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Contents

1. Introduction

This thesis concerns with the theoretical and numerical study of the distributed optimal control of a flow of two incompressible, immiscible fluids.
From the mathematical point of view, the studied problems can be formulated as an *abstract optimal control problem* with the following structure

Problem 1.1. *Find* $\bar{y} \in Y, \bar{u} \in U$ *such that*

$$J(\bar{y}, \bar{u}) = \left\{ \min_{(y,u) \in Y \times U} J(y,u) \quad \text{subject to} \quad e(y,u) = 0,\ c(y) \in \mathcal{K},\ u \in U_{ad} \right\},$$

where $J : Y \times U \to \mathbb{R}$ is the *objective function*, $e : Y \times U \to Z$, $c : Y \to R$ are operators, Y, U, Z, R are real Banach spaces, $\mathcal{K} \subset R$ is a closed convex cone and $U_{ad} \subset U$ is a closed convex set. Furthermore, $e(w) = 0$ stands for a general equality constraint and the condition $c(y) \in \mathcal{K}$ represents an abstract inequality constraint. In these settings, the variables $u \in U, y \in Y$ represent, respectively, the *control* and the *state* of the system. For a general introduction about optimization problems, we refer the reader to [68].
In the problems considered in the present thesis, W, Z and R are function spaces and the *state equation* $e(y,u) = 0$ represents a system of *Partial Differential Equations* (PDEs).
Optimal control problems where the solution is constrained by partial differential equations, are very interesting from mathematical point of view and have important and practical applications in many disciplines such as physics, engineering, mechanics, chemistry, medicine, finance and industry in general. For a general overview about PDEs-constrained optimal control problems, we refer the reader to [58]. For examples and applications, we refer to [65], where are collected several papers which describe the efficiency of the optimal control strategies to deal with radio frequency ablation, electro-mechanical smart structures, freezing of living cells, nanoscale particles production, radiative heat transfer, shape of artificial blood pumps.
The standard approach to solve problems like Problem 1.1 above, is to use the tools of *Mathematical Programming in Banach spaces*, see [58], [70], [87]. If the mappings J, e, c are continuously Fréchet differentiable and the constraints $e(y,u) = 0$, $c(y) \in \mathcal{K}$ satisfy a regularity condition called *constraint qualification* at the solution $(\bar{y}, \bar{u})$, then the following *first order optimality conditions* or *Karush-Kuhn-Tucker* (KKT) *conditions* hold true at $(\bar{y}, \bar{u})$:

There exists Lagrange multipliers $\bar{p} \in Z^*, \bar{\lambda} \in R^*$ *such that*

$$e(\bar{y}, \bar{u}) = 0, \tag{1.1}$$

$$c(\bar{y}) \in \mathcal{K}, \tag{1.2}$$

$$\bar{u} \in U_{ad}, \tag{1.3}$$

$$\bar{\lambda} \in \mathcal{K}^o, \quad \langle \bar{\lambda}, c(\bar{y}) \rangle_{R^*,R} = 0, \tag{1.4}$$

$$L_y(\bar{y}, \bar{u}, \bar{p}) + c'(\bar{y})^* \bar{\lambda} = 0, \tag{1.5}$$

$$\langle L_u(\bar{y}, \bar{u}, \bar{p}), u - \bar{u} \rangle_{U^*,U} \geq 0, \qquad \forall\, u \in U_{ad}, \tag{1.6}$$

where the Lagrangian function $L : Y \times U \times Z^* \to \mathbb{R}$ is defined as

$$L(y, u, p) := J(y, u) + \langle p, e(y, u) \rangle_{Z^*,Z}, \tag{1.7}$$

L_y, L_u are its partial Fréchet derivative and

$$\mathcal{K}^o = \{ \lambda \in R^* : \ \langle \lambda, r \rangle_{R^*,R} \leq 0, \ \forall\, r \in \mathcal{K} \}. \tag{1.8}$$

In the present work the set of PDEs which represents the constraints

$$e(y, u) = 0, \ c(y) \in \mathcal{K},$$

of the optimal control problems under investigation is the Cahn-Hilliard-Navier-Stokes system which models the flow of two immiscible, incompressible fluids.
The incompressible Navier-Stokes and Stokes equations represent the central models in fluid mechanics. They can be derived considering a Newtonian fluid with constant viscosity coefficients and assuming mass conservation, proper evolution of linear momentum and total energy and divergence-free velocity field (see [20], [67], [81] and the references therein for further details). We refer to [20], [58] (Section 1.8), [80] for analytical results and to [37], [73], [80] for numerical approaches.
The Cahn-Hilliard equations [21], [22], [23], is a model which was originally derived to describe phase transition in binary alloys. In this first approach, the model consider a fluid where there is coexistence of two species A and B. If the temperature of the system is greater than a critical temperature T_c, the fluid manifests a state where the two species, called *phases*, are uniformly mixed. When one perform a deep quenching (rapid reduction of the temperature), the system performs a *spinodal decomposition*, i.e. it moves towards a state where the two species are spatially separated and the *interface*, the surface which separates the two phases, has a minimum area. In order to describe this behaviour, considering a fluid in a spatial domain Ω and denoting by x and t the space and time coordinates, the Cahn-Hilliard model use a function $y(x, t)$. This variable is called *phase-field* or *order parameter* and it has the following structure

$$y(x, t) = \frac{c_A(x, t) - c_B(x, t)}{c_A(x, t) + c_B(x, t)}, \tag{1.9}$$

where c_A and c_B are the concentrations of the two species. Then, if $T > T_c$, the order parameter is constant, uniform and such that $-1 < y < 1$; conversely, when $T < T_c$, $y(x, t)$ converges to a state where it assumes its extremal values $-1, 1$ in the major part of the domain

$$y(x, t) = 1 \qquad \Rightarrow \qquad c_B(x, t) = 0 \qquad \text{pure phase A},$$

$$y(x,t) = -1 \qquad \Rightarrow \qquad c_A(x,t) = 0 \qquad \text{pure phase B},$$

with a thin interface where $-1 < y(x,t) < 1$ and the two species are mixed. Subsequent to its original formulation, Cahn-Hilliard model was used to deal with other physical systems showing analogous phase separation behaviour, including, for example, problems in image processing [14], [24] and in fluid mechanics [5]. Furthermore, Cahn-Hilliard model have provided an efficient option, from mathematical point of view, to deal with *interfaces dynamics* (see [77] for a review and also [8], [11]). The structure of the Cahn-Hilliard system is the following

$$y_t - \gamma \Delta w = 0, \tag{1.10a}$$

$$y(0) = y_0, \tag{1.10b}$$

$$w + \varepsilon^2 \Delta y \in \partial \Phi(y), \tag{1.10c}$$

$$\frac{\partial y}{\partial \mathbf{n}}\Big|_{\Omega} = \frac{\partial w}{\partial \mathbf{n}}\Big|_{\Omega} = 0. \tag{1.10d}$$

It is a fourth order system of parabolic type with Neumann boundary conditions. The function w is the *chemical potential* and $\frac{1}{\gamma} = Pe > 0$ is the Péclet's number which is related to the mobility of the fluid. ε is a constant parameter which is tipically small $0 < \varepsilon << 1$: its value is connected with the thickness of the interface which is of order $O(\varepsilon)$. The function $\Phi = \Phi(y)$ is the *homogeneous free energy density* and $\partial\Phi$ stands for its generalized derivative [28] (see also Section 2.4.4 in [58]). This generalized derivative is single-valued if Φ is differentiable at y. For this reason, in general, equation (1.10c) is a variational inclusion. The Cahn-Hilliard system (1.10) above, comes from the minimization [17] of a *Ginzburg-Landau* type energy functional $E_\varepsilon(y)$, which is such that

$$E_\varepsilon(y) = \frac{\varepsilon^2}{2} \int_\Omega |\nabla y|^2 \, dx + \int_\Omega \Phi(y) \, dx.$$

Then, the analytical form of the homogeneous free energy density $\Phi(y)$ is crucial in order to establish the proper behaviour of the system. Basically, the choice of Φ depends on the context of application of the model but, in general, the free energy density Φ is such that it penalizes the deviation from the physically meaningful values $[-1, 1]$. In literature, several types of Φ has been considered. A widely studied version for the homogeneous free energy density is the double-well potential

$$\Phi(y) = \frac{1}{4}\left(1 - y^2\right)^2, \tag{1.11}$$

for example in [32], [34], [71]. Also the case where Φ is an arbitrary polynomial is analysed in [69], [76], [79], [82]. A logarithmic form of the homogeneous free energy density is studied in the original paper of Cahn and Hilliard [22] and in [4]. We emphasize that the logarithmic potential bounds the phase-field in the interval $(-1, 1)$, while the double-well does not. However, they are both differentiable and, in these cases, equation (1.10c) is an equality. In order to deal with the case of a deep quench of a binary alloy, in [72] it is proposed the following form of the homogeneous free energy density

$$\Phi(y) := \begin{cases} \frac{1}{2}\left(1 - y^2\right), & \text{if} \quad y \in [-1, 1], \\ +\infty, & \text{otherwise}, \end{cases} \tag{1.12}$$

that is the so-called *double-obstacle potential.* This form of Φ allows a better description of the underlying physical phenomena, because it bounds the order parameter in the meaningful interval $[-1,1]$. Concerning the Cahn-Hilliard equations with Φ equal to the double-obstacle potential, we refer the reader to [17], [54], [61] for analytical results and to [7], [8], [9], [10], [15], [18], [38], [39], [49] for numerical and discrete approaches.

The flow of two immiscible, incompressible fluids can be described by coupling the Cahn-Hilliard system with the Navier-Stokes system

$$\mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} + \nabla p + \rho\, y\cdot\nabla w = \mathbf{u}, \tag{1.13a}$$

$$\mathbf{v}|_\Omega = 0, \tag{1.13b}$$

$$\mathbf{v}(0) = \mathbf{v}_0, \tag{1.13c}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{1.13d}$$

$$y_t - \gamma\Delta w + \mathbf{v}\cdot\nabla y = 0, \tag{1.14a}$$

$$y(0) = y_0, \tag{1.14b}$$

$$w + \varepsilon^2\Delta y \in \partial\Phi(y), \tag{1.14c}$$

$$\frac{\partial y}{\partial\mathbf{n}}\Big|_\Omega = \frac{\partial w}{\partial\mathbf{n}}\Big|_\Omega = 0. \tag{1.14d}$$

In the Navier-Stokes system (1.13), p represents the pressure, $\mathbf{u}$ is an external volume force and $Re = \frac{1}{\nu}$ is the Reynold's number. The mean velocity field $\mathbf{v}$, is defined [62] to be

$$\mathbf{v} = \frac{1+y}{2}\mathbf{v}_A + \frac{1-y}{2}\mathbf{v}_B,$$

where $\mathbf{v}_i, i = A, B$, is the velocity field of the fluid component i. The constant parameter ρ is the *capillarity number.* Equations (1.13), (1.14) represent a model which is related to the so-called model 'H' in the nomenclature of Hohenberg and Halperin [42], [45], [59], [64]. Concerning the analysis of this model, we refer the reader to [1], [2], [3], [19], [25], [33], [35], [41], [43], [44], [48], [60], [62], [63] and the references therein. In particular, among the references above, [35] and [62] contain a comprehensive of analytic and numerical results, for the double-well potential in the Cahn-Hilliard part. In [48], the authors consider the Cahn–Hilliard-Navier–Stokes system with a double-obstacle homogeneous free energy density. Then, they perform a Moreau–Yosida regularization of the double-obstacle potential and find a solution of the regularized system. In this way, the phase-field is not confined to the physical interval $[-1,1]$, but may overshoot the values ± 1 by a small amount which depends on a regularization parameter.

In this thesis we study the following type of optimal control problem

Problem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain and $y_d : \Omega_T := \Omega\times(0,T) \to \mathbb{R}$ be given. Let $\alpha > 0$ and $T > 0$ be fixed. Find a control $\mathbf{u} : \Omega_T \to \mathbb{R}^2$ and a state $y : \Omega_T \to \mathbb{R}$ such that*

$$J(y,\mathbf{u}) = \int_0^T \left[\frac{1}{2}\int_\Omega (y-y_d)^2\,dx + \frac{\alpha}{2}\int_\Omega |\mathbf{u}|^2\,dx\right] dt,$$

is minimized subject to the Cahn-Hilliard-Navier-Stokes system (1.13)*,* (1.14)*.*

Problem 1.2 can be classified as a *distributed optimal control problem*. Indeed, the control **u**, i.e. the external volume force in the Navier-Stokes equations (1.13), is distributed on the whole domain. The control acts on the system with the purpose of driving the state y, that is the phase-field in Cahn-Hilliard equation (1.14), as close as possible to a *desired state* y_d. The structure of the objective function J is standard: the first term in J measure the distance between the state y and the desired state y_d; the second is a regularization term which guarantees well-posedness of the problem. The constant parameter α is usually small ($\alpha \in [10^{-5}, 10^{-3}]$).
In literature, optimal control problems involving multiphase fluids flow, are studied in relatively few papers. In several works, the authors consider just the optimal control of the Cahn-Hilliard system without any coupling with the Navies-Stokes equations: in [29], [30] a boundary control problem with Φ equal to the double-obstacle potential (1.12) is studied; in [84], [86] a distributed optimal control problem, where the free energy density correspond, respectively, to a general polynomial and to the double-well potential (1.11) is analysed; in [85], the control of a viscous Cahn-Hilliard system is considered; in [54] a distributed optimal control problem with Φ equal to the double-obstacle potential is assessed; in [31], the authors study a problem involving non-local interactions.
Concerning the contributions to the analysis of the optimal control of the complete Cahn-Hilliard-Navier-Stokes system, in [50], [55], a mathematical analysis of a semi-discrete (in time) problem is performed. In [57], the authors study a fully discretized version of the model, where the free energy density corresponds to the double-obstacle potential: they perform a Moreau-Yosida regularization of the resulting state equations and then they obtain the solution of the problem applying the *instantaneous control* [26], [56] strategy. In [78], a distributed optimal control problem is considered, taking into account the effect of a disturbance which destabilizes the control effects. In [36], the case of non-local interactions is considered.

In the mathematical analysis of Problem 1.2, the main issue is the structure of the homogeneous free energy density Φ in the Cahn-Hilliard equations. From physical point of view, the most meaningful analytical form for the function Φ corresponds to the double-obstacle potential. Unfortunately, that makes the problem very challenging. Indeed, due to the non-smooth nature of the double-obstacle potential, in this case equation (1.14c) in the Cahn-Hilliard system is a variational inequality. Optimal control problems with variational inequalities are related to the mathematical programs with equilibrium constraints (MPECs), which *do not satisfy any kind of constraints qualifications* [51], [52], [54]. Then, in Problem 1.2, if Φ is the double-obstacle potential, it is not possible to apply the standard tools of mathematical programming in Banach spaces.
Below we give an overview of the structure of the thesis and briefly explain how to overcome the difficulties that arise in the optimal control of considered problems.

1.1. Structure of the Thesis

The thesis is organized in two main parts. In the first part, which includes Chapters 2 and 3, we consider the distributed optimal control problem of the *non-smooth* Cahn-Hilliard-Stokes system. We assume that the homogeneous free energy density in the Cahn-Hilliard equations, corresponds to the double-obstacle potential (1.12). The analysis is performed at continuous level in Chapter 2 and by a finite dimensional approach in Chapter 3. In the second part, which encompasses Chapters 4 and 5, we study the distributed optimal control problem of the *smooth* Cahn-Hilliard-Navier-Stokes system. In this case the homogeneous free energy density is equal to the double-well potential (1.11). We assess this problem considering infinite dimensional settings in Chapter 4 and a discrete approach in Chapter 5.

In **Chapter 2**, we perform a mathematical analysis of Problem 1.2 above, replacing the Navier-Stokes equations (1.13) with the Stokes equations

$$\mathbf{v}_t - \nu\Delta\mathbf{v} + \nabla p = \mathbf{u}, \tag{1.15a}$$

$$\mathbf{v}|_\Omega = 0, \tag{1.15b}$$

$$\mathbf{v}(0) = \mathbf{v}_0, \tag{1.15c}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{1.15d}$$

Hence, we do not consider the effects of the inertia term and furthermore, we set the capillarity number $\rho = 0$, neglecting the surface tension in equation (1.13a). As a consequence, the two state equations of the problem are decoupled. The first assumption does not compromise the results we establish in the thesis, i.e., they remain valid for the Navier-Stokes system. The second assumption is crucial: considering $\rho \neq 0$ cause severe difficulties concerning the derivation of the optimality conditions of the optimal control problem. For this reason, the case $\rho \neq 0$ with Φ equal to the double-obstacle potential remains an open problem.
The optimal control problem under investigation is challenging. Indeed, as we explained in the previous section, it has a lack of constraints qualification. Then, it is not possible to solve it applying directly the tools of mathematical programming in Banach space. In order to overcome this difficult, we adapt the idea from [54]: we regularize the problem, so that it is possible to apply the tools of mathematical programming in Banach spaces; we derive the optimality conditions of the regularized problem; we obtain the optimality conditions of the original problem as a limit with respect to the regularization parameter of the optimality conditions of the regularized problem. This last result is an original contribution of this thesis.

In **Chapter 3**, at discrete level, we study the optimal control problem following the same procedure applied in Chapter 2. In this way, we derive three new results: a set of optimality conditions of the problem; the convergence of the discrete optimality conditions to the continuous optimality conditions, as the discretization parameters go to zero; an efficient algorithm for the solution of the discrete optimality conditions. Finally, in order to show the effectiveness of our approach, we perform some computations.

In **Chapter 4**, we assess the Cahn-Hilliard-Navier-Stokes optimal control Problem 1.2, where we assume the homogeneous free energy density Φ equal to the double-well potential (1.11).
Compared to problem analysed in Chapter 2, there are three main differences. First, we consider in the Cahn-Hilliard equation a *smooth* free energy density. This assumption simplifies the study of the problem, because, in this way, the constraints qualification is satisfied and it is possible to apply the tools of mathematical programming in Banach spaces. Secondly, we do not neglect the effect of the inertia term in the Navier-Stokes equations. Finally, we take into account the surface tension effects ($\rho \neq 0$). Consequently, the Navier-Stokes equations and the Cahn-Hilliard equations (as well as the corresponding system of optimality conditions) contain rather complicated nonlinear terms which complicate the analysis of the problem.
In this chapter, we get original contributions of the thesis: the first order optimality conditions of the problem and regularity properties for the adjoint variables.

In **Chapter 5** we propose and analyse a fully discrete approximation of the Cahn-Hilliard-Navier-Stokes optimal control problem. We establish new results: the discrete first order optimality conditions, the convergence of the discrete optimality conditions to the continuous optimality conditions, as the discretization parameters go to zero. Finally, we construct a practical algorithm for the solution of the discrete optimality conditions and perform some numerical experiments.

In **Appendix A**, we present the notation and the basic results used in the thesis.
In **Appendix B**, we show some of the longer proofs of the results established in the thesis.

2. Optimal Control of the Non-Smooth Cahn-Hilliard-Stokes System

2.1. Introduction

In this chapter, we study the optimal control problem which concerns the flow of a mixture of two incompressible, immiscible fluids. The evolution of the system is described by the Stokes equations (1.15) and the Cahn-Hilliard equations (1.14), where the free energy density corresponds to the double-obstacle potential (1.12). In order to state the problem under investigation properly, we make some preliminary assumptions. We denote by: $\Omega \in \mathbb{R}^2$ an open, bounded, convex polygonal domain; $T > 0$ a fixed time horizon; $\Omega_T = \Omega \times (0, T)$; $\alpha > 0$ a positive small constant. The setting and the notation used throughout this Chapter is presented in Appendix A.2.1, A.2.2. In particular, we consider L^2_0, the space of the L^2-functions with zero mean, $H_0 = L^2_0 \cap H^1$ and the associated Bochner's space

$$W_0 = \left\{ y \in L^2\left(H_0\right) : y_t \in L^2\left(H_0^*\right) \right\}.$$

In addition, we assume that $\boldsymbol{\mathcal{D}}$ is the space of the vector-valued, divergence-free, $\mathbf{H}^1_0$-functions and we consider the associated Bochner's space

$$\mathbf{W}_0 = \left\{ \mathbf{v} \in L^2(\boldsymbol{\mathcal{D}}) : \mathbf{v}_t \in L^2(\boldsymbol{\mathcal{D}}^*) \right\}.$$

We define the following space,

$$\mathbf{X} = \mathbf{W}_0 \times W_0 \times L^2\left(H^1\right), \tag{2.1}$$

with element

$$\mathbf{x} = (\mathbf{v}, y, w).$$

The spaces $\mathbf{X}$ and $\mathbf{X} \times L^2\left(\mathbf{L}^2\right)$ are endowed with the following norms,

$$\|\mathbf{x}\|_{\mathbf{X}} = \left[\|\mathbf{v}\|^2_{\mathbf{W}_0} + \|y\|^2_{W_0} + \|w\|^2_{L^2(H^1)} \right]^{\frac{1}{2}},$$

$$\|\left(\mathbf{x}, \mathbf{u}\right)\|_{\mathbf{X} \times L^2(\mathbf{L}^2)} = \left[\|\mathbf{x}\|^2_{\mathbf{X}} + \|\mathbf{u}\|^2_{L^2(\mathbf{L}^2)} \right]^{\frac{1}{2}}.$$

Moreover, we define the following set

$$\mathcal{K} = \left\{ \theta \in L^2\left(H^1\right) : -1 \leq \theta \leq 1, \text{ a.e. on } \Omega_T \right\}. \tag{2.2}$$

We consider the following objective function

$$J : \mathbf{X} \times L^2\left(\mathbf{L}^2\right) \to \mathbb{R}, \tag{2.3}$$

such that

$$J(\mathbf{x}, \mathbf{u}) := \int_0^T \left[\frac{1}{2} \int_\Omega (y - y_d)^2 \, dx + \frac{\alpha}{2} \int_\Omega \mathbf{u}^2 \, dx \right] dt, \tag{2.4}$$

where we assume $y_d \in \mathcal{C}\left([0,T]\,; L_0^2\right)$. Then, we consider the following optimal control problem:

Problem 2.1. *Given* $\mathbf{v}_0 \in \boldsymbol{\mathcal{D}} \cap \mathbf{H}^2$, $y_0 \in L_0^2 \cap H^2 \cap \mathcal{K}$, *find* $(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathbf{X} \times L^2\left(\mathbf{L}^2\right)$, *such that*

$$\min_{(\mathbf{x},\mathbf{u}) \in \mathbf{X} \times L^2(\mathbf{L}^2)} J\left(\mathbf{x}, \mathbf{u}\right) = J\left(\bar{\mathbf{x}}, \bar{\mathbf{u}}\right),$$

subject to

$$\int_0^T \left[(\mathbf{v}_t, \boldsymbol{\psi}) + \nu \left(\nabla \mathbf{v}, \nabla \boldsymbol{\psi}\right) - (\mathbf{u}, \boldsymbol{\psi}) \right] \, dt = 0, \tag{2.5a}$$

$$\mathbf{v}(0) = \mathbf{v}_0, \quad \text{in } \Omega, \tag{2.5b}$$

$$\int_0^T \left[\langle y_t, \eta \rangle_{H^{1*}, H^1} + \gamma \left(\nabla w, \nabla \eta\right) - (y, \mathbf{v} \cdot \nabla \eta) \right] \, dt = 0, \tag{2.6a}$$

$$y(0) = y_0, \quad \text{in } \Omega, \tag{2.6b}$$

$$\int_0^T \left[-\left(w, \theta - y\right) + \varepsilon^2 \left(\nabla y, \nabla \theta - \nabla y\right) - \left(y, \theta - y\right) \right] \, dt \geq 0, \tag{2.6c}$$

$$y \in \mathcal{K}, \tag{2.6d}$$

for all $\boldsymbol{\psi} \in L^2\left(\boldsymbol{\mathcal{D}}\right), \eta \in L^2\left(H^1\right), \theta \in \mathcal{K}$.

In Problem 2.1 above, (2.5) are the weak form of the non-stationary Stokes equations for incompressible fluid (1.15) and (2.6) are the weak form of the Cahn-Hilliard system (1.14), where (1.14c) is reformulated as a variational inequality. Indeed, if Φ corresponds to the double-obstacle potential (1.12), we can write

$$\Phi(y) = \frac{1}{2}\left(1 - y^2\right) + I_{[-1,1]},$$

where $I_{[-1,1]}$ is the indicator function of the interval $[-1,1]$. Then, if $\partial\Phi\left(y\right)$ is the generalized derivative of Φ calculated in y, we have

$$\partial I_{[-1,1]}(y) = \left\{ v : v(\theta - y) \leq 0, \forall \theta : -1 \leq \theta \leq 1 \right\},$$

for all y such that $-1 \leq y \leq 1$. Hence, (1.14) can be reformulated in the following equivalent form

$$y_t - \gamma \Delta w + \mathbf{v} \cdot \nabla y = 0, \quad \text{in } \Omega_T, \tag{2.7a}$$

(2.7b) $$y(0) = y_0, \qquad \text{in } \Omega,$$

(2.7c) $$-\left(w + \varepsilon^2 \Delta y + y\right)(\theta - y) \geq 0, \quad \forall \theta : -1 \leq \theta \leq 1, \qquad \text{in } \Omega_T,$$

(2.7d) $$-1 \leq y \leq 1, \qquad \text{in } \Omega_T,$$

(2.7e) $$\frac{\partial y}{\partial \mathbf{n}}\Big|_{\Omega} = \frac{\partial w}{\partial \mathbf{n}}\Big|_{\Omega} = 0, \qquad \text{in } \partial\Omega \times [0, T],$$

Therefore, (2.6) is just a weak formulation of (1.14).
We stress a property of the Cahn-Hilliard system (2.7). Assuming $y, w, \mathbf{v}$ smooth enough and integrating in Ω in (2.7a), we get

$$\int_{\Omega} y_t \; dx = \gamma \int_{\Omega} \Delta w \; dx - \int_{\Omega} \mathbf{v} \cdot \nabla y \; dx.$$

Thus, using the boundary conditions (2.7e) and the divergence theorem, we derive

$$\frac{d}{dt} \int_{\Omega} y \; dx = -\gamma \int_{\partial\Omega} \frac{\partial w}{\partial \mathbf{n}} \; d\sigma + \int_{\partial\Omega} y \; \mathbf{v} \cdot \mathbf{n} \; d\sigma + \int_{\Omega} y \; \nabla \cdot \mathbf{v} \; dx = 0.$$

Therefore, the Cahn-Hilliard system (2.7) is *mass preserving*

$$\int_{\Omega} y\,(x,t) \;\; dx = \int_{\Omega} y_0\,(x) \;\; dx = m.$$

Hence, if given $\mathbf{v}, y_0$, the solution of (1.14) are y, w, then denoting with $\hat{y}_0 = y_0 - \frac{m}{|\Omega|}$, the functions $\hat{y} = y - \frac{m}{|\Omega|}, w$ are solution of

(2.8a) $$\hat{y}_t - \gamma \Delta w + \mathbf{v} \cdot \nabla \hat{y} = 0, \qquad \text{in } \Omega_T,$$

(2.8b) $$\hat{y}(0) = \hat{y}_0, \qquad \text{in } \Omega,$$

(2.8c) $$w + \varepsilon^2 \Delta \hat{y} \in \partial \hat{\Phi}\,(\hat{y})\,, \qquad \text{in } \Omega_T,$$

(2.8d) $$\frac{\partial \hat{y}}{\partial \mathbf{n}}\Big|_{\Omega} = \frac{\partial w}{\partial \mathbf{n}}\Big|_{\Omega} = 0, \qquad \text{in } \partial\Omega \times [0, T],$$

where

$$\hat{\Phi}\,(\hat{y}) = \Phi\left(\hat{y} + \frac{m}{|\Omega|}\right).$$

Thus, the difference between the systems (1.14) and (2.8) is just a translation in the free energy density Φ. So, in order to simplify the analysis of the problem, without loss of generality, we assumed the following *zero mass condition* on the initial data

(2.9) $$\int_{\Omega} y_0\,(x) \;\; dx = 0.$$

The optimal control Problem 2.1 is very challenging. Indeed, it does not fulfils any kind of *constraint qualification* and this fact prevents the application of the standard theory of mathematical programming in Banach spaces [51], [52], [54]. It means that it is not possible to derive, *directly*, a set of first order optimality condition to solve the problem. Therefore, to deal with Problem 2.1, we regularize the double-obstacle potential in the constraint (2.6), by introducing a regularization parameter δ. In this way we define a regularized version of Problem 2.1 which satisfy the constraint qualification. Then, we derive the first order optimality conditions of Problem 2.1 as a limit of the first order optimality conditions of the regularized problem, for the regularization parameter $\delta \to 0^+$.

2.2. Regularized Optimal Control Problem

This section is devoted to the analysis of the regularized version of the non-smooth optimal control Problem 2.1: we show that this problem is well-posed and then we derive the first order optimality conditions.
The regularization of Problem 2.1 is defined as follows. We consider a parameter $\delta \in \left(0, \frac{1}{4}\right)$ and a function $\Phi_\delta(r) \in \mathcal{C}^2(\mathbb{R})$ such that

$$\Phi_\delta(r) := \frac{1}{2}\left(1 - y^2\right) + f_\delta(y), \tag{2.10}$$

where

$$f_\delta(r) := \begin{cases} \dfrac{1}{2\delta}\left[r + \left(1 + \dfrac{\delta}{2}\right)\right]^2 + \dfrac{\delta}{24} & \text{if } r \leq -1-\delta, \\ -\dfrac{1}{6\delta^2}\left(r+1\right)^3 & \text{if } -1-\delta < r < -1, \\ 0 & \text{if } -1 \leq r \leq 1, \\ \dfrac{1}{6\delta^2}\left(r-1\right)^3 & \text{if } 1 < r < 1+\delta, \\ \dfrac{1}{2\delta}\left[r - \left(1 + \dfrac{\delta}{2}\right)\right]^2 + \dfrac{\delta}{24} & \text{if } r \geq 1+\delta. \end{cases} \tag{2.11}$$

Direct calculation shows

$$f'_\delta(r) := \frac{1}{\delta}\beta_\delta\left(r\right) := \begin{cases} \dfrac{1}{\delta}\left[r + \left(1 + \dfrac{\delta}{2}\right)\right] & \text{if } r \leq -1-\delta, \\ -\dfrac{1}{2\delta^2}\left(r+1\right)^2 & \text{if } -1-\delta < r < -1, \\ 0 & \text{if } -1 \leq r \leq 1, \\ \dfrac{1}{2\delta^2}\left(r-1\right)^2 & \text{if } 1 < r < 1+\delta, \\ \dfrac{1}{\delta}\left[r - \left(1 + \dfrac{\delta}{2}\right)\right] & \text{if } r \geq 1+\delta, \end{cases} \tag{2.12}$$

and

$$f''_\delta(r) := \frac{1}{\delta}\beta'_\delta\left(r\right) := \begin{cases} \dfrac{1}{\delta} & \text{if } r \leq -1-\delta, \\ -\dfrac{1}{\delta^2}\left(r+1\right) & \text{if } -1-\delta < r < -1, \\ 0 & \text{if } -1 \leq r \leq 1, \\ \dfrac{1}{\delta^2}\left(r-1\right) & \text{if } 1 < r < 1+\delta, \\ \dfrac{1}{\delta} & \text{if } r \geq 1+\delta. \end{cases} \tag{2.13}$$

The function Φ_δ defined in (2.10) is, for any fixed $\delta \in \left(0, \frac{1}{4}\right)$ a regularization of the double-obstacle potential (1.12) (see [17] for a picture of it). It is such that

$$\Phi_\delta(r) \to \Phi(r) \quad \text{as } \delta \to 0^+, \quad \forall r \in \mathbb{R},$$

$$\Phi_\delta(r) \to +\infty \quad \text{as } r \to \pm\infty, \quad \forall \delta \in \left(0, \frac{1}{4}\right)$$

and furthermore there exists a positive constant C_0, such that

$$\Phi_\delta(r) \geq -C_0\ \delta, \quad \forall \delta \in \left[0, \frac{1}{4}\right). \tag{2.14}$$

By the definition of f_δ it follows that

$$f_\delta(r) \geq \frac{1}{2\delta}\beta_\delta(r)^2, \tag{2.15}$$

and from its convexity

$$f_\delta(r) \geq f_\delta(s) + \frac{1}{\delta}\beta_\delta(s)(r-s), \tag{2.16}$$

for all $r, s \in \mathbb{R}$. Moreover β_δ is a Lipschitz continuous function

$$0 \leq \beta_\delta' \leq 1, \tag{2.17}$$

such that

$$|\beta_\delta'(r) - \beta_\delta'(s)| \leq \frac{1}{\delta}|r-s|. \tag{2.18}$$

for all $\delta \in \left(0, \frac{1}{4}\right)$ and $r, s \in \mathbb{R}$.
In order to represent the regularized version of the non-smooth optimal control Problem 2.1 in a more compact, general form, we consider the following map

$$e_\delta : \mathbf{X} \times L^2\left(\mathbf{L}^2\right) \to \mathbf{Z} = \left[\ L^2\left(\boldsymbol{\mathcal{D}}\right) \times L^2\left(H_0\right) \times L^2\left(H^1\right) \times \boldsymbol{\mathcal{S}} \times L_0^2\ \right]^*, \tag{2.19}$$

where the space $\boldsymbol{\mathcal{S}}$ is defined in (A.3). The map e_δ is such that, for all $\mathbf{p} = (\boldsymbol{\psi}, \eta, \theta, \boldsymbol{\xi}, \varphi) \in \mathbf{Z}^*$,

$$\begin{aligned} \langle \mathbf{p}, e_\delta(\mathbf{x}, \mathbf{u})\rangle_{Z^*,Z} &= \langle a(\mathbf{v}, \mathbf{u}), \boldsymbol{\psi}\rangle_{\mathbf{L}^2(\boldsymbol{\mathcal{D}}^*),\mathbf{L}^2(\boldsymbol{\mathcal{D}})} + \langle b(\mathbf{v}, y, w), \eta\rangle_{L^2\left(H_0^*\right),L^2(H_0)} \\ &\quad + \langle c_\delta(y, w), \eta\rangle_{L^2(H^{1*}),L^2(H^1)} + (\boldsymbol{\xi}, \mathbf{v}(0) - \mathbf{v}_0) \\ &\quad + (\varphi, y(0) - y_0), \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} \langle a(\mathbf{v}, \mathbf{u}), \boldsymbol{\psi}\rangle_{\mathbf{L}^2(\boldsymbol{\mathcal{D}}^*),\mathbf{L}^2(\boldsymbol{\mathcal{D}})} &= \int_0^T \left[(\mathbf{v}_t, \boldsymbol{\psi}) + \nu(\nabla\mathbf{v}, \nabla\boldsymbol{\psi}) - (\mathbf{u}, \boldsymbol{\psi})\right]\ dt, \\ \langle b(\mathbf{v}, y, w), \eta\rangle_{L^2\left(H_0^*\right),L^2(H_0)} &= \int_0^T \left[(y_t, \eta) + \gamma(\nabla w, \nabla\eta) - (y, \mathbf{v}\cdot\nabla\eta)\right]\ dt, \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} &\langle c_\delta(y, w), \eta\rangle_{L^2(H^{1*}),L^2(H^1)} \\ &= \int_0^T \left[(w + y, \theta) - \varepsilon^2(\nabla y, \nabla\theta) - \frac{1}{\delta}(\beta_\delta(y), \theta)\right]\ dt. \end{aligned} \tag{2.22}$$

Moreover, with $\mathbf{z} = (\mathbf{z}_1, z_2, z_3, \mathbf{z}_4, z_5) \in \mathbf{Z}$, we define the norm

$$\|\mathbf{z}\|_{\mathbf{Z}} = \left[\|\mathbf{z}_1\|^2_{\mathbf{L}^2(\boldsymbol{\mathcal{D}}^*)} + \|z_2\|^2_{L^2\left(H_0^*\right)} + \|z_3\|^2_{L^2(H^{1*})} + \|\mathbf{z}_4\|^2_{\boldsymbol{\mathcal{S}}} + \|z_5\|^2_{L_0^2}\right]^{\frac{1}{2}}.$$

So, the regularized version of the non-smooth optimal control Problem 2.1 is the following:

Problem 2.2. *Given* $\mathbf{v}_0 \in \boldsymbol{\mathcal{D}} \cap \mathbf{H}^2$, $y_0 \in L_0^2 \cap H^2 \cap \mathcal{K}$, *find* $(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathbf{X} \times L^2(\mathbf{L}^2)$, *such that*

$$\min_{(\mathbf{x},\mathbf{u}) \in \mathbf{X} \times L^2(\mathbf{L}^2)} J(\mathbf{x}, \mathbf{u}) = J(\bar{\mathbf{x}}, \bar{\mathbf{u}}),$$

subject to

$$e_\delta(\mathbf{x}, \mathbf{u}) = 0. \tag{2.23}$$

Using the definition (2.19), (2.20) of the map e_δ, we note that the regularization process acts just on the Cahn-Hilliard equations, where the generalized derivative of the non-smooth double-obstacle potential (1.12) is replaced by the standard derivative of the potential Φ_δ (2.10).

2.2.1. Properties of the Regularized State Equations

From the definition (2.19), (2.20) of the map e_δ, we derive that the weak form of the state equations (2.23) of the regularized optimal control Problem 2.2 read as follows:

$$\int_0^T \left[(\mathbf{v}_t, \boldsymbol{\psi}) + \nu(\nabla\mathbf{v}, \nabla\boldsymbol{\psi}) - (\mathbf{u}, \boldsymbol{\psi})\right] dt = 0, \tag{2.24a}$$

$$\mathbf{v}(0) = \mathbf{v}_0, \quad \text{in } \Omega, \tag{2.24b}$$

$$\int_0^T \left[\langle y_t, \eta\rangle_{H^{1*},H^1} + \gamma(\nabla w, \nabla\eta) - (y, \mathbf{v}\cdot\nabla\eta)\right] dt = 0, \tag{2.25a}$$

$$y(0) = y_0, \quad \text{in } \Omega, \tag{2.25b}$$

$$\int_0^T \left[(w, \theta) - \varepsilon^2(\nabla y, \nabla\theta) + (y, \theta) - \frac{1}{\delta}(\beta_\delta(y), \theta)\right] dt = 0, \tag{2.25c}$$

for all $\boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}}), \eta, \theta \in L^2(H^1)$. In the next Lemma 2.3, we derive existence, uniqueness and regularity properties of the solution of (2.24), (2.25).

Lemma 2.3 (existence, uniqueness, regularity). *For any fixed* $\delta \in \left(0, \frac{1}{4}\right)$, $\mathbf{v}_0 \in \boldsymbol{\mathcal{D}} \cap \mathbf{H}^2$, $y_0 \in L_0^2 \cap H^2$, $-1 \le y_0 \le 1$ *a.e. in* Ω, $\mathbf{u} \in L^2(\mathbf{L}^2)$ *the system* (2.24), (2.25) *has a unique solution*

$$(\mathbf{v}, y, w) \in \left(H^1(\boldsymbol{\mathcal{S}}) \cap L^\infty(\boldsymbol{\mathcal{D}})\right) \times \left(W_0 \cap L^\infty(H_0) \cap L^2\left(H^2\right)\right) \times L^2\left(H^1\right),$$

which satisfies

$$\|\mathbf{v}_t\|^2_{L^2(\boldsymbol{\mathcal{S}})} + \|\mathbf{v}\|^2_{L^\infty(\boldsymbol{\mathcal{D}})} + \|y\|^2_{W_0} + \|y\|^2_{L^\infty(H_0)} + \|y\|^2_{L^2(H^2)} + \|w\|^2_{L^2(H^1)} \le C(\mathbf{u}), \tag{2.26}$$

$$\left.\frac{\partial y}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0, \quad \text{a.e. on} \quad (0,T)\,, \tag{2.27}$$

$$\left\| \frac{1}{\delta}\beta_\delta\left(y\right) \right\|^2_{L^2(L^2)} \leq C\left(\mathbf{u}\right), \tag{2.28}$$

where the constant $C\left(\mathbf{u}\right)$ depends continuously on $\|\mathbf{u}\|_{L^2(\mathbf{L}^2)}$ and data problem (initial conditions and constant parameters), but it is independent of δ.

The proof of the Lemma is given in Appendix B, Section B.1.

Remark 2.4. It is obvious that the solution $y \in W_0$. In fact setting $\eta = \chi_{[0,t]}$ in (2.25a), where

$$\chi_{[0,t]}\left(s\right) := \begin{cases} 1 & \text{if} \quad s \in [0,t]\,, \\ 0 & \text{otherwise} \end{cases}$$

and integrating by parts in time, we have

$$\left(y\left(t\right),1\right) = \left(y\left(0\right),1\right) = 0, \quad \forall\, t \in (0,T].$$

As a consequence of the results of Lemma 2.3, associated to the state equations of the regularized optimal control Problem 2.2

$$e_\delta\left(\mathbf{x},\mathbf{u}\right) = 0,$$

we can define a bounded solution operator $s_\delta : L^2\left(\mathbf{L}^2\right) \to \mathbf{X}$, which such that

$$e_\delta\left(s_\delta\left(\mathbf{u}\right),\mathbf{u}\right) = 0, \quad \forall\, \mathbf{u} \in L^2\left(\mathbf{L}^2\right). \tag{2.29}$$

2.2.2. Well-Posedness of the Regularized Optimal Control Problem

We note that the map $J : \mathbf{X} \times L^2\left(\mathbf{L}^2\right) \to \mathbb{R}$, defined in (2.4), in the regularized optimal control Problem 2.2 is continuous, convex and bounded from below. Thus, it is *weakly lower semicontinuous.* We use the weakly lower semicontinuity of J to get the following result, which ensures that Problem 2.2 is well posed.

Theorem 2.5 (existence of minimizers). *For any fixed $\delta \in \left(0,\frac{1}{4}\right)$, the regularized optimal control Problem 2.2 admits a solution.*

Proof. For any $\mathbf{u} \in L^2\left(\mathbf{L}^2\right)$, Lemma 2.3 ensures the existence and the uniqueness of the solution $\mathbf{x} = \left(\mathbf{v},y,w\right) \in \mathbf{X}$. Therefore the feasible set

$$F_{ad} = \left\{\left(\mathbf{x},\mathbf{u}\right) \in \mathbf{X} \times L^2\left(\mathbf{L}^2\right) : e_\delta\left(\mathbf{x},\mathbf{u}\right) = 0\right\},$$

is not empty. Then there exists

$$\inf_{(\mathbf{x},\mathbf{u})\in F_{ad}} J\left(\mathbf{x},\mathbf{u}\right) = \hat{J} > -\infty.$$

and a sequence $\{(\mathbf{x}_n, \mathbf{u}_n)\}_{n\in\mathbb{N}} \subset F_{ad}$, such that

$$J(\mathbf{x}_n, \mathbf{u}_n) \to \hat{J}. \tag{2.30}$$

By the definition (2.4) of the cost functional J, the sequence $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbf{L}^2)$ and so, by Lemma 2.3, there exists a constant C such that

$$\|\mathbf{x}_n\|_{\mathbf{X}} \le C,$$

and furthermore

$$\|\mathbf{v}_n\|_{L^\infty(\boldsymbol{\mathcal{D}})} + \|y_n\|_{L^\infty(H_0)} \le C.$$

Then, we can extract a subsequence (labelled with index m), such that

$$\mathbf{v}_m \rightharpoonup \mathbf{v}, \qquad \text{in } \mathbf{W}_0, \tag{2.31}$$
$$\mathbf{v}_m \overset{*}{\rightharpoonup} \mathbf{v}, \qquad \text{in } L^\infty(\boldsymbol{\mathcal{D}}), \tag{2.32}$$
$$\mathbf{v}_m \to \mathbf{v}, \qquad \text{in } L^2(\boldsymbol{\mathcal{S}}), \tag{2.33}$$
$$y_m \rightharpoonup y, \qquad \text{in } W_0, \tag{2.34}$$
$$y_m \overset{*}{\rightharpoonup} y, \qquad \text{in } L^\infty(H_0), \tag{2.35}$$
$$y_m \to y, \qquad \text{in } L^2\left(L^2_0\right), \tag{2.36}$$
$$w_m \rightharpoonup w, \qquad \text{in } L^2\left(H^1\right), \tag{2.37}$$
$$\mathbf{u}_m \rightharpoonup \mathbf{u}, \qquad \text{in } L^2\left(\mathbf{L}^2\right), \tag{2.38}$$

where (2.33) and (2.36) follow, respectively, from (2.31) and (2.34), using the Aubin-Lions-Simon Theorem (see for example Theorem II.5.16 in [20]). So, we have

$$(\mathbf{x}_m, \mathbf{u}_m) \rightharpoonup (\mathbf{x}, \mathbf{u}), \qquad \text{in } \mathbf{X} \times L^2\left(\mathbf{L}^2\right).$$

The subsequence $\{(\mathbf{x}_m, \mathbf{u}_m)\}_m \subset F_{ad}$, therefore $e_\delta(\mathbf{x}_m, \mathbf{u}_m) = 0$. We show in the following that $e_\delta(\mathbf{x}, \mathbf{u}) = 0$. From (2.31), (2.34), (2.37) and (2.38), we get

$$\begin{aligned}
\langle a(\mathbf{v}_m, \mathbf{u}_m), \boldsymbol{\psi}\rangle_{L^2(\boldsymbol{\mathcal{D}}^*),L^2(\boldsymbol{\mathcal{D}})} &\to \langle a(\mathbf{v}, \mathbf{u}), \boldsymbol{\psi}\rangle_{L^2(\boldsymbol{\mathcal{D}}^*),L^2(\boldsymbol{\mathcal{D}})},\\
\int_0^T [(y_{mt}, \eta) + \gamma(\nabla w_m, \nabla\eta)]\, dt &\to \int_0^T [(y_t, \eta) + \gamma(\nabla w, \nabla\eta)]\, dt,\\
\int_0^T \left[(w_m + y_m, \theta) - \varepsilon^2(\nabla y_m, \nabla\theta)\right] dt &\to \int_0^T \left[(w + y, \theta) - \varepsilon^2(\nabla y, \nabla\theta)\right] dt.
\end{aligned}$$

as $m \to +\infty$, for all $(\boldsymbol{\psi}, \eta, \theta) \in L^2(\boldsymbol{\mathcal{D}}) \times L^2(H_0) \times L^2(H^1)$. Concerning the remaining term in the functional b (2.21), we have that

$$\begin{aligned}
&\left|\int_0^T (y_m, \mathbf{v}_m\cdot\nabla\eta) - (y, \mathbf{v}\cdot\nabla\eta)\, dt\right|\\
&\le \int_0^T |(y_m - y, \mathbf{v}_m\cdot\nabla\eta)|\, dt + \int_0^T |(y, [\mathbf{v}_m - \mathbf{v}]\cdot\nabla\eta)|\, dt = D_1 + D_2,
\end{aligned}$$

where using the inequalities (A.17), (A.18) and (2.33), (2.36), we derive

$$D_1 \le C\int_0^T \|y - y_m\|^{\frac{1}{2}} \|y - y_m\|^{\frac{1}{2}}_{H_0} \|\mathbf{v}_m\|^{\frac{1}{2}} \|\mathbf{v}_m\|^{\frac{1}{2}}_{\boldsymbol{\mathcal{D}}} \|\eta\|_{H_0}\, dt$$

$$
\begin{aligned}
&\leq C \|y-y_m\|_{L^\infty(H_0)}^{\frac{1}{2}} \|\mathbf{v}_m\|_{L^\infty(\boldsymbol{\mathcal{D}})}^{\frac{1}{2}} \|y-y_m\|_{L^2(L^2)}^{\frac{1}{2}} \|\mathbf{v}_m\|_{L^2(\boldsymbol{\mathcal{S}})}^{\frac{1}{2}} \|\eta\|_{L^2(H_0)} \to 0,\\
D_2 &\leq C\int_0^T \|y\|^{\frac{1}{2}} \|y\|_{H_0}^{\frac{1}{2}} \|\mathbf{v}_m-\mathbf{v}\|^{\frac{1}{2}} \|\mathbf{v}_m-\mathbf{v}\|_{\boldsymbol{\mathcal{D}}}^{\frac{1}{2}} \|\eta\|_{H_0}\, dt\\
&\leq C \|y\|_{L^\infty(H_0)}^{\frac{1}{2}} \|\mathbf{v}_m-\mathbf{v}\|_{L^\infty(\boldsymbol{\mathcal{D}})}^{\frac{1}{2}} \|y\|_{L^2(L^2)}^{\frac{1}{2}} \|\mathbf{v}_m-\mathbf{v}\|_{L^2(\boldsymbol{\mathcal{S}})}^{\frac{1}{2}} \|\eta\|_{L^2(H_0)} \to 0,
\end{aligned}
$$

as $m\to+\infty$. In order to manage the remaining term in c_δ (2.22), from (2.17), we note that β_δ is a Lipschitz function and therefore

$$
\int_0^T |\ (\beta_\delta(y_m)-\beta_\delta(y),\theta)\,|\ dt \leq \|y_m-y\|_{L^2(L^2)} \|\theta\|_{L^2(L^2)} \to 0,
$$

as $m\to+\infty$. So, we can claim that

$$
\langle a(\mathbf{v},\mathbf{u}),\boldsymbol{\psi}\rangle_{L^2(\boldsymbol{\mathcal{D}}^*),L^2(\boldsymbol{\mathcal{D}})}+\langle b(\mathbf{v},y,w),\eta\rangle_{L^2\left(H_0^*\right),L^2(H_0)}+\langle c_\delta(y,w),\eta\rangle_{L^2(H^{1*}),L^2(H^1)}=0,
$$

for all $(\boldsymbol{\psi},\eta,\theta)\in L^2(\boldsymbol{\mathcal{D}})\times L^2(H_0)\times L^2(H^1)$. With $\boldsymbol{\psi}=\boldsymbol{\xi}(1-t/T),\boldsymbol{\xi}\in\boldsymbol{\mathcal{S}}$ and $\eta=\varphi(1-t/T),\varphi\in L_0^2$, integrating by parts and using the previous results, it is easy to realize that

$$
\begin{aligned}
(\mathbf{v}_m(0)-\mathbf{v}(0),\boldsymbol{\xi}) &= -\int_0^T (\mathbf{v}_{mt}-\mathbf{v}_t,\boldsymbol{\psi})\ dt-\int_0^T (\mathbf{v}_m-\mathbf{v},\boldsymbol{\psi}_t)\ dt\to 0,\\
(y_m(0)-y(0),\varphi) &= -\int_0^T \langle y_{mt}-y_t,\eta\rangle_{H_0^*,H_0}\ dt-\int_0^T (y_m-y,\eta_t)\ dt,\to 0,
\end{aligned}
$$

as $m\to+\infty$. Furthermore, for all m, we have $\mathbf{v}_m(0)=\mathbf{v}_0$ and $y_m(0)=y_0$. Therefore

$$
\mathbf{v}(0)=\mathbf{v}_0,\quad y(0)=y_0.
$$

Thus, we have

$$
e_\delta(\mathbf{x},\mathbf{u})=0,
$$

that is $(\mathbf{x},\mathbf{u})\in F_{ad}$. Then, using that J is weakly lower semicontinuous, we can write

$$
J(\mathbf{x},\mathbf{u})\leq \liminf_{m\to+\infty} J(\mathbf{x}_m,\mathbf{u}_m)=\hat{J}.
$$

Hence, $(\mathbf{x},\mathbf{u})$ is a solution of the optimal control Problem 2.2. □

2.2.3. Optimality Conditions of the Regularized Optimal Control Problem

In this section, we show that Problem 2.2 satisfies the conditions needed to apply the standard theory of mathematical programming in Banach spaces (see Assumptions 1.47 in [58]). Subsequently, we derive the first order optimality conditions of the regularized optimal control Problem 2.2 (see Theorem 1.48, Corollary 1.3 in [58]). We need to verify that the regularized optimal control problem satisfies the following conditions:

- the continuous Fréchet differentiability of the cost functional $J:\mathbf{X}\times L^2(\mathbf{L}^2)\to\mathbb{R}$ defined in (2.4);

- the continuous Fréchet differentiability of the constraint $e_\delta : \mathbf{X} \times L^2(\mathbf{L}^2) \to Z$ defined in (2.19), (2.20);
- the existence of the inverse of the mapping $e_{\delta \mathbf{x}}(s_\delta(\mathbf{u}), \mathbf{u})$, where s_δ is the bounded solution operator defined in (2.29).

It is easy to realize that the mapping $J : \mathbf{X} \times L^2(\mathbf{L}^2) \to \mathbb{R}$ is continuously Fréchet differentiable. Indeed, the Fréchet derivative

$$J' : \mathbf{X} \times L^2\left(\mathbf{L}^2\right) \to \mathcal{L}\left(\mathbf{X} \times L^2\left(\mathbf{L}^2\right), \mathbb{R}\right),$$

and J has partial Fréchet derivatives

$$\begin{aligned}
\langle\ J_{\mathbf{v}}(\mathbf{x}, \mathbf{u}), \mathbf{d}_{\mathbf{v}}\ \rangle_{\mathbf{W}_0^*, \mathbf{W}_0} &= 0, \\
\langle\ J_y(\mathbf{x}, \mathbf{u}), d_y\ \rangle_{W_0^*, W_0} &= \int_0^T (y - y_d, d_y)\ dt, \\
\langle\ J_w(\mathbf{x}, \mathbf{u}), d_w\ \rangle_{L^2(H^{1*}), L^2(H^1)} &= 0, \\
(J_{\mathbf{u}}(\mathbf{x}, \mathbf{u}), \mathbf{d}_{\mathbf{u}})_{L^2(\mathbf{L}^2)} &= \int_0^T (\alpha\ \mathbf{u}, \mathbf{d}_{\mathbf{u}})\ dt,
\end{aligned}$$

such that

$$\langle\ J'(\mathbf{x}, \mathbf{u}), (\mathbf{d}_{\mathbf{x}}, \mathbf{d}_{\mathbf{u}})\ \rangle_{(\mathbf{X} \times L^2(\mathbf{L}^2))^*, \mathbf{X} \times L^2(\mathbf{L}^2)} = \int_0^T [(y - y_d, d_y) + \alpha(\mathbf{u}, \mathbf{d}_{\mathbf{u}})]\ dt,$$

for all $(\mathbf{d}_{\mathbf{x}}.\mathbf{d}_{\mathbf{u}}) \in \mathbf{X} \times L^2(\mathbf{L}^2)$. Therefore

$$\left| J(\mathbf{x} + \mathbf{d}_{\mathbf{x}}, \mathbf{u} + \mathbf{d}_{\mathbf{u}}) - J(\mathbf{x}, \mathbf{u}) - \left\langle J'(\mathbf{x}, \mathbf{u}), (\mathbf{d}_{\mathbf{x}}, \mathbf{d}_{\mathbf{u}}) \right\rangle_{(\mathbf{X} \times L^2(\mathbf{L}^2))^*, \mathbf{X} \times L^2(\mathbf{L}^2)} \right| = 0$$

i.e., J is Fréchet differentiable. Moreover, J is continuously Fréchet differentiable, since

$$\begin{aligned}
&\left| \left\langle J'(\mathbf{x} + \mathbf{d}_{\mathbf{x}}, \mathbf{u} + \mathbf{d}_{\mathbf{u}}) - J'(\mathbf{x}, \mathbf{u}), (\mathbf{h}_{\mathbf{x}}, \mathbf{h}_{\mathbf{u}}) \right\rangle_{(\mathbf{X} \times L^2(\mathbf{L}^2))^*, \mathbf{X} \times L^2(\mathbf{L}^2)} \right| \\
&\qquad = \left| \int_0^T [(d_y, h_y) + \alpha(\mathbf{d}_{\mathbf{u}}, \mathbf{h}_{\mathbf{u}})]\ dt \right| \\
&\qquad \le \|d_y\|_{L^2(L^2)} \|h_y\|_{L^2(L^2)} + \alpha \|\mathbf{d}_{\mathbf{u}}\|_{L^2(\mathbf{L}^2)} \|\mathbf{h}_{\mathbf{u}}\|_{L^2(\mathbf{L}^2)} \\
&\qquad \le \|\,(\mathbf{d}_{\mathbf{x}}, \mathbf{d}_{\mathbf{u}})\,\|_{\mathbf{X} \times L^2(\mathbf{L}^2)} \left[\|h_y\|_{L^2(L^2)} + \alpha \|\mathbf{h}_{\mathbf{u}}\|_{L^2(\mathbf{L}^2)} \right] \to 0,
\end{aligned}$$

as $(\mathbf{d}_{\mathbf{x}}, \mathbf{d}_{\mathbf{u}}) \to 0$ in $\mathbf{X} \times L^2(\mathbf{L}^2)$, for all $(\mathbf{h}_{\mathbf{x}}, \mathbf{h}_{\mathbf{u}}) \in \mathbf{X} \times L^2(\mathbf{L}^2)$.

The differentiation properties of the map e_δ are summarized in the following lemma.

Lemma 2.6. *For any fixed $\delta \in \left(0, \frac{1}{4}\right)$, the map $e_\delta : \mathbf{X} \times L^2(\mathbf{L}^2) \to Z$ is continuously Fréchet differentiable.*

Proof. We have

$$e_\delta' : \mathbf{X} \times L^2\left(\mathbf{L}^2\right) \to \mathcal{L}\left(\mathbf{X} \times L^2\left(\mathbf{L}^2\right), \mathbf{Z}\right),$$

with partial Fréchet derivatives

$$\begin{aligned}
\langle \mathbf{p}, e_{\delta \mathbf{v}}(\mathbf{x},\mathbf{u})\,\mathbf{d}_{\mathbf{v}}\rangle_{\mathbf{Z}^*,\mathbf{Z}} &= \int_0^T \left[(\mathbf{d}_{\mathbf{v}t},\boldsymbol{\psi}) + \nu(\nabla \mathbf{d}_{\mathbf{v}}, \nabla\boldsymbol{\psi}) - (y, \mathbf{d}_{\mathbf{v}}\cdot\nabla\eta)\right]\,dt + (\boldsymbol{\xi}, \mathbf{d}_{\mathbf{v}}(0)),\\
\langle \mathbf{p}, e_{\delta y}(\mathbf{x},\mathbf{u})\,d_y\rangle_{\mathbf{Z}^*,\mathbf{Z}} &= \int_0^T \Big[\langle d_{yt},\eta\rangle_{H^{1*},H^1} - (d_y, \mathbf{v}\cdot\nabla\eta) - \varepsilon^2(\nabla d_y,\nabla\theta)\\
&\quad + (d_y,\theta) - \frac{1}{\delta}\left(\beta'_\delta(y)\,d_y,\theta\right)\Big]\,dt + (\varphi, d_y(0)),\\
\langle \mathbf{p}, e_{\delta w}(\mathbf{x},\mathbf{u})\,d_w\rangle_{\mathbf{Z}^*,\mathbf{Z}} &= \int_0^T \left[\gamma(\nabla d_w,\nabla\eta) + (d_w,\theta)\right]\,dt,\\
\langle \mathbf{p}, e_{\delta \mathbf{u}}(\mathbf{x},\mathbf{u})\,\mathbf{d}_{\mathbf{u}}\rangle_{\mathbf{Z}^*,\mathbf{Z}} &= -\int_0^T (\mathbf{d}_{\mathbf{u}},\boldsymbol{\psi})\,dt,
\end{aligned}$$

where $\mathbf{p} = (\boldsymbol{\psi},\eta,\theta,\boldsymbol{\psi},\varphi) \in \mathbf{Z}^*$. We have Fréchet differentiability if

$$\left\| e_\delta(\mathbf{x}+\mathbf{d}_{\mathbf{x}},\mathbf{u}+\mathbf{d}_{\mathbf{u}}) - e_\delta(\mathbf{x},\mathbf{u}) - e'_\delta(\mathbf{x},\mathbf{u})(\mathbf{d}_{\mathbf{x}},\mathbf{d}_{\mathbf{u}})\right\|_Z = o\left(\|(\mathbf{d}_{\mathbf{x}},\mathbf{d}_{\mathbf{u}})\|_{\mathbf{X}\times\mathbf{L}^2(\mathbf{L}^2)}\right),$$

as $\|(\mathbf{d}_{\mathbf{x}},\mathbf{d}_{\mathbf{u}})\|_{\mathbf{X}\times\mathbf{L}^2(\mathbf{L}^2)} \to 0$. It is easy to realize that

$$\begin{aligned}
&\left|\left\langle \mathbf{p}, e_\delta(\mathbf{x}+\mathbf{d}_{\mathbf{x}},\mathbf{u}+\mathbf{d}_{\mathbf{u}}) - e_\delta(\mathbf{x},\mathbf{u}) - e'_\delta(\mathbf{x},\mathbf{u})(\mathbf{d}_{\mathbf{x}},\mathbf{d}_{\mathbf{u}})\right\rangle_{\mathbf{Z}^*,\mathbf{Z}}\right|\\
\le\ &\left|\int_0^T (d_y,\mathbf{d}_{\mathbf{v}}\cdot\nabla\eta)\,dt\right| + \left|\frac{1}{\delta}\int_0^T \left(\beta_\delta(y+d_y) - \beta_\delta(y) - \beta'_\delta(y)\,d_y,\theta\right)dt\right|\\
=\ &E_1 + E_2.
\end{aligned}$$

With $\mathbf{p} = (\boldsymbol{\psi},\eta,\theta,\boldsymbol{\psi},\varphi) \in \mathbf{Z}^*$, we have, using the embeddings (A.5), (A.6),

$$\begin{aligned}
E_1 &\le C\int_0^T \|d_y\|^{\frac12}\,\|d_y\|^{\frac12}_{H_0}\,\|\mathbf{d}_{\mathbf{v}}\|^{\frac12}\,\|\mathbf{d}_{\mathbf{v}}\|^{\frac12}_{\boldsymbol{\mathcal{D}}}\,\|\eta\|_{H_0}\,dt\\
&\le C\,\|d_y\|^{\frac12}_{\mathcal{C}\left([0,T];L^2_0\right)}\,\|\mathbf{d}_{\mathbf{v}}\|^{\frac12}_{\boldsymbol{\mathcal{C}}([0,T];\boldsymbol{\mathcal{S}})}\,\|d_y\|^{\frac12}_{L^2(H_0)}\,\|\mathbf{d}_{\mathbf{v}}\|^{\frac12}_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})}\,\|\eta\|_{L^2(H_0)}\\
&\le C\,\|\eta\|_{L^2(H_0)}\left(\|d_y\|_{W_0} + \|\mathbf{d}_{\mathbf{v}}\|_{\mathbf{W}_0}\right)\left(\|d_y\|_{L^2(H_0)} + \|\mathbf{d}_{\mathbf{v}}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})}\right)\\
&\le C\,\|\eta\|_{L^2(H_0)}\,\|(\mathbf{d}_{\mathbf{x}},\mathbf{d}_{\mathbf{u}})\|^2_{\mathbf{X}\times\mathbf{L}^2(\mathbf{L}^2)}.
\end{aligned}$$

Next, we note that

$$\frac{1}{\|d_y\|_{W_0}}\int_0^T \left(\beta_\delta(y+d_y) - \beta_\delta(y),\theta\right)\,dt \to \frac{1}{\|d_y\|_{W_0}}\int_0^T \left(\beta'_\delta(y)\,d_y,\theta\right)\,dt,$$

as $d_y \to 0$ in W_0. Hence, for all $(\mathbf{d}_{\mathbf{x}},\mathbf{d}_{\mathbf{u}}) \in \mathbf{X}\times\mathbf{L}^2(\mathbf{L}^2)$, we derive

$$\frac{E_2}{\|(\mathbf{d}_{\mathbf{x}},\mathbf{d}_{\mathbf{u}})\|_{\mathbf{X}\times\mathbf{L}^2(\mathbf{L}^2)}} \le \frac{\left|\frac{1}{\delta}\int_0^T \left(\beta_\delta(y+d_y) - \beta_\delta(y) - \beta'_\delta(y)\,d_y,\theta\right)\,dt\right|}{\|d_y\|_{W_0}} \to 0,$$

as $(\mathbf{d}_{\mathbf{x}},\mathbf{d}_{\mathbf{u}}) \to 0$ in $\mathbf{X}\times\mathbf{L}^2(\mathbf{L}^2)$. Thus, we have shown that e_δ is Fréchet differentiable.

Next, we show that e_δ is continuosly Fréchet differentiable, i.e. for all $(\mathbf{x},\mathbf{u})\,,(\mathbf{d_x},\mathbf{d_u})\in\mathbf{X}\times L^2\left(\mathbf{L}^2\right)$,

$$\|e'_\delta\left(\mathbf{x}+\mathbf{d_x},\mathbf{u}+\mathbf{d_u}\right)-e'_\delta\left(\mathbf{x},\mathbf{u}\right)\|_{\mathcal{L}(\mathbf{X}\times\mathbf{L}^2(\mathbf{L}^2),\mathbf{Z})}\to 0,$$

as $(\mathbf{d_x},\mathbf{d_u})\to 0$. We have,

$$\begin{aligned}
&\Big|\left\langle\ \mathbf{p},\left[\ e'_\delta\left(\mathbf{x}+\mathbf{d_x},\mathbf{u}+\mathbf{d_u}\right)-e'_\delta\left(\mathbf{x},\mathbf{u}\right)\ \right]\left(\mathbf{h_x},\mathbf{h_u}\right)\right\rangle_{\mathbf{Z}^*,\mathbf{Z}}\Big|\\
=&\ \Big|\int_0^T\left[\left(d_y,\mathbf{h_v}\cdot\nabla\eta\right)+\left(h_y,\mathbf{d_v}\cdot\nabla\eta\right)+\frac{1}{\delta}\left(\beta'_\delta\left(y+d_y\right)-\beta'_\delta\left(y\right),h_y\ \theta\right)\right]\ dt\Big|\\
=&\ F_1+F_2+F_3,
\end{aligned}$$

for all $\mathbf{p}=(\boldsymbol{\psi},\eta,\theta,\boldsymbol{\psi},\varphi)\in Z^*$. As well as in the estimate for E_1, we derive

$$\begin{aligned}
F_1&\le C\ \|d_y\|^{\frac12}_{\mathcal{C}\left([0,T];L^2_0\right)}\ \|\mathbf{h_v}\|^{\frac12}_{\boldsymbol{\mathcal{C}}([0,T];\boldsymbol{\mathcal{S}})}\ \|d_y\|^{\frac12}_{L^2(H_0)}\ \|\mathbf{h_v}\|^{\frac12}_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})}\ \|\eta\|_{L^2(H_0)}\to 0,\\
F_2&\le C\ \|h_y\|^{\frac12}_{\mathcal{C}\left([0,T];L^2_0\right)}\ \|\mathbf{d_v}\|^{\frac12}_{\boldsymbol{\mathcal{C}}([0,T];\boldsymbol{\mathcal{S}})}\ \|h_y\|^{\frac12}_{L^2(H_0)}\ \|\mathbf{d_v}\|^{\frac12}_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})}\ \|\eta\|_{L^2(H_0)}\to 0,
\end{aligned}$$

as $(\mathbf{d_x},\mathbf{d_u})\to 0$. Moreover, using the property (2.18) of β'_δ,

$$F_3\le\frac{C}{\delta^2}\int_0^T\|d_y\|\ \|h_y\|_{H_0}\ \|\theta\|_{H_0}\ dt\le\frac{C}{\delta^2}\ \|d_y\|_{\mathcal{C}\left([0,T];L^2_0\right)}\ \|h_y\|_{L^2(H_0)}\ \|\theta\|_{L^2(H_0)}\to 0,$$

as $(\mathbf{d_x},\mathbf{d_u})\to 0$. It follows that e_δ is continuously Fréchet differentiable. □

Theorem 2.7. *For any fixed* $\delta\in\left(0,\frac14\right)$, $\mathbf{u}\in L^2\left(\mathbf{L}^2\right)$, *we have that*

$$e_{\delta\mathbf{x}}\left(s_\delta\left(\mathbf{u}\right),\mathbf{u}\right)\in\mathcal{L}\left(\mathbf{X},\mathbf{Z}\right),$$

has a bounded inverse.

The proof of the Theorem is given in Appendix B, Section B.1

Note that, by Theorem 2.7, we have that for all $\mathbf{u}\in L^2\left(\mathbf{L}^2\right)$,

$$\left[\ e_{\delta\mathbf{x}}\left(s_\delta\left(\mathbf{u}\right),\mathbf{u}\right)\ \right]^{-1}\in\mathcal{L}\left(\mathbf{Z},\mathbf{X}\right).\tag{2.39}$$

The continuous Fréchet differentiability of the cost functional $J:\mathbf{X}\times L^2\left(\mathbf{L}^2\right)\to\mathbb{R}$, Lemma 2.6 and Theorem 2.7 guarantee that all the solutions $(\mathbf{x}.\mathbf{u})$ of the regularized optimal control Problem 2.2 satisfy, together an adjoint variable $\mathbf{q}\in\mathbf{Z}^*$, a set of first order optimality conditions (see Theorem 1.48 and Corollary 1.3 in [58]). To derive the first order optimality conditions, it is convenient to define the Lagrange functional $L_\delta:\mathbf{X}\times L^2\left(\mathbf{L}^2\right)\times\mathbf{Z}^*\to\mathbb{R}$,

$$L_\delta\left(\mathbf{x},\mathbf{u},\mathbf{q}\right)=J\left(\mathbf{x},\mathbf{u}\right)+\left\langle\ \mathbf{q},e_\delta\left(\mathbf{x},\mathbf{u}\right)\ \right\rangle_{\mathbf{Z}^*,\mathbf{Z}},\tag{2.40}$$

where $\mathbf{q}=\left(\mathbf{q_v},q_y,q_w,\mathbf{q_{v0}},q_{y0}\right)\in\mathbf{Z}^*$. Thus, the optimality conditions of Problem 2.2 can be formulated as follows: *find* $(\mathbf{x},\mathbf{u},\mathbf{q})\in\mathbf{X}\times L^2\left(\mathbf{L}^2\right)\times\mathbf{Z}^*$ *such that*

$$L_{\delta\mathbf{q}}\left(\mathbf{x},\mathbf{u},\mathbf{q}\right)=0,\qquad\text{in }\mathbf{Z},\tag{2.41}$$

$$(2.42) \qquad L_{\delta\mathbf{x}}(\mathbf{x},\mathbf{u},\mathbf{q}) = 0, \qquad \text{in } \mathbf{X}^*,$$
$$(2.43) \qquad L_{\delta\mathbf{u}}(\mathbf{x},\mathbf{u},\mathbf{q}) = 0, \qquad \text{in } L^2\left(\mathbf{L}^2\right).$$

It is easy to realize that (2.41) are the state equations $e_\delta(\mathbf{x},\mathbf{u}) = 0$. Relation (2.42) corresponds to the so-called *adjoint equations* and (2.43) is a further *optimality relation*.
In the following lemma, we show that given a solution $\mathbf{x} = s_\delta(\mathbf{u})$ of the state equations (2.41), the adjoint equations (2.42) have a unique solution $\mathbf{q} \in \mathbf{Z}^*$.

Lemma 2.8. *Let $\mathbf{u} \in L^2(\mathbf{L}^2)$ and $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{x} = s_\delta(\mathbf{u})$ be given. Then, the adjoint equations* (2.42) *have a unique solution $\mathbf{q} \in \mathbf{Z}^*$, for any fixed $\delta \in \left(0, \frac{1}{4}\right)$.*

Proof. For all $\mathbf{d_x} \in \mathbf{X}$, we have

$$\left\langle L_{\delta\mathbf{x}}(\mathbf{x},\mathbf{u},\mathbf{q}), \mathbf{d_x}\right\rangle_{\mathbf{X}^*,\mathbf{X}} = \left\langle J_{\mathbf{x}}(\mathbf{x},\mathbf{u}), \mathbf{d_x}\right\rangle_{\mathbf{X}^*,\mathbf{X}} + \left\langle \mathbf{q}, e_{\delta\mathbf{x}}(\mathbf{x},\mathbf{u})\,\mathbf{d_x}\right\rangle_{\mathbf{Z}^*,\mathbf{Z}},$$

thus the adjoint equations (2.42) are equivalent to

$$e_{\delta\mathbf{x}}(\mathbf{x},\mathbf{u})^*\,\mathbf{q} = -J_{\mathbf{x}}(\mathbf{x},\mathbf{u}), \quad \text{in} \quad \mathbf{X}^*.$$

Then, if $\mathbf{x} = s_\delta(\mathbf{u})$, $\mathbf{q} = \mathbf{q}(\mathbf{u})$ is given by

$$\mathbf{q}(\mathbf{u}) = -\left[e_{\delta\mathbf{x}}(s(\mathbf{u}),\mathbf{u})\right]^{-*} J_{\mathbf{x}}(s_\delta(\mathbf{u}),\mathbf{u}).$$

By Theorem 2.7, we know that $\left[e_{\delta x}(s_\delta(\mathbf{u}),\mathbf{u})\right]^{-*} \in \mathcal{L}(\mathbf{X}^*,\mathbf{Z}^*)$. So, the proof is complete. ☐

The first order optimality conditions (2.41)-(2.43) are written in terms of the abstract variables $(\mathbf{x},\mathbf{u},\mathbf{q}) \in \mathbf{X} \times L^2(\mathbf{L}^2) \times \mathbf{Z}^*$. In the following Corollary 2.9, from the definitions of the spaces $\mathbf{X}$ in (2.1) and $\mathbf{Z}$ in (2.19), we write these optimality conditions explicitly, using the state variables

$$(\mathbf{v}, y, w) \in \mathbf{W}_0 \times W_0 \times L^2\left(H^1\right),$$

and the adjoint variables

$$(\mathbf{q_v}, q_y, q_w, \mathbf{q_{v0}}, q_{y0}) \in L^2(\boldsymbol{\mathcal{D}}) \times L^2(H_0) \times L^2\left(H^1\right) \times \boldsymbol{\mathcal{S}} \times L_0^2.$$

Corollary 2.9 (optimality conditions). *For any given $\delta \in \left(0, \frac{1}{4}\right)$, the first order optimality conditions (2.41)-(2.43) of the regularized optimal control Problem 2.2 read as follows:*

$$(2.44a) \qquad \int_0^T \left[(\mathbf{v}_t, \boldsymbol{\psi}) + \nu(\nabla\mathbf{v}, \nabla\boldsymbol{\psi}) - (\mathbf{u}, \boldsymbol{\psi})\right]\,dt = 0,$$

$$(2.44b) \qquad \mathbf{v}(0) = \mathbf{v}_0, \qquad \text{in } \Omega,$$

$$(2.44c) \qquad \int_0^T \left[\langle y_t, \eta\rangle_{H^{1*},H^1} + \gamma(\nabla w, \nabla\eta) - (y, \mathbf{v}\cdot\nabla\eta)\right]\,dt = 0,$$

$$(2.44d) \qquad y(0) = y_0, \qquad \text{in } \Omega,$$

$$(2.44e) \qquad \int_0^T \left[(w,\theta) - \varepsilon^2(\nabla y, \nabla\theta) + (y,\theta) - \frac{1}{\delta}(\beta_\delta(y),\theta)\right]\,dt = 0,$$

for all $\boldsymbol{\psi} \in \mathbf{L}^2(\boldsymbol{\mathcal{D}}), \eta, \theta \in L^2(H^1)$,

$$\int_0^T [-\langle \mathbf{q}_{\mathbf{v}t}, \boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}} + (\nabla \mathbf{q}_{\mathbf{v}}, \nabla \boldsymbol{\psi}) - (y, \nabla q_y \cdot \boldsymbol{\psi})] \ dt = 0, \tag{2.45a}$$

$$\mathbf{q}_{\mathbf{v}}(T) = 0, \quad \text{in } \Omega, \tag{2.45b}$$

$$\int_0^T \left[\langle \eta_t, q_y\rangle_{H_0^*,H_0} - \varepsilon^2 (\nabla q_w, \nabla \eta) + (q_w, \eta) - (\mathbf{v} \cdot \nabla q_y, \eta) + (y - y_d, \eta)\right] \ dt$$
$$+ (q_{y0}, \eta(0)) - \frac{1}{\delta} \int_0^T (\beta'_\delta(y) q_w, \eta) \ dt = 0, \tag{2.45c}$$

$$\int_0^T [\ (q_w, \theta) + \gamma (\nabla q_y, \nabla \theta)\ ] \ dt = 0. \tag{2.45d}$$

for all $\boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}}), \eta \in W_0, \theta \in L^2(H^1)$,

$$\int_0^T (\alpha \mathbf{u} - \mathbf{q}_{\mathbf{v}}, \boldsymbol{\varphi}) \ dt = 0, \tag{2.46}$$

for all $\boldsymbol{\varphi} \in L^2(\mathbf{L}^2)$.

Proof. Direct calculation shows that equations (2.44) and (2.46) can be derived, respectively, from (2.41) and (2.43). From (2.42), we get (2.45c), (2.45d) and the following equation

$$\int_0^T [\langle \boldsymbol{\psi}_t, \mathbf{q}_{\mathbf{v}}\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}} + (\nabla \mathbf{q}_{\mathbf{v}}, \nabla \boldsymbol{\psi}) - (y, \nabla q_y \cdot \boldsymbol{\psi})] \ dt + (\mathbf{q}_{\mathbf{v}0}, \boldsymbol{\psi}(0)) = 0. \tag{2.47}$$

In (2.47), we have $\mathbf{q}_{\mathbf{v}} \in L^2(\boldsymbol{\mathcal{D}})$ and $\mathbf{q}_{\mathbf{v}0} \in \boldsymbol{\mathcal{S}}$. If we assume $\mathbf{q}_{\mathbf{v}} \in \mathbf{W}_0$ and integrate by parts in time, from (2.47) we obtain

$$\int_0^T [-\langle \mathbf{q}_{\mathbf{vt}}, \boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}} + (\nabla \mathbf{q}_{\mathbf{v}}, \nabla \boldsymbol{\psi}) - (y, \nabla q_y \cdot \boldsymbol{\psi})] \ dt$$
$$+ (\mathbf{q}_{\mathbf{v}}(T), \boldsymbol{\psi}(T)) - (\mathbf{q}_{\mathbf{v}}(0), \boldsymbol{\psi}(0)) + (\mathbf{q}_{\mathbf{v}0}, \boldsymbol{\psi}(0)) = 0. \tag{2.48}$$

Thus, setting $\mathbf{q}_{\mathbf{v}}(T) = 0$ and $\mathbf{q}_{\mathbf{v}}(0) = \mathbf{q}_{\mathbf{v}0}$ in (2.48), we get that $\mathbf{q}_{\mathbf{v}}$ satisfies

$$\int_0^T [-\langle \mathbf{q}_{\mathbf{vt}}, \boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}} + (\nabla \mathbf{q}_{\mathbf{v}}, \nabla \boldsymbol{\psi}) - (y, \nabla q_y \cdot \boldsymbol{\psi})] \ dt = 0, \tag{2.49}$$
$$\mathbf{q}_{\mathbf{v}}(T) = 0, \qquad \text{in } \Omega$$

for all $\boldsymbol{\psi} \in \mathbf{W}_0$. In (2.49), $q_y \in L^2(H_0)$ and from (2.26), $y \in L^\infty(H_0)$. Therefore, it is easy to prove that

$$\left| \int_0^T (y, \nabla q_y \cdot \boldsymbol{\psi}) \ dt \right| \leq C \, \|y\|_{L^\infty(H_0)} \, \|q_y\|_{L^2(H_0)} \, \|\boldsymbol{\psi}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})}, \quad \forall \ \boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}}). \tag{2.50}$$

Thus, from (2.50), using a density argument, we obtain that (2.47) is equivalent to (2.45a), (2.45b), with test functions $\boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}})$. In fact, equations (2.45a), (2.45b) have a unique solution $\mathbf{q}_{\mathbf{v}} \in \mathbf{W}_0$ which is, by Lemma 2.8, the unique solution of (2.47). □

We conclude this section with Lemma 2.10, that provides regularity results and $\delta-$independent stability estimates for the adjoint variables

$$\mathbf{q_v} \in L^2(\boldsymbol{\mathcal{D}}), q_y \in L^2(H_0), q_w \in L^2(H^1), q_{y0} \in L_0^2.$$

These results will be used in the next section, where we perform the limit of the optimality conditions system (2.44)-(2.46) for the regularization parameter $\delta \to 0^+$.

Lemma 2.10. *For any fixed* $\delta \in \left(0, \frac{1}{4}\right)$*, let us assume that*

$$\mathbf{v} \in \mathbf{W}_0, y \in W_0, w \in L^2(H^1),$$
$$\mathbf{u} \in L^2(\mathbf{L}^2),$$
$$\mathbf{q_v} \in L^2(\boldsymbol{\mathcal{D}}), q_y \in L^2(H_0), q_w \in L^2(H_0), q_{y0} \in L_0^2,$$

are a solution of the optimality conditions (2.44)-(2.46). Then, the adjoint variables have improved regularity properties

$$\mathbf{q_v} \in H^1(\boldsymbol{\mathcal{S}}) \cap L^\infty(\boldsymbol{\mathcal{D}}), \tag{2.51}$$

$$q_y \in L^\infty(H_0) \cap L^2(H^2), \tag{2.52}$$

$$q_{y0} \in H_0. \tag{2.53}$$

$$\left.\frac{\partial q_y}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0, \quad \text{a.e. on } (0,T), \tag{2.54}$$

and

$$\begin{aligned} &\|\mathbf{q_v}_t\|^2_{L^2(\boldsymbol{\mathcal{S}})} + \|\mathbf{q_v}\|^2_{L^\infty(\boldsymbol{\mathcal{D}})} + \|q_y\|^2_{L^\infty(H_0)} \\ &+\|q_y\|^2_{L^2(H^2)} + \|q_{y0}\|^2_{H_0} + \|q_w\|^2_{L^2(H_0)} + \left\|\frac{1}{\delta}\beta'_\delta(y)\right\|^2_{W_0^*} \leq C(\mathbf{u}), \end{aligned} \tag{2.55}$$

where the constant $C(\mathbf{u})$ *depends continuously on* $\|\mathbf{u}\|_{L^2(\mathbf{L}^2)}$ *and data problem (initial conditions and constant parameters), but it is independent of* δ.

The proof of the Lemma is shown in Appendix B, Section B.1.

2.3. Non-Smooth Optimal Control Problem

Using the results obtained in Section 2.2, we study the non-smooth optimal control Problem 2.1. In particular, we derive the first order optimality conditions of Problem 2.1 as a limit of the first order optimality conditions (2.44)-(2.46) of the regularized Problem 2.2, for the regularization parameter $\delta \to 0^+$.

2.3.1. Properties of the State Equations of the Non-Smooth Optimal Control Problem

In this section, we consider the state equations (2.5), (2.6) of the non-smooth optimal control Problem 2.1. In Theorem 2.11 below, we get that these equations can be derived as limit of the state equations (2.24), (2.25) of the regularized optimal control Problem 2.2, for the regularization parameter $\delta \to 0^+$. Next, in Lemma 2.12, we show existence, uniqueness and regularity properties of the solution of (2.5), (2.6).

Theorem 2.11. *Consider a sequence* $\{\delta_n\}_{n\in\mathbb{N}} \subset \left(0, \frac{1}{4}\right)$ *such that* $\delta_n \to 0^+$, *a bounded sequence* $\{\mathbf{u}_n\}_{n\in N} \subset L^2(\mathbf{L}^2)$ *and the corresponding sequence of solutions* $\{(\mathbf{v}_n, y_n, w_n)\}_{n\in N} \subset \mathbf{W}_0 \times W_0 \times L^2(H^1)$ *of the state equations (2.24), (2.25) of the regularized optimal control Problem 2.2. Then, there exists a subsequence (labelled by index* m*), such that*

$$\mathbf{u}_m \rightharpoonup \mathbf{u}, \quad \text{in } L^2\left(\mathbf{L}^2\right), \tag{2.56}$$
$$\mathbf{v}_m \rightharpoonup \mathbf{v}, \quad \text{in } H^1(\boldsymbol{\mathcal{S}}), \tag{2.57}$$
$$\mathbf{v}_m \overset{*}{\rightharpoonup} \mathbf{v}, \quad \text{in } L^\infty(\boldsymbol{\mathcal{D}}) \tag{2.58}$$
$$\mathbf{v}_m \to \mathbf{v}, \quad \text{in } L^2(\boldsymbol{\mathcal{S}}), \tag{2.59}$$
$$y_m \rightharpoonup y, \quad \text{in } W_0, \tag{2.60}$$
$$y_m \overset{*}{\rightharpoonup} y, \quad \text{in } L^\infty(H_0) \tag{2.61}$$
$$y_m \rightharpoonup y, \quad \text{in } L^2\left(H^2\right), \tag{2.62}$$
$$y_m \to y, \quad \text{in } L^2(H_0), \tag{2.63}$$
$$w_m \rightharpoonup w, \quad \text{in } L^2\left(H^1\right). \tag{2.64}$$

Moreover, there exists a constant C, *such that*

$$\|\mathbf{v}_t\|^2_{L^2(\boldsymbol{\mathcal{S}})} + \|\mathbf{v}\|^2_{L^\infty(\boldsymbol{\mathcal{D}})} + \|y\|^2_{W_0} + \|y\|^2_{L^\infty(H_0)} + \|y\|^2_{L^2(H^2)} + \|w\|^2_{L^2(H^1)} \leq C. \tag{2.65}$$

Furthermore $(\mathbf{v}, y, w, \mathbf{u})$ *satisfies the state equations (2.5), (2.6) of the non-smooth optimal control Problem 2.1 and*

$$\left.\frac{\partial y}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0, \quad \text{a.e. on} \quad (0, T), \tag{2.66}$$

Proof. The results (2.56)-(2.65) and (2.66) are direct consequence of the Lemma 2.3. Indeed, since the sequence $\{\mathbf{u}_n\}_{n\in N}$ is bounded in $L^2(\mathbf{L}^2)$, we can extract a subsequence (labelled with an index l) $\{\mathbf{u}_l\}_l$, such that

$$\mathbf{u}_l \rightharpoonup \mathbf{u}, \quad \text{in} \quad L^2\left(\mathbf{L}^2\right).$$

Hence, considering the corresponding sequence of solutions $\{(\mathbf{v}_l, y_l, w_l)\}_l$ of the regularized state equations (2.24), (2.25) and using the $\delta-$independent estimate (2.26), we infer that there exists a further subsequence (labelled by an index m) $\{(\mathbf{v}_m, y_m, w_m)\}_m$ which fulfils (2.57), (2.58), (2.60)-(2.62), (2.64) and (2.65). Then, the strong convergence results (2.59) and (2.63) are given by the Aubin-Lions-Simon Theorem (see for example Theorem II.5.16 in [20]). Furthermore, from (2.27), it holds

$$\left.\frac{\partial y_m}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0, \quad \text{a.e. on} \quad (0, T).$$

for all m. Thus (2.66) follows from (2.62). Next, we show that $(\mathbf{v}, y, w, \mathbf{u})$ satisfies the state equations (2.5), (2.6) of the non-smooth optimal control Problem 2.1. We have that $(\mathbf{v}_m, y_m, w_m, \mathbf{u}_m)$ in (2.56)-(2.64) is such that

$$\int_0^T \left[(\mathbf{v}_{mt}, \boldsymbol{\psi}) + \nu(\nabla \mathbf{v}_m, \nabla \boldsymbol{\psi}) - (\mathbf{u}_m, \boldsymbol{\psi})\right] dt = 0, \tag{2.67a}$$

$$\mathbf{v}_m(0) = \mathbf{v}_0, \quad \text{in } \Omega, \tag{2.67b}$$

$$\int_0^T \left[\langle y_{mt}, \eta \rangle_{H^{1*},H^1} + \gamma \left(\nabla w_m, \nabla \eta \right) - \left(y_m, \mathbf{v}_m \cdot \nabla \eta \right) \right] \, dt = 0, \tag{2.67c}$$

$$y_m(0) = y_0, \quad \text{in } \Omega, \tag{2.67d}$$

$$\int_0^T \left[\left(w_m + y_m, \theta \right) - \varepsilon^2 \left(\nabla y_m, \nabla \theta \right) - \frac{1}{\delta_m} \left(\beta_{\delta_m} \left(y_m \right), \theta \right) \right] \, dt = 0, \tag{2.67e}$$

for all $\boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}})$, $\eta, \theta \in L^2(H^1)$. As $m \to +\infty$ the convergence of (2.67a) to (2.5a) is straightforward. The same holds concerning the convergence of the linear terms in (2.67c) to the corresponding terms in (2.6a). The convergence of the nonlinear term in (2.67c) to the corresponding term in (2.6a), is derived noting that, as $\to +\infty$,

$$\begin{aligned}
& \Big| \int_0^T \left(y_m, \mathbf{v}_m \cdot \nabla \eta \right) \, dt - \int_0^T \left(y, \mathbf{v} \cdot \nabla \eta \right) \, dt \Big| \\
\leq \; & \Big| \int_0^T \left(y_m - y, \mathbf{v}_m \cdot \nabla \eta \right) \, dt \Big| + \Big| \int_0^T \left(y, [\mathbf{v}_m - \mathbf{v}] \cdot \nabla \eta \right) \, dt \Big| \\
\leq \; & C \Big[\| y_m - y \|_{L^\infty(H^1)}^{\frac{1}{2}} \| \mathbf{v}_m \|_{L^\infty(\boldsymbol{\mathcal{D}})}^{\frac{1}{2}} \| y_m - y \|_{L^2(L^2)}^{\frac{1}{2}} \| \mathbf{v}_m \|_{L^2(\boldsymbol{\mathcal{S}})}^{\frac{1}{2}} \\
+ \; & \| y \|_{L^\infty(H^1)}^{\frac{1}{2}} \| \mathbf{v}_m - \mathbf{v} \|_{L^\infty(\boldsymbol{\mathcal{D}})}^{\frac{1}{2}} \| y \|_{L^2(L^2)}^{\frac{1}{2}} \| \mathbf{v}_m - \mathbf{v} \|_{L^2(\boldsymbol{\mathcal{S}})}^{\frac{1}{2}} \Big] \| \eta \|_{L^2(H^1)} \to 0,
\end{aligned}$$

where we used (2.58), (2.59), (2.61) and (2.62). Next, given $\theta \in \mathcal{K}$, by the definition (2.12) of β_δ, we have $\beta_\delta(\theta) \equiv 0$. Then, by the property (2.17), we get

$$\begin{aligned}
& \int_0^T \left[- \left(w_m + y_m, \theta - y_m \right) + \varepsilon^2 \left(\nabla y_m, \nabla \theta - \nabla y_m \right) \right] \, dt \\
= \; & \frac{1}{\delta_m} \int_0^T \left[\left(\beta_{\delta_m}(\theta) - \beta_{\delta_m}(y_m), \theta - y_m \right) \right] \, dt \geq 0,
\end{aligned} \tag{2.68}$$

for all $\theta \in \mathcal{K}$. Hence, using the convergence properties of y_m, w_m, from (2.68), we derive (2.6b). In order to show (2.6c), we define a function $\hat{f} : \mathbb{R} \to \mathbb{R}$,

$$\hat{f}(r) = \lim_{\delta \to 0^+} \beta_\delta(r) = \begin{cases} r + 1, & \text{if} \quad r \leq -1, \\ 0, & \text{if} \quad |r| \leq 1, \\ r - 1, & \text{if} \quad r \geq 1. \end{cases} \tag{2.69}$$

We note that $\hat{f}$ is a Lipschitz function such that

$$|\hat{f}(r) - \beta_\delta(r)| \leq \frac{\delta}{2}, \quad |\hat{f}(r) - \hat{f}(s)| \leq |r - s|, \quad \forall r, s \in \mathbb{R}. \tag{2.70}$$

From (2.28) in Theorem (2.3) we have that

$$\| \beta_{\delta_m}(y_m) \|_{L^2(L^2)} \leq C(\mathbf{u}_m) \, \delta_m \leq C_1 \, \delta_m,$$

and therefore

$$\lim_{m \to +\infty} \| \beta_{\delta_m}(y_m) \|_{L^2(L^2)} = 0. \tag{2.71}$$

Thus, using (2.70) and (2.71), it holds

$$\begin{aligned}
&\Big|\int_0^T \left(\hat{f}(y),\theta\right)\,dt\Big| \\
\leq\ &\int_0^T \Big[\|\hat{f}(y)-\hat{f}(y_m)\|+\|\hat{f}(y_m)-\beta_{\delta_m}(y_m)\|+\|\beta_{\delta_m}(y_m)\|\Big]\,\|\theta\|\,dt \\
\leq\ &C\left[\|y-y_m\|_{L^2(L^2)}+\delta_m\right]\|\theta\|_{L^2(L^2)}.
\end{aligned}$$

for all $\theta\in L^2(L^2)$. Therefore, from the strong convergence result (2.62), we derive (2.6d). Finally, as well as in the proof of Theorem 2.5, we can realize that

$$\mathbf{v}(0)=\mathbf{v}_0,\quad y(0)=y_0.$$

So, the proof is concluded. □

In Lemma 2.12 below, we show the properties of the solution of the state equations (2.5)-(2.6) of the non-smooth optimal control Problem 2.1.

Lemma 2.12. *For any given* $\mathbf{u}\in L^2(\mathbf{L}^2)$*, the state equations (2.5)-(2.6) of the non-smooth optimal control Problem 2.1 have a unique solution* $(\mathbf{v},y,w)$*, which is such that*

$$\text{(2.72)}\quad \mathbf{v}\in H^1(\boldsymbol{\mathcal{S}})\cap L^\infty(\boldsymbol{\mathcal{D}}),\quad y\in W_0\cap L^\infty(H_0)\cap L^2\left(H^2\right)\cap\mathcal{K},\quad w\in L^2\left(H^1\right),$$

and satisfies the estimate

$$\text{(2.73)}\quad \|\mathbf{v}_t\|^2_{L^2(\mathbf{L}^2)}+\|\mathbf{v}\|^2_{L^\infty(\boldsymbol{\mathcal{D}})}+\|y\|^2_{W_0}+\|y\|^2_{L^\infty(H_0)}+\|y\|^2_{L^2(H^2)}+\|w\|^2_{L^2(H^1)}\leq C(\mathbf{u}),$$

where $C(\mathbf{u})$ *is a constant that depends continuously on* $\|\mathbf{u}\|_{L^2(L^2)}$ *and data problem (initial conditions and constant parameters).*

Proof. Applying Theorem 2.11 in the case of a sequence $\{\mathbf{u}_n\}_{n\in\mathbb{N}}\subset L^2(\mathbf{L}^2)$, such that

$$\mathbf{u}_n(t)=\mathbf{u}\in L^2\left(\mathbf{L}^2\right),\quad\forall n\in\mathbb{N},\quad\forall t\in(0,T),$$

we derive the existence of a solution $(\mathbf{v},y,w)$ of (2.5)-(2.6) which satisfies (2.72) and (2.73). Next, we show the uniqueness of this solution. From the same arguments used in the proof of Lemma 2.3 (see Appendix B), we get that $\mathbf{v}\in H^1(\boldsymbol{\mathcal{S}})\cap L^\infty(\boldsymbol{\mathcal{D}})$ is unique. Then, we prove the uniqueness of $y\in W_0\cap L^\infty(H_0)\cap L^2(H^2)\cap\mathcal{K}$ and $w\in L^2(H^1)$. We assume that for a given $\mathbf{v}$, there are two solutions (y_1,w_1), (y_2,w_2) of (2.6). Therefore, $d_y=y_2-y_1$ and $d_w=w_2-w_1$ satisfy

$$\text{(2.74)}\qquad -\gamma\int_0^T(\nabla d_w,\nabla\eta)\,dt=\int_0^T\left[\langle d_{yt},\eta\rangle_{H^{1*},H^1}-(d_y,\mathbf{v}\cdot\nabla\eta)\right]\,dt,$$
$$d_y(0)=0,$$

for all $\eta\in L^2(H^1)$ and

$$\text{(2.75)}\qquad \int_0^T e^{-\mu t}\left[-(d_w,d_y)+\varepsilon^2\|\nabla d_y\|^2\right]\,dt\leq\int_0^T e^{-\mu t}\|d_y\|^2\,dt,$$

where (2.75) is obtained setting in (2.6b), respectively, $\theta = e^{-\mu t} d_y + y_1 \in \mathcal{K}$ when the solution is (y_1, w_1) and $\theta = -e^{-\mu t} d_y + y_2 \in \mathcal{K}$ when the solution is (y_2, w_2) and then adding the equations obtained. Above $\mu > 0$ is a constant. From (2.74), (2.75) we can prove the uniqueness of y as well as in the proof of Theorem 2.3. In order to show the uniqueness of w, i.e. $d_w = 0$, we set $\eta = d_w$ in (2.74). In this way, we get

$$\|\nabla d_w\|_{L^2(L^2)} = 0. \tag{2.76}$$

Then, following [17], we can define a.e. in $(0,T)$,

$$\Omega_0\left(t\right) = \left\{x \in \Omega : \left|y\left(x,t\right)\right| < 1\right\}.$$

As $(y,1) = 0$, $\Omega_0(t)$ is not empty. Given $\phi \in \mathcal{C}_c^\infty\left(\Omega_0\left(t\right)\right)$ we consider $\theta_\pm = y \pm \sigma\phi$, with σ such that $\theta_\pm \in \mathcal{K}$. Substituting $\theta = \theta_\pm$ in (2.6c), we derive that

$$\varepsilon^2 \int_0^T \left(\nabla y, \nabla \phi\right)\, dt = \int_0^T \left(w + y, \phi\right), \quad \forall\, \phi \in \mathcal{C}_c^\infty\left(\Omega_0\left(t\right)\right), \tag{2.77}$$

and (2.77) holds for $w = w_1$ and for $w = w_2$. Hence,

$$\int_0^T \left(d_w, \phi\right)\, dt = 0, \quad \forall\, \phi \in \mathcal{C}_c^\infty\left(\Omega_0\left(t\right)\right). \tag{2.78}$$

From (2.76), we know that d_w is a constant. Then, using (2.78), we infer that $d_w = 0$. ☐

2.3.2. Minimizers of the Non-Smooth Optimal Control Problem

In Theorem 2.13 below, we show an essential property of the solutions of the non-smooth optimal control Problem 2.1: there exists a sequence of solutions of the regularized optimal control Problem 2.2, which converges to a solution of the non-smooth Problem 2.1, for the regularization parameter $\delta \to 0^+$.

Theorem 2.13. *Consider a sequence $\{\delta_n\}_{n\in\mathbb{N}} \subset \left(0, \frac{1}{4}\right)$ such that $\delta_n \to 0^+$ and the corresponding sequence of solutions of the regularized optimal control Problem 2.2,*

$$\left\{\left(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n\right)\right\}_{n\in\mathbb{N}} = \left\{\left(s_{\delta_n}\left(\bar{\mathbf{u}}_n\right), \bar{\mathbf{u}}_n\right)\right\}_{n\in\mathbb{N}} \subset \mathbf{X} \times L^2\left(\mathbf{L}^2\right).$$

Then, it is possible to extract a subsequence (labelled by index m), such that as $m \to +\infty$

$$\left(\bar{\mathbf{x}}_m, \bar{\mathbf{u}}_m\right) \rightharpoonup \left(\bar{\mathbf{x}}, \bar{\mathbf{u}}\right), \quad \text{in} \quad \mathbf{X} \times L^2\left(\mathbf{L}^2\right),$$

where $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ is a solution of the non-smooth optimal control Problem 2.1.

Proof. Given the sequences $\{\delta_n\}_{n\in\mathbb{N}}$, $\left\{\left(s_{\delta_n}\left(\bar{\mathbf{u}}_n\right), \bar{\mathbf{u}}_n\right)\right\}_{n\in\mathbb{N}}$ and some $\mathbf{u} \in L^2\left(\mathbf{L}^2\right)$, by the definition of the cost functional J and the results of Lemma 2.3

$$\frac{\alpha}{2}\|\bar{\mathbf{u}}_n\|^2_{L^2(\mathbf{L}^2)} \le J\left(s_{\delta_n}\left(\bar{\mathbf{u}}_n\right), \bar{\mathbf{u}}_n\right) \le J\left(s_{\delta_n}\left(\mathbf{u}\right), \mathbf{u}\right) \le \|y_d\|^2_{L^2(L^2)} + C\left(\mathbf{u}\right) + \frac{\alpha}{2}\|\mathbf{u}\|^2_{L^2(\mathbf{L}^2)},$$

for all $n \in \mathbb{N}$. Therefore the sequence $\{\bar{\mathbf{u}}_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbf{L}^2)$ and using Theorem 2.11, we can consider a subsequence (labelled by index m) such that

$$(s_{\delta_m}(\bar{\mathbf{u}}_m), \bar{\mathbf{u}}_m) = (\bar{\mathbf{x}}_m, \bar{\mathbf{u}}_m) \rightharpoonup (\bar{\mathbf{x}}, \bar{\mathbf{u}}), \quad \text{in} \quad \mathbf{X} \times L^2\left(\mathbf{L}^2\right),$$

where $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ is a solution of the state equations (2.5)-(2.6) of the non-smooth optimal control Problem 2.1. It remains to prove that $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ solves the optimal control Problem 2.1. Let $(\mathbf{x}^*, \mathbf{u}^*)$ be a solution of Problem 2.1. Considering the sequence $\{(s_{\delta_m}(\mathbf{u}^*), \mathbf{u}^*)\}_m$, by theorem 2.11, there exists a further subsequence (labelled by index l), such that

$$(s_{\delta_l}(\mathbf{u}^*), \mathbf{u}^*) \rightharpoonup (\mathbf{x}^*, \mathbf{u}^*), \quad \text{in} \quad \mathbf{X} \times L^2\left(\mathbf{L}^2\right),$$

as $l \to +\infty$. Then, using that $(\mathbf{x}^*, \mathbf{u}^*)$ is a solution of (2.1) and the weak lower semicontinuity of J, we have

(2.79)
$$J(\mathbf{x}^*, \mathbf{u}^*) \leq J(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \leq \liminf_{m\to+\infty} J(s_{\delta_m}(\bar{\mathbf{u}}_m), \bar{\mathbf{u}}_m) \leq \limsup_{m\to+\infty} J(s_{\delta_m}(\bar{\mathbf{u}}_m), \bar{\mathbf{u}}_m).$$

Obviously

$$\limsup_{m\to+\infty} J(s_{\delta_m}(\bar{\mathbf{u}}_m), \bar{\mathbf{u}}_m) = \limsup_{l\to+\infty} J(s_{\delta_l}(\bar{\mathbf{u}}_l), \bar{\mathbf{u}}_l),$$

and furthermore

$$J(s_{\delta_l}(\bar{\mathbf{u}}_l), \bar{\mathbf{u}}_l) \leq J(s_{\delta_l}(\mathbf{u}^*), \mathbf{u}^*),$$

because $\{(s_{\delta_l}(\bar{\mathbf{u}}_l), \bar{\mathbf{u}}_l)\}_l$ is a sequence of minimizers for the regularized optimal control Problem 2.2. So

$$\limsup_{m\to+\infty} J(s_{\delta_m}(\bar{\mathbf{u}}_m), \bar{\mathbf{u}}_m) \leq \limsup_{l\to+\infty} J(s_{\delta_l}(\mathbf{u}^*), \mathbf{u}^*) = J(\mathbf{x}^*, \mathbf{u}^*). \tag{2.80}$$

Using together (2.79) and (2.80), we infer

$$J(\mathbf{x}^*, \mathbf{u}^*) \leq J(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \leq J(\mathbf{x}^*, \mathbf{u}^*),$$

which means that $J(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ is a solution of the non-smooth optimal control Problem 2.1. This concludes the proof. □

In the following, we state an equivalent formulation of the non-smooth optimal control Problem 2.1. We introduce two Lagrange multipliers $\beta_r, \beta_l \in L^2(L^2)$ in the state equations so that we obtain a problem which has the form of a mathematical program with complementarity constraints. In the next sections we will observe that the Lagrange multipliers β_r, β_l will be linked to the adjoint variables which satisfy the first order optimality conditions for the non-smooth Problem 2.1.
We define the space

$$\mathbf{R} = \mathbf{X} \times L^2\left(L^2\right) \times L^2\left(L^2\right),$$

with elements

$$\mathbf{r} = (\mathbf{x}, \beta_r, \beta_l),$$

and

$$\mathcal{K}^+ = \left\{\varphi \in L^2\left(L^2\right) : \ \varphi \geq 0 \text{ a.e. on } \Omega_T\right\}.$$

Furthermore, we consider the cost functional $\tilde{J} : \mathbf{R} \times L^2(\mathbf{L}^2) \to \mathbb{R}$, such that

$$\tilde{J}(\mathbf{r}, \mathbf{u}) \equiv J(\mathbf{x}, \mathbf{u}).$$

Thus, we consider the following problem:

Problem 2.14. *Find $(\bar{\mathbf{r}}, \bar{\mathbf{u}}) \in \mathbf{R} \times L^2(\mathbf{L}^2)$ such that*

$$\min_{(\mathbf{r},\mathbf{u}) \in \mathbf{R} \times L^2(\mathbf{L}^2)} \tilde{J}(\mathbf{r}, \mathbf{u}) = \tilde{J}(\bar{\mathbf{r}}, \bar{\mathbf{u}}),$$

subject to

$$\int_0^T \left[(\mathbf{v}_t, \boldsymbol{\psi}) + \nu (\nabla \mathbf{v}, \nabla \boldsymbol{\psi}) - (\mathbf{u}, \boldsymbol{\psi}) \right] dt = 0, \tag{2.81a}$$

$$\mathbf{v}(0) = \mathbf{v}_0, \qquad \text{in } \Omega, \tag{2.81b}$$

$$\int_0^T \left[\langle y_t, \eta \rangle_{H^{1*}, H^1} + \gamma (\nabla w, \nabla \eta) - (y, \mathbf{v} \cdot \nabla \eta) \right] dt = 0, \tag{2.82a}$$

$$y(0) = y_0, \qquad \text{in } \Omega, \tag{2.82b}$$

$$\int_0^T \left[-(w + y, \theta) + \varepsilon^2 (\nabla y, \nabla \theta) + (\beta, \theta) \right] dt = 0, \tag{2.82c}$$

$$y \in \mathcal{K}, \tag{2.82d}$$

$$\beta = \beta_r - \beta_l, \quad \text{with} \quad \beta_r,\ \beta_l \in \mathcal{K}^+, \tag{2.82e}$$

$$\int_0^T (\beta_r, 1 - y)\ dt = 0, \tag{2.82f}$$

$$\int_0^T (\beta_l, 1 + y)\ dt = 0. \tag{2.82g}$$

for all $\boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}})$, $\eta, \theta \in L^2(H^1)$.

