\hat{\rho}_k^{(+,\sigma_0)}(t)) \leq \log(\hat{\rho}_k^{(+,\mathrm{h})}(t)) + \log(\hat{\beta}_k), \quad \forall t \geq 0, \tag{6.254}$$

which implies that

$$\left|\log(\hat{\rho}_k^{(+,\sigma_0)}(t)) - \log(\hat{\rho}_k^{(+,\mathrm{h})}(t))\right| \leq \max\left\{\left|\log(\check{\beta}_k)\right|, \left|\log(\hat{\beta}_k)\right|\right\} < \infty, \quad \forall t \geq 0. \tag{6.255}$$

□

It therefore only remains for us to study the asymptotic behavior of $\hat{\rho}_k^{(+,\mathrm{h})}(t)$. Recall that for $k \in \{0, n/2\}$, we have $\hat{\rho}_k^{(+,\mathrm{h})}(t) := \hat{\rho}_k^{(+,0)}(t)$, c.f. Definition 6.48, i.e. this case is already understood. For $k \in \{1, ..., n-1\}\backslash\{n/2\}$, however, we are facing the additional problem of an $\hat{r}_k^{(\sigma_0)}(t)$ influence (c.f. Eq. (6.225)), which is why $\hat{\rho}_k^{(+,\mathrm{h})}(t)$ does *not* coincide with $\hat{\rho}_k^{(+,0)}(t)$. In order to proceed to an asymptotic characterization, we thus first need to study $\hat{r}_k^{(\sigma_0)}(t)$ and relate it to $\hat{r}_k^{(0)}(t)$.

6.5.2.1. Ratio process

In the following proposition, we show that the ratio process $\hat{r}_k^{(\sigma_0)}(t)$ coincides in law with a *time-changed* version of $\hat{r}_k^{(0)}(t)$. In this section we will always assume that $k \in \{1, ..., n-1\} \setminus \{n/2\}$ and restrict ourselves to the case of a symmetric coupling, i.e. $(\mathcal{P}_-\,\tilde{\mathfrak{l}})_k = 0$.

Proposition 6.58 (Additive-noise influence on ratio process)
The influence of the additive-noise perturbation on the ratio process can be captured by a time-change, i.e. $\hat{r}_k^{(\sigma_0)}(t)$ can be represented as

$$\hat{r}_k^{(\sigma_0)}(t) = \hat{r}_k^{(\mathrm{TC})}(\tau_k^{\sigma_0}(t)), \quad \forall t \geq 0, \tag{6.256}$$

where $(\hat{r}_k^{(\mathrm{TC})}(t))_{t\geq 0}$ is a process coinciding in law with $(\hat{r}_k^{(0)}(t))_{t\geq 0}$ and $\tau_k^{\sigma_0}(t)$ is a time-change given by

$$\tau_k^{\sigma_0}(t) := t + \sigma_0^2\,\tau_k^{\hat{\rho}^{(+,\sigma_0)}}(t), \tag{6.257}$$

$$\tau_k^{\hat{\rho}^{(+,\sigma_0)}}(t) := \frac{2}{\frac{n^2}{4}\,[|\widetilde{\nu}_k|^2 + \sigma_r^2]} \int_0^t \frac{1}{\left[2 - \left(1 - [\hat{r}_k^{(\sigma_0)}(s)]^2\right) \mathsf{c}_k \cos(2\tilde{\gamma}_k)\right]\, \hat{\rho}_k^{(+,\sigma_0)}(s)}\, \mathrm{d}s. \tag{6.258}$$

Note that $\tau_k^{\hat{\rho}^{(+,\sigma_0)}}(t)$ depends on the trajectories of both $\hat{r}_k^{(\sigma_0)}(t)$ and $\hat{\rho}_k^{(+,\sigma_0)}(t)$, which is why Proposition 6.58 does not provide us with a direct solution of $\hat{r}_k^{(\sigma_0)}(t)$ in terms of $\hat{r}_k^{(0)}(t)$. Under certain assumptions, it will nevertheless enable us to estimate the asymptotic behavior of $\hat{\rho}_k^{(+,\sigma_0)}(t)$, c.f. Lemma 6.60 below.

Proof. We introduce the functions $f_k : [-1,1] \to \mathbb{R}$, $g_k : \mathbb{R} \to \mathbb{R}$,

$$f_k(r) := \left(\frac{n}{2}\sqrt{|\tilde{\nu}_k|^2 + \sigma_r^2}\right)\sqrt{2 - (1-r^2)\,\mathsf{c}_k\cos(2\tilde{\gamma}_k)}, \tag{6.259}$$

$$g_k(\rho) := \sqrt{\frac{2}{\rho}}. \tag{6.260}$$

From the symmetry assumption of $(\mathcal{P}_-\,\tilde{\mathfrak{l}})_k = 0$ and from Eq. (6.219) we can conclude that

$$\begin{aligned}
\mathrm{d}\hat{r}_k^{(\sigma_0)}(t) =& \left(\frac{n}{2}\sqrt{|\tilde{\nu}_k|^2 + \sigma_r^2}\right)\sqrt{2 - \left(1 - [\hat{r}_k^{(\sigma_0)}(t)]^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)}\,\sqrt{1 - [\hat{r}_k^{(\sigma_0)}(t)]^2}\;\mathrm{d}(\breve{B}^r_{\mathrm{hom}})_k(t)\\
&- \left(\frac{n^2}{4}\,\left[|\tilde{\nu}_k|^2 + \sigma_r^2\right]\right)\left[2 - \left(1 - [\hat{r}_k^{(\sigma_0)}(t)]^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right]\hat{r}_k^{(\sigma_0)}(t)\,\mathrm{d}t\\
&+ \sigma_0\sqrt{\frac{2}{\hat{\rho}_k^{(+,\sigma_0)}(t)}}\,\sqrt{1 - [\hat{r}_k^{(\sigma_0)}(t)]^2}\,\mathrm{d}(\breve{B}^r_{\mathrm{add}})_k(t) - |\sigma_0|^2\,\frac{2}{\hat{\rho}_k^{(+,\sigma_0)}(t)}\,\hat{r}_k^{(\sigma_0)}(t)\,\mathrm{d}t\\
=& \sqrt{1 - [\hat{r}_k^{(\sigma_0)}(t)]^2}\,f_k\left(\hat{r}_k^{(\sigma_0)}(t)\right)\,\mathrm{d}(\breve{B}^r_{\mathrm{hom}})_k(t) - \hat{r}_k^{(\sigma_0)}(t)\,\left[f_k\left(\hat{r}_k^{(\sigma_0)}(t)\right)\right]^2\mathrm{d}t\\
&+ \sqrt{1 - [\hat{r}_k^{(\sigma_0)}(t)]^2}\,\sigma_0\,g_k\left(\hat{\rho}_k^{(+,\sigma_0)}(t)\right)\mathrm{d}(\breve{B}^r_{\mathrm{add}})_k(t) - \hat{r}_k^{(\sigma_0)}(t)\,\sigma_0^2\,\left[g_k\left(\hat{\rho}_k^{(+,\sigma_0)}(t)\right)\right]^2\mathrm{d}t\\
=& \sqrt{1 - [\hat{r}_k^{(\sigma_0)}(t)]^2}\,\sqrt{\left[f_k\left(\hat{r}_k^{(\sigma_0)}(t)\right)\right]^2 + \sigma_0^2\,\left[g_k\left(\hat{\rho}_k^{(+,\sigma_0)}(t)\right)\right]^2}\,\mathrm{d}\breve{B}^r_k(t)\\
&- \hat{r}_k^{(\sigma_0)}(t)\left(\left[f_k\left(\hat{r}_k^{(\sigma_0)}(t)\right)\right]^2 + \sigma_0^2\,\left[g_k\left(\hat{\rho}_k^{(+,\sigma_0)}(t)\right)\right]^2\right)\mathrm{d}t,
\end{aligned}$$

where $\breve{B}^r_k(t)$, defined by

$$\mathrm{d}\breve{B}^r_k(t) := \frac{f_k\left(\hat{r}_k^{(\sigma_0)}(t)\right)\,\mathrm{d}(\breve{B}^r_{\mathrm{hom}})_k(t) + \sigma_0\,g_k\left(\hat{\rho}_k^{(+,\sigma_0)}(t)\right)\mathrm{d}(\breve{B}^r_{\mathrm{add}})_k(t)}{\sqrt{\left[f_k\left(\hat{r}_k^{(\sigma_0)}(t)\right)\right]^2 + \sigma_0^2\,\left[g_k\left(\hat{\rho}_k^{(+,\sigma_0)}(t)\right)\right]^2}} \tag{6.261}$$

is a Brownian motion, since $(\breve{B}^r_{\mathrm{hom}})_k$ and $(\breve{B}^r_{\mathrm{add}})_k$ are independent. We introduce the time-change

$$\tau_k^{\sigma_0}(t) := \int_0^t \frac{\left[f_k\left(\hat{r}_k^{(\sigma_0)}(s)\right)\right]^2 + \left[g_k\left(\hat{\rho}_k^{(+,\sigma_0)}(s)\right)\right]^2}{\left[f_k\left(\hat{r}_k^{(\sigma_0)}(s)\right)\right]^2}\,\mathrm{d}s = t + \sigma_0^2\,\tau_k^{\hat{\rho}^{(+,\sigma_0)}}(t), \tag{6.262}$$

$$\tau_k^{\hat{\rho}^{(+,\sigma_0)}}(t) := \int_0^t \frac{\left[g_k\left(\hat{\rho}_k^{(+,\sigma_0)}(s)\right)\right]^2}{\left[f_k\left(\hat{r}_k^{(\sigma_0)}(s)\right)\right]^2}\,\mathrm{d}s = \int_0^t \frac{2\left(\frac{n^2}{4}\left[|\tilde{\nu}_k|^2 + \sigma_r^2\right]\right)^{-1}}{\left[2 - \left(1 - [\hat{r}_k^{(\sigma_0)}(s)]^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right]\,\hat{\rho}_k^{(+,\sigma_0)}(s)}\,\mathrm{d}s.$$

Note that Eq. (6.262) is well defined, since f_k is a strictly positive function. Since $\tau_k^{\sigma_0}(t)$ is $\mathbb{P}$-a.s. strictly increasing, it can $\mathbb{P}$-a.s. be pathwise inverted and we find that

$$\hat{r}_k^{(\mathrm{TC})}(t) := \hat{r}_k^{(\sigma_0)}((\tau_k^{\sigma_0})^{-1}(t)) \tag{6.263}$$

satisfies[25]

$$\mathrm{d}\hat{r}_k^{(\mathrm{TC})}(t) = \sqrt{1-[\hat{r}_k^{(\mathrm{TC})}(t)]^2}\sqrt{\left[f_k\left(\hat{r}_k^{(\mathrm{TC})}(t)\right)\right]^2}\,\mathrm{d}B_k(t) - \hat{r}_k^{(\mathrm{TC})}(t)\left[f_k\left(\hat{r}_k^{(\mathrm{TC})}(t)\right)\right]^2\mathrm{d}t, \tag{6.264}$$

where $B_k(t)$ is a real-valued Brownian motion.[26] Apart from being driven by a different Brownian motion, Eq. (6.264) coincides with the $\sigma_0 = 0$ case of SDE (6.219), describing the evolution of $\hat{r}_k^{(0)}(t)$. By the weak uniqueness of solutions to this SDE, we can therefore conclude that $(\hat{r}_k^{(\mathrm{TC})}(t))_{t\geq 0} \overset{d}{=} (\hat{r}_k^{(0)}(t))_{t\geq 0}$. Solving Eq. (6.263) for $\hat{r}_k^{(\sigma_0)}(t)$, we $\mathbb{P}$-a.s. find that

$$\hat{r}_k^{(\sigma_0)}(t) = \hat{r}_k^{(\mathrm{TC})}(\tau_k^{\sigma_0}(t)), \tag{6.265}$$

which verifies the statement of Eq. (6.256). □

Note that $(\hat{r}_k^{(\mathrm{TC})}(t))_{t\geq 0} \overset{d}{=} (\hat{r}_k^{(0)}(t))_{t\geq 0}$ implies that we can apply the same steps as in Section 6.3.2, leading to Corollary 6.19, in order to show that the asymptotic time averages of $\hat{r}_k^{(\mathrm{TC})}(t)$ and $\hat{r}_k^{(0)}(t)$ coincide, i.e.

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \hat{r}_k^{(\mathrm{TC})}(s)\,\mathrm{d}s = \psi_k^{(1)}, \quad \mathbb{P}\text{-a.s.}, \tag{6.266a}$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t [\hat{r}_k^{(\mathrm{TC})}(s)]^2\,\mathrm{d}s = \psi_k^{(2)}, \quad \mathbb{P}\text{-a.s.} \tag{6.266b}$$

Ultimately, we want to show that these explicitly known asymptotic time averages are good approximations for the ones of $\hat{r}_k^{(\sigma_0)}(t)$ in the inhomogeneous case, c.f. Lemma 6.60. To this end, we are thus left with proving that the time averages of $\hat{r}_k^{(\mathrm{TC})}(t)$ closely approximate those of $\hat{r}_k^{(\sigma_0)}(t)$. Recalling Proposition 6.58, we find that this can be achieved if we can bound the clock $\tau_k^{\hat{\rho}^{(+,\sigma_0)}}$, which is the only element distinguishing $\hat{r}_k^{(\sigma_0)}(t)$ from $\hat{r}_k^{(\mathrm{TC})}(t)$, c.f. Eq. (6.257).

