Quasi-hereditary algebras and the geometry of representations of algebras

Dissertation zur Erlangung des Doktorgrades
der Fakultät für Mathematik
der Universität Bielefeld

vorgelegt von
Ögmundur Eiríksson

Juni 2018

Abstract

Chapter 2: We construct two functors from the submodule category of a self-injective representation-finite algebra Λ to the module category of the stable Auslander algebra of Λ. They factor through the module category of the Auslander algebra of Λ. Moreover they induce equivalences from the quotient categories of the submodule category modulo their respective kernels and said kernels have finitely many indecomposable objects up to isomorphism. We show how this interacts with an idempotent recollement of the module category of the Auslander algebra of Λ, and get a characterisation of the self-injective Nakayama algebras as a byproduct.

Chapter 3: We recall how dense GL_d-orbits in quiver flag varieties correspond to rigid objects in monomorphism categories. In order to identify rigid objects via the AR-formula we show how the AR-translate of a representation category of a quiver can be used to calculate the AR-translates of objects in the monomorphism categories of the corresponding path algebra. We also illustrate other methods to find rigid objects in monomorphism categories; via a long exact sequence, and so called Ext-directed decompositions.

Chapter 4: We introduce the notion of a quiver-graded Richardson orbit, generalising the notion of a dense orbit of a parabolic subgroup of GL_d acting on the nilpotent radical of its Lie algebra. In this generalised setting dense orbits do not exist in general. We introduce the nilpotent quiver algebra, which is simultaneously left strongly quasi-hereditary and right ultra strongly quasi-hereditary. We show there is a one-to-one correspondence between rigid objects in the subcategory of standard filtered modules up to isomorphism and quiver-graded Richardson orbits.

Acknowledgements

First and foremost I would like to thank my supervisor Dr. Julia Sauter for support and advice throughout my PhD studies. Additionally I thank Dr. Teresa Conde for useful observations and suggestions, Prof. Claus M. Ringel for useful discussions and suggestions and my de jure supervisor Prof. Henning Krause for proofreading and advice on practical matters. Finally I want to thank an unnamed referee of my article [28] for careful proofreading and many suggestions for improvements.

Contents

Introduction

The representation theory of finite-dimensional algebras may be considered as a generalisation of classical linear algebra. These studies go back to the latter half of the 19th century with the theory of semi-simple algebras. In the latter half of the 20th century there has been a systematic effort in the studies of finite dimensional modules over finite dimensional algebras over fields, or more generally finitely generated modules over Artin algebras. This effort has introduced powerful categorical methods, but despite them the tame-wild dichotomy shows that classifications in representation categories is in many cases not possible. Hence it is often necessary to add extra conditions, or to restrict to particular algebras to classify representations. To this end various special families of finite dimensional algebras have been studied. These include the path algebras if quivers, Nakayama algebras, self-injective algebras and quasi-hereditary algebras, to name a few.

Quasi-hereditary algebras came up in an effort by Cline-Parshall-Scott [16] to stratify derived categories of rational representations of semi-simple algebraic groups. The algebras themselves are found in [58], but a Morita equivalent version was developed independently by Donkin [27]. Soon it became clear that those algebras are abundant and interesting in many more situations, and in particular they have been extensively studied in the representation theory of finite dimensional algebras, after much foundational work by Dlab-Ringel [25, 26, 51].

The Auslander algebra of the truncated polynomial ring. The truncated polynomial ring $\Lambda := k[x]/\langle x^N \rangle$ has many of the properties above, being a commutative, representation finite and self-injective Nakayama algebra. The category of finitely generated modules over Λ is well understood, and since it is representation finite we can define the Auslander algebra Γ of Λ. It is the endomorphism ring of the finite dimensional module given as the direct sum of representatives for the isomorphism classes of indecomposable modules. The module category of Γ is understood to a large extent; like all Auslander algebras, Γ comes with a quasi-hereditary structure, and in this case the quasi-hereditary structure is unique. The main results of this thesis are inspired by certain aspects of the category Γ-mod of finitely generated Γ-modules.

Firstly consider the category of morphisms of Λ-modules. It has as objects morphisms between finitely generated Λ-modules, the submodule cate-

gory $\mathcal{S}(\Lambda)$ is the full subcategory given by the monomorphism. Ringel-Zhang [55] described a functor α from the homomorphism category to Γ-mod. By either restricting α to the submodule category or pre-composing with the cokernel functor on the submodule category we get two different functors from the submodule category to Γ-mod. Ringel-Zhang compose these with a functor from Γ-mod to the module category of the preprojective algebra Π of type A_{N-1}, this functor is an adjoint to a functor given by an epimorphism of rings $\Gamma \to \Pi$. It turns out that the compositions can be described in terms of the objects they kill, and those objects are easy to describe. In this case there is a connection to the quasi-hereditary structure on Γ, the essential image of α restricted to the submodule category is the subcategory $\mathcal{F}(\Delta)$ of modules filtered by standard modules of the quasi-hereditary structure on Γ.

Another aspect comes from actions of parabolic subgroups of the general linear group GL_d. Let $P \subset \mathrm{GL}_d$ be a parabolic subgroup and let $\mathfrak{n}$ denote the nilpotent radical of the Lie-algebra of P. Hille-Röhrle [38] and Brüstle-Hille-Ringel-Röhrle [12] showed how the dense orbit of P acting on $\mathfrak{n}$ corresponds to a rigid object of $\mathcal{F}(\Delta)$, whose dimension vector is determined by P. These dense orbits are called Richardson orbits.

We consider generalisations of the aspects above, where the different generalisations concentrate on different aspects. Hence they only overlap to a limited extent.

Other Auslander algebras. To generalise results of Ringel-Zhang [55] we simply take the Auslander algebra of general self-injective representation finite algebras and study accordingly generalised versions of the functors in [55]. Our results in this genralised setting have already been published in [28], and our coverage will follow that article closely. The generalised version of the functors from [55] are the functors $F, G \colon \mathcal{S}(\Lambda) \to \underline{\Gamma}$-mod, where $\underline{\Gamma}$ is the stable Auslander algebra of Λ, those are constructed in Section 2.4. Our generalisation of [55, Theorem 1] is the following.

Theorem 2.23. *Let Λ be a basic, self-injective and representation finite algebra and let m be the number of isomorphism classes of $\mathrm{ind}(\Lambda)$. Then $\ker(F)$ and $\ker(G)$ have $2m$ indecomposable objects up to isomorphism, moreover*

(i) *F induces an equivalence of categories $\mathcal{S}(\Lambda)/\ker(F) \to \underline{\Gamma}$-mod;*

(ii) *G induces an equivalence of categories $\mathcal{S}(\Lambda)/\ker(G) \to \underline{\Gamma}$-mod.*

We also investigate the interplay of the syzygy in the stable module category of the stable Auslander algebra with the functors that we generalise, a connection illustrated in Theorem 2.24.

In this generality the Auslander algebra does not have a unique quasi-hereditary structure, and we show that the property that $\mathcal{F}(\Delta)$ gives the essential image of the submodule category under α is actually something unique to the self-injective Nakayama algebras. More precisely we have the following:

Theorem 2.26. *Let Λ be a basic representation-finite algebra and let Γ be its Auslander algebra. Then Γ has a quasi-hereditary structure such that the full subcategory of torsionless modules is precisely the Δ-filtered Γ-modules if and only if Λ is uniserial.*

Quiver-graded Richardson orbits. In order to generalise the construction of Richardson orbits from [12] we construct the algebra $N_s(Q)$, for Q an arbitrary finite quiver and $s \in \mathbb{N}$. This algebra is given as the path algebra of a quiver $Q^{(s)}$ modulo specific relations. The quiver $Q^{(s)}$ is given by taking a linearly oriented quiver of type A_s for each vertex of Q, and adding $s-1$ arrows between them for each arrow of Q in a specific way. If Q is the Jordan quiver, then $N_s(Q)$ is actually the Auslander algebra of $k[x]/\langle x^s \rangle$, but note that $N_s(Q)$ is not an Auslander algebra in general. It has more similarities to the Auslander-Dlab-Ringel-algebras (shortened to ADR-algebras) studied by Conde and Erdmann [20, 21], and $N_s(Q)$ actually arises as an ADR-algebra if Q has no sinks. Despite being something of an ad hoc construction, $N_s(Q)$ arises as a tensor algebra, and it has a natural quasi-hereditary structure with similar properties as the quasi-hereditary structure of ADR-algebras. Fix a dimension vector $d \in \mathbb{N}^{Q_0}$ and a Q_0-tuple of flags $(F_i)_{i \in Q_0}$ of length s on the vector spaces $(k^{d_i})_{i \in Q_0}$. This determines a dimension filtration $\mathbf{d}$ of $d \in \mathbb{N}^{Q_0}$, i.e. $\mathbf{d}$ is a sequence of dimension vectors $\mathbf{d}^{(1)}, \ldots, \mathbf{d}^{(s)} = d$ with $\mathbf{d}^{(t)} \leq \mathbf{d}^{(t+1)}$ pointwise.

The flags determine a parabolic subgroup $P_{\mathbf{d}} \subset GL_d$, where $GL_d := \prod_{i \in Q_0} GL_{d_i}$. We can consider $\mathbf{d}$ as a dimension vector for $N_s(Q)$-modules in a canonical way. The parabolic group $P_{\mathbf{d}}$ acts on the closed subvariety $R_d^{\mathbf{d}}$ of the representation variety $\text{Rep}_d(Q)$, and we prove the following analogue of the correspondence in [12].

Theorem 4.30. *Consider $(N_s(Q), \Delta)$, where Δ is the canonical quasi-hereditary structure. The following are equivalent.*

(i) *There is a rigid Δ-filtered $N_s(Q)$-module of dimension $\mathbf{d}$.*

(ii) *There is a dense $P_{\mathbf{d}}$-orbit in $R_d^{\mathbf{d}}$.*

Thus $R_d^{\mathbf{d}}$ can be considered as a quiver graded version of $\mathfrak{n}$, we call dense $P_{\mathbf{d}}$-orbits of $R_d^{\mathbf{d}}$ quiver-graded Richardson orbits. In our more general setting, these dense orbits do not exist in general, as we will discuss in examples.

Representations fixing a flag. Monomorphism categories $\text{mon}_s(A)$ of an algebra A are a straightforward generalisation of submodule categories, and they can be realised as full subcategories of the categories of finitely generated modules over the ring of upper triangular matrices with coefficients in A, denoted by $T_s(A)$ for $s \times s$ matrices.

If Q is an acyclic quiver, Sauter [57] has shown that $T_s(Q)$ has a quasi-hereditary structure which has the monomorphism category as the category of

Δ-filtered modules. We recall this quasi-hereditary structure and show that if A is a quasi-hereditary algebra, then $T_s(A)$ is also quasi-hereditary. The standard filtered modules with respect to this structure are always included in the monomorphism categories but the other inclusion does not hold in general.

Now we restrict our attention to path algebras of acyclic quivers Q, this restriction means that what follows can not be considered as a generalisation of [12], because that setting would be correspond to taking the Jordan quiver.

Quiver Grassmannians are varieties parametrising the submodules of a given dimension vector of a fixed Q-representation M. More generally quiver flag varieties parametrise flags of submodules of a fixed module, thus generalising quiver Grassmannians along similar lines as monomorphism categories generalise submodule categories. We may consider the other end of this situation, hence we fix a flag of vector spaces at each vertex of Q, determining a dimension filtration $\mathbf{d}$ of $d \in \mathbb{N}^{Q_0}$. Then we study the closed subvariety $\mathrm{Rep}_d^{\mathbf{d}} \subset \mathrm{Rep}_d(Q)$ of representations fixing the flag, i.e. representations that make our distinguished flag a flag of submodules. The parabolic subgroup $P_{\mathbf{d}} \subset \mathrm{GL}_d$ fixing the flag acts on $\mathrm{Rep}_d^{\mathbf{d}}$, and we prove an analogue to Theorem 4.30 in this setting.

Theorem 3.7. *The following are equivalent:*

- *The variety $\mathrm{Rep}_d^{\mathbf{d}}$ has a dense $P_{\mathbf{d}}$-orbit.*

- *There exists a rigid object in $\mathrm{mon}_s(Q)$ of dimension vector $\mathbf{d}$.*

Note that for many $\mathbf{d}$ there is no dense orbit as in the Theorem. We are able to prove a slightly weaker version of Theorem 3.7 over algebras given by quivers with relations, this is Theorem 3.9. There the situation is more complicated, in particular the varieties involved are not necessarily irreducible.

Rigid modules. Theorem 3.7 is our motivation to try to identify or construct rigid objects in monomorphism categories. In this pursuit we are interested to calculate AR-translates of modules in the monomorphism category. The following theorem can bee seen as a refinement of a result by Ringel-Schmidmeier [54] and Xiong-Zhang-Zhang [66]. It gives the AR-translate τ_Γ of objects in the monomorphism category in terms of a functor τ'_Q given in a straightforward way by the relatively simple AR-translate of Q-representations, along with the cokernel functor on monomorphism categories.

Theorem 3.4. *Let $\mathbf{M} \in \mathrm{mon}_s(Q)$. Then*

$$\tau_\Gamma \, \mathbf{M} \cong \tau'_Q \, \mathbf{Cok} \, \mathbf{M}.$$

As a method to calculate extensions of $T_s(A)$-modules we introduce a long exact sequence, allowing us to break the calculations into calculating extension- and homomorphism-spaces of A-modules cf. Section 3.4.1.

We introduce Ext-directed decompositions as a tool to construct rigid objects in monomorphism categories. We show that for any A-module M that

has such a decomposition, for sufficiently large s there is a rigid object in $\mathrm{mon}_s(A)$ given by a flag of submodules of M.

Note that our neither our long exact sequence nor the construction based on Ext-directed partitions rely on A being a path algebra, it may be given by a quiver with relations.

Outline

The content of this thesis is organised as follows.

In Chapter 1 we present various preliminaries that are used in later chapters. All of them are well known by experts in the field and easily found in the literature. They are included here to fix notation and for the convenience of the reader.

The results of Chapter 2 were already published in the Jorunal of Algebra [28], they are presented here with only minor changes in organisation. We recall the theory of submodule categories and Auslander algebras. Then we introduce the generalised versions of the functors studied in [55] enabling us to state and prove Theorems 2.23 and 2.24, generalising Theorem 1 and Theorem 2 in [55] respectively. Finally we study quasi-hereditary structures of Auslander algebras in order to prove Theorem 2.26.

In Chapter 3 we start with introductions to monomorphism categories and quiver flag varieties. Then we outline connections between the homological properties of the monomorphism categories and the geometric properties of the quiver flag varieties, these are summarised in Theorem 3.7. We show how a quasi-hereditary structure on A induces one on $\mathrm{T}_s(A)$, and that if A is hereditary then $\mathcal{F}(\Delta)$ is the monomorphism category. We also consider approximations of monomorphism categories and use them to calculate AR-translates in monomorphism categories over hereditary algebras, slightly elaborating on results by Ringel-Schmidmeier [54] and Xiong-Zhang-Zhang [66]. Finally we give methods to construct rigid objects in monomorphism categories given certain conditions.

Chapter 4 mostly contains results from [29]. They have been chosen to emphasize the contributions of the author of this thesis while still maintaining continuity. In Section 4.2 we describe the variety $\mathrm{R}_d^{\mathbf{d}}$ and an action of a parabolic subgroup $\mathrm{P}_{\mathbf{d}}$ of GL_d on it. We give several equivalent conditions for this group action to act with a dense orbit, called a quiver-graded Richardson orbit, in Theorem 4.2. In Section 4.3 we construct the nilpotent quiver algebra $\mathrm{N}_s(Q)$ and the subcategory $\mathcal{N}$ of $\mathrm{N}_s(Q)$-mod. We show how rigid objects in $\mathcal{N}$ give Richardson orbits and vice versa. Section 4.4 is dedicated to a quasi-hereditary structure on $\mathrm{N}_s(Q)$, and to show how $\mathcal{N}$ is given by the standard filtered modules in $\mathrm{N}_s(Q)$-mod. There are also additions not included in [29], where we calculate the Ringel-dual of $\mathrm{N}_s(Q)$, and construct $\mathrm{N}_s(Q)$ as the non-negatively graded part of a graded endomorphism ring. Finally Section 4.5 has a small collection of examples to illustrate the theory.

Chapter 1

Preliminaries

This chapter does not contain any original work, but introduces various notions used in the later chapters of this thesis. It can be read as a whole but is mainly meant to fix notation and conventions, and for reference when reading the later chapters. Throughout the thesis k will denote a field. In general k can be arbitrary, but in some sections we add the assumption that k is algebraically closed or of characteristic 0.

We assume some general knowledge on modules, abelian and triangulated categories, and k-varieties, all notions widely covered in the literature.

1.1 Additive categories

Recall that an *additive category* is a category where all homomorphism sets have the structure of an abelian group, composition of maps is bilinear, and all finite (co)products exist. Note that this includes the empty (co)product, which is a zero object. We say a category is *k-linear* if all homomorphism sets are vector spaces over k and composition of maps is k-bilinear.

Let $\mathcal{A}$ be a k-linear additive category. We write $M \in \mathcal{A}$ to indicate that M is an object of $\mathcal{A}$, and we often write $(X,Y)_{\mathcal{A}} := \operatorname{Hom}_{\mathcal{A}}(X,Y)$ for shorthand. An *additive subcategory* of $\mathcal{A}$ is a full subcategory closed under taking finite direct sums and direct summands. For an object $M \in \mathcal{A}$ we let $\operatorname{add}(M)$ denote the smallest additive subcategory of $\mathcal{A}$ containing M. The *kernel* of an additive functor $F \colon \mathcal{A} \to \mathcal{B}$ is the full subcategory of all objects $X \in \mathcal{A}$ such that $F(X) = 0$. The *essential image* of F is the full subcategory of all objects $Y \in \mathcal{B}$ such that $Y \simeq F(X)$ for some $X \in \mathcal{A}$. Let $\mathcal{B}$ be a full subcategory of $\mathcal{A}$. For objects $X, Y \in \mathcal{A}$ we let $R_{\mathcal{B}}(X,Y) \subset (X,Y)_{\mathcal{A}}$ denote the subspace of maps that factor through an object in $\mathcal{B}$. The *quotient category* $\mathcal{A}/\mathcal{B}$ has the same objects as $\mathcal{A}$, and the homomorphism spaces are given by the vector space quotients $\operatorname{Hom}_{\mathcal{A}/\mathcal{B}}(X,Y) := (X,Y)_{\mathcal{A}}/R_{\mathcal{B}}(X,Y)$.

Let $\mathcal{A}$ be an abelian category, we say an additive subcategory $\mathcal{C}$ of $\mathcal{A}$ is *extension closed* if, for a short exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$, $X', X'' \in \mathcal{C}$ implies $X \in \mathcal{C}$.

We say an object X in $\mathcal{A}$ is *rigid* if $\operatorname{Ext}^1_{\mathcal{A}}(X,X) = 0$.

1.1.1 Approximations

The concepts of left and right approximations as well as covariantly and contravariantly finite classes of objects go back to Auslander-Smalo [7] and Enochs [30]. We will recall the definitions briefly, they are stated for additive categories even though the same definitions work for any category.

Let $\mathcal{A}$ be an additive category, a morphism $\varphi\colon X \to Y$ in $\mathcal{A}$ is *left minimal* if $\psi \circ \varphi = \varphi$ implies that ψ is an isomorphism. Dually we say φ is *right minimal* if $\varphi \circ \psi = \varphi$ implies ψ is an isomorphism.

Let $\mathcal{C}$ be a class of objects in $\mathcal{A}$. We say a map $\varphi\colon X \to Y$ in $\mathcal{A}$ is a left $\mathcal{C}$-approximation of X if Y belongs to $\mathcal{C}$ and the induced morphism $\mathrm{Hom}_{\mathcal{A}}(Y, Z) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$ is surjective for all $Z \in \mathcal{C}$. We say $\mathcal{C}$ is *contravariantly finite* in $\mathcal{A}$ if all objects in $\mathcal{A}$ have a left $\mathcal{C}$-approximation. Dually we say $\varphi\colon Y \to X$ is a right $\mathcal{C}$ approximation of X if Y belongs to $\mathcal{C}$ and the induced morphism $\mathrm{Hom}_{\mathcal{A}}(Z, Y) \to \mathrm{Hom}_{\mathcal{A}}(Z, X)$ is surjective for all objects $Z \in \mathcal{A}$. If all objects in $\mathcal{A}$ have a right $\mathcal{C}$ approximation we say $\mathcal{C}$ is *covariantly finite* in $\mathcal{A}$. We say $\mathcal{C}$ is *functorially finite* in $\mathcal{A}$ if it is both covariantly and contravariantly finite in $\mathcal{A}$.

We say $\varphi\colon X \to Y$ is a minimal left (resp. right) $\mathcal{C}$ approximation of X (resp. Y) if it is a left $\mathcal{C}$ approximation of X (resp. right $\mathcal{C}$ approximation of Y) and left (resp. right) minimal.

1.2 Module categories

Let A be an associative k-algebra with unit. We denote the category of left A-modules by A-Mod. Accordingly denote the category of right A-modules by Mod-A. We denote the full subcategory of finitely generated left (resp. right) A-modules by A-mod (resp. mod-A), if A is left (resp. right) noetherian it is an abelian subcategory. By a module we will always mean a finitely generated left-module, unless specified otherwise. From now on we assume A is a finite-dimensional algebra, our main reference for module categories of finite-dimensional algebras is [6]. The opposite algebra of A is denoted by A^{op}. The vector space duality $\mathrm{Hom}_k(-, k)$ induces a duality $\mathrm{D}\colon A^{\mathrm{op}}\text{-mod} \to A\text{-mod}$, [6, II.3]. Of course we can identify A^{op}-mod with mod-A. As with additive categories we often use the shorthand notation $(M, N)_A := \mathrm{Hom}_A(M, N)$.

Let M be an A-module, the *radical* $\mathrm{rad}(M)$ of M is the intersection of all maximal proper submodules of M. The *socle* $\mathrm{soc}(M)$ of M is the maximal semi-simple submodule of M. Dually, the *top* of M, denoted $\mathrm{top}(M)$, is the maximal semi-simple factor module of M, or equivalently the quotient $M/\mathrm{rad}(M)$. Since A is finite-dimensional, every finitely generated A-module M is finite-dimensional. Thus there is a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

of submodules of M such that $S_i := M_i/M_{i-1} \neq 0$ is simple for $i = 1, \ldots, n$, this is a *composition series* of M. By the Jordan-Hölder theorem the module

$S := \oplus_{i=1}^{n} S_i$ and the number n are uniquely determined up to isomorphism of S. We call the simple summands of S the *composition factors* of M and n the *length* of M.

Define the *radical filtration* of M inductively via

$$\operatorname{rad}_0 M := M, \quad \operatorname{rad}_{n+1} M := \operatorname{rad}(\operatorname{rad}_n M), \; n \in \mathbb{N}.$$

By the above observation $\operatorname{rad}_n = 0$ for n high enough, and we call the smallest n such that this holds the *Loewy-length* of M.

We say a module M in A-mod is *generated* by N if there exists an epimorphism $N^n \twoheadrightarrow M$ for some $n \in \mathbb{N}$. Dually we say M is *cogenerated* by N if there is a monomorphism $M \hookrightarrow N^n$ for some $n \in \mathbb{N}$. We denote by $\operatorname{gen}(N)$ (resp. $\operatorname{cogen}(N)$) the full subcategory of modules generated by N (resp. cogenerated by N).

We let $\operatorname{End}_A(M) := \operatorname{Hom}_A(M, M)$ denote the *endomorphism ring* of M, with multiplication given by the usual concatenation of maps $(f \circ g)(m) = f(g(m))$. Since A-mod is hom-finite, this is a finite-dimensional algebra. Observe that M is naturally a left $\operatorname{End}_A(M)$-module. We say a module M is *indecomposable* if $M \neq 0$ and $M \cong M_1 \oplus M_2$ implies that either $M \cong M_1$ or $M \cong M_2$. We say an algebra is *representation finite* if, for each $d \in \mathbb{N}$, there are only finitely many isomorphism classes of modules M of k-dimension d. For an abelian category $\mathcal{A}$ we let $\operatorname{ind}(\mathcal{A})$ denote the class of indecomposable objects in $\mathcal{A}$, and we write $\operatorname{ind}(A) := \operatorname{ind}(A\text{-mod})$ for a finite-dimensional algebra A. The following theorem is well known, for proof we refer to [6, Thm. II.2.2]

Theorem 1.1 (Krull-Remak-Schmidt)**.** *Let A be a finite-dimensional k-algebra. A finitely generated A-module M is indecomposable if and and only if $\operatorname{End}_A(M)$ is local.*

Let $(M_i)_{i \in I}$ and $(N_j)_{j \in J}$ be finite families of indecomposable finitely generated modules such that

$$\bigoplus_{i \in I} M_i \cong \bigoplus_{j \in J} N_j.$$

Then there is a bijection $\sigma \colon I \to J$ such that $M_i \cong N_{\sigma(i)}$ for all $i \in I$.

We say a module M is *basic* if all of its indecomposable summands are pairwise non-isomorphic. We write $_A A$, A_A or $_A A_A$, respectively, to indicate that we consider A as a left-, right- or bi-module, respectively, over itself. We say the algebra A is basic if $_A A$ is basic.

We denote the full subcategory of finitely generated projective (resp. injective) modules by A-proj (resp. A-inj). We call the quotient category $A\text{-}\underline{\mathrm{mod}} := A\text{-mod}/A\text{-proj}$ the *stable module category* of A. Dually we define the *costable module category* of A as $A\text{-}\overline{\mathrm{mod}} := A\text{-mod}/A\text{-inj}$. We denote the homomorphism spaces in $A\text{-}\underline{\mathrm{mod}}$ by $\underline{\operatorname{Hom}}_A(-, -)$, and those of $A\text{-}\overline{\mathrm{mod}}$ by $\overline{\operatorname{Hom}}_A(-, -)$.

We say A is *self-injective* if $_A A$ is an injective module. In this case A-proj $=$ A-inj. Thus A-mod is a *Frobenius category* , i.e. an abelian category where the injective and projective objects coincide.

The category A-mod has enough projective and injective objects. Hence each A-module M has a *projective cover* $\pi_M \colon P_M \to M$ and an *injective envelope* $\iota_M \colon M \to I_M$. Recall that the projective cover is minimal in the following sense. If $P' \subset P$ is a proper submodule of P, then $\pi_M(P')$ is a proper submodule of M. Dually the injective envelope is minimal in the sense that if $N \cap M = 0$ for a submodule N of I, then $N = 0$. Both I_M and P_M are unique up to a non-unique isomorphism. We call an exact sequence

$$P_1 \overset{\pi_1}{\to} P_0 \overset{\pi_0}{\to} M \to 0,$$

with P_0 and P_1 projective a *projective presentation* of M. A projective presentation is *minimal* if π_0 is a projective cover of M and π_1 is a projective cover of $\ker \pi_0$. An *injective co-presentation* of M is defined dually.

A *projective resolution* of M is an exact sequence

$$\cdots \overset{\pi_3}{\to} P_2 \overset{\pi_2}{\to} P_1 \overset{\pi_1}{\to} P_0 \overset{\pi_0}{\to} M \to 0,$$

with P_n projective for all $n \in \mathbb{N}_0$. Dually an *injective co-resolution* is an exact sequence

$$0 \to M \to I_0 \to I_1 \to I_2 \to \cdots ,$$

with I_n injective for all $n \in \mathbb{N}_0$. We define the *projective dimension* pdim M (resp. *injective dimension* idim M) of M as the maximal n such that there exists a projective resolution with $P_m = 0$ (resp. an injective co-resolution with $I_m = 0$) for all $m > n$. The *global dimension* of A, denoted gldim A, is the supremum of the projective dimension of all A-modules, or equivalently the supremum of all injective dimensions of A-modules.

We say an abelian category $\mathcal{A}$ is *hereditary* if $\operatorname{Ext}^n_{\mathcal{A}}(M, N) = 0$ for all objects $M, N \in \mathcal{A}$ and all $n \geq 2$. We say A is a *hereditary algebra* if the following equivalent conditions hold.

(1) A-mod is a hereditary category.

(2) Any submodule of a projective A-module is projective.

(3) A has global dimension ≤ 1.

We say two algebras A and B are *Morita equivalent* if the categories A-mod and B-mod are equivalent as abelian categories. Let $D(A)$ denote the derived category of A-mod, for the construction we refer to [36]. We say A and B are derived equivalent if $D(A)$ and $D(B)$ are equivalent as triangulated categories.

1.2.1 Auslander-Reiten theory

We still assume A is a finite-dimensional algebra.

Definition 1.2. Let $f\colon M \to N$ be a map in A-mod. We say f is *left almost split* if f is not a split monomorphism and any map $h\colon M \to L$ which is not a split monomorphism factors through f. *Right almost split* is defined dually. We say an exact sequence

$$0 \longrightarrow L \xrightarrow{\ f\ } M \xrightarrow{\ g\ } N \longrightarrow 0$$

is an almost split sequence (or AR-sequence) if f is left almost split and g is right almost split.

Let $M \in A$-mod and let $P_1 \xrightarrow{\pi_1} P_0 \xrightarrow{\pi_0} M \to 0$ be a minimal projective presentation of M. The duality gives an induced map $\mathrm{D}\,(\pi_1)\colon \mathrm{D}\,P_0 \to \mathrm{D}\,P_1$. We define the *transpose* of M as $\mathrm{Tr}\,M := \mathrm{coker}\,\mathrm{D}\,(\pi_1)$. The *Auslander-Reiten translate* (or AR-translate) on A-mod is defined as the composition $\tau := \mathrm{D}\,\mathrm{Tr}$.

We define the *inverse Auslander-Reiten translate* as $\tau^{-1} := \mathrm{Tr}\,\mathrm{D}$. We list some basic facts about this translate, for details and further information we refer to Chapters IV and V in [6].

Proposition 1.3. (1) *Let $M \in A$-mod be indecomposable. If M is not projective, then there exists an almost split exact sequence*

$$0 \longrightarrow \tau\,M \longrightarrow N \longrightarrow M \longrightarrow 0.$$

Dually, if M is not injective, then there is an almost split sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow \tau^{-1}M \longrightarrow 0.$$

(2) *The AR-translate τ induces an equivalence of categories:*

$$\tau\colon A\text{-}\underline{\mathrm{mod}} \to A\text{-}\overline{\mathrm{mod}}.$$

The inverse AR-translate τ^{-1} induces the inverse.

(3) *Let $M, N \in A$-mod and assume M has no projective summands. There are natural isomorphisms of vector spaces:*

$$\overline{\mathrm{Hom}}_A(N, \tau M) \cong \mathrm{D}\,\mathrm{Ext}^1_A(M, N),$$
$$\mathrm{D}\,\underline{\mathrm{Hom}}_A(M, N) \cong \mathrm{Ext}^1_A(N, \tau M).$$

These identities are collectively known as the Auslander-Reiten formula.

(4) *If we assume additionally that A is hereditary, then τ induces an endofunctor on A-mod isomorphic to the functor $\mathrm{D}\,\mathrm{Ext}^1_A(-, A)$.*

1.3 Representations of quivers

A *quiver* $Q = (Q_0, Q_1, s, t)$ consists of a finite set Q_0 of vertices and a finite set Q_1 of arrows along with maps $s, t \colon Q_1 \to Q_0$. For an arrow $a \in Q_1$ we call $s(a)$ the *source* of a and $t(a)$ the *target* of a, and we say a is an arrow from $s(a)$ to $t(a)$, written $a \colon s(a) \to t(a)$. We often suppress the notation for s and t when designating a quiver. We say an arrow a is a *loop* at i if $i = s(a) = t(a)$. A *path* in Q is is a sequence $a_n a_{n-1} \cdots a_2 a_1$ of arrows with $t(a_i) = s(a_{i+1})$ for $i = 1, \ldots, n-1$. We say such a path has *length* n. We say $s(a_n a_{n-1} \cdots a_2 a_1) := s(a_1)$ (resp. $t(a_n a_{n-1} \cdots a_2 a_1) := t(a_n)$) is the source (resp. target) of the path $a_n a_{n-1} \cdots a_2 a_1$. For each $i \in Q_0$ we have the trivial path e_i with $s(e_i) = t(e_i) = i$ of length 0. If α, β are paths with $t(\alpha) = s(\beta)$ then we write $\beta\alpha$ for the concatenation.

The *path algebra* kQ of Q is a k-vector space with all paths in Q as a basis. The multiplication on kQ is the bilinear map $kQ \times kQ \to kQ$ defined on paths via

$$\beta \cdot \alpha := \begin{cases} \beta\alpha & s(\beta) = t(\alpha); \\ 0 & \text{otherwise.} \end{cases}$$

This makes kQ an associative algebra with unit $1_Q := \sum_{i \in Q_0} e_i$. We say Q has an *oriented cycle* if there is a non-trivial path α in Q such that $s(\alpha) = t(\alpha)$. It is easy to check that kQ is finite-dimensional if and only if Q has no oriented cycles.

A *representation* M of Q is a Q_0-tuple $(M_i)_{i \in Q_0}$ of finite-dimensional k-vector spaces along with maps of vector spaces $M_a \colon M_i \to M_j$ for each arrow $(a \colon i \to j) \in Q_1$. Let M, N be representations of Q, a morphism $f \colon M \to N$ of representations is a Q_0-tuple of linear maps $f_i \colon M_i \to N_i$ such that $N_a \circ f_i = f_j \circ M_a$ for all $(a \colon i \to j) \in Q_1$. Of course the identity is a morphism of quiver representations, and concatenating morphisms of Q-representations gives another morphism of Q-representations. Hence quiver representations and morphisms of quiver representations form an additive k-linear category Q-rep. This category is equivalent to the full subcategory kQ-fdmod of finite-dimensional kQ-modules. If kQ is finite-dimensional, then this is the same as kQ-mod, and hence an abelian category. For the equivalence we refer to [6, III.1]. From now on we will often not distinguish between a finite-dimensional kQ-module and the corresponding Q-representation. For a Q-representation M we define the *dimension vector* of M as $\underline{\dim} M := (\dim_k M_i)_{Q_0}$. The equivalence of kQ-fdmod and Q-rep then gives a notion of a dimension vector of a kQ-module M, it is given by $\underline{\dim}(M) = (\dim_k e_i M)_{i \in Q_0}$.

A *quiver with relations* (Q, R) is a quiver Q along with a set R of elements in the path algebra Q, called the set of relations. We say a representation M of Q is a (Q, R)-representation if $rM = 0$ for all $r \in R$, where we consider M as a finite-dimensional kQ-module. We denote the ideal of kQ generated by the arrows by $\mathfrak{m}$, and we say an ideal I is *admissible* if there is n such that $\mathfrak{m}^n \subset \langle I \rangle \subset \mathfrak{m}^2$. We let $\langle R \rangle$ denote the two sided ideal in kQ generated by R. We say R is admissible if $\langle R \rangle$ is admissible, then $kQ/\langle R \rangle$ is finite-dimensional,

and so $kQ/\langle R \rangle$-mod is equivalent to the category (Q, R)-rep of representations of (Q, R). We will often not distinguish between a (Q, R)-representation and the corresponding $kQ/\langle R \rangle$-module.

The *opposite quiver* of Q, denoted by Q^{op}, has the vertex set Q_0 and the arrows $Q_1^{\mathrm{op}} = \{a^{\mathrm{op}} \mid a \in Q_1\}$, with $s(a^{\mathrm{op}}) = t(a)$ and $t(a^{\mathrm{op}}) = s(a)$. Then $k(Q^{\mathrm{op}}) \cong (kQ)^{\mathrm{op}}$.

Fix a quiver with admissible relations (Q, R), possibly with $R = \emptyset$, and $A := kQ/\langle R \rangle$. For each vertex $i \in Q_0$ we have the simple representation $S(i)$, it is given by $S(i)_i = k$ and $S(i)_j = 0$ for all $j \neq i$, and by taking the zero map for any arrow of Q_1. In our case all simple A-modules are of this form. The projective (resp. injective) representation $P(i)$ at i is the A-projective cover (resp. injective envelope) of $S(i)$. Observe that $\mathrm{top} P(i) \cong S(i) \cong \mathrm{soc} I(i)$, in particular these are indecomposable. We have the following decomposition of A as a module over itself

$$_A A \cong \bigoplus_{i \in Q_0} {}_A A e_i \cong \bigoplus_{i \in Q_0} P(i).$$

This shows A is a basic algebra, similarly the right A-module D A is basic.