Lemma 2.15. *Problem 2.1 and Problem 2.14 are equivalent.*

Proof. We proceed in the following way: we show that (2.81)-(2.82) can be obtained as limit of the state equations (2.24), (2.25) of the regularized optimal control Problem 2.2, for the regularization parameter $\delta \to 0^+$. Using Theorem 2.11, we need just to prove that there exist y, w which together β_r, β_l satisfy (2.82c), (2.82e)-(2.82g). We can write the regularized state equation (2.25c) in the following way

$$\int_0^T \left[-(w + y, \theta) + \varepsilon^2 (\nabla y, \nabla \theta) + \frac{1}{\delta} \left(\beta_{r\delta}(y) - \beta_{l\delta}(y), \theta \right) \right] dt = 0, \tag{2.83}$$

where,

$$\frac{1}{\delta} \beta_{r\delta}(s) := \begin{cases} 0, & \text{if } s \leq 1, \\ \dfrac{1}{2\delta^2} (s-1)^2, & \text{if } 1 < s < 1 + \delta, \\ \dfrac{1}{\delta} \left[s - \left(1 + \dfrac{\delta}{2} \right) \right], & \text{if } s \geq 1 + \delta. \end{cases} \tag{2.84}$$

$$\frac{1}{\delta}\beta_{l\delta}(s) := \begin{cases} -\dfrac{1}{\delta}\left[s + \left(1+\dfrac{\delta}{2}\right)\right], & \text{if } s \leq -1-\delta, \\ \dfrac{1}{2\delta^2}(s+1)^2, & \text{if } -1-\delta < s < -1, \\ 0, & \text{if } -1 \leq s, \end{cases} \tag{2.85}$$

and

$$\begin{aligned} &\beta_\delta(s) = \beta_{r\delta}(s) - \beta_{l\delta}(s), \\ &\beta_{r\delta}(s)\,\beta_{l\delta}(s) = 0, \\ &\beta_{r\delta}(s) \geq 0,\ \beta_{l\delta}(s) \geq 0, \quad \forall s \in \mathbb{R}, \quad \forall \delta \in \left(0, \frac{1}{4}\right). \end{aligned}$$

As in Theorem 2.11, given $\mathbf{u} \in L^2(\mathbf{L}^2)$, we consider a sequence $\{\delta_n\}_{n\in\mathbb{N}} \subset \left(0, \frac{1}{4}\right)$ such that $\delta_n \to 0^+$ and the corresponding sequence of solution of the regularized state equations $\{(\mathbf{v}_n, y_n, w_n)\}_{n\in N} \subset \mathbf{W}_0 \times W_0 \times L^2(H^1)$. By (2.28) in Lemma 2.3, there exists a subsequence (labelled with index m), such that

$$\frac{1}{\delta_m}\beta_{r\delta_m}(y_m) \rightharpoonup \beta_r, \quad \text{in } L^2(L^2), \tag{2.86}$$

$$\frac{1}{\delta_m}\beta_{l\delta_m}(y_m) \rightharpoonup \beta_l, \quad \text{in } L^2(L^2), \tag{2.87}$$

$$\beta_r,\ \beta_l \in \mathcal{K}^+, \tag{2.88}$$

$$y_m \to y, \quad \text{in } L^2(L^2_0), \tag{2.89}$$

$$w_m \rightharpoonup w, \quad \text{in } L^2(H^1), \tag{2.90}$$

$$y \in \mathcal{K}, \tag{2.91}$$

and (y, w, β) satisfies (2.82c). In order to prove (2.82f), (2.82g), using (2.86), (2.87), (2.89), we note that,

$$\frac{1}{\delta_m}\int_0^T \left(\beta_{r\delta_m}(y_m), 1-y_m\right)\,dt \ \to\ \int_0^T (\beta_r, 1-y)\,dt, \tag{2.92}$$

$$\frac{1}{\delta_m}\int_0^T \left(\beta_{l\delta_m}(y_m), 1+y_m\right)\,dt \ \to\ \int_0^T (\beta_l, 1+y)\,dt \tag{2.93}$$

as $m \to +\infty$. Furthermore, from (2.88), (2.91), it follows that

$$\int_0^T (\beta_r, 1-y)\,dt \geq 0, \quad \int_0^T (\beta_l, 1+y)\,dt \geq 0. \tag{2.94}$$

Conversely, from the definitions of $\beta_{r\delta_m}, \beta_{l\delta_m}$ in (2.85), (2.84),

$$\frac{1}{\delta_m}\int_0^T \left(\beta_{r\delta_m}(y_m), 1-y_m\right)\,dt \leq 0, \quad \frac{1}{\delta_m}\int_0^T \left(\beta_{l\delta_m}(y_m), 1+y_m\right)\,dt \leq 0. \tag{2.95}$$

Thus, from (2.92)-(2.95), we obtain

$$\int_0^T (\beta_r, 1-y)\,dt = 0, \quad \int_0^T (\beta_l, 1+y)\,dt = 0.$$

Hence, the proof is concluded. □

2.3.3. Optimality Conditions of the Non-Smooth Optimal Control Problem

In this section we show the main result in this Chapter: we derive the first order optimality conditions of the Problem 2.14 (and hence for the equivalent non-smooth optimal control Problem 2.1) as limit of the optimality conditions (2.44), (2.45), (2.46) of the regularized Problem 2.2, for the regularization parameter $\delta \to 0^+$.

Theorem 2.16. *Let* $\{\delta_n\}_{n\in\mathbb{N}} \subset \left(0, \frac{1}{4}\right)$ *be a sequence such that* $\delta_n \to 0^+$ *and*

$$\{(\mathbf{x}_n, \mathbf{u}_n)\}_{n\in\mathbb{N}} = \{(\mathbf{v}_n, y_n, w_n, \mathbf{u}_n)\}_{n\in\mathbb{N}} \subset \mathbf{X} \times L^2\left(\mathbf{L}^2\right).$$

the corresponding sequence of solutions of the regularized optimal control Problem 2.2. Further, let

$$\{\mathbf{q}_n\}_{n\in\mathbb{N}} = \{(\mathbf{q}_{\mathbf{v}n}, q_{yn}, q_{wn}, \mathbf{q}_{\mathbf{v}n}(0), q_{y0n})\}_{n\in\mathbb{N}} \subset \mathbf{Z}^*,$$

be the sequence of the adjoint variables such that triple $\mathbf{x}_n, \mathbf{u}_n, \mathbf{q}_n$ *satisfies the optimality conditions* (2.44), (2.45), (2.46) *of the regularized optimal control Problem 2.2 for all* $n \in \mathbb{N}$. *Then, there exists a subsequence (labelled by an index* m*)* $\{(\mathbf{x}_m, \mathbf{u}_m, \mathbf{q}_m)\}_m$*, a solution of the non-smooth optimal control Problem 2.14*

$$(\mathbf{r}, \mathbf{u}) = (\mathbf{v}, y, w, \beta_r, \beta_l, \mathbf{u}) \in \mathbf{R} \times L^2\left(\mathbf{L}^2\right),$$

and a set of variables

$$(\mathbf{q}_\mathbf{v}, q_y, q_w, \mathbf{q}_\mathbf{v}(0), q_{y0}, \lambda) \in \mathbf{Z}^* \times W_0^*,$$

such that, as $m \to +\infty$*,*

$$\mathbf{v}_m \rightharpoonup \mathbf{v}, \quad \text{in } H^1(\boldsymbol{\mathcal{S}}), \tag{2.96}$$

$$\mathbf{v}_m \overset{*}{\rightharpoonup} \mathbf{v}, \quad \text{in } L^\infty(\boldsymbol{\mathcal{D}}) \tag{2.97}$$

$$\mathbf{v}_m \to \mathbf{v}, \quad \text{in } L^2(\boldsymbol{\mathcal{S}}), \tag{2.98}$$

$$y_m \rightharpoonup y, \quad \text{in } W_0, \tag{2.99}$$

$$y_m \overset{*}{\rightharpoonup} y, \quad \text{in } L^\infty(H_0) \tag{2.100}$$

$$y_m \rightharpoonup y, \quad \text{in } L^2\left(H^2\right), \tag{2.101}$$

$$y_m \to y, \quad \text{in } L^2(H_0), \tag{2.102}$$

$$w_m \rightharpoonup w, \quad \text{in } L^2\left(H^1\right) \tag{2.103}$$

$$\mathbf{q}_{\mathbf{v}m} \rightharpoonup \mathbf{q}_\mathbf{v}, \quad \text{in } H^1(\boldsymbol{\mathcal{S}}), \tag{2.104}$$

$$\mathbf{q}_{\mathbf{v}m} \overset{*}{\rightharpoonup} \mathbf{q}_\mathbf{v}, \quad \text{in } L^\infty(\boldsymbol{\mathcal{D}}) \tag{2.105}$$

$$\mathbf{q}_{\mathbf{v}m} \to \mathbf{q}_\mathbf{v}, \quad \text{in } L^2(\boldsymbol{\mathcal{S}}), \tag{2.106}$$

$$q_{ym} \overset{*}{\rightharpoonup} q_y, \quad \text{in } L^\infty(H_0) \tag{2.107}$$

$$q_{ym} \rightharpoonup q_y, \quad \text{in } L^2\left(H^2\right) \tag{2.108}$$

$$q_{y0m} \rightharpoonup q_{y0}, \quad \text{in } H_0, \tag{2.109}$$

$$q_{wm} \rightharpoonup q_w, \quad \text{in } L^2\left(H^1\right), \tag{2.110}$$

(2.111) $$\mathbf{u}_m \rightharpoonup \mathbf{u}, \quad \text{in } H^1(\boldsymbol{\mathcal{S}}),$$

(2.112) $$\mathbf{u}_m \overset{*}{\rightharpoonup} \mathbf{u}, \quad \text{in } L^\infty(\boldsymbol{\mathcal{D}})$$

(2.113) $$\mathbf{u}_m \to \mathbf{u}, \quad \text{in } L^2(\boldsymbol{\mathcal{S}}),$$

(2.114) $$\frac{1}{\delta_m}\beta_{\delta_m}(y_m) \rightharpoonup \beta = \beta_r - \beta_l, \quad \text{in } L^2\left(L^2\right)$$

(2.115) $$\frac{1}{\delta_m}\beta'_{\delta_m}(y_m)\, q_{wm} \overset{*}{\rightharpoonup} \lambda, \quad \text{in } W_0^*.$$

Furthermore

$$(\mathbf{v}, y, w, \beta_r, \beta_l, \mathbf{u}, \mathbf{q_v}, q_y, q_w, \lambda),$$

satisfies the following system of optimality conditions

(2.116a) $$\int_0^T \left[(\mathbf{v}_t, \boldsymbol{\psi}) + \nu(\nabla\mathbf{v}, \nabla\boldsymbol{\psi}) - (\mathbf{u}, \boldsymbol{\psi})\right]\, dt = 0,$$

(2.116b) $$\mathbf{v}(0) = \mathbf{v}_0, \quad \text{in } \Omega$$

(2.116c) $$\int_0^T \left[\langle y_t, \eta\rangle_{H^{1*},H^1} + \gamma(\nabla w, \nabla\eta) - (y, \mathbf{v}\cdot\nabla\eta)\right]\, dt = 0,$$

(2.116d) $$y(0) = y_0, \quad \text{in } \Omega$$

(2.116e) $$\int_0^T \left[-(w + y, \theta) + \varepsilon^2(\nabla y, \nabla\theta) + (\beta_r - \beta_l, \theta)\right]\, dt = 0,$$

(2.116f) $$y \in \mathcal{K},$$

(2.116g) $$\beta_r,\ \beta_l \in \mathcal{K}^+,$$

(2.116h) $$\int_0^T (\beta_r, 1 - y)\, dt = 0,$$

(2.116i) $$\int_0^T (\beta_l, 1 + y)\, dt = 0,$$

for all $\boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}}),\ \eta, \theta \in L^2(H^1)$,

(2.117a) $$\int_0^T \left[-(\mathbf{q}_{\mathbf{v}t}, \boldsymbol{\psi}) + (\nabla\mathbf{q_v}, \nabla\boldsymbol{\psi}) - (y, \nabla q_y \cdot \boldsymbol{\psi})\right]\, dt = 0,$$

(2.117b) $$\mathbf{q_v}(T) = 0,$$

(2.117c) $$\int_0^T \left[\langle \eta_t, q_y\rangle_{H_0^*,H_0} - \varepsilon^2(\nabla q_w, \nabla\eta) + (q_w, \eta) - (\mathbf{v}\cdot\nabla q_y, \eta) + (y - y_d, \eta)\right]\, dt + (q_{y0}, \eta(0)) - \langle\lambda, \eta\rangle_{W_0^*,W_0} = 0,$$

(2.117d) $$\int_0^T \left[\ (q_w, \theta) + \gamma(\nabla q_y, \nabla\theta)\ \right]\, dt = 0,$$

for all $\boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}}), \eta \in W_0, \theta \in L^2(H^1)$, *and*

(2.118) $$\int_0^T (\alpha\mathbf{u} - \mathbf{q_v}, \boldsymbol{\varphi})\, dt = 0.$$

for all $\boldsymbol{\varphi} \in L^2(\mathbf{L}^2)$. *Moreover, if* $\frac{1}{\delta_m}\beta'_{\delta_m}(y_m)\, q_{wm}$ *is bounded in* $L^2(H^{1*})$, *then for all Lipschitz functions* $g : \mathbb{R} \to \mathbb{R}$ *such that* $g(-1) = g(1) = 0$, *we get*

$$\lim_{m\to+\infty} \int_0^T \left(\frac{1}{\delta_m}\beta'_{\delta_m}(y_m)\, q_{wm}, g(y_m) \right) dt = 0, \tag{2.119a}$$

$$\lim_{m\to+\infty} \int_0^T \left(\frac{1}{\delta_m}\beta_{\delta_m}(y_m), q_{wm} \right) dt = 0, \tag{2.119b}$$

$$\liminf_{m\to+\infty} \int_0^T \left(\frac{1}{\delta_m}\beta'_{\delta_m}(y_m)\, q_{wm}, q_{wm} \right) dt \geq 0, \tag{2.119c}$$

Proof. Given the sequence of solutions $\{(\mathbf{v}_n, y_n, w_n, \mathbf{u}_n)\}_{n\in\mathbb{N}}$ of the regularized optimal control Problem 2.2, we can consider the sequence of the adjoint variables $\{(\mathbf{q}_{\mathbf{v}n}, q_{yn}, q_{wn})\}_{n\in\mathbb{N}}$, such that $\mathbf{v}_n, y_n, w_n, \mathbf{u}_n, \mathbf{q}_{\mathbf{v}n}, q_{yn}, q_{wn}$ solve, for all $n \in \mathbb{N}$, the optimality conditions (2.44)-(2.46) of the regularized optimal control Problem 2.2. From the results of Lemmas 2.3, 2.10, Theorem 2.11, Lemma 2.12, Theorem 2.13 and Lemma 2.15, we derive the existence of a convergent subsequence (labelled by an index m) $\{(\mathbf{v}_m, y_m, w_m, \mathbf{u}_m, \mathbf{q}_{\mathbf{v}m}, q_{ym}, q_{wm})\}_m$ and a set of limit variables $\mathbf{v}, y, w, \beta_r, \beta_l, \mathbf{u}, \mathbf{q}_\mathbf{v}, q_y, q_w$, such that

- the functions $\mathbf{v}_m, y_m, w_m, \mathbf{u}_m, \mathbf{q}_{\mathbf{v}m}, q_{ym}, q_{wm}$ are, for all m, solution of the optimality conditions (2.44)-(2.46) of the regularized optimal control Problem 2.2;
- the limits (2.96)-(2.114) above are satisfied;
- the state variables $\mathbf{v}, y, w, \beta_r, \beta_l, \mathbf{u}, \mathbf{q}_\mathbf{v}$ satisfy the optimality conditions (2.116), (2.118) above.

Next, we show that there exists $\lambda \in W_0^*$ as a result of the limit (2.114) and that $\mathbf{v}, y, \beta_r, \beta_l, \mathbf{u}, \mathbf{q}_\mathbf{v}, q_y, q_w, \lambda$ are solution of the optimality conditions (2.117). It hold, for all m,

$$\int_0^T \left[-(\mathbf{q}_{\mathbf{v}mt}, \boldsymbol{\psi}) + (\nabla \mathbf{q}_{\mathbf{v}m}, \nabla\boldsymbol{\psi}) - (y_m, \nabla q_{ym}\cdot\boldsymbol{\psi}) \right] dt = 0, \tag{2.120a}$$

$$\mathbf{q}_{\mathbf{v}m}(T) = 0, \tag{2.120b}$$

$$\begin{aligned}\int_0^T &\Big[\langle \eta_t, q_{ym}\rangle_{H_0^*,H_0} - \varepsilon^2 (\nabla q_{wm}, \nabla\eta) + (q_{wm}, \eta) - (\mathbf{v}_m\cdot\nabla q_{ym}, \eta) \\ &+ (y_m - y_d, \eta) - \frac{1}{\delta_m}\left(\beta'_{\delta_m}(y_m)\, q_{wm}, \eta\right) \Big] dt + (q_{y0m}, \eta(0)) = 0,\end{aligned} \tag{2.120c}$$

$$\int_0^T \left[(q_{wm}, \theta) + \gamma(\nabla q_{ym}, \nabla\theta) \right] dt = 0, \tag{2.120d}$$

for all $\boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}}), \eta \in W_0, \theta \in L^2(H^1), \boldsymbol{\varphi} \in L^2(\mathbf{L}^2)$. From (2.96)-(2.110) we infer that all linear terms in (2.120) converge to the corresponding limits in (2.117). For the nonlinear terms, we derive that they converge observing that,

$$\left| \int_0^T (y_m, \nabla q_{ym}\cdot\boldsymbol{\psi})\, dt - \int_0^T (y, \nabla q_y\cdot\boldsymbol{\psi})\, dt \right|$$

$$
\begin{aligned}
&\leq \left| \int_0^T (y_m - y, \nabla q_{ym} \cdot \boldsymbol{\psi})\ dt \right| + \left| \int_0^T (y, [\nabla q_{ym} - \nabla q_y] \cdot \boldsymbol{\psi})\ dt \right| \\
&\qquad \leq C\ \|y_m - y\|_{L^\infty(H^1)}^{\frac{1}{2}}\ \|q_{ym}\|_{L^\infty(H_0)}\ \|y_m - y\|_{L^2(L^2)}\ \|\boldsymbol{\psi}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})} \\
&\qquad\qquad + \left| \int_0^T (y, [\nabla q_{ym} - \nabla q_y] \cdot \boldsymbol{\psi})\ dt \right| \ \to\ 0,
\end{aligned}
$$

$$
\begin{aligned}
&\left| \int_0^T (\mathbf{v}_m \cdot \nabla q_{ym}, \eta)\ dt - \int_0^T (\mathbf{v} \cdot \nabla q_y, \eta)\ dt \right| \\
&\leq \left| \int_0^T ([\mathbf{v}_m - \mathbf{v}] \cdot \nabla q_{ym}, \eta)\ dt \right| + \left| \int_0^T (\mathbf{v} \cdot [\nabla q_{ym} - \nabla q_y], \eta)\ dt \right| \\
&\qquad \leq C\ \|\mathbf{v}_m - \mathbf{v}\|_{\mathbf{L}^\infty(\boldsymbol{\mathcal{D}})}^{\frac{1}{2}}\ \|q_{ym}\|_{L^\infty(H_0)}\ \|\mathbf{v}_m - \mathbf{v}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{S}})}\ \|\eta\|_{\mathbf{L}^2(H_0)} \\
&\qquad\qquad + \left| \int_0^T (\mathbf{v} \cdot [\nabla q_{ym} - \nabla q_y], \eta)\ dt \right| \ \to\ 0.
\end{aligned}
$$

as $m \to +\infty$. From the convergence of the terms in (2.120c), we infer that there exists $\lambda \in W_0$, such that

$$
\frac{1}{\delta_m} \beta'_{\delta_m}(y_m)\, q_{wm} \overset{*}{\rightharpoonup} \lambda,
$$

and that the optimality condition (2.117c) above holds. Furthermore, with $\boldsymbol{\psi} = t/T \cdot \boldsymbol{\xi}, \boldsymbol{\xi} \in \boldsymbol{\mathcal{S}}$, using integration by parts in time and (2.120b), we can write

$$
(\mathbf{q_v}(T), \boldsymbol{\xi}) = \int_0^T (\mathbf{q}_{\mathbf{v}mt} - \mathbf{q}_{\mathbf{v}t}, \boldsymbol{\psi})\ dt + \int_0^T (\mathbf{q}_{\mathbf{v}m} - \mathbf{q_v}, \boldsymbol{\psi}_t)\ dt \to 0,
$$

as $m \to +\infty$. Then $\mathbf{q_v}(T) = 0$. Finally, we prove the complementarity conditions (2.119a)-(2.119c). We define the following metric projection operator

$$
\mathcal{P}s = \begin{cases} -1 & \text{if} \quad s < -1, \\ s & \text{if} \quad -1 \leq s \leq 1, \\ 1 & \text{if} \quad s > 1. \end{cases} \tag{2.121}
$$

Then, with g Lipschitz and such that $g(-1) = g(1) = 0$, we derive

$$
\begin{aligned}
&\int_0^T \left(\frac{1}{\delta_m} \beta'_{\delta_m}(y_m)\, q_{wm}, g(y_m) \right)\ dt \\
= &\int_0^T \left(\frac{1}{\delta_m} \beta'_{\delta_m}(y_m)\, q_{wm}, g(y_m) - g(\mathcal{P}y_m) \right)\ dt + \int_0^T \left(\frac{1}{\delta_m} \beta'_{\delta_m}(y_m)\, q_{wm}, g(\mathcal{P}y_m) \right)\ dt = \\
= &\ I_1 + I_2.
\end{aligned}
$$

From the properties of β'_δ and g, it is easy to realize that $I_2 = 0$. Furthermore using the boundedness of $\frac{1}{\delta_m} \beta'_{\delta_m}(y_m)\, q_{wm}$ in $L^2(H^{1*})$ and the strong convergence of y_m to y in $L^2(H^1)$ (stated in (2.102)),

$$
I_1 \leq \left\| \frac{1}{\delta_m} \beta'_{\delta_m}(y_m)\, q_{wm} \right\|_{L^2(H^{1*})} \|g(y_m) - g(\mathcal{P}y_m)\|_{L^2(H^1)} \to 0,
$$

as $m \to +\infty$. This proves (2.119a). We have

$$\beta_\delta(s) = l_\delta(s)\,\beta'_\delta(s),$$

where

$$l_\delta(s) := \begin{cases} s+1+\dfrac{\delta}{2}, & \text{if } s \le -1-\delta, \\ \dfrac{1}{2}(s+1), & \text{if } -1-\delta < s < -1, \\ 0, & \text{if } -1 \le s \le 1, \\ \dfrac{1}{2}(s-1), & \text{if } 1 < s < 1+\delta, \\ s-\left(1+\dfrac{\delta}{2}\right), & \text{if } s \ge 1+\delta. \end{cases} \tag{2.122}$$

Thus

$$\int_0^T \left(\frac{1}{\delta_m}\beta_{\delta_m}(y_m), q_{wm}\right) dt = \int_0^T \left(\frac{1}{\delta_m}\beta'_{\delta_m}(y_m)\,q_{wm}, l_{\delta_m}(y_m)\right) dt. \tag{2.123}$$

l_δ is a Lipschitz continuous function with constant 1. Furthermore $l_{\delta_m}(y) = 0$, for all m. Then,

$$\|l_{\delta_m}(y_m)\|_{L^2(L^2)} = \|l_{\delta_m}(y_m) - l_{\delta_m}(y)\|_{L^2(L^2)} \le \|y_m - y\|_{L^2(L^2)} \to 0,$$

as $m \to +\infty$. Moreover (see Theorem 4.6 in [54]),

$$\begin{aligned} \|\nabla l_{\delta_m}(y_m)\|_{L^2(L^2)} &= \|l'_{\delta_m}(y_m)\nabla y_m\|_{L^2(L^2)} \le \|\hat{f}'(y_m)\nabla y_m\|_{L^2(L^2)} \le \\ &\le \|\nabla \hat{f}(y_m)\|_{L^2(L^2)} = \|\nabla(y_m - \mathcal{P}y_m)\|_{L^2(L^2)} \to 0, \end{aligned}$$

as $m \to +\infty$, where $\hat{f}$ is the function defined in (2.69). So, $l_{\delta_m}(y_m)$ strongly converges to zero in $L^2(H^1)$. Then, using the boundedness of $\frac{1}{\delta_m}\beta'_{\delta_m}(y_m)\,q_{wm}$ in $L^2(H^{1*})$ in (2.123), we have (2.119b). Finally, by definition

$$\int_0^T \left(\frac{1}{\delta_m}\beta'_{\delta_m}(y_m)\,q_{wm}, q_{wm}\right) dt \ge 0,$$

for all m. Consequently (2.119c) holds. □

Remark 2.17. The complementarity conditions (2.119a)-(2.119c) establish a connection between the state variables β_r, β_l and the variable λ. We will show that, at discrete level, these complementarity conditions will be essential for the numerical solution of the non-smooth optimal control problem.

Remark 2.18. Equations (2.116)-(2.119) in Theorem 2.16, are a set of first optimality conditions for the non-smooth optimal control Problem 2.1 and they represent a function space version of the so-called C-Stationarity conditions [75] (see also [51], [54]).

3. Optimal Control of the Discrete Non-Smooth Cahn-Hilliard-Stokes System

3.1. Introduction

In this Chapter, we study the fully discretized version (in space and time) of the non-smooth optimal control Problem 2.1. We adapt the analysis from Chapter 2 to the discrete setting and show that the discrete problem converges to the continuous one, as the discretization parameters go to zero.
Technical details of the discretization are collected in Appendix A.3. In particular, we denote with $h, k = T/N$, respectively, the space and time discretization parameters, which are defined in Appendix A.3.1. Also the definitions of the discrete function spaces $\mathbf{S}_h, \mathbf{V}_h, \mathbf{D}_h, P_h, Y_h$ are given in Appendix A.3.1. Moreover, if Z_h is a discrete functions space, given $Z^n \in Z_h$ for $n = 1, \dots, N$, we denote by the corresponding calligraphic letter the associated vector variable

$$\mathcal{Z} = (Z^n)_{n=1}^N \in Z_h^N,$$

and with $d_t Z^n$ the discrete time derivative at time level n,

$$d_t Z^n = \frac{Z^n - Z^{n-1}}{k}.$$

We use $(\cdot, \cdot)_h$ to denote the mass-lumped scalar product defined in (A.29). We define the following discrete spaces

$$\mathbf{X}_{h,k} = \mathbf{V}_h^{N+1} \times P_h^N \times P_h^{N+1} \times Y_h^N, \tag{3.1}$$

with elements

$$\boldsymbol{\mathcal{X}} = (\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}), \tag{3.2}$$

and

$$K_h = \{Z \in Y_h : -1 \le Z \le 1\}. \tag{3.3}$$

Given h, k, we consider the following discretized version of the objective function J stated in (2.4),

$$J_{h,k} : \mathbf{X}_{h,k} \times L^2\left(\mathbf{L}^2\right) \to \mathbb{R},$$

where

$$J_{h,k}\left(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{U}}\right) := \sum_{n=1}^{N}\left[\frac{k}{2}\|Y^n - y_{d,h}^n\|^2 + \frac{\alpha}{2}\int_{t_{n-1}}^{t_n} \|\boldsymbol{\mathcal{U}}\|^2 \, dt\right]. \tag{3.4}$$

where the functions $y_{d,h}^n \in P_h$ and $t_n = n \cdot k$ for $n = 1, \ldots, N$. Then, we study the following discrete non-smooth optimal control problem:

Problem 3.1. *Given h, k, $\mathbf{v}_{0,h} \in \mathbf{D}_h, y_{0,h} \in P_h \cap K_h$, $y_{d,h}^n \in P_h$ for $n = 1, \ldots, N$, find $\left(\bar{\boldsymbol{\mathcal{X}}},\bar{\boldsymbol{\mathcal{U}}}\right) \in \mathbf{X}_{h,k} \times L^2\left(\mathbf{L}^2\right)$ such that*

$$\min_{(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{U}})\in\mathbf{X}_{h,k}\times L^2(\mathbf{L}^2)} J_{h,k}\left(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{U}}\right) = J_{h,k}\left(\bar{\boldsymbol{\mathcal{X}}},\bar{\boldsymbol{\mathcal{U}}}\right),$$

subject to

$$(d_t\mathbf{V}^n, \boldsymbol{\psi}) + \nu\left(\nabla\mathbf{V}^n, \nabla\boldsymbol{\psi}\right) - \left(P^n, \nabla\cdot\boldsymbol{\psi}\right) - \frac{1}{k}\int_{t_{n-1}}^{t_n} \left(\boldsymbol{\mathcal{U}}, \boldsymbol{\psi}\right) \, dt = 0, \tag{3.5a}$$

$$\mathbf{V}^0 = \mathbf{v}_{0,h}, \tag{3.5b}$$

$$\left(\nabla\cdot\mathbf{V}^n, \phi\right) = 0, \tag{3.5c}$$

$$\left(d_tY^n, \eta\right)_h + \gamma\left(\nabla W^n, \nabla\eta\right) - \left(Y^{n-1}\mathbf{V}^{n-1}, \nabla\eta\right) = 0, \tag{3.6a}$$

$$Y^0 = y_{0,h}, \tag{3.6b}$$

$$-\left(W^n + Y^{n-1}, \theta - Y^n\right)_h + \varepsilon^2\left(\nabla Y^n, \nabla\theta - \nabla Y^n\right) \geq 0, \tag{3.6c}$$

$$Y^n \in K_h \tag{3.6d}$$

for all $\boldsymbol{\psi} \in \mathbf{V}_h$, $\phi \in P_h$, $\eta \in Y_h$, $\theta \in K_h$, $n = 1, \ldots, N$.

We emphasize that Problem 3.1 corresponds to a fully discretized version of the continuous non-smooth Problem 2.1. Indeed, equations (3.5), (3.6) are discrete versions, respectively, of the state equations (2.5), (2.6) of Problem 2.1.
Optimal control Problem 3.1, as well as Problem 2.1, does not satisfy any kind of constraint qualification, So, even in the discrete settings, it is not possible to directly derive a system of first order optimality condition to solve the problem. Hence, to deal with it, we follow the same procedure applied in Chapter 2. We consider a discretized version of the regularized optimal control Problem 2.2 studied in Section 2.2. Then, we derive the first order optimality conditions of the non-smooth discrete Problem 3.1 as limit of the first order optimality conditions of the regularized discrete problem, for the regularization parameter $\delta \to 0^+$. Then, we show that these optimality conditions converge to the optimality conditions of the non-smooth continuous Problem 2.1, for the discretization parameters $h \to 0, k \to 0$. Finally, we formulate an algorithm for the numerical solution of the non-smooth discrete problem and we perform some computation studies.

3.2. Regularized Discrete Optimal Control Problem

This section is devoted to the analysis of the fully discretized version of the regularized optimal control Problem 2.2. For this problem, we show that it is well-posed and then we derive the first order optimality conditions.
In order to represent the problem under investigation in a more compact, general form, we define the following map

$$e_{\delta,h,k} : \mathbf{X}_{h,k} \times L^2\left(\mathbf{L}^2\right) \to \mathbf{X}_{h,k}, \tag{3.7}$$

where, for all $\boldsymbol{\mathcal{Z}} = (\boldsymbol{\psi}, \phi, \eta, \theta) \in \mathbf{X}_{h,k}$,

$$\begin{aligned} \langle \boldsymbol{\mathcal{Z}}, e_{\delta,h,k}(\mathcal{X},\boldsymbol{\mathcal{U}})\rangle_{X^*_{h,k},X_{h,k}} &= \langle \boldsymbol{\psi}, a_{1,h,k}(\boldsymbol{\mathcal{V}},\mathcal{P},\boldsymbol{\mathcal{U}})\rangle + \langle \phi, a_{2,h,k}(\boldsymbol{\mathcal{V}})\rangle \\ &+ \langle \eta, b_{h,k}(\boldsymbol{\mathcal{V}},\mathcal{Y},\mathcal{W})\rangle + \langle \theta, c_{\delta,h,k}(\mathcal{Y},\mathcal{W})\rangle \\ &+ \left(\boldsymbol{\psi}^0, \mathbf{V}^0 - \mathbf{v}_{0,h}\right) + \left(\eta^0, Y^0 - y_{0,h}\right), \end{aligned} \tag{3.8}$$

with

$$\begin{aligned} \langle \boldsymbol{\psi}, a_{1,h,k}(\boldsymbol{\mathcal{V}},\mathcal{P},\boldsymbol{\mathcal{U}})\rangle &= \sum_{n=1}^{N} \Big[k\left(d_t \mathbf{V}^n, \boldsymbol{\psi}^n\right) + k\nu\left(\nabla \mathbf{V}^n, \nabla\boldsymbol{\psi}^n\right) \\ &\quad - k\left(P^n, \nabla\cdot\boldsymbol{\psi}^n\right) - \int_{t_{n-1}}^{t_n} \left(\boldsymbol{\mathcal{U}}, \boldsymbol{\psi}^n\right)\, dt \Big], \\ \langle \phi, a_{2,h,k}(\boldsymbol{\mathcal{V}})\rangle &= \sum_{n=1}^{N} k\left(\nabla\cdot\mathbf{V}^n, \phi^n\right), \\ \langle \eta, b_{h,k}(\boldsymbol{\mathcal{V}},\mathcal{Y},\mathcal{W})\rangle &= \sum_{n=1}^{N} \Big[k\left(d_t Y^n, \eta^n\right)_h + k\gamma\left(\nabla W^n, \nabla\eta^n\right) \\ &\quad - k\left(Y^{n-1}, \mathbf{V}^{n-1}\cdot\nabla\eta^n\right)\Big], \\ \langle \theta, c_{\delta,h,k}(\mathcal{Y},\mathcal{W})\rangle &= \sum_{n=1}^{N} k\left[\left(W^n + Y^{n-1} - \frac{1}{\delta}\beta_\delta(Y^n), \theta^n\right)_h - \varepsilon^2\left(\nabla Y^n, \nabla\theta^n\right)\right]. \end{aligned}$$

Thus, we consider the following regularized discrete optimal control problem:

Problem 3.2. *Given h, k, $\mathbf{v}_{0,h} \in \mathbf{D}_h, y_{0,h} \in P_h \cap K_h$, $y^n_{d,h} \in P_h$ for $n = 1, \ldots, N$, find $\left(\bar{\boldsymbol{\mathcal{X}}},\bar{\boldsymbol{\mathcal{U}}}\right) \in \mathbf{X}_{h,k} \times L^2\left(\mathbf{L}^2\right)$ such that*

$$\min_{(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{U}})\in\mathbf{X}_{h,k}\times L^2(\mathbf{L}^2)} J_{h,k}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{U}}) = J_{h,k}\left(\bar{\boldsymbol{\mathcal{X}}},\bar{\boldsymbol{\mathcal{U}}}\right),$$

subject to

$$e_{\delta,h,k}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{U}}) = 0. \tag{3.9}$$

We note, by the definition (3.8) of the map $e_{\delta,h,k}$, that the state equation (3.9) in Problem 3.2, represents just a discretized version of the state equations (2.24), (2.25) of the continuous regularized optimal control Problem 2.2.

Remark 3.3. In the setting of the optimal control Problem 3.2, we choose $\boldsymbol{\mathcal{U}} \in L^2\left(\mathbf{L}^2\right)$ for the control variable. However, as a consequence of the first order optimality conditions of the problem, that we will derive in Section 3.2.3, we will get $\boldsymbol{\mathcal{U}} \in \mathbf{V}^N_h$. For this reason, we prefer denote the control variable as a fully discrete function, using a calligraphic capital letter.

3.2.1. Properties of the Regularized Discrete State Equations

By the definition (3.8) of the map $e_{\delta,h,k}$, the state equations for the regularized discrete optimal control Problem 3.2 read as follows:

$$(d_t\mathbf{V}^n, \boldsymbol{\psi}) + \nu(\nabla\mathbf{V}^n, \nabla\boldsymbol{\psi}) - (P^n, \nabla\cdot\boldsymbol{\psi}) - \frac{1}{k}\int_{t_{n-1}}^{t_n}(\boldsymbol{\mathcal{U}}, \boldsymbol{\psi})\ dt = 0, \tag{3.10a}$$

$$\mathbf{V}^0 = \mathbf{v}_{0,h}, \tag{3.10b}$$

$$(\nabla\cdot\mathbf{V}^n, \phi) = 0, \tag{3.10c}$$

$$(d_tY^n, \eta)_h + \gamma(\nabla W^n, \nabla\eta) - \left(Y^{n-1}\mathbf{V}^{n-1}, \nabla\eta\right) = 0, \tag{3.11a}$$

$$Y^0 = y_{0,h}, \tag{3.11b}$$

$$(W^n, \theta)_h - \varepsilon^2(\nabla Y^n, \nabla\theta) + \left(Y^{n-1}, \theta\right)_h - \frac{1}{\delta}(\beta_\delta(Y^n), \theta)_h = 0, \tag{3.11c}$$

for all $\boldsymbol{\psi} \in \mathbf{V}_h,\ \phi \in P_h,\ \eta, \theta \in Y_h,\ n = 1, \ldots, N$. We observe that equation (3.11a) is *mass preserving*, that is

$$(Y^n, 1)_h = \ldots = (Y^0, 1)_h = (y_{0,h}, 1)_h = 0, \quad \forall n = 1, \ldots, N. \tag{3.12}$$

In the following Lemma 3.4 we derive existence, uniqueness of the solution of state equations (3.10), (3.11) of the regularized discrete optimal control Problem 3.2.

Lemma 3.4 (existence, uniqueness). *For any fixed* $h, k, \delta \in \left(0, \frac{1}{4}\right), \boldsymbol{\mathcal{U}} \in L^2(\mathbf{L}^2)$, *the system of the discrete, regularized state equations* (3.10), (3.11) *has a unique solution* $(\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}) \in \mathbf{V}_h^{N+1} \times P_h^N \times P_h^{N+1} \times Y_h^N$.

Proof. Using standard arguments, it is possible to prove that (3.10) has a unique solution $(\boldsymbol{\mathcal{V}}, \mathcal{P}) \in \mathbf{V}_h^{N+1} \times P_h^N$.
We follow [62] to prove the existence and the uniqueness of the solution $(Y^n, W^n) \in P_h \times Y_h$ at a time level n: we demonstrate that, given n, the state equations (3.11) are equivalent to a strictly convex optimization problem which has a unique solution.
Let us suppose that Y^n, W^n are solutions at the time step n of (3.11). Setting $W^n = \hat{W}^n + \frac{1}{|\Omega|}(W^n, 1)_h$ in (3.11a) and integrating by parts in the advection term, we have

$$\left(\nabla\hat{W}^n, \nabla\eta\right) = -\frac{1}{\gamma}\left(\nabla\cdot\left[Y^{n-1}\mathbf{V}^{n-1}\right], \eta\right) - \frac{1}{k\gamma}\left(Y^n - Y^{n-1}, \eta\right)_h. \tag{3.13}$$

By the definitions of the discrete Green's operators defined in (A.32), (A.33), from (3.13), we derive

$$\hat{W}^n = -\frac{1}{\gamma}\mathcal{G}^h\left[\nabla\cdot\left(Y^{n-1}\mathbf{V}^{n-1}\right)\right] - \frac{1}{k\gamma}\hat{\mathcal{G}}^h\left[Y^n - Y^{n-1}\right]. \tag{3.14}$$

So, from (3.14), we can write (3.11c) in the following way

$$\frac{1}{k\gamma}\left(\hat{\mathcal{G}}^h\left[Y^n - Y^{n-1}\right],\theta\right)_h + \frac{1}{\gamma}\left(\mathcal{G}^h[\nabla\cdot\left(Y^{n-1}\mathbf{V}^{n-1}\right)],\theta\right)_h + \varepsilon^2\left(\nabla Y^n,\nabla\theta\right)$$
$$-\left(Y^{n-1},\theta\right)_h + \frac{1}{\delta}\left(\beta_\delta\left(Y^n\right),\theta\right) - \frac{1}{|\Omega|}(W^n,1)_h\left(\theta,1\right)_h = 0. \tag{3.15}$$

If in (3.15) $\hat{\theta}\in P_h$, it holds

$$\frac{1}{k\gamma}\left(\hat{\mathcal{G}}^h\left[Y^n - Y^{n-1}\right],\hat{\theta}\right)_h + \frac{1}{\gamma}\left(\mathcal{G}^h[\nabla\cdot\left(Y^{n-1}\mathbf{V}^{n-1}\right)],\hat{\theta}\right)_h$$
$$+\varepsilon^2\left(\nabla Y^n,\nabla\hat{\theta}\right) - \left(Y^{n-1},\hat{\theta}\right)_h + \frac{1}{\delta}\left(\beta_\delta\left(Y^n\right),\hat{\theta}\right) = 0. \tag{3.16}$$

From (3.16), we infer that Y^n is a solution of the following minimization problem

$$Y^n = \arg\min_{Z\in P_h}\left[\frac{\varepsilon^2}{2}\|\nabla Z\|^2 + \left(f_\delta\left(Z\right),1\right)_h + \frac{1}{2k\gamma}\left\|\nabla\hat{\mathcal{G}}^h\left(Z - Y^{n-1}\right)\right\|^2\right.$$
$$\left.-\left(Y^{n-1} - \frac{1}{\gamma}\mathcal{G}^h\left[\nabla\cdot\left(Y^{n-1}\mathbf{V}^{n-1}\right)\right],Z\right)_h\right]. \tag{3.17}$$

Thus, we have shown that, if Y^n, W^n are solutions at a time step n of (3.11), then Y^n is solution of (3.17).
Conversely, let us suppose that Y^n is solution of (3.17) above. Then, Y^n satisfies (3.16). By definitions of operators $\hat{\mathcal{G}}^h, \mathcal{G}^h$, we have

$$Z\in P_h \quad\Longrightarrow\quad \hat{\mathcal{G}}^h Z,\ \mathcal{G}^h Z\in P_h,$$

and furthermore for all $\theta\in Y_h$, we can define

$$\hat{\theta} = \theta - \frac{1}{|\Omega|}\left(\theta,1\right)\in P_h.$$

Therefore, if Y^n satisfies (3.16), it holds

$$\frac{1}{k\gamma}\left(\hat{\mathcal{G}}^h\left[Y^n - Y^{n-1}\right],\theta\right)_h + \frac{1}{\gamma}\left(\mathcal{G}^h[\nabla\cdot\left(Y^{n-1}\mathbf{V}^{n-1}\right)],\theta\right)_h$$
$$+\varepsilon^2\left(\nabla Y^n,\nabla\theta\right) - \left(Y^{n-1},\theta\right)_h + \frac{1}{\delta}\left(\beta_\delta\left(Y^n\right),\theta\right)$$
$$= -\left(Y^{n-1},\frac{1}{|\Omega|}\left(\theta,1\right)_h\right)_h + \frac{1}{\delta}\left(\beta_\delta\left(Y^n\right),\frac{1}{|\Omega|}\left(\theta,1\right)_h\right), \tag{3.18}$$

for all $\theta\in Y_h$. Then, we define

$$W^n = \hat{W}^n + \frac{1}{|\Omega|}\left[-\left(Y^{n-1},1\right)_h + \frac{1}{\delta}\left(\beta_\delta\left(Y^n\right)_h,1\right)\right],$$

where $\hat{W}^n\in P_h$ is such that

$$\hat{W}^n = -\frac{1}{\gamma}\mathcal{G}^h\left[\nabla\cdot\left(Y^{n-1}\mathbf{V}^{n-1}\right)\right] - \frac{1}{k\gamma}\hat{\mathcal{G}}^h\left[Y^n - Y^{n-1}\right], \tag{3.19}$$

and

$$(W^n, 1)_h = -\left(Y^{n-1}, 1\right)_h + \frac{1}{\delta}\left(\beta_\delta\left(Y^n\right)_h, 1\right). \tag{3.20}$$

In this way, from (3.19), integrating by parts, we have

$$\left(Y^n - Y^{n-1}, \eta\right)_h + k\gamma\left(\nabla W^n, \nabla\eta\right) - k\left(Y^{n-1}\mathbf{V}^{n-1}, \nabla\eta\right) = 0,$$

for all $\eta \in Y_h$. Finally, using (3.19), we have that (3.18) reads as

$$\begin{aligned} & -\left(\hat{W}^n, \theta\right)_h + \varepsilon^2\left(\nabla Y^n, \nabla\theta\right) - \left(Y^{n-1}, \theta\right)_h + \frac{1}{\delta}\left(\beta_\delta\left(Y^n\right), \theta\right) \\ = & \left[-\left(Y^{n-1}, 1\right)_h + \frac{1}{\delta}\left(\beta_\delta\left(Y^n\right), 1\right)_h\right]\frac{1}{|\Omega|}\left(\theta, 1\right)_h, \end{aligned}$$

for all $\theta \in Y_h$. Then from (3.20), we get

$$-\left(W^n, \theta\right)_h + \varepsilon^2\left(\nabla Y^n, \nabla\theta\right) - \left(Y^{n-1}, \theta\right)_h + \frac{1}{\delta}\left(\beta_\delta(Y^n), \theta\right)_h = 0,$$

which is (3.11c). Thus, we have shown that if Y^n is solution of (3.17), then Y^n and W^n are solutions at the time step n of (3.11).
We conclude that the equations (3.11) and the minimization problem (3.17) are equivalent. The latter is a strictly convex minimization problem and then it has a unique solution. The same holds for the equations (3.11). □

As a consequence of Lemma 3.4 above, associated to the discrete state equations of Problem 3.2,

$$e_{\delta,h,k}\left(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}}\right) = 0,$$

we can define a *solution operator* $s_{\delta,h,k} : L^2\left(\mathbf{L}^2\right) \to \mathbf{X}_{h,k}$, which is such that

$$e_{\delta,h,k}\left(s_{\delta,h,k}\left(\boldsymbol{\mathcal{U}}\right), \boldsymbol{\mathcal{U}}\right) = 0, \quad \forall\, \boldsymbol{\mathcal{U}} \in L^2\left(\mathbf{L}^2\right). \tag{3.21}$$

In the following Lemmas 3.5, 3.6, 3.7, 3.8 we derive stability estimates for the solution of the state equations (3.10), (3.11) of the regularized discrete optimal control Problem 3.2. These estimates are independent of the discretization parameters h, k and also of the regularization parameter δ.

Lemma 3.5. *Let us assume that there exists a constant $\tilde{C}$ independent of $h, k, \delta \in \left(0, \frac{1}{4}\right)$, such that*

$$\|\nabla \mathbf{v}_{0,h}\| \le \tilde{C},$$

Then, for any fixed $\boldsymbol{\mathcal{U}} \in L^2\left(\mathbf{L}^2\right)$ the solution $(\boldsymbol{\mathcal{V}}, \mathcal{P}) \in \mathbf{V}_h^{N+1} \times P_h^N$ of (3.10) satisfies

$$\sup_{n=0,\ldots,N} \|\nabla \mathbf{V}^n\| \le C\left(\boldsymbol{\mathcal{U}}\right), \tag{3.22}$$

$$\sum_{n=1}^{N} k\,\|d_t \mathbf{V}^n\|^2 \le C\left(\boldsymbol{\mathcal{U}}\right), \tag{3.23}$$

$$\sum_{n=1}^{N} \|\nabla \mathbf{V}^n - \nabla \mathbf{V}^{n-1}\|^2 \le C\left(\boldsymbol{\mathcal{U}}\right), \tag{3.24}$$

$$\sum_{n=1}^{N} k\ \|\tilde{\Delta}_h \mathbf{V}^n\|^2 \ \leq\ C\,(\boldsymbol{\mathcal{U}})\,, \tag{3.25}$$

$$\sup_{n=1,\ldots,N} \left\| \sum_{i=1}^{n} k\ P^i \right\| \ \leq\ C\,(\boldsymbol{\mathcal{U}})\,, \tag{3.26}$$

where the constant $C\,(\boldsymbol{\mathcal{U}})$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$ but it is independent of $h, k, \delta \in \left(0, \frac{1}{4}\right)$ and $\tilde{\Delta}_h$ is the discrete Laplacian operator defined in (A.37).

The proof of the Lemma is shown in Appendix B, Section B.2.

Lemma 3.6. *Let us assume that there exists a constant $\tilde{C}$ independent of $h, k, \delta \in \left(0, \frac{1}{4}\right)$ such that*

$$\|\nabla \mathbf{v}_{0,h}\| \leq \tilde{C},$$

Then, for any fixed $\delta \in \left(0, \frac{1}{4}\right)$ and $\boldsymbol{\mathcal{U}} \in L^2\left(\mathbf{L}^2\right)$, the solution $(\mathcal{Y}, \mathcal{W}) \in P_h^{N+1} \times Y_h^N$ of (3.11) satisfies

$$\begin{aligned} & E_\delta(Y^n) + \frac{\varepsilon^2}{2}\|\nabla Y^n - \nabla Y^{n-1}\|^2 + \frac{1}{2}\|Y^n - Y^{n-1}\|_h^2 + k\ \frac{\gamma}{2}\|\nabla W^n\|^2 \\ \leq\ & E_\delta(Y^{n-1}) + k\ C\,(\boldsymbol{\mathcal{U}})\,\|\nabla Y^{n-1}\|^2, \end{aligned} \tag{3.27}$$

for all $n = 1, \ldots, N$, where

$$E_\delta(Y^n) = \frac{\varepsilon^2}{2}\|\nabla Y^n\|^2 + (\Phi_\delta(Y^n), 1)_h\,, \tag{3.28}$$

and the constant $C\,(\boldsymbol{\mathcal{U}})$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$ but it is independent of $h, k, \delta \in \left(0, \frac{1}{4}\right)$.

Proof. At a time level n, setting $\eta = W^n$ and $\theta = Y^n - Y^{n-1}$ in (3.11), we derive

$$\begin{aligned} & k\gamma\|W^n\|^2 - k\left(Y^{n-1}, \mathbf{V}^{n-1}\cdot\nabla W^n\right) + \varepsilon^2\left(\nabla Y^n, \nabla Y^n - \nabla Y^{n-1}\right) \\ - \ & \left(Y^{n-1}, Y^n - Y^{n-1}\right)_h + \frac{1}{\delta}\left(\beta_\delta(Y^n), Y^n - Y^{n-1}\right)_h = 0. \end{aligned} \tag{3.29}$$

Expanding the third and the fourth term in (3.29) and using the convexity of f_δ (2.16), we have

$$\begin{aligned} & k\gamma\|W^n\|^2 - k\left(Y^{n-1}, \mathbf{V}^{n-1}\cdot\nabla W^n\right) + \frac{\varepsilon^2}{2}\|\nabla Y^n\|^2 - \frac{\varepsilon^2}{2}\|\nabla Y^{n-1}\|^2 \\ + \ & \frac{\varepsilon^2}{2}\|\nabla Y^n - \nabla Y^{n-1}\|^2 + \frac{1}{2}\|Y^{n-1}\|_h^2 - \frac{1}{2}\|Y^n\|_h^2 + \frac{1}{2}\|Y^n - Y^{n-1}\|_h^2 \\ + \ & \left(f_\delta(Y^n) - f_\delta(Y^{n-1}), 1\right)_h \leq 0, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} & \frac{\varepsilon^2}{2}\|\nabla Y^n\|^2 + (f_\delta(Y^n), 1)_h - \frac{1}{2}\|Y^n\|_h^2 + k\gamma\|W^n\|^2 + \frac{\varepsilon^2}{2}\|\nabla Y^n - \nabla Y^{n-1}\|^2 \\ + \ & \frac{1}{2}\|Y^n - Y^{n-1}\|_h^2 \leq \frac{\varepsilon^2}{2}\|\nabla Y^{n-1}\|^2 + \left(f_\delta(Y^{n-1}), 1\right)_h - \frac{1}{2}\|Y^{n-1}\|_h^2 \end{aligned} \tag{3.30}$$

$$+ \quad k\left(Y^{n-1}, \mathbf{V}^{n-1}\cdot\nabla W^n\right).$$

Adding $\frac{1}{2}(1,1)_h$ to left and right hand sides of (3.30) and using the definition (2.10) of Φ_δ, we can write

$$\begin{aligned} &E_\delta\left(Y^n\right) + k\gamma\|W^n\|^2 + \frac{\varepsilon^2}{2}\|\nabla Y^n - \nabla Y^{n-1}\|^2 + \frac{1}{2}\|Y^n - Y^{n-1}\|_h^2 \\ \leq\ &E_\delta\left(Y^{n-1}\right) + k\left(Y^{n-1}, \mathbf{V}^{n-1}\cdot\nabla W^n\right) = E_\delta\left(Y^{n-1}\right) + I_1. \end{aligned} \tag{3.31}$$

From the generalized Holder's inequality (A.14), (A.17), Poincaré's inequality (A.15), Young's inequality (A.13) and the result (3.22) in the previous Lemma (3.5), we derive that I_1 in (3.31) satisfies

$$\begin{aligned} I_1 \ &\leq\ k\,\|Y^{n-1}\|_{L^4}\|\mathbf{V}^{n-1}\|_{\mathbf{L}^4}\|\nabla W^n\| \leq k\ C\left(\boldsymbol{\mathcal{U}}\right)\|Y^{n-1}\|_{H^1}\|\nabla W^n\| \\ &\leq\ k\ C\left(\boldsymbol{\mathcal{U}}\right)\frac{1}{2\sigma}\|\nabla Y^{n-1}\|^2 + k\ \frac{\sigma}{2}\|\nabla W^n\|^2. \end{aligned}$$

Therefore, with σ small enough

$$I_1 \leq k\ \frac{\gamma}{2}\|\nabla W^n\|^2 + k\ C\left(\boldsymbol{\mathcal{U}}\right)\|\nabla Y^{n-1}\|^2. \tag{3.32}$$

Then, using together (3.31), (3.32), we obtain the final result (3.27). □

Lemma 3.7. *Let us assume that there exists a constant $\tilde{C}$, independent of $h, k, \delta \in \left(0, \frac{1}{4}\right)$, such that*

$$\|\nabla \mathbf{v}_{0,h}\|^2 + \|\nabla y_{0,h}\| \leq \tilde{C}, \tag{3.33}$$

Then, for any fixed $\delta \in \left(0, \frac{1}{4}\right)$ and $\boldsymbol{\mathcal{U}} \in L^2\left(\mathbf{L}^2\right)$, the solution $(\mathcal{Y}, \mathcal{W}) \in P_h^{N+1} \times Y_h^N$ of the state equations (3.11) satisfies

$$\sup_{n=0,\ldots,N} \|Y^n\|_{H_0} \ \leq\ C\left(\boldsymbol{\mathcal{U}}\right), \tag{3.34}$$

$$\sum_{n=1}^{N} k\ \left\|\nabla \mathcal{G} d_t Y^n\right\|^2 \ \leq\ C\left(\boldsymbol{\mathcal{U}}\right), \tag{3.35}$$

$$\sum_{n=1}^{N} k\ \|\hat{\Delta}_h Y^n\|_h^2 \ \leq\ C\left(\boldsymbol{\mathcal{U}}\right), \tag{3.36}$$

$$\sum_{n=1}^{N} \left\|Y^n - Y^{n-1}\right\|_{H_0}^2 \ \leq\ C\left(\boldsymbol{\mathcal{U}}\right), \tag{3.37}$$

$$\sum_{n=1}^{N} k\ \|W^n\|_{H^1}^2 \ \leq\ C\left(\boldsymbol{\mathcal{U}}\right). \tag{3.38}$$

where the constant $C\left(\boldsymbol{\mathcal{U}}\right)$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$ but it is independent of $h, k, \delta \in \left(0, \frac{1}{4}\right)$, $\mathcal{G}$ is the Green's operator defined in (A.20) and $\hat{\Delta}_h$ is the discrete Laplacian operator defined in (A.36).

Proof. In the estimate (3.27) in Lemma 3.6 we sum on the index n. We derive

$$E_\delta(Y^m) + \frac{\varepsilon^2}{2}\sum_{n=1}^{m}\|\nabla Y^n - \nabla Y^{n-1}\|^2 + \frac{1}{2}\sum_{n=1}^{m}\|Y^n - Y^{n-1}\|_h^2 + \frac{\gamma}{2}\sum_{n=1}^{m} k\;\|\nabla W^n\|^2$$

$$\leq E_\delta(y_{0,h}) + C\,(\boldsymbol{\mathcal{U}})\sum_{n=1}^{m} k\;\|\nabla Y^{n-1}\|^2, \tag{3.39}$$

for all $m = 1, \ldots, N$. By the definitions of the discrete energy E_δ (3.28) and the function Φ_δ (2.10), we have

$$E_\delta\left(y_{0,h}\right) = \frac{\varepsilon^2}{2}\|\nabla y_{0,h}\|^2 + \frac{1}{2}\left(1 - y_{0,h}^2, 1\right)_h + \left(f_\delta\left(y_{0,h}\right), 1\right)_h. \tag{3.40}$$

Since $-1 \leq y_{0,h} \leq 1$, the function f_δ (2.11) is such that $f_\delta\left(y_{0,h}\right) \equiv 0$. Hence, from (3.40), we get

$$\begin{aligned} E_\delta\left(y_{0,h}\right) &= \frac{\varepsilon^2}{2}\|\nabla y_{0,h}\|^2 + \frac{1}{2}\left(1,1\right)_h - \frac{1}{2}\left(y_{0,h}^2, 1\right)_h \\ &\leq \frac{\varepsilon^2}{2}\|\nabla y_{0,h}\|^2 + \frac{1}{2}\|1\|_h^2 \leq \frac{\varepsilon^2}{2}\|\nabla y_{0,h}\|^2 + C|\Omega|. \end{aligned} \tag{3.41}$$

Then, inserting (3.41) in (3.41) and using the assumption (3.33), we have

$$E_\delta\left(Y^m\right) = \frac{\varepsilon^2}{2}\|\nabla Y^m\|^2 + \left(\Phi_\delta\left(Y^m\right), 1\right)_h \leq C\,(\boldsymbol{\mathcal{U}})\sum_{n=1}^{m} k\;\left[1 + \|\nabla Y^{n-1}\|^2\right]. \tag{3.42}$$

Using the property (2.14) of the potential Φ_δ, from (3.42), we can write

$$\frac{\varepsilon^2}{2}\|\nabla Y^m\|^2 \leq C\,(\boldsymbol{\mathcal{U}})\sum_{n=1}^{m} k\;\left[1 + \|\nabla Y^{n-1}\|^2\right]. \tag{3.43}$$

Applying the discrete Gronwall's Lemma (see for example Lemma 1.4.2 in [73]) to (3.43), we obtain

$$\|\nabla Y^m\| \leq C\,(\boldsymbol{\mathcal{U}}), \tag{3.44}$$

for all $m = 0, \ldots, N$. Hence, from Poincaré-Wirtinger inequality (A.15), we have that the result (3.34) holds. From (3.34) together (3.39), we derive the further result (3.37) and moreover

$$\sum_{n=1}^{m} k\;\|\nabla W^n\|^2 \leq C\,(\boldsymbol{\mathcal{U}}), \tag{3.45}$$

for all $m = 1, \ldots, N$. Setting $\theta = 1$ in (3.11c), we have

$$\left(W^n, 1\right)_h = \frac{1}{\delta}\left(\beta_\delta\left(Y^n\right), 1\right)_h. \tag{3.46}$$

Since $|\beta_\delta\left(r\right)| \leq \beta_\delta\left(r\right) r$, from (3.46), we derive

$$|\left(W^n, 1\right)_h| \leq \frac{1}{\delta}\left(\beta_\delta\left(Y^n\right), Y^n\right)_h. \tag{3.47}$$

Substituting $\theta = Y^n$ in (3.11c), we have

$$\frac{1}{\delta}\left(\beta_\delta\left(Y^n\right), Y^n\right)_h = \left(W^n, Y^n\right)_h - \varepsilon^2\|\nabla Y^n\|^2 + \left(Y^{n-1}, Y^n\right)_h . \tag{3.48}$$

Hence, using together (3.47) and (3.48), we can write

$$|\left(W^n, 1\right)_h| \leq \left(W^n, Y^n\right)_h - \varepsilon^2\|\nabla Y^n\|^2 + \left(Y^{n-1}, Y^n\right)_h . \tag{3.49}$$

From the definition (A.32) of the discrete Green operator $\hat{\mathcal{G}}_h$, Cauchy-Schwarz inequality and (A.35), we have

$$\left(Y^n, W^n\right)_h = \left(\nabla\hat{\mathcal{G}}_h Y^n, \nabla W^n\right) \leq \|\nabla\hat{\mathcal{G}}_h Y^n\|\|\nabla W^n\| \leq C\ \|Y^n\|_h\|\nabla W^n\|.$$

Hence, from (3.49), we get

$$|\left(W^n, 1\right)_h| \leq C\ \|Y^n\|_h\|\nabla W^n\| - \varepsilon^2\|\nabla Y^n\|^2 + \left(Y^{n-1}, Y^n\right)_h ,$$

which implies, using the equivalence between the L^2 norm and the h-norm, Cauchy-Schwarz and Young's inequality (A.13),

$$|\left(W^n, 1\right)_h| \leq C\ \left[\|Y^n\|^2 + \|Y^{n-1}\|^2 + \|\nabla W^n\|^2\right] . \tag{3.50}$$

Summing on the index n in (3.50), taking into account of the result (3.34) and using (3.45), we derive

$$\sum_{n=1}^{m} k\ |\left(W^n, 1\right)_h| \leq C\left(\boldsymbol{\mathcal{U}}\right) . \tag{3.51}$$

for all $m = 1, \ldots, N$. Therefore, from (3.45), (3.51) and the discrete Poincaré's inequality (A.50), we infer that the result (3.38) holds.
By the definition of the Green's operator $\mathcal{G}$ in (A.20), the first state equation in (3.11) and the definition of the projection operator Q^h in (A.41), we have

$$\begin{aligned} \left(\nabla\mathcal{G}d_tY^n, \nabla\eta\right) = \left(d_tY^n, \eta\right) = \left(d_tY^n, Q^h\eta\right)_h \\ = \quad -\gamma\left(\nabla W^n, \nabla Q^h\eta\right) + \left(Y^{n-1}, \mathbf{V}^{n-1}\cdot\nabla Q^h\eta\right), \end{aligned} \tag{3.52}$$

for all $\eta \in H^1$. Using in (3.52) the generalized Holder's inequality (A.14), (A.17) and the property (A.42) of the projection operator Q^h, we derive

$$\begin{aligned} \left(\nabla\mathcal{G}d_tY^n, \nabla\eta\right) &\leq \left[\gamma\|\nabla W^n\| + \|Y^{n-1}\|_{L^4}\|\mathbf{V}^{n-1}\|_{L^4}\right]\|\nabla Q^h\eta\| \\ &\leq C\ \left[\|\nabla W^n\| + \|Y^{n-1}\|_{H^1}\|\nabla\mathbf{V}^{n-1}\|\right]\|\nabla\eta\|, \end{aligned}$$

which implies, setting $\eta = \mathcal{G}d_tY^n$ and taking into account the results (3.22), (3.34), (3.38), the desired estimate (3.35).
Setting $\theta = -\hat{\Delta}_hY^n$ in the second state equation in (3.10), we get

$$\left(W^n + Y^{n-1}, -\hat{\Delta}_hY^n\right)_h + \varepsilon^2\left(\nabla Y^n, \nabla\hat{\Delta}_hY^n\right) + \frac{1}{\delta}\left(I_h\beta_\delta(Y^n), \hat{\Delta}_hY^n\right)_h \tag{3.53}$$

$$= \quad (\nabla W^n, \nabla Y^n) - \varepsilon^2 \|\hat{\Delta}_h Y^n\|_h^2 + \left(\nabla Y^{n-1}, \nabla Y^n\right) - \frac{1}{\delta}\left(\nabla\left[I_h \beta_\delta(Y^n)\right], \nabla Y^n\right) = 0,$$

where I_h is the interpolation operator defined in (A.27). Using the following property (see inequality (4.3) in [41]),

$$\varepsilon^2 \left(\nabla Y^n, \nabla I_h \beta_\delta\left(Y^n\right)\right) \geq 0, \tag{3.54}$$

from (3.53), we can write

$$\varepsilon^2 \|\hat{\Delta}_h Y^n\|_h^2 \leq (\nabla W^n, \nabla Y^n) + \left(\nabla Y^{n-1}, \nabla Y^n\right), \tag{3.55}$$

which implies, multiplying by k, using Young's inequality (A.13) and summing on the index n

$$\varepsilon^2 \sum_{n=1}^{m} k \ \|\hat{\Delta}_h Y^n\|_h^2 \leq \frac{1}{2} \sum_{n=1}^{m} k \ \left[\|\nabla W^n\|^2 + 2\|\nabla Y^n\|^2 + \|\nabla Y^{n-1}\|^2\right],$$

for all $m = 1, \ldots, N$. Hence from the previous results (3.34) and (3.38), we obtain the estimate (3.36). □

Lemma 3.8. *Let us assume that there exists a constant $\tilde{C}$, independent of $h, k, \delta \in \left(0, \frac{1}{4}\right)$, such that*

$$\|\nabla \mathbf{v}_{0,h}\|^2 + \|\nabla y_{0,h}\| \leq \tilde{C},$$

Then, for any fixed $\delta \in \left(0, \frac{1}{4}\right)$ and $\boldsymbol{\mathcal{U}} \in L^2\left(\mathbf{L}^2\right)$, the solution $\mathcal{Y} \in P_h^{N+1}$ of the state equations (3.11) satisfies the following estimate

$$\sum_{n=1}^{N} k \ \left\|\frac{1}{\delta} \beta_\delta\left(Y^n\right)\right\|_h^2 \leq C\left(\boldsymbol{\mathcal{U}}\right), \tag{3.56}$$

where the constant $C\left(\boldsymbol{\mathcal{U}}\right)$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}^2$ but it is independent of $h, k, \delta \in \left(0, \frac{1}{4}\right)$.

Proof. Setting in (3.11c) $\theta = I^h \beta_\delta\left(Y^n\right) \in Y_h$, where I^h is the interpolation operator (A.27), we derive

$$\begin{aligned} (3.57) \qquad & \varepsilon^2 \left(\nabla Y^n, \nabla I^h \beta_\delta\left(Y^n\right)\right) + \frac{1}{\delta}\|I^h \beta_\delta(Y^n)\|_h = \left(W^n + Y^{n-1}, \beta_\delta\left(Y^n\right)\right)_h \\ \leq \quad & \|W^n + Y^{n-1}\|_h \|I^h \beta_\delta\left(Y^n\right)\|_h \leq \left[\|W^n\|_h + \|Y^{n-1}\|_h\right] \|I^h \beta_\delta\left(Y^n\right)\|_h \\ \leq \quad & \delta\left[\|W^n\|_h^2 + \|Y^{n-1}\|_h^2\right] + \frac{1}{2\delta}\|I^h \beta_\delta\left(Y^n\right)\|_h^2, \end{aligned}$$

where we used the useful inequality

$$(a+b)\,c \leq \mu\left(a^2+b^2\right) + \frac{1}{2\mu}c^2, \ \forall \mu \geq 0.$$

Rearranging (3.57), we get

$$\varepsilon^2 \left(\nabla Y^n, \nabla I^h \beta_\delta\left(Y^n\right)\right) + \frac{1}{2\delta}\|I^h \beta_\delta(Y^n)\|_h^2 \leq \delta\left[\|W^n\|_h^2 + \|Y^{n-1}\|_h^2\right].$$

which implies, using (3.54),

$$\frac{1}{2\delta}\|I^h \beta_\delta(Y^n)\|_h^2 \leq \delta\left[\|W^n\|_h^2 + \|Y^{n-1}\|_h^2\right]. \tag{3.58}$$

In (3.58), we divide by δ, multiply by k and sum on the index n. In this way, using the estimates (3.34), (3.38) in Lemma 3.7, we get the result (3.56). □

3.2.2. Well-Posedness of the Regularized Discrete Optimal Control Problem

The regularized discrete optimal control Problem 3.2 has the form of an abstract optimal control problem and it is straightforward to prove, in the following Lemma 3.9, the existence of solutions.

Lemma 3.9 (existence of minimizers). *For any fixed $h, k, \delta \in \left(0, \frac{1}{4}\right)$, the regularized discrete optimal control Problem 3.2 admits a solution.*

Proof. The map $J_{h,k} : \mathbf{X}_{h,k} \times L^2(\mathbf{L}^2) \to \mathbb{R}$ is continuous, convex and bounded from below. Thus, it is *weakly lower semicontinuous*. Then, the proof of the Lemma is analogous to the one of Theorem 2.5 in Chapter 2. □

3.2.3. Optimality Conditions for the Regularized Discrete Optimal Control Problem

As in Chapter 2, we show that the regularized Problem 3.2, satisfies the conditions needed to apply the standard theory of mathematical programming in Banach spaces (see Assumptions 1.47 in [58]) and next, we derive the first order optimality conditions (see Theorem 1.48 and Corollary 1.3 in [58]).
We need to verify that the discrete regularized optimal control Problem 3.2 satisfies the following conditions:

- the continuous differentiability of the cost functional $J_{h,k} : \mathbf{X}_{h,k} \times L^2(\mathbf{L}^2) \to \mathbb{R}$;
- the continuous differentiability of the constraint $e_{\delta,h,k} : \mathbf{X}_{h,k} \times L^2(\mathbf{L}^2) \to \mathbf{X}_{h,k}$ defined in (3.7).
- the existence of the inverse of the map $\frac{\partial e_{\delta,h,k}}{\partial \boldsymbol{\mathcal{X}}}\left(s_{\delta,h,k}(\boldsymbol{\mathcal{U}}), \boldsymbol{\mathcal{U}}\right)$.

It is straightforward to check that the first two conditions above are satisfied. Then, we skip the proofs. In the following Theorem 3.10, we prove that also the last condition is verified.

Theorem 3.10. *For any fixed $h, k, \delta \in \left(0, \frac{1}{4}\right)$ and $\boldsymbol{\mathcal{U}} \in L^2(\mathbf{L}^2)$, the operator*

$$\frac{\partial e_{\delta,h,k}}{\partial \boldsymbol{\mathcal{X}}}\left(s_{\delta,h,k}(\boldsymbol{\mathcal{U}}), \boldsymbol{\mathcal{U}}\right) \in \mathcal{L}\left(\mathbf{X}_{h,k}, \mathbf{X}_{h,k}\right)$$

is invertible.

Proof. We need to prove that for all $\boldsymbol{\mathcal{Z}} \in \mathbf{X}_{h,k}$ there exists a unique $\mathbf{d}_{\boldsymbol{\mathcal{X}}} \in \mathbf{X}_{h,k}$ such that

$$\frac{\partial e_{\delta,h,k}}{\partial \boldsymbol{\mathcal{X}}}\left(s_{\delta,h,k}(\boldsymbol{\mathcal{U}}), \boldsymbol{\mathcal{U}}\right) \mathbf{d}_{\boldsymbol{\mathcal{X}}} = \boldsymbol{\mathcal{Z}}. \tag{3.59}$$

Equation (3.59) is equivalent to demonstrate that given $(\mathbf{Z}_{\mathbf{V}}, Z_P, Z_Y, Z_W) \in \mathbf{V}_h^{N+1} \times P_h^N \times P_h^{N+1} \times Y_h^N$, the following system of equations

$$\left(\mathbf{d}_{\mathbf{V}}^n - \mathbf{d}_{\mathbf{V}}^{n-1}, \boldsymbol{\psi}\right) + k\nu\left(\nabla \mathbf{d}_{\mathbf{V}}^n, \nabla \boldsymbol{\psi}\right) - k\left(d_P^n, \nabla \cdot \boldsymbol{\psi}\right) \quad = \quad \left(\mathbf{Z}_{\mathbf{V}}^n, \boldsymbol{\psi}\right), \tag{3.60}$$

$$
\begin{aligned}
&(3.61) & (\nabla \cdot \mathbf{d}_{\mathbf{V}}^n, \phi) &= (Z_P^n, \phi), \\
&(3.62) & \mathbf{d}_{\mathbf{V}}^0 &= \mathbf{Z}_{\mathbf{V}}^0, \\
& & \left(d_Y^n - d_Y^{n-1}, \eta\right)_h + k\gamma \left(\nabla d_W^n, \nabla \eta\right) & \\
&(3.63) & -k\left(d_Y^{n-1}\mathbf{V}^{n-1} + Y^{n-1}\mathbf{d}_{\mathbf{V}}^{n-1}, \nabla\eta\right) &= \left(Z_Y^n, \eta\right)_h, \\
&(3.64) & \left(d_W^n + d_Y^{n-1}, \theta\right)_h - \varepsilon^2 \left(\nabla d_Y^n, \nabla\theta\right) - \frac{1}{\delta}\left(\beta_\delta'(Y^n) d_Y^n, \theta\right)_h &= \left(Z_W^n, \theta\right)_h, \\
&(3.65) & d_Y^0 &= Z_Y^0,
\end{aligned}
$$

where $n = 1, \ldots, N$, has a unique solution $(\mathbf{d}_{\mathbf{V}}, d_P, d_Y, d_W) \in \mathbf{V}_h^{N+1} \times P_h^N \times P_h^{N+1} \times Y_h^N$, for all $(\boldsymbol{\psi}, \phi, \eta, \theta) \in \mathbf{V}_h \times P_h \times Y_h \times Y_h$. By standard arguments, it is possible to derive that (3.60), (3.61), (3.62) have a unique solution $(\mathbf{d}_{\mathbf{V}}, d_P) \in \mathbf{V}_h^{N+1} \times P_h^N$. It remains to show the existence and the uniqueness of the solution d_Y, d_W of (3.63), (3.64), (3.65). At each time level n, rearranging (3.63), (3.64), we have

$$
\varepsilon^2 \left(\nabla d_Y^n, \nabla\theta\right) + \frac{1}{\delta}\left(\beta_\delta'(Y^n) d_Y^n, \theta\right)_h - \left(d_W^n, \theta\right)_h = -\left(Z_W^n, \theta\right)_h + \left(d_Y^{n-1}, \theta\right)_h,
$$

$$
\left(d_Y^n, \eta\right)_h + k\gamma \left(\nabla d_W^n, \nabla\eta\right) = \left(Z_Y^n + d_Y^{n-1}, \eta\right)_h + k\left(d_Y^{n-1}\mathbf{V}^{n-1} + Y^{n-1}\mathbf{d}_{\mathbf{V}}^{n-1}, \nabla\eta\right).
$$

We write last two equations in a matrix-vector form. In this way, they read

$$
\begin{aligned}
&(3.66) & E\,\underline{d_Y^n} - M_h\,\underline{d_W^n} &= \underline{f_1^n}, \\
&(3.67) & M_h\,\underline{d_Y^n} + k\,\gamma\,A\,\underline{d_W^n} &= \underline{f_2^n},
\end{aligned}
$$

where

$$
\underline{d_Y^n}_i = d_Y^n(x_i), \quad \underline{d_W^n}_i = d_W^n(x_i),
$$

$$
A_{ij} = \left(\nabla\eta_j, \nabla\eta_i\right), \quad M_{ij} = \left(\eta_j, \eta_i\right), \quad E_{ij} = \frac{1}{\delta}\left(\beta_\delta'(Y^n)\eta_j, \eta_i\right)_h + \varepsilon^2\,A_{i,j},
$$

$$
\begin{aligned}
\underline{f_1^n}_i &= -\left(Z_W^n, \eta_i\right)_h + \left(d_Y^{n-1}, \eta_i\right)_h, \\
\underline{f_2^n}_i &= \left(Z_Y^n, \eta_i\right)_h + \left(d_Y^{n-1}, \eta_i\right)_h + k\left(d_Y^{n-1}, \mathbf{V}^{n-1}\cdot\nabla\eta_i\right) + k\left(Y^{n-1}, \mathbf{d}_{\mathbf{V}}^{n-1}\cdot\nabla\eta_i\right),
\end{aligned}
$$

for $i, j = 1, \ldots, N_h$, using the Lagrange basis $\{\eta_1, \ldots, \eta_{N_h}\}$ of Y_h. The solution of (3.66), (3.67) is given by the following Schur-complement based scheme

$$
\begin{aligned}
\underline{d_W^n} &= M_h^{-1}\left(E\,\underline{d_Y^n} - \underline{f_1^n}\right), \\
\underline{d_Y^n} &= \left(M_h + k\,\gamma\,A\,M_h^{-1}\,E\right)^{-1}\left(k\,\gamma\,A\,M_h^{-1}\,\underline{f_1^n} + \underline{f_2^n}\right),
\end{aligned}
$$

which is well-posed if the matrix $\left(M_h + k\,\gamma\,A\,M_h^{-1}\,E\right)^{-1}$ exists. In order to show that, we note that M_h is diagonal with positive elements and A is symmetric and positive semi-definite. Moreover

$$
\begin{aligned}
&E = \frac{1}{\delta}\,\mathrm{diag}\left(\ldots, \beta_\delta'(Y^n)(x_j), \ldots\right) M_h + \varepsilon^2\,A, \\
&\text{with} \quad \beta_\delta'(Y^n)(x_j) \geq 0, \quad \forall j = 1, \ldots, N_h.
\end{aligned}
$$

Therefore E is symmetric and positive definite. Obviously $M_h^{-1} A M_h^{-1}$ is symmetric and positive semidefinite and $M_h^{-1}\ A\ M_h^{-1}\ E$ is positive semi-definite (see Prop. 6.1 in [?]). Noting that

$$M_h + k\,\gamma\ A\ M_h^{-1}\ E = M_h\ \left(I + k\,\gamma\ M_h^{-1}\ A\ M_h^{-1}\ E\right),$$

and using the previous considerations, we infer that $I + k\,\gamma\ M_h^{-1}\ A\ M_h^{-1}\ E$ is positive definite. Then, we conclude that $M_h\ \left(I + k\,\gamma\ M_h^{-1}\ A\ M_h^{-1}\ E\right)$ is positive definite too. Hence, the matrix $\left(M_h + k\,\gamma\ A\ M_h^{-1}\ E\right)^{-1}$ exists and the proof is completed. □

The continuous differentiability of the maps $J_{h,k} : \mathbf{X}_{h,k} \times L^2(\mathbf{L}^2) \to \mathbb{R}$, $e_{\delta,h,k} : \mathbf{X}_{h,k} \times L^2(\mathbf{L}^2) \to \mathbf{X}_{h,k}$ and Theorem 3.10 guarantee that all the solutions of the regularized optimal control Problem 3.2 can be derived solving a set of first order optimality conditions (see Theorem 1.48 and Corollary 1.3 in [58]). As in Chapter 2, for any fixed h, k and $\delta \in \left(0, \frac{1}{4}\right)$, we define the discrete Lagrange functional $L_{\delta,h,k} : \mathbf{X}_{h,k} \times L^2(\mathbf{L}^2) \times \mathbf{X}_{h,k} \to \mathbb{R}$,

$$L_{\delta,h,k}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}) = J_{h,k}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}}) + \langle \boldsymbol{\mathcal{Q}}, e_{\delta,h,k}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}})\rangle_{\mathbf{X}^*_{h,k}, \mathbf{X}_{h,k}}, \tag{3.68}$$

where

$$\boldsymbol{\mathcal{Q}} = (\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}}, \mathcal{Q}_{\mathcal{P}}, \mathcal{Q}_{\mathcal{Y}}, \mathcal{Q}_{\mathcal{W}}) \in \mathbf{X}_{h,k}.$$

Thus, the first order optimality conditions of the discrete regularized optimal control Problem 3.2 correspond to *find* $(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}) \in \mathbf{X}_{h,k} \times L^2(\mathbf{L}^2) \times \mathbf{X}_{h,k}$ *such that*

$$\frac{\partial L_{\delta,h,k}}{\partial \boldsymbol{\mathcal{Q}}}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}) = 0, \tag{3.69}$$

$$\frac{\partial L_{\delta,h,k}}{\partial \boldsymbol{\mathcal{X}}}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}) = 0, \tag{3.70}$$

$$\frac{\partial L_{\delta,h,k}}{\partial \boldsymbol{\mathcal{U}}}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}) = 0. \tag{3.71}$$

Equations (3.69) are just the discrete state equations $e_{\delta,h,k}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}}) = 0$ of Problem 3.2, (3.70) corresponds to the *discrete adjoint equations* and (3.71) is another *optimality relation.*
In the next Lemma 3.11, we prove that given a solution $\boldsymbol{\mathcal{X}} = s_{\delta,h,k}(\boldsymbol{\mathcal{U}})$ of the discrete state equations (3.69), the discrete adjoint equations (3.70) have a unique solution $\boldsymbol{\mathcal{Q}} \in \mathbf{X}_{h,k}$.

Lemma 3.11. *Let* $h, k, \boldsymbol{\mathcal{U}} \in L^2(\mathbf{L}^2)$ *and* $\boldsymbol{\mathcal{X}} = s_{\delta,h,k}(\boldsymbol{\mathcal{U}}) \in \mathbf{X}_{h,k}$ *be given. Then, the discrete adjoint equations* (3.70) *have a unique solution* $\boldsymbol{\mathcal{Q}} \in \mathbf{X}_{h,k}$*, for any fixed* $\delta \in \left(0, \frac{1}{4}\right)$.

Proof. As a result of Theorem 3.10, we have that the map

$$\left[\frac{\partial e_{\delta,h,k}}{\partial \boldsymbol{\mathcal{X}}}\left(s_{\delta,h,k}(\boldsymbol{\mathcal{U}}), \boldsymbol{\mathcal{U}}\right)\right]^{-1} \in \mathcal{L}(\mathbf{X}_{h,k}, \mathbf{X}_{h,k}),$$

exists. Thus, the proof of the Lemma is analogous to the one of Lemma 2.8 in Chapter 2. □

The first order optimality conditions (3.69)-(3.71) are written in terms of the abstract variables $(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{Q}}) \in \mathbf{X}_{h,k} \times L^2(\mathbf{L}^2) \times \mathbf{X}_{h,k}$. Using the definition of the discrete space $\mathbf{X}_{h,k} = \mathbf{V}_h^{N+1} \times P_h^N \times P_h^{N+1} \times Y_h^N$ we write these optimality conditions explicitly, using the state and the adjoint variables

$$
\begin{aligned}
(\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}) &= \boldsymbol{\mathcal{X}},\\
(\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}}, \mathcal{Q}_{\mathcal{P}}, \mathcal{Q}_{\mathcal{Y}}, \mathcal{Q}_{\mathcal{W}}) &= \boldsymbol{\mathcal{Q}}.
\end{aligned}
$$

Corollary 3.12 (optimality conditions). *The first order optimality conditions* (3.69)-(3.71) *of the regularized optimal control Problem 3.2 read as follows. For all* $n = 1, \ldots, N$*:*

$$(d_t \mathbf{V}^n, \boldsymbol{\psi}) + \nu (\nabla \mathbf{V}^n, \nabla \boldsymbol{\psi}) - (P^n, \nabla \cdot \boldsymbol{\psi}) - (\mathbf{U}^n, \boldsymbol{\psi}) = 0, \tag{3.72a}$$

$$(\nabla \cdot \mathbf{V}^n, \phi) = 0, \tag{3.72b}$$

$$\mathbf{V}^0 = \mathbf{v}_{0,h}, \tag{3.72c}$$

$$(d_t Y^n, \eta)_h + \gamma (\nabla W^n, \nabla \eta) - \left(Y^{n-1} \mathbf{V}^{n-1}, \nabla \eta\right) = 0, \tag{3.72d}$$

$$(W^n, \theta)_h - \varepsilon^2 (\nabla Y^n, \nabla \theta) + \left(Y^{n-1}, \theta\right)_h - \frac{1}{\delta} (\beta_\delta(Y^n), \theta)_h = 0, \tag{3.72e}$$

$$Y^0 = y_{0,h}, \tag{3.72f}$$

for all $\boldsymbol{\psi} \in \mathbf{V}_h,\ \phi \in P_h,\ \eta, \theta \in Y_h,$

$$(-d_t \mathbf{Q}_{\mathbf{V}}^n, \boldsymbol{\psi}) + \nu \left(\nabla \mathbf{Q}_{\mathbf{V}}^{n-1}, \nabla \boldsymbol{\psi}\right) + \left(Q_P^{n-1}, \nabla \cdot \boldsymbol{\psi}\right) - (Y^n \nabla Q_Y^n, \boldsymbol{\psi}) = 0, \tag{3.73a}$$

$$\mathbf{Q}_{\mathbf{V}}^N = 0, \tag{3.73b}$$

$$\left(\nabla \cdot \mathbf{Q}_{\mathbf{V}}^{n-1}, \phi\right) = 0, \tag{3.73c}$$

$$
\begin{aligned}
&(-d_t Q_Y^n, \eta)_h - \varepsilon^2 \left(\nabla Q_W^{n-1}, \nabla \eta\right) + (Q_W^n, \eta)_h\\
&- (\nabla Q_Y^n \cdot \mathbf{V}^n, \eta) - \frac{1}{\delta} \left(\beta_\delta' (Y^n)\, Q_W^{n-1}, \eta\right)_h + \left(Y^n - y_{d,h}^n, \eta\right) = 0,
\end{aligned} \tag{3.73d}
$$

$$Q_Y^N = 0, \tag{3.73e}$$

$$\left(Q_W^{n-1}, \theta\right)_h + \gamma \left(\nabla Q_Y^{n-1}, \nabla \theta\right) = 0. \tag{3.73f}$$

$$Q_W^N = 0, \tag{3.73g}$$

for all $\boldsymbol{\psi} \in \mathbf{V}_h, \phi, \eta \in P_h, \theta \in Y_h,$

$$\alpha \mathbf{U}^n - \mathbf{Q}_{\mathbf{V}}^{n-1} = 0. \tag{3.74}$$

Proof. By direct calculation, equations (3.72b)-(3.72f) and (3.73), can be derived, respectively, from (3.69) and (3.70). From (3.71), we have that

$$\sum_{i=1}^{N} \int_{t_{n-1}}^{t_n} \left(\alpha \boldsymbol{\mathcal{U}} - \mathbf{Q}_{\mathbf{V}}^{n-1}, \boldsymbol{\varphi}\right)\, dt = 0,$$

for all $\boldsymbol{\varphi} \in L^2(\mathbf{L}^2)$. Thus, we get $\boldsymbol{\mathcal{U}} \in \mathbf{V}_h^N$,

$$\boldsymbol{\mathcal{U}}(t) = \mathbf{U}^n \in \mathbf{V}_h, \quad \forall t \in (t_{n-1}, t_n),$$

and consequently (3.74) and (3.72a). □

Remark 3.13. As a consequence of the optimality conditions (3.73f), we infer that $Q_W^n \in P_h$, for all $n = 0, \dots, N-1$.

In the following Lemma 3.14, we derive δ-independent stability estimates for the adjoint variables $(\boldsymbol{\mathcal{Q}}_{\mathcal{V}}, \mathcal{Q}_{\mathcal{P}}, \mathcal{Q}_{\mathcal{Y}}, \mathcal{Q}_{\mathcal{W}}) \in \mathbf{V}_h^{N+1} \times P_h^N \times P_h^{N+1} \times Y_h^N$. These estimates are used in the next sections, where, to deal with the discrete formulation of the non-smooth optimal control Problem 2.1, we perform the limit of the optimality conditions system (3.72)-(3.74) for the regularization parameter $\delta \to 0^+$.

Lemma 3.14. *Let us assume there exists a constant $\tilde{C}$ independent of h, k and $\delta \in \left(0, \frac{1}{4}\right)$, such that*

$$\|\nabla \mathbf{v}_{0,h}\| + \|\nabla y_{0,h}\| + \sum_{n=1}^{N} k\ \|y_{d,h}^n\|^2 \leq \tilde{C}. \tag{3.75}$$

Then, if $(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}) \in \boldsymbol{\mathcal{X}}_{h,k} \times L^2(\mathbf{L}^2) \times \boldsymbol{\mathcal{X}}_{h,k}$ is a solution of the adjoint equations (3.72)-(3.74) *for fixed h, k and $\delta \in \left(0, \frac{1}{4}\right)$,*

$$\sup_{n=0,\dots,N} \|\nabla \mathbf{Q}_{\mathbf{V}}^n\| \leq C(\boldsymbol{\mathcal{U}}), \tag{3.76}$$

$$\sum_{n=1}^{N} \|d_t \mathbf{Q}_{\mathbf{V}}^n\|^2 \leq C(\boldsymbol{\mathcal{U}}), \tag{3.77}$$

$$\sum_{n=1}^{N} \|\mathbf{Q}_{\mathbf{V}}^{n-1} - \mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2 \leq C(\boldsymbol{\mathcal{U}}), \tag{3.78}$$

$$\sum_{n=0}^{N} k\ \|\tilde{\Delta}_h \mathbf{Q}_{\mathbf{V}}^n\|^2 \leq C(\boldsymbol{\mathcal{U}}), \tag{3.79}$$

$$\sup_{n=0,\dots,N-1} \left\| \sum_{i=0}^{n} k\ Q_P^i \right\| \leq C(\boldsymbol{\mathcal{U}}), \tag{3.80}$$

$$\sup_{n=0,\dots,N} \|Q_Y^n\|_{H_0} \leq C(\boldsymbol{\mathcal{U}}), \tag{3.81}$$

$$\sum_{n=1}^{N} \|Q_Y^{n-1} - Q_Y^n\|_{H_0}^2 \leq C(\boldsymbol{\mathcal{U}}), \tag{3.82}$$

$$\sum_{n=0}^{N} k\ \|\Delta_h Q_Y^n\|_{H_0}^2 \leq C(\boldsymbol{\mathcal{U}}), \tag{3.83}$$

$$\sum_{n=0}^{N} k\ \|Q_W^n\|_{H_0}^2 \leq C(\boldsymbol{\mathcal{U}}), \tag{3.84}$$

and

$$0 \leq \sum_{n=1}^{N} k\ \left(\frac{1}{\delta}\beta_\delta'(Y^n)\, Q_W^{n-1}, Q_W^{n-1}\right)_h \leq C(\boldsymbol{\mathcal{U}}). \tag{3.85}$$

where the constant $C(\boldsymbol{\mathcal{U}})$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$ but it is independent of δ, h, k and $\tilde{\Delta}_h$, Δ_h are the discrete Laplacian defined, respectively, in (A.37), (A.36).

Proof. For a given $n = 1, \ldots, N$, we set $\eta = kQ_W^{n-1} \in P_h, \theta = k\left(Q_Y^{n-1} - Q_Y^n\right)$ in (3.73d), (3.73f). In this way, we derive two relation that, used together, produce

$$\begin{aligned}
&\frac{\gamma}{2}\|\nabla Q_Y^{n-1}\|^2 - \frac{\gamma}{2}\|\nabla Q_Y^n\|^2 + \frac{\gamma}{2}\|\nabla\left(Q_Y^{n-1} - Q_Y^n\right)\|^2 + k\varepsilon^2\|\nabla Q_W^{n-1}\|^2 \\
&\qquad + \frac{k}{\delta}\left(\beta'_\delta(Y^n)Q_W^{n-1}, Q_W^{n-1}\right)_h \\
&= -k\ \left(\mathbf{V}^n \cdot \nabla Q_Y^n, Q_W^{n-1}\right) + k\left(Q_W^n, Q_W^{n-1}\right)_h + k\left(Y^n - y_{d,h}^n, Q_W^{n-1}\right) \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{3.86}$$

Regarding I_1, I_2, I_3 in (3.86), we derive:

- $$\begin{aligned} I_3 &\le k\ \|Y^n - y_{d,h}^n\|\|Q_W^{n-1}\| \le k\ C\ \|Y^n - y_{d,h}^n\|\|\nabla Q_W^{n-1}\| \\ &\le k\ \sigma\ \|\nabla Q_W^{n-1}\|^2 + k\ C(\sigma)\ \|Y^n - y_{d,h}^n\|^2, \end{aligned}$$

  using Cauchy-Schwartz, Poincaré-Wirtinger inequality (A.15) and Young's inequality (A.13);

- $$\begin{aligned} I_2 &= -k\ \gamma\left(\nabla Q_Y^n, \nabla Q_W^{n-1}\right) \le k\ \gamma\ \|\nabla Q_W^{n-1}\|\|\nabla Q_Y^n\| \\ &\le k\ \gamma\ \sigma\ \|\nabla Q_W^{n-1}\|^2 + k\ \gamma\ C(\sigma)\ \|\nabla Q_Y^n\|^2, \end{aligned}$$

  setting $\theta = kQ_W^{n-1}$ in (3.73f) evaluated at n, using the generalized Holder's inequality (A.14), (A.17) and Young's inequality (A.13);

- $$I_1 \le k\ C\ \|\nabla \mathbf{V}^n\|\|\nabla Q_Y^n\|\|\nabla Q_W^{n-1}\| \le k\ \sigma\ \|\nabla Q_W^{n-1}\|^2 + k\ C(\sigma)\ C_1\left(\boldsymbol{\mathcal{U}}\right)\|\nabla Q_Y^n\|^2,$$

  from the generalized Holder's inequality (A.14), (A.17), Young's inequality (A.13), Poincaré-Wirtinger inequality (A.15) and the estimate on $\|\nabla \mathbf{V}^n\|$ (3.22) derived in Lemma 3.5.

Inserting the estimate of I_1, I_2, I_3 in (3.86), with σ small enough, we get

$$\begin{aligned}
&\frac{\gamma}{2}\|\nabla Q_Y^{n-1}\|^2 - \frac{\gamma}{2}\|\nabla Q_Y^n\|^2 + \frac{\gamma}{2}\|\nabla Q_Y^{n-1} - \nabla Q_Y^n\|^2 + k\ C_1(\sigma)\ \|\nabla Q_W^{n-1}\|^2 \\
&+ \frac{k}{\delta}\left(\beta'_\delta(Y^n)Q_W^{n-1}, Q_W^{n-1}\right)_h \le k\ C_2(\sigma,\boldsymbol{\mathcal{U}})\ \|\nabla Q_Y^n\|^2 + k\ C_3(\sigma)\ \|Y^n - y_{d,h}^n\|^2,
\end{aligned} \tag{3.87}$$

where $C_2(\sigma,\boldsymbol{\mathcal{U}})$ is a constant which depends on σ and $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$. Summing over the index $n = N, \ldots, m$ in (3.87), we derive

$$\begin{aligned}
&\frac{\gamma}{2}\|\nabla Q_Y^{m-1}\|^2 + \frac{\gamma}{2}\sum_{n=N}^{m}\|\nabla Q_Y^{n-1} - \nabla Q_Y^n\|^2 + C_1(\sigma)\sum_{n=N}^{m} k\ \|\nabla Q_W^{n-1}\|^2 \\
&+ \sum_{n=N}^{m}\frac{k}{\delta}\left(\beta'_\delta(Y^n)Q_W^{n-1}, Q_W^{n-1}\right)_h \le C_2(\sigma,\boldsymbol{\mathcal{U}})\sum_{n=N}^{m} k\ \left[\|\nabla Q_Y^n\|^2 + \|Y^n - y_{d,h}^n\|^2\right],
\end{aligned} \tag{3.88}$$

for all $m = 1, \ldots, N$. In (3.88), by the definition (2.13), $0 \le \beta'_\delta \le 1$. Therefore

$$\left(\beta'_\delta(Y^n) Q_W^{n-1}, Q_W^{n-1}\right)_h \ge 0, \quad \forall n = 1, \ldots, N.$$

Moreover, from the assumption (3.75) and the estimate (3.34) in Lemma 3.7 of $\|Y^n\|$, the last term on the r.h.s. in (3.88) is such that

$$\sum_{n=N}^{m} k \, \|Y^n - y_{d,h}^n\|^2 \le C\left(\boldsymbol{\mathcal{U}}\right).$$

Thus, we can apply to (3.88) the discrete Gronwall's Lemma (see for example Lemma 1.4.2 in [73]). In this way, we derive the results (3.81), (3.82), (3.84) and (3.85). The optimality condition (3.73f) is equivalent to the following

$$Q_W^{n-1} = \gamma \tilde{\Delta}_h Q_Y^{n-1}, \quad \forall n = 1, \ldots, N.$$

Hence, the result (3.83) is just a consequence of the result (3.84).
Setting $\boldsymbol{\psi} = -k d_t \mathbf{Q}_\mathbf{V}^n$ in the optimality condition (3.73a), we derive

$$\begin{aligned} k\|d_t\mathbf{Q}_\mathbf{V}^n\|^2 + \frac{\nu}{2}\|\nabla\mathbf{Q}_\mathbf{V}^{n-1}\|^2 - \frac{\nu}{2}\|\nabla\mathbf{Q}_\mathbf{V}^n\|^2 + \frac{\nu}{2}\|\nabla\mathbf{Q}_\mathbf{V}^{n-1} - \nabla\mathbf{Q}_\mathbf{V}^n\|^2 \\ = -k\left(Y^n, \nabla Q_Y^n \cdot d_t\mathbf{Q}_\mathbf{V}^n\right). \end{aligned} \tag{3.89}$$

Using the generalized Holder's inequality (A.14), (A.17), interpolation inequality (A.51) and Young's inequality (A.13), we can estimate the r.h.s. in (3.89). We have

$$\begin{aligned} k \mid \left(Y^n, \nabla Q_Y^n \cdot d_t\mathbf{Q}_\mathbf{V}^{n-1}\right) \mid &\le k \, \|Y^n\|_{L^4}\|\nabla Q_Y^n\|_{L^4}\|d_t\mathbf{Q}_\mathbf{V}^n\| \\ &\le k \, C\|Y^n\|_{H_0}\left[\, \|\nabla Q_Y^n\| + \|\Delta_h Q_Y^n\| \,\right] \, \|d_t\mathbf{Q}_\mathbf{V}^n\| \\ \le k \, \sigma\|d_t\mathbf{Q}_\mathbf{V}^{n-1}\|^2 &+ k \, C(\sigma)\, \|Y^n\|_{H_0}^2 \left[\, \|\nabla Q_Y^n\|^2 + \|\Delta_h Q_Y^n\|^2\right]. \end{aligned} \tag{3.90}$$

Hence, using (3.90) with σ small enough, from (3.89) we get

$$\begin{aligned} k \, C_1(\sigma)\, \|d_t\mathbf{Q}_\mathbf{V}^n\|^2 + \frac{\nu}{2}\|\nabla\mathbf{Q}_\mathbf{V}^{n-1}\|^2 - \frac{\nu}{2}\|\nabla\mathbf{Q}_\mathbf{V}^n\|^2 + \frac{\nu}{2}\|\nabla\mathbf{Q}_\mathbf{V}^{n-1} - \nabla\mathbf{Q}_\mathbf{V}^n\|^2 \\ \le k \, C_2(\sigma)\, \|Y^n\|_{H_0}^2 \left[\, \|\nabla Q_Y^n\|^2 + \|\Delta_h Q_Y^n\|^2\right]. \end{aligned} \tag{3.91}$$

Summing over the index $n = N, \ldots, m$ in (3.91), we derive

$$\begin{aligned} C_1(\sigma) \sum_{n=N}^{m} k\|d_t\mathbf{Q}_\mathbf{V}^n\|^2 + \frac{\nu}{2}\|\nabla\mathbf{Q}_\mathbf{V}^{m-1}\|^2 + \frac{\nu}{2}\sum_{n=N}^{m} \|\nabla\mathbf{Q}_\mathbf{V}^{n-1} - \nabla\mathbf{Q}_\mathbf{V}^n\|^2 \\ \le C_2(\sigma) \sum_{n=N}^{m} k \, \|Y^n\|_{H_0}^2 \left[\, \|\nabla Q_Y^n\|^2 + \|\Delta_h Q_Y^n\|^2\right], \end{aligned} \tag{3.92}$$

for all $m = 1, \ldots, N$. Then, from the estimate (3.34) in Lemma 3.7, (3.82), (3.83), we realize that the results (3.76)-(3.78) hold.
We set $\boldsymbol{\psi} = k\mathbf{A}^h\mathbf{Q}_\mathbf{V}^{n-1}$ in (3.73a), where $\mathbf{A}^h$ is the discrete Stokes operator (A.40). In this way, we have

$$k \, \nu\|\mathbf{A}^h\mathbf{Q}_\mathbf{V}^{n-1}\|^2 \; = \; k \, \left(d_t\mathbf{Q}_\mathbf{V}^n, \mathbf{A}^h\mathbf{Q}_\mathbf{V}^{n-1}\right) + k \, \left(Y^n, \nabla Q_Y^n \cdot \mathbf{A}^h\mathbf{Q}_\mathbf{V}^{n-1}\right) \tag{3.93}$$

$$= \quad M_1 + M_2.$$

From generalized Holder's inequality (A.14), (A.17), interpolation inequality (A.51) and Young's inequality (A.13), we derive

$$\begin{aligned}
|M_1| &\leq k\ \sigma \|\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + C\,(\sigma)\,k\ \|d_t \mathbf{Q}_{\mathbf{V}}^n\|^2, \\
|M_2| &\leq k\ \|Y^n\|_{L^4} \|\nabla Q_Y^n\|_{L^4} \|\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\| \\
&\leq k\ C\ \|Y^n\|_{H_0} \left[\ \|\nabla Q_Y^n\| + \|\Delta_h Q_Y^n\|\ \right] \|\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\| \\
&\leq k\ \sigma \|\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + k\ C\,(\sigma)\,\|Y^n\|_{H_0}^2 \left[\ \|\nabla Q_Y^n\|^2 + \|\Delta_h Q_Y^n\|^2\right].
\end{aligned}$$

Then, using the estimates for M_1, M_2 in (3.93), with σ sufficiently small and summing on the index $n = 1, \ldots, N$, we get

$$C_1\,(\sigma) \sum_{n=1}^{N} k\ \|\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 \tag{3.94}$$

$$\leq C_2\,(\sigma) \sum_{n=1}^{N} k\ \|d_t \mathbf{Q}_{\mathbf{V}}^n\|^2 + C_3\,(\sigma) \sum_{n=1}^{N} k\ \|Y^n\|_{H_0}^2 \left[\ \|\nabla Q_Y^n\|^2 + \|\Delta_h Q_Y^n\|^2\right].$$

The results (3.34) in Lemma 3.7, (3.82), (3.83), (3.77), guarantee that the r.h.s in (3.94) is bounded. Hence, it hold

$$\sum_{n=1}^{N} k\ \|\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 \leq C\,(\boldsymbol{\mathcal{U}})\,.$$

Then, following [46] as in the proof of Lemma 3.5, we derive the result (3.79). The proof of the last estimate (3.80) is analogous to the one given in Lemma 3.5. ☐

3.3. Discrete Non-Smooth Optimal Control Problem

In this section, we study the non-smooth discrete optimal control Problem 3.1, which represents a discretized version of the non-smooth optimal control Problem 2.1. Using the results obtained in Section 3.2, we derive a system of first order optimality conditions of this problem as limit of the first order optimality conditions (3.72)-(3.74) of the regularized discrete optimal control Problem 3.2, for the regularization parameter $\delta \to 0^+$.

3.3.1. Properties of the State Equations of the Discrete Non-Smooth Optimal Control Problem

In the next Lemma 3.15, we show that the state equations (3.5),(3.6) of the the non-smooth discrete optimal control Problem 3.1 can be derived as limit of the state equations (3.10), (3.11) of the regularized discrete Problem 3.2, for the regularization parameter $\delta \to 0^+$. Next, in Lemma 3.16, we show that the equations derived have a unique solution.

Lemma 3.15. *Let us assume that there exists a constant $\tilde{C}$ independent of h, k and $\delta \in \left(0, \frac{1}{4}\right)$ such that*

$$\|\nabla \mathbf{v}_{0,h}\| + \|\nabla y_{0,h}\| \le \tilde{C}.$$

For any fixed h, k, consider a sequence $\{\delta_l\}_{l\in\mathbb{N}} \subset \left(0, \frac{1}{4}\right)$ such that $\delta_l \to 0^+$, a bounded sequence $\{\boldsymbol{\mathcal{U}}_l\}_{l\in\mathbb{N}} \subset L^2\left(\mathbf{L}^2\right)$ and the corresponding sequence of solution $\{(\boldsymbol{\mathcal{V}}_l, \mathcal{Y}_l, \mathcal{W}_l)\}_{l\in\mathbb{N}}$ of the state equations (3.10) (3.11) *of the regularized discrete optimal control Problem 3.2. Then, there exists a subsequence (labelled by index m), such that*

$$\boldsymbol{\mathcal{U}}_m \rightharpoonup \boldsymbol{\mathcal{U}}, \quad \text{in } L^2\left(\mathbf{L}^2\right), \tag{3.95}$$

$$\boldsymbol{\mathcal{V}}_m \to \boldsymbol{\mathcal{V}}, \quad \text{in } \mathbf{V}_h^{N+1}, \tag{3.96}$$

$$\mathcal{P}_m \to \mathcal{P}, \quad \text{in } P_h^N, \tag{3.97}$$

$$\mathcal{Y}_m \to \mathcal{Y}, \quad \text{in } P_h^{N+1}, \tag{3.98}$$

$$\mathcal{W}_m \to \mathcal{W}, \quad \text{in } Y_h^N. \tag{3.99}$$

and the limit variables satisfy

$$\sup_{n=0,\dots,N} \|\mathbf{V}^n\|_{\boldsymbol{\mathcal{D}}} \le C(\boldsymbol{\mathcal{U}}), \tag{3.100}$$

$$\sum_{n=1}^{N} k\, \|d_t \mathbf{V}^n\|^2 \le C(\boldsymbol{\mathcal{U}}), \tag{3.101}$$

$$\sum_{n=1}^{N} \|\mathbf{V}^n - \mathbf{V}^{n-1}\|_{\boldsymbol{\mathcal{D}}}^2 \le C(\boldsymbol{\mathcal{U}}), \tag{3.102}$$

$$\sum_{n=1}^{N} k\, \|\tilde{\Delta}_h \mathbf{V}^n\|^2 \le C(\boldsymbol{\mathcal{U}}), \tag{3.103}$$

$$\sup_{n=1,\dots,N} \left\| \sum_{i=1}^{n} k\ P^i \right\| \le C(\boldsymbol{\mathcal{U}}). \tag{3.104}$$

$$\sup_{n=0,\dots,N} \|Y^n\|_{H_0} \le C(\boldsymbol{\mathcal{U}}), \tag{3.105}$$

$$\sum_{i=1}^{N} k\, \|\nabla \mathcal{G} d_t Y^i\|^2 \le C(\boldsymbol{\mathcal{U}}), \tag{3.106}$$

$$\sum_{i=1}^{N} k\, \|\hat{\Delta}_h Y^i\|_h^2 \le C(\boldsymbol{\mathcal{U}}), \tag{3.107}$$

$$\sum_{i=1}^{N} \left\|Y^i - Y^{i-1}\right\|_{H_0}^2 \le C(\boldsymbol{\mathcal{U}}), \tag{3.108}$$

$$\sum_{i=1}^{N} k\, \|W^i\|_{H^1}^2 \le C(\boldsymbol{\mathcal{U}}). \tag{3.109}$$

where the constant $C(\boldsymbol{\mathcal{U}})$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$ but it is independent of $h, k, \delta \in \left(0, \frac{0}{4}\right)$. Furthermore, $(\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}, \boldsymbol{\mathcal{U}})$ satisfies the state equations (3.5), (3.6) of the discrete non-smooth optimal control Problem 3.1.

Proof. The statements (3.95)-(3.109) are a direct consequence of the results obtained in Lemma 3.5 and Lemma 3.7. We prove that the limit variables $\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}, \boldsymbol{\mathcal{U}}$ satisfy the state equations (3.5), (3.6) of the discrete non-smooth optimal control Problem 3.1. Considering the subsequences in (3.95)-(3.99), we have

$$
\begin{aligned}
&(3.110a) && (d_t\mathbf{V}_m^n, \boldsymbol{\psi}) + \nu\left(\nabla\mathbf{V}_m^n, \nabla\boldsymbol{\psi}\right) - \left(P_m^n, \nabla\cdot\boldsymbol{\psi}\right) - \frac{1}{k}\int_{t_{n-1}}^{t_n} \left(\boldsymbol{\mathcal{U}}_m, \boldsymbol{\psi}\right)\, dt = 0,\\
&(3.110b) && \mathbf{V}_m^0 = \mathbf{v}_{0,h},\\
&(3.110c) && \left(\nabla\cdot\mathbf{V}_m^n, \phi\right) = 0,\\
&(3.110d) && \left(d_tY_m^n, \eta\right)_h + \gamma\left(\nabla W_m^n, \nabla\eta\right) - \left(Y_m^{n-1}\mathbf{V}_m^{n-1}, \nabla\eta\right) = 0,\\
&(3.110e) && Y_m^0 = y_{0,h},\\
&(3.110f) && -\left(W_m^n + Y_m^{n-1}, \theta\right)_h + \varepsilon^2\left(\nabla Y_m^n, \nabla\theta\right) + \frac{1}{\delta_m}\left(\beta_{\delta_m}\left(Y_m^n\right), \theta\right) = 0,
\end{aligned}
$$

As $m \to +\infty$, the convergence of the equations (3.110a), (3.110c) to equations (3.5a), (3.5c) is straightforward. This is true also about the convergence of the nonlinear term in (3.110d), (3.110f) to the corresponding terms in (3.6a), (3.6c). We show the convergence of the nonlinear terms. Regarding the third term in (3.110d), we note that

$$
\begin{aligned}
O_1 &= \Big|\sum_{n=1}^{N} k\ \left(Y_m^{n-1}, \mathbf{V}_m^{n-1}\cdot\nabla\eta^n\right) - \sum_{n=1}^{N} k\ \left(Y^{n-1}, \mathbf{V}^{n-1}\cdot\nabla\eta^n\right)\Big|\\
&\le \sum_{n=1}^{N} k\ \Big|\left(Y_m^{n-1} - Y^{n-1}, \mathbf{V}_m^{n-1}\cdot\nabla\eta^n\right) + \left(Y^{n-1}, \left[\mathbf{V}_m^{n-1} - \mathbf{V}^{n-1}\right]\cdot\nabla\eta^n\right)\Big|\\
&\le C\sum_{n=1}^{N} k\ \left[\|Y_m^{n-1} - Y^{n-1}\|_{H_0}\|\mathbf{V}_m^{n-1}\|_{\mathbf{H}_0^1} + \|Y_m^{n-1}\|_{H_0}\|\mathbf{V}_m^{n-1} - \mathbf{V}^{n-1}\|_{\mathbf{H}_0^1}\right]\|\nabla\eta^n\|.
\end{aligned}
$$

Then, using $\mathcal{Y}_m \to \mathcal{Y}$ in Y_h^{N+1} and $\boldsymbol{\mathcal{V}}_m \to \boldsymbol{\mathcal{V}}$ in $\mathbf{V}_h^{N+1}$, we infer that

$$
O_1 \to 0, \tag{3.111}
$$

as $m \to +\infty$. Therefore (3.110d) converges to (3.6a) as $m \to +\infty$. We set $\theta \in K_h$ in (3.110f). Then, as in the proof of Theorem 2.11, using the definition (2.12) and the property (2.17) of the function β_δ, we can write

$$
\begin{aligned}
&-\left(W_m^n + Y_m^{n-1}, \theta - Y_m^n\right)_h + \varepsilon^2\left(\nabla Y_m^n, \nabla\theta - \nabla Y_m^n\right)\\
&\qquad = \frac{1}{\delta_m}\left(\beta_{\delta_m}(\theta) - \beta_{\delta_m}\left(Y_m^n\right), \theta - Y_m^n\right) \ge 0.
\end{aligned}
$$

Last equation yields (3.6c) as $m \to +\infty$. Finally we prove that (3.6d) holds, i.e., $-1 \le Y^n \le 1$, for all $n = 1, \ldots, N$. From (3.56) in Theorem 3.8, we have

$$
\sum_{n=1}^{N} k\ \left\|\beta_{\delta_m}\left(Y_m^n\right)\right\|^2 \le C\ \delta_m^2
$$

and consequently

$$\lim_{m\to+\infty}\sum_{n=1}^{N} k\left\|\beta_{\delta_m}\left(Y_m^n\right)\right\|^2 = 0.$$

Then, using the function $\hat{f}$ defined in (2.69), we can write

$$\left|\sum_{n=1}^{N} k\left(\hat{f}\left(Y^n\right),\theta^n\right)\right|$$

$$\leq \sum_{n=1}^{N} k\left[\|\hat{f}\left(Y^n\right)-\hat{f}\left(Y_m^n\right)\|+\|\hat{f}\left(Y_m^n\right)-\beta_{\delta_m}\left(Y_m^n\right)\|+\|\beta_{\delta_m}\left(Y_m^n\right)\|\right]\|\theta^n\| \leq$$

$$\leq C\left[\left(\sum_{n=1}^{N} k\,\|Y^n-Y_m^n\|^2\right)^{\frac{1}{2}}+T^{\frac{1}{2}}\,\delta_m\right]\left(\sum_{n=1}^{N} k\,\|\theta^n\|^2\right)^{\frac{1}{2}} \to 0,$$

as $m\to+\infty$, for all $\theta^n\in Y_h,\ n=1,\ldots,N$. Therefore

$$\hat{f}\left(Y^n\right)\equiv 0,\quad \forall n=1,\ldots,N.$$

Hence, from the definition (2.69) of the function $\hat{f}$, we infer that (3.6d) holds. □

In the next Lemma 3.16, we derive the properties of the solution of the state equations (3.5), (3.6) of the discrete non-smooth optimal control Problem 3.1.