Lemma 6.59 (Bound on $\tau_k^{\hat{\rho}^{(+,\sigma_0)}}$)
For every $k \in \{1,...,n-1\}\setminus\{n/2\}$, there is an $\mathbb{R}^+$-valued random variable $\tilde{C}_k$, s.t. $\mathbb{P}$-a.s. we obtain the upper bound

$$\tau_k^{\hat{\rho}^{(+,\sigma_0)}}(t) \leq \tilde{C}_k < \infty, \quad \forall t > 0. \tag{6.267}$$

[25] c.f. [KS91],Section 3.4.A, in particular Theorem 3.4.6, Problem 3.4.7 and Proposition 3.4.8
[26] c.f. [KS91], Theorem 5.4.6, Eq. (4.16)

Proof. Plugging in the decomposition of $\hat{\rho}_k^{(+,\sigma_0)}(t)$ provided by Proposition 6.53 into the definition Eq. (6.258) of $\tau_k^{\hat{\rho}^{(+,\sigma_0)}}(t)$, yields an upper bound of the form

$$\begin{aligned} \tau_k^{\hat{\rho}^{(+,\sigma_0)}}(t) &:= \frac{2}{\frac{n^2}{4}\,[|\widetilde{\nu}_k|^2+\sigma_r^2]} \int_0^t \frac{1}{\left[2-\left(1-[\hat{r}_k^{(\sigma_0)}(s)]^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right]\,\hat{\rho}_k^{(+,\sigma_0)}(s)}\,\mathrm{d}s \\ &\le \frac{2}{\frac{n^2}{4}\,[|\widetilde{\nu}_k|^2+\sigma_r^2]} \int_0^t \frac{1}{\hat{\rho}_k^{(+,\mathrm{h})}(s)\,\beta_k^{(1)}(\tau_k(s))}\,\mathrm{d}s, \end{aligned}$$

where we have employed the estimate $\left[2-\left(1-[\hat{r}_k^{(\sigma_0)}(s)]^2\right)\mathsf{c}_k\cos(2\tilde{\gamma}_k)\right]\ge 1$. Since by assumption we have $k\in\{1,...,n-1\}\setminus\{n/2\}$, we can conclude that $\beta_k^{(1)}(t)$ is a 4-dimensional squared Bessel process (recall Proposition 6.53), which is why the origin is *nonattainable*,[27] i.e.

$$\mathbb{P}\left[\beta_k^{(1)}(t)>0,\forall\, 0<t<\infty\right]=1, \tag{6.268}$$

and the process is *transient*,[28] i.e.

$$\mathbb{P}\left[\lim_{t\to\infty}\beta_k^{(1)}(t)=\infty\right]=1. \tag{6.269}$$

This allows us to conclude that $\mathbb{P}$-a.a. paths are bounded away from zero, i.e.

$$\inf_{t>0}\beta_k^{(1)}(t)>0, \quad \mathbb{P}\text{-a.s.}, \tag{6.270}$$

from which we $\mathbb{P}$-a.s. obtain the upper bound

$$\begin{aligned} \tau_k^{\hat{\rho}^{(+,\sigma_0)}}(t) &\le \frac{2}{\frac{n^2}{4}\,[|\widetilde{\nu}_k|^2+\sigma_r^2]}\,\frac{1}{\inf_{s>0}\beta_k^{(1)}(s)} \int_0^t \frac{1}{\hat{\rho}_k^{(+,\mathrm{h})}(s)}\,\mathrm{d}s \\ &\le \frac{2}{\frac{n^2}{4}\,[|\widetilde{\nu}_k|^2+\sigma_r^2]}\,\frac{1}{\inf_{s>0}\beta_k^{(1)}(s)}\,C_k =: \tilde{C}_k<\infty, \end{aligned}$$

where we have employed

$$\int_0^t \frac{1}{\hat{\rho}_k^{(+,\mathrm{h})}(s)}\,\mathrm{d}s \le C_k, \quad \forall t>0, \tag{6.271}$$

as given by Lemma 6.55. □

Combining the previous estimates, we can show that choosing a suitably small additive-noise parameter $\sigma_0>0$ allows us to obtain an arbitrarily good approximation of the time averages in the inhomogeneous case by the time averages $\psi_k^{(1)}$, $\psi_k^{(2)}$ of the homogeneous system's ratio process.

[27] c.f. [KS91], Proposition 3.3.22
[28] c.f. [KS91], Problem 3.3.24

Lemma 6.60 (Asymptotic time averages of the ratio process)
For all $\delta^{(\hat{r})} > 0$, there is a $\sigma_0^* = \sigma_0^*(\delta^{(\hat{r})}) > 0$, s.t. for all $0 < \sigma_0 < \sigma_0^*$ and all indices $k \in \{1, ..., n-1\} \setminus \{n/2\}$, we $\mathbb{P}$-a.s. obtain the asymptotic approximations

$$\limsup_{t\to\infty} \left| \frac{1}{t} \int_0^t \hat{r}_k^{(\sigma_0)}(s)\,\mathrm{d}s - \psi_k^{(1)} \right| < \delta^{(\hat{r})}, \tag{6.272}$$

$$\limsup_{t\to\infty} \left| \frac{1}{t} \int_0^t [\hat{r}_k^{(\sigma_0)}(s)]^2\,\mathrm{d}s - \psi_k^{(2)} \right| < \delta^{(\hat{r})}. \tag{6.273}$$

Proof. Proposition 6.58 yields $\hat{r}_k^{(\sigma_0)}(t) = \hat{r}_k^{(\mathrm{TC})}(\tau_k^{\sigma_0}(t))$, where $\tau_k^{\sigma_0}(t) := t + \sigma_0^2\, \tau_k^{\hat{\rho}^{(+,\sigma_0)}}(t)$ and in Lemma 6.59 we have obtained an $\mathbb{P}$-a.s. upper bound of the form $\tau_k^{\hat{\rho}^{(+,\sigma_0)}}(t) \le \tilde{C}_k$. The map $t \to \tau_k^{\sigma_0}(t)$ is $\mathbb{P}$-a.s. differentiable with

$$\begin{aligned}
1 \le [\tau_k^{\sigma_0}(t)]' &= 1 + \sigma_0^2 \frac{2}{\frac{n^2}{4}\,[|\tilde{\nu}_k|^2 + \sigma_r^2]} \frac{1}{\left[2 - \left(1 - [\hat{r}_k^{(\sigma_0)}(t)]^2\right) \mathsf{c}_k \cos(2\tilde{\gamma}_k)\right]\, \hat{\rho}_k^{(+,\sigma_0)}(t)} \\
&\le 1 + \sigma_0^2 \frac{2}{\frac{n^2}{4}\,[|\tilde{\nu}_k|^2 + \sigma_r^2]} \frac{1}{\beta_k^{(1)}(\tau_k(t))} \frac{1}{\hat{\rho}_k^{(+,\mathrm{h})}(t)} \\
&\le 1 + \sigma_0^2 \frac{2}{\frac{n^2}{4}\,[|\tilde{\nu}_k|^2 + \sigma_r^2]} \frac{1}{\check{\beta}_k} \frac{1}{c_k' \exp\left(\frac{c_k}{2}\, t\right)} \\
&\le 1 + \sigma_0^2 \frac{2}{\frac{n^2}{4}\,[|\tilde{\nu}_k|^2 + \sigma_r^2]} \frac{1}{\check{\beta}_k\, c_k'},
\end{aligned} \tag{6.274}$$

where in the second line we have made use of the decomposition $\hat{\rho}_k^{(+,\sigma_0)}(t) = \hat{\rho}_k^{(+,\mathrm{h})}(t)\, \beta_k^{(1)}(\tau_k(t))$ given by Proposition 6.53 and in the third line we have employed lower bounds on $\beta_k^{(1)}(\tau_k(t))$ and $\hat{\rho}_k^{(+,\mathrm{h})}(t)$ via Lemma 6.56 and Eq. (6.246). In the last step we have applied $c_k > 0$, as was noted in the proof of Lemma 6.55. Note that Eq. (6.274) yields the estimate

$$\left| [\tau_k^{\sigma_0}(t)]' - 1 \right| \le \sigma_0^2 \frac{2}{\frac{n^2}{4}\,[|\tilde{\nu}_k|^2 + \sigma_r^2]} \frac{1}{\check{\beta}_k\, c_k'}. \tag{6.275}$$

Let $j \in \{1, 2\}$ and employ Eqs. (6.256) and (6.266) in order to deduce that

$$\begin{aligned}
&\limsup_{t\to\infty} \left| \frac{1}{t} \int_0^t \left[\hat{r}_k^{(\sigma_0)}(s)\right]^j \mathrm{d}s - \psi_k^{(j)} \right| \\
&= \limsup_{t\to\infty} \left| \frac{1}{t} \int_0^t \left[\hat{r}_k^{(\sigma_0)}(s)\right]^j - \left[\hat{r}_k^{(\mathrm{TC})}(s)\right]^j \mathrm{d}s \right| \\
&= \limsup_{t\to\infty} \left| \frac{1}{t} \int_0^t \left[\hat{r}_k^{(\mathrm{TC})}(\tau_k^{\sigma_0}(s))\right]^j - \left[\hat{r}_k^{(\mathrm{TC})}(s)\right]^j \mathrm{d}s \right| \\
&= \limsup_{t\to\infty} \left| \frac{1}{t} \int_0^{\tau_k^{\sigma_0}(t)} \left[\hat{r}_k^{(\mathrm{TC})}(\tau)\right]^j \frac{\mathrm{d}\tau}{(\tau_k^{\sigma_0})'(s(\tau))} - \frac{1}{t} \int_0^t \left[\hat{r}_k^{(\mathrm{TC})}(s)\right]^j \mathrm{d}s \right| \\
&\le \limsup_{t\to\infty} \frac{1}{t} \int_0^t \left| \frac{\left[\hat{r}_k^{(\mathrm{TC})}(\tau)\right]^j}{(\tau_k^{\sigma_0})'(s(\tau))} - \left[\hat{r}_k^{(\mathrm{TC})}(\tau)\right]^j \right| \mathrm{d}\tau + \limsup_{t\to\infty} \frac{1}{t} \int_t^{\tau_k^{\sigma_0}(t)} \left| \frac{\left[\hat{r}_k^{(\mathrm{TC})}(\tau)\right]^j}{(\tau_k^{\sigma_0})'(s(\tau))} \right| \mathrm{d}\tau.
\end{aligned} \tag{6.276}$$

Application of $\left|\hat{r}_k^{(\mathrm{TC})}(\tau)\right| \leq 1$ and $[\tau_k^{\sigma_0}(t)]' \geq 1$ (c.f. Eq. (6.274)), allows us to show that the second term vanishes:

$$\limsup_{t\to\infty} \frac{1}{t} \int_t^{\tau_k^{\sigma_0}(t)} \left| \frac{\left[\hat{r}_k^{(\mathrm{TC})}(\tau)\right]^j}{(\tau_k^{\sigma_0})'(s(\tau))} \right| \mathrm{d}\tau \leq \limsup_{t\to\infty} \frac{1}{t} \int_t^{\tau_k^{\sigma_0}(t)} \mathrm{d}\tau = \limsup_{t\to\infty} \frac{\tau_k^{\sigma_0}(t) - t}{t}$$
$$= \sigma_0^2 \limsup_{t\to\infty} \frac{\tau_k^{\hat{\rho}^{(+,\sigma_0)}}(t)}{t} \leq \sigma_0^2 \limsup_{t\to\infty} \frac{\tilde{C}_k}{t} = 0,$$

where in the second line we have employed Eq. (6.257) and Lemma 6.59. For the first term on the right-hand side of Eq. (6.276) we observe that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \left| \frac{\left[\hat{r}_k^{(\mathrm{TC})}(\tau)\right]^j}{(\tau_k^{\sigma_0})'(s(\tau))} - \left[\hat{r}_k^{(\mathrm{TC})}(\tau)\right]^j \right| \mathrm{d}\tau = \limsup_{t\to\infty} \frac{1}{t} \int_0^t \left|\hat{r}_k^{(\mathrm{TC})}(\tau)\right|^j \left| \frac{1 - (\tau_k^{\sigma_0})'(s(\tau))}{(\tau_k^{\sigma_0})'(s(\tau))} \right| \mathrm{d}\tau$$
$$\leq \limsup_{t\to\infty} \frac{1}{t} \int_0^t \frac{\left|(\tau_k^{\sigma_0})'(s(\tau)) - 1\right|}{(\tau_k^{\sigma_0})'(s(\tau))} \mathrm{d}\tau$$
$$\leq \sigma_0^2 \frac{2}{\frac{n^2}{4}\,[|\tilde{\nu}_k|^2 + \sigma_r^2]} \frac{1}{\breve{\beta}_k\, c_k'},$$

where again we have employed $\left|\hat{r}_k^{(\mathrm{TC})}(\tau)\right| \leq 1$ and also made use of Eq. (6.275). Applying these estimates to Eq. (6.276), yields

$$\limsup_{t\to\infty} \left| \frac{1}{t} \int_0^t \left[\hat{r}_k^{(\sigma_0)}(s)\right]^j - \psi_k^{(j)} \right| \leq \sigma_0^2 \frac{2}{\frac{n^2}{4}\,[|\tilde{\nu}_k|^2 + \sigma_r^2]} \frac{1}{\breve{\beta}_k\, c_k'}. \tag{6.277}$$

Now Eqs. (6.272) and (6.273) are satisfied for all $0 < \sigma_0 < \sigma_0^*$, where

$$\sigma_0^* := \sqrt{\frac{\delta^{(\hat{r})}\, \frac{n^2}{4}\, [|\tilde{\nu}_k|^2 + \sigma_r^2]\, \breve{\beta}_k\, c_k'}{2}}. \qquad \square$$

6.5.2.2. Asymptotic bounds on (in-)homogeneous process

The estimates on the time-averages of the ratio process allow us to state the main result of this section, relating the asymptotic growth of $(\hat{\rho}_k^{(+,\sigma_0)}(t))_{t\geq 0}$ to the growth of $(\hat{\rho}_k^{(+,0)}(t))_{t\geq 0}$. As in the previous section, we assume that $\mathcal{P}_-\,\tilde{\mathfrak{l}} = 0$.