If kQ is finite-dimensional, then $P(i)$ has as basis all paths beginning in i. This shows that all submodules of a projective kQ-module are projective, i.e. kQ is hereditary.

1.4 Algebraic group actions

In this section we assume k is an algebraically closed field. For basic properties of schemes and varieties we refer to [37]. All varieties considered here are k-varieties. In our convention k-varieties are given by the closed points of reduced, separated k-schemes of finite type. Note that we do not require varieties to be irreducible, so our convention is different from that of [37] and many other authors.

An *algebraic group* G is a k-variety G with a group structure such that both the multiplication $G \times G \to G$ and the map $G \to G$ taking the inverse are maps of algebraic varieties.

A *left G-action* on a variety X over k is a left group action $\rho \colon G \times X \to X$ which is a map of varieties, we write $g \cdot x := \rho(g, x)$. We call such X a *G-variety* . The *orbit* of $x \in X$ under G, denoted $G \cdot x$, is the subset $\{y \in X \mid \exists g \in G, \ g \cdot x = y\}$. When we talk about orbits $G \cdot v$ under a group operation, we always assume that the multiplication map $G \to G \cdot x, g \mapsto gx$ is separated. It is well known that the closure of any G-orbit is a union of G-orbits and each G-orbit of X is locally closed, i.e. an intersection of an open and a closed subset of X.

1.4.1 Representations varieties and their group actions

Let $Q = (Q_0, Q_1)$ be a quiver and let $\mathbf{d} = (d_i) \in \mathbb{N}_0^{Q_0}$ be a dimension vector. A Q_0-graded vector space is simply a Q_0-tuple of vector spaces. The homo-

morphisms between Q_0-graded vector spaces $U = (U_i)_{Q_0}$ and $V = (V_i)_{Q_0}$ are given by

$$\operatorname{Hom}_{Q_0}(U, V) = \bigoplus_{i \in Q_0} \operatorname{Hom}_k(U_i, V_i).$$

We let $k^{\mathbf{d}}$ denote the Q_0-graded vector space $\oplus_{i \in Q_0} k^{d_i}$. We write $\mathcal{M}_{n \times m}(k)$ for the variety of $n \times m$ matrices over k. For dimension vectors $\mathbf{d}$ and $\mathbf{e}$ we define

$$\mathcal{M}_{\mathbf{e}, \mathbf{d}}(k) := \prod_{i \in Q_0} \mathcal{M}_{e_i \times d_i}(k) = \operatorname{Hom}_{Q_0}(k^{\mathbf{d}}, k^{\mathbf{e}}).$$

The *representation space* of Q is the variety

$$\operatorname{Rep}_{\mathbf{d}}(Q) := \prod_{(a \colon i \to j) \in Q_1} \operatorname{Hom}_k(k^{d_i}, k^{d_j}) = \prod_{(a \colon i \to j) \in Q_1} \mathcal{M}_{d_j \times d_i}(k).$$

This parametrizes all Q-representations with the underlying Q_0-graded vector space $k^{\mathbf{d}}$. We let $\operatorname{Rep}_{\mathbf{d}}(Q, R)$ denote the closed subvariety of representations that satisfy the relations R. If $A = kQ/\langle R \rangle$ is given by a quiver with relations we write $\operatorname{Rep}_{\mathbf{d}}(A) := \operatorname{Rep}_{\mathbf{d}}(Q, R)$.

We let GL_n denote the algebraic group of $n \times n$ matrices over k, and we define

$$\operatorname{GL}_{\mathbf{d}} := \prod_{i \in Q_0} \operatorname{GL}_{d_i}.$$

The group $\operatorname{GL}_{\mathbf{d}}$ acts on both $\operatorname{Rep}_{\mathbf{d}}(Q)$ and $\operatorname{Rep}_{\mathbf{d}}(Q, R)$ by conjugation, i.e.

$$(g_i)_{Q_0} \cdot (M_a)_{Q_1} := \left(g_{t(a)} M_a g_{s(a)}^{-1} \right)_{Q_1}.$$

The orbits of this action are in bijection with isomorphism classes of Q-representations with dimension vector $\mathbf{d}$. The stabilizer of $M \in \operatorname{Rep}_{\mathbf{d}}(Q)$ or $M \in \operatorname{Rep}_{\mathbf{d}}(Q, R)$ with respect to this group action is $\operatorname{Aut}_Q(M)$, the *automorphism group* of M. This can be identified with invertible maps in $\operatorname{End}_Q(M)$.

Let $A = kQ/\langle R \rangle$ for (Q, R) a quiver with admissible relations. Consider the bijection between isomorphism classes of A-modules of dimension vector $\mathbf{d}$ and $\operatorname{GL}_{\mathbf{d}}$-orbits in $\operatorname{Rep}_{\mathbf{d}}(A)$. A version of Voigt's lemma stated in [35] has the following corollary:

Proposition 1.4 (Corollary 1.2 [35]). *Consider the following two statements for a point $M \in \operatorname{Rep}_{\mathbf{d}}(A)$:*

(1) *M is a smooth point and the $\operatorname{GL}_{\mathbf{d}}$-orbit of M is an open subset of $\operatorname{Rep}_{\mathbf{d}}(A)$.*

(2) *The module M is rigid.*

In general we have (2) $\implies$ (1). *The converse holds if $A = kQ$ is a path algebra.*

Remark 1.5. The statement of Corollary 1.4 is modified to account for the fact that we only consider reduced scheme structures and representation spaces for a fixed dimension vector, while Gabriel writes the statement for the canonical scheme structure of the module variety of modules of a fixed k-dimension. The converse holds for $\mathrm{Rep}_d(Q)$ because its canonical scheme structure is reduced, but for general A we lose information by taking the reduced structure.

Example 1.6. The implication $(2) \implies (1)$ does not hold in general. Consider

the Jordan quiver $Q = \overset{\alpha}{\underset{\bullet}{\circlearrowright}}$ with the relation α^2, so $A = k[\alpha]/\alpha^2$. Take the dimension vector $\mathbf{d} = 1$. Clearly $\mathrm{Rep}_d(A)$ only has one orbit, which is one point given by the simple A-module S. Thus $\mathrm{Rep}_1(A)$ is just one point, which is of course smooth because we have chosen the reduced scheme structure on $\mathrm{Rep}_d(A)$. However it is easy to see that S is not rigid.

1.4.2 Fibre bundles

Let G be an algebraic group with a closed subgroup H, and let X be an H-variety. Then H acts on $G \times X$ from the right via $(g, x) \cdot h = (gh, h^{-1}x)$. We define the *associated fibre bundle* as the quotient

$$G \times^H X := (G \times X)/H.$$

This quotient is a variety cf. [63, Section 3.7], we denote the right H-orbit of (g, x) by $[g, x]$. The $G \times^H X$ has a natural left G-action given by multiplication on G from the left, i.e. $g \cdot [g', x] = [gg', x]$. We have a canonical map $\iota \colon X \to G \times^H X$, $x \mapsto [1, x]$, this is a closed embedding of varieties, and it is invariant under the left action of H.

Lemma 1.7. *Let H be a closed algebraic subgroup of G, X be an H-variety. We let $\iota \colon X \to G \times^H X$ be the canonical inclusion, and we identify X to the image of ι. The assignments*

$$H \cdot [1, x] \mapsto G \cdot [1, x] \quad \text{and} \quad G \cdot [g, x] \mapsto X \cap G \cdot [g, x]$$

are mutually inverse and give a one-to-one correspondence between the H-orbits of X and the G-orbits of $[1, x]$. Open orbits correspond to open orbits under this correspondence. In particular, if X is irreducible, then $G \times^H X$ has a dense G-orbit if and only if X has a dense H-orbit.

Proof. Let $\mathcal{O}$ be a G-orbit in $G \times^H X$, with $[g, x] \in \mathcal{O}$. Then $[1, x] = g^{-1}[g, x] \in \mathcal{O}$, so any G-orbit in $G \times^H X$ has the form $G \cdot [1, x]$ for some $x \in X$. Assume $[g, y] \in \mathcal{O} \cap X$. Then $[g, y] = [1, x]$ for some $x \in X$, i.e. there is $h \in H$ such that $(gh, h^{-1}y) = (1, x)$ for $x \in X$. This shows $g \in H$, so $\mathcal{O} \cap X \subset H \cdot [1, x]$, and the inclusion $H \cdot [1, x] \subset \mathcal{O} \cap X$ is obvious.

Assume $\mathcal{O} = G \cdot [1, x]$ is an open G-orbit of $G \times^H X$. That implies $\mathcal{O} \cap X$ is a non-empty open subset of X.

Conversely, assume $H \cdot [1, x] \subset X$ is open, then there is an irreducible component $X' \subset X$ such that $\overline{H \cdot [1, x]} = X'$. Now $H \cdot [1, x] \subset G \cdot [1, x]$ implies

$$X' = \overline{H \cdot [1, x]} \subset \overline{G \cdot [1, x]}.$$

But all G-orbits intersect X, and the closure of $G \cdot [1, x]$ is a union of G-orbits, thus it must be the union of all G-orbits of $G \times^H X'$, which is of course all $G \times^H X'$. T hen X' is dense in an irreducible component, and hence open. $\quad \square$

If we take X to be an H-stable subvariety of a G-variety Y, then we define the *collapsing map*

$$G \times^H X \to Y, \quad [g, x] \mapsto g \cdot x.$$

Let X and Y be irreducible varieties, we say a map of varieties $f \colon X \to Y$ is a *resolution of singularities* of Y if X is smooth and f is surjective, birational, and restricts to an isomorphism over the non-singular locus of Y. We have the following general easy lemma, for a proof cf. [57, Lemma 39 p. 148].

Lemma 1.8. *Let G be a connected algebraic group, $H \subset G$ a closed subgroup and Y a G-variety with a smooth H-subvariety X. Assume $G \cdot X \subset Y$ has a dense G-orbit $\mathcal{O}$. Then, the fibres of the collapsing map $\pi \colon G \times^H X \to G \cdot X$ over $\mathcal{O}$ are smooth, pairwise isomorphic and irreducible of dimension $\dim G \times^H X - \dim \mathcal{O}$. Furthermore, the following are equivalent*

(1) *The collapsing map $\pi \colon G \times^H X \to G \cdot X$ is a resolution of singularities of $\overline{\mathcal{O}}$.*

(2) *$\mathcal{O} \subset G \cdot X$ and $\dim G \times^H X = \dim \mathcal{O}$.*

1.4.3 Flag varieties

Let $d, s \in \mathbb{N}$ and consider the vector space k^d. Let

$$\mathbf{d} = (\mathbf{d}^{(1)}, \dots, \mathbf{d}^{(s)} = d) \in \mathbb{N}^s, \quad \mathbf{d}^{(t)} \leq \mathbf{d}^{(t+1)}, \; t = 1, \dots, s - 1.$$

We call this a *dimension filtration* of d of length s. A flag F on k^d for the filtration $\mathbf{d}$ is a sequence

$$F^{(1)} \subset \cdots \subset F^{(s)} = k^d$$

of subspaces of V such that $\dim_k F^{(t)} = \mathbf{d}^{(t)}$ for $1 \leq t \leq s$. Consider the parabolic subgroup

$$\mathrm{P}_{\mathbf{d}} := \{ g \in \mathrm{GL}_d \mid gF^{(t)} \subset F^{(t)}, \; 1 \leq t \leq s \} \subset \mathrm{GL}_d$$

We denote the projective variety of flags on k^d for the dimension filtration $\mathbf{d}$ by $\mathrm{Fl}\binom{k^d}{\mathbf{d}}$, we call this a *flag variety*. The quotient $\mathrm{GL}_d / \mathrm{P}_{\mathbf{d}}$ is isomorphic as a variety to $\mathrm{Fl}\binom{k^d}{\mathbf{d}}$ via the bijection $g\mathrm{P}_{\mathbf{d}} \leftrightarrow gFg^{-1}$. We will often not distinguish between those two varieties.

Let $Q = (Q_0, Q_1)$ be a finite quiver and let $A = kQ$ be the path algebra. We fix a dimension vector $d = (d_i)_{i \in Q_0} \in \mathbb{N}_0^{Q_0}$ and let

$$\mathbf{d} = (\mathbf{d}^{(1)}, \ldots, \mathbf{d}^{(s)} = d)$$

be a sequence of dimension vectors, where $\mathbf{d}^{(t)} = (d_i^{(t)})_{i \in Q_0} \in \mathbb{N}_0^{Q_0}$ for $1 \le t \le s$, and $d_i^{(t)} \le d_i^{(t+1)}$ pointwise for all $i \in Q_0$ and $1 \le t \le s-1$. In other words this is a dimension filtration $\mathbf{d}_i$ of d_i for each vertex $i \in Q_0$. Just like above we call such a $\mathbf{d}$ a dimension filtration of the dimension vector d of length s. We may consider $\mathbf{d}^{(0)} = 0$ as part of the dimension filtration. For each vertex $i \in Q_0$, fix a flag $F_i = (F_i^{(1)} \subset \cdots \subset F_i^{(s)} = k^{d_i})$ for the dimension filtration $\mathbf{d}_i$. We say the Q_0-graded flag $F := (F_i)_{i \in Q_0}$ is a flag for the dimension filtration $\mathbf{d}$, and write $\underline{\dim} F = \mathbf{d}$. By convention we set $F_i^{(t)} = 0$ for all $t \le 0$ and all $i \in Q_0$. We have a parabolic subgroup of GL_d fixing our flag, defined as

$$\mathrm{P}_{\mathbf{d}} := \prod_{i \in Q_0} \mathrm{P}_{\mathbf{d}_i} \subset \mathrm{GL}_d.$$

We write

$$\mathrm{Fl}\binom{k^d}{\mathbf{d}} := \left(\mathrm{Fl}\binom{k^{d_i}}{\mathbf{d}_i} \right)_{i \in Q_0}$$

for the Q_0-graded flag variety . Of course the identification $\mathrm{GL}_{d_i}/\mathrm{P}_{\mathbf{d}_i} = \mathrm{Fl}\binom{k^{d_i}}{\mathbf{d}_i}$ at each vertex $i \in Q_0$ allows us to identify:

$$\mathrm{Fl}\binom{k^d}{\mathbf{d}} = \mathrm{GL}_d/\mathrm{P}_{\mathbf{d}}.$$

1.5 Quasi-hereditary algebras

The notion of quasi-hereditary algebras goes back to Cline-Parshall-Scott [17],[58]. They had motivation from highest weight categories arising from semi-simple complex Lie algebras, but there are many other naturally arising examples.

Let A be a finite-dimensional algebra. Our main reference on quasi-hereditary algebras is [26]. Let $S(\xi)$, $\xi \in \Xi$ be the isomorphism classes of simple A-modules. We assume $(\Xi, \le)$ is a partially ordered set. Let $P(\xi)$ (resp. $I(\xi)$) denote the projective cover (resp. injective envelope) of $S(\xi)$. The *standard module* $\Delta(\xi)$ at ξ is the maximal factor module of $P(\xi)$ such that for every composition factor $S(\rho)$ of $\Delta(\xi)$ we have $\rho \le \xi$. Dually, the *costandard module* $\nabla(\xi)$ at ξ is the maximal submodule of $I(\xi)$ that only has composition factors $S(\rho)$ with $\rho \le \xi$. We denote the class of standard modules by Δ, and the class of costandard modules by ∇. For a class Θ in A-mod we denote by $\mathcal{F}(\Theta)$ the full subcategory in A-mod of modules that have a filtration by modules in Θ. This means all modules M such that there is a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

such that M_i/M_{i-1} is isomorphic to a module in Θ for $i = 1, \ldots, n$. We call $\mathcal{F}(\Delta)$ the Δ-*filtered* modules and $\mathcal{F}(\nabla)$ the ∇-*filtered* modules. We say our partial order on Ξ is *adapted*, if for every A-module M with $\mathrm{top}(M) \cong S(\xi_1)$ and $\mathrm{soc}(M) \cong S(\xi_2)$, where ξ_1 and ξ_2 are incomparable, there is $\rho \in \Xi$ such that either $\rho > \xi_1$ or $\rho > \xi_2$, and $S(\rho)$ is a composition factor of M. If a partial ordering on Ξ is adapted, it implies any refinement of that partial ordering gives the same standard and costandard modules, cf. [26, Section 1]. If all the standard modules have an endomorphism ring isomorphic to k, and $_AA \in \mathcal{F}(\Delta)$, we say the $\mathcal{F}(\Delta)$ gives a *quasi-hereditary structure* on A. Then A along with the standard modules Δ is a *quasi-hereditary algebra*. If it is clear what Δ should be, we may simply say that A is quasi-hereditary, without specifying Δ.

Proposition 1.9. *The following hold for a quasi-hereditary algebra* (A, Δ).

(1) $\mathcal{F}(\Delta) = \{M \mid \mathrm{Ext}_A^1(\nabla, M) = 0\} = \{M \mid \mathrm{Ext}_A^i(\nabla, M) = 0, \forall i \geq 1\}$;

(2) $\mathcal{F}(\nabla) = \{M \mid \mathrm{Ext}_A^1(M, \Delta) = 0\} = \{M \mid \mathrm{Ext}_A^i(M, \Delta) = 0, \forall i \geq 1\}$;

(3) $\mathrm{gldim}\, A < \infty$.

Proof. The assertions (1) and (2) are given in [26, Theorem 1]. Condition (3) is a corollary of Lemma 2.2. ibid. $\square$

Remark 1.10. The proposition shows that $\mathcal{F}(\nabla)$ is determined by Δ. Furthermore the costandard modules are determined up to isomorphism by the condition that $\nabla(i)$ is the minimal submodule of $I(i)$ which belongs to $\mathcal{F}(\nabla)$. Hence the costandard modules ∇ are determined by the pair (A, Δ). Dually ∇ determines Δ.

We say two quasi-hereditary algebras (A, Δ) and (A', Δ') are isomorphic if there is a ring isomorphism $A \simeq A'$ such that the standard modules of Δ and Δ' are isomorphic as A-modules with respect to this ring isomorphism.

Definition 1.11. An A-module T is a *generalized tilting module* if there is $m \geq 1$ such that the following holds:

(a) $\mathrm{pdim}\, T \leq m$;

(b) T is rigid;

(c) there is an exact sequence $0 \to {_AA} \to T^0 \to T^1 \to \cdots \to T^m \to 0$, with $T^0, T^1, \ldots, T^m \in \mathrm{add}(T)$.

We say T is a *tilting module* if these properties hold for $m = 1$.

A module $T \in A$-mod is a *generalized cotilting module* if there is $m \geq 1$ such that:

(a') $\mathrm{idim}\, T \leq m$;

(b') T is rigid;

12

(c') there is an exact sequence $0 \to T_m \to \cdots \to T_1 \to T_0 \to \mathrm{D}\,({}_A A) \to 0$, with $T^0, T^1, \ldots, T^m \in \mathrm{add}(T)$.

We say T is a *cotilting module* if this holds for $m = 1$.

We define the *characteristic module* C_A of the quasi-hereditary algebra (A, Δ) as the unique basic A-module such that $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = \mathrm{add}(C_A)$. By [26, Prp. 3.1] this module is unique up to isomorphism and has indecomposable summands parametrized by Ξ in a canonical way. We denote those summands by $C_A(\xi)$ for $\xi \in \Xi$. Moreover C_A is a generalized tilting and cotilting module.

For a quasi-hereditary algebra (A, Δ), we consider the endomorphism ring $\mathcal{R}_A := \mathrm{End}(C_A)^{\mathrm{op}}$. There is a functor $\mathrm{Hom}_A(C_A, -) \colon A\text{-mod} \to \mathcal{R}_A\text{-mod}$. We define standard modules $\Delta'(i) := \mathrm{Hom}_A(C_A, \nabla(i))$. This gives a quasi-hereditary algebra $(\mathcal{R}_A, \Delta')$ which we call the *Ringel dual* of A, this notion goes back to [51]. The primitive idempotents of $\mathcal{R}_A$ are $e_\xi = \mathrm{id} \in \mathrm{End}_A(T(\xi))^{\mathrm{op}}$ for $\xi \in \Xi$, and these are in bijection with the isomorphisms classes of simple $\mathcal{R}_A$-modules. If the standard modules Δ are determined by a partial ordering $\leq$ on Ξ, then the inverse ordering on Ξ gives the quasi-hereditary algebra $(\mathcal{R}_A, \Delta')$.

Proposition 1.12 (Theorem 7, [51])**.** *If (A, Δ) is a quasi-hereditary algebra and A is basic, then A is isomorphic to the Ringel dual of $(\mathcal{R}_A, \Delta')$ as a quasi-hereditary algebra.*

We say (A, Δ) is *Ringel self-dual* if $(\mathcal{R}_A, \Delta')$ and (A, Δ) are isomorphic as quasi-hereditary algebras.

Remark 1.13. Similarly as for the Ringel dual, the opposite algebra A^{op} of a quasi-hereditary algebra (A, Δ) has a quasi-hereditary structure Δ' determined by Δ. It is given by taking $\Delta'(\xi) := \mathrm{D}\,\nabla(\xi)$.

1.5.1 Strongly quasi-hereditary algebras

We say a module is *divisible* if it is a factor module of an injective module, and *torsionless* if it is a submodule of a projective module.

Lemma 1.14 (Lemma 4.1 & 4.1* [26])**.** *Let (A, Δ) be a quasi-hereditary algebra, the following are equivalent:*

(a) *The projective dimension of any standard module is at most 1.*

(b) *The characteristic module C_A has projective dimension at most 1.*

(c) *The category $\mathcal{F}(\nabla)$ is closed under cokernels of monomorphisms.*

(d) *All divisible modules belong to $\mathcal{F}(\nabla)$.*

The following dual conditions are also equivalent.

(a') *The injective dimension of any costandard module is at most 1.*

(b') *The characteristic module C_A has injective dimension at most 1.*

(c') *The category $\mathcal{F}(\Delta)$ is closed under kernels of epimorphisms.*

(d') *All torsionless modules belong to $\mathcal{F}(\Delta)$.*

We say (A, Δ) is *left strongly quasi-hereditary* if one and therefore all of the equivalent conditions $(a), (b), (c), (d)$ hold. Dually we say it is *right strongly quasi-hereditary* when conditions $(a'), (b'), (c'), (d')$ hold.

Let Δ' be the quasi-hereditary structure on A^{op} as in Remark 1.13. The dual sends an injective coresolution of $\nabla(i)$ to a projective resolution of $\Delta'(i)$, thus a right strongly quasi-hereditary structure on A corresponds to a left strongly quasi-hereditary structure on A^{op}.

The lemma above is already found in the survey [26], and much of the properties of strongly quasi-hereditary algebras are found in [24]. However the term strongly quasi-hereditary is more recent, coming from [52]. In that article there is also an alternate method to construct left strongly quasi-hereditary algebras, namely in terms of a layer function.

Definition 1.15. A *layer function* on Ξ is a function $\ell\colon \Xi \to \mathbb{N}$ such that for each $\xi \in \Xi$ there is an exact sequence

$$0 \longrightarrow P_1 \longrightarrow P(\xi) \longrightarrow \Delta(\xi) \longrightarrow 0$$

with the following properties.

(1) P_1 is a direct sum of projective modules $P(\rho)$ with $\ell(\rho) > \ell(\xi)$;

(2) for every composition factor $S(\rho)$ of $\mathrm{rad}(\Delta(\xi))$ we have $\ell(\rho) < \ell(\xi)$.

The modules $\Delta(\xi)$ are in fact the standard modules of a left strongly quasi-hereditary structure on A, cf. [52, Section 4].

In [20], Conde introduced the notion of *ultra strongly quasi-hereditary* algebras. We define right ultra strongly quasi-hereditary algebras as quasi-hereditary algebras satisfying

(US1) $\mathrm{rad}(\Delta(\xi))$ is either a standard module or zero;

(US2) if $\mathrm{rad}(\Delta(\xi)) = 0$, then $I(\xi)$ has a filtration by standard modules.

These algebras are in particular right strongly quasi-hereditary. Moreover, by [20, Proposition 5.3], for every ξ such that $\Delta(\xi)$ is simple, the injective hull $I(\xi)$ is in $\mathcal{F}(\Delta)$, and hence in $\mathrm{add}(C_A)$. We say (A, Δ) is left ultra strongly quasi-hereditary if the induced structure $(A^{\mathrm{op}}, \Delta')$ is right ultra strongly quasi-hereditary.

1.6 Tensor algebras

Let A_0 be an algebra and let A_1 be a finitely generated A_0-A_0-bimodule. With this data we define a tensor algebra

$$T_{A_0} A_1 := \bigoplus_{n \geq 0} (\underbrace{A_1 \otimes_{A_0} \otimes \cdots \otimes_{A_0} A_1}_{n-\text{times}}).$$

The empty tensor product is A_0, and we give $A_1^{\otimes n}$ the grade n. This makes $T_{A_0} A_1$ a positively graded algebra with multiplication given by tensor products. The following is a general fact of graded algebras.

Lemma 1.16. *Let $A \cong A_0 \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$, as a $\mathbb{Z}$-graded algebra. Assume x_i, r_j are of degree 1 for all i, j. Then A is isomorphic to $T_{A_0} A_1$ as a graded algebra.*

Proof. Since all the r_j are homogeneously graded we have a well defined $\mathbb{Z}$-grading on A. By the universal property of tensor algebras, the inclusion of A_1 in A as an A_0-module induces a unique ring homomorphism $\phi \colon T_{A_0} A_1 \to A$. Now consider the inclusion map $\{x_1, \ldots, x_n\} \to A_1 \subset T_{A_0} A_1$. This induces an A_0-algebra homomorphism $A_0 \langle x_1, \cdots, x_n \rangle \to T_{A_0} A_1$, with all the r_j in the kernel. Hence this induces a graded ring homomorphism $A \to T_{A_0} A_1$, which is inverse to ϕ. $\qquad\square$

For the rest of this subsection we assume $A \cong T_{A_0} A_1$ and we identify them as graded algebras. However we only consider ungraded modules. Let A_+ be the positively graded part of A and let $M \in A$-mod. The following lemma is a known fact for general tensor algebras. The following is a corollary of a standard sequence for tensor algebras, cf. [19, Chapter 2, Prp 2.6], although we give a more explicit proof here, adapted from [65, Theorem B.2].

Lemma 1.17. *There is an exact sequence of A-modules*

$$0 \longrightarrow A_+ \otimes_{A_0} M \xrightarrow{\delta_M} A \otimes_{A_0} M \xrightarrow{\epsilon_M} M \longrightarrow 0. \qquad (\text{Std})$$

We call it the standard sequence. Let $a \in A$ and $a_1 \in A_1$. The maps are given by

$$\epsilon_M(a \otimes m) := a \cdot m,$$
$$\delta_M((a \otimes a_1) \otimes m) := (a \otimes a_1) \otimes m - a \otimes a_1 \cdot m.$$

Dually, let M be a right A-module. Then there is an exact sequence of left A-modules:

$$0 \longrightarrow \mathrm{D}(M) \longrightarrow \mathrm{D}(M \otimes_{A_0} A) \longrightarrow \mathrm{D}(M \otimes_{A_0} A_+) \longrightarrow 0. \qquad (\text{DStd})$$

Proof. All tensor products are over A_0. Clearly the composition $\epsilon_M \delta_M$ is zero and ϵ_M is an epimorphism. Let us decompose

$$A \otimes M = \bigoplus_{n=1}^{t} A_n \otimes M,$$

and similar for $A_+ \otimes M$. Now δ_M decomposes into maps

$$A_n \otimes M \to (A_n \otimes M) \oplus (A_{n-1} \otimes M),$$

where the first component is the identity. That shows that if $x = \sum_{n=1}^{t} x_n$ is in the kernel of δ_M, then $x_t = 0$. By induction on t we get $x = 0$, thus δ_M is injective. Moreover this shows $\operatorname{Im} \delta_M \cap A_0 \otimes M = 0$.

It remains to show that $A \otimes M = A_0 \otimes M \oplus \operatorname{Im} \delta_M$. Let $x = \sum_{n=0}^{t} x_n \in A \otimes M$, we show $x \in \operatorname{Im} \delta_M \oplus A_0 \otimes M$ by induction on t, the case $t = 0$ is trivial. Let $t \geq 1$, then $x_t \in A_+ \otimes M$ and $x - \delta(x) = \sum_{n=0}^{t-1} x'_n$, so $x \in A_0 \otimes M \oplus \operatorname{Im} \delta_M$ by induction hypothesis.

There is an analogous version of the standard sequence for right-modules, or equivalently A^{op}-modules. The sequence (DStd) is then obtained by applying D to that sequence. $\qquad\square$

16

Chapter 2

Recollements of Auslander algebras

2.1 Overview

This chapter has the results of the article [28] already published in the Journal of Algebra. The content is mostly the same as in that article, although some changes have been made to organisation and notation to make it compatible with other parts of this thesis.

Let Λ be a basic finite dimensional k-algebra of finite representation type. In this chapter we outline a connection between the submodule category $\mathcal{S}(\Lambda)$ and a recollement of the Auslander algebra of Λ, a connection described by Ringel-Zhang [55] in a particular case.

Studies of submodule categories go back to Birkhoff [10]. Recently they have been a subject to active research, including work of Simson about their tame-wild dichotomy [59, 60, 61]. Also Ringel and Schmidmeier have studied their Auslander-Reiten theory [53], as well as some particular cases of wild type [54]. Moreover Luo-Zhang [46, 67] studied them with respect to Gorenstein-projective modules and tilting theory. The homological properties of submodule categories give extensive information on quiver Grassmannians, in particular their isomorphism classes correspond to strata in certain stratifications [13]. If Λ is self-injective, the submodule category is a Frobenius category, and Chen has shown its stable category is equivalent to the singularity category of $T_2(\Lambda)$ [15]. In [42] Kussin, Lenzing and Meltzer give a connection of submodule categories to weighted projective lines, which again connects them to singularity categories [43].

The content is organized as follows. In Section 2.2 we recall the notion of a category of finitely presented functors, and that the module category of an Auslander algebra is equivalent to the category of finitely presented additive contravariant functors from Λ-mod to the category of abelian groups. We also recall the basic properties of a functor α from $S(\Lambda)$ to the category of finitely presented functors, and characterizations of the projective and injective objects in that category.

In Section 2.3 we restrict our attention to the Auslander algebras of self-injective algebras. Then we study the recollement induced by a certain idempotent of the Auslander algebra, and introduce an induced tilting and cotilting module T. Moreover we recall some properties of the stable Auslander algebra.

In Section 2.4 we consider functors F and G that arise as compositions of functors studied in the previous sections. We prove Theorems 2.23 and 2.24 for these functors and thereby generalise the situation in [55].

Section 2.5 is dedicated to the proof of Theorem 2.26. First we give some properties of the Nakayama algebras. In Subsection 2.5.2 we describe a quasi-hereditary structure on the Auslander algebras of the Nakayama algebras, which fulfils the conditions of Theorem 2.26. For the Auslander algebras of self-injective Nakayama algebras, Tan [64] has already described this structure in detail. In Subsection 2.5.3 we prove that no other Auslander algebras of representation finite algebras have quasi-hereditary structures that satisfy those conditions.

2.2 Submodule categories

Let A be a finite dimensional algebra. Denote the algebra of upper triangular 2×2 matrices with coefficients in A by $T_2(A)$. The *category of morphisms* in A-mod is the category which has maps $(M_1 \overset{f_M}{\to} M_0)$ of A-modules as objects, and where a morphism of two objects $(M_1 \overset{f_M}{\to} M_0)$ and $(N_1 \overset{f_N}{\to} N_0)$ is a pair $(g_1, g_0) \in (M_1, N_1)_A \times (M_0, N_0)_A$ such that $f_N g_1 = g_0 f_M$. We will identify $T_2(A)$-mod with the category of morphisms in A-mod. The *submodule category* of A, denoted $\mathcal{S}(A)$, is the full subcategory of monomorphisms in $T_2(A)$-mod. We denote the full subcategory of epimorphisms by $\mathcal{E}(A)$.

Definition 2.1. We define functors:

$$\eta \colon \mathcal{S}(A) \to T_2(A)\text{-mod}, \quad f \mapsto f,$$
$$\epsilon \colon \mathcal{S}(A) \to T_2(A)\text{-mod}, \quad f \mapsto \operatorname{coker}(f).$$

The functor η is simply the inclusion of $\mathcal{S}(A)$ in $T_2(A)$-mod. On morphisms, ϵ is given by the induced maps of cokernels. Note that ϵ is full and faithful and its essential image is the full subcategory $\mathcal{E}(A)$ of $T_2(A)$, hence we can consider ϵ as a composition of an equivalence $\mathcal{S}(A) \to \mathcal{E}(A)$ followed by η.

Now we recall some well known facts on representable functors. These go back to Auslander [2], Freyd [32, 33] and Gabriel [34], while [44] contains a handy summary of those techniques.

Let $\mathcal{A}$ be an essentially small additive category. We consider the category $\operatorname{Fun}(\mathcal{A})$ of additive functors from $\mathcal{A}^{\mathrm{op}}$ to the category **Ab** of abelian groups, with morphisms given by natural transformations. We say a functor $F \in \operatorname{Fun}(\mathcal{A})$ is *representable* if F is isomorphic to $(-, M)_{\mathcal{A}}$ for some $M \in \operatorname{Ob}(\mathcal{A})$. We say F is *finitely presented* if there exist representable functors

$(-, M)_{\mathcal{A}}, (-, N)_{\mathcal{A}}$ and an exact sequence

$$(-, M)_{\mathcal{A}} \longrightarrow (-, N)_{\mathcal{A}} \longrightarrow F \longrightarrow 0.$$

We denote the full subcategory of finitely presented functors by $\mathrm{fun}(\mathcal{A})$. The category $\mathrm{fun}(\mathcal{A})$ is abelian, cf. [32, Theorem 5.11]. To reduce encumbrance we write $\mathrm{fun}(A) := \mathrm{fun}(A\text{-mod})$.

The following lemma comes from applying [32, Theorem 5.35] to the opposite of A-mod.

Lemma 2.2. *The functor $M \mapsto (-, M)_A$ from A-mod to $\mathrm{fun}(A)$ induces an equivalence of categories from A-mod to the full subcategory of representable functors in the category $\mathrm{fun}(A)$. Moreover, the representable functors are the projective objects of $\mathrm{fun}(A)$.*

If we apply [32, Theorem 5.35] to A-mod and then apply the vector space duality we obtain the following dual statement to Lemma 2.2, cf. [32, Exercise A. Chapter 5].

Lemma 2.3. *The functor $M \mapsto D(M, -)_A$ from A-mod to $\mathrm{fun}(A)$ induces an equivalence from A-mod to the full subcategory of injective objects in $\mathrm{fun}(A)$.*

2.2.1 Representations of the Auslander algebra

Let Λ be a finite dimensional basic k-algebra of finite representation type. Let E be the additive generator of Λ-mod, i.e. the basic Λ-module such that $\mathrm{add}(E) = \Lambda$-mod. The *Auslander algebra* of Λ is $\mathrm{Aus}(\Lambda) := \mathrm{End}_{\Lambda}(E)^{\mathrm{op}}$. Write $\Gamma := \mathrm{Aus}(\Lambda)$ and let $e \in \Gamma$ be the idempotent given by the projection onto the summand Λ of E. Write $\Gamma e \Gamma$ for the two sided ideal generated by e, we define the *stable Auslander algebra* as the algebra $\underline{\Gamma} := \Gamma / \Gamma e \Gamma$.

Any functor in $\mathrm{fun}(\Lambda)$ is determined by its value on the Auslander generator and its endomorphisms, so $\mathrm{fun}(\Lambda)$ is equivalent to Γ-mod, this is [34, Chapitre II, Proposition 2]. Note that the representable functors $(-, M)_{\Lambda}$ correspond to the right $\mathrm{End}(E)$-modules $(E, M)_{\Lambda}$ acted upon by pre-composition, but these may also be viewed as left Γ-modules.

We will consider the functor

$$\alpha := \mathrm{coker}(E, -)_{\Lambda} \colon T_2(\Lambda)\text{-mod} \to \Gamma\text{-mod},$$

which was already studied by Auslander and Reiten in [5].