Lemma 3.16. *For any fixed h,k, $\boldsymbol{\mathcal{U}}\in L^2\left(\mathbf{L}^2\right)$, the system of the state equations (3.5), (3.6) has a unique solution $(\boldsymbol{\mathcal{V}},\mathcal{P},\mathcal{Y},\mathcal{W})\in\mathbf{V}_h^{N+1}\times P_h^N\times P_h^{N+1}\times Y_h^N$. Furthermore, if there exists a constant $\tilde{C}$ independent of h,k, such that*

$$\|\nabla\mathbf{v}_{0,h}\|+\|\nabla y_{0,h}\|\leq\tilde{C}, \tag{3.112}$$

there exists a constant $C\left(\boldsymbol{\mathcal{U}}\right)$ which depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$ but it is independent of h,k, such that $(\boldsymbol{\mathcal{V}},\mathcal{P},\mathcal{Y},\mathcal{W})$ satisfies the estimates (3.100)-(3.109) in Lemma 3.15.

Proof. As a consequence of Lemma 3.15, the system of equations (3.5), (3.6) has a solution $(\boldsymbol{\mathcal{V}},\mathcal{P},\mathcal{Y},\mathcal{W})\in\mathbf{V}_h^{N+1}\times P_h^N\times P_h^{N+1}\times Y_h^N$. Moreover, if the assumption (3.112) above holds, this solution satisfies the estimates (3.100)-(3.109) in Lemma 3.15. It remains to show the uniqueness of the solution. Given $\boldsymbol{\mathcal{U}}\in L^2\left(\mathbf{L}^2\right)$, using the linearity of the equations (3.5), it is straightforward to prove that $(\boldsymbol{\mathcal{V}},\mathcal{P})\in\mathbf{V}_h^{N+1}\times P_h^N$ is unique. To prove the uniqueness of $(\mathcal{Y},\mathcal{W})\in P_h^{N+1}\times Y_h^N$, we proceed by induction. Let us suppose that, given a time level n, the solution (Y^{n-1},W^{n-1}) of (3.6) at the time level $n-1$ is unique. Then, at the time level n, we consider two possible solutions $\left(Y_1^n,W_1^n\right),\left(Y_2^n,W_2^n\right)\in P_h\times Y_h$ of (3.6) and we define

$$S_Y^n=Y_2^n-Y_1^n,\qquad S_W^n=W_2^n-W_1^n.$$

We subtract (3.6a) with $Y^n=Y_1^n$ to (3.6a) with $Y^n=Y_2^n$. In this way, we get

$$\left(S_Y^n,\eta\right)_h+k\gamma\left(\nabla S_W^n,\nabla\eta\right)=0. \tag{3.113}$$

We add (3.6c) with $Y^n = Y_1^n, W^n = W_1^n, \theta = Y_2^n$ to (3.6c) with $Y^n = Y_2^n, W^n = W_2^n, \theta = Y_1^n$. Thus, we have

$$-\left(S_W^n, S_Y^n\right)_h + \varepsilon^2\|\nabla S_Y^n\|^2 \leq \|S_Y^n\|_h^2. \tag{3.114}$$

Substituting $\eta = \hat{\mathcal{G}}_h S_Y^n$ in (3.113), we have

$$\left(S_Y^n, \hat{\mathcal{G}}_h S_Y^n\right)_h = -k\ \gamma \left(\nabla S_W^n, \nabla \hat{\mathcal{G}}_h S_Y^n\right),$$

which is equivalent, by the definition (A.33) of the discrete Green's operator $\hat{\mathcal{G}}_h$, to

$$\|\nabla \hat{\mathcal{G}}_h S_Y^n\|^2 = -k\gamma \left(S_Y^n, S_W^n\right)_h. \tag{3.115}$$

Multiplying (3.114) by $k\gamma$ and using (3.115), we can write

$$\|\nabla \hat{\mathcal{G}}_h S_Y^n\|^2 + k\gamma\varepsilon^2\|\nabla d_Y^n\|^2 \leq k\gamma\|S_Y^n\|_h^2. \tag{3.116}$$

By Young's inequality and the definition (A.33) of the discrete Green's operator $\hat{\mathcal{G}}_h$, we infer that

$$\|S_Y^n\|_h^2 = \left(\nabla S_Y^n, \nabla \hat{\mathcal{G}}_h S_Y^n\right) \leq \frac{\varepsilon^2}{2}\|\nabla S_Y^n\|^2 + \frac{1}{2\varepsilon^2}\|\nabla \hat{\mathcal{G}}_h S_Y^n\|^2. \tag{3.117}$$

Hence, using (3.117) in (3.116), we have that, for all k

$$\|\nabla \hat{\mathcal{G}}_h S_Y^n\|^2 \leq k\frac{\gamma}{2\varepsilon^2}\|\nabla \hat{\mathcal{G}}_h S_Y^n\|^2.$$

Therefore $\nabla \hat{\mathcal{G}}_h S_Y^n = 0$, which means $S_Y^n = Y_2^n - Y_1^n = 0$. Moreover, setting $\eta = S_W^n$ in (3.113), we derive

$$\nabla S_W^n = 0,$$

i.e., S_W^n is equal to some constant. In order to show that this constant is indeed zero, we consider $\xi \in Y_h$, such that

$$Y^n(x_j) = \pm 1 \quad \Rightarrow \quad \xi(x_j) = 0,$$

for all x_j vertices of $\mathcal{T}_h$. Then, we substitute $\theta_\pm = Y^n \pm \rho\xi$ in (3.6c), with ρ constant and small enough so that $-1 \leq \theta \leq 1$. In this way, we get

$$\begin{aligned} \varepsilon^2\left(\nabla Y^n, \nabla\xi\right) &\geq \left(Y^{n-1} + W^n, \xi\right)_h, \\ -\quad \varepsilon^2\left(\nabla Y^n, \nabla\xi\right) &\geq -\left(Y^{n-1} + W^n, \xi\right)_h, \end{aligned}$$

which imply

$$\left(\nabla Y^n, \nabla\xi\right) = \left(Y^{n-1} + W^n, \xi\right)_h. \tag{3.118}$$

Subtracting (3.118) with $W^n = W_1^n$ to (3.118) with $W^n = W_2^n$, we have

$$\left(S_W^n, \xi\right)_h = S_W^n\left(1, \xi\right)_h = 0.$$

Thus, we infer $S_W^n = W_2^n - W_1^n = 0$. We have shown that, if (Y^{n-1}, W^{n-1}) is the unique solution of (3.6) at a time level $n-1$, then (3.6) have a unique solution (Y^n, W^n) even at a time level n. Hence, using the initial condition $Y^0 = y_{0,h}$, by induction, we derive that the solution $(\mathcal{Y}, \mathcal{W}) \in P_h^{N+1} \times Y_h^N$ of (3.6) is unique. □

3.3.2. Minimizers of the Discrete Non-Smooth Optimal Control Problem

As in Chapter 2, in the next Lemma 3.17, we prove the existence of solutions of the discrete non-smooth optimal control Problem Problem 3.1.

Lemma 3.17 (existence of minimizers). *For any given h, k, the optimal control Problem 3.1 admits a solution.*

Proof. The proof is analogous to the one of Theorem 2.5 in Chapter 2. □

In the next Theorem 3.18 we show the relationship between the solutions of the regularized discrete optimal control Problem 3.2 and the solutions of non-smooth discrete optimal control Problem 3.1: there exists a sequence of solutions of the regularized Problem 3.2, which converges to a solution of the non-smooth Problem 3.1, for the regularization parameter $\delta \to 0^+$.

Theorem 3.18. *Let us assume that there exists a constant $\tilde{C}$ independent on $h, k, \delta \in \left(0, \frac{1}{4}\right)$, such that*

$$\|\nabla \mathbf{v}_{0,h}\| + \|\nabla y_{0,h}\| + \sum_{n=1}^{N} k \; \|y_{d,h}^n\|^2 \leq \tilde{C}. \tag{3.119}$$

Furthermore, for any fixed h, k, let us consider a sequence $\{\delta_l\}_{l\in\mathbb{N}} \subset \left(0, \frac{1}{4}\right)$, such that $\delta_l \to 0^+$ and the corresponding sequence of solutions of the regularized discrete optimal control problem 3.2,

$$\left\{\left(\bar{\boldsymbol{\mathcal{X}}}_l, \bar{\boldsymbol{\mathcal{U}}}_l\right)\right\}_{l\in\mathbb{N}} \subset \mathbf{X}_{h,k} \times L^2\left(\mathbf{L}^2\right).$$

Then, it is possible to extract a subsequence (labelled with an index m), such that, as $m \to +\infty$,

$$\begin{aligned} \bar{\boldsymbol{\mathcal{X}}}_m &\to \bar{\boldsymbol{\mathcal{X}}}, && \text{in } \mathbf{X}_{h,k}, \\ \bar{\boldsymbol{\mathcal{U}}}_m &\rightharpoonup \bar{\boldsymbol{\mathcal{U}}}, && \text{in } L^2\left(\mathbf{L}^2\right), \end{aligned}$$

where $\left(\bar{\boldsymbol{\mathcal{X}}}, \bar{\boldsymbol{\mathcal{U}}}\right)$ is a solution of the discrete non-smooth optimal control Problem 3.1.

Proof. Given the sequences $\{\delta_l\}_{l\in\mathbb{N}}$ and $\left\{\left(\bar{\boldsymbol{\mathcal{X}}}_l, \bar{\boldsymbol{\mathcal{U}}}_l\right)\right\}_{l\in\mathbb{N}} = \left\{\left(s_{\delta_l,h,k}\left(\bar{\boldsymbol{\mathcal{U}}}_l\right), \bar{\boldsymbol{\mathcal{U}}}_l\right)\right\}_{l\in\mathbb{N}}$, where $s_{\delta,h,k} : L^2\left(\mathbf{L}^2\right) \to \mathbf{X}_{h,k}$ is the solution operator (3.21) associated to the state equations of the regularized Problem 3.2, we consider $\boldsymbol{\mathcal{U}} \in L^2\left(\mathbf{L}^2\right)$ such that

$$\boldsymbol{\mathcal{U}}(t) = \mathbf{u} \in \mathbf{L}^2, \quad \forall\, t \in [0, T].$$

Then, by the definition (3.4) of the discrete cost functional $J_{h,k}$, the results of Lemma 3.7 and the assumption (3.119), there exists a constant $C(\boldsymbol{\mathcal{U}})$, such that

$$\|\bar{\boldsymbol{\mathcal{U}}}_l\|^2_{L^2(\mathbf{L}^2)} \leq J_{h,k}\left(s_{\delta_l,h,k}\left(\bar{\boldsymbol{\mathcal{U}}}_l\right), \bar{\boldsymbol{\mathcal{U}}}_l\right) \leq J_{h,k}\left(s_{\delta_l,h,k}\left(\boldsymbol{\mathcal{U}}\right), \boldsymbol{\mathcal{U}}\right) \leq C\left(\boldsymbol{\mathcal{U}}\right) + \frac{\alpha}{2}\|\boldsymbol{\mathcal{U}}\|^2_{L^2(\mathbf{L}^2)}.$$

The constant $C(\boldsymbol{\mathcal{U}})$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$ and it is independent of $h, k, \delta_l \in \left(0, \frac{1}{4}\right)$. Therefore, the sequence $\left\{\bar{\boldsymbol{\mathcal{U}}}_l\right\}_{l\in\mathbb{N}}$ is bounded in $L^2\left(\mathbf{L}^2\right)$ and by the estimates established in Lemmas 3.5, 3.7, the sequence $\left\{\left(s_{\delta_l,h,k}\left(\bar{\boldsymbol{\mathcal{U}}}_l\right),\bar{\boldsymbol{\mathcal{U}}}_l\right)\right\}_{l\in\mathbb{N}}$ is bounded in $\mathbf{X}_{h,k}\times L^2\left(\mathbf{L}^2\right)$. Hence, by Theorem 3.15, there exists a subsequence (labelled by index m), which is such that

$$\left(s_{\delta_m,h,k}\left(\bar{\boldsymbol{\mathcal{U}}}_m\right),\bar{\boldsymbol{\mathcal{U}}}_m\right)=\left(\bar{\boldsymbol{\mathcal{X}}}_m,\bar{\boldsymbol{\mathcal{U}}}_m\right)\rightharpoonup\left(\bar{\boldsymbol{\mathcal{X}}},\bar{\boldsymbol{\mathcal{U}}}\right),\quad\text{in }\mathbf{X}_{h,k}\times L^2\left(\mathbf{L}^2\right),$$

and $\left(\bar{\boldsymbol{\mathcal{X}}},\bar{\boldsymbol{\mathcal{U}}}\right)$ is a solution of the state equations (3.5), (3.6) of the discrete non-smooth optimal control Problem 3.1. Then, using the same procedure applied in the proof of Theorem 2.13, it easy to prove that $\left(\bar{\boldsymbol{\mathcal{X}}},\bar{\boldsymbol{\mathcal{U}}}\right)$ is also a minimizer of Problem 3.1. □

As we have done in Chapter 2, in the following we present an equivalent formulation of the discrete non-smooth optimal control Problem 3.1. In this formulation, we introduce two Lagrange multipliers in the state equations

$$\mathcal{B}_r,\mathcal{B}_l\in Y_h^N.$$

In this way, the optimal control Problem 3.1 will assume the structure of a mathematical program with complementarity constraints. In the optimality conditions for this problem, the Lagrange multipliers $\mathcal{B}_r,\mathcal{B}_l$ will be related to a variable $\varLambda\in Y_h^N$. Then, just the relationship between $\mathcal{B}_r,\mathcal{B}_l$ and $\varLambda$ will be one of the key issue for the numerical solution of the optimality conditions of non-smooth optimal control Problem 3.1. We define the spaces

$$\mathbf{R}_{h,k}=\mathbf{X}_{h,k}\times Y_h^N\times Y_h^N,$$

with elements $\boldsymbol{\mathcal{R}}=\left(\boldsymbol{\mathcal{X}},\mathcal{B}_r,\mathcal{B}_l\right)$ and

$$K_h^+=\left\{Z\in Y_h:\ Z\geq 0\right\}.$$

Furthermore we consider the cost functional $\tilde{J}_{h,k}:\mathbf{R}_{h,k}\times L^2\left(\mathbf{L}^2\right)\to\mathbb{R}$, which is such that

$$\tilde{J}_{h,k}\left(\boldsymbol{\mathcal{R}},\boldsymbol{\mathcal{U}}\right)\equiv J_{h,k}\left(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{U}}\right).$$

Then, we consider the following problem:

Problem 3.19. *Find* $\left(\bar{\boldsymbol{\mathcal{R}}},\bar{\boldsymbol{\mathcal{U}}}\right)\in\mathbf{R}_{h,k}\times L^2\left(\mathbf{L}^2\right)$, *such that*

$$\min_{\left(\boldsymbol{\mathcal{R}},\boldsymbol{\mathcal{U}}\right)\in\mathbf{R}_{h,k}\times L^2\left(\mathbf{L}^2\right)}\tilde{J}_{h,k}\left(\boldsymbol{\mathcal{R}},\boldsymbol{\mathcal{U}}\right)=\tilde{J}_{h,k}\left(\bar{\boldsymbol{\mathcal{R}}},\bar{\boldsymbol{\mathcal{U}}}\right),$$

subject to

$$\text{(3.120a)}\qquad\left(d_t\mathbf{V}^n,\boldsymbol{\psi}\right)+\nu\left(\nabla\mathbf{V}^n,\nabla\boldsymbol{\psi}\right)-\left(P^n,\nabla\cdot\boldsymbol{\psi}\right)-\frac{1}{k}\int_{t_{n-1}}^{t_n}\left(\boldsymbol{\mathcal{U}},\boldsymbol{\psi}\right)\,dt=0,$$

$$\text{(3.120b)}\qquad\mathbf{V}^0=\mathbf{v}_{0,h},$$

$$\text{(3.120c)}\qquad\left(\nabla\cdot\mathbf{V}^n,\phi\right)=0,$$

$$\text{(3.121a)}\qquad\left(d_tY^n,\eta\right)_h+\gamma\left(\nabla W^n,\nabla\eta\right)-\left(Y^{n-1}\mathbf{V}^{n-1},\nabla\eta\right)=0,$$

(3.121b) $$Y^0 = y_{0,h},$$

(3.121c) $$-\left(W^n + Y^{n-1}, \theta\right)_h + \varepsilon^2 \left(\nabla Y^n, \nabla\theta\right) + \left(B^n, \theta\right)_h = 0,$$

(3.121d) $$Y^n \in K_h,$$

(3.121e) $$B^n = B_r^n - B_l^n, \quad \text{with} \quad B_r^n, B_l^n \in K_h^+,$$

(3.121f) $$\left[B_r^n \left(1 - Y^n\right)\right](x_j) = 0,$$

(3.121g) $$\left[B_l^n \left(1 + Y^n\right)\right](x_j) = 0,$$

for all $\boldsymbol{\psi} \in \mathbf{V}_h,\ \phi \in P_h,\ \eta, \theta \in Y_h,\ j = 1, \ldots, \mathcal{N}_h,\ n = 1, \ldots, N.$

Lemma 3.20. *Problems 3.1 and 3.19 are equivalent.*

Proof. We need to show the equivalence between the state equations (3.5),(3.6) and the state equations (3.120),(3.121).
First we prove that every solution of (3.5),(3.6) is also a solution of (3.120),(3.121). Given $\boldsymbol{\mathcal{U}} \in L^2(\mathbf{L}^2)$, we consider a sequence $\{\delta_i\}_{i\in\mathbb{N}} \subset \left(0, \frac{1}{4}\right)$, such that $\delta_i \to 0^+$ and the corresponding sequence of solutions of the regularized discrete state equations (3.10), (3.11),

$$\left\{\left(\boldsymbol{\mathcal{V}}_i, \mathcal{P}_i, \mathcal{Y}_i, \mathcal{W}_i\right)\right\}_{i\in\mathbb{N}} = \left\{\boldsymbol{\mathcal{X}}_i\right\}_{i\in\mathbb{N}} = \left\{s_{\delta_i,h,k}\left(\boldsymbol{\mathcal{U}}\right)\right\}_{i\in\mathbb{N}}.$$

By Theorem 3.15, we know that there exists a subsequence (labelled with index m) such that

$$\boldsymbol{\mathcal{X}}_m \to \boldsymbol{\mathcal{X}} \in \mathbf{X}_{h,k},$$

where $\boldsymbol{\mathcal{X}} = (\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W})$ is the unique solution of (3.5),(3.6). It easy to realize that $\boldsymbol{\mathcal{X}}$, together some $\mathcal{B}_r, \mathcal{B}_l \in Y_h^N$ satisfy (3.120),(3.121). Indeed, using the result (3.56) in Lemma 3.8, there exist $\mathcal{B}_r, \mathcal{B}_l \in Y_h^N$, such that

(3.122) $$\frac{1}{\delta_m}\beta_{r\delta_m}\left(Y_m^n\right)(x_j) \quad\to\quad B_r^n(x_j) \geq 0,$$

(3.123) $$\frac{1}{\delta_m}\beta_{l\delta_m}\left(Y_m^n\right)(x_j) \quad\to\quad B_l^n(x_j) \geq 0,$$

as $m \to +\infty$, for all $j = 1, \ldots, \mathcal{N}_h$, $n = 1, \ldots, N$, where the functions $\beta_{r\delta}, \beta_{l\delta}$ are defined in (2.84), (2.85). Furthermore $(\mathcal{Y}, \mathcal{W}, \mathcal{B})$ satisfies (3.121c). In order to prove that (3.121f) is satisfied, using $Y^n \in K_h$, we note that

(3.124) $$\frac{1}{\delta_m}\left[\beta_{r\delta_m}\left(Y_m^n\right)\left(1 - Y_m^n\right)\right](x_j) \to \left[B_r^n\left(1 - Y^n\right)\right](x_j) \geq 0,$$

as $m \to +\infty$, for all $j = 1, \ldots, \mathcal{N}_h$, $n = 1, \ldots, N$. Noting

$$\beta_{r\delta}(1) = 0,$$

and that $\beta_{r\delta}$ is monotone increasing function, we get

(3.125) $$\left[\beta_{r\delta_m}\left(Y_m^n\right)\left(1 - Y_m^n\right)\right](x_j) = -\left[\beta_{r\delta_m}(1) - \beta_{r\delta_m}\left(Y_m^n\right)\right]\left[\left(1 - Y_m^n\right)\right](x_j) \leq 0.$$

So, by comparison between (3.124) and (3.125), we infer that (3.121f) holds. In the same way, it is possible to derive that (3.121g) is satisfied.

We perform the second step of the proof demonstrating that every solution $(\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}, \mathcal{B})$ of (3.120),(3.121) is also a solution of (3.5),(3.6). We need to prove just that $(\mathcal{Y}, \mathcal{W})$ solves (3.6c). Setting in (3.121c) $\theta = \tilde{\theta} - Y^n$, with $\tilde{\theta} \in K_h$, we have

$$(3.126) \quad -\left(W^n + Y^{n-1}, \tilde{\theta} - Y^n\right)_h + \varepsilon^2 \left(\nabla Y^n, \nabla\tilde{\theta} - \nabla Y^n\right) = -\left(B^n, \tilde{\theta} - Y^n\right)_h,$$

where, using a quadrature formula with weights $\omega_j \geq 0$,

$$-\left(B^n, \tilde{\theta} - Y^n\right)_h = -\sum_{j=1}^{N_h} \omega_j \; B^n(x_j) \left[\tilde{\theta} - Y^n\right](x_j).$$

From (3.121f), (3.121g), we get that for all x_j vertices of $\mathcal{T}_h$

$$-B^n(x_j)\left[\tilde{\theta} - Y^n\right](x_j) = \begin{cases} B_l^n(x_j)\left[\tilde{\theta}(x_j) + 1\right] \geq 0, & \text{if } Y^n(x_j) = -1, \\ 0, & \text{if } -1 < Y^n(x_j) < 1, \\ -B_r^n(x_j)\left[\tilde{\theta}(x_j) - 1\right] \geq 0, & \text{if } Y^n(x_j) = 1. \end{cases}$$

Hence, in (3.126) $-\left(B^n, \tilde{\theta} - Y^n\right)_h \geq 0$ and equation (3.6c) holds. □

3.3.3. Optimality Conditions for the Discrete Non-Smooth Optimal Control Problem

In this section we derive the first order optimality conditions of the discrete non-smooth Problem 3.1 as limit of the optimality conditions (3.72)-(3.74) of the regularized discrete Problem 3.2, for the regularization parameter $\delta \to 0^+$.

Theorem 3.21. *Let us assume that there is a constant $\tilde{C}$, independent of $h, k, \delta \in \left(0, \frac{1}{4}\right)$, such that*

$$(3.127) \qquad \|\nabla y_{0,h}\| + \|\nabla \mathbf{v}_{0,h}\| + \sum_{n=1}^{N} k \; \|y_{d,h}^n\|^2 \leq \tilde{C}.$$

Let $\{\delta_i\}_{i\in\mathbb{N}} \subset \left(0, \frac{1}{4}\right)$ be a sequence such that $\delta_i \to 0^+$ and

$$\{(\boldsymbol{\mathcal{X}}_i, \boldsymbol{\mathcal{U}}_i)\}_{i\in\mathbb{N}} = \{(\boldsymbol{\mathcal{V}}_i, \mathcal{P}_i, \mathcal{Y}_i, \mathcal{W}_i, \boldsymbol{\mathcal{U}}_i)\}_{i\in\mathbb{N}} \subset \mathbf{X}_{h,k} \times L^2\left(\mathbf{L}^2\right),$$

be the corresponding sequence of solution of the regularized discrete optimal control Problem 3.2. Let

$$\{\boldsymbol{\mathcal{Q}}_i\}_{i\in\mathbb{N}} = \{(\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}i}, \mathcal{Q}_{\mathcal{P}i}, \mathcal{Q}_{\mathcal{Y}i}, \mathcal{Q}_{\mathcal{W}i})\}_{i\in\mathbb{N}} \subset \mathbf{X}_{h,k},$$

be the sequence of adjoint variables such that the triple $\boldsymbol{\mathcal{X}}_i, \boldsymbol{\mathcal{U}}_i, \boldsymbol{\mathcal{Q}}_i$ satisfies the optimality conditions (3.72)-(3.74) of the regularized Problem 3.2, for all $i \in \mathbb{N}$. Then, there exists a subsequence (labelled by the index m) $\{(\boldsymbol{\mathcal{X}}_m, \boldsymbol{\mathcal{U}}_m, \boldsymbol{\mathcal{Q}}_m)\}_m$, a solution of the discrete non-smooth optimal control Problem 3.19,

$$(\boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{U}}) = (\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}, \mathcal{B}_r, \mathcal{B}_l, \boldsymbol{\mathcal{U}}) \in \mathbf{R}_{h,k} \times L^2\left(\mathbf{L}^2\right),$$

and a set of variables

$$(\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}}, \mathcal{Q}_{\mathcal{P}}, \mathcal{Q}_{\mathcal{Y}}, \mathcal{Q}_{\mathcal{W}}, \varLambda) \in \mathbf{X}_{h,k} \times Y_h^N,$$

such that, as $m \to +\infty$,

$$\boldsymbol{\mathcal{V}}_m \to \boldsymbol{\mathcal{V}}, \quad \text{in } \mathbf{V}_h^{N+1}, \tag{3.128}$$
$$\mathcal{P}_m \to \mathcal{P}, \quad \text{in } P_h^N, \tag{3.129}$$
$$\mathcal{Y}_m \to \mathcal{Y}, \quad \text{in } Y_h^{N+1}, \tag{3.130}$$
$$\mathcal{W}_m \to \mathcal{W}, \quad \text{in } Y_h^N. \tag{3.131}$$
$$\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}m} \to \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}}, \quad \text{in } \mathbf{V}_h^{N+1}, \tag{3.132}$$
$$\mathcal{Q}_{\mathcal{P}m} \to \mathcal{Q}_{\mathcal{P}}, \quad \text{in } P_h^N, \tag{3.133}$$
$$\mathcal{Q}_{\mathcal{Y}m} \to \mathcal{Q}_{\mathcal{Y}}, \quad \text{in } P_h^{N+1}, \tag{3.134}$$
$$\mathcal{Q}_{\mathcal{W}m} \to \mathcal{Q}_{\mathcal{W}}, \quad \text{in } Y_h^N. \tag{3.135}$$
$$\boldsymbol{\mathcal{U}}_m \to \boldsymbol{\mathcal{U}}, \quad \text{in } \mathbf{V}_h^N, \tag{3.136}$$

$$\frac{1}{\delta_m}\left[\beta_{\delta_m}\left(Y_m^n\right)\right](x_j) \quad \to \quad B^n(x_j) = B_r^n(x_j) - B_l^n(x_j), \tag{3.137}$$
$$\frac{1}{\delta_m}\left[\beta'_{\delta_m}\left(Y_m^n\right) Q_W^{n-1}\right](x_j) \quad \to \quad \Lambda^{n-1}(x_j), \tag{3.138}$$

for all $j = 1, \ldots, \mathcal{N}_h$, $n = 1, \ldots, N$. *Furthermore*

$$(\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}, \mathcal{B}_r, \mathcal{B}_l, \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}}, \mathcal{Q}_{\mathcal{P}}, \mathcal{Q}_{\mathcal{Y}}, \mathcal{Q}_{\mathcal{W}}, \varLambda),$$

satisfies the following system of optimality conditions

$$(d_t\mathbf{V}^n, \boldsymbol{\psi}) + \nu(\nabla\mathbf{V}^n, \nabla\boldsymbol{\psi}) - (P^n, \nabla\cdot\boldsymbol{\psi}) - (\mathbf{U}^n, \boldsymbol{\psi}) = 0, \tag{3.139a}$$
$$\mathbf{V}^0 = \mathbf{v}_{0,h}, \tag{3.139b}$$
$$(\nabla\cdot\mathbf{V}^n, \phi) = 0, \tag{3.139c}$$
$$(d_tY^n, \eta)_h + \gamma(\nabla W^n, \nabla\eta) - \left(Y^{n-1}\mathbf{V}^{n-1}, \nabla\eta\right) = 0, \tag{3.139d}$$
$$Y^0 = y_{0,h}, \tag{3.139e}$$
$$-\left(W^n + Y^{n-1}, \theta\right)_h + \varepsilon^2(\nabla Y^n, \nabla\theta) + (B^n, \theta)_h = 0, \tag{3.139f}$$
$$Y^n \in K_h, \tag{3.139g}$$
$$B^n = B_r^n - B_l^n, \quad \text{with} \quad B_r^n, B_l^n \in K_h^+, \tag{3.139h}$$
$$\left[B_r^n\left(1 - Y^n\right)\right](x_j) = 0, \tag{3.139i}$$
$$\left[B_l^n\left(1 + Y^n\right)\right](x_j) = 0, \tag{3.139j}$$

for all $\boldsymbol{\psi} \in \mathbf{V}_h$, $\phi \in P_h$, $\eta, \theta \in Y_h$, $j = 1, \ldots, \mathcal{N}_h$, $n = 1, \ldots, N$,

$$-(d_t\mathbf{Q}_{\mathbf{V}}^n, \boldsymbol{\psi}) + \nu\left(\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}, \nabla\boldsymbol{\psi}\right) + \left(Q_P^{n-1}, \nabla\cdot\boldsymbol{\psi}\right) - (Y^n\nabla Q_Y^n, \boldsymbol{\psi}) = 0, \tag{3.140a}$$
$$\mathbf{Q}_{\mathbf{V}}^N = 0, \tag{3.140b}$$
$$\left(\nabla\cdot\mathbf{Q}_{\mathbf{V}}^{n-1}, \phi\right) = 0, \tag{3.140c}$$

$$(3.140\text{d})\qquad \begin{aligned} -\left(d_t Q_Y^n, \eta\right)_h - \varepsilon^2\left(\nabla Q_W^{n-1}, \nabla\eta\right) + \left(Q_W^n, \eta\right)_h \\ -\left(\nabla Q_Y^n \cdot \mathbf{V}^n, \eta\right) + \left(Y^n - y_{d,h}^n, \eta\right) - \left(\Lambda^{n-1}, \eta\right)_h = 0, \end{aligned}$$

$$(3.140\text{e})\qquad Q_Y^N = 0,$$

$$(3.140\text{f})\qquad Q_W^N = 0,$$

$$(3.140\text{g})\qquad \left(Q_W^{n-1}, \theta\right)_h + \gamma\left(\nabla Q_Y^{n-1}, \nabla\theta\right) = 0.$$

for all $\boldsymbol{\psi} \in \mathbf{V}_h,\ \phi, \eta \in P_h, \theta \in Y_h,\ n = 1, \ldots, N,$

$$(3.141)\qquad \alpha \mathbf{U}^n - \mathbf{Q}_{\mathbf{V}}^{n-1} = 0,$$

for all $n = 1, \ldots, N$. *Moreover for all Lipschitz functions* $g : \mathbb{R} \to \mathbb{R}$, *with constant* L_g, *such that* $g(1) = g(-1) = 0$,

$$(3.142\text{a})\qquad \left[g\left(Y^n\right)\Lambda^{n-1}\right](x_j) = 0,$$

$$(3.142\text{b})\qquad \left[B^n Q_W^{n-1}\right](x_j) = 0,$$

$$(3.142\text{c})\qquad \left[\Lambda^{n-1} Q_W^{n-1}\right](x_j) \geq 0,$$

for all $j = 1, \ldots, \mathcal{N}_h,\ n = 1, \ldots, N$. *Finally there exists a constant* C, *independent on* h, k, *such that the following estimates are satisfied*

$$(3.143\text{a})\qquad \sup_{n=0,\ldots,N} \|\mathbf{V}^n\|_{\mathbf{H}_0^1} \leq C,$$

$$(3.143\text{b})\qquad \sum_{n=1}^{N} k\ \|d_t \mathbf{V}^n\|^2 \leq C,$$

$$(3.143\text{c})\qquad \sum_{n=1}^{N} \|\mathbf{V}^n - \mathbf{V}^{n-1}\|_{\mathbf{H}_0^1}^2 \leq C,$$

$$(3.143\text{d})\qquad \sum_{n=1}^{N} k\ \|\tilde{\Delta}_h \mathbf{V}^n\|^2 \leq C,$$

$$(3.143\text{e})\qquad \sup_{n=1,\ldots,N} \left\| \sum_{i=1}^{n} k\ P^i \right\| \leq C,$$

$$(3.143\text{f})\qquad \sup_{n=0,\ldots,N} \|Y^n\|_{H_0} \leq C,$$

$$(3.143\text{g})\qquad \sum_{n=1}^{N} k\ \left\|\nabla \mathcal{G} d_t Y^n\right\|^2 \leq C,$$

$$(3.143\text{h})\qquad \sum_{n=1}^{N} k\ \|\hat{\Delta}_h Y^n\|_h^2 \leq C,$$

$$(3.143\text{i})\qquad \sum_{n=1}^{N} \left\|Y^n - Y^{n-1}\right\|_{H_0}^2 \leq C,$$

$$(3.143\text{j})\qquad \sum_{n=1}^{N} k\ \|W^n\|_{H^1}^2 \leq C,$$

(3.143k) $$\sum_{n=1}^{N} k \ \left[\|B_r^n\|^2 + \|B_l^n\|^2\right] \leq C,$$

(3.143l) $$\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)} \leq C,$$

and

(3.144a) $$\sup_{n=0,\dots,N} \|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1} \leq C,$$

(3.144b) $$\sum_{n=1}^{N} \|d_t \mathbf{Q}_{\mathbf{V}}^n\|^2 \leq C,$$

(3.144c) $$\sum_{n=1}^{N} \|\mathbf{Q}_{\mathbf{V}}^n - \mathbf{Q}_{\mathbf{V}}^{n-1}\|_{\mathbf{H}_0^1}^2 \leq C,$$

(3.144d) $$\sum_{n=0}^{N} k \ \|\tilde{\Delta}_h \mathbf{Q}_{\mathbf{V}}^n\|^2 \leq C,$$

(3.144e) $$\sup_{n=0,\dots,N} \left\| \sum_{i=0}^{n} k \ Q_P^n \right\| \leq C,$$

(3.144f) $$\sup_{n=0,\dots,N} \|Q_Y^n\|_{H_0} \leq C,$$

(3.144g) $$\sum_{n=1}^{N} \|Q_Y^n - Q_Y^{n-1}\|_{H_0}^2 \leq C,$$

(3.144h) $$\sum_{n=0}^{N} k \ \|\Delta_h Q_Y^n\|_{H_0}^2 \leq C,$$

(3.144i) $$\sum_{n=0}^{N} k \ \|Q_W^n\|_{H_0}^2 \leq C,$$

(3.144j) $$\sum_{n=0}^{N} k \ (\Lambda^n, Q_W^n)_h \leq C.$$

Proof. Given a sequence of solutions $\{(\boldsymbol{\mathcal{V}}_i, \mathcal{P}_i, \mathcal{Y}_i, \mathcal{W}_i, \boldsymbol{\mathcal{U}}_i)\}_{i\in\mathbb{N}}$ of the regularized discrete optimal control Problem 3.2, we consider the sequence $\{(\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}i}, \mathcal{Q}_{\mathcal{P}i}, \mathcal{Q}_{\mathcal{Y}i}, \mathcal{Q}_{\mathcal{W}i})\}_{i\in\mathbb{N}}$ of the adjoint variables, such that $\boldsymbol{\mathcal{V}}_i, \mathcal{P}_i, \mathcal{Y}_i, \mathcal{W}_i, \boldsymbol{\mathcal{U}}_i, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}i}, \mathcal{Q}_{\mathcal{P}i}, \mathcal{Q}_{\mathcal{Y}i}, \mathcal{Q}_{\mathcal{W}i}$ is, for all i, a solution of the optimality conditions (3.72)-(3.74) of the regularized discrete optimal control Problem 3.2. Then, from the results in Lemmas 3.5, 3.7, 3.8, 3.14, Theorem 3.18 and Lemma 3.20, we realize that there exist a convergent subsequence (labelled by an index m) $\{(\boldsymbol{\mathcal{V}}_m, \mathcal{P}_m, \mathcal{Y}_m, \mathcal{W}_m, \boldsymbol{\mathcal{U}}_m, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}m}, \mathcal{Q}_{\mathcal{P}m}, \mathcal{Q}_{\mathcal{Y}m}, \mathcal{Q}_{\mathcal{W}m})\}_m$ and a set of limit variables $\{(\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}, \mathcal{B}_r, \mathcal{B}_l, \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}}, \mathcal{Q}_{\mathcal{P}}, \mathcal{Q}_{\mathcal{Y}}, \mathcal{Q}_{\mathcal{W}})\}$ such that:

- the variables $\boldsymbol{\mathcal{V}}_m, \mathcal{P}_m, \mathcal{Y}_m, \mathcal{W}_m, \boldsymbol{\mathcal{U}}_m, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}m}, \mathcal{Q}_{\mathcal{P}m}, \mathcal{Q}_{\mathcal{Y}m}, \mathcal{Q}_{\mathcal{W}m}$ are, for all m, a solution of the optimality conditions (3.72)-(3.74) of the regularized discrete optimal control Problem 3.2;
- the limits (3.128)-(3.137) stated above hold;

- the limit variables $\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}, \mathcal{B}_r, \mathcal{B}_l, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}}, \mathcal{Q}_{\mathcal{P}}, \mathcal{Q}_{\mathcal{Y}}, \mathcal{Q}_{\mathcal{W}}$ satisfy the estimates (3.143a)-(3.144i);

- the state and control limit variables $\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}, \mathcal{B}_r, \mathcal{B}_l, \boldsymbol{\mathcal{U}}$ are a solution of the non-smooth optimal control Problem 3.19 and the optimality conditions (3.139), (3.141) hold.

In order to show that (3.140) are satisfied, we consider that for all m,

$$-\left(d_t\mathbf{Q}_{\mathbf{V}m}^n, \boldsymbol{\psi}\right) + \nu\left(\nabla\mathbf{Q}_{\mathbf{V}m}^{n-1}, \nabla\boldsymbol{\psi}\right) + \left(Q_{Pm}^{n-1}, \nabla\cdot\boldsymbol{\psi}\right) - \left(Y_m^n\nabla Q_{Ym}^n, \boldsymbol{\psi}\right) = 0, \tag{3.145a}$$

$$\mathbf{Q}_{\mathbf{V}m}^N = 0, \tag{3.145b}$$

$$\left(\nabla\cdot\mathbf{Q}_{\mathbf{V}m}^{n-1}, \phi\right) = 0, \tag{3.145c}$$

$$\begin{aligned} &-\left(d_tQ_{Ym}^n, \eta\right)_h - \varepsilon^2\left(\nabla Q_{Wm}^{n-1}, \nabla\eta\right) + \left(Q_{Wm}^n, \eta\right)_h \\ &-\left(\nabla Q_{Ym}^n\cdot\mathbf{V}_m^n, \eta\right) - \frac{1}{\delta}\left(\beta'_{\delta_m}\left(Y_m^n\right)Q_{Wm}^{n-1}, \eta\right)_h + \left(Y_m^n - y_{d,h}^n, \eta\right) = 0, \end{aligned} \tag{3.145d}$$

$$Q_{Ym}^N = 0, \tag{3.145e}$$

$$Q_{Wm}^N = 0, \tag{3.145f}$$

$$\left(Q_{Wm}^{n-1}, \theta\right)_h + \gamma\left(\nabla Q_{Ym}^{n-1}, \nabla\theta\right) = 0. \tag{3.145g}$$

As $m \to +\infty$, all the linear terms in (3.145) converge to the corresponding terms in (3.140). Concerning the nonlinear term in (3.145), we need to show that, as $m \to +\infty$,

$$\sum_{n=1}^{N} k\ \left(Y_m^n\nabla Q_{Ym}^n, \boldsymbol{\psi}^n\right) \to \sum_{n=1}^{N} k\ \left(Y^n\nabla Q_Y^n, \boldsymbol{\psi}^n\right), \tag{3.146}$$

$$\sum_{n=1}^{N} k\ \left(\nabla Q_{Ym}^n\cdot\mathbf{V}_m^n, \eta^n\right) \to \sum_{n=1}^{N} k\ \left(\nabla Q_Y^n\cdot\mathbf{V}^n, \eta^n\right). \tag{3.147}$$

We have that:

- using the generalized Holder's inequality (A.14), (A.17), the convergence and the boundedness of $\mathcal{Y}_m$ and $\mathcal{Q}_{\mathcal{Y}m}$ in P_h^{N+1},

$$\begin{aligned} &\left|\sum_{n=1}^{N} k\ \left(Y_m^n\nabla Q_{Ym}^n, \boldsymbol{\psi}^n\right) - \sum_{n=1}^{N} k\ \left(Y^n\nabla Q_Y^n, \boldsymbol{\psi}^n\right)\right| \\ \le C\ &\sum_{n=1}^{N} k\left[\|Y_m^n - Y^n\|_{H^1}\|\nabla Q_{Ym}^n\| + \|Y_m^n\|_{H^1}\|\nabla Q_{Ym}^n - \nabla Q_Y^n\|\right]\|\nabla\boldsymbol{\psi}^n\| \to 0, \end{aligned}$$

  as $m \to +\infty$;

- using the generalized Holder's inequality (A.14), (A.17), the convergence and the boundedness of $\mathcal{Q}_{\mathcal{Y}m}$ and $\boldsymbol{\mathcal{V}}_m$, respectively in P_h^{N+1} and $\mathbf{V}_h^{N+1}$

$$\left|\sum_{n=1}^{N} k\ \left(\nabla Q_{Ym}^n\cdot\mathbf{V}_m^n, \eta^n\right) - \sum_{n=1}^{N} k\ \left(\nabla Q_Y^n\cdot\mathbf{V}^n, \eta^n\right)\right| \le$$

$$\leq C \sum_{n=1}^{N} k \left[\|\nabla Q_{Ym}^n - \nabla Q_Y^n\| \|\nabla \mathbf{V}_m^n\| + \|\nabla Q_Y^n\| \|\nabla \mathbf{V}_m^n - \nabla \mathbf{V}^n\| \right] \|\eta^n\| \to 0,$$

as $m \to +\infty$.

Hence (3.146), (3.147) hold. From the convergence of the other terms in (3.145d), we infer that there exists

$$\varLambda \in Y_h^N,$$

such that, as $m \to +\infty$,

$$\sum_{n=1}^{N} k \left(\frac{1}{\delta_m} \beta'_{\delta_m} \left(Y_m^n\right) Q_{Wm}^{n-1}, \eta^n \right)_h \to \sum_{n=1}^{N} k \left(\Lambda^{n-1}, \eta^n \right)_h, \tag{3.148}$$

where

$$\Lambda^{n-1}\left(x_j\right) = \lim_{\delta_m \to 0^+} \frac{1}{\delta_m} \left[\beta'_{\delta_m}\left(Y_m^n\right) Q_{Wm}^{n-1} \right] \left(x_j\right),$$

for all $j = 1, \ldots, \mathcal{N}_h$, $n = 1, \ldots, N$. Therefore $\boldsymbol{\mathcal{V}}, \mathcal{Y}, \boldsymbol{\mathcal{Q}_{\mathcal{V}}}, \mathcal{Q}_{\mathcal{P}}, \mathcal{Q}_{\mathcal{Y}}, \mathcal{Q}_{\mathcal{W}}, \varLambda$ solve the optimality conditions (3.140) above. Furthermore from the estimate (3.85) in Lemma 3.14, we infer that the result (3.144j) holds. Finally, we prove (3.142). We observe that for all $\eta \in Y_h$,

$$\left| \left(g(Y_m^n) - g\left(Y^n\right), \eta\right)_h \right| \leq \|g\left(Y_m^n\right) - g\left(Y^n\right)\|_h \|\eta\|_h \leq L_g \; \|Y_m^n - Y^n\|_h \|\eta\|_h \to 0,$$

as $m \to +\infty$. So, in this limit, we have $\left[g\left(Y_m^n\right)\right]\left(x_j\right) \to \left[g\left(Y^n\right)\right]\left(x_j\right)$. Then, using the projection operator $\mathcal{P}$, defined in (2.121),

$$\left| \; [g(Y^n)\Lambda^{n-1}](x_j) \; \right| = \lim_{\delta_m \to 0^+} \left| \left[\frac{1}{\delta_m} \beta'_{\delta_m}\left(Y_m^n\right) Q_{Wm}^{n-1} \; g(Y_m^n) \right] \left(x_j\right) \right|, \tag{3.149}$$

and furthermore

$$\begin{aligned} &\left| \left[\frac{1}{\delta_m} \beta'_{\delta_m}\left(Y_m^n\right) Q_{Wm}^{n-1} \; g(Y_m^n) \right] \left(x_j\right) \right| \\ &= \left| \left[\frac{1}{\delta_m} \beta'_{\delta_m}\left(Y_m^n\right) Q_{Wm}^{n-1} \right] \left[g(Y_m^n) - g(\mathcal{P}Y_m^n) + g(\mathcal{P}Y_m^n) \right] \left(x_j\right) \right| \\ &\leq \left| \left[\frac{1}{\delta_m} \beta'_{\delta_m}\left(Y_m^n\right) Q_{Wm}^{n-1} \; g(\mathcal{P}Y_m^n) \right] \left(x_j\right) \right| + \left| \left[\frac{1}{\delta_m} \beta'_{\delta_m}\left(Y_m^n\right) Q_{Wm}^{n-1} \right] \left[g(Y_m^n) - g(Y_m^n) \right] \left(x_j\right) \right| \\ &= M_1 + M_2. \end{aligned}$$

From the definition (2.13) of the function β'_δ and the properties of g, it easy to check that the term M_1 is zero. Moreover,

$$M_2 \leq \left| \left[\frac{1}{\delta_m} \beta'_{\delta_m}\left(Y_m^n\right) Q_{Wm}^{n-1} \right] \left(x_j\right) \right| \cdot L_g \; \left| \left[Y_m^n - \mathcal{P}Y_{m}^n \right] \left(x_j\right) \right| \to 0,$$

as $m \to +\infty$. Thus, from (3.149), we have that $\left[g(Y^n)\Lambda^{n-1}\right]\left(x_j\right) = 0$, for all $j = 1, \ldots, \mathcal{N}_h$, $n = 1, \ldots, N$. Hence, (3.142a) hold. In order to prove (3.142b), we observe that

$$\left[B^n Q_W^{n-1} \right] \left(x_j\right) = \lim_{\delta_m \to 0^+} \left[\frac{1}{\delta_m} \beta_{\delta_m}\left(Y_m^n\right) Q_{Wm}^n \right] \left(x_j\right). \tag{3.150}$$

Therefore, as well as in the proof of Theorem 2.16, we can write

$$\left[\frac{1}{\delta_m}\beta_{\delta_m}\left(Y_m^n\right)Q_{Wm}^{n-1}\right](x_j) = \left[\frac{1}{\delta_m}\beta'_{\delta_m}\left(Y_m^n\right)Q_{Wm}^{n-1}\, l_{\delta_m}\left(Y_m^n\right)\right](x_j),$$

where the function l_δ is defined in (2.122): it is a Lipschitz function with constant 1 and such that $l_\delta\left(Y^n\right)(x_j) = 0$, for all $j = 1, \ldots, \mathcal{N}_h, n = 1, \ldots, N$. Hence,

$$\begin{aligned}
&\left|\left[\frac{1}{\delta_m}\beta'_{\delta_m}\left(Y_m^n\right)Q_{Wm}^{n-1}\, l_{\delta_m}\left(Y_m^n\right)\right](x_j)\right| \\
&\left|\left[\frac{1}{\delta_m}\beta'_{\delta_m}\left(Y_m^n\right)Q_{Wm}^{n-1}\left(l_{\delta_m}\left(Y_m^n\right) - l_{\delta_m}\left(Y^n\right)\right)\right](x_j)\right| \\
&\leq \left|\left[\frac{1}{\delta_m}\beta'_{\delta_m}\left(Y_m^n\right)Q_{Wm}^{n-1}\right](x_j)\right| \cdot \left|\left[Y_m^n - Y^n\right](x_j)\right| \to 0,
\end{aligned}$$

as $m \to +\infty$. Thus, from (3.150), we get that (3.142b) holds.
Finally, we show (3.142c). We have, as $m \to +\infty$,

$$0 \leq \left[\frac{1}{\delta_m}\beta'_{\delta_m}\left(Y_m^n\right)Q_{Wm}^{n-1}\right](x_j) \cdot Q_{Wm}^{n-1}(x_j) \to [\Lambda^{n-1}Q_W^{n-1}](x_j),$$

for all for all $j = 1, \ldots, \mathcal{N}_h, n = 1, \ldots, N$. Then, (3.142c) is satisfied. □

3.4. Convergence of the Solutions of the Discrete Optimal Control Problem

In this section we study, as $h, k \to 0$, the convergence of the solution of the optimality conditions (3.139)-(3.142) of the discrete non-smooth optimal control Problem 3.19, to the solution of the optimality conditions (2.116)-(2.119) of non-smooth Problem 2.14.
We introduce some notations. If Z_h is a discrete functions space, given a discrete vector function

$$\mathcal{Z} = \left(Z^n\right)_{n=0}^N \in Z_h^{N+1},$$

we use $\mathcal{Z}_{h,k}$ to generically denote the following three different kinds of time interpolated variable

$$\mathcal{Z}_{h,k}^{\bullet}(t) := \frac{t - t_{n-1}}{k}Z^n + \frac{t_n - t}{k}Z^{n-1}, \qquad t \in \left[t_{n-1}, t_n\right], \tag{3.151}$$

$$\mathcal{Z}_{h,k}^{+}(t) := Z^n, \qquad t \in \left(t_{n-1}, t_n\right], \tag{3.152}$$

$$\mathcal{Z}_{h,k}^{-}(t) := Z^{n-1}, \qquad t \in \left[t_{n-1}, t_n\right), \tag{3.153}$$

where

$$t_n = nk, \quad n = 0, \ldots, N.$$

Concerning the initial conditions $\mathbf{v}_{0,h}, y_{0,h}$ and the desired state $y_{d,h}^n, n = 1, \ldots, N$, in the discrete non-smooth optimal control Problem 3.19, given

$$\mathbf{v}_0 \in \boldsymbol{\mathcal{D}} \cap \mathbf{H}^2, \quad y_0 \in H_0 \cap H^2 \cap \mathcal{K}, \quad y_d \in \mathcal{C}\left([0,T]; L_0^2\right),$$

we assume

$$\mathbf{v}_{0,h} = \mathbf{Q}_s^h \mathbf{v}_0, \quad y_{0,h} = Q^h y_0, \quad y_{d,h}^n = Q_0^h \; y_d\left(t_n\right). \tag{3.154}$$

In (3.154), the projection operator $\mathbf{Q}_s^h, Q^h, Q_0^h$, are defined, respectively, in (A.48), (A.41), (A.43). It is easy to realize that there exists a constant C, independent of h, k, such that

$$\|\nabla \mathbf{v}_{0,h}\| + \|\nabla y_{0,h}\| + \sum_{n=1}^{N} k \; \|y_{d,h}^n\|^2 \leq C. \tag{3.155}$$

Hence, from Theorem 3.21, we have that the estimates (3.143), (3.144) hold.

Remark 3.22. In the following we consider sequences $\{h_n\}_{n\in\mathbb{N}}, \{k_m\}_{m\in\mathbb{N}}$ of the discretization parameters such that

$$h_n \to 0^+, \quad k_m \to 0^+,$$

as $n, m \to +\infty$. In order to make the reading more fluent, we skip the indices n, m and we simply write

$$h, k \to 0. \tag{3.156}$$

Even in the case of extracted subsequences, we use the notation (3.156), without relabelling.

Theorem 3.23. *Consider a sequence $h, k \to 0$ and let*

$$\left\{\left(\boldsymbol{\mathcal{V}}_{h,k}, \mathcal{P}_{h,k}, \mathcal{Y}_{h,k}, \mathcal{W}_{h,k}, \mathcal{B}_{r,h,k}, \mathcal{B}_{l,h,k}, \boldsymbol{\mathcal{U}}_{h,k}\right)\right\}_{h,k},$$

be a corresponding sequence of the time interpolation of the solutions of the discrete optimal control Problem 3.19. Then, there exist functions

$$\mathbf{v} \in H^1\left(\mathbf{L}^2\right) \cap L^\infty\left(\mathbf{H}_0^1\right), \quad \int_0^t p\left(s\right)\; ds \in L^\infty\left(L_0^2\right)$$

$$y \in W_0 \cap L^\infty\left(H_0\right), \quad w \in L^2\left(H^1\right), \quad \beta_r, \beta_l \in L^2\left(L^2\right)$$

and a subsequence (not relabelled), such that,

$$\boldsymbol{\mathcal{V}}_{h,k}^\bullet \rightharpoonup \mathbf{v}, \qquad \text{in } H^1\left(\mathbf{L}^2\right), \tag{3.157}$$

$$\boldsymbol{\mathcal{V}}_{h,k}^{\bullet,\pm} \overset{*}{\rightharpoonup} \mathbf{v}, \qquad \text{in } L^\infty\left(\mathbf{H}_0^1\right), \tag{3.158}$$

$$\boldsymbol{\mathcal{V}}_{h,k}^{\bullet,\pm} \to \mathbf{v}, \qquad \text{in } L^2\left(\mathbf{H}_0^1\right). \tag{3.159}$$

$$\int_0^t \mathcal{P}_{h,k}^+\left(s\right)\; ds \overset{*}{\rightharpoonup} \int_0^t p\left(s\right)\; ds, \qquad \text{in } L^\infty\left(L_0^2\right), \tag{3.160}$$

$$\mathcal{Y}_{h,k}^\bullet \rightharpoonup y, \qquad \text{in } W_0, \tag{3.161}$$

$$\mathcal{Y}_{h,k}^{\bullet,\pm} \overset{*}{\rightharpoonup} y, \qquad \text{in } L^\infty\left(H_0\right), \tag{3.162}$$

$$\mathcal{Y}_{h,k}^{\bullet,\pm} \to y, \qquad \text{in } L^2\left(L_0^2\right), \tag{3.163}$$

(3.164) $$\mathcal{W}^+_{h,k} \rightharpoonup w, \qquad \text{in } L^2\left(H^1\right),$$

(3.165) $$\mathcal{B}^+_{r,h,k} \rightharpoonup \beta_r, \qquad \text{in } L^2\left(L^2\right),$$

(3.166) $$\mathcal{B}^+_{l,h,k} \rightharpoonup \beta_l, \qquad \text{in } L^2\left(L^2\right),$$

(3.167) $$\boldsymbol{\mathcal{U}}^+_{h,k} \rightharpoonup \mathbf{u}, \qquad \text{in } L^2\left(\mathbf{L}^2\right).$$

as $h, k \to 0$.

Proof. Using standard compactness arguments, by the estimates (3.143), we get the results (3.157), (3.158), (3.160)-(3.162), (3.164)-(3.166) and (3.167). From (3.143a), (3.143b), (3.143d), we have that

$$\|\boldsymbol{\mathcal{V}}_{h,k}\|_{H^1(\mathbf{L}^2)} + \|\boldsymbol{\mathcal{V}}_{h,k}\|_{L^2\left(\mathbf{H}^1_0\right)} + \|\tilde{\Delta}_h \boldsymbol{\mathcal{V}}_{h,k}\|_{L^2(\mathbf{L}^2)} \leq C,$$

uniformly in h, k. So, using the results obtained in [13] (Lemma 2.4) or [83] (Lemma 4.9), we derive (3.159). The strong convergence result $\mathcal{Y}^{\bullet,\pm}_{h,k}$ to y in $L^2\left(L^2\right)$ in (3.163), follows by Aubin-Lions-Simon Theorem (see for example Theorem II.5.16 in [20]). Next, we prove that $\boldsymbol{\mathcal{V}}^{\bullet,\pm}_{h,k}$ and $\mathcal{Y}^{\bullet,\pm}_{h,k}$ converge, respectively, to the same limit. We have,

$$\left\|\boldsymbol{\mathcal{V}}^{\bullet}_{h,k} - \boldsymbol{\mathcal{V}}^{+}_{h,k}\right\|^2_{L^2\left(\mathbf{H}^1_0\right)} = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left\| \frac{t-t_{n-1}}{k}\nabla \mathbf{V}^n + \frac{t_n - t}{k}\nabla \mathbf{V}^{n-1} - \nabla \mathbf{V}^n \right\|^2 dt =$$
$$= \sum_{n=1}^{N} \|\nabla \mathbf{V}^n - \nabla \mathbf{V}^{n-1}\|^2 \int_{t_{n-1}}^{t_n} \left(\frac{t-t_n}{k}\right)^2 dt = \frac{k}{3}\sum_{n=1}^{N} \|\nabla \mathbf{V}^n - \nabla \mathbf{V}^{n-1}\|^2,$$

and

$$\left\|\boldsymbol{\mathcal{V}}^{\bullet}_{h,k} - \boldsymbol{\mathcal{V}}^{-}_{h,k}\right\|^2_{L^2\left(\mathbf{H}^1_0\right)} = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left\| \frac{t-t_{n-1}}{k}\nabla \mathbf{V}^n + \frac{t_n - t}{k}\nabla \mathbf{V}^{n-1} - \nabla \mathbf{V}^{n-1} \right\|^2 dt =$$
$$= \sum_{n=1}^{N} \|\nabla \mathbf{V}^n - \nabla \mathbf{V}^{n-1}\|^2 \int_{t_{n-1}}^{t_n} \left(\frac{t-t_{n-1}}{k}\right)^2 dt = \frac{k}{3}\sum_{n=1}^{N} \|\nabla \mathbf{V}^n - \nabla \mathbf{V}^{n-1}\|^2.$$

Therefore, by the estimate (3.143c), we derive

$$\left\|\boldsymbol{\mathcal{V}}_{h,k} - \boldsymbol{\mathcal{V}}^{\pm}_{h,k}\right\|_{L^2\left(\mathbf{H}^1_0\right)} \to 0, \quad \text{as} \quad h, k \to 0,$$

that is $\boldsymbol{\mathcal{V}}^{\bullet,\pm}_{h,k}$ converge to the same limit. Moreover,

$$\left\|\boldsymbol{\mathcal{V}}^{\pm}_{h,k} - \mathbf{v}\right\|_{L^2\left(\mathbf{H}^1_0\right)} \leq \left\|\boldsymbol{\mathcal{V}}^{\pm}_{h,k} - \boldsymbol{\mathcal{V}}_{h,k}\right\|_{L^2\left(\mathbf{H}^1_0\right)} + \left\|\boldsymbol{\mathcal{V}}_{h,k} - \mathbf{v}\right\|_{L^2\left(\mathbf{H}^1_0\right)}.$$

Hence, also $\boldsymbol{\mathcal{V}}^{\pm}_{h,k}$, up to subsequences, converge strongly to $\mathbf{v}$ in $\mathbf{L}^2\left(\mathbf{H}^1_0\right)$. Using the same strategy, it easy to check that $\mathcal{Y}^{\bullet,\pm}_{h,k}$ converge to the same limit y and that this convergence is strong in $L^2\left(L^2_0\right)$. ☐

Theorem 3.24. *Consider a sequence* $h, k \to 0$ *and a constant* $\hat{C}$ *such that*

(3.168) $$\frac{h^2}{k} \leq \hat{C}.$$

Let

$$\{(\boldsymbol{\mathcal{V}}_{h,k}, \mathcal{P}_{h,k}, \mathcal{Y}_{h,k}, \mathcal{W}_{h,k}, \mathcal{B}_{r,h,k}, \mathcal{B}_{l,h,k}, \boldsymbol{\mathcal{U}}_{h,k}, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}, \mathcal{Q}_{\mathcal{P},h,k}, \mathcal{Q}_{\mathcal{Y},h,k}, \mathcal{Q}_{\mathcal{W},h,k}, \Lambda_{h,k})\}_{h,k},$$

be a corresponding sequence of the time interpolation of the solutions of the optimality conditions (3.139)-(3.142) *where in particular*

$$\{(\boldsymbol{\mathcal{V}}_{h,k}, \mathcal{P}_{h,k}, \mathcal{Y}_{h,k}, \mathcal{W}_{h,k}, \mathcal{B}_{r,h,k}, \mathcal{B}_{l,h,k}, \boldsymbol{\mathcal{U}}_{h,k})\}_{h,k},$$

is a sequence of solutions of of the discrete non-smooth optimal control Problem 3.19. Then, there exist functions

$$\mathbf{q_v} \in H^1\left(\mathbf{L}^2\right) \cap L^\infty\left(\boldsymbol{\mathcal{D}}\right), \quad \int_0^t q_p\left(s\right)\ ds \in L^\infty\left(L_0^2\right)$$

$$q_y \in L^\infty\left(H_0\right), \quad q_{y0} \in H_0, \quad q_w \in L^2\left(H^1\right), \quad \lambda \in W_0^*,$$

and a subsequence (not relabelled) such that, as $h, k \to 0$,

$$\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k} \rightharpoonup \mathbf{q_v}, \quad \text{in } H^1\left(\mathbf{L}^2\right), \tag{3.169}$$

$$\boldsymbol{\mathcal{Q}}^{\bullet,\pm}_{\boldsymbol{\mathcal{V}},h,k} \overset{*}{\rightharpoonup} \mathbf{q_v}, \quad \text{in } L^\infty\left(\mathbf{H}_0^1\right), \tag{3.170}$$

$$\boldsymbol{\mathcal{Q}}^{\bullet,\pm}_{\boldsymbol{\mathcal{V}},h,k} \to \mathbf{q_v}, \quad \text{in } L^2\left(\mathbf{H}_0^1\right), \tag{3.171}$$

$$\int_0^t \mathcal{Q}^{\pm}_{\mathcal{P},h,k}\left(s\right)\ ds \overset{*}{\rightharpoonup} \int_0^t q_p\left(s\right)\ ds, \quad \text{in } L^\infty\left(L_0^2\right), \tag{3.172}$$

$$\mathcal{Q}^{\bullet,\pm}_{\mathcal{Y},h,k} \overset{*}{\rightharpoonup} q_y, \quad \text{in } L^\infty\left(H_0\right), \tag{3.173}$$

$$\mathcal{Q}^{\bullet,-}_{\mathcal{Y},h,k}\left(0\right) \rightharpoonup q_{y0}, \quad \text{in } H_0, \tag{3.174}$$

$$\mathcal{Q}^{\bullet,\pm}_{\mathcal{W},h,k} \rightharpoonup q_w, \quad \text{in } L^2\left(H^1\right), \tag{3.175}$$

$$\bar{\Lambda}^{-}_{h,k} \overset{*}{\rightharpoonup} \lambda, \quad \text{in } W_0^*, \tag{3.176}$$

where $Q^h\bar{\Lambda}^-_{h,k} = \Lambda^-_{h,k}$ *and* Q^h *is the projection operator defined in* (A.41).

Proof. As in the previous Theorem 3.23, from the estimates (3.144) and using standard compactness argument, we can prove (3.169)-(3.175). Moreover, it is easy to derive that $\boldsymbol{\mathcal{Q}}^{\bullet,\pm}_{\boldsymbol{\mathcal{V}},h,k}$ and $\mathcal{Q}^{\bullet,\pm}_{\mathcal{Y},h,k}$ converge, respectively, to the same limit. In order to show that $\mathcal{Q}^{\bullet,\pm}_{\mathcal{W},h,k}$ converge to the same limit q_w as in (3.175), using the optimality condition (3.140g), we note that for all $\theta \in L^2\left(H^1\right)$,

$$\begin{aligned}\left|\int_0^T \left(\mathcal{Q}^{\bullet}_{\mathcal{W},h,k} - \mathcal{Q}^{\pm}_{\mathcal{W},h,k}, \theta\right)\ dt\right| &= \left|\int_0^T \left(\mathcal{Q}^{\bullet}_{\mathcal{W},h,k} - \mathcal{Q}^{\pm}_{\mathcal{W},h,k}, Q^h\theta\right)_h\ dt\right| \\ &= \gamma\left|\int_0^T \left(\nabla\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} - \nabla\mathcal{Q}^{\pm}_{\mathcal{Y},h,k}, \nabla Q^h\theta\right)\ dt\right| \\ &\le \gamma\ \|\nabla\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} - \nabla\mathcal{Q}^{\pm}_{\mathcal{Y},h,k}\|_{L^2(L^2)}\ \|\nabla Q^h\theta\|_{L^2(L^2)}.\end{aligned}$$

Hence, from the estimate (3.144g) and using the property (A.42) of the operator Q^h, we derive that, as $h, k \to 0$,

$$\left|\int_0^T \left(\mathcal{Q}_{\mathcal{W},h,k} - \mathcal{Q}^{\pm}_{\mathcal{W},h,k}, \theta\right)\ dt\right| \to 0, \quad \forall\ \theta \in L^2\left(H^1\right).$$

Therefore, $\mathcal{Q}_{\mathcal{W},h,k}^{\bullet,\pm}$ converge to the same limit q_w. It remains to show (3.176). From the optimality condition (3.140d), we have

$$\int_0^T \left(\Lambda_{h,k}^-,\eta\right)\ dt = \int_0^T \left(\Lambda_{h,k}^-,Q^h\eta\right)_h\ dt$$

$$= \int_0^T \Big[\left(\left(\mathcal{Q}_{\mathcal{Y},h,k}\right)_t, Q^h\eta\right)_h - \varepsilon^2\left(\nabla\mathcal{Q}_{\mathcal{W},h,k}^-,\nabla Q^h\eta\right) + \left(\mathcal{Q}_{\mathcal{W},h,k}^+,Q^h\eta\right)_h$$
$$- \left(\boldsymbol{\mathcal{V}}_{h,k}^+\cdot\nabla\mathcal{Q}_{\mathcal{Y},h,k}^+,Q^h\eta\right) + \left(\mathcal{Y}_{h,k}^+ - \mathcal{Y}_{d,h,k}^+,Q^h\eta\right)\Big]\ dt$$
$$= O_1 + O_2 + O_3 + O_4 + O_5,$$

for all $\eta \in W_0$. Using $\mathcal{Q}_{\mathcal{Y},h,k}(T) = 0$, the embedding $W_0 \hookrightarrow \mathcal{C}\left([0,T],L_0^2\right)$, the estimate (A.42) on the projection operator Q^h, the generalized Holder's inequality (A.14) and (A.17), we get

$$\begin{aligned}
O_1 &= \int_0^T \left(\left(\mathcal{Q}_{\mathcal{Y},h,k}\right)_t,\eta\right)\ dt \\
&\le \left|\int_0^T \langle\eta_t,\mathcal{Q}_{\mathcal{Y},h,k}\rangle_{H^{1*},H^1}\ dt + \left(\mathcal{Q}_{\mathcal{Y},h,k}(0),\eta(0)\right)\right| \\
&\le \|\eta_t\|_{L^2(H^{1*})}\|\mathcal{Q}_{\mathcal{Y},h,k}\|_{L^2(H^1)} + \|\mathcal{Q}_{\mathcal{Y},h,k}(0)\|\|\eta(0)\| \le C\ \|\eta\|_{W_0}.
\end{aligned}$$

$$O_2 \le \varepsilon^2\int_0^T \|\nabla\mathcal{Q}_{\mathcal{W},h,k}^-\|\|\nabla Q^h\eta\|\ dt \le C\ \varepsilon^2\int_0^T \|\nabla\mathcal{Q}_{\mathcal{W},h,k}^-\|\|\nabla\eta\|\ dt \le C\ \|\eta\|_{W_0},$$

$$O_3 = \int_0^T \left(\mathcal{Q}_{\mathcal{W},h,k}^+,\eta\right)\ dt \le \|\mathcal{Q}_{\mathcal{W},h,k}^+\|_{L^2(L^2)}\|\eta\|_{L^2(L^2)} \le C\ \|\eta\|_{W_0},$$

$$\begin{aligned}
O_4 &\le \int_0^T \|\boldsymbol{\mathcal{V}}_{h,k}^+\|_{L^4}\|\|\nabla\mathcal{Q}_{\mathcal{Y},h,k}^+\|\|Q^h\eta\|_{L^4}\ dt \\
&\le C\ \int_0^T \|\boldsymbol{\mathcal{V}}_{h,k}^+\|_{H^1}\|\|\nabla\mathcal{Q}_{\mathcal{Y},h,k}^+\|\|Q^h\eta\|_{H^1}\ dt \\
&\le C\ \|Q^h\eta\|_{L^2(H^1)} \le C\ \|\eta\|_{L^2(H^1)} \le C\ \|\eta\|_{W_0},
\end{aligned}$$

$$O_5 \le \|\mathcal{Y}_{h,k}^+ - \mathcal{Y}_{d,h,k}^+\|_{L^2(L^2)}\|Q^h\eta\|_{L^2(L^2)} \le C\|Q^h\eta\|_{L^2(H^1)} \le C\ \|\eta\|_{W_0}.$$

Hence, for all $\eta \in W_0$,

$$\left|\int_0^T \left(\Lambda_{hk}^-,\eta\right)\ dt\right| \le C\|\eta\|_{W_0}. \tag{3.177}$$

It is easy to realize that the projection operator Q^h restricted on the discrete space Y_h is an isomorphism. Then, given $\Lambda_{h,k}^-$ there exists $\bar{\Lambda}_{h,k}^-$, such that $\Lambda_{h,k}^- = Q^h\bar{\Lambda}_{h,k}^-$. In order to show the result (3.176), we need to prove that (3.177) holds with $\Lambda_{h,k}^-$ replaced by $\bar{\Lambda}_{h,k}^-$. Using the estimate (A.42) of the projection operator Q^h and

(3.177), we can write

$$\tag{3.178} \begin{aligned} &\Big| \int_0^T \left(\bar{\varLambda}_{hk}^{-}, \eta\right)\ dt \Big| \\ &= \Big| \int_0^T \left(\bar{\varLambda}_{hk}^{-}, \eta\right)\ dt - \int_0^T \left(\varLambda_{hk}^{-}, \eta\right)\ dt + \int_0^T \left(\varLambda_{hk}^{-}, \eta\right)\ dt \Big| \\ &\leq \int_0^T \Big| \left(\bar{\varLambda}_{hk}^{-} - Q^h \bar{\varLambda}_{hk}^{-}, \eta\right) \Big| + \Big| \int_0^T \left(\varLambda_{hk}^{-}, \eta\right)\ dt \Big| \\ &\leq C\ \left[\ h\ \|\bar{\varLambda}_{hk}^{-}\|_{L^2(L^2)} + 1\ \right]\ \|\eta\|_{W_0}. \end{aligned}$$

We note that

$$\left(\Lambda^{n-1}, \eta\right)_h = \left(Q^h \bar{\Lambda}^{n-1}, \eta\right)_h = \left(\bar{\Lambda}^{n-1}, \eta\right), \quad \forall\, \eta \in P_h,$$

Then, with $\eta = \bar{\Lambda}^{n-1}$ in (3.140d), we have

$$\tag{3.179} \begin{aligned} k\ \|\bar{\Lambda}^{n-1}\|^2 = k\ &\left(\frac{Q_Y^{n-1} - Q_Y^n}{k}, \bar{\Lambda}^{n-1}\right)_h + k\varepsilon^2 \left(\hat{\Delta}_h Q_W^{n-1}, \bar{\Lambda}^{n-1}\right)_h + k \left(Q_W^n, \bar{\Lambda}^{n-1}\right)_h \\ &- k \left(\mathbf{V}^n \cdot \nabla Q_Y^n, \bar{\Lambda}^{n-1}\right) + k \left(Y^n - y_{d,h}^n, \bar{\Lambda}^{n-1}\right). \end{aligned}$$

Using Young's inequality, the uniform estimate $\|\mathbf{V}^n\|_{\mathbf{H}_0^1} \leq C$ and multiplying by h^2, from (3.179), we derive

$$\tag{3.180} \begin{aligned} h^2\ k\ \|\bar{\Lambda}^{n-1}\|_h^2 \ &\leq\ C_1 \left[\frac{h^2}{k} \|Q_Y^{n-1} - Q_Y^n\|^2 + k\ h^2 \|\hat{\Delta}_h Q_W^{n-1}\|_h^2\right] + \\ &+\ C_2\ k\ h^2 \left[\|Q_W^n\|_h^2 + \|\nabla Q_Y^n\|^2 + \|\hat{\Delta}_h Q_Y^n\|_h^2 + \|Y^n - y_{d,h}^n\|^2\right]. \end{aligned}$$

Thus, if the assumption (3.168) holds, using the well known inverse inequality

$$\|\hat{\Delta}_h Z\|_h \leq \frac{C}{h}\ \|\nabla Z\|, \quad \forall\ Z \in Y_h,$$

the estimates (3.143), (3.144) and the definition (3.154) of $y_{d,h}^n, n = 1, \dots, N$, from (3.180), we can write

$$\tag{3.181} h\ \|\bar{\Lambda}_{hk}^{-}\|_{L^2(L^2)} \leq C.$$

Taking into account of (3.181) in (3.178), we derive the result (3.176). □

In the next Theorem 3.25, we provides regularity properties for the functions

$$\mathbf{v},\ y,\ w,\ \beta_r,\ \beta_l,\ \mathbf{u},\ \mathbf{q_v},\ q_y,\ q_w,\ \lambda,$$

considered in the previous Theorems 3.23, 3.24. Furthermore, we show that these functions satisfy the optimality conditions (2.116)-(2.118) of the continuous non-smooth optimal control Problem 2.14.

Theorem 3.25. *The functions*

$$\mathbf{v},\ y,\ w,\ \beta_r,\ \beta_l,\ \mathbf{u},\ \mathbf{q_v},\ q_y,\ q_w,\ \lambda,$$

in Theorems 3.23, 3.24 are such that

$$\mathbf{v} \in H^1(\boldsymbol{\mathcal{S}}) \cap L^\infty(\boldsymbol{\mathcal{D}}), \tag{3.182}$$
$$y \in W_0 \cap L^\infty(H_0) \cap L^2\left(H^2\right), \tag{3.183}$$
$$w \in L^2\left(H^1\right), \tag{3.184}$$
$$\beta_r, \beta_l \in L^2\left(L^2\right), \tag{3.185}$$
$$\mathbf{u} \in H^1(\boldsymbol{\mathcal{S}}) \cap L^\infty(\boldsymbol{\mathcal{D}}), \tag{3.186}$$
$$\mathbf{q_v} \in H^1(\boldsymbol{\mathcal{S}}) \cap L^\infty(\boldsymbol{\mathcal{D}}), \tag{3.187}$$
$$q_y \in L^\infty(H_0), \tag{3.188}$$
$$q_{y0} \in H_0, \tag{3.189}$$
$$q_w \in L^2\left(H^1\right), \tag{3.190}$$
$$\lambda \in W_0^*. \tag{3.191}$$

Furthermore, they satisfy the optimality conditions (2.116)-(2.118) of the continuous non-smooth optimal control Problem 2.14.

Proof. We divide the proof in several steps.

i) Results (3.182), (2.116a), (2.116b).

From the discrete optimality conditions (3.139a), (3.139b), we can write that

$$\int_0^T \left[\left(\left(\boldsymbol{\mathcal{V}}^\bullet_{h,k}\right)_t, \boldsymbol{\psi}_h\right) + \nu\left(\nabla\boldsymbol{\mathcal{V}}^+_{h,k}, \nabla\boldsymbol{\psi}_h\right) - \left(\boldsymbol{\mathcal{U}}^+_{h,k}, \boldsymbol{\psi}_h\right)\right] = 0, \tag{3.192}$$
$$\boldsymbol{\mathcal{V}}^\bullet_{h,k}(0) = \mathbf{Q}^h_s\mathbf{v}_0. \tag{3.193}$$

for all $\boldsymbol{\psi}_h \in \mathcal{C}^\infty_c((0,T);\mathbf{D}_h)$. Given $\boldsymbol{\psi} \in \mathcal{C}^\infty_c((0,T);\boldsymbol{\mathcal{D}})$, we set in (3.192) $\boldsymbol{\psi}_h = \mathbf{Q}^h_s\boldsymbol{\psi}$. Using the property (A.49) of the Stokes projection operator $\mathbf{Q}^h_s$, we note that

$$\|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|^2_{L^2(\boldsymbol{\mathcal{D}})} = \int_0^T \|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|^2_{\boldsymbol{\mathcal{D}}}\, dt \le C\, h^2 \int_0^T \|\boldsymbol{\psi}\|^2_{\mathbf{H}^2}\, dt \to 0, \tag{3.194}$$

as $h \to 0$. Hence, from the convergence results of Theorem 3.23 and (3.194), it easy to realize that, as $h, k \to 0$, equation (3.192) converges to

$$\int_0^T \left[(\mathbf{v}_t, \boldsymbol{\psi}) + \nu(\nabla\mathbf{v}, \nabla\boldsymbol{\psi}) - (\mathbf{u}, \boldsymbol{\psi})\right]\, dt = 0,$$

with $\boldsymbol{\psi} \in \mathcal{C}^\infty_c((0,T);\boldsymbol{\mathcal{D}})$. Moreover

$$\boldsymbol{\mathcal{V}}^\bullet_{h,k}(0) = \mathbf{Q}^h_s\mathbf{v}_0 \to \mathbf{v}_0, \qquad \text{in } \boldsymbol{\mathcal{D}}. \tag{3.195}$$

as $h \to 0$. With $\boldsymbol{\psi} = \boldsymbol{\xi}(1 - t/T)$, where $\boldsymbol{\xi} \in \mathbf{L}^2$, using integration by parts in time, we derive

(3.196)
$$\left(\boldsymbol{\mathcal{V}}^{\bullet}_{h,k}(0) - \mathbf{v}(0), \boldsymbol{\xi}\right) = -\int_0^T \left(\left(\boldsymbol{\mathcal{V}}^{\bullet}_{h,k} - \mathbf{v}\right)_t, \boldsymbol{\psi}\right) \, dt \; - \int_0^T \left(\boldsymbol{\mathcal{V}}^{\bullet}_{h,k} - \mathbf{v}, \boldsymbol{\psi}_t\right) \, dt \; \to 0.$$

as $h, k \to 0$. So, $\boldsymbol{\mathcal{V}}_{h,k}(0) \rightharpoonup \mathbf{v}(0)$ in $\mathbf{L}^2$. Hence, using (3.195), (3.196) and the uniqueness of the weak limit, we can claim that $\mathbf{v}(0) = \mathbf{v}_0$. Therefore, we have that

$$\int_0^T \left[(\mathbf{v}_t, \boldsymbol{\psi}) + \nu(\nabla \mathbf{v}, \nabla \boldsymbol{\psi}) - (\mathbf{u}, \boldsymbol{\psi})\right] \, dt = 0,$$
$$\mathbf{v}(0) = \mathbf{v}_0,$$

for all $\boldsymbol{\psi} \in \mathcal{C}_c^\infty((0,T); \boldsymbol{\mathcal{D}})$. Thus, from the density result (A.8), we infer that $\mathbf{v}, \mathbf{u}$ satisfy the optimality conditions (2.116a), (2.116b) of the continuous non-smooth optimal control Problem 2.14, for all $\boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}})$. Finally, using the results of Lemma 2.12, we realize that (3.182) holds.

ii) Results (3.183), (2.116c)-(2.116i).

From the discrete optimality conditions (3.139d)-(3.139f), we have that

(3.197)
$$\int_0^T \left[\left(\left(\mathcal{Y}^{\bullet}_{h,k}\right)_t, \eta_h\right)_h + \gamma\left(\nabla \mathcal{W}^{+}_{h,k}, \nabla \eta_h\right) - \left(\mathcal{Y}^{-}_{h,k}, \boldsymbol{\mathcal{V}}^{-}_{h,k} \cdot \nabla \eta_h\right)\right] \, dt = 0,$$

(3.198)
$$\mathcal{Y}^{\bullet,-}_{h,k}(0) = Q^h y_0,$$

(3.199)
$$\int_0^T \left[-\left(\mathcal{W}^{+}_{h,k} + \mathcal{Y}^{-}_{h,k}, \theta_h\right)_h + \varepsilon^2\left(\nabla \mathcal{Y}^{+}_{h,k}, \nabla \theta_h\right) + \left(\mathcal{B}^{+}_{h,k}, \theta_h\right)_h\right] \, dt = 0,$$

for all $\eta_h,\ \theta_h \in \mathcal{C}_c^\infty((0,T); Y_h)$. Given $\eta,\ \theta \in \mathcal{C}_c^\infty\left((0,T); \mathcal{C}_c^\infty\left(\bar{\Omega}\right)\right)$, we set in the system (3.197)-(3.199) $\eta_h = Q_1^h \eta,\ \theta_h = Q_1^h \theta$. Using the property (A.47) of the projection operator Q_1^h, it holds

$$\|\eta_h - \eta\|^2_{L^2(H^1)} = \int_0^T \|\eta_h - \eta\|^2_{H^1} \, dt \le C\, h^2 \int_0^T \|\eta\|^2_{H^2} \, dt.$$

Hence,

(3.200)
$$\eta_h \to \eta, \quad \theta_h \to \theta, \quad \text{in } L^2\left(H^1\right).$$

as $h \to 0$. Let $\mathbf{v}, y, w, \beta_r, \beta_l$ be the limiting functions in Theorem 3.23. We have

(3.201)
$$\left|\int_0^T \left(\left(\mathcal{Y}_{h,k}\right)_t, \eta_h\right)_h \, dt - \int_0^T \langle y_t, \eta\rangle_{H^{1*},H^1} dt\right| \le A_1 + A_2,$$

$$A_1 = \left|\int_0^T \left(\left(\mathcal{Y}_{h,k}\right)_t, \eta_h\right)_h \, dt - \int_0^T \left(\left(\mathcal{Y}_{h,k}\right)_t, \eta_h\right) \, dt\right|,$$

$$A_2 = \Big| \int_0^T \left(\left(\mathcal{Y}_{h,k}\right)_t, \eta_h \right) \, dt - \int_0^T \langle y_t, \eta \rangle_{H^{1*}, H^1} dt \Big|.$$

Using (A.31) and integration by parts in time, we can write

(3.202)

$$\begin{aligned} A_1 = \Big| &- \int_0^T \left(\mathcal{Y}_{h,k}, \eta_{ht}\right)_h \, dt + \left(\mathcal{Y}_{h,k}(T), \eta_h(T)\right)_h - \left(\mathcal{Y}_{h,k}(0), \eta_h(0)\right)_h \\ &+ \int_0^T \left(\mathcal{Y}_{h,k}, \eta_{ht}\right) \, dt - \left(\mathcal{Y}_{h,k}(T), \eta_h(T)\right) + \left(\mathcal{Y}_{h,k}(0), \eta_h(0)\right) \Big| \\ &\leq \Big| \int_0^T \left(\mathcal{Y}_{h,k}, \eta_{ht}\right)_h \, dt - \int_0^T \left(\mathcal{Y}_{h,k}, \eta_{ht}\right) \, dt \Big| \\ &+ \Big| \left(\mathcal{Y}_{h,k}(T), \eta_h(T)\right)_h - \left(\mathcal{Y}_{h,k}(T), \eta_h(T)\right) \Big| \\ &+ \Big| \left(\mathcal{Y}_{h,k}(0), \eta_h(0)\right)_h - \left(\mathcal{Y}_{h,k}(0), \eta_h(0)\right) \Big| \\ &\leq Ch \left[\int_0^T \|\nabla \mathcal{Y}_{h,k}\| \|\eta_{ht}\| \, dt + \|\nabla \mathcal{Y}_{h,k}(T)\| \|\eta_h(T)\| + \|\nabla \mathcal{Y}_{h,k}(0)\| \|\eta_h(0)\| \right] \\ &\leq C \, h \, \|\mathcal{Y}_{h,k}\|_{L^\infty(H^1)} \left[\int_0^T \|\eta_{ht}\| \, dt + \|\eta_h(T)\| + \|\eta_h(0)\| \right] \\ &\leq C \, h \left(1 + h^2\right) \, \|\mathcal{Y}_{h,k}\|_{L^\infty(H^1)} \left[\int_0^T \|\eta_t\|_{H^2} \, dt + \|\eta(T)\|_{H^2} + \|\eta(0)\|_{H^2} \right] \to 0, \end{aligned}$$

as $h, k \to 0$. Moreover, using (3.161) and (3.200), we derive

$$A_2 \to 0, \tag{3.203}$$

as $h, k \to 0$. Taking into account of (3.202), (3.203) in (3.201), we infer that

$$\Big| \int_0^T \left(\left(\mathcal{Y}_{h,k}\right)_t, \eta_h \right)_h \, dt - \int_0^T \langle y_t, \eta \rangle_{H^{1*}, H^1} dt \Big| \to 0, \tag{3.204}$$

as $h, k \to 0$. From the results of Theorem 3.23 and (3.200), it easy to realize that

$$\int_0^T \left(\nabla \mathcal{W}^+_{h,k}, \nabla \eta_h \right) \, dt \to \int_0^T \left(\nabla w, \nabla \eta \right) \, dt, \tag{3.205}$$

$$\int_0^T \left(\nabla \mathcal{Y}^+_{h,k}, \nabla \theta_h \right) \, dt \to \int_0^T \left(\nabla y, \nabla \theta \right) \, dt. \tag{3.206}$$

as $h, k \to 0$. We have

$$\Big| \int_0^T \left(\mathcal{W}^+_{h,k} + \mathcal{Y}^-_{h,k} - \mathcal{B}^+_{h,k}, \theta_h \right)_h \, dt - \int_0^T \left(w + y - \beta, \theta \right) \, dt \Big| \leq D_1 + D_2, \tag{3.207}$$

where $\beta = \beta_r - \beta_l$ and

$$D_1 = \Big| \int_0^T \left(\mathcal{W}^+_{h,k} + \mathcal{Y}^-_{h,k} - \mathcal{B}^+_{h,k}, \theta_h \right)_h \, dt - \int_0^T \left(\mathcal{W}^+_{h,k} + \mathcal{Y}^-_{h,k} - \mathcal{B}^+_{h,k}, \theta_h \right) \, dt \Big|,$$

$$D_2 = \left| \int_0^T \left(\mathcal{W}^+_{h,k} + \mathcal{Y}^-_{h,k} - \mathcal{B}^+_{h,k}, \theta_h\right) \; dt - \int_0^T (w + y - \beta, \theta) \; dt \; \right|.$$

Using the results of Theorem 3.23 and (A.31), we have

$$\begin{aligned}(3.208)\quad D_1 &\le C\ h \int_0^T \left[\|\mathcal{W}^+_{h,k}\| + \|\mathcal{Y}^-_{h,k}\| + \|\mathcal{B}^+_{h,k}\|\right] \|\nabla\theta_h\| \; dt \\ &\le C\ h \; \left[\|\mathcal{W}^+_{h,k}\|_{L^2(H^1)} + \|\mathcal{Y}^-_{h,k}\|_{L^\infty(H^1)} + \|\mathcal{B}^+_{h,k}\|_{L^2(L^2)}\right] \|\theta_h\|_{L^2(H^1)} \to 0,\end{aligned}$$

$$(3.209)\quad D_2 \to 0,$$

as $h, k \to 0$. Inserting (3.208), (3.209) in (3.207) produces, as $h, k \to 0$,

$$(3.210)\qquad \left| \int_0^T \left(\mathcal{W}^+_{h,k} + \mathcal{Y}^-_{h,k} - \mathcal{B}^+_{h,k}, \theta_h\right)_h \; dt - \int_0^T (w + y - \beta, \theta) \; dt \; \right| \to 0.$$

We have

$$(3.211)\qquad \left| \int_0^T \left(\mathcal{Y}^-_{h,k}, \boldsymbol{\mathcal{V}}^-_{h,k} \cdot \nabla\eta_h\right) \; dt - \int_0^T (y, \mathbf{v} \cdot \nabla\eta) \; dt \right| \le P_1 + P_2 + P_3.$$

where

$$\begin{aligned} P_1 &\le \left| \int_0^T \left(\mathcal{Y}^-_{h,k}, \left[\boldsymbol{\mathcal{V}}^-_{h,k} - \mathbf{v}\right] \cdot \nabla\eta_h\right) \; dt \; \right|, \\ P_2 &\le \left| \int_0^T \left(\mathcal{Y}^-_{h,k}, \mathbf{v} \cdot \nabla\left[\eta_h - \eta\right]\right) \; dt \; \right|, \\ P_3 &\le \left| \int_0^T \left(\mathcal{Y}^-_{h,k} - y, \mathbf{v} \cdot \nabla\eta\right) \; dt \; \right|. \end{aligned}$$

From the generalized Holder's inequality (A.14) and (A.17), we infer

$$(3.212)\qquad P_1 \le \|\mathcal{Y}^-_{h,k}\|_{L^\infty(H^1)} \|\boldsymbol{\mathcal{V}}^-_{h,k} - \mathbf{v}\|_{L^2\left(\mathbf{H}^1_0\right)} \|\eta_h\|_{L^2(H^1)} \to 0,$$

$$(3.213)\qquad P_2 \le \|\mathcal{Y}^-_{h,k}\|_{L^\infty(H^1)} \|\mathbf{v}\|_{L^2(\boldsymbol{\mathcal{D}})} \|\eta_h - \eta\|_{L^2(H^1)} \to 0,$$

$$(3.214)\qquad P_3 \le \left(\max_{t\in[0,T]} \|\nabla\eta\,(t)\,\|_{\mathcal{C}\left(\bar{\Omega}\right)}\right) \|\mathcal{Y}^-_{h,k} - y\|_{L^2(L^2)} \|\mathbf{v}\|_{L^2(\boldsymbol{\mathcal{S}})} \to 0.$$

as $h, k \to 0$. Taking into account of (3.212)-(3.214) in (3.211), we derive

$$(3.215)\qquad \left| \int_0^T \left(\mathcal{Y}^-_{h,k}, \boldsymbol{\mathcal{V}}^-_{h,k} \cdot \nabla\eta_h\right) \; dt - \int_0^T (y, \mathbf{v} \cdot \nabla\eta) \; dt \right| \to 0,$$

From the property (A.42) of the projection operator Q^h, we have

$$(3.216)\qquad \mathcal{Y}_{h,k}\,(0) = Q^h\; y_0 \to y_0, \qquad\qquad \text{in } L^2_0.$$

as $h \to 0$. Furthermore, with $\eta = \xi\,(1 - t/T)$, where $\xi \in L^2$, using integration by parts in time, we get

(3.217)

$$\left(\mathcal{Y}_{h,k}\,(0) - y\,(0)\,, \xi\right) = -\int_0^T \langle \left(\mathcal{Y}_{h,k} - y\right)_t, \eta\rangle_{H^{1*},H^1} dt \; - \int_0^T \left(\mathcal{Y}_{h,k} - y, \eta_t\right) \; dt \; \to 0,$$

as $h, k \to 0$. Therefore $\mathcal{Y}_{h,k}(0) \rightharpoonup y(0)$ in L^2. Hence, considering (3.216), (3.217) and the uniqueness of the weak limit, we can claim that

$$\mathcal{Y}_{h,k}(0) \to y(0) = y_0, \tag{3.218}$$

as $h, k \to 0$. From the discrete optimality conditions (3.139g), (3.139h), we have

$$\mathcal{Y}_{h,k}^{\bullet,\pm} \in \mathcal{K},$$
$$\mathcal{B}_{h,k}^{+} = \mathcal{B}_{l,h,k}^{+} - \mathcal{B}_{l,h,k}^{+}, \quad \mathcal{B}_{r,h,k}^{+}, \mathcal{B}_{l,h,k}^{+} \in \mathcal{K}^{+},$$

Then, from the results of Theorem 3.23, it is easy to realize that $\left(\mathcal{Y}_{h,k}^{\bullet,\pm}, \mathcal{B}_{l,h,k}^{+}, \mathcal{B}_{l,h,k}^{+}\right)$ converge to (y, β_r, β_l), which is such that

$$y \in \mathcal{K},\ \ \beta_r, \beta_l \in \mathcal{K}^{+}. \tag{3.219}$$

From the discrete optimality conditions (3.139i) and (3.139j), they hold

$$\int_0^T \left(\mathcal{B}_{r,h,k}^{+}, 1 - \mathcal{Y}_{h,k}^{+}\right)_h \ dt = 0, \tag{3.220}$$

$$\int_0^T \left(\mathcal{B}_{l,h,k}^{+}, 1 + \mathcal{Y}_{h,k}^{+}\right)_h \ dt = 0. \tag{3.221}$$

We have

$$\left| \int_0^T \left(\mathcal{B}_{r,h,k}^{+}, 1 - \mathcal{Y}_{h,k}^{+}\right)_h dt - \int_0^T (\beta_r, 1 - y) \ dt \right| \leq F_1 + F_2, \tag{3.222}$$

where

$$F_1 = \left| \int_0^T \left(B_{r,h,k}^{+}, 1 - \mathcal{Y}_{h,k}^{+}\right)_h \ dt - \int_0^T \left(B_{r,h,k}^{+}, 1 - \mathcal{Y}_{h,k}^{+}\right) \ dt \right|,$$
$$F_2 = \left| \int_0^T \left(\mathcal{B}_{r,h,k}^{+}, 1 - \mathcal{Y}_{h,k}^{+}\right) \ dt - \int_0^T (\beta_r, 1 - y) \ dt \right|.$$

Using (A.31) and the results of Theorem 3.23, we get

$$F_1 \leq C\ h \int_0^T \|B_{r,h,k}^{+}\| \|\nabla \mathcal{Y}_{h,k}^{+}\| \ dt \leq C\ h\ \|B_{r,h,k}^{+}\|_{L^2(L^2)} \|\mathcal{Y}_{h,k}^{+}\|_{L^2(H^1)} \to 0, \tag{3.223}$$

$$F_2 \to 0, \tag{3.224}$$

as $h, k \to 0$. Inserting (3.223), (3.224) in (3.222), we derive that

$$\left| \int_0^T \left(\mathcal{B}_{r,h,k}^{+}, 1 - \mathcal{Y}_{h,k}^{+}\right)_h dt - \int_0^T (\beta_r, 1 - y) \ dt \right| \to 0, \tag{3.225}$$

as $h, k \to 0$. By similar arguments, we infer

$$\left| \int_0^T \left(\mathcal{B}_{l,h,k}^{+}, 1 + \mathcal{Y}_{h,k}^{+}\right)_h dt - \int_0^T (\beta_l, 1 + y) \ dt \right| \to 0, \tag{3.226}$$

as $h, k \to 0$. From (3.204), (3.205), (3.206), (3.210), (3.215), (3.218), (3.219), (3.225) and (3.230), we can claim that the functions $\mathbf{v}, y, w, \beta_r, \beta_l$ in Theorem 3.23, satisfy

$$\int_0^T \left[\langle y_t, \eta \rangle_{H^{1*},H^1} + \gamma (\nabla w, \nabla \eta) - (y, \mathbf{v} \cdot \nabla \eta)\right] \, dt = 0,$$
$$y(0) = y_0,$$
$$\int_0^T \left[-(w + y, \theta) + \varepsilon^2 (\nabla y, \nabla \theta) + (\beta_r - \beta_l, \theta)\right] \, dt = 0,$$
$$y \in \mathcal{K},$$
$$\beta_r, \ \beta_l \in \mathcal{K}^+,$$
$$\int_0^T (\beta_r, 1 - y) \, dt = 0,$$
$$\int_0^T (\beta_l, 1 + y) \, dt = 0,$$

for all $\eta, \ \theta \in \mathcal{C}_c^\infty \left((0,T); \mathcal{C}_c^\infty \left(\bar{\Omega}\right)\right)$. Hence, by the density result (A.7), we can say that $\mathbf{v}, y, w, \beta_r, \beta_l$ solve the optimality conditions (2.116c)-(2.116i) of the continuous non-smooth optimal control Problem 2.14, for all $\eta, \ \theta \in L^2(H^1)$. Finally, using the results of Lemma 2.12, we realize that (3.183) above hold.

iii) Results (3.187), (2.117a), (2.117b).

From the discrete optimality conditions (3.140a), (3.140b), we can write

(3.227)
$$\int_0^T \left[-\left(\left(\boldsymbol{\mathcal{Q}}^{\bullet}_{\boldsymbol{\mathcal{V}},h,k}\right)_t, \boldsymbol{\psi}_h\right) + \left(\nabla \boldsymbol{\mathcal{Q}}^{-}_{\boldsymbol{\mathcal{V}},h,k}, \nabla \boldsymbol{\psi}_h\right) - \left(\mathcal{Y}^+_{h,k}, \nabla \mathcal{Q}^+_{\mathcal{Y},h,k} \cdot \boldsymbol{\psi}_h\right)\right] \, dt = 0,$$

(3.228)
$$\boldsymbol{\mathcal{Q}}^{\bullet;+}_{\boldsymbol{\mathcal{V}},h,k}(T) = 0.$$

for all $\boldsymbol{\psi}_h \in \mathcal{C}_c^\infty((0,T); \mathbf{D}_h)$. For any given $\boldsymbol{\psi} \in \mathcal{C}_c^\infty((0,T); \boldsymbol{\mathcal{D}})$, we set in (3.227) $\boldsymbol{\psi}_h = \mathbf{Q}_s^h \boldsymbol{\psi}$. Then, from property (A.49) of the Stokes projection operator, we derive

(3.229)
$$\boldsymbol{\psi}_h \to \boldsymbol{\psi}, \qquad \text{in } L^2(\boldsymbol{\mathcal{D}}),$$

as $h \to 0$. From the results of Theorem 3.24, we have

(3.230)
$$\int_0^T \left(\left(\boldsymbol{\mathcal{Q}}^{\bullet}_{\boldsymbol{\mathcal{V}},h,k}\right)_t, \boldsymbol{\psi}_h\right) \, dt \to \int_0^T (\mathbf{q}_{\mathbf{v}t}, \boldsymbol{\psi}) \, dt,$$

(3.231)
$$\int_0^T \left(\nabla \boldsymbol{\mathcal{Q}}^{-}_{\boldsymbol{\mathcal{V}},h,k}, \nabla \boldsymbol{\psi}_h\right) \, dt \to \int_0^T (\nabla \mathbf{q}_{\mathbf{v}}, \nabla \boldsymbol{\psi}) \, dt,$$

as $h, k \to 0$. We note that

(3.232)
$$\left| \int_0^T \left(\mathcal{Y}^+_{h,k}, \nabla \mathcal{Q}^+_{\mathcal{Y},h,k} \cdot \boldsymbol{\psi}_h\right) \, dt - \int_0^T (y, \nabla q_y \cdot \boldsymbol{\psi}) \, dt \right| \leq Q_1 + Q_2 + Q_3,$$

where

$$Q_1 = \left| \int_0^T \left(\mathcal{Y}_{h,k}^+, \nabla \mathcal{Q}_{\mathcal{Y},h,k}^+ \cdot [\boldsymbol{\psi}_h - \boldsymbol{\psi}] \right) \, dt \right|,$$

$$Q_2 = \left| \int_0^T \left(\left[\mathcal{Y}_{h,k}^+ - y \right], \nabla \mathcal{Q}_{\mathcal{Y},h,k}^+ \cdot \boldsymbol{\psi} \right) \, dt \right|,$$

$$Q_3 = \left| \int_0^T \left(y, \nabla \left[\mathcal{Q}_{\mathcal{Y},h,k}^+ - q_y \right] \cdot \boldsymbol{\psi} \right) \, dt \right|.$$

Using the results of Theorems 3.23, 3.24 and (3.229) above, we get

$$(3.233) \quad Q_1 \le C \int_0^T \|\mathcal{Y}_{h,k}^+\|_{H^1} \|\nabla \mathcal{Q}_{\mathcal{Y},h,k}^+\| \|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|_{\mathbf{H}_0^1} dt \le C \|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|_{\mathbf{L}^2\left(\mathbf{H}_0^1\right)} \to 0,$$

$$(3.234) \quad Q_2 \le C \int_0^T \|\mathcal{Y}_{h,k}^+ - y\|_{L^2} \|\nabla \mathcal{Q}_{\mathcal{Y},h,k}^+\| \|\boldsymbol{\psi}\|_{\mathbf{L}^\infty} dt \le C \|\mathcal{Y}_{h,k}^+ - y\|_{L^2\left(L_0^2\right)} \to 0,$$

as $h, k \to 0$. Moreover

$$\left| \int_0^T \left(y, \nabla \mathcal{Q}_{\mathcal{Y},h,k}^+ \cdot \boldsymbol{\psi} \right) \, dt \right| \le \int_0^T \|y\| \|\nabla \mathcal{Q}_{\mathcal{Y},h,k}^+\| \|\boldsymbol{\psi}\|_{\mathbf{L}^\infty} dt \le C \|\mathcal{Q}_{\mathcal{Y},h,k}^+\|_{L^2(H_0)}.$$

Therefore, using the weak convergence of $\mathcal{Q}_{\mathcal{Y},h,k}^+$, we can claim that

$$(3.235) \qquad Q_3 \to 0,$$

as $h, k \to 0$. Inserting (3.233), (3.234), (3.235) in (3.232), we realize that

$$(3.236) \qquad \left| \int_0^T \left(\mathcal{Y}_{h,k}^+, \nabla \mathcal{Q}_{\mathcal{Y},h,k}^+ \cdot \boldsymbol{\psi}_h \right) \, dt - \int_0^T \left(y, \nabla q_y \cdot \boldsymbol{\psi} \right) \, dt \right| \to 0.$$

as $h, k \to 0$. With $\boldsymbol{\psi} = \boldsymbol{\xi} \cdot t/T$, where $\boldsymbol{\xi} \in \mathbf{L}^2$, using integration by parts in time, we infer

$$\left(\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{\bullet} (T) - \mathbf{q}_{\mathbf{v}} (T), \boldsymbol{\xi} \right) = $$
$$= \int_0^T \left(\left(\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{\bullet} - \mathbf{q}_{\mathbf{v}} \right)_t, \boldsymbol{\psi} \right) \, dt + \int_0^T \left(\boldsymbol{\psi}_t, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{\bullet} - \mathbf{q}_{\mathbf{v}} \right) \, dt \to 0,$$

as $h, k \to 0$. Therefore,

$$(3.237) \qquad \mathbf{q}_{\mathbf{v}} (T) = 0.$$

Using (3.230), (3.231), (3.236) and (3.237), we derive that $y, \mathbf{q}_{\mathbf{v}}, q_y$ in Theorems 3.23, 3.24 satisfy

$$\int_0^T \left[- \left(\mathbf{q}_{\mathbf{v}t}, \boldsymbol{\psi} \right) + \left(\nabla \mathbf{q}_{\mathbf{v}}, \nabla \boldsymbol{\psi} \right) - \left(y, \nabla q_y \cdot \boldsymbol{\psi} \right) \right] \, dt = 0,$$
$$\mathbf{q}_{\mathbf{v}} (T) = 0,$$

for all $\boldsymbol{\psi} \in \mathcal{C}_c^\infty\left((0,T);\boldsymbol{\mathcal{D}}\right)$. Thus, from the density result (A.8), we infer that $y, \mathbf{q_v}, q_y$ satisfy the optimality conditions (2.117a), (2.117b) of the continuous non-smooth optimal control Problem 2.14, for all $\boldsymbol{\psi} \in L^2\left(\boldsymbol{\mathcal{D}}\right)$. Finally, using the results of Theorem 2.16, we conclude that also (3.187) above hold.

iv) Results (2.117c), (2.117d)

From the discrete optimality conditions (3.140d)-(3.140g), we have

$$\begin{aligned}
&\int_0^T \Big[-\left(\left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k}\right)_t, \eta_h\right)_h - \varepsilon^2\left(\nabla \mathcal{Q}^{-}_{\mathcal{W},h,k}, \nabla\eta_h\right) + \left(\mathcal{Q}^{+}_{\mathcal{W},h,k}, \eta_h\right)_h \\
&\quad - \left(\boldsymbol{\mathcal{V}}^{+}_{h,k}\cdot\nabla\mathcal{Q}^{+}_{\mathcal{Y},h,k}, \eta_h\right) - \left(\Lambda^{-}_{h,k}, \eta_h\right)_h + \left(\mathcal{Y}^{+}_{h,k} - \mathcal{Y}^{+}_{d,h,k}, \eta_h\right)\Big]\, dt = 0, \qquad (3.238)
\end{aligned}$$

$$\mathcal{Q}^{\bullet,+}_{\mathcal{Y},h,k}(T) = 0, \qquad (3.239)$$

$$\int_0^T \Big[\left(\mathcal{Q}^{-}_{\mathcal{W},h,k}, \theta_h\right)_h + \gamma\left(\nabla\mathcal{Q}^{-}_{\mathcal{Y},h,k}, \nabla\theta_h\right)\Big]\, dt = 0. \qquad (3.240)$$

for all $\eta_h \in \mathcal{C}^\infty\left([0,T];P_h\right), \theta_h \in \mathcal{C}_c^\infty\left((0,T);Y_h\right)$. For any given $\eta \in \mathcal{C}^\infty\left([0,T];\mathcal{C}_c^\infty(\bar{\Omega}) \cap L_0^2\right)$, $\theta \in \mathcal{C}_c^\infty\left((0,T);\mathcal{C}_c^\infty(\bar{\Omega})\right)$, we set $\eta_h = Q_1^h\eta, \theta_h = Q_1^h\theta$ in (3.238), (3.240). Then, from the property (A.47) of the projection operator Q_1^h, we get

$$\eta_h \to \eta, \qquad \text{in } L^2\left(H_0\right), \qquad (3.241)$$

$$\theta_h \to \theta, \qquad \text{in } L^2\left(H^1\right), \qquad (3.242)$$

as $h \to 0$. Moreover, using the definition (3.154) of $\mathcal{Y}^{+}_{d,h,k}$ and the property (A.44) of the projection operator Q_0^h, we get

$$\mathcal{Y}^{+}_{d,h,k} \to y_d, \qquad \text{in } L^2\left(L_0^2\right), \qquad (3.243)$$

From the results of Theorems 3.23, 3.24, (3.241), (3.242) and (3.243), we realize that

$$\int_0^T \left(\nabla\mathcal{Q}^{-}_{\mathcal{W},h,k}, \nabla\eta_h\right)\, dt \to \int_0^T \left(\nabla q_w, \nabla\eta\right)\, dt, \qquad (3.244)$$

$$\int_0^T \left(\nabla\mathcal{Q}^{-}_{\mathcal{Y},h,k}, \nabla\theta_h\right)\, dt \to \int_0^T \left(\nabla q_y, \nabla\theta\right)\, dt, \qquad (3.245)$$

$$\int_0^T \left(\mathcal{Y}^{+}_{h,k} - \mathcal{Y}^{+}_{d,h,k}, \eta_h\right)\, dt \to \int_0^T \left(y - y_d, \eta\right)\, dt, \qquad (3.246)$$

as $h, k \to 0$. We have

(3.247)

$$\left| \int_0^T \left(-\left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k}\right)_t, \eta_h\right)_h\, dt - \left[\int_0^T \langle \eta_t, q_y\rangle_{H_0^*,H_0} dt + \left(q_{y0}, \eta(0)\right)\right] \right| \le G_1 + G_2,$$

where

$$G_1 = \left| \int_0^T \left(-\left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k}\right)_t, \eta_h\right)_h\, dt + \int_0^T \left(\left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k}\right)_t, \eta_h\right) dt \right|,$$

$$G_2 = \Big| \int_0^T \Big(- \left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} \right)_t , \eta_h \Big) \; dt - \Big[\int_0^T \langle \eta_t, q_y \rangle_{H_0^*, H_0} dt + \left(q_{y0}, \eta \left(0 \right) \right) \Big] \Big|,$$

Using the property (A.47) for the projection operator Q_1^h, the relation (A.31), the results of Theorem 3.24 and integration by parts in time, we derive

$$\begin{aligned} G_1 = \Big| & \int_0^T \left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k}, \eta_{ht} \right)_h \; dt - \left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} \left(T \right), \eta_h \left(T \right) \right)_h + \left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} \left(0 \right), \eta_h \left(0 \right) \right)_h \\ & + \int_0^T - \left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k}, \eta_{ht} \right) \; dt + \left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} \left(T \right), \eta_h \left(T \right) \right) - \left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} \left(0 \right), \eta_h \left(0 \right) \right) \Big| \\ & \leq \Big| \int_0^T \left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k}, \eta_{ht} \right)_h \; dt - \int_0^T \left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k}, \eta_{ht} \right) \; dt \Big| \\ & \quad + \Big| \left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} \left(0 \right), \eta_h \left(0 \right) \right)_h - \left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} \left(0 \right), \eta_h \left(0 \right) \right) \Big| \\ & \leq C \; h \Big[\int_0^T \| \nabla \mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} \| \| \eta_{ht} \| \; dt + \| \nabla \mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} \left(0 \right) \| \| \eta_h \left(0 \right) \| \Big] \\ & \leq C \; h \; \| \mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} \|_{L^\infty(H_0)} \Big[\int_0^T \| \eta_{ht} \| \; dt + \| \eta_h \left(0 \right) \| \Big] \\ & \leq C \; h \left(1 + h^2 \right) \; \| \mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} \|_{L^\infty(H_0)} \Big[\int_0^T \| \eta_t \|_{H^2} \; dt + \| \eta \left(0 \right) \|_{H^2} \Big] \to 0, \end{aligned} \tag{3.248}$$

and

(3.249)
$$G_2 = \Big| \int_0^T \left[\left(\eta_{ht}, \mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} \right) - \langle \eta_t, q_y \rangle_{H_0^*, H_0} \right] \; dt + \left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} \left(0 \right), \eta_h \left(0 \right) \right) - \left(q_{y0}, \eta \left(0 \right) \right) \Big| \to 0,$$

as $h, k \to 0$. Inserting (3.248), (3.249) in (3.247), we get

$$\Big| \int_0^T \Big(- \left(\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} \right)_t , \eta_h \Big)_h \; dt - \Big[\int_0^T \langle \eta_t, q_y \rangle_{H_0^*, H_0} dt + \left(q_{y0}, \eta \left(0 \right) \right) \Big] \Big| \to 0, \tag{3.250}$$

as $h, k \to 0$. We have

$$\Big| \int_0^T \left(\mathcal{Q}^{+}_{\mathcal{W},h,k}, \eta_h \right)_h dt - \int_0^T \left(q_w, \eta \right) dt \Big| \leq I_1 + I_2, \tag{3.251}$$

where

$$\begin{aligned} I_1 &= \Big| \int_0^T \left(\mathcal{Q}^{+}_{\mathcal{W},h,k}, \eta_h \right)_h dt - \int_0^T \left(\mathcal{Q}^{+}_{\mathcal{W},h,k}, \eta_h \right) \; dt \Big|, \\ I_2 &= \Big| \int_0^T \left(\mathcal{Q}^{+}_{\mathcal{W},h,k}, \eta_h \right) dt - \int_0^T \left(q_w, \eta \right) dt \Big|. \end{aligned}$$

From (A.31), the results of Theorem 3.24 and (3.241), we can claim that

$$I_1 \leq C \; h \; \left(1 + h \right) \| \eta \|_{L^2(H_0)} \| \mathcal{Q}^{+}_{\mathcal{W},h,k} \|_{L^2(H^1)} \to 0, \tag{3.252}$$

$$I_2 \to 0, \tag{3.253}$$

as $h,k\to 0$. Thus, inserting (3.252), (3.253) in (3.251), we realize

$$\Big|\int_0^T \left(\mathcal{Q}^+_{\mathcal{W},h,k},\eta_h\right)_h dt - \int_0^T (q_w,\eta)\,dt\Big| \to 0, \tag{3.254}$$

We have

$$\Big|\int_0^T \left(\boldsymbol{\mathcal{V}}^+_{h,k}\cdot\nabla\mathcal{Q}^+_{\mathcal{Y},h,k},\eta_h\right)\ dt - \int_0^T (\mathbf{v}\cdot\nabla q_y,\eta)\ dt\ \Big| \le R_1+R_2+R_3, \tag{3.255}$$

where

$$\begin{aligned} R_1 &= \Big|\int_0^T \left(\left[\boldsymbol{\mathcal{V}}^+_{h,k}-\mathbf{v}\right]\cdot\nabla\mathcal{Q}^+_{\mathcal{Y},h,k},\eta_h\right)\ dt\ \Big|,\\ R_2 &= \Big|\int_0^T \left(\mathbf{v}\cdot\nabla\mathcal{Q}^+_{\mathcal{Y},h,k},\eta_h-\eta\right)\ dt\ \Big|,\\ R_3 &= \Big|\int_0^T \left(\mathbf{v}\cdot\left[\nabla\mathcal{Q}^+_{\mathcal{Y},h,k}-\nabla q_y\right],\eta\right)\ dt\ \Big|, \end{aligned}$$

From the results of Theorems 3.23, 3.24, we infer

$$R_1 \le C\ \|\boldsymbol{\mathcal{V}}^+_{h,k}-\mathbf{v}\|_{\mathbf{L}^2\left(\mathbf{H}^1_0\right)}\ \|\mathcal{Q}^+_{\mathcal{Y},h,k}\|_{L^\infty(H_0)}\ \|\eta_h\|_{L^2(H_0)} \to 0, \tag{3.256}$$

$$R_2 \le C\ \|\mathbf{v}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})}\ \|\mathcal{Q}^+_{\mathcal{Y},h,k}\|_{L^\infty(H_0)}\ \|\eta_h-\eta\|_{L^2(H_0)} \to 0, \tag{3.257}$$

as $h,k\to 0$. Furthermore, we note that

$$\Big|\int_0^T \left(\mathbf{v}\cdot\nabla\mathcal{Q}^+_{\mathcal{Y},h,k},\eta\right)\ dt\ \Big| \le C\ \|\mathbf{v}\|_{L^\infty(\boldsymbol{\mathcal{D}})}\ \|\mathcal{Q}^+_{\mathcal{Y},h,k}\|_{L^2(H_0)}\ \|\eta\|_{L^2(H^1)},$$

therefore, using the weak convergence of $\mathcal{Q}^+_{\mathcal{Y},h,k}$, we derive

$$R_3 \to 0, \tag{3.258}$$

as $h,k\to 0$. Hence, using (3.256), (3.257) and (3.258) in (3.255), we get

$$\Big|\int_0^T \left(\boldsymbol{\mathcal{V}}^+_{h,k}\cdot\nabla\mathcal{Q}^+_{\mathcal{Y},h,k},\eta_h\right)\ dt - \int_0^T (\mathbf{v}\cdot\nabla q_y,\eta)\ dt\ \Big| \to 0, \tag{3.259}$$

as $h,k\to 0$. From the results of Theorem 3.24, we have

$$\int_0^T \left(\Lambda^-_{h,k},\eta_h\right)_h\ dt = \int_0^T \left(\bar{\Lambda}^-_{h,k},\eta_h\right)\ dt \to \langle\lambda,\eta\rangle_{W_0^*,W_0}. \tag{3.260}$$

as $h,k\to 0$. Using (3.244)-(3.246), (3.250), (3.254), (3.259) and (3.260), we derive that $\mathbf{v},y,q_y,q_w,\lambda$ satisfies

$$\begin{aligned} &\int_0^T \left[\langle\eta_t,q_y\rangle_{H_0^*,H_0} - \varepsilon^2\left(\nabla q_w,\nabla\eta\right) + (q_w,\eta)\right.\\ &\left.-\left(\mathbf{v}\cdot\nabla q_y,\eta\right) + (y-y_d,\eta)\right]\ dt + \left(q_{y0},\eta\left(0\right)\right) - \langle\lambda,\eta\rangle_{W_0^*,W_0} = 0, \end{aligned}$$

$$\int_0^T [\ (q_w, \theta) + \gamma\,(\nabla q_y, \nabla \theta)\ ]\ dt = 0,$$

for all $\eta \in \mathcal{C}^\infty\left([0,T]\,;\mathcal{C}_c^\infty(\bar{\Omega}) \cap L_0^2\right)$, $\theta \in \mathcal{C}_c^\infty\left((0,T)\,;\mathcal{C}_c^\infty(\bar{\Omega})\right)$. Finally, by the density arguments (A.7), (A.12), we get that $\mathbf{v}, y, q_y, q_w, \lambda$ satisfies (2.117c), (2.117d) for all $\eta \in W_0,\ \theta \in L^2(H^1)$.

v) Results (2.118), (3.186)

From the discrete optimality condition (3.141), we have

$$\alpha\, \boldsymbol{\mathcal{U}}_{h,k}^{+} = \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{-}.$$

Then, up to a multiplicative constant, we can identify $\boldsymbol{\mathcal{U}}_{h,k}^{+}$ with $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}$. Hence, using the results of Theorem 3.24, we derive

$$\begin{aligned} &\boldsymbol{\mathcal{U}}_{h,k}^{+} \overset{*}{\rightharpoonup} \mathbf{u}, && \text{in } L^\infty\left(\mathbf{H}_0^1\right), \\ &\boldsymbol{\mathcal{U}}_{h,k}^{+} \to \mathbf{u}, && \text{in } L^2\left(\mathbf{H}_0^1\right), \end{aligned}$$

as $h, k \to 0$. Furthermore, $(\mathbf{u}, \mathbf{q_v})$ satisfies (2.118) and, from (3.187), we get that (3.186) holds. □

In the next Lemma we prove additional optimality conditions which represent the discrete counterpart of the relations (2.119) in Theorem 2.16.