Theorem 6.61 (Asymptotic growth of (in-)homogeneous process)
For $k \in \{0, n/2\}$, the asymptotic evolution of $\hat{\rho}_k^{(+,\sigma_0)}(t)$ and $\hat{\rho}_k^{(+,\mathrm{h})}(t)$ is determined by

$$(\hat{\rho}_k^{(+,\sigma_0)}(t))_{t\geq 0} \sim \left(\hat{\rho}_k^{(+,\mathrm{h})}(t)\right)_{t\geq 0} \sim (\exp(\xi_k\, t))_{t\geq 0}\,, \tag{6.278}$$

where the growth rate is given by

$$\xi_k := \frac{n^2}{4} |\tilde{\nu}_k|^2 (-\cos(2\tilde{\gamma}_k)) + \sqrt{n}\, \tilde{\mathfrak{l}}'_k. \tag{6.279}$$

For any $\delta^{(\hat{r})} > 0$, there is a $\sigma_0^* > 0$, s.t. for all $0 < \sigma_0 < \sigma_0^*$ and all $k \in \{1, ..., n-1\} \setminus \{n/2\}$ we obtain the following asymptotic bounds

$$\left(\exp\left(\left(\xi_k - \delta^{(\hat{r})}\, \hat{c}_k\right) t\right)\right)_{t \geq 0} \lesssim \left(\hat{\rho}_k^{(+,\sigma_0)}(t)\right)_{t \geq 0} \lesssim \left(\exp\left(\left(\xi_k + \delta^{(\hat{r})}\, \hat{c}_k\right) t\right)\right)_{t \geq 0}, \tag{6.280}$$

where

$$\xi_k := \frac{n^2}{4} \left[|\tilde{\nu}_k|^2 + \sigma_r^2\right] \left[1 - \frac{1}{2}\left(1 - \psi_k^{(2)}\right) \mathsf{c}_k \cos(2\tilde{\gamma}_k)\right] + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k, \tag{6.281a}$$

$$\hat{c}_k := \frac{n^2}{8} \left[|\tilde{\nu}_k|^2 + \sigma_r^2\right] \mathsf{c}_k \cos(2\tilde{\gamma}_k). \tag{6.281b}$$

Proof. Lemma 6.57 yields the asymptotic equivalence of $\hat{\rho}_k^{(+,\sigma_0)}(t)$ and $\hat{\rho}_k^{(+,\mathrm{h})}(t)$. As shown in Lemma 6.49, $\hat{\rho}_k^{(+,\mathrm{h})}(t)$ in turn can be represented in terms of an exponential solution. In Remark 6.50, we have noted that for all $k \in \{0, ..., n-1\}$, the martingales do not yield a contribution to the asymptotic growth rate, i.e.

$$\lim_{t \to \infty} \frac{\breve{M}_k^+(t)}{t} = 0. \tag{6.282}$$

For $k \in \{0, n/2\}$, we can thus employ Eq. (6.223) in order to show that the growth rate ξ_k coincides with the one in the homogeneous case, c.f. Theorem 6.24.

For $k \in \{1, ..., n-1\} \setminus \{n/2\}$, however, Eq. (6.225) implies that the asymptotic growth of $\hat{\rho}_k^{(+,\mathrm{h})}(t)$ is affected by time averages of the ratio process $\hat{r}_k^{(\sigma_0)}(t)$. While $\hat{r}_k^{(\sigma_0)}(t)$ differs from $\hat{r}_k^{(0)}(t)$, application of Lemma 6.60 allows us to control the deviation of their time averages. This enables us to relate the growth rates in the inhomogeneous case to the ones in the homogeneous setup. Employing the notation of Eq. (6.281), we observe that Lemma 6.49 yields

$$\begin{aligned} &\frac{\log\left(\hat{\rho}_k^{(+,\mathrm{h})}(t)\right) - \left(\xi_k - \delta^{(\hat{r})}\, \hat{c}_k\right) t}{t} \\ &= \frac{\breve{M}_k^+(t)}{t} + \frac{n^2}{4} \left[|\tilde{\nu}_k|^2 + \sigma_r^2\right] \left[1 - \frac{1}{2}\left(1 - \frac{1}{t}\int_0^t [\hat{r}_k^{(\sigma_0)}(s)]^2\, \mathrm{d}s\right) \mathsf{c}_k \cos(2\tilde{\gamma}_k)\right] + \sqrt{n}\, (\mathcal{P}_+ \tilde{\mathfrak{l}}')_k \\ &\quad + \frac{\ln\left(\hat{\rho}_k^{(+,\mathrm{h})}(0)\right)}{t} - \left(\xi_k - \delta^{(\hat{r})}\, \hat{c}_k\right) \\ &= \frac{\breve{M}_k^+(t)}{t} + \frac{\ln\left(\hat{\rho}_k^{(+,\mathrm{h})}(0)\right)}{t} + \hat{c}_k \left(\frac{1}{t}\int_0^t [\hat{r}_k^{(\sigma_0)}(s)]^2\, \mathrm{d}s - \psi_k^{(2)} + \delta^{(\hat{r})}\right), \end{aligned} \tag{6.283}$$

where the first two terms vanish in the limit of $t \to \infty$, c.f. Remark 6.50.

We conclude that (recall $\hat{\rho}_k^{(+,\sigma_0)}(t) \sim \hat{\rho}_k^{(+,\mathrm{h})}(t)$, c.f. Lemma 6.57)

$$\frac{\hat{\rho}_k^{(+,\sigma_0)}(t)}{\exp\left(\left(\xi_k - \delta^{(\hat{r})}\,\hat{c}_k\right)t\right)} \sim \exp\left(\hat{c}_k\left(\delta^{(\hat{r})} + \frac{1}{t}\int_0^t [\hat{r}_k^{(\sigma_0)}(s)]^2\,\mathrm{d}s - \psi_k^{(2)}\right)t\right) =: Z(t) > 0, \tag{6.284}$$

where by Lemma 6.60,

$$\liminf_{t\to\infty} Z(t) \geq \exp\left(\hat{c}_k\left(\delta^{(\hat{r})} - \limsup_{t\to\infty}\left|\frac{1}{t}\int_0^t [\hat{r}_k^{(\sigma_0)}(s)]^2\,\mathrm{d}s - \psi_k^{(2)}\right|\right)\right) \geq 1. \tag{6.285}$$

By definition, c.f. Eq. (6.124), it follows that $\exp\left(\left(\xi_k - \delta^{(\hat{r})}\,\hat{c}_k\right)t\right) \lesssim \hat{\rho}_k^{(+,\sigma_0)}(t)$, confirming the asymptotic lower bound of Eq. (6.280). The upper bound follows analogously. □

6.5.2.3. Ordering of growth rates and synchronization in the presence of additive noise

Note that any strict ordering of growth rates in the absence of additive noise will be preserved in the presence of a small additive-noise perturbation. If for instance $\xi_k < \xi_l$,[29] we can choose a suitably small parameter $\delta^{(\hat{r})} > 0$, s.t. $\xi_k + \delta^{(\hat{r})}\,\hat{c}_k < \xi_l - \delta^{(\hat{r})}\,\hat{c}_l$. For all $\sigma_0 < \sigma_0^*(\delta^{(\hat{r})})$, the process $\hat{\rho}_l^{(+,\sigma_0)}(t)$ will thus still asymptotically dominate $\hat{\rho}_k^{(+,\sigma_0)}(t)$. This remark, of course, extends to larger sets of ordered growth rates.

Assumption 6.62 (Ordering of growth rates in presence of noise perturbations)
Let $\mathfrak{I} \subset \{0, ..., \lfloor \frac{n}{2} \rfloor\}$ denote the index set of all largest growth rates, i.e.

$$\xi_l = \xi_{l'} > \xi_k, \quad \forall l, l' \in \mathfrak{I},\ \forall k \in \left\{0, ..., \left\lfloor \frac{n}{2} \right\rfloor\right\} \setminus \mathfrak{I}. \tag{6.286}$$

Choose $\delta^{(\hat{r})} > 0$ small enough, s.t.

$$\xi_l - \delta^{(\hat{r})}\,\hat{c}_l > \xi_k + \delta^{(\hat{r})}\,\hat{c}_k, \quad \forall l \in \mathfrak{I},\ \forall k \in \left\{0, ..., \left\lfloor \frac{n}{2} \right\rfloor\right\} \setminus \mathfrak{I} \tag{6.287}$$

and assume that $0 < \sigma_0 < \sigma_0^*$, where $\sigma_0^* = \sigma_0^*(\delta^{(\hat{r})})$, as given by Theorem 6.61. Furthermore, the deterministic coupling is required to be symmetric, i.e. $\mathcal{P}_- \tilde{\mathfrak{l}} = 0$.

This implies that for sufficiently small σ_0, the synchronization statement of Theorem 6.33 remains valid.

Theorem 6.63 (Synchronization in the presence of additive noise)
If Assumption 6.62 is satisfied, then the normalized eigenmodes corresponding to non-maximal growth rates vanish asymptotically, i.e.

$$\lim_{t\to\infty} \sum_{k\in\{0,...,\lfloor \frac{n}{2} \rfloor\}\setminus\mathfrak{I}} \mathsf{e}_k^+(t) = 0, \quad \mathbb{P}\text{-a.s.} \tag{6.288}$$

[29]Recall that ξ_k and ξ_l denote the growth rates of the *homogeneous* system, i.e. without additive noise, c.f. Theorem 6.61.

This implies that the system *pathwise synchronizes* towards the convex subset $C_{\mathfrak{J}}$ corresponding to the maximal growth rates, i.e.

$$\lim_{t\to\infty} \operatorname{dist}\left(\mathsf{e}(t), C_{\mathfrak{J}}\right) = 0, \quad \mathbb{P}\text{-a.s.} \tag{6.289}$$

Here $\mathsf{e}(t)$ and $\mathsf{e}_k^+(t)$ are defined as in Definition 6.30, now denoting the energy normalized eigenmode vectors of the *inhomogeneous* system.

Proof. Employing Theorem 6.61, this proof can be carried out analogously to the proof of Theorem 6.33. □

6.6. Evolution of original system

In this final section we analyze the implications of the asymptotic synchronization result for the averaged system (Theorem 6.63) on the synchronization behavior of the original 'unaveraged' system $\mathsf{p}^\varepsilon(t) := (\tilde{y}^\varepsilon(\tilde{y}^\varepsilon)^\dagger)(t)$ given by Proposition 3.39. For this purpose, we employ the averaging result of Theorem 4.71, which establishes a 'link' between $(\mathsf{p}^\varepsilon(t))_{t\geq 0}$ and the effective system $(\hat{p}(t))_{t\geq 0}$.

6.6.1. Convergence in $\mathcal{C}([0,\infty),\mathbb{R}^m)$

The averaging theorem is stated in terms of a *weak convergence* of $\mathbb{C}^{n,n}$-valued, continuous processes. Making use of the following definition, we can identify their paths with elements of $\mathcal{C}([0,\infty),\mathbb{R}^m)$, where $m := (2n)^2$.

Definition 6.64 (The space $\mathcal{C}([0,\infty),\mathbb{R}^m)$)
For $m \in \mathbb{N}$, we denote the space of continuous, $\mathbb{R}^m$-valued paths by[a]

$$\mathcal{C}([0,\infty),\mathbb{R}^m) := \{f : [0,\infty) \to \mathbb{R}^m \mid f \text{ continuous}\}. \tag{6.290}$$

On $\mathcal{C}([0,\infty),\mathbb{R}^m)$ we define the metric

$$\mathsf{d}(f,g) := \sum_{m=1}^{\infty} \frac{1}{2^m} \min\left\{\sup_{0\leq t\leq m} |f(t)-g(t)|, 1\right\}, \tag{6.291}$$

w.r.t. which $\mathcal{C}([0,\infty),\mathbb{R}^m)$ becomes a *complete, separable metric space.*[b]
For all $T \geq 0$, we define the time-shift Θ_T as

$$\Theta_T : \mathcal{C}([0,\infty),\mathbb{R}^m) \to \mathcal{C}([0,\infty),\mathbb{R}^m), \quad f \to (\Theta_T f) := (f(T+t))_{t\geq 0}. \tag{6.292}$$

[a] The following definitions are given in [KS91], Section 2.4, c.f. in particular Eq. (4.1) and Eq. (5.15).
[b] c.f. [KS91], Section 2.4, Problem 4.1

Note that we can characterize convergence w.r.t. the metric d as uniform convergence on all compact time intervals, i.e. for $f^{(N)}, \hat{f} \in \mathcal{C}([0,\infty), \mathbb{R}^m)$ we have the equivalence

$$\mathsf{d}(f^{(N)}, \hat{f}) \to 0, \quad \text{if and only if} \quad \forall T > 0: \sup_{0\le t\le T} \left| f^{(N)}(t) - \hat{f}(t) \right| \to 0. \tag{6.293}$$

Recall that we want to transfer *asymptotic* synchronization results, i.e. result in the $t \to \infty$ limit, to the original process. For this purpose, it will prove useful to study the relation of the time-shift operator and the metric d in the case of a process that $\mathbb{P}$-a.s. converges to zero. We show that a suitably large time-shift of such a process can, with high probability, be found in a small neighborhood of the zero process.