Remark 2.4. The Gabriel quiver of Γ is the opposite quiver of the Auslander-Reiten quiver of Λ-mod with relations given by the Auslander-Reiten translate. The indecomposable projective Γ-modules are represented by the indecomposable objects of Λ-mod. More precisely, given $M \in \mathrm{ind}(\Lambda)$, then $(E, M)_{\Lambda}$ is the indecomposable projective Γ-module arising as the projective representation of the opposite of the Auslander-Reiten quiver of Λ-mod generated at the vertex of M.

Definition 2.5. We call a category a *Krull-Schmidt category* if every object decomposes into a finite direct sum of indecomposable objects in a unique way up to isomorphism.

A functor $F\colon \mathcal{A} \to \mathcal{B}$ between Krull-Schmidt categories is called *objective* if the induced functor $\mathcal{A}/\ker(F) \to \mathcal{B}$ is faithful.

Our notion of an objective functor is equivalent to that used in [55]. For more information on this property we refer to [56].

Proposition 2.6. *The functor α is full, dense and objective. Its kernel is* $\mathrm{add}((E \overset{\mathrm{id}}{\to} E) \oplus (E \to 0))$.

Remark 2.7. The indecomposable objects of $\ker(\alpha)$ are either of the form $(M \overset{\mathrm{id}}{\to} M)$ or $(M \to 0)$ for $M \in \mathrm{ind}(\Lambda)$. Since Λ is of finite representation type, say with m indecomposable objects up to isomorphism, this means $\ker(\alpha)$ has exactly $2m$ indecomposable objects up to isomorphism.

Proof of Proposition 2.6. We imitate the proof of [55, Proposition 3]. Let X be an object in Γ-mod, it has a projective presentation

$$(E, M_1)_\Lambda \xrightarrow{\ p_1\ } (E, M_0)_\Lambda \xrightarrow{\ p_0\ } X \longrightarrow 0.$$

By Lemma 2.2 there is $f \in (M_1, M_0)_\Lambda$ such that $p_1 = (E, f)_\Lambda$. But then $\alpha(f) \simeq X$, so α is dense. Let $\Phi \in \mathrm{Hom}_\Gamma(X, Y)$ and let $f \in (M_1, M_0)_\Lambda$ and $g \in (N_1, N_0)_\Lambda$ be such that $\alpha(f) \cong X$ and $\alpha(g) \cong Y$. Now Φ can be extended to a map (Φ_1, Φ_0) of the projective presentations of X and Y. There are ϕ_i for $i = 0, 1$ such that $(E, \phi_i) \cong \Phi_i$. But then clearly $\alpha(\phi_1, \phi_0) \cong \Phi$, thus α is full.

Clearly $\alpha(M \overset{\mathrm{id}}{\to} M) \cong 0 \cong \alpha(M \to 0)$. Let

$$(g_1, g_0) \in \mathrm{Hom}_{T_2(\Lambda)}((M_1 \overset{f_M}{\to} M_0), (N_1 \overset{f_N}{\to} N_0))$$

be such that $\alpha(g_1, g_0) = 0$. We want to show that (g_1, g_0) factors through a $T_2(\Lambda)$-module of the form $(M \overset{\mathrm{id}}{\to} M) \oplus (N \to 0)$.

Consider the following commutative diagram:

$$
\begin{array}{ccccccc}
(E, M_1)_\Lambda & \xrightarrow{(E, f_M)_\Lambda} & (E, M_0)_\Lambda & \longrightarrow & \alpha(f_M) & \longrightarrow & 0 \\
{\scriptstyle (E, g_1)_\Lambda} \downarrow & {\scriptstyle h'} \nearrow\!\!\!\!\!\!\!\diagdown & \downarrow {\scriptstyle (E, g_0)_\Lambda} & & \downarrow {\scriptstyle \alpha(g_1, g_0) = 0} & & \\
(E, N_1)_\Lambda & \xrightarrow[(E, f_N)_\Lambda]{} & (E, N_0)_\Lambda & \xrightarrow[c]{} & \alpha(f_N) & \longrightarrow & 0.
\end{array}
$$

The rows are projective presentations. Now $c \circ (E, g_0)_\Lambda = 0$ and hence there is h' such that $(E, g_0)_\Lambda = (E, f_N)_\Lambda \circ h'$. Since the functor $(E, -)_\Lambda$ is full there is

a map $h\colon M_0 \to N_1$ such that $h' = (E, h)_\Lambda$ and $g_0 = f_N h$. Then the following diagram in Λ-mod is commutative:

$$
\begin{array}{ccccc}
M_1 & \xrightarrow{[f_M, \mathrm{id}]} & M_0 \oplus M_1 & \xrightarrow{[h, g_1 - h f_M]} & N_1 \\
\downarrow{\scriptstyle f_M} & & \downarrow{\scriptstyle [\mathrm{id},0]} & & \downarrow{\scriptstyle f_N} \\
M_0 & \xrightarrow{\ \ \mathrm{id}\ \ } & M_0 & \xrightarrow{\ \ g_0\ \ } & N_0.
\end{array}
$$

Note that the compositions of the rows are g_1 and g_0, and hence (g_1, g_0) factors through the $T_2(\Lambda)$-module $(M_0 \oplus M_1 \xrightarrow{[\mathrm{id},0]} M_0)$. $\qquad\square$

Remark 2.8. The functors ϵ and η are faithful and hence objective. The composition $\alpha\eta$ is also objective since it is just a restriction of the objective functor α to an additive subcategory. Moreover $\alpha\epsilon$ is objective because ϵ is fully faithful and the image of ϵ contains all objects of $\ker\alpha$.

The following corollary of Proposition 2.6 describes the composition $\alpha\eta$.

Corollary 2.9. *Let* $\chi := \mathrm{add}(E \xrightarrow{\mathrm{id}} E)$. *Let* Γ-torsl *denote the full subcategory of* Γ-mod *consisting of objects of projective dimension* ≤ 1. *The functor* $\alpha\eta$ *induces an equivalence of categories*

$$
\mathcal{S}(\Lambda)/\chi \to \Gamma\text{-torsl}.
$$

Proof. We know already that $\alpha\eta$ is full and objective and by Proposition 2.6 the kernel of $\alpha\eta$ is χ. It remains to show that the essential image of $\alpha\eta$ is Γ-torsl. Let $f \in (M_1, M_0)_\Lambda$ be a monomorphism. Since Hom-functors are left-exact, (E, f) is a monomorphism and $\alpha(f)$ has a projective resolution

$$
0 \longrightarrow (E, M_1)_\Lambda \xrightarrow{(E, f)_\Lambda} (E, M_0)_\Lambda \longrightarrow \alpha(f) \longrightarrow 0.
$$

Using that $(E, -)_\Lambda\colon \Lambda$-mod $\to \Gamma$-proj is an equivalence, we see any object in Γ-torsl has a projective resolution of this form, where $f\colon M_1 \to M_0$ is a monomorphism. $\qquad\square$

Remark 2.10. We know α behaves really well with respect to the additive structure on $T_2(\Lambda)$-mod and Γ-mod, and these are both abelian categories. However α is far from being exact, in fact it preserves neither epimorphisms nor monomorphisms. Take for example $\Lambda = k[x]/\langle x^2 \rangle$ and let $_\Lambda k$ be the simple Λ-module. Consider a monomorphism $f\colon (0 \to {}_\Lambda\Lambda) \to ({}_\Lambda\Lambda \xrightarrow{\mathrm{id}} {}_\Lambda\Lambda)$. Since $\alpha({}_\Lambda\Lambda \xrightarrow{\mathrm{id}} {}_\Lambda\Lambda) = 0$ but $\alpha(0 \to {}_\Lambda\Lambda) \neq 0$, $\alpha(f)$ is not a monomorphism. Also there is an epimorphism $g\colon ({}_\Lambda\Lambda \xrightarrow{\mathrm{id}} {}_\Lambda\Lambda) \to ({}_\Lambda\Lambda \to {}_\Lambda k)$, but $\alpha({}_\Lambda\Lambda \to {}_\Lambda k) \neq 0$, thus $\alpha(g)$ is no epimorphism.

There are several characterisations of the subcategory Γ-torsl, one of which also justifies the notation we use for it.

Proposition 2.11. *The following are equivalent for an object* $X \in \Gamma$-mod.

(i) X is in Γ-torsl.

(ii) The injective envelope of X is projective.

(iii) X is torsionless, i.e. a submodule of a projective module.

Proof. (i) $\implies$ (ii). Let X be of projective dimension ≤ 1, so it has a projective resolution $0 \longrightarrow P_1 \overset{u}{\longrightarrow} P_0 \longrightarrow X \longrightarrow 0$. Let $v_i \colon P_i \to I(P_i)$ be the injective envelope of P_i for $i = 0, 1$ and consider the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \overset{u}{\longrightarrow} & P_0 & \longrightarrow & X & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle v_0} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & I(P_1) & \overset{u'}{\longrightarrow} & I(P_0) & \longrightarrow & X' & \longrightarrow & 0.
\end{array}
$$

Here X' is defined as the module making the diagram commutative with exact rows. Since v_0 is injective the snake lemma yields a monomorphism $\ker(f) \to \operatorname{coker}(v_1)$, but since any Auslander algebra has dominant dimension ≥ 2 we can embed $\operatorname{coker}(v_1)$ into a projective-injective module. Thus $\ker(f)$ embeds in a projective-injective module $I(\ker(f))$. Again using that the dominant dimension of Γ is ≥ 2, we know $I(P_i)$ is projective for $i = 0, 1$. Thus the lower sequence splits and X' is projective-injective. The inclusion $\ker(f) \to I(\ker(f))$ factors through X, because $I(\ker(f))$ is injective, and thus we get a monomorphism $X \to X' \oplus I(\ker(f))$.

(ii) $\implies$ (iii). Clear.

(iii) $\implies$ (i). We have an exact sequence $0 \longrightarrow X \longrightarrow P \overset{\pi}{\longrightarrow} C \longrightarrow 0$, where P is projective. Then C has a projective resolution

$$
0 \longrightarrow P_2 \overset{p_2}{\longrightarrow} P_1 \overset{p_1}{\longrightarrow} P \overset{\pi}{\longrightarrow} C \longrightarrow 0,
$$

with $\operatorname{Im} p_1 \cong X$. Thus X has a projective resolution of length ≤ 1. $\qquad\square$

We say a module is *divisible* if it is a factor module of an injective module. We denote the full subcategory of divisible Γ-modules by Γ-divbl. We get the following dual statement to Proposition 2.11.

Proposition 2.12. *The following are equivalent for an object $X \in \Gamma$-mod.*

(i) X has injective dimension ≤ 1.

(ii) The the projective cover of X is injective.

(iii) X is in Γ-divbl.

Later we will have use for the following lemma, which is due to Auslander and Reiten, see [4, Propositon 4.1]. A proof of the version stated here is found in [55, Section 6].

Lemma 2.13. *Let f be a morphism in Λ-mod. Then f is an epimorphism if and only if $((E, P)_\Lambda, \alpha(f))_\Gamma = 0$ for any projective module P in Λ-mod.*

2.2.2 Relative projective and injective objects of $\mathcal{S}(\Lambda)$

The submodule category $\mathcal{S}(\Lambda)$ is additive and by the five lemma it is an extension closed subcategory of $T_2(\Lambda)$-mod. The projective and injective objects of $T_2(\Lambda)$-mod are known, a classification can for example be found in [66, Lemma 1.1]. All projective $T_2(\Lambda)$-modules are a direct sum of modules of the form $(P \xrightarrow{\mathrm{id}} P)$ or $(0 \rightarrow P)$, where P is a projective Λ-module. In particular all projective $T_2(\Lambda)$-modules belong to $\mathcal{S}(\Lambda)$, and they are the relative projective modules of that exact subcategory.

Dually, the injective $T_2(\Lambda)$-modules are a direct sum of modules of the form $(I \xrightarrow{\mathrm{id}} I)$ or $(I \rightarrow 0)$, where I is an injective Λ-module. The relative injective objects of $\mathcal{S}(\Lambda)$ can be written as direct sums of objects of the form $(I \xrightarrow{\mathrm{id}} I)$ or $(0 \rightarrow I)$, with I an injective Λ-module.

If additionally Λ is self-injective, i.e. Λ-mod is a Frobenius category, the proposition below, found in [15, Lemma 2.1], is an easy consequence.

Proposition 2.14. *Let Λ be a self-injective algebra of finite representation type. Then $\mathcal{S}(\Lambda)$ is a Frobenius category and the projective-injective objects are exactly those in* $\mathrm{add}((\Lambda \xrightarrow{\mathrm{id}} \Lambda) \oplus (0 \rightarrow \Lambda))$.

Remark 2.15. If Λ is self-injective the submodule category $\mathcal{S}(\Lambda)$ is precisely the full subcategory of Gorenstein projective $T_2(\Lambda)$-modules, cf. [45, Theorem 1.1]. Thus η is the inclusion of the Gorenstein projective modules of $T_2(\Lambda)$-mod.

2.3 The Auslander algebra of self-injective algebras

In this section we fix Λ as a finite-dimensional basic self-injective k-algebra of finite representation type.

Let $\nu := \mathrm{D}\,(-, {}_\Lambda\Lambda)_\Lambda$ be the Nakayama functor on Λ-mod. Its restriction to projective modules is an equivalence from the projective Λ-modules to the injective Λ-modules with inverse $\nu^{-1} := (\mathrm{D}\,(-), \Lambda_\Lambda)_\Lambda$. Recall that e denotes the idempotent of Γ given by the opposite of the projection onto the summand Λ of E. Let Γe denote the left ideal generated by e. The following lemma describes the projective-injective objects of Γ-mod explicitly.

Lemma 2.16. *The projective-injective objects of Γ-mod are precisely the objects of* $\mathrm{add}(\Gamma e)$. *Moreover* $\Gamma e \cong (E, \Lambda)_\Lambda \cong \mathrm{D}\,(\Lambda, E)_\Lambda$.

Proof. It is clear that $\Gamma e \cong (E, \Lambda)_\Lambda$. Recall that there is an equivalence $\mathrm{D}\,({}_\Lambda\Lambda, -)_\Lambda \cong (-, \nu_\Lambda\Lambda)_\Lambda$, and $\nu_\Lambda\Lambda = {}_\Lambda\Lambda$ because Λ is self-injective. Hence $(E, \Lambda)_\Lambda \cong \mathrm{D}\,(\Lambda, E)_\Lambda$, and by lemmas 2.2 and 2.3 it is a projective-injective module.

Let $(E, M)_\Lambda$ be a projective-injective Γ-module. Then every monomorphism $(E, M)_\Lambda \rightarrow (E, N)_\Lambda$ is a split monomorphism, but that implies any monomorphism $M \rightarrow N$ in Λ-mod is a split monomorphism. Thus M is a projective-injective Λ-module. $\qquad\square$

Remark 2.17. This means the indecomposable projective-injective Γ-modules are the projective modules at vertices corresponding to indecomposable projective-injective Λ-modules, when we consider the Gabriel quiver of Γ as the opposite of the Auslander-Reiten quiver of Λ-mod.

2.3.1 Recollement

Notice that $e\Gamma e = \operatorname{End}(\Lambda)^{\mathrm{op}} \cong \Lambda$, hence Λ embeds into Γ. Let $\Gamma e \Gamma$ denote the two sided ideal of Γ generated by e and denote the quotient $\Gamma/\Gamma e \Gamma$ by $\underline{\Gamma}$, we call this the *stable Auslander algebra* of Λ. Consider the diagram

$$\underline{\Gamma}\text{-mod} \xleftarrow[\quad p \quad]{\overset{q}{\underset{\iota}{\xleftarrow{\quad\quad}}}} \Gamma\text{-mod} \xleftarrow[\quad r \quad]{\overset{l}{\underset{e}{\xleftarrow{\quad\quad}}}} \Lambda\text{-mod}$$

of functors, where the functors are defined as follows:

$$
\begin{aligned}
q &:= \Gamma/\Gamma e \Gamma \otimes_\Gamma -, & l &:= \Gamma e \otimes_\Lambda -, \\
\iota &:= \text{Inclusion}, & e &:= (\Gamma e, -)_\Gamma, \\
p &:= (\Gamma/\Gamma e \Gamma, -)_\Gamma, & r &:= (e\Gamma, -)_\Lambda.
\end{aligned}
$$

This construction goes back to Cline,Parshall and Scott [17, 18], and it gives a recollement of abelian categories. In other words the functors above satisfy the following conditions:

(a) The functor l is a left adjoint of e and r is a right adjoint of e.

(b) The unit $\operatorname{id}_\Lambda \to el$ and the counit $er \to \operatorname{id}_\Lambda$ are isomorphisms.

(c) The functor q is a left adjoint of ι and p is a right adjoint of ι.

(d) The unit $\operatorname{id}_{\underline{\Gamma}} \to p\iota$ and the counit $q\iota \to \operatorname{id}_{\underline{\Gamma}}$ are isomorphisms.

(e) The functor ι is an embedding onto the full subcategory $\ker(e)$.

Remark 2.18. Since Λ is self-injective, Γ can be identified with the projective quotient algebra introduced in [22, Section 5] and the recollement above is the same as the main recollement from [22, Section 4].

We construct the *intermediate extension functor* $c\colon \Lambda\text{-mod} \to \Gamma\text{-mod}$ as follows. Since the counit $er \to \operatorname{id}_\Lambda$ is an isomorphism we have an inverse $\operatorname{id}_\Lambda \to er$. If we apply the adjunction (l, e) to the inverse we get a natural transformation $\gamma\colon l \to r$. Then we define $c := \operatorname{Im}\gamma$.

Consider the Γ-module $T := c(E)$.

Lemma 2.19. *The module T is a tilting and cotilting module. Moreover the following conditions hold.*

(i) *The kernel of p is* $\ker(p) = \operatorname{cogen}(T) = \Gamma\text{-torsl}$.

(ii) *The kernel of q is* $\ker(q) = \operatorname{gen}(T) = \Gamma\text{-divbl}$.

Proof. To see that T is a tilting and cotilting module we refer to [22, Section 5]. There it is also shown that $\ker(p) = \operatorname{cogen}(T)$ and $\ker(q) = \operatorname{gen}(T)$. Since T is tilting, all projective Γ-modules are in $\operatorname{cogen}(T)$. Hence Γ-torsl is contained in $\operatorname{cogen}(T)$. Since T is a tilting module it is of projective dimension at most 1 and hence torsionless by Proposition 2.11. But Γ-torsl is closed under taking submodules, thus $\operatorname{cogen}(T) \subset \Gamma$-torsl, this proves (i). The proof of $\operatorname{gen}(T) = \Gamma$-divbl goes dually. $\qquad\square$

2.3.2 The stable Auslander algebra

The algebra $\underline{\Gamma}$ has an alternative description. Notice that $\Gamma e \Gamma \subset \Gamma$ is given by all maps in $\operatorname{End}(E)^{\mathrm{op}}$ that factor through a projective-injective Λ-module. Therefore $\underline{\Gamma} = \underline{\operatorname{End}}_\Lambda(E)^{\mathrm{op}}$, the opposite of the endomorphism ring of E in the stable category $\Lambda\text{-}\underline{\mathrm{mod}}$. Thus $\underline{\Gamma}$-mod is equivalent to the category of finitely presented additive functors from $(\Lambda\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}$ to k-vector spaces.

The following proposition is classical. It follows from [33, Theorem 1.7] and the fact that every map in a triangulated category is a weak kernel and weak cokernel.

Proposition 2.20 (Freyd's Theorem)**.** *Let $\mathcal{T}$ be a triangulated category. Then* $\operatorname{fun}(\mathcal{T})$ *is a Frobenius category.*

Since Λ is self-injective, $\Lambda\text{-}\underline{\mathrm{mod}}$ is a triangulated category. Hence the following corollary.

Corollary 2.21. *The category $\underline{\Gamma}$-mod is a Frobenius category.*

2.4 From submodule categories to representations of the stable Auslander algebra

Here we follow the story of [55] in a more general setting for any basic self-injective algebra Λ of finite representation type. We have already studied the functor $\alpha\eta\colon \mathcal{S}(\Lambda) \to \Gamma$-mod in Section 2.2 and $q\colon \Gamma$-mod $\to \underline{\Gamma}$-mod in Section 2.3. We use what we have gathered about those functors to study the compositions

$$\mathcal{S}(\Lambda) \xrightarrow[\epsilon]{\eta} T_2(\Lambda)\text{-mod} \xrightarrow{\alpha} \Gamma\text{-mod} \xrightarrow{q} \underline{\Gamma}\text{-mod}.$$

The functors F and G are given by $F := q\alpha\eta$ and $G := q\alpha\epsilon$. The functor F was already studied by Li and Zhang in [45]. In [5] Auslander and Reiten considered the composition G, based on previous work by Gabriel [34].

2.4.1 Induced equivalences

We have already established that η and α as well as the compositions $\alpha\eta$ and $\alpha\epsilon$ are objective. In corollary 2.9 we established that the essential image of

$\alpha\eta$ is Γ-torsl. For now we shall consider $\alpha\eta$ as a functor to Γ-torsl. Moreover we write q_t for the restriction of q to Γ-torsl.

Proposition 2.22. *The functor q_t is objective. More precisely it induces an equivalence from Γ-torsl/$\mathrm{add}(T)$ to $\underline{\Gamma}$-mod.*

Proof. Since $\ker q = \mathrm{gen}(T)$ and Γ-torsl $= \mathrm{cogen}(T)$ we know that the kernel of q_t is $\mathrm{cogen}(T) \cap \mathrm{gen}(T) = \mathrm{add}(T)$. First we show that the induced functor Γ-torsl/$\mathrm{add}(T) \to \underline{\Gamma}$-mod is faithful. By [31, Proposition 4.2] there is an exact sequence of functors

$$le \xrightarrow{\ \psi\ } \mathrm{id}_\Gamma \xrightarrow{\ \phi\ } \iota q \longrightarrow 0.$$

Let $Y \in \Gamma$-torsl, there is an epimorphism $\phi_Y \colon Y \to \iota q(Y)$ and the morphism $\psi_Y \colon le(Y) \to Y$ factors through $\ker(\phi_Y)$ via an epimorphism, in particular $\ker(\phi_Y)$ is in $\mathrm{gen}(T)$. Since $\ker(\phi_Y)$ is a submodule of Y it belongs to $\mathrm{cogen}(T)$, thus $\ker(\phi_Y) \in \mathrm{add}(T)$. Now let $f \colon X \to Y$ be a morphism in Γ-torsl such that $q_t(f) = 0$. Then $\phi_Y f = 0$ and thus f factors through $\ker(\phi_Y)$.

To show q_t is full we consider the adjoint pair (q, ι). Let $X, Y \in \Gamma$-torsl, we want to show the map q_{XY} induced by the functor q in the following sequence is surjective.

$$(X,Y)_\Gamma \xrightarrow{\ q_{XY}\ } (qX, qY)_{\underline{\Gamma}} \xrightarrow{\ \Phi\ } (X, \iota qY)_\Gamma.$$

Here Φ is the isomorphism given by the adjunction (q, ι). Let $\phi_Y := \Phi(\mathrm{id}_{qY})$, by [31, Proposition 4.2] this is an epimorphism, so we get an exact sequence

$$0 \longrightarrow K \longrightarrow Y \xrightarrow{\ \phi_Y\ } \iota qY \longrightarrow 0.$$

By our argument above we know $K \in \mathrm{add}(T)$. Apply $(X, -)_\Gamma$ to the exact sequence above and get the exact sequence

$$0 \longrightarrow (X,K)_\Gamma \longrightarrow (X,Y)_\Gamma \xrightarrow{\ (X,\phi_Y)\ } (X, \iota qY)_\Gamma \longrightarrow \mathrm{Ext}^1_\Gamma(X,K).$$

Now $X \in \mathrm{cogen}(T)$ and $K \in \mathrm{add}(T)$ so $\mathrm{Ext}^1(X,K) = 0$, thus $(X, \phi_Y) = \Phi \circ q_{XY}$ is an epimorphism. Since Φ is an isomorphism that implies q_{XY} is an epimorphism.

To show denseness we adapt the proof of [55, Proposition 5]. Let $\underline{X} \in \underline{\Gamma}$-mod and write $X := \iota(\underline{X})$. We let $u \colon X \to I(X)$ be the injective envelope and $p \colon PI(X) \to I(X)$ be a projective cover with kernel K. We get an induced diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\ v'\ } & Y & \xrightarrow{\ p'\ } & X & \longrightarrow & 0 \\
  &   & \| &   & \downarrow u' & & \downarrow u & & \\
0 & \longrightarrow & K & \xrightarrow{\ v\ } & PI(X) & \xrightarrow{\ p\ } & I(X) & \longrightarrow & 0.
\end{array}
$$

26

Since u' is a monomorphism, Y embeds into a projective-injective Γ-module, and hence $Y \in \Gamma$-torsl. Moreover K has injective dimension at most 1, hence $q_t(K) = 0$. By the defining properties of a recollement we have $q_t(X) \cong q\iota(\underline{X}) \cong \underline{X}$, and q_t is right-exact because q is a left adjoint. It follows that $q_t(Y) \cong q_t(X) \cong \underline{X}$. We have shown that q_t is dense. $\qquad\square$

We know $\alpha\eta$ is objective and dense when considered as a functor to Γ-torsl and thus the functor F is objective. Moreover F is full and dense because q_t and $\alpha\eta$ are full and dense.

Let $f \in T_2(\Lambda)$-mod, then Lemma 2.13 implies that f is an epimorphism if and only if $e(\alpha(f)) = (\Gamma e, \alpha(f))_\Gamma = 0$. Since α is dense this means the essential image of $\alpha\epsilon$ is $\ker(e)$, but we can identify $\ker(e)$ with $\underline{\Gamma}$-mod via ι. We have established $\alpha\epsilon$ is an objective functor so we conclude G is objective. The functor G is also full and dense because $\alpha\epsilon$ is full and dense when considered as a functor to $\ker(e)$.

Now we can prove our first main theorem.

Theorem 2.23. *Let Λ be a basic, self-injective and representation finite algebra. Let $\mathcal{U}$ denote the smallest additive subcategory of $\mathcal{S}(\Lambda)$ containing $(E \xrightarrow{\mathrm{id}} E)$ and all objects of the form $(M \xrightarrow{f} I)$, where I is a projective-injective Λ-module. Also define $\mathcal{V} := \mathrm{add}((E \xrightarrow{\mathrm{id}} E) \oplus (0 \to E))$. Let m be the number of isomorphism classes of $\mathrm{ind}(\Lambda)$. Then $\mathcal{U}$ and $\mathcal{V}$ have $2m$ indecomposable objects up to isomorphism and the following holds.*

(i) The functor F induces an equivalence of categories $\mathcal{S}(\Lambda)/\mathcal{U} \to \underline{\Gamma}$-mod.

(ii) The functor G induces an equivalence of categories $\mathcal{S}(\Lambda)/\mathcal{V} \to \underline{\Gamma}$-mod.

Proof. The indecomposable objects of $\mathcal{V}$ are $(M \xrightarrow{\mathrm{id}} M)$ and $(0 \to M)$ for any $M \in \mathrm{ind}(\Lambda)$. Hence $\mathcal{V}$ clearly has $2m$ indecomposable objects up to isomorphism. The indecomposable objects of $\mathcal{U}$ are $(M \xrightarrow{\mathrm{id}} M)$ and the injective envelope $(M \to I(M))$ for each $M \in \mathrm{ind}(\Lambda)$, as well as the objects $(0 \to I)$ for each injective object $I \in \mathrm{ind}(\Lambda)$. Since the objects $(I \xrightarrow{\mathrm{id}} I)$ for $I \in \mathrm{ind}(\Lambda)$ injective appear twice in this list, $\mathcal{U}$ has $2m$ indecomposable objects up to isomorphism.

Next we prove (i). Let $(M \xrightarrow{f} N) \in \mathcal{S}(\Lambda)$, and assume $F(f) = 0$. Consider the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P & \xrightarrow{\;\sim\;} & P & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (E,M)_\Lambda & \xrightarrow{(E,f)_\Lambda} & (E,N)_\Lambda & \longrightarrow & \alpha(f) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & P_1 & \xrightarrow{\;p_1\;} & P_0 & \longrightarrow & \alpha(f) & \longrightarrow & 0.
\end{array}
$$

Here the bottom row is a minimal projective resolution and all rows and columns are exact, thus P is projective and the first two columns are split exact

sequences of projective modules. Since Λ-mod is equivalent to the full subcategory of projective Γ-modules this shows f is a direct sum of an isomorphism $(M' \xrightarrow{f'} N')$, corresponding to $P \simeq P$, and a monomorphism $(M'' \xrightarrow{f''} N'')$, corresponding to the map p_1. Now $F(f) = q(\alpha(f)) = 0$ if and only if P_0 is projective-injective by Lemma 2.19, which is if and only if N'' is projective-injective by Lemma 2.16.

We already know F is objective, and thus the functor $\mathcal{S}(\Lambda)/\mathcal{U} \to \underline{\Gamma}$-mod induced by F is faithful. Since F is full and dense the induced functor is also full and dense.

Now to (ii). The kernel of α is $\mathrm{add}((E \xrightarrow{\mathrm{id}} E) \oplus (E \to 0))$. But

$$\epsilon((E \xrightarrow{\mathrm{id}} E) \oplus (0 \to E)) = (E \xrightarrow{\mathrm{id}} E) \oplus (E \to 0),$$

hence $\mathcal{V} = \mathrm{add}((E \xrightarrow{\mathrm{id}} E) \oplus (0 \to E)) = \ker(\alpha\epsilon)$. Moreover the restriction of q to the essential image $\ker(e)$ of $\alpha\epsilon$ is an equivalence by the defining properties of a recollement. We have shown G is full, dense and objective, thus (ii) holds. $\qquad\square$

2.4.2 Interplay with triangulated structure

We have already established that the categories $\mathcal{S}(\Lambda)$ and $\underline{\Gamma}$-mod are Frobenius categories. Then it is natural to ask whether the triangulated structure of the stable category $\underline{\Gamma}$-mod interacts nicely with that functors F and G.

Let $\pi\colon \underline{\Gamma}$-mod $\to \underline{\Gamma}$-$\underline{\mathrm{mod}}$ be the projection to the stable category. We denote the syzygy functor on $\underline{\Gamma}$-$\underline{\mathrm{mod}}$ by Ω. The following was proven in a special case in [55, Section 7], and we prove this more general statement analogously.

Theorem 2.24. *The functors πF and πG differ by the syzygy functor on $\underline{\Gamma}$-mod, more precisely $\pi F = \Omega \pi G$.*

Proof. Let $(L \xrightarrow{f} M)$ be an object in $\mathcal{S}(\Lambda)$. We have the corresponding exact sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0.$$

Notice that $g = \epsilon(f)$. Apply $(E, -)_\Lambda$ to this sequence and obtain the exact sequence

$$0 \longrightarrow (E, L)_\Lambda \xrightarrow{(E,f)} (E, M)_\Lambda \xrightarrow{(E,g)} (E, N)_\Lambda.$$

The cokernel of (E, f), and hence the image of (E, g), is by definition $\alpha\eta(f)$. Also the cokernel of (E, g) is $\alpha\epsilon(f)$. Thus we get an exact sequence

$$0 \longrightarrow \alpha\eta(f) \xrightarrow{\mathrm{Im}\,(E,g)} (E, N)_\Lambda \longrightarrow \alpha\epsilon(f) \longrightarrow 0.$$

From [31, Proposition 4.2] we know there is an exact sequence of functors $le \longrightarrow \mathrm{id}_\Gamma \longrightarrow \iota q \longrightarrow 0$. We obtain a commutative diagram with exact rows

and columns:

$$
\begin{array}{ccccc}
le\alpha\eta(f) & \xrightarrow{\phi} & le(E,N)_\Lambda & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow \alpha\eta(f) & \xrightarrow{\mathrm{Im}\,(E,g)} & (E,N)_\Lambda & \longrightarrow \alpha\epsilon(f) \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\iota F(f) & \xrightarrow{\iota q(\mathrm{Im}\,(E,g))} & \iota q(E,N)_\Lambda & \longrightarrow \iota G(f) \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

Since e is exact and $e\alpha\epsilon = 0$ the map ϕ is an isomorphism. But then we can extend the top row to a short exact sequence and apply the snake lemma to see that $\iota q(\mathrm{Im}\,(E,g))$ is a monomorphism.

Since ι is fully faithful and exact this implies we have the following exact sequence in $\underline{\Gamma}$-mod:

$$
0 \longrightarrow F(f) \longrightarrow q((E,N)_\Lambda) \longrightarrow G(f) \longrightarrow 0.
$$

Now ι preserves epimorphisms and q is its left adjoint, thus q preserves projective objects. We know $(E,N)_\Lambda$ is a projective Γ-module and hence $q((E,N)_\Lambda)$ is projective, this shows $\pi F(f) \cong \Omega\pi G(f)$ in $\underline{\Gamma}$-$\underline{\mathrm{mod}}$. $\qquad\square$

Remark 2.25. By Proposition 2.14 $\mathcal{S}(\Lambda)$ is also a Frobenius category, so the stable category $\underline{S}(\Lambda)$ is a triangulated category. Hence one might ask whether F and G induce a triangle functor from $\underline{S}(\Lambda)$ to $\underline{\Gamma}$-$\underline{\mathrm{mod}}$. However all maps factoring through projective objects in $\mathcal{S}(\Lambda)$ factor through both $\mathcal{U}$ and $\mathcal{V}$, thus any induced triangle functor would have to factor through the abelian category $\underline{\Gamma}$-mod, which renders any such functor trivial.

2.5 Auslander algebras of Nakayama algebras

A finite length module is said to be *uniserial* if it has a unique composition series. We say an algebra A is uniserial, or a *Nakayama algebra* , if all indecomposable A-modules have a unique composition series. In this section we prove Theorem 2.26.

Theorem 2.26. *Let Λ be a basic representation-finite algebra and let Γ be its Auslander algebra. Then Γ has a quasi-hereditary structure such that the objects of Γ-torsl are precisely the Δ-filtered Γ-modules if and only if Λ is uniserial.*

The only if part is proven in Subsection 2.5.3, but before that we describe a quasi-hereditary structure with the properties from Theorem 2.26 for the Auslander algebras of Nakayama algebras. First, however, we consider the example of self-injective Nakayama algebras over an algebraically closed field explicitly, to get some picture of the situation.
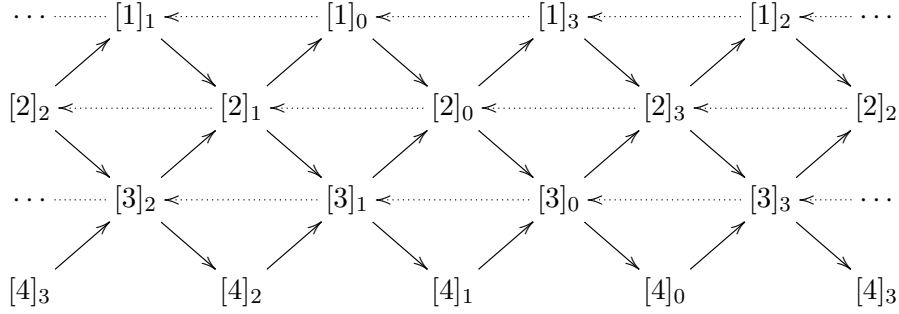
$$\cdots \quad [1]_1 \leftarrow [1]_0 \leftarrow [1]_3 \leftarrow [1]_2 \leftarrow \cdots$$
$$[2]_2 \leftarrow [2]_1 \leftarrow [2]_0 \leftarrow [2]_3 \leftarrow [2]_2$$
$$\cdots \quad [3]_2 \leftarrow [3]_1 \leftarrow [3]_0 \leftarrow [3]_3 \quad \cdots$$
$$[4]_3 \quad [4]_2 \quad [4]_1 \quad [4]_0 \quad [4]_3$$

Figure 2.1: Auslander-Reiten quiver of $A(4,3)$

2.5.1 Self-injective Nakayama algebras

The classification of Nakayama algebras over algebraically closed fields is well known, and can for example be found in [1, V.3]. We recall the self-injective case, which is of particular interest to us in the context of Theorems 2.23 and 2.24, to get an explicit description of an example. Let $\tilde{\mathbb{A}}_m$ denote the quiver with vertices $\mathbb{Z}/m\mathbb{Z}$ and arrows $i \to i+1$ for all $i \in \mathbb{Z}/m\mathbb{Z}$. Write $k\tilde{\mathbb{A}}_m$ for the path algebra of this quiver and let $J(k\tilde{\mathbb{A}}_m)$ denote the ideal generated by the arrows. For $m, N \in \mathbb{N}$ we define $A(m,N) \coloneqq k\tilde{\mathbb{A}}_m / J(k\tilde{\mathbb{A}}_m)^{N+1}$, these are precisely the basic connected self-injective Nakayama algebras. Note that $A(1,N) \cong k[x]/\langle x^{N+1} \rangle$, and hence the case studied in detail in [55] is included. We parametrize the simple $A(m,N)$ modules by the vertices $j \in \mathbb{Z}/m\mathbb{Z}$ of $\tilde{\mathbb{A}}_m$. The category $A(m,N)$-mod has indecomposable objects $[i]_j$ for $j \in \mathbb{Z}/m\mathbb{Z}$ and $i = 1, ..., N+1$, where $\mathrm{soc}([i]_j) = S(j)$ and $[i]_j$ has Loewy length i.