Lemma 3.26. *Given a sequence $h, k \to 0$, let us consider a subsequence (not relabelled) such that the results of the Theorems 3.23, 3.24 and 3.25 hold. Then*

$$\lim_{h,k} \int_0^T \left(g\left(\mathcal{Y}_{h,k}^{+}\right), \bar{\Lambda}_{h,k}^{-}\right)\ dt = 0, \tag{3.261}$$

$$\lim_{h,k} \int_0^T \left(\mathcal{B}_{h,k}^{+}, \mathcal{Q}_{\mathcal{W},h,k}^{-}\right)\ dt = 0, \tag{3.262}$$

and

$$0 \le \liminf_{h,k} \int_0^T \left(\bar{\Lambda}_{h,k}^{-}, \mathcal{Q}_{\mathcal{W},h,k}^{-}\right)\ dt \le C, \tag{3.263}$$

for all $g : \mathbb{R} \to \mathbb{R}$ Lipschitz such that $g(-1) = g(1) = 0$, where C is a constant independent of h, k.

Proof. From (3.142a), we have

$$\sum_{n=1}^{N} k\ \left(g\left(Y^n\right), \Lambda^{n-1}\right)_h = \int_0^T \left(g\left(\mathcal{Y}_{h,k}^{+}\right), \bar{\Lambda}_{h,k}^{-}\right)\ dt = 0,$$

for all h, k. Then, (3.261) hold. Using (3.142b) we derive

$$\left(B^n, Q_W^{n-1}\right)_h = 0, \quad \forall\ n = 1, \dots, N.$$

Hence, from (A.31), we get

$$\Big|\int_0^T \left(\mathcal{B}^+_{h,k}, \mathcal{Q}^-_{\mathcal{W},h,k}\right)\, dt\Big| = \Big|\sum_{n=1}^{n} k\, \left(B^n, Q_W^{n-1}\right)\Big|$$

$$\leq \sum_{n=1}^{n} k\, \Big|\left(B^n, Q_W^{n-1}\right) - \left(B^n, Q_W^{n-1}\right)_h\Big| \leq C\, h \sum_{n=1}^{n} k\, \|B^n\|\, \|\nabla Q_W^{n-1}\|$$

$$\leq C\, h\, \|\mathcal{B}^+_{h,k}\|_{L^2(L^2)}\, \|\mathcal{Q}^-_{\mathcal{W},h,k}\|_{L^2(H^1)} \to 0,$$

as $h, k \to 0$. So, (3.262) is satisfied. Using (3.142c) and the stability estimate (3.144j), we can write

$$0 \leq \int_0^T \left(\bar{\Lambda}^-_{h,k}, \mathcal{Q}^-_{\mathcal{W},h,k}\right)\, dt \leq C,$$

which implies (3.263). □

3.5. Numerical Solution of the Discrete Optimal Control Problem

In this section we show the strategy we use for the numerical solution of the non-smooth discrete optimal control Problem 3.1.

In order to justify our approach, we need to perform some preliminary considerations. Let $\{\delta_n\}_n$ be a sequence of the regularization parameter such that $\delta_n \to 0^+$ and Theorem 3.18 holds and let $\{P_n\}_n$ the corresponding sequence of the discrete regularized optimal control Problems 3.2. For any fixed n there exists a sequence $\left\{\left(\boldsymbol{\mathcal{X}}_{h,k,(n),(i)}, \boldsymbol{\mathcal{U}}_{h,k,(n),(i)}\right)\right\}_i$, such that

$$\left(\boldsymbol{\mathcal{X}}_{h,k,(n),(i)}, \boldsymbol{\mathcal{U}}_{h,k,(n),(i)}\right) \to \left(\boldsymbol{\mathcal{X}}_{h,k,(n)}, \boldsymbol{\mathcal{U}}_{h,k,(n)}\right),$$

as $i \to +\infty$, where $\left(\boldsymbol{\mathcal{X}}_{h,k,(n)}, \boldsymbol{\mathcal{U}}_{h,k,(n)}\right)$ is a solution of the regularized Problem 3.2. For instance, the sequences $\left\{\left(\boldsymbol{\mathcal{X}}_{h,k,(n),(i)}, \boldsymbol{\mathcal{U}}_{h,k,(n),(i)}\right)\right\}_i$ can be obtained by the following *steepest descent algorithm* (see for example [58], Section 2.2.1):

Algorithm 3.27 (Steepest Descent). *Perform the following steps:*

1. *choose an initial guess* $\boldsymbol{\mathcal{U}}_{h,k,(n),(0)}$ *and set* $i = 0$*;*
2. *solve the discrete state equations* (3.72) *to get* $\boldsymbol{\mathcal{X}}_{h,k,(n),(i)}$*;*
3. *solve the discrete adjoint equations* (3.73) *to get* $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k,(n),(i)}$*;*
4. *given* $\tilde{J}_{\delta,h,k}\left(\boldsymbol{\mathcal{U}}_{h,k}\right) = J_{h,k}\left(s_{\delta,h,k}\left(\boldsymbol{\mathcal{U}}_{h,k}\right), \boldsymbol{\mathcal{U}}_{h,k}\right)$*, calculate*

$$\nabla_{\boldsymbol{\mathcal{U}}_{h,k}} \tilde{J}_{\delta_n,h,k}\left(\boldsymbol{\mathcal{U}}_{h,k,(n),(i)}\right) = \alpha\, \boldsymbol{\mathcal{U}}_{h,k,(n),(i)} - \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k,(n),(i)},$$

choose an admissible step size $\sigma_{(i)}$ *and set*

$$\begin{aligned}\boldsymbol{\mathcal{U}}_{h,k,(n),(i+1)} &= \boldsymbol{\mathcal{U}}_{h,k,(n),(i)} - \sigma_{(i)} \nabla_{\boldsymbol{\mathcal{U}}_{h,k}} \tilde{J}_{\delta,h,k}\left(\boldsymbol{\mathcal{U}}_{h,k,(n),(i)}\right),\\ i &= i + 1,\end{aligned}$$

and go to step 2.

Once we get the sequence $\left\{\left(\boldsymbol{\mathcal{X}}_{h,k,(n)}, \boldsymbol{\mathcal{U}}_{h,k,(n)}\right)\right\}_n$ of the solution of the regularized problems P_n, Theorem 3.18 guarantees

$$\left(\boldsymbol{\mathcal{X}}_{h,k,(n)}, \boldsymbol{\mathcal{U}}_{h,k,(n)}\right) \to \left(\boldsymbol{\mathcal{X}}_{h,k}, \boldsymbol{\mathcal{U}}_{h,k}\right),$$

as $n \to +\infty$, where $\left(\boldsymbol{\mathcal{X}}_{h,k}, \boldsymbol{\mathcal{U}}_{h,k}\right)$ is a solution of the non-smooth Problem 3.1.
The approach above described is not, in practice, numerically realizable. So, to overcome this difficult, we use the continuity of $\tilde{J}_{\delta,h,k}\left(\boldsymbol{\mathcal{U}}_{h,k}\right) = J_{h,k}\left(s_{\delta,h,k}\left(\boldsymbol{\mathcal{U}}_{h,k}\right), \boldsymbol{\mathcal{U}}_{h,k}\right)$ with respect to the control $\boldsymbol{\mathcal{U}}_{h,k}$ and the regularization parameter δ: first, we perform the limit with respect to the regularization parameter $\delta \to 0$ and then we apply the steepest descent algorithm above directly to the non-smooth Problem 3.19. In order to do that, we briefly introduce the following notation: given a discrete control $\boldsymbol{\mathcal{U}}_{h,k}$, we denote by $\mathcal{Y}_{h,k} = \mathcal{Y}_{h,k}\left(\boldsymbol{\mathcal{U}}_{h,k}\right)$ the corresponding discrete phase-field solution of the discrete state equations (3.139) and by $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k} = \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}\left(\boldsymbol{\mathcal{U}}_{h,k}\right)$ the corresponding variable given by the discrete optimality conditions (3.140) and the complementarity relations (3.142). Furthermore, we define

$$\boldsymbol{\mathcal{G}}_{h,k} := \alpha\, \boldsymbol{\mathcal{U}}_{h,k} - \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}.$$

We use the following algorithm to solve the optimality conditions (3.139)-(3.142).

Algorithm 3.28. *We perform the following steps:*

1. *we choose an initial guess for the control* $\boldsymbol{\mathcal{U}}_{h,k,(0)}$*, a constant* $TOL > 0$*, an integer* N_{max} *and set* $i = 0$*;*

2. *given* $\boldsymbol{\mathcal{U}}_{h,k,(i)}$*, we solve the discrete state equations (3.139) to get* $\boldsymbol{\mathcal{V}}_{h,k,(i)}$*,* $\mathcal{Y}_{h,k,(i)}$*,* $\mathcal{W}_{h,k,(i)}$*;*

3. *given* $\boldsymbol{\mathcal{V}}_{h,k,(i)}$*,* $\mathcal{Y}_{h,k,(i)}$*,* $\mathcal{W}_{h,k,(i)}$*, we solve the optimality conditions (3.140) and the complementarity conditions (3.142) to derive* $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k,(i)}$*;*

4. *we calculate*

$$\|\boldsymbol{\mathcal{G}}_{h,k,(i)}\|_{L^2(\mathbf{L}^2)} = \left[\sum_{n=1}^{N} k\|\alpha \mathbf{U}^n_{(i)} - \mathbf{Q}^{n-1}_{\mathbf{V}(i)}\|^2\right]^{\frac{1}{2}},$$

IF $\|\boldsymbol{\mathcal{G}}_{h,k,(i)}\| < TOL$ *or* $i > N_{max}$*, then STOP;*
ELSE we choose a stepsize $\sigma_{(i)}$*, set*

$$\begin{aligned}\boldsymbol{\mathcal{U}}_{h,k,(i+1)} &= \boldsymbol{\mathcal{U}}_{h,k,(i)} - \sigma_{(i)}\boldsymbol{\mathcal{G}}_{h,k,(i)}\\ i &= i+1,\end{aligned}$$

and go to step 2*;*

We perform the second and the third steps of the Algorithm 1 by the so called *Primal Dual Active Set Strategy* (PDAS), (see [16] for details). In order to do that we make the following assumption.

Assumption 3.29 (Strict Complementarity).

$$Y^n(x_j) = \pm 1, \quad \Rightarrow \quad B^n(x_j) \neq 0, \tag{3.264}$$

for all $j = 1, \ldots, \mathcal{N}_h,\ n = 1, \ldots, N$.

The above strict complementarity assumption is commonly used in the solution of problems which involve complementarity conditions like (3.139i), (3.139j) and (3.142). We refer the reader to [47], [53], [75] and the references therein for further details.
In the next sections we explain in details of second and third steps of Algorithm 3.28.

Algorithm 3.28: Step 2

We solve the discrete Stokes equations (3.139a)-(3.139c) to get $\boldsymbol{\mathcal{V}}_{(i)}$. Then, we apply the PDAS to solve the discrete Cahn-Hilliard equations (3.139d)-(3.139j) to obtain $\mathcal{Y}_{(i)}$. In order to do that, given the set of the indices of the vertices of the triangulation of the domain Ω,

$$\mathcal{J}_h = \{j \in \{1, \ldots, N_h\} : x_j \text{ is a vertex of } \mathcal{T}_h\},$$

we define, at each time level $n = 1, \ldots, N$,

$$\begin{aligned} \mathcal{A}_+^n &= \{j \in \mathcal{J}_h : c\,(Y^n(x_j) - 1) + B^n(x_j) > 0\}, \\ \mathcal{A}_-^n &= \{j \in \mathcal{J}_h : c\,(Y^n(x_j) + 1) + B^n(x_j) < 0\}, \\ \mathcal{I}^n &= \mathcal{J}_h \setminus \left(\mathcal{A}_+^n \cup \mathcal{A}_-^n\right), \end{aligned}$$

where $c > 0$ is a constant. $\mathcal{A}_\pm^n$ are called the *active sets*; $\mathcal{I}^n$ are the *inactive sets.* It is easy to realize that, under the strict complementarity assumption (3.264), the following equivalence holds

$$\begin{cases} Y^n(x_j) = \pm 1, & \text{if } j \in \mathcal{A}_\pm^n, \\ B^n(x_j) = 0, & \text{if } j \in \mathcal{I}^n, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} -1 \le Y^n(x_j) \le 1, \\ B_r^n(x_j) \ge 0,\ B_l^n(x_j) \ge 0, \\ B_r^n(x_j)\,(1 - Y^n(x_j)) = 0, \\ B_l^n(x_j)\,(1 + Y^n(x_j)) = 0. \end{cases}$$

Then, to solve the discrete Cahn-Hilliard equations (3.139d)-(3.139j) and derive $\mathcal{Y}_{(i)}$, we use the following algorithm.

Algorithm 3.30 (PDAS). *For all* $n = 1, \ldots, N$*:*

1. *we initialize* $\mathcal{A}_{+(0)}^n, \mathcal{A}_{-(0)}^n$ *by*

$$\mathcal{A}_{\pm(0)}^n = \left\{j \in \mathcal{J}_h : Y^{n-1}(x_j) = \pm 1\right\},$$

   calculate $\mathcal{I}_{(0)}^n$ *and set* $m = 0$*;*

2. *we set* $Y_{(m)}^n(x_j) = \pm 1, \forall j \in \mathcal{A}_{\pm(m)}^n$ *and* $B_{(m)}^n(x_j) = 0, \forall j \in \mathcal{I}_{(m)}^n$*;*

3. *we solve the following linear system*

$$M^{nI}_{h(m)}\ \underline{Y^n_{I(m)}} + k\gamma A\ \underline{W^n_{(m)}} = \underline{f_1}\left(Y^n_{A(m)}, Y^{n-1}, \mathbf{V}^{n-1}\right), \tag{3.265}$$

$$-\varepsilon^2 A^{nI}_{(m)}\ \underline{Y^n_{I(m)}} + M_h\ \underline{W^n_{(m)}} - M^{nA}_{h(m)}\ \underline{B^n_{A(m)}} = \underline{f_2}\left(Y^n_{A(m)}, Y^{n-1}\right), \tag{3.266}$$

to obtain $Y^n_{(m)}(x_j)$ *for* $j \in \mathcal{I}^n_{(m)}$ *by the vector* $\underline{Y^n_{I(m)}}$, $B^n_{(m)}(x_j)$ *for* $j \in \mathcal{A}^n_{+(m)} \cup \mathcal{A}^n_{-(m)}$ *by the vector* $\underline{B^n_{A(m)}}$ *and* $W^n_{(m)}$ *by the vector* $\underline{W^n_{(m)}}$.

4. *we set*

$$\begin{aligned}
\mathcal{A}^n_{+(m+1)} &= \{j \in \mathcal{J}_h : c\left(Y^n(x_j) - 1\right) + B^n(x_j) > 0\}, \\
\mathcal{A}^n_{-(m+1)} &= \{j \in \mathcal{J}_h : c\left(Y^n(x_j) + 1\right) + B^n(x_j) < 0\}, \\
\mathcal{I}^n_{(m+1)} &= \mathcal{J}_h \setminus \left(\mathcal{A}^n_{+(m+1)} \cup \mathcal{A}^n_{-(m+1)}\right).
\end{aligned}$$

5. *IF* $\mathcal{A}^n_{\pm(m+1)} = \mathcal{A}^n_{\pm(m)}$, *we set* $Y^n = Y^n_{(m)}$, *then STOP;*
   ELSE we set

$$m = m + 1$$

   and go to step 2.

In the linear system (3.265), (3.266) above, we use the following matrices

$$\begin{aligned}
M_{hij} &= (\eta_i, \eta_j)_h, & A_{ij} &= (\nabla\eta_i, \nabla\eta_j), & & i, j \in \mathcal{J}_h, \\
M^{nI}_{h(m)ij} &= (\eta_i, \eta_j)_h, & A^{nI}_{(m)ij} &= (\nabla\eta_i, \nabla\eta_j), & & i \in \mathcal{J}_h,\ j \in \mathcal{I}^n_{(m)}, \\
M^{nA}_{h(m)ij} &= (\eta_i, \eta_j)_h, & A^{nA}_{(m)ij} &= (\nabla\eta_i, \nabla\eta_j), & & i \in \mathcal{J}_h,\ j \in \mathcal{A}^n_{+(m)} \cup \mathcal{A}^n_{-(m)},
\end{aligned}$$

and the following vectors

$$\begin{aligned}
\underline{Y^n_{I(m)}}_j &= Y^n_{(m)}(x_j), & & j \in \mathcal{I}^n_{(m)}, \\
\underline{Y^n_{A(m)}}_j &= Y^n_{(m)}(x_j), & & j \in \mathcal{A}^n_{+(m)} \cup \mathcal{A}^n_{-(m)}, \\
\underline{W^n_{(m)}}_j &= W^n_{(m)}(x_j), & & j \in \mathcal{J}_h, \\
\underline{B^n_{A(m)}}_j &= B^n_{(m)}(x_j), & & j \in \mathcal{A}^n_{+(m)} \cup \mathcal{A}^n_{-(m)},
\end{aligned}$$

$$\begin{aligned}
\underline{f_1}_i\left(Y^n_{A(m)}, Y^{n-1}, \mathbf{V}^{n-1}\right) &= -M^{nA}_{h(m)}\ \underline{Y^n_{A(m)}} + \left(Y^{n-1}, \eta_i\right)_h - k\left(Y^{n-1}, \mathbf{V}^{n-1}\cdot\nabla\eta_i\right), \\
\underline{f_2}_i\left(Y^n_{A(m)}, Y^{n-1}\right) &= \varepsilon^2 A^{nA}_{(m)}\ \underline{Y^n_{A(m)}} - \left(Y^{n-1}, \eta_i\right)_h,
\end{aligned}$$

where $\{\eta_1, \ldots, \eta_{N_h}\}$ is a Lagrange basis for Y_h.

Algorithm 3.28: Step 3

We solve, for all $n = 1, \ldots, N$, the discrete backward equations (3.140d)-(3.140g) to derive $\mathcal{Q}_{\mathcal{Y}(i)}$. In order to do that, we note from step 2 that we know $\boldsymbol{\mathcal{V}}_{(i)}, \mathcal{Y}_{(i)}$ and the sets $\mathcal{A}^n_{\pm}, \mathcal{I}^n$, for all $n = 1, \ldots, N$. For any given n, in (3.140d)-(3.140g), we have three unknowns $Q_Y^{n-1}, Q_W^{n-1}, \Lambda^{n-1}$ and just two equations. So, we consider the complementarity conditions (3.142a)-(3.142c), which are such that

$$\Lambda^{n-1}(x_j) = 0, \qquad \text{if } -1 < Y^n(x_j) < 1, \tag{3.267}$$

$$\left[B^n Q_W^{n-1}\right](x_j) = 0, \tag{3.268}$$

$$\left[\Lambda^{n-1} Q_W^{n-1}\right](x_j) \geq 0,$$

for all $j \in \mathcal{J}_h$, $n = 1, \ldots, N$. Above, (3.267) is just a reformulation of (3.142a) and it easy to realize that it is equivalent to

$$\Lambda^{n-1}(x_j) = 0, \quad \forall j \in \mathcal{I}^n.$$

Moreover from (3.268), using the strict complementarity assumption (3.264), we derive

$$Q_W^{n-1}(x_j) = 0, \quad \forall j \in \mathcal{A}_+^n \cup \mathcal{A}_-^n.$$

So, given $n = 1, \ldots, N$, we use (3.140d)-(3.140f) to get just:

$$\begin{aligned} &Q_Y^{n-1}(x_j), && \forall j \in \mathcal{J}_h, \\ &Q_W^{n-1}(x_j), && \forall j \in \mathcal{I}^{n-1}, \\ &\Lambda^{n-1}(x_j), && \forall j \in \mathcal{A}_+^{n-1} \cup \mathcal{A}_-^{n-1}. \end{aligned}$$

We get them solving the following linear system

$$\begin{aligned} M_h\, \underline{Q_Y^{n-1}} - k\varepsilon^2 A^{nI}\, \underline{Q_{WI}^{n-1}} - k\, M_h^{nA}\, \underline{\Lambda_A^{n-1}} &= \underline{r}\left(Q_Y^n, Q_W^n, \mathbf{V}^n, Y^n, y_{d,h}^n\right), \\ \gamma A\, \underline{Q_Y^{n-1}} + M_h^{nI}\, \underline{Q_{WI}^{n-1}} &= 0, \end{aligned}$$

where

$$\begin{aligned} \underline{Q_Y^{n-1}}_j &= Q_Y^{n-1}(x_j), && j \in \mathcal{J}_h, \\ \underline{Q_{WI}^{n-1}}_j &= Q_W^{n-1}(x_j), && j \in \mathcal{I}^n, \\ \underline{\Lambda_A^{n-1}}_j &= \Lambda^{n-1}(x_j), && j \in \mathcal{A}_+^n \cup \mathcal{A}_-^n, \end{aligned}$$

$$\underline{r}_i\left(Q_Y^n, Q_W^n, \mathbf{V}^n, Y^n, y_{d,h}^n\right) = \left(Q_Y^n - kQ_W^n, \eta_i\right)_h + k\left(\nabla Q_Y^n \cdot \mathbf{V}^n, \eta_i\right) - k\left(Y^n - y_{d,h}^n, \eta_i\right),$$

where $\{\eta_1, \ldots, \eta_{N_h}\}$ is a Lagrange basis for Y_h. Once we get $\mathcal{Q}_{\mathcal{Y},h,k,(i)}$, we solve the discrete backward equations (3.140a)-(3.140c), to derive $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k,(i)}$.

3.6. Numerical Experiments

In order to show the effectiveness of our method, we consider two numerical experiments.

3.6.1. Circle to Square 1

The domain is the unit square $\Omega = (0,1)^2$ in the two dimensional plane $(x_1, x_2) = x$. The initial condition $y_{0,h}$ for the phase-field y is given by the linear interpolation of the following function

$$y_0(r) = \begin{cases} -1 & \text{if} \quad r - R \leq -\dfrac{\pi\varepsilon}{2}, \\ \sin\left(\dfrac{r-R}{\varepsilon}\right) & \text{if} \quad |r-R| < \dfrac{\pi\varepsilon}{2}, \\ 1 & \text{if} \quad r - R \geq \dfrac{\pi\varepsilon}{2}, \end{cases} \tag{3.269}$$

where $r = r(x_1, x_2) = \sqrt{(x_1 - x_{c1})^2 + (x_2 - x_{c2})^2}$, $R = 0.2$ and $(x_{c1}, x_{c2}) = (0.5, 0.5)$. We emphasize that the function y_0 corresponds to a stationary solution of the Cahn-Hilliard equation with double obstacle potential, see fig. 3.1.

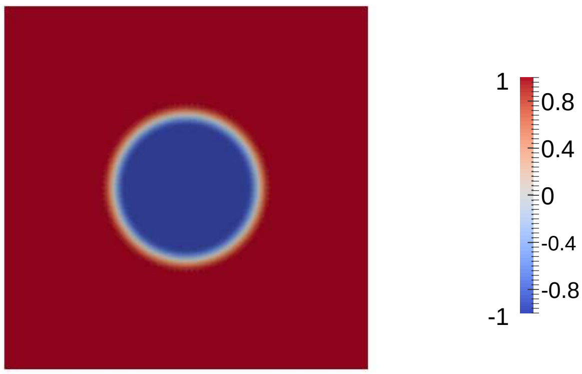


Figure 3.1.: Initial phase-field $y_0(x)$

The values of the constants parameter in the model are $\alpha = 10^{-5}, \nu = 0.1, \gamma = 0.005, \varepsilon = 0.02$. Furthermore, the time step $k = 0.01$ and the time horizon is $T = 100k$. The desired state y_d is represented in fig. 3.2. It is independent on time and the two phases fluid are separated by a vanishing interface which has the shape of a square. We emphasize that, in order to make the desired state reachable, we have chosen y_0 and y_d such that

$$\int_\Omega y_0(x)\, dx = \int_\Omega y_d(x)\, dx. \tag{3.270}$$

In the Algorithm 3.28, we assume as initial step for the control $\boldsymbol{\mathcal{U}}_{h,k,(0)} \equiv 0$, the tolerance $TOL = 10^{-9}$ and the maximum number of iterations $N_{\max} = 10^3$. Moreover, the step size $\sigma_{(i)}$ in is derived according to the Barzilai-Borwein method, see [12] for details . In particular, with $\sigma_{\text{init}} = 4 \cdot 10^3$, $\sigma_{\min} = 2 \cdot 10^3$, $\sigma_{\max} = 4 \cdot 10^3$ and denoting by i the iteration index, we assume:

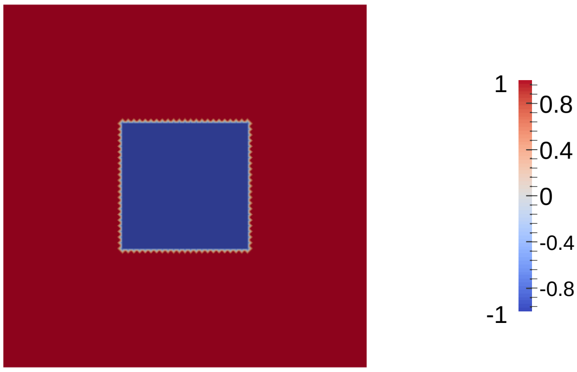


Figure 3.2.: Desired state distribution $y_d(x)$

- if $i = 0$, $\sigma_{(i)} = \sigma_{\text{init}}$;
- for $i \geq 1$

(3.271) $$\sigma_{(i)} = \frac{\int_0^T \left(\mathcal{U}_{h,k,(i)} - \mathcal{U}_{h,k,(i-1)}, \mathcal{G}_{h,k,(i)} - \mathcal{G}_{h,k,(i-1)}\right) dt}{\|\mathcal{G}_{h,k,(i)} - \mathcal{G}_{h,k,(i-1)}\|^2_{L^2(\mathbf{L}^2)}},$$

- if $\sigma_{(i)} < 0$ or $\sigma_{(i)} > \sigma_{\max}$, then $\sigma_{(i)} = \sigma_{\min}$.

Figures 3.3, 3.4 show the efficiency of the Algorithm 3.28. In about 400 iterations the system seems approaching to a minimum of the cost functional. Moreover, $\|\mathcal{G}_{h,k,(i)}\|_{L^2(\mathbf{L}^2)}$ decreases apparently with a logarithmic rate, with respect to the number of iterations.

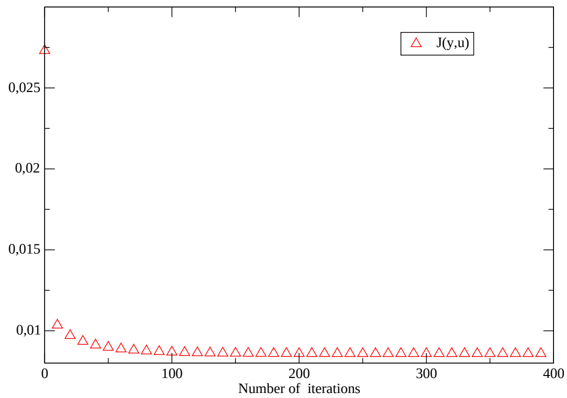


Figure 3.3.: behaviour of $J_{h,k}\left(\mathcal{Y}_{h,k,(i)}, \mathcal{U}_{h,k,(i)}\right)$, with i index of iterations

In figures 3.5, it is depicted the evolution in time of the optimal phase-field $\mathcal{Y}_{h,k}\left(x,t\right)$ and velocity $\boldsymbol{\mathcal{V}}_{h,k}\left(x,t\right)$, derived by the application of the Algorithm 3.28. The shape of the state changes in the first few time steps. Then, the velocity field keeps the

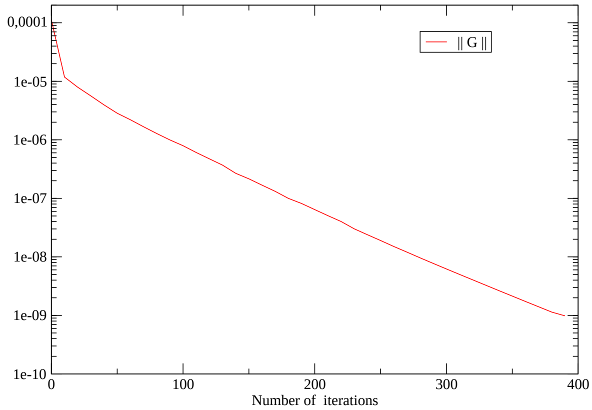
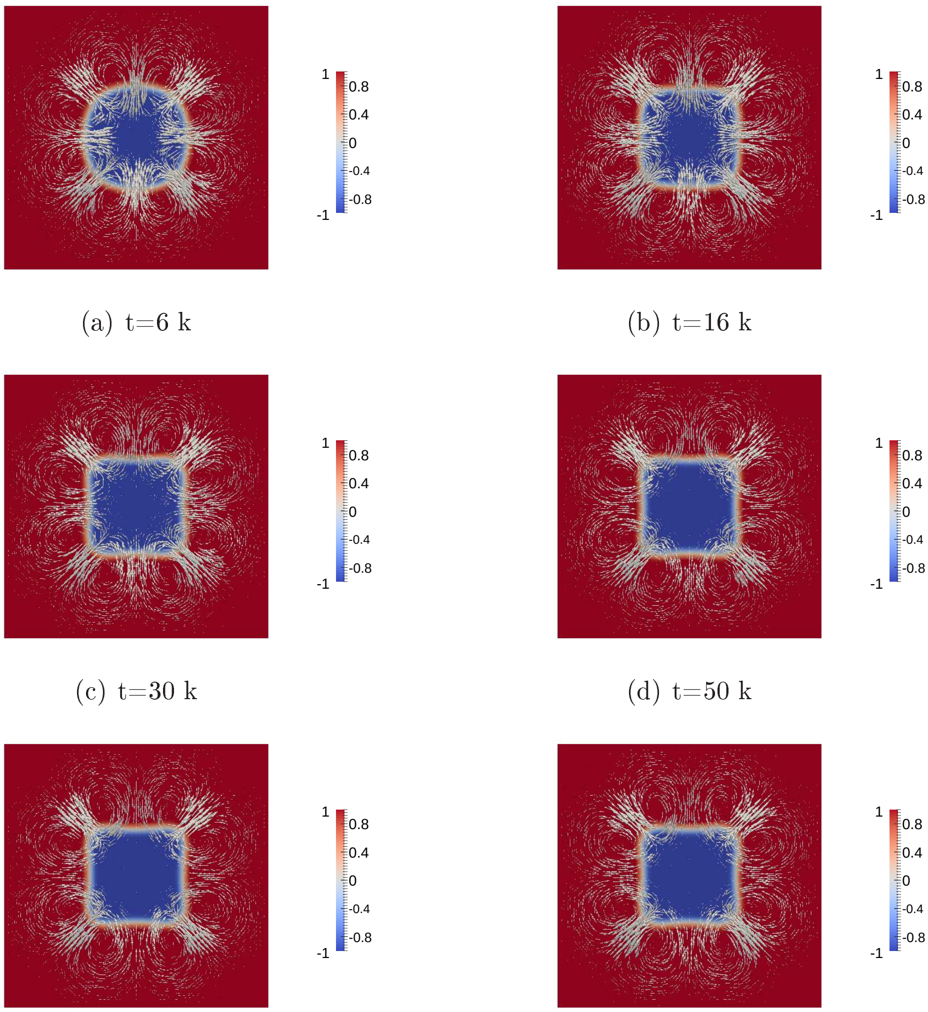


Figure 3.4.: behaviour of $\|\boldsymbol{\mathcal{G}}_{h,k,(i)}\|_{L^2(\mathbf{L}^2)}$, with i index of iterations



(a) t=6 k

(b) t=16 k

(c) t=30 k

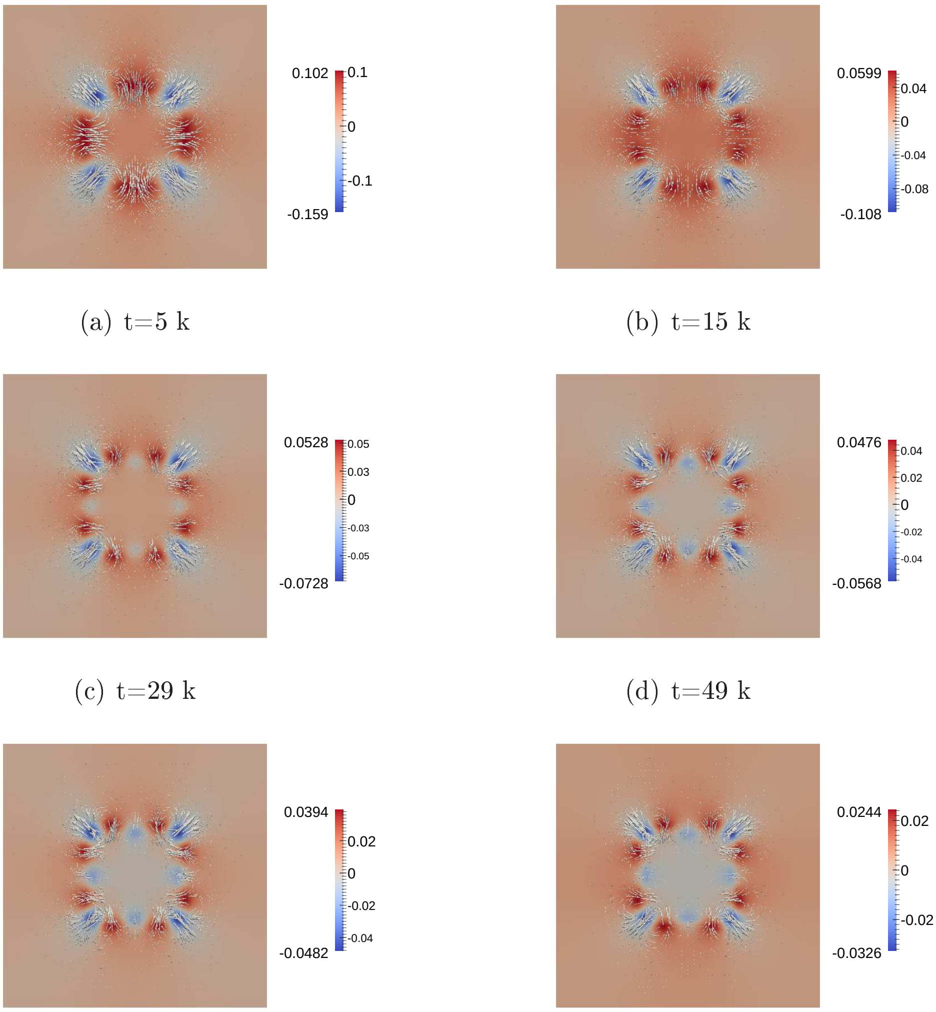
(d) t=50 k

(e) t=74 k

(f) t=90 k

Figure 3.5.: Time evolution of state $\mathcal{Y}_{h,k}(x,t)$ and velocity $\boldsymbol{\mathcal{V}}_{h,k}(x,t)$

distribution of the phase-field *close* as much as possible to the desired state.
Finally in figures 3.6, it is possible to see the evolution in time of the optimal function $\mathcal{Q}_{\mathcal{Y},h,k}\,(x,t)$ and the control $\boldsymbol{\mathcal{U}}_{h,k}\,(x,t)$: in the last time steps, the control acts on the velocity field in a such a way that the phase-field keeps the desired shape.



(a) t=5 k (b) t=15 k

(c) t=29 k (d) t=49 k

(e) t=73 k (f) t=89 k

Figure 3.6.: Time evolution of the optimal $\mathcal{Q}_{\mathcal{Y},h,k}\,(x,t)$ and the control $\boldsymbol{\mathcal{U}}_{h,k}\,(x,t)$

3.6.2. Circle to Square 2

As in the previous case, the domain is the unit square $\Omega = (0,1)^2$ in the two dimensional plane $(x_1, x_2) = x$. The initial condition has the form depicted in (3.269), but it is "shifted" toward the right side of the domain and centred around the point $(x_{c1}, x_{c2}) = (0.7, 0.5)$, as shown in figure 3.7. Even in this case the desired state is time-independent and it has a shape analogous to the previous case, but it is centred on the left of the domain, around the point $(\tilde{x}_{c1}, \tilde{x}_{c2}) = (0.3, 0.5)$, as shown

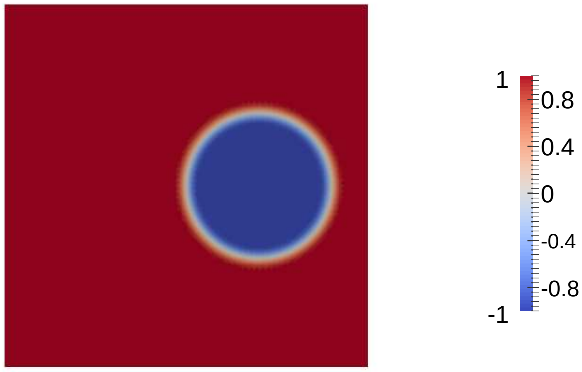


Figure 3.7.: Initial phase-field $y_0(x)$

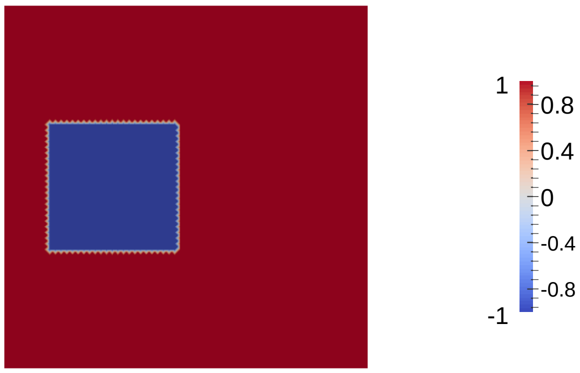


Figure 3.8.: Desired state distribution $y_d(x)$

in figure 3.8.
The values of the constant parameters in the model are $\alpha = 10^{-5}, \nu = 0.1, \gamma = 0.005, \varepsilon = 0.02$. The timestep $k = 0.005$ and the time horizon is $T = 400k$. Also in this case, condition (3.270) is fulfilled and then the desired state is reachable. In Algorithm 3.28, we assume $TOL = 10^{-9}$, $N_{\max} = 1000$ and the initial guess for the control $\boldsymbol{\mathcal{U}}_{h,k,(0)} \equiv 0$. The step size is chosen, as well as the previous experiment, using the Barzilai-Borwein method [12], with the following settings: $\sigma_{\text{init}} = 10^5$, $\sigma_{\min} = 10^3$, $\sigma_{\max} = 10^5$, see (3.271). In figures 3.9 and 3.10 are depicted the values of the cost functional with respect to the number of iterations: apart the first iterations, the decreasing is slower than the previous numerical experiment.

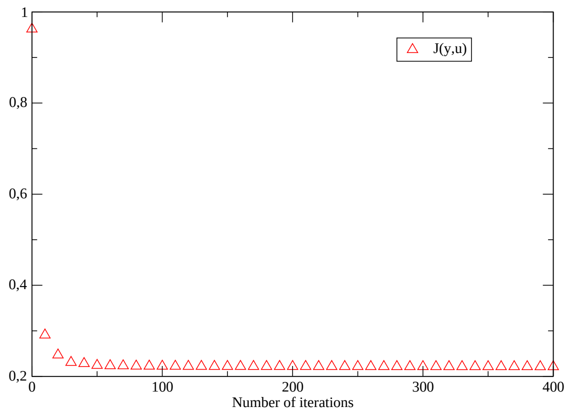


Figure 3.9.: behaviour of $J_{h,k}\left(\mathcal{Y}_{h,k,(i)}, \boldsymbol{\mathcal{U}}_{h,k,(i)}\right)$, with $0 < i \leq 400$ index of iterations

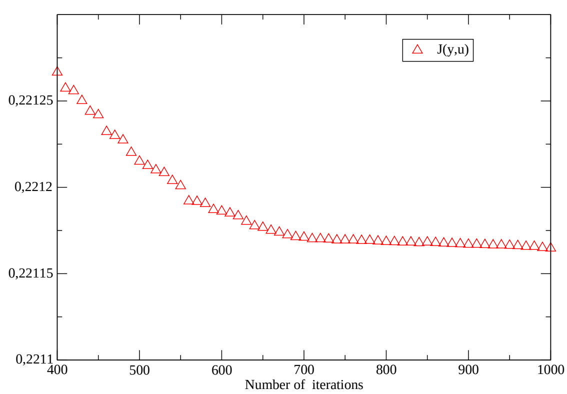


Figure 3.10.: behaviour of $J_{h,k}\left(\mathcal{Y}_{h,k,(i)}, \boldsymbol{\mathcal{U}}_{h,k,(i)}\right)$, with $400 \leq i \leq 1000$ index of iterations

The behaviour of the system is also displayed in figure 3.11: $\|\boldsymbol{\mathcal{G}}_{h,k,(i)}\|_{L^2(\mathbf{L}^2)}$ decreases with less regularity with respect to the previous case and in 1000 steepest descent iterations it does not reaches the proposed tolerance $TOL = 10^{-9}$.
In figures 3.12, it is shown the evolution in time of the optimal phase-field $\mathcal{Y}_{h,k}(x,t)$ and velocity $\boldsymbol{\mathcal{V}}_{h,k}(x,t)$ derived by the application of the Algorithm 3.28. The behaviour of the system is the one expected: starting from the initial distribution, the fluid is driven toward a final state which is close to the desired state.

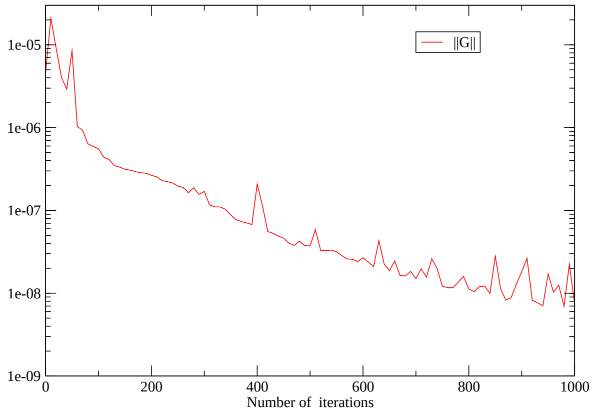
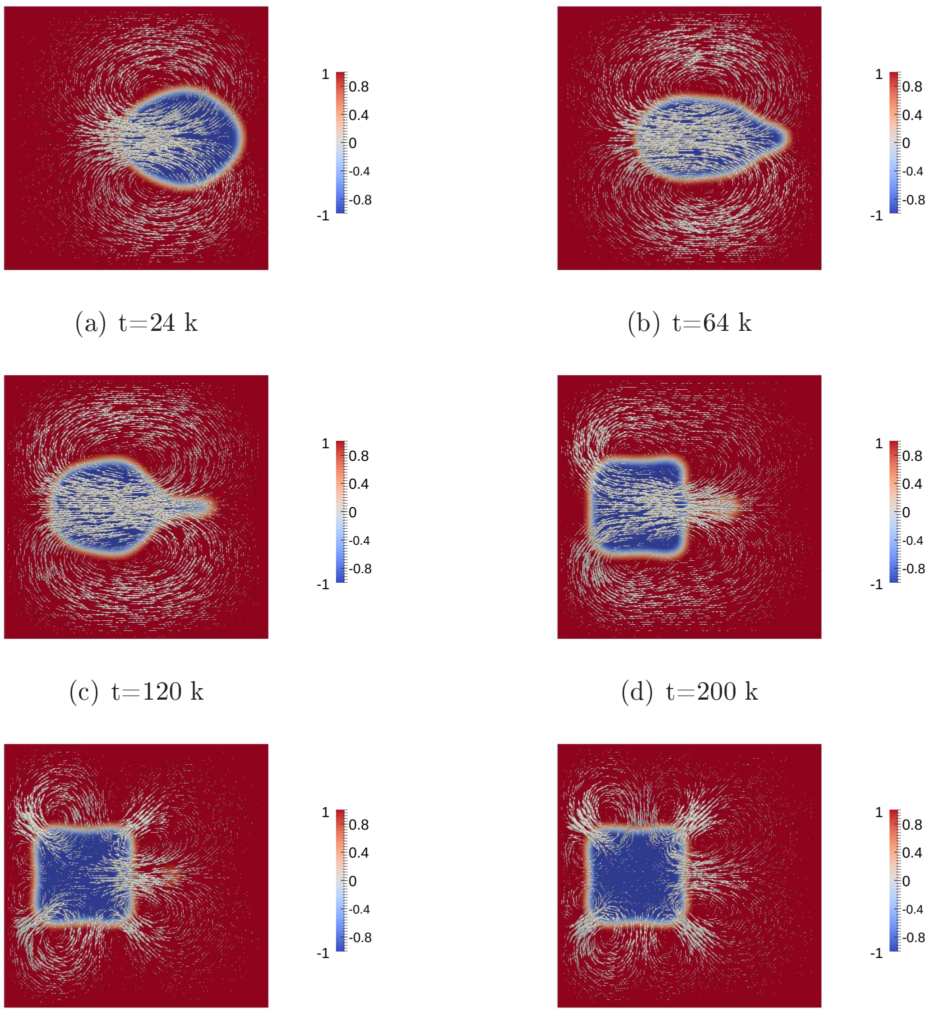


Figure 3.11.: behaviour of $\|\boldsymbol{\mathcal{G}}_{h,k,(i)}\|_{L^2(\mathbf{L}^2)}$, with $400 \leq i \leq 1000$ index of iterations



(a) t=24 k

(b) t=64 k

(c) t=120 k

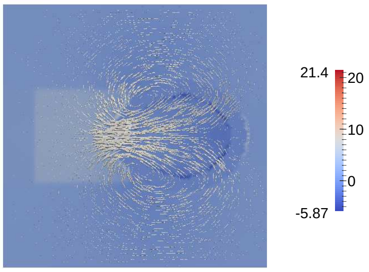
(d) t=200 k

(e) t=296 k

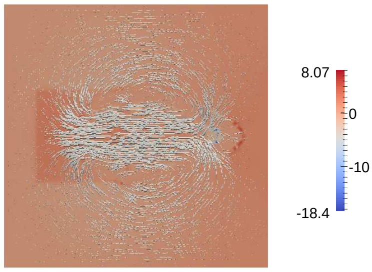
(f) t=360 k

Figure 3.12.: Time evolution of the optimal state $\mathcal{Y}_{h,k}\,(x,t)$ and velocity $\boldsymbol{\mathcal{V}}_{h,k}\,(x,t)$

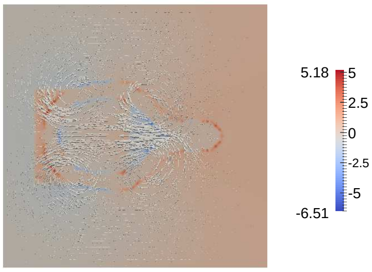
In figures 3.13, it is displayed the evolution in time of the optimal Lagrange multiplier $\Lambda_{h,k}(x,t)$ and control $\boldsymbol{\mathcal{U}}_{h,k}(x,t)$: it is possible to see the lack of regularity of $\Lambda_{h,k}(x,t)$ which is, in our opinion, the reason of the non optimal behaviour of the steepest descent approach.



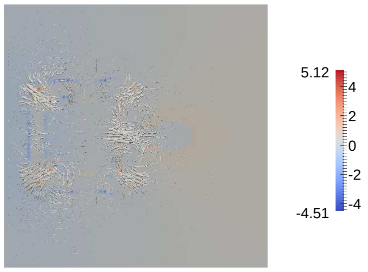
(a) t=23 k



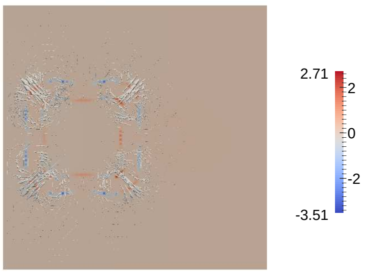
(b) t=63 k



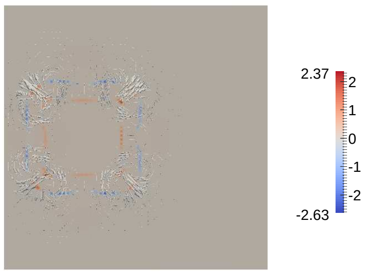
(c) t=119 k



(d) t=199 k



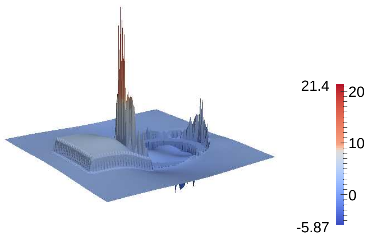
(e) t=295 k



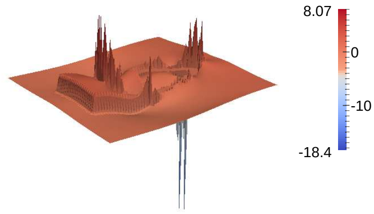
(f) t=359 k

Figure 3.13.: Time evolution of the optimal lagrange multiplier $\Lambda_{h,k}(x,t)$ and control $\boldsymbol{\mathcal{U}}_{h,k}(x,t)$

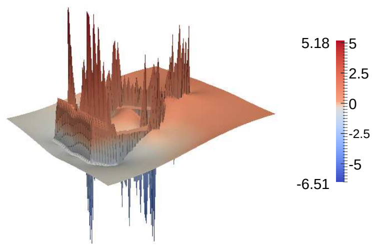
The non regularity of the lagrange multiplier $\Lambda_{h,k}$ is also displayed in figures 3.14



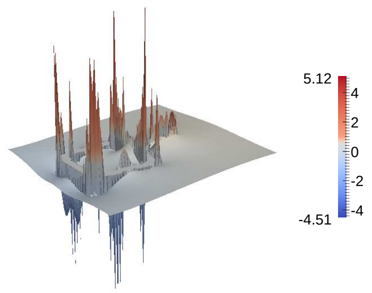
(a) t=23 k



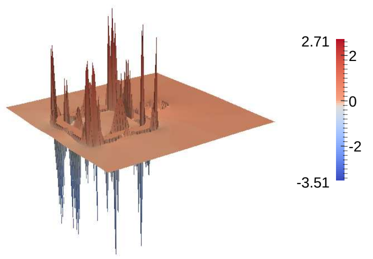
(b) t=63 k



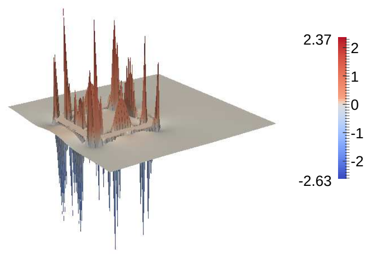
(c) t=119 k



(d) t=199 k



(e) t=295 k



(f) t=359 k

Figure 3.14.: Time evolution in 3d of the optimal lagrange multiplier $\Lambda_{h,k}(x,t)$

4. Optimal Control of the Cahn-Hilliard-Navier-Stokes System

4.1. Introduction

In this Chapter, we analyse the optimal control problem of the flow of two incompressible, immiscible fluids with surface tension effects. In contrast to the previous two Chapters we consider the full Cahn-Hilliard-Navier-Stokes system, i.e., we include the nonlinarity (inertia effects) in the Navier-Stokes equations and take the surface tension coefficient $\rho \neq 0$. More precisely, the considered Cahn-Hilliard-Navier-Stokes system consists of the system (1.13), (1.14), where the potential in the free energy density associated with the Cahn-Hilliard equation (1.14) is given by the double-well potential (1.11).

Below we introduce the mathematical setting for the considered problem. We denote by: $\Omega \in \mathbb{R}^2$ an open, bounded, convex polygonal domain; $T > 0$ a fixed time horizon; $\Omega_T = \Omega \times (0,T)$; $\alpha > 0$ a positive small constant. We assume all the settings and the notation stated in Appendix A.2.1, A.2.2. In particular, we consider L^2_0, the space of the L^2-functions with zero mean, $H_0 = L^2_0 \cap H^1$ and the associated Bochner's space

$$W_0 = \left\{ y \in L^2\left(H_0\right) : y_t \in L^2\left(H_0^*\right) \right\}.$$

In addition, we assume that $\boldsymbol{\mathcal{D}}$ is the space of the vector-valued, divergence-free, $\mathbf{H}^1_0$-functions and we consider the associated Bochner's space

$$\mathbf{W}_0 = \left\{ \mathbf{v} \in L^2(\boldsymbol{\mathcal{D}}) : \mathbf{v}_t \in L^2(\boldsymbol{\mathcal{D}}^*) \right\}.$$

We define

$$H_\Delta = \left\{ z \in H^2 : \Delta z \in H^1 \right\}, \tag{4.1}$$

and the associated Bochner's space $L^2\left(H_\Delta\right)$. The spaces H_Δ and $L^2\left(H_\Delta\right)$ are endowed with the following norms

$$\|z\|_{H_\Delta} = \left[\|z\|^2_{H^2} + \|\Delta z\|^2_{H^1} \right]^{\frac{1}{2}},$$
$$\|y\|_{L^2(H_\Delta)} = \left[\|y\|^2_{L^2(H^2)} + \|\Delta y\|^2_{L^2(H^1)} \right]^{\frac{1}{2}}.$$

It easy to realize that H_Δ and $L^2\left(H_\Delta\right)$ are Banach spaces. Furthermore, we consider the space

$$\mathbf{X} = \mathbf{W}_0 \times W_{0,\Delta}, \quad \text{where} \quad W_{0,\Delta} = W_0 \cap L^\infty\left(H_0\right) \cap L^2\left(H_\Delta\right), \tag{4.2}$$

with elements $\mathbf{x} = (\mathbf{v}, y)$. The spaces $\mathbf{X}$ and $\mathbf{X} \times L^2(\mathbf{L}^2)$ are endowed with the following norms

$$\|\mathbf{x}\|_{\mathbf{X}} = \left[\|\mathbf{v}\|^2_{\mathbf{W}_0} + \|y\|^2_{W_0} + \|y\|^2_{L^\infty(H_0)} + \|y\|^2_{L^2(H_\Delta)}\right]^{\frac{1}{2}},$$
$$\|(\mathbf{x}, \mathbf{u})\|_{\mathbf{X}\times L^2(\mathbf{L}^2)} = \left[\|\mathbf{x}\|^2_{\mathbf{X}} + \|\mathbf{u}\|^2_{L^2(\mathbf{L}^2)}\right]^{\frac{1}{2}}.$$

Moreover, we define the following set

(4.3) $$\mathcal{K} = \left\{\theta \in L^2\left(H^1\right) : -1 \le \theta \le 1, \text{ a.e. on } \Omega_T\right\}.$$

We consider the following objective function

(4.4) $$J : \mathbf{X} \times L^2\left(\mathbf{L}^2\right) \to \mathbb{R},$$

such that

(4.5) $$J(\mathbf{x}, \mathbf{u}) := \int_0^T \left[\frac{1}{2}\int_\Omega (y - y_d)^2\, dx + \frac{\alpha}{2}\int_\Omega \mathbf{u}^2\, dx\right] dt,$$

where we assume $y_d \in \mathcal{C}([0, T]; L^2_0)$. In order to represent the optimal control problem under investigation in a more compact, general form, we define the following map

(4.6) $$e : \mathbf{X} \times L^2\left(\mathbf{L}^2\right) \to \mathbf{Z} = \left[\, L^2(\boldsymbol{\mathcal{D}}) \times L^2(H_0) \times \boldsymbol{\mathcal{S}}\right]^* \times H_0.$$

The map e in (4.6) is such that, for all $\mathbf{p} = (\boldsymbol{\psi}, \eta, \boldsymbol{\xi}, \varphi) \in \mathbf{Z}^*$,

(4.7) $$\begin{aligned}\langle \mathbf{p}, e(\mathbf{v}, y, \mathbf{u})\rangle_{\mathbf{Z}^*,\mathbf{Z}} &= \langle a(\mathbf{v}, y, \mathbf{u}), \boldsymbol{\psi}\rangle_{L^2(\boldsymbol{\mathcal{D}}^*),L^2(\boldsymbol{\mathcal{D}})} + \langle c(\mathbf{v}, y), \eta\rangle_{L^2\left(H_0^*\right),L^2(H_0)}\\ &\quad + (\boldsymbol{\xi}, \mathbf{v}(0) - \mathbf{v}_0) + \langle \varphi, y(0) - y_0\rangle_{H_0^*,H_0},\end{aligned}$$

where

$$\begin{aligned}&\langle a(\mathbf{v}, y, \mathbf{u}), \boldsymbol{\psi}\rangle_{L^2(\boldsymbol{\mathcal{D}}^*),L^2(\boldsymbol{\mathcal{D}})}\\ &= \int_0^T \left[\langle \mathbf{v}_t, \boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}} + \nu(\nabla\mathbf{v}, \nabla\boldsymbol{\psi}) + b(\mathbf{v}, \mathbf{v}, \boldsymbol{\psi}) + \rho(y, \nabla w\cdot\boldsymbol{\psi}) - (\mathbf{u}, \boldsymbol{\psi})\right] dt,\end{aligned}$$

and

$$\langle c(\mathbf{v}, y), \eta\rangle_{L^2\left(H_0^*\right),L^2(H_0)} = \int_0^T \left[\langle y_t, \eta\rangle_{H_0^*,H_0} + \gamma(\nabla w, \nabla\eta) - (y, \mathbf{v}\cdot\nabla\eta)\right] dt,$$

with

(4.8) $$w := -\varepsilon^2\Delta y - y + y^3.$$

Furthermore, given $\mathbf{z} = (\mathbf{z}_1, z_2, \mathbf{z}_3, z_4) \in \mathbf{Z}$, we assume

$$\|z\|_Z = \left[\|\mathbf{z}_1\|^2_{\mathbf{L}^2(\boldsymbol{\mathcal{D}}^*)} + \|z_2\|^2_{L^2\left(H_0^*\right)} + \|\mathbf{z}_3\|^2_{\boldsymbol{\mathcal{S}}} + \|z_4\|^2_{H_0}\right]^{\frac{1}{2}}.$$

Then, we study the following *smooth* optimal control problem:

Problem 4.1. *Given* $\mathbf{v}_0 \in \boldsymbol{\mathcal{D}} \cap \mathbf{H}^2$, $y_0 \in L_0^2 \cap H^2 \cap \mathcal{K}$, *find* $(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathbf{X} \times L^2(\mathbf{L}^2)$, *such that*

$$\min_{(\mathbf{x},\mathbf{u}) \in \mathbf{X} \times L^2(\mathbf{L}^2)} J(\mathbf{x}, \mathbf{u}) = J(\bar{\mathbf{x}}, \bar{\mathbf{u}}),$$

subject to

$$e(\mathbf{x}, \mathbf{u}) = 0. \tag{4.9}$$

From the definition (4.6), (4.7) of the map e and by the definition (4.8) of the chemical potential w, we can write the state equations (4.9) in the following way

(4.10a)
$$\int_0^T \left[(\mathbf{v}_t, \boldsymbol{\psi}) + \nu (\nabla \mathbf{v}, \nabla \boldsymbol{\psi}) + b(\mathbf{v}, \mathbf{v}, \boldsymbol{\psi}) + \rho (y, \nabla w \cdot \boldsymbol{\psi}) - (\mathbf{u}, \boldsymbol{\psi}) \right] \, dt = 0,$$

$$\mathbf{v}(0) = \mathbf{v}_0, \quad \text{in } \Omega, \tag{4.10b}$$

$$\int_0^T \left[(y_t, \eta) + \gamma (\nabla w, \nabla \eta) - (y, \mathbf{v} \cdot \nabla \eta) \right] \, dt = 0, \tag{4.11a}$$

$$y(0) = y_0, \quad \text{in } \Omega \tag{4.11b}$$

$$\int_0^T \left[(w, \theta) - \varepsilon^2 (\nabla y, \nabla \theta) + (y, \theta) - \left(y^3, \theta\right) \right] \, dt = 0, \tag{4.11c}$$

for all $\boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}}), \eta, \theta \in L^2(H^1)$.
In (4.10a) above, $b(\cdot,\cdot,\cdot)$ is the canonical trilinear form associated to the nonlinearity in the Navier-Stokes equations

$$b : \mathbf{H}_0^1 \times \mathbf{H}_0^1 \times \mathbf{H}_0^1 \to \mathbb{R}, \tag{4.12}$$
$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} \, dx,$$

which is such that

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) + b(\mathbf{u}, \mathbf{w}, \mathbf{v}) = 0, \tag{4.13}$$

for all $\mathbf{u} \in \boldsymbol{\mathcal{D}},\ \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1$.
Optimal control Problem 4.1 concerns the flow of a mixture of two immiscible, incompressible fluids. Compared to Problem 2.1, the phase dynamics in the Cahn-Hilliard equations (4.11) is determined by the double-well potential $\Phi(y)$ (1.11), which is such that

$$\Phi'(y) = -y + y^3,$$

see last two terms in (4.11c). This assumption makes Problem 4.1 smooth and allows a direct application of the tools of mathematical programming in Banach spaces. Conversely, two issues make make the mathematical analysis of Problem 4.1 more challenging than Problem 2.1: the equations in (4.10),(4.11) are coupled by the last term in (4.10a), where the capillarity number $\rho > 0$; the fluids hydrodynamics is governed by the Navier-Stokes equation (4.10a), without neglecting the advection

effects described by the trilinear form b (4.12).
In the next sections, we study the properties of the state equations (4.10), (4.11), then we show that Problem 4.1 has solutions, that it satisfies the conditions needed to apply the standard theory of mathematical programming in Banach spaces (see Assumptions 1.47 in [58]) and we get the first order optimality conditions (see Theorem 1.48 and Corollary 1.3 in [58]).

4.2. Properties of the State Equations

In the following theorem, we derive existence, uniqueness and regularity properties of the solution $(\mathbf{v}, y, w)$ of the state equations (4.10), (4.11).

Theorem 4.2 (**existence, uniqueness, regularity**). *For any fixed* $\mathbf{v}_0 \in \boldsymbol{\mathcal{D}} \cap \mathbf{H}^2$, $y_0 \in L_0^2 \cap H^2 \cap \mathcal{K}$ *and* $\mathbf{u} \in L^2(\mathbf{L}^2)$, *the system of the state equations* (4.10), (4.11) *has a unique solution*

$$(\mathbf{v}, y, w) \in \left(H^1\left(\mathbf{L}^2\right) \cap L^\infty\left(\boldsymbol{\mathcal{D}} \cap \mathbf{H}^2\right)\right) \times \left(H^1\left(L_0^2\right) \cap L^\infty\left(H^2\right)\right) \times \left(L^\infty\left(L^2\right) \cap L^2\left(H^2\right)\right),$$

which is such that

$$\|\mathbf{v}\|_{H^1(\mathbf{L}^2)} + \|\mathbf{v}\|_{L^\infty(\boldsymbol{\mathcal{D}})} + \|\mathbf{v}\|_{L^\infty(\mathbf{H}^2)} + \|y\|_{H^1\left(L_0^2\right)} + \|y\|_{L^\infty(H^2)} + \|w\|_{L^\infty(L^2)} + \|w\|_{L^2(H^2)} \le C(\mathbf{u}), \tag{4.14}$$

where the constant $C(\mathbf{u})$ *depends continuously on* $\|\mathbf{u}\|_{L^2(\mathbf{L}^2)}$ *and data problem (initial conditions and constant parameters).*

Proof. Concerning the existence and uniqueness of the solution $\mathbf{v} \in H^1(\mathbf{L}^2) \cap L^\infty(\boldsymbol{\mathcal{D}} \cap \mathbf{H}^2)$, $y \in H^1(L^2) \cap L^\infty(L_0^2 \cap H^2)$, $w \in L^\infty(L^2) \cap L^2(H^2)$, see Remark 2.2 in [62] and also [27], [74]. Then, the estimate (4.14) can be obtained by standard procedures. □

Remark 4.3. Obviously, the solution $y(t) \in L_0^2$, for all $t \in (0, T]$. In fact, with $\eta = \chi_{[0,t]}$ in (4.11a), where

$$\chi_{[0,t]}(s) := \begin{cases} 1 & \text{if} \quad s \in [0,t], \\ 0 & \text{otherwise,} \end{cases}$$

using integration by parts in time, we have

$$(y(t), 1) = (y(0), 1) = 0, \quad \forall\, t \in (0, T].$$

From Theorem 4.2, we derive that associated to the state equations of the optimal control Problem 4.1

$$e(\mathbf{x}, \mathbf{u}) = 0,$$

there exists a bounded solution operator $s : L^2(\mathbf{L}^2) \to \mathbf{X}$, which such that

$$e(s(\mathbf{u}), \mathbf{u}) = 0, \quad \forall\, \mathbf{u} \in L^2\left(\mathbf{L}^2\right). \tag{4.15}$$

4.3. Well-Posedness of the Optimal Control Problem

The map $J : \mathbf{X} \times L^2(\mathbf{L}^2) \to \mathbb{R}$ defined in (4.5), is continuous, convex and bounded from below. Hence, it is *weakly lower semicontinuous*. Hence, we can prove the following results.

Theorem 4.4 (existence of minimizers). *The regularized optimal control problem (4.1) admits solutions.*

Proof. The proof is analogous to the one of Theorem 2.5. □

4.4. Optimality Conditions of the Optimal Control Problem

In this section, we show that the cost functional J and the map e defined, respectively, in (4.5) and (4.6), (4.7), satisfy the conditions needed to apply the standard theory of mathematical programming in Banach spaces (see Assumptions 1.47 in [58]). Next, we derive the first order optimality conditions of the optimal control Problem 4.1 (see Theorem 1.48 and Corollary 1.3 in [58]).
We need to check that the following conditions hold:

- the cost functional $J : \mathbf{X} \times L^2(\mathbf{L}^2) \to \mathbb{R}$ is continuously Fréchet differentiable;
- the map $e : \mathbf{X} \times L^2(\mathbf{L}^2) \to Z$ is continuously Fréchet differentiable;
- there exists the inverse of the map $e_{\mathbf{x}}(s(\mathbf{u}), \mathbf{u})$, where s is the bounded solution operator defined in (4.15).

The Fréchet derivative of the mapping J is such that

$$J' : \mathbf{X} \times L^2\left(\mathbf{L}^2\right) \to \mathcal{L}\left(\mathbf{X} \times L^2\left(\mathbf{L}^2\right), \mathbb{R}\right),$$

with partial derivatives

$$\begin{aligned}
\langle\ J_{\mathbf{v}}(\mathbf{x}, \mathbf{u}), \mathbf{d}_{\mathbf{v}}\ \rangle_{\mathbf{W}_0^*, \mathbf{W}_0} &= 0, \\
\langle\ J_y(\mathbf{x}, \mathbf{u}), d_y\ \rangle_{W_{0,\Delta}^*, W_{0,\Delta}} &= \int_0^T (y - y_d, d_y)\ dt, \\
(J_{\mathbf{u}}(\mathbf{x}, \mathbf{u}), \mathbf{d}_{\mathbf{u}})_{L^2(\mathbf{L}^2)} &= \alpha \int_0^T (\mathbf{u}, \mathbf{d}_{\mathbf{u}})\ dt,
\end{aligned}$$

Therefore,

$$\langle\ J'(\mathbf{x}, \mathbf{u}), (\mathbf{d}_{\mathbf{x}}, \mathbf{d}_{\mathbf{u}})\ \rangle_{(\mathbf{X} \times L^2(\mathbf{L}^2))^*, \mathbf{X} \times L^2(\mathbf{L}^2)} = \int_0^T \left[(y - y_d, d_y) + \alpha(\mathbf{u}, \mathbf{d}_{\mathbf{u}})\right]\ dt,$$

for all $(\mathbf{d}_{\mathbf{x}}.\mathbf{d}_{\mathbf{u}}) \in \mathbf{X} \times L^2(\mathbf{L}^2)$. Hence J is Fréchet differentiable. Moreover, we have

$$\left|\left\langle J'(\mathbf{x} + \mathbf{d}_{\mathbf{x}}, \mathbf{u} + \mathbf{d}_{\mathbf{u}}) - J'(\mathbf{x}, \mathbf{u}), (\mathbf{h}_{\mathbf{x}}, \mathbf{h}_{\mathbf{u}})\right\rangle_{(\mathbf{X} \times L^2(\mathbf{L}^2))^*, \mathbf{X} \times L^2(\mathbf{L}^2)}\right|$$

$$
\begin{aligned}
&= \left| \int_0^T \left[(d_y, h_y) + \alpha (\mathbf{d_u}, \mathbf{h_u}) \right] \, dt \right| \\
&\leq \|d_y\|_{L^2(L^2)} \|h_y\|_{L^2(L^2)} + \alpha \|\mathbf{d_u}\|_{L^2(\mathbf{L}^2)} \|\mathbf{h_u}\|_{L^2(\mathbf{L}^2)} \\
&\leq \| (\mathbf{d_x}, \mathbf{d_u}) \|_{\mathbf{X} \times L^2(\mathbf{L}^2)} \left[\|h_y\|_{L^2(L^2)} + \alpha \|\mathbf{h_u}\|_{L^2(\mathbf{L}^2)} \right] \to 0,
\end{aligned}
$$

as $(\mathbf{d_x}, \mathbf{d_u}) \to 0$ in $\mathbf{X} \times L^2(\mathbf{L}^2)$, for all $(\mathbf{h_x}, \mathbf{h_u}) \in \mathbf{X} \times L^2(\mathbf{L}^2)$. Then, J is continuously Fréchet differentiable. Concerning the properties of the map $e : \mathbf{X} \times L^2(\mathbf{L}^2) \to \mathbf{Z}$, we have the following result.

Lemma 4.5. *The map $e : \mathbf{X} \times L^2(\mathbf{L}^2) \to \mathbf{Z}$ is continuously Fréchet differentiable.*

Proof. We have

$$
e' : \mathbf{X} \times L^2\left(\mathbf{L}^2\right) \to \mathcal{L}\left(\mathbf{X} \times L^2\left(\mathbf{L}^2\right), \mathbf{Z}\right),
$$

with partial Fréchet derivatives

$$
\begin{aligned}
&\left\langle \mathbf{p}, e_{\mathbf{v}}(\mathbf{x}, \mathbf{u}) \mathbf{d_v} \right\rangle_{\mathbf{Z}^*, \mathbf{Z}} = (\boldsymbol{\xi}, \mathbf{d_v}(0)) \\
&+ \int_0^T \left[\langle \mathbf{d}_{\mathbf{v}t}, \boldsymbol{\psi} \rangle_{\boldsymbol{\mathcal{D}}^*, \boldsymbol{\mathcal{D}}} + \nu (\nabla \mathbf{d_v}, \nabla \boldsymbol{\psi}) + b(\mathbf{d_v}, \mathbf{v}, \boldsymbol{\psi}) + b(\mathbf{v}, \mathbf{d_v}, \boldsymbol{\psi}) - (y, \mathbf{d_v} \cdot \nabla \eta) \right] dt,
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \mathbf{p}, e_y(\mathbf{x}, \mathbf{u}) d_y \right\rangle_{\mathbf{Z}^*, \mathbf{Z}} = \int_0^T & \left[\langle d_{yt}, \eta \rangle_{H_0^*, H_0} + \gamma \left(\nabla \left[-\varepsilon^2 \Delta d_y - d_y + 3y^2 d_y \right], \nabla \eta \right) \right. \\
& - (d_y, \mathbf{v} \cdot \nabla \eta) + \rho (d_y, \nabla w \cdot \boldsymbol{\psi}) \\
& \left. + \rho \left(y, \nabla \left[-\varepsilon^2 \Delta d_y - d_y + 3y^2 d_y \right] \cdot \boldsymbol{\psi} \right) \right] dt + \langle \varphi, d_y(0) \rangle_{H_0^*, H_0}
\end{aligned}
$$

and

$$
\left\langle \mathbf{p}, e_{\mathbf{u}}(\mathbf{x}, \mathbf{u}) \mathbf{d_u} \right\rangle_{\mathbf{Z}^*, \mathbf{Z}} = - \int_0^T (\mathbf{d_u}, \boldsymbol{\psi}) \, dt,
$$

for all $\mathbf{p} = (\boldsymbol{\psi}, \eta, \boldsymbol{\xi}, \varphi) \in \mathbf{Z}^*$, $(\mathbf{x}, \mathbf{u}) = (\mathbf{v}, y, \mathbf{u}), (\mathbf{d_x}, \mathbf{d_u}) = (\mathbf{d_v}, d_y, \mathbf{d_u}) \in \mathbf{X} \times L^2(\mathbf{L}^2)$. The map e is Fréchet differentiable if

$$
\begin{aligned}
\left\| e(\mathbf{x} + \mathbf{d_x}, \mathbf{u} + \mathbf{d_u}) - e(\mathbf{x}, \mathbf{u}) - e'(\mathbf{x}, \mathbf{u})(\mathbf{d_x}, \mathbf{d_u}) \right\|_{\mathbf{Z}} \\
= o\left(\| (\mathbf{d_x}, \mathbf{d_u}) \|_{\mathbf{X} \times L^2(\mathbf{L}^2)} \right),
\end{aligned} \tag{4.16}
$$

as $(\mathbf{d_x}, \mathbf{d_u}) \to 0$ in $\mathbf{X} \times L^2(\mathbf{L}^2)$. For all $\mathbf{p} = (\boldsymbol{\psi}, \eta, \boldsymbol{\xi}, \varphi) \in \mathbf{Z}^*$, $(\mathbf{x}, \mathbf{u}), (\mathbf{d_x}, \mathbf{d_u}) \in \mathbf{X} \times L^2(\mathbf{L}^2)$, we realize that

$$
\begin{aligned}
&\left| \left\langle \mathbf{p}, e(\mathbf{x} + \mathbf{d_x}, \mathbf{u} + \mathbf{d_u}) - e(\mathbf{x}, \mathbf{u}) - e'(\mathbf{x}, \mathbf{u})(\mathbf{d_x}, \mathbf{d_u}) \right\rangle_{\mathbf{Z}^*, \mathbf{Z}} \right| \\
&\leq \left| \int_0^T b(\mathbf{d_v}, \mathbf{d_v}, \boldsymbol{\psi}) \, dt \right| + \left| \int_0^T (d_y, \mathbf{d_v} \cdot \nabla \eta) \, dt \right| \\
&+ \left| \int_0^T \rho \left(y, \nabla \left[d_y^3 + 3y d_y^2 \right] \cdot \boldsymbol{\psi} \right) dt \right| + \left| \int_0^T \rho \left(d_y, \nabla \left[d_y^3 + 3y d_y^2 \right] \cdot \boldsymbol{\psi} \right) dt \right| \\
&+ \left| \int_0^T \rho \left(d_y, \nabla \left[-\varepsilon^2 \Delta d_y - d_y + 3y^2 d_y \right] \cdot \boldsymbol{\psi} \right) dt \right| + \left| \int_0^T \gamma \left(\nabla \left[d_y^3 + 3y d_y^2 \right], \nabla \eta \right) dt \right| \\
&= S_1 + S_2 + S_3 + S_4 + S_5 + S_6.
\end{aligned}
$$

Using the property (4.13) of the trilinear form $b(\cdot,\cdot,\cdot)$, the interpolation inequality (A.18) and the embeddings (A.5), (A.6), we derive

$$\begin{aligned}
S_1 &\le \int_0^T \|\mathbf{d_v}\|_{\mathbf{L}^4}\ \|\nabla \boldsymbol{\psi}\|\ \|\mathbf{d_v}\|_{\mathbf{L}^4}\ dt \le C \int_0^T \|\mathbf{d_v}\|\ \|\mathbf{d_v}\|_{\boldsymbol{\mathcal{D}}}\ \|\boldsymbol{\psi}\|_{\boldsymbol{\mathcal{D}}}\ dt \\
&\le C\ \|\mathbf{d_v}\|_{\mathcal{C}([0,T];\boldsymbol{\mathcal{S}})}\ \|\mathbf{d_v}\|_{L^2(\boldsymbol{\mathcal{D}})}\ \|\boldsymbol{\psi}\|_{L^2(\boldsymbol{\mathcal{D}})} \le C\ \|\mathbf{d_v}\|_{\mathbf{W}_0}\ \|\mathbf{d_v}\|_{L^2(\boldsymbol{\mathcal{D}})}\ \|\boldsymbol{\psi}\|_{L^2(\boldsymbol{\mathcal{D}})} \\
&\le C\ \|\left(\mathbf{d_x},\mathbf{d_u}\right)\|^2_{\mathbf{X}\times L^2(\mathbf{L}^2)}\ \|\boldsymbol{\psi}\|_{L^2(\boldsymbol{\mathcal{D}})},
\end{aligned}$$

$$\begin{aligned}
S_2 &\le \int_0^T \|d_y\|_{L^4}\ \|\mathbf{d_v}\|_{\mathbf{L}^4}\ \|\nabla \eta\|\ dt \le C\ \|d_y\|_{L^\infty(H_0)}\ \|\mathbf{d_v}\|_{L^2(\boldsymbol{\mathcal{D}})}\ \|\eta\|_{L^2(H_0)} \\
&\le C\ \|\left(\mathbf{d_x},\mathbf{d_u}\right)\|^2_{\mathbf{X}\times L^2(\mathbf{L}^2)}\ \|\eta\|_{L^2(H_0)},
\end{aligned}$$

$$\begin{aligned}
S_3 &\le \rho \int_0^T \|y\|_{L^4}\ \|\nabla\left[d_y^3 + 3\ y\ d_y^2\right]\|\ \|\boldsymbol{\psi}\|_{L^4}\ dt \le \\
&\le C\ \|y\|_{L^\infty(H_0)} \int_0^T \|3\ d_y^2\ \nabla d_y + 3\ d_y^2\ \nabla y + 6\ y\ d_y\ \nabla d_y\|\ \|\boldsymbol{\psi}\|_{\boldsymbol{\mathcal{D}}}\ dt \\
&\le C\ \|y\|_{L^\infty(H_0)} \\
&\times \int_0^T \left[\|d_y\|^2_{L^6}\ \|\nabla d_y\|_{L^6} + \|d_y\|^2_{L^6}\ \|\nabla y\|_{L^6} + \|y\|^2_{L^6}\ \|d_y\|_{L^6}\ \|\nabla d_y\|_{L^6}\right]\ \|\boldsymbol{\psi}\|_{\boldsymbol{\mathcal{D}}}\ dt \\
&\le C\ \|y\|_{L^\infty(H_0)} \\
&\times \int_0^T \left[\|d_y\|^2_{H_0}\ \|d_y\|_{H^2} + \|d_y\|^2_{H_0}\ \|y\|_{H^2} + \|y\|^2_{H_0}\ \|d_y\|_{H_0}\ \|d_y\|_{H^2}\right]\ \|\boldsymbol{\psi}\|_{\boldsymbol{\mathcal{D}}}\ dt \\
&\le C\ \|y\|_{L^\infty(H_0)}\ \|d_y\|_{L^\infty(H_0)}\ \|\boldsymbol{\psi}\|_{L^2(\boldsymbol{\mathcal{D}})} \\
&\times \left[\|d_y\|_{L^\infty(H_0)}\ \|d_y\|_{L^2(H^2)} + \|d_y\|_{L^\infty(H_0)}\ \|y\|_{L^2(H^2)} + \|y\|_{L^\infty(H_0)}\ \|d_y\|_{L^2(H^2)}\right] \\
&= o\left(\|\left(\mathbf{d_x},\mathbf{d_u}\right)\|_{\mathbf{X}\times L^2(\mathbf{L}^2)}\right)\ \|\boldsymbol{\psi}\|_{L^2(\boldsymbol{\mathcal{D}})},
\end{aligned}$$

$$\begin{aligned}
S_4 &\le \rho \int_0^T \|d_y\|_{L^4}\ \|\nabla\left[d_y^3 + 3\ y\ d_y^2\right]\|\ \|\boldsymbol{\psi}\|_{L^4}\ dt \\
&\le C\ \|d_y\|^2_{L^\infty(H_0)} \times \|\boldsymbol{\psi}\|_{L^2(\boldsymbol{\mathcal{D}})} \\
&\times \left[\|d_y\|_{L^\infty(H_0)}\ \|d_y\|_{L^2(H^2)} + \|d_y\|_{L^\infty(H_0)}\ \|y\|_{L^2(H^2)} + \|y\|_{L^\infty(H_0)}\ \|d_y\|_{L^2(H^2)}\right] \times \\
&= o\left(\|\left(\mathbf{d_x},\mathbf{d_u}\right)\|_{X\times L^2(\mathbf{L}^2)}\right)\ \|\boldsymbol{\psi}\|_{L^2(\boldsymbol{\mathcal{D}})},
\end{aligned}$$

$$\begin{aligned}
S_5 &\le \rho \int_0^T \|d_y\|_{L^4}\ \|\nabla\left[-\varepsilon^2 \Delta d_y - d_y + 3\ y^2\ d_y\right]\|\ \|\boldsymbol{\psi}\|_{L^4}\ dt \\
&\le C\ \|d_y\|_{L^\infty(H_0)} \int_0^T \|-\varepsilon^2 \nabla\Delta d_y - \nabla d_y + 6\ y\ d_y\ \nabla y + 3\ y^2\ \nabla d_y\|\ \|\boldsymbol{\psi}\|_{\boldsymbol{\mathcal{D}}}\ dt \\
&\le C\ \|d_y\|_{L^\infty(H_0)} \\
&\times \int_0^T \left[\|\nabla\Delta d_y\| + \|\nabla d_y\| + \|y\|_{L^6}\ \|d_y\|_{L^6}\ \|\nabla y\|_{L^6} + \|y\|^2_{L^6}\ \|\nabla d_y\|_{L^6}\right]\ \|\boldsymbol{\psi}\|_{\boldsymbol{\mathcal{D}}}\ dt \\
&\le C\ \|d_y\|_{L^\infty(H_0)}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^T \left[\|\Delta d_y\|_{H_0} + \|d_y\|_{H_0} + \|y\|_{H_0}\ \|d_y\|_{H_0}\ \|y\|_{H^2} + \|y\|_{H_0}^2\ \|d_y\|_{H^2}\right]\ \|\boldsymbol{\psi}\|_{\boldsymbol{\mathcal{D}}}\ dt \\
&\leq C\ \|d_y\|_{L^\infty(H_0)} \|\boldsymbol{\psi}\|_{L^2(\boldsymbol{\mathcal{D}})} \\
&\times \left[\|\Delta d_y\|_{L^2(H_0)} + \|d_y\|_{L^2(H_0)} + \|y\|_{L^\infty(H_0)}\ \|d_y\|_{L^\infty(H_0)}\ \|y\|_{L^2(H^2)} + \|y\|_{L^\infty(H_0)}^2\ \|d_y\|_{L^2(H^2)}\right] \\
&= o\left(\|\left(\mathbf{d_x}, \mathbf{d_u}\right)\|_{\mathbf{X}\times L^2(\mathbf{L}^2)}\right)\ \|\boldsymbol{\psi}\|_{L^2(\boldsymbol{\mathcal{D}})},
\end{aligned}
$$

$$
\begin{aligned}
S_6 &\leq \gamma \int_0^T \|\nabla\left[d_y^3 + 3\ y\ d_y^2\right]\|\ \|\nabla\eta\|\ dt \\
&\leq C \int_0^T \left[\|d_y\|_{L^6}^2\ \|\nabla d_y\|_{L^6} + \|d_y\|_{L^6}^2\ \|\nabla y\|_{L^6} + \|y\|_{L^6}\ \|d_y\|_{L^6}\ \|\nabla d_y\|_{L^6}\right]\ \|\eta\|_{H_0}\ dt \\
&\leq C \int_0^T \left[\|d_y\|_{H_0}^2\ \|d_y\|_{H^2} + \|d_y\|_{H_0}^2\ \|y\|_{H^2} + \|y\|_{H_0}\ \|d_y\|_{H_0}\ \|d_y\|_{H^2}\right]\ \|\eta\|_{H_0}\ dt \\
&\leq C\ \|d_y\|_{L^\infty(H_0)} \\
&\times \left[\|d_y\|_{L^\infty(H_0)}\ \|d_y\|_{H^2} + \|d_y\|_{L^\infty(H_0)}\ \|y\|_{H^2} + \|y\|_{L^\infty(H_0)}\ \|d_y\|_{H^2}\right]\ \|\eta\|_{L^2(H_0)} \leq \\
&= o\left(\|\left(\mathbf{d_x}, \mathbf{d_u}\right)\|_{\mathbf{X}\times L^2(\mathbf{L}^2)}\right)\ \|\eta\|_{L^2(H_0)}.
\end{aligned}
$$

So, using the above estimates of $S_1, \ldots, S_6$ in (4.16), we infer that the mapping $e : \mathbf{X} \times L^2(\mathbf{L}^2) \to \mathbf{Z}$ is Fréchet differentiable.

The map $e : \mathbf{X} \times L^2(\mathbf{L}^2) \to \mathbf{Z}$ is continuously Fréchet differentiable if, for all $(\mathbf{x}, \mathbf{u}), (\mathbf{d_x}, \mathbf{d_u}) \in \mathbf{X} \times L^2(\mathbf{L}^2)$,

$$
\|e'\left(\mathbf{x} + \mathbf{d_x}, \mathbf{u} + \mathbf{d_u}\right) - e'\left(\mathbf{x}, \mathbf{u}\right)\|_{\mathcal{L}(\mathbf{X}\times L^2(\mathbf{L}^2), \mathbf{Z})} \to 0,
$$

as $(\mathbf{d_x}, \mathbf{d_u}) \to 0$ in $\mathbf{X}\times L^2(\mathbf{L}^2)$. For all $(\mathbf{h_x}, \mathbf{h_u}) \in \mathbf{X}\times L^2(\mathbf{L}^2)$, $\mathbf{p} = (\boldsymbol{\psi}, \eta, \boldsymbol{\xi}, \varphi) \in \mathbf{Z}^*$, we get

$$
\begin{aligned}
&\left|\left\langle\ \mathbf{p}, \left[\ e'\left(\mathbf{x} + \mathbf{d_x}, \mathbf{u} + \mathbf{d_u}\right) - e'\left(\mathbf{x}, \mathbf{u}\right)\ \right]\left(\mathbf{h_x}, \mathbf{h_u}\right)\right\rangle_{\mathbf{Z}^*, \mathbf{Z}}\right| \qquad (4.17) \\
&= \left|\int_0^T \left[b\left(\mathbf{h_v}, \mathbf{d_v}, \boldsymbol{\psi}\right) + b\left(\mathbf{d_v}, \mathbf{h_v}, \boldsymbol{\psi}\right) - \left(d_y, \mathbf{h_v}\cdot\nabla\eta\right) - \left(h_y, \mathbf{d_v}\cdot\nabla\eta\right)\right.\right. \\
&\quad + 3\gamma\left(\nabla\left[d_y^2\ h_y + 2\ y\ d_y\ h_y\right], \nabla\eta\right) + 3\rho\left(y, \nabla\left[d_y^2\ h_y + 2\ y\ d_y\ h_y\right]\cdot\boldsymbol{\psi}\right) \\
&\quad + \rho\left(h_y, \nabla\left[-\varepsilon^2\Delta d_y - d_y + d_y^3 + 3y^2\ d_y + 3y\ d_y^2\right]\cdot\boldsymbol{\psi}\right) \\
&\quad \left.\left. + \rho\left(d_y, \nabla\left[-\varepsilon^2\Delta h_y - h_y + 3y^2\ h_y\right]\cdot\boldsymbol{\psi}\right) + 3\rho\left(d_y, \nabla\left[d_y^2\ h_y + 2\ y\ d_y\ h_y\right]\cdot\boldsymbol{\psi}\right)\right] dt\ \right|
\end{aligned}
$$

Working in (4.17) as well as in the derivation of the estimates of $S_1, \ldots, S_6$ above, we have

$$
\left|\left\langle\ \mathbf{p}, \left[\ e'\left(\mathbf{x} + \mathbf{d_x}, \mathbf{u} + \mathbf{d_u}\right) - e'\left(\mathbf{x}, \mathbf{u}\right)\ \right]\left(\mathbf{h_x}, \mathbf{h_u}\right)\right\rangle_{\mathbf{Z}^*, \mathbf{Z}}\right| \to 0,
$$

as $(\mathbf{d_x}, \mathbf{d_u}) \to 0$ in $\mathbf{X}\times L^2(\mathbf{L}^2)$, for all $\mathbf{p} \in Z^*$, $(\mathbf{x}, \mathbf{u})$, $(\mathbf{h_x}, \mathbf{h_u}) \in \mathbf{X}\times L^2(\mathbf{L}^2)$. Then $e : \mathbf{X}\times L^2(\mathbf{L}^2) \to \mathbf{Z}$ is continuously Fréchet differentiable. □

Theorem 4.6. *For any fixed* $\mathbf{u} \in L^2(\mathbf{L}^2)$,

$$
e_{\mathbf{x}}\left(s\left(\mathbf{u}\right), \mathbf{u}\right) \in \mathcal{L}\left(\mathbf{X}, \mathbf{Z}\right),
$$

has a bounded inverse.

The proof of the Theorem is given in Appendix B, Section B.3.

Remark 4.7. As a consequence of Theorem 4.6, we can say that

$$[e_{\mathbf{x}}(s(\mathbf{u}), \mathbf{u})]^{-1} \in \mathcal{L}(\mathbf{Z}, \mathbf{X}),$$

for all $\mathbf{u} \in L^2(\mathbf{L}^2)$.

The continuous Fréchet differentiability of the cost functional $J : \mathbf{X} \times L^2(\mathbf{L}^2) \to \mathbb{R}$, Lemma 4.5 and Theorem 4.6 ensure that all the solutions $(\mathbf{x}, \mathbf{u})$ of the optimal control Problem 4.1 satisfy, together an adjoint variable $\mathbf{q} \in \mathbf{Z}^*$, a set of first order optimality conditions (see Theorem 1.48 and Corollary 1.3 in [58]). In order to get the first order optimality conditions, we define the following Lagrange functional $L : \mathbf{X} \times L^2(\mathbf{L}^2) \times \mathbf{Z}^* \to \mathbb{R}$,

$$L(\mathbf{x}, \mathbf{u}, \mathbf{q}) = J(\mathbf{x}, \mathbf{u}) + \langle\, \mathbf{q}, e(\mathbf{x}, \mathbf{u})\, \rangle_{\mathbf{Z}^*, \mathbf{Z}}, \tag{4.18}$$

where $\mathbf{q} = (\mathbf{q_v}, q_y, \mathbf{q_{v0}}, q_{y0}) \in \mathbf{Z}^*$. Then, the optimality conditions of Problem 4.1 correspond to: *find* $(\mathbf{x}, \mathbf{u}, \mathbf{q}) \in \mathbf{X} \times L^2(\mathbf{L}^2) \times \mathbf{Z}^*$, *such that*

$$L_{\mathbf{q}}(\mathbf{x}, \mathbf{u}, \mathbf{q}) = 0, \qquad \text{in } \mathbf{Z}, \tag{4.19}$$

$$L_{\mathbf{x}}(\mathbf{x}, \mathbf{u}, \mathbf{q}) = 0, \qquad \text{in } \mathbf{X}^*, \tag{4.20}$$

$$L_{\mathbf{u}}(\mathbf{x}, \mathbf{u}, \mathbf{q}) = 0, \qquad \text{in } L^2\left(\mathbf{L}^2\right). \tag{4.21}$$

It is straightforward to check that (4.19) are the state equations $e(\mathbf{x}, \mathbf{u}) = 0$. The second equation (4.20) represents the *adjoint equations* and (4.21) is a further *optimality relation*.
In the next Lemma 4.8, we show that given a solution $\mathbf{x} = s(\mathbf{u})$ of the state equations (4.19), the adjoint equations (4.20) have a unique solution $\mathbf{q} \in \mathbf{Z}^*$.

Lemma 4.8. *Let* $\mathbf{u} \in L^2(\mathbf{L}^2)$ *and* $\mathbf{x} \in \mathbf{X}$ *such that* $\mathbf{x} = s(\mathbf{u})$ *be given. Then, the adjoint equations* (4.20) *have a unique solution* $\mathbf{q} \in \mathbf{Z}^*$.

Proof. The proof of the Lemma is analogous to the one of Lemma 2.8. □

The first order optimality conditions (4.19)-(4.21) are written in terms of the variables $(\mathbf{x}, \mathbf{u}, \mathbf{q}) \in \mathbf{X} \times L^2(\mathbf{L}^2) \times \mathbf{Z}^*$. In the next Theorem 4.9, using the definitions (4.2), (4.6) of the spaces $\mathbf{X}$ and $\mathbf{Z}$, we write these optimality conditions explicitly, in terms of the state variables

$$(\mathbf{v}, y) \in \mathbf{W}_0 \times W_{0,\Delta} \quad \text{and} \quad w = -\varepsilon^2 \Delta y - y + y^3,$$

and the adjoint variables

$$(\mathbf{q_v}, q_y, \mathbf{q_{v0}}, q_{y0}) \in L^2(\boldsymbol{\mathcal{D}}) \times L^2(H_0) \times \boldsymbol{\mathcal{S}} \times H_0^* \quad \text{and} \quad q_w = \gamma \Delta q_y + \rho \nabla y \cdot \mathbf{q_v}.$$

Note that w is the chemical potential defined in (4.8) and q_w is a further adjoint variable. Moreover, still in Theorem 4.9, we derive regularity properties for the adjoint variables.

Theorem 4.9 (optimality conditions, regularity of the adjoint variables). *The first order optimality conditions (4.19)-(4.21) of the optimal control Problem 4.1 read as follows:*

(4.22a)
$$\int_0^T \left[(\mathbf{v}_t, \boldsymbol{\psi}) + \nu (\nabla \mathbf{v}, \nabla \boldsymbol{\psi}) + b(\mathbf{v}, \mathbf{v}, \boldsymbol{\psi}) + \rho (y, \nabla w \cdot \boldsymbol{\psi}) - (\mathbf{u}, \boldsymbol{\psi})\right] \, dt = 0,$$
(4.22b)
$$\mathbf{v}(0) = \mathbf{v}_0,$$
(4.22c)
$$\int_0^T \left[(y_t, \eta) + \gamma (\nabla w, \nabla \eta) - (y, \mathbf{v} \cdot \nabla \eta)\right] \, dt = 0,$$
(4.22d)
$$y(0) = y_0,$$
(4.22e)
$$\int_0^T \left[(w, \theta) - \varepsilon^2 (\nabla y, \nabla \theta) + (y, \theta) - \left(y^3, \theta\right)\right] \, dt = 0,$$

for all $\boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}})$, $\eta, \theta \in L^2(H^1)$,

(4.23a)
$$\int_0^T \left[(-\mathbf{q}_{\mathbf{v}t}, \boldsymbol{\psi}) + \nu (\nabla \mathbf{q}_{\mathbf{v}}, \nabla \boldsymbol{\psi}) + b(\boldsymbol{\psi}, \mathbf{v}, \mathbf{q}_{\mathbf{v}}) + b(\mathbf{v}, \boldsymbol{\psi}, \mathbf{q}_{\mathbf{v}}) - (y, \nabla q_y \cdot \boldsymbol{\psi})\right] \, dt = 0,$$
(4.23b)
$$\mathbf{q}_{\mathbf{v}}(T) = 0,$$
(4.23c)
$$\int_0^T \left[(-q_{yt}, \eta) - \varepsilon^2 (\nabla q_w, \nabla \eta) + \rho (\nabla w \cdot \mathbf{q}_{\mathbf{v}}, \eta) - (\mathbf{v} \cdot \nabla q_y, \eta) + (q_w, \eta) - \left(3y^2 q_w, \eta\right) + (y - y_d, \eta)\right] dt = 0,$$
(4.23d)
$$q_y(T) = 0,$$
(4.23e)
$$\int_0^T \left[(q_w, \theta) + \gamma (\nabla q_y, \nabla \theta) + \rho (y, \mathbf{q}_{\mathbf{v}} \cdot \nabla \theta)\right] \, dt = 0,$$

for all $\boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}})$, $\eta \in L^2(H_0)$, $\theta \in L^2(H^1)$,

(4.24)
$$\int_0^T (\alpha \, \mathbf{u} - \mathbf{q}_{\mathbf{v}}, \boldsymbol{\varphi}) \, dt = 0,$$

for all $\boldsymbol{\varphi} \in L^2(\mathbf{L}^2)$. *Furthermore, any solution* $(\mathbf{v}, y, w, \mathbf{q}_{\mathbf{v}}, q_y, q_w)$ *of (4.22)-(4.24) is such that*

(4.25) $$\mathbf{v} \in H^1\left(\boldsymbol{\mathcal{S}}^2\right) \cap L^\infty\left(\boldsymbol{\mathcal{D}} \cap \mathbf{H}^2\right),$$
(4.26) $$y \in H^1\left(L_0^2\right) \cap L^\infty\left(H^2\right),$$
(4.27) $$w \in L^\infty\left(L^2\right) \cap L^2\left(H^2\right),$$
(4.28) $$\mathbf{q}_{\mathbf{v}} \in H^1\left(\boldsymbol{\mathcal{S}}^2\right) \cap L^\infty(\boldsymbol{\mathcal{D}}),$$
(4.29) $$q_y \in H^1\left(L_0^2\right) \cap L^\infty(H_0),$$
(4.30) $$q_w \in L^2(H_0),$$
(4.31) $$\mathbf{u} \in H^1\left(\mathbf{L}^2\right) \cap L^\infty(\boldsymbol{\mathcal{D}}),$$

and

(4.32) $$\mathbf{q}_{\mathbf{v}0} = \mathbf{q}_{\mathbf{v}}(0), \qquad \text{in } \boldsymbol{\mathcal{D}},$$

$$q_{y0} = q_y(0), \qquad \text{in } H_0. \tag{4.33}$$

Finally,

$$\|\mathbf{q}_{\mathbf{v}t}\|_{L^2(\mathbf{L}^2)} + \|\mathbf{q}_{\mathbf{v}}\|_{L^\infty(\boldsymbol{\mathcal{D}})} + \|q_{yt}\|_{L^2\left(L_0^2\right)} + \|q_y\|_{L^\infty(H_0)} + \|q_w\|_{L^2(H_0)} \le C(\mathbf{u}), \tag{4.34}$$

where the constant $C(\mathbf{u})$ depends continuously on $\|\mathbf{u}\|_{L^2(\mathbf{L}^2)}$ and data (initial conditions and constant parameters) in Problem 4.1.