Lemma 6.65 (Time shift and $\mathcal{C}([0,\infty), \mathbb{R}^m)$ metric)
If $\mathcal{X} = (\mathcal{X}(t))_{t\ge 0}$ is a continuous, $\mathbb{R}^m$-valued, non-negative stochastic process, s.t.

$$\lim_{t\to\infty} \mathcal{X}(t) = 0, \quad \mathbb{P}\text{-a.s.}, \tag{6.294}$$

then for all $\Delta, \delta > 0$, there is a $T_* > 0$, s.t. for all $T \ge T_*$ we have

$$\mathbb{P}\left(\Theta_T \mathcal{X} \notin \mathsf{D}_\delta(0)\right) \le \Delta. \tag{6.295}$$

Proof. Choose $N \in \mathbb{N}$, s.t. $2^{-N} < \delta/2$. Assumption (6.294) now implies that

$$\lim_{T\to\infty} \left(\sup_{0\le t\le N} \mathcal{X}(T+t) \right) = 0, \quad \mathbb{P}\text{-a.s.}, \tag{6.296}$$

i.e. the process $\mathcal{X}'(T) := \sup_{0\le t\le N} \Theta_T \mathcal{X}(t)$ converges $\mathbb{P}$-a.s. to zero and thus also in probability. Employing the non-negativity of $\mathcal{X}(t)$, we can conclude that

$$\begin{aligned} \lim_{T\to\infty} \mathbb{P}\left(\sup_{0\le t\le N} |\Theta_T \mathcal{X}(t)| \ge \frac{\delta}{2} \right) &= \lim_{T\to\infty} \mathbb{P}\left(\left| \sup_{0\le t\le N} \Theta_T \mathcal{X}(t) \right| \ge \frac{\delta}{2} \right) \\ &= \lim_{T\to\infty} \mathbb{P}\left(|\mathcal{X}'(T)| \ge \frac{\delta}{2} \right) = 0. \end{aligned} \tag{6.297}$$

This result allows us to find a $T_* > 0$, s.t. for all $T \ge T_*$,

$$\mathbb{P}\left(\sup_{0\le t\le N} |\Theta_T \mathcal{X}(t)| \ge \frac{\delta}{2} \right) \le \Delta. \tag{6.298}$$

We therefore obtain that

$$\begin{aligned} \mathbb{P}\left(\Theta_T \mathcal{X} \notin \mathsf{D}_\delta(0)\right) &= \mathbb{P}\left(\mathsf{d}(\Theta_T \mathcal{X}, 0) \ge \delta\right) \\ &\le \mathbb{P}\left(\sup_{0\le t\le N} |\Theta_T \mathcal{X}(t)| \ge \frac{\delta}{2} \right) + \mathbb{P}\left(\mathsf{d}(\Theta_T \mathcal{X}, 0) \ge \delta, \sup_{0\le t\le N} |\Theta_T \mathcal{X}(t)| < \frac{\delta}{2} \right) \\ &\le \Delta, \end{aligned}$$

where the second probability vanishes, since $\sup_{0\le t\le N} |\Theta_T \mathcal{X}(t)| < \frac{\delta}{2}$ implies that

$$\begin{aligned}\mathsf{d}(\Theta_T\mathcal{X},0) &= \sum_{m=1}^{N} \frac{1}{2^m}\min\left\{\sup_{0\le t\le m} |\Theta_T\mathcal{X}(t)|\,,1\right\} + \sum_{m=N+1}^{\infty}\frac{1}{2^m}\min\left\{\sup_{0\le t\le m}|\Theta_T\mathcal{X}(t)|\,,1\right\}\\ &\le \left(\sum_{m=1}^{N}\frac{1}{2^m}\right)\sup_{0\le t\le N}|\Theta_T\mathcal{X}(t)| + \sum_{m=N+1}^{\infty}\frac{1}{2^m} < \frac{\delta}{2}+\frac{1}{2^N} < \delta,\end{aligned}$$

in contradiction to $\mathsf{d}(\Theta_T\mathcal{X},0)\ge\delta$. □

In the next step, we observe that weak convergence of a series of processes implies the weak convergence of all time-shifted versions of these processes.

Lemma 6.66 (Weak convergence persists under time shift)
Let $(\mathcal{X}(t))_{t\ge0}, (\mathcal{X}^\varepsilon(t))_{t\ge0}$ be $\mathbb{R}^m$-valued, continuous stochastic processes, s.t.

$$(\mathcal{X}^\varepsilon(t))_{t\ge0} \Rightarrow (\mathcal{X}(t))_{t\ge0}, \quad \text{as } \varepsilon\to0, \tag{6.299}$$

then for all $T>0$, it follows that

$$\Theta_T(\mathcal{X}^\varepsilon(t))_{t\ge0} \Rightarrow \Theta_T(\mathcal{X}(t))_{t\ge0}, \quad \text{as } \varepsilon\to0. \tag{6.300}$$

Proof. By [KS91], Section 2.4.C, a weak convergence of stochastic processes implies a weak convergence of all finite dimensional distributions, i.e. for all $0\le t_1<\ldots<t_N$,

$$\begin{pmatrix}\mathcal{X}^\varepsilon(t_1)\\ \vdots\\ \mathcal{X}^\varepsilon(t_N)\end{pmatrix} \Rightarrow \begin{pmatrix}\mathcal{X}(t_1)\\ \vdots\\ \mathcal{X}(t_N)\end{pmatrix}, \quad \text{as } \varepsilon\to0. \tag{6.301}$$

Clearly, the convergence of the finite dimensional distributions holds for the time-shifted processes $\Theta_T(\mathcal{X}^\varepsilon(t))_{t\ge0}$ as well.

According to *Prohorov's theorem*,[30] a family of measures on the complete, separable metric space $\mathcal{C}([0,\infty),\mathbb{R}^m)$ is tight if and only if it is relative compact. Since by assumption, the processes $(\mathcal{X}^\varepsilon(t))_{t\ge0}$ weakly converge as $\varepsilon\to0$, we have relative compactness and thus tightness of the measures $\{\mathbb{P}((\mathcal{X}^\varepsilon(t))_{t\ge0})^{-1} \mid \varepsilon>0\}$. We show that this implies a tightness of the 'time-shifted' measures $\{\mathbb{P}(\Theta_T(\mathcal{X}^\varepsilon(t))_{t\ge0})^{-1} \mid \varepsilon>0\}$. By definition, the family of measures $\{\mathbb{P}((\mathcal{X}^\varepsilon(t))_{t\ge0})^{-1} \mid \varepsilon>0\}$ is tight, if and only if for all $\alpha>0$, there is a compact set $K_\alpha\subset\mathcal{C}([0,\infty),\mathbb{R}^m)$, s.t.

$$\mathbb{P}\left((\mathcal{X}^\varepsilon(t))_{t\ge0}\in K_\alpha\right)\ge1-\alpha, \quad \forall\varepsilon>0. \tag{6.302}$$

Since for any $\alpha>0$,

$$\Theta_TK_\alpha := \{\Theta_Tf \mid f\in K_\alpha\} \tag{6.303}$$

[30] c.f. [KS91], Section 2.4.B, Theorem 4.7

yields

$$\mathbb{P}\left(\Theta_T(\mathcal{X}^\varepsilon(t))_{t\geq 0} \in \Theta_T K_\alpha\right) \geq \mathbb{P}\left((\mathcal{X}^\varepsilon(t))_{t\geq 0} \in K_\alpha\right) \geq 1-\alpha, \quad \forall \varepsilon > 0, \tag{6.304}$$

we obtain the aspired tightness result for the family of time-shifted measures, provided that $\Theta_T K_\alpha$ is compact, which can be seen as follows:

Let $(g_N)_N \subset \Theta_T K_\alpha$. Then there are $(f_N)_N \subset K_\alpha$, s.t. for all $N \in \mathbb{N}$ we have $g_N = \Theta_T f_N$. Compactness of K_α yields the existence of a convergent subsequence $f_{N_k} \Rightarrow \hat{f}$, where $\hat{f} \in K_\alpha$. This in turn allows us to conclude that g_{N_k} converges towards $\hat{g} := \Theta_T \hat{f} \in \Theta_T K_\alpha$:

$$\mathsf{d}\left(g_{N_k}, \hat{g}\right) = \mathsf{d}\left(\Theta_T f_{N_k}, \Theta_T \hat{f}\right) \leq 2^{\lceil T\rceil} \mathsf{d}\left(f_{N_k}, \hat{f}\right) \to 0, \quad k \to \infty, \tag{6.305}$$

where the inequality follows from the definition of the metric on $\mathcal{C}\left([0,\infty),\mathbb{R}^m\right)$, since for any $f, g \in \mathcal{C}\left([0,\infty),\mathbb{R}^m\right)$ we have

$$\begin{aligned}
\mathsf{d}(\Theta_T f, \Theta_T g) &= \sum_{m=1}^{\infty} \frac{1}{2^m} \min\left\{\sup_{0\leq t\leq m} |f_{T+t} - g_{T+t}|, 1\right\} \\
&\leq 2^{\lceil T\rceil} \sum_{m=1}^{\infty} \frac{1}{2^{m+\lceil T\rceil}} \min\left\{\sup_{0\leq t\leq m+\lceil T\rceil} |f(t) - g(t)|, 1\right\} \\
&\leq 2^{\lceil T\rceil} \mathsf{d}(f,g).
\end{aligned}$$

Now, by [KS91], Section 2.4.C, Theorem 4.15, convergence of the finite dimensional distributions of $\mathbb{P}(\Theta_T(\mathcal{X}^\varepsilon(t))_{t\geq 0})^{-1}$ and tightness together yield Eq. (6.300). □

6.6.2. Synchronization of original system

In analogy to Definitions 6.3, 6.6 and 6.30, we introduce notations for the (normalized) eigenmode amplitudes of the original system.

Definition 6.67 (Eigenmode amplitudes of original system)
For each $k \in \{0, ..., n-1\}$, we define $\hat{\rho}_k^\varepsilon(t)$ as

$$\hat{\rho}_k^\varepsilon(t) := (\tilde{y}^\varepsilon(t)(\tilde{y}^\varepsilon(t))^\dagger)_{kk} \equiv |\tilde{y}_k^\varepsilon(t)|^2. \tag{6.306}$$

For $k \in \{1, ..., n-1\} \setminus \{n/2\}$, we introduce the corresponding amplitude sum

$$\rho_k^{+,\varepsilon}(t) := \hat{\rho}_k^\varepsilon(t) + \hat{\rho}_{n-k}^\varepsilon(t). \tag{6.307}$$

For each $l \in \{0, ..., n-1\}$, we define the normalized eigenmode amplitude $\mathsf{e}_l^\varepsilon(t)$ as

$$\mathsf{e}_l^\varepsilon(t) := \frac{\hat{\rho}_l^\varepsilon(t)}{\sum_{k=0}^{n-1} \hat{\rho}_k^\varepsilon(t)} = \frac{\hat{\rho}_l^\varepsilon(t)}{\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \rho_k^{+,\varepsilon}(t)} \in [0,1], \tag{6.308}$$

and denote the corresponding 'pair-sum' by

$$\mathsf{e}_l^{+,\varepsilon}(t) := \begin{cases} \mathsf{e}_l^{\varepsilon}(t), & \text{if } l \in \{0, n/2\}, \\ \mathsf{e}_l^{\varepsilon}(t) + \mathsf{e}_{n-l}^{\varepsilon}(t), & \text{if } l \in \{1, ..., n-1\} \setminus \{n/2\}. \end{cases} \tag{6.309}$$

Now we are in a position to state the synchronization result for the original system.

Theorem 6.68 (Synchronization for original system)
Let Assumption 6.62 be satisfied and let $0 < \Delta, \delta << 1$. Then there is a $T_* > 0$, s.t. for all $T \geq T_*$ there is an $\varepsilon_*^{(T)} > 0$, s.t. for all $0 < \varepsilon < \varepsilon_*^{(T)}$ we have

$$\mathbb{P}\left(\sum_{k \in \{0,...,\lfloor \frac{n}{2} \rfloor\} \setminus \mathfrak{I}} \Theta_T \mathsf{e}_k^{+,\varepsilon} \notin \mathsf{D}_\delta(0) \right) \leq \Delta. \tag{6.310}$$

This implies that

$$\mathbb{P}\left(\sum_{k \in \mathfrak{I}} \Theta_T \mathsf{e}_k^{+,\varepsilon} \in \mathsf{D}_\delta(1) \right) \geq 1 - \Delta, \tag{6.311}$$

i.e. with high probability of at least $1 - \Delta$, the sum of all normalized synchronization eigenmodes $\mathsf{e}_k^{+,\varepsilon}$, $k \in \mathfrak{I}$, *eventually* lies in an arbitary small δ-ball of the '1'-process. Here *eventually* encompasses a suitably large time-shift Θ_T and a suitably small scaling factor ε.

Proof. Setting

$$\chi := \sum_{k \in \{0,...,\lfloor \frac{n}{2} \rfloor\} \setminus \mathfrak{I}} \mathsf{e}_k^{+}, \quad \chi^\varepsilon := \sum_{k \in \{0,...,\lfloor \frac{n}{2} \rfloor\} \setminus \mathfrak{I}} \mathsf{e}_k^{+,\varepsilon}, \tag{6.312}$$

application of Theorem 6.63 yields

$$\lim_{t \to \infty} \chi(t) = 0, \quad \mathbb{P}\text{-a.s.}, \tag{6.313}$$

which allows us to employ Lemma 6.65 in order to find a $T_* > 0$, s.t. for all $T \geq T_*$ we have

$$\mathbb{P}\left(\Theta_T \chi \notin \mathsf{D}_\delta(0)\right) \leq \frac{\Delta}{2}. \tag{6.314}$$

The averaging result of Theorem 4.71 yields

$$(\tilde{y}^\varepsilon(t)(\tilde{y}^\varepsilon(t))^\dagger)_{t \geq 0} \Rightarrow (\widehat{\mathsf{p}}(t))_{t \geq 0}, \quad \text{as } \varepsilon \to 0, \tag{6.315}$$

which by Lemma 6.66 implies that for all $T \geq T_*$, we have

$$\Theta_T(\tilde{y}^\varepsilon(t)(\tilde{y}^\varepsilon(t))^\dagger)_{t \geq 0} \Rightarrow \Theta_T(\widehat{\mathsf{p}}(t))_{t \geq 0}, \quad \text{as } \varepsilon \to 0. \tag{6.316}$$

By the *continuous mapping theorem*[31] we therefore find that

$$\Theta_T \mathcal{X}^\varepsilon = \sum_{k\in\left\{0,\ldots,\left\lfloor \frac{n}{2}\right\rfloor\right\}\setminus\mathfrak{I}} \Theta_T \mathsf{e}_k^{+,\varepsilon} \Rightarrow \sum_{k\in\left\{0,\ldots,\left\lfloor \frac{n}{2}\right\rfloor\right\}\setminus\mathfrak{I}} \Theta_T \mathsf{e}_k^{+} = \Theta_T \mathcal{X}. \tag{6.317}$$

Since the complement of the open ball $\mathsf{D}_\delta(0)$ is a closed subset of $\mathcal{C}\left([0,\infty),\mathbb{R}^m\right)$, application of *Portmanteau's theorem*[32] yields

$$\limsup_{\varepsilon\to 0} \mathbb{P}\left(\Theta_T \mathcal{X}^\varepsilon \notin \mathsf{D}_\delta(0)\right) \leq \mathbb{P}\left(\Theta_T \mathcal{X} \notin \mathsf{D}_\delta(0)\right), \tag{6.318}$$

i.e. there is an $\varepsilon_*^{(T)} > 0$, s.t. for all $0 < \varepsilon < \varepsilon_*^{(T)}$ we have

$$\mathbb{P}\left(\Theta_T \mathcal{X}^\varepsilon \notin \mathsf{D}_\delta(0)\right) \leq \mathbb{P}\left(\Theta_T \mathcal{X} \notin \mathsf{D}_\delta(0)\right) + \frac{\Delta}{2}, \tag{6.319}$$

which together with Eq. (6.314) gives rise to Eq. (6.310). In order to prove Eq. (6.311), we just need to make use of $\mathsf{d}(f,g) = \mathsf{d}(|f-g|,0)$, in order to obtain that

$$\mathsf{d}\left(\sum_{k\in\mathfrak{I}} \Theta_T \mathsf{e}_k^{+,\varepsilon}, 1\right) = \mathsf{d}\left(1 - \sum_{k\in\mathfrak{I}} \Theta_T \mathsf{e}_k^{+,\varepsilon}, 0\right) = \mathsf{d}\left(\sum_{k\in\left\{0,\ldots,\left\lfloor \frac{n}{2}\right\rfloor\right\}\setminus\mathfrak{I}} \Theta_T \mathsf{e}_k^{+,\varepsilon}, 0\right). \tag{6.320}$$

Employing Eq. (6.310), we can thus conclude that

$$\mathbb{P}\left(\sum_{k\in\mathfrak{I}} \Theta_T \mathsf{e}_k^{+,\varepsilon} \in \mathsf{D}_\delta(1)\right) = 1 - \mathbb{P}\left(\sum_{k\in\left\{0,\ldots,\left\lfloor \frac{n}{2}\right\rfloor\right\}\setminus\mathfrak{I}} \Theta_T \mathsf{e}_k^{+,\varepsilon} \notin \mathsf{D}_\delta(0)\right) \geq 1 - \Delta. \tag{6.321}$$

□

[31] [Dur10], Theorem 3.2.4.
[32] [Dur10], Theorem 3.2.5.