To get an idea of the general shape of the Auslander-Reiten quiver, take for example the Auslander-Reiten quiver of $A(4,3)$ in figure 2.1. The Gabriel quiver of Γ is the opposite of this quiver.

Remark. We may consider $A(m,N)$-mod as a $\mathbb{Z}/m\mathbb{Z}$-fold cover of $A(1,N)$-mod. Namely, if we give $A(1,N) = k[x]/\langle x^{N+1} \rangle$ the $\mathbb{Z}$-grading given by monomial degrees, it induces a $\mathbb{Z}/m\mathbb{Z}$ grading and the categories $k[x]/\langle x^{N+1} \rangle\text{-mod}^{\mathbb{Z}/m\mathbb{Z}}$ and $A(m,N)$-mod are isomorphic.

2.5.2 Auslander algebras of Nakayama algebras

We refer to Section 1.5 of the preliminaries for our notation for quasi-hereditary algebras. Ringel has shown that any Auslander algebra of a representation-finite algebra has a left strongly quasi-hereditary structure cf. [52, Section 5]. A large part of the quasi-hereditary structures described here were already studied in [64]

Let Λ be a Nakayama algebra and let Γ be its Auslander algebra. The isomorphism classes of simple Γ-modules are in a canonical bijection with the isomorphism classes of $\mathrm{ind}(\Lambda)$, denoted by $\mathrm{ind}(\Lambda)/\sim$. For any $M \in \mathrm{ind}(\Lambda)$,

let $p_M \colon E \to M$ be the projection, and $i_M \colon M \to E$ be the inclusion. Then M corresponds to the idempotent $(i_M p_M)^{\mathrm{op}} \in \Gamma$, which corresponds to an isomorphism class of simple Γ-modules. We use this bijection to parametrize the simple Γ-modules. For $M \in \mathrm{ind}(\Lambda)$ we let $[M]$ denote its isomorphism class in $\mathrm{ind}(\Lambda)/\sim$, although we write $S(M), P(M), I(M), \Delta(M), \nabla(M)$ resp. instead of $S([M]), P([M]), I([M]), \Delta([M]), \nabla([M])$ resp.

Let $\ell(M)$ denote the Loewy-length of a Λ-module M. We consider a partial ordering on $\mathrm{ind}(\Lambda)/\sim$ given by the Loewy length: For $M, N \in \mathrm{ind}(\Lambda)$, say $[M] > [N]$ if $\ell(M) < \ell(N)$, but $[M] \neq [N]$ are incomparable if $\ell(M) = \ell(N)$.

Remark 2.27. Notice that modules with greater Loewy-length are smaller in our partial ordering. Thus the simple modules are maximal.

Let X be an indecomposable Γ-module with $\mathrm{top}(X) = S(M)$ and $\mathrm{soc}(X) = S(N)$. If $M \not\cong N$ and $\ell(M) = \ell(N)$, then there is a non-trivial map $f \colon N \to M$ such that for every indecomposable summand M' of $\mathrm{Im}\, f$, $S(M')$ is in the composition series of X. We have $\ell(M) > \ell(M')$ for every such summand M' of $\mathrm{Im}\, f$, i.e. $[M'] > [M]$. Thus our partial order is adapted.

Let $M \in \mathrm{ind}(\Lambda)$. If M is simple then there are no non-trivial homomorphisms from other simple Λ modules to M. Hence $\mathrm{Hom}_\Gamma(P(N), P(M)) = 0$ for all simple $N \not\cong M$, which implies $\Delta(M) = P(M)$. Similarly we have $\mathrm{Hom}_\Gamma(I(M), I(N)) = 0$ for all simple $N \not\cong M$, and thus $\nabla(M) \cong I(M)$.

Now Λ is uniserial, so if M is not simple it has a unique maximal proper submodule M', and clearly $\ell(M') = \ell(M) - 1$. Recall that $P(M) \cong (E, M)_\Lambda$. Let $N \in \mathrm{ind}(\Lambda)$, for any map in $(N, M')_\Lambda$, composition with the inclusion of M' in M gives a map in $(N, M)_\Lambda$. In this way $P(M')$ embeds in $P(M)$ as a Γ-submodule. Any non-surjective map to M factors through M', in particular, if $N \in \mathrm{ind}(\Lambda)$ with $\ell(N) < \ell(M)$, then any map in $(N, M)_\Lambda$ factors through M'. This shows that $\Delta(M) \cong P(M)/P(M')$ and thus $\Delta(M)$ has projective dimension 1. Consequently all modules in $\mathcal{F}(\Delta)$ have projective dimension at most 1.

We proceed in a similar way for the costandard modules. If M is non-simple, then it has a unique maximal proper factor module M''. The projection $M \to M''$ induces an epimorphism $I(M) \to I(M'')$. We know any non-injective map from M to N factors through M''. In particular, if $\ell(N) < \ell(M)$ and $N \in \mathrm{ind}(\Lambda)$, then any map in $(M, N)_\Lambda$ factors through the projection $M \to M''$. Thus the kernel of the map $I(M) \to I(M'')$ has no composition factors $S(N)$ such that $\ell(N) < \ell(M)$. Together this implies that we have an exact sequence

$$ 0 \longrightarrow \nabla(M) \longrightarrow I(M) \longrightarrow I(M'') \longrightarrow 0. $$

In particular $\nabla(M)$ has injective dimension 1 and thus any module in $\mathcal{F}(\nabla)$ has injective dimension at most 1. Taking everything together we get the following proposition.

Proposition 2.28. *Let Γ be the Auslander algebra of a Nakayama algebra. The partial ordering above gives Γ a quasi-hereditary structure with $\mathcal{F}(\Delta) = \Gamma$-torsl and $\mathcal{F}(\nabla) = \Gamma$-divbl.*

Proof. The Γ-module $_\Gamma\Gamma$ is in $\mathcal{F}(\Delta)$ and all the standard modules are Schurian. Thus our partial ordering gives a quasi-hereditary structure on Γ. Since all objects of $\mathcal{F}(\Delta)$ have projective dimension at most 1 we see $\mathcal{F}(\Delta) \subset \Gamma$-torsl. Also all the costandard modules have injective dimension at most 1, so by 1.14 and Proposition 2.11 we have Γ-torsl $\subset \mathcal{F}(\Delta)$.

We show $\mathcal{F}(\nabla) = \Gamma$-divbl dually using [26, Lemma 4.1] and Proposition 2.12. $\qquad\square$

2.5.3 Other representation-finite algebras

Quasi-hereditary structures on Auslander algebras of representation-finite algebras that have the property given in Proposition 2.28 are rare in general. Indeed, the examples illustrated in Subsection 2.5.2 are the only cases.

Proposition 2.29. *Let Λ be a basic representation-finite algebra and let Γ be its Auslander algebra. If Γ has a quasi-hereditary structure such that the Δ-filtered modules coincide with Γ-torsl, then Λ is uniserial.*

Proof. Let Γ have a quasi-hereditary structure such that $\mathcal{F}(\Delta) = \Gamma$-torsl. It suffices to show that all indecomposable projective and all indecomposable injective Λ-modules have a unique composition series. Let $M \in \mathrm{ind}(\Lambda)$ be a submodule of an indecomposable injective module. Let $\Delta(M)$ be the standard module generated by $(E, M)_\Lambda$. By assumption we have a projective resolution

$$0 \longrightarrow P_1 \xrightarrow{\ \pi_1\ } (E, M)_\Lambda \xrightarrow{\ \pi_0\ } \Delta(M) \longrightarrow 0.$$

Since Γ-proj is equivalent to Λ-mod we get a monomorphism $f\colon N \to M$ in Λ-mod such that $\pi_1 = (E, f)_\Lambda$. Let M' be any proper submodule of M and let ι denote its inclusion. Then $\alpha(\iota)$ is Δ-filtered, hence it has $\Delta(M)$ as a factor module. Thus there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (E, M')_\Lambda & \xrightarrow{(E,\iota)} & (E, M)_\Lambda & \longrightarrow & \alpha(\iota) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\psi} & & \| & & \downarrow & & \\
0 & \longrightarrow & (E, N)_\Lambda & \xrightarrow{\ \pi_1\ } & (E, M)_\Lambda & \xrightarrow{\ \pi_0\ } & \Delta(M) & \longrightarrow & 0.
\end{array}$$

We get an induced map $\psi\colon (E, M')_\Lambda \to (E, N)_\Lambda$ making the diagram above commutative. This yields a monomorphism $g\colon M' \to N$ such that $\iota = fg$. If we identify N with its image in M via f this shows every proper submodule M' of M is a submodule of N. Since N is a submodule of an indecomposable injective module it is also indecomposable. This shows that any indecomposable injective Λ-module has a unique composition series. Dually, using that Γ-divbl $= \mathcal{F}(\nabla)$ we can show all indecomposable projective Λ-modules have a unique composition series. $\qquad\square$

Combining Proposition 2.29 with Proposition 2.28 now yields Theorem 2.26.

Given a quasi-hereditary structure on Γ the condition $\mathcal{F}(\Delta) = \Gamma$-torsl is the same as conditions (a) and (d') in Lemma 1.14 combined. Thus all the conditions of this lemma hold, in particular $\mathcal{F}(\nabla) = \Gamma$-divbl and thus $\mathrm{add}(T) = \mathrm{add}(C_\Gamma)$, where C_Γ is the characteristic module of Γ. Since both T and C_Γ are basic this implies they are isomorphic. Conversely, if $T \cong C_\Gamma$, then $\mathcal{F}(\Delta) = \mathrm{cogen}(C_\Gamma) = \mathrm{cogen}(T) = \Gamma$-torsl. Hence Theorem 2.26 yields the following corollary.

Corollary 2.30. *Let Λ be a basic self-injective algebra of finite representation type and let Γ be the Auslander algebra of Λ. We let $T = c(E)$ be the canonical tilting and cotilting module as defined in Subsection 2.3.1. Then T is a characteristic module of a quasi-hereditary structure on Γ if and only if Λ is a Nakayama algebra.*

Chapter 3

Quiver flag varieties

3.1 Overview

The monomorphism categories are a straightforward generalisation of the sub-module categories that we have covered in Section 2.2, given by a sequence of monomorphisms in A-mod. Similarly quiver flag varieties are a straightforward generalisation of quiver Grassmannians, parameterising flags of submodules.

 The aim of this chapter is to outline those two generalisations in parallel and see how they are connected. We will follow similar lines for a different setting in Chapter 4 and [29], which allows some comparison of the merits of the different settings.

 We fix a flag of vector spaces on each vertex of a quiver Q, giving a dimension filtration $\mathbf{d}$ of a dimension vector d. This determines a closed subvariety $\mathrm{Rep}_d^{\mathbf{d}} \subset \mathrm{Rep}_d(Q)$, parametrising representations that fix the flag. Let $\mathrm{P}_{\mathbf{d}} \subset \mathrm{GL}_d$ be the parabolic subgroup that fixes the flags, then $\mathrm{Rep}_d^{\mathbf{d}}$ has a $\mathrm{P}_{\mathbf{d}}$-action. We also consider a GL_d-variety $\mathrm{Repfl}_{\mathbf{d}}$ parametrising representations along with flags of submodules, as well as the $\mathrm{GL}_{\mathbf{d}}$-variety $\mathrm{Repinj}_{\mathbf{d}}(Q)$. These varieties are all irreducible, and their respective group actions are closely related via maps

$$\mathrm{Repinj}_{\mathbf{d}}(Q) \xrightarrow{\pi_{\mathbf{d}}^{\mathrm{mon}}} \mathrm{Repfl}_{\mathbf{d}} \xrightarrow{\mathrm{co}_{\mathbf{d}}} \mathrm{Rep}_d(Q).$$

The map $\mathrm{co}_{\mathbf{d}}$ can be considered as a quiver graded analogue of the springer map, the same setting is studied for Dynkin quivers in [48]. There it is shown how this map can give desingularisations of GL_d-orbit closures in $\mathrm{Rep}_d(Q)$. In fact it is equivalent, given a flag, for each of the group actions above to act with a dense orbit. We are of course interested to find out when this is the case, and we show that this condition translates into the existence of a rigid object in the monomorphism category. The quiver flag varieties arise as the fibres $\mathrm{co}_{\mathbf{d}}^{-1}(M)$ over points in $\mathrm{Rep}_d(Q)$. The fibres $\mathrm{co}_{\mathbf{d}}^{-1}(M)$ that are contained within dense GL_d-orbits of $\mathrm{Repfl}_{\mathbf{d}}$ have dense $\mathrm{Aut}_Q(M)$-orbits.

 If we take $A := kQ/\langle R \rangle$ for (Q, R) a quiver with admissible relations the situation is usually more complicated. The same constructions give varieties

that are not irreducible in general. We will see what we can salvage of the results about the varieties coming from path algebras of acyclic quivers, and give some counterexamples when they do not generalise.

Let A be a finite dimensional algebra. If A is quasi-hereditary, we get an induced quasi-hereditary structure on the algebra $\mathrm{T}_s(A)$ of upper triangular $s \times s$ matrices over A. If $A = kQ$ is a path algebra with the simple modules as standard modules the full subcategory of Δ-filtered modules in $\mathrm{T}_s(Q)$ is precisely the monomorphism category $\mathrm{mon}_s(Q)$.

Our geometric considerations are a motivation to look for rigid objects in $\mathrm{mon}_s(A)$. For $\mathrm{mon}_s(Q)$ we have some explicit description of the AR-translate, that might help with this. The relative AR-translate in $\mathrm{mon}_s(Q)$ has already been calculated explicitly in [54] and [66]. We show how we can calculate the AR-translates within $\mathrm{T}_s(Q)$-mod of objects in the monomorphism category $\mathrm{mon}_s(Q)$. We observe that this formula implies the formulas of Ringel-Schmidmeier [54] and Xiong-Zhang-Zhang [66].

The algebra $\mathrm{T}_s(A)$ arises as a tensor algebra in a canonical way, and we use this fact to write out a long exact sequence that allows us to calculate extensions in $\mathrm{T}_s(A)$-mod from homomorphisms and extensions over A-mod.

Another method to construct rigid objects is essentially given by Reineke [48]. We define Ext-directed decompositions and show how an Ext-directed decomposition of M allows us to construct a flag of submodules on M which gives a rigid object in $\mathrm{mon}_s(Q)$.

The content of this chapter is organized as follows. In Section 3.2 we define the monomorphism category. In Section 3.2.1 we show how a quasi-hereditary structure on A induces one on $\mathrm{T}_s(A)$. In Section 3.2.2 we give our formula for the AR-translate of modules in $\mathrm{mon}_s(A)$.

Section 3.3 is devoted to the geometric side of our story. There we define the various varieties to consider and how they are related. We also show how dense orbits arise from rigid objects in the monomorphism category. In Section 3.3.1 we discuss generalisations to quivers with relations.

Section 3.4 gives some techniques to calculate extensions and construct rigid objects in $\mathrm{mon}_s(A)$. Section 3.4.1 gives a long exact sequence that may be useful to calculate extensions and Section 3.4.2 shows how Ext-directed decompositions in A-mod allow us to construct rigid objects in $\mathrm{mon}_s(A)$.

3.2   Monomorphism categories

Let (Q, R) be a quiver with admissible relations and $A := kQ/\langle R \rangle$.

Let $T_s(A)$ denote the algebra of upper triangular $s \times s$ matrices over A. The category $T_s(A)$-mod is equivalent to the following category. It has as objects s-tuples of A-modules $\mathbf{M} = (M_1, \ldots, M_s)$ along with A-module homomorphisms $\phi_t \colon M_t \to M_{t+1}$ for $t = 1, \ldots, s-1$. A morphisms $f \colon \mathbf{M} \to \mathbf{M}'$ is a tuple $(f_1, \ldots, f_s)$ of morphisms of A-modules compatible with the maps ϕ_t and ϕ_t', i.e. $f_{t+1}\phi_t = \phi_t' f_t$ for $t = 1, \ldots, s-1$.

The *monomorphism category* $\mathrm{mon}_s(A)$ is the full subcategory of objects such that $\phi_1, \ldots, \phi_{s-1}$ are all monomorphisms. This is the same as the definition of monomorphism categories in [66], and generalises the submodule categories that we introduced in Section 2.2. In the case of path algebras we often write $\mathrm{T}_s(Q) := T_s(kQ)$ and $\mathrm{mon}_s(Q) := \mathrm{mon}_s(kQ)$.

Iterated application of the 5-lemma shows that $\mathrm{mon}_s(A)$ is an extension closed subcategory of $T_s(A)$-mod. The irreducible idempotents of $T_s(A)$ are given by the irreducible idempotents of A in the diagonal coordinates. We let $e(i_t)$ denote the idempotent given by the idempotent e_i of A in the t-th diagonal coordinate.

3.2.1 Quasi-hereditary structure

Let A as above and assume A has a quasi-hereditary structure Δ_A, and that this structure is induced by a an adapted partial ordering $\leq_A$ on Q_0. Let $\mathrm{pdim}\,\Delta(i) \leq d$ for all the standard modules. The irreducible idempotents of $T_s(A)$ are in a canonical bijection to isomorphism classes of simple $T_s(A)$-modules, we write $S(i_t)$ for the simple module corresponding to $e(i_t)$. Thus the set $\Xi = \{i_t \mid (i,t) \in Q_0 \times \{1, \ldots s\}\}$ parametrizes the simple modules. We define an adapted partial order on Ξ as follows:

$$i_t \leq j_{t'} :\Longleftrightarrow t > t' \text{ or } t = t', \ i \leq_A j.$$

Let $i \in Q_0$ and take a projective resolution

$$0 \longrightarrow P_d \longrightarrow \cdots \longrightarrow \bigoplus_{j \in J} P(j) \longrightarrow P(i) \longrightarrow \Delta(i) \longrightarrow 0,$$

where J is a multiset of elements from Q_0. Since $\Delta(i)$ is a standard module we may assume $j \geq i$ for all $j \in J$.

The form of the indecomposable projective $T_s(A)$-modules is known from [66, Lemma 1.1], they have the form

$$P(i_t) \cong (0 \to \cdots \to 0 \to \underbrace{P(i) \overset{\mathrm{id}}{\to} \cdots \overset{\mathrm{id}}{\to} P(i)}_{s-t+1\text{-times}}),$$

where $P(i)$ is the indecomposable corresponding to $i \in Q_0$.

Define

$$\Delta(i_t) := (0 \to \cdots \to 0 \to \underbrace{\Delta(i) \overset{\mathrm{id}}{\to} \cdots \overset{\mathrm{id}}{\to} \Delta(i)}_{s-t+1\text{-times}}).$$

Clearly the projective resoulution of $\Delta(i)$ induces a projective resolution

$$0 \longrightarrow P'_d \longrightarrow \cdots \longrightarrow \bigoplus_{j \in J} P(j_t) \longrightarrow P(i_t) \longrightarrow \Delta(i_t) \longrightarrow 0.$$

By our assumption on J we have $j_t \geq i_t$ for all $j \in J$, and $\Delta(i_t)$ has no composition factors greater than i_t. Thus $\Delta(i_t)$ is really a standard module

36

with respect to the partial ordering on Ξ. We denote the family of these standard modules by $\Delta_{T_s(A)}$.

Since the projective module $P(i)$ is in $\mathcal{F}(\Delta_A)$ for all $i \in Q_0$, $P(i_t)$ has a filtration of standard modules for all $i_t \in \Xi$, so $T_s(A)$ is also quasi hereditary. Moreover, if $\Delta(i)$ has projective dimension at most d, then the same holds for $\Delta(i_t)$. We summarise this in the following proposition

Proposition 3.1. *Let $A = kQ/\langle R \rangle$ with R admissible relations. If (A, Δ_A) is quasi-hereditary, that induces a quasi hereditary structure $\Delta_{T_s(A)}$ on $T_s(A)$, with the standard modules given by the construction above. Moreover we have the following properties.*

(i) *If modules in Δ_A have projective dimension at most d, then the same holds for all modules in $\mathcal{F}(\Delta_{T_s(A)})$.*

(ii) *Let $\mathrm{mon}_s(\mathcal{F}(\Delta_A))$ denote the full subcategory of objects in $\mathrm{mon}_s(A)$ of the form $(M_1 \overset{\phi_1}{\to} \cdots \overset{\phi_{s-1}}{\to} M_s)$ where all the M_s are in $\mathcal{F}(\Delta_A)$. We have*

$$\mathcal{F}(\Delta_{T_s(A)}) \subset \mathrm{mon}_s(\mathcal{F}(\Delta_A)).$$

Proof. We have already given the quasi-hereditary structure $\Delta_{\mathrm{T}_s(A)}$ above. We have shown (i) for all the $\Delta(i_t)$, and it follows immediately for all modules in $\mathcal{F}(\Delta_{\mathrm{T}_s(A)})$.

All modules in $\Delta_{T_s(A)}$ are in $\mathrm{mon}_s(\mathcal{F}(\Delta_A))$. Since $\mathrm{mon}_s(\mathcal{F}(\Delta_A))$ is closed under taking extensions (ii) follows. $\qquad\square$

The case of path algebras Consider the special case $A = kQ$ for a connected acyclic quiver Q. Since Q is acyclic there is a total ordering $\leq$ on Q_0 such that $i \leq j$ whenever there exists an arrow $i \to j$ in Q_1. It is easy to check that the induced standard modules are the simple modules, so in particular this ordering gives a quasi-hereditary structure on kQ. By Proposition 3.1 the algebra $\mathrm{T}_s(A)$ is left-strongly quasi-hereditary, and the standard modules are of the form

$$\Delta(i_t) \cong (0 \to \cdots \to 0 \to \underbrace{S(i) \overset{\mathrm{id}}{\to} \cdots \overset{\mathrm{id}}{\to} S(i)}_{s-t+1\text{-times}}),$$

for $i \in Q_0$ and $t = 1, \ldots, s$. In this case $\mathcal{F}(\Delta) = A\text{-mod}$ is closed under quotients, and hence the inclusion from (ii) in Proposition 3.1 becomes

$$\mathcal{F}(\Delta_{T_s(A)}) = \mathrm{mon}_s(A).$$

In particular all modules in $\mathrm{mon}_s(A)$ have projective dimension at most 1. Julia Sauter already gave this realisation of the monomorphism category of a path algebra as the subcategory of Δ-filtered objects in [57].

3.2.2 AR-translates in monomorphism categories

Theorem 3.7 shows the geometric meaning of rigid objects in $\mathrm{mon}_s(A)$, which urges us to find ways to calculate extensions in the monomorphism categories. The Auslander-Reiten formula, which is part (3) in Proposition 1.3, suggests a method to calculate the dimension of Ext^1 in terms of dimensions of homomorphisms, if we can calculate some AR-translates. We have some methods for this if we restrict to the case of path algebras $A = kQ$. Then we are able to calculate the AR-translate of $M \in \mathrm{mon}_s(Q)$ in terms of the usually simpler AR-translate in kQ-mod.

Let us fix a finite quiver Q and positive integer s, for this section we write $\Gamma := T_s(Q)$. The monomorphism category is functorially finite, i.e. all objects of Γ-mod have left and right $\mathrm{mon}_s(Q)$-approximations. These approximations are constructed explicitly in [66], and generalise the approximations already constructed for submodule categories by Ringel-Schmidmeier [54]. We outline the construction of the left approximations, since it is relevant to what follows. Note that our labeling of the submodules is inverse to the one used in [66].

Let $\mathbf{M} \in \Gamma$-mod, we use the notation

$$\mathbf{M} = \left(M^{(1)} \overset{\phi_1}{\to} \cdots \overset{\phi_{s-1}}{\to} M^{(s)} \right).$$

We define a functor

$$\mathbf{Cok}\colon \Gamma\text{-mod} \to \Gamma\text{-mod},$$

$$\mathbf{Cok}(\mathbf{M}) := \left(M^{(s)} \to \mathrm{coker}(\phi_{s-1} \cdots \phi_1) \to \cdots \to \mathrm{coker}(\phi_{s-1}) \right).$$

The maps defining $\mathbf{Cok}(\mathbf{M})$ are the natural projections to cokernels, and a map $\mathbf{M} \to \mathbf{N}$ of Γ-modules induces a map $\mathbf{Cok}(\mathbf{M}) \to \mathbf{Cok}(\mathbf{N})$ in the obvious way from the properties of cokernels. Similarly we define

$$\mathbf{Ker}\colon \Gamma\text{-mod} \to \Gamma\text{-mod},$$

$$\mathbf{Ker}(\mathbf{M}) := \left(\ker \phi_1 \to \cdots \to \ker(\phi_{n-1} \cdots \phi_1) \to M^{(1)} \right).$$

All the maps defining $\mathbf{Ker}(\mathbf{M})$ are given by inclusions of kernels, and maps of Γ-modules $\mathbf{Ker}(M) \to \mathbf{Ker}(N)$ are the obvious induced maps. Note that $\mathrm{mon}_s(Q)$ is the essential image of $\mathbf{Ker}$.

We define the composition $\mathbf{Mono} := \mathbf{Ker}\,\mathbf{Cok}$.

Lemma 3.2 (Lemma 1.2 [66]). *For any $\mathbf{M} \in \Gamma$-mod, there is a minimal left $\mathrm{mon}_s(Q)$-approximation $\mathbf{M} \to \mathbf{Mono}(\mathbf{M})$.*

Remark 3.3. There is also a minimal right $\mathrm{mon}_s(Q)$-approximation $\mathbf{Mimo}(\mathbf{M}) \to \mathbf{M}$ for any $\mathbf{M} \in \Gamma$-mod. For proof and construction of $\mathbf{Mimo}$ cf. [66, Section 1.4].

It was established in [8] that functorially finite subcategories of module categories have almost split sequences. In [41] Kleiner described what we call a *relative AR-translate* for a functorially finite subcategory, which induces the almost split sequences in that subcategory. More precisely we have the analog of condition (1) in Proposition 1.3. For clarity we let τ_Γ denote the AR-translate in Γ-mod. By [41, Theorem 2.3] the relative AR-translate τ_{mon} for $\mathrm{mon}_s(Q)$ is given by

$$\tau_{\mathrm{mon}}\, \mathbf{M} = \mathbf{Mimo}\, \tau_\Gamma\, \mathbf{M}. \tag{3.1}$$

The AR-translate on kQ-mod induces an endofunctor on kQ-mod, denoted by τ_Q, because kQ is hereditary. This allows us to define a functor $\tau_Q' \colon \Gamma\text{-mod} \to \Gamma\text{-mod}$ as follows:

$$\tau_Q'\left(M^{(1)} \overset{\phi_1}{\to} \cdots \overset{\phi_{s-1}}{\to} M^{(s)}\right) := \left(\tau_Q M^{(1)} \overset{\tau_Q\, \phi_1}{\to} \cdots \overset{\tau_Q\, \phi_{s-1}}{\to} \tau_Q\, M^{(s)}\right).$$

Maps of Γ-modules are sent to the obvious induced maps. In [66] Xiong-Zhang-Zhang give a different formula for τ_{mon}, which generalises a formula from [54].

$$\tau_{\mathrm{mon}}\, \mathbf{M} \cong \mathbf{Mimo}\, \tau_Q'\, \mathbf{Cok}\, \mathbf{M}. \tag{3.2}$$

This shows that the relative AR-translate on $\mathrm{mon}_s(Q)$ is to an extent given by the simpler AR-translate on kQ-mod.

For our application of the Auslander-Reiten formula we are more interested in calculating τ_Γ than τ_{mon}. The identities (3.1) and (3.2) suggest the following way to calculate it:

Theorem 3.4. *Let* $\mathbf{M} \in \mathrm{mon}_s(Q)$. *Then*

$$\tau_\Gamma\, \mathbf{M} \cong \tau_Q'\, \mathbf{Cok}\, \mathbf{M}.$$

For the proof we recall the Nakayama functor $\nu_A \colon A\text{-mod} \to A\text{-mod}$ for a finite dimensional algebra A, it is defined as $\nu_A := \mathrm{D}\,(-, {}_A A)_A$. Its restriction to projective modules is an equivalence from the projective A-modules to the injective A-modules with inverse $\nu_A^{-1} := (\mathrm{D}\,(-), A_A)_A$. This is well known, for a proof cf. [62, III.5].

Proof of Theorem 3.4. Recall that $\mathrm{pdim}\,\mathbf{M} \leq 1$. Fix a (minimal) projective resolution of $\mathbf{M}$ of the form

$$0 \longrightarrow \bigoplus_{j \in J'} \mathbf{P}_j \overset{\Psi}{\longrightarrow} \bigoplus_{j \in J} \mathbf{P}_j \longrightarrow \mathbf{M} \longrightarrow 0,$$

where J, J' are some indexing sets and each $\mathbf{P}_j$ is an indecomposable projective module. Apply the Nakayama-functor ν_Γ to obtain

$$0 \longrightarrow \tau_\Gamma\, \mathbf{M} \longrightarrow \bigoplus_{j \in J'} \nu_\Gamma\, \mathbf{P}_j \overset{\nu\Psi}{\longrightarrow} \bigoplus_{j \in J} \nu_\Gamma\, \mathbf{P}_j.$$

Recall that each indecomposable projective module P_j has the form

$$\mathbf{P}_j \cong \left(0 \to \cdots \to 0 \to P_j^{(t)} \overset{\mathrm{id}}{\to} \cdots \overset{\mathrm{id}}{\to} P_j^{(s)}\right),$$

where the $P_j^{(u)}$, $u = t, \ldots, s$ are copies of the same indecomposable projective kQ-module P_j. We write $P_j^{(u)} = 0$ for $u = 1, \ldots, t-1$. Then $\nu_\Lambda \mathbf{P}_j$ is an injective module of the form

$$\nu_\Gamma \, \mathbf{P}_j \cong \left((\nu_Q P_j)^{(1)} \overset{\mathrm{id}}{\to} \cdots \overset{\mathrm{id}}{\to} (\nu_Q P_j)^{(t)} \to 0 \to \cdots \to 0\right),$$

where $(\nu_Q \, P_j)^{(u)}$ are copies of $\nu_Q \, P_j$.

For $1 \le t \le s$ we define subsets $J_t \subset J$ such that $j \in J_t$ if and only if $P_j^{(t)} \neq 0$, and similarly $J_t' := \{j \in J' \mid P_j^{(t)} \neq 0\}$.

The $t+1$-st line of the latter exact sequence above gives the following exact sequence of kQ modules:

$$0 \longrightarrow (\tau_\Gamma \, M)^{(t+1)} \longrightarrow \bigoplus_{j \in J' \setminus J_t'} \nu \, P_j \overset{\Psi_t'}{\longrightarrow} \bigoplus_{j \in J \setminus J_t} \nu \, P_j.$$

Here the matrix for Ψ_t' is given by restricting to the columns and rows in the matrix for $\nu\Psi$ corresponding to elements in $J' \setminus J_t'$ and $J \setminus J_t$ respectively.

Now we calculate $\tau_Q' \mathbf{Cok} \, M$. For each $1 \le t < s$ we get the following diagram by filling in via the snake lemma.

Here Ψ_t is the matrix obtained by taking columns and rows in Ψ corresponding to elements in J_t' and J_t respectively. From the diagram we see the map ψ must be given by the matrix Ψ_t' defined earlier. If we apply ν_Q to the last row we get the exact sequence

$$0 \longrightarrow \tau_Q \, \mathrm{coker}(\phi_t \cdots \phi_{s-1}) \longrightarrow \bigoplus_{j \in J' \setminus J_t'} \nu_Q \, P_j^{(s)} \overset{\Psi_t'}{\longrightarrow} \bigoplus_{j \in J \setminus J_t} \nu_Q \, P_j^{(s)}.$$

Note that this implies $\tau_Q \, \mathrm{coker}(\phi_t \cdots \phi_{s-1}) = (\tau_Q' \, \mathbf{Cok} \, \mathbf{M})^{(t+1)}$ is isomorphic to $(\tau_\Gamma \, \mathbf{M})^{(t+1)}$. This still holds for $t = 0$ if we put $J_0 = J_0' = \emptyset$ and $M_0 = 0$ and $\phi_0 = 0$ in the diagram above.

The maps $(\tau_Q' \, \mathbf{Cok} \, \mathbf{M})^{(t)} \to (\tau_Q' \, \mathbf{Cok} \, \mathbf{M})^{(t+1)}$ and $(\tau_\Gamma \, \mathbf{M})^{(t)} \to (\tau_\Gamma \, \mathbf{M})^{(t+1)}$ that belong to the Γ-module structure on $\tau_Q' \, \mathbf{Cok} \, \mathbf{M}$ and $\tau_\Gamma \, \mathbf{M}$ respectively are induced by the same maps on the corresponding injective coresolutions. Hence $\tau_Q' \mathbf{Cok} \, \mathbf{M}$ and $\tau_\Gamma \, \mathbf{M}$ are isomorphic as Γ-modules. $\qquad\square$

Remark 3.5. The identity (3.2) is proved with direct calculations in [54] and [66], but it follows directly from (3.1) using Theorem 3.4.

3.3   Quiver flag varieties

In this section we assume k is an algebraically closed field. We use the notions of Section 1.4, in particular we do not require varieties to be irreducible. For preliminaries on flag varieties we refer to Section 1.4.3.

Let (Q_0, Q_1) be a finite quiver. For a dimension vector $d = (d_i)_{i \in Q_0} \in \mathbb{N}_0^{Q_0}$ a *dimension filtration* of d of length s is an s-tuple of dimension vectors

$$\mathbf{d} = (\mathbf{d}^{(1)}, \ldots, \mathbf{d}^{(s)} = d),$$

where $\mathbf{d}^{(t-1)} \leq \mathbf{d}^{(t)}$ pointwise for $t = 2, \ldots, s$. Let M be a Q-representation of dimension vector d, and let $\mathbf{d}$ be a dimension filtration of d of length s. The *quiver flag variety* is the closed subvariety of $\mathrm{Fl}\binom{k^d}{\mathbf{d}}$ given by:

$$\mathrm{Fl}_Q \begin{pmatrix} M \\ \mathbf{d} \end{pmatrix} := \left\{ F \in \mathrm{Fl}\begin{pmatrix} k^d \\ \mathbf{d} \end{pmatrix} \mid M_a(F_i^{(t)}) \subset F_j^{(t)}, \, \forall (a \colon i \to j) \in Q_1 \right\}.$$

We see that the points of $\mathrm{Fl}_Q \binom{M}{\mathbf{d}}$ correspond to a flag of Q-subrepresentations of M.

Remark 3.6. The quiver flag varieties slightly generalize the notion of quiver Grassmanninas, which have been researched extensively. If M is a Q-representation with $\underline{\dim}\,(M) = d$ and $\mathbf{d} = (\mathbf{d}^{(1)}, \mathbf{d}^{(2)}) = (e, d)$ is a dimension filtration of d of length 2, then

$$\mathrm{Gr}_Q \begin{pmatrix} M \\ e \end{pmatrix} \cong \mathrm{Fl}_Q \begin{pmatrix} M \\ \mathbf{d} \end{pmatrix}.$$

It has been shown that every projective variety may arise as a quiver Grassmannian [49], indicating that quiver flag varieties can become very complicated.