Proof. Equations (4.22) are the state equations $e(\mathbf{x}, \mathbf{u}) = 0$ in terms of $(\mathbf{v}, y, w)$, that we derived in (4.10), (4.11). The last optimality condition (4.24) is given by direct calculation from (4.21). Moreover, the results (4.25)-(4.27) follow from Theorem 4.2. In Theorem 5.26, we will prove that given

$$\begin{aligned}
\mathbf{v} &\in H^1\left(\mathbf{L}^2\right) \cap L^\infty\left(\boldsymbol{\mathcal{D}} \cap \mathbf{H}^2\right),\\
y &\in H^1\left(L_0^2\right) \cap L^\infty\left(H^2\right),\\
w &\in L^\infty\left(L^2\right) \cap L^2\left(H^2\right).
\end{aligned}$$

which solve the state equations (4.22), there exist $\mathbf{q}_{\mathbf{v}} \in H^1(\mathbf{L}^2) \cap L^\infty(\boldsymbol{\mathcal{D}}), q_y \in H^1(L_0^2) \cap L^\infty(H_0), q_w \in L^2(H_0)$ that satisfy the optimality conditions (4.23) and the estimate (4.34). Hence, from the optimality relation (4.24), we get that (4.31) hold. By direct calculation, we derive that the adjoint equations (4.20), in terms of the variables $(\mathbf{q}_{\mathbf{v}}, q_y, \mathbf{q}_{\mathbf{v}0}, q_{y0})$ have the following form

$$\begin{aligned}
&\int_0^T \Big[\langle\, \boldsymbol{\psi}_t, \mathbf{q}_{\mathbf{v}} \,\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}} + \nu\left(\nabla \mathbf{q}_{\mathbf{v}}, \nabla \boldsymbol{\psi}\right) + b\left(\boldsymbol{\psi}, \mathbf{v}, \mathbf{q}_{\mathbf{v}}\right)\\
&\qquad + b\left(\mathbf{v}, \boldsymbol{\psi}, \mathbf{q}_{\mathbf{v}}\right) - \left(y, \nabla q_y \cdot \boldsymbol{\psi}\right)\Big]\, dt + \left(\mathbf{q}_{\mathbf{v}0}, \boldsymbol{\psi}(0)\right) = 0,
\end{aligned} \tag{4.35a}$$

$$\begin{aligned}
&\int_0^T \Big[\langle\, \eta_t, q_y \,\rangle_{H_0^*,H_0} + \gamma\left(\nabla q_y, \nabla\left[-\varepsilon^2 \Delta\eta - \eta + 3\, y^2\, \eta\right]\right)\\
&\qquad - \left(\mathbf{v} \cdot \nabla q_y, \eta\right) + \rho\left(y, \mathbf{q}_{\mathbf{v}} \cdot \nabla\left[-\varepsilon^2 \Delta\eta - \eta + 3\, y^2\, \eta\right]\right)\\
&+ \rho\left(\nabla\left[-\varepsilon^2 \Delta y - y + y^3\right] \cdot \mathbf{q}_{\mathbf{v}}, \eta\right) + \left(y - y_d, \eta\right)\Big]\, dt + \langle\, q_{y0}, \eta(0) \,\rangle_{H_0^*,H_0} = 0,
\end{aligned} \tag{4.35b}$$

for all $\boldsymbol{\psi} \in \mathbf{W}_0$, $\eta \in W_{0,\Delta} = W_0 \cap L^\infty(H_0) \cap L^2(H_\Delta)$. In the following we prove the equivalence between the adjoint equations (4.35) and the system (4.23).
Setting $\boldsymbol{\psi} \in \mathbf{W}_0$ in (4.23a), taking into account (4.23b) and using integration by parts in time, we have

$$\begin{aligned}
&\int_0^T \Big[\langle\, \boldsymbol{\psi}_t, \mathbf{q}_{\mathbf{v}} \,\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}} + \nu\left(\nabla \mathbf{q}_{\mathbf{v}}, \nabla \boldsymbol{\psi}\right) + b\left(\boldsymbol{\psi}, \mathbf{v}, \mathbf{q}_{\mathbf{v}}\right)\\
&+ b\left(\mathbf{v}, \boldsymbol{\psi}, \mathbf{q}_{\mathbf{v}}\right) - \left(y, \nabla q_y \cdot \boldsymbol{\psi}\right)\Big]\, dt + \left(\mathbf{q}_{\mathbf{v}}(0), \boldsymbol{\psi}(0)\right) = 0,
\end{aligned} \tag{4.36}$$

which is, assuming $\mathbf{q}_{\mathbf{v}0} = \mathbf{q}_{\mathbf{v}}(0)$, the first adjoint equation (4.35a). Furthermore, with

$$\boldsymbol{\psi} \in \mathbf{W}_0, \quad \eta \in W_0 \cap L^\infty(H_0) \cap L^2(H_\Delta),$$

in (4.23c) and using integration by parts in space and time, we derive

$$\int_0^T \Big[\langle \eta_t, q_y \rangle_{H_0^*,H_0} + \left(q_w, \varepsilon^2 \Delta\eta + \eta - 3y^2\eta\right) \tag{4.37}$$
$$- \left(\mathbf{v}\cdot\nabla q_y, \eta\right) + \rho\left(\nabla w \cdot \mathbf{q_v}, \eta\right) + \left(y - y_d, \eta\right)\Big]\, dt + \left(q_y(0), \eta(0)\right) = 0.$$

Assuming $\theta = \varepsilon^2\Delta\eta + \eta - 3y^2\eta$ in (4.23e), we get

$$\int_0^T \left(q_w, \varepsilon^2\Delta\eta + \eta - 3y^2\eta\right)\, dt \tag{4.38}$$
$$= \int_0^T \left[\gamma\left(\nabla q_y, \nabla\left[-\varepsilon^2\Delta\eta - \eta + 3y^2\eta\right]\right) + \rho\left(y\cdot\mathbf{q_v}, \nabla\left[-\varepsilon^2\Delta\eta - \eta + 3y^2\eta\right]\right)\right]\, dt$$

Then, using (4.38) in (4.37), setting $w = -\varepsilon^2\Delta y - y + y^3$ and assuming $q_{y0} = q_y(0)$, we have just the second adjoint equation (4.35b). So, we can claim that:

- given a solution $(\mathbf{q_v}, q_y, q_w)$ of (4.23), then $(\mathbf{q_v}, q_y)$ and $\mathbf{q_{v0}} = \mathbf{q_v}(0)$, $q_{y0} = q_y(0)$ is a solution of the adjoint equations (4.35);
- the spaces $H^1(\mathbf{L}^2) \cap L^\infty(\boldsymbol{\mathcal{D}})$ and $H^1(L_0^2) \cap L^\infty(H_0)$ are, respectively, compactly embedded in $\mathcal{C}([0,T];\boldsymbol{\mathcal{D}})$ and $\mathcal{C}([0,T];H_0)$ (see for example Theorem II.5.16 by Aubin-Lions-Simon in [20]); then, $\mathbf{q_v}(0) \in \boldsymbol{\mathcal{D}}$ and $q_y(0) \in H_0$;
- given the state variables $(\mathbf{v}, y, w)$, the solution $(\mathbf{q_v}, \mathbf{q_{v0}}, q_y, q_{y0})$ of the adjoint equations (4.35) is unique, then also the solution $(\mathbf{q_v}, q_y, q_w)$ of (4.23) is unique;
- given the state variables $(\mathbf{v}, y, w)$, the adjoint equations (4.35) are equivalent to (4.23).

We prove last statement above by contradiction. We suppose that there is $(\mathbf{q_v}, \mathbf{q_{v0}}, q_y, q_{y0})$ which is the unique solution of the adjoint equations (4.35) and does not satisfy (4.23). However, (4.23) has a solution, we say $(\bar{\mathbf{q}}_\mathbf{v}, \bar{q}_y)$ and we know that $(\bar{\mathbf{q}}_\mathbf{v}, \bar{q}_y)$, together $\bar{\mathbf{q}}_{\mathbf{v}0} = \bar{\mathbf{q}}_\mathbf{v}(0)$ and $\bar{q}_{y0} = \bar{q}_y(0)$ is also a solution of (4.35). Then we have a contradiction, because we obtain, given $(\mathbf{v}, y, w)$, two different solution of the adjoint equations (4.35). Hence, the system (4.22)-(4.24) is equivalent to the first order optimality conditions (4.19)-(4.21). □

5. Optimal Control of the Discrete Cahn-Hilliard-Navier-Stokes System

5.1. Introduction

In this Chapter, we study the fully discrete version (in space and time) of the optimal control Problem 4.1. We adapt the analysis from Chapter 4 to the discrete setting and show that the discrete problem converges to the continuous one, as the discretization parameters go to zero.

Technical details of the discretization are collected in Appendix A.3. In particular, we denote with $h, k = T/N$, respectively, the space and time discretization parameters, which are defined in A.3.1. Also the definitions of the discrete function spaces $\mathbf{S}_h, \mathbf{V}_h, \mathbf{D}_h, P_h, Y_h$ are given in A.3.1. Moreover, if Z_h is a discrete functions space, given $Z^n \in Z_h$ for $n = 1, \ldots, N$, we denote by the corresponding calligraphic letter the associated vector variable

$$\mathcal{Z} = (Z^n)_{n=1}^N \in Z_h^N.$$

and with $d_t Z^n$ the discrete time derivative at time level n,

$$d_t Z^n = \frac{Z^n - Z^{n-1}}{k}.$$

We use $(\cdot,\cdot)_h$ to denote the mass-lumped scalar product defined in (A.29). We define the following discrete spaces

$$\mathbf{X}_{h,k} = \mathbf{V}_h^{N+1} \times P_h^N \times P_h^{N+1} \times Y_h^N, \tag{5.1}$$

with elements

$$\boldsymbol{\mathcal{X}} = (\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}), \tag{5.2}$$

and

$$K_h = \{Z \in Y_h : -1 \leq Z \leq 1\}. \tag{5.3}$$

Given h, k, we consider the following discretized version of the objective function J stated in (4.5),

$$J_{h,k} : \mathbf{X}_{h,k} \times L^2(\mathbf{L}^2) \to \mathbb{R},$$

where

$$J_{h,k}\left(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{U}}\right) := \sum_{n=1}^{N}\left[\frac{k}{2}\|Y^n - y_{d,h}^n\|^2 + \frac{\alpha}{2}\int_{t_{n-1}}^{t_n} \|\boldsymbol{\mathcal{U}}\|^2 \, dt\right]. \tag{5.4}$$

where the functions $y_{d,h}^n \in P_h$ and $t_n = n \cdot k$ for $n = 1, \ldots, N$.
In order to represent the problem under investigation in a more compact, general form, we define the following map:

$$e_{h,k} : X_{h,k} \times L^2\left(\mathbf{L}^2\right) \to X_{h,k}, \tag{5.5}$$

where, for all $\boldsymbol{\mathcal{Z}} = (\boldsymbol{\psi}, \phi, \eta, \theta) \in X_{h,k}$,

$$\begin{aligned}\langle \boldsymbol{\mathcal{Z}}, e_{h,k}\left(\mathcal{X},\boldsymbol{\mathcal{U}}\right)\rangle_{X_{h,k}^*,X_{h,k}} &= \langle \boldsymbol{\psi}, a_{1,h,k}\left(\boldsymbol{\mathcal{V}},\mathcal{P},\boldsymbol{\mathcal{U}}\right)\rangle + \langle \phi, a_{2,h,k}\left(\boldsymbol{\mathcal{V}}\right)\rangle \\ &+ \langle \eta, c_{h,k}\left(\boldsymbol{\mathcal{V}},\mathcal{Y},\mathcal{W}\right)\rangle + \langle \theta, d_{h,k}\left(\mathcal{Y},\mathcal{W}\right)\rangle \\ &+ \left(\boldsymbol{\psi}^0, \mathbf{V}^0 - \mathbf{v}_{0,h}\right) + \left(\eta^0, Y^0 - y_{0,h}\right),\end{aligned} \tag{5.6}$$

with

$$\begin{aligned}\langle \boldsymbol{\psi}, a_{1,h,k}\left(\boldsymbol{\mathcal{V}},\mathcal{P},\boldsymbol{\mathcal{U}}\right)\rangle &= \sum_{n=1}^{N}\Big[k\left(d_t\mathbf{V}^n, \boldsymbol{\psi}^n\right) + k\nu\left(\nabla\mathbf{V}^n, \nabla\boldsymbol{\psi}^n\right) + kB\left(\mathbf{V}^{n-1}, \mathbf{V}^n, \boldsymbol{\psi}\right) \\ &\quad -k\left(P^n, \nabla\cdot\boldsymbol{\psi}^n\right) + k\rho\left(Y^{n-1}, \nabla W^n \cdot \boldsymbol{\psi}\right) - \int_{t_{n-1}}^{t_n}\left(\boldsymbol{\mathcal{U}}, \boldsymbol{\psi}^n\right) dt\Big], \\ \langle \phi, a_{2,h,k}\left(\boldsymbol{\mathcal{V}}\right)\rangle &= \sum_{n=1}^{N} k\left(\nabla\cdot\mathbf{V}^n, \phi^n\right), \\ \langle \eta, c_{h,k}\left(\boldsymbol{\mathcal{V}},\mathcal{Y},\mathcal{W}\right)\rangle &= \sum_{n=1}^{N}\left[k\left(d_tY^n, \eta^n\right)_h + k\gamma\left(\nabla W^n, \nabla\eta^n\right) - k\left(Y^{n-1}, \mathbf{V}^{n-1}\cdot\nabla\eta^n\right)\right], \\ \langle \theta, d_{h,k}\left(\mathcal{Y},\mathcal{W}\right)\rangle &= \sum_{n=1}^{N} k\left[\left(W^n + Y^{n-1} - \left(Y^n\right)^3, \theta^n\right)_h - \varepsilon^2\left(\nabla Y^n, \nabla\theta^n\right)\right].\end{aligned}$$

In (5.6), the trilinear form $B(\cdot,\cdot,\cdot)$ corresponds to a discretization of the trilinear form $b(\cdot,\cdot,\cdot)$ defined in (4.12). It reads

$$B\left(\mathbf{V},\mathbf{U},\mathbf{W}\right) = \frac{1}{2}\int_\Omega \left(\mathbf{V}\cdot\nabla\right)\mathbf{U}\cdot\mathbf{W}\, dx - \frac{1}{2}\int_\Omega \left(\mathbf{U}\cdot\nabla\right)\mathbf{V}\cdot\mathbf{W}\, dx, \tag{5.7}$$

for all $\mathbf{V},\mathbf{U},\mathbf{W} \in \mathbf{V}_h$. Then, we consider the following fully discretized version of the continuous optimal control Problem 4.1:

Problem 5.1. *Given h, k, $\mathbf{v}_{0,h} \in \mathbf{D}_h, y_{0,h} \in P_h \cap K_h$, $y_{d,h}^n \in P_h$ for $n = 1, \ldots, N$, find $\left(\bar{\boldsymbol{\mathcal{X}}},\bar{\boldsymbol{\mathcal{U}}}\right) \in \mathbf{X}_{h,k} \times L^2\left(\mathbf{L}^2\right)$ such that*

$$\min_{(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{U}})\in\mathbf{X}_{h,k}\times L^2(\mathbf{L}^2)} J_{h,k}\left(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{U}}\right) = J_{h,k}\left(\bar{\boldsymbol{\mathcal{X}}},\bar{\boldsymbol{\mathcal{U}}}\right),$$

subject to

$$e_{h,k}\left(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{U}}\right) = 0. \tag{5.8}$$

We emphasize that the constraint (5.8) in Problem 5.1 is a discretized version of the state equations (4.9) of the continuous optimal control Problem 4.1. In the following section, we derive existence, uniqueness and regularity properties of the solution of (5.8).

5.2. Properties of the Discrete State Equations

Using the definition (5.5), (5.6) of the map $e_{h,k}$, we can write the state equations (5.8) of the discrete optimal control Problem 5.1 in the following way:

$$\begin{aligned}
&(5.9a) && (d_t\mathbf{V}^n, \boldsymbol{\psi}) + \nu(\nabla\mathbf{V}^n, \nabla\boldsymbol{\psi}) + B\left(\mathbf{V}^{n-1}, \mathbf{V}^n, \boldsymbol{\psi}\right) - (P^n, \nabla\cdot\boldsymbol{\psi}) \\
& && \quad +\rho\left(Y^{n-1}, \nabla W^n\cdot\boldsymbol{\psi}\right) - \frac{1}{k}\int_{t_{n-1}}^{t_n} (\boldsymbol{\mathcal{U}}, \boldsymbol{\psi})\; dt = 0, \\
&(5.9b) && \mathbf{V}^0 = \mathbf{v}_{0,h}, \\
&(5.9c) && (\nabla\cdot\mathbf{V}^n, \phi) = 0,
\end{aligned}$$

$$\begin{aligned}
&(5.10a) && (d_tY^n, \eta)_h + \gamma(\nabla W^n, \nabla\eta) - \left(Y^{n-1}\mathbf{V}^{n-1}, \nabla\eta\right) = 0, \\
&(5.10b) && Y^0 = y_{0,h}, \\
&(5.10c) && (W^n, \theta)_h - \varepsilon^2(\nabla Y^n, \nabla\theta) + \left(Y^{n-1}, \theta\right)_h - \left((Y^n)^3, \theta\right)_h = 0,
\end{aligned}$$

for all $\boldsymbol{\psi}\in\mathbf{V}_h,\ \phi\in P_h,\ \eta,\theta\in Y_h,\ n = 1,\dots,N$. We note that equation (5.10a) above is *mass preserving*:

$$(5.11)\qquad (Y^n, 1)_h = \ldots = (Y^0, 1)_h = (y_{0,h}, 1)_h = 0, \quad \forall n = 1,\dots,N.$$

In the following Lemma 5.2 we show existence and uniqueness of the solution of state equations (5.9), (5.10) of the discrete optimal control Problem 5.1.

Lemma 5.2 (existence, uniqueness). *For any fixed h, k and $\boldsymbol{\mathcal{U}}\in L^2(\mathbf{L}^2)$, the system of the state equations* (5.9), (5.10) *has a unique solution* $(\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W})\in\mathbf{V}_h^{N+1}\times P_h^N\times P_h^{N+1}\times Y_h^N$.

Proof. See Lemma 4.1 in [62]. □

As a consequence of Lemma (5.2) above, associated to the discrete state equations of the optimal control Problem 5.1,

$$e_{h,k}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}}) = 0,$$

we can define a *solution operator* $s_{h,k} : L^2(\mathbf{L}^2)\to\mathbf{X}_{h,k}$, which is such that

$$(5.12)\qquad e_{h,k}(s_{h,k}(\boldsymbol{\mathcal{U}}), \boldsymbol{\mathcal{U}}) = 0, \quad \forall\,\boldsymbol{\mathcal{U}}\in L^2\left(\mathbf{L}^2\right).$$

Given the system (5.9), (5.10), we consider, at each time level $n = 1,\dots,N$ the following, associated *discrete energy*

$$(5.13)\qquad E(\mathbf{V}^n, Y^n) = \frac{1}{2}\|\mathbf{V}^n\|^2 + \frac{\rho\varepsilon^2}{2}\|\nabla Y^n\|^2 + \rho\left(\tilde{\Phi}(Y^n), 1\right)_h$$

where $\Phi(\cdot)$ is the double well potential defined in (1.11), which is such that

$$(5.14)\qquad \tilde{\Phi}(y) = \tilde{\Phi}_+(y) + \tilde{\Phi}_-(y),$$

where

$$\tilde{\Phi}_+(y) = \frac{1}{4}y^4, \quad \tilde{\Phi}_-(y) = \frac{1}{4}\left(1 - 2y^2\right).$$

In the following Lemma 5.3, we derive a property of the discrete energy (5.13) associated to the state equations (5.9), (5.10). We use this property later in the document, to get stability estimates for the solution $(\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W})$ of the state equations (5.9), (5.10).

Lemma 5.3. *For any fixed h, k and $\boldsymbol{\mathcal{U}} \in L^2(\mathbf{L}^2)$, the solution $(\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}) \in \mathbf{V}_h^{N+1} \times P_h^N \times P_h^{N+1} \times Y_h^N$ of the state equations (5.9), (5.10) is such that, for all $n = 1, \ldots, N$,*

$$\begin{aligned} E(\mathbf{V}^n, Y^n) - E\left(\mathbf{V}^{n-1}, Y^{n-1}\right) + \frac{1}{4}\|\mathbf{V}^n - \mathbf{V}^{n-1}\|^2 \\ + k\frac{\nu}{2}\|\nabla\mathbf{V}^n\|^2 + \frac{\rho\varepsilon^2}{2}\|\nabla Y^n - \nabla Y^{n-1}\|^2 + \frac{k\rho\gamma}{2}\|\nabla W^n\|^2 \\ \le k^2\,\frac{\rho^2}{\gamma^2}\,\tilde{C}\left(\left(Y^{n-1}\right)^4, 1\right)_h \left(\|\nabla\mathbf{V}^n\|^2 + \|\nabla\mathbf{V}^{n-1}\|^2\right) + \frac{C^*}{2\nu}\int_{t_{n-1}}^{t_n} \|\boldsymbol{\mathcal{U}}\|^2 dt. \end{aligned} \tag{5.15}$$

where $\tilde{C} = \tilde{C}(\Omega)$ and $C^* = C^*(\Omega)$.

Proof. Setting $\boldsymbol{\psi} = \mathbf{V}^n$ in (5.9a) and using (5.9c), we have

$$\begin{aligned} \frac{1}{2}\|\mathbf{V}^n\|^2 - \frac{1}{2}\|\mathbf{V}^{n-1}\|^2 + \frac{1}{2}\|\mathbf{V}^n - \mathbf{V}^{n-1}\|^2 + k\nu\|\nabla\mathbf{V}^n\|^2 + k\rho\left(Y^{n-1}, \nabla W^n \cdot \mathbf{V}^n\right) \\ = \int_{t_{n-1}}^{t_n} (\boldsymbol{\mathcal{U}}, \mathbf{V}^n)\ dt. \end{aligned} \tag{5.16}$$

Substituting $\eta = W^n$ in (5.10a) and $\theta = Y^n - Y^{n-1}$ in (5.10b) we derive

$$\left(Y^n - Y^{n-1}, W^n\right)_h + k\gamma\|\nabla W^n\|^2 - k\left(Y^{n-1}\mathbf{V}^{n-1}, \nabla W^n\right) = 0, \tag{5.17}$$

and

$$\begin{aligned} \left(W^n, Y^n - Y^{n-1}\right)_h = \frac{\varepsilon^2}{2}\|\nabla Y^n\|^2 - \frac{\varepsilon^2}{2}\|\nabla Y^{n-1}\|^2 + \frac{\varepsilon^2}{2}\|\nabla Y^n - \nabla Y^{n-1}\|^2 \\ - \left(Y^{n-1}, Y^n - Y^{n-1}\right)_h + \left((Y^n)^3, Y^n - Y^{n-1}\right)_h. \end{aligned} \tag{5.18}$$

Using (5.18) in (5.17), we can write

$$\begin{aligned} \frac{\varepsilon^2}{2}\|\nabla Y^n\|^2 - \frac{\varepsilon^2}{2}\|\nabla Y^{n-1}\|^2 + \frac{\varepsilon^2}{2}\|\nabla Y^n - \nabla Y^{n-1}\|^2 + k\gamma\|\nabla W^n\|^2 + \\ + \left((Y^n)^3, Y^n - Y^{n-1}\right)_h - \left(Y^{n-1}, Y^n - Y^{n-1}\right)_h - k\left(Y^{n-1}\mathbf{V}^{n-1}, \nabla W^n\right) = 0, \end{aligned} \tag{5.19}$$

In (5.19), using the convexity of the functions $\tilde{\Phi}_+(\cdot)$ and $-\tilde{\Phi}_-(\cdot)$, we note that

$$\left((Y^n)^3, Y^n - Y^{n-1}\right)_h = \left(\tilde{\Phi}'_+(Y^n), Y^n - Y^{n-1}\right)_h \ge \left(\tilde{\Phi}_+(Y^n) - \tilde{\Phi}_+\left(Y^{n-1}\right), 1\right)_h,$$

$$-\left(Y^{n-1}, Y^n - Y^{n-1}\right)_h = \left(\tilde{\Phi}'_-\left(Y^{n-1}\right), Y^n - Y^{n-1}\right)_h \geq \left(\tilde{\Phi}_-\left(Y^n\right) - \tilde{\Phi}_-\left(Y^{n-1}\right), 1\right)_h.$$

Then, multiplying (5.19) by ρ, we get

$$\begin{aligned}(5.20)\quad & \frac{\rho\varepsilon^2}{2}\|\nabla Y^n\|^2 - \frac{\rho\varepsilon^2}{2}\|\nabla Y^{n-1}\|^2 + \frac{\rho\varepsilon^2}{2}\|\nabla Y^n - \nabla Y^{n-1}\|^2 + k\rho\gamma\|\nabla W^n\|^2 \\ & + \rho\left(\tilde{\Phi}\left(Y^n\right) - \tilde{\Phi}\left(Y^{n-1}\right), 1\right)_h - k\rho\left(Y^{n-1}\mathbf{V}^{n-1}, \nabla W^n\right) \leq 0.\end{aligned}$$

Therefore, using together (5.16) and (5.20), we derive

$$\begin{aligned}(5.21)\quad & \frac{1}{2}\|\mathbf{V}^n\|^2 - \frac{1}{2}\|\mathbf{V}^{n-1}\|^2 + \frac{1}{2}\|\mathbf{V}^n - \mathbf{V}^{n-1}\|^2 + k\nu\|\nabla\mathbf{V}^n\|^2 \\ & + \frac{\rho\varepsilon^2}{2}\|\nabla Y^n\|^2 - \frac{\rho\varepsilon^2}{2}\|\nabla Y^{n-1}\|^2 + \frac{\rho\varepsilon^2}{2}\|\nabla Y^n - \nabla Y^{n-1}\|^2 \\ & + k\rho\gamma\|\nabla W^n\|^2 + \rho\left(\tilde{\Phi}\left(Y^n\right) - \tilde{\Phi}\left(Y^{n-1}\right), 1\right)_h \\ & + k\rho\left(Y^{n-1}, \nabla W^n \cdot \left[\mathbf{V}^n - \mathbf{V}^{n-1}\right]\right) \leq \int_{t_{n-1}}^{t_n} \left(\boldsymbol{\mathcal{U}}, \mathbf{V}^n\right)\, dt.\end{aligned}$$

Rearranging (5.21), we have

$$\begin{aligned}(5.22)\quad & E\left(\mathbf{V}^n, Y^n\right) - E\left(\mathbf{V}^{n-1}, Y^{n-1}\right) + \frac{1}{2}\|\mathbf{V}^n - \mathbf{V}^{n-1}\|^2 \\ & + k\nu\|\nabla\mathbf{V}^n\|^2 + \frac{\rho\varepsilon^2}{2}\|\nabla Y^n - \nabla Y^{n-1}\|^2 + k\rho\gamma\|\nabla W^n\|^2 \\ & \leq k\rho\left|\left(Y^{n-1}, \nabla W^n \cdot \left[\mathbf{V}^n - \mathbf{V}^{n-1}\right]\right)\right| + \left|\int_{t_{n-1}}^{t_n} \left(\boldsymbol{\mathcal{U}}, \mathbf{V}^n\right)\, dt\right| = A_1^n + A_2^n.\end{aligned}$$

The two quantities A_1^n, A_2^n in (5.22) can be estimated using interpolation of $\mathbf{L}^4$ in $\mathbf{L}^2$, Poincaré's inequality, Poincaré-Wirtinger's inequality and Young's inequality. In this way, we derive

$$\begin{aligned}(5.23)\quad A_1^n & \leq k\rho\|Y^{n-1}\|_{L^4}\|\nabla W^n\|\|\mathbf{V}^n - \mathbf{V}^{n-1}\|_{L^4} \\ & \leq k\sigma\|\nabla W^n\|^2 + k\,\frac{\rho^2}{4\sigma}\|Y^{n-1}\|_{L^4}^2\|\mathbf{V}^n - \mathbf{V}^{n-1}\|_{L^4}^2 \\ & \leq k\sigma\|\nabla W^n\|^2 + k\,\frac{\rho^2 C_1(\Omega)}{\sigma}\|Y^{n-1}\|_{L^4}^2\|\mathbf{V}^n - \mathbf{V}^{n-1}\|\|\nabla\mathbf{V}^n - \nabla\mathbf{V}^{n-1}\| \\ & \leq k\sigma\|\nabla W^n\|^2 + k^2\,\frac{\rho^4 C_2(\Omega)}{\sigma^2\mu}\|Y^{n-1}\|_{L^4}^4\|\nabla\mathbf{V}^n - \nabla\mathbf{V}^{n-1}\|^2 + \mu\|\mathbf{V}^n - \mathbf{V}^{n-1}\|^2 \\ & \leq k\sigma\|\nabla W^n\|^2 + k^2\,\frac{\rho^4 C_3(\Omega)}{\sigma^2\mu}\|Y^{n-1}\|_{L^4}^4\left(\|\nabla\mathbf{V}^n\|^2 + \|\nabla\mathbf{V}^{n-1}\|^2\right) + \mu\|\mathbf{V}^n - \mathbf{V}^{n-1}\|^2,\end{aligned}$$

$$(5.24)\quad A_2^n \leq \frac{\nu}{2}k\|\nabla\mathbf{V}^n\|^2 + \frac{C^*}{2\nu}\int_{t_{n-1}}^{t_n} \|\boldsymbol{\mathcal{U}}\|^2 dt.$$

Substituting (5.23), (5.24) in (5.22) we can write

$$(5.25)\quad E\left(\mathbf{V}^n, Y^n\right) - E\left(\mathbf{V}^{n-1}, Y^{n-1}\right) + \frac{1}{2}\|\mathbf{V}^n - \mathbf{V}^{n-1}\|^2$$

$$+k\nu\|\nabla \mathbf{V}^n\|^2 + \frac{\rho\varepsilon^2}{2}\|\nabla Y^n - \nabla Y^{n-1}\|^2 + k\rho\gamma\|\nabla W^n\|^2$$
$$\leq k\sigma\|\nabla W^n\|^2 + k^2\,\frac{\rho^4 C_3(\Omega)}{\sigma^2\mu}\|Y^{n-1}\|_{L^4}^4\left(\|\nabla \mathbf{V}^n\|^2 + \|\nabla \mathbf{V}^{n-1}\|^2\right) + \mu\|\mathbf{V}^n - \mathbf{V}^{n-1}\|^2$$
$$+\frac{\nu}{2}k\|\nabla \mathbf{V}^n\|^2 + \frac{C^*}{2\nu}\int_{t_{n-1}}^{t_n}\|\boldsymbol{\mathcal{U}}\|^2 dt,$$

for all $n = 1, \ldots, N$. In (5.25), setting $\sigma = \frac{\gamma\rho}{2}$, $\mu = \frac{1}{4}$ and rearranging we derive

$$\begin{aligned} &E(\mathbf{V}^n, Y^n) - E\left(\mathbf{V}^{n-1}, Y^{n-1}\right) + \frac{1}{4}\|\mathbf{V}^n - \mathbf{V}^{n-1}\|^2 \\ &+ k\frac{\nu}{2}\|\nabla \mathbf{V}^n\|^2 + \frac{\rho\varepsilon^2}{2}\|\nabla Y^n - \nabla Y^{n-1}\|^2 + \frac{k\rho\gamma}{2}\|\nabla W^n\|^2 \\ &\leq k^2\,\frac{\rho^2 C_4(\Omega)}{\gamma^2}\|Y^{n-1}\|_{L^4}^4\left(\|\nabla \mathbf{V}^n\|^2 + \|\nabla \mathbf{V}^{n-1}\|^2\right) + \frac{C^*}{2\nu}\int_{t_{n-1}}^{t_n}\|\boldsymbol{\mathcal{U}}\|^2 dt. \end{aligned} \tag{5.26}$$

Finally, using (A.56), from (5.26) we derive the result (5.15). □

In the following, using the property (5.15) above, we derive stability estimates for the solution $(\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}) \in \mathbf{V}_h^{N+1} \times P_h^N \times P_h^{N+1} \times Y_h^N$ of the state equations (5.9), (5.10).

Lemma 5.4. *Let us assume that there exists a constant C_B independent of h, k, such that*

$$E(\mathbf{v}_{0,h}, y_{0,h}) + \|\nabla \mathbf{v}_{0,h}\| \leq C_B. \tag{5.27}$$

Then, for any fixed $\boldsymbol{\mathcal{U}} \in L^2(\mathbf{L}^2)$, there exists a constant

$$C_1 = \min\left(\frac{\nu}{8\frac{\rho^2}{\gamma^2}\tilde{C}\left(\left(y_{0,h}\right)^4, 1\right)_h}, \frac{\nu}{8\frac{\rho^2}{\gamma^2}\tilde{C}\left[\frac{8}{\rho}\left(E(\mathbf{v}_{0,h}, y_{0,h}) + \frac{\nu}{8}\|\nabla \mathbf{v}_{0,h}\|^2 + \frac{C^*}{2\nu}\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}^2\right) + 2|\Omega|\right]}\right),$$

such that, if

$$k \leq C_1, \tag{5.28}$$

the solution $(\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}) \in \mathbf{V}_h^{N+1} \times P_h^N \times P_h^{N+1} \times Y_h^N$ of (5.9), (5.10) satisfies:

$$\sup_{n=0,\ldots,N}\|\mathbf{V}^n\| \leq C(\boldsymbol{\mathcal{U}}), \tag{5.29}$$

$$\sum_{n=1}^{N} k\|\mathbf{V}^n\|_{\mathbf{H}_0^1}^2 \leq C(\boldsymbol{\mathcal{U}}), \tag{5.30}$$

$$\sum_{n=1}^{N}\|\mathbf{V}^n - \mathbf{V}^{n-1}\|^2 \leq C(\boldsymbol{\mathcal{U}}), \tag{5.31}$$

$$\sup_{n=0,\ldots,N}\|Y^n\|_{H_0} \leq C(\boldsymbol{\mathcal{U}}), \tag{5.32}$$

$$\sum_{n=1}^{N} \|Y^n - Y^{n-1}\|_{H_0} \le C\,(\boldsymbol{\mathcal{U}})\,, \tag{5.33}$$

$$\sum_{n=1}^{N} k\|\nabla W^n\|^2 \le C\,(\boldsymbol{\mathcal{U}})\,, \tag{5.34}$$

where the constant $C\,(\boldsymbol{\mathcal{U}})$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$, data problem and constant parameters, but it is independent of h, k.

Proof. First we prove that (5.27) implies that there exists a constant C_A independent of h, k such that

$$\left((y_{0,h})^4\,,1\right)_h \le C_A. \tag{5.35}$$

Using the definition of the discrete energy (5.13), from (5.27) we infer

$$\left(1 + (y_{0,h})^4 - 2\,(y_{0,h})^2\,,1\right)_h \le \frac{4C_B}{\rho},$$

which implies

$$\left((y_{0,h})^4\,,1\right)_h \le \frac{4C_B}{\rho} + 2\left((y_{0,h})^2\,,1\right)_h - |\Omega|. \tag{5.36}$$

Using Young's inequality (A.13), we derive

$$\left((y_{0,h})^2\,,1\right)_h \le \frac{1}{4}\left((y_{0,h})^4\,,1\right)_h + |\Omega|, \tag{5.37}$$

and inserting (5.37) in (5.36) and rearranging, we have

$$\left((y_{0,h})^4\,,1\right)_h \le \frac{8C_B}{\rho} + 2|\Omega| = C_A.$$

Next, we perform the proof of the Lemma by induction. We assume that for all $i = 1, \ldots, n$, the time step k is such that

$$k\frac{\rho^2}{\gamma^2}\tilde{C}\left(\left(Y^{i-1}\right)^4,1\right)_h \le \frac{\nu}{8}. \tag{5.38}$$

Setting in (5.15) $n = i$ and the summing on $i = 1, \ldots, n$, we have

$$\begin{aligned} E\left(\mathbf{V}^n, Y^n\right) + \frac{1}{4}\sum_{i=1}^{n}\|\mathbf{V}^i - \mathbf{V}^{i-1}\|^2 \\ +\frac{\nu}{2}\sum_{i=1}^{n} k\|\nabla\mathbf{V}^i\|^2 + \frac{\rho\varepsilon^2}{2}\sum_{i=1}^{n}\|\nabla Y^i - \nabla Y^{i-1}\|^2 + \frac{\rho\gamma}{2}\sum_{i=1}^{n} k\|\nabla W^i\|^2 \\ \le E\left(\mathbf{v}_{0,h}, y_{0,h}\right) + \sum_{i=1}^{n} k^2\frac{\rho^2}{\gamma^2}\tilde{C}\left(\left(Y^{i-1}\right)^4,1\right)_h\left(\|\nabla\mathbf{V}^i\|^2 + \|\nabla\mathbf{V}^{i-1}\|^2\right) + \frac{C^*}{2\nu}\|\boldsymbol{\mathcal{U}}\|^2_{L^2(\mathbf{L}^2)}. \end{aligned} \tag{5.39}$$

Using in (5.39) the assumption (5.38), $k \leq 1$ and rearranging, we derive

$$\begin{aligned}
(5.40) \qquad & E\left(\mathbf{V}^n, Y^n\right) + \frac{1}{4}\sum_{i=1}^{n}\|\mathbf{V}^i - \mathbf{V}^{i-1}\|^2 \\
& + \frac{\nu}{4}\sum_{i=1}^{n} k\|\nabla \mathbf{V}^i\|^2 + \frac{\rho\varepsilon^2}{2}\sum_{i=1}^{n}\|\nabla Y^i - \nabla Y^{i-1}\|^2 + \frac{\rho\gamma}{2}\sum_{i=1}^{n} k\|\nabla W^i\|^2 \\
& \leq E\left(\mathbf{v}_{0,h}, y_{0,h}\right) + \frac{\nu}{8}\|\nabla \mathbf{v}_{0,h}\|^2 + \frac{C^*}{2\nu}\|\boldsymbol{\mathcal{U}}\|^2_{L^2(\mathbf{L}^2)}.
\end{aligned}$$

From (5.40), using the procedure applied above to derive (5.35), we get

$$(5.41) \qquad \left(\left(Y^n\right)^4, 1\right)_h \leq \frac{8}{\rho}\left(E\left(\mathbf{v}_{0,h}, y_{0,h}\right) + \frac{\nu}{8}\|\nabla \mathbf{v}_{0,h}\|^2 + \frac{C^*}{2\nu}\|\boldsymbol{\mathcal{U}}\|^2_{L^2(\mathbf{L}^2)}\right) + 2|\Omega|.$$

Hence, setting

$$(5.42) \qquad k \leq \frac{\nu}{8\frac{\rho^2}{\gamma^2}\tilde{C}\left[\frac{8}{\rho}\left(E\left(\mathbf{v}_{0,h}, y_{0,h}\right) + \frac{\nu}{8}\|\nabla \mathbf{v}_{0,h}\|^2 + \frac{C^*}{2\nu}\|\boldsymbol{\mathcal{U}}\|^2_{L^2(\mathbf{L}^2)}\right) + 2|\Omega|\right]},$$

we have, at time level n,

$$k\frac{\rho^2}{\gamma^2}\tilde{C}\left(\left(Y^n\right)^4, 1\right)_h \leq \frac{\nu}{8}.$$

Therefore the condition (5.28), ensures that (5.40) holds for all $n = 1, \ldots, N$. Then, using the hypothesis (5.27), Poincaré's inequality (A.16), Poincaré'-Wirtinger's inequality (A.15) and the definition of the discrete energy (5.13), we derive the results (5.29)-(5.34). □

Later in the chapter, we show that the solutions of the discrete Problem 5.1 converge to the solution of the continuous Problem 4.1. In order to do that, we need stronger estimates for the discrete variables $(\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}) \in \mathbf{V}_h^{N+1} \times P_h^N \times P_h^{N+1} \times Y_h^N$. We establish these estimates in the following lemmas.

Lemma 5.5. *Under the same hypothesis of lemma 5.4, the solution $\mathcal{Y} \in P_h^{N+1}$ of (5.9), (5.10) is such that*

$$(5.43) \qquad \sum_{n=1}^{N} k\|\hat{\Delta}_h Y^i\|_h^2 \leq C\left(\boldsymbol{\mathcal{U}}\right),$$

$$(5.44) \qquad \sum_{n=1}^{N} k\|\nabla Y^i\|_{L^p}^2 \leq C\left(\boldsymbol{\mathcal{U}}\right), \qquad \forall\, p \in [1, +\infty),$$

where the constant $C\left(\boldsymbol{\mathcal{U}}\right)$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$, data problem and constant parameters, but it is independent of h, k and $\hat{\Delta}_h$ is the discrete Laplacian defined in (A.36).

Proof. With $\theta = \hat{\Delta}_h Y^n$ in the discrete state equation (5.10c), we can write

(5.45)
$$\left(W^n, \hat{\Delta}_h Y^n\right)_h - \varepsilon^2 \left(\nabla Y^n, \nabla \hat{\Delta}_h Y^n\right) + \left(Y^{n-1}, \hat{\Delta}_h Y^n\right)_h - \left((Y^n)^3, \hat{\Delta}_h Y^n\right)_h = 0.$$

From (5.45), using the definition of the discrete Laplacian (A.36), we get

$$\varepsilon^2 \|\hat{\Delta}_h Y^n\|_h^2 + \left(\nabla I^h\left[(Y^n)^3\right], \nabla Y^n\right) = (\nabla W^n, \nabla Y^n) + \left(\nabla Y^{n-1}, \nabla Y^n\right), \tag{5.46}$$

where I_h is the interpolation operator defined in (A.27). In (5.46) (see [41], inequality (4.3)), we have

$$\left(\nabla I^h\left[(Y^n)^3\right], \nabla Y^n\right) \geq 0. \tag{5.47}$$

Hence, applying Young's inequality (A.13) with $\sigma = 1/2$ in (5.46), we infer

$$\varepsilon^2 \|\hat{\Delta}_h Y^n\|_h^2 \leq \frac{1}{2}\|\nabla W^n\|^2 + \|\nabla Y^n\|^2 + \frac{1}{2}\|Y^{n-1}\|^2.$$

Then, from the results of lemma 5.4, we realize that (5.43) is satisfied. Finally, from the inequality (A.39), we conclude that (5.44) holds □

In the following, we use the same notations of Section 3.4. If Z_h is a discrete functions space, given a discrete vector function

$$\mathcal{Z} = (Z^n)_{n=0}^N \in Z_h^{N+1},$$

we use $\mathcal{Z}_{h,k}$ to generically denote the following three different kinds of time interpolated variable

$$\mathcal{Z}_{h,k}^{\bullet}(t) := \frac{t - t_{n-1}}{k} Z^n + \frac{t_n - t}{k} Z^{n-1}, \qquad t \in [t_{n-1}, t_n], \tag{5.48}$$
$$\mathcal{Z}_{h,k}^{+}(t) := Z^n, \qquad t \in (t_{n-1}, t_n], \tag{5.49}$$
$$\mathcal{Z}_{h,k}^{-}(t) := Z^{n-1}, \qquad t \in [t_{n-1}, t_n), \tag{5.50}$$

where
$$t_n = nk, \quad n = 0, \ldots, N.$$

Lemma 5.6. *Under the same hypothesis of lemma 5.4, the solution $\mathcal{Y} \in P_h^{N+1}$ of (5.9), (5.10) is such that*

$$\sum_{n=1}^N k\|Y^n\|_{\mathcal{C}(\bar{\Omega})}^p \leq C(\boldsymbol{\mathcal{U}}), \qquad \forall\, p \in [1, +\infty). \tag{5.51}$$

Proof. From (5.32) and (5.44), we can write

$$\|\nabla \mathcal{Y}_{h,k}\|_{L^\infty(L^2)} \leq C(\boldsymbol{\mathcal{U}}), \tag{5.52}$$
$$\|\nabla \mathcal{Y}_{h,k}\|_{L^2(L^p)} \leq C(\boldsymbol{\mathcal{U}}), \qquad \forall\, p \in [1, \infty). \tag{5.53}$$

Then, from (5.52), (5.53), using an interpolation argument (see [20], Theorem II.5.5), we get

$$\forall\, p \in [1,+\infty)\ \exists\, q > 2, \quad \text{such that} \quad \|\nabla \mathcal{Y}_{h,k}\|_{L^p(L^q)} \le C(\boldsymbol{\mathcal{U}}). \tag{5.54}$$

Therefore, applying Poincaré-Wirtinger inequality (A.15) in (5.54), we derive

$$\|\mathcal{Y}_{h,k}\|_{L^p(W^{1,q})} \le C(\boldsymbol{\mathcal{U}}), \qquad \forall\, p \in [1,+\infty),\ q > 2.$$

So, from the embedding $W^{1,q} \hookrightarrow \mathcal{C}(\bar{\Omega})$, which holds in $d = 2$ if $q > 2$, we observe that

$$\|\mathcal{Y}_{h,k}\|_{L^p(\mathcal{C}(\bar{\Omega}))} \le C(\boldsymbol{\mathcal{U}}), \qquad \forall\, p \in [1,+\infty).$$

So, the result (5.51) holds. □

Lemma 5.7. *Let us assume that there exists a constant $\tilde{C}$ independent of h, k, such that*

$$E(\mathbf{v}_{0,h}, y_{0,h}) + \|\mathbf{v}_{0,h}\|_{\mathbf{H}_0^1} + \|\hat{\Delta}_h y_{0,h}\|_h \le \tilde{C}. \tag{5.55}$$

Then, for any fixed $\boldsymbol{\mathcal{U}} \in L^2(\mathbf{L}^2)$ and k such that

$$k \le C_1,$$

the solution $\mathcal{Y} \in P_h^{N+1}$ of (5.9), (5.10) satisfies:

$$\sup_{n=0,\dots,N} \|\hat{\Delta}_h Y^n\|_h \le C(\boldsymbol{\mathcal{U}}), \tag{5.56}$$

$$\sum_{n=1}^{N} k\|d_t Y^n\|_h^2 \le C(\boldsymbol{\mathcal{U}}), \tag{5.57}$$

$$\sum_{n=1}^{N} \|\hat{\Delta}_h Y^n - \hat{\Delta}_h Y^{n-1}\|_h^2 \le C(\boldsymbol{\mathcal{U}}), \tag{5.58}$$

$$\sup_{n=0,\dots,N} \|Y^n\|_{\mathcal{C}(\bar{\Omega})} \le C(\boldsymbol{\mathcal{U}}), \tag{5.59}$$

$$\sup_{n=0,\dots,N} \|Y^n\|_{W^{1,4}} \le C(\boldsymbol{\mathcal{U}}), \tag{5.60}$$

where the constant $C(\boldsymbol{\mathcal{U}})$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$, data problem and constant parameters, but it is independent of h, k.

Proof. Setting $\eta = d_t Y^n$ in (5.10a), $\theta = \hat{\Delta}_h d_t Y^n$ in (5.10c) and using the definition (A.36) of the discrete Laplacian , we can write

$$-\gamma\left(W^n, \hat{\Delta}_h d_t Y^n\right)_h + \|d_t Y^n\|_h^2 = \left(Y^{n-1}, \mathbf{V}^{n-1}\cdot \nabla d_t Y^n\right), \tag{5.61}$$

$$\left(W^n, \hat{\Delta}_h d_t Y^n\right)_h = -\varepsilon^2\left(\hat{\Delta}_h Y^n, \hat{\Delta}_h d_t Y^n\right)_h - \left(Y^{n-1} - (Y^n)^3, \hat{\Delta}_h d_t Y^n\right)_h. \tag{5.62}$$

Substituting (5.62) in (5.61) and rearranging we derive

$$\varepsilon^2\gamma\left(\hat{\Delta}_h Y^n, \hat{\Delta}_h d_t Y^n\right)_h + \|d_t Y^n\|_h^2 = R_1^n + R_2^n, \tag{5.63}$$

where, using integration by parts (in space),

$$R_1^n = \left(Y^{n-1}, \mathbf{V}^{n-1}\cdot\nabla d_t Y^n\right) = -\left(Y^{n-1}\cdot\mathbf{V}^{n-1} + Y^{n-1}\left[\nabla\cdot\mathbf{V}^{n-1}\right], d_t Y^n\right), \tag{5.64}$$

and

$$R_2^n = -\gamma\left(Y^{n-1} - (Y^n)^3, \hat{\Delta}_h d_t Y^n\right)_h, \tag{5.65}$$

for all $n = 1,\ldots,N$. For any fixed n, such that $1 \leq n \leq N$, we have

$$\sum_{i=1}^{n} kR_2^i = -\gamma\sum_{i=1}^{n}\left(\hat{\Delta}_h Y^i - \hat{\Delta}_h Y^{i-1}, Y^{i-1} - \left(Y^i\right)^3\right)_h = R_{21} + R_{22} + R_{23}, \tag{5.66}$$

where

$$\begin{aligned}
R_{21} &= -\gamma\sum_{i=1}^{n-1} k\left(\hat{\Delta}_h Y^i, \frac{Y^{i-1} - (Y^i)^3 - \left[Y^i - (Y^{i+1})^3\right]}{k}\right)_h,\\
R_{22} &= \gamma\left(\hat{\Delta}_h Y^0, Y^0 - \left(Y^1\right)^3\right)_h,\\
R_{23} &= -\gamma\left(\hat{\Delta}_h Y^n, Y^{n-1} - (Y^n)^3\right)_h,
\end{aligned}$$

Using the definition (A.36) of the discrete Laplacian and the Young's inequality (A.13), we get

$$\begin{aligned}
R_{23} &= -\gamma\left(\hat{\Delta}_h Y^n, Y^{n-1}\right)_h + \gamma\left(\hat{\Delta}_h Y^n, (Y^n)^3\right)_h\\
&= \gamma\left(\nabla Y^n, \nabla Y^{n-1}\right) - \gamma\left(\nabla Y^n, \nabla I^h\left(Y^n\right)^3\right)_h\\
&\leq \frac{\gamma}{2}\|\nabla Y^n\|^2 + \frac{\gamma}{2}\|\nabla Y^{n-1}\|^2 - \gamma\left(\nabla Y^n, \nabla I^h\left(Y^n\right)^3\right)_h.
\end{aligned} \tag{5.67}$$

From the definition (A.36) of the discrete Laplacian, Young's inequality (A.13), the definition (A.27) of the interpolation operator I^h, the equivalence between the h-norm and the L^2-norm (A.30), the generalized Holder's inequality (A.14) and the inequality (A.17), we can write

$$\begin{aligned}
R_{22} &= \gamma\left(\hat{\Delta}_h Y^0, Y^0\right)_h - \gamma\left(\hat{\Delta}_h Y^0, \left(Y^1\right)^3\right)_h\\
&= -\gamma\|\nabla y_{0,h}\|^2 + \frac{\gamma}{2}\|\hat{\Delta}_h y_{0,h}\|_h^2 + \frac{\gamma}{2}\|I^h\left(Y^1\right)^3\|_h^2\\
&\leq -\gamma\|\nabla y_{0,h}\|^2 + \frac{\gamma}{2}\|\hat{\Delta}_h y_{0,h}\|_h^2 + \frac{C\gamma}{2}\|\left(I^h Y^1\right)^3\|^2\\
&\leq -\gamma\|\nabla y_{0,h}\|^2 + \frac{\gamma}{2}\|\hat{\Delta}_h y_{0,h}\|_h^2 + \frac{C\gamma}{2}\|Y^1\|_{L^6}^6\\
&\leq -\gamma\|\nabla y_{0,h}\|^2 + \frac{\gamma}{2}\|\hat{\Delta}_h y_{0,h}\|_h^2 + \frac{C\gamma}{2}\|Y^1\|_{H_0}^6.
\end{aligned} \tag{5.68}$$

Furthermore,

$$R_{21} = \gamma \sum_{i=1}^{n-1} k \left(\hat{\Delta}_h Y^i, d_t Y^i + \frac{(Y^i)^3 - (Y^{i+1})^3}{k} \right)_h \tag{5.69}$$
$$\leq \gamma \sum_{i=1}^{n-1} k \|\hat{\Delta}_h Y^i\|_h \left[\|d_t Y^i\|_h + \left\| \frac{(Y^i)^3 - (Y^{i+1})^3}{k} \right\|_h \right].$$

Noting that for all $a, b \in \mathbb{R}$,

$$|a^3 - b^3| \leq \frac{3}{2} |a - b| |a^2 + b^2|.$$

we derive

$$\left\| \frac{(Y^i)^3 - (Y^{i+1})^3}{k} \right\|_h \leq \frac{3}{2} \|d_t Y^{i+1}\|_h \| \left(Y^i\right)^2 + \left(Y^{i+1}\right)^2 \|_{\mathcal{C}(\bar{\Omega})}. \tag{5.70}$$

Therefore, using (5.70) and Young's inequality (A.13) in (5.69), we get

$$R_{21} \leq \gamma \sum_{i=1}^{n-1} k \|\hat{\Delta}_h Y^i\|_h \left[\|d_t Y^i\|_h + \frac{3}{2} \|d_t Y^{i+1}\|_h \left(\|Y^i\|^2_{\mathcal{C}(\bar{\Omega})} + \|Y^{i+1}\|^2_{\mathcal{C}(\bar{\Omega})} \right) \right]$$
$$\leq 2\sigma\gamma \sum_{i=1}^{n} k \|d_t Y^i\|_h^2 + C(\sigma) \sum_{i=1}^{n-1} k \|\hat{\Delta}_h Y^i\|_h^2 \left(1 + \|Y^i\|^4_{\mathcal{C}(\bar{\Omega})} + \|Y^{i+1}\|^4_{\mathcal{C}(\bar{\Omega})} \right). \tag{5.71}$$

Thus, inserting (5.67), (5.68) and (5.71) in (5.66), we realize

$$\sum_{i=1}^{n} k \; R_2^i \leq -\gamma \|\nabla y_{0,h}\|^2 - \gamma \left(\nabla Y^n, \nabla I^h \left(Y^n\right)^3 \right)_h \tag{5.72}$$
$$+ \frac{\gamma}{2} \|\nabla Y^n\|^2 + \frac{\gamma}{2} \|\nabla Y^{n-1}\|^2 + \frac{\gamma}{2} \|\hat{\Delta}_h y_{0,h}\|_h^2 + \frac{C\gamma}{2} \|Y^1\|^6_{H_0}$$
$$+ 2\sigma\gamma \sum_{i=1}^{n} k \|d_t Y^i\|_h^2 + C(\sigma) \sum_{i=1}^{n-1} k \|\hat{\Delta}_h Y^i\|_h^2 \left(1 + \|Y^i\|^4_{\mathcal{C}(\bar{\Omega})} + \|Y^{i+1}\|^4_{\mathcal{C}(\bar{\Omega})} \right).$$

Concerning R_1^n in (5.64), using the generalized Holder's inequality (A.14), inequality (A.17), Poincaré's inequality (A.16) and Young's inequality (A.13), we infer

$$\sum_{i=1}^{n} k \; R_1^i \leq \sum_{i=1}^{n} k \left[\|Y^{i-1}\|_{L^4} \|\mathbf{V}^{i-1}\|_{\mathbf{L}^4} \|d_t Y^i\| + \|Y^{i-1}\|_{\mathcal{C}(\bar{\Omega})} \|\nabla \cdot \mathbf{V}^{i-1}\| \|d_t Y^i\| \right]$$
$$\leq C \sum_{i=1}^{n} k \left[\|Y^{i-1}\|_{H_0} \|\mathbf{V}^{i-1}\|_{\mathbf{H}_0^1} \|d_t Y^i\| + \|Y^{i-1}\|_{\mathcal{C}(\bar{\Omega})} \|\mathbf{V}^{i-1}\|_{\mathbf{H}_0^1} \|d_t Y^i\| \right]$$
$$\leq \sum_{i=1}^{n} k \left[2\sigma \|d_t Y^i\|^2 + C(\sigma) \|\mathbf{V}^{i-1}\|^2_{\mathbf{H}_0^1} \left(\|Y^{i-1}\|^2_{H_0} + \|Y^{i-1}\|^2_{\mathcal{C}(\bar{\Omega})} \right) \right]. \tag{5.73}$$

By the embedding $W^{1,4} \hookrightarrow \mathcal{C}\left(\bar{\Omega}\right)$, the Poincaré-Wirtinger inequality (A.15) and the discrete interpolation inequality (A.51), we realize

$$\|Y^{i-1}\|^2_{\mathcal{C}(\bar{\Omega})} \leq C \|Y^{i-1}\|^2_{W^{1,4}} \leq \|Y^{i-1}\|^2_{L^4} + \|\nabla Y^{i-1}\|^2_{L^4} \tag{5.74}$$

$$\leq C_1\|Y^{i-1}\|^2_{H_0} + C_2\left[\|\nabla Y^{i-1}\|^2 + \|\hat{\Delta}_h Y^{i-1}\|^2_h\right] \leq C\left[\|Y^{i-1}\|^2_{H_0} + \|\hat{\Delta}_h Y^{i-1}\|^2_h\right].$$

So, using (5.74) in (5.73), we have

$$\sum_{i=1}^{n} k\ R_1^i \leq \sum_{i=1}^{n} k\left[2\sigma\|d_tY^i\|^2 + C(\sigma)\|\mathbf{V}^{i-1}\|^2_{\mathbf{H}_0^1}\left(\|Y^{i-1}\|^2_{H_0} + \|\hat{\Delta}_h Y^{i-1}\|^2_h\right)\right]. \tag{5.75}$$

Setting $n = i$ in (5.63), summing over the index $i = 1, \ldots, n$, with $1 \leq n \leq N$ and taking into account of (5.72) and (5.73), we get

$$\begin{aligned}
&\frac{\varepsilon^2\gamma}{2}\|\hat{\Delta}_h Y^n\|^2_h - \frac{\varepsilon^2\gamma}{2}\|\hat{\Delta}_h Y^0\|^2_h + \frac{\varepsilon^2\gamma}{2}\sum_{i=1}^{n} k\left[\|\hat{\Delta}_h Y^i - \hat{\Delta}_h Y^{i-1}\|^2_h + \|d_tY^n\|^2_h\right]\\
&\qquad \leq -\gamma\|\nabla y_{0,h}\|^2 - \gamma\left(\nabla Y^n, \nabla I^h\left[(Y^n)^3\right]\right)_h\\
&\qquad + \frac{\gamma}{2}\|\nabla Y^n\|^2 + \frac{\gamma}{2}\|\nabla Y^{n-1}\|^2 + \frac{\gamma}{2}\|\hat{\Delta}_h y_{0,h}\|^2_h + \frac{C_1\gamma}{2}\|Y^1\|^6_{H_0}\\
&\qquad + 2\sigma\gamma\sum_{i=1}^{n} k\|d_tY^i\|^2_h + C_2(\sigma)\sum_{i=1}^{n-1} k\|\hat{\Delta}_h Y^i\|^2_h\left(1 + \|Y^i\|^4_{\mathcal{C}(\bar{\Omega})} + \|Y^{i+1}\|^4_{\mathcal{C}(\bar{\Omega})}\right)\\
&\qquad + \sum_{i=1}^{n} k\left[2\sigma\|d_tY^i\|^2 + C_3(\sigma)\|\mathbf{V}^{i-1}\|^2_{\mathbf{H}_0^1}\left(\|Y^{i-1}\|^2_{H_0} + \|\hat{\Delta}_h Y^{i-1}\|^2_h\right)\right].
\end{aligned} \tag{5.76}$$

Rearranging and using $\left(\nabla Y^n, \nabla I^h\left[(Y^n)^3\right]\right)_h \geq 0$ (see [41], equality (4.3)), we can write

$$\begin{aligned}
&\frac{\varepsilon^2\gamma}{2}\|\hat{\Delta}_h Y^n\|^2_h + \frac{\varepsilon^2\gamma}{2}\sum_{i=1}^{n} k\left[\|\hat{\Delta}_h Y^i - \hat{\Delta}_h Y^{i-1}\|^2_h + \|d_tY^n\|^2_h\right]\\
&\qquad \leq \frac{\gamma}{2}\|\nabla Y^n\|^2 + \frac{\gamma}{2}\|\nabla Y^{n-1}\|^2 + \frac{C_1\gamma}{2}\|Y^1\|^6_{H_0} + \frac{\gamma}{2}\left(\varepsilon^2 + 1\right)\|\hat{\Delta}_h y_{0,h}\|^2_h\\
&\qquad + 2\sigma(\gamma + 1)\sum_{i=1}^{n} k\|d_tY^i\|^2_h + C_2(\sigma)\sum_{i=1}^{n-1} k\|\hat{\Delta}_h Y^i\|^2_h\left(1 + \|Y^i\|^4_{\mathcal{C}(\bar{\Omega})} + \|Y^{i+1}\|^4_{\mathcal{C}(\bar{\Omega})}\right)\\
&\qquad + C_3(\sigma)\sum_{i=1}^{n} k\|\mathbf{V}^{i-1}\|^2_{\mathbf{H}_0^1}\left(\|Y^{i-1}\|^2_{H_0} + \|\hat{\Delta}_h Y^{i-1}\|^2_h\right),
\end{aligned} \tag{5.77}$$

for all $1 \leq n \leq N$. Hence, with σ such that

$$2\sigma(\gamma + 1) < \frac{\varepsilon^2\gamma}{2},$$

from (5.77) we derive

$$\begin{aligned}
&\|\hat{\Delta}_h Y^n\|^2_h + \sum_{i=1}^{n} k\left[\|\hat{\Delta}_h Y^i - \hat{\Delta}_h Y^{i-1}\|^2_h + \|d_tY^n\|^2_h\right]\\
&\qquad \leq C_1\left[\|\nabla Y^n\|^2 + \|\nabla Y^{n-1}\|^2 + \|Y^1\|^6_{H_0} + \|\hat{\Delta}_h y_{0,h}\|^2_h\right]
\end{aligned} \tag{5.78}$$

$$+C_2 \sum_{i=1}^{n-1} k\|\hat{\Delta}_h Y^i\|_h^2 \left(1 + \|Y^i\|^4_{\mathcal{C}(\bar{\Omega})} + \|Y^{i+1}\|^4_{\mathcal{C}(\bar{\Omega})}\right)$$
$$+C_3 \sum_{i=1}^{n} k\|\mathbf{V}^{i-1}\|^2_{\mathbf{H}_0^1} \left(\|Y^{i-1}\|^2_{H_0} + \|\hat{\Delta}_h Y^{i-1}\|_h^2\right),$$

for all $1 \leq n \leq N$. Then, using the assumption (5.55), the statements (5.30), (5.32), (5.51) established in the previous lemmas and applying the discrete Gronwall's inequality (see for example [73], Lemma 1.4.2), we get the results (5.56)-(5.58). Finally, as in (5.74), we derive

$$\|Y^n\|_{\mathcal{C}(\bar{\Omega})} \leq C\|Y^n\|_{W^{1,4}} \leq C\left[\|Y^n\|_{H_0} + \|\hat{\Delta}_h Y^n\|_h\right].$$

for all $1 \leq n \leq N$. So, (5.59) and (5.60) hold. The proof is complete. □

Lemma 5.8. *Under the same hypothesis of Lemma 5.7, the solution $\mathcal{W} \in Y_h^N$ of (5.9), (5.10) is such that*

$$\sum_{n=1}^{N} k\ \|\hat{\Delta}_h W^n\|_h^2 \leq C(\boldsymbol{\mathcal{U}}), \tag{5.79}$$

$$\sum_{n=1}^{N} k\ \|\nabla W^n\|_{L^p}^2 \leq C(\boldsymbol{\mathcal{U}}), \qquad \forall\ p \in [1, +\infty). \tag{5.80}$$

where the constant $C(\boldsymbol{\mathcal{U}})$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$, data problem and constant parameters, but it is independent of h, k.

Proof. Setting $\eta = -\hat{\Delta}_h W^n$ in (5.10a), using the definition (A.36) of the discrete Laplacian and integrating by parts in space, we get

$$\gamma\|\hat{\Delta}_h W^n\|_h^2 = \left(d_t Y^n, \hat{\Delta}_h W^n\right)_h + \left(\nabla Y^{n-1}\cdot \mathbf{V}^{n-1}, \hat{\Delta}_h W^n\right) + \left(Y^{n-1}\left[\nabla\cdot\mathbf{V}^{n-1}\right], \hat{\Delta}_h W^n\right). \tag{5.81}$$

From (5.81), applying the generalized Holder's inequality (A.14) and the equivalence between the h-norm and the L^2-norm, we can write

$$\gamma\|\hat{\Delta}_h W^n\|_h^2 = \left[\|d_t Y^n\|_h + C\left(\|\nabla Y^{n-1}\|_{L^4}\|\mathbf{V}^{n-1}\|_{L^4} + \|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}\|\nabla\cdot\mathbf{V}^{n-1}\|\right)\right]\|\hat{\Delta}_h W^n\|_h,$$

which implies, using Young's inequality (A.13), inequality (A.17), Poincaré's inequality and discrete interpolation inequality (A.51),

$$\gamma\|\hat{\Delta}_h W^n\|_h^2 \leq 3\sigma\|\hat{\Delta}_h W^n\|_h^2 + C(\sigma)\left[\|d_t Y^n\|_h^2 + \left(\|\hat{\Delta}_h Y^{n-1}\|_h^2 + \|\nabla Y^{n-1}\|^2 + \|Y^{n-1}\|^2_{\mathcal{C}(\bar{\Omega})}\right)\|\mathbf{V}^{n-1}\|^2_{\mathbf{H}_0^1}\right]. \tag{5.82}$$

Assuming

$$3\sigma < \gamma,$$

in (5.82), rearranging the terms, multiplying by k and summing up over the index n, we derive

$$\sum_{n=1}^{N} k\|\hat{\Delta}_h W^n\|_h^2$$
$$\leq C\sum_{n=1}^{N} k\left[\|d_t Y^n\|_h^2 + \left(\|\hat{\Delta}_h Y^{n-1}\|_h^2 + \|\nabla Y^{n-1}\|^2 + \|Y^{n-1}\|^2_{\mathcal{C}(\bar{\Omega})}\right)\|\mathbf{V}^{n-1}\|^2_{\mathbf{H}_0^1}\right],$$

So, from the results (5.30), (5.32), (5.56), (5.57), (5.59) of the previous lemmas, we realize that (5.79) holds. Finally, applying the inequality (A.39) to (5.79), we have (5.80). □

Lemma 5.9. *Under the same hypothesis of Lemma 5.7, the solution $\mathcal{W} \in Y_h^N$ of (5.9), (5.10) is such that, for all $q \in [1, +\infty), p \in [1, 3)$,*

$$\sup_{n=1,\dots,N} \|W^n\|_h \leq C(\boldsymbol{\mathcal{U}}), \tag{5.83}$$

$$\sum_{n=1}^{N} k\,\|W^n\|^2_{\mathcal{C}(\bar{\Omega})} \leq C(\boldsymbol{\mathcal{U}}), \tag{5.84}$$

$$\sum_{n=1}^{N} k\,\left[\|\nabla W^n\|^4 + \|\nabla W^n\|^2_{L^q} + \|\nabla W^n\|^p_{L^3}\right] \leq C(\boldsymbol{\mathcal{U}}), \tag{5.85}$$

where the constant $C(\boldsymbol{\mathcal{U}})$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$, data problem and constant parameters, but it is independent of h, k.

Proof. With $\theta = W^n$ in (5.10c), using the definition (A.36) of the discrete Laplacian, we can write

$$\|W^n\|_h^2 = -\varepsilon^2\left(\hat{\Delta}_h Y^n, W^n\right)_h - \left(Y^{n-1}, W^n\right)_h + \left((Y^n)^3, W^n\right)_h. \tag{5.86}$$

By the generalized Holder's inequality (A.14), the equivalence between the h-norm and the L^2-norm, the definition (A.27) of the interpolation operator I^h, the inequality (A.17) and the Young's inequality (A.13), from (5.86), we derive

$$\|W^n\|_h^2 \leq 3\sigma\|W^n\|_h + C(\sigma)\left[\|\hat{\Delta}_h Y^n\|_h^2 + \|Y^{n-1}\|^2_{H_0} + \|Y^n\|^6_{H_0}\right],$$

which implies, with $3\sigma < 1$,

$$\|W^n\|_h^2 \leq C\left[\|\hat{\Delta}_h Y^n\|_h^2 + \|Y^{n-1}\|^2_{H_0} + \|Y^n\|^6_{H_0}\right]. \tag{5.87}$$

Using in (5.87) the results (5.32) and (5.56) established in the previous lemmas, we infer that (5.83) holds. From (5.34) and (5.83), we get

$$\|\mathcal{W}_{h,k}\|_{L^\infty(L^2)} + \|\mathcal{W}_{h,k}\|_{L^2(H^1)} \leq C(\boldsymbol{\mathcal{U}}),$$

and subsequently, by an interpolation argument (see [41], pag. 3051),

$$\|\mathcal{W}_{h,k}\|_{L^4(L^4)} \leq C(\boldsymbol{\mathcal{U}}). \tag{5.88}$$

So, taking into account of (5.80) with $p = 4$ and (5.88) above, we have

$$\|\mathcal{W}_{h,k}\|_{L^2(W^{1,4})} \leq C\left(\boldsymbol{\mathcal{U}}\right),$$

which implies, using the embedding $W^{1,4} \hookrightarrow \mathcal{C}\left(\bar{\Omega}\right)$, the result (5.84). From the definition (A.36) of the discrete Laplacian, we infer

$$\|\nabla\mathcal{W}_{h,k}\|^4_{L^4(L^2)} = \int_0^T \|\nabla\mathcal{W}_{h,k}\|^4_{L^2}\, dt = \int_0^T \left|\left(\nabla\mathcal{W}_{h,k}, \nabla\mathcal{W}_{h,k}\right)\right|^2 dt =$$

$$= -\int_0^T \left|\left(\hat{\Delta}_h\mathcal{W}_{h,k}, \mathcal{W}_{h,k}\right)_h\right|^2 dt \leq \int_0^T \|\hat{\Delta}_h\mathcal{W}_{h,k}\|^2_h\|\mathcal{W}_{h,k}\|^2_h\, dt. \tag{5.89}$$

Using (5.79), (5.83) in (5.89) above, we realize

$$\|\nabla\mathcal{W}_{h,k}\|_{L^4(L^2)} \leq C\left(\boldsymbol{\mathcal{U}}\right). \tag{5.90}$$

Thus, from (5.80) and (5.90), we can write

$$\|\nabla\mathcal{W}_{h,k}\|_{L^4(L^2)} + \|\nabla\mathcal{W}_{h,k}\|_{L^2(L^q)} \leq C\left(\boldsymbol{\mathcal{U}}\right), \tag{5.91}$$

for all $q \in [1, \infty)$. Then, applying interpolation (see [20], Theorem II.5.5) to (5.91), we derive

$$\|\nabla\mathcal{W}_{h,k}\|_{L^p(L^3)} \leq C\left(\boldsymbol{\mathcal{U}}\right), \tag{5.92}$$

for all $p \in [1, 3)$. Using together (5.90) and (5.92), we get the result (5.85). □

Lemma 5.10. *Under the same hypothesis of Lemma 5.7, the solution $\mathcal{W} \in Y_h^N$ of (5.9), (5.10) is such that*

$$\sum_{n=1}^{N-1} \|W^n - W^{n+1}\|^2 \leq C\left(\boldsymbol{\mathcal{U}}\right), \tag{5.93}$$

where the constant $C\left(\boldsymbol{\mathcal{U}}\right)$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$, data problem and constant parameters, but it is independent of h, k.

Proof. The discrete state equation (5.10c) implies

$$\begin{aligned} &\left(W^n - W^{n+1}, \theta\right)_h - \varepsilon^2\left(\nabla Y^n - \nabla Y^{n+1}, \nabla\theta\right) \\ &+ \left(Y^{n-1} - Y^n, \theta\right)_h - \left(\left(Y^n\right)^3 - \left(Y^{n+1}\right)^3, \theta\right)_h = 0. \end{aligned} \tag{5.94}$$

With $\theta = W^n - W^{n+1}$ in (5.94) above, using the definition (A.36) of the discrete Laplacian $\hat{\Delta}_h$, we have

$$\|W^n - W^{n+1}\|^2 = E_1^n + E_2^n + E_3^n, \tag{5.95}$$

where

$$E_1^n = -\varepsilon^2\left(\hat{\Delta}_h Y^n - \hat{\Delta}_h Y^{n+1}, W^n - W^{n+1}\right),$$

$$E_2^n = -\left(Y^{n-1} - Y^n, W^n - W^{n+1}\right)_h,$$
$$E_3^n = \left((Y^n)^3 - \left(Y^{n+1}\right)^3, W^n - W^{n+1}\right)_h.$$

From the generalized Holder's inequality (A.14) and Young's inequality (A.13), we can write

$$\begin{aligned}E_1^n &\le \varepsilon^2 \|\hat{\Delta}_h Y^n - \hat{\Delta}_h Y^{n+1}\|_h \|W^n - W^{n+1}\|_h \\ &\le \sigma \|W^n - W^{n+1}\|^2 + C(\sigma) \|\hat{\Delta}_h Y^n - \hat{\Delta}_h Y^{n+1}\|^2,\end{aligned}$$

$$\begin{aligned}E_2^n &\le \|Y^{n-1} - Y^n\|_h \|W^n - W^{n+1}\|_h \\ &\le \sigma \|W^n - W^{n+1}\|_h^2 + C(\sigma) \|Y^{n-1} - Y^n\|_h^2,\end{aligned}$$

$$\begin{aligned}E_3^n &\le \left\| (Y^n)^3 - \left(Y^{n+1}\right)^3 \right\|_h \|W^n - W^{n+1}\|_h \\ &= \left\| \left(Y^n - Y^{n+1}\right) \left[(Y^n)^2 + Y^n Y^{n+1} + \left(Y^{n+1}\right)^2\right] \right\|_h \|W^n - W^{n+1}\|_h \\ &\le C\, \|Y^n - Y^{n+1}\|_h \left\| (Y^n)^2 + Y^n Y^{n+1} + \left(Y^{n+1}\right)^2 \right\|_{\mathcal{C}(\bar{\Omega})} \|W^n - W^{n+1}\|_h \\ &\le \sigma \|W^n - W^{n+1}\|_h^2 \\ &+ C(\sigma) \|Y^n - Y^{n+1}\|_h^2 \left\| (Y^n)^2 + Y^n Y^{n+1} + \left(Y^{n+1}\right)^2 \right\|_{\mathcal{C}(\bar{\Omega})}^2 \\ &\le \sigma \|W^n - W^{n+1}\|_h^2 \\ &+ C(\sigma) \left[2\|Y^n\|_{\mathcal{C}(\bar{\Omega})} + \|Y^n\|_{\mathcal{C}(\bar{\Omega})} \|Y^{n+1}\|_{\mathcal{C}(\bar{\Omega})} + 2\|Y^{n+1}\|_{\mathcal{C}(\bar{\Omega})}\right]^2 \|Y^n - Y^{n+1}\|_h^2.\end{aligned}$$

Then, inserting the above estimates for $E_1^n, \dots, E_3^n$ in (5.95), we derive

$$\begin{aligned}\|W^n - W^{n+1}\|^2 &\le 3\sigma \|W^n - W^{n+1}\|^2 + \\ &+ C_1(\sigma) \|\hat{\Delta}_h Y^n - \hat{\Delta}_h Y^{n+1}\|^2 + C_2(\sigma) \|Y^{n-1} - Y^n\|_h^2 \\ &+ C_3(\sigma) \left[2\|Y^n\|_{\mathcal{C}(\bar{\Omega})} + \|Y^n\|_{\mathcal{C}(\bar{\Omega})} \|Y^{n+1}\|_{\mathcal{C}(\bar{\Omega})} + 2\|Y^{n+1}\|_{\mathcal{C}(\bar{\Omega})}\right]^2 \|Y^n - Y^{n+1}\|_h^2,\end{aligned} \tag{5.96}$$

which implies, with σ small enough,

$$\begin{aligned}\|W^n - W^{n+1}\|^2 &\le C_1 \|\hat{\Delta}_h Y^n - \hat{\Delta}_h Y^{n+1}\|^2 + C_2 \|Y^{n-1} - Y^n\|_h^2 \\ &+ C_3 \left[2\|Y^n\|_{\mathcal{C}(\bar{\Omega})} + \|Y^n\|_{\mathcal{C}(\bar{\Omega})} \|Y^{n+1}\|_{\mathcal{C}(\bar{\Omega})} + 2\|Y^{n+1}\|_{\mathcal{C}(\bar{\Omega})}\right]^2 \|Y^n - Y^{n+1}\|_h^2.\end{aligned} \tag{5.97}$$

Summing up over $n = 1, \dots, N-1$ in (5.97), we infer

$$\begin{aligned}\sum_{n=1}^{N-1} \|W^n - W^{n+1}\|^2 &\le C_1 \sum_{n=1}^{N-1} \|\hat{\Delta}_h Y^n - \hat{\Delta}_h Y^{n+1}\|^2 + C_2 \sum_{n=1}^{N-1} \|Y^{n-1} - Y^n\|_h^2 \\ &+ C_3 \sum_{n=1}^{N-1} \left[2\|Y^n\|_{\mathcal{C}(\bar{\Omega})} + \|Y^n\|_{\mathcal{C}(\bar{\Omega})} \|Y^{n+1}\|_{\mathcal{C}(\bar{\Omega})} + 2\|Y^{n+1}\|_{\mathcal{C}(\bar{\Omega})}\right]^2 \|Y^n - Y^{n+1}\|_h^2.\end{aligned}$$

Hence, using the results (5.33), (5.58), (5.59) established in the previous lemmas, we realize that (5.93) holds. □

Lemma 5.11. *Let us assume that there exists a constant $\tilde{C}$ independent of h, k, such that*

$$E(\mathbf{v}_{0,h}, y_{0,h}) + \|\mathbf{v}_{0,h}\|_{\mathbf{H}_0^1} + \|\hat{\Delta}_h y_{0,h}\|_h + \|\tilde{\Delta}_h \mathbf{v}_{0,h}\| \leq \tilde{C}. \tag{5.98}$$

Then, for any fixed $\boldsymbol{\mathcal{U}} \in L^2(\mathbf{L}^2)$ and k such that

$$k \leq C_1,$$

the solution $\boldsymbol{\mathcal{V}} \in \mathbf{V}_h^{N+1}$ of (5.9), (5.10) satisfies:

$$\sup_{n=0,\dots,N} \|\mathbf{V}^n\|_{\mathbf{H}_0^1} \leq C(\boldsymbol{\mathcal{U}}), \tag{5.99}$$

$$\sum_{n=1}^{N} \|\mathbf{V}^n - \mathbf{V}^{n-1}\|_{\mathbf{H}_0^1}^2 \leq C(\boldsymbol{\mathcal{U}}), \tag{5.100}$$

$$\sum_{n=1}^{N} k\,\|d_t\mathbf{V}^n\|^2 \leq C(\boldsymbol{\mathcal{U}}), \tag{5.101}$$

$$\sum_{n=1}^{N} k\,\|\tilde{\Delta}_h\mathbf{V}^n\|^2 \leq C(\boldsymbol{\mathcal{U}}), \tag{5.102}$$

where the constant $C(\boldsymbol{\mathcal{U}})$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$, data problem and constant parameters, but it is independent of h, k.

Proof. With $\boldsymbol{\psi} = k d_t\mathbf{V}^n$ in (5.9a), (5.9c), we have

$$k\|d_t\mathbf{V}^n\|^2 + k\nu(\nabla\mathbf{V}^n, \nabla d_t\mathbf{V}^n) = A_1^n + A_2^n + A_3^n, \tag{5.103}$$

where

$$\begin{aligned} A_1^n &= -kB\left(\mathbf{V}^{n-1}, \mathbf{V}^n, d_t\mathbf{V}^n\right), \\ A_2^n &= -k\rho\left(Y^{n-1}, \nabla W^n \cdot d_t\mathbf{V}^n\right), \\ A_3^n &= k\left(\mathbf{U}^n, d_t\mathbf{V}^n\right). \end{aligned}$$

From the definition (5.7) of the trilinear form $B(\cdot,\cdot,\cdot)$ and performing integration by parts in space, we get

$$\begin{aligned} A_1^n = \frac{k}{2}\left(\left[\mathbf{V}^{n-1}\cdot\nabla\right]\mathbf{V}^n, d_t\mathbf{V}^n\right) - \frac{k}{2}\left(\left[\mathbf{V}^{n-1}\cdot\nabla\right]d_t\mathbf{V}^n, \mathbf{V}^n\right) = \\ = k\left(\left[\mathbf{V}^{n-1}\cdot\nabla\right]\mathbf{V}^n, d_t\mathbf{V}^n\right) + \frac{k}{2}\left(\left[\nabla\cdot\mathbf{V}^{n-1}\right]d_t\mathbf{V}^n, \mathbf{V}^n\right). \end{aligned} \tag{5.104}$$

Using in (5.104) generalized Holder's inequality (A.14), Young's inequality (A.13), Poincaré's inequality (A.16), inequality (A.18) and the discrete interpolation inequality (A.54), we can write

$$\begin{aligned} A_1^n \leq\ & k\|\mathbf{V}^{n-1}\|_{\mathbf{L}^4}\|\nabla\mathbf{V}^{n-1}\|_{\mathbf{L}^4}\|d_t\mathbf{V}^n\| + \frac{k}{2}\|\nabla\cdot\mathbf{V}^{n-1}\|_{\mathbf{L}^4}\|d_t\mathbf{V}^n\|\|\mathbf{V}^n\|_{\mathbf{L}^4} \\ &\leq 2k\sigma\|d_t\mathbf{V}^n\|^2 + k\,C(\sigma)\left[\|\nabla\mathbf{V}^n\|_{\mathbf{L}^4}^2\|\mathbf{V}^{n-1}\|_{\mathbf{L}^4}^2 + \|\nabla\mathbf{V}^{n-1}\|_{\mathbf{L}^4}^2\|\mathbf{V}^n\|_{\mathbf{L}^4}^2\right] \end{aligned}$$

$$\leq 2k\sigma\|d_t\mathbf{V}^n\|^2 + k\ C\,(\sigma)\,\|\nabla\mathbf{V}^{n-1}\|\|\nabla\mathbf{V}^n\|\left[\|\mathbf{V}^{n-1}\|\|\tilde{\Delta}_h\mathbf{V}^n\| + \|\mathbf{V}^n\|\|\tilde{\Delta}_h\mathbf{V}^{n-1}\|\right],$$

which implies, still applying Young's inequality (A.13), with $\mu = \sigma$,

$$A_1^n \leq 2k\sigma\|d_t\mathbf{V}^n\|^2 + k\mu\left[\|\tilde{\Delta}_h\mathbf{V}^{n-1}\|^2 + \|\tilde{\Delta}_h\mathbf{V}^n\|^2\right] + k\ C\,(\sigma,\mu)\left[\|\mathbf{V}^{n-1}\|^2 + \|\mathbf{V}^n\|^2\right]\|\nabla\mathbf{V}^{n-1}\|^2\|\nabla\mathbf{V}^n\|^2. \tag{5.105}$$

In the same way, we derive

$$A_2^n \leq k\rho\|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}\|\nabla W^n\|\|d_t\mathbf{V}^n\| \leq k\sigma\|d_t\mathbf{V}^n\|^2 + kC\,(\sigma)\,\|Y^{n-1}\|^2_{\mathcal{C}(\bar{\Omega})}\|\nabla W^n\|^2 \tag{5.106}$$

and

$$A_3^n \leq k\sigma\|d_t\mathbf{V}^n\|^2 + kC\,(\sigma)\,\|\mathbf{U}^n\|^2. \tag{5.107}$$

Using (5.105)-(5.107) in (5.103)-, we infer

$$\begin{aligned} k\|d_t\mathbf{V}^n\|^2 + k\nu\,(\nabla\mathbf{V}^n, \nabla d_t\mathbf{V}^n) &\leq 4k\sigma\|d_t\mathbf{V}^n\|^2 + k\mu\left[\|\tilde{\Delta}_h\mathbf{V}^{n-1}\|^2 + \|\tilde{\Delta}_h\mathbf{V}^n\|^2\right] \\ &+ k\ C_1\,(\sigma,\mu)\left[\|\mathbf{V}^{n-1}\|^2 + \|\mathbf{V}^n\|^2\right]\|\nabla\mathbf{V}^{n-1}\|^2\|\nabla\mathbf{V}^n\|^2 \\ &+ k\ C_2\,(\sigma)\,\|Y^{n-1}\|^2_{\mathcal{C}(\bar{\Omega})}\|\nabla W^n\|^2 + k\ C_3\,(\sigma)\,\|\mathbf{U}^n\|^2. \end{aligned} \tag{5.108}$$

Setting $n = i$ in (5.108), summing up over the index $i = 1, \ldots, n$, with $1 \leq n \leq N$ and rearranging, we realize

$$\begin{aligned} &\frac{\nu}{2}\|\nabla\mathbf{V}^n\|^2 + \sum_{i=1}^{n}\left[k\|d_t\mathbf{V}^i\|^2 + \frac{\nu}{2}\|\nabla\mathbf{V}^i - \nabla\mathbf{V}^{i-1}\|^2\right] \\ &\leq \frac{\nu}{2}\|\nabla\mathbf{V}^0\|^2 + 4\sigma\sum_{i=1}^{n}k\|d_t\mathbf{V}^i\|^2 + \mu\sum_{i=1}^{n}k\left[\|\tilde{\Delta}_h\mathbf{V}^{i-1}\|^2 + \|\tilde{\Delta}_h\mathbf{V}^i\|^2\right] \\ &+ C_1\,(\sigma,\mu)\sum_{i=1}^{n}k\left[\|\mathbf{V}^{i-1}\|^2 + \|\mathbf{V}^i\|^2\right]\|\nabla\mathbf{V}^{i-1}\|^2\|\nabla\mathbf{V}^i\|^2 \\ &+ C_2\,(\sigma)\sum_{i=1}^{n}k\left[\|Y^{i-1}\|^2_{\mathcal{C}(\bar{\Omega})}\|\nabla W^i\|^2 + \|\mathbf{U}^i\|^2\right]. \end{aligned} \tag{5.109}$$

Noting that there exists a constant C such that

$$\sum_{i=1}^{n}k\|\tilde{\Delta}_h\mathbf{V}^{i-1}\|^2 \leq C\|\tilde{\Delta}_h\mathbf{V}^0\|^2 + \sum_{i=1}^{n}k\|\tilde{\Delta}_h\mathbf{V}^i\|^2, \tag{5.110}$$

from (5.109) we have

$$\frac{\nu}{2}\|\nabla\mathbf{V}^n\|^2 + \sum_{i=1}^{n}\left[k\|d_t\mathbf{V}^i\|^2 + \frac{\nu}{2}\|\nabla\mathbf{V}^i - \nabla\mathbf{V}^{i-1}\|^2\right] \tag{5.111}$$

$$
\begin{aligned}
&\le \frac{\nu}{2}\|\nabla \mathbf{v}_{0,h}\|^2 + \mu C\|\tilde{\Delta}_h \mathbf{v}_{0,h}\|^2 + 4\sigma \sum_{i=1}^{n} k\|d_t \mathbf{V}^i\|^2 + 2\mu \sum_{i=1}^{n} k\|\tilde{\Delta}_h \mathbf{V}^i\|^2 \\
&\qquad + C_1(\sigma,\mu)\sum_{i=1}^{n} k\left[\|\mathbf{V}^{i-1}\|^2 + \|\mathbf{V}^i\|^2\right]\|\nabla \mathbf{V}^{i-1}\|^2\|\nabla \mathbf{V}^i\|^2 \\
&\qquad + C_2(\sigma)\sum_{i=1}^{n} k\left[\|Y^{i-1}\|^2_{\mathcal{C}(\bar{\Omega})}\|\nabla W^i\|^2 + \|\mathbf{U}^i\|^2\right].
\end{aligned}
$$

for all $n = 1,\dots,N$. With $\boldsymbol{\psi} = k\mathbf{A}^h\mathbf{V}^n$ in (5.9a), (5.9c), where $\mathbf{A}^h$ is the discrete Stokes operator defined in (A.40), we get

$$
k\nu\left(\nabla \mathbf{V}^n, \nabla \mathbf{A}^h\mathbf{V}^n\right) = D_1^n + D_2^n + D_3^n + D_4^n, \tag{5.112}
$$

where

$$
\begin{aligned}
D_1^n &= -k\left(d_t\mathbf{V}^n, \mathbf{A}^h\mathbf{V}^n\right),\\
D_2^n &= -kB\left(\mathbf{V}^{n-1}, \mathbf{V}^n, \mathbf{A}^h\mathbf{V}^n\right),\\
D_3^n &= -k\rho\left(Y^{n-1}, \nabla W^n \cdot \mathbf{A}^h\mathbf{V}^n\right),\\
D_4^n &= k\left(\mathbf{U}^n, \mathbf{A}^h\mathbf{V}^n\right).
\end{aligned}
$$

Using the definition (A.40) of the discrete Stokes operator, we note that the left hand side of (5.112) reads

$$
\begin{aligned}
k\nu\left(\nabla \mathbf{V}^n, \nabla \mathbf{A}^h\mathbf{V}^n\right) &= k\nu\left(\nabla \mathbf{V}^n, \nabla\left[-\mathbf{T}^h\tilde{\Delta}_h\mathbf{V}^n\right]\right) = k\nu\left(\tilde{\Delta}_h\mathbf{V}^n, \mathbf{T}^h\tilde{\Delta}_h\mathbf{V}^n\right)\\
&= k\nu\left(\mathbf{T}^h\tilde{\Delta}_h\mathbf{V}^n, \mathbf{T}^h\tilde{\Delta}_h\mathbf{V}^n\right) = \|\mathbf{T}^h\tilde{\Delta}_h\mathbf{V}^n\|^2 = \|\mathbf{A}^h\mathbf{V}^n\|^2.
\end{aligned} \tag{5.113}
$$

Furthermore (see [6]), there exists a constant C such that

$$
C\|\tilde{\Delta}_h\mathbf{V}^n\| \le \|\mathbf{A}^h\mathbf{V}^n\| \le \|\tilde{\Delta}_h\mathbf{V}^n\| \tag{5.114}
$$

Hence, taking into account of (5.113) and (5.114) in (5.112), we can write

$$
k\ \nu\ C\|\tilde{\Delta}_h\mathbf{V}^n\|^2 \le D_1^n + D_2^n + D_3^n + D_4^n. \tag{5.115}
$$

Using Young's inequality (A.13), (5.114) above, integration by parts in space, generalized Holder's inequality (A.14), Poincaré's inequality (A.16), inequality (A.18) and discrete interpolation inequality (A.54), we derive

$$
D_1^n \le k\sigma\|\tilde{\Delta}_h\mathbf{V}^n\| + k\ C(\sigma)\|d_t\mathbf{V}^n\|^2,
$$

$$
\begin{aligned}
D_2^n &= -k\left(\left[\mathbf{V}^{n-1}\cdot\nabla\right]\mathbf{V}^n, \mathbf{A}^h\mathbf{V}^n\right) - \frac{k}{2}\left(\left[\nabla\cdot\mathbf{V}^{n-1}\right]\mathbf{V}^n, \mathbf{A}^h\mathbf{V}^n\right)\\
&\le k\|\mathbf{V}^{n-1}\|_{\mathbf{L}^4}\|\nabla\mathbf{V}^n\|_{\mathbf{L}^4}\|\tilde{\Delta}_h\mathbf{V}^n\| + \frac{kC}{2}\|\nabla\mathbf{V}^{n-1}\|_{\mathbf{L}^4}\|\mathbf{V}^n\|_{\mathbf{L}^4}\|\tilde{\Delta}_h\mathbf{V}^n\|,\\
&\le 2k\sigma\|\tilde{\Delta}_h\mathbf{V}^n\|^2 + k\ C(\sigma)\left[\|\mathbf{V}^{n-1}\|^2_{\mathbf{L}^4}\|\nabla\mathbf{V}^n\|^2_{\mathbf{L}^4} + \|\mathbf{V}^n\|^2_{\mathbf{L}^4}\|\nabla\mathbf{V}^{n-1}\|^2_{\mathbf{L}^4}\right]\\
&\le 2k\sigma\|\tilde{\Delta}_h\mathbf{V}^n\|^2 + k\ C(\sigma)\left[\|\mathbf{V}^{n-1}\|\|\nabla\mathbf{V}^{n-1}\|\|\nabla\mathbf{V}^n\|\|\tilde{\Delta}_h\mathbf{V}^n\|\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ k\ C\,(\sigma)\left[\|\mathbf{V}^n\|\|\nabla\mathbf{V}^n\|\|\nabla\mathbf{V}^{n-1}\|\|\tilde{\Delta}_h\mathbf{V}^{n-1}\|\right] \\
&\leq 3k\sigma\|\tilde{\Delta}_h\mathbf{V}^n\|^2 + k\sigma\|\tilde{\Delta}_h\mathbf{V}^{n-1}\| \\
&+ k\ C\,(\sigma)\left[\|\mathbf{V}^{n-1}\|^2 + \|\mathbf{V}^n\|^2\right]\|\nabla\mathbf{V}^{n-1}\|^2\|\nabla\mathbf{V}^n\|^2,
\end{aligned}
$$

$$
\begin{aligned}
D_3^n &\leq k\rho\|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}\|\nabla W^n\|\|\tilde{\Delta}_h\mathbf{V}^n\| \\
&\leq k\sigma\|\tilde{\Delta}_h\mathbf{V}^n\|^2 + kC\,(\sigma)\,\|Y^{n-1}\|^2_{\mathcal{C}(\bar{\Omega})}\|\nabla W^n\|^2,
\end{aligned}
$$

$$
D_4^n \leq k\sigma\|\tilde{\Delta}_h\mathbf{V}^n\|^2 + kC\,(\sigma)\,\|\mathbf{U}^n\|^2.
$$

Hence, inserting the estimates for $D_1^n, \ldots, D_4^n$ in (5.115), we infer

$$
\begin{aligned}
k\ \nu\ C\|\tilde{\Delta}_h\mathbf{V}^n\|^2 & \\
\leq 6k\sigma\|\tilde{\Delta}_h\mathbf{V}^n\|^2 + k\sigma\|\tilde{\Delta}_h\mathbf{V}^{n-1}\| + k\ C_1\,(\sigma)\,\|d_t\mathbf{V}^n\|^2 & \\
+k\ C_2\,(\sigma)\left[\|\mathbf{V}^{n-1}\|^2 + \|\mathbf{V}^n\|^2\right]\|\nabla\mathbf{V}^{n-1}\|^2\|\nabla\mathbf{V}^n\|^2 & \\
+k\ C_3\,(\sigma)\,\|Y^{n-1}\|^2_{\mathcal{C}(\bar{\Omega})}\|\nabla W^n\|^2 + k\ C_4\,(\sigma)\,\|\mathbf{U}^n\|^2. &
\end{aligned} \tag{5.116}
$$

Setting $n = i$ in (5.116), summing up over the index $i = 1, \ldots, n$, with $1 \leq n \leq N$ and rearranging, we realize

$$
\begin{aligned}
\nu\ C\sum_{i=1}^n k\|\tilde{\Delta}_h\mathbf{V}^i\|^2 \leq & \\
\leq \sigma C_1\|\tilde{\Delta}_h\mathbf{v}_{0,h}\|^2 + 7\sigma\sum_{i=1}^n k\|\tilde{\Delta}_h\mathbf{V}^i\|^2 + & \\
+C_2\,(\sigma)\sum_{i=1}^n k\left[\|Y^{i-1}\|^2_{\mathcal{C}(\bar{\Omega})}\|\nabla W^i\|^2 + \|\mathbf{U}^i\|^2\right] + & \\
+C_3\,(\sigma)\sum_{i=1}^n k\left[\|d_t\mathbf{V}^i\|^2 + \left(\|\mathbf{V}^{i-1}\|^2 + \|\mathbf{V}^i\|^2\right)\|\nabla\mathbf{V}^{i-1}\|^2\|\nabla\mathbf{V}^i\|^2\right], &
\end{aligned} \tag{5.117}
$$

which implies, with σ small enough,

$$
\begin{aligned}
\sum_{i=1}^n k\|\tilde{\Delta}_h\mathbf{V}^i\|^2 \leq & \\
\leq C_1\|\tilde{\Delta}_h\mathbf{v}_{0,h}\|^2 + C_2\sum_{i=1}^n k\left[\|Y^{i-1}\|^2_{\mathcal{C}(\bar{\Omega})}\|\nabla W^i\|^2 + \|\mathbf{U}^i\|^2\right] + & \\
+C_3\sum_{i=1}^n k\left[\|d_t\mathbf{V}^i\|^2 + \left(\|\mathbf{V}^{i-1}\|^2 + \|\mathbf{V}^i\|^2\right)\|\nabla\mathbf{V}^{i-1}\|^2\|\nabla\mathbf{V}^i\|^2\right], &
\end{aligned} \tag{5.118}
$$

for all $n = 1, \ldots, N$. Inserting (5.118) in (5.111) and rearranging, we have

$$
\frac{\nu}{2}\|\nabla\mathbf{V}^n\|^2 + \sum_{i=1}^n\left[k\|d_t\mathbf{V}^i\|^2 + \frac{\nu}{2}\|\nabla\mathbf{V}^i - \nabla\mathbf{V}^{i-1}\|^2\right] \leq \tag{5.119}
$$

$$\leq \frac{\nu}{2}\|\nabla \mathbf{v}_{0,h}\|^2 + \mu C_1 \|\tilde{\Delta}_h \mathbf{v}_{0,h}\|^2 + (\sigma+\mu)\, C_2 \sum_{i=1}^{n} k\|d_t \mathbf{V}^i\|^2$$
$$+ (1+\mu)\, C_3\,(\sigma,\mu) \sum_{i=1}^{n} k\left[\|\mathbf{V}^{i-1}\|^2 + \|\mathbf{V}^i\|^2\right] \|\nabla \mathbf{V}^{i-1}\|^2 \|\nabla \mathbf{V}^i\|^2$$
$$+ (1+\mu)\, C_4\,(\sigma) \sum_{i=1}^{n} k\left[\|Y^{i-1}\|^2_{\mathcal{C}(\bar{\Omega})} \|\nabla W^i\|^2 + \|\mathbf{U}^i\|^2\right].$$

Hence, assuming in (5.119) σ, μ small enough, we get

$$\frac{\nu}{2}\|\nabla \mathbf{V}^n\|^2 + \sum_{i=1}^{n}\left[k\|d_t \mathbf{V}^i\|^2 + \frac{\nu}{2}\|\nabla \mathbf{V}^i - \nabla \mathbf{V}^{i-1}\|^2\right] \tag{5.120}$$
$$\leq \frac{\nu}{2}\|\nabla \mathbf{v}_{0,h}\|^2 + C_1\|\tilde{\Delta}_h \mathbf{v}_{0,h}\|^2 + C_2 \sum_{i=1}^{n} k\left[\|Y^{i-1}\|^2_{\mathcal{C}(\bar{\Omega})}\|\nabla W^i\|^2 + \|\mathbf{U}^i\|^2\right]$$
$$+ C_3 \sum_{i=1}^{n} k\left[\|\mathbf{V}^{i-1}\|^2 + \|\mathbf{V}^i\|^2\right]\|\nabla \mathbf{V}^{i-1}\|^2\|\nabla \mathbf{V}^i\|^2.$$

So, using the assumption (5.98), the results (5.29), (5.30), (5.34) (5.59) established in the previous lemmas and the discrete Gronwall's inequality (see for example [73], Lemma 1.4.2), we conclude that (5.99)-(5.101) hold. Finally, from (5.118), we derive that (5.102) holds. ☐

Corollary 5.12. *Under the same hypothesis of Lemma 5.11, the solution $\boldsymbol{\mathcal{V}} \in \mathbf{V}_h^{N+1}$ of (5.9), (5.10) is such that*

$$\sum_{n=0}^{N} k\|\mathbf{V}^i\|^2_{\boldsymbol{\mathcal{C}}(\bar{\Omega})} \leq C\,(\boldsymbol{\mathcal{U}}), \tag{5.121}$$

where the constant $C\,(\boldsymbol{\mathcal{U}})$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$, data problem and constant parameters, but it is independent of h, k.

Proof. Using Young's inequality (A.13), Poincaré's inequality (A.16), inequalities (A.17), (A.54) and the embedding $\mathbf{W}^{1,4} \hookrightarrow \boldsymbol{\mathcal{C}}\left(\bar{\Omega}\right)$ we have,

$$\begin{aligned}\sum_{i=0}^{n} k\|\mathbf{V}^i\|^2_{\boldsymbol{\mathcal{C}}(\bar{\Omega})} &\leq C\sum_{i=0}^{n} k\left[\|\mathbf{V}^i\|^2_{\mathbf{L}^4} + \|\nabla \mathbf{V}^i\|^2_{\mathbf{L}^4}\right] \leq \\ &\leq C\sum_{i=0}^{n} k\left[\|\mathbf{V}^i\|^2_{\mathbf{H}_0^1} + \|\nabla \mathbf{V}^i\|\|\tilde{\Delta}_h \mathbf{V}^i\|\right] \leq \\ &\leq C\sum_{i=0}^{n} k\left[\|\mathbf{V}^i\|^2_{\mathbf{H}_0^1} + \|\tilde{\Delta}_h \mathbf{V}^i\|^2\right].\end{aligned}$$

Hence, by the results (5.30), (5.102) established the previous lemmas, we get (5.121). ☐

5.3. Well-Posedness of the Discrete Optimal Control Problem

Problem 5.1 has the form of an abstract optimal control problem where the cost functional $J_{h,k} : X_{h,k} \times (\mathbf{L}^2)^N \to \mathbb{R}$ defined in (5.4) is continuous, convex and bounded from below, i.e. *weakly lower semicontinuous*. Then, it is easy to get the following result.

Theorem 5.13 (existence of minimizers). *The discrete optimal control problem (5.1) admits a solution.*

Proof. The proof is analogous to the one of Theorem 2.5 in Chapter 2. □

5.4. Optimality Conditions for the Discrete Optimal Control Problem

In the following, we show that the regularized Problem 5.1 satisfies the conditions needed to apply the standard theory of mathematical programming in Banach spaces (see Assumptions 1.47 in [58]). Then, we derive the first order optimality conditions (see Theorem 1.48 and Corollary 1.3 in [58]).
We need to verify that the discrete optimal control Problem 5.1 is such that

- the cost functional $J_{h,k} : \mathbf{X}_{h,k} \times L^2(\mathbf{L}^2) \to \mathbb{R}$ is continuously differentiable;
- the map $e_{h,k} : \mathbf{X}_{h,k} \times L^2(\mathbf{L}^2) \to \mathbf{X}_{h,k}$ defined in (5.6) is continuously differentiable;
- the map $\frac{\partial e_{\delta,h,k}}{\partial \boldsymbol{\mathcal{X}}}(s_{h,k}(\boldsymbol{\mathcal{U}}), \boldsymbol{\mathcal{U}})$ has an inverse, where $s_{h,k} : L^2(\mathbf{L}^2) \to \mathbf{X}_{h,k}$ is the solution operator defined in (5.12).

It is straightforward to realize that two conditions above are verified. So, we skip the corresponding proofs. In the following Theorem 5.14, we prove that also the last condition holds.

Theorem 5.14. *For any fixed h, k and $\boldsymbol{\mathcal{U}} \in L^2(\mathbf{L}^2)$, the operator*

$$\frac{\partial e_{h,k}}{\partial \boldsymbol{\mathcal{X}}}(s_{h,k}(\boldsymbol{\mathcal{U}}), \boldsymbol{\mathcal{U}}) \in \mathcal{L}(\mathbf{X}_{h,k}, \mathbf{X}_{h,k})$$

is invertible.

Proof. We need to prove that for all $\boldsymbol{\mathcal{Z}} \in \mathbf{X}_{h,k}$ there exists a unique $\mathbf{d}_{\boldsymbol{\mathcal{X}}} \in \mathbf{X}_{h,k}$ such that

$$\frac{\partial e_{h,k}}{\partial \boldsymbol{\mathcal{X}}}(s_{h,k}(\boldsymbol{\mathcal{U}}), \boldsymbol{\mathcal{U}})\, \mathbf{d}_{\boldsymbol{\mathcal{X}}} = \boldsymbol{\mathcal{Z}}. \tag{5.122}$$

Equation (5.122) is equivalent to demonstrate that, given $(\mathbf{Z}_{\mathbf{V}}, Z_P, Z_Y, Z_W) \in \mathbf{V}_h^{N+1} \times P_h^N \times P_h^{N+1} \times Y_h^N$ and $(\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}) \in \mathbf{V}_h^{N+1} \times P_h^N \times P_h^{N+1} \times Y_h^N$ solution of the state equations (5.9)-(5.10), the following system of equations

$$\left(\mathbf{d}_{\mathbf{V}}^n - \mathbf{d}_{\mathbf{V}}^{n-1}, \boldsymbol{\psi}\right) + k\nu\left(\nabla \mathbf{d}_{\mathbf{V}}^n, \nabla \boldsymbol{\psi}\right) - k\left(d_P^n, \nabla \cdot \boldsymbol{\psi}\right) \tag{5.123}$$

$$+kB\left(\mathbf{d}_{\mathbf{V}}^{n-1}, \mathbf{V}^n, \boldsymbol{\psi}\right) + kB\left(\mathbf{V}^{n-1}, \mathbf{d}_{\mathbf{V}}^n, \boldsymbol{\psi}\right)$$
$$+ k\rho\left(d_Y^{n-1}, \nabla W^n \cdot \boldsymbol{\psi}\right) + k\rho\left(Y^{n-1}, \nabla d_W^n \cdot \boldsymbol{\psi}\right) = \left(\mathbf{Z}_{\mathbf{V}}^n, \boldsymbol{\psi}\right),$$

(5.124) $$\mathbf{d}_{\mathbf{V}}^0 = \mathbf{Z}_{\mathbf{V}}^0,$$

(5.125) $$\left(\nabla \cdot \mathbf{d}_{\mathbf{V}}^n, \phi\right) = \left(Z_P^n, \phi\right),$$

(5.126) $$\left(d_Y^n - d_Y^{n-1}, \eta\right)_h + k\gamma\left(\nabla d_W^n, \nabla \eta\right)$$
$$-k\left(d_Y^{n-1}\mathbf{V}^{n-1} + Y^{n-1}\mathbf{d}_{\mathbf{V}}^{n-1}, \nabla \eta\right) = \left(Z_Y^n, \eta\right)_h,$$

(5.127) $$d_Y^0 = Z_Y^0,$$

(5.128) $$\left(d_W^n + d_Y^{n-1}, \theta\right)_h - \varepsilon^2\left(\nabla d_Y^n, \nabla \theta\right) - \left(3\left(Y^n\right)^2 d_Y^n, \theta\right)_h = \left(Z_W^n, \theta\right)_h,$$

with $n = 1, \ldots, N$, has a unique solution $(\mathbf{d}_{\mathbf{V}}, d_P, d_Y, d_W) \in \mathbf{V}_h^{N+1} \times P_h^N \times P_h^{N+1} \times Y_h^N$. At each time level n, we can show the existence and the uniqueness of the solution for (5.126), (5.127), (5.128) using exactly the procedure performed in the proof of Theorem 3.10. The only difference is that, in this case, the elements of the matrix E are the following

$$E_{ij} = \left(3\left(Y^n\right)^3 \eta_j, \eta_i\right)_h + \varepsilon^2\ A_{i,j},$$

Finally, given $d_W^n \in Y_h$, using standard arguments, we can claim that (5.123), (5.124), (5.125) have a unique solution $(\mathbf{d}_{\mathbf{V}}^n, d_P^n) \in \mathbf{V}_h \times P_h$. ☐

The continuous differentiability of the maps $J_{h,k} : \mathbf{X}_{h,k} \times L^2(\mathbf{L}^2) \to \mathbb{R}$, $e_{h,k} : \mathbf{X}_{h,k} \times L^2(\mathbf{L}^2) \to \mathbf{X}_{h,k}$ and Theorem 5.14 guarantee that all the solutions of the optimal control Problem 5.1 can be derived solving a set of first order optimality conditions (see Theorem 1.48 and Corollary 1.3 in [58]). In order to get these equations, for any fixed h, k, we define the discrete Lagrange functional $L_{h,k} : \mathbf{X}_{h,k} \times L^2(\mathbf{L}^2) \times \mathbf{X}_{h,k} \to \mathbb{R}$,

(5.129) $$L_{h,k}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}) = J_{h,k}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}}) + \langle \boldsymbol{\mathcal{Q}}, e_{h,k}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}})\rangle_{\mathbf{X}_{h,k}^*, \mathbf{X}_{h,k}},$$

where

$$\boldsymbol{\mathcal{Q}} = (\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}}, \mathcal{Q}_{\mathcal{P}}, \mathcal{Q}_{\mathcal{Y}}, \mathcal{Q}_{\mathcal{W}}) \in \mathbf{X}_{h,k}.$$

The first order optimality conditions of the discrete optimal control Problem 5.1 correspond to *find* $(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}) \in \mathbf{X}_{h,k} \times L^2(\mathbf{L}^2) \times \mathbf{X}_{h,k}$ *such that*

(5.130) $$\frac{\partial L_{h,k}}{\partial \boldsymbol{\mathcal{Q}}}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}) = 0,$$

(5.131) $$\frac{\partial L_{h,k}}{\partial \boldsymbol{\mathcal{X}}}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}) = 0,$$

(5.132) $$\frac{\partial L_{h,k}}{\partial \boldsymbol{\mathcal{U}}}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}) = 0.$$

Equation (5.130) corresponds to the discrete state equations $e_{h,k}(\mathcal{X}, \boldsymbol{\mathcal{U}}) = 0$ of Problem 5.1, (5.131) are the *discrete adjoint equations* and (5.132) is another *optimality relation.*
In the next Lemma 5.15, we prove that given a solution $\boldsymbol{\mathcal{X}} = s_{h,k}(\boldsymbol{\mathcal{U}})$ of the discrete state equations (5.130), the discrete adjoint equations (5.131) have a unique solution $\boldsymbol{\mathcal{Q}} \in \mathbf{X}_{h,k}$.