7 Conclusion

7.1. Results

In this thesis we have investigated a general circulant network of weakly coupled oscillators, allowing for both deterministic and multiplicative-noise coupling.

We were able to exploit the scale hierarchy by proving an *averaging theorem* for the complex outer-product process. This result extends existing averaging principles by allowing for *resonant* oscillators as well as a class of certain non-bounded drift- and dispersion terms (Theorem 4.71).

Utilizing the averaging theorem has enabled us to provide a detailed characterization of the system's asymptotic evolution. A weak *deterministic* coupling, for instance, was shown to exert a twofold influence on the oscillator system (Section 6.1). A unidirectional momentum coupling and a symmetric (bidirectional) space coupling were seen to induce a periodic exchange of energy between the oscillators, along with a periodic evolution of the phase differences (Section 6.1.1, i.p. Proposition 6.1; Section 6.1.2). A symmetric momentum coupling and a unidirectional space coupling on the other hand were observed to give rise to an exponential amplification or decay of the system's *eigenmode amplitudes* (Remark 6.4).

These observations were extended to the case of a general interplay of deterministic and multiplicative-noise coupling topologies. In this setup, we have proven that the asymptotic evolution of the eigenmode amplitudes can be captured by exponential *growth rates* (Theorem 6.24 and Lemma 6.28). The magnitude of these growth rates has been characterized in terms of so-called *noise-coupling angles* which represent the relative strength of momentum- and space-like noise coupling (Fig. 6.2 and Remark 6.29). As a main result, the energy-normalized system was shown to pathwise *synchronize* to a superposition of those eigenmodes corresponding to the maximal growth rates (Theorem 6.33). This *eigenmode synchronization* was proven to asymptotically exhibit an 'exponentially fast' rate of convergence, thereby conforming with a generalized notion of *complete stochastic synchronization* (Remark 6.34).

As an application of this synchronization theorem, we have provided a complete classification of the synchronization states induced by a general *distance-l* noise coupling of n oscillators (Theorem 6.45). This has allowed us to identify critical numbers of oscillators (Lemma 6.44) which determine whether or not in-phase synchronization can be achieved. In particular, we have proven that a nearest-neighbor coupling is unable to induce an in-phase synchronization in a ring of $n = 13$ or more oscillators (c.f. Theorem 6.45 and Table 6.1), thereby providing a noise-coupling analogue to the deterministic result of [CM09]. On the other hand, a global all-to-all noise coupling was seen to always induce in-phase synchronization in a network of more than two oscillators (Example 6.46).

We were furthermore able to observe a multitude of *averaging effects*, in particular generalizing the ones outlined in [BE11]. We haven proven that linear coupling terms involving the 'real-part' or the 'imaginary-part' of the state variable, effectively amount to a complex linear coupling of the averaged system (Corollary 5.4). In the same spirit, we have observed that an additive noise corresponding to a *real-valued* Brownian motion, effectively acts on the averaged system as a (rescaled) *complex* Brownian motion (Remark 5.23). Moreover, we have determined a class of *divergence free*, nonlinear perturbations (Definition 3.20) which yield a vanishing contribution to the effective system (Lemmas 5.7 and 5.8). These perturbations were constructed in terms of antihermitian, nonlinear coupling matrices, whose components are given by holomorphic and antisymmetric functions. As a special subclass, we have identified *nonlinear pair-coupling* matrices (Remark 3.22) which in the case of a sinusodial weight function give rise to a complex Kuramoto-like coupling (Example 3.23).

Finally, all of the presented results on noise-induced synchronization have been shown to persist in the presence of an isotropic *additive-noise perturbation* (Theorems 6.61 and 6.63).

7.2. Outlook

The research presented in this thesis offers numerous aspects that merit further investigations.

We have considered a system of *identical* oscillators. Following [AW02], this can naturally be generalized to the case of oscillators with *rationally dependent* frequencies. As a key step, one could employ a state vector $x(t)$ of the form presented in Eq. (1.11).[1] By virtue of this definition, one could retrace the steps of Section 3.1.2, in order to show that the conserved quantities of the uncoupled system (c.f. [AW02], Eqs. (69)-(71)) can again be identified with the components of the complex outer-product process $x(t)x(t)^\dagger$.

The additive noise under consideration was chosen to be *isotropic*, representing a 'uniform' environmental-noise perturbation. This can be generalized to a *circulant additive noise*,[2] which under the discrete Fourier transform simplifies to a diagonal noise term. Furthermore, one could investigate effects of a *directed* noise-coupling.[3]

We have analyzed *circulant* coupling graphs, i.e. weighted and directed graphs whose adjacency matrix is given by a *circulant matrix*. We have seen that such coupling matrices can be diagonalized by means of a discrete Fourier transform. This enabled us to investigate rotationally invariant coupling topologies for *rings* of oscillators. Extending these concepts, one can analyze *circulant hypergraphs* ([CQ13]), i.e. coupling graphs corresponding to *circulant tensors* ([CQ13]). These circulant tensors can be diagonalized by a *multidimensional* discrete Fourier transform ([RE11]). Such a generalization would allow us study to *cyclic lattices* ([Mic07],[FS14]) of oscillators.

[1] c.f. [AW02], Eq. (67) for the general n-dimensional case

[2] i.e. to a circulant additive-noise dispersion matrix

[3] Recall that we have studied directed coupling topologies in the context of the deterministic coupling, while assuming that the noise-coupling vector is even.

Future work could also focus on extending the class of admissible nonlinear perturbations. For this purpose, one in particular has to impose certain restrictions to ensure the non-explosion of the resulting stochastic process.[4] Regarding the evolution of the effective system, one could employ the fact that the averaging procedure has a linearizing effect on a general holomorphic coupling function (c.f. Proposition 5.2).

Finally, systems with a more involved scale hierarchy could be studied by iterated application of the averaging principle. For instance, we have shown that a weakly coupled two-oscillator system exhibits a periodic exchange of energy (Section 6.1.2), i.e. it can be interpreted as an 'energy'-oscillator. If on a still weaker scale, several of such two-oscillator systems are now coupled to one another, one could employ a second averaging procedure in order to investigate the system of weakly coupled 'energy'-oscillators.

[4]In this thesis this is achieved by studying functions which satisfy a *weak coercivity* condition.

A Nomenclature

A.1. Spaces, measures and metrics

Space	Description	Reference
$\mathbb{N}$	natural numbers, including zero	
$\mathbb{Z}$	integers	
$\mathbb{Z}_n := \mathbb{Z}/(n\mathbb{Z})$	cyclic group of order n, generated by cyclic permutations; represented by the set $\{0, ..., n-1\}$	Def. 2.24
D_n	dihedral group generated by cyclic permutations and the reflection mapping	Sec. 1.2.1
S_n	symmetric group of all permutations of n elements	Sec. 1.2.1
$\mathbb{R}$	real numbers	
$\mathbb{C}$	complex numbers	
$\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$	field, representing real or complex numbers	
$\mathbb{K}^n$	space of n-dim. vectors with components in $\mathbb{K}$	
$\mathbb{K}^{n,m}$	space of $n \times m$ matrices with components in $\mathbb{K}$	

Table A.1.: Basic spaces and discrete symmetry groups

Space	Description	Reference
$\Gamma := (\mathbb{R}^N / \sim_{\mathsf{P}})$	quotient space induced by P-equivalence	Eq. (4.91)
$\Gamma_{\mathbb{C}} := (\mathbb{C}^n / \sim_{\mathsf{p}})$	quotient space induced by p-equivalence	Eq. (4.98)
I_Γ^l	partition of quotient space Γ, i.e. $\Gamma = \bigcup_{l=0}^n I_\Gamma^l$	Eq. (4.104)
$I_{\mathsf{P}}^l := \psi^{-1}(I_\Gamma^l)$	partition of $\mathsf{P}(\mathbb{R}^N)$, i.e. $\mathsf{P}(\mathbb{R}^N) = \bigcup_{l=0}^n I_{\mathsf{P}}^l$	Eq. (4.105)
$R^l := \pi^{-1}(I_\Gamma^l)$	partition of $\mathbb{R}^N$, i.e. $\mathbb{R}^N = \bigcup_{l=0}^n R^l$	Def. 4.36

Table A.2.: Quotient spaces and partitions

Set	Description	Reference
$S^1 := \{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$	unit circle	
$\mathbb{C}_* := \mathbb{C} \setminus \{0\}$	complex numbers without origin	
$K_r := \{z \in \mathbb{C} \mid \lvert z\rvert < r\}$	disc of radius r	
$K_{*,r} := K_r \cap \mathbb{C}_*$	disc of radius r without origin	Eq. (2.57)
$\operatorname{supp}(f)$	support of a function f	
▱	k-dimensional parallelepiped in $\mathbb{R}^N$	Eq. (4.116)
$\widehat{▱}$	matrix representation of ▱	Eq. (4.117)
$▱_\varphi$	rotated version of the parellelepiped ▱	Eq. (4.122)
$\mathcal{S}$	standard $(n-1)$-simplex	Def. 6.30
C_l	convex subset of $\mathcal{S}$, spanned by l'th unit-vector pair	Def. 6.32
$C_{\mathfrak{J}}$	convex subset of $\mathcal{S}$, spanned by pairs indexed in $\mathfrak{J}$	Def. 6.32
$\mathfrak{J} \subset \{0, ..., \lfloor \frac{n}{2} \rfloor\}$	index set, representing index chosen pairs	Def. 6.32
$\mathfrak{J}_0 := \{0, n/2\} \cap \mathbb{N}$	all indices $k \in \{0, ..., n-1\}$, s.t. $k \equiv n - k \bmod n$	Def. 6.32
$\mathfrak{J}^{(n)}$	set of all indices $m^* \in \{1, ..., n-1\} \setminus \{n/2\}$ which maximize the factor $\cos^2\left(\frac{2\pi}{n} m\right)$	Lem. 6.40
$\mathfrak{J}^{(l,n)}$	set of all indices $k^* \in \{1, ..., n-1\} \setminus \{n/2\}$ which maximize the factor $\cos^2\left(\frac{2\pi\, k\, l}{n}\right)$	Def. 6.41, Prop. 6.43

Table A.3.: Sets

Measure	Description	Reference
μ	Lebesgue measure on $\mathbb{R}^N = \mathbb{R}^{2n}$ or $\mathbb{C}^n$	
$\pi_*\mu$	measure on Γ, defined as the pushforward of the Lebesgue measure under the map π	Lem. 4.47
$\mathsf{P}_*\mu$	measure on $\mathsf{P}(\mathbb{R}^N)$, defined as the pushforward of the Lebesgue measure under the map P	
$\mathcal{H}_k$	k-dimensional Hausdorff measure	Lem. 4.40
$\mathbb{P}$	probability measure of implicit probability space $(\Omega, \mathcal{F}, \mathbb{P})$	
$\mu_\varepsilon := \mathbb{P}(Y_0^\varepsilon)^{-1}$	initial distribution of the process $(Y^\varepsilon(t))_{t\geq 0}$	Prop. 4.67
$\zeta_\varepsilon := \frac{\mathrm{d}\mu_\varepsilon}{\mathrm{d}\mu}$	density of the initial condition with respect to the Lebesgue measure	Prop. 4.67
$\widehat{\mu}$	weak limit of the initial distributions μ_ε, as $\varepsilon \to 0$	Prop. 4.67
ζ	density of $\pi_*\widehat{\mu}$, i.e. $\frac{\mathrm{d}\widehat{\mu}}{\mathrm{d}\mu} = \zeta \circ \pi$	Prop. 4.67