Fix a flag $F \in \mathrm{Fl}\binom{k^d}{\mathbf{d}}$ on the Q_0-graded vector space $k^d = \oplus_{i \in Q_0} k^{d_i}$. We consider the subspace

$$\mathrm{Rep}_d^{\mathbf{d}} := \left\{ M \in \mathrm{Rep}_d(Q) \mid M_a(F_i^{(t)}) \subset F_j^{(t)}, \forall\, (a \colon i \to j) \in Q_1, \, 1 \leq t \leq s \right\}$$

of $\mathrm{Rep}_d(Q)$. It is a vector space, in particular it is smooth and irreducible. Note that $\mathrm{Rep}_d^{\mathbf{d}}$ depends on Q and the flag F, but we will always refer to it for a fixed Q and F, so we suppress them in our notation. Let $\mathrm{P}_{\mathbf{d}} \subset \mathrm{GL}_d$ be the parabolic subgroup fixing the flag F. Restricting the GL_d-action gives a

$P_{\mathbf{d}}$-action on $\mathrm{Rep}_d(Q)$. The subvariety $\mathrm{Rep}_d^{\mathbf{d}}$ is invariant under this $P_{\mathbf{d}}$-action, i.e. $\mathrm{Rep}_d^{\mathbf{d}}$ is a $P_{\mathbf{d}}$-subrepresentation of $\mathrm{Rep}_d(Q)$.

Consider the associated fibre bundle for this subrepresentation, along with the collapsing map

$$\mathrm{co}_{\mathbf{d}} \colon \mathrm{GL}_d \times^{P_{\mathbf{d}}} \mathrm{Rep}_d^{\mathbf{d}} \to \mathrm{Rep}_d(Q), \quad [g,x] \mapsto g \cdot x.$$

Since GL_d and $\mathrm{Rep}_d^{\mathbf{d}}$ are smooth and irreducible the associated fibre bundle $\mathrm{GL}_d \times^{P_{\mathbf{d}}} \mathrm{Rep}_d^{\mathbf{d}}$ is smooth and irreducible. The map $\pi_{\mathbf{d}}$ is projective and GL_d-equivariant. Therefore, the image $\mathrm{Im}\,\pi_{\mathbf{d}}$ is a closed GL_d-invariant subset of $\mathrm{Rep}_d(Q)$. Recall from Section 1.4.3 that we identify $\mathrm{GL}_d/P_{\mathbf{d}}$ and $\mathrm{Fl}\binom{k^d}{\mathbf{d}}$, hence we define

$$\mathrm{Repfl}_{\mathbf{d}} := \big\{(M, U) \in \mathrm{Rep}_d(Q) \times \mathrm{GL}_d/P_{\mathbf{d}} \mid M_a(U_i^{(t)}) \subset U_j^{(t)},$$
$$\forall\, (a \colon i \to j) \in Q_1,\ 1 \le t \le s\big\}.$$

This is a GL_d-invariant subvariety of $\mathrm{Rep}_d(Q) \times \mathrm{GL}_d/P_{\mathbf{d}}$ with the diagonal GL_d-action, again we leave Q out if the notation since it is fixed. If we apply [63, Lemma 4, p.26] to the projection $\mathrm{pr}_2 \colon \mathrm{Repfl}_{\mathbf{d}} \to \mathrm{GL}_d/P_{\mathbf{d}}$ we get a GL_d-equivariant isomorphism of varieties $\varphi \colon \mathrm{Repfl}_{\mathbf{d}} \to \mathrm{GL}_d \times^{P_{\mathbf{d}}} \mathrm{R}_d^{\mathbf{d}}$ such that the following diagram commutes:

Note that the fibres of $\mathrm{co}_{\mathbf{d}}$ over each point $M \in \mathrm{Rep}_d(Q)$ are quiver flag varieties, more precisely

$$\mathrm{co}_{\mathbf{d}}^{-1}(M) \cong \mathrm{Fl}_Q\binom{M}{\mathbf{d}}.$$

Let d be a dimension vector for Q with dimension filtration $\mathbf{d}$ of length s. For now we can consider an algebra $A := kQ/\langle R\rangle$ where R are admissible relations in kQ. We can view $\mathbf{d} = (d_i^t)_{i \in Q_0, 1 \le t \le s}$ as a dimension vector for $T_s(A)$, where d_i^t is the dimension at the idempotent $e(i_t)$.

We define

$$\mathrm{Repinj}_{\mathbf{d}}(A) := \{M \in \mathrm{Rep}_{\mathbf{d}}(T_s(A)) \mid M \in \mathrm{mon}_s(A)\}.$$

In other words these are the modules such that $\phi_t \colon M_t \to M_{t+1}$ is injective for $t = 1, \ldots, s-1$. Since being injective is an open condition on linear maps, $\mathrm{Repinj}_{\mathbf{d}}(A)$ is an open subvariety of $\mathrm{Rep}_{\mathbf{d}}(T_s(A))$.

We will now again restrict our attention to path algebras, if $A = kQ$ we write $\mathrm{Repinj}_{\mathbf{d}}(Q) := \mathrm{Repinj}_{\mathbf{d}}(kQ)$. There is a natural map

$$\pi_{\mathbf{d}}^{\mathrm{mon}} \colon \mathrm{Repinj}_{\mathbf{d}}(Q) \to \mathrm{Repfl}_{\mathbf{d}},$$
$$((M_1, \ldots, M_s), (\phi_1, \ldots, \phi_{s-1})) \mapsto (M_s, \mathrm{Im}\,(\phi_1 \circ \cdots \circ \phi_{s-1}) \subset \cdots \subset \mathrm{Im}\,\phi_{s-1} \subset M_s).$$

Note that $\mathrm{Repinj}_{\mathbf{d}}(Q)$ is a $\mathrm{GL}_{\mathbf{d}}$-invariant subset of $\mathrm{Rep}_{\mathbf{d}}(T_s(Q))$, and the map $\pi_{\mathbf{d}}^{\mathrm{mon}}$ is GL_d invariant as well. In fact $\pi_{\mathbf{d}}^{\mathrm{mon}}$ is a principal $\prod_{t=1}^{s-1} \mathrm{GL}_{\mathbf{d}(t)}$-bundle; we have required all the ϕ_t's to be monomorphisms so the map amounts to forgetting the choice of inclusions of subspaces. Thus $\pi_{\mathbf{d}}^{\mathrm{mon}}$ maps $\mathrm{GL}_{\mathbf{d}}$-orbits of $\mathrm{Repinj}_{\mathbf{d}}(Q)$ to GL_d-orbits of $\mathrm{Repfl}_{\mathbf{d}}$, and an orbit in $\mathrm{Repinj}_{\mathbf{d}}(Q)$ is dense if and only if its image is a dense orbit.

Now it is a natural question to ask if the group actions introduced here have a dense orbit. Many of those questions are equivalent:

Theorem 3.7. *Let $(Q, \mathbf{d})$ and $\mathrm{co}_{\mathbf{d}}$ be as above. Then, the following five statements are equivalent:*

(1) *The variety $\mathrm{Rep}_d^{\mathbf{d}}$ has a dense $\mathrm{P}_{\mathbf{d}}$-orbit.*

(2) *The variety $\mathrm{Repfl}_{\mathbf{d}}$ has a dense GL_d-orbit.*

(3) *The variety $\mathrm{Imco}_{\mathbf{d}}$ has a dense GL_d-orbit $\mathcal{O}$, and for every point $M \in \mathcal{O}$, the variety $\mathrm{Fl}_Q\binom{M}{\mathbf{d}}$ has a dense $\mathrm{Aut}_Q(M)$-orbit.*

(4) *The variety $\mathrm{Repinj}_{\mathbf{d}}(Q)$ has a dense $\mathrm{GL}_{\mathbf{d}}$-orbit.*

(5) *There exists a rigid object in $\mathrm{mon}_s(Q)$ of dimension vector $\mathbf{d}$.*

For the proof we need Lemma 1.7 from Section 1.4.

Proof of Theorem 3.7. Conditions (1) and (2) are equivalent by Lemma 1.7.
Let $M \in \mathrm{Rep}_d^{\mathbf{d}}$, we have the isomorphisms

$$\mathrm{co}_{\mathbf{d}}^{-1}(\mathrm{GL}_d M) \cong \mathrm{GL}_d \times^{\mathrm{stab}_{\mathrm{GL}_d}(M)} \mathrm{co}_{\mathbf{d}}^{-1}(M) = \mathrm{GL}_d \times^{\mathrm{Aut}_Q(M)} \mathrm{Fl}_Q\binom{M}{\mathbf{d}}.$$

If $\mathrm{GL}_d \cdot M$ is a dense orbit of $\mathrm{Im}\,\mathrm{co}_{\mathbf{d}}$, then $\mathrm{Fl}_Q\binom{M}{\mathbf{d}}$ is the generic fibre of $\mathrm{co}_{\mathbf{d}}$. That implies $\mathrm{Fl}_Q\binom{M}{\mathbf{d}}$ is irreducible because $\mathrm{Repfl}_{\mathbf{d}}$ is irreducible. Then Lemma 1.7 shows $\mathrm{co}_{\mathbf{d}}^{-1}(\mathrm{GL}_d \cdot M)$ has a dense GL_d-orbit if and only if $\mathrm{Fl}_Q\binom{M}{\mathbf{d}}$ has a dense $\mathrm{Aut}_Q(M)$-orbit.

Now assume $\mathrm{Repfl}_{\mathbf{d}}$ has a dense GL_d-orbit $\mathrm{GL}_d \cdot [1, M]$. Then $\mathrm{GL}_d \cdot M$ is a dense orbit of $\mathrm{Im}\,\mathrm{co}_{\mathbf{d}}$, because $\mathrm{co}_{\mathbf{d}}$ gives a dominant map to the image. Moreover, $\mathrm{GL}_d \cdot [1, M]$ is a dense orbit of $\mathrm{co}_{\mathbf{d}}^{-1}(\mathrm{GL}_d \cdot M) \subset \mathrm{Repfl}_{\mathbf{d}}$. By the argument above that implies $\mathrm{Fl}_Q\binom{M}{\mathbf{d}}$ has a dense $\mathrm{Aut}_Q(M)$-orbit, and this is independent of the choice of M.

Conversely, assume condition (3) holds and let $M \in \mathcal{O}$. Then $\mathrm{co}_{\mathbf{d}}^{-1}(\mathcal{O})$ is an open, and hence dense, subset of $\mathrm{Repfl}_{\mathbf{d}}$. Furthermore $\mathrm{co}_{\mathbf{d}}^{-1}(\mathcal{O})$ has a dense GL_d-orbit by the argument above, but that orbit is a dense orbit of a dense subset of $\mathrm{Repfl}_{\mathbf{d}}$, hence a dense GL_d-orbit in $\mathrm{Repfl}_{\mathbf{d}}$.

We have noted that an orbit in $\mathrm{Repinj}_{\mathbf{d}}(Q)$ is dense if and only if its image via $\pi_{\mathbf{d}}^{\mathrm{mon}}$ is dense, thus (2) and (4) are equivalent.

Now $\mathrm{Repfl}_{\mathbf{d}}$ is smooth and irreducible, and $\pi_{\mathbf{d}}^{\mathrm{mon}}$ is a principal fibre bundle. In particular $\mathrm{Repinj}_{\mathbf{d}}(Q)$ is a smooth irreducible open subset of $\mathrm{Rep}_{\mathbf{d}}(T_s(Q))$. Then Proposition 1.4 implies that an orbit in $\mathrm{Repinj}_{\mathbf{d}}(Q)$ is dense if and only if it has a rigid object in $\mathrm{mon}_s(kQ)$, thus (4) and (5) are equivalent. $\square$

The collapsing map can under certain conditions be a desingularisation of an orbit closure. Lemma 1.8 has the following corollary which tells us exactly when this is the case.

Corollary 3.8. *Let $M \in \text{Rep}_d(Q)$ and $\mathbf{d}$ be a filtration of d. The map $\text{co}_{\mathbf{d}} \colon \text{Repfl}_{\mathbf{d}} \to \text{Im co}_{\mathbf{d}}$ is a resolution of singularities of $\overline{\mathcal{O}_M}$ if and only if the following two conditions are fulfilled:*

(D1) $\text{Fl}\binom{M}{\mathbf{d}} \neq \emptyset$;

(D2) $\dim_k \text{Ext}^1_Q(M, M) = \dim \text{Rep}_d(Q) - \dim \text{Repfl}_{\mathbf{d}}$.

In particular the conditions imply $\text{Im co}_{\mathbf{d}} = \overline{\mathcal{O}_M}$ and that the restriction $\text{co}_{\mathbf{d}}^{-1}(\mathcal{O}_M) \to \mathcal{O}_M$ is an isomorphism.

3.3.1 Adding relations

Now we let Q be a finite quiver, not necessarily acyclic. As before we fix a dimension vector d and a Q_0-graded flag F on k^d with dimension filtration $\mathbf{d}$. That determines the parabolic subgroup $\text{P}_{\mathbf{d}} \subset \text{GL}_d$. Let $A := kQ/\langle R \rangle$, where R are admissible relations.

Recall that $\text{Rep}_d(A)$ is a closed subvariety of $\text{Rep}_d(Q)$, and it is closed under the action of GL_d. Thus we get a closed $\text{P}_{\mathbf{d}}$-invariant subvariety

$$\text{Rep}_d^{\mathbf{d}}(A) := \text{Rep}_d^{\mathbf{d}} \cap \text{Rep}_d(A)$$

of $\text{Rep}_d^{\mathbf{d}}$. By restricting the collapsing map $\text{co}_{\mathbf{d}}$ to $\text{Repfl}_{\mathbf{d}}(A) := \text{co}_{\mathbf{d}}^{-1}(\text{Rep}_d(A))$ we we get a collapsing map

$$\text{co}_{\mathbf{d}}^A \colon \text{GL}_d \times^{\text{P}_{\mathbf{d}}} \text{Rep}_d^{\mathbf{d}}(A) \to \text{Rep}_d(A).$$

Moreover we have

$$(\pi_{\mathbf{d}}^{\text{mon}})^{-1}(\text{Repfl}_{\mathbf{d}}(A)) = \text{Repinj}_{\mathbf{d}}(A).$$

Unlike $\text{Rep}_d(Q)$, $\text{Rep}_d(A)$ is not necessarily smooth or irreducible, so $\text{Repfl}_{\mathbf{d}}(A)$ and $\text{Repinj}_{\mathbf{d}}(A)$ are not necessarily irreducible or smooth either. Thus they can each have several open orbit under their respective group actions. Since $\pi_{\mathbf{d}}^{\text{mon}}$ is GL_d-invariant, it induces a map from the set of $\text{GL}_{\mathbf{d}}$-orbits of $\text{Repinj}_{\mathbf{d}}(A)$ to the set of GL_d-orbits of $\text{Repfl}_{\mathbf{d}}(A)$.

With these maps we get a slightly modified version of Theorem 3.7.

Theorem 3.9. *Let $A, d, \mathbf{d}$ and $\text{co}_{\mathbf{d}}^A$ be as above. The map of orbits induced by $\pi_{\mathbf{d}}^{\text{mon}}$ above along with the correspondence from Lemma 1.7 give bijections between the following sets:*

(1) The set of open $\text{P}_{\mathbf{d}}$-orbits in $\text{Rep}_d^{\mathbf{d}}(A)$.

(2) The set of open GL_d-orbits in $\text{Repfl}_{\mathbf{d}}(A)$.

(3) *The set of open* $\mathrm{GL}_{\mathbf{d}}$*-orbits in* $\mathrm{Repinj}_{\mathbf{d}}(A)$.

Consider additionally the following sets:

(4) *The set of isomorphism classes of rigid object in* $\mathrm{mon}_s(A)$ *of dimension vector* $\mathbf{d}$.

(5) *The set of open* GL_d*-orbits* $\mathcal{O}$ *in* $\mathrm{Imco}_{\mathbf{d}}^A$ *such that for every point* $M \in \mathcal{O}$, *the variety* $\mathrm{Fl}_Q\binom{M}{\mathbf{d}}$ *has a dense* $\mathrm{Aut}_Q(M)$*-orbit.*

There are canonical injective maps from (4) *to* (3) *and from* (5) *to* (2).

Proof. We salvage what we can from the proof of Theorem 3.7. The bijection between (1) and (2) is an immediate corollary of Lemma 1.7.

The restriction of π_d^{mon} is a principal fibre bundle, and hence it gives a one-to-one correspondence between open orbits of $\mathrm{Repfl}_{\mathbf{d}}(A)$ and open orbits of $\mathrm{Repinj}_{\mathbf{d}}(A)$, which gives our correspondence between (2) and (3).

By Proposition 1.4 rigid objects in $\mathrm{mon}_s(A)$ give open orbits in $\mathrm{Repinj}_{\mathbf{d}}(A)$, and two objects induce the same orbit if and only if they are isomorphic. This gives our map from (4) to (3).

To see that an element of (5) induces an orbit in (2), consider an orbit $\mathcal{O}$ as in (5) and let $M \in \mathcal{O}$. Clearly $(\mathrm{co}_{\mathbf{d}}^A)^{-1}(\mathcal{O})$ is an open subset of $\mathrm{Repfl}_{\mathbf{d}}(A)$. We still have

$$(\mathrm{co}_{\mathbf{d}}^A)^{-1}(\mathcal{O}) \cong \mathrm{GL}_d \times^{\mathrm{Aut}_Q(M)} \mathrm{Fl}_Q\binom{M}{\mathbf{d}},$$

and by Lemma 1.7 and our assumption that $\mathrm{Fl}_Q\binom{M}{\mathbf{d}}$ has a dense $\mathrm{Aut}_Q(M)$-orbit, $(\mathrm{co}_{\mathbf{d}}^A)^{-1}(\mathcal{O})$ has a dense GL_d-orbit. This orbit is also dense in $\mathrm{Repfl}_{\mathbf{d}}(A)$, so an element of (2). $\qquad\square$

Remark 3.10. In general, not all orbits in (3) are given by the map from (4) to (3). For a counterexample we can take the zero flag in Example 1.6. Also the map from (5) to (2) is not surjective in general, as can be seen from Example 3.11 below. Note that the orbit in Example 3.11 does arise from a rigid $\mathrm{T}_s(A)$-module via the map (4) to (3).

Example 3.11. Consider the quiver

$$Q = \; 1 \xrightarrow{\;\alpha\;} 2 \xrightarrow{\;\beta\;} 3$$

with the relation $\beta\alpha$, we write $A := kQ/\langle\beta\alpha\rangle$. We take the dimension filtration $\mathbf{d} = ((1,1,1),(1,2,1))$ of the dimension vector $d = (1,2,1)$. Take the module $X \in \mathrm{mon}_2(A)$ defined by

$$S(1) \oplus P(2) \xrightarrow{\begin{bmatrix} \mathrm{id} & 0 \\ 0 & 0 \\ 0 & \mathrm{id} \end{bmatrix}} S(1) \oplus S(2) \oplus P(2).$$

This happens to be a rigid $\mathrm{T}_s(A)$-module, so $\mathcal{O}_X$ is a dense orbit of $\mathrm{Repinj}_{\mathbf{d}}(A)$. On the other hand the image of $\mathcal{O}_X$ under $\mathrm{co}_{\mathbf{d}}^A \circ \pi_{\mathbf{d}}^{\mathrm{mon}}$ is contained in the orbit of $M := S(1) \oplus S(2) \oplus P(2)$. But M is a degeneration of $M' := P(1) \oplus P(2)$, and $\mathcal{O}_{M'}$ is also in $\mathrm{Im} \, \mathrm{co}_{\mathbf{d}}^A$. Thus M is not in an open orbit of $\mathrm{Im} \, \mathrm{co}_{\mathbf{d}}^A$.

3.4 Rigid objects in monomorphism categories

In this section we consider some tools that may be used to construct or identify rigid objects in monomorphism categories. We still assume $A = kQ/\langle R \rangle$ where (Q, R) is a quiver with admissible relations.

3.4.1 Long exact sequence

The algebra $T_s(A)$ arises as a tensor algebra. To see this we give coordinate i, j in an upper triangular matrix the grading $j - i$. In particular $T_s(A)$ is positively graded, we denote this graded algebra by Λ. Then Λ_0 is the subalgebra of diagonal matrices. We let e_t denote the idempotent of $\mathrm{T}_s(A)$ with the unit of A in the t-th diagonal coordinate and all other entries zero.

Now we have

$$T_s(A) \cong \Lambda_0 \langle x_1, \ldots, x_{s-1} \mid e_i x_j = \delta_{i,j+1} x_j, x_j e_i = \delta_{ij} x_j \rangle.$$

Here the generator x_j corresponds to the unit of A in coordinate $(j, j+1)$ of a matrix, so we have generators and relations in degree 1. Then Lemma 1.16 shows that in fact $T_s(A) \cong T_{\Lambda_0} \Lambda_1$ as a graded algebra.

Let M, N be ungraded Λ-modules. We write $M_t := e_t M$ for the $e_t \Lambda e_t$-module given by M, and similarly $N_t := e_t N$.

Let Λ_+ denote the strictly positively graded part of Λ. We apply the standard sequence (Std) to M and get

$$0 \longrightarrow \Lambda_+ \otimes_{\Lambda_0} M \xrightarrow{\delta_M} \Lambda \otimes_{\Lambda_0} M \xrightarrow{\epsilon_M} M \longrightarrow 0.$$

Lemma 3.12. *We have the following identities of vector spaces:*

$$\mathrm{Ext}_\Lambda^n(\Lambda \otimes_{\Lambda_0} M, N) \cong \bigoplus_{t=1}^{s} \mathrm{Ext}_A^n(M_t, N_t), \quad n \geq 0;$$

$$\mathrm{Ext}_\Lambda^n(\Lambda_+ \otimes_{\Lambda_0} M, N) \cong \bigoplus_{t=2}^{s} \mathrm{Ext}_A^n(M_{t-1}, N_t), \quad n \geq 0.$$

Proof. By the hom-tensor adjunction we have

$$\mathrm{Hom}_\Lambda(\Lambda \otimes_{\Lambda_0} M, N) \cong \mathrm{Hom}_{\Lambda_0}(M, \mathrm{Hom}_\Lambda(\Lambda, N)).$$

Then the first identity follows for $n = 0$ when we observe that $e_t \mathrm{Hom}_\Lambda(\Lambda, N)) \cong N_t$. The formula also holds for $n > 0$ because Λ_{Λ_0} is flat.

For the second identity observe that $\Lambda_+ = \Lambda \otimes \Lambda_1$ and

$$e_t \Lambda_1 \otimes_{\Lambda_0} M \cong e_{t-1} M = M_{t-1}, \quad 2 \leq t \leq s.$$

Then the second identity follows from the first. $\qquad \square$

We apply the functor $\text{Hom}_\Lambda(-, N)$ to obtain a long exact sequence, and using the identities from Lemma 3.12 it has the form

$$0 \longrightarrow \text{Hom}_\Lambda(M, N) \longrightarrow \bigoplus_{t=1}^{s} \text{Hom}_A(M_t, N_t) \longrightarrow \bigoplus_{t=2}^{s} \text{Hom}_A(M_{t-1}, N_t) \longrightarrow$$

$$\longrightarrow \text{Ext}_\Lambda^1(M, N) \longrightarrow \bigoplus_{t=1}^{s} \text{Ext}_A^1(M_t, N_t) \longrightarrow \bigoplus_{t=2}^{s} \text{Ext}_A^1(M_{t-1}, N_t) \longrightarrow$$

$$\longrightarrow \text{Ext}_\Lambda^2(M, N) \longrightarrow \cdots .$$

Even though Λ is graded, all homomorphism spaces here are taken over ungraded modules.

3.4.2 Ext-directed decompositions

Let A be a finite-dimensional algebra and let $M \in \text{Rep}_d(A)$. Consider the task to find a dimension filtration $\mathbf{d}$ of d such that $\text{Repfl}_{\mathbf{d}}(A)$ has an open non-empty orbit that maps to the orbit $\text{GL}_d \cdot M$ under the collapsing map $\text{co}_{\mathbf{d}}^A$. This is achieved by finding a rigid object in $\text{mon}_s(A)$ with dimension vector $\mathbf{d}$. It turns out that in certain cases the necessary tools are already given by Reineke [48].

Recall that $\text{ind}(A)$ denotes the set of indecomposable isomorphism classes in A-mod. A partition of $\text{ind}(A)$ is a finite sequence $I_* = (I_1, \ldots, I_s)$ of subsets such that $\text{ind}(A) = I_1 \cup \ldots \cup I_s$. There already exists the notion of directed partitions of $\text{ind}(A)$, cf. [48, Defn. 2.1] for the case of Dynkin quivers.

Definition 3.13. The partition $I_* = (I_1, \ldots, I_s)$ is *directed* if for any $M \in I_t$ and $N \in I_{t'}$ the following holds:

(a) $\text{Ext}_A^1(M, N) = 0$ if $t \le t'$;

(b) $\text{Hom}_A(M, N) = 0$ if $t > t'$.

We make the following definition inspired by the definition above.

Definition 3.14. Let M be a module. We say a decomposition $M = M_{(1)} \oplus \cdots \oplus M_{(s)}$ of M is Ext-*directed* if $\text{Ext}_A^1(M_{(t)}, M_{(t')}) = 0$ if $t \le t'$.

Our Ext-directed decomposition is weaker than a directed partition in two ways. Firstly we have dropped the condition of hom-vanishing from aforementioned definition, and secondly we consider decompositions of individual modules instead of partitions of all of $\text{ind}(A)$. In particular, if $\text{ind}(A)$ has a directed partition, then every finitely generated A-module has an Ext-directed decomposition. Reineke remarks that directed partitions exist for $\text{ind}(kQ)$ for all Dynkin quivers Q. However Ext-directed decompositions do not exist in general, for example they don't exist if M has a non-rigid indecomposable summand.

Example 3.15. Ext-directedness is strictly weaker than directedness in the sense of directed partitions. Let A be the algebra given by the quiver

$$Q = \quad 1 \underset{b}{\overset{a}{\rightleftarrows}} 2,$$

with the relations $ba = 0$. Let $M = I(2) \oplus S(2)$, then M has an ext-directed decomposition $M_{(1)} = S(2), M_{(2)} = I(2)$. However there is a non-zero homomorphism $I(2) \to S(1)$, so the condition of Hom-vanishing from the definition of directed partitions cannot hold.

Let M be an A-module with dimension vector d and let $M_{(1)} \oplus \cdots \oplus M_{(s)}$ be an Ext-directed decomposition of M.

We define the $T_s(A)$-modules

$$\mathbf{M}_{(t)} := \left(0 \to \cdots \to 0 \to \underbrace{M_{(t)} \overset{\text{id}}{\to} \cdots \overset{\text{id}}{\to} M_{(t)}}_{t \text{ times}} \right).$$

Lemma 3.16. *Let M be an A-module with* Ext*-directed decomposition $M_{(1)} \oplus \cdots \oplus M_{(s)}$. We let $M_{(t)}$ and $\mathbf{M}_{(t)}$ be determined by the decomposition of M as above. Then the* $\mathrm{T}_s(A)$*-module*

$$\mathbf{M} := \bigoplus_{t=1}^{s} \mathbf{M}_{(t)}$$

is rigid and belongs in $\mathrm{mon}_s(A)$.

Proof. It is clear that $\mathbf{M}$ belongs to $\mathrm{mon}_s(A)$, thus it suffices to show that $\mathrm{Ext}^1_{T_s(A)}(\mathbf{M}_{(t)}, \mathbf{M}_{(t')})^1 = 0$ for all $t, t' \in \{1, \ldots, s\}$. If $t = t'$ this holds because

$$\mathrm{Ext}^1_{T_s(A)}(\mathbf{M}_{(t)}, \mathbf{M}_{(t)}) = \mathrm{Ext}^1_A(M_{(t)}, M_{(t)}) = 0.$$

Consider an exact sequence:

$$0 \longrightarrow \mathbf{M}_{(t')} \overset{f}{\longrightarrow} E \overset{g}{\longrightarrow} \mathbf{M}_{(t)} \longrightarrow 0.$$

Assume for now that $t > t'$, then the map $g_{s-t+1} : E_{s-t+1} \to M_{(t)}$ is an isomorphism of A-modules. The inverse of g_{s-t+1} induces a map $\mathbf{M}_{(t)} \to E$, which is a splitting of g.

Recall that $\mathrm{mon}_s(A)$ is extension closed, so $E \in \mathrm{mon}_s(A)$. Now assume $t < t'$, then $\mathrm{Ext}^1_A(M_{(t)}, M_{(t')}) = 0$. That implies there is a splitting ψ_s of the map $f_s : M_{(t')} \to E_s$. Since E is in the monomorphism category, ψ_s induces uniquely determined maps $\psi_u : E_u \to M_{(t')}$ for $u = s - t' + 1, \ldots, s$. By our construction of $\mathbf{M}_{(t')}$ these give a morphism $\psi : E \to \mathbf{M}_{(t')}$ of $T_s(A)$-modules which is a splitting of f.

$\square$

The lemma above along with Theorem 3.7 immediately gives the following corollary.

Corollary 3.17. *Let $A = kQ$ for Q Dynkin. Then there exists $s \in \mathbb{N}$ such that for any $M \in A$-mod there is an Ext-directed decomposition*

$$M = M_{(1)} \oplus \cdots \oplus M_{(s)}.$$

Let $\mathbf{M}$ be constructed from this decomposition as above. Let F be the corresponding flag on the vector space underlying M and let $\mathrm{P}_{\mathbf{d}} \subset \mathrm{GL}_d$ be the subgroup fixing F. Then $\mathrm{P}_{\mathbf{d}} \cdot M$ is a dense orbit of $\mathrm{Rep}_d^{\mathbf{d}}$.

By the above we have a method for the following task: Given an Ext-directed decomposition of $M \in kQ$-mod of dimension vector d we can find a flag F with dimension filtration $\mathbf{d}$ on M such that $\mathrm{P}_{\mathbf{d}} \cdot M$ is the dense orbit of $\mathrm{Rep}_d^{\mathbf{d}}$. Thus we have a method to provide many examples of dense orbits of $\mathrm{Repfl}_{\mathbf{d}}(Q)$, but we do not control the length of the flag or the dimension vector $\mathbf{d}$ of the flag.

Example 3.18. Let $Q = \quad 1 \longleftarrow 2 \longleftarrow 3 \quad$ and

$$M := P(3) \oplus P(2) \oplus P(1) \oplus I(2) \oplus S(2) \oplus S(3)$$

in kQ-mod, then $d := \underline{\dim}\, M = (1, 2, 3)$. In fact M is an additive generator of kQ-mod and it has an Ext-directed decomposition

$$\begin{aligned}
M_{(1)} &= P(1) \oplus P(2) \oplus P(3), \\
M_{(2)} &= S(2) \oplus I(2), \\
M_{(3)} &= S(3).
\end{aligned}$$

Our construction above gives the $\mathrm{T}_s(Q)$-module

$$\mathbf{M} := M_{(3)} \xrightarrow{\begin{bmatrix} 0 \\ \mathrm{id} \end{bmatrix}} M_{(2)} \oplus M_{(3)} \xrightarrow{\begin{bmatrix} 0 & 0 \\ \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{bmatrix}} M_{(1)} \oplus M_{(2)} \oplus M_{(3)}.$$

The corresponding dimension filtration of d is $\mathbf{d} = ([0, 0, 1], [0, 1, 2], [1, 2, 3])$. Then Corollary 3.17 implies that $\mathrm{P}_{\mathbf{d}} \cdot M \subset \mathrm{Rep}_d^{\mathbf{d}}$ is a dense orbit.

Chapter 4

Quiver-graded Richardson orbits

This chapter contains parts of joint work with Sauter [29]. It is only possible to separate my contributions and those of Sauter to a limited extent, in particular if we are to present the results in a coherent manner. The contents of [29, Section 4] have been left out, since that section was mainly written by Sauter, the remaining material all has contributions from the author of this thesis. However some results of that project that where omitted from [29] have been added in this thesis. They concern properties of the quasi-hereditary structure of the algebra $N_s(Q)$.

4.1 Overview

The quiver-graded Richardson orbits presented here are a generalisation of the setting of Hille-Röhrle [12], which is a follows. Let $d \in \mathbb{N}$ and consider a parabolic subgroup $P \subset \mathrm{GL}_d$. This subgroup acts on the nilpotent radical $\mathfrak{n}$ of the lie algebra $\mathrm{Lie}(P)$ with the adjoint action. It was already proven by Richardson [50] that $\mathfrak{n}$ always has a dense orbit with respect to this action. Brüstle-Hille-Ringel-Röhrle [12] classified the dense orbits for groups P in terms of the category of Δ-filtered modules over the Auslander algebra of $k[x]/\langle x^n \rangle$ with its unique quasi-hereditary structure. This is outlined in more detail in Example 4.31. Brüstle-Hille [11] slightly generalised this to include the action of P on $\mathfrak{n}^{(l)}$ for $l > 0$, i.e. members of the descending central series of $\mathfrak{n}$. For this they introduced for each l a corresponding quasi-hereditary algebra whose Δ-filtered modules corresponds to P-orbits on $\mathfrak{n}^{(l)}$.

Jensen-Su-Yu [39] used similar ideas for seaweed Lie algebras, and they also have similarities to the methods used by Baur to construct standard Richardson elements in classical Lie-algebra [9].

We consider the action of the general linear group GL_d acting on a representation space $\mathrm{Rep}_d(Q)$ of a quiver Q for some dimension vector $d \in \mathbb{N}^{Q_0}$. Fixing a flag of vector spaces at each vertex determines a parabolic subgroup

which acts on a closed subvariety of $\text{Rep}_d(Q)$, which plays the role of $\mathfrak{n}$ in the setting above. Apart from the case studied in [12], Lusztig [47] and Reineke [48] have studied this setting for complete flags and acyclic quivers, as a quiver graded analog of the Springer map.

In this more general setting the group action does not always act with a dense orbit, but we are able to give several equivalent criteria for this in Theorem 4.2. However these criteria are nothing close to a complete answer when this is the case.

To get a connection comparable to the one in [12] we construct the algebra $N_s(Q)$, and the subcategory subcategory $\mathcal{N}$ of $N_s(Q)$-mod, whoso rigid modules correspond to dense orbits, cf. Theorems 4.25 and 4.30. We go on to show that $N_s(Q)$ has a quasi-hereditary structure that makes it simultaneously left and right strongly quasi-hereditary, and that $\mathcal{N}$ is actually the subcategory $\mathcal{F}(\Delta)$ of standard filtered modules. We are able to calculate the Ringel dual of $N_s(Q)$, this is given by Theorem 4.28, and we also provide an alternative construction of $N_s(Q)$ as a subalgebra of a graded endomorphism ring cf. Theorem 4.20.

The subcategory $\mathcal{F}(\Delta)$ of $N_s(Q)$-mod has some similarities with the monomorphism categories of Xiong-Zhang-Zhang [66], which we discuss in Section 3.2.

If Q has no sinks, then $N_s(Q)$ coincides with the Auslander-Dlab-Ringel (ADR) algebra studied by Conde in [20]. Moreover Kalck-Karmazyn [40] have observed that both coincide with their algebra E_R (up to taking opposite) if Q has neither sinks nor sources.

In Section 4.2 we establish our setting and define what we call a Richardson orbit. We also prove some basic geometric properties of the varieties involved. In Section 4.3 we construct the nilpotent quiver algebra $N_s(Q)$ for a quiver Q and $s \in \mathbb{N}$. We consider it as a tensor algebras and introduce the subcategory $\mathcal{N}$ of $N_s(Q)$-mod, which can be embedded into monomorphism categories. We also prove Theorem 4.20, which gives $N_s(Q)$ as a graded subalgebra of a graded endomorphisms algebra. Finally we show that $N_s(Q)$ is isomorphic to an ADR-algebra if Q has no sinks. In Section 4.4 we describe a left and right strong quasi-hereditary structure on $N_s(Q)$ and show that with this structure $\mathcal{N} = \mathcal{F}(\Delta)$. We also calculate the Ringel dual and show how the rigid modules in $\mathcal{F}(\Delta)$ correspond to Richardson orbits. In Section 4.5 we provide some examples, both where there exists a Richardson orbit and where there doesn't. We also provide an algorithm that gives the dense orbits for quivers of type A_2.