Lemma 5.15. *Let $h, k, \boldsymbol{\mathcal{U}} \in L^2(\mathbf{L}^2)$ and $\boldsymbol{\mathcal{X}} = s_{h,k}(\boldsymbol{\mathcal{U}}) \in \mathbf{X}_{h,k}$ be given. Then, the discrete adjoint equations* (5.131) *have a unique solution $\boldsymbol{\mathcal{Q}} \in \mathbf{X}_{h,k}$.*

Proof. As a consequence of Theorem 5.14 above, we have

$$\left[\frac{\partial e_{h,k}}{\partial \boldsymbol{\mathcal{X}}}(s_{h,k}(\boldsymbol{\mathcal{U}}), \boldsymbol{\mathcal{U}})\right]^{-1} \in \mathcal{L}(\mathbf{X}_{h,k}, \mathbf{X}_{h,k}).$$

So, the proof is analogous to the one of Lemma 2.8 in Chapter 2. □

In the following Corollary 5.16, we derive the explicit form of the optimality conditions (5.130)-(5.132) in terms of the state and the adjoint variables

$$\begin{aligned} (\boldsymbol{\mathcal{V}}, \mathcal{P}, \mathcal{Y}, \mathcal{W}) &= \boldsymbol{\mathcal{X}}, \\ (\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}}, \mathcal{Q}_{\mathcal{P}}, \mathcal{Q}_{\mathcal{Y}}, \mathcal{Q}_{\mathcal{W}}) &= \boldsymbol{\mathcal{Q}}. \end{aligned}$$

Corollary 5.16 (**optimality conditions**). *The first order optimality conditions (5.130)-(5.132) of the discrete optimal control Problem 5.1 read as follows. For all $n = 1, \ldots, N$:*

$$(d_t \mathbf{V}^n, \boldsymbol{\psi}) + \nu(\nabla \mathbf{V}^n, \nabla \boldsymbol{\psi}) + B(\mathbf{V}^{n-1}, \mathbf{V}^n, \boldsymbol{\psi}) - (P^n, \nabla \cdot \boldsymbol{\psi}) + \rho(Y^{n-1}, \nabla W^n \cdot \boldsymbol{\psi}) - (\mathbf{U}^n, \boldsymbol{\psi}) = 0, \tag{5.133a}$$

$$\mathbf{V}^0 = \mathbf{v}_{0,h}, \tag{5.133b}$$

$$(\nabla \cdot \mathbf{V}^n, \phi) = 0, \tag{5.133c}$$

$$(d_t Y^n, \eta)_h + \gamma(\nabla W^n, \nabla \eta) - (Y^{n-1}\mathbf{V}^{n-1}, \nabla \eta) = 0, \tag{5.133d}$$

$$Y^0 = y_{0,h}, \tag{5.133e}$$

$$(W^n, \theta)_h - \varepsilon^2(\nabla Y^n, \nabla \theta) + (Y^{n-1}, \theta)_h - ((Y^n)^3, \theta)_h = 0, \tag{5.133f}$$

for all $\boldsymbol{\psi} \in \mathbf{V}_h$, $\phi \in P_h$, $\eta, \theta \in Y_h$,

$$-(d_t \mathbf{Q}_{\mathbf{V}}^n, \boldsymbol{\psi}) + \nu(\nabla \mathbf{Q}_{\mathbf{V}}^{n-1}, \nabla \boldsymbol{\psi}) + (Q_P^{n-1}, \nabla \cdot \boldsymbol{\psi}) + B(\boldsymbol{\psi}, \mathbf{V}^{n+1}, \mathbf{Q}_{\mathbf{V}}^n) + B(\mathbf{V}^{n-1}, \boldsymbol{\psi}, \mathbf{Q}_{\mathbf{V}}^{n-1}) - (Y^n, \nabla Q_Y^n \cdot \boldsymbol{\psi}) = 0, \tag{5.134a}$$

$$\mathbf{Q}_{\mathbf{V}}^N = 0, \tag{5.134b}$$

$$(\nabla \cdot \mathbf{Q}_{\mathbf{V}}^{n-1}, \phi) = 0, \tag{5.134c}$$

$$-(d_t Q_Y^n, \eta)_h - \varepsilon^2(\nabla Q_W^{n-1}, \nabla \eta) + (Q_W^n, \eta)_h - (\nabla Q_Y^n \cdot \mathbf{V}^n, \eta) + \rho(\nabla W^{n+1} \cdot \mathbf{Q}_{\mathbf{V}}^n, \eta) - (3(Y^n)^2 Q_W^{n-1}, \eta)_h + (Y^n - y_{d,h}^n, \eta) = 0, \tag{5.134d}$$

$$Q_Y^N = 0, \tag{5.134e}$$

$$Q_W^N = 0, \tag{5.134f}$$

$$(Q_W^{n-1}, \theta)_h + \gamma(\nabla Q_Y^{n-1}, \nabla \theta) + \rho(Y^{n-1}, \mathbf{Q}_{\mathbf{V}}^{n-1} \cdot \nabla \theta) = 0. \tag{5.134g}$$

for all $\boldsymbol{\psi} \in \mathbf{V}_h$, $\phi \in P_h$, $\eta \in P_h, \theta \in Y_h$,

$$\alpha \mathbf{U}^n - \mathbf{Q}_{\mathbf{V}}^{n-1} = 0. \tag{5.135}$$

Proof. Equations (5.133b)-(5.133e) and (5.134) can be derived by direct calculation from, respectively, (5.130) and (5.131). The optimality relation (3.71) implies

$$\sum_{i=1}^{N}\int_{t_{n-1}}^{t_n}\left(\alpha\boldsymbol{\mathcal{U}}-\mathbf{Q}_{\mathbf{V}}^{n-1},\boldsymbol{\varphi}\right)\,dt=0,$$

for all $\boldsymbol{\varphi}\in L^2\left(\mathbf{L}^2\right)$. Then, we have $\boldsymbol{\mathcal{U}}\in\mathbf{V}_h^N$,

$$\boldsymbol{\mathcal{U}}\left(t\right)=\mathbf{U}^n\in\mathbf{V}_h,\quad\forall t\in\left(t_{n-1},t_n\right),$$

and also (5.135) and (5.133a). □

Remark 5.17. From (5.134g), we realize that $Q_W^n\in P_h$, for all $n=0,\ldots,N-1$.

Later in the document, we prove that the solutions of the discrete optimality conditions (5.133)-(5.135) above, converge to the solution of the continuous optimality conditions (4.22)-(4.24) of Problem 4.1 as the discretization parameter go to zero. In order to do that, in the following lemmas, we derive (h,k)-independent stability estimates for the adjoint variables $(\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}},\mathcal{Q}_{\mathcal{P}},\mathcal{Q}_{\mathcal{Y}},\mathcal{Q}_{\mathcal{W}})\in\mathbf{V}_h^{N+1}\times P_h^N\times P_h^{N+1}\times Y_h^N$.

Theorem 5.18. *Let us assume there exists a constant $\tilde{C}$ independent of h,k, such that*

$$E\left(\mathbf{v}_{0,h},y_{0,h}\right)+\|\mathbf{v}_{0,h}\|_{\mathbf{H}_0^1}+\|\hat{\Delta}_h y_{0,h}\|_h+\|\tilde{\Delta}_h\mathbf{v}_{0,h}\|+\sum_{n=1}^{N}k\|y_{d,h}^n\|^2\le\tilde{C}.$$

Then, there exist a time step k_{max} such that for all $k\le k_{max}$, if $(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{Q}})\in\boldsymbol{\mathcal{X}}_{h,k}\times L^2\left(\mathbf{L}^2\right)\times\boldsymbol{\mathcal{X}}_{h,k}$ is a solution of the optimality conditions (5.133)-(5.135),

$$\sup_{n=1,\ldots,N}\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|_{\mathbf{H}_0^1}\le C\left(\boldsymbol{\mathcal{U}}\right),\tag{5.136}$$

$$\sum_{n=1}^{N}k\|d_t\mathbf{Q}_{\mathbf{V}}^n\|^2\le C\left(\boldsymbol{\mathcal{U}}\right),\tag{5.137}$$

$$\sum_{n=1}^{N}\|\mathbf{Q}_{\mathbf{V}}^{n-1}-\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\le C\left(\boldsymbol{\mathcal{U}}\right),\tag{5.138}$$

$$\left\|\sum_{n=1}^{N}k\ Q_P^{n-1}\right\|\le C\left(\boldsymbol{\mathcal{U}}\right),\tag{5.139}$$

$$\sup_{n=1,\ldots,N}\|Q_Y^{n-1}\|_{H_0}\le C\left(\boldsymbol{\mathcal{U}}\right),\tag{5.140}$$

$$\sum_{n=1}^{N}\|Q_Y^{n-1}-Q_Y^n\|_{H_0}^2\le C\left(\boldsymbol{\mathcal{U}}\right),\tag{5.141}$$

$$\sum_{n=1}^{N}k\ \|\hat{\Delta}_h Q_Y^{n-1}\|^2\le C\left(\boldsymbol{\mathcal{U}}\right),\tag{5.142}$$

$$\sum_{n=1}^{N}k\ \|Q_W^{n-1}\|_{H_0}^2\le C\left(\boldsymbol{\mathcal{U}}\right).\tag{5.143}$$

where the constant $C\left(\boldsymbol{\mathcal{U}}\right)$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$, data problem and constant parameters, but it is independent of h,k.

Proof. We divide the proof in several steps.

i) With $\boldsymbol{\psi} = k\ \mathbf{Q}_{\mathbf{V}}^{n-1}$ in (5.134a), (5.134c), we have

$$-k\left(d_t\mathbf{Q}_{\mathbf{V}}^{n}, \mathbf{Q}_{\mathbf{V}}^{n-1}\right) + k\nu\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 = F_1^{n-1} + F_2^{n-1}, \tag{5.144}$$

where

$$\begin{aligned} F_1^{n-1} &= -kB\left(\mathbf{Q}_{\mathbf{V}}^{n-1}, \mathbf{V}^{n+1}, \mathbf{Q}_{\mathbf{V}}^{n}\right),\\ F_2^{n-1} &= k\left(Y^n\nabla Q_Y^n, \mathbf{Q}_{\mathbf{V}}^{n-1}\right). \end{aligned}$$

Using integration by parts in space, the generalized Holder's inequality (A.14), Poincaré's inequality (A.16) and the inequalities (A.17), (A.18), we get

$$\begin{aligned} F_1^n &= -k\left(\left[\mathbf{Q}_{\mathbf{V}}^{n-1}\cdot\nabla\right]\mathbf{V}^{n+1}, \mathbf{Q}_{\mathbf{V}}^{n}\right) - \frac{k}{2}\left(\left[\nabla\cdot\mathbf{Q}_{\mathbf{V}}^{n-1}\right]\mathbf{Q}_{\mathbf{V}}^{n}, \mathbf{V}^{n+1}\right)\\ &\le k\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|_{\mathbf{L}^4}\|\nabla\mathbf{V}^{n+1}\|\|\mathbf{Q}_{\mathbf{V}}^{n}\|_{\mathbf{L}^4} + \frac{k}{2}\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|\|\mathbf{Q}_{\mathbf{V}}^{n}\|_{\mathbf{L}^4}\|\mathbf{V}^{n+1}\|_{\mathbf{L}^4}\\ &\le Ck\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|\|\nabla\mathbf{V}^{n+1}\|\|\mathbf{Q}_{\mathbf{V}}^{n}\|_{\mathbf{L}^4}\\ &\le k\sigma\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + k\ C(\sigma)\|\nabla\mathbf{V}^{n+1}\|^2\|\mathbf{Q}_{\mathbf{V}}^{n}\|_{\mathbf{L}^4}^2\\ &\le k\sigma\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + k\ C(\sigma)\|\nabla\mathbf{V}^{n+1}\|^2\|\mathbf{Q}_{\mathbf{V}}^{n}\|\|\nabla\mathbf{Q}_{\mathbf{V}}^{n}\|\\ &\le k\sigma\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + k\sigma\|\nabla\mathbf{Q}_{\mathbf{V}}^{n}\|^2 + k\ C(\sigma)\|\nabla\mathbf{V}^{n+1}\|^4\|\mathbf{Q}_{\mathbf{V}}^{n}\|^2, \end{aligned}$$

$$\begin{aligned} F_2^n &\le k\|Y^n\|_{\mathcal{C}(\bar\Omega)}\|\nabla Q_Y^n\|\|\mathbf{Q}_{\mathbf{V}}^{n-1}\| \le kC\|Y^n\|_{\mathcal{C}(\bar\Omega)}\|\nabla Q_Y^n\|\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|\\ &\le k\sigma\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + kC(\sigma)\|Y^n\|_{\mathcal{C}(\bar\Omega)}^2\|\nabla Q_Y^n\|^2. \end{aligned}$$

Inserting the above estimates of F_1^{n-1}, F_2^{n-1} in (5.144), we can write

$$\begin{aligned} -k\left(d_t\mathbf{Q}_{\mathbf{V}}^{n}, \mathbf{Q}_{\mathbf{V}}^{n-1}\right) + k\nu\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\| \le 2k\sigma\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2\\ +k\sigma\|\nabla\mathbf{Q}_{\mathbf{V}}^{n}\|^2 + k\ C_1(\sigma)\|\nabla\mathbf{V}^{n+1}\|^4\|\mathbf{Q}_{\mathbf{V}}^{n}\|^2 + kC_2(\sigma)\|Y^n\|_{\mathcal{C}(\bar\Omega)}^2\|\nabla Q_Y^n\|^2. \end{aligned} \tag{5.145}$$

Setting $n = i$ in (5.145), summing up over the index $i = N, \dots, n$, with $1 \le n \le N$ and rearranging, we derive

$$\begin{aligned} \frac{1}{2}\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + \sum_{i=n}^{N}\left[\frac{1}{2}\|\mathbf{Q}_{\mathbf{V}}^{i-1} - \mathbf{Q}_{\mathbf{V}}^{i}\|^2 + k\nu\|\nabla\mathbf{Q}_{\mathbf{V}}^{i-1}\|^2\right] \le\\ \le \sum_{i=n}^{N} k\left[3\sigma\|\nabla\mathbf{Q}_{\mathbf{V}}^{i-1}\|^2 + C_1(\sigma)\|\nabla\mathbf{V}^{i+1}\|^4\|\mathbf{Q}_{\mathbf{V}}^{i}\|^2 + C_2(\sigma)\|Y^i\|_{\mathcal{C}(\bar\Omega)}^2\|\nabla Q_Y^i\|^2\right]. \end{aligned} \tag{5.146}$$

which implies, with σ small enough,

$$\begin{aligned} \|\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + \sum_{i=n}^{N}\left[\|\mathbf{Q}_{\mathbf{V}}^{i-1} - \mathbf{Q}_{\mathbf{V}}^{i}\|^2 + k\|\nabla\mathbf{Q}_{\mathbf{V}}^{i-1}\|^2\right]\\ \le C\sum_{i=n}^{N} k\left[\|\nabla\mathbf{V}^{i+1}\|^4\|\mathbf{Q}_{\mathbf{V}}^{i}\|^2 + \|Y^i\|_{\mathcal{C}(\bar\Omega)}^2\|\nabla Q_Y^i\|^2\right], \end{aligned} \tag{5.147}$$

for all $n = 1, \ldots, N$.

ii) With $\theta = \hat{\Delta}_h Q_Y^{n-1}$ in (5.134g), using the definition (A.36) of the discrete Laplacian, we infer

(5.148) $$\|\hat{\Delta}_h Q_Y^{n-1}\|_h^2 = G_1^{n-1} + G_2^{n-1},$$

where

$$G_1^{n-1} = -\frac{1}{\gamma}\left(\nabla Q_W^{n-1}, \nabla Q_Y^{n-1}\right),$$
$$G_2^{n-1} = \frac{\rho}{\gamma}\left(Y^{n-1}, \mathbf{Q}_{\mathbf{V}}^{n-1} \cdot \nabla \hat{\Delta}_h Q_Y^{n-1}\right).$$

From the generalized Holder's inequality (A.14), Young's inequality (A.13), integration by parts in space, Poincaré's inequality (A.16) and inequality (A.17), we realize

$$G_1^{n-1} \leq \sigma \|\nabla Q_W^{n-1}\|^2 + C_1(\sigma)\|\nabla Q_Y^{n-1}\|^2,$$

$$\begin{aligned} G_2^{n-1} &= -\frac{\rho}{\gamma}\left(\nabla Y^{n-1} \cdot \mathbf{Q}_{\mathbf{V}}^{n-1}, \hat{\Delta}_h Q_Y^{n-1}\right) - \frac{\rho}{\gamma}\left(\nabla \cdot \mathbf{Q}_{\mathbf{V}}^{n-1}, Y^{n-1}\hat{\Delta}_h Q_Y^{n-1}\right) \\ &\leq \frac{\rho}{\gamma}\|\nabla Y^{n-1}\|_{L^4}\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|_{\mathbf{L}^4}\|\hat{\Delta}_h Q_Y^{n-1}\|_h + \frac{\rho}{\gamma}\|\nabla \cdot \mathbf{Q}_{\mathbf{V}}^{n-1}\|\|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}\|\hat{\Delta}_h Q_Y^{n-1}\|_h \\ &\leq 2\sigma\|\hat{\Delta}_h Q_Y^{n-1}\|_h^2 + C_1(\sigma)\left[\|\nabla Y^{n-1}\|_{L^4}^2 + \|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}^2\right]\|\nabla \mathbf{Q}_{\mathbf{V}}^{n-1}\|^2. \end{aligned}$$

Hence, inserting the above estimates of G_1^{n-1}, G_2^{n-1} in (5.148), we have

(5.149) $$\|\hat{\Delta}_h Q_Y^{n-1}\|_h^2 \leq 2\sigma\|\hat{\Delta}_h Q_Y^{n-1}\|_h^2 + \sigma\|\nabla Q_W^{n-1}\|^2$$
$$+C(\sigma)\left[\left(\|\nabla Y^{n-1}\|_{L^4}^2 + \|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}^2\right)\|\nabla \mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + \|\nabla Q_Y^{n-1}\|^2\right].$$

which implies, with σ small enough,

(5.150) $$\|\hat{\Delta}_h Q_Y^{n-1}\|_h^2$$
$$\leq C\left[\|\nabla Q_W^{n-1}\|^2 + \left(\|\nabla Y^{n-1}\|_{L^4}^2 + \|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}^2\right)\|\nabla \mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + \|\nabla Q_Y^{n-1}\|^2\right].$$

Setting in (5.149) $n = i$, multiplying by k and summing up over $i = n, \ldots, N$, with $1 \leq n \leq N$, we get

(5.151) $$\sum_{i=n}^{N} k\|\hat{\Delta}_h Q_Y^{i-1}\|_h^2$$
$$+C\sum_{i=n}^{N} k\left[\left(\|\nabla Y^{i-1}\|_{L^4}^2 + \|Y^{i-1}\|_{\mathcal{C}(\bar{\Omega})}^2\right)\|\nabla \mathbf{Q}_{\mathbf{V}}^{i-1}\|^2 + \|\nabla Q_Y^{i-1}\|^2 + \|\nabla Q_W^{i-1}\|^2\right],$$

for all $n = 1, \ldots, N$.

iii) Setting $\eta = Q_W^n, \theta = -k d_t Q_Y^n$ in (5.134d), (5.134g), we can write

(5.152)
$$-k\left(d_t Q_Y^n, Q_W^n\right)_h - k\varepsilon^2\|\nabla Q_W^{n-1}\|^2 + k\left(Q_W^n, Q_W^{n-1}\right)_h - k\left(\nabla Q_Y^n \cdot \mathbf{V}^n, Q_W^{n-1}\right)$$

$$+k\rho\left(\nabla W^{n+1}\cdot\mathbf{Q}_{\mathbf{V}}^{n},Q_{W}^{n-1}\right)-k\left(3\left(Y^{n}\right)^{2},\left(Q_{W}^{n-1}\right)^{2}\right)_{h}+k\left(Y^{n}-y_{d,h}^{n},Q_{W}^{n-1}\right)=0,$$

(5.153)

$$-k\left(Q_{W}^{n-1},d_{t}Q_{Y}^{n}\right)_{h}=k\gamma\left(\nabla Q_{Y}^{n-1},\nabla d_{t}Q_{Y}^{n}\right)+k\rho\left(Y^{n-1},\mathbf{Q}_{\mathbf{V}}^{n-1}\cdot\nabla d_{t}Q_{Y}^{n}\right).$$

Substituting (5.153) in (5.152) and rearranging, we derive

$$\begin{aligned}-k\gamma\left(\nabla Q_{Y}^{n-1},\nabla d_{t}Q_{Y}^{n}\right)+k\varepsilon^{2}\|\nabla Q_{W}^{n-1}\|^{2}+k\left(3\left(Y^{n}\right)^{2}Q_{W}^{n-1},Q_{W}^{n-1}\right)_{h}\\=H_{1}^{n-1}+\ldots+H_{5}^{n-1},\end{aligned}\tag{5.154}$$

where

$$\begin{aligned}H_{1}^{n-1}&=-k\left(\nabla Q_{Y}^{n}\cdot\mathbf{V}^{n},Q_{W}^{n-1}\right),\\H_{2}^{n-1}&=k\rho\left(\nabla W^{n+1}\cdot\mathbf{Q}_{\mathbf{V}}^{n},Q_{W}^{n-1}\right),\\H_{3}^{n-1}&=k\left(Y^{n}-y_{d,h}^{n},Q_{W}^{n-1}\right),\\H_{4}^{n-1}&=k\left(Q_{W}^{n},Q_{W}^{n-1}\right)_{h},\\H_{5}^{n-1}&=k\rho\left(Y^{n-1},\mathbf{Q}_{\mathbf{V}}^{n-1}\cdot\nabla d_{t}Q_{Y}^{n}\right).\end{aligned}$$

In addition, from (5.134g), we note that

$$H_{4}^{n-1}=k\left(Q_{W}^{n},Q_{W}^{n-1}\right)_{h}=-k\gamma\left(\nabla Q_{Y}^{n},\nabla Q_{W}^{n-1}\right)-k\rho\left(Y^{n},\mathbf{Q}_{\mathbf{V}}^{n}\cdot\nabla Q_{W}^{n-1}\right).$$

Using the generalized Holder's inequality (A.14), the Poincaré's-Wirtinger inequality (A.15), the Poincaré's inequality (A.16), the inequalities (A.17) and (A.18), Young's inequality (A.13), we derive

$$\begin{aligned}H_{1}^{n-1}&\le k\|\nabla Q_{Y}^{n}\|\|\mathbf{V}^{n}\|_{\mathbf{L}^{4}}\|Q_{W}^{n-1}\|_{L^{4}}\le kC\|\nabla Q_{Y}^{n}\|\|\nabla\mathbf{V}^{n}\|\|\nabla Q_{W}^{n-1}\|\\&\le k\sigma\|\nabla Q_{W}^{n-1}\|^{2}+kC\left(\sigma\right)\|\nabla\mathbf{V}^{n}\|^{2}\|\nabla Q_{Y}^{n}\|^{2},\end{aligned}$$

$$\begin{aligned}H_{2}^{n-1}&\le k\rho\|\nabla W^{n+1}\|_{L^{4}}\|\mathbf{Q}_{\mathbf{V}}^{n}\|_{L^{4}}\|Q_{W}^{n-1}\|\\&\le k\rho C\|\nabla W^{n+1}\|_{L^{4}}\|\mathbf{Q}_{\mathbf{V}}^{n}\|^{\frac{1}{2}}\|\nabla\mathbf{Q}_{\mathbf{V}}^{n}\|^{\frac{1}{2}}\|\nabla Q_{W}^{n-1}\|\\&\le k\sigma\|\nabla Q_{W}^{n-1}\|^{2}+kC\left(\sigma\right)\|\nabla W^{n+1}\|_{L^{4}}^{2}\|\mathbf{Q}_{\mathbf{V}}^{n}\|\|\nabla\mathbf{Q}_{\mathbf{V}}^{n}\|\\&\le k\sigma\|\nabla Q_{W}^{n-1}\|^{2}+k\sigma\|\nabla\mathbf{Q}_{\mathbf{V}}^{n}\|^{2}+kC\left(\sigma\right)\|\nabla W^{n+1}\|_{L^{4}}^{4}\|\mathbf{Q}_{\mathbf{V}}^{n}\|^{2},\end{aligned}$$

$$\begin{aligned}H_{3}^{n-1}&\le k\|Y^{n}-y_{d,h}^{n}\|\|Q_{W}^{n-1}\|\le kC\|Y^{n}-y_{d,h}^{n}\|\|\nabla Q_{W}^{n-1}\|\\&\le k\sigma\|\nabla Q_{W}^{n-1}\|^{2}+kC\left(\sigma\right)\|Y^{n}-y_{d,h}^{n}\|^{2},\end{aligned}$$

$$\begin{aligned}H_{4}^{n-1}&\le k\gamma\|\nabla Q_{Y}^{n}\|\|\nabla Q_{W}^{n-1}\|+k\rho\|Y^{n}\|_{\mathcal{C}(\bar{\Omega})}\|\mathbf{Q}_{\mathbf{V}}^{n}\|\|\nabla Q_{W}^{n-1}\|\\&\le 2k\sigma\|\nabla Q_{W}^{n-1}\|^{2}+kC_{1}\left(\sigma\right)\|\nabla Q_{Y}^{n}\|^{2}+kC_{2}\left(\sigma\right)\|Y^{n}\|_{\mathcal{C}(\bar{\Omega})}^{2}\|\mathbf{Q}_{\mathbf{V}}^{n}\|^{2}.\end{aligned}$$

Furthermore, applying discrete integration by parts in time, we infer

$$\sum_{i=n}^{N}H_{5}^{i-1}=\sum_{i=n}^{N}k\rho\left(Y^{i-1},\mathbf{Q}_{\mathbf{V}}^{i-1}\cdot\nabla d_{t}Q_{Y}^{i}\right)=I_{1}^{n-1}+I_{2}+I_{3},\tag{5.155}$$

where

$$
\begin{aligned}
I_1^{n-1} &= -\rho \left(Y^{n-1}, \mathbf{Q}_{\mathbf{V}}^{n-1} \cdot \nabla Q_Y^{n-1}\right), \\
I_2 &= -\rho \sum_{i=n}^{N-1} k \left(Y^{i-1}, d_t \mathbf{Q}_{\mathbf{V}}^{i} \cdot \nabla Q_Y^{i}\right), \\
I_3 &= -\rho \sum_{i=n}^{N-1} k \left(d_t Y^{i}, \mathbf{Q}_{\mathbf{V}}^{i} \cdot \nabla Q_Y^{i}\right).
\end{aligned}
$$

Using the generalized Holder's inequality (A.14), Young's inequality (A.13), (5.147), discrete interpolation inequality (A.51) and (5.150), we realize

$$
\begin{aligned}
I_1^{n-1} &\leq \rho \|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})} \|\mathbf{Q}_{\mathbf{V}}^{n-1}\| \|\nabla Q_Y^{n-1}\| \\
&\leq \sigma \|\nabla Q_Y^{n-1}\|^2 + C(\sigma) \|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}^2 \|\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 \\
&\leq \sigma \|\nabla Q_Y^{n-1}\|^2 + C(\sigma) \|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}^2 \sum_{i=n}^{N} k \left[\|\nabla \mathbf{V}^{i+1}\|^4 \|\mathbf{Q}_{\mathbf{V}}^{i}\|^2 + \|Y^{i}\|_{\mathcal{C}(\bar{\Omega})}^2 \|\nabla Q_Y^{i}\|^2\right].
\end{aligned}
$$

$$
\begin{aligned}
I_2 &\leq \rho \sum_{i=n}^{N-1} k \|Y^{i-1}\|_{\mathcal{C}(\bar{\Omega})} \|d_t \mathbf{Q}_{\mathbf{V}}^{i}\| \|\nabla Q_Y^{i}\| \\
&\leq \sum_{i=n}^{N-1} k \left[\sigma \|d_t \mathbf{Q}_{\mathbf{V}}^{i}\|^2 + C(\sigma) \|Y^{i-1}\|_{\mathcal{C}(\bar{\Omega})}^2 \|\nabla Q_Y^{i}\|^2\right]
\end{aligned}
$$

$$
\begin{aligned}
I_3 &\leq \rho \sum_{i=n}^{N-1} k \|d_t Y^{i}\| \|\mathbf{Q}_{\mathbf{V}}^{i}\|_{\mathbf{L}^4} \|\nabla Q_Y^{i}\|_{L^4} \\
&\leq \rho C \sum_{i=n}^{N-1} k \|d_t Y^{i}\| \|\nabla \mathbf{Q}_{\mathbf{V}}^{i}\| \left(\|\hat{\Delta}_h Q_Y^{i}\| + \|\nabla Q_Y^{i}\|\right) \\
&\leq \sum_{i=n}^{N-1} k \left[\sigma \left(\|\hat{\Delta}_h Q_Y^{i}\|_h^2 + \|\nabla Q_Y^{i}\|^2\right) + C(\sigma) \|d_t Y^{i}\|^2 \|\nabla \mathbf{Q}_{\mathbf{V}}^{i}\|^2\right] \\
&\leq C_1 \sum_{i=n}^{N-1} k\sigma \left[\|\nabla Q_W^{i}\|^2 + \left(\|\nabla Y^{i}\|_{L^4}^2 + \|Y^{i}\|_{\mathcal{C}(\bar{\Omega})}^2\right) \|\nabla \mathbf{Q}_{\mathbf{V}}^{i}\|^2 + \|\nabla Q_Y^{i}\|^2\right] \\
&+ C_2(\sigma) \sum_{i=n}^{N-1} k \|d_t Y^{i}\|^2 \|\nabla \mathbf{Q}_{\mathbf{V}}^{i}\|^2.
\end{aligned}
$$

Hence, inserting the estimates of I_1^{n-1}, I_2, I_3 in (5.155) and rearranging, we conclude

$$
\begin{aligned}
\sum_{i=n}^{N} H_5^{i-1} \leq C_1(\sigma) \|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}^2 \sum_{i=n}^{N} k \left[\|\nabla \mathbf{V}^{i+1}\|^4 \|\mathbf{Q}_{\mathbf{V}}^{i}\|^2 + \|Y^{i}\|_{\mathcal{C}(\bar{\Omega})}^2 \|\nabla Q_Y^{i}\|^2\right] \\
+\sigma \|\nabla Q_Y^{n-1}\|^2 + \sum_{i=n}^{N} k \left[\sigma \|d_t \mathbf{Q}_{\mathbf{V}}^{i}\|^2 + C_2(\sigma) \|Y^{i-1}\|_{\mathcal{C}(\bar{\Omega})}^2 \|\nabla Q_Y^{i}\|^2\right]
\end{aligned}
$$

$$+C_1 \sum_{i=n}^{N} k\sigma \left[\|\nabla Q_W^{i-1}\|^2 + \left(\|\nabla Y^i\|_{L^4}^2 + \|Y^i\|_{\mathcal{C}(\bar{\Omega})}^2\right) \|\nabla \mathbf{Q}_{\mathbf{V}}^i\|^2 + \|\nabla Q_Y^i\|^2\right]$$

$$+C_2(\sigma) \sum_{i=n}^{N} k\|d_t Y^i\|^2 \|\nabla \mathbf{Q}_{\mathbf{V}}^i\|^2. \tag{5.156}$$

So, setting in (5.154) $n = i$, using the estimates of $H_1^{n-1}, \ldots, H_5^{n-1}$ and summing up over $i = n, \ldots, N$, with $1 \leq n \leq N$, we have

$$\frac{\gamma}{2}\|\nabla Q_Y^{n-1}\|^2 + \sum_{i=n}^{N} \left[\frac{\gamma}{2}\|\nabla Q_Y^{n-1} - \nabla Q_Y^n\|^2 + k\varepsilon^2\|\nabla Q_W^{i-1}\|^2 + k\left(3\left(Y^i\right)^2 Q_W^{i-1}, Q_W^{i-1}\right)_h\right]$$

$$\leq \sum_{i=n}^{N} k\left[\sigma\|\nabla \mathbf{Q}_{\mathbf{V}}^i\|^2 + \sigma\|d_t \mathbf{Q}_{\mathbf{V}}^i\|^2 + 5\sigma\|\nabla Q_W^{i-1}\|^2 + C_1(\sigma)\|\nabla \mathbf{V}^i\|^2\|\nabla Q_Y^i\|^2\right]$$

$$+\sum_{i=n}^{N} k\left[C_2(\sigma)\|\nabla W^{i+1}\|_{L^4}^4\|\mathbf{Q}_{\mathbf{V}}^i\|^2 + C_3(\sigma)\|Y^i - y_{d,h}^i\|^2\right]$$

$$+\sum_{i=n}^{N} k\left[C_4(\sigma)\|\nabla Q_Y^n\|^2 + C_5(\sigma)\|Y^n\|_{\mathcal{C}(\bar{\Omega})}^2\|\mathbf{Q}_{\mathbf{V}}^n\|^2\right]$$

$$+C_6(\sigma)\|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}^2 \sum_{i=n}^{N} k\left[\|\nabla \mathbf{V}^{i+1}\|^4\|\mathbf{Q}_{\mathbf{V}}^i\|^2 + \|Y^i\|_{\mathcal{C}(\bar{\Omega})}^2\|\nabla Q_Y^i\|^2\right]$$

$$+\sigma\|\nabla Q_Y^{n-1}\|^2 + \sum_{i=n}^{N} k\left[C_7(\sigma)\|Y^{i-1}\|_{\mathcal{C}(\bar{\Omega})}^2\|\nabla Q_Y^i\|^2\right]$$

$$+C_8 \sum_{i=n}^{N} k\sigma\left[\|\nabla Q_W^{i-1}\|^2 + \left(\|\nabla Y^i\|_{L^4}^2 + \|Y^i\|_{\mathcal{C}(\bar{\Omega})}^2\right)\|\nabla \mathbf{Q}_{\mathbf{V}}^i\|^2 + \|\nabla Q_Y^i\|^2\right]$$

$$+C_9(\sigma)\sum_{i=n}^{N} k\|d_t Y^i\|^2\|\nabla \mathbf{Q}_{\mathbf{V}}^i\|^2, \tag{5.157}$$

for all $n = 1, \ldots, N$.

iv) With $\boldsymbol{\psi} = -d_t \mathbf{Q}_{\mathbf{V}}^n$ in (5.134a), (5.134c), we get

$$-k\nu\left(\nabla \mathbf{Q}_{\mathbf{V}}^{n-1}, \nabla d_t \mathbf{Q}_{\mathbf{V}}^n\right) + k\|d_t \mathbf{Q}_{\mathbf{V}}^n\|^2 = L_1^{n-1} + L_2^{n-1} + L_3^{n-1}, \tag{5.158}$$

where

$$L_1^{n-1} = kB\left(\mathbf{V}^{n-1}, d_t \mathbf{Q}_{\mathbf{V}}^n, \mathbf{Q}_{\mathbf{V}}^{n-1}\right),$$
$$L_2^{n-1} = kB\left(d_t \mathbf{Q}_{\mathbf{V}}^n, \mathbf{V}^{n+1}, \mathbf{Q}_{\mathbf{V}}^n\right),$$
$$L_3^{n-1} = -k\left(Y^n, \nabla Q_Y^n \cdot d_t \mathbf{Q}_{\mathbf{V}}^n\right).$$

Applying integration by parts in space, generalized Holder's inequality (A.14), Young's inequality (A.13), Poincaré's inequality (A.16), inequalities (A.17), (A.18), interpolation inequality (A.54) and the embedding $\mathbf{W}^{1,4} \hookrightarrow \boldsymbol{\mathcal{C}}(\bar{\Omega})$, we can write

$$L_1^{n-1} = -\frac{k}{2}\left(\nabla \cdot \mathbf{V}^{n-1}, d_t \mathbf{Q}_{\mathbf{V}}^n \cdot \mathbf{Q}_{\mathbf{V}}^{n-1}\right) - k\left(\left[\mathbf{V}^{n-1} \cdot \nabla\right] \mathbf{Q}_{\mathbf{V}}^{n-1}, d_t \mathbf{Q}_{\mathbf{V}}^n\right)$$

$$
\begin{aligned}
&\leq \frac{k}{2}\|\nabla\cdot\mathbf{V}^{n-1}\|_{\mathbf{L}^4}\|d_t\mathbf{Q}_{\mathbf{V}}^n\|\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|_{\mathbf{L}^4} + k\|\mathbf{V}^{n-1}\|_{\boldsymbol{\mathcal{C}}(\bar\Omega)}\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|\|d_t\mathbf{Q}_{\mathbf{V}}^n\| \\
&\leq 2k\sigma\|d_t\mathbf{Q}_{\mathbf{V}}^n\|^2 + kC(\sigma)\left[\|\nabla\mathbf{V}^{n-1}\|_{\mathbf{L}^4}^2\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|_{\mathbf{L}^4}^2 + \|\mathbf{V}^{n-1}\|_{\boldsymbol{\mathcal{C}}(\bar\Omega)}^2\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2\right] \\
&\leq 2k\sigma\|d_t\mathbf{Q}_{\mathbf{V}}^n\|^2 + kC_1(\sigma)\|\nabla\mathbf{V}^{n-1}\|\|\tilde\Delta_h\mathbf{V}^{n-1}\|\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\| \\
&+ kC_2(\sigma)\left[\|\mathbf{V}^{n-1}\|_{\mathbf{L}^4}^2 + \|\nabla\mathbf{V}^{n-1}\|\|\tilde\Delta_h\mathbf{V}^{n-1}\|\right]\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 \\
&\leq 2k\sigma\|d_t\mathbf{Q}_{\mathbf{V}}^n\|^2 + 2k\mu\|\tilde\Delta_h\mathbf{V}^{n-1}\|^2\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 \\
&+ kC_1(\sigma,\mu)\|\nabla\mathbf{V}^{n-1}\|^2\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + kC_2(\sigma,\mu)\|\nabla\mathbf{V}^{n-1}\|^2\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2,
\end{aligned}
$$

$$
\begin{aligned}
L_2^{n-1} &= \frac{k}{2}\left(\left[d_t\mathbf{Q}_{\mathbf{V}}^n\cdot\nabla\right]\mathbf{V}^{n+1},\mathbf{Q}_{\mathbf{V}}^n\right) - \frac{k}{2}\left(\left[d_t\mathbf{Q}_{\mathbf{V}}^n\cdot\nabla\right]\mathbf{Q}_{\mathbf{V}}^n,\mathbf{V}^{n+1}\right) \\
&= \frac{k}{2}\|d_t\mathbf{Q}_{\mathbf{V}}^n\|\|\nabla\mathbf{V}^{n+1}\|_{\mathbf{L}^4}\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{L}^4} + \frac{k}{2}\|d_t\mathbf{Q}_{\mathbf{V}}^n\|\|\nabla\mathbf{Q}_{\mathbf{V}}^n\|\|\mathbf{V}^{n+1}\|_{\boldsymbol{\mathcal{C}}(\bar\Omega)} \\
&\leq 2k\sigma\|d_t\mathbf{Q}_{\mathbf{V}}^n\|^2 + kC(\sigma)\left[\|\nabla\mathbf{V}^{n+1}\|_{\mathbf{L}^4}^2\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{L}^4}^2 + \|\mathbf{V}^{n+1}\|_{\boldsymbol{\mathcal{C}}(\bar\Omega)}^2\|\nabla\mathbf{Q}_{\mathbf{V}}^n\|^2\right] \\
&\leq 2k\sigma\|d_t\mathbf{Q}_{\mathbf{V}}^n\|^2 + kC_1(\sigma)\|\nabla\mathbf{V}^{n+1}\|\|\tilde\Delta_h\mathbf{V}^{n+1}\|\|\mathbf{Q}_{\mathbf{V}}^n\|\|\nabla\mathbf{Q}_{\mathbf{V}}^n\| \\
&+ kC_2(\sigma)\left[\|\mathbf{V}^{n+1}\|_{\mathbf{L}^4}^2 + \|\nabla\mathbf{V}^{n+1}\|\|\tilde\Delta_h\mathbf{V}^{n+1}\|\right]\|\nabla\mathbf{Q}_{\mathbf{V}}^n\|^2 \\
&\leq 2k\sigma\|d_t\mathbf{Q}_{\mathbf{V}}^n\|^2 + 2k\mu\|\tilde\Delta_h\mathbf{V}^{n+1}\|^2\|\nabla\mathbf{Q}_{\mathbf{V}}^n\|^2 \\
&+ kC_1(\sigma,\mu)\|\nabla\mathbf{V}^{n+1}\|^2\|\nabla\mathbf{Q}_{\mathbf{V}}^n\|^2 + kC_2(\sigma,\mu)\|\nabla\mathbf{V}^{n+1}\|^2\|\mathbf{Q}_{\mathbf{V}}^n\|^2,
\end{aligned}
$$

$$
\begin{aligned}
L_3^{n-1} &\leq k\|Y^n\|_{\mathcal{C}(\bar\Omega)}\|\nabla Q_Y^n\|\|d_t\mathbf{Q}_{\mathbf{V}}^n\| \\
&\leq k\sigma\|d_t\mathbf{Q}_{\mathbf{V}}^n\|^2 + kC(\sigma)\|Y^n\|_{\mathcal{C}(\bar\Omega)}^2\|\nabla Q_Y^n\|^2.
\end{aligned}
$$

Inserting the above estimates of $L_1^{n-1}, L_2^{n-1}, L_3^{n-1}$ in (5.158), we derive

$$
\begin{aligned}
&\frac{\nu}{2}\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 - \frac{\nu}{2}\|\nabla\mathbf{Q}_{\mathbf{V}}^n\|^2 + \frac{\nu}{2}\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1} - \nabla\mathbf{Q}_{\mathbf{V}}^n\|^2 + k\|d_t\mathbf{Q}_{\mathbf{V}}^n\|^2 \\
&\leq 5k\sigma\|d_t\mathbf{Q}_{\mathbf{V}}^n\|^2 + 2k\mu\left[\|\tilde\Delta_h\mathbf{V}^{n-1}\|^2\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + \|\tilde\Delta_h\mathbf{V}^{n+1}\|^2\|\nabla\mathbf{Q}_{\mathbf{V}}^n\|^2\right] \\
&\qquad + kC_1(\sigma,\mu)\|\nabla\mathbf{V}^{n+1}\|^2\left[\|\nabla\mathbf{Q}_{\mathbf{V}}^n\|^2 + \|\mathbf{Q}_{\mathbf{V}}^n\|^2\right] \\
&\qquad + kC_2(\sigma,\mu)\|\nabla\mathbf{V}^{n-1}\|^2\left[\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + \|\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2\right] \\
&\qquad + kC_3(\sigma)\|Y^n\|_{\mathcal{C}(\bar\Omega)}^2\|\nabla Q_Y^n\|^2.
\end{aligned}
\tag{5.159}
$$

From (5.159), noting that

$$
\|\tilde\Delta_h\mathbf{V}^{n-1}\|^2 \leq \sum_{j=n}^{N} k\|\tilde\Delta_h\mathbf{V}^{j-1}\|^2,
$$

we infer

$$
\begin{aligned}
&\frac{\nu}{2}\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 - \frac{\nu}{2}\|\nabla\mathbf{Q}_{\mathbf{V}}^n\|^2 + \frac{\nu}{2}\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1} - \nabla\mathbf{Q}_{\mathbf{V}}^n\|^2 + k\|d_t\mathbf{Q}_{\mathbf{V}}^n\|^2 \\
&\leq 5k\sigma\|d_t\mathbf{Q}_{\mathbf{V}}^n\|^2 + 2k\mu\left[\sum_{j=n}^{N} k\|\tilde\Delta_h\mathbf{V}^{j-1}\|^2\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + \|\tilde\Delta_h\mathbf{V}^{n+1}\|^2\|\nabla\mathbf{Q}_{\mathbf{V}}^n\|^2\right]
\end{aligned}
\tag{5.160}
$$

$$+kC_1(\sigma,\mu)\|\nabla\mathbf{V}^{n+1}\|^2\left[\|\nabla\mathbf{Q}_{\mathbf{V}}^n\|^2+\|\mathbf{Q}_{\mathbf{V}}^n\|^2\right]$$
$$+kC_2(\sigma,\mu)\|\nabla\mathbf{V}^{n-1}\|^2\left[\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2+\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2\right]$$
$$+kC_3(\sigma)\|Y^n\|^2_{\mathcal{C}(\bar\Omega)}\|\nabla Q_Y^n\|^2.$$

So, setting in (5.160) $n=i$, summing up over $i=n,\dots,N$ and rearranging, we realize

$$\begin{aligned}
&\frac{\nu}{2}\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2+\sum_{i=n}^{N}\left[\frac{\nu}{2}\|\nabla\mathbf{Q}_{\mathbf{V}}^{i-1}-\nabla\mathbf{Q}_{\mathbf{V}}^{i}\|^2+k\|d_t\mathbf{Q}_{\mathbf{V}}^i\|^2\right]\\
&\le 5\sigma\sum_{i=n}^{N}k\|d_t\mathbf{Q}_{\mathbf{V}}^i\|^2+2\mu\left(\sum_{i=n}^{N}k\|\tilde\Delta_h\mathbf{V}^{i-1}\|^2\right)\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2\\
&\quad+kC_1(\sigma,\mu)\|\nabla\mathbf{V}^{n-1}\|^2\left[\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2+\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2\right]\\
&\quad+2\mu\sum_{i=n}^{N}k\left[\|\tilde\Delta_h\mathbf{V}^i\|^2+\|\tilde\Delta_h\mathbf{V}^{i+1}\|^2\right]\|\nabla\mathbf{Q}_{\mathbf{V}}^i\|^2\\
&\quad+C_2(\sigma,\mu)\sum_{i=n}^{N}k\left[\|\nabla\mathbf{V}^i\|^2+\|\nabla\mathbf{V}^{i+1}\|^2\right]\left[\|\mathbf{Q}_{\mathbf{V}}^i\|^2+\|\nabla\mathbf{Q}_{\mathbf{V}}^i\|^2\right]\\
&\quad+C_3(\sigma)\sum_{i=n}^{N}k\|Y^i\|^2_{\mathcal{C}(\bar\Omega)}\|\nabla Q_Y^i\|^2,
\end{aligned}\tag{5.161}$$

for all $n=1,\dots,N$.

v) We sum (5.146), (5.157) and (5.161). Then, we conclude

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2+\frac{\nu}{2}\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2+\frac{1}{2}\sum_{i=n}^{N}\left[\|\mathbf{Q}_{\mathbf{V}}^{i-1}-\mathbf{Q}_{\mathbf{V}}^i\|^2+\nu\|\nabla\mathbf{Q}_{\mathbf{V}}^{i-1}-\nabla\mathbf{Q}_{\mathbf{V}}^i\|^2\right]\\
&\quad+\sum_{i=n}^{N}k\left[\|d_t\mathbf{Q}_{\mathbf{V}}^i\|^2+\nu\|\nabla\mathbf{Q}_{\mathbf{V}}^{i-1}\|^2\right]+\frac{\gamma}{2}\|\nabla Q_Y^{n-1}\|^2\\
&\quad+\frac{\gamma}{2}\sum_{i=n}^{N}\|\nabla Q_Y^{n-1}-\nabla Q_Y^n\|^2+\sum_{i=n}^{N}k\left[\varepsilon^2\|\nabla Q_W^{i-1}\|^2+\left(3\left(Y^i\right)^2Q_W^{i-1},Q_W^{i-1}\right)_h\right]\\
&\le\left[k\,C_1(\sigma,\mu)\|\nabla\mathbf{V}^{n-1}\|^2+2\mu\sum_{i=n}^{N}k\|\tilde\Delta_h\mathbf{V}^{i-1}\|^2\right]\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2\\
&\quad+k\,C_2(\sigma,\mu)\|\nabla\mathbf{V}^{n-1}\|^2\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2+3\sigma\sum_{i=n}^{N}k\|\nabla\mathbf{Q}_{\mathbf{V}}^{i-1}\|^2+\sigma\|\nabla Q_Y^{n-1}\|^2\\
&\quad+C_3(\sigma)\sum_{i=n}^{N}k\left[\left(1+\|Y^{n-1}\|^2_{\mathcal{C}(\bar\Omega)}\right)\|\nabla\mathbf{V}^{i+1}\|^4+\|Y^i\|^2_{\mathcal{C}(\bar\Omega)}+\|\nabla W^{i+1}\|^4_{L^4}\right]\|\mathbf{Q}_{\mathbf{V}}^i\|^2\\
&\quad+C_4(\sigma,\mu)\sum_{i=n}^{N}k\left[\|\nabla\mathbf{V}^i\|^2+\|\nabla\mathbf{V}^{i+1}\|^2\right]\left[\|\mathbf{Q}_{\mathbf{V}}^i\|^2+\|\nabla\mathbf{Q}_{\mathbf{V}}^i\|^2\right]
\end{aligned}\tag{5.162}$$

$$+\sum_{i=n}^{N} k\left[\sigma\left(1+C_5\right)+\|Y^i\|^2_{\mathcal{C}(\bar{\Omega})}+\|\nabla Y^i\|^2_{L^4}+C_6\left(\sigma\right)\|d_tY^i\|^2\right]\|\nabla \mathbf{Q}^i_{\mathbf{V}}\|^2$$
$$+2\mu\sum_{i=n}^{N} k\left[\|\tilde{\Delta}_h\mathbf{V}^i\|^2+\|\tilde{\Delta}_h\mathbf{V}^{i+1}\|^2\right]\|\nabla \mathbf{Q}^i_{\mathbf{V}}\|^2+6\sigma\sum_{i=n}^{N} k\|d_t\mathbf{Q}^i_{\mathbf{V}}\|^2$$
$$+C_7\left(\sigma\right)\sum_{i=n}^{N} k\left[1+\|\nabla\mathbf{V}^i\|^2+\left(1+\|Y^{n-1}\|^2_{\mathcal{C}(\bar{\Omega})}\right)\|Y^i\|^2_{\mathcal{C}(\bar{\Omega})}+\|Y^{i-1}\|^2_{\mathcal{C}(\bar{\Omega})}\right]\|\nabla Q^i_Y\|^2$$
$$+\sigma C_8\sum_{i=n}^{N} k\|\nabla Q^i_Y\|^2+\sigma\left(5+C_9\right)\sum_{i=n}^{N} k\|\nabla Q^{i-1}_W\|^2+C_{10}\left(\sigma\right)\sum_{i=n}^{N} k\|Y^i-y^i_{d,h}\|^2.$$

Using the results of Lemmas 5.4, 5.5, 5.6, 5.7 5.8, 5.9 and 5.11, there exist $\tilde{\sigma},\tilde{\mu},k_{max}$ such that, in (5.162),

$$k_{max}C_1\left(\tilde{\sigma},\tilde{\mu}\right)\|\nabla\mathbf{V}^{n-1}\|^2+2\tilde{\mu}\sum_{i=n}^{N} k_{max}\|\tilde{\Delta}_h\mathbf{V}^{i-1}\|^2+3\tilde{\sigma}k_{max}<\frac{\nu}{2},$$
$$k_{max}C_2\left(\tilde{\sigma},\tilde{\mu}\right)\|\nabla\mathbf{V}^{n-1}\|^2<\frac{1}{2},$$
$$\tilde{\sigma}<\frac{\gamma}{2},$$
$$6\tilde{\sigma}<1,$$
$$\tilde{\sigma}\left(5+C_9\right)<\varepsilon^2,$$

for all $n=1,\ldots,N$. Then, assuming $\sigma=\tilde{\sigma},\mu=\tilde{\mu},k\leq k_{max}$, from (5.162), we have

$$\|\mathbf{Q}^{n-1}_{\mathbf{V}}\|^2+\|\nabla\mathbf{Q}^{n-1}_{\mathbf{V}}\|^2+\sum_{i=n}^{N}\left[\|\mathbf{Q}^{i-1}_{\mathbf{V}}-\mathbf{Q}^i_{\mathbf{V}}\|^2+\|\nabla\mathbf{Q}^{i-1}_{\mathbf{V}}-\nabla\mathbf{Q}^i_{\mathbf{V}}\|^2\right] \tag{5.163}$$
$$+\sum_{i=n}^{N} k\left[\|d_t\mathbf{Q}^i_{\mathbf{V}}\|^2+\|\nabla\mathbf{Q}^{i-1}_{\mathbf{V}}\|^2\right]+\|\nabla Q^{n-1}_Y\|^2$$
$$+\sum_{i=n}^{N}\|\nabla Q^{n-1}_Y-\nabla Q^n_Y\|^2+\sum_{i=n}^{N} k\left[\|\nabla Q^{i-1}_W\|^2+\left(\left(Y^i\right)^2Q^{i-1}_W,Q^{i-1}_W\right)_h\right]$$
$$\leq C_1\left(\boldsymbol{\mathcal{U}}\right)\sum_{i=n}^{N} k\left[\left(1+\|Y^{n-1}\|^2_{\mathcal{C}(\bar{\Omega})}\right)\|\nabla\mathbf{V}^{i+1}\|^4+\|Y^i\|^2_{\mathcal{C}(\bar{\Omega})}+\|\nabla W^{i+1}\|^4_{L^4}\right]\|\mathbf{Q}^i_{\mathbf{V}}\|^2$$
$$+C_2\left(\boldsymbol{\mathcal{U}}\right)\sum_{i=n}^{N} k\left[\|\nabla\mathbf{V}^i\|^2+\|\nabla\mathbf{V}^{i+1}\|^2\right]\left[\|\mathbf{Q}^i_{\mathbf{V}}\|^2+\|\nabla\mathbf{Q}^i_{\mathbf{V}}\|^2\right]$$
$$+C_3\left(\boldsymbol{\mathcal{U}}\right)\sum_{i=n}^{N} k\left[1+\|Y^i\|^2_{\mathcal{C}(\bar{\Omega})}+\|\nabla Y^i\|^2_{L^4}+\|d_tY^i\|^2\right]\|\nabla\mathbf{Q}^i_{\mathbf{V}}\|^2$$
$$+C_4\left(\boldsymbol{\mathcal{U}}\right)\sum_{i=n}^{N} k\left[\|\tilde{\Delta}_h\mathbf{V}^i\|^2+\|\tilde{\Delta}_h\mathbf{V}^{i+1}\|^2\right]\|\nabla\mathbf{Q}^i_{\mathbf{V}}\|^2$$
$$+C_5\left(\boldsymbol{\mathcal{U}}\right)\sum_{i=n}^{N} k\left[1+\|\nabla\mathbf{V}^i\|^2+\left(1+\|Y^{n-1}\|^2_{\mathcal{C}(\bar{\Omega})}\right)\|Y^i\|^2_{\mathcal{C}(\bar{\Omega})}+\|Y^{i-1}\|^2_{\mathcal{C}(\bar{\Omega})}\right]\|\nabla Q^i_Y\|^2$$

$$+C_6(\boldsymbol{\mathcal{U}})\sum_{i=n}^{N}k\|\nabla Q_Y^i\|^2+C_7(\boldsymbol{\mathcal{U}})\sum_{i=n}^{N}k\|Y^i-y_{d,h}^i\|^2,$$

for all $n=1,\dots,N$. Note that the constants $C_i(\boldsymbol{\mathcal{U}}), i=1,\dots,7$, depend just on on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$, data problem and constant parameters, but they are independent of h,k. So, taking into account of the results of 5.4, 5.5, 5.6, 5.7 5.8, 5.9, 5.11 and applying discrete Gronwall's inequality (see for example [73], Lemma 1.4.2) we get (5.136), (5.137), (5.138). Then, from the Poincaré's-Wirtinger inequality (A.15), we derive (5.140), (5.141) and (5.143). Next, by (5.151), we infer that (5.142) holds. Finally, we have the estimate (5.139) for the discrete adjoint pressure using the same procedure performed in the proof of lemma 3.5. ☐

Lemma 5.19. *Under the same hypothesis of lemma 5.18 and with $k\le k_{max}$, the solution $\mathcal{Q}_{\mathcal{Y}}$ of* (5.133)-(5.135) *is such that*

$$\sum_{n=1}^{N}k\|\nabla Q_Y\|_{L^p}^2\le C(\boldsymbol{\mathcal{U}}), \tag{5.164}$$

$$\sum_{n=1}^{N}k\|\nabla Q_Y\|_{W^{1,q}}^p\le C(\boldsymbol{\mathcal{U}}), \tag{5.165}$$

$$\sum_{n=1}^{N}k\|\nabla Q_Y\|_{\mathcal{C}(\bar\Omega)}^p\le C(\boldsymbol{\mathcal{U}}), \tag{5.166}$$

for all $1\le p<\infty, q>2$, where the constant $C(\boldsymbol{\mathcal{U}})$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$, data problem and constant parameters, but it is independent of h,k.

Proof. Applying (A.39) to (5.142), we have (5.164). Then using (5.140) and an interpolation argument (see [20], Theorem II.5.5), from

$$\|\nabla\mathcal{Q}_{\mathcal{Y},h,k}\|_{L^2(L^p)}+\|\nabla\mathcal{Q}_{\mathcal{Y},h,k}\|_{L^\infty(L^2)}\le C(\boldsymbol{\mathcal{U}}), \tag{5.167}$$

we get (5.165). Finally, (5.166) is a consequence of Sobolev embedding theorem. ☐

Lemma 5.20. *Under the same hypothesis of lemma 5.18 and with $k\le k_{max}$, the solution $(\mathcal{Q}_{\mathcal{Y}},\mathcal{Q}_{\mathcal{W}})$ of* (5.133)-(5.135) *is such that*

$$\sum_{n=1}^{N}k\|d_tQ_Y^n\|_h^2\le C(\boldsymbol{\mathcal{U}}), \tag{5.168}$$

$$\sup_{n=1\dots,N}\|Q_W^{n-1}\|_h\le C(\boldsymbol{\mathcal{U}}), \tag{5.169}$$

$$\sum_{n=1}^{N}\|Q_W^{n-1}-Q_W^n\|_h^2\le C(\boldsymbol{\mathcal{U}}), \tag{5.170}$$

$$\sum_{n=1}^{N}k\|\hat\Delta_hQ_W^{n-1}\|_h^2\le C(\boldsymbol{\mathcal{U}}), \tag{5.171}$$

$$\sum_{n=1}^{N}k\|\nabla Q_W^{n-1}\|_{\mathbf{L}^p}^2\le C(\boldsymbol{\mathcal{U}}), \tag{5.172}$$

for all $1 \leq p < \infty$*, where the constant* $C(\boldsymbol{\mathcal{U}})$ *depends just on* $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$*, data problem and constant parameters, but it is independent of* h, k*.*

Proof. We divide the proof in several steps.

i) With $\eta = -d_t Q_Y^n$ in the adjoint equation (5.134d), we have

$$k\|d_t Q_Y^n\|_h^2 + k\varepsilon^2 \left(\nabla Q_W^{n-1}, \nabla d_t Q_Y^n\right) - k\left(Q_W^n, d_t Q_Y^n\right)_h + k\left(\nabla Q_Y^n \cdot \mathbf{V}^n, d_t Q_Y^n\right) - k\rho\left(\nabla W^{n+1} \cdot \mathbf{Q}_\mathbf{V}^n, d_t Q_Y^n\right) + k\left(3\left(Y^n\right)^2 Q_W^{n-1}, d_t Q_Y^n\right)_h - k\left(Y^n - y_{d,h}^n, d_t Q_Y^n\right) = 0. \tag{5.173}$$

Using the adjoint equation (5.134g), we get

$$\begin{aligned} k\varepsilon^2 \left(\nabla Q_W^{n-1}, \nabla d_t Q_Y^n\right) &= -k\frac{\varepsilon^2}{\gamma}\left(Q_W^{n-1}, d_t Q_W^n\right)_h - k\frac{\varepsilon^2\rho}{\gamma}\left(d_t\left[Y^n \mathbf{Q}_\mathbf{V}^n\right], \nabla Q_W^{n-1}\right) \\ &= \frac{\varepsilon^2}{2\gamma}\|Q_W^{n-1}\|_h - \frac{\varepsilon^2}{2\gamma}\|Q_W^n\|_h + \frac{\varepsilon^2}{2\gamma}\|Q_W^{n-1} - Q_W^{n-1}\|_h \\ &\quad -k\frac{\varepsilon^2\rho}{\gamma}\left(Y^{n-1}, d_t\mathbf{Q}_\mathbf{V}^n \cdot \nabla Q_W^{n-1}\right) - k\frac{\varepsilon^2\rho}{\gamma}\left(d_t Y^n, \mathbf{Q}_\mathbf{V}^n \cdot \nabla Q_W^{n-1}\right). \end{aligned} \tag{5.174}$$

So, substituting (5.174) in (5.173), we can write

$$k\|d_t Q_Y^n\|_h^2 + \frac{\varepsilon^2}{2\gamma}\|Q_W^{n-1}\|_h - \frac{\varepsilon^2}{2\gamma}\|Q_W^n\|_h + \frac{\varepsilon^2}{2\gamma}\|Q_W^{n-1} - Q_W^{n-1}\|_h = M_1^{n-1} + \ldots + M_7^{n-1}, \tag{5.175}$$

where

$$\begin{aligned} M_1^{n-1} &= k\frac{\varepsilon^2\rho}{\gamma}\left(Y^{n-1}, d_t\mathbf{Q}_\mathbf{V}^n \cdot \nabla Q_W^{n-1}\right), \\ M_2^{n-1} &= k\frac{\varepsilon^2\rho}{\gamma}\left(d_t Y^n, \mathbf{Q}_\mathbf{V}^n \cdot \nabla Q_W^{n-1}\right), \\ M_3^{n-1} &= k\rho\left(\nabla W^{n+1} \cdot \mathbf{Q}_\mathbf{V}^n, d_t Q_Y^n\right), \\ M_4^{n-1} &= -k\left(\nabla Q_Y^n \cdot \mathbf{V}^n, d_t Q_Y^n\right), \\ M_5^{n-1} &= +k\left(Q_W^n, d_t Q_Y^n\right)_h, \\ M_6^{n-1} &= -k\left(3\left(Y^n\right)^2 Q_W^{n-1}, d_t Q_Y^n\right)_h, \\ M_7^{n-1} &= k\left(Y^n - y_{d,h}^n, d_t Q_Y^n\right). \end{aligned}$$

Using the generalized Holder's inequality (A.14), Young's inequality (A.13), inequality (A.17), the Poicaré's-Wirtinger inequality (A.15) and the discrete interpolation inequality (A.52), we derive

$$\begin{aligned} M_1^{n-1} &\leq k\frac{\varepsilon^2\rho}{\gamma}\|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}\|d_t\mathbf{Q}_\mathbf{V}^n\|\|\nabla Q_W^{n-1}\| \\ &\leq k\sigma\|d_t\mathbf{Q}_\mathbf{V}^n\|^2 + kC(\sigma)\|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}^2\|Q_W^{n-1}\|_{H_0}^2, \end{aligned}$$

$$
\begin{aligned}
M_2^{n-1} &\le k\frac{\varepsilon^2\rho}{\gamma}\|d_tY^n\|\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{L}^4}\|\nabla Q_W^{n-1}\|_{\mathbf{L}^4}\\
&\le k\sigma\|\nabla Q_W^{n-1}\|_{\mathbf{L}^4}^2 + k\ C(\sigma)\,\|d_tY^n\|^2\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{L}^4}^2\\
&\le k\sigma C_1\|\nabla Q_W^{n-1}\|\left[\|\nabla Q_W^{n-1}\| + \|\hat{\Delta}_h Q_W^{n-1}\|\right] + k\ C_2(\sigma)\,\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\|d_tY^n\|^2\\
&\le k\sigma C_1\|Q_W^{n-1}\|_{H_0}^2 + k\sigma C_2\|\hat{\Delta}_h Q_W^{n-1}\|^2 + k\ C_3(\sigma)\,\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\|d_tY^n\|^2,
\end{aligned}
$$

$$
\begin{aligned}
M_3^{n-1} &\le k\rho\|\nabla W^{n+1}\|_{\mathbf{L}^4}\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{L}^4}\|d_tQ_Y^n\|\\
&\le k\sigma\|d_tQ_Y^n\|_h^2 + k\ C(\sigma)\,\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\|\nabla W^{n+1}\|_{\mathbf{L}^4}^2,
\end{aligned}
$$

$$
\begin{aligned}
M_4^{n-1} &\le k\|\nabla Q_Y^n\|_{\mathbf{L}^4}\|\mathbf{V}^n\|_{\mathbf{L}^4}\|d_tQ_Y^n\|\\
&\le k\sigma\|d_tQ_Y^n\|_h^2 + k\ C(\sigma)\,\|\mathbf{V}^n\|_{\mathbf{H}_0^1}^2\|\nabla Q_Y^n\|_{\mathbf{L}^4}^2,
\end{aligned}
$$

$$
M_5^{n-1} \le k\sigma\|d_tQ_Y^n\|_h^2 + k\ C(\sigma)\,\|Q_W^n\|_{H_0}^2,
$$

$$
\begin{aligned}
M_6^{n-1} &\le 3Ck\|Y^n\|_{\mathcal{C}(\bar{\Omega})}^2\|Q_W^{n-1}\|\|d_tQ_Y^n\|_h\\
&\le k\sigma\|d_tQ_Y^n\|_h^2 + k\ C(\sigma)\,\|Y^n\|_{\mathcal{C}(\bar{\Omega})}^4\|Q_W^{n-1}\|_{H_0}^2,
\end{aligned}
$$

$$
M_7^{n-1} \le k\sigma\|d_tQ_Y^n\|_h^2 + k\ C(\sigma)\,\|Y^n - y_{d,h}^n\|^2.
$$

Inserting the estimates of $M_1^{n-1} + \ldots + M_7^{n-1}$ in (5.175), we infer

(5.176)
$$
\begin{aligned}
&k\|d_tQ_Y^n\|_h^2 + \frac{\varepsilon^2}{2\gamma}\|Q_W^{n-1}\|_h^2 - \frac{\varepsilon^2}{2\gamma}\|Q_W^n\|_h^2 + \frac{\varepsilon^2}{2\gamma}\|Q_W^{n-1} - Q_W^{n-1}\|_h^2\\
&\le k\ C_1(\sigma)\left[\|Y^n - y_{d,h}^n\|^2 + \|d_t\mathbf{Q}_{\mathbf{V}}^n\|^2 + \|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\left(\|d_tY^n\|^2 + \|\nabla W^{n+1}\|_{\mathbf{L}^4}^2\right)\right]\\
&\quad + k\ C_2(\sigma)\left[\|\mathbf{V}^n\|_{\mathbf{H}_0^1}^2\|\nabla Q_Y^n\|_{\mathbf{L}^4}^2 + \left(1 + \|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}^2 + \|Y^n\|_{\mathcal{C}(\bar{\Omega})}^4\right)\|Q_W^{n-1}\|_{H_0}^2\right]\\
&\quad + k\ C_3(\sigma)\,\|Q_W^n\|_{H_0}^2 + k\sigma C_4\|\hat{\Delta}_h Q_W^{n-1}\|_h^2 + 5k\sigma\|d_tQ_Y^n\|_h^2.
\end{aligned}
$$

ii) With $\eta = \hat{\Delta}_h Q_W^{n-1}$ in the adjoint equation (5.134d), by the definition (A.36) of the discrete Laplacian, we realize

(5.177)
$$
k\varepsilon^2\|\hat{\Delta}_h Q_W^{n-1}\|_h^2 = N_1^{n-1} + N_2^{n-1} + N_3^{n-1} + N_4^{n-1} + N_5^{n-1} + N_6^{n-1},
$$

where

$$
\begin{aligned}
N_1^{n-1} &= -k\left(d_tQ_Y^n, \hat{\Delta}_h Q_W^{n-1}\right)_h,\\
N_2^{n-1} &= -k\left(Q_W^n, \hat{\Delta}_h Q_W^{n-1}\right)_h,\\
N_3^{n-1} &= k\left(\nabla Q_Y^n \cdot \mathbf{V}^n, \hat{\Delta}_h Q_W^{n-1}\right),\\
N_4^{n-1} &= -k\rho\left(\nabla W^{n+1} \cdot \mathbf{Q}_{\mathbf{V}}^n, \hat{\Delta}_h Q_W^{n-1}\right),
\end{aligned}
$$

$$N_5^{n-1} = k\left(3\,(Y^n)^2\,Q_W^{n-1}, \hat{\Delta}_h Q_W^{n-1}\right)_h,$$
$$N_6^{n-1} = -k\left(Y^n - y_{d,h}^n, \hat{\Delta}_h Q_W^{n-1}\right).$$

Using generalized Holder's inequality (A.14), Young's inequality (A.13), the equivalence (A.30) between the $h-$norm and the L^2-norm and the inequality (A.17), we conclude

$$N_1^{n-1} \le k\|d_t Q_Y^n\|_h\|\hat{\Delta}_h Q_W^{n-1}\|_h \le k\sigma\|\hat{\Delta}_h Q_W^{n-1}\|_h^2 + k\ C\,(\sigma)\,\|d_t Q_Y^n\|_h^2,$$

$$N_2^{n-1} \le k\|Q_W^n\|_h\|\hat{\Delta}_h Q_W^{n-1}\|_h \le k\sigma\|\hat{\Delta}_h Q_W^{n-1}\|_h^2 + k\ C\,(\sigma)\,\|Q_W^n\|_{H_0}^2,$$

$$\begin{aligned} N_3^{n-1} &\le k\ C\|\nabla Q_Y^n\|_{\mathbf{L}^4}\|\mathbf{V}^n\|_{\mathbf{L}^4}\|\hat{\Delta}_h Q_W^{n-1}\|_h \\ &\le k\sigma\|\hat{\Delta}_h Q_W^{n-1}\|_h^2 + k\ C\,(\sigma)\,\|\mathbf{V}^n\|_{\mathbf{H}_0^1}^2\|\nabla Q_Y^n\|_{\mathbf{L}^4}^2, \end{aligned}$$

$$\begin{aligned} N_4^{n-1} &\le k\rho\|\nabla W^{n+1}\|_{\mathbf{L}^4}\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{L}^4}\|\hat{\Delta}_h Q_W^{n-1}\| \\ &\le k\sigma\|\hat{\Delta}_h Q_W^{n-1}\|_h^2 + k\ C\,(\sigma)\,\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\|\nabla W^{n+1}\|_{\mathbf{L}^4}^2, \end{aligned}$$

$$\begin{aligned} N_5^{n-1} &\le 3\ C\ k\|Y^n\|_{\mathcal{C}(\bar{\Omega})}^2\|Q_W^{n-1}\|\|\hat{\Delta}_h Q_W^{n-1}\|_h \\ &\le k\sigma\|\hat{\Delta}_h Q_W^{n-1}\|_h^2 + k\ C\,(\sigma)\,\|Y^n\|_{\mathcal{C}(\bar{\Omega})}^4\|Q_W^{n-1}\|_{H_0}^2, \end{aligned}$$

$$\begin{aligned} N_6^{n-1} &\le k\ C\|Y^n - y_{d,h}^n\|\|\hat{\Delta}_h Q_W^{n-1}\|_h \\ &\le k\sigma\|\hat{\Delta}_h Q_W^{n-1}\|_h^2 + k\ C\,(\sigma)\,\|Y^n - y_{d,h}^n\|^2. \end{aligned}$$

Inserting the estimates of $N_1^{n-1} + \ldots + N_6^{n-1}$ in (5.177), we have

$$\begin{aligned} k\varepsilon^2\|\hat{\Delta}_h Q_W^{n-1}\|_h^2 &\le 6k\sigma\|\hat{\Delta}_h Q_W^{n-1}\|_h^2 \\ &+k\ C_1\,(\sigma)\left[\|d_t Q_Y^n\|_h^2 + \|Q_W^n\|_{H_0}^2 + \|\mathbf{V}^n\|_{\mathbf{H}_0^1}^2\|\nabla Q_Y^n\|_{\mathbf{L}^4}^2\right] \\ &+k\ C_2\,(\sigma)\left[\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\|\nabla W^{n+1}\|_{\mathbf{L}^4}^2 + \|Y^n\|_{\mathcal{C}(\bar{\Omega})}^4\|Q_W^{n-1}\|_{H_0}^2 + \|Y^n - y_{d,h}^n\|^2\right]. \end{aligned} \tag{5.178}$$

Thus, with σ small enough, from (5.178), we get

$$\begin{aligned} k\|\hat{\Delta}_h Q_W^{n-1}\|_h^2 &\le k\ C_1\left[\|d_t Q_Y^n\|_h^2 + \|Q_W^n\|_{H_0}^2 + \|\mathbf{V}^n\|_{\mathbf{H}_0^1}^2\|\nabla Q_Y^n\|_{\mathbf{L}^4}^2\right] \\ &+k\ C_2\left[\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\|\nabla W^{n+1}\|_{\mathbf{L}^4}^2 + \|Y^n\|_{\mathcal{C}(\bar{\Omega})}^4\|Q_W^{n-1}\|_{H_0}^2 + \|Y^n - y_{d,h}^n\|^2\right]. \end{aligned} \tag{5.179}$$

Setting in (5.179) $n = i$, summing up over $i = n, \ldots, N$, with $1 \le n \le N$, we can write

$$\sum_{i=n}^{N} k\|\hat{\Delta}_h Q_W^{i-1}\|_h^2 \le C_1 \sum_{i=n}^{N} k\left[\|d_t Q_Y^i\|_h^2 + \|Q_W^i\|_{H_0}^2 + \|\mathbf{V}^i\|_{\mathbf{H}_0^1}^2\|\nabla Q_Y^i\|_{\mathbf{L}^4}^2\right] \tag{5.180}$$

$$+C_2 \sum_{i=n}^{N} \Big[\|\mathbf{Q}_{\mathbf{V}}^i\|_{\mathbf{H}_0^1}^2 \|\nabla W^{i+1}\|_{\mathbf{L}^4}^2 + \|Y^i\|_{\mathcal{C}(\bar\Omega)}^4 \|Q_W^{i-1}\|_{H_0}^2 + \|Y^i - y_{d,h}^i\|^2 \Big].$$

iii) Substituting the estimate (5.179) in (5.176) we derive

$$k\|d_t Q_Y^n\|_h^2 + \frac{\varepsilon^2}{2\gamma}\|Q_W^{n-1}\|_h^2 - \frac{\varepsilon^2}{2\gamma}\|Q_W^n\|_h^2 + \frac{\varepsilon^2}{2\gamma}\|Q_W^{n-1} - Q_W^{n-1}\|_h^2$$
$$\leq k\ C_1(\sigma) \Big[\|Y^n - y_{d,h}^n\|^2 + \|d_t \mathbf{Q}_{\mathbf{V}}^n\|^2 + \|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2 \left(\|d_t Y^n\|^2 + \|\nabla W^{n+1}\|_{\mathbf{L}^4}^2 \right) \Big]$$
$$+k\ C_2(\sigma) \Big[\|\mathbf{V}^n\|_{\mathbf{H}_0^1}^2 \|\nabla Q_Y^n\|_{\mathbf{L}^4}^2 + \left(1 + \|Y^{n-1}\|_{\mathcal{C}(\bar\Omega)}^2 + \|Y^n\|_{\mathcal{C}(\bar\Omega)}^4 \right) \|Q_W^{n-1}\|_{H_0}^2 \Big]$$
$$+k\ C_3(\sigma)\, \|Q_W^n\|_{H_0}^2 + k\ C_4\ \sigma \|d_t Q_Y^n\|_h^2.$$

which implies, with σ small enough,

(5.181)
$$k\|d_t Q_Y^n\|_h^2 + \|Q_W^{n-1}\|_h^2 - \|Q_W^n\|_h^2 + \|Q_W^{n-1} - Q_W^{n-1}\|_h^2 \leq k\ C_1 \|Q_W^n\|_{H_0}^2$$
$$+k\ C_2 \Big[\|Y^n - y_{d,h}^n\|^2 + \|d_t \mathbf{Q}_{\mathbf{V}}^n\|^2 + \|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2 \left(\|d_t Y^n\|^2 + \|\nabla W^{n+1}\|_{\mathbf{L}^4}^2 \right) \Big]$$
$$+k\ C_3 \Big[\|\mathbf{V}^n\|_{\mathbf{H}_0^1}^2 \|\nabla Q_Y^n\|_{\mathbf{L}^4}^2 + \left(1 + \|Y^{n-1}\|_{\mathcal{C}(\bar\Omega)}^2 + \|Y^n\|_{\mathcal{C}(\bar\Omega)}^4 \right) \|Q_W^{n-1}\|_{H_0}^2 \Big].$$

Setting in (5.181) $n = i$, summing up over $i = n, \dots, N$, with $1 \leq n \leq N$, we infer

(5.182)
$$\|Q_W^{n-1}\|_h^2 + \sum_{i=n}^{N} k \left[\|d_t Q_Y^n\|_h^2 + \|Q_W^{n-1} - Q_W^{n-1}\|_h^2 \right] \leq C_1 \sum_{i=n}^{N} k \|Q_W^i\|_{H_0}^2$$
$$+C_2 \sum_{i=n}^{N} k \Big[\|Y^i - y_{d,h}^i\|^2 + \|d_t \mathbf{Q}_{\mathbf{V}}^i\|^2 + \|\mathbf{Q}_{\mathbf{V}}^i\|_{\mathbf{H}_0^1}^2 \left(\|d_t Y^i\|^2 + \|\nabla W^{i+1}\|_{\mathbf{L}^4}^2 \right) \Big]$$
$$+C_3 \sum_{i=n}^{N} k \Big[\|\mathbf{V}^i\|_{\mathbf{H}_0^1}^2 \|\nabla Q_Y^i\|_{\mathbf{L}^4}^2 + \left(1 + \|Y^{i-1}\|_{\mathcal{C}(\bar\Omega)}^2 + \|Y^i\|_{\mathcal{C}(\bar\Omega)}^4 \right) \|Q_W^{i-1}\|_{H_0}^2 \Big],$$

for all $n = 1, \dots, N$. From lemma assumption and the results established in Lemmas 5.4, 5.5, 5.6, 5.7 5.8, 5.9, 5.11, 5.19 and Theorem 5.18, we observe that all term at r.h.s. in (5.182) are bounded by a constant where the constant $C(\boldsymbol{\mathcal{U}})$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$. Hence, (5.168), (5.169) and (5.170) hold. Then, by (5.180), we note that also the result (5.171) is satisfied. Finally, using Theorem 6.4 in [41], we have the result (5.172). □

Lemma 5.21. *Under the same hypothesis of lemma 5.18 and with $k \leq k_{max}$, the solution $\mathcal{Q}_{\mathcal{Y}}$ of* (5.133)-(5.135) *is such that*

(5.183)
$$\sup_{n=1\dots,N} \|\hat\Delta_h Q_Y^{n-1}\|_h \leq C(\boldsymbol{\mathcal{U}}),$$

(5.184)
$$\sup_{n=1\dots,N} \|\nabla Q_Y^{n-1}\|_{L^p} \leq C(\boldsymbol{\mathcal{U}}),$$

(5.185)
$$\sup_{n=1\dots,N} \|Q_Y^{n-1}\|_{W^{1,4}} \leq C(\boldsymbol{\mathcal{U}}).$$

(5.186)
$$\sup_{n=1\dots,N} \|Q_Y^{n-1}\|_{\mathcal{C}(\bar\Omega)} \leq C(\boldsymbol{\mathcal{U}}),$$

for all $1 \leq p < \infty$, where the constant $C(\boldsymbol{\mathcal{U}})$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$, data problem and constant parameters, but it is independent of h, k.

Proof. With $\theta = -\hat{\Delta}_h Q_Y^{n-1}$ in the discrete adjoint equation (5.134g), using the definition (A.36) of the discrete Laplacian and integrating by parts in space, we have

$$\gamma \|\hat{\Delta}_h Q_Y^{n-1}\|_h^2 = O_1^{n-1} + O_2^{n-1} + O_3^{n-1}, \tag{5.187}$$

where

$$\begin{aligned} O_1^{n-1} &= \left(Q_W^{n-1}, \hat{\Delta}_h Q_Y^{n-1}\right)_h, \\ O_2^{n-1} &= -\rho\left(\nabla Y^{n-1} \cdot \mathbf{Q}_{\mathbf{V}}^{n-1}, \hat{\Delta}_h Q_Y^{n-1}\right), \\ O_3^{n-1} &= -\rho\left(Y^{n-1}\left[\nabla \cdot \mathbf{Q}_{\mathbf{V}}^{n-1}\right], \hat{\Delta}_h Q_Y^{n-1}\right). \end{aligned}$$

By the generalized Holder's inequality (A.14), Young's inequality (A.13), inequality (A.17)

$$O_1^{n-1} \leq \|Q_W^{n-1}\|_h \|\hat{\Delta}_h Q_Y^{n-1}\|_h \leq \sigma \|\hat{\Delta}_h Q_Y^{n-1}\|_h^2 + C(\sigma)\|Q_W^{n-1}\|_h^2,$$

$$\begin{aligned} O_2^{n-1} &\leq \rho\, C \|\nabla Y^{n-1}\|_{\mathbf{L}^4} \|\mathbf{Q}_{\mathbf{V}}^{n-1}\|_{\mathbf{L}^4} \|\hat{\Delta}_h Q_Y^{n-1}\|_h \\ &\leq \sigma \|\hat{\Delta}_h Q_Y^{n-1}\|_h^2 + C(\sigma)\|\nabla Y^{n-1}\|_{\mathbf{L}^4}^2 \|\mathbf{Q}_{\mathbf{V}}^{n-1}\|_{\mathbf{H}_0^1}^2, \end{aligned}$$

$$\begin{aligned} O_3^{n-1} &\leq \rho \|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})} \|\nabla \cdot \mathbf{Q}_{\mathbf{V}}^{n-1}\| \|\hat{\Delta}_h Q_Y^{n-1}\|_h \\ &\leq \sigma \|\hat{\Delta}_h Q_Y^{n-1}\|_h^2 + C(\sigma)\|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}^2 \|\mathbf{Q}_{\mathbf{V}}^{n-1}\|_{\mathbf{H}_0^1}^2. \end{aligned}$$

Hence, inserting the estimates of $O_1^{n-1}, O_2^{n-1}, O_3^{n-1}$ in (5.187), we get

$$\begin{aligned} \gamma \|\hat{\Delta}_h Q_Y^{n-1}\|_h^2 &\leq 3\sigma \|\hat{\Delta}_h Q_Y^{n-1}\|_h^2 \\ +C(\sigma)\Big[\|Q_W^{n-1}\|_h^2 &+ \left(\|\nabla Y^{n-1}\|_{\mathbf{L}^4}^2 + \|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}^2\right)\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|_{\mathbf{H}_0^1}^2\Big]. \end{aligned}$$

which implies, with σ small enough,

$$\|\hat{\Delta}_h Q_Y^{n-1}\|_h^2 \leq C\left[\|Q_W^{n-1}\|_h^2 + \left(\|\nabla Y^{n-1}\|_{\mathbf{L}^4}^2 + \|Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}^2\right)\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|_{\mathbf{H}_0^1}^2\right], \tag{5.188}$$

for all $n = 1, \ldots, N$. From (5.188), applying the results (5.59), (5.60), (5.136), (5.169) established in the previous lemmas and theorems, we derive that (5.183) holds. Then, inequality (A.39) implies (5.184). Finally, using the embedding $W^{1,4} \hookrightarrow \mathcal{C}(\bar{\Omega})$ and the interpolation inequality (A.51), we can write

$$\begin{aligned} \|Q_Y^{n-1}\|_{\mathcal{C}(\bar{\Omega})}^4 \leq C\|Q_Y^{n-1}\|_{W^{1,4}}^4 &= C\left[\|Q_Y^{n-1}\|_{L^4}^4 + \|\nabla Q_Y^{n-1}\|_{L^4}^4\right] \\ &\leq C\left[\|Q_Y^{n-1}\|_{H_0}^4 + \|\hat{\Delta}_h Q_Y^{n-1}\|_h^4\right]. \end{aligned}$$

Hence, using (5.140) established in Theorem 5.18 and (5.183) above, we infer that (5.185), (5.186) are satisfied. □

Lemma 5.22. *Under the same hypothesis of lemma 5.18 and with* $k \leq k_{max}$*, the solution* $\mathcal{Q}_{\mathcal{V}}$ *of* (5.133)-(5.135) *is such that*

$$\sum_{n=1}^{N} k\|\tilde{\Delta}_h \mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 \leq C(\boldsymbol{\mathcal{U}}), \tag{5.189}$$

where the constant $C(\boldsymbol{\mathcal{U}})$ *depends just on* $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$*, data problem and constant parameters, but it is independent of* h, k.

Proof. With $\boldsymbol{\psi} = \mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}$ in (5.134a), we have

$$\begin{aligned} &k\nu\left(\nabla \mathbf{Q}_{\mathbf{V}}^{n-1}, \nabla \mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\right) = k\left(d_t \mathbf{Q}_{\mathbf{V}}^{n}, \mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\right) \\ &-kB\left(\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}, \mathbf{V}^{n+1}, \mathbf{Q}_{\mathbf{V}}^{n}\right) - kB\left(\mathbf{V}^{n-1}, \mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}, \mathbf{Q}_{\mathbf{V}}^{n-1}\right) + k\left(Y^n \nabla Q_Y^n, \mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\right), \end{aligned} \tag{5.190}$$

where the discrete Stokes operator $\mathbf{A}^h$ is defined in (A.40).
In (5.190), using the properties of $\mathbf{A}^h$, it holds

$$\begin{aligned} k\nu\left(\nabla \mathbf{Q}_{\mathbf{V}}^{n-1}, \nabla \mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\right) &= k\nu\left(\nabla \mathbf{Q}_{\mathbf{V}}^{n-1}, -\nabla \mathbf{T}^h \tilde{\Delta}_h \mathbf{Q}_{\mathbf{V}}^{n-1}\right) \\ &= k\nu\left(\tilde{\Delta}_h \mathbf{Q}_{\mathbf{V}}^{n-1}, \mathbf{T}^h \tilde{\Delta}_h \mathbf{Q}_{\mathbf{V}}^{n-1}\right) \\ &= k\nu\left(\mathbf{T}^h \tilde{\Delta}_h \mathbf{Q}_{\mathbf{V}}^{n-1}, \mathbf{T}^h \tilde{\Delta}_h \mathbf{Q}_{\mathbf{V}}^{n-1}\right) \\ &= k\nu\|\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\|^2. \end{aligned} \tag{5.191}$$

Substituting (5.191) in (5.190), we get

$$k\nu\|\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 = P_1^{n-1} + P_2^{n-1} + P_3^{n-1} + P_4^{n-1}, \tag{5.192}$$

where

$$\begin{aligned} P_1^{n-1} &= k\left(d_t \mathbf{Q}_{\mathbf{V}}^{n}, \mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\right), \\ P_2^{n-1} &= -kB\left(\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}, \mathbf{V}^{n+1}, \mathbf{Q}_{\mathbf{V}}^{n}\right), \\ P_3^{n-1} &= -kB\left(\mathbf{V}^{n-1}, \mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}, \mathbf{Q}_{\mathbf{V}}^{n-1}\right), \\ P_4^{n-1} &= k\left(Y^n \nabla Q_Y^n, \mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\right). \end{aligned}$$

Using the generalized Holder's inequality (A.14), Young's inequality(A.13), the embedding $\mathbf{W}^{1,4} \hookrightarrow \boldsymbol{\mathcal{C}}(\bar{\Omega})$, discrete embedding inequality (A.54) and inequality (A.17), we can write

$$\begin{aligned} P_1^{n-1} &\leq k\|d_t \mathbf{Q}_{\mathbf{V}}^{n}\|\|\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\| \\ &\leq k\sigma\|\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + kC(\sigma)\|d_t \mathbf{Q}_{\mathbf{V}}^{n}\|^2, \end{aligned}$$

$$\begin{aligned} P_2^{n-1} &= -\frac{k}{2}\left(\left[\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1} \cdot \nabla\right] \mathbf{V}^{n+1}, \mathbf{Q}_{\mathbf{V}}^{n}\right) + \frac{k}{2}\left(\left[\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1} \cdot \nabla\right] \mathbf{Q}_{\mathbf{V}}^{n}, \mathbf{V}^{n+1}\right) \\ &\leq \frac{k}{2}\|\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\|\|\nabla \mathbf{V}^{n+1}\|_{\mathbf{L}^4}\|\mathbf{Q}_{\mathbf{V}}^{n}\|_{\mathbf{L}^4} + \frac{k}{2}\|\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\|\|\nabla \mathbf{Q}_{\mathbf{V}}^{n}\|\|\mathbf{V}^{n+1}\|_{\boldsymbol{\mathcal{C}}(\bar{\Omega})} \\ &\leq kC_1\|\mathbf{A}^h \mathbf{Q}_{\mathbf{V}}^{n-1}\|\|\tilde{\Delta}_h \mathbf{V}^{n+1}\|^{\frac{1}{2}}\|\nabla \mathbf{V}^{n+1}\|^{\frac{1}{2}}\|\mathbf{Q}_{\mathbf{V}}^{n}\|_{\mathbf{H}_0^1} \end{aligned}$$

$$
\begin{aligned}
&+ kC_2\|\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\|\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}\left[\|\mathbf{V}^{n+1}\|_{\mathbf{L}^4}^4 + \|\nabla\mathbf{V}^{n+1}\|_{\mathbf{L}^4}^4\right]^{\frac{1}{4}} \\
&\leq k\sigma\|\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + kC_1(\sigma)\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\|\tilde{\Delta}_h\mathbf{V}^{n+1}\|\|\nabla\mathbf{V}^{n+1}\| \\
&+ k\sigma\|\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + kC_2(\sigma)\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\left[\|\mathbf{V}^{n+1}\|_{\mathbf{L}^4}^4 + \|\nabla\mathbf{V}^{n+1}\|_{\mathbf{L}^4}^4\right]^{\frac{1}{2}} \\
&\leq 2k\sigma\|\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + kC_1(\sigma)\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\left[\|\tilde{\Delta}_h\mathbf{V}^{n+1}\|^2 + \|\mathbf{V}^{n+1}\|_{\mathbf{H}_0^1}^2\right] \\
&+ kC_2(\sigma)\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\left[\|\mathbf{V}^{n+1}\|_{\mathbf{L}^4}^2 + \|\nabla\mathbf{V}^{n+1}\|_{\mathbf{L}^4}^2\right] \\
&\leq 2k\sigma\|\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + kC_1(\sigma)\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\left[\|\tilde{\Delta}_h\mathbf{V}^{n+1}\|^2 + \|\mathbf{V}^{n+1}\|_{\mathbf{H}_0^1}^2\right] \\
&+ kC_2(\sigma)\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\|\tilde{\Delta}_h\mathbf{V}^{n+1}\|\|\nabla\mathbf{V}^{n+1}\| \\
&\leq 2k\sigma\|\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + kC(\sigma)\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\left[\|\tilde{\Delta}_h\mathbf{V}^{n+1}\|^2 + \|\mathbf{V}^{n+1}\|_{\mathbf{H}_0^1}^2\right],
\end{aligned}
$$

$$
\begin{aligned}
P_3^{n-1} &\leq -\frac{k}{2}\left(\left[\mathbf{V}^{n-1}\cdot\nabla\right]\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}, \mathbf{Q}_{\mathbf{V}}^{n-1}\right) + \frac{k}{2}\left(\left[\mathbf{V}^{n-1}\cdot\nabla\right]\mathbf{Q}_{\mathbf{V}}^{n-1}, \mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\right) \\
&= k\left(\left[\mathbf{V}^{n-1}\cdot\nabla\right]\mathbf{Q}_{\mathbf{V}}^{n-1}, \mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\right) + \frac{k}{2}\left(\nabla\cdot\mathbf{V}^{n-1}, \mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\cdot\mathbf{Q}_{\mathbf{V}}^{n-1}\right) \\
&\leq k\|\mathbf{V}^{n-1}\|_{\boldsymbol{\mathcal{C}}(\bar{\Omega})}\|\nabla\mathbf{Q}_{\mathbf{V}}^{n-1}\|\|\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\| + \frac{k}{2}\|\nabla\cdot\mathbf{V}^{n-1}\|_{\mathbf{L}^4}\|\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\|\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|_{\mathbf{L}^4} \\
&\leq 2k\sigma\|\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + kC(\sigma)\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|_{\mathbf{H}_0^1}^2\left[\|\tilde{\Delta}_h\mathbf{V}^{n-1}\|^2 + \|\mathbf{V}^{n-1}\|_{\mathbf{H}_0^1}^2\right],
\end{aligned}
$$

$$
\begin{aligned}
P_4^{n-1} &\leq k\|Y^n\|_{\mathcal{C}(\bar{\Omega})}\|\nabla Q_Y^n\|\|\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\| \\
&\leq k\sigma\|\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + kC(\sigma)\|Y^n\|_{\mathcal{C}(\bar{\Omega})}^2\|Q_Y^n\|_{H_0}^2.
\end{aligned}
$$

Inserting the estimates of $P_1^{n-1}, \ldots, P_4^{n-1}$ in (5.192), we derive

(5.193)
$$
\begin{aligned}
k\nu\|\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 \leq\ & 6k\sigma\|\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 + k\ C_1(\sigma)\|d_t\mathbf{Q}_{\mathbf{V}}^n\|^2 \\
&+ k\ C_2(\sigma)\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\left[\|\tilde{\Delta}_h\mathbf{V}^{n+1}\|^2 + \|\mathbf{V}^{n+1}\|_{\mathbf{H}_0^1}^2\right] \\
&+ k\ C_3(\sigma)\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|_{\mathbf{H}_0^1}^2\left[\|\tilde{\Delta}_h\mathbf{V}^{n-1}\|^2 + \|\mathbf{V}^{n-1}\|_{\mathbf{H}_0^1}^2\right] \\
&+ k\ C_4(\sigma)\|Y^n\|_{\mathcal{C}(\bar{\Omega})}^2\|Q_Y^n\|_{H_0}^2,
\end{aligned}
$$

which implies, with σ small enough,

(5.194)
$$
\begin{aligned}
k\|\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 \leq\ & k\ C_1\|d_t\mathbf{Q}_{\mathbf{V}}^n\|^2 + k\ C_2\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\left[\|\tilde{\Delta}_h\mathbf{V}^{n+1}\|^2 + \|\mathbf{V}^{n+1}\|_{\mathbf{H}_0^1}^2\right] \\
&+k\ C_3\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|_{\mathbf{H}_0^1}^2\left[\|\tilde{\Delta}_h\mathbf{V}^{n-1}\|^2 + \|\mathbf{V}^{n-1}\|_{\mathbf{H}_0^1}^2\right] + k\ C_4\|Y^n\|_{\mathcal{C}(\bar{\Omega})}^2\|Q_Y^n\|_{H_0}^2.
\end{aligned}
$$

Summing up over $n = 1, \ldots, N$ in (5.194), we infer

(5.195)
$$
\sum_{n=1}^N k\|\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2 \leq C_1\sum_{n=1}^N k\|d_t\mathbf{Q}_{\mathbf{V}}^n\|^2 + C_2\sum_{n=1}^N k\|\mathbf{Q}_{\mathbf{V}}^n\|_{\mathbf{H}_0^1}^2\left[\|\tilde{\Delta}_h\mathbf{V}^{n+1}\|^2 + \|\mathbf{V}^{n+1}\|_{\mathbf{H}_0^1}^2\right]
$$

$$+C_3\sum_{n}^{N}k\|\mathbf{Q}_{\mathbf{V}}^{n-1}\|_{\mathbf{H}_0^1}^2\left[\|\tilde{\Delta}_h\mathbf{V}^{n-1}\|^2+\|\mathbf{V}^{n-1}\|_{\mathbf{H}_0^1}^2\right]+C_4\sum_{n=1}^{N}k\|Y^n\|_{\mathcal{C}(\bar{\Omega})}^2\|Q_Y^n\|_{H_0}^2.$$

Then, from the results established in the previous lemmas, theorems and corollaries, we realize that

$$\sum_{n=1}^{N}k\|\mathbf{A}^h\mathbf{Q}_{\mathbf{V}}^{n-1}\|^2\leq C\left(\boldsymbol{\mathcal{U}}\right).$$

So, see [6], from the following inequality

$$C\|\tilde{\Delta}_h\mathbf{V}\|\leq\|\mathbf{A}^h\mathbf{V}\|\leq\|\tilde{\Delta}_h\mathbf{V}\|,$$

which is valid for all $\mathbf{V}\in\mathbf{V}_h$, we conclude that the result (5.189) holds. □

5.5. Convergence of the Solutions of the Discrete Optimal Control Problem

In this section we study, as $h,k\to 0$, the convergence of the solution of the optimality conditions (5.133)-(5.135) of the discrete optimal control problem 5.1, to the solution of the optimality conditions (4.22)-(4.24) of the continuous optimal control Problem 4.1.
Regarding the initial conditions $\mathbf{v}_{0,h},y_{0,h}$ and the desired state $y_{d,h}^n, n=1,\ldots,N$, in the discrete non-smooth optimal control Problem 5.1, given

$$\mathbf{v}_0\in\boldsymbol{\mathcal{D}}\cap\mathbf{H}^2,\quad y_0\in H_0\cap H^2\cap\mathcal{K},\quad y_d\in\mathcal{C}\left([0,T];L_0^2\right),$$

we assume

$$\mathbf{v}_{0,h}=\mathbf{Q}_s^h\mathbf{v}_0,\quad y_{0,h}=Q^hy_0,\quad y_{d,h}^n=Q_0^h\,y_d\left(t_n\right),\tag{5.196}$$

where the projection operator $\mathbf{Q}_s^h,Q^h,Q_0^h$, are defined, respectively, in (A.48), (A.41), (A.43). In this way, we can suppose that there exists a constant $\tilde{C}$, such that

$$E\left(\mathbf{v}_{0,h},y_{0,h}\right)+\|\mathbf{v}_{0,h}\|_{\mathbf{H}_0^1}+\|\tilde{\Delta}_h\mathbf{v}_{0,h}\|+\|\hat{\Delta}_hy_{0,h}\|_h+\sum_{n=1}^{N}k\|y_{d,h}^n\|^2\leq\tilde{C},\tag{5.197}$$

independently of h,k. With this assumptions, from the results established in the previous sections, any solution of the discrete optimality conditions (5.133)-(5.135)

$$\left(\boldsymbol{\mathcal{V}},\mathcal{P},\mathcal{Y},\mathcal{W},\boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}}},\mathcal{Q}_{\mathcal{P}},\mathcal{Q}_{\mathcal{Y}},\mathcal{Q}_{\mathcal{W}}\right),$$

is such that

$$\sup_{n,m,i,j=1,\ldots,N}\left[\|\mathbf{V}^n\|_{\mathbf{H}_0^1}+\|Y^m\|_{H_0}+\|\Delta_hY^i\|^2+\|W^j\|\right]\leq C\left(\boldsymbol{\mathcal{U}}\right),\tag{5.198}$$

$$\sup_{n=1,\dots,N} \left\| \sum_{i=1}^{n} k\ P^i \right\| \le C\left(\boldsymbol{\mathcal{U}}\right), \tag{5.199}$$

$$\sum_{n=1}^{N} k \left[\|d_t \mathbf{V}^n\|^2 + \frac{1}{k}\|\mathbf{V}^n - \mathbf{V}^{n-1}\|^2_{\mathbf{H}^1_0} + \|\tilde{\Delta}_h \mathbf{V}^n\|^2 \right] + \tag{5.200}$$
$$+ \sum_{n=1}^{N} k \ \left[\|d_t Y^n\|^2 + \frac{1}{k}\left\|Y^n - Y^{n-1}\right\|^2_{H_0} + \|W^n\|^2_{H^1} + \frac{1}{k}\|W^n - W^{n+1}\|^2 \right] \le C\left(\boldsymbol{\mathcal{U}}\right),$$

$$\sup_{n,m,i,j=1,\dots,N} \left[\|\mathbf{Q}^{n-1}_{\mathbf{V}}\|_{\mathbf{H}^1_0} + \|Q^{m-1}_Y\|_{H_0} + \|\Delta_h Q^{i-1}_Y\|^2 + \|Q^{j-1}_W\| \right] \le C\left(\boldsymbol{\mathcal{U}}\right), \tag{5.201}$$

$$\sup_{i=1,\dots,n} \left\| \sum_{i=1}^{n} k\ Q^{i-1}_P \right\| \le C\left(\boldsymbol{\mathcal{U}}\right), \tag{5.202}$$

$$\sum_{n=1}^{N} k \left[\|d_t \mathbf{Q}^n_{\mathbf{V}}\|^2 + \frac{1}{k}\|\mathbf{Q}^{n-1}_{\mathbf{V}} - \mathbf{Q}^n_{\mathbf{V}}\|^2_{\mathbf{H}^1_0} + \|\tilde{\Delta}_h \mathbf{Q}^{n-1}_{\mathbf{V}}\|^2 \right] + \tag{5.203}$$
$$+ \sum_{n=1}^{n} k \left[\|d_t Q^n_Y\|^2 + \frac{1}{k}\|Q^{n-1}_Y - Q^n_Y\|^2_{H_0} \right] +$$
$$+ \sum_{n=1}^{N} k \left[\|Q^{n-1}_W\|^2_{H_0} + \frac{1}{k}\|Q^{n-1}_W - Q^n_W\|^2 + \|\Delta_h Q^{n-1}_W\|^2 \right] \le C\left(\boldsymbol{\mathcal{U}}\right),$$

for all $h, k \le k_{max}$, where the constant $C\left(\boldsymbol{\mathcal{U}}\right)$ depends just on $\|\boldsymbol{\mathcal{U}}\|_{L^2(\mathbf{L}^2)}$, data problem and constant parameters, but it is independent of h, k.

Remark 5.23. In the following theorems consider sequences of the discretization parameters

$$\{h_n\}_{n\in\mathbb{N}} \quad \text{and} \quad \{k_m\}_{m\in\mathbb{N}} \subset (0, k_{max}),$$

such that

$$h_n \to 0^+, \quad k_m \to 0^+,$$

as $n, m \to +\infty$. In this way, the estimates (5.198)-(5.203) are satisfied for all h_n, k_m. In order to make the reading more fluent, we skip the indices n, m and we simply write

$$h, k \to 0. \tag{5.204}$$

Even in the case of extracted subsequences, we use the notation (5.204), without relabelling.

Theorem 5.24. *Consider a sequence $h, k \to 0$ and let*

$$\{(\boldsymbol{\mathcal{V}}_{h,k}, \mathcal{P}_{h,k}, \mathcal{Y}_{h,k}, \mathcal{W}_{h,k}, \boldsymbol{\mathcal{U}}_{h,k})\}_{h,k},$$

be a corresponding sequence of the time interpolation of the solutions of the discrete optimal control Problem 5.1. Then, there exist functions

$$\mathbf{v} \in H^1\left(\mathbf{L}^2\right) \cap L^\infty\left(\mathbf{H}_0^1\right), \quad \int_0^t p(s)\ ds \in L^\infty\left(L_0^2\right)$$

$$y \in H^1\left(L^2\right) \cap L^\infty\left(H_0\right), \quad w \in L^2\left(H^1\right) \cap L^\infty\left(L^2\right), \quad \mathbf{u} \in L^2\left(\mathbf{L}^2\right)$$

and a subsequence (not relabeled), such that,

$$\boldsymbol{\mathcal{V}}_{h,k}^{\bullet} \rightharpoonup \mathbf{v}, \quad \text{in } H^1\left(\mathbf{L}^2\right), \tag{5.205}$$

$$\boldsymbol{\mathcal{V}}_{h,k}^{\bullet,\pm} \overset{*}{\rightharpoonup} \mathbf{v}, \quad \text{in } L^\infty\left(\mathbf{H}_0^1\right), \tag{5.206}$$

$$\boldsymbol{\mathcal{V}}_{h,k}^{\bullet,\pm} \rightarrow \mathbf{v}, \quad \text{in } L^2\left(\mathbf{H}_0^1\right), \tag{5.207}$$

$$\int_0^t \mathcal{P}_{h,k}^{+}(s)\ ds \overset{*}{\rightharpoonup} \int_0^t p(s)\ ds, \quad \text{in } L^\infty\left(L_0^2\right), \tag{5.208}$$

$$\mathcal{Y}_{h,k}^{\bullet} \rightharpoonup y, \quad \text{in } H^1\left(L^2\right), \tag{5.209}$$

$$\mathcal{Y}_{h,k}^{\bullet,\pm} \overset{*}{\rightharpoonup} y, \quad \text{in } L^\infty\left(H_0\right), \tag{5.210}$$

$$\mathcal{Y}_{h,k}^{\bullet,\pm} \rightarrow y, \quad \text{in } L^2\left(H_0\right), \tag{5.211}$$

$$\mathcal{W}_{h,k}^{+} \rightharpoonup w, \quad \text{in } L^2\left(H^1\right), \tag{5.212}$$

$$\mathcal{W}_{h,k}^{+} \overset{*}{\rightharpoonup} w, \quad \text{in } L^\infty\left(L^2\right), \tag{5.213}$$

$$\boldsymbol{\mathcal{U}}_{h,k}^{+} \rightharpoonup \mathbf{u}, \quad \text{in } L^2\left(\mathbf{L}^2\right). \tag{5.214}$$

Proof. We consider a function $\bar{\boldsymbol{\mathcal{U}}} \in L^2(\mathbf{L}^2)$. Then, by the definition (5.4) of the cost functional $J_{h,k} : \mathbf{X}_{h,k} \times L^2(\mathbf{L}^2) \rightarrow \mathbb{R}$, the assumption (5.197), the estimate (5.198), we have, for all h, k,

$$\frac{\alpha}{2}\|\boldsymbol{\mathcal{U}}_{h,k}\|_{L^2(\mathbf{L}^2)}^2 \leq J_{h,k}\left(s_{h,k}\left(\boldsymbol{\mathcal{U}}_{h,k}\right), \boldsymbol{\mathcal{U}}_{h,k}\right) \leq J_{h,k}\left(s_{h,k}\left(\bar{\boldsymbol{\mathcal{U}}}\right), \bar{\boldsymbol{\mathcal{U}}}\right) \leq C\left(\bar{\boldsymbol{\mathcal{U}}}\right) + \frac{\alpha}{2}\|\bar{\boldsymbol{\mathcal{U}}}\|_{L^2(\mathbf{L}^2)}^2, \tag{5.215}$$

where the map $s_{h,k} : L^2(\mathbf{L}^2) \rightarrow \mathbf{X}_{h,k}$ is the state equations solution operator defined in (5.12) and

$$\left(s_{h,k}\left(\boldsymbol{\mathcal{U}}_{h,k}\right), \boldsymbol{\mathcal{U}}_{h,k}\right) = \left(\boldsymbol{\mathcal{X}}_{h,k}, \boldsymbol{\mathcal{U}}_{h,k}\right) = \left(\boldsymbol{\mathcal{V}}_{h,k}, \mathcal{P}_{h,k}, \mathcal{Y}_{h,k}, \mathcal{W}_{h,k}, \boldsymbol{\mathcal{U}}_{h,k}\right),$$

is a solution of the optimal control Problem 5.1. Using (5.215) above, we realize that the sequence $\{\boldsymbol{\mathcal{U}}_{h,k}\}_{h,k}$ is bounded by a constant which is independent of h, k. So, using the estimates (5.198)-(5.200), there exists a convergent subsequence such that the limits (5.214), (5.205), (5.206), (5.208)-(5.210), (5.212) and (5.213) hold. Furthermore, by the estimates (5.198), (5.200), we have

$$\begin{aligned} &\|\boldsymbol{\mathcal{V}}_{h,k}^{\bullet}\|_{H^1(\mathbf{L}^2)} + \|\boldsymbol{\mathcal{V}}_{h,k}^{\bullet}\|_{L^\infty\left(\mathbf{H}_0^1\right)} + \|\tilde{\Delta}_h \boldsymbol{\mathcal{V}}_{h,k}^{\bullet}\|_{L^2(\mathbf{L}^2)} \\ &+\|\mathcal{Y}_{h,k}^{\bullet}\|_{H^1(L^2)} + \|\mathcal{Y}_{h,k}^{\bullet}\|_{L^\infty(H_0)} + \|\Delta_h \mathcal{Y}_{h,k}^{\bullet}\|_{L^2(L^2)} \leq C, \end{aligned}$$

uniformly in h, k. So, using the results established in [13] (Lemma 2.4) or [83] (Lemma 4.9), we derive the strong convergence statements (5.207) and (5.211). It remains to prove that $\boldsymbol{\mathcal{V}}_{h,k}^{\bullet,\pm}$ and $\mathcal{Y}_{h,k}^{\bullet,\pm}$ converge, respectively, to the same limit. It can be done as in the proof of Theorem 3.23 and we skip this part of the proof. □

Theorem 5.25. *Consider a sequence $h, k \to 0$ and let*

$$\{(\boldsymbol{\mathcal{V}}_{h,k}, \mathcal{P}_{h,k}, \mathcal{Y}_{h,k}, \mathcal{W}_{h,k}, \boldsymbol{\mathcal{U}}_{h,k}, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}, \mathcal{Q}_{\mathcal{P},h,k}, \mathcal{Q}_{\mathcal{Y},h,k}, \mathcal{Q}_{\mathcal{W},h,k})\}_{h,k},$$

be a corresponding sequence of the time interpolation of the solutions of the optimality conditions (5.133)-(5.135)*, where in particular*

$$\{(\boldsymbol{\mathcal{V}}_{h,k}, \mathcal{P}_{h,k}, \mathcal{Y}_{h,k}, \mathcal{W}_{h,k}, \boldsymbol{\mathcal{U}}_{h,k})\}_{h,k},$$

is a sequence of solutions of the discrete optimal control Problem 5.1. Then, there exist functions

$$\mathbf{q}_{\mathbf{v}} \in H^1\left(\mathbf{L}^2\right) \cap L^\infty\left(\mathbf{H}_0^1\right), \quad \int_0^t q_p(s)\ ds \in L^\infty\left(L_0^2\right)$$

$$q_y \in H^1\left(L^2\right) \cap L^\infty\left(H_0\right), \quad q_w \in L^2\left(H_0\right) \cap L^\infty\left(L^2\right),$$

and a subsequence (not relabeled) such that,

$$\boldsymbol{\mathcal{Q}}^{\bullet}_{\boldsymbol{\mathcal{V}},h,k} \rightharpoonup \mathbf{q}_{\mathbf{v}}, \quad \text{in } H^1\left(\mathbf{L}^2\right), \tag{5.216}$$

$$\boldsymbol{\mathcal{Q}}^{\bullet;\pm}_{\boldsymbol{\mathcal{V}},h,k} \overset{*}{\rightharpoonup} \mathbf{q}_{\mathbf{v}}, \quad \text{in } L^\infty\left(\boldsymbol{\mathcal{D}}\right), \tag{5.217}$$

$$\boldsymbol{\mathcal{Q}}^{\bullet;\pm}_{\boldsymbol{\mathcal{V}},h,k} \to \mathbf{q}_{\mathbf{v}}, \quad \text{in } L^2\left(\boldsymbol{\mathcal{D}}\right), \tag{5.218}$$

$$\int_0^t \mathcal{Q}^{+}_{\mathcal{P},h,k}(s)\ ds \overset{*}{\rightharpoonup} \int_0^t q_p(s)\ ds, \quad \text{in } L^\infty\left(L_0^2\right), \tag{5.219}$$

$$\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k} \rightharpoonup q_y, \quad \text{in } H^1\left(L^2\right), \tag{5.220}$$

$$\mathcal{Q}^{\bullet;\pm}_{\mathcal{Y},h,k} \overset{*}{\rightharpoonup} q_y, \quad \text{in } L^\infty\left(H_0\right), \tag{5.221}$$

$$\mathcal{Q}^{\bullet;\pm}_{\mathcal{Y},h,k} \to q_y, \quad \text{in } L^2\left(H_0\right), \tag{5.222}$$

$$\mathcal{Q}^{\bullet;\pm}_{\mathcal{W},h,k} \overset{*}{\rightharpoonup} q_w, \quad \text{in } L^\infty\left(L^2\right), \tag{5.223}$$

$$\mathcal{Q}^{\bullet;\pm}_{\mathcal{W},h,k} \rightharpoonup q_w, \quad \text{in } L^2\left(H_0\right). \tag{5.224}$$

Proof. From (5.214) established in Theorem 5.24 and by the estimates (5.201) and (5.203), we have the results (5.216),(5.217), (5.219)-(5.221), (5.223) and (5.224). Moreover, from (5.201), (5.203), we get

$$\begin{aligned}
&\|\boldsymbol{\mathcal{Q}}^{\bullet}_{\boldsymbol{\mathcal{V}},h,k}\|_{H^1(\mathbf{L}^2)} + \|\boldsymbol{\mathcal{Q}}^{\bullet}_{\boldsymbol{\mathcal{V}},h,k}\|_{L^\infty\left(\mathbf{H}_0^1\right)} + \|\tilde{\Delta}_h \boldsymbol{\mathcal{Q}}^{\bullet}_{\boldsymbol{\mathcal{V}},h,k}\|_{L^2(\mathbf{L}^2)} \\
&+\|\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k}\|_{H^1(L^2)} + \|\mathcal{Q}^{\bullet}_{\mathcal{Y},h,k}\|_{L^\infty(H_0)} + \|\Delta_h \mathcal{Q}^{\bullet}_{\mathcal{Y},h,k}\|_{L^2(L^2)} \le C.
\end{aligned}$$

Then, by the results in [13] (Lemma 2.4) or [83] (Lemma 4.9), we derive the strong convergence statements (5.218) and (5.222). Finally, as in the proof of Theorem 3.23, by (5.201), (5.203), we can show that $\boldsymbol{\mathcal{Q}}^{\bullet;\pm}_{\boldsymbol{\mathcal{V}},h,k}$, $\mathcal{Q}^{\bullet;\pm}_{\mathcal{Y},h,k}$, $\mathcal{Q}^{\bullet;\pm}_{\mathcal{W},h,k}$ converge, respectively, to the same limit. □

In the next Theorem 5.26, we derive regularity properties for the functions

$$\mathbf{v},\ y,\ w,\ \mathbf{u},\ \mathbf{q}_{\mathbf{v}},\ q_y,\ q_w,$$

considered in the previous Theorems 3.23, 3.24. Moreover, we show that these functions are solution of the optimality conditions (4.22)-(4.24) of the continuous optimal control Problem 4.1.