Table A.4.: Measures

Space	Description	Reference
$\mathrm{Lin}_{\mathbb{K}}(U,V)$	set of $\mathbb{K}$-linear maps $f: U \to V$, where U,V are vector spaces over $\mathbb{K}$	Lem. 2.3
$C(\mathbb{R}^N)$	set of real-valued, continuous functions on $\mathbb{R}^N$	
$C^k(\mathbb{R}^N)$	set of real-valued, k-times continuously differentiable functions, defined on $\mathbb{R}^N$	
$C_c^k(\mathbb{R}^N)$	functions from $C^k(\mathbb{R}^N)$ with compact support	Thm. 4.25
$C^k(\Gamma)$	set of real-valued, k-times continuous differentiable functions, defined on Γ	Lem. 4.49
$(\mathcal{C}([0,\infty),\mathbb{R}^m), \mathsf{d}(\cdot,\cdot))$	space of continuous $\mathbb{R}^m$-valued paths with metric d	Def. 6.64
$(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$	real Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$	Sec. 4.2
$\left(L^2(\mathbb{R}^N), \langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^N)}\right)$	space of (equivalence classes of) square-integrable functions with its canonical scalar product	Sec. 4.2
$\left(L^2_{\pi_*\mu}(\Gamma), \langle\cdot,\cdot\rangle_{L^2_{\pi_*\mu}(\Gamma)}\right)$	L^2-space on Γ defined via pullback π_*	Def. 4.50, Lem. 4.52
$\left(L^2_{\mathsf{P}_*\mu}(\mathsf{P}(\mathbb{R}^N)), \langle\cdot,\cdot\rangle_{L^2_{\mathsf{P}_*\mu}}\right)$	L^2-space on $\mathsf{P}(\mathbb{R}^N)$ defined via pullback P_*	Eq. (4.148)
$\left(H^1(\mathbb{R}^N), \langle\cdot,\cdot\rangle_{H^1(\mathbb{R}^N)}\right)$	Sobolev space $W^{k,p}$ in the case of $k=1$ and $p=2$	Rem. 4.26
$\left(H^1_{\pi_*\mu}(\Gamma), \langle\cdot,\cdot\rangle_{H^1_{\pi_*\mu}(\Gamma)}\right)$	H^1-space on Γ defined via pullback π_*	Eq. (4.145)
$\left(\mathcal{D}(E_\varepsilon), \tilde{E}^1_\varepsilon(\cdot,\cdot)\right)$	domain of the Dirichlet form E_ε, equipped with inner product arising from $\tilde{E}^1_\varepsilon$	Thm. 4.25
$\left(\mathcal{D}(\mathcal{E}), \tilde{\mathcal{E}}^1(\cdot,\cdot)\right)$	domain of the projected Dirichlet form $\mathcal{E}$, equipped with inner product arising from $\tilde{\mathcal{E}}^1$	Lem. 4.52
$\left(\mathcal{D}(\mathcal{E}_{\mathsf{P}}), \tilde{\mathcal{E}}^1_{\mathsf{P}}(\cdot,\cdot)\right)$	domain of the projected Dirichlet form $\mathcal{E}_{\mathsf{P}}$, equipped with inner product arising from $\tilde{\mathcal{E}}^1_{\mathsf{P}}$	Eq. (4.147)
$\mathcal{D}(\mathcal{A})$	domain of the infinitesimal generator $\mathcal{A}$	Prop. 4.59
$\mathcal{D}(\mathcal{A}_{\mathsf{P}}) \coloneqq \psi_*(\mathcal{D}(\mathcal{A}))$	domain of the infinitesimal generator $\mathcal{A}_{\mathsf{P}}$	Cor. 4.60

Table A.5.: Function spaces, metrics and inner products

Element	Description	Reference
i,j,k,l	indices from the set $\{0,...,n-1\}$	
e_l	l'th unit vector of $\mathbb{R}^n$, where $l=0,...,n-1$	Def. 6.32
z	generic element of $S^1 \subset \mathbb{C}$	
x,y	generic elements of $\mathbb{C}^n$	
X,Y	generic elements of $\mathbb{R}^N$, e.g. $X=\mathsf{j}(x), Y=\mathsf{j}(y)$	
$\hat{P}$	generic element of $\mathsf{P}(\mathbb{R}^N) \subset \mathbb{C}^{n,n}$	

Element	Description	Reference
$\hat{p}$	generic element of $\mathsf{p}(\mathbb{C}^n)$	
y_*	generic element of $\mathsf{p}^{-1}(\hat{p})$, where $\hat{p} \in \mathsf{p}(\mathbb{C}^n)$	

Table A.6.: Elements

A.2. Maps and operators

Object	Description	Reference
$\mathbb{1}_{n\times n}$	$n \times n$ unit matrix	
$\delta_{k,l}$	Kronecker delta; $\delta_{k,l} = 1$ if $k = 0$, while $\delta_{k,l} = 0$ otherwise	
$\epsilon_{a,b,c}$	Levi-Civita symbol	Eq. (3.27)
$I := \mathfrak{J}(i\,\mathbb{1}_{n\times n})$	$\mathbb{R}^{2n,2n}$-representation of multiplying an element of $\mathbb{C}^n$ by the imaginary unit i	Eq. (2.15)
$\operatorname{cycl}(a) := (a_{k-l})_{k,l}$	circulant $\mathbb{C}^{n,n}$ matrix arising from a vector $a \in \mathbb{C}^n$	Eq. (2.69)
$\mathfrak{u} := \exp\left(\frac{2\pi i}{n}\right)$	n-th unit root	Sec. 2.2.1
$Q := \frac{1}{\sqrt{n}}\left(\mathfrak{u}^{k\cdot l}\right)_{k,l}$	transformation matrix of the inverse DFT	Eq. (2.60), Eq. (2.66)
$R := (\delta_{k,n-l})_{k,l} = Q^2$	reflection matrix	Eq. (2.79), Lem. 2.29
$\mathcal{P}_\pm := \frac{1}{2}(\mathbb{1}_{n\times n} \pm R)$	projection matrices to the spaces of even and odd vectors	Eq. (2.80)
$\mathcal{Q}_k := \mathbb{1}_{k\times k} \otimes Q$	block-diagonal matrix, containing k copies of Q	Eq. (3.175)
$\mathcal{R}_k := \mathbb{1}_{k\times k} \otimes R$	block-diagonal matrix, containing k copies of R	Eq. (3.177)
$U(n)$	Lie group of all unitary $n \times n$ matrices	Eq. (3.12)
$SU(n)$	Lie group of all special unitary $n \times n$ matrices	Eq. (3.18)
$u(n)$	Lie algebra corresponding to the Lie group $U(n)$	Sec. 3.1.2.2
$su(n)$	Lie algebra corresponding to the Lie group $SU(n)$	Sec. 3.1.2.2
$\mathsf{T}^{(0)}$	infinitesimal generator of $U(1)$; element of $u(1)$	Sec. 3.1.2.2
$\mathsf{T}^{(a)}$, $a = 1, ..., n^2 - 1$	infinitesimal generators of $SU(n)$; elements of $su(n)$	Eq. (3.20)
$\sigma^{(a)}$, $a = 1, 2, 3$	Pauli matrices	Ex. 3.3
$\mathsf{T}^{(1;k,l)}, \mathsf{T}^{(2;k,l)}, \mathsf{T}^{(3;k)}$	specific choice of $SU(n)$ generators	Eq. (3.46)

Table A.7.: Matrices, tensors and continuous symmetry groups

Relation	Description	Reference
$A^\top$	transpose of matrix A, i.e. $A^\top_{k,l} := A_{l,k}$	
$\overline{A}$	complex conjugate of matrix A	
$A^\dagger$	hermitian conjugate of matrix A, i.e. $A^\top_{k,l} := \overline{A}_{l,k}$	
$(A + \text{h.c.})$	shorthand for $A + A^\dagger$	Eq. (2.109)
$\tilde{x} := Q^\dagger x$	DFT of a vector-valued quantity	Def. 2.23
$\tilde{A} := Q^\dagger A\, Q$	DFT of a matrix-valued quantity	Lem. 2.27
$x^\dagger\, y = \sum_k \overline{x}_k y_k$	complex scalar product of vectors $x, y \in \mathbb{K}^d$	Lem. 2.10
$xy^\dagger$	complex outer product of vectors $x, y \in \mathbb{K}^d$ which gives a $\mathbb{K}^{d,d}$ matrix defined by $(xy^\dagger)_{k,l} := x_k \overline{y}_l$	Def. 2.11
$\text{tr}(A) := \sum_k A_{k,k}$	trace of matrix A	
$A : B := \text{tr}(A^\dagger B)$	Frobenius product of matrices A and B	Def. 2.8
$A \otimes B$	Kronecker product of matrices A and B	Lem. 3.37
$[A, B] := A\,B - B\,A,$	commutator of matrices A and B	Def. 2.4
$\{A, B\} := A\,B + B\,A.$	anticommutator of matrices A and B	Def. 2.4
$a \circledast b$	circular convolution of the vectors a and b defined by $(a \circledast b)_k := \sum_{l=0}^{n-1} a_{k-l}\, b_l$	Eq. (2.68)
$x \odot y$	pointwise multiplication (Hadamard product) of the vectors a and b, defined by $(x \odot y)_k := x_k\, y_k$	Eq. (2.71)
$\left\langle M, N^\dagger \right\rangle (t)$	covariation process of two square-integrable martingales $(M(t))_{t\geq 0}, (N(t))_{t\geq 0}$	Def. 2.32

Table A.8.: Matrix operations and product constructions

Decomposition	Description	Reference
$M = M_+ + M_-$	decomposition of an $\mathbb{R}^{N,N}$ matrix into a $\mathbb{C}$-linear and $\mathbb{C}$-antilinear part via $M_\pm := \frac{1}{2}\,(M \mp IMI)$	Lem. 2.6
$M = M^{(+)} + M^{(-)}$	decomposition of an $\mathbb{R}^{N,N}$ matrix into a symmetric and antisymmetric part via $M^{(\pm)} := \frac{1}{2}\left(M \pm M^\top\right)$	Lem. 4.31
$x = x_+ + x_-$	decomposition of a $\mathbb{C}^n$ vector into an even and odd part via $x_\pm := \mathcal{P}_\pm x$	Eq. (2.80)
$\mathcal{E} = \tilde{\mathcal{E}} + \check{\mathcal{E}}$	decomposition of a bilinear form $\mathcal{E}$ into symmetric and antisymmetric part via $\tilde{\mathcal{E}}(f, g) := \frac{1}{2}\,(\mathcal{E}(f, g) + \mathcal{E}(g, f))$	Def. 4.14
$f = f^+ + f^-$	decomposition of a real-valued function into positiv and negativ part via $f^\pm := \frac{1}{2}\,(f \pm \|f\|)$	Def. 4.27

Table A.9.: Decompositions

Map	Description	Reference
$\mathsf{j}^{(n)} : \mathbb{C}^n \to \mathbb{R}^{2n}$	identifies a complex $\mathbb{C}^n$ vector with a real $\mathbb{R}^{2n}$ vector	Eq. (2.1)
$\mathsf{j}_{\#}^{(n',n)}$	identifies a map between complex spaces with a map between the corresponding real spaces	Eq. (2.2)
$\mathfrak{J}^{(n,n')} : \mathbb{C}^{n,n'} \hookrightarrow \mathbb{R}^{2n,2n'}$	identifies a $\mathbb{C}^{n,n'}$ matrix with a representing $\mathbb{R}^{2n,2n'}$ matrix, which for $n = n'$ commutes with I	Eq. (2.3)
$\breve{\mathfrak{J}}^{(n,n')} : \mathbb{C}^{n,n'} \hookrightarrow \mathbb{R}^{2n,2n'}$	identifies a $\mathbb{C}^{n,n'}$ matrix with a representing $\mathbb{R}^{2n,2n'}$ matrix, which for $n = n'$ anticommutes with I	Eq. (2.4)
$\mathfrak{J}_{1/2}^{(n,n')} : \mathbb{C}^{n,n'} \to \mathbb{R}^{2n,n'}$	identifies a $\mathbb{C}^{n,n'}$ matrix with a representing $\mathbb{R}^{2n,n'}$ matrix	Eq. (2.5)
$\widehat{\psi}_{\mathbb{K}}^{(d)} : \mathbb{K}^{d,d} \to \mathbb{K}^{d^2}$	identifies a matrix with a representing vector	Def. 2.2
$\bmod n : \mathbb{Z} \to \mathbb{Z}_n$	maps an integer to its representative in $\mathbb{Z}_n$, which we have identified with the set $\{0, ..., n-1\}$	Def. 3.9
$\mathcal{F} : \mathbb{C}^n \to \mathbb{C}^n$	discrete Fourier transformation	Eq. (2.65)
$d : \mathbb{Z}^2 \to \mathbb{Z}_n$	directed coupling distance	Eq. (3.66)
$d_s : \mathbb{Z}^2 \to \mathbb{Z}_n$	symmetric coupling distance	Eq. (3.67)
$\mathsf{p} : \mathbb{C}^n \to \mathbb{C}^{n,n}$	complex outer-product map	Eq. (2.107)
$\mathsf{P} : \mathbb{R}^N \to \mathbb{R}^m$	real-valued representation of complex outer product map	Def. 4.30
$\pi : \mathbb{R}^N \to \Gamma$	projection onto quotient space Γ	Eq. (4.92)
$\pi_{\mathbb{C}} := \psi_{\mathbb{C}} \circ \mathsf{p} : \mathbb{C}^n \to \Gamma_{\mathbb{C}}$	projection onto complex quotient space $\Gamma_{\mathbb{C}}$	Eq. (4.100)
$\psi : \mathsf{P}(\mathbb{R}^N) \to \Gamma$	identifies the image $\mathsf{P}(\mathbb{R}^N)$ with the quotient space Γ	Lem. 4.34
$\psi_{\mathbb{C}} : \mathsf{p}(\mathbb{C}^n) \to \Gamma_{\mathbb{C}}$	identifies the image $\mathsf{p}(\mathbb{C}^n)$ with the quotient space $\Gamma_{\mathbb{C}}$	Eq. (4.99)
$\pi_* : \mathbb{R}^{\Gamma} \to \mathbb{R}^{\mathbb{R}^N}$	pullback function; lifts a function defined on the quotient space Γ to a function defined on $\mathbb{R}^N$	Eq. (4.94)
$\mathsf{P}_* : \mathbb{R}^{\mathsf{P}(\mathbb{R}^N)} \to \mathbb{R}^{\mathbb{R}^N}$	pullback function; lifts a function defined on the image $\mathsf{P}(\mathbb{R}^N)$ to a function defined on $\mathbb{R}^N$	Eq. (4.149)
$n(\mathsf{c}, a)$	winding number of c around a	Eq. (2.59)
$\mathrm{Res}\|_{z_0}(h)$	residue of h at z_0	Thm. 2.20
$\beta : \mathsf{p}(\mathbb{C}^n) \to \mathbb{C}^n$	decomposition of complex outer product	Lem. 5.21
$\phi : \mathbb{N} \to \mathbb{N}$	Euler's totient function	Eq. (6.191)
Θ_T	time shift	Eq. (6.292)

Table A.10.: Maps

Relation	Description	Reference
$\propto$	proportional to	Eq. (3.70)
$\sim_{\mathsf{P}}$	equivalence relation on $\mathbb{R}^N$, which identifies all elements within a connected component of a P level set	Def. 4.33
$\sim_{\mathsf{p}}$	equivalence relation on $\mathbb{C}^n$ which identifies all elements that only differ by a global phase factor	Lem. 4.35
$(X(t))_{t\geq 0} \sim (Y(t))_{t\geq 0}$	asymptotic equivalence of the stochastic processes $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$	Def. 6.22
$(X(t))_{t\geq 0} \lesssim (Y(t))_{t\geq 0}$	process $(Y(t))_{t\geq 0}$ asymptotically dominates $(X(t))_{t\geq 0}$	Def. 6.22