4.2 Quiver-graded Richardson orbits

Let k be an algebraically closed field and $Q = (Q_0, Q_1)$ a finite quiver with path algebra $A = kQ$. We use the notation from Section 1.4.3, we recall some

of it briefly here. Fix a dimension vector $d = (d_i)_{i \in Q_0} \in \mathbb{N}_0^{Q_0}$ and let

$$\mathbf{d} = (\mathbf{d}^{(1)}, \dots, \mathbf{d}^{(s)} = d)$$

be a dimension filtration of $\mathbf{d}$. For each vertex $i \in Q_0$, fix a flag

$$F_i = (F_i^{(1)} \subset \cdots \subset F_i^{(s)} = k^{d_i}),$$

where $\dim F_i^{(t)} = d_i^{(t)}$, i.e. $\underline{\dim} F = \mathbf{d}$. By convention we set $F_i^{(t)} = 0$ for all $t \leq 0$ and all $i \in Q_0$. We denote by

$$\mathrm{P}_{\mathbf{d}} := \prod_{i \in Q_0} \left\{ g \in \mathrm{GL}_{d_i} \mid g F_i^{(t)} \subset F_i^{(t)}, \ 1 \leq t \leq s \right\} \subset \mathrm{GL}_d$$

the parabolic subgroup that is the stabiliser of F in the Q_0-graded vector space $k^d := \bigoplus_{i \in Q_0} k^{d_i}$.

Consider the subspace

$$\mathrm{R}_d^{\mathbf{d}} := \{ M \in \mathrm{Rep}_d(Q) \mid M_a(F_i^{(t)}) \subset F_j^{(t-1)}, \forall \, (a \colon i \to j) \in Q_1, \ 1 \leq t \leq s \}$$

of $\mathrm{Rep}_d(Q)$. It is a vector space, in particular it is smooth and irreducible. Note that $\mathrm{R}_d^{\mathbf{d}}$ depends on Q and the choice of F, but we suppress them in the notation because both are fixed. We get a $\mathrm{P}_{\mathbf{d}}$-action on $\mathrm{Rep}_d(Q)$ by restricting the action of GL_d via conjugation, the variety $\mathrm{R}_d^{\mathbf{d}}$ is invariant under this $\mathrm{P}_{\mathbf{d}}$-action. Hence we can see $\mathrm{R}_d^{\mathbf{d}}$ as a $\mathrm{P}_{\mathbf{d}}$-subrepresentation of $\mathrm{Rep}_d(Q)$.

Definition 4.1. We say there is a *quiver-graded Richardson orbit for* $(Q, \mathbf{d})$ if there is a dense $\mathrm{P}_{\mathbf{d}}$-orbit in $\mathrm{R}_d^{\mathbf{d}}$. In this case we call the dense $\mathrm{P}_{\mathbf{d}}$-orbit the *quiver-graded Richardson orbit*.

Consider the associated fibre bundle for the subgroup $\mathrm{P}_{\mathbf{d}}$ of GL_d acting on $\mathrm{R}_d^{\mathbf{d}}$, along with the collapsing map

$$\pi_{\mathbf{d}} \colon \mathrm{GL}_d \times^{\mathrm{P}_{\mathbf{d}}} \mathrm{R}_d^{\mathbf{d}} \to \mathrm{Rep}_d(Q), \quad [g, x] \mapsto g \cdot x.$$

Since GL_d and $\mathrm{R}_d^{\mathbf{d}}$ are smooth and irreducible the associated fibre bundle $\mathrm{GL}_d \times^{\mathrm{P}_{\mathbf{d}}} \mathrm{R}_d^{\mathbf{d}}$ is smooth and irreducible. The map $\pi_{\mathbf{d}}$ is projective and GL_d-equivariant. Therefore, the image $\mathrm{Im}\, \pi_{\mathbf{d}}$ is a closed GL_d-invariant subset of $\mathrm{Rep}_d(Q)$. We identify $\mathrm{GL}_d / \mathrm{P}_{\mathbf{d}}$ with the Q_0-graded flags with dimension filtration $\mathbf{d}$ via the bijection $g\mathrm{P}_{\mathbf{d}} \leftrightarrow gFg^{-1}$ and define

$$\mathrm{RF}_{\mathbf{d}} := \big\{ (M, U) \in \mathrm{Rep}_d(Q) \times \mathrm{GL}_d / \mathrm{P}_{\mathbf{d}} \mid M_a(U_i^{(t)}) \subset U_j^{(t-1)},$$
$$\forall \, (a \colon i \to j) \in Q_1, \ 1 \leq t \leq s \big\}.$$

This is a GL_d-invariant subvariety of $\mathrm{Rep}_d(Q) \times \mathrm{GL}_d / \mathrm{P}_{\mathbf{d}}$ with the diagonal action, and again we leave Q out if the notation since it is fixed. If we apply [63,

Lemma 4, p.26] to $\mathrm{pr}_2\colon \mathrm{RF}_{\mathbf{d}} \to \mathrm{GL}_d/\mathrm{P}_{\mathbf{d}}$ we get a GL_d-equivariant isomorphism $\varphi\colon \mathrm{RF}_{\mathbf{d}} \to \mathrm{GL}_d \times^{\mathrm{P}_{\mathbf{d}}} \mathrm{R}_d^{\mathbf{d}}$ such that the following diagram commutes:

$$\mathrm{RF}_{\mathbf{d}} \xrightarrow{\quad\varphi\quad} \mathrm{GL}_d \times^{\mathrm{P}_{\mathbf{d}}} \mathrm{R}_d^{\mathbf{d}}$$
$$\mathrm{pr}_1 \searrow \qquad \swarrow \pi_{\mathbf{d}}$$
$$\mathrm{Rep}_d(Q).$$

We say a Q_0-graded flag $U = (U^{(1)} \subset \cdots \subset U^{(s)} = M)$ is a *flag of submodules* of M if all the $U^{(t)}$ are a kQ-submodule of M. We denote the set of flags of submodules of M of dimensions filtration $\mathbf{d}$ by $\mathrm{Fl}_Q\binom{M}{\mathbf{d}}$, and call it a *quiver flag variety* . It is easy to see that the fibre of pr_1 over $M \in \mathrm{Rep}_d(Q)$ is contained in a quiver flag variety, and hence the same holds for the fibre via $\pi_{\mathbf{d}}$. We fix the following notation for the fibre of $\pi_{\mathbf{d}}$ over M:

$$\mathrm{Fl}_Q\binom{M}{\mathbf{d}}^{(1)} := \pi_{\mathbf{d}}^{-1}(M) = \left\{ U \in \mathrm{Fl}_Q\binom{M}{\mathbf{d}} \mid U^{(t)}/U^{(t-1)} \text{ semi-simple }, 1 \le t \le s \right\}.$$

$$\mathrm{Fl}_Q\binom{M}{\mathbf{d}}^{(1)} := \pi_{\mathbf{d}}^{-1}(M) = \left\{ U \in \mathrm{Fl}_Q\binom{M}{\mathbf{d}} \mid M_a(U^{(t)} \subset M_a(U^{(t-1)}), \forall a \in Q_1 \right\}$$
$$= \left\{ U \in \mathrm{Fl}_Q\binom{M}{\mathbf{d}} \mid U^{(t)}/U^{(t-1)} \text{ semi-simple over } kQ, 1 \le t \le s \right\}.$$

This is a closed subvariety of $\mathrm{Fl}_Q\binom{M}{\mathbf{d}}$. The stabilizer of M with respect to the action of GL_d is the automorphism group $\mathrm{Aut}_Q(M)$ of M.

Theorem 4.2. *Let $(Q, \mathbf{d})$ and $\pi_{\mathbf{d}}$ be as above. Then, the following three statements are equivalent:*

(1) *The variety $\mathrm{R}_d^{\mathbf{d}}$ has a dense $\mathrm{P}_{\mathbf{d}}$-orbit.*

(2) *The variety $\mathrm{RF}_{\mathbf{d}}$ has a dense GL_d-orbit.*

(3) *The variety $\mathrm{Im}\,\pi_{\mathbf{d}}$ has a dense GL_d-orbit $\mathcal{O}$, and for every point $M \in \mathcal{O}$ the variety $\mathrm{Fl}_Q\binom{M}{\mathbf{d}}^{(1)}$ has a dense $\mathrm{Aut}_Q(M)$-orbit.*

The following proof is essentially the same as the proof of Theorem 3.7.

Proof of Theorem 4.2. Conditions (1) and (2) are equivalent by Lemma 1.7.
Let $M \in \mathrm{R}_d^{\mathbf{d}}$, we have the identities

$$\pi_{\mathbf{d}}^{-1}(\mathrm{GL}_d \cdot M) = \mathrm{GL}_d \times^{\mathrm{stab}_{\mathrm{GL}_d}(M)} \pi_{\mathbf{d}}^{-1}(M) = \mathrm{GL}_d \times^{\mathrm{Aut}_Q(M)} \mathrm{Fl}_Q\binom{M}{\mathbf{d}}^{(1)}.$$

Thus Lemma 1.7 shows that if $\mathrm{Fl}_Q\binom{M}{\mathbf{d}}^{(1)}$ is irreducible, then $\pi_{\mathbf{d}}^{-1}(\mathrm{GL}_d \cdot M)$ has a dense GL_d-orbit if and only if $\mathrm{Fl}_Q\binom{M}{\mathbf{d}}^{(1)}$ has a dense $\mathrm{Aut}_Q(M)$-orbit.

Now assume $\mathrm{RF_d}$ has a dense GL_d-orbit $\mathrm{GL}_d \cdot [1, M]$. Then $\mathrm{GL}_d \cdot M$ is a dense orbit of $\mathrm{Im}\,\pi_\mathbf{d}$, because $\pi_\mathbf{d}$ is a dominant map to $\mathrm{Im}\,\pi_\mathbf{d}$, and $\mathrm{GL}_d \cdot [1, M]$ is a dense orbit of $\pi_\mathbf{d}^{-1}(\mathrm{GL}_d \cdot M) \subset \mathrm{RF_d}$. Moreover, $\mathrm{Fl}_Q\binom{M}{\mathbf{d}}^{(1)}$ is the generic fibre, which implies that it is irreducible. By the argument above that implies $\mathrm{Fl}_Q\binom{M}{\mathbf{d}}^{(1)}$ has a dense $\mathrm{Aut}_Q(M)$-orbit, and this is independent of the choice of M.

Conversely, assume condition (3) holds and let $M \in \mathcal{O}$. Then $\pi_\mathbf{d}^{-1}(\mathcal{O})$ is an open, and hence dense, subset of $\mathrm{RF_d}$. Furthermore $\pi_\mathbf{d}^{-1}(\mathcal{O})$ has a dense GL_d-orbit by the argument above, but that orbit is a dense orbit of a dense subset of $\mathrm{RF_d}$, hence a dense orbit in $\mathrm{RF_d}$.

$\square$

It is straightforward to calculate the dimension of $\mathrm{RF_d}$ and $\mathrm{Rep}_d(Q)$ in terms of Q and $\mathbf{d}$.

$$\dim \mathrm{R}_d^\mathbf{d} = \sum_{(a\,:\,i\to j)\in Q_1} \sum_{t=1}^{s} d_j^{t-1}(d_i^t - d_i^{t-1});$$

$$\dim \mathrm{RF_d} = \dim \mathrm{R}_d^\mathbf{d} + \dim \mathrm{GL}_d/\mathrm{P}_\mathbf{d}$$

$$= \sum_{(a\,:\,i\to j)\in Q_1} \sum_{t=1}^{s} d_j^{t-1}(d_i^t - d_i^{t-1}) + \sum_{i\in Q_0} \sum_{t=1}^{s-1}(d_i^s - d_i^t)(d_i^t - d_i^{t-1}).$$

We set $d \cdot d := \sum_{i\in Q_0} d_i^2$.

$$\langle \mathbf{d}, \mathbf{d} \rangle^{(1)} := d \cdot d - \dim \mathrm{RF_d}.$$

In Section 4.3 we will construct a finite-dimensional algebra of global dimension at most 2, and in Section 4.3.2 we show that $\langle \mathbf{d}, \mathbf{d} \rangle^{(1)}$ is actually the Euler form for its module category.

If we apply Lemma 1.8 to our collapsing map $\pi_\mathbf{d} \colon \mathrm{GL}_d \times^{\mathrm{P}_\mathbf{d}} \mathrm{R}_d^\mathbf{d} \to \mathrm{Rep}_d(Q)$ we get the following:

Corollary 4.3. *Let $M \in \mathrm{Rep}_d(Q)$, $\mathcal{O}_M := \mathrm{GL}_d \cdot M$ and $\mathbf{d}$ a filtration of d.*

(a) *Assume that $\mathrm{Im}\,\pi_\mathbf{d} = \overline{\mathcal{O}_M}$. Then the varieties $\mathrm{Fl}_Q\binom{N}{\mathbf{d}}^{(1)}$ for $N \in \mathcal{O}_M$ are pairwise isomorphic, smooth and irreducible of dimension*

$$\dim \mathrm{RF_d} - \dim \mathrm{Rep}_d(Q) + \dim \mathrm{Ext}_Q^1(M, M).$$

(b) *The map $\pi_\mathbf{d} \colon \mathrm{RF_d} \to \mathrm{Im}\,\pi_\mathbf{d}$ is a resolution of singularities of $\overline{\mathcal{O}_M}$ if and only if the following two conditions are fulfilled:*

(D1) $\mathrm{Fl}_Q\binom{M}{\mathbf{d}}^{(1)} \neq \emptyset$;

(D2) $\dim_k \mathrm{Ext}_Q^1(M, M) = \dim \mathrm{Rep}_d(Q) - \dim \mathrm{RF_d}$
(or equivalently $\dim_k \mathrm{Hom}_Q(M, M) = \langle \mathbf{d}, \mathbf{d} \rangle^{(1)}$).

Conditions (D1) and (D2) in the corollary imply that $\operatorname{Im} \pi_{\mathbf{d}} = \overline{\mathcal{O}_M}$. If $\pi_{\mathbf{d}} \colon \mathrm{RF}_{\mathbf{d}} \to \operatorname{Im} \pi_{\mathbf{d}}$ is resolution of singularities of an orbit closure, then the fibres over the dense orbit in $\operatorname{Im} \pi_{\mathbf{d}}$ consist only of a point. Together this implies condition (3) from Theorem 4.2, we conclude:

Corollary 4.4. *If $\pi_{\mathbf{d}} \colon \mathrm{RF}_{\mathbf{d}} \to \operatorname{Im} \pi_{\mathbf{d}}$ is a resolution of singularities of an orbit closure, then there is a quiver-graded Richardson orbit for $(Q, \mathbf{d})$.*

Remark 4.5. The corollary provides a lot of examples of quiver graded Richardson orbits. If Q is a Dynkin quiver, Reineke found for every point $M \in \mathrm{Rep}_d(Q)$ a dimension filtration giving a resolution of singularities in his setting in [48]. Small modifications to his construction give desingularisations in our setting, and hence Richardson orbits.

4.3 The nilpotent quiver algebra

Our aim is to describe an algebra whose homological properties can be used to study the variety $\mathrm{RF}_{\mathbf{d}}$. As usually we fix a field k, only in Section 4.4.2 we have to add the condition that k is algebraically closed.

4.3.1 The nilpotent quiver algebra

Let $Q = (Q_0, Q_1)$ be an arbitrary finite quiver and let kQ be its path algebra. Fixing $s \in \mathbb{Z}_+$ we define the *staircase quiver* $Q^{(s)}$ of Q. It has vertices i_t for $i \in Q_0$ and $t = 1, \ldots, s$. It has two families of arrows, firstly there is an arrow $b(i_t) \colon i_t \to i_{t+1}$ for each $i \in Q_0$ and $t = 1, \ldots, s-1$. We call these arrows the *vertical arrows*. Also for each $(a \colon i \to j) \in Q_1$ and $t = 2, \ldots, s$ there is an arrow $(a_t \colon i_t \to j_{t-1})$, these arrows are called the *diagonal arrows*. Consider the following relations of paths in $kQ^{(s)}$:
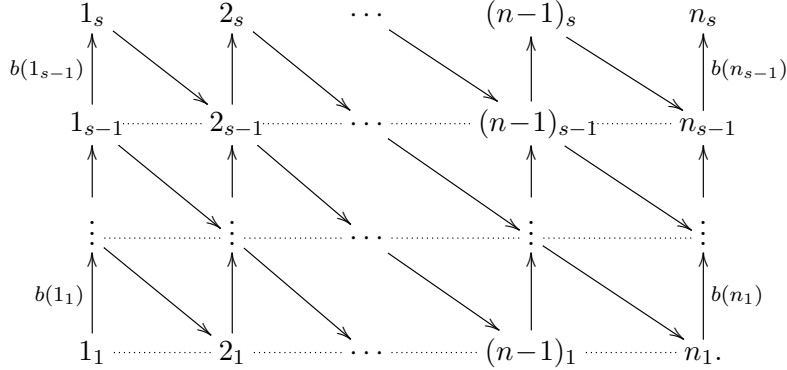
$$a_{t+1}b(i_t) = b(j_{t-1})a_t, \qquad \forall (a \colon i \to j) \in Q_1, 1 < t < s, \qquad \text{(R1)}$$
$$a_2 b(i_1) = 0, \qquad \forall (a \colon i \to j) \in Q_1. \qquad \text{(R2)}$$

We denote the linearly oriented quiver of type A_n by $\mathbb{A}_n$, more precisely

$$\mathbb{A}_n := 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n .$$

To clarify we draw the quiver $\mathbb{A}_n^{(s)}$, the relations (R1) and (R2) are shown with dashed lines. Note that there are no relations in the top row.

Let $\mathfrak{I} := \langle (R1) \cup (R2) \rangle \subset kQ^{(s)}$ be the ideal generated by the relations (R1) and (R2). The *nilpotent quiver algebra* is defined as $N_s(Q) := kQ^{(s)}/\mathfrak{I}$. We will denote the idempotent of $N_s(Q)$ corresponding to the vertex i_t by $e(i_t)$.

Every path in the path algebra $kQ^{(s)}$ can, up to the given relations, be written in a standard form. Namely, if γ is a path in $Q^{(s)}$, we have a unique path α of diagonal arrows and a unique path β of vertical arrows such that $\gamma \in \beta\alpha + \mathfrak{I}$. All different paths of this form are linearly independent in the vector space $N_s(Q)$, so these form a basis. We call it the standard basis of $N_s(Q)$.

4.3.2 Tensor structures

The nilpotent quiver algebra can arise as a tensor algebra in two different ways by considering different gradings. We outline both of those.

The nilpotent quiver algebra as a tensor algebra I

We put the following grading on $N_s(Q)$. We give each diagonal arrow of $Q^{(s)}$ the grade 1 and each vertical arrow the grading 0. The relations (R1) and (R2) are homogeneous of degree 1 with respect to this grading, so it induces a grading on $N_s(Q)$. We let Λ denote the resulting graded algebra. Observe that Λ satisfies the conditions of Lemma 1.16, thus $\Lambda \cong T_{\Lambda_0}\Lambda_1$ as a graded algebra. From now on we identify those algebras.

The algebra Λ_0 has the form $\Lambda_0 \simeq (k\mathbb{A}_s)^{Q_0}$, i.e. the disjoint union of $|Q_0|$ components, each isomorphic to the path algebra $k\mathbb{A}_s$. Moreover $\Lambda_t = 0$ for $t \geq s$.

The nilpotent quiver algebra as a tensor algebra II

We put the following grading on $N_s(Q)$. We give each vertical arrow of $Q^{(s)}$ the grading 1 and each diagonal arrow the grading 0. The relations (R1) and (R2) are homogeneous of degree one with respect to this grading, so it induces a grading on $N_s(Q)$. We let Γ denote the resulting graded algebra, where Γ_r

denotes the degree r part of Γ. Similarly as for Λ, the conditions of Lemma 1.16 hold, so $\Gamma \cong T_{\Gamma_0}\Gamma_1$ as graded algebras. From now on we identify Γ with $T_{\Gamma_0}\Gamma_1$.

Remark 4.6. We have $\Gamma_t = 0$ for $t \geq s$. The algebra Γ_0 is actually the path algebra of the subquiver that has only diagonal arrows. In particular it is hereditary. Even if Q is connected, Γ_0 will typically have many different components.

Lemma 4.7. Γ_{Γ_0} *is projective as a right* Γ_0*-module. Moreover both* Γ_1 *and* Γ *are flat as right* Γ_0*-modules.*

Proof. We know Γ is isomorphic to the tensor algebra $T_{\Gamma_0}\Gamma_1$. Thus Γ_{Γ_0} is flat if Γ_1 is flat as a right Γ_0-module. Since projective modules are flat it suffices to show that Γ_1 is projective as a right Γ_0-module. For this it is sufficient to show that Γ_1 is a right submodule of Γ_0, because Γ_0 is hereditary.

We know Γ_0 has a basis given by all non-trivial paths of the form $a_n \cdots a_1$, where the a_m are diagonal arrows of $Q^{(s)}$. Similarly Γ_1 has a basis given by all non-trivial paths in $Q^{(s)}$ of the form $ba_n \cdots a_1$, where $a_n \cdots a_1$ is a basis element of Γ_0, and b is a vertical arrow such that $ba_n \neq 0$. Clearly there exists at most one such arrow b. Consider the linear map

$$\iota \colon \Gamma_1 \to \Gamma_0, \quad ba_n \cdots a_1 \mapsto a_n \cdots a_1.$$

This is a well defined injective linear map. Let a_0 be an arrow of degree zero. Then

$$\iota(ba_n \cdots a_1 a_0) = a_n \cdots a_1 \cdot a_0 = \iota(ba_n \cdots a_1)a_0.$$

Hence ι is compatible with right multiplication by arrows of degree one. It is also clearly compatible with right multiplication by trivial paths, thus ι is a homomorphism of right Γ_0-modules. $\qquad\square$

Proposition 4.8. *Let* M, N *be in* Γ*-mod. We consider them as left* Γ_0*-modules where the context requires. Then there is an exact sequence:*

$$0 \longrightarrow \operatorname{Hom}_\Gamma(M,N) \longrightarrow \operatorname{Hom}_{\Gamma_0}(M,N) \longrightarrow \operatorname{Hom}_{\Gamma_0}(\Gamma_1 \otimes_{\Gamma_0} M, N) \longrightarrow$$

$$\longrightarrow \operatorname{Ext}^1_\Gamma(M,N) \longrightarrow \operatorname{Ext}^1_{\Gamma_0}(M,N) \longrightarrow \operatorname{Ext}^1_{\Gamma_0}(\Gamma_1 \otimes_{\Gamma_0} M, N) \longrightarrow$$

$$\longrightarrow \operatorname{Ext}^2_\Gamma(M,N) \longrightarrow 0.$$

Proof. Let $P_\bullet$ be a projective resolution of M as a Γ_0-module. Now Γ_{Γ_0} is flat so $\Gamma \otimes_{\Gamma_0} P_\bullet$ is a projective resolution of $\Gamma \otimes_{\Gamma_0} M$. Also observe that

$\operatorname{Hom}_\Gamma(\Gamma, N) \cong N$ as Γ_0-modules. We obtain the following identities by the hom-tensor adjunction:

$$
\begin{aligned}
\operatorname{Ext}_\Gamma^n(\Gamma \otimes_{\Gamma_0} M, N) \cong{} & H^n \operatorname{Hom}_\Gamma(\Gamma \otimes_{\Gamma_0} P_\bullet, N) \\
\cong{} & H^n \operatorname{Hom}_{\Gamma_0}(P_\bullet, \operatorname{Hom}_\Gamma(\Gamma, N)) \cong \operatorname{Ext}_{\Gamma_0}^n(M, N).
\end{aligned}
$$

Similarly we get the identity:

$$
\operatorname{Ext}_\Gamma^n(\Gamma \otimes_{\Gamma_0} \Gamma_1 \otimes_{\Gamma_0} M, N) \cong \operatorname{Ext}_{\Gamma_0}^n(\Gamma_1 \otimes_{\Gamma_0} M, N).
$$

We apply those identities to the long exact sequence obtained by applying $\operatorname{Hom}_\Gamma(-, N)$ to the exact sequence (Std). Since Γ_0 is hereditary the terms $\operatorname{Ext}_{\Gamma_0}^n(M, N)$ and $\operatorname{Ext}_{\Gamma_0}^n(\Gamma_1 \otimes_{\Gamma_0} M, N)$ vanish for $n \geq 2$ and we have an exact sequence of the form stated. $\qquad\square$

The exact sequence implies all n-th extensions vanish for $n > 2$, hence we get:

Corollary 4.9. *The algebra* $\mathrm{N}_s(Q)$ *has global dimension at most 2.*

The Euler form

Definition 4.10. Let $\mathbf{d}, \mathbf{e} \in \mathbb{N}^{Q_0^{(s)}}$ be dimension vectors, we define

$$
\begin{aligned}
\langle \mathbf{d}, \mathbf{e} \rangle^{(1)} := {} & \sum_{i \in Q_0} \sum_{t=1}^s \mathbf{d}_{i_t} \mathbf{e}_{i_t} - \sum_{(i \to j) \in Q_1} \sum_{t=2}^s \mathbf{d}_{i_t} \mathbf{e}_{j_{t-1}} \\
& - \sum_{i \in Q_0} \sum_{t=2}^s \mathbf{d}_{i_{t-1}} \mathbf{e}_{i_t} + \sum_{(i \to j) \in Q_1} \sum_{t=1}^{s-1} \mathbf{d}_{i_t} \mathbf{e}_{j_t}.
\end{aligned}
$$

Straightforward calculations show that if $\mathbf{d} = \mathbf{e}$ is a dimension filtration and we replace $\mathbf{d}_{i_t}$ by d_i^t, then this coincides with the definition of $\langle \mathbf{d}, \mathbf{d} \rangle^{(1)}$ from Section 4.2 .

Let A be a finite-dimensional algebra of finite global dimension. The Euler form for A-modules M and N is defined as

$$
\langle M, N \rangle_A := \sum_{i=0}^\infty (-1)^i \dim_k \operatorname{Ext}_A^i(M, N).
$$

Recall that we denote the idempotent of Γ corresponding to the vertex i_t by $e(i_t)$. For a Γ_0-module M we write $[M]_{i_t} := \dim_k M_{i_t}$, where M_{i_t} denotes the i_t-th vector space of M. We denote the dimension vector $([M]_{i_t})$ of M by $[M]$. For any $i \in Q_0$ we have $e(i_1)\Gamma_1 \otimes_{\Gamma_0} M = 0$. For $t \in \{2, \ldots, s\}$ we have

$$
e(i_t)\Gamma_1 \otimes_{\Gamma_0} M = b(i_{t-1})\Gamma_0 \otimes_{\Gamma_0} M.
$$

Thus $[\Gamma_1 \otimes_{\Gamma_0} M]_{i_t} = [M]_{i_{t-1}}$. In this way we can describe the dimension vector of $\Gamma_1 \otimes_{\Gamma_0} M$ in terms of the dimension vector of M.

Now Γ_0 is an hereditary algebra so we know how to calculate Euler forms of Γ_0-modules. We have

$$\langle \Gamma_1 \otimes M, N \rangle_{\Gamma_0} = \sum_{i \in Q_0} \sum_{t=1}^{s} [\Gamma_1 \otimes M]_{i_t}[N]_{i_t} - \sum_{(i \to j) \in Q_1} \sum_{t=2}^{s} [\Gamma_1 \otimes M]_{i_t}[N]_{j_{t-1}}$$

$$= \sum_{i \in Q_0} \sum_{t=2}^{s} [M]_{i_{t-1}}[N]_{i_t} - \sum_{(i \to j) \in Q_1} \sum_{t=1}^{s-1} [M]_{i_t}[N]_{j_t}.$$

From the long exact sequence in Proposition 4.8 and using what we know about Euler-forms for Γ_0 we get:

$$\langle M, N \rangle_\Gamma = \langle M, N \rangle_{\Gamma_0} - \langle \Gamma_1 \otimes M, N \rangle_{\Gamma_0}$$

$$= \sum_{i \in Q_0} \sum_{t=1}^{s} [M]_{i_t}[N]_{i_t} - \sum_{(i \to j) \in Q_1} \sum_{t=2}^{s} [M]_{i_t}[N]_{j_{t-1}}$$

$$- \sum_{i \in Q_0} \sum_{t=2}^{s} [M]_{i_{t-1}}[N]_{i_t} + \sum_{(i \to j) \in Q_1} \sum_{t=1}^{s-1} [M]_{i_t}[N]_{j_t}$$

$$= \langle [M], [N] \rangle^{(1)}.$$

Thus $\langle -, - \rangle^{(1)}$ is actually the Euler form on $\mathrm{N}_s(Q)$-mod.

Remark 4.11. Let $Q^{(s)} = (Q_0^{(s)}, Q_1^{(s)})$ as before. We realized $\mathrm{N}_s(Q)$ as the path algebra $kQ^{(s)}$ modulo the ideal $\mathfrak{I}$. The generators of $\mathfrak{I}$ can be seen as extra arrows $(i_t \to j_t)$ for each $t = 1, \ldots, s-1$ and $(i \to j) \in Q_1$, we denote those arrows by $Q_2^{(s)}$. Then the Euler form can be written as

$$\langle \mathbf{d}, \mathbf{e} \rangle^{(1)} = \sum_{i_t \in Q_0^{(s)}} \mathbf{d}_{i_t}\mathbf{e}_{i_t} - \sum_{(i_t \to j_u) \in Q_1^{(s)}} \mathbf{d}_{i_t}\mathbf{e}_{j_u} + \sum_{(i_t \to j_t) \in Q_2^{(s)}} \mathbf{d}_{i_t}\mathbf{e}_{j_t}.$$

4.3.3 Monomorphism categories and the category $\mathcal{N}$

We are interested in $\mathrm{N}_s(Q)$-modules that are related to Richardson orbits.

Definition 4.12. The category $\mathcal{N}$ is the full subcategory of $\mathrm{N}_s(Q)$-mod given by $Q^{(s)}$ representations that satisfy the relations (R1) and (R2) with the additional property that all maps corresponding to vertical arrows are injective.

For definition of $\mathrm{T}_s(Q)$ and the full subcategroy $\mathrm{mon}_s(Q) \subset \mathrm{T}_s(Q)$-mod we refer to Section 3.2. The subcategory $\mathcal{N}$ can be considered as a nilpotent analogue to the monomorphism categories

For a path α in kQ we let $\alpha(q, t)$ denote the matrix in $T_s(Q)$ that has α in coordinate (q, t) and all other coordinates trivial.

Lemma 4.13. *There is a ring homomorphism* $\Phi\colon T_s(Q) \to N_s(Q)$, *determined by the following data:*

$$\Phi(a(t,t)) = \begin{cases} 0, & t = 1, \\ b(j_{t-1})a_t, & t = 2, \ldots, s. \end{cases} \quad \forall (a\colon i \to j) \in Q_1,$$

$$\Phi(e_i(q,t)) = \begin{cases} e(i_t), & q = t, \\ b(i_{q-1}) \cdots b(i_t), & q > t. \end{cases} \quad \forall i \in Q_0.$$

In the proof δ_{ij} denotes the Kronecker-delta function, i.e.

$$\delta_{ij} := \begin{cases} 1, & i = j; \\ 0, & \text{else}. \end{cases}$$

Proof. Clearly the elements whose value is determined above generate $T_s(Q)$ as a k-algebra, so there is at most one k-algebra homomorphism satisfying those conditions. It is easy to check the homomorphism conditions on the generators $e_i(q,t)$, namely

$$\begin{aligned} \Phi(e_i(q,t))\Phi(e_j(q',t')) &= b(i_{q-1}) \cdots b(i_t) \cdot b(j_{q'-1}) \cdots b(j_{t'}) \\ &= \delta_{ij}\delta_{q't}b(i_{q-1}) \cdots b(j_{t'}) \\ &= \Phi(e_i(q,t)e_j(q',t')). \end{aligned}$$

We also have to show that the relations of $T_s(Q)$ are sent to zero. Firstly, if $a\colon i \to j$ and $a'\colon i' \to j'$ are arrows of Q_1 and if $s(a) \neq t(a')$ or $t \neq q$, then $\Phi(a(t,t))\Phi(a'(q,q)) = b(j_{t-1})a_t b(j'_{q-1})a'_q = 0$. Finally observe that

$$\begin{aligned} \Phi(a(t,t)&e_i(t,t-1)) - \Phi(e_j(t,t-1)a(t-1,t-1)) \\ &= b(j_{t-1})a_t b(i_{t-1}) - b(j_{t-1})b(j_{t-2})a_{t-1} \\ &= b(j_{t-1})(a_t b(i_{t-1}) - b(j_{t-2})a_{t-1}) \in \mathfrak{I}. \end{aligned}$$

$\square$

Of course Φ induces a restriction functor $\Phi^*\colon N_s(Q)\text{-mod} \to T_s(Q)\text{-mod}$.

Proposition 4.14. *The functor* Φ^* *restricts to a fully faithful functor from* $\mathcal{N}$ *to* $\text{mon}_s(Q)$.

Proof. The functor Φ^* is faithful because it is a restriction functor. Let N be an object of $\mathcal{N}$. Then $x \mapsto \Phi(e_i(q,t))x = b(i_{q-1}) \cdots b(i_t)x$ is an injective linear map $e_{i_t}N \to e_{i_q}N$ for all $i \in Q_0$ and $1 \leq t < q \leq s$ because $N \in \mathcal{N}$. But we can characterize $\text{mon}_s(Q)$ as the full subcategory of $T_s(Q)\text{-mod}$ with objects M such that $x \mapsto e_i(q,t)x$ is a monomorphism of vector spaces $e_i(q,q)M \to e_i(t,t)M$ for all $i \in Q_0$ and $1 \leq t \leq q \leq s$, thus $\Phi^*(N) \in \text{mon}_s(Q)$.

Let $N, N' \in \mathcal{N}$ and let $f \in \text{Hom}_{T_s(Q)}(\Phi^*(N), \Phi^*(N'))$. We know f is compatible with multiplication with all elements in the image of Φ. The image

contains all the trivial paths and all paths that have only vertical arrows. It only remains to show that f is compatible with a_t for all $a \in Q_1$ and $t \in \{2, \ldots, s\}$. Let $m \in N$, since $\Phi(a(t,t)) = b(j_{t-1})a_t$ we know

$$b(j_{t-1})a_t \cdot f(m) = f(b(j_{t-1})a_t \cdot m) = b(j_{t-1})f(a_t \cdot m).$$

But multiplication by $b(j_{t-1})$ on $e_{i_t}N'$ is a monomorphism of vector spaces by assumption, which implies $f(a_t \cdot m) = a_t \cdot f(m)$. Thus f is compatible with multiplication by a set of generators of $N_s(Q)$, which shows it is a morphism of $N_s(Q)$-modules. $\qquad\square$

Remark 4.15. The essential image of the restriction of Φ^* to $\mathcal{N}$ can be characterized as the full subcategory of objects M in $\mathrm{mon}_s(Q)$ such that the quotient M_{t+1}/M_t is a semi-simple kQ-module for $t = 1, \ldots, s-1$.

The category $\mathcal{N}$ and tensor algebra structure

We give other characterizations of the modules in $\mathcal{N}$. Recall that $N_s(Q)$ arises as a tensor algebra in two different ways, as Λ (cf. Section 4.3.2) and as Γ (cf. Section 4.3.2). Here we use the latter construction.

For a tensor algebra $\Gamma = T_{\Gamma_0}(\Gamma_1)$ the category of left Γ-modules is equivalent to a category with objects pairs (M, φ), where M is a Γ_0-module and $\varphi \colon \Gamma_1 \otimes_{\Gamma_0} M \to M$ is a Γ_0-linear map. The morphisms from (M, φ) to (N, ψ) in this category are given by a Γ_0-homomorphism $f \colon M \to N$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\Gamma_1 \otimes_{\Gamma_0} M & \xrightarrow{\;\;\varphi\;\;} & M \\
{\scriptstyle \mathrm{id}\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\
\Gamma_1 \otimes_{\Gamma_0} N & \xrightarrow{\;\;\psi\;\;} & N.
\end{array}
$$

Denote this category by (Γ_0, Γ_1)-mod. The direction Γ-mod $\to (\Gamma_0, \Gamma_1)$-mod is given by restricting the Γ-module structure on M to the Γ_0-module structure, which we denote by $_{\Gamma_0}M$, and restricting the scalar multiplication $\Gamma \otimes_\Gamma M \to M$ to $\Gamma_1 \otimes_\Gamma M$ to obtain the map φ.

Proposition 4.16. *The full subcategory of pairs (M, φ) in (Γ_0, Γ_1)-mod such that φ is a monomorphism corresponds to $\mathcal{N}$ under the equivalence above.*

Proof. Let (M, φ) be an object of (Γ_0, Γ_1)-mod and let $m \in M$. Then $b(i_t) \otimes m = b(i_t) \otimes e(i_t)m$, so $\varphi(b(i_t) \otimes m) = b(i_t)e(i_t)m = 0$ if and only if $b(i_t)m = 0$. Thus φ is injective if and only if the map $e(i_t)M \to e(i_{t+1})M$, given by left multiplication by $b(i_t)$, is injective for all $t = 1, \ldots, s-1$ and all $i \in Q_0$. But that precisely the criterion for the Γ-module M to belong to $\mathcal{N}$.