Theorem 5.26. *The functions*

$$\mathbf{v},\ y,\ w,\ \mathbf{u},\ \mathbf{q}_{\mathbf{v}},\ q_y,\ q_w,$$

considered in Theorems 5.24, 5.25 are such that

$$\mathbf{v} \in H^1(\boldsymbol{\mathcal{S}}) \cap L^\infty(\boldsymbol{\mathcal{D}}), \tag{5.225}$$
$$y \in H^1\left(L_0^2\right) \cap L^\infty(H_0), \tag{5.226}$$
$$w \in L^2\left(H^1\right) \cap L^\infty\left(L^2\right), \tag{5.227}$$
$$\mathbf{u} \in H^1(\boldsymbol{\mathcal{S}}) \cap L^\infty(\boldsymbol{\mathcal{D}}), \tag{5.228}$$
$$\mathbf{q}_{\mathbf{v}} \in H^1(\boldsymbol{\mathcal{S}}) \cap L^\infty(\boldsymbol{\mathcal{D}}), \tag{5.229}$$
$$q_y \in H^1\left(L_0^2\right) \cap L^\infty(H_0), \tag{5.230}$$
$$q_w \in L^2(H_0) \cap L^\infty\left(L^2\right), \tag{5.231}$$

and they satisfy the optimality conditions (4.22)-(4.24) of the continuous optimal control Problem 4.1. Furthermore, it holds

$$\|\mathbf{q}_{\mathbf{v}t}\|_{L^2(\boldsymbol{\mathcal{S}})} + \|\mathbf{q}_{\mathbf{v}}\|_{L^\infty(\boldsymbol{\mathcal{D}})} + \|q_{yt}\|_{L^2\left(L_0^2\right)} + \|q_y\|_{L^\infty(H_0)} + \|q_w\|_{L^2(H_0)} \leq C(\mathbf{u}), \tag{5.232}$$

where the constant $C(\mathbf{u})$ depends just on $\|\mathbf{u}\|_{L^2(\mathbf{L}^2)}$, data problem and constant parameters.

Proof. We divide the proof in several steps.

i) Results (5.225), (4.22a), (4.22b).

From the discrete state equations (5.133a)-(5.133c), we have that

$$\begin{aligned} \int_0^T \Big[\left((\boldsymbol{\mathcal{V}}_{h,k})_t , \boldsymbol{\psi}_h \right) + \nu \left(\nabla \boldsymbol{\mathcal{V}}_{h,k}^+ , \nabla \boldsymbol{\psi}_h \right) + B\left(\boldsymbol{\mathcal{V}}_{h,k}^- , \boldsymbol{\mathcal{V}}_{h,k}^+ , \boldsymbol{\psi}_h \right) \\ + \rho \left(\mathcal{Y}_{h,k}^- , \nabla \mathcal{W}_{h,k}^+ \cdot \boldsymbol{\psi}_h \right) - \left(\boldsymbol{\mathcal{U}}_{h,k}^+ , \boldsymbol{\psi}_h \right) \Big] \ dt = 0, \end{aligned} \tag{5.233}$$

$$\boldsymbol{\mathcal{V}}_{h,k}(0) = \mathbf{Q}_s^h \mathbf{v}_0, \tag{5.234}$$

$$\int_0^T \left(\nabla \cdot \boldsymbol{\mathcal{V}}_{h,k}^+ , \phi_h \right) \ dt = 0, \tag{5.235}$$

for all $\boldsymbol{\psi}_h \in \mathcal{C}_c^\infty((0,T);\mathbf{D}_h)$, $\phi_h \in \mathcal{C}_c^\infty((0,T);P_h)$ Given $\boldsymbol{\psi} \in \boldsymbol{\mathcal{C}}_c^\infty((0,T);\boldsymbol{\mathcal{D}})$, $\phi \in \mathcal{C}_c^\infty((0,T);L_0^2)$, we set in (5.233) $\boldsymbol{\psi} = \mathbf{Q}_s^h \boldsymbol{\psi}$ and $\phi_h = Q_0^h \phi$. From the property (A.49) of the Stokes projection operator $\mathbf{Q}_s^h$ and the relation (A.44) valid for the projection operator Q_0^h, we note that

$$\|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|_{L^2\left(\mathbf{H}_0^1\right)}^2 = \int_0^T \|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|_{\mathbf{H}_0^1}^2 \ dt \leq C\ h^2 \int_0^T \|\boldsymbol{\psi}\|_{\mathbf{H}^2}^2 \ dt \to 0, \tag{5.236}$$

$$\|\phi - \phi_h\|_{L^2\left(L_0^2\right)} = \int_0^T \|\phi - \phi_h\|_{L_0^2} dt \ \to 0, \tag{5.237}$$

as $h \to 0$. Using the results of Theorem 5.24 and (5.236) above, we get

$$\int_0^T \left((\boldsymbol{\mathcal{V}}_{h,k})_t , \boldsymbol{\psi}_h \right) dt \to \int_0^T (\mathbf{v}_t, \boldsymbol{\psi})\, dt, \tag{5.238}$$

$$\int_0^T \left(\nabla \boldsymbol{\mathcal{V}}_{h,k}^{+}, \nabla \boldsymbol{\psi}_h\right) dt \to \int_0^T (\nabla \mathbf{v}, \nabla \boldsymbol{\psi})\, dt \tag{5.239}$$

$$\int_0^T \left(\boldsymbol{\mathcal{U}}_{h,k}^{+}, \boldsymbol{\psi}_h\right) dt \to \int_0^T (\mathbf{u}, \boldsymbol{\psi})\, dt, \tag{5.240}$$

as $h \to 0$. From the definition of the discrete trilinear form $B(\cdot,\cdot,\cdot)$, we can write

$$\int_0^T B\left(\boldsymbol{\mathcal{V}}_{h,k}^{-}, \boldsymbol{\mathcal{V}}_{h,k}^{+}, \boldsymbol{\psi}_h\right) dt = Q_1 + Q_2, \tag{5.241}$$

where

$$\begin{aligned}
Q_1 &= \frac{1}{2}\int_0^T \left(\left[\boldsymbol{\mathcal{V}}_{h,k}^{-} \cdot \nabla\right] \boldsymbol{\mathcal{V}}_{h,k}^{+}, \boldsymbol{\psi}_h\right) dt,\\
Q_2 &= -\frac{1}{2}\int_0^T \left(\left[\boldsymbol{\mathcal{V}}_{h,k}^{-} \cdot \nabla\right] \boldsymbol{\psi}_h, \boldsymbol{\mathcal{V}}_{h,k}^{+}\right) dt,
\end{aligned}$$

which are such that

$$\begin{aligned}
Q_1 = \frac{1}{2}\int_0^T \left(\left[\left(\boldsymbol{\mathcal{V}}_{h,k}^{-} - \mathbf{v}\right) \cdot \nabla\right] \boldsymbol{\mathcal{V}}_{h,k}^{+}, \boldsymbol{\psi}_h\right) dt &+ \frac{1}{2}\int_0^T \left([\mathbf{v} \cdot \nabla]\left(\boldsymbol{\mathcal{V}}_{h,k}^{+} - \mathbf{v}\right), \boldsymbol{\psi}_h\right) dt\\
+\frac{1}{2}\int_0^T \left([\mathbf{v} \cdot \nabla]\, \mathbf{v}, \boldsymbol{\psi}_h - \boldsymbol{\psi}\right) dt &+ \frac{1}{2}\int_0^T b(\mathbf{v}, \mathbf{v}, \boldsymbol{\psi})\, dt\\
&= Q_{11} + Q_{12} + Q_{13} + Q_{14},
\end{aligned}$$

$$\begin{aligned}
Q_2 = \frac{1}{2}\int_0^T \left(\left[\left(\boldsymbol{\mathcal{V}}_{h,k}^{-} - \mathbf{v}\right) \cdot \nabla\right] \boldsymbol{\psi}_h, \boldsymbol{\mathcal{V}}_{h,k}^{+}\right) dt &+ \frac{1}{2}\int_0^T \left([\mathbf{v} \cdot \nabla](\boldsymbol{\psi}_h - \boldsymbol{\psi}), \boldsymbol{\mathcal{V}}_{h,k}^{+}\right) dt\\
+\frac{1}{2}\int_0^T \left([\mathbf{v} \cdot \nabla]\, \boldsymbol{\psi}, \boldsymbol{\mathcal{V}}_{h,k}^{+} - \mathbf{v}\right) dt &+ \frac{1}{2}\int_0^T b(\mathbf{v}, \boldsymbol{\psi}, \mathbf{v})\, dt\\
&= Q_{21} + Q_{22} + Q_{23} + Q_{24}.
\end{aligned}$$

where $b(\cdot,\cdot,\cdot,)$ is the trilinear form defined in (4.12). Using the generalized Holder's inequality (A.14), Young's inequality (A.13), inequality (A.17), the results of Theorem 5.24 and (5.236) above, we derive

$$\begin{aligned}
|Q_{11}| &\le \frac{1}{2}\int_0^T \|\boldsymbol{\mathcal{V}}_{h,k}^{-} - \mathbf{v}\|_{\mathbf{L}^4} \|\nabla \boldsymbol{\mathcal{V}}_{h,k}^{+}\| \|\boldsymbol{\psi}_h\|_{\mathbf{L}^4}\, dt\\
&\le C\int_0^T \|\boldsymbol{\mathcal{V}}_{h,k}^{-} - \mathbf{v}\|_{\boldsymbol{\mathcal{D}}} \|\boldsymbol{\mathcal{V}}_{h,k}^{+}\|_{\mathbf{H}_0^1} \|\boldsymbol{\psi}_h\|_{\mathbf{H}_0^1}\, dt\\
&\le C\|\boldsymbol{\mathcal{V}}_{h,k}^{+}\|_{L^\infty\left(\mathbf{H}_0^1\right)} \int_0^T \|\boldsymbol{\mathcal{V}}_{h,k}^{-} - \mathbf{v}\|_{\mathbf{H}_0^1} \|\boldsymbol{\psi}_h\|_{\mathbf{H}_0^1}\, dt\\
&\le C\|\boldsymbol{\mathcal{V}}_{h,k}^{+}\|_{L^\infty\left(\mathbf{H}_0^1\right)} \|\boldsymbol{\mathcal{V}}_{h,k}^{-} - \mathbf{v}\|_{L^2\left(\mathbf{H}_0^1\right)} \|\boldsymbol{\psi}_h\|_{L^2\left(\mathbf{H}_0^1\right)} \to 0,
\end{aligned}$$

$$|Q_{12}| \le \frac{1}{2}\int_0^T \|\mathbf{v}\|_{\mathbf{L}^4} \|\nabla \boldsymbol{\mathcal{V}}_{h,k}^{+} - \nabla \mathbf{v}\| \|\boldsymbol{\psi}_h\|_{\mathbf{L}^4}\, dt$$

$$
\begin{aligned}
&\le C \int_0^T \|\mathbf{v}\|_{\mathbf{H}_0^1} \|\boldsymbol{\mathcal{V}}_{h,k}^+ - \mathbf{v}\|_{\mathbf{H}_0^1} \|\boldsymbol{\psi}_h\|_{\mathbf{H}_0^1} \; dt \\
&\le C \|\mathbf{v}\|_{L^\infty(\mathbf{H}_0^1)} \int_0^T \|\boldsymbol{\mathcal{V}}_{h,k}^+ - \mathbf{v}\|_{\mathbf{H}_0^1} \|\boldsymbol{\psi}_h\|_{\mathbf{H}_0^1} \; dt \\
&\le C \|\mathbf{v}\|_{L^\infty(\mathbf{H}_0^1)} \|\boldsymbol{\mathcal{V}}_{h,k}^+ - \mathbf{v}\|_{L^2(\mathbf{H}_0^1)} \|\boldsymbol{\psi}_h\|_{\mathbf{L}^2(\mathbf{H}_0^1)} \to 0,
\end{aligned}
$$

$$
\begin{aligned}
|Q_{13}| &\le \frac{1}{2} \int_0^T \|\mathbf{v}\|_{\mathbf{L}^4} \|\nabla \mathbf{v}\| \|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|_{\mathbf{L}^4} \; dt \\
&\le C \int_0^T \|\mathbf{v}\|_{\mathbf{H}_0^1} \|\mathbf{v}\|_{\mathbf{H}_0^1} \|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|_{\mathbf{H}_0^1} \; dt \\
&\le C \|\mathbf{v}\|_{L^\infty(\mathbf{H}_0^1)} \int_0^T \|\mathbf{v}\|_{\mathbf{H}_0^1} \|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|_{\mathbf{H}_0^1} \; dt \\
&\le C \|\mathbf{v}\|_{L^\infty(\mathbf{H}_0^1)} \|\mathbf{v}\|_{L^2(\mathbf{H}_0^1)} \|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|_{L^2(\mathbf{H}_0^1)} \to 0,
\end{aligned}
$$

$$
\begin{aligned}
|Q_{21}| &\le \frac{1}{2} \int_0^T \|\boldsymbol{\mathcal{V}}_{h,k}^- - \mathbf{v}\|_{\mathbf{L}^4} \|\nabla \boldsymbol{\psi}_h\| \|\boldsymbol{\mathcal{V}}_{h,k}^+\|_{\mathbf{L}^4} \; dt \\
&\le C \int_0^T \|\boldsymbol{\mathcal{V}}_{h,k}^- - \mathbf{v}\|_{\mathbf{H}_0^1} \|\boldsymbol{\psi}_h\|_{\mathbf{H}_0^1} \|\boldsymbol{\mathcal{V}}_{h,k}^+\|_{\mathbf{H}_0^1} \; dt \\
&\le C \|\boldsymbol{\mathcal{V}}_{h,k}^+\|_{L^\infty(\mathbf{H}_0^1)} \int_0^T \|\boldsymbol{\mathcal{V}}_{h,k}^- - \mathbf{v}\|_{\mathbf{H}_0^1} \|\boldsymbol{\psi}_h\|_{\mathbf{H}_0^1} \; dt \\
&\le C \|\boldsymbol{\mathcal{V}}_{h,k}^+\|_{L^\infty(\mathbf{H}_0^1)} \|\boldsymbol{\mathcal{V}}_{h,k}^- - \mathbf{v}\|_{L^2(\mathbf{H}_0^1)} \|\boldsymbol{\psi}_h\|_{L^2(\mathbf{H}_0^1)} \to 0,
\end{aligned}
$$

$$
\begin{aligned}
|Q_{22}| &\le \frac{1}{2} \int_0^T \|\mathbf{v}\|_{\mathbf{L}^4} \|\nabla \boldsymbol{\psi}_h - \nabla \boldsymbol{\psi}\| \|\boldsymbol{\mathcal{V}}_{h,k}^+\|_{\mathbf{L}^4} \; dt \\
&\le C \int_0^T \|\mathbf{v}\|_{\mathbf{H}_0^1} \|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|_{\mathbf{H}_0^1} \|\boldsymbol{\mathcal{V}}_{h,k}^+\|_{\mathbf{H}_0^1} \; dt \\
&\le C \|\boldsymbol{\mathcal{V}}_{h,k}^+\|_{L^\infty(\mathbf{H}_0^1)} \int_0^T \|\mathbf{v}\|_{\mathbf{H}_0^1} \|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|_{\mathbf{H}_0^1} \; dt \\
&\le C \|\boldsymbol{\mathcal{V}}_{h,k}^+\|_{L^\infty(\mathbf{H}_0^1)} \|\mathbf{v}\|_{L^2(\mathbf{H}_0^1)} \|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|_{L^2(\mathbf{H}_0^1)} \to 0,
\end{aligned}
$$

$$
\begin{aligned}
|Q_{23}| &\le \frac{1}{2} \int_0^T \|\mathbf{v}\|_{\mathbf{L}^4} \|\nabla \boldsymbol{\psi}\| \|\boldsymbol{\mathcal{V}}_{h,k}^+ - \mathbf{v}\|_{\mathbf{L}^4} \; dt \\
&\le C \int_0^T \|\mathbf{v}\|_{\mathbf{H}_0^1} \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1} \|\boldsymbol{\mathcal{V}}_{h,k}^+ - \mathbf{v}\|_{\mathbf{H}_0^1} \; dt \\
&\le C \|\mathbf{v}\|_{L^\infty(\mathbf{H}_0^1)} \int_0^T \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1} \|\boldsymbol{\mathcal{V}}_{h,k}^+ - \mathbf{v}\|_{\mathbf{H}_0^1} \; dt \\
&\le C \|\mathbf{v}\|_{L^\infty(\mathbf{H}_0^1)} \|\boldsymbol{\psi}\|_{L^2(\mathbf{H}_0^1)} \|\boldsymbol{\mathcal{V}}_{h,k}^+ - \mathbf{v}\|_{L^2(\mathbf{H}_0^1)} \to 0,
\end{aligned}
$$

as $h, k \to 0$. Hence from (5.241), we infer

$$\int_0^T B\left(\boldsymbol{\mathcal{V}}_{h,k}^-, \boldsymbol{\mathcal{V}}_{h,k}^+, \boldsymbol{\psi}_h\right) dt \to \frac{1}{2}\int_0^T b\left(\mathbf{v}, \mathbf{v}, \boldsymbol{\psi}\right) dt + \frac{1}{2}\int_0^T b\left(\mathbf{v}, \boldsymbol{\psi}, \mathbf{v}\right) dt, \tag{5.242}$$

as $h, k \to 0$. Using the strong convergence statement (5.237), the results of Theorem 5.24 and the equation (5.235) above, we realize

$$\int_0^T \left(\nabla \cdot \boldsymbol{\mathcal{V}}_{h,k}^+, \phi_h\right)\ dt \to \int_0^T \left(\nabla \cdot \mathbf{v}, \phi\right)\ dt = 0, \tag{5.243}$$

for all $\phi \in \mathcal{C}_c^\infty\left((0,T); L_0^2\right)$. By a density argument, we note that (5.243) is satisfied for all $\phi \in L^2\left(L_0^2\right)$. Then, $\mathbf{v} \in L^2\left(\boldsymbol{\mathcal{D}}\right)$. Therefore, using the property (4.13) of the trilinear form $b\left(\cdot,\cdot,\cdot\right)$, we can replace (5.242) above by

$$\int_0^T B\left(\boldsymbol{\mathcal{V}}_{h,k}^-, \boldsymbol{\mathcal{V}}_{h,k}^+, \boldsymbol{\psi}_h\right) dt \to \int_0^T b\left(\mathbf{v}, \mathbf{v}, \boldsymbol{\psi}\right) dt. \tag{5.244}$$

We note that

$$\int_0^T \left(\mathcal{Y}_{h,k}^-, \nabla \mathcal{W}_{h,k}^+ \cdot \boldsymbol{\psi}_h\right) dt = R_1 + R_2 + R_3 + R_4, \tag{5.245}$$

where

$$\begin{aligned}
R_1 &= \int_0^T \left(\mathcal{Y}_{h,k}^-, \nabla \mathcal{W}_{h,k}^+ \cdot \left[\boldsymbol{\psi}_h - \boldsymbol{\psi}\right]\right) dt,\\
R_2 &= \int_0^T \left(\mathcal{Y}_{h,k}^- - y, \nabla \mathcal{W}_{h,k}^+ \cdot \boldsymbol{\psi}\right) dt,\\
R_3 &= \int_0^T \left(y, \nabla\left[\mathcal{W}_{h,k}^+ - w\right] \cdot \boldsymbol{\psi}\right) dt,\\
R_4 &= \int_0^T \left(y, \nabla w \cdot \boldsymbol{\psi}\right) dt.
\end{aligned}$$

Using the generalized Holder's inequality (A.14), Young's inequality (A.13), inequality (A.17), the results of Theorem 5.24 and the strong convergence statement (5.236) above, we have

$$\begin{aligned}
|R_1| &\le \int_0^T \|\mathcal{Y}_{h,k}^-\|_{L^4} \|\nabla \mathcal{W}_{h,k}^+\| \|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|_{\mathbf{L}^4}\ dt\\
&\le C\int_0^T \|\mathcal{Y}_{h,k}^-\|_{H_0} \|\mathcal{W}_{h,k}^+\|_{H^1} \|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|_{\mathbf{H}_0^1}\ dt\\
&\le C\|\mathcal{Y}_{h,k}^-\|_{L^\infty(H_0)} \int_0^T \|\mathcal{W}_{h,k}^+\|_{H^1} \|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|_{\mathbf{H}_0^1}\ dt\\
&\le C\|\mathcal{Y}_{h,k}^-\|_{L^\infty(H_0)} \|\mathcal{W}_{h,k}^+\|_{L^2(H^1)} \|\boldsymbol{\psi}_h - \boldsymbol{\psi}\|_{L^2\left(\mathbf{H}_0^1\right)}\ dt \to 0,
\end{aligned}$$

$$|R_2| \le \int_0^T \|\mathcal{Y}_{h,k}^- - y\|_{L^4} \|\nabla \mathcal{W}_{h,k}^+\| \|\boldsymbol{\psi}\|_{\mathbf{L}^4}\ dt \le$$

$$\leq \int_0^T \|\mathcal{Y}_{h,k}^- - y\|_{H_0} \|\mathcal{W}_{h,k}^+\|_{H^1} \|\boldsymbol{\psi}\|_{\boldsymbol{\mathcal{D}}}\, dt \leq$$
$$\leq C\|\boldsymbol{\psi}\|_{\mathcal{C}([0,T];\boldsymbol{\mathcal{D}})} \int_0^T \|\mathcal{Y}_{h,k}^- - y\|_{H_0} \|\mathcal{W}_{h,k}^+\|_{H^1}\, dt$$
$$\leq C\|\boldsymbol{\psi}\|_{\mathcal{C}([0,T];\boldsymbol{\mathcal{D}})} \|\mathcal{Y}_{h,k}^- - y\|_{L^2(H_0)} \|\mathcal{W}_{h,k}^+\|_{L^2(H^1)} \to 0,$$

as $h,k \to 0$. Furthermore for all $\eta \in L^2(H^1)$,

$$\left| \int_0^T (y, \nabla\eta \cdot \boldsymbol{\psi})\, dt \right| \leq \|\boldsymbol{\psi}\|_{\boldsymbol{\mathcal{C}}([0,T];\boldsymbol{\mathcal{D}})} \|y\|_{L^2(H_0)} \|\eta\|_{L^2(H^1)}.$$

Hence by the weak convergence of $\mathcal{W}_{h,k}^+$ to w, as stated in (5.212), we get

$$|R_3| \to 0,$$

as $h,k \to 0$. Inserting the results for R_1, R_2, R_3 in (5.245), we can write

$$\int_0^T \left(\mathcal{Y}_{h,k}^-, \nabla\mathcal{W}_{h,k}^+ \cdot \boldsymbol{\psi}_h\right) dt \to \int_0^T (y, \nabla w \cdot \boldsymbol{\psi})\, dt, \tag{5.246}$$

as $h,k \to 0$. From equation (5.234) and the property (A.49) of the Stokes projection operator $\mathbf{Q}_s^h$, we derive

$$\boldsymbol{\mathcal{V}}_{h,k}(0) = \mathbf{Q}_s^h \mathbf{v}_0 \to \mathbf{v}_0 \quad \text{in} \quad \mathbf{H}_0^1. \tag{5.247}$$

Furthermore, with $\boldsymbol{\psi} = \boldsymbol{\xi}(1 - t/T)$, where $\boldsymbol{\xi} \in \mathbf{L}^2$, using integration by parts in time, we infer

$$(\boldsymbol{\mathcal{V}}_{h,k}(0) - \mathbf{v}(0), \boldsymbol{\xi}) = -\int_0^T \left((\boldsymbol{\mathcal{V}}_{h,k} - \mathbf{v})_t, \boldsymbol{\psi}\right) dt - \int_0^T (\boldsymbol{\mathcal{V}}_{h,k} - \mathbf{v}, \boldsymbol{\psi}_t)\, dt \to 0,$$

which implies

$$\boldsymbol{\mathcal{V}}_{h,k}(0) \rightharpoonup \mathbf{v}(0), \tag{5.248}$$

as $h,k \to 0$. So, from the results of Theorem 5.24 and (5.238)-(5.240), (5.244), (5.246), (5.247) and (5.248), we realize that

$$\begin{aligned} \mathbf{v} &\in H^1(\boldsymbol{\mathcal{S}}) \cap L^\infty(\boldsymbol{\mathcal{D}}), \\ y &\in H^1\left(L_0^2\right) \cap L^\infty(H_0), \\ w &\in L^2\left(H^1\right) \cap L^\infty\left(L^2\right), \\ \mathbf{u} &\in L^2\left(\mathbf{L}^2\right), \end{aligned}$$

satisfy

$$\int_0^T \left[(\mathbf{v}_t, \boldsymbol{\psi}) + \nu(\nabla\mathbf{v}, \nabla\boldsymbol{\psi}) + b(\mathbf{v},\mathbf{v},\boldsymbol{\psi}) + \rho(y, \nabla w \cdot \boldsymbol{\psi}) - (\mathbf{u}, \boldsymbol{\psi})\right] dt = 0,$$
$$\mathbf{v}(0) = \mathbf{v}_0,$$

for all $\boldsymbol{\psi} \in \mathcal{C}_c^\infty((0,T);\boldsymbol{\mathcal{D}})$. Thus, from the density result (A.8), we can say that (4.22a), (4.22b) are satisfied for all $\boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}})$.

ii) Equations (4.22c), (4.22d), (4.22e).

From the discrete state equations (5.10a)-(5.10b), we have

$$\int_0^T \left[\left((\mathcal{Y}_{h,k})_t, \eta_h \right)_h + \gamma \left(\nabla \mathcal{W}_{h,k}^+, \nabla \eta_h \right) - \left(\mathcal{Y}_{h,k}^-, \boldsymbol{\mathcal{V}}_{h,k}^- \cdot \nabla \eta_h \right) \right] \, dt = 0, \tag{5.249}$$

$$\mathcal{Y}_{h,k}(0) = Q^h y_0, \tag{5.250}$$

$$\int_0^T \left[\left(\mathcal{W}_{h,k}^+, \theta_h \right)_h - \varepsilon^2 \left(\nabla \mathcal{Y}_{h,k}^+, \nabla \theta_h \right) + \left(\mathcal{Y}_{h,k}^- - \left(\mathcal{Y}_{h,k}^+ \right)^3, \theta_h \right)_h \right] \, dt = 0, \tag{5.251}$$

for all $\eta_h, \theta_h \in \mathcal{C}_c^\infty((0,T);Y_h)$. Given $\eta, \theta \in \mathcal{C}_c^\infty\left((0,T);\mathcal{C}_c^\infty\left(\bar{\Omega}\right)\right)$, we set in (5.249) and (5.251)

$$\eta_h = Q_1^h \eta, \quad \theta_h = Q_1^h \theta.$$

Then, using the property (A.47) of the projection operator Q_1^h, it is easy to get that

$$\eta_h \to \eta, \quad \theta_h \to \theta, \quad \text{in} \quad L^2\left(H^1\right). \tag{5.252}$$

From the results of Theorem 5.24 and (5.252) above, we derive

$$\int_0^T \left(\nabla \mathcal{W}_{h,k}^+, \nabla \eta_h \right) dt \to \int_0^T \left(\nabla w, \nabla \eta \right) dt, \tag{5.253}$$

$$\int_0^T \left(\nabla \mathcal{Y}_{h,k}^+, \nabla \theta_h \right) dt \to \int_0^T \left(\nabla y, \nabla \theta \right) dt, \tag{5.254}$$

as $h, k \to 0$. Furthermore, we realize that

$$S_1 = \left| \int_0^T \left((\mathcal{Y}_{h,k})_t, \eta_h \right)_h \, dt - \int_0^T \left(y_t, \eta \right) \, dt \right| \to 0, \tag{5.255}$$

$$S_2 = \left| \int_0^T \left(\mathcal{W}_{h,k}^+, \theta_h \right)_h \, dt - \int_0^T (w, \theta) \, dt \right| \to 0, \tag{5.256}$$

$$S_3 = \left| \int_0^T \left(\mathcal{Y}_{h,k}^-, \theta_h \right)_h \, dt - \int_0^T (y, \theta) \, dt \right| \to 0, \tag{5.257}$$

$$S_4 = \left| \int_0^T \left(\left(\mathcal{Y}_{h,k}^+ \right)^3, \theta_h \right)_h \, dt - \int_0^T \left(y^3, \theta \right) dt \right| \to 0. \tag{5.258}$$

as $h, k \to 0$. In fact, noting that

$$\begin{aligned} S_1 \leq & \left| \int_0^T \left((\mathcal{Y}_{h,k})_t, \eta_h \right)_h \, dt - \int_0^T \left((\mathcal{Y}_{h,k})_t, \eta_h \right) dt \right| \\ & + \left| \int_0^T \left((\mathcal{Y}_{h,k})_t, \eta_h \right) dt - \int_0^T \left(y_t, \eta \right) dt \right| = S_{11} + S_{12}, \end{aligned}$$

$$S_2 \leq \left| \int_0^T \left(\mathcal{W}_{h,k}^+, \theta_h\right)_h \, dt - \int_0^T \left(\mathcal{W}_{h,k}^+, \theta_h\right) dt \right| + \left| \int_0^T \left(\mathcal{W}_{h,k}^+, \theta_h\right) dt - \int_0^T (w, \theta)\, dt \right| = S_{21} + S_{22},$$

$$S_3 \leq \left| \int_0^T \left(\mathcal{Y}_{h,k}^-, \theta_h\right)_h \, dt - \int_0^T \left(\mathcal{Y}_{h,k}^-, \theta_h\right) dt \right| + \left| \int_0^T \left(\mathcal{Y}_{h,k}^-, \theta_h\right) dt - \int_0^T (y, \theta)\, dt \right| = S_{31} + S_{32},$$

$$S_4 \leq \left| \int_0^T \left(\left(\mathcal{Y}_{h,k}^+\right)^3, \theta_h\right)_h \, dt - \int_0^T \left(\left(\mathcal{Y}_{h,k}^+\right)^3, \theta_h\right) dt \right| + \left| \int_0^T \left(\left(\mathcal{Y}_{h,k}^+\right)^3, \theta_h\right) dt - \int_0^T \left(y^3, \theta\right) dt \right| = S_{41} + S_{42},$$

and using the results of Theorem 5.24, the relation (A.31), the generalized Holder's inequality (A.14), the inequality (A.17) and the relation (5.252) above, we note that

$$S_{11} \leq Ch \int_0^T \|\left(\mathcal{Y}_{h,k}\right)_t\| \|\nabla \eta_h\| \, dt \leq Ch \|\left(\mathcal{Y}_{h,k}\right)_t\|_{L^2(L^2)} \|\eta_h\|_{L^2(H^1)} \to 0,$$

$$S_{21} \leq Ch \int_0^T \|\mathcal{W}_{h,k}^+\| \|\nabla \theta_h\| \, dt \leq Ch \|\mathcal{W}_{h,k}^+\|_{L^2(L^2)} \|\theta_h\|_{L^2(H^1)} \to 0,$$

$$S_{31} \leq Ch \int_0^T \|\mathcal{Y}_{h,k}^-\| \|\nabla \theta_h\| \, dt \leq Ch \|\mathcal{Y}_{h,k}^-\|_{L^2(L^2)} \|\theta_h\|_{L^2(H^1)} \to 0,$$

$$\begin{aligned} S_{41} &\leq Ch \int_0^T \|\left(\mathcal{Y}_{h,k}^+\right)^3\| \|\nabla \theta_h\| \, dt \leq Ch \int_0^T \|\mathcal{Y}_{h,k}^+\|_{L^6} \|\theta_h\|_{H^1} \, dt \\ &\leq Ch \int_0^T \|\mathcal{Y}_{h,k}^+\|_{H_0} \|\theta_h\|_{H^1} \, dt \leq Ch \|\mathcal{Y}_{h,k}^+\|_{L^2(H_0)} \|\theta_h\|_{L^2(H^1)} \to 0, \end{aligned}$$

and

$$S_{12} \to 0, \quad S_{22} \to 0, \quad S_{32} \to 0, \quad S_{42} \to 0,$$

as $h, k \to 0$. We have,

$$\int_0^T \left(\mathcal{Y}_{h,k}^-, \boldsymbol{\mathcal{V}}_{h,k}^- \cdot \nabla \eta_h\right) dt = A_1 + A_2 + A_3 + A_4, \tag{5.259}$$

where

$$A_1 = \int_0^T \left(\mathcal{Y}_{h,k}^- - y, \boldsymbol{\mathcal{V}}_{h,k}^- \cdot \nabla \eta_h\right) dt,$$

$$A_2 = \int_0^T \left(y, \left[\boldsymbol{\mathcal{V}}_{h,k}^- - \mathbf{v}\right] \cdot \nabla \eta_h\right) dt,$$
$$A_3 = \int_0^T \left(y, \mathbf{v} \cdot \nabla \left[\eta_h - \eta\right]\right) dt,$$
$$A_4 = \int_0^T \left(y, \mathbf{v} \cdot \nabla \eta\right) dt.$$

Then, using the results of Theorem 5.24, generalized Holder's inequality (A.14) and inequality (A.17), we get

$$\begin{aligned}
|A_1| &\leq \int_0^T \|\mathcal{Y}_{h,k}^- - y\|_{L^4} \|\boldsymbol{\mathcal{V}}_{h,k}^-\|_{\mathbf{L}^4} \|\nabla \eta_h\| \; dt \\
&\leq C \int_0^T \|\mathcal{Y}_{h,k}^- - y\|_{H_0} \|\boldsymbol{\mathcal{V}}_{h,k}^-\|_{\boldsymbol{\mathcal{D}}} \|\eta_h\|_{H^1} \; dt \\
&\leq C \|\boldsymbol{\mathcal{V}}_{h,k}^-\|_{L^\infty\left(\mathbf{H}_0^1\right)} \int_0^T \|\mathcal{Y}_{h,k}^- - y\|_{H_0} \|\eta_h\|_{H^1} \; dt \\
&\leq C \|\boldsymbol{\mathcal{V}}_{h,k}^-\|_{L^\infty\left(\mathbf{H}_0^1\right)} \|\mathcal{Y}_{h,k}^- - y\|_{L^2(H_0)} \|\eta_h\|_{L^2(H^1)} \to 0,
\end{aligned}$$

$$\begin{aligned}
|A_2| &\leq \int_0^T \|y\|_{L^4} \|\boldsymbol{\mathcal{V}}_{h,k}^- - \mathbf{v}\|_{\mathbf{L}^4} \|\nabla \eta_h\| \; dt \\
&\leq C \int_0^T \|y\|_{H_0} \|\boldsymbol{\mathcal{V}}_{h,k}^- - \mathbf{v}\|_{\mathbf{H}_0^1} \|\eta_h\|_{H^1} \; dt \\
&\leq C \|y\|_{L^\infty(H_0)} \int_0^T \|\boldsymbol{\mathcal{V}}_{h,k}^- - \mathbf{v}\|_{\mathbf{H}_0^1} \|\eta_h\|_{H^1} \; dt \\
&\leq C \|y\|_{L^\infty(H_0)} \|\boldsymbol{\mathcal{V}}_{h,k}^- - \mathbf{v}\|_{L^2\left(\mathbf{H}_0^1\right)} \|\eta_h\|_{L^2(H^1)} \to 0,
\end{aligned}$$

$$\begin{aligned}
|A_3| &\leq \int_0^T \|y\|_{L^4} \|\mathbf{v}\|_{\mathbf{L}^4} \|\nabla \eta_h - \eta\| \; dt \\
&\leq C \int_0^T \|y\|_{H_0} \|\mathbf{v}\|_{\mathbf{H}_0^1} \|\eta_h - \eta\|_{H^1} \; dt \\
&\leq C \|y\|_{L^\infty(H_0)} \int_0^T \|\mathbf{v}\|_{\mathbf{H}_0^1} \|\eta_h - \eta\|_{H^1} \; dt \\
&\leq C \|y\|_{L^\infty(H_0)} \|\mathbf{v}\|_{L^2\left(\mathbf{H}_0^1\right)} \|\eta_h - \eta\|_{L^2(H^1)} \to 0,
\end{aligned}$$

as $h, k \to 0$. Hence, from (5.259), we can write

$$\int_0^T \left(\mathcal{Y}_{h,k}^-, \boldsymbol{\mathcal{V}}_{h,k}^- \cdot \nabla \eta_h\right) dt \to \int_0^T \left(y, \mathbf{v} \cdot \nabla \eta\right) dt, \tag{5.260}$$

as $h, k \to 0$. Concerning the initial condition, using the property (A.42) of the L^2-projection operator Q^h, we derive

$$\mathcal{Y}_{h,k}(0) = Q^h y_0 \to y_0, \quad \text{in} \quad L^2. \tag{5.261}$$

Furthermore, with $\eta = \xi\,(1 - t/T)$, where $\xi \in L^2$, integrating by parts in time, we infer

$$\left(\mathcal{Y}_{h,k}(0) - y(0), \xi\right) = -\int_0^T \left(\left(\mathcal{Y}_{h,k} - y\right)_t, \eta\right)\, dt \;-\int_0^T \left(\mathcal{Y}_{h,k} - y, \eta_t\right)\, dt \;\to 0,$$

as $h, k \to 0$. Therefore $\mathcal{Y}_{h,k}(0) \rightharpoonup y(0)$ in L^2. Thus, using (5.261) and the uniqueness of the weak limit, we realize that

$$y(0) = y_0. \tag{5.262}$$

Therefore, from (5.253)-(5.258), (5.260) and (5.262), we observe that

$$\begin{aligned}
&\mathbf{v} \in H^1(\boldsymbol{\mathcal{S}}) \cap L^\infty(\boldsymbol{\mathcal{D}}),\\
&y \in H^1\left(L_0^2\right) \cap L^\infty(H_0),\\
&w \in L^2\left(H^1\right) \cap L^\infty\left(L^2\right),\\
&\mathbf{u} \in L^2\left(\mathbf{L}^2\right),
\end{aligned}$$

satisfy

$$\int_0^T \left[(y_t, \eta) + \gamma\,(\nabla w, \nabla\eta) - (y, \mathbf{v}\cdot\nabla\eta)\right]\, dt = 0,$$

$$y(0) = y_0,$$

$$\int_0^T \left[(w, \theta) - \varepsilon^2\,(\nabla y, \nabla\theta) + (y, \theta) - \left(y^3, \theta\right)\right]\, dt = 0,$$

for all $\eta, \theta \in \mathcal{C}_c^\infty\left((0,T); \mathcal{C}_c^\infty\left(\bar{\Omega}\right)\right)$. So, using the density result (A.7), we can claim that (4.22c), (4.22d), (4.22e) hold for all $\eta, \theta \in L^2\left(H^1\right)$.

iii) Results (5.229), (4.23a), (4.23a).

From the discrete adjoint equations (5.134a)-(5.134b) we have

$$\begin{aligned}
&\int_0^T \Big[\left(-\left(\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}\right)_t, \boldsymbol{\psi}_h\right) + \nu\left(\nabla\boldsymbol{\mathcal{Q}}^-_{\boldsymbol{\mathcal{V}},h,k}, \nabla\boldsymbol{\psi}_h\right) + \\
&+ B\left(\boldsymbol{\psi}_h, \boldsymbol{\mathcal{V}}^{++}_{h,k}, \boldsymbol{\mathcal{Q}}^+_{\boldsymbol{\mathcal{V}},h,k}\right) + B\left(\boldsymbol{\mathcal{V}}^-_{h,k}, \boldsymbol{\psi}_h, \boldsymbol{\mathcal{Q}}^-_{\boldsymbol{\mathcal{V}},h,k}\right) - \left(\mathcal{Y}^+_{h,k}, \nabla\mathcal{Q}^+_{\mathcal{Y},h,k}\cdot\boldsymbol{\psi}_h\right)\Big]\, dt = 0,
\end{aligned} \tag{5.263}$$

$$\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}(T) = 0, \tag{5.264}$$

$$\int_0^T \left(\nabla\cdot\boldsymbol{\mathcal{Q}}^-_{\boldsymbol{\mathcal{V}},h,k}, \phi_h\right)\, dt = 0, \tag{5.265}$$

for all $\boldsymbol{\psi}_h \in \mathcal{C}_c^\infty((0,T); \mathbf{D}_h)$, $\phi_h \in \mathcal{C}_c^\infty((0,T); P_h)$. In (5.263) the function $\boldsymbol{\mathcal{V}}^{++}_{h,k}$ is defined as follows

$$\boldsymbol{\mathcal{V}}^{++}_{h,k} := \begin{cases} \mathbf{V}^{n+1}, & \text{if} \quad t \in (t_{n-1}, t_n], \quad n = 1, \ldots, N-1,\\ \mathbf{V}^N, & \text{if} \quad t \in (t_{N-1}, t_N], \end{cases} \tag{5.266}$$

and we note that

$$\|\nabla\boldsymbol{\mathcal{V}}^+_{h,k} - \nabla\boldsymbol{\mathcal{V}}^{++}_{h,k}\|^2_{L^2(\mathbf{L}^2)} = \sum_{n=1}^{N-1}\int_{t_{n-1}}^{t_n} \|\nabla\mathbf{V}^n - \mathbf{V}^{n+1}\|^2 dt$$

$$\leq \sum_{n=1}^{N-1} k\|\nabla \mathbf{V}^n - \mathbf{V}^{n+1}\|^2 dt$$

$$\leq \sum_{n=1}^{N} k\|\nabla \mathbf{V}^{n-1} - \mathbf{V}^n\|^2 dt. \tag{5.267}$$

Hence, from (5.267) above, the estimate (5.200) and the result (5.207) established in Theorem 5.24, we get

$$\boldsymbol{\mathcal{V}}_{h,k}^{++} \to \mathbf{v} \quad \text{in} \quad L^2\left(\mathbf{H}_0^1\right). \tag{5.268}$$

as $h, k \to 0$. We consider $\boldsymbol{\psi} \in \mathcal{C}_c^\infty\left((0,T); \boldsymbol{\mathcal{D}}\right)$, $\phi \in \mathcal{C}_c^\infty\left((0,T); L_0^2\right)$ and we set $\boldsymbol{\psi}_h = \mathbf{Q}_s^h \boldsymbol{\psi}$ in (5.263) and $\phi_h = Q_0^h \phi$. Then, using the results of Theorem 5.25 and the strong convergence of $\boldsymbol{\psi}_h$ to $\boldsymbol{\psi}$ (see (5.236) in Step 1), we can write

$$\int_0^T \left(-\left(\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}\right)_t, \boldsymbol{\psi}_h\right) dt \to \int_0^T \left(-\mathbf{q}_{\mathbf{v}t}, \boldsymbol{\psi}\right) dt, \tag{5.269}$$

$$\int_0^T \left(\nabla \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^-, \nabla \boldsymbol{\psi}_h\right) dt \to \int_0^T \left(\nabla \mathbf{q}_{\mathbf{v}}, \nabla \boldsymbol{\psi}\right) dt. \tag{5.270}$$

as $h, k \to 0$. Regarding the third term in (5.263), we derive

$$\int_0^T B\left(\boldsymbol{\psi}_h, \boldsymbol{\mathcal{V}}_{h,k}^{++}, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^+\right) dt = D_1 + D_2, \tag{5.271}$$

where

$$D_1 = \frac{1}{2}\int_0^T \left(\left[\boldsymbol{\psi}_h \cdot \nabla\right] \boldsymbol{\mathcal{V}}_{h,k}^{++}, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^+\right) dt,$$

$$D_2 = -\frac{1}{2}\int_0^T \left(\left[\boldsymbol{\psi}_h \cdot \nabla\right] \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^+, \boldsymbol{\mathcal{V}}_{h,k}^{++}\right) dt,$$

It is easy to realize that

$$\begin{aligned} D_1 = {} & \frac{1}{2}\int_0^T \left(\left[\left(\boldsymbol{\psi}_h - \boldsymbol{\psi}\right) \cdot \nabla\right] \boldsymbol{\mathcal{V}}_{h,k}^{++}, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^+\right) dt \\ & + \frac{1}{2}\int_0^T \left(\left[\boldsymbol{\psi} \cdot \nabla\right] \left[\boldsymbol{\mathcal{V}}_{h,k}^{++} - \mathbf{v}\right], \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^+\right) dt \\ & + \frac{1}{2}\int_0^T \left(\left[\boldsymbol{\psi} \cdot \nabla\right] \mathbf{v}, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^+ - \mathbf{q}_{\mathbf{v}}\right) dt \\ & + \frac{1}{2}\int_0^T \left(\left[\boldsymbol{\psi} \cdot \nabla\right] \mathbf{v}, \mathbf{q}_{\mathbf{v}}\right) dt = D_{11} + D_{12} + D_{13} + D_{14}, \end{aligned} \tag{5.272}$$

$$\begin{aligned} D_2 = {} & \frac{1}{2}\int_0^T \left(\left[\left(\boldsymbol{\psi}_h - \boldsymbol{\psi}\right) \cdot \nabla\right] \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^+, \boldsymbol{\mathcal{V}}_{h,k}^{++}\right) dt \\ & + \frac{1}{2}\int_0^T \left(\left[\boldsymbol{\psi} \cdot \nabla\right] \left[\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^+ - \mathbf{q}_{\mathbf{v}}\right], \boldsymbol{\mathcal{V}}_{h,k}^{++}\right) dt \end{aligned}$$

$$+\frac{1}{2}\int_0^T \left([\boldsymbol{\psi}\cdot\nabla]\,\mathbf{q}_{\mathbf{v}}, \boldsymbol{\mathcal{V}}_{h,k}^{++}-\mathbf{v}\right)dt$$

$$+\frac{1}{2}\int_0^T \left([\boldsymbol{\psi}\cdot\nabla]\,\mathbf{q}_{\mathbf{v}}, \mathbf{v}\right)dt = D_{21}+D_{22}+D_{23}+D_{24}. \tag{5.273}$$

Using the generalized Holder's inequality (A.14), Young's inequality (A.13), inequality (A.17), the results of Theorem 5.25, the strong convergence statement (5.236) in Step 1 and (5.268) above, we infer

$$\begin{aligned}
|D_{11}| &\leq \frac{1}{2}\int_0^T \|\boldsymbol{\psi}_h-\boldsymbol{\psi}\|_{\mathbf{L}^4}\|\nabla\boldsymbol{\mathcal{V}}_{h,k}^{++}\|\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{\mathbf{L}^4}\,dt\\
&\leq C\int_0^T \|\boldsymbol{\psi}_h-\boldsymbol{\psi}\|_{\mathbf{H}_0^1}\|\boldsymbol{\mathcal{V}}_{h,k}^{++}\|_{\mathbf{H}_0^1}\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{\mathbf{H}_0^1}\,dt\\
&\leq C\|\boldsymbol{\mathcal{V}}_{h,k}^{++}\|_{L^\infty(\mathbf{H}_0^1)}\int_0^T \|\boldsymbol{\psi}_h-\boldsymbol{\psi}\|_{\mathbf{H}_0^1}\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{\mathbf{H}_0^1}\,dt\\
&\leq C\|\boldsymbol{\mathcal{V}}_{h,k}^{++}\|_{L^\infty(\mathbf{H}_0^1)}\|\boldsymbol{\psi}_h-\boldsymbol{\psi}\|_{L^2(\mathbf{H}_0^1)}\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{L^2(\mathbf{H}_0^1)}\to 0,
\end{aligned}$$

$$\begin{aligned}
|D_{12}| &\leq \frac{1}{2}\int_0^T \|\boldsymbol{\psi}\|_{\mathbf{L}^4}\|\nabla\boldsymbol{\mathcal{V}}_{h,k}^{++}-\nabla\mathbf{v}\|\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{\mathbf{L}^4}\,dt\\
&\leq C\int_0^T \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1}\|\boldsymbol{\mathcal{V}}_{h,k}^{++}-\mathbf{v}\|_{\mathbf{H}_0^1}\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{\mathbf{H}_0^1}\,dt\\
&\leq C\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{L^\infty(\mathbf{H}_0^1)}\int_0^T \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1}\|\boldsymbol{\mathcal{V}}_{h,k}^{++}-\mathbf{v}\|_{\mathbf{H}_0^1}\,dt\\
&\leq C\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{L^\infty(\mathbf{H}_0^1)}\|\boldsymbol{\psi}\|_{L^2(\mathbf{H}_0^1)}\|\boldsymbol{\mathcal{V}}_{h,k}^{++}-\mathbf{v}\|_{L^2(\mathbf{H}_0^1)}\to 0,
\end{aligned}$$

$$\begin{aligned}
|D_{13}| &\leq \frac{1}{2}\int_0^T \|\boldsymbol{\psi}\|_{\mathbf{L}^4}\|\nabla\mathbf{v}\|\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}-\mathbf{q}_{\mathbf{v}}\|_{\mathbf{L}^4}\,dt\\
&\leq C\int_0^T \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1}\|\mathbf{v}\|_{\mathbf{H}_0^1}\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}-\mathbf{q}_{\mathbf{v}}\|_{\mathbf{H}_0^1}\,dt\\
&\leq C\|\mathbf{v}\|_{L^\infty(\mathbf{H}_0^1)}\int_0^T \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1}\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}-\mathbf{q}_{\mathbf{v}}\|_{\mathbf{H}_0^1}\,dt\\
&\leq C\|\mathbf{v}\|_{L^\infty(\mathbf{H}_0^1)}\|\boldsymbol{\psi}\|_{L^2(\mathbf{H}_0^1)}\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}-\mathbf{q}_{\mathbf{v}}\|_{L^2(\mathbf{H}_0^1)}\to 0,
\end{aligned}$$

$$\begin{aligned}
|D_{21}| &\leq \frac{1}{2}\int_0^T \|\boldsymbol{\psi}_h-\boldsymbol{\psi}\|_{\mathbf{L}^4}\|\nabla\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|\|\boldsymbol{\mathcal{V}}_{h,k}^{++}\|_{\mathbf{L}^4}\,dt\\
&\leq C\int_0^T \|\boldsymbol{\psi}_h-\boldsymbol{\psi}\|_{\mathbf{H}_0^1}\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{\mathbf{H}_0^1}\|\boldsymbol{\mathcal{V}}_{h,k}^{++}\|_{\mathbf{H}_0^1}\,dt\\
&\leq C\|\boldsymbol{\mathcal{V}}_{h,k}^{++}\|_{L^\infty(\mathbf{H}_0^1)}\int_0^T \|\boldsymbol{\psi}_h-\boldsymbol{\psi}\|_{\mathbf{H}_0^1}\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{\mathbf{H}_0^1}\,dt\\
&\leq C\|\boldsymbol{\mathcal{V}}_{h,k}^{++}\|_{L^\infty(\mathbf{H}_0^1)}\|\boldsymbol{\psi}_h-\boldsymbol{\psi}\|_{L^2(\mathbf{H}_0^1)}\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{L^2(\mathbf{H}_0^1)}\to 0,
\end{aligned}$$

$$\begin{aligned}|D_{22}| &\leq \frac{1}{2}\int_0^T \|\boldsymbol{\psi}\|_{\mathbf{L}^4}\|\nabla\boldsymbol{\mathcal{Q}}^+_{\boldsymbol{\mathcal{V}},h,k} - \nabla\mathbf{q_v}\|\|\boldsymbol{\mathcal{V}}^{++}_{h,k}\|_{\mathbf{L}^4}\ dt\\ &\leq C\int_0^T \|\boldsymbol{\psi}\|_{\mathbf{H}^1_0}\|\boldsymbol{\mathcal{Q}}^+_{\boldsymbol{\mathcal{V}},h,k} - \mathbf{q_v}\|_{\mathbf{H}^1_0}\|\boldsymbol{\mathcal{V}}^{++}_{h,k}\|_{\mathbf{H}^1_0}\ dt\\ &\leq C\|\boldsymbol{\mathcal{V}}^{++}_{h,k}\|_{L^\infty(\mathbf{H}^1_0)}\int_0^T \|\boldsymbol{\psi}\|_{\mathbf{H}^1_0}\|\boldsymbol{\mathcal{Q}}^+_{\boldsymbol{\mathcal{V}},h,k} - \mathbf{q_v}\|_{\mathbf{H}^1_0}\ dt \leq\\ &\leq C\|\boldsymbol{\mathcal{V}}^{++}_{h,k}\|_{L^\infty(\mathbf{H}^1_0)}\|\boldsymbol{\psi}\|_{L^2(\mathbf{H}^1_0)}\|\boldsymbol{\mathcal{Q}}^+_{\boldsymbol{\mathcal{V}},h,k} - \mathbf{q_v}\|_{L^2(\mathbf{H}^1_0)} \to 0,\end{aligned}$$

$$\begin{aligned}|D_{23}| &\leq \frac{1}{2}\int_0^T \|\boldsymbol{\psi}\|_{\mathbf{L}^4}\|\nabla\mathbf{q_v}\|\|\boldsymbol{\mathcal{V}}^{++}_{h,k} - \mathbf{v}\|_{\mathbf{L}^4}\ dt\\ &\leq C\int_0^T \|\boldsymbol{\psi}\|_{\mathbf{H}^1_0}\|\mathbf{q_v}\|_{\mathbf{H}^1_0}\|\boldsymbol{\mathcal{V}}^{++}_{h,k} - \mathbf{v}\|_{\mathbf{H}^1_0}\ dt\\ &\leq C\|\mathbf{q_v}\|_{L^\infty(\mathbf{H}^1_0)}\int_0^T \|\boldsymbol{\psi}\|_{\mathbf{H}^1_0}\|\boldsymbol{\mathcal{V}}^{++}_{h,k} - \mathbf{v}\|_{\mathbf{H}^1_0}\ dt\\ &\leq C\|\mathbf{q_v}\|_{L^\infty(\mathbf{H}^1_0)}\|\boldsymbol{\psi}\|_{L^2(\mathbf{H}^1_0)}\|\boldsymbol{\mathcal{V}}^{++}_{h,k} - \mathbf{v}\|_{L^2(\mathbf{H}^1_0)} \to 0,\end{aligned}$$

as $h, k \to 0$. Inserting the previous relations in (5.272) and (5.273), from (5.271) we observe

$$(5.274)\quad \int_0^T B\left(\boldsymbol{\psi}_h, \boldsymbol{\mathcal{V}}^{++}_{h,k}, \boldsymbol{\mathcal{Q}}^+_{\boldsymbol{\mathcal{V}},h,k}\right) dt \to \frac{1}{2}\int_0^T b\left(\boldsymbol{\psi}, \mathbf{v}, \mathbf{q_v}\right) dt + \frac{1}{2}\int_0^T b\left(\boldsymbol{\psi}, \mathbf{q_v}, \mathbf{v}\right) dt,$$

as $h, k \to 0$. Using the strong convergence statement (5.237), the results of Theorem 5.25 and the equation (5.265) above, we have

$$(5.275)\qquad \int_0^T \left(\nabla\cdot\boldsymbol{\mathcal{Q}}^-_{\boldsymbol{\mathcal{V}},h,k}, \phi_h\right)\ dt \to \int_0^T \left(\nabla\cdot\mathbf{q_v}, \phi\right)\ dt = 0,$$

for all $\phi \in \mathcal{C}^\infty_c\left((0,T); L^2_0\right)$.Moreover, by a density argument, we note that (5.275) hold for all $\phi \in L^2\left(L^2_0\right)$. Then, $\mathbf{q_v} \in L^2\left(\boldsymbol{\mathcal{D}}\right)$. Therefore, using the property (4.13) of the trilinear form $b\left(\cdot,\cdot,\cdot\right)$, we can replace (5.274) above by

$$(5.276)\qquad \int_0^T B\left(\boldsymbol{\psi}_h, \boldsymbol{\mathcal{V}}^{++}_{h,k}, \boldsymbol{\mathcal{Q}}^+_{\boldsymbol{\mathcal{V}},h,k}\right) dt \to \int_0^T b\left(\boldsymbol{\psi}, \mathbf{v}, \mathbf{q_v}\right) dt.$$

Considering the fourth term in (5.263), we can write

$$(5.277)\qquad \int_0^T B\left(\boldsymbol{\mathcal{V}}^-_{h,k}, \boldsymbol{\psi}_h, \boldsymbol{\mathcal{Q}}^-_{\boldsymbol{\mathcal{V}},h,k}\right) dt = E_1 - E_2,$$

where

$$\begin{aligned}E_1 &= \frac{1}{2}\int_0^T \left(\left[\boldsymbol{\mathcal{V}}^-_{h,k}\cdot\nabla\right]\boldsymbol{\psi}_h, \boldsymbol{\mathcal{Q}}^-_{\boldsymbol{\mathcal{V}},h,k}\right) dt,\\ E_2 &= \frac{1}{2}\int_0^T \left(\left[\boldsymbol{\mathcal{V}}^-_{h,k}\cdot\nabla\right]\boldsymbol{\mathcal{Q}}^-_{\boldsymbol{\mathcal{V}},h,k}, \boldsymbol{\psi}_h\right) dt.\end{aligned}$$

and

$$E_1 = \frac{1}{2}\int_0^T \left(\left[\left(\boldsymbol{\mathcal{V}}_{h,k}^- - \mathbf{v}\right)\cdot\nabla\right]\boldsymbol{\psi}_h, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^-\right) dt + \frac{1}{2}\int_0^T \left(\left[\mathbf{v}\cdot\nabla\right]\left[\boldsymbol{\psi}_h - \boldsymbol{\psi}\right], \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^-\right) dt + \frac{1}{2}\int_0^T \left(\left[\mathbf{v}\cdot\nabla\right]\boldsymbol{\psi}, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^- - \mathbf{q_v}\right) dt = E_{11} + E_{12} + E_{13} + E_{14}, \tag{5.278}$$

$$E_2 = \frac{1}{2}\int_0^T \left(\left[\left(\boldsymbol{\mathcal{V}}_{h,k}^- - \mathbf{v}\right)\cdot\nabla\right]\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^-, \boldsymbol{\psi}_h\right) dt + \frac{1}{2}\int_0^T \left(\left[\mathbf{v}\cdot\nabla\right]\left[\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^- - \mathbf{q_v}\right], \boldsymbol{\psi}_h\right) dt + \frac{1}{2}\int_0^T \left(\left[\mathbf{v}\cdot\nabla\right]\mathbf{q_v}, \boldsymbol{\psi}_h - \boldsymbol{\psi}\right) dt = E_{21} + E_{22} + E_{23} + E_{24}. \tag{5.279}$$

From the generalized Holder's inequality (A.14), Young's inequality (A.13), inequality (A.17), the results of Theorem 5.25 and the strong convergence statement (5.236) in Step 1, we get

$$E_{11} \to 0, \quad E_{12} \to 0, \quad E_{13} \to 0,$$
$$E_{21} \to 0, \quad E_{22} \to 0, \quad E_{23} \to 0,$$

as $h, k \to 0$. Hence, using the above relations in (5.278), (5.279), from (5.277), we derive

$$\int_0^T B\left(\boldsymbol{\mathcal{V}}_{h,k}^-, \boldsymbol{\psi}_h, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^-\right) dt \to \int_0^T b\left(\mathbf{v}, \boldsymbol{\psi}, \mathbf{q_v}\right) dt. \tag{5.280}$$

It remains to show the convergence of the last term in (5.263). It reads

$$\int_0^T \left(\mathcal{Y}_{h,k}^+, \nabla\mathcal{Q}_{\mathcal{Y},h,k}^+ \cdot \boldsymbol{\psi}_h\right) dt = F_1 + F_2 + F_3 + F_4, \tag{5.281}$$

where

$$F_1 = \int_0^T \left(\mathcal{Y}_{h,k}^+, \nabla\left[\mathcal{Q}_{\mathcal{Y},h,k}^+ - \mathbf{q_v}\right]\cdot\boldsymbol{\psi}_h\right) dt,$$
$$F_2 = \int_0^T \left(\mathcal{Y}_{h,k}^+ - y, \nabla\mathbf{q_v}\cdot\boldsymbol{\psi}_h\right) dt,$$
$$F_3 = \int_0^T \left(y, \nabla\mathbf{q_v}\cdot\left[\boldsymbol{\psi}_h - \boldsymbol{\psi}\right]\right) dt,$$
$$F_4 = \int_0^T \left(y, \nabla\mathbf{q_v}\cdot\boldsymbol{\psi}\right) dt.$$

Using the generalized Holder's inequality (A.14), inequality (A.17), the results of Theorems 5.25, 5.24 and the strong convergence statement (5.236), we can write

$$|F_1| \le \int_0^T \|\mathcal{Y}_{h,k}^+\|_{L^4}\|\nabla\mathcal{Q}_{\mathcal{Y},h,k}^+ - \nabla\mathbf{q_v}\|\|\boldsymbol{\psi}_h\|_{L^4}\; dt$$

$$\begin{aligned}
&\leq C\int_0^T \|\mathcal{Y}_{h,k}^{+}\|_{H_0}\|\mathcal{Q}_{\mathcal{Y},h,k}^{+}-\mathbf{q_v}\|_{\mathbf{H}_0^1}\|\boldsymbol{\psi}_h\|_{\mathbf{H}_0^1}\,dt\\
&\leq C\|\mathcal{Y}_{h,k}^{+}\|_{L^\infty(H_0)}\int_0^T \|\mathcal{Q}_{\mathcal{Y},h,k}^{+}-\mathbf{q_v}\|_{\mathbf{H}_0^1}\|\boldsymbol{\psi}_h\|_{\mathbf{H}_0^1}\,dt\\
&\leq C\|\mathcal{Y}_{h,k}^{+}\|_{L^\infty(H_0)}\|\mathcal{Q}_{\mathcal{Y},h,k}^{+}-\mathbf{q_v}\|_{L^2\left(\mathbf{H}_0^1\right)}\|\boldsymbol{\psi}_h\|_{L^2\left(\mathbf{H}_0^1\right)}\to 0,
\end{aligned}$$

$$\begin{aligned}
|F_2|&\leq \int_0^T \|\mathcal{Y}_{h,k}^{+}-y\|_{L^4}\|\nabla\mathbf{q_v}\|\|\boldsymbol{\psi}_h\|_{L^4}\,dt\\
&\leq C\int_0^T \|\mathcal{Y}_{h,k}^{+}-y\|_{H_0}\|\mathbf{q_v}\|_{\mathbf{H}_0^1}\|\boldsymbol{\psi}_h\|_{\mathbf{H}_0^1}\,dt\\
&\leq C\|\mathbf{q_v}\|_{L^\infty\left(\mathbf{H}_0^1\right)}\int_0^T \|\mathcal{Y}_{h,k}^{+}-y\|_{H_0}\|\boldsymbol{\psi}_h\|_{\mathbf{H}_0^1}\,dt\\
&\leq C\|\mathbf{q_v}\|_{L^\infty\left(\mathbf{H}_0^1\right)}\|\mathcal{Y}_{h,k}^{+}-y\|_{L^2(H_0)}\|\boldsymbol{\psi}_h\|_{L^2\left(\mathbf{H}_0^1\right)}\to 0,
\end{aligned}$$

$$\begin{aligned}
|F_3|&\leq \int_0^T \|y\|_{L^4}\|\nabla\mathbf{q_v}\|\|\boldsymbol{\psi}_h-\boldsymbol{\psi}\|_{L^4}\,dt\\
&\leq C\int_0^T \|y\|_{H_0}\|\mathbf{q_v}\|_{\mathbf{H}_0^1}\|\boldsymbol{\psi}_h-\boldsymbol{\psi}\|_{\mathbf{H}_0^1}\,dt\\
&\leq C\|\mathbf{q_v}\|_{L^\infty\left(\mathbf{H}_0^1\right)}\int_0^T \|y\|_{H_0}\|\boldsymbol{\psi}_h-\boldsymbol{\psi}\|_{\mathbf{H}_0^1}\,dt\\
&\leq C\|\mathbf{q_v}\|_{L^\infty\left(\mathbf{H}_0^1\right)}\|y\|_{L^2(H_0)}\|\boldsymbol{\psi}_h-\boldsymbol{\psi}\|_{L^2\left(\mathbf{H}_0^1\right)}\to 0.
\end{aligned}$$

as $h,k\to 0$. Therefore, from (5.281), we derive

$$\int_0^T \left(\mathcal{Y}_{h,k}^{+},\nabla\mathcal{Q}_{\mathcal{Y},h,k}^{+}\cdot\boldsymbol{\psi}_h\right)dt\to\int_0^T (y,\nabla q_y\cdot\boldsymbol{\psi})\,dt, \tag{5.282}$$

as $h,k\to 0$. Finally, we prove that $\mathbf{q_v}(T)=0$. With $\boldsymbol{\psi}=\boldsymbol{\xi}\ t/T$, where $\boldsymbol{\xi}\in\mathbf{L}^2$, integrating by parts in time, we realize that

$$\begin{aligned}
&\left(\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}(T)-\mathbf{q_v}(T),\boldsymbol{\xi}\right)=\\
&=\int_0^T\left(\left(\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}-\mathbf{q_v}\right)_t,\boldsymbol{\psi}\right)\,dt+\int_0^T\left(\boldsymbol{\psi}_t,\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}-\mathbf{q_v}\right)\,dt\to 0,
\end{aligned}$$

as $h,k\to 0$. Therefore

$$\mathbf{q_v}(T)=0. \tag{5.283}$$

Hence, from (5.269), (5.270), (5.276), (5.280), (5.282) and (5.283), we claim that

$$\begin{aligned}
\mathbf{v}&\in H^1(\boldsymbol{\mathcal{S}})\cap L^\infty(\boldsymbol{\mathcal{D}}),\\
y&\in H^1\left(L_0^2\right)\cap L^\infty(H_0),\\
\mathbf{q_v}&\in H^1(\boldsymbol{\mathcal{S}})\cap L^\infty(\boldsymbol{\mathcal{D}}),
\end{aligned}$$

$$q_y \in H^1\left(L_0^2\right) \cap L^\infty\left(H_0\right),$$
$$q_w \in L^2\left(H^1\right) \cap L^\infty\left(L_0^2\right),$$

satisfy

$$\int_0^T \left[\left(-\mathbf{q}_{\mathbf{v}t}, \boldsymbol{\psi}\right) + \nu\left(\nabla \mathbf{q}_{\mathbf{v}}, \nabla \boldsymbol{\psi}\right) + b\left(\boldsymbol{\psi}, \mathbf{v}, \mathbf{q}_{\mathbf{v}}\right) + b\left(\mathbf{v}, \boldsymbol{\psi}, \mathbf{q}_{\mathbf{v}}\right) - \left(y, \nabla q_y \cdot \boldsymbol{\psi}\right)\right] dt = 0,$$
$$\mathbf{q}_{\mathbf{v}}(T) = 0,$$

for all $\boldsymbol{\psi} \in \mathcal{C}_c^\infty\left((0,T);\boldsymbol{\mathcal{D}}\right)$. Then, from the density result (A.8), we conclude that (4.23a), (4.23b) hold for all $\boldsymbol{\psi} \in L^2\left(\boldsymbol{\mathcal{D}}\right)$.

iv) Results (4.23c)-(4.23e)

From the discrete adjoint equations (5.134d)-(5.134f), we have

(5.284)
$$\int_0^T \Big[\left(-\left(\mathcal{Q}_{\mathcal{Y},h,k}\right)_t, \eta_h\right)_h - \varepsilon^2\left(\nabla \mathcal{Q}_{\mathcal{W},h,k}^-, \nabla \eta_h\right) + \left(\mathcal{Q}_{\mathcal{W},h,k}^+, \eta_h\right)_h - \left(\nabla \mathcal{Q}_{\mathcal{Y},h,k}^+ \cdot \boldsymbol{\mathcal{V}}_{h,k}^+, \eta_h\right) + \rho\left(\nabla \mathcal{W}_{h,k}^{++} \cdot \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^+, \eta_h\right) - 3\left(\left(\mathcal{Y}_{h,k}^+\right)^2 \mathcal{Q}_{\mathcal{W},h,k}^-, \eta_h\right)_h + \left(\mathcal{Y}_{h,k}^+ - \mathcal{Y}_{d,h,k}^+, \eta_h\right)\Big] dt = 0,$$

(5.285)
$$\mathcal{Q}_{\mathcal{Y},h,k}(T) = 0,$$

(5.286)
$$\int_0^T \Big[\left(\mathcal{Q}_{\mathcal{W},h,k}^-, \theta_h\right)_h + \gamma\left(\nabla \mathcal{Q}_{\mathcal{Y},h,k}^-, \nabla \theta_h\right) + \rho\left(\mathcal{Y}_{h,k}^-, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^- \cdot \nabla \theta_h\right)\Big] dt = 0.$$

for all $\eta_h \in \mathcal{C}_c^\infty\left((0,T);P_h\right), \theta_h \in \mathcal{C}_c^\infty\left((0,T);Y_h\right)$. In (5.284) $\mathcal{Y}_{d,h,k}^+$ is the time interpolation of the values $y_d^n = Q_0^h y_d\left(t_n\right), n = 1, \ldots, N$. By the property (A.44) of the projection operator Q_0^h and using $y_d \in \mathcal{C}\left([0,T];L^2\right)$, we get

(5.287)
$$\begin{aligned}
\|\mathcal{Y}_{d,h,k}^+ - y_d\|_{L^2(L^2)}^2 &= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|Q_0^h y_d\left(t_n\right) - y_d(t)\|^2 dt \\
&\le 2\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left[\|Q_0^h y_d\left(t_n\right) - y_d\left(t_n\right)\|^2 + \|y_d\left(t_n\right) - y_d(t)\|^2\right] dt \\
&= 2\sum_{n=1}^N k\|Q_0^h y_d\left(t_n\right) - y_d\left(t_n\right)\|^2 + 2\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|y_d\left(t_n\right) - y_d(t)\|^2 dt \\
&\le 2\sum_{n=1}^N k\|Q_0^h y_d\left(t_n\right) - y_d\left(t_n\right)\|^2 + 2\sum_{n=1}^N k \max_{t\in[t_{n-1},t_n]} \|y_d\left(t_n\right) - y_d(t)\|^2 \\
&\le 2\sum_{n=1}^N k\|Q_0^h y_d\left(t_n\right) - y_d\left(t_n\right)\|^2 + 2T \max_{n=1,\ldots,N} \max_{t\in[t_{n-1},t_n]} \|y_d\left(t_n\right) - y_d(t)\|^2 \to 0,
\end{aligned}$$

as $h,k \to 0$. Given $\eta \in \mathcal{C}_c^\infty\left((0,T);\mathcal{C}_c^\infty\left(\bar{\Omega}\right) \cap L_0^2\right), \theta \in \mathcal{C}_c^\infty\left((0,T);\mathcal{C}_c^\infty\left(\bar{\Omega}\right)\right)$ we set in (5.284), (5.286)

$$\eta_h = Q_1^h \eta, \quad \theta_h = Q_1^h \theta.$$

From the results of Theorem 5.25, the strong convergence statement (5.252) and (5.287) above, we derive

$$\int_0^T \left(\nabla \mathcal{Q}^-_{\mathcal{W},h,k}, \nabla \eta_h\right) dt \to \int_0^T (\nabla q_w, \nabla \eta)\, dt, \tag{5.288}$$

$$\int_0^T \left(\nabla \mathcal{Q}^-_{\mathcal{Y},h,k}, \nabla \theta_h\right) \to \int_0^T (\nabla q_y, \nabla \theta)\, dt, \tag{5.289}$$

$$\int_0^T \left(\mathcal{Y}^+_{h,k} - \mathcal{Y}^+_{d,h,k}, \eta_h\right) dt \to \int_0^T (y - y_d, \eta)\, dt, \tag{5.290}$$

as $h, k \to 0$. Moreover, it is easy to show that

$$G_1 = \left| \int_0^T \left(\left(\mathcal{Q}_{\mathcal{Y},h,k}\right)_t, \eta_h\right)_h dt - \int_0^T (q_{yt}, \eta)\, dt \right| \to 0, \tag{5.291}$$

$$G_2 = \left| \int_0^T \left(\mathcal{Q}^+_{\mathcal{W},h,k}, \eta_h\right)_h dt - \int_0^T (q_w, \eta)\, dt \right| \to 0, \tag{5.292}$$

$$G_3 = \left| \int_0^T \left(\left(\mathcal{Y}^+_{h,k}\right)^2 \mathcal{Q}^-_{\mathcal{W},h,k}, \eta_h\right)_h dt - \int_0^T \left(y^2 q_w, \eta\right) dt \right| \to 0, \tag{5.293}$$

$$G_4 = \left| \int_0^T \left(\mathcal{Q}^-_{\mathcal{W},h,k}, \theta_h\right)_h dt - \int_0^T (q_w, \theta)\, dt \right| \to 0, \tag{5.294}$$

as $h, k \to 0$. Indeed, we can write

$$\begin{aligned} G_1 \leq & \left| \int_0^T \left(\left(\mathcal{Q}_{\mathcal{Y},h,k}\right)_t, \eta_h\right) dt - \int_0^T \left(\left(\mathcal{Q}_{\mathcal{Y},h,k}\right)_t, \eta_h\right) dt \right| \\ & + \left| \int_0^T \left(\left(\mathcal{Q}_{\mathcal{Y},h,k}\right)_t, \eta_h\right) dt - \int_0^T (q_{yt}, \eta)\, dt \right| = G_{11} + G_{12}, \end{aligned}$$

$$\begin{aligned} G_2 \leq & \left| \int_0^T \left(\mathcal{Q}^+_{\mathcal{W},h,k}, \eta_h\right)_h dt - \int_0^T \left(\mathcal{Q}^+_{\mathcal{W},h,k}, \eta_h\right) dt \right| \\ & + \left| \int_0^T \left(\mathcal{Q}^+_{\mathcal{W},h,k}, \eta_h\right) dt - \int_0^T (q_w, \eta)\, dt \right| = G_{21} + G_{22}, \end{aligned}$$

$$\begin{aligned} G_3 \leq & \left| \int_0^T \left(\left(\mathcal{Y}^+_{h,k}\right)^2 \mathcal{Q}^-_{\mathcal{W},h,k}, \eta_h\right)_h dt - \int_0^T \left(\left(\mathcal{Y}^+_{h,k}\right)^2 \mathcal{Q}^-_{\mathcal{W},h,k}, \eta_h\right) dt \right| \\ & + \left| \int_0^T \left(\left(\mathcal{Y}^+_{h,k}\right)^2 \mathcal{Q}^-_{\mathcal{W},h,k}, \eta_h\right) dt - \int_0^T \left(y^2 q_w, \eta\right) dt \right| = G_{31} + G_{32}, \end{aligned}$$

$$\begin{aligned} G_4 \leq & \left| \int_0^T \left(\mathcal{Q}^-_{\mathcal{W},h,k}, \theta_h\right)_h dt - \int_0^T \left(\mathcal{Q}^-_{\mathcal{W},h,k}, \theta_h\right) dt \right| \\ & + \left| \int_0^T \left(\mathcal{Q}^-_{\mathcal{W},h,k}, \theta_h\right) dt - \int_0^T (q_w, \theta)\, dt \right| = G_{41} + G_{42}, \end{aligned}$$

and using the results of Theorems 5.24, 5.25, relation (A.31), generalized Holder's inequality (A.14), inequality (A.17), relation (5.252) above and the estimate (5.198), we infer

$$G_{11} \leq Ch \int_0^T \| \left(\mathcal{Q}_{\mathcal{Y},h,k}\right)_t \| \|\nabla \eta_h\| \; dt \leq Ch \| \left(\mathcal{Q}_{\mathcal{Y},h,k}\right)_t \|_{L^2(L^2)} \|\eta_h\|_{L^2(H_0)} \to 0,$$

$$G_{21} \leq Ch \int_0^T \|\mathcal{Q}^+_{\mathcal{W},h,k}\| \|\nabla \eta_h\| \; dt \leq \|\mathcal{Q}^+_{\mathcal{W},h,k}\|_{L^2(L^2)} \|\eta_h\|_{L^2(H_0)} \to 0,$$

$$\begin{aligned} G_{31} &\leq Ch \int_0^T \| \left(\mathcal{Y}^+_{h,k}\right)^2 \mathcal{Q}^-_{\mathcal{W},h,k}\| \|\nabla \eta_h\| \; dt \\ &\leq Ch \int_0^T \|\mathcal{Y}^+_{h,k}\|^2_{\mathcal{C}(\bar{\Omega})} \|\mathcal{Q}^-_{\mathcal{W},h,k}\| \|\eta_h\|_{H_0} dt \\ &\leq Ch \max_{t\in[0,T]} \|\mathcal{Y}^+_{h,k}\left(t\right)\|^2_{\mathcal{C}\left(\bar{\Omega}\right)} \|\mathcal{Q}^-_{\mathcal{W},h,k}\|_{L^2(L^2)} \|\eta_h\|_{L^2(H_0)} \to 0, \end{aligned}$$

$$G_{41} \leq Ch \int_0^T \|\mathcal{Q}^-_{\mathcal{W},h,k}\| \|\nabla \theta_h\| \leq Ch \|\mathcal{Q}^-_{\mathcal{W},h,k}\|_{L^2(L^2)} \|\eta\|_{L^2(H^1)} \to 0,$$

and

$$G_{12} \to 0, \quad G_{22} \to 0, \quad G_{32} \to 0, \quad G_{42} \to 0,$$

as $h, k \to 0$. In the fifth term in (5.284) the function $\mathcal{W}^{++}_{h,k}$ is defined as follows

$$\mathcal{W}^{++}_{h,k} := \begin{cases} W^{n+1}, & \text{if} \quad t \in \left(t_{n-1}, t_n\right], \quad n = 1, \ldots, N-1, \\ W^N, & \text{if} \quad t \in \left(t_{N-1}, t_N\right], \end{cases} \tag{5.295}$$

and using the results of Lemmas 5.10 and Theorem 5.24, we realize that

$$\mathcal{W}^{++}_{h,k} \rightharpoonup w, \qquad \text{in } L^2\left(H^1\right). \tag{5.296}$$

Integrating by parts in space, it is easy to show

$$\int_0^T \left(\nabla \mathcal{W}^{++}_{h,k} \cdot \boldsymbol{\mathcal{Q}}^+_{\boldsymbol{\mathcal{V}},h,k}, \eta_h\right) dt = H_1 + H_2, \tag{5.297}$$

where

$$\begin{aligned} H_1 &= \int_0^T \left(\mathcal{W}^{++}_{h,k} \left[\nabla \cdot \boldsymbol{\mathcal{Q}}^+_{\boldsymbol{\mathcal{V}},h,k}\right], \eta_h\right) dt, \\ H_2 &= \int_0^T \left(\mathcal{W}^{++}_{h,k}, \boldsymbol{\mathcal{Q}}^+_{\boldsymbol{\mathcal{V}},h,k} \cdot \nabla \eta_h\right) dt, \end{aligned}$$

and

$$\begin{aligned} H_1 = &\int_0^T \left(\mathcal{W}^{++}_{h,k} \left[\nabla \cdot \boldsymbol{\mathcal{Q}}^+_{\boldsymbol{\mathcal{V}},h,k}\right], \eta_h - \eta\right) dt \\ &+ \int_0^T \left(\mathcal{W}^{++}_{h,k} \left[\nabla \cdot \boldsymbol{\mathcal{Q}}^+_{\boldsymbol{\mathcal{V}},h,k} - \nabla \cdot \mathbf{q_v}\right], \eta\right) dt \end{aligned} \tag{5.298}$$

$$+\int_0^T \left(\mathcal{W}_{h,k}^{++} - w, [\nabla\cdot\mathbf{q_v}]\,\eta\right) dt + \int_0^T (w, [\nabla\cdot\mathbf{q_v}]\,\eta)\, dt$$
$$= H_{11} + H_{12} + H_{13} + H_{14},$$

(5.299)
$$H_2 = \int_0^T \left(\mathcal{W}_{h,k}^{++}, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\cdot[\nabla\eta_h - \nabla\eta]\right) dt$$
$$+\int_0^T \left(\mathcal{W}_{h,k}^{++}, \left[\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+} - \mathbf{q_v}\right]\cdot\nabla\eta\right) dt$$
$$+\int_0^T \left(\mathcal{W}_{h,k}^{++} - w, \mathbf{q_v}\cdot\nabla\eta\right) dt + \int_0^T (w, \mathbf{q_v}\cdot\nabla\eta)\, dt$$
$$= H_{21} + H_{22} + H_{23} + H_{24}.$$

Using the results of Theorems 5.24, 5.25, generalized Holder's inequality (A.14), inequality (A.17), relation (5.252) above and the estimates (5.198), (5.200), we can show

$$\begin{aligned}
|H_{11}| &\leq \int_0^T \|\mathcal{W}_{h,k}^{++}\|_{L^4}\|\nabla\cdot\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|\|\eta_h - \eta\|_{L^4} dt \leq \\
&\leq C\int_0^T \|\mathcal{W}_{h,k}^{++}\|_{H^1}\|\nabla\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|\|\eta_h - \eta\|_{H_0} dt \\
&\leq C\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{L^\infty\left(\mathbf{H}_0^1\right)}\int_0^T \|\mathcal{W}_{h,k}^{++}\|_{H^1}\|\eta_h - \eta\|_{H_0} dt \\
&\leq C\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{L^\infty\left(\mathbf{H}_0^1\right)}\|\mathcal{W}_{h,k}^{++}\|_{L^2(H^1)}\|\eta_h - \eta\|_{L^2(H_0)} dt \to 0,
\end{aligned}$$

$$\begin{aligned}
|H_{12}| &\leq \int_0^T \|\mathcal{W}_{h,k}^{++}\|_{L^4}\|\nabla\cdot\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+} - \nabla\cdot\mathbf{q_v}\|\|\eta\|_{L^4} dt \\
&\leq C\int_0^T \|\mathcal{W}_{h,k}^{++}\|_{H^1}\|\nabla\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+} - \nabla\mathbf{q_v}\|\|\eta\|_{H_0} dt \\
&\leq C\left(\max_{t\in[0,T]}\|\eta\,(t)\,\|_{H_0}\right)\int_0^T \|\mathcal{W}_{h,k}^{++}\|_{H^1}\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+} - \mathbf{q_v}\|_{\mathbf{H}_0^1} dt \\
&\leq C\left(\max_{t\in[0,T]}\|\eta\,(t)\,\|_{H_0}\right)\|\mathcal{W}_{h,k}^{++}\|_{L^2(H^1)}\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+} - \mathbf{q_v}\|_{L^2\left(\mathbf{H}_0^1\right)} \to 0,
\end{aligned}$$

$$\begin{aligned}
|H_{21}| &\leq \int_0^T \|\mathcal{W}_{h,k}^{++}\|_{L^4}\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{L^4}\|\nabla\eta_h - \nabla\eta\| dt \\
&\leq C\int_0^T \|\mathcal{W}_{h,k}^{++}\|_{H^1}\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{\mathbf{H}_0^1}\|\eta_h - \eta\|_{H_0} dt \\
&\leq C\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{L^\infty\left(\mathbf{H}_0^1\right)}\int_0^T \|\mathcal{W}_{h,k}^{++}\|_{H^1}\|\eta_h - \eta\|_{H_0} dt \\
&\leq C\|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}\|_{\mathbf{L}^\infty(\boldsymbol{\mathcal{D}})}\|\mathcal{W}_{h,k}^{++}\|_{L^2(H^1)}\|\eta_h - \eta\|_{L^2(H_0)} \to 0,
\end{aligned}$$

$$\begin{aligned}
|H_{22}| &\leq \int_0^T \|\mathcal{W}_{h,k}^{++}\|_{L^4} \|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+} - \mathbf{q_v}\|_{\mathbf{L}^4} \|\nabla \eta\| dt \\
&\leq C \int_0^T \|\mathcal{W}_{h,k}^{++}\|_{H^1} \|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+} - \mathbf{q_v}\|_{\mathbf{H}_0^1} \|\eta\|_{H_0} dt \\
&\leq C \left(\max_{t\in[0,T]} \|\eta(t)\|_{H_0} \right) \int_0^T \|\mathcal{W}_{h,k}^{++}\|_{H^1} \|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+} - \mathbf{q_v}\|_{\mathbf{H}_0^1} \, dt \\
&\leq C \left(\max_{t\in[0,T]} \|\eta(t)\|_{H_0} \right) \|\mathcal{W}_{h,k}^{++}\|_{L^2(H^1)} \|\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+} - \mathbf{q_v}\|_{L^2\left(\mathbf{H}_0^1\right)} \to 0.
\end{aligned}$$

as $h, k \to 0$. Furthermore, for all $w \in L^2(H^1)$

$$\begin{aligned}
\left| \int_0^T (w, [\nabla \cdot \mathbf{q_v}] \, \eta) \, dt \right| &\leq \|\mathbf{q_v}\|_{\mathbf{L}^\infty(\boldsymbol{\mathcal{D}})} \|\eta\|_{L^2(H_0)} \|w\|_{L^2(H^1)}, \\
\left| \int_0^T (w, \mathbf{q_v} \cdot \nabla \eta) \, dt \right| &\leq \|\mathbf{q_v}\|_{\mathbf{L}^\infty(\boldsymbol{\mathcal{D}})} \|\eta\|_{L^2(H_0)} \|w\|_{L^2(H^1)},
\end{aligned}$$

hence

$$|H_{13}| \to 0, \quad |H_{23}| \to 0, \tag{5.300}$$

as $h, k \to 0$.
Therefore, using the previous relations in (5.298), (5.299), we have

$$\begin{aligned}
\int_0^T \left(\mathcal{W}_{h,k}^{++} \left[\nabla \cdot \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+} \right], \eta_h \right) dt &\to \int_0^T \left(w \left[\nabla \cdot \mathbf{q_v} \right], \eta \right) dt, \\
\int_0^T \left(\mathcal{W}_{h,k}^{++}, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+} \cdot \nabla \eta_h \right) dt &\to \int_0^T \left(w, \mathbf{q_v} \cdot \nabla \eta \right) dt,
\end{aligned}$$

and then, using (5.297) and integrating by parts in space, we get

$$\int_0^T \left(\nabla \mathcal{W}_{h,k}^{++} \cdot \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{+}, \eta_h \right) dt \to \int_0^T \left(\nabla w \cdot \mathbf{q_v}, \eta \right) dt, \tag{5.301}$$

as $h, k \to 0$. Concerning the fifth term in (5.284), we derive

$$I = \left| \int_0^T \left(\left(\mathcal{Y}_{h,k}^{+} \right)^2 \mathcal{Q}_{\mathcal{W},h,k}^{-}, \eta_h \right)_h dt - \int_0^T \left(y^2 q_w, \eta \right) dt \right| \to 0, \tag{5.302}$$

as $h, k \to 0$. Indeed, we note

$$\begin{aligned}
I \leq& \left| \int_0^T \left(\left(\mathcal{Y}_{h,k}^{+} \right)^2 \mathcal{Q}_{\mathcal{W},h,k}^{-}, \eta_h \right)_h dt - \int_0^T \left(\left(\mathcal{Y}_{h,k}^{+} \right)^2 \mathcal{Q}_{\mathcal{W},h,k}^{-}, \eta_h \right) dt \right| \\
&+ \left| \int_0^T \left(\left(\mathcal{Y}_{h,k}^{+} \right)^2 \mathcal{Q}_{\mathcal{W},h,k}^{-}, \eta_h \right) dt - \int_0^T \left(y^2 q_w, \eta \right) dt \right| = I_1 + I_2,
\end{aligned} \tag{5.303}$$

where

$$I_2 \leq \left| \int_0^T \left(\left(\mathcal{Y}_{h,k}^{+} \right)^2 \mathcal{Q}_{\mathcal{W},h,k}^{-}, \eta_h - \eta \right) dt \right|$$

$$+\left|\int_0^T \left(\left(\mathcal{Y}_{h,k}^+\right)^2 \left[\mathcal{Q}_{\mathcal{W},h,k}^- - q_w\right], \eta\right) dt\right|$$
$$+\left|\int_0^T \left(\left(\mathcal{Y}_{h,k}^+\right)^2 - y^2, q_w\ \eta\right) dt\right| = I_{21} + I_{22} + I_{23}.$$

Using the relation (A.31), generalized Holder's inequality (A.14), estimate (5.198) and the results of Theorems 5.24, 5.25, we can write

$$\begin{aligned}
I_1 &\le Ch \int_0^T \| \left(\mathcal{Y}_{h,k}^+\right)^2 \mathcal{Q}_{\mathcal{W},h,k}^- \| \|\nabla \eta_h\| dt \\
&\le Ch \int_0^T \|\mathcal{Y}_{h,k}^+\|_{\mathcal{C}(\bar{\Omega})} \|\mathcal{Q}_{\mathcal{W},h,k}^-\| \|\eta_h\|_{H_0} dt \\
&\le Ch \|\mathcal{Y}_{h,k}^+\|_{L^\infty(\mathcal{C}(\bar{\Omega}))} \int_0^T \|\mathcal{Q}_{\mathcal{W},h,k}^-\| \|\eta_h\|_{H_0} dt \\
&\le Ch \|\mathcal{Y}_{h,k}^+\|_{L^\infty(\mathcal{C}(\bar{\Omega}))} \|\mathcal{Q}_{\mathcal{W},h,k}^-\|_{L^2(L^2)} \|\eta_h\|_{L^2(H_0)} dt \to 0,
\end{aligned}$$

$$\begin{aligned}
I_{21} &\le \int_0^T \| \left(\mathcal{Y}_{h,k}^+\right)^2 \|_{\mathcal{C}(\bar{\Omega})} \|\mathcal{Q}_{\mathcal{W},h,k}^-\| \|\eta_h - \eta\| dt \\
&\le 2\|\mathcal{Y}_{h,k}^+\|_{L^\infty(\mathcal{C}(\bar{\Omega}))} \int_0^T \|\mathcal{Q}_{\mathcal{W},h,k}^-\| \|\eta_h - \eta\| dt \\
&\le 2\|\mathcal{Y}_{h,k}^+\|_{L^\infty(\mathcal{C}(\bar{\Omega}))} \|\mathcal{Q}_{\mathcal{W},h,k}^-\|_{L^2(L^2)} \|\eta_h - \eta\|_{L^2(L^2)} \to 0,
\end{aligned}$$

$$\begin{aligned}
I_{23} &= \int_0^T \left(\left[\mathcal{Y}_{h,k}^+ - y\right] \left[\mathcal{Y}_{h,k}^+ + y\right], q_w\ \eta\right) dt \\
&\le \int_0^T \|\mathcal{Y}_{h,k}^+ - y\|_{L^4} \|\mathcal{Y}_{h,k}^+ + y\|_{L^4} \|q_w\|_{L^2} \|\eta\|_{\mathcal{C}(\bar{\Omega})} dt \\
&\le C \int_0^T \|\mathcal{Y}_{h,k}^+ - y\|_{H_0} \|\mathcal{Y}_{h,k}^+ + y\|_{H_0} \|q_w\|_{L^2} \|\eta\|_{\mathcal{C}(\bar{\Omega})} dt \\
&\le C \left(\max_{t\in[0,T]} \|\eta(t)\|_{\mathcal{C}(\bar{\Omega})}\right) \|q_w\|_{L^\infty(L^2)} \int_0^T \|\mathcal{Y}_{h,k}^+ - y\|_{H_0} \|\mathcal{Y}_{h,k}^+ + y\|_{H_0} dt \\
&\le C \left(\max_{t\in[0,T]} \|\eta(t)\|_{\mathcal{C}(\bar{\Omega})}\right) \|q_w\|_{L^\infty(L^2)} \|\mathcal{Y}_{h,k}^+ - y\|_{L^2(H_0)} \|\mathcal{Y}_{h,k}^+ + y\|_{L^2(H_0)} \to 0,
\end{aligned}$$

as $h, k \to 0$. Moreover for all $q_w \in L^2(H_0)$,

$$\left|\int_0^T \left(\left(\mathcal{Y}_{h,k}^+\right)^2 q_w, \eta\right) dt\right| \le 2\|\mathcal{Y}_{h,k}^+\|_{L^\infty(\mathcal{C}(\bar{\Omega}))} \|q_w\|_{L^2(H_0)} \|\eta\|_{L^2(H_0)}.$$

Therefore,

$$I_{22} \to 0,$$

as $h, k \to 0$. Hence, using the previous relations in (5.303) (5.303), we infer that (5.302) holds. It is easy to realize

$$\left|\int_0^T \left(\mathcal{Y}_{h,k}^-, \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^- \cdot \nabla \theta_h\right) dt - \int_0^T \left(y, \mathbf{q}_{\mathbf{v}} \cdot \nabla \theta\right) dt\right| \to 0, \tag{5.304}$$

as $h, k \to 0$. Indeed

$$\left| \int_0^T \left(\mathcal{Y}_{h,k}^-, \mathcal{Q}_{\mathcal{V},h,k}^- \cdot \nabla\theta_h\right) dt - \int_0^T (y, \mathbf{q_v} \cdot \nabla\theta)\, dt \right| \le L_1 + L_2 + L_3, \tag{5.305}$$

where

$$\begin{aligned}
L_1 &= \left| \int_0^T \left(\mathcal{Y}_{h,k}^-, \mathcal{Q}_{\mathcal{V},h,k}^- \cdot [\nabla\theta_h - \nabla\theta]\right) dt \right|, \\
L_2 &= \left| \int_0^T \left(\mathcal{Y}_{h,k}^-, \left[\mathcal{Q}_{\mathcal{V},h,k}^- - \mathbf{q_v}\right] \cdot \nabla\theta\right) dt \right|, \\
L_3 &= \left| \int_0^T \left(\mathcal{Y}_{h,k}^- - y, \mathbf{q_v} \cdot \nabla\theta\right) dt \right|,
\end{aligned}$$

and using the generalized Holder's inequality (A.14), inequality (A.17), the strong convergence statement (5.252) and the results of Theorems 5.24, 5.25, we observe

$$\begin{aligned}
L_1 &\le \int_0^T \|\mathcal{Y}_{h,k}^-\|_{L^4} \|\mathcal{Q}_{\mathcal{V},h,k}^-\|_{\mathbf{L}^4} \|\nabla\theta_h - \nabla\theta\| dt \\
&\le C \int_0^T \|\mathcal{Y}_{h,k}^-\|_{H_0} \|\mathcal{Q}_{\mathcal{V},h,k}^-\|_{\mathbf{H}_0^1} \|\theta_h - \theta\|_{H^1} dt \\
&\le C \|\mathcal{Y}_{h,k}^-\|_{L^\infty(H_0)} \int_0^T \|\mathcal{Q}_{\mathcal{V},h,k}^-\|_{\mathbf{H}_0^1} \|\theta_h - \theta\|_{H^1} dt \\
&\le C \|\mathcal{Y}_{h,k}^-\|_{L^\infty(H_0)} \|\mathcal{Q}_{\mathcal{V},h,k}^-\|_{L^2\left(\mathbf{H}_0^1\right)} \|\theta_h - \theta\|_{L^2(H^1)} dt \to 0,
\end{aligned}$$

$$\begin{aligned}
L_2 &\le \int_0^T \|\mathcal{Y}_{h,k}^-\|_{L^4} \|\mathcal{Q}_{\mathcal{V},h,k}^- - \mathbf{q_v}\|_{\mathbf{L}^4} \|\nabla\theta\| dt \\
&\le C \int_0^T \|\mathcal{Y}_{h,k}^-\|_{H_0} \|\mathcal{Q}_{\mathcal{V},h,k}^- - \mathbf{q_v}\|_{\mathbf{H}_0^1} \|\theta\|_{H^1} dt \\
&\le C \|\mathcal{Y}_{h,k}^-\|_{L^\infty(H_0)} \int_0^T \|\mathcal{Q}_{\mathcal{V},h,k}^- - \mathbf{q_v}\|_{\mathbf{H}_0^1} \|\theta\|_{H^1} dt \\
&\le C \|\mathcal{Y}_{h,k}^-\|_{L^\infty(H_0)} \|\mathcal{Q}_{\mathcal{V},h,k}^- - \mathbf{q_v}\|_{L^2\left(\mathbf{H}_0^1\right)} \|\theta\|_{L^2(H^1)} \to 0,
\end{aligned}$$

$$\begin{aligned}
L_3 &\le \int_0^T \|\mathcal{Y}_{h,k}^- - y\|_{L^4} \|\mathbf{q_v}\|_{\mathbf{L}^4} \|\nabla\theta\| dt \\
&\le C \int_0^T \|\mathcal{Y}_{h,k}^- - y\|_{H_0} \|\mathbf{q_v}\|_{\mathbf{H}_0^1} \|\theta\|_{H^1} dt \\
&\le C \|\mathbf{q_v}\|_{L^\infty\left(\mathbf{H}_0^1\right)} \int_0^T \|\mathcal{Y}_{h,k}^- - y\|_{H_0} \|\theta\|_{H^1} dt \\
&\le C \|\mathbf{q_v}\|_{L^\infty\left(\mathbf{H}_0^1\right)} \|\mathcal{Y}_{h,k}^- - y\|_{L^2(H_0)} \|\theta\|_{L^2(H^1)} \to 0,
\end{aligned}$$

as $h, k \to 0$. Thus, using the previous relations in (5.305), we have that (5.304) is satisfied. Next, we prove that $q_y(T) = 0$. With $\eta = \xi\ t/T$, where $\xi \in L^2$, integrating by parts in time, we get

$$\begin{aligned}&\left(\mathcal{Q}_{\mathcal{Y},h,k}\left(T\right)-q_y\left(T\right),\xi\right)\\&=\int_0^T\left(\left(\mathcal{Q}_{\mathcal{Y},h,k}-q_y\right)_t,\eta\right)\,dt+\int_0^T\left(\eta_t,\mathcal{Q}_{\mathcal{Y},h,k}-q_y\right)\,dt\to 0,\end{aligned}$$

as $h,k\to 0$, for all $\xi\in L^2$. Therefore,

$$q_y\left(T\right)=0. \tag{5.306}$$

Hence, from (5.288)-(5.294), (5.301), (5.302), (5.304) and (5.306), we derive that

$$\begin{aligned}\mathbf{v}&\in H^1\left(\boldsymbol{\mathcal{S}}\right)\cap L^\infty\left(\boldsymbol{\mathcal{D}}\right),\\ y&\in H^1\left(L_0^2\right)\cap L^\infty\left(H_0\right),\\ \mathbf{q_v}&\in H^1\left(\boldsymbol{\mathcal{S}}\right)\cap L^\infty\left(\boldsymbol{\mathcal{D}}\right),\\ w&\in L^2\left(H^1\right)\cap L^\infty\left(L^2\right),\\ q_y&\in H^1\left(L_0^2\right)\cap L^\infty\left(H_0\right),\\ q_w&\in L^2\left(H^1\right)\cap L^\infty\left(L_0^2\right),\end{aligned}$$

satisfy

$$\begin{aligned}\int_0^T\Big[\left(-q_{yt},\eta\right)-\varepsilon^2\left(\nabla q_w,\nabla\eta\right)+\rho\left(\nabla w\cdot\mathbf{q_v},\eta\right)-\left(\mathbf{v}\cdot\nabla q_y,\eta\right)&\\+\left(q_w,\eta\right)-\left(3y^2q_w,\eta\right)+\left(y-y_d,\eta\right)\Big]\,dt&=0,\\ q_y(T)&=0,\\ \int_0^T\left[\left(q_w,\theta\right)+\gamma\left(\nabla q_y,\nabla\theta\right)+\rho\left(y,\mathbf{q_v}\cdot\nabla\theta\right)\right]\,dt&=0,\end{aligned}$$

for all $\eta\in\mathcal{C}_c^\infty\left((0,T);\mathcal{C}_c^\infty\left(\bar\Omega\right)\cap L_0^2\right),\theta\in\mathcal{C}_c^\infty\left((0,T);\mathcal{C}_c^\infty\left(\bar\Omega\right)\right)$. So, from the density result (A.7), we infer that (4.23c)-(4.23e) hold for all $\eta\in L^2\left(H_0\right),\theta\in L^2\left(H^1\right)$. Finally, the estimate (5.232), is a direct consequence of the estimates (5.201), (5.203) and the results of Theorem 5.25.

v) Results (5.228), (4.24)

From the discrete variational equality (5.135), we can write

$$\alpha\,\boldsymbol{\mathcal{U}}_{h,k}^{+}=\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{-}.$$

Then, up to a multiplicative constant, we can identify the control $\boldsymbol{\mathcal{U}}_{h,k}^{+}$ with the adjoint variable $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k}^{-}$. So, as $h,k\to 0$,

$$\begin{aligned}\boldsymbol{\mathcal{U}}_{h,k}^{+}&\overset{*}{\rightharpoonup}\mathbf{u}, &&\text{in } L^\infty\left(\mathbf{H}_0^1\right),\\ \boldsymbol{\mathcal{U}}_{h,k}^{+}&\to\mathbf{u}, &&\text{in } L^2\left(\mathbf{H}_0^1\right).\end{aligned}$$

Furthermore, equation (4.24) hold and $\mathbf{u}\in H^1\left(\boldsymbol{\mathcal{S}}\right)\cap L^\infty\left(\boldsymbol{\mathcal{D}}\right)$. □

5.6. Numerical Solution of the Discrete Optimal Control Problem

In order to solve the discrete optimality conditions (5.133)-(5.135) of the optimal control Problem 5.1, we apply the same procedure performed in Section 3.5, i.e., we use the steepest descent approach described in Algorithm 3.28. We emphasize that in this case, where we are dealing with a smooth problem, Algorithm 3.28 represents a true steepest descent method, where given

$$\tilde{J}_{h,k}\left(\boldsymbol{\mathcal{U}}_{h,k}\right) = J_{h,k}\left(s_{h,k}\left(\boldsymbol{\mathcal{U}}_{h,k}\right), \boldsymbol{\mathcal{U}}_{h,k}\right),$$

we have

$$\boldsymbol{\mathcal{G}}_{h,k} := \alpha\, \boldsymbol{\mathcal{U}}_{h,k} - \boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k} = \nabla_{\boldsymbol{\mathcal{U}}_{h,k}} \tilde{J}_{h,k}\left(\boldsymbol{\mathcal{U}}_{h,k}\right).$$

Furthermore, concerning the steps 2 and 3 of Algorithm 3.28 there are several differences between the case here discussed and the one presented in Section 3.5. We show them in the following.

Algorithm 3.28: Step 2

Let us assume that i is the steepest descent iteration index. The state equations in system (5.133) are coupled but there are not any kind of complementarity conditions which complicate matters. So, in order to get $\boldsymbol{\mathcal{V}}_{h,k,(i)}, \mathcal{Y}_{h,k,(i)}, \mathcal{W}_{h,k,(i)}$, we need to solve, at each time level $n = 1, \ldots, N$ a unique linear system resulting from the discrete Navier-Stokes equations (5.133a)-(5.133c) and Cahn-Hilliard equations (5.133d)-(5.133f).

Algorithm 3.28: Step 3

Given $\boldsymbol{\mathcal{V}}_{h,k,(i)}, \mathcal{Y}_{h,k,(i)}, \mathcal{W}_{h,k,(i)}$ we calculate $\boldsymbol{\mathcal{Q}}_{\boldsymbol{\mathcal{V}},h,k,(i)}$. To do that, we take into account that also the discrete adjoint equations in system (5.134) are coupled but the complementarity conditions are missing. Then, we need just to solve a unique linear system built from the backward adjoint equations (5.134a)-(5.134c) and (5.134d)-(5.134f).

5.7. Numerical Experiments

In the following, in order to show the efficiency of our approach, we show two numerical experiment.