Table A.11.: Relations

Operator	Description	Reference
$\dot{x}(t) := \frac{\mathrm{d}}{\mathrm{d}t} x(t)$	first order time derivative of a function $x(t)$	
$\ddot{x}(t) := (\frac{\mathrm{d}}{\mathrm{d}t})^2 \, x(t)$	second order time derivative of a function $x(t)$	
∇	Nabla operator; vector of partial derivatives ∇_k in $\mathbb{R}^{2n}$	Def. 2.14
$\nabla^{(\mathrm{Re})} := (\nabla_k)_{k=0,\dots,n-1}$	vector of 'first n' partial derivatives in $\mathbb{R}^{2n}$	Def. 2.14
$\nabla^{(\mathrm{Im})} := (\nabla_k)_{k=n,\dots,2n-1}$	vector of 'last n' partial derivatives in $\mathbb{R}^{2n}$	Def. 2.14
$\overline{\nabla} := I \, \nabla$	skew Gradient	Eq. (1.75)
$\partial := \frac{1}{2}(\nabla^{(\mathrm{Re})} - i \, \nabla^{(\mathrm{Im})})$	'complex' Nabla operator; vector of partial complex derivatives ∂_k	Eq. (2.40)
$\overline{\partial} := \frac{1}{2}(\nabla^{(\mathrm{Re})} + i \, \nabla^{(\mathrm{Im})})$	vector of partial complex conjugate derivatives $\overline{\partial}_k$	Eq. (2.40)
$\nabla\nabla^\top$	real Hessian operator	Def. 2.14
$\partial\partial^\dagger$	complex Hessian operator	Def. 2.14
$\overleftarrow{\nabla}$	Nabla operator acting to the left	Def. 2.14
$\overleftarrow{\partial}$	complex Nabla operator acting to the left	Def. 2.14
$\mathrm{D}\,\mathsf{P}$	Jacobian matrix, defined by $(\mathrm{D}\,\mathsf{P})_{k,l} := \nabla_l \mathsf{P}_k$	Eq. (4.83)
$\mathrm{D}_k\,\mathsf{P}$	k-dimensional Jacobian	Def. 4.41
$\mathrm{D}_k^\Gamma\,\mathsf{P}$	k-dimensional Jacobian on Γ	Def. 4.46
∇_{P}	Nabla operator on $\mathsf{P}(\mathbb{R}^N)$	Lem. 4.55

Table A.12.: Derivatives

A.3. Parameters

Parameter	Description	Reference
n	number of oscillators; system evolves in $\mathbb{C}^n$	Sec. 3.1.1
$N := 2n$	real-valued system evolves in $\mathbb{R}^{2n}$	Def. 4.4
$N' := 8n$	dimension of combined Brownian motion $\boldsymbol{B}$	Def. 4.4
$m := N^2$	number of components of $\mathsf{P}(X) \in \mathbb{R}^{N,N} \cong \mathbb{R}^m$	Def. 4.30
d	dimension of squared Bessel process $\beta^{(x)}$	Def. 6.51

Table A.13.: Dimensions

Parameter	Description	Reference
$\mathfrak{m}$	mass of oscillating particles	Sec. 3.1.1
K	strength of harmonic oscillator coupling	Sec. 3.1.1
$\kappa := \sqrt{\frac{\mathsf{K}}{\mathfrak{m}}}$	frequency of harmonic oscillation	Sec. 3.1.1
$x_k(0) = \lvert x_k(0)\rvert\, \mathrm{e}^{i\phi_k}$	initial condition of k'th oscillator	Eq. (3.7)
ϕ_k	initial phase of the k'th oscillator	Eq. (3.7)
ε	scaling parameter, which captures 'weakness' of interactions between oscillators	Sec. 3.5
$\breve{A}$	pair-coupling matrix for unrescaled system	Eq. (3.59)
A	pair-coupling matrix (for rescaled system)	Eq. (3.60)
$\alpha_k := \sum_{j=0}^{n-1} A_{kj}$	row sums of pair-coupling matrix A	Eq. (3.64)
$L := A - \operatorname{diag}(\alpha)$	Laplacian matrix	Eq. (3.65)
$\mathfrak{l}$	space-coupling vector for deterministic coupling	Ass. 3.10
$L = \operatorname{cycl}(\mathfrak{l})$	circulant space-coupling matrix	Ass. 3.10
$\mathfrak{l}'$	momentum-coupling vector for deterministic coupling	Ass. 3.14
$L' = \operatorname{cycl}(\mathfrak{l}')$	circulant momentum-coupling matrix	Ass. 3.14
$\lambda := \mathfrak{l}' - i\,\mathfrak{l}$	combined deterministic-coupling vector	Def. 3.16
$\Lambda := \operatorname{cycl}(\lambda)$	circulant deterministic-coupling matrix	Def. 3.16
$\mathfrak{n}$	space-coupling vector for multiplicative noise	Def. 3.24
$\mathfrak{n}'$	momentum-coupling vector for multiplicative noise	Def. 3.24
$\nu := \mathfrak{n}' - i\,\mathfrak{n}$	combined multiplicative-noise coupling vector	Def. 3.24
$\mathcal{V} := \operatorname{cycl}(\nu)$	circulant multiplicative-noise coupling matrix	Def. 3.24
γ_k	noise-coupling angles, defined by $\nu_k = \lvert\nu_k\rvert\, \mathrm{e}^{i\,\gamma_k}$	Def. 3.26

Parameter	Description	Reference
$\tilde{\gamma}_k$	noise-coupling angles for transformed system, defined by $\widetilde{\nu}_k = \lvert\widetilde{\nu}_k\rvert\, \mathrm{e}^{i\,\tilde{\gamma}_k}$	Def. 3.29
σ_r	strength of regularizing noise	Def. 3.31
σ_0	strength of additive noise	Def. 3.33
$\mathsf{c}_k := \frac{\lvert\widetilde{\nu}_k\rvert^2}{\lvert\widetilde{\nu}_k\rvert^2+\sigma_r^2}$	relative strength of multiplicative and regularizing noise	Eq. (6.52)
$\mathsf{f}_k \propto (\mathcal{P}_-\tilde{\mathfrak{l}})_k$	asymmetry factor, proportional to odd component of transformed space-coupling vector	Eq. (6.89)
ξ_k	asymptotic growth rates	Thm. 6.24

Table A.14.: Coupling parameters

Parameter	Description	Reference
$\delta^{(\hat{r})}$	upper bound on precision with which $\psi_k^{(1)}$ approximates the time average of the ratio process	Lem. 6.60
$\sigma_0^* = \sigma_0^*(\delta^{(\hat{r})})$	upper bound on admissible additive noise strength for achieving precision $\delta^{(\hat{r})}$ in ratio process approximation	Lem. 6.60
δ	'spatial precision', i.e. radius of admissible deviation from synchronization state w.r.t. metric d	Thm. 6.68
Δ	'stochastic precision'; with probability of at least $1-\Delta$ the system synchronizes with spacial precision δ	Thm. 6.68

Table A.15.: Precision parameters

A.4. Drift terms, dispersion and diffusion matrices

Object	Description	Reference
$H(x) := \frac{\kappa}{2}\,\lVert x\rVert^2$	Hamiltonian of uncoupled system	Eq. (3.10)
$u(x) := u_{\mathrm{lin}}(x) + u_{\mathrm{nl}}(x)$	drift term for $x^\varepsilon(t)$ evolution, capturing deterministic interactions	Eq. (3.54)
$u_{\mathrm{lin}}(x) := \mathrm{Re}(\lambda \circledast x)$	linear part of drift term $u(x)$	Eq. (3.161)
$u_{\mathrm{nl}}(x)$	nonlinear part of drift term $u(x)$	
$u_{\mathsf{p}}(\tilde{y})$	drift term for evolution of complex outer-product process $\mathsf{p}(\tilde{y}^\varepsilon(t))$	Eq. (3.182)

Object	Description	Reference
$\widehat{u}_{\mathsf{p}}(\hat{p})$	averaged version of u_{p}	Eq. (4.171), Eq. (4.230)
$\widehat{u}_{\mathsf{p},\mathrm{det}}(\hat{p})$	contribution from deterministic interactions to $\widehat{u}_{\mathsf{p}}$	Eq. (4.231)
$\widehat{u}_{\mathsf{p},\mathrm{Itô}}(\hat{p})$	contribution from Itô correction to $\widehat{u}_{\mathsf{p}}$	Eq. (4.232)
$V(X) \coloneqq \mathfrak{J}(i)$	real-valued representation of drift induced by Hamiltonian $H(x)$	Eq. (4.9)
$U(X) \coloneqq \mathsf{j} \circ \tilde{u} \circ \mathsf{j}^{-1}(X)$	drift term for $Y^{\varepsilon}(t)$ evolution; real-valued representation of $\tilde{u}$	Eq. (4.10)
$U_{\mathsf{P}}(Y)$	real-valued representation of u_{p}; drift term for $\mathsf{P}(Y^{\varepsilon})(t)$ evolution	Eq. (4.154)
$\widehat{U}_{\mathsf{P}}(\hat{P})$	averaged version of U_{P}, defined on $\mathsf{P}(\mathbb{R}^N)$	Eq. (4.157)
$\widehat{U}_{\Gamma}(\gamma)$	averaged version of U_{P}, defined on Γ	Eq. (4.159)
$b_k(r)$	drift term for evolution of ratio process $\hat{r}_k(t)$	Eq. (6.86)

Table A.16.: Hamiltonian and drift terms

Object	Description	Reference
$\boldsymbol{\sigma}(x)$	dispersion matrix for $x^{\varepsilon}(t)$ evolution	Eq. (3.163)
$\sigma_{\mathrm{mult}}(x)$	multiplicative noise part of $\boldsymbol{\sigma}(x)$	Eq. (3.164)
$\boldsymbol{\sigma}_{\mathrm{reg}}(x)$	regularizing noise part of $\boldsymbol{\sigma}(x)$	Eq. (3.165)
$\boldsymbol{\sigma}_{\mathrm{add}}$	additive noise part of $\boldsymbol{\sigma}(x)$	Eq. (3.166)
$\sigma_{\mathsf{p}}(\tilde{y})$	dispersion term for evolution of complex outer-product process $\mathsf{p}(\tilde{y}^{\varepsilon}(t))$	Eq. (3.183)
$\widehat{a}_{\mathsf{p}}(\hat{p})$	averaged version of $\sigma_{\mathsf{p}}{\sigma_{\mathsf{p}}}^{\dagger}$	Eq. (4.172)
$\widehat{a}'_{\mathsf{p}}(\hat{p})$	averaged version of $\sigma_{\mathsf{p}}{\sigma_{\mathsf{p}}}^{\top}$; reordered version of $\widehat{a}_{\mathsf{p}}$	Eq. (4.172), Eq. (4.176)
$\widehat{a}_{\mathsf{p},\mathrm{mult}}(\hat{p})$	multiplicative noise part of $\widehat{a}_{\mathsf{p}}$	Eq. (4.237)
$\widehat{a}_{\mathsf{p},\mathrm{reg}}(\hat{p})$	regularizing noise part of $\widehat{a}_{\mathsf{p}}$	Eq. (4.237)
$\widehat{a}_{\mathsf{p},\mathrm{add}}(\hat{p})$	additive noise part of $\widehat{a}_{\mathsf{p}}$	Eq. (4.237)
$\Sigma(Y)$	real-valued representation of $\widetilde{\sigma}$; dispersion term for $Y^{\varepsilon}(t)$ evolution	Eq. (4.11)
$A(Y) \coloneqq (\Sigma\Sigma^{\top})(Y)$	diffusion matrix corresponding to Σ	Eq. (4.14)
$\Sigma_{\mathsf{P}}(Y)$	real-valued representation of σ_{p}; dispersion term for $\mathsf{P}(Y^{\varepsilon})(t)$ evolution	Eq. (4.155)
$A_{\mathsf{P}}(Y) \coloneqq (\Sigma_{\mathsf{P}}{\Sigma_{\mathsf{P}}}^{\top})(Y)$	diffusion term for $\mathsf{P}(Y^{\varepsilon})(t)$ evolution	Eq. (4.156)

Object	Description	Reference
$\widehat{A}_{\mathsf{P}}(\hat{P})$	averaged version of A_{P}, defined on $\mathsf{P}(\mathbb{R}^N)$	Eq. (4.158)
$\widehat{A}_{\Gamma}(\hat{P})$	averaged version of A_{P}, defined on Γ	Eq. (4.160)
$a_k(r)$	diffusion term for evolution of ratio process $\hat{r}_k(t)$	Eq. (6.87)

Table A.17.: Dispersion and diffusion matrices

Function	Description	Reference
$\mathfrak{D}(X)$	diagonal matrix for uniform ellipticity bound	Eq. (4.16)
$\mathfrak{d}(x)$	vector generating $\mathfrak{D}(X)$	Eq. (4.17)
$\Phi(X)$	vector field for $Y^\varepsilon(t)$ evolution without ε-dependent term	Eq. (4.52)
$\Phi_\varepsilon(X)$	vector field for $Y^\varepsilon(t)$ evolution including ε-dependent term	Eq. (4.53)
$\mathsf{s}_k(r)$	scale function of ratio process $\hat{r}_k(t)$	Eq. (6.88)
$m_k(r)$	speed measure density for ratio process $\hat{r}_k(t)$	Eq. (6.94)
$\breve{m}_k(r)$	normalized speed measure density	Eq. (6.97)
$\mathsf{t}(r, \tilde{\gamma}_k)$	$\tanh^{-1}$ or $\tan^{-1}$ factor, depending on sign of $\cos(2\tilde{\gamma}_k)$	Eq. (6.90)