$\qquad\square$

Embedding monomorphism categories in $\mathcal{N}$

Consider the following example. Take the quiver $Q = 1 \longrightarrow 2 \longleftarrow 3$. We can embed the quiver of $kQ \otimes_k k\mathbb{A}_2$ into $Q^{(s)}$ as the full subquiver of bold dots in the following diagram:
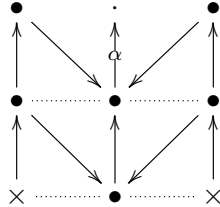
Observe that the commutativity relations in $Q^{(3)}$ give the commutativity relations of the subquiver $kQ \otimes k\mathbb{A}_2$. By adding the identity for the arrow labelled by α, and placing the zero vector space on the vertices marked $\times$, one can view an object of $\mathrm{mon}_2(Q)$ as an object in $\mathcal{N} \subset N_3(Q)$.

More generally, if Q is acyclic with all paths of length $\leq n$, then one can embed the monomorphism category $\mathrm{mon}_{s-n}(kQ)$ in $\mathcal{N} \subset \mathrm{N}_s(Q)$-mod. On the geometrical side this means we can embed any quiver flag variety of an acyclic quiver in a fibre of the collapsing map $\pi_{\mathbf{d}}$ for some dimension filtration $\mathbf{d}$. Since every projective variety arises as a quiver Grassmannian of an acyclic quiver, this demonstrates that these fibres can be rather complicated, and general questions about orbits in $\mathrm{RF}_{\mathbf{d}}$ can be expected to be impossibly difficult.

Be aware that this embedding is not an inverse to the functor Φ^* in any sensible way.

4.3.4 $\mathrm{N}_s(Q)$ as an endomorphism ring

Let $J \subset kQ$ be the ideal generated by the arrows of Q, and let $A := kQ/J^s$ with a $\mathbb{Z}$-grading given by length of paths. In particular $A_{\geq t} = J^t/J^s$. Let $A(t)$ denote the t-shift of A, i.e. the A-module with $A(t)_l = A_{l-t}$. Consider the algebra

$$B(A) = \mathrm{End}_A^{\mathbb{Z}} \left(\bigoplus_{t=0}^{s-1} A(t) \right)^{\mathrm{op}} \simeq \begin{bmatrix} A_0 & A_1 & \cdots & A_{s-1} \\ 0 & A_0 & \ddots & \vdots \\ \vdots & \ddots & A_0 & A_1 \\ 0 & \cdots & 0 & A_0 \end{bmatrix}.$$

This is very similar to the Beilinson algebra of a positively graded Artin algebra, cf. [14]. It is obtained by removing the last row and the last column from the matrix above.

Here $\mathrm{End}_A^{\mathbb{Z}}$ denotes endomorphisms of graded modules. Define

$$E(t) := A(s-t)/A(s-t)_{\geq s}, \quad t = 1, \ldots, s.$$

In particular $E(t) \cong kQ/J^t$ as an ungraded kQ-module. We write

$$E := \bigoplus_{t=1}^{s} E(t).$$

For a path $\alpha\colon i \to j$ in A we let $\ell(\alpha)$ denote the length of the path, and we let $\alpha^*\colon A \to A$ denote the map given by multiplication from the right.

Also let $\alpha_t^*\colon P(j)/J^t \to P(i)/J^{t+\ell(\alpha)}$ be the map induced by α^*, we consider it as an element of $\operatorname{End}_A(E)$ via the inclusions $P(j)/J^t \subset E(t)$ and $P(i)/J^{t+\ell(\alpha)} \subset E(t+\ell(\alpha))$. Observe that this is a map of graded modules, i.e. $\alpha_t^* \in \operatorname{End}_A^{\mathbb{Z}}(E)$. Recall the grading on $\mathrm{N}_s(Q)$ from Section 4.3.2, with vertical arrows of grade 1. As before we denote $\mathrm{N}_s(Q)$ with this grading by Γ.

Lemma 4.17. *We have*

$$B(A) = \operatorname{End}_A^{\mathbb{Z}} \left(\bigoplus_{t=0}^{s-1} A(t) \right)^{\mathrm{op}} \cong \operatorname{End}_A^{\mathbb{Z}}(E)^{\mathrm{op}} \cong \Gamma_0.$$

Proof. All summands of $A(t)$ are projective as non graded modules, hence

$$\operatorname{Hom}_A^{\mathbb{Z}}(A(t), A(t')) = A_{t-t'}^{\mathrm{op}} = \operatorname{Hom}_A^{\mathbb{Z}}(E(s-t), E(s-t')).$$

Now $A_{t-t'}^{\mathrm{op}}$ has as basis all paths $\alpha^{\mathrm{op}}\colon j \to i$ of length $t - t'$ in Q^{op}, and the latter identity is given by $\alpha^{\mathrm{op}} \mapsto \alpha_{s-t}^*$. We associate this with a path $\alpha_{s-t}\colon i_{s-t} \to j_{s-t'}$ in $Q^{(s)}$, which is in the standard basis of Γ_0. Running t and t' through $\{0, \ldots, s-1\}$ gives a bijection between the standard basis of Γ_0 and a basis of $\operatorname{End}_A^{\mathbb{Z}}(E)$. Moreover it is clear that multiplication of basis elements in the ring $\operatorname{End}_A^{\mathbb{Z}}(E)^{\mathrm{op}}$ is compatible with the multiplication of the corresponding paths in $Q^{(s)}$. This shows that $B(A)$ is isomorphic to Γ_0. $\square$

Next consider the endomorphism ring $\operatorname{End}_A(E)$ of ungraded A-modules. The grading of A induces a grading on the endomorphism ring via the degree of maps, in particular
$$\operatorname{End}_A^{\mathbb{Z}}(E) = \operatorname{End}_A(E)_0.$$

Note that $\operatorname{End}_A(E)_l = 0$ for $l \geq s$. We write $\pi_t\colon E(t+1) \to E(t)$ for the canonical projection. These projections all have grade 1.

Lemma 4.18. *The elements of the form $\alpha_t^* \pi_t \cdots \pi_{t+l-1}$, where $t = 1, \ldots, s-l$ and α is a path in Q of length at most $s - t$, form a basis of $\operatorname{End}_A(E)_l$ for $l = 0, \cdots, s-1$.*

Proof. Clearly $\alpha_t^* \pi_t \cdots \pi_{t+l-1} \in \operatorname{End}_A(E)_l$ is non-zero for $l \in \mathbb{N}_0$, $t \in \mathbb{Z}_+$, such that $l + t \leq s$. The canonical quotient maps $\pi_t\colon E(t+1) \to E(t)$ have degree 1, and they generate $\operatorname{End}_A(E)_1$ as a left $\operatorname{End}_A(E)_0$-module because of the universal property of quotients. More precisely, let $f\colon E(t+1) \to E$ be a map of degree 1, then the image of f is concentrated in degrees at least $s - t$, and hence $J^t \operatorname{Im} f = 0$. Therefore f factors through the map $\pi_t\colon E(t+1) \to$

$E(t)$. The analogous argument along with induction shows that any map $f\colon E(t+l) \to E$ of degree l can be written on the form $f'\pi_t \cdots \pi_{t+l-1}$, where f has degree 0.

Now E decomposes as a vector space into summands of the form

$$(P(i) + J^{t+\ell(\alpha)})/J^{t+\ell(\alpha)} \subset E(t + \ell(\alpha)).$$

Therefore it is sufficient to check that elements of the form $\alpha_t^* \pi_t \cdots \pi_{t+l-1}$ for different $\alpha\colon i \to j$ of the same length $\ell(\alpha) \leq s-t$ are linearly independent. But this holds because different such α_t^* are linearly independent in $\mathrm{End}_A(E)_0$. $\quad\square$

Remark 4.19. The degree l part of $\mathrm{End}_A(E)$ has the form

$$\mathrm{End}_A(E)_l \simeq \begin{bmatrix} A_l & A_{l+1} & \cdots & A_{s-1} & 0 & \cdots & 0 \\ A_{l-1} & A_l & & A_{s-2} & \vdots & & \vdots \\ \vdots & \ddots & & \vdots & \vdots & & \vdots \\ A_0 & A_1 & \cdots & A_{s-l-1} & \vdots & & \vdots \\ 0 & A_0 & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & A_1 & \vdots & & \vdots \\ 0 & \cdots & 0 & A_0 & 0 & \cdots & 0 \end{bmatrix}$$

That gives the dimension formula

$$\dim \mathrm{End}_A(E)_l = \sum_{m=l}^{s-1} \sum_{t=0}^{m} \dim A_t.$$

Theorem 4.20. *There is an isomorphism of graded algebras*

$$\mathrm{End}_A(E)_{\geq 0}^{\mathrm{op}} \cong \Gamma.$$

Proof. Let $\beta\alpha$ be a standard basis element of Γ, with α a diagonal path and $\beta\colon j_t \to j_{t+l}$ a vertical path. Define

$$\phi\colon \Gamma \to \mathrm{End}_A(E)_{\geq 0}^{\mathrm{op}}, \quad \phi(\beta\alpha) := (\alpha_t^* \pi_t \cdots \pi_{t+l-1})^{\mathrm{op}}.$$

By Lemma 4.18 this gives an isomorphism of graded vector spaces.

It remains to show that ϕ is a homomorphism of algebras. Since Γ is generated as an algebra by arrows and the trivial paths in $Q^{(s)}$, it is sufficient to check the condition on these generators. Let $\phi_0\colon \Gamma_0 \to \mathrm{End}_A(E)_0$ be the map given by restricting ϕ to Γ_0. By Lemma 4.17 ϕ_0 is a ring isomorphism, in particular ϕ preserves multiplication of diagonal arrows and idempotents. We identify the idempotents of Γ and $\mathrm{End}_A(E)$ via ϕ_0.

By construction it is clear that $\phi(\beta\alpha) = \phi(\beta)\phi(\alpha)$ for α diagonal and β a vertical path. We check the condition for products of vertical paths:

$$\phi(b(i_t)b(j_{t'})) = \phi(\delta_{ij}\delta_{t,t'+1}b(i_t)b(j_{t-1})) = \delta_{ij}\delta_{t,t'+1}\pi_t^{\mathrm{op}}\pi_{t-1}^{\mathrm{op}}e(j_{t-1})$$
$$= \pi_t^{\mathrm{op}}e(i_t)\pi_{t'}^{\mathrm{op}}e(j_{t'}) = \phi(b(i_t))\phi(b(j_{t'})).$$

Clearly the same argument works for longer paths. We have shown ϕ is a bijective homomorphism of algebras. $\qquad\square$

Connection to ADR algebras In [3] Auslander constructed an algebra of finite global dimension for any finite dimensional algebra A, which carries over some of the structure of A-mod. Later Dlab-Ringel [23] showed it is a quasi-hereditary algebra. We will consider the basic version $\mathcal{R}_A$ of this algebra, following [20] we call it the Auslander-Dlab-Ringel-algebra for A, or ADR-algebra for short.

Let A be an algebra with finite Loewy length as a left module, and let s be such that $\mathrm{rad}(A)^s = 0$. Define

$$M_A := \bigoplus_{t=1}^{s} A/\mathrm{rad}(A)^t,$$

and let $\bar{M}_A$ be the basic A-module with $\mathrm{add}(\bar{M}_A) = \mathrm{add}(M_A)$. The ADR-algebra is defined as $\mathcal{R}_A := \mathrm{End}_A(\bar{M}_A)^{\mathrm{op}}$.

Let us now consider $A = kQ/J^s$. Note that the module $E(t)$ defined above is isomorphic to $A/\mathrm{rad}(A)^t$ as an ungraded A-module for $t = 1, \ldots, s$; thus $E \cong M_A$. Now E is basic if and only if all summands if $E(t)$ have Loewy-length t. If this is the case then $\mathrm{End}_A(E)_{<0} = 0$, i.e. $\mathrm{End}_A(E)_{\geq 0} = \mathrm{End}_A(E)$, so $\mathrm{N}_s(Q)$ coincides with the ADR-algebra $\mathcal{R}_A$. It has already been proven by Conde in [20] that ADR-algebras are right ultra strongly quasi-hereditary. These similarities might suggest some of the the properties of a quasi-hereditary structure of $\mathrm{N}_s(Q)$, which we describe in Section 4.4.

4.4 Quasi-hereditary structure of the nilpotent quiver algebra

Keep in mind that the categories Γ-mod, Λ-mod and $\mathrm{N}_s(Q)$-mod are all equivalent. For a finite dimensional algebra A we let $S(A) = {}_A S(A)$ (resp. $S(A)_A$) denote the isomorphism classes of simple left (resp. right) A-modules. The sets $S(\mathrm{N}_s(Q)), S(\Gamma), S(\Lambda), S(\Gamma_0), S(\Lambda_0), S(\mathrm{N}_s(Q))_{\mathrm{N}_s(Q)}, S(\Gamma)_\Gamma, S(\Lambda)_\Lambda, S(\Gamma_0)_{\Gamma_0}$ and $S(\Lambda_0)_{\Lambda_0}$ are all in a canonical bijection with the vertices of $Q^{(s)}$. Accordingly, if $A = \Gamma, \Gamma_0, \Lambda$ or Λ_0, we write ${}_A S(i_t)$ (resp. $S(i_t)_A$) for the simple left (resp. right) A-module corresponding to i_t, and ${}_A P(i_t), {}_A I(i_t)$ (resp. $P(i_t)_A, I(i_t)_A$) for the projective cover and injective envelope of that simple. Note that even though Λ and Γ are graded algebras, we always consider ungraded modules unless explicitly stated otherwise. Now Λ-mod $\cong \mathrm{N}_s(Q)$-mod $\cong \Gamma$-mod and we will not distinguish between those categories. Accordingly we write

$$P(i_t) := {}_\Lambda P(i_t) = {}_\Gamma P(i_t) = {}_{\mathrm{N}_s(Q)} P(i_t),$$
$$I(i_t) := {}_\Lambda I(i_t) = {}_\Gamma I(i_t) = {}_{\mathrm{N}_s(Q)} I(i_t),$$

as our default setting. Let us consider the function

$$\ell \colon S(\mathrm{N}_s(Q)) \to \mathbb{N}, \quad \ell(i_t) := \ell(S(i_t)) := t.$$

If $e \in \mathrm{N}_s(Q)$ is an idempotent, we denote the two-sided ideal generated by e by (e). Let $\mathrm{Span}(i) := \{(a\colon i \to j) \in Q_1\}$, and dually $\mathrm{Cosp}(i) := \{(a\colon j \to i) \in Q_1\}$.

Lemma 4.21. *Let $t \in \{2, \ldots, s\}$. Then*

$$\Lambda_+ \otimes_{\Lambda_0} {}_{\Lambda_0} P(i_t) \cong \bigoplus_{\substack{(a\colon i \to j) \\ \in \mathrm{Span}(i)}} {}_{\Lambda} P(j_{t-1}). \tag{Syz}$$

Also

$$P(i_t)_{\Gamma_0} \otimes_{\Gamma_0} \Gamma_+ \cong P(i_{t-1})_\Gamma. \tag{Cosyz}$$

Let $e_1 = \sum_{i \in Q_0} e(i_1)$, there is a canonical inclusion $\iota \colon \mathrm{mod}\text{-}\Lambda/(e_1) \to \mathrm{mod}\text{-}\Lambda$. Then we have the following identity of right Λ-modules.

$$P(i_s)_{\Lambda_0} \otimes_{\Lambda_0} \Lambda_+ \cong \bigoplus_{\substack{(a\colon j \to i) \\ \in \mathrm{Cosp}(i)}} \iota(P(j_s)_{\Lambda/(e_1)}). \tag{Emb}$$

Proof. The first identity (Syz) can be seen from the following identities of Λ_0-modules.

$$\Lambda_1 \otimes_{\Lambda_0} {}_{\Lambda_0} P(i_t) = \bigoplus_{\substack{(a\colon i \to j) \\ \in \mathrm{Span}(i)}} \Lambda_0 a_t e(i_t) = \bigoplus_{\substack{(a\colon i \to j) \\ \in \mathrm{Span}(i)}} \Lambda_0 e(j_{t-1}) = \bigoplus_{\substack{(a\colon i \to j) \\ \in \mathrm{Span}(i)}} {}_{\Lambda_0} P(j_{t-1}).$$

Since $\Lambda_+ = \Lambda \otimes_{\Lambda_0} \Lambda_1$ we get the identity (Syz) by applying $\Lambda \otimes_{\Lambda_0} -$ to the identity above.

In a similar way we get

$$P(i_t)_{\Gamma_0} \otimes_{\Gamma_0} \Gamma_1 \cong e(i_t)\Gamma_0 \otimes_{\Gamma_0} \Gamma_1 \cong e(i_t)\Gamma_1 \cong b(i_{t-1})\Gamma_0 \cong P(i_{t-1})_{\Gamma_0}.$$

Applying $- \otimes_{\Gamma_0} \Gamma$ to this gives the identity (Cosyz).

Finally we prove (Emb): First observe that we have the following identity of right Λ_0-modules.

$$e(i_s)\Lambda_1 \cong \bigoplus_{\substack{(a\colon j \to i) \\ \in \mathrm{Cosp}(i)}} b(j_{s-1})a_s \Lambda_0 \cong \bigoplus_{\substack{(a\colon j \to i) \\ \in \mathrm{Cosp}(i)}} e(j_s)\Lambda_0/(e_1).$$

That gives the following isomorphisms of right Λ-modules:

$$P(i_s)_{\Lambda_0} \otimes_{\Lambda_0} \Lambda_+ \cong e(i_s)\Lambda_1 \otimes_{\Lambda_0} \Lambda$$
$$\cong \bigoplus_{\substack{(a\colon j \to i) \\ \in \mathrm{Cosp}(i)}} e(j_s)\Lambda_0/(e_1) \otimes_{\Lambda_0} \Lambda \cong \bigoplus_{\substack{(a\colon j \to i) \\ \in \mathrm{Cosp}(i)}} \iota(P(j_s)_{\Lambda/(e_1)}).$$

$\square$

Let $\Delta(i_t)$ denote the maximal factor module of $P(i_t)$ such that all the composition factors $S(j_u)$ of $\mathrm{rad}\Delta(i_t)$ have layer $\ell(j_u) < t$. We write $\Delta := \{\Delta(i_t) \mid i_t \in Q_0^{(s)}\}$.

Proposition 4.22. *Let $t \in \{2, \ldots, s\}$. There is an exact sequence*

$$0 \longrightarrow \bigoplus_{\substack{(a\,:\,i \to j) \\ \in \mathrm{Span}(i)}} P(j_{t-1}) \longrightarrow P(i_t) \longrightarrow \Delta(i_t) \longrightarrow 0. \qquad \text{(Res)}$$

of $\mathrm{N}_s(Q)$-modules. This sequence is a projective resolution of $\Delta(i_t)$. Hence the conditions of Definition 1.15 are satisfied, i.e. ℓ is a layer function . In particular $(\mathrm{N}_s(Q), \Delta)$ is quasi-hereditary. The corresponding costandard modules have the following injective coresolution:

$$0 \longrightarrow \nabla(i_t) \longrightarrow I(i_t) \longrightarrow I(i_{t-1}) \longrightarrow 0. \qquad \text{(Cores)}$$

Proof. Apply the sequence (Std) to the Λ-module $\Lambda/\Lambda_+ \otimes_{\Lambda_0} {}_{\Lambda_0}P(i_t)$. Using the identity (Syz) we get the sequence

$$0 \longrightarrow \bigoplus_{\substack{(a\,:\,i \to j) \\ \in \mathrm{Span}(i)}} P(j_{t-1}) \longrightarrow P(i_t) \longrightarrow \Lambda/\Lambda_+ \otimes_{\Lambda_0} {}_{\Lambda_0}P(i_t) \longrightarrow 0.$$

Now observe that all the composition factors of $\Lambda/\Lambda_+ \otimes_{\Lambda_0} {}_{\Lambda_0}P(i_t)$ have layer higher than or equal to t. Thus $\Lambda/\Lambda_+ \otimes_{\Lambda_0} {}_{\Lambda_0}P(i_t)$ is a factor module of $\Delta(i_t)$. On the other hand the top of each summand of the kernel of the sequence has layer $t-1 < t$, hence $\Delta(i_t)$ is a factor module of $\Lambda/\Lambda_+ \otimes_{\Lambda_0} {}_{\Lambda_0}P(i_t)$. Together this shows $\Lambda/\Lambda_+ \otimes_{\Lambda_0} {}_{\Lambda_0}P(i_t) \cong \Delta(i_t)$.

Next we apply the sequence (DStd) to the right Γ-module $P(i_t)_{\Gamma_0} \otimes_{\Gamma_0} \Gamma/\Gamma_+$. By the identity (Cosyz) we get the following sequence of left Γ-modules.

$$0 \longrightarrow \mathrm{D}\left(P(i_t)_{\Gamma_0} \otimes_{\Gamma_0} \Gamma/\Gamma_+\right) \longrightarrow I(i_t) \longrightarrow I(i_{t-1}) \longrightarrow 0.$$

Now $\mathrm{D}\left(P(i_t)_{\Gamma_0}\right) \cong {}_{\Gamma_0}I(i_t)$, and thus $\mathrm{D}\left(P(i_t)_{\Gamma_0} \otimes_{\Gamma_0} \Gamma/\Gamma_+\right) \cong \Gamma/\Gamma_+ \otimes_{\Gamma_0} {}_{\Gamma_0}I(i_t)$. Observe that all composition factors of $\Gamma/\Gamma_+ \otimes_{\Gamma_0 \Gamma_0} I(i_t)$ have layer greater than or equal to t. Moreover, any factor module of $I(i_t)$ that properly contains this submodule, would have $S(i_{t-1})$ as composition factor. Together this shows that $\Gamma/\Gamma_+ \otimes_{\Gamma_0} {}_{\Gamma_0}I(i_t) \cong \nabla(i_t)$, the costandard module at i_t. $\square$

Let $\mathrm{add}(\Delta)$ (resp. $\mathrm{add}(\nabla)$) denote the full subcategory of $\mathrm{N}_s(Q)$-mod given by finite direct sums of standard modules (resp. costandard modules).

Corollary 4.23. *The quasi-hereditary algebra $(\mathrm{N}_s(Q), \Delta)$ is both left and right strongly quasi-hereditary. Moreover there are equivalences of categories*

$$\Lambda_0\text{-proj} \mathrel{\substack{\xrightarrow{\Lambda/\Lambda_+ \otimes_{\Lambda_0} -} \\ \xleftarrow{\mathrm{res}_{\Lambda_0}}}} \mathrm{add}(\Delta), \qquad\qquad \Gamma_0\text{-inj} \mathrel{\substack{\xrightarrow{\Gamma/\Gamma_+ \otimes_{\Gamma_0} -} \\ \xleftarrow{\mathrm{res}_{\Gamma_0}}}} \mathrm{add}(\nabla).$$

Here res_{Λ_0} and res_{Γ_0} are the restriction functors induced by the embeddings $\Lambda_0 \to \Lambda$ and $\Gamma_0 \to \Gamma$ respectively.

For $t = 1, \ldots, s$ we define $e_t := \sum_{i \in Q_0} e(i_t)$, and $E_t := \sum_{u=1}^{t} e_u$, and $E_0 := 0$. Let us write (E_t) for the two sided ideal of $\mathrm{N}_s(Q)$ generated by the idempotent E_t. For each t we have the quotient $\mathrm{N}_s(Q) \to \mathrm{N}_s(Q)/(E_t)$, observe that $\mathrm{N}_s(Q)/(E_t) \cong N_{s-t}(Q)$. The quotient induces a fully faithful functor $\iota_t \colon \mathrm{N}_s(Q)/(E_t)\text{-mod} \to \mathrm{N}_s(Q)\text{-mod}$. For $t = 0, \ldots, s-1$ we define $T(i_{t+1}) := \iota_t({}_{N_{s-t}(Q)}I(i_s))$.

Proposition 4.24. *There are sequences*

$$0 \longrightarrow \Delta(i_t) \longrightarrow T(i_t) \longrightarrow \bigoplus_{\substack{(a \colon j \to i) \\ \in \mathrm{Cosp}(i)}} T(j_{t+1}) \longrightarrow 0. \qquad \text{(Filt)}$$

$$0 \longrightarrow T(i_{t+1}) \longrightarrow T(i_t) \longrightarrow \nabla(i_t) \longrightarrow 0. \qquad \text{(Filt*)}$$

Moreover the characteristic tilting module for $(\mathrm{N}_s(Q), \Delta)$ is

$$T := \bigoplus_{t=1}^{s} \bigoplus_{i \in Q_0} T(i_t).$$

Proof. All composition factors of $T(i_t), \Delta(i_t)$ and $T(j_{t+1})$ have layer $\geq t$. Thus we can work in the full subcategory $\mathrm{N}_s(Q)/(E_{t-1})\text{-mod}$, so we assume $t = 1$ without loss of generality. Let us apply (DStd) to the right $\mathrm{N}_s(Q)$-module $P(i_s)_{\Lambda_0} \otimes_{\Lambda_0} \Lambda/\Lambda_+$. By the identity (Emb) we get the exact sequence

$$0 \longrightarrow \mathrm{D}(P(i_s)_{\Lambda_0} \otimes_{\Lambda_0} \Lambda/\Lambda_+) \longrightarrow I(i_s) \longrightarrow \bigoplus_{\substack{(a \colon j \to i) \\ \in \mathrm{Cosp}(i)}} \iota_1({}_{\Lambda/(e_1)}I(j_s)) \longrightarrow 0.$$

Now $\mathrm{D}\left(P(i_s)_{\Lambda_0}\right) = {}_{\Lambda_0}I(i_s)$, and since tensoring with Λ/Λ_+ does not affect the underlying vector space we get

$$\mathrm{D}\left(P(i_s)_{\Lambda_0} \otimes_{\Lambda_0} \Lambda/\Lambda_+\right) \cong \Lambda/\Lambda_+ \otimes_{\Lambda_0} {}_{\Lambda_0}I(i_s) \cong \Lambda/\Lambda_+ \otimes_{\Lambda_0} {}_{\Lambda_0}P(i_1).$$

Since we assume $t = 1$ the kernel of the sequence is isomorphic to $\Delta(i_1)$. Moreover, $I(i_s) \cong T(i_1)$ and $\iota_1({}_{\Lambda/(e_1)}I(j_s)) = T(j_2)$ by construction, hence this is the sequence (Filt). By induction with the sequence (Filt) we see the modules $T(i_t)$ are in $\mathcal{F}(\Delta)$.

Consider the quotient map $\Gamma \to \Gamma/(e_1)$. We multiply from the left with the idempotent $e(i_s)$ to get a short exact sequence of right Γ-modules

$$0 \longrightarrow e(i_s)\Gamma_{s-1} \longrightarrow e(i_s)\Gamma \longrightarrow e(i_s)\Gamma/\Gamma_{s-1} \longrightarrow 0.$$

Observe that $e(i_s)\Gamma_{s-1} \cong e(i_1)\Gamma_0$ as a right Γ-module (or a Γ/Γ_+-module) and, by the choice of idempotent, we have $e(i_s)\Gamma/\Gamma_{s-1} \cong e(i_s)\Gamma/(e_1)$. We take the dual of the sequence above to obtain the sequence of Γ_0-modules

$$0 \longrightarrow \mathrm{D}\left(e(i_s)\Gamma/(e_1)\right) \longrightarrow I(i_s) \longrightarrow {}_{\Gamma_0}I(i_1) \longrightarrow 0.$$

But by the construction of $T(i_t)$ and Corollary 4.23 of the costandard modules this yields the exact sequence

$$0 \longrightarrow T(i_2) \longrightarrow T(i_1) \longrightarrow \nabla(i_1) \longrightarrow 0,$$

of Γ-modules. To get the sequence (Filt*) for arbitrary t we apply the same argument to $\Gamma/(E_{t-1})$.

To summarize, all the modules $T(i_t)$ are pairwise distinct, indecomposable, and belong to $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. That shows T is the characteristic tilting module of $N_s(Q)$. $\qquad\square$

We summarize the most important properties of $N_s(Q)$ and the subcategories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ in Theorem 4.25. Keep in mind that we have established two different $\mathbb{Z}$-gradings on $N_s(Q)$, producing the graded algebras Λ and Γ. Hence any $N_s(Q)$-module can equivalently be considered as an ungraded Λ- or Γ-module. We define

$$I := D\left(N_s(Q)e_s\right) = \bigoplus_{i \in Q_0} I(i_s).$$

Theorem 4.25. *The pair $(N_s(Q), \Delta)$, with Δ given by the layer function ℓ, is both left strongly quasi-hereditary and right ultra strongly quasi-hereditary algebra. In particular the category $\mathcal{F}(\Delta)$ is closed under taking submodules and $\mathcal{F}(\nabla)$ is closed under taking factor modules. Moreover the following are equivalent for an $N_s(Q)$-module M.*

(a) $M \in \mathcal{N}$.

(b) $M \in \mathrm{cogen}(I)$.

(c) $M \in \mathcal{F}(\Delta)$.

(d) The underlying Λ_0-module of M is projective.

(e) For the corresponding (Γ_0, Γ_1)-module (M, φ), the map φ is a monomorphism.

Also the following conditions are equivalent:

(b') $M \in \mathrm{gen}(I)$.

(c') $M \in \mathcal{F}(\nabla)$.

(d') The underlying Γ_0-module of M is injective.

Proof. Corollary 4.23 already shows $N_s(Q)$ is left and right strongly quasi-hereditary. Recall that $(N_s(Q), \Delta)$ is right ultra strongly quasi-hereditary if it satisfies conditions (US1) and (US2) from Section 1.5.1. By Corollary 4.23 we see $N_s(Q)$ fulfils condition (US1). Moreover $\mathrm{rad}(\Delta(i_t)) = 0$ if and only if

$t = s$, in which case $I(i_t) = T(i_1)$. Clearly $T(i_1)$ has a filtration by standard modules, thus $N_s(Q)$ satisfies (US2).

Observe that $\mathcal{N}$ can be characterized as containing exactly the modules such that all summands of the socle have the form $S(i_s)$ for some $i \in Q_0$. Since the indecomposable summands of I are $I(i_s)$ for $i \in Q_0$, we have $\mathcal{N} = \operatorname{cogen}(I)$.

Since $N_s(Q)$ is right strongly quasi-hereditary, $\mathcal{F}(\Delta)$ is closed under taking submodules. By Proposition 4.24 we know $I(i_s) = T(i_1)$ is in $\mathcal{F}(\Delta)$ for all $i \in Q_0$, thus $\operatorname{cogen}(I) \subset \mathcal{F}(\Delta)$. Now the standard modules are in $\mathcal{N}$ and, since $\mathcal{N}$ is closed under taking extensions, that shows $\mathcal{F}(\Delta) = \mathcal{N}$.

Conditions (c) and (d) are equivalent by Corollary 4.23, and Condition (e) is equivalent to (a) by Proposition 4.16.

Since $N_s(Q)$ is left strongly quasi-hereditary we have $\mathcal{F}(\nabla) = \operatorname{gen}(T)$ by the appendix of [52]. But from the sequence (Cores) we see $\operatorname{gen}(T) = \operatorname{gen}(I)$, hence (b') and (c') are equivalent. The equivalence of (c') and (d') is clear from the equivalence in Corollary 4.23.

$\square$

4.4.1 The Ringel-dual

Let us denote $N_s(Q^{\mathrm{op}})$ by Γ, accordingly we let $\mathcal{R}_\Gamma$ denote the Ringel dual of $N_s(Q^{\mathrm{op}})$, i.e. $\mathcal{R}_\Gamma = \operatorname{End}_\Gamma(T)^{\mathrm{op}}$, where T is the characteristic module of Γ. We want to describe $\mathcal{R}_\Gamma$ explicitly. This has already been done by Conde-Erdmann for the case where Q has no sinks and no sources. In that case $N_s(Q)$ and $N_s(Q^{\mathrm{op}})$ are both ADR-algebras, and so $\mathcal{R}_\Gamma \cong N_s(Q)^{\mathrm{op}}$ by the main theorem of [21]. We will show that this formula holds more generally.

Consider the following diagram for $r = 1, \ldots, s - 1$, using the notation $T(i_{s+1}) := 0$.

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Delta(i_{t+1}) & \longrightarrow & T(i_{t+1}) & \longrightarrow & \bigoplus_{a \in \operatorname{Span}(i)} T(t(a)_{t+2}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Delta(i_t) & \longrightarrow & T(i_t) & \longrightarrow & \bigoplus_{a \in \operatorname{Span}(i)} T(t(a)_{t+1}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S(i_t) & \longrightarrow & \nabla(i_t) & \longrightarrow & \bigoplus_{a \in \operatorname{Span}(i)} \nabla(t(a)_{t+1}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

By the sequences (Filt) and (Filt*) the first two rows and last two columns are exact, and the first column is clearly exact. Note that we have modified those sequences to take into account the fact that we consider $N_s(Q^{\mathrm{op}})$ instead

of $N_s(Q)$. The upper right square is commutative by the construction of the sequences (Filt) and (Filt*), and it is easy to check that the upper left square is commutative. Thus the snake lemma implies that the last row is exact and the whole diagram is commutative.

The module $I = D(e_s\Gamma)$ is a summand of T, and we let e denote the corresponding idempotent of $\mathcal{R}_\Gamma$. Also let T' be such that $T = I \oplus T'$.

Lemma 4.26. *The endomorphism ring $\mathrm{End}(T)$ is generated by the following maps:*

(a) *The identity $\mathrm{id}_{i_t}\colon T(i_t) \to T(i_t)$ for all vertices i_t of $Q^{(s)}$;*

(b) *maps $a_t^*\colon T(i_{t-1}) \to T(j_t)$ for all arrows $a\colon i \to j$ of Q, and all $t = 2, \ldots s$;*

(c) *the canonical inclusions $b(i_t)^*\colon T(i_{t+1}) \to T(i_t)$ for $i \in Q_0$ and $t = 1, \ldots, s-1$.*

The maps a^* are given by projecting to the corresponding summand in a sequence of the form (Filt), and the maps $b(i_t)^*$ occur in sequences of the form (Filt).*

Moreover, we have relations

$$b(j_t)^* a_{t+1}^* = a_t^* b(i_{t-1})^*,$$

for $t = 2, \ldots, s-1$, and $\mathrm{Hom}_\Gamma(T(i_s), T(j_s)) = 0$ for $i \neq j$.

Proof. To prove that the generators given above generate all of $\mathcal{R}_\Gamma$ we use induction on s. It is clearly true for $s = 1$, because then Γ is semi-simple and the $T(i_1)$-s represent the isomorphism classes of simple Γ-modules.

Assume our statement is true for $s - 1$, then $\mathrm{End}_\Gamma(T')$ is generated by the generators $(a), (b)$ and (c). It remains to show that they generate $\mathrm{Hom}_\Gamma(I, T')$, $\mathrm{Hom}(T', I)$ and $\mathrm{End}_\Gamma(I)$.

Let $T(j_t)$ be a summand of T', so $t \geq 2$, and let $f\colon T(j_t) \to I(i_s)$ be a homomorphism of Γ-modules. Consider the following diagram, where the bottom row is given by (Filt*).

$$
\begin{array}{ccccccccc}
 & & & & T(j_t) & & & & \\
 & & & {\scriptstyle f'} \nearrow & \downarrow {\scriptstyle f} & & & & \\
0 & \longrightarrow & T(i_2) & \xrightarrow{b(i_1)^*} & I(i_s) & \xrightarrow{\pi} & \nabla(i_1) & \longrightarrow & 0
\end{array}
$$

Since $S(i_1)$ is not a composition factor of $T(j_t)$ the composition πf is zero, so f factors through $b(i_1)^*$ as shown in the diagram. But then f is generated by $b(i_1)^*$ and maps in $\mathrm{End}_\Gamma(T)$. Doing this for all summands of T' and all $i \in Q_0$ shows that $\mathrm{Hom}(T', I)$ is generated as stated.