5.7.1. Circle to Square 1

We propose a numerical experiment which is similar to the one presented in Section 3.6. So, the domain is still the unit square $\Omega = (0,1)^2$ in the two dimensional plane (x_1, x_2) and the initial condition y_0 for the phase-field has the form (3.269) and it is shown in figure 3.1. The values of the constants parameter in the model are $\alpha = 10^{-4}, \nu = 0.1, \gamma = 0.005, \rho = 0.1, \varepsilon = 0.02$, the timestep $k = 0.01$ and the time horizon is $T = 100k$. Even the desired state y_d is the same represented in figure 3.2. Concerning the settings of the steepest descent Algorithm 3.28, we consider as initial guess for the control $\boldsymbol{\mathcal{U}}_{h,k,(0)} \equiv 0$, the tolerance $TOL = 10^{-9}$ and the maximum number of s.d. descent iterations $N_{\max} = 10^3$. Furthermore, also in this case the steepest descent step size $\sigma_{(i)}$ is established according to the Barzilai-Borwein method [12], with the following settings (see section 3.5, in particular (3.271) for details): $\sigma_{\text{init}} = 10^3$, $\sigma_{\min} = 300$, $\sigma_{\max} = 800$.

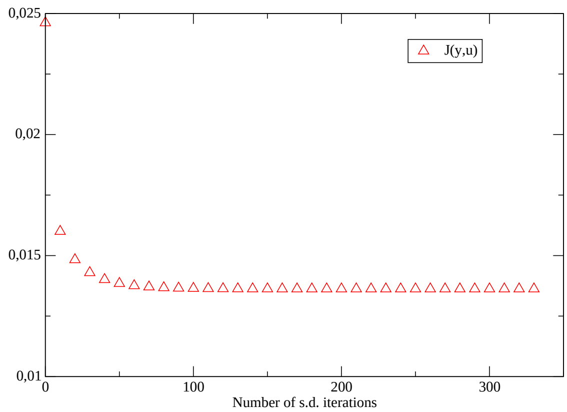


Figure 5.1.: behaviour of $J_{h,k}\left(\mathcal{Y}_{h,k,(i)}, \boldsymbol{\mathcal{U}}_{h,k,(i)}\right)$, with i index of s.d. iterations

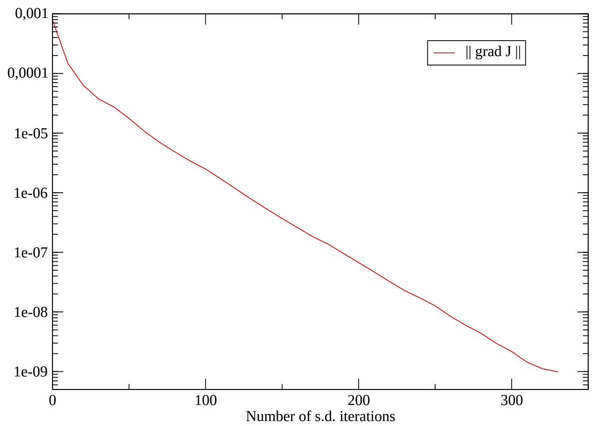
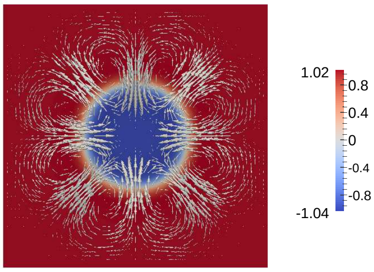


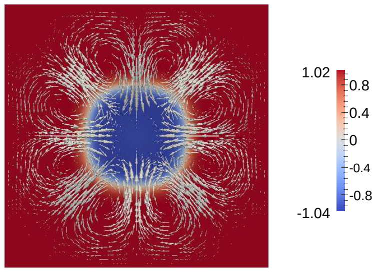
Figure 5.2.: behaviour of $\|\boldsymbol{\mathcal{G}}_{h,k,(i)}\|_{L^2(\mathbf{L}^2)}$, with i index of s.d. iterations

Figures 5.1, 5.2 show the good behaviour of the steepest descent algorithm: in about 330 iterations the system seems approaching to a minimum of the cost functional, see fig. 5.1. Moreover $\|\boldsymbol{\mathcal{G}}_{h,k,(i)}\|_{L^2(\mathbf{L}^2)}$ decreases apparently with a logarithmic rate, with respect to the number of steepest descent iterations, see figure 5.2.
In figures 5.3, it is depicted the evolution in time of the *optimal* phase-field $\mathcal{Y}_{h,k}(x,t)$ and velocity $\boldsymbol{\mathcal{V}}_{h,k}(x,t)$ (i.e. at the end of steepest descent algorithm). The behaviour is the one desired: the velocity is such that the phase-field distribution changes in the first time steps and then it keeps its shape close to the desired state.

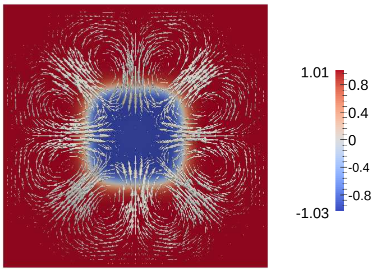
As expected there are overshoots, however relatively small, of the phase-field outside the interval $[-1, 1]$.



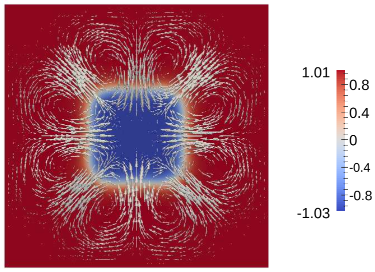
(a) t=6 k



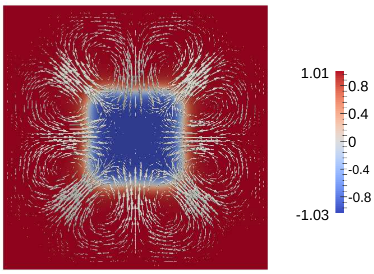
(b) t=16 k



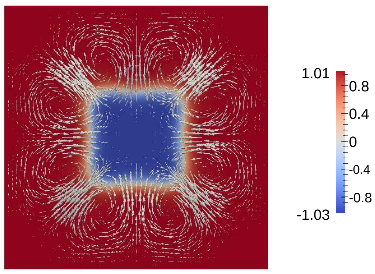
(c) t=30 k



(d) t=50 k



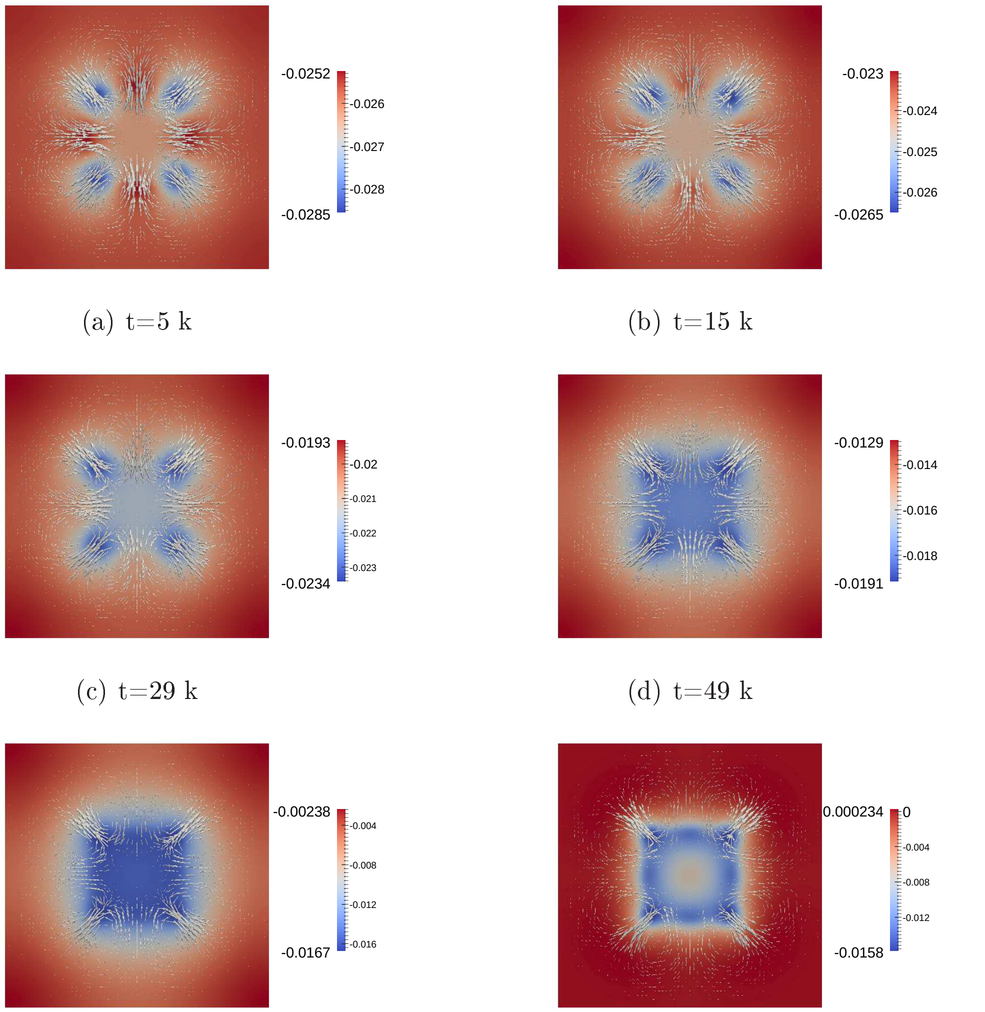
(e) t=74 k



(f) t=90 k

Figure 5.3.: Time evolution of optimal state $\mathcal{Y}_{h,k}(x,t)$ and velocity $\boldsymbol{\mathcal{V}}_{h,k}(x,t)$

In figures 5.4, it is shown the evolution in time of the optimal adjoint state $\mathcal{Q}_{\mathcal{Y},h,k}\,(x,t)$ and the control $\boldsymbol{\mathcal{U}}_{h,k}\,(x,t)$: in the last time steps, they become time by time less intense when the phase-field distribution is closer to the desired state.



(a) t=5 k (b) t=15 k

(c) t=29 k (d) t=49 k

(e) t=73 k (f) t=89 k

Figure 5.4.: Time evolution of the optimal adjoint state $\mathcal{Q}_{\mathcal{Y},h,k}(x,t)$ and the control $\boldsymbol{\mathcal{U}}_{h,k}(x,t)$

5.7.2. Circle to Square 2

Even in this case, the domain is the unit square $\Omega = (0,1)^2$ in the two dimensional plane (x_1, x_2). The initial condition corresponds to the linear interpolation of (3.269) but it is shifted on the right of the domain, around the point $(x_{c1}, x_{c2}) = (0.7, 0.5)$, as shown in figure 3.7. The values of the constant parameters in the model are $\alpha = 10^{-4}, \nu = 0.1, \gamma = 0.005, \rho = 0.1, \varepsilon = 0.02$. The timestep $k = 0.005$ and the time horizon is $T = 400k$. In this numerical experiment we consider a time-dependent desired state. In particular, $y_d(x_1, x_2, t)$ is a state where the two phases are separated by a vanishing interface which has exactly the shape of the square considered in the first numerical experiment, such that:

- at $t = 0$ it is centred around $(x_{c1}, x_{c2}) = (0.7, 0.5)$;
- for $t \in [0, 300k]$ it performs a horizontal uniform motion toward the left hand side of the domain;
- for $t \in [300k, 400k]$ it is centered around the point $(\tilde{x}_{c1}, \tilde{x}_{c2}) = (0.3, 0.5)$, see figure 3.8 in Section 3.6.

Also in this case, condition

$$\int_\Omega y_d(x,t)\,dx = \int_\Omega y_0(x),$$

is, for all $t \in [0, T]$, satisfied and then the desired state is reachable. The settings for the steepest descent Algorithm 3.28 are $TOL = 10^{-9}$, $N_{\max} = 1000$ and the initial guess for the control is $\boldsymbol{\mathcal{U}}_{h,k,(0)} \equiv 0$. Furthermore, even in this case, the s.d. step size is chosen according to the Barzilai-Borwein method [12], with: $\sigma_{\text{init}} = 300$, $\sigma_{\min} = 300$, $\sigma_{\max} = 800$.
Figures 5.5 and 5.6 show the efficiency of our method: in about 420 iterations the cost functional approaches to the minimum and the decreasing of $\|\boldsymbol{\mathcal{G}}_{h,k,(i)}\|_{L^2(\mathbf{L}^2)}$ looks at a logarithmic rate. In figures 5.7, it is depicted the evolution in time of

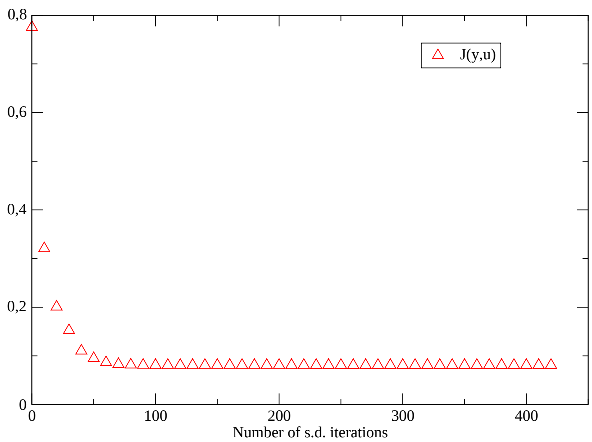


Figure 5.5.: behaviour of $J_{h,k}\left(\mathcal{Y}_{h,k,(i)}, \boldsymbol{\mathcal{U}}_{h,k,(i)}\right)$, with i index of s.d. iterations

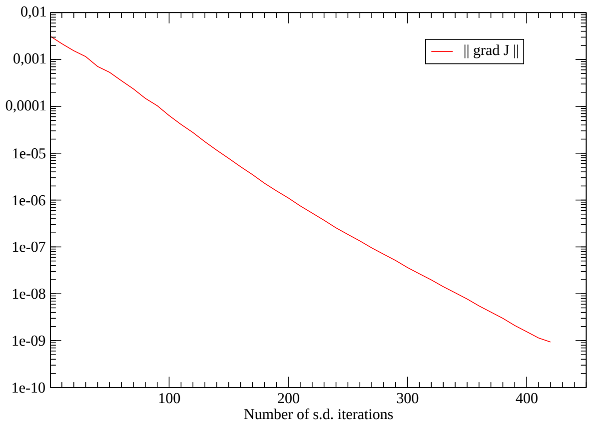
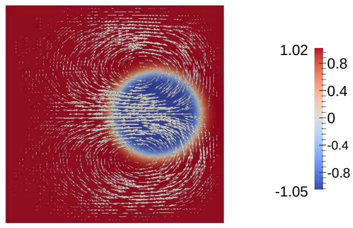
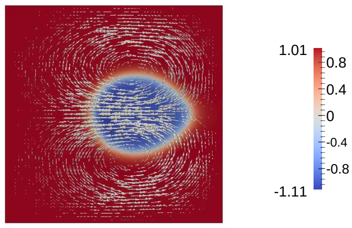


Figure 5.6.: behaviour of $\|\boldsymbol{\mathcal{G}}_{h,k,(i)}\|_{L^2(\mathbf{L}^2)}$, with i index of s.d. iterations

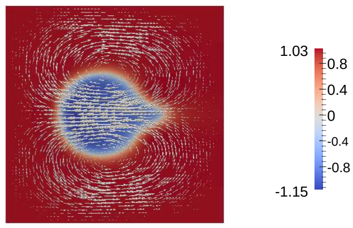
the optimal phase-field $\mathcal{Y}_{h,k}(x,t)$ and velocity $\boldsymbol{\mathcal{V}}_{h,k}(x,t)$ (i.e. at the end of steepest descent algorithm).
We get the expected overshoot for the phase-field distribution values, but the overall behaviour is good: the state of the system follows the movements of the desired state and at the end of the evolution it assumes the shape of a square.
Finally, in figures 5.8, it is possible to observe the evolution in time of the optimal adjoint state $\mathcal{Q}_{\mathcal{Y},h,k}(x,t)$ and the control $\boldsymbol{\mathcal{U}}_{h,k}(x,t)$: it is possible to see that the control in the last time steps drives the velocity and then the phase-field so that it assumes the shape of a square in the exact position.



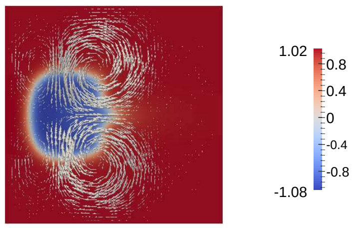
(a) t=24 k



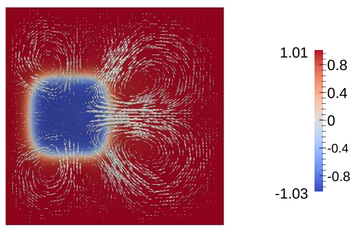
(b) t=64 k



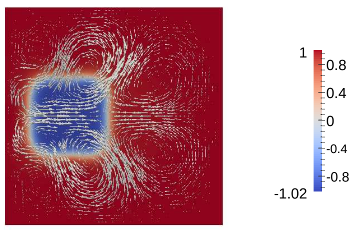
(c) t=120 k



(d) t=200 k

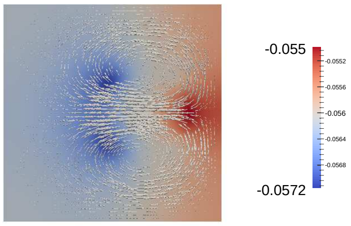


(e) t=296 k

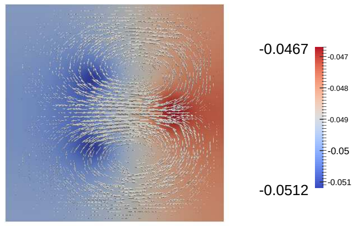


(f) t=360 k

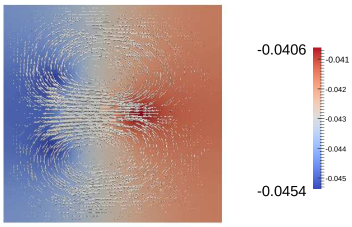
Figure 5.7.: Time evolution of state $\mathcal{Y}_{h,k}(x,t)$ and velocity $\boldsymbol{\mathcal{V}}_{h,k}(x,t)$



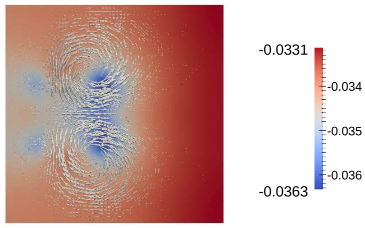
(a) t=23 k



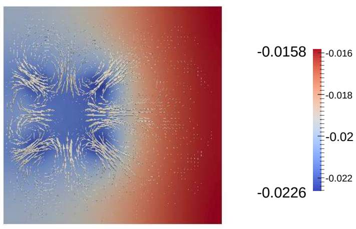
(b) t=63 k



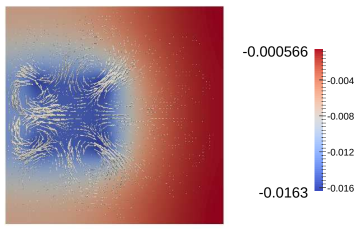
(c) t=119 k



(d) t=199 k



(e) t=295 k



(f) t=359 k

Figure 5.8.: Time evolution of the optimal adjoint state $\mathcal{Q}_{\mathcal{Y},h,k}(x,t)$ and control $\boldsymbol{\mathcal{U}}_{h,k}(x,t)$

Appendix A.

Notations and Basic Results

A.1. Main Notations

We use C to indicate a generic nonnegative constant, which can change its value in the different steps of a same calculation or proof. In the case of dependencies we write $C(\cdot)$. Given a function or map or operator $f = f(t, x, y, z, \dots)$, we denote its partial derivative in the following ways

$$\frac{\partial f}{\partial x} = \partial_x f = f_x,$$

Given a spatial bounded domain Ω, we use $\mathbf{n}$ to denote the outer normal boundary vector. Then, given a function $g : \Omega \to \mathbb{R}$,

$$\frac{\partial f}{\partial \mathbf{n}}\Big|_{\Omega},$$

is used to denote its outer normal boundary derivative.

A.2. Banach Spaces

A.2.1. General Notation

Given a Banach space B, we denote by B^* the corresponding dual space. We use $\|\cdot\|_B$, $|\cdot|_B$ and $\langle \cdot, \cdot \rangle_{B^*,B}$ to denote, respectively, the norm, the seminorm and the dual pairing in B. In the case of a Hilbert space, $(\cdot, \cdot)_B$ denotes the scalar product. Where no confusion arises, we use $(\cdot, \cdot)$ and $\|\cdot\|$ to denote, respectively, the scalar product and the norm in L^2; in the other case we add the corresponding index. If X, Y are two Banach spaces, we use

$$\mathcal{L}(X, Y),$$

to denote the Banach space of the bounded, linear map from X to Y.

A.2.2. Sobolev and Bochner spaces

Let Ω an open and bounded domain in $\mathbb{R}^d$. We use $W^{m,p} := W^{m,p}(\Omega)$ and $H^m := W^{m,2}$ to denote the standard *Sobolev spaces* and by $W^{m,p}\left(W^{k,q}\right) := W^{m,p}\left(0, T; W^{k,q}\right)$ we refer to standard *Bochner spaces*. In the case of vector valued functions and

spaces containing such functions we write them in bold-face notation.
We frequently use the following spaces of *zero mean* functions

$$L_0^2 := \left\{ z \in L^2(\Omega) : \int_\Omega z \, dx = 0 \right\}, \qquad \| \cdot \|_{L_0^2} = \| \cdot \|_{L^2}, \tag{A.1}$$

$$H_0 := H^1(\Omega) \cap L_0^2(\Omega), \qquad \| \cdot \|_{H_0} = \| \cdot \|_{H^1}, \tag{A.2}$$

and the following Hilbert space

$$W_0 := \left\{ y \in L^2(H_0) : y_t \in L^2(H_0^*) \right\},$$

endowed with the following norm

$$\|y\|_{W_0} = \left[\|y\|_{L^2(H^1)}^2 + \|y_t\|_{L^2(H^{1*})}^2 \right]^{\frac{1}{2}}, \quad \forall\, y \in W_0.$$

Regarding *vector valued functions* in Stokes and Navier-Stokes equations, given

$$\boldsymbol{\mathcal{M}} := \{ \mathbf{v} \in \boldsymbol{\mathcal{C}}_c^\infty(\Omega) : \nabla \cdot \mathbf{v} = 0 \},$$

we consider the following Hilbert spaces (see for example [20], Section 3.3, for a characterization of these spaces)

$$\boldsymbol{\mathcal{S}} := \left\{ \text{closure of } \boldsymbol{\mathcal{M}} \text{ in } \mathbf{L}^2 \right\}, \qquad \| \cdot \|_{\boldsymbol{\mathcal{S}}} = \| \cdot \|_{\mathbf{L}^2}, \tag{A.3}$$

$$\boldsymbol{\mathcal{D}} := \left\{ \text{closure of } \boldsymbol{\mathcal{M}} \text{ in } \mathbf{H}_0^1 \right\} = \left\{ \mathbf{v} \in \mathbf{H}_0^1 : \nabla \cdot \mathbf{v} = 0 \right\}, \quad \| \cdot \|_{\boldsymbol{\mathcal{D}}} = \| \cdot \|_{\mathbf{H}^1}, \tag{A.4}$$

and

$$\mathbf{W}_0 := \left\{ \mathbf{v} \in L^2(\boldsymbol{\mathcal{D}}) : \mathbf{v}_t \in L^2(\boldsymbol{\mathcal{D}}^*) \right\},$$

with norm

$$\|\mathbf{v}\|_{\mathbf{W}_0} = \left[\|\mathbf{v}\|_{L^2(\boldsymbol{\mathcal{D}})}^2 + \|\mathbf{v}\|_{L^2(\boldsymbol{\mathcal{D}}^*)}^2 \right]^{\frac{1}{2}}, \quad \forall \mathbf{v} \in \mathbf{W}_0.$$

A.2.3. Useful Embeddings

We have the following continuous embedding (see for example Theorem 1.32 in [58] or Theorem II.5.13 in [20]):

$$W_0 \hookrightarrow \mathcal{C}\left([0,T]; L_0^2\right), \tag{A.5}$$

$$\mathbf{W}_0 \hookrightarrow \mathcal{C}\left([0,T]; \boldsymbol{\mathcal{S}}\right), \tag{A.6}$$

Furthermore we use the following results:

Lemma A.1. *The following embeddings are dense*

$$\mathcal{C}_c^\infty\left((0,T); \mathcal{C}_c^\infty\left(\bar{\Omega}\right)\right) \hookrightarrow L^2\left(H^1\right), \tag{A.7}$$

$$\mathcal{C}_c^\infty\left((0,T); \boldsymbol{\mathcal{D}}\right) \hookrightarrow L^2\left(\boldsymbol{\mathcal{D}}\right). \tag{A.8}$$

Proof. In order to show (A.7), we need to prove that given $\eta \in L^2(H^1)$, for all $n \in \mathbb{N}$ there exists $\eta_n \in \mathcal{C}_c^\infty\left((0,T);\mathcal{C}_c^\infty\left(\bar{\Omega}\right)\right)$ such that

$$\|\eta_n - \eta\|_{L^2(H^1)} \leq \frac{1}{n}. \tag{A.9}$$

The space $\mathcal{C}_c^\infty((0,T);H^1)$ is dense in $L^2(H^1)$ (see for example [58], Lemma 1.9), therefore given $\eta \in L^2(H^1)$, for all $n \in \mathbb{N}$ there exists $\theta_n \in \mathcal{C}_c^\infty((0,T);H^1)$ such that

$$\|\theta_n - \eta\|_{L^2(H^1)} \leq \frac{1}{2n}. \tag{A.10}$$

For all $t \in [0,T]$, $\theta_n(t) \in H^1$. Then, for all $m \in \mathbb{N}$, we can consider a mollifying operator $\mathcal{S}_m$ and the function

$$\hat{\theta}_{mn}(t) = \mathcal{S}_m[\theta_n(t)] \in \mathcal{C}_c^\infty\left(\bar{\Omega}\right),$$

so that

$$\|\hat{\theta}_{mn}(t) - \theta_n(t)\|_{H^1} \to 0, \quad \text{as} \quad m \to +\infty,$$

for all $t \in [0,T]$. The mollifier $\mathcal{S}_m$ acts just on the spatial variables (see also Section 2.2 in [20]), therefore

$$\hat{\theta}_{mn} \in \mathcal{C}_c^\infty\left((0,T);\mathcal{C}_c^\infty\left(\bar{\Omega}\right)\right)$$

and

$$\|\hat{\theta}_{mn} - \theta_n\|_{L^2(H^1)}^2 = \int_0^T \|\hat{\theta}_{mn}(t) - \theta_n(t)\|_{H^1}^2\, dt \leq T \max_{t\in[0,T]} \|\hat{\theta}_{mn}(t) - \theta_n(t)\|_{H^1}^2 \to 0,$$

as $m \to +\infty$. Hence, given $\theta_n \in \mathcal{C}_c^\infty((0,T);H^1)$, there exists $\eta_n \in \mathcal{C}_c^\infty\left((0,T);\mathcal{C}_c^\infty\left(\bar{\Omega}\right)\right)$ such that

$$\|\eta_n - \theta_n\|_{L^2(H^1)} \leq \frac{1}{2n}. \tag{A.11}$$

Using together (A.10) and (A.11), we get (A.9).
The second embedding (A.8) is a direct consequence of Lemma 1.9 in [58]. □

Lemma A.2. *The following embedding is dense*

$$\mathcal{C}^\infty\left([0,T];\mathcal{C}_c^\infty\left(\bar{\Omega}\right) \cap L_0^2\right) \hookrightarrow W_0, \tag{A.12}$$

Proof. Given the Gelfand triple

$$H_0 \hookrightarrow L_0^2 \simeq \left(L_0^2\right)^* \hookrightarrow H_0^*,$$

where both embeddings are continuous and dense, by Lemma II.5.10 in [20], we have that

$$\mathcal{C}^\infty([0,T];H_0) \hookrightarrow W_0,$$

is a dense embedding. Thus, working as in the proof of Lemma A.1 above, using a mollifying operator, it is possible to show (A.12). □

A.2.4. Useful Inequalities

Very often, we use the following:

- *Young's inequality*

(A.13)
$$ab \le \sigma a^2 + \frac{b^2}{4\sigma} = \sigma a^2 + C(\sigma) b^2, \quad \forall\ a, b \ge 0,\ \sigma > 0;$$

- generalized *Holder's inequality* (see for example Lemma 1.13 in [58])

(A.14)
$$\|u_1 \cdots u_k\|_{L^p} \le \|u_1\|_{L^{p_1}} \cdots \le \|u_k\|_{L^{p_k}}, \quad \forall u_i \in L^{p_i},\ \text{with } 1/p_1 + \ldots 1/p_k = 1/p, \quad p_i, p \in [1, +\infty];$$

- *Poincaré-Wirtinger inequality* (see for example Proposition III.2.39 in [20])

(A.15)
$$\|\eta\|_{L^p} \le C \left[\|\nabla \eta\|_{L^p} + \frac{1}{|\Omega|} |\, (\eta, 1)\, | \right], \quad \forall\ \eta \in W^{1,p},\ p \in [1, +\infty)\,;$$

- *Poincaré's inequality*

(A.16)
$$\|z\| \le C\ \|\nabla z\|, \quad \forall\ z \in H^1_0;$$

- *special inequalities*

(A.17)
$$\|u\|_{L^p} \le C\ \|u\|_{H^1}, \qquad \forall\ u \in H^1, \quad p \in [2, +\infty),$$

(A.18)
$$\|u\|_{L^4} \le C\ \|u\|^{\frac{1}{2}}\ \|u\|^{\frac{1}{2}}_{H^1}, \qquad \forall\ u \in H^1.$$

A.2.5. Green's Operator

Given the space

(A.19)
$$\mathcal{F} = \left\{ f \in H^{1*} : \langle f, 1 \rangle_{H^{1*}, H^1} = 0 \right\},$$

we can define the *Green's operator* $\mathcal{G} : \mathcal{F} \to H^1$ in the following way: given $f \in \mathcal{F}$ then $\mathcal{G} f \in H^1$ is the unique solution of

(A.20)
$$(\nabla \mathcal{G} f, \nabla \eta) = \langle f, \eta \rangle_{H^{1*}, H^1}, \qquad \forall\ \eta \in H^1,$$
$$(\mathcal{G} f, 1) = 0.$$

The existence and uniqueness of $\mathcal{G} f$ is given by the Lax-Milgram theorem and the Poincaré's-Wirtinger inequality (A.15). It is possible to show that if $f \in \mathcal{F}$, we can set

(A.21)
$$\|f\|_{H^{1*}} = \|\nabla \mathcal{G} f\|.$$

Furthermore, if $f \in \mathcal{F} \cap L^2$, by (A.15) and (A.20), we have

(A.22)
$$\|f\|_{H^{1*}} = (\mathcal{G} f, f)^{\frac{1}{2}}\,,$$

(A.23)
$$\|f\|_{H^{1*}} \le C \|f\|.$$

A.3. Discrete Settings

Given an open, bounded, Lipschitz domain $\Omega \subset \mathbb{R}^2$ and a time interval $[0, T]$, with $T > 0$, we assume in the document the following discrete settings. Let:

- $\{t_0, t_1, \ldots, t_N\}$ be a partition of $[0, T]$ in N sub-intervals of length $k = T/N$;
- $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω in disjoint rectangular triangles τ, such that
$$\bar{\Omega} = \cup_{\tau \in \mathcal{T}_h} \bar{\tau},$$
with mesh size
$$h := \max_{\tau \in \mathcal{T}_h} \operatorname{diam}(\tau), \quad h \in (0,1). \tag{A.24}$$
- $x_j, j \in \mathcal{J}_h = \{1, \ldots, N_h\}$ be, respectively, the vertices of the triangulation $\mathcal{T}_h$ and set of their indices.
- $\mathcal{P}_r(\tau)$ be the space of polynomials of degree less than or equal to r on τ and $\boldsymbol{\mathcal{P}}_r(\tau)$ the corresponding 2-dimensional space.

A.3.1. Discrete Spaces

We associate to the triangulation $\mathcal{T}_h$ the following finite dimensional spaces:

$$\begin{aligned} \mathbf{S}_h &:= \left\{ \mathbf{S} \in \boldsymbol{\mathcal{C}}(\bar{\Omega}) : \mathbf{S}|_\tau \in \boldsymbol{\mathcal{P}}_2(\tau) \right\}, \\ \mathbf{V}_h &:= \mathbf{S}_h \cap \mathbf{H}_0^1, \\ Y_h &:= \left\{ Y \in \mathcal{C}(\bar{\Omega}) : Y|_\tau \in \mathcal{P}_1(\tau) \right\}, \\ P_h &:= Y_h \cap L_0^2. \end{aligned}$$

Furthermore we consider the space of the divergence-free functions

$$\mathbf{D}_h := \{ \mathbf{V} \in \mathbf{V}_h : (\nabla \cdot \mathbf{V}, P) = 0, \forall\, P \in P_h \}. \tag{A.25}$$

We emphasize (see for example page 310 in [73]) that the $P_2 - P_1$ mixed finite element space $(\mathbf{V}_h, P_h)$ for the Stokes equation is *stable*, in the sense that it satisfies the following *inf-sup condition*

$$\sup_{\mathbf{V} \in \mathbf{V}_h} \frac{(\nabla \cdot \mathbf{V}, P)}{\|\nabla \mathbf{V}\|} \geq C \|P\|, \quad \forall P \in P_h. \tag{A.26}$$

where the constant C does not depend on h.

A.3.2. Interpolation Operator

The interpolation operator $I^h : \mathcal{C}(\bar{\Omega}) \to Y_h$, is such that

$$\left[I^h(\chi) \right](x_j) = \chi(x_j), \tag{A.27}$$

for all x_j vertex of the triangulation $\mathcal{T}_h$. It holds (see for example Section 3.4.1 in [73]),

$$\|\chi - I^h \chi\| + h \left\| \nabla \left(\chi - I^h \chi \right) \right\| \leq C\, h^2 |\chi|_{H^2}. \tag{A.28}$$

A.3.3. Mass Lumping and h-Norm

The *mass-lumped* scalar product and associated h-norm are defined as follows

$$(\chi, \eta)_h = \int_\Omega I^h(\chi\eta)\ dx, \quad \|\chi\|_h = \sqrt{(\chi, \chi)_h}, \quad \forall\ \chi, \eta \in \mathcal{C}\left(\bar{\Omega}\right). \tag{A.29}$$

There exist two constant C_1, C_2, which depend just on the domain Ω, such that the h-norm and the L^2-norm satisfy the following equivalence relation

$$C_1\|Z\|_h \leq \|Z\| \leq C_2\|Z\|_h, \quad \forall\ Z \in Y_h. \tag{A.30}$$

Moreover,

$$\left|\,(Y, Z)_h - (Y, Z)\,\right| \leq C\ h\ \|Y\|\ \|\nabla Z\|, \quad \forall\ Y, Z \in Y_h. \tag{A.31}$$

A.3.4. Discrete Green's Operators

As well as in [62], we introduce the following discrete Green's operators

$$\mathcal{G}^h : \mathcal{F} \to P_h,$$
$$\hat{\mathcal{G}}^h : P_h \to P_h,$$

such that for all $Z \in Y_h$, we have

$$\left(\nabla \mathcal{G}^h \eta, \nabla Z\right) = \langle \eta, Z\rangle_{H^{1*},H^1}, \tag{A.32}$$

$$\left(\nabla \hat{\mathcal{G}}^h Y, \nabla Z\right) = (Y, Z)_h\,. \tag{A.33}$$

The operator $\mathcal{G}^h, \hat{\mathcal{G}}^h$ satisfy the following inequalities (see for example [18]):

$$\|\nabla \mathcal{G}^h \eta\| \leq C\|\eta\|, \qquad \forall\ \eta \in \mathcal{F} \cap L^2, \tag{A.34}$$

$$\|\nabla \hat{\mathcal{G}}^h Z\| \leq C\|Z\|_h, \qquad \forall\ Z \in P_h \tag{A.35}$$

A.3.5. Discrete Laplacian and Stokes Operators

We define the followings *discrete Laplacian operators*

$$\Delta_h : Y_h \to Y_h,$$
$$\hat{\Delta}_h : Y_h \to Y_h,$$
$$\tilde{\Delta}_h : \mathbf{V}_h \to \mathbf{V}_h.$$

They are such that

$$(-\Delta_h Y, Z) = (\nabla Y, \nabla Z) = \left(-\hat{\Delta}_h Y, Z\right)_h, \quad \forall Z \in Y_h, \tag{A.36}$$

and

$$\left(-\tilde{\Delta}_h \mathbf{V}, \mathbf{Z}\right) = (\nabla \mathbf{V}, \nabla \mathbf{Z}), \quad \forall \mathbf{Z} \in \mathbf{V}_h. \tag{A.37}$$

Moreover, there exist a constant $C = C(\Omega)$, so that

$$\|\hat{\Delta}_h Y\|_h^2 \le \|\Delta_h Y\|^2 \le C\|\hat{\Delta}_h Y\|_h^2. \tag{A.38}$$

The following inequality (see [41], Theorem 6.4) holds

$$\|\nabla Z\|_{L^p} \le C(p)\,\|\Delta_h Z\|, \tag{A.39}$$

for all $Z \in P_h$ and $1 \le p < 2d/(d-2)$, where d is the space dimension. Finally, use the *discrete Stokes operator* $\mathbf{A}^h$ defined as follows

$$\mathbf{A}^h := -\mathbf{T}^h \tilde{\Delta}_h, \tag{A.40}$$

where $\mathbf{T}^h : \mathbf{L}^2 \to \mathbf{D}_h$ denotes the $\mathbf{L}^2$ projection.

A.3.6. Projection Operators

In the document we use the following four projection operator.

- The L^2-**projection operator** $Q^h : L^2 \to Y_h$,

  $$\left(Q^h \eta, Z\right)_h = (\eta, Z), \quad \forall\, Z \in Y_h, \tag{A.41}$$

  which is such that (see for example [62])

  $$\left\| \left(I - Q^h\right)\eta \right\| + h\left\|\nabla\left(I - Q^h\right)\eta\right\| \le Ch\|\nabla\eta\|, \quad \forall \eta \in H^1 \tag{A.42}$$

- The L^2-**projection operator** $Q_0^h : L^2 \to Y_h$,

  $$\left(Q_0^h \eta, Z\right) = (\eta, Z), \quad \forall\, Z \in Y_h. \tag{A.43}$$

  It is possible to prove (see for example [41], condition (S6), p. 3041),

  $$\lim_{h\to 0} \|\eta - Q_0^h \eta\| = 0, \quad \forall\, \eta \in L^2. \tag{A.44}$$

- The H^1-**projection operator** $Q_1^h : H^1 \to Y_h$,

  $$\left(Q_1^h \eta, Z\right)_{H^1} = (\eta, Z)_{H^1}, \quad \forall Z \in Y_h, \tag{A.45}$$

  which is such that (see for example Section 3.5 in [73])

  $$\|\eta - Q_1^h \eta\| \le C\, h^{l+1}\, |\eta|_{H^{l+1}}, \qquad \forall\, \eta \in H^{l+1},\ 0 \le l \le 1, \tag{A.46}$$

  $$\|\eta - Q_1^h \eta\|_{H^1} \le C\, h\, |\eta|_{H^2}, \qquad \forall\, \eta \in H^2. \tag{A.47}$$

- The **Stokes projection** $\mathbf{Q}_s^h : \boldsymbol{\mathcal{D}} \to \mathbf{D}_h$,

  $$\left(\nabla \mathbf{Q}_s^h \mathbf{v}, \nabla \mathbf{Z}\right) = (\nabla \mathbf{v}, \nabla \mathbf{Z}), \quad \forall \mathbf{Z} \in \mathbf{D}_h \tag{A.48}$$

  which is such that (see [41]),

  $$\|\mathbf{Q}_s^h \mathbf{v} - \mathbf{v}\| + h\|\nabla\left(\mathbf{Q}_s^h \mathbf{v} - \mathbf{v}\right)\| \le C\, h^l \|\mathbf{v}\|_{\mathbf{H}^l}, \tag{A.49}$$

  for all $\mathbf{v} \in \mathbf{H}^l \cap \boldsymbol{\mathcal{D}}, l = 1, 2$.

A.3.7. Useful Discrete Inequalities

We often use the *discrete Poincaré inequality*

$$\|Z\|_h \leq C\left(\|\nabla Z\| + |(Z,1)_h|\right), \quad \forall Z \in Y_h, \tag{A.50}$$

and the following *discrete embedding and interpolation inequalities* (see [46]):

$$\|\nabla Z\|_{L^4} \leq C\ \left(\|\Delta_h Z\| + \|\nabla Z\|\right), \tag{A.51}$$

$$\|\nabla Z\|_{L^4} \leq C\ \|\nabla Z\|^{\frac{1}{2}}\left(\|\Delta_h Z\| + \|\nabla Z\|\right)^{\frac{1}{2}}, \tag{A.52}$$

$$\|\nabla \mathbf{Z}\|_{\mathbf{L}^4} \leq C\ \|\tilde{\Delta}_h \mathbf{Z}\|, \tag{A.53}$$

$$\|\nabla \mathbf{Z}\|_{\mathbf{L}^4} \leq C\ \|\nabla \mathbf{Z}\|^{\frac{1}{2}}\|\tilde{\Delta}_h \mathbf{Z}\|^{\frac{1}{2}}. \tag{A.54}$$

Furthermore, given a triangulation of a domain Ω with mesh size h, it hold (see for example [62]) the following inverse inequality

$$\|\mathbf{V}\|_{L^4} \leq \frac{C}{h}\|\mathbf{V}\|, \tag{A.55}$$

for all $\mathbf{V} \in \mathbf{S}_h$, where C is a constant which is independent on h.

Lemma A.3. *For all* $Y \in Y_h := \left\{Y \in \mathcal{C}(\bar{\Omega}) : Y|_\tau \in \mathcal{P}_1(\tau)\right\}$, *it holds*

$$\|Y\|_{L^4}^4 \leq 5\left(Y^4, 1\right)_h. \tag{A.56}$$

Proof. Let $Y \in Y_h$. On each mesh triangle $\tau \in \mathcal{T}_h$, we have

$$Y\Big|_\tau = Y_{1,\tau}\ \varphi_{1,\tau} + Y_{2,\tau}\ \varphi_{2,\tau} + Y_{3,\tau}\ \varphi_{3,\tau},$$

where $\varphi_{i,\tau} \in Y_h$, $i = 1,2,3$ are the basis functions associated with the three vertices of the mesh triangle $\tau \in \mathcal{T}_h$. Calculations produce

$$\begin{aligned}\|Y\|_{L^4}^4 = \sum_{\tau\in\mathcal{T}_h} \frac{|\tau|}{15}\Big[&Y_{1,\tau}^4 + Y_{2,\tau}^4 + Y_{3,\tau}^4 + Y_{1,\tau}^2 Y_{2,\tau}^2 + Y_{1,\tau}^2 Y_{3,\tau}^2 + Y_{2,\tau}^2 Y_{3,\tau}^2 \\ &+ Y_{1,\tau}^3 Y_{2,\tau} + Y_{1,\tau}^3 Y_{3,\tau} + Y_{1,\tau} Y_{2,\tau}^3 + Y_{2,\tau}^3 Y_{3,\tau} + Y_{1,\tau}\ Y_{3,\tau}^3 + Y_{2,\tau}\ Y_{3,\tau}^3 \\ &+ Y_{1,\tau}^2\ Y_{2,\tau}\ Y_{3,\tau} + Y_{1,\tau}\ Y_{2,\tau}^2\ Y_{3,\tau} + Y_{1,\tau}\ Y_{2,\tau}\ Y_{3,\tau}^2\Big],\end{aligned} \tag{A.57}$$

$$\left(Y^4, 1\right)_h = \sum_{\tau\in\mathcal{T}_h} \frac{|\tau|}{3}\left[Y_{1,\tau}^4 + Y_{2,\tau}^4 + Y_{3,\tau}^4\right]. \tag{A.58}$$

Using the Young's inequality

$$ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2, \quad \text{for all } a, b \in \mathbb{R},$$

it is easy to realize that

$$\begin{aligned}&\Big[Y_{1,\tau}^4 + Y_{2,\tau}^4 + Y_{3,\tau}^4 + Y_{1,\tau}^2 Y_{2,\tau}^2 + Y_{1,\tau}^2 Y_{3,\tau}^2 + Y_{2,\tau}^2 Y_{3,\tau}^2 \\ &+ Y_{1,\tau}^3 Y_{2,\tau} + Y_{1,\tau}^3 Y_{3,\tau} + Y_{1,\tau} Y_{2,\tau}^3 + Y_{2,\tau}^3 Y_{3,\tau} + Y_{1,\tau}\ Y_{3,\tau}^3 + Y_{2,\tau}\ Y_{3,\tau}^3 \\ &+ Y_{1,\tau}^2\ Y_{2,\tau}\ Y_{3,\tau} + Y_{1,\tau}\ Y_{2,\tau}^2\ Y_{3,\tau} + Y_{1,\tau}\ Y_{2,\tau}\ Y_{3,\tau}^2\Big] \\ &\leq 5\left[Y_{1,\tau}^4 + Y_{2,\tau}^4 + Y_{3,\tau}^4\right],\end{aligned} \tag{A.59}$$

for all $\tau \in \mathcal{T}_h$. Hence, from (A.57), (A.58) and (A.59) we get that the result (A.56) holds. ☐

Appendix B.

Proofs

B.1. Proofs of Chapter 2

Proof of Lemma 2.3

Proof. First, we prove that the Stokes state equations (2.24) have a unique solution $\mathbf{v} \in H^1(\boldsymbol{\mathcal{S}}) \cap L^\infty(\boldsymbol{\mathcal{D}})$, which satisfies

$$\|\mathbf{v}_t\|^2_{L^2(\boldsymbol{\mathcal{S}})} + \|\mathbf{v}\|^2_{L^\infty(\boldsymbol{\mathcal{D}})} \leq C\left[\|\mathbf{v}_0\|^2_{\boldsymbol{\mathcal{D}}} + \|\mathbf{u}\|^2_{L^2(\mathbf{L}^2)}\right]. \tag{B.1}$$

In order to show that, we use a Galerkin's approximation (see for example page 44 in [80], page 45 in [58], [66]). The space $\boldsymbol{\mathcal{D}}$ is separable, then we can consider an orthogonal dense subset $\{\boldsymbol{\xi}_j\}_{j\in\mathbb{N}} \subset \boldsymbol{\mathcal{D}}$, normalized such that

$$(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j) = \delta_{ij}.$$

For all $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}$, we have

$$\left\|\sum_{j=1}^{k} (\boldsymbol{\psi}, \boldsymbol{\xi}_j)\, \boldsymbol{\xi}_j - \boldsymbol{\psi}\right\|_{\boldsymbol{\mathcal{D}}} \to 0, \tag{B.2}$$

as $k \to +\infty$. A suitable dense subset $\{\boldsymbol{\xi}_j\}_{j\in\mathbb{N}}$ of $\boldsymbol{\mathcal{D}}$ can be derived considering the eigenfunctions of the Stokes operator, as in Paragraph 5.2 and Theorem IV.5.5 in [20]. Let $\mathbf{W}_k$ denote the finite dimensional subspace of $\boldsymbol{\mathcal{D}}$ spanned by $\{\boldsymbol{\xi}_j\}_{j=1,\dots,k}$. We define a projection operator $\mathbf{P}^k : \boldsymbol{\mathcal{D}} \to \mathbf{W}_k$,

$$\mathbf{P}^k\boldsymbol{\psi} = \sum_{j=1}^{k} (\boldsymbol{\psi}, \boldsymbol{\xi}_j)\, \boldsymbol{\xi}_j, \tag{B.3}$$

which is such that

$$(\mathbf{P}^k\boldsymbol{\psi}, \boldsymbol{\xi}) = (\boldsymbol{\psi}, \boldsymbol{\xi}), \quad (\nabla\mathbf{P}^k\boldsymbol{\psi}, \nabla\boldsymbol{\xi}) = (\nabla\boldsymbol{\psi}, \nabla\boldsymbol{\xi}), \quad \forall\, \boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}, \boldsymbol{\xi} \in \mathbf{W}_k.$$

In this way, for any fixed $k \in \mathbb{N}$, the Galerkin's approximation of the time dependent Stokes equations (2.24a)-(2.24b), consists in finding $\mathbf{v}^k$, such that

$$(\mathbf{v}^k_t, \boldsymbol{\psi}^k) + \nu\,(\nabla\mathbf{v}^k, \nabla\boldsymbol{\psi}^k) - (\mathbf{u}, \boldsymbol{\psi}^k) = 0, \qquad \text{a.e. on } (0,T) \tag{B.4}$$

$$\mathbf{v}^k(0) = \mathbf{P}^k\mathbf{v}_0, \qquad \text{in } \Omega, \tag{B.5}$$

for all $\boldsymbol{\psi}^k \in \mathbf{W}_k$. Setting

$$\mathbf{v}^k = \sum_{j=1}^{k} b_j\left(t\right)\boldsymbol{\xi}_j, \tag{B.6}$$

it is possible to prove that the linear system associated to (B.4), (B.5) has a unique solution

$$\mathbf{b}^k\left(t\right) = \left(b_1\left(t\right), \ldots, b_k\left(t\right)\right)^T,$$

such that $b_i, \in H^1\left(0, T\right)$ for all $i = 1, \ldots, k$.
So, we can claim that for any fixed $k \in \mathbb{N}$, $\mathbf{v}^k \in H^1\left(\mathbf{W}_k\right)$ solves (B.4), (B.5), for all $\boldsymbol{\psi}^k \in \mathcal{C}\left([0,T]; \mathbf{W}_k\right)$. Substituting $\boldsymbol{\psi}^k = \mathbf{v}_t^k$ in (B.4), we get

$$\|\mathbf{v}_t^k\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{v}^k\|^2 = \left(\mathbf{u}, \mathbf{v}_t^k\right). \tag{B.7}$$

Hence, setting $s = t$ in (B.7) above and integrating in time from 0 to t, with $t \in (0, T]$, we can write

$$\int_0^t \|\mathbf{v}_s^k\|^2 ds + \frac{\nu}{2}\|\nabla\mathbf{v}^k\left(t\right)\|^2 = \int_0^t \left(\mathbf{u}, \mathbf{v}_s^k\right) ds + \frac{\nu}{2}\|\mathbf{v}^k\left(0\right)\|^2. \tag{B.8}$$

From (B.8), applying Young's inequality (A.13) with with $\sigma = 1/2$ in the integral at the r.h.s, we derive

$$\frac{1}{2}\int_0^t \|\mathbf{v}_s^k\|^2 ds + \frac{\nu}{2}\|\nabla\mathbf{v}^k\left(t\right)\|^2 = \frac{1}{2}\int_0^t \|\mathbf{u}\|^2 ds + \frac{\nu}{2}\|\mathbf{v}^k\left(0\right)\|^2, \tag{B.9}$$

which implies, using Poincaré's inequality (A.16),

$$\|\mathbf{v}_t^k\|^2_{\mathbf{L}^2(\boldsymbol{\mathcal{S}})} + \|\mathbf{v}^k\|^2_{\mathbf{L}^\infty(\boldsymbol{\mathcal{D}})} \leq C\left[\|\mathbf{v}^k\left(0\right)\|^2 + \|\mathbf{u}\|^2_{\mathbf{L}^2(\mathbf{L}^2)}\right], \tag{B.10}$$

where C is a constant which depends just on the constant parameter ν. By the definition (B.3) of the projection operator $\mathbf{P}^k$, we realize that

$$\|\mathbf{v}^k\left(0\right)\| = \|\mathbf{P}^k\mathbf{v}_0\| \leq \|\mathbf{v}_0\|,$$

and therefore, from (B.10) above, we infer

$$\|\mathbf{v}_t^k\|^2_{L^2(\boldsymbol{\mathcal{S}})} + \|\mathbf{v}^k\|^2_{L^\infty(\boldsymbol{\mathcal{D}})} \leq C\left[\|\mathbf{v}_0\|^2_{\boldsymbol{\mathcal{D}}} + \|\mathbf{u}\|^2_{L^2(\mathbf{L}^2)}\right]. \tag{B.11}$$

Hence, given the sequence $\{\mathbf{v}^k\}_{k\in\mathbb{N}}$, it is possible to extract a subsequence, labelled with index m, such that

$$\mathbf{v}^m \rightharpoonup \mathbf{v}, \qquad \text{in } H^1\left(\boldsymbol{\mathcal{S}}\right), \tag{B.12}$$

$$\mathbf{v}^m \overset{*}{\rightharpoonup} \mathbf{v}, \qquad \text{in } L^\infty\left(\boldsymbol{\mathcal{D}}\right), \tag{B.13}$$

$$\mathbf{v}^m \to \mathbf{v}, \qquad \text{in } L^2\left(\boldsymbol{\mathcal{S}}\right), \tag{B.14}$$

where (B.14) follows from (B.12),(B.13) using a compactness theorem, see [17] and [66]. As a consequence of (B.4), (B.5), we have that

$$\int_0^T \left[(\mathbf{v}_t^m, \boldsymbol{\psi}^m) + \nu \left(\nabla \mathbf{v}^m, \nabla \boldsymbol{\psi}^m \right) - (\mathbf{u}, \boldsymbol{\psi}^m) \right] \; dt = 0, \tag{B.15}$$

$$\mathbf{v}^m(0) = \mathbf{P}^m \mathbf{v}_0, \qquad \text{in } \Omega, \tag{B.16}$$

for all $\boldsymbol{\psi}^m \in \mathcal{C}_c^\infty\left((0,T)\,;\mathbf{W}_m\right)$. For any $\boldsymbol{\psi} \in \mathcal{C}_c^\infty\left((0,T)\,;\boldsymbol{\mathcal{D}}\right)$, we set in (B.15) $\boldsymbol{\psi}^m = \mathbf{P}^m \boldsymbol{\psi}$, which is such that

$$\|\boldsymbol{\psi}^m - \boldsymbol{\psi}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})} \to 0, \tag{B.17}$$

as $m \to \infty$. Then, from (B.12)-(B.14) and (B.17), we get

$$\left| \int_0^T (\mathbf{v}_t^m, \boldsymbol{\psi}^m) \; dt - \int_0^T (\mathbf{v}_t, \boldsymbol{\psi}) \; dt \right| \to 0, \tag{B.18}$$

$$\left| \int_0^T (\nabla \mathbf{v}^m, \nabla \boldsymbol{\psi}^m) \; dt - \int_0^T (\nabla \mathbf{v}, \nabla \boldsymbol{\psi}) \; dt \right| \to 0, \tag{B.19}$$

$$\left| \int_0^T (\mathbf{u}, \boldsymbol{\psi}^m) \; dt - \int_0^T (\mathbf{u}, \boldsymbol{\psi}) \; dt \right| \to 0, \tag{B.20}$$

as $m \to +\infty$. Furthermore, considering $\boldsymbol{\psi} = \boldsymbol{\xi}\,(1 - t/T)$, with $\boldsymbol{\xi} \in \boldsymbol{\mathcal{D}}$, using integration by parts in time, we can write

$$\left(\mathbf{v}^m(0) - \mathbf{v}(0)\,, \boldsymbol{\xi}\right) = \int_0^T \left[(-\mathbf{v}_t^m + \mathbf{v}_t, \boldsymbol{\psi}) + (-\mathbf{v}^m + \mathbf{v}, \boldsymbol{\psi}_t) \right] \; dt \to 0,$$

as $m \to +\infty$, for all $\boldsymbol{\xi} \in \boldsymbol{\mathcal{D}}$. Therefore

$$\mathbf{v}^m(0) \rightharpoonup \mathbf{v}(0)\,, \qquad \text{in } \boldsymbol{\mathcal{D}}.$$

Moreover, from (B.2) and the definition (B.3) of the projection operator $\mathbf{P}^m$, we note

$$\mathbf{P}^m \mathbf{v}_0 = \mathbf{v}^m(0) \to \mathbf{v}_0, \qquad \text{in } \boldsymbol{\mathcal{D}}.$$

Hence, we conclude that

$$\mathbf{v}(0) = \mathbf{v}_0. \tag{B.21}$$

So, from (B.18)-(B.21), we can say that $\mathbf{v} \in H^1(\boldsymbol{\mathcal{S}}) \cap L^\infty(\boldsymbol{\mathcal{D}})$ satisfy

$$\int_0^T \left[(\mathbf{v}_t, \boldsymbol{\psi}) + \nu \left(\nabla \mathbf{v}, \nabla \boldsymbol{\psi} \right) - (\mathbf{u}, \boldsymbol{\psi}) \right] \; dt = 0,$$

$$\mathbf{v}(0) = \mathbf{v}_0, \quad \text{in } \Omega,$$

for all $\boldsymbol{\psi} \in \mathcal{C}_c^\infty\left((0,T)\,;\boldsymbol{\mathcal{D}}\right)$. So, from the embedding (A.8), we infer that $\mathbf{v} \in H^1(\boldsymbol{\mathcal{S}}) \cap L^\infty(\boldsymbol{\mathcal{D}})$ solves the state equations (2.24), for all $\boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}})$. Furthermore, using the linearity of the equations, it is easy to realize that this solution is unique.

Next, we demonstrate that given $\mathbf{v} \in H^1(\boldsymbol{\mathcal{S}}) \cap L^\infty(\boldsymbol{\mathcal{D}})$, the Cahn-Hilliard state equations (2.25), have a unique solution $(y, w) \in W_0 \cap L^\infty(H_0) \cap L^2(H^2) \times L^2(H^1)$. As in the previous part of the proof, following the authors of [17], we apply a Galerkin's method. Let $\{\phi_j\}_{j\in\mathbb{N}}$ be an orthogonal dense subset of H^1, normalized in the following way

$$(\phi_i, \phi_j) = \delta_{ij},$$

and consisting in the eigenfunctions for

$$-\Delta\phi + \phi = \mu\,\phi, \quad \left.\frac{\partial \phi}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0. \tag{B.22}$$

For all $\varphi \in H^1$, we have

$$\left\| \sum_{j=1}^{k} (\varphi, \phi_j)\,\phi_j - \varphi \right\|_{H^1} \to 0, \quad \text{as} \quad k \to +\infty.$$

Let V_k denote the finite dimensional subspace of H^1 spanned by $\{\phi_j\}_{j=1,\ldots,k}$. We define the following projection $P^k : H^1 \to V_k$

$$P^k\varphi = \sum_{j=1}^{k} (\varphi, \phi_j)\,\phi_j, \tag{B.23}$$

which is such that

$$\left(P^k\varphi, \zeta\right) = (\varphi, \zeta), \quad \left(\nabla P^k\varphi, \nabla\zeta\right) = (\nabla\varphi, \nabla\zeta), \quad \forall\varphi \in H^1, \zeta \in V_k.$$

For any fixed $k \in \mathbb{N}$, the Galerkin's approximations of (2.25) consists in finding y^k, w^k, such that

$$\left(y_t^k, \eta^k\right) + \gamma\left(\nabla w^k, \nabla\eta^k\right) - \left(y^k, \mathbf{v}\cdot\nabla\eta^k\right) = 0, \qquad \text{a.e. on } (0,T), \tag{B.24}$$

$$y^k(0) = P^k y_0, \qquad \text{in } \Omega \tag{B.25}$$

$$\left(w^k + y^k - \frac{1}{\delta}\beta_\delta\left(y^k\right), \theta^k\right) - \varepsilon^2\left(\nabla y^k, \nabla\theta^k\right) = 0, \qquad \text{a.e. on } (0,T), \tag{B.26}$$

for all $\eta^k, \theta^k \in V_k$. In order to solve to solve (B.24)-(B.26), we set

$$y^k = \sum_{j=1}^{k} c_j(t)\phi_j, \quad w^k = \sum_{j=1}^{k} l_j(t)\phi_j$$

and we look for solutions $\mathbf{c}^k(t) = (c_1(t), \ldots, c_k(t))^T, \mathbf{l}^k(t) = (l_1(t), \ldots, l_k(t))^T$ of the following linear system

$$\frac{d\mathbf{c}^k}{dt}(t) + \gamma A\;\mathbf{l}^k(t) - D(t)\;\mathbf{c}^k(t) = 0, \tag{B.27}$$

$$\mathbf{c}^k(0) = \left((y_0, \varphi_1), \ldots, (y_0, \varphi_k)\right)^T. \tag{B.28}$$

$$\mathbf{l}^k(t) - \varepsilon^2 A\;\mathbf{c}^k(t) + \mathbf{c}^k(t) - \frac{1}{\delta}\mathbf{r}\left(\mathbf{c}^k(t)\right) = 0, \tag{B.29}$$

where the matrices $A, D(t)$ and the vector $\mathbf{r}(t)$ read

$$A_{ij} = (\nabla\phi_j, \nabla\phi_i), \quad D_{ij}(t) = (\phi_j, \mathbf{v}(t)\cdot\nabla\phi_i), \quad \mathbf{r}_i(\mathbf{c}(t)) = \left(\beta_\delta\left(\sum_{j=1}^{k} c_j(t)\phi_j\right), \phi_i\right),$$

for $i, j = 1, \ldots, k$. From (B.29), we get

$$\mathbf{l}^k(t) = \left[\varepsilon^2 A - I\ \right]\mathbf{c}^k(t) + \frac{1}{\delta}\mathbf{r}\left(\mathbf{c}^k(t)\right),$$

which, substituted in (B.27), produce

$$\frac{d\mathbf{c}^k}{dt}(t) + \left[\gamma\ \varepsilon^2\ A^2 - \gamma\ A - D(t)\ \right]\mathbf{c}^k(t) + \frac{\gamma}{\delta}\ A\ \mathbf{r}\left(\mathbf{c}^k(t)\right) = 0.$$

By definitions, $A \in L^\infty\left(0, T; \mathbb{R}^{k\times k}\right)$ and using $\mathbf{v} \in L^\infty(\boldsymbol{\mathcal{D}})$, we also have $D \in L^\infty\left(0, T; \mathbb{R}^{k\times k}\right)$. Furthermore, denoting with $\|\cdot\|_2$ the euclidean norm and using that β_δ is Lipschitz function, we note that

$$\|\mathbf{r}(\mathbf{c}_2) - \mathbf{r}(\mathbf{c}_1)\|_2^2 \le \sum_{i,j=1}^{k} \|\phi_i\|^2\ \|\phi_j\|^2\ \|\mathbf{c}_2 - \mathbf{c}_1\|_2^2 = L_k\ \|\mathbf{c}_2 - \mathbf{c}_1\|_2^2.$$

which implies that $\mathbf{r} : \mathbb{R}^k \to \mathbb{R}^k$ is a Lipschitz continuous function. So, by standard theory for ODEs with measurable coefficients, for any fixed k, there is a unique solution $\left(\mathbf{c}^k(t), \mathbf{l}^k(t)\right)$ of (B.27)-(B.29), which is such that $c_i(t), l_i(t) \in H^1(0, T)$ for all $i = 1, \ldots, k$. Therefore, we can say that y^k, w^k are solutions of

(B.30) $$\left(y_t^k, \eta^k\right) + \gamma\left(\nabla w^k, \nabla\eta^k\right) - \left(y^k, \mathbf{v}\cdot\nabla\eta^k\right) = 0, \qquad \text{a.e. on } (0, T),$$

(B.31) $$y^k(0) = P^k y_0,$$

(B.32) $$\left(w^k + y^k, \theta^k\right) - \varepsilon^2\left(\nabla y^k, \nabla\theta^k\right) - \frac{1}{\delta}\left(\beta_\delta\left(y^k\right), \theta^k\right) = 0, \qquad \text{a.e. on } (0, T),$$

for all $\eta^k, \theta^k \in \mathcal{C}([0, T]; V_k)$. We note that setting $\eta^k = 1$ in (B.30), we have

$$(y_t^k, 1) = 0 \quad \Longrightarrow \quad (y^k(t), 1) = (y^k(0), 1) = (y_0, 1) = 0, \quad \forall t \in (0, T].$$

Considering the *Ginzburg-Landau energy functional* of the Cahn-Hilliard system

$$E_\delta(y) = \frac{\varepsilon^2}{2}\int_\Omega |\nabla y|^2\ dx + \int_\Omega \Phi_\delta(y)\ dx,$$

from (B.32), we get

(B.33) $$\frac{dE_\delta\left(y^k\right)}{dt} = \varepsilon^2\left(\nabla y^k, \nabla y_t^k\right) - \left(y^k, y_t^k\right) + \frac{1}{\delta}\left(\beta_\delta\left(y^k\right), y_t^k\right) = \left(w^k, y_t^k\right).$$

Using together (B.30) with $\eta^k = w^k$ and (B.33), we can write

$$\frac{dE_\delta\left(y^k\right)}{dt} + \gamma\|\nabla w^k\|^2 = \left(y^k, \mathbf{v}\cdot\nabla w^k\right).$$

which implies, integrating in time,

$$E_\delta\left(y^k(s)\right)+\gamma\int_0^t\|\nabla w^k\|^2\,ds=\int_0^t\left(y^k,\mathbf{v}\cdot\nabla w^k\right)\,ds+E_\delta\left(P^k y_0\right). \tag{B.34}$$

Using the convexity of the function f_δ stated in (2.16) and $f_\delta(y_0)=0$, we can set in (B.34),

$$\begin{aligned}E_\delta\left(P^k y_0\right)&=\frac{\varepsilon^2}{2}\|\nabla P^k y_0\|^2+\frac{1}{2}\left(1-\left(P^k y_0\right)^2,1\right)+\left(f_\delta\left(P^k y_0\right),1\right)\\&\le\frac{\varepsilon^2}{2}\|\nabla P^k y_0\|^2+\frac{1}{2}\left(1-\left(P^k y_0\right)^2,1\right)+\frac{1}{\delta}\left(\beta_\delta\left(P^k y_0\right),P^k y_0-y_0\right).\end{aligned} \tag{B.35}$$

So, from (B.35), taking into account that $P^k y_0\to y_0$ in H^1, we derive

$$E_\delta\left(P^k y_0\right)\le\frac{\varepsilon^2}{2}\|\nabla y_0\|^2+\frac{1}{2}\left(1-y_0^2,1\right)=E\left(y_0\right) \tag{B.36}$$

Furthermore, applying the generalized Holder's inequality (A.14) and Poincaré's inequality (A.15), we infer

$$\begin{aligned}\left|\int_0^t\left(y^k,\mathbf{v}\cdot\nabla\eta^k\right)\,ds\right|&\le\int_0^t\|y^k\|_{L^4}\;\|\mathbf{v}\|_{\mathbf{L}^4}\;\|\nabla w\|\,ds\\&\le C\;\|\mathbf{v}\|_{\mathbf{L}^\infty(\boldsymbol{\mathcal{D}})}\;\|\nabla y^k\|_{L^2(0,t;L^2)}\;\|\nabla w\|_{L^2(0,t;L^2)},\end{aligned} \tag{B.37}$$

for all $t\in(0,T)$. From the property (2.14), it holds

$$\left(\Phi_\delta\left(y^k\right),1\right)\ge -C_0\;|\Omega|\;\delta. \tag{B.38}$$

Thus, using together (B.34) (B.36), (B.37), (B.38) and applying Young's inequality (A.13), we realize that

$$\begin{aligned}\frac{\varepsilon^2}{2}\|\nabla y^k(t)\|^2+\gamma\int_0^t\|\nabla w^k\|^2\,ds&\le C_1(\sigma)\;\|\mathbf{v}\|^2_{\mathbf{L}^\infty(\boldsymbol{\mathcal{D}})}\;\|\nabla y^k\|^2_{L^2(0,t;L^2)}\\&\quad+\sigma\|\nabla w\|^2_{L^2(0,t;L^2)}+C_2\|y_0\|^2_{H^1}+C_3.\end{aligned}$$

which implies, with σ small enough and using (B.1),

$$\|\nabla y^k(t)\|^2+\int_0^t\|\nabla w^k\|^2\,ds\le C_1\;\left[1+\|\mathbf{u}\|^2_{\mathbf{L}^2(\mathbf{L}^2)}\right]\int_0^t\|\nabla y^k\|^2\,dt+C_2\left[1+\|y_0\|^2_{H^1}\right],$$

where the constants C_1, C_2 depends on initial conditions and on fixed parameters. So, applying Gronwall's Lemma (see for example Lemma 1.4.1 [73]), we have

$$\|\nabla y^k(t)\|^2\le C(\mathbf{u}),\qquad\forall\,t\in(0,T), \tag{B.39}$$

$$\|\nabla w^k\|^2_{L^2(L^2)}\le C(\mathbf{u}). \tag{B.40}$$

From (B.39), using Poincare's-Wirtinger inequality (A.15), we get

$$\|y^k(t)\|^2_{H_0}\le C(\mathbf{u}),\qquad\forall\,t\in(0,T). \tag{B.41}$$

Setting $\eta = P^k \mathcal{G} y_t^k$ in (B.30) and using the definitions and the properties of the Green's $\mathcal{G}$ and projection operator P^k, we can write

$$\begin{aligned} \|\nabla \mathcal{G} y_t^k\|^2 &= \left(y_t^k, \mathcal{G} y_t^k\right) \\ &= -\gamma \left(\nabla w^k, \nabla \left[P^k \mathcal{G} y_t^k\right]\right) + \left(y^k, \mathbf{v} \cdot \nabla \left[P^k \mathcal{G} y_t^k\right]\right) \\ &= -\gamma \left(\nabla w^k, \nabla \left[\mathcal{G} y_t^k\right]\right) + \left(y^k, \mathbf{v} \cdot \nabla \left[P^k \mathcal{G} y_t^k\right]\right) \\ &= A_1 + A_2. \end{aligned} \tag{B.42}$$

Taking into account that, for all k,

$$\left\|\nabla P^k \varphi\right\|^2 \leq \|\nabla \varphi\|^2, \quad \forall \varphi \in H^1$$

using Young's inequality (A.13) and Holder's inequality (A.14), we derive

$$\begin{aligned} A_1 &\leq \gamma\sigma \|\nabla \mathcal{G} y_t^k\|^2 + \gamma\, C(\sigma)\, \|\nabla w^k\|^2, \\ A_2 &\leq \sigma \|\nabla \mathcal{G} y_t^k\|^2 + C(\sigma)\, \|\mathbf{v}\|^2_{\boldsymbol{\mathcal{D}}}\, \|y^k\|_{H^1}. \end{aligned}$$

Inserting the estimates of A_1, A_2 in (B.42), with σ small enough and integrating in time, we infer

$$\|\nabla \mathcal{G} y_t^k\|^2_{L^2(L^2)} \leq C_1 \left[\|\nabla w^k\|^2_{L^2(L^2)} + \|\mathbf{v}\|^2_{\mathbf{L}^\infty(\boldsymbol{\mathcal{D}})}\, \|y^k\|_{L^2(H^1)}\right]. \tag{B.43}$$

Therefore, from (A.21), (B.40), (B.41) and (B.43), we realize that

$$\|y_t^k\|_{L^2(H^{1*})} \leq C(\mathbf{u}). \tag{B.44}$$

With $\theta^k = 1$ in (B.32) and using $|\beta_\delta(r)| \leq \beta_\delta(r)\, r$, we observe

$$\left|\left(w^k, 1\right)\right| \leq \frac{1}{\delta}\left(\beta_\delta\left(y^k\right), y^k\right). \tag{B.45}$$

Then, substituting $\theta^k = y^k$ in (B.32), using the definition (A.20) of the Green's operator $\mathcal{G}$ and inequality (A.23), from (B.45) we have

$$\left|\left(w^k, 1\right)\right| \leq \|y^k\|^2 - \varepsilon^2 \|\nabla y^k\|^2 + \left(w^k, y^k\right) \leq \|y^k\|^2 - \varepsilon^2 \|\nabla y^k\|^2 + \|\nabla w^k\|\, \|y^k\|. \tag{B.46}$$

So, using (B.39), (B.40), (B.46) and the Poincare's-Wirtinger inequality (A.15), it holds

$$\|w^k\|^2_{L^2(H^1)} \leq C(\mathbf{u}). \tag{B.47}$$

With $\theta^k = -\gamma \Delta y^k$ in (B.32), we derive

$$\varepsilon^2 \gamma \|\Delta y^k\|^2 - \gamma \|\nabla y^k\|^2 + \frac{\gamma}{\delta}\left(\nabla \beta_\delta\left(y^k\right), \nabla y^k\right) = \gamma\left(w^k, -\Delta y^k\right) = \gamma\left(\nabla w^k, \nabla y^k\right)$$

and with $\eta^k = y^k$ (B.30), we can write

$$\gamma\left(\nabla w^k, \nabla y^k\right) = -\left(y_t^k, y_k\right) + \left(y^k, \mathbf{v} \cdot \nabla y^k\right) = -\frac{1}{2}\frac{d}{dt}\|y^k\|^2 + \left(y^k, \mathbf{v} \cdot \nabla y^k\right).$$

Using together last two relations, we observe

$$\varepsilon^2\gamma\|\Delta y^k\|^2 + \frac{\gamma}{\delta}\left(\nabla\beta_\delta\left(y^k\right), \nabla y^k\right) + \frac{1}{2}\frac{d}{dt}\|y^k\|^2 = \gamma\|\nabla y^k\|^2 + \left(y^k, \mathbf{v}\cdot\nabla y^k\right). \tag{B.48}$$

Noting that

$$\left(\nabla\beta_\delta\left(y^k\right), \nabla y^k\right) = \left(\beta'_\delta\left(y^k\right), \nabla y^k\cdot\nabla y^k\right) \geq 0,$$

integrating in time in $(0,t)$, in (B.48) and using (B.1), (B.41), we infer

$$\|\Delta y^k\|^2_{L^2(0,t;L^2)} \leq C(\mathbf{u}), \qquad \forall\, t\in(0,T]. \tag{B.49}$$

Since the domain Ω is convex polygonal (see [17] and [40]), (B.49) implies

$$\|y^k\|_{L^2(H^2)} \leq C(\mathbf{u}). \tag{B.50}$$

From (B.41), (B.44), (B.47) and (B.50), given the sequences $\left\{y^k\right\}_{k\in\mathbb{N}}, \left\{w^k\right\}_{k\in\mathbb{N}}$, it is possible to extract a subsequence (labelled by an index m), such that

$$y^m \rightharpoonup y, \qquad \text{in } W_0, \tag{B.51}$$

$$y^m \overset{*}{\rightharpoonup} y, \qquad \text{in } L^\infty(H_0), \tag{B.52}$$

$$y^m \rightharpoonup y, \qquad \text{in } L^2\left(H^2\right), \tag{B.53}$$

$$y^m \to y, \qquad \text{in } L^2\left(L^2_0\right), \tag{B.54}$$

$$w^m \rightharpoonup w, \qquad \text{in } L^2\left(H^1\right), \tag{B.55}$$

where (y,w) together the velocity field $\mathbf{v}$ satisfy the estimate (2.26). Note that (B.54) is a consequence of a compactness theorem (see [17] and [66]). Using

$$\left.\frac{\partial y^m}{\partial\mathbf{n}}\right|_{\partial\Omega} = 0, \quad \text{a.e. on } (0,T),\ \forall\, m$$

and (B.53), we have the result (2.27). As a consequence of the Galerkin's approximation (B.30),(B.32), we can claim that

$$\int_0^T \left[(y^m_t, \eta^m) + \gamma\left(\nabla w^m, \nabla\eta^m\right) - (y^m, \mathbf{v}\cdot\nabla\eta^m)\right]\, dt = 0, \tag{B.56}$$

$$\int_0^T \left[(w^m, \theta^m) - \varepsilon^2\left(\nabla y^m, \nabla\theta^m\right) + (y^m, \theta^m) - \frac{1}{\delta}\left(\beta_\delta\left(y^m\right), \theta^m\right)\right]\, dt = 0. \tag{B.57}$$

for all $\eta^m, \theta^m \in \mathcal{C}^\infty_c((0,T);V_m)$. So, given $\eta,\theta\in\mathcal{C}^\infty_c((0,T);H^1)$, we set in (B.56), (B.57) $\eta^m = P^m\eta, \theta^m = P^m\theta$, which are such that

$$\|P^m\eta-\eta\|_{L^2(H^1)} \to 0, \quad \|P^m\theta-\theta\|_{L^2(H^1)} \to 0,$$

as $m\to+\infty$. In this way, performing the limit on m in (B.56), (B.57), we get

$$\int_0^T \left[(y_t,\eta) + \gamma\left(\nabla w, \nabla\eta\right) - (y, \mathbf{v}\cdot\nabla\eta)\right]\, dt = 0 \tag{B.58}$$

$$\int_0^T \left[(w,\theta) - \varepsilon^2\left(\nabla y, \nabla\theta\right) + (y,\theta) - \frac{1}{\delta}\left(\beta_\delta\left(y\right), \theta\right)\right]\, dt = 0. \tag{B.59}$$

for all $\eta, \theta \in \mathcal{C}_c^\infty((0,T);H^1)$. Indeed, the convergence of the linear terms in (B.56), (B.57) to the corresponding terms in (B.58), (B.59) is straightforward. Concerning the nonlinear terms, we derive

$$\begin{aligned}
&\Big|\int_0^T (y^m, \mathbf{v}\cdot\nabla\eta^m)\ dt - \int_0^T (y, \mathbf{v}\cdot\nabla\eta)\ dt\Big| \\
\leq& \Big|\int_0^T (y^m - y, \mathbf{v}\cdot\nabla\eta^m)\ dt\Big| + \Big|\int_0^T (y, \mathbf{v}\cdot[\nabla\eta^m - \nabla\eta])\ dt\Big| = B_1 + B_2,
\end{aligned}$$

where

$$\begin{aligned}
B_1 &\leq C\ \|y^m - y\|_{L^\infty(H^1)}^{\frac{1}{2}}\ \|\mathbf{v}\|_{L^\infty(\boldsymbol{\mathcal{D}})}^{\frac{1}{2}}\ \|y^m - y\|_{L^2(L^2)}\ \|\mathbf{v}\|_{L^2(\boldsymbol{\mathcal{S}})}\ \|\nabla\eta\|_{L^2(L^2)} \to 0, \\
B_2 &\leq C\ \|y\|_{L^\infty(H^1)}^{\frac{1}{2}}\ \|\mathbf{v}\|_{L^\infty(\boldsymbol{\mathcal{D}})}^{\frac{1}{2}}\ \|y\|_{L^2(L^2)}\ \|\mathbf{v}\|_{L^2(\boldsymbol{\mathcal{S}})}\ \|\nabla\eta^m - \nabla\eta\|_{L^2(L^2)} \to 0,
\end{aligned}$$

as $m \to +\infty$. Moreover, using $0 \leq \beta_\delta' \leq 1$ and (B.54),

$$\begin{aligned}
&\Big|\int_0^T (\beta_\delta(y^m), \theta^m)\ dt - \int_0^T (\beta_\delta(y), \theta)\ dt\Big| \\
&\leq \Big|\int_0^T (\beta_\delta(y^m) - \beta_\delta(y), \theta^m)\ dt\Big| + \Big|\int_0^T (\beta_\delta(y), \theta^m - \theta)\ dt\Big| \\
&\leq \|y^m - y\|_{L^2(L^2)}^2\ \|\theta\|_{L^2(L^2)} + \|\beta_\delta(y)\|_{L^2(L^2)}^2\ \|\theta^m - \theta\|_{L^2(L^2)} \to 0, \qquad \text{(B.60)}
\end{aligned}$$

as $m \to +\infty$. Therefore equations (B.58), (B.59) are satisfied by (y, w). Noting that $\mathcal{C}_c^\infty((0,T);H^1) \hookrightarrow L^2(H^1)$ is a dense embedding (see for example Lemma 1.9 in [58]), we can claim also that (y, w) satisfies (2.25a), (2.25c). We prove the initial condition $y(0) = y_0$. With $\eta = \zeta\ (1 - t/T), \zeta \in H^1$, integrating by parts in time, we note that

$$(y^m(0) - y(0), \zeta) = \int_0^T [-(y_t^m, \eta) + \langle y_t, \eta\rangle_{H^{1*},H^1} - (y^m, \eta_t) + (y^m, \eta_t)]\ dt \to 0,$$

as $m \to +\infty$. hence, $P^m y_0 = y^m(0) \rightharpoonup y(0)$ in H^1. So, using $P^m y_0 \to y_0$ in H^1, we infer $y_0 = y(0)$. In order to prove the estimate (2.28), we set $\theta = \beta_\delta(y) \in H^1$ in (2.25c). Using $(\nabla y, \nabla\beta_\delta(y)) = (\nabla y \cdot \nabla y, \beta_\delta'(y)) \geq 0$ and Young's inequality (A.13) with $\sigma = \delta/2$, we derive

$$\begin{aligned}
\frac{1}{\delta}\|\beta_\delta(y)\|_{L^2(L^2)}^2 &\leq \int_0^T (w + y, \beta_\delta(y))\ dt \\
&\leq \frac{\delta}{2}\left[\|w\|_{L^2(L^2)}^2 + \|y\|_{L^2(L^2)}^2\right] + \frac{1}{2\delta}\|\beta_\delta(y)\|_{L^2(L^2)}^2,
\end{aligned}$$

which implies, from (B.41), (B.47)

$$\Big\|\frac{1}{\delta}\beta_\delta(y)\Big\|_{L^2(L^2)}^2 \leq C(\mathbf{u}).$$

It remains to show the uniqueness of the solution (y, w) of (2.25). We assume that given $\mathbf{v}$, there are two solutions (y_1, w_1), (y_2, w_2) of (2.25). Hence, $d_y = y_2 - y_1$ and $d_w = w_2 - w_1$ satisfy

$$-\gamma\int_0^T (\nabla d_w, \nabla\eta)\ dt = \int_0^T [\langle d_{yt}, \eta\rangle_{H^{1*},H^1} - (d_y, \mathbf{v}\cdot\nabla\eta)]\ dt, \qquad \text{(B.61)}$$

$$d_y(0) = 0,$$

for all $\eta \in L^2(H^1)$. Furthermore, setting in (2.25c), $\theta = d_y e^{-\mu t}$ when the solution is (y_1, w_1) and $\theta = -d_y e^{-\mu t}$ when the solution is (y_2, w_2) and summing the equations obtained, we have

$$\int_0^T e^{-\mu t}\left[-(d_w + d_y, d_y) + \varepsilon^2\|\nabla d_y\|^2 + \frac{1}{\delta}\left(\beta_\delta(y_2) - \beta_\delta(y_1), d_y\right)\right] dt = 0. \tag{B.62}$$

where μ is a positive real constant. We note that $\beta_\delta(\cdot)$ is monotone increasing, so from (B.62) we get

$$\int_0^T e^{-\mu t}\left[-(d_w, d_y) + \varepsilon^2\|\nabla d_y\|^2\right] dt \leq \int_0^T e^{-\mu t}\|d_y\|^2\, dt. \tag{B.63}$$

Inserting in (B.61) $\eta = e^{-\mu t}\, \mathcal{G} d_y$ and using the definition (A.20) of the Green's operator $\mathcal{G}$, we can write

$$-\gamma\int_0^T e^{-\mu t}(d_w, d_y)\, dt = \int_0^T e^{-\mu t}\left[\, \langle d_{yt}, \mathcal{G} d_y\rangle_{H^{1*},H^1} - (d_y, \mathbf{v}\cdot\nabla\mathcal{G} d_y)\,\right] dt,$$

that substituted in (B.63) produces

$$\begin{aligned}\int_0^T e^{-\mu t}\ &\left[\langle d_{yt}, \mathcal{G} d_y\rangle_{H^{1*},H^1} + \gamma\varepsilon^2\|\nabla d_y\|^2\right] dt \\ &\leq \int_0^T e^{-\mu t}\ \left[\,\gamma\,\|d_y\|^2 + (d_y, \mathbf{v}\cdot\nabla\mathcal{G} d_y)\,\right] dt.\end{aligned} \tag{B.64}$$

In (B.64), using Young's, Holder's, Poincare's and (A.17) inequalities, the definition of $\mathcal{G}$ and $\mathbf{v} \in L^\infty(\boldsymbol{\mathcal{D}})$, we derive

$$\begin{aligned}\int_0^T \langle d_{yt}, \mathcal{G} d_y\rangle_{H^{1*},H^1}\, dt &= \int_0^T \frac{1}{2}\frac{d}{dt}\|\nabla\mathcal{G} d_y\|^2\, dt,\\ \gamma\int_0^T \|d_y\|^2\, dt &= \gamma\int_0^T (\nabla d_y, \nabla\mathcal{G} d_y)\, dt\\ &\leq \int_0^T\left[\,\gamma\sigma\|\nabla d_y\|^2 + \frac{\gamma}{4\sigma}\|\nabla\mathcal{G} d_y\|^2\,\right] dt,\\ \int_0^T (d_y, \mathbf{v}\cdot\nabla\mathcal{G} d_y)\, dt &\leq C\int_0^T \|d_y\|_{H^1}\,\|\mathbf{v}\|_{\boldsymbol{\mathcal{D}}}\,\|\nabla\mathcal{G} d_y\|\, dt\\ &\leq \int_0^T\left[\,\sigma\|\nabla d_y\|^2 + \frac{\hat{C}}{4\sigma}\|\nabla\mathcal{G} d_y\|^2\,\right] dt,\end{aligned}$$

where the constant $\hat{C}$ depends on data problem and $\|\mathbf{v}\|_{L^\infty(\boldsymbol{\mathcal{D}})}$. Then, from the previous result, we can assume $\hat{C} = \hat{C}(\mathbf{u})$. So, from (B.64), with σ such that

$$\sigma(1+\gamma) < \gamma\varepsilon^2 \tag{B.65}$$

we infer

$$\int_0^T e^{-\mu t} \left[\frac{d}{dt}\|\nabla \mathcal{G} d_y\|^2 + 2\left[\gamma\varepsilon^2 - \sigma\left(1+\gamma\right)\right]\|\nabla d_y\|^2\right] dt$$
$$\leq \frac{\gamma + \hat{C}}{2\sigma} \int_0^T e^{-\mu t} \|\nabla \mathcal{G} d_y\|^2 \, dt.$$

Hence, from (B.64), assuming $\mu = \frac{\gamma+\hat{C}}{2\sigma}$ and integrating by parts in the first term on the r.h.s., we realize

$$e^{-\mu T}\|\nabla \mathcal{G} d_y\left(T\right)\|^2 + 2\int_0^T e^{-\mu t}\left[\gamma\varepsilon^2 - \sigma\left(1+\gamma\right)\right]\|\nabla d_y\|^2 \, dt \leq \|\nabla \mathcal{G} d_y\left(0\right)\|^2 = 0,$$

which implies $\|\nabla d_y\|_{L^2(L^2)} = 0$. Then, applying Poincaré's-Wirtinger inequality (A.15), we conclude $d_y = 0$, that is $y_1 = y_2$. With this result, looking at the state equation (2.25c), we can say that d_w satisfies

$$\int_0^T \left(d_w, 1\right) \, dt = 0,$$

and setting $\eta = d_w$ in (B.61), we have $\|\nabla d_w\|_{L^2(L^2)} = 0$. Therefore, from the Poincaré's-Wirtinger inequality (A.15), we get the uniqueness of w. ☐

Proof of Lemma 2.7

Proof. In order to show that $e_{\delta\mathbf{x}}\left(s_\delta\left(\mathbf{u}\right), \mathbf{u}\right)$ has a bounded inverse, we need to prove that for all $\mathbf{z} \in Z$, there exists a unique $\mathbf{d_x} \in \mathbf{X}$ such that

$$e_{\delta\mathbf{x}}\left(s_\delta\left(\mathbf{u}\right), \mathbf{u}\right)\mathbf{d_x} = \mathbf{z} \tag{B.66}$$

and furthermore

$$\|\mathbf{d_x}\|_{\mathbf{X}} \leq C\|\mathbf{z}\|_{\mathbf{Z}}. \tag{B.67}$$

Equation (B.66) is equivalent to find $(\mathbf{d_v}, d_y, d_w) \in \mathbf{W}_0 \times W_0 \times L^2\left(H^1\right)$ which satisfy

$$\int_0^T \left[\langle \mathbf{d_v}_t, \boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}} + \nu\left(\nabla\mathbf{d_v}, \nabla\boldsymbol{\psi}\right)\right] \, dt = \int_0^T \langle \mathbf{z}_1, \boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}} \, dt, \tag{B.68}$$

$$\mathbf{d_v}(0) = \mathbf{z}_4 \in \boldsymbol{\mathcal{S}} \tag{B.69}$$

$$\int_0^T \left[\langle d_{yt}, \eta\rangle_{H_0^*,H_0} + \gamma\left(\nabla d_w, \nabla\eta\right)\right.$$
$$\left. - \left(d_y\mathbf{v} + y\ \mathbf{d_v}, \nabla\eta\right)\right] dt = \int_0^T \langle z_2, \eta\rangle_{H_0^*,H_0} dt, \tag{B.70}$$

$$d_y(0) = z_5 \in L_0^2, \tag{B.71}$$

$$\int_0^T \left[\left(d_w + d_y, \theta\right) - \varepsilon^2\left(\nabla d_y, \nabla\theta\right)\right.$$
$$\left. - \frac{1}{\delta}\left(\beta_\delta'\left(y\right) d_y, \theta\right)\right] \, dt = \int_0^T \langle z_3, \theta\rangle_{H^{1*},H^1} \, dt. \tag{B.72}$$

for all $\boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}}), \eta \in L^2(H_0), \theta \in L^2(H^1)$. Note that we assume that $(\mathbf{v}, y, w)$ in (B.68)-(B.72) are solutions of the regularized state equations (2.24), (2.25), for a given $\mathbf{u} \in L^2(\mathbf{L}^2)$.
By standard arguments (see for example Theorem 1.37 in [58]), it is easy to realize that (B.68)-(B.69) has a unique solution $\mathbf{d}_\mathbf{v} \in \mathbf{W}_0$, which is such that

$$\|\mathbf{d}_\mathbf{v}\|^2_{\mathbf{W}_0} \leq C\left[\|\mathbf{z}_4\|^2_{\boldsymbol{\mathcal{S}}} + \|\mathbf{z}_1\|^2_{L^2(\boldsymbol{\mathcal{D}}^*)}\right]. \tag{B.73}$$

In order to show the existence of the solutions d_y, d_w of (B.70)-(B.72), first we note that $y\ \mathbf{d}_\mathbf{v} \in L^2(\mathbf{L}^2)$ and therefore, in (B.70), we can absorb the last term at l.h.s. in the linear functional at r.h.s. Second, we can replace (B.70) with the following

$$\int_0^T \left[\langle d_{yt}, \eta\rangle_{H^{1*},H^1} + \gamma(\nabla d_w, \nabla\eta) - (d_y, \mathbf{v}\cdot\nabla\eta)\right]\, dt = \int_0^T \langle \tilde{z}_2, \eta\rangle_{H^{1*},H^1}\, dt, \tag{B.74}$$

where $\tilde{z}_2 \in L^2(H^{1*})$ is such that

$$\int_0^T \langle \tilde{z}_2, \eta\rangle_{H^{1*},H^1} = \int_0^T \langle\ z_2, \eta - \frac{1}{|\Omega|}(\eta, 1)\ \rangle_{H_0^*,H_0}\, dt,$$

and $\|z_2\|_{L^2\left(H_0^*\right)} = \|\tilde{z}_2\|_{L^2(H^{1*})}$. In the following, we show the existence and the uniqueness of the solution of (B.70)-(B.72) applying the same Galerkin's approximation used in the proof of Theorem 2.3. In this way, we derive that there exist $f_j, g_j \in H^1(0,T), j = 1, \ldots, k$, such that

$$d_y^k = \sum_{j=1}^k f_j(t)\phi_j, \quad d_w^k = \sum_{j=1}^k g_j(t)\phi_j,$$

are, for all $k \in \mathbb{N}$, solution of

$$\left(d_{yt}^k, \eta\right) + \gamma\left(\nabla d_w^k, \nabla\eta\right) - \left(d_y^k, \mathbf{v}\cdot\nabla\eta\right) = \langle \tilde{z}_2, \eta\rangle_{H^{1*},H^1}, \tag{B.75}$$

$$d_y^k(0) = P^k z_5, \tag{B.76}$$

$$\left(d_w^k, \theta\right) - \varepsilon^2\left(\nabla d_y^k, \nabla\theta\right) + \left(d_y^k, \theta\right) - \frac{1}{\delta}\left(\beta'_\delta(y)\, d_y^k, \theta\right) = \langle z_3, \theta\rangle_{H^{1*},H^1}, \tag{B.77}$$

for all $\eta, \theta \in \mathcal{C}([0,T]; V_k)$. From (B.75), with $\theta = -d_y$ in (B.77) and using $0 \leq \beta'_\delta \leq 1$, we infer

$$-\gamma\left(\nabla d_w^k, \nabla\eta\right) = \left(d_{yt}^k, \eta\right) - \left(d_y^k, \mathbf{v}\cdot\nabla\eta\right) - \langle \tilde{z}_2, \eta\rangle_{H^{1*},H^1}, \tag{B.78}$$

$$-\left(d_w^k, d_y\right) + \varepsilon^2\|\nabla d_y^k\|^2 \leq \|d_y^k\|^2 - \langle z_3, d_y^k\rangle_{H^{1*},H^1}. \tag{B.79}$$

Substituting $\eta = P^k \mathcal{G} d_y^k$ in (B.78), using the definitions (A.20), (B.23) of the Green's operator $\mathcal{G}$ and projection operator P^k, we have

$$-\gamma\left(d_w^k, d_y^k\right) = \frac{1}{2}\frac{d}{dt}\|\nabla\mathcal{G}d_y^k\|^2 - \left(d_y^k, \mathbf{v}\cdot\nabla\left[P^k\mathcal{G}d_y^k\right]\right) - \langle \tilde{z}_2, P^k\mathcal{G}d_y^k\rangle_{H^{1*},H^1},$$

which produces, substituted in (B.79) and integrating in time,

$$\int_0^t \left[\frac{1}{2}\frac{d}{ds}\|\nabla\mathcal{G}d_y^k\|^2 + \gamma\varepsilon^2\|\nabla d_y^k\|^2\right]\, dt \tag{B.80}$$

$$\leq \int_0^t \left[\gamma\|d_y^k\|^2 - \gamma\langle z_3, d_y^k\rangle_{H^{1*},H^1} + \left(d_y^k, \mathbf{v}\cdot\nabla\left[P^k\mathcal{G}d_y^k\right]\right) + \langle\tilde{z}_2, P^k\mathcal{G}d_y^k\rangle_{H^{1*},H^1}\right]\ dt$$
$$= F_1 + F_2 + F_3 + F_4.$$

Using Young's inequality (A.13), Poincare's inequality (A.15), Holder's inequality (A.14) and the definition of the operator $\mathcal{G}$, we get

$$F_1 = \gamma\int_0^t \left(\nabla\mathcal{G}d_y^k, \nabla d_y^k\right)\ dt \leq \gamma\int_0^t \left[\sigma\|\nabla d_y^k\|^2 + C(\sigma)\|\nabla\mathcal{G}d_y^k\|^2\right]\ dt,$$
$$F_2 \leq \gamma C_1\sigma\int_0^t \|\nabla d_y^k\|^2\ dt + \gamma C(\sigma)\|z_3\|^2_{L^2(H^{1*})}$$
$$F_3 \leq C_2\|\mathbf{v}\|_{\mathbf{L}^\infty(\boldsymbol{\mathcal{D}})}\int_0^t \left[\sigma\|\nabla d_y^k\|^2 + C(\sigma)\|\nabla\mathcal{G}d_y^k\|^2\right]\ dt,$$
$$F_4 \leq C_3\int_0^t \|\nabla\mathcal{G}d_y^k\|^2\ dt + C_3\|z_2\|_{L^2\left(H_0^*\right)}$$

Inserting the estimates of $F_1,\ldots,F_4$ above in (B.80), assuming σ small enough, applying Gronwall's lemma (see for example Lemma 1.4.1 in [73]) and the following

$$\|\nabla\mathcal{G}\phi\| \leq C\|\phi\|, \quad \forall\phi\in L_0^2,$$

we derive,

$$\|\nabla\mathcal{G}d_y^k(t)\|^2 + \|d_y^k\|^2_{L^2(0,t;H_0)} \leq C\left[\|z_2\|^2_{L^2\left(H_0^*\right)} + \|z_3\|^2_{L^2(H^{1*})} + \|z_5\|^2\right]. \tag{B.81}$$

for all $t\in(0,T]$. With $\eta = P^k\mathcal{G}d_{yt}^k$ in (B.78), we can write

$$\|\nabla\mathcal{G}d_{yt}^k\|^2 + \gamma\left(d_w, \mathcal{G}d_{yt}^k\right) - \left(d_y^k, \mathbf{v}\cdot\nabla\left[P^k\mathcal{G}d_{yt}^k\right]\right) = \langle\tilde{z}_2, P^k\mathcal{G}d_{yt}^k\rangle_{H^{1*},H^1}, \tag{B.82}$$

and with $\theta = P^k\mathcal{G}d_{yt}^k$ in (B.77), we derive

$$\begin{aligned}\gamma\left(d_w^k, \mathcal{G}d_{yt}^k\right) &= \gamma\varepsilon^2\left(\nabla d_y^k, \nabla\mathcal{G}d_{yt}^k\right) - \gamma\left(d_y^k, \mathcal{G}d_{yt}^k\right)\\ &\quad + \frac{\gamma}{\delta}\left(\beta_\delta'(y)\,d_y^k, P^k\mathcal{G}d_{yt}^k\right) + \gamma\langle z_3, P^k\mathcal{G}d_{yt}^k\rangle_{H^{1*},H^1}.\end{aligned} \tag{B.83}$$

Substituting (B.83) in (B.82) produces

$$\begin{aligned}\|\nabla\mathcal{G}d_{yt}^k\|^2_{L^2(L^2)} &= \int_0^T \left[-\gamma\varepsilon^2\left(\nabla d_y^k, \nabla\mathcal{G}d_{yt}^k\right) + \gamma\left(d_y^k, \mathcal{G}d_{yt}^k\right) - \frac{\gamma}{\delta}\left(\beta_\delta'(y)\,d_y^k, P^k\mathcal{G}d_{yt}^k\right)\right]\ dt\\ &\quad + \int_0^T \left[-\gamma\langle z_3, P^k\mathcal{G}d_{yt}^k\rangle_{H^{1*},H^1} + \left(d_y^k, \mathbf{v}\cdot\nabla\left[P^k\mathcal{G}d_{yt}^k\right]\right) + \langle\tilde{z}_2, P^k\mathcal{G}d_{yt}^k\rangle_{H^{1*},H^1}\right]\ dt\\ &= G_1 + G_2 + G_3 + G_4 + G_5 + G_6.\end{aligned} \tag{B.84}$$

Using $0 \leq \beta_\delta' \leq 1$, we infer

$$G_1 \leq \gamma\varepsilon^2\left[\sigma\|\nabla\mathcal{G}d_{yt}^k\|^2_{L^2(L^2)} + C(\sigma)\|\nabla d_y^k\|^2_{L^2(L^2)}\right],$$
$$G_2 \leq \gamma C_1\left[\sigma\|\nabla\mathcal{G}d_{yt}^k\|^2_{L^2(L^2)} + C(\sigma)\|\nabla d_y^k\|^2_{L^2(L^2)}\right],$$

$$G_3 \le \frac{\gamma}{\delta} C_1 \left[\sigma \|\nabla \mathcal{G} d_{yt}^k\|_{L^2(L^2)}^2 + C(\sigma) \|\nabla d_y^k\|_{L^2(L^2)}^2 \right],$$
$$G_4 \le \gamma \left[\sigma \|\nabla \mathcal{G} d_{yt}^k\|_{L^2(L^2)}^2 + C(\sigma) \|z_3\|_{L^2(H^{1*})}^2 \right],$$
$$G_5 \le C_1 \|\mathbf{v}\|_{\mathbf{L}^\infty(\mathcal{D})} \left[\sigma \|\nabla \mathcal{G} d_{yt}^k\|_{L^2(L^2)}^2 + C(\sigma) \|\nabla d_y^k\|_{L^2(L^2)}^2 \right],$$
$$G_6 \le C_1 \left[\sigma \|\nabla \mathcal{G} d_{yt}^k\|_{L^2(L^2)}^2 + C(\sigma) \|z_2\|_{L^2\left(H_0^*\right)}^2 \right].$$

Inserting the estimates of $G_1, \ldots, G_6$ in (B.84), with σ small enough, we realize that uniformly in k, but not in δ,

$$\|\nabla \mathcal{G} d_{yt}^k\|_{L^2(L^2)}^2 \le C(\delta) \left[\|\nabla d_y^k\|_{L^2(L^2)}^2 + \|z_2\|_{L^2\left(H_0^*\right)}^2 + \|z_3\|_{L^2(H^{1*})}^2 \right]. \tag{B.85}$$

Therefore, using (B.81) and (B.85), we can say that

$$\|d_{yt}^k\|_{L^2(H^{1*})} \le C(\delta) \left[\|z_2\|_{L^2\left(H_0^*\right)}^2 + \|z_3\|_{L^2(H^{1*})}^2 + \|z_5\|^2 \right]. \tag{B.86}$$

Substituting $\eta = d_w$ in (B.75) and using (B.81), (B.86), we have

$$\|\nabla d_w^k\|_{L^2(L^2)} \le C(\delta) \left[\|z_2\|_{L^2\left(H_0^*\right)}^2 + \|z_3\|_{L^2(H^{1*})}^2 + \|z_5\|^2 \right]. \tag{B.87}$$

Furthermore, with $\theta = 1$ in (B.77), we get

$$(d_w, 1) = \frac{1}{\delta} \left(\beta_\delta'(y), d_y^k \right) + \langle z_3, 1 \rangle_{H^{1*}, H^1}, \tag{B.88}$$

which implies that $\left(d_w^k, 1\right)$ is bounded uniformly in k. So, by (B.87), (B.88) and Poincaré-Wirtinger's inequality (A.15), we can write

$$\|d_w^k\|_{L^2(H^1)} \le C(\delta) \left[\|z_2\|_{L^2\left(H_0^*\right)}^2 + \|z_3\|_{L^2(H^{1*})}^2 + \|z_5\|^2 \right]. \tag{B.89}$$

Given the sequences $\left\{d_y^k\right\}_{k\in\mathbb{N}}, \left\{d_w^k\right\}_{k\in\mathbb{N}}$, using (B.81), (B.86) and (B.89), there exist a subsequence (labelled by an index m), such that

$$\begin{aligned} d_{yt}^m &\rightharpoonup d_{yt}, && \text{in } L^2\left(H^{1*}\right), \\ d_y^m &\rightharpoonup d_y, && \text{in } L^2\left(H_0\right), \\ d_w^m &\rightharpoonup d_w, && \text{in } L^2\left(H^1\right). \end{aligned}$$

where (d_y, d_w) satisfies

$$\int_0^T \left[\langle d_{yt}, \eta \rangle_{H^{1*}, H^1} + \gamma (\nabla d_w, \nabla \eta) - (d_y, \mathbf{v} \cdot \nabla \eta) - \langle \tilde{z}_2, \eta \rangle_{H^{1*}, H^1} \right] dt = 0,$$
$$\int_0^T \left[(d_w + d_y, \theta) - \varepsilon^2 (\nabla d_y, \nabla \theta) - \frac{1}{\delta} \left(\beta_\delta'(y) d_y, \theta \right) - \langle z_3, \theta \rangle_{H^{1*}, H^1} \right] dt = 0,$$

for all $\eta, \theta \in L^2(H^1)$. Moreover, as in the proof of Theorem 2.3, using integration by parts in time, we derive that the initial condition $d_y(0) = z_5$ is satisfied. Summarizing, we can say that (B.70)-(B.72) have solution (d_y, d_w), such that

$$\|d_y\|_{W_0}^2 + \|d_w\|_{L^2(H^1)}^2 \le C(\delta) \left[\|z_2\|_{L^2\left(H_0^*\right)}^2 + \|z_3\|_{L^2(H^{1*})}^2 + \|z_5\|^2 \right].$$

It remains to show uniqueness. Let us assume that, given $z_2 \in L^2(H_0^*)$, $z_4 \in L^2(H_0^*)$, $z_5 \in L_0^2$ and $\mathbf{d_v} \in \mathbf{W}_0$, we have two solutions $(d_{y1}, d_{w1}), (d_{y2}, d_{w2}) \in W_0 \times L^2(H^1)$ of (B.70)-(B.72). Then, $h_y = d_{y2} - d_{y1}$ and $h_w = d_{w2} - d_{w1}$ satisfy

$$-\gamma \int_0^T (\nabla h_w, \nabla \eta)\ dt = \int_0^T \left[\langle h_{yt}, \eta\rangle_{H_0^*, H_0} - (h_y, \mathbf{v}\cdot\nabla\eta)\right]\ dt, \tag{B.90}$$

$$h_y(0) = 0, \tag{B.91}$$

$$\int_0^T \left[-(h_w, \theta) + \varepsilon^2 (\nabla h_y, \nabla\theta) - (h_y, \theta) + \frac{1}{\delta}(\beta_\delta'(y)\, h_y, \theta)\right]\ dt = 0. \tag{B.92}$$

With $\theta = h_y\ e^{-\mu t}$ in (B.92) and using $\beta_\delta' \geq 0$, we infer

$$\int_0^T e^{-\mu t}\ \left[-(h_w, h_y) + \varepsilon^2 \|\nabla h_y\|^2\right]\ dt \leq \int_0^T e^{-\mu t}\ \|h_y\|^2\ dt. \tag{B.93}$$

So, from (B.90), (B.93), applying the same procedure performed in the proof of Theorem 2.3, we have uniqueness of (d_w, d_y). □

Proof of Lemma 2.10

Proof. In order to demonstrate the Lemma, we formulate a Galerkin's approximation of the adjoint equations (2.45c), (2.45d). Given the spatial domain Ω, let $\{\phi_j\}_{j\in\mathbb{N}}$ be the orthogonal dense subset of H^1 defined in (B.22). We have that

$$\left\{\tilde{\phi}_j\right\}_{j\in\mathbb{N}} = \{\phi_j\}_{j\in\mathbb{N}} \setminus \{\phi_1\},$$

where $\phi_1 = 1/|\Omega|$, is an orthogonal dense subset of H_0, normalized such that

$$\left(\tilde{\phi}_i, \tilde{\phi}_j\right) = \delta_{ij}.$$

Even in this case, we can define the following projection $\tilde{P}^k : H_0 \to \tilde{V}_k$

$$\tilde{P}^k \varphi = \sum_{j=1}^{k} \left(\varphi, \tilde{\phi}_j\right) \tilde{\phi}_j, \tag{B.94}$$

which is such that

$$\left\|\tilde{P}^k\varphi - \varphi\right\|_{H_0} \to 0, \quad \text{as} \quad k \to +\infty.$$

Let $\tilde{V}_k$ denote the finite dimensional subspace of H_0 spanned by $\left\{\tilde{\phi}_j\right\}_{j=1,\dots,k}$. Considering the associated Galerkin's approximations of the adjoint equations (2.45c), (2.45d), it is possible to show that exist

$$q_y^k = \sum_{j=1}^{k} a_j(t)\, \tilde{\phi}_j, \quad q_w^k = \sum_{j=1}^{k} b_j(t)\, \tilde{\phi}_j,$$

with $a_i, b_i \in H^1(0,T)$, $i = 1, \ldots, k$, such that

$$-\left(q_{yt}^k, \eta^k\right) - \varepsilon^2 \left(\nabla q_w^k, \nabla \eta^k\right) - \left(q_w^k - \mathbf{v} \cdot \nabla q_y^k + y - y_d, \eta^k\right) - \frac{1}{\delta}\left(\beta_\delta'(y)\, q_w^k, \eta^k\right) = 0, \tag{B.95}$$

$$q_y^k(T) = 0, \tag{B.96}$$

$$\left(q_w^k, \theta^k\right) + \gamma\left(\nabla q_y^k, \nabla \theta^k\right) = 0. \tag{B.97}$$

for all $\eta^k, \theta^k \in \mathcal{C}\left([0,T]; \tilde{V}_k\right)$. Substituting $\eta^k = -q_w$ in (B.95) and $\theta^k = q_{yt}^k$ in (B.97), we get two relations that used together produce

$$\begin{aligned} -\frac{\gamma}{2}\frac{d}{dt}\|\nabla q_y\|^2 + \varepsilon^2 \|\nabla q_w^k\|^2 + \frac{1}{\delta}\left(\beta_\delta'(y)\, q_w^k, q_w^k\right) \\ = \|q_w^k\|^2 - \left(\mathbf{v} \cdot \nabla q_y^k, q_w^k\right) + \left(y - y_d, q_w^k\right). \end{aligned} \tag{B.98}$$

From (B.97), we derive

$$\|q_w^k\|^2 = -\gamma\left(\nabla q_y^k, \nabla q_w^k\right)$$

and moreover, it holds

$$\left(\beta_\delta'(y)\, q_w^k, q_w^k\right) \geq 0.$$

Thus, (B.98) implies

$$\begin{aligned} -\frac{\gamma}{2}\frac{d}{dt}\|\nabla q_y^k\|^2 + \varepsilon^