Table A.18.: Functions depending on drift and diffusion terms

A.5. Random variables, processes and Dirichlet forms

Random variable	Description	Reference
$S_k^{(m)}$	(first) exit time of the ratio process $\hat{r}_k(t)$ from the interval (l_m, r_m)	Eq. (6.102)
S_k	exit time of the ratio process $\hat{r}_k(t)$ from the interval $(-1, +1)$	Eq. (6.103)
$\psi_k^{(1)}$	asymptotic time average of $\hat{r}_k(t)$	Eq. (6.110a)
$\psi_k^{(2)}$	asymptotic time average of $[\hat{r}_k(t)]^2$	Eq. (6.110b)
C_k	upper bound on time change $\tau_k(t)$	Lem. 6.55
$\check{\beta}_k$	lower bound on time-changed squared Bessel process $\beta_k^{(1)}(\tau_k(t))$	Lem. 6.56
$\hat{\beta}_k$	upper bound on time-changed squared Bessel process $\beta_k^{(1)}(\tau_k(t))$	Lem. 6.56

Random variable	Description	Reference
$\tilde{C}_k$	upper bound on time change $\tau_k^{\hat{\rho}^{(+,\sigma_0)}}(t)$	Lem. 6.59

Table A.19.: Random variables

Process	Description	Reference
$\theta_k(t), \theta'_k(t)$	position of k'th pair of strongly coupled massive particles, which together constitute the k'th oscillator	Sec. 3.1.1
$\check{\eta}_k(t) := \theta_k(t) - \theta'_k(t)$	elongation of k'th oscillator (uncoupled system)	Sec. 3.1.1
$\check{p}_k(t) := \mathfrak{m}\,\dot{\check{\eta}}_k(t)$	momentum of k'th oscillator (uncoupled system)	Sec. 3.1.1
$\eta_k(t) := \sqrt{\frac{\mathsf{K}}{2}}\,\check{\eta}_k(t)$	space-rescaled version of $\check{\eta}_k(t)$	Sec. 3.1.1
$p_k(t) := \sqrt{\frac{1}{2\mathfrak{m}}}\,\check{p}_k(t)$	space-rescaled version of $\check{p}_k(t)$	Sec. 3.1.1
$x(t) := p(t) + i\,\eta(t)$	$\mathbb{C}^n$-valued process, describing uncoupled oscillator system	Eq. (3.4)
$c^{(a)}(t) := x^\dagger(t)\mathsf{T}^{(a)}x(t)$	conserved quantity of the uncoupled system, corresponding to the symmetry generated by $\mathsf{T}^{(a)}$	Eq. (3.30)

Table A.20.: Stochastic processes describing uncoupled oscillator system

Process	Description	Reference
$\check{\eta}_k^\varepsilon(t)$	elongation of k'th oscillator (weakly coupled system)	Sec. 3.2.1.1
$\check{p}_k^\varepsilon(t)$	momentum of k'th oscillator (weakly coupled system)	Sec. 3.2.1.1
$\eta_k^\varepsilon(t) := \sqrt{\frac{\mathsf{K}}{2}}\,\check{\eta}_k^\varepsilon(t)$	space-rescaled version of $\check{\eta}_k^\varepsilon(t)$	Sec. 3.2.1.1
$p_k^\varepsilon(t) := \sqrt{\frac{1}{2\mathfrak{m}}}\,\check{p}_k^\varepsilon(t)$	space-rescaled version of $\check{p}_k^\varepsilon(t)$	Sec. 3.2.1.1
$x^\varepsilon(t) := p^\varepsilon(t) + i\,\eta^\varepsilon(t)$	$\mathbb{C}^n$-valued process, describing weakly coupled oscillator system	Sec. 3.2.1.1
$\tilde{y}^\varepsilon(t) := \tilde{x}^\varepsilon(\frac{t}{\kappa\,\varepsilon})$	time-rescaled version of $\tilde{x}^\varepsilon(t)$	Lem. 3.38
$\mathsf{p}^\varepsilon(t) := \mathsf{p}(\tilde{y}^\varepsilon(t))$	complex outer-product process	Prop. 3.39
$\hat{\rho}_k^\varepsilon(t) := \mathsf{p}_{k,k}^\varepsilon(t)$	'amplitude' (squared absolute value) of k'th eigenmode	Eq. (6.306)
$\rho_k^{+,\varepsilon}(t)$	sum of eigenmode amplitudes for an index pair $k \neq n-k$	Eq. (6.307)
$\mathsf{e}_k^\varepsilon(t)$	energy normalized k'th eigenmode amplitude	Eq. (6.308)
$\mathsf{e}_k^{+,\varepsilon}(t)$	sum of energy normalized eigenmode amplitudes for an index pair $k \neq n-k$	Eq. (6.309)

Process	Description	Reference
$Y^\varepsilon(t)$	real-valued representation of $\tilde{y}^\varepsilon(t)$	Eq. (4.6)
$\mathsf{P}^\varepsilon(t) \coloneqq \mathsf{P}(Y^\varepsilon(t))$	real-valued representation of $\mathsf{p}^\varepsilon(t)$	Eq. (4.191)
$\pi^\varepsilon(t) \coloneqq \pi(Y^\varepsilon(t))$	representation of $\mathsf{p}^\varepsilon(t)$ in Γ	Eq. (4.191)

Table A.21.: Stochastic processes describing weakly coupled oscillator system

Process	Description	Reference
$\widehat{\mathsf{p}}(t)$	weak limit of the processes p^ε, as $\varepsilon \to 0$	Thm. 4.71
$\widehat{\mathsf{P}}(t)$	real-valued representation of $\widehat{\mathsf{p}}(t)$; weak limit of the processes P^ε, as $\varepsilon \to 0$	Cor. 4.70
$\widehat{\pi}(t)$	representation of $\widehat{\mathsf{p}}(t)$ in Γ; weak limit of the processes π^ε, as $\varepsilon \to 0$	Prop. 4.58
$\hat{\rho}_k(t) \coloneqq \hat{p}_{k,k}(t)$	'amplitude' (squared absolute value) of k'th eigenmode of averaged system	Def. 6.3
$E(t) \coloneqq \mathrm{tr}(\hat{p}(t))$	total energy of oscillator system	Def. 6.3
$\hat{\rho}_k^\pm(t) \coloneqq \hat{\rho}_k(t) \pm \hat{\rho}_{n-k}(t)$	sum and difference of eigenmode amplitudes for an index pair $k \neq (n-k)$	Def. 6.6
$\hat{r}_k(t) \coloneqq \frac{\hat{\rho}_k^-(t)}{\hat{\rho}_k^+(t)}$	ratio process	Def. 6.6
$\mathsf{e}_k(t) \coloneqq \frac{\hat{\rho}_k(t)}{E(t)}$	energy normalized k'th eigenmode amplitude	Def. 6.30
$\mathsf{e}_k^+(t) \coloneqq \mathsf{e}_k(t) + \mathsf{e}_{n-k}(t)$	sum of energy normalized eigenmode amplitudes for an index pair $k \neq (n-k)$	Def. 6.30
$\hat{\rho}_k^{(+,\sigma_0)}(t)$	notation for $\hat{\rho}_k^+(t)$ with explicit reference to dependence on additive noise strength σ_0	Def. 6.47
$\hat{r}_k^{(\sigma_0)}(t)$	notation for $\hat{r}_k(t)$ with explicit reference to dependence on additive noise strength σ_0	Def. 6.47
$\hat{\rho}_k^{(+,\mathrm{h})}(t)$	version of $\hat{\rho}_k^{(+,0)}(t)$ which depends on $\hat{r}_k^{(\sigma_0)}(t)$	Def. 6.48
$\hat{r}_k^{(\mathrm{TC})}(t)$	time-changed version of $\hat{r}_k^{(\sigma_0)}(t)$	Eq. (6.263)

Table A.22.: Stochastic processes describing averaged oscillator system

Brownian motion	Description	Reference
$\boldsymbol{B}(t)$	$5n$-dim. real-valued Brownian motion, representing combined noise influence on oscillator system	Eq. (3.167)
$B_{\mathrm{mult}}(t)$	n-dim. real-valued Brownian motion; multiplicative noise part of $\boldsymbol{B}(t)$	Eq. (3.132)
$\boldsymbol{B}_{\mathrm{reg}}(t)$	$2n$-dim. real-valued Brownian motion; regularizing noise part of $\boldsymbol{B}(t)$	Eq. (3.146)

Brownian motion	Description	Reference
$B_{\text{reg}}(t)$	'first' n components of $\boldsymbol{B}_{\text{reg}}(t)$	Eq. (3.146)
$B'_{\text{reg}}(t)$	'last' n components of $\boldsymbol{B}_{\text{reg}}(t)$	Eq. (3.146)
$\boldsymbol{B}_{\text{add}}(t)$	$2n$-dim. real-valued Brownian motion; additive noise part of $\boldsymbol{B}(t)$	Eq. (3.153)
$B_{\text{add}}(t)$	'first' n components of $\boldsymbol{B}_{\text{add}}(t)$	Eq. (3.153)
$B'_{\text{add}}(t)$	'last' n components of $\boldsymbol{B}_{\text{add}}(t)$	Eq. (3.153)
$\widetilde{\boldsymbol{B}}(t) \coloneqq \mathcal{Q}_5^{\dagger}\boldsymbol{B}(t)$	blockwise DFT of Brownian motion $\boldsymbol{B}(t)$	Eq. (3.175)
$(\hat{B}_{\text{add}})_k(t)$	real-valued BM arising from B_{add} and B'_{add}	Eq. (6.40)
$(\breve{B}_{\text{hom}})_k(t)$	real-valued BM arising from B_{mult}, B'_{mult}, B_{reg}, B'_{reg}; $(\breve{B}_{\text{hom}})_k$ and $(\breve{B}_{\text{hom}})_{n-k}$ are *not* independent	Lem. 6.5, Lem. 6.8
$(\breve{B}^{\pm}_{\text{hom}})_k(t)$	real-valued BM arising from $(\breve{B}_{\text{hom}})_k, (\breve{B}_{\text{hom}})_{n-k}$; $(\breve{B}^{+}_{\text{hom}})_k$ and $(\breve{B}^{-}_{\text{hom}})_k$ are *not* independent	Eq. (6.55), Eq. (6.80)
$(\breve{B}^{\pm}_{\text{add}})_k(t)$	real-valued BM arising from $(\hat{B}_{\text{add}})_k, (\hat{B}_{\text{add}})_{n-k}$; $(\breve{B}^{+}_{\text{add}})_k$ and $(\breve{B}^{-}_{\text{add}})_k$ are *not* independent	Eq. (6.56), Eq. (6.81)
$(\breve{B}^{r}_{\text{hom}})_k$	real-valued BM arising from $(\breve{B}^{+}_{\text{hom}})_k, (\breve{B}^{-}_{\text{hom}})_k$	Eq. (6.77)
$(\breve{B}^{r}_{\text{add}})_k$	real-valued BM arising from $(\breve{B}^{+}_{\text{add}})_k, (\breve{B}^{-}_{\text{add}})_k$	Eq. (6.78)
$\breve{B}^{r}_k(t)$	real-valued BM arising from $(\breve{B}^{r}_{\text{hom}})_k(t), (\breve{B}^{r}_{\text{add}})_k(t)$	Eq. (6.261)
$\breve{M}^{+}_k(t)$	real-valued martingale arising from $(\breve{B}^{+}_{\text{hom}})_k$	Prop. 6.20

Table A.23.: Martingales

Process	Description	Reference
$\beta^{(1)}_k(t)$	squared Bessel process starting at $x = 1$	Prop. 6.53
$\tau_k(t)$	time-change which is applied to $\beta^{(1)}_k(t)$	Eq. (6.232)
$\tau^{\sigma_0}_k(t)$	time-change used for representing $\hat{r}^{(\sigma_0)}_k$ in terms of $\hat{r}^{(0)}_k$	Eq. (6.257)
$\tau^{\hat{\rho}^{(+,\sigma_0)}}_k(t)$	increment between $\tau^{\sigma_0}_k(t)$ and the identity time-change	Eq. (6.258)

Table A.24.: Time-changes and (squared) Bessel processes

Object	Description	Reference
$(T^{\varepsilon}(t))_{t>0}$	strongly continuous contraction semigroup corresponding to $(Y^{\varepsilon}(t))_{t\geq 0}$	Thm. 4.11
$\mathcal{A}_{\varepsilon}$	infinitesimal generator corresponding to $(Y^{\varepsilon}(t))_{t\geq 0}$	Def. 4.2

Object	Description	Reference
$E_\varepsilon(\cdot,\cdot)$	Dirichlet form corresponding to $\mathcal{A}_\varepsilon$	Def. 4.12
$E_\varepsilon^\alpha(\cdot,\cdot)$	defined by $E_\varepsilon^\alpha(f,g) \coloneqq E_\varepsilon(f,g) + \alpha \langle f,g\rangle_{L^2(\mathbb{R}^N)}$	Def. 4.20
$\mathcal{E}(\cdot,\cdot)$	projected Dirichlet form, defined by $\mathcal{E}(f,g) \coloneqq E_\varepsilon(\underline{f},\underline{g})$	Lem. 4.51
$\mathcal{E}^\alpha(\cdot,\cdot)$	defined by $\mathcal{E}^\alpha(f,g) \coloneqq \mathcal{E}(f,g) + \alpha \left\langle \underline{f},\underline{g}\right\rangle_{L^2(\mathbb{R}^N)}$	Def. 4.20
$\mathcal{E}_{\mathsf{P}}(\cdot,\cdot)$	projected Dirichlet form, related to $\mathcal{E}(\cdot,\cdot)$ by $\mathcal{E}_{\mathsf{P}}(f,g) \coloneqq \mathcal{E}(f\circ\psi^{-1}, g\circ\psi^{-1})$	Lem. 4.54
$(T(t))_{t\geq 0}$	contraction semigroup associated to $\mathcal{E}(\cdot,\cdot)$	Prop. 4.59
$\mathcal{A}$	infinitesimal generator associated to $\mathcal{E}(\cdot,\cdot)$	Prop. 4.59
$\mathcal{A}_{\mathsf{P}}$	infinitesimal generator associated to $\mathcal{E}_{\mathsf{P}}(\cdot,\cdot)$	Cor. 4.60

Table A.25.: Semigroups, generators and Dirichlet forms

Notation	Description	Reference
“direct quote”	marks a direct quote from a cited source	
‘heuristic expression’	marks an expression which is not to be interpreted literally, but rather as a heuristic description	
ODE	ordinary differential equation	
SDE	stochastic differential equations	
BM	Brownian motion	
DFT	discrete Fourier transform	

Table A.26.: Quotation signs and shorthands

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