Let $g\colon I(i_s) \to T(j_t)$ be a map of Γ-modules with $T(j_t) \in \mathrm{add}(T')$. Consider the following diagram, with the exact sequence given by (Filt).

$$0 \longrightarrow \Delta(i_1) \overset{\iota}{\longrightarrow} I(i_s) \longrightarrow \bigoplus_{a \in \mathrm{Span}(i)} T(t(a)_2) \longrightarrow 0$$

with vertical map $g\colon I(i_s) \to T(j_t)$ and diagonal map g'.

Since $T(j_t)$ does not have $S(i_1)$ as composition factor we have $g\iota = 0$, so there is a map g' as above making the diagram commutative. Thus g is generated by $\mathrm{End}_\Gamma(T)$ and maps of the form a_2^*.

Finally let $h\colon I(j_s) \to I(i_s)$, consider the diagram

$$0 \longrightarrow T(i_2) \overset{b(i_1)^*}{\longrightarrow} I(i_s) \overset{\pi}{\longrightarrow} \nabla(i_1) \longrightarrow 0.$$

with vertical map $h\colon I(j_s) \to I(i_s)$ and diagonal map h'.

If $\pi h = 0$ then we get a map h' as in the diagram, and h is thus generated by $\mathrm{Hom}_\Gamma(T', I)$ and $b(i_1)^*$ analogously to the argument above. If $\pi h \neq 0$, then clearly $i = j$, and there is a scalar c s.t. $\pi(h - c\,\mathrm{id}_{i_1}) = 0$, because $\dim_k \mathrm{Hom}(I(i_s), \nabla(i_1)) = 1$. But then h is generated by id_{i_1} along with $\mathrm{Hom}_\Gamma(T', I)$ and $b(i_1)^*$. We have shown our distinguished generators generate all of $\mathrm{End}_\Gamma(T)$.

The relations follow for the commutativity of the diagram at the beginning of this subsection, and the fact that the modules $T(i_s)$ are simple. $\qquad\square$

Corollary 4.27. *The assignments*

$$\Theta(e(i_{s-t+1})) := \mathrm{id}_{i_t}, \quad \Theta(a_{s-t+1}) := a_t^*, \quad \Theta(b(i_{s-t+1})) := b(i_t)^*,$$

determine a surjective ring homomorphism $\Theta\colon N_s(Q) \to \mathrm{End}_\Gamma(T)$.

Proof. It is clear that this preserves multiplication with idempotents, and the relations $b(j_t)^* a_{t+1}^* = a_t^* b(i_{t-1})^*$ and $\mathrm{Hom}_\Gamma(T(i_s), T(j_s)) = 0$ show that all necessary relations are satisfied. Since we have shown that the image of this map generates $\mathrm{End}_\Gamma(T)$, we know Θ is surjective. $\qquad\square$

Theorem 4.28. *The map* $\Theta\colon N_s(Q) \to \mathrm{End}_\Gamma(T)$ *is an isomorphism. In particular* $\mathcal{R}_\Gamma \cong N_s(Q)^{\mathrm{op}}$.

Proof. We argue by induction on s using k-dimension. Clearly our statement holds for $s = 1$. Let $e(1) = \sum_{i \in Q_0} e(i_1)$ and assume

$$(1 - e)\mathcal{R}_\Gamma(1 - e) \cong (1 - e_1)N_s(Q^{\mathrm{op}})(1 - e_1),$$

via an isomorphism of the form Θ. In particular $\dim_k(1-e)\mathcal{R}_\Gamma(1-e) = \dim_k(1-e_1)N_s(Q^{\mathrm{op}})(1-e_1)$. Let $t \in \{2,\ldots,s\}$ and apply $(-,T(j_t))_\Gamma$ to a sequence of the form (Filt) to obtain an exact sequence

$$0 \longrightarrow (\bigoplus_{a\in\mathrm{Span}(i)} T(t(a)_2), T(j_t))_\Gamma \longrightarrow (I(i_s), T(j_2))_\Gamma \longrightarrow (\Delta(i_1), T(j_2))_\Gamma = 0.$$

This shows.

$$\dim_k \mathrm{id}_{j_t}\,\mathcal{R}_\Gamma\,\mathrm{id}_{i_1} = \dim_k(T(i_1),T(j_t))_\Gamma = \dim_k \bigoplus_{a\in\mathrm{Span}(i)}(T(t(a)_2),T(j_t))_\Gamma$$

$$= \dim_k \bigoplus_{a\in\mathrm{Span}(i)} \mathrm{id}_{j_t}\,\mathcal{R}_\Gamma\,\mathrm{id}_{t(a)_2}.$$

But by the identity (Res) we have the identity

$$\dim_k e(j_t)N_s(Q)e(i_s) = \dim_k \bigoplus_{a\in\mathrm{Span}(i)} e(j_t)N_s(Q)e(i_{s-1}).$$

If we combine this with the induction hypothesis we get

$$\dim_k e(j_{s-t+1})N_s(Q)e(i_s) = \dim_k \mathrm{id}_{j_t}\,\mathcal{R}_\Gamma\,\mathrm{id}_{i_1}.$$

Similarly apply $(T(j_t),-)_\Gamma$ to a sequence of the form (Filt*) to obtain

$$0 \longrightarrow (T(j_2)),T(i_2))_\Gamma \longrightarrow (T(j_2),T(i_1))_\Gamma \longrightarrow (T(j_2),\nabla(i_1))_\Gamma = 0.$$

Thus

$$\dim_k \mathrm{id}_{j_t}\,\mathcal{R}_\Gamma\,\mathrm{id}_{i_1} = \dim_k \mathrm{id}_{j_t}\,\mathcal{R}_\Gamma\,\mathrm{id}_{i_2}.$$

From the identity (Cores) and induction hypothesis we get

$$\dim_k e(i_s)N_s(Q)e(j_{s-t+1}) = \dim_k e(i_{s-1})N_s(Q)e(j_{s-t+1})$$
$$= \dim_k \mathrm{id}_{i_2}\,\mathcal{R}_\Gamma\,\mathrm{id}_{j_t} = \dim_k \mathrm{id}_{i_1}\,\mathcal{R}_\Gamma\,\mathrm{id}_{j_t}.$$

Moreover, by similar methods and using our observations from the proof of Lemma 4.26, we get

$$\dim_k e(i_s)N_s(Q)e(j_s) = \dim_k e(i_{s-1})N_s(Q)e(j_s) + \delta_{ij}$$
$$= \dim_k \mathrm{id}_{i_2}\,\mathcal{R}_\Gamma\,\mathrm{id}_{j_1} + \delta_{ij} = \dim_k \mathrm{id}_{i_1}\,\mathcal{R}_\Gamma\,\mathrm{id}_{j_1}.$$

But then adding over all idempotents on both sides gives $\dim_k N_s(Q) = \dim_k \mathrm{End}_\Gamma(T)$. That concludes the proof. $\square$

Remark 4.29. Equivalently we can state the theorem as: The Ringel dual of $N_s(Q)$ is $N_s(Q^{\mathrm{op}})^{\mathrm{op}}$. This better conforms with the notation of the rest of this chapter.

4.4.2 Rigid Δ-filtered modules correspond to Richardson orbits

Let k be an algebraically closed field. Let $A = \mathrm{N}_s(Q)$ and take a dimension vector $\mathbf{d} = (\mathbf{d}^{(1)}, \ldots, \mathbf{d}^{(s)} = d)$ in $\mathbb{N}^{Q_0^{(s)}}$, following the convention of Section 4.3.1. We denote by $\mathrm{Rep}_{\mathbf{d}}(\mathcal{N}) \subset \mathrm{Rep}_{\mathbf{d}}(A)$ the subset corresponding to $\mathbf{d}$-dimensional A-modules which are in the subcategory $\mathcal{N}$. This is an open $\mathrm{GL}_{\mathbf{d}}$-invariant subset. By restricting the map $\pi_{\mathbf{d}}^{\mathrm{mon}}$ from Section 3.3 we get a map

$$\pi_{\mathcal{N}} \colon \mathrm{Rep}_{\mathbf{d}}(\mathcal{N}) \to \mathrm{RF}_{\mathbf{d}}.$$

Since $\pi_{\mathbf{d}}^{\mathrm{mon}}$ is a GL_d-equivariant principal bundle for the group $\prod_{t=1}^{s-1} \mathrm{GL}_{\mathbf{d}^{(t)}}$, $\pi_{\mathcal{N}}$ is so too. Therefore, taking image and preimage gives a bijection between dense GL_d-orbits in $\mathrm{RF}_{\mathbf{d}}$ and dense $\mathrm{GL}_{\mathbf{d}}$-orbits in $\mathrm{Rep}_{\mathbf{d}}(\mathcal{N})$.

Recall from the previous subsection that the category $\mathcal{N}$ is the category of Δ-filtered modules for a quasi-hereditary structure. We can use Proposition 1.4 to identify dense orbits in $\mathrm{Rep}_{\mathbf{d}}(\mathcal{N})$, thus yielding the following:

Theorem 4.30. *Consider* $(\mathrm{N}_s(Q), \Delta)$, *where* Δ *is the quasi-hereditary structure from Section 4.4. The following are equivalent.*

(i) *There is a rigid* Δ*-filtered* $\mathrm{N}_s(Q)$*-module of dimension* $\mathbf{d}$.

(ii) *There is a dense* $\mathrm{P}_{\mathbf{d}}$*-orbit in* $\mathrm{R}_d^{\mathbf{d}}$.

More precisely, if M *is a rigid* Δ*-filtered module of dimension* $\mathbf{d}$, *we can consider it as a point in* $\mathrm{Rep}_{\mathbf{d}}(\mathcal{N})$. *Then the* GL_d*-orbit of* $\pi_{\mathcal{N}}(M) \in \mathrm{RF}_{\mathbf{d}}$ *is dense, and hence it gives a Richardson orbit.*

Example 4.31. We fix a parabolic subgroup of GL_n which is stabilising a chosen partial flag $0 = U^{(0)} \subset U^{(1)} \subset \cdots \subset U^{(s)} = k^n$ of subspaces. In [38] Hille and Röhrle studied the action of this parabolic subgroup on the Lie algebra of its unipotent radical. It was already shown by Richardson in [50] that the parabolic subgroup acts with a dense orbit. They found that questions regarding the orbits of this group action can be translated into questions about Δ-filtered objects of the Auslander algebra of $k[a]/\langle a^s \rangle$ (recall that this Auslander algebra has a unique quasi-hereditary structure). This is a special case of the setting of this Chapter.

Consider the Jordan quiver Q with one vertex 1 and one arrow $a \colon 1 \to 1$. We draw that quiver along with the corresponding staircase quiver. Since any reference to the only vertex of Q in the staircase quiver is redundant, we leave it out and write $b_t := b(i_t)$.

$$1\,\circlearrowleft\, a \qquad 1 \underset{a_2}{\overset{b_1}{\rightleftarrows}} 2 \underset{a_3}{\overset{b_2}{\rightleftarrows}} \cdots \underset{a_{s-1}}{\overset{b_{s-2}}{\rightleftarrows}} s{-}1 \underset{a_s}{\overset{b_{s-1}}{\rightleftarrows}} s.$$

The relations of $\mathrm{N}_s(Q)$ are the commutativity relations $b_{t-1}a_t = a_{t+1}b_t$ for $t = 2, \ldots, s-1$, and the zero relation $a_2 b_1 = 0$. That shows the algebra $\mathrm{N}_s(Q)$ is isomorphic to the Auslander algebra of $k[a]/\langle a^s \rangle$.

There is a unique indecomposable projective-injective $N_s(Q)$-module, namely $P(s) = I(s)$. We have

$$P(t) = (1, 2, \ldots, t, \ldots, t) = I(t),$$
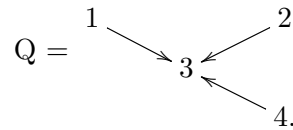$$T(s-t+1) = (0, \ldots, 0, 1, 2, \ldots, t).$$

From Section 4.4 there are exact sequences

$$0 \longrightarrow T(t) \longrightarrow I(t) \longrightarrow I(t-1) \longrightarrow 0,$$

$$0 \longrightarrow P(t-1) \longrightarrow P(s) \longrightarrow T(t) \longrightarrow 0.$$

In [12] Brüstle-Hille-Röhrle-Ringel studied the Δ-filtered objects in the module category $N_s(Q)\rangle$-mod for this particular Q. Since the Auslander algebra of $k[a]/\langle a^s \rangle$ has a unique quasi-hereditary structure it coincides with the one we constructed on $N_s(Q)$. They found there are, up to isomorphism, 2^s indecomposable rigid Δ-filtered modules. Moreover, they give an explicit method to find a rigid Δ-filtered module for any Δ-dimension vector, cf. [12, Theorem 1]. The fact that this is always possible for the Jordan quiver translates to the aforementioned classical theorem of Richardson, cf. [50].

4.5 Examples

Let $(Q, \mathbf{d})$ be a quiver and a dimension filtration such that there is a dense GL_d-orbit in the image of $\pi_{\mathbf{d}}$. Let M be a point in this orbit and assume that the kQ/J^s-module M is indecomposable, rigid and fulfils $\mathrm{End}_{kQ/J^s}(Mr) = k$. Then the operation of $\mathrm{Aut}_{kQ/J^s}(M) = k^*$ on the fibre $\pi_{\mathbf{d}}^{-1}(M)$ is trivial. This means that if there is a Richardson orbit for $(Q, \mathbf{d})$, then $\pi_{\mathbf{d}}$ is a desingularisation of $\mathrm{Im}\, \pi_{\mathbf{d}}$. This does however not have to be the case, as we see in the following example.
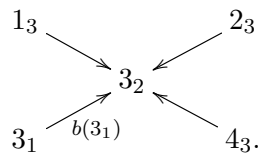
Q **of type** D_4. Let Q be the quiver of type D_4 with a sink in the middle

$$Q = \quad \begin{array}{ccc} 1 & & 2 \\ & \searrow \;\; 3 \;\; \swarrow & \\ & \uparrow & \\ & 4. & \end{array}$$

We consider $s = 3$ and the dimension filtration

$$\mathbf{d} = \left(\begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 0 & & 2 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & & 2 \\ & 1 & \end{smallmatrix} \right).$$

Since $kQ/J^3 = kQ$ is representation-finite, we can always find a dense orbit in $\mathrm{Im}\, \pi_{\mathbf{d}}$. Consider the following subquiver Q' of $Q^{(3)}$:

$$\begin{array}{ccc} 1_3 & & 2_3 \\ \searrow & & \swarrow \\ & 3_2 & \\ \nearrow & & \nwarrow \\ 3_1 \;\; {\scriptstyle b(3_1)} & & 4_3. \end{array}$$

75

For any representation M of Q' we get an $N_3(Q)$-module N by adding the identity for the arrow $b(3_2)$, and the zero vector space at the vertices i_t for $i = 1, 2, 4$, $t = 1, 2$.

This gives a faithful embedding of kQ'-mod in $N_3(Q)$-mod, and N is Δ-filtered if the map $M_{b(3_1)}$ is injective. But there does not exist a Richardson orbit for $(Q, \mathbf{d})$, because the quiver representations of type $\widetilde{D_4}$ of dimension $d' = \left(\begin{smallmatrix} 1 & & 1 \\ & 2 & \\ 1 & & 1 \end{smallmatrix}\right)$ have no dense orbit, and $M_{b(3_1)}$ is injective on an open subset of $\operatorname{Rep}_{d'}(Q')$.

The Kronecker quiver. Let Q be the 2-Kronecker quiver $1 \rightrightarrows 2$. If we take $\mathbf{d} = ((0, 1), (1, 1))$, then we claim $\operatorname{Im} \pi_{\mathbf{d}} = \operatorname{Rep}_{(1,1)}(Q)$: Since for every kQ-module $M = (\, k \underset{\mu}{\overset{\lambda}{\rightrightarrows}} k \,)$ we can find a $\mathbf{d}$-dimensional Δ-filtered $N_2(Q)$-module

$$
\mathrm{N} = \begin{matrix} k & & k \\ \uparrow & \overset{\lambda}{\underset{\mu}{\searrow\searrow}} & \uparrow {\scriptstyle 1} \\ 0 & & k \end{matrix}
$$

with $eN = M$. It is well-known that there is no dense $\operatorname{GL}_{(1,1)}$-orbit in $\operatorname{Rep}_{(1,1)}$, thus there exists no Richardson orbit for $(Q, \mathbf{d})$. If we assume λ and μ are not both zero, then the fibre $\pi_{\mathbf{d}}^{-1}(M)$ is just one point, in particular it has a dense $\operatorname{GL}_{\mathbf{d}}$-orbit, thus condition (3) in Theorem 4.2 is partially satisfied, even though $\operatorname{Im} \pi_{\mathbf{d}}$ has no dense orbit.

Oriented cycle. Let $n \geq 1$ and let Q be an oriented cycle with n vertices. For every $s \geq 1$ the algebra $N_s(Q)$ is isomorphic to the Auslander algebra of the representation-finite self-injective algebra $A = kQ/J^s$. Let

$$
e := \sum_{i \in Q_0} e(i_s)
$$

be the idempotent corresponding to the top layer. Note that we can identify A with $eN_s(Q)e$. For every summand $T(i_t)$ (cf. Section 4.4 for notation) of the characteristic tilting module, $eT(i_t)$ can be seen as the indecomposable A-module of length $s - t$ with socle $S(i)$. Thus every indecomposable A-module M arises, up to isomorphism, as $e\hat{M}$ with $\hat{M} \in \operatorname{add}(T)$. Since T is rigid, this means that for every M in A-mod, there is a rigid $N_s(Q)$-module $\hat{M}$ such that $M = e\hat{M}$. In particular, if $\mathbf{d} := \underline{\dim} \hat{M}$ and $d = \underline{\dim} M$, the map $\pi_{\mathbf{d}}$ is a desingularisation of $\overline{\mathcal{O}}_M \subset \operatorname{Rep}_d(A)$.

4.5.1 Algorithm for $Q = \mathbb{A}_2$

For the quiver $Q = \mathbb{A}_2$ $s \in \mathbb{Z}_+$ the category $\mathcal{N} \subset N_s(Q)$-mod has only finitely many indecomposable objects. This implies that for all dimension filtrations $\mathbf{d}$ there exists a Richardson orbit for $(Q, \mathbf{d})$, we give an algorithm calculating it.

We write $\mathbb{A}_2 = x \longrightarrow y$ and let $\mathbf{d} = (d_1, \ldots, d_s)$ be a dimension filtration, i.e. a dimension vector for $N_s(\mathbb{A}_2)$ with $d_t \leq d_{t+1}$ pointwise for $t = 1, \ldots, s-1$. For $N_s(\mathbb{A}_2)$-modules M, N, we write

$$[M] := \underline{\dim} M, \quad [M, N]^1 := \dim \mathrm{Ext}^1_{N_s(\mathbb{A}_2)}(M, N).$$

Remark. Due to an overwhelming need for distinct coefficients, i and j do not denote vertices of Q as usual, but will typically denote the layer of a vertex in $\mathrm{N}_s(Q)$.

We let $\Delta(x_i)$ (resp. $\Delta(y_j)$)) denote the standard module at x_i (resp. y_j). There is unique vector $\delta_{\mathbf{d}} = ((\hat{x}_i)^s_{i=1}, (\hat{y}_j)^s_{j=1}) \in \mathbb{N}^{2s}$ such that

$$\mathbf{d} = \sum_{i=1}^s \hat{x}_i [\Delta(x_i)] + \sum_{j=1}^s \hat{y}_j [\Delta(y_j)].$$

We call this the Δ-dimension vector corresponding to $\mathbf{d}$. We write $\delta_{x_i} := \delta_{[\Delta(x_i)]}$ and $\delta_{y_j} := \delta_{[\Delta(y_j)]}$. For $i > j$ we denote by $E(i, j)$ the indecomposable $N_s(Q)$-module with Δ-dimension vector $\delta_{x_i} + \delta_{y_j}$.

Proposition 4.32. *The following algorithm returns a rigid $N_s(\mathbb{A}_2)$-module.*

Let $\mathbf{d}$ be a dimension filtration and let $\delta_{\mathbf{d}} = ((\hat{x}_i)^s_{i=1}, (\hat{y}_j)^s_{j=1})$ denote the corresponding Δ-dimension vector. Let $M = 0$ be the trivial $N_s(\mathbb{A}_2)$-module. We execute the following steps:

(1) *If $\hat{x}_i = 0$ for all $i = 1, \ldots, s$ go to step (3). Otherwise let i be minimal such that $\hat{x}_i \neq 0$ and go to step (2).*

(2) *If $\hat{y}_j = 0$ for all $j < i$, replace M with $M \oplus \Delta(x_i)$ and $\delta_{\mathbf{d}}$ with $\delta_{\mathbf{d}} - \delta_{x_i}$, then go back to step (1).*

   Otherwise let j be maximal such that $j < i$ and $\hat{y}_j \neq 0$. Replace M with $M \oplus E(i, j)$ and $\delta_{\mathbf{d}}$ with $\delta_{\mathbf{d}} - \delta_{x_i} - \delta_{y_j}$. Then go back to step (1).

(3) *Return the module $M \oplus \bigoplus_{j=1}^s \Delta(y_j)^{\hat{y}_j}$.*

Proof. Denote the module returned by the algorithm by M. We write $M = M_1 \oplus M_2 \oplus M_3$, with $M_1 \in \mathrm{add} \bigoplus_{i=1}^s \Delta(x_i)$, $M_2 \in \mathrm{add} \bigoplus_{i>j} E(i, j)$ and $M_3 \in \mathrm{add} \bigoplus_{j=1}^s \Delta(y_j)$.
We observe that $T = \bigoplus_{i=1}^s \Delta(x_i) \oplus \bigoplus_{i=1}^{s-1} E(i+1, i) \oplus \Delta(y_s)$ is the characteristic tilting module of $N_s(\mathbb{A}_2)$-mod, and that $\Delta(y_j)$ is projective for $j = 1, \ldots, s$. We use these properties and apply appropriate Hom-functors to short exact sequences of the form

$$0 \to \Delta(y_j) \to E(i, j) \to \Delta(x_i) \to 0,$$

to calculate the dimensions of Ext-groups.

We know $\mathrm{Ext}^1(M, M_1) = 0$, because M is in $\mathcal{F}(\Delta)$ and $M_1 \in \mathrm{add}(T)$. We also have $\mathrm{Ext}^1(M_3, M) = 0$, because M_3 is projective. It remains to show:

$$\textbf{(a) } \mathrm{Ext}^1(M_1, M_3) = 0, \quad \textbf{(c) } \mathrm{Ext}^1(M_2, M_3) = 0,$$
$$\textbf{(b) } \mathrm{Ext}^1(M_1, M_2) = 0, \quad \textbf{(d) } \mathrm{Ext}^1(M_2, M_2) = 0.$$

Case (a): We have $[\Delta(x_i), \Delta(y_j)]^1 = 1$ if and only if $i > j$. If $\Delta(x_i) \in \text{add}(M_1)$ and $i > j$ we see that during the iteration of step (2) that yields $\Delta(x_i)$, we have $\hat{y}_j = 0$. But then $\Delta(y_j) \notin \text{add}(M_3)$, because $\hat{y}_j$ is still zero in step (3).

Case (b): From the long exact sequence obtained by applying $\text{Hom}_{\mathcal{N}}(\Delta(x_t), -)$ to the short exact sequence above, we have $[\Delta(x_t), E(i, j)]^1 = 1$ if and only if $i > t > j$. Let $\Delta(x_t) \in \text{add}(M_1)$ and let $i > t > j$. By the condition in step (2) we have $\hat{y}_j = 0$ during the step that yields $\Delta(x_t)$. If $E(i, j) \in \text{add}(M_2)$, then that summand is yielded in a later iteration of step (2), because $i > t$. But that is impossible because $\hat{y}_j$ is still zero in that step.

Case (c): From the long exact sequence that we obtain by applying the hom functor $\text{Hom}_{\mathcal{N}}(-, \Delta(y_t))$ to the short exact sequence above we have $[E(i, j), \Delta(y_t)]^1 = 1$ if and only if $i > t > j$. Let $E(i, j) \in \text{add}(M_2)$ and let $i > t > j$. In the step that yields $E(i, j)$ we must have $\hat{y}_t = 0$, otherwise j is not maximal such that $i > j$ and $\hat{y}_j \neq 0$. But then $\hat{y}_t = 0$ in step (3) also, thus $\Delta(y_t) \notin \text{add} M$.

Case (d): We set $[X, Y] := \dim \text{Hom}(X, Y)$ and $[X, Y]^1 := \dim \text{Ext}^1(X, Y)$. We apply $\text{Hom}_{\mathcal{N}}(-, E(t, l))$ to the short exact sequence above to obtain

$$[E(i, j), E(t, l)]^1 = [E(i, j), E(t, l)] - [\Delta(y_j), E(t, l)] + [\Delta(x_i), E(t, l)]^1.$$

We first note $[E(i, j), E(t, l)] = 1$ if and only if one has $i \geq t, j \geq l$. Also we have $[\Delta(y_j), E(t, l)] = 1$ if and only if $j \geq l$. From case (b) we know $[\Delta(x_i), E(t, l)]^1 = 1$ if and only if $t > i > l$. We conclude

$$[E(i, j), E(t, l)]^1 = \begin{cases} 1, & \text{if } t > i > l > j, \\ 0, & \text{else.} \end{cases}$$

Let $E(i, j), E(t, l) \in \text{add}(M_2)$ and assume $t > i > l > j$. From step (1) in the algorithm we see $E(i, j)$ is obtained before $E(t, l)$. Thus $\hat{y}_j, \hat{y}_l > 0$ at the start of the iteration of step (2) that yields $E(i, j)$. But since $i > l > j$ that is a contradiction to j being maximal such that $\hat{y}_j > 0$ and $j > i$. Thus $E(i, j)$ and $E(t, l)$ cannot both be summands of M. $\qquad\square$

Bibliography

[1] I. Assem, D. Simson, and A. Skowronski. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory.* Elements of the Representation Theory of Associative Algebras. Cambridge University Press, 2006.

[2] M. Auslander. Coherent functors. In S. Eilenberg, D. Harrison, S. MacLane, and H. Röhrl, editors, *Proceedings of the Conference on Categorical Algebra*, pages 189–231. Springer Berlin Heidelberg, 1966.

[3] M. Auslander. Representation theory of Artin algebras. I, II. *Comm. Algebra*, 1:177–268; ibid. 1 (1974), 269–310, 1974.

[4] M. Auslander and I. Reiten. Stable equivalence of dualizing R-varieties. *Adv. Math.*, 12(3):306 – 366, 1974.

[5] M. Auslander and I. Reiten. On the representation type of triangular matrix rings. *J. Lond. Math. Soc.*, s2-12(3):371–382, 1976.

[6] M. Auslander, I. Reiten, and S. O. Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[7] M. Auslander and S. O. Smalø. Preprojective modules over Artin algebras. *J. Algebra*, 66(1):61–122, 1980.

[8] M. Auslander and S. O. Smalø. Almost split sequences in subcategories. *J. Algebra*, 69(2):426–454, 1981.

[9] K. Baur. Richardson elements for classical Lie algebras. *J. Algebra*, 297(1):168–185, 2006.

[10] G. Birkhoff. Subgroups of abelian groups. *Proc. Lond. Math. Soc.*, s2-38(1):385–401, 1935.

[11] T. Brüstle and L. Hille. Finite, tame, and wild actions of parabolic subgroups in $GL(V)$ on certain unipotent subgroups. *J. Algebra*, 226(1):347–360, 2000.

[12] T. Brüstle, L. Hille, C. M. Ringel, and G. Röhrle. The Δ-filtered modules without self-extensions for the Auslander algebra of $k[T]/\langle T^n \rangle$. *Algebr. Represent. Theory*, 2(3):295–312, 1999.

[13] G. Cerulli Irelli, E. Feigin, and M. Reineke. Quiver Grassmannians and degenerate flag varieties. *Algebra Number Theory*, 6(1):165–194, 2012.

[14] X.-W. Chen. Graded self-injective algebras "are" trivial extensions. *J. Algebra*, 322(7):2601–2606, 2009.

[15] X.-W. Chen. The stable monomorphism category of a Frobenius category. *Math. Res. Lett.*, 18(1):125–137, 2011.

[16] E. Cline, B. Parshall, and L. Scott. Derived categories and Morita theory. *J. Algebra*, 104(2):397–409, 1986.

[17] E. Cline, B. Parshall, and L. Scott. Algebraic stratification in representation categories. *J. Algebra*, 117(2):504–521, 1988.

[18] E. Cline, B. Parshall, and L. Scott. Finite-dimensional algebras and highest weight categories. *J. Reine Angew. Math.*, 391:85–99, 1988.

[19] P. Cohn. *Algebra*, volume 3. J. Wiley & Sons, 1991.

[20] T. Conde. The quasihereditary structure of the Auslander-Dlab-Ringel algebra. *J. Algebra*, 460:181–202, 2016.

[21] T. Conde and K. Erdmann. The Ringel dual of the Auslander-Dlab-Ringel algebra. *ArXiv e-prints*, Aug. 2017.

[22] W. Crawley-Boevey and J. Sauter. On quiver Grassmannians and orbit closures for representation-finite algebras. *Math. Z.*, 285(1):367–395, 2017.

[23] V. Dlab and C. M. Ringel. Every semiprimary ring is the endomorphism ring of a projective module over a quasihereditary ring. *Proc. Amer. Math. Soc.*, 107(1):1–5, 1989.

[24] V. Dlab and C. M. Ringel. Filtrations of right ideals related to projectivity of left ideals. In *Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année (Paris, 1987/1988)*, volume 1404 of *Lecture Notes in Math.*, pages 95–107. Springer, Berlin, 1989.

[25] V. Dlab and C. M. Ringel. Quasi-hereditary algebras. *Illinois J. Math.*, 33(2):280–291, 1989.

[26] V. Dlab and C. M. Ringel. The module theoretical approach to quasihereditary algebras. In H. Tachikawa and S. Brenner, editors, *Representations of Algebras and Related Topics*, pages 200–224. Cambridge University Press, 1992. Cambridge Books Online.

[27] S. Donkin. On Schur algebras and related algebras. I. *J. Algebra*, 104(2):310–328, 1986.

[28] Ö. Eiríksson. From submodule categories to the stable Auslander algebra. *J. Algebra*, 486:98–118, 2017.

[29] Ö. Eiríksson and J. Sauter. Quiver-graded Richardson orbits. *ArXiv e-prints*, July 2017.

[30] E. E. Enochs. Injective and flat covers, envelopes and resolvents. *Israel J. Math.*, 39(3):189–209, 1981.

[31] V. Franjou and T. Pirashvili. Comparison of abelian categories recollements. *Doc. Math.*, 9:41–56 (electronic), 2004.

[32] P. Freyd. *Abelian categories: An introduction to the theory of functors.* Harper's series in modern mathematics. Harper & Row, 1964.

[33] P. Freyd. Representations in abelian categories. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 95–120. Springer, New York, 1966.

[34] P. Gabriel. Des catégories abéliennes. *Bull. Soc. math. France*, 90:323–448, 1962.

[35] P. Gabriel. Finite representation type is open. *Representations of Algebras, Springer Lecture Notes in Mathematics*, 488:132–155, 1975.

[36] D. Happel. *Triangulated Categories in the Representation of Finite Dimensional Algebras.* Ideas in Context. Cambridge University Press, 1988.

[37] R. Hartshorne. *Algebraic Geometry.* Encyclopaedia of mathematical sciences. Springer, 1977.

[38] L. Hille and G. Röhrle. A classification of parabolic subgroups of classical groups with a finite number of orbits on the unipotent radical. *Transform. Groups*, 4(1):35–52, 1999.

[39] B. T. Jensen, X. Su, and R. W. T. Yu. Rigid representations of a double quiver to type A, and Richardson elements in seaweed Lie algebras. *Bull. Lond. Math. Soc.*, 41(1):1–15, 2009.

[40] M. Kalck and J. Karmazyn. Ringel duality for certain strongly quasi-hereditary algebras. *ArXiv e-prints*, Nov. 2017.

[41] M. Kleiner. Approximations and almost split sequences in homologically finite subcategories. *J. Algebra*, 198(1):135 – 163, 1997.

[42] D. Kussin, H. Lenzing, and H. Meltzer. Nilpotent operators and weighted projective lines. *J. Reine Angew. Math.*, 685:33–71, 2013.

[43] D. Kussin, H. Lenzing, and H. Meltzer. Triangle singularities, ADE-chains, and weighted projective lines. *Adv. Math.*, 237:194–251, 2013.

[44] H. Lenzing. Auslander's work on Artin algebras. In *Algebras and modules, I (Trondheim, 1996)*, volume 23 of *CMS Conf. Proc.*, pages 83–105. Amer. Math. Soc., Providence, RI, 1998.

[45] Z.-W. Li and P. Zhang. A construction of Gorenstein-projective modules. *J. Algebra*, 323(6):1802 – 1812, 2010.

[46] X.-H. Luo and P. Zhang. Separated monic representations I: Gorenstein-projective modules. *J. Algebra*, 479:1 – 34, 2017.

[47] G. Lusztig. Quivers, perverse sheaves, and quantized enveloping algebras. *J. Amer. Math. Soc.*, 4(2):365–421, 1991.

[48] M. Reineke. Quivers, desingularizations and canonical bases. In A. Joseph, A. Melnikov, and R. Rentschler, editors, *Studies in Memory of Issai Schur*, volume 210 of *Progress in Mathematics*, pages 325–344. Birkhäuser Boston, 2003.

[49] M. Reineke. Every projective variety is a quiver Grassmannian. *Algebr. Represent. Theory*, 16(5):1313–1314, 2013.

[50] R. W. Richardson, Jr. Conjugacy classes in parabolic subgroups of semisimple algebraic groups. *Bull. London Math. Soc.*, 6:21–24, 1974.

[51] C. M. Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Math. Z.*, 208(2):209–223, 1991.

[52] C. M. Ringel. Iyama's finiteness theorem via strongly quasi-hereditary algebras. *J. Pure Appl. Algebra*, 214(9):1687 – 1692, 2010.

[53] C. M. Ringel and M. Schmidmeier. Submodule categories of wild representation type. *J. Pure Appl. Algebra*, 205(2):412 – 422, 2006.

[54] C. M. Ringel and M. Schmidmeier. The Auslander-Reiten translation in submodule categories. *Trans. Amer. Math. Soc.*, 360(2):691–716 (electronic), 2008.

[55] C. M. Ringel and P. Zhang. From submodule categories to preprojective algebras. *Math. Z.*, 278(1-2):55–73, 2014.

[56] C. M. Ringel and P. Zhang. Objective triangle functors. *Sci. China Math.*, 58(2):221–232, 2015.

[57] J. A. Sauter. *Springer theory and the geometry of quiver flag varieties*. PhD thesis, The University of Leeds, 2013.

[58] L. L. Scott. Simulating algebraic geometry with algebra. I. The algebraic theory of derived categories. In *The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986)*, volume 47 of *Proc. Sympos. Pure Math.*, pages 271–281. Amer. Math. Soc., Providence, RI, 1987.

[59] D. Simson. Chain categories of modules and subprojective representations of posets over uniserial algebras. *Rocky Mountain J. Math.*, 32(4):1627–1650, 12 2002.

[60] D. Simson. Representation types of the category of subprojective representations of a finite poset over and a solution of a Birkhoff type problem. *J. Algebra*, 311(1):1 – 30, 2007.

[61] D. Simson. Tame–wild dichotomy of Birkhoff type problems for nilpotent linear operators. *J. Algebra*, 424:254 – 293, 2015.

[62] A. Skowroński and K. Yamagata. *Frobenius algebras. I.* EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2011. Basic representation theory.

[63] P. Slodowy. The quotient of the adjoint action. In *Simple Singularities and Simple Algebraic Groups*, volume 815 of *Lecture Notes in Mathematics*, pages 17–42. Springer Berlin Heidelberg, 1980.

[64] R. Tan. Auslander algebras of self-injective Nakayama algebras. *Pure Math. Sci.*, 2(1-4):89–108, 2013.

[65] S. Wolf. *The Hall Algebra and the Composition Monoid.* PhD thesis, Universität Paderborn, 2009.

[66] B.-L. Xiong, P. Zhang, and Y.-H. Zhang. Auslander–Reiten translations in monomorphism categories. *Forum Math.*, 26(3):863–912, 2014.

[67] P. Zhang. Monomorphism categories, cotilting theory, and Gorenstein-projective modules. *J. Algebra*, 339:181–202, 2011.

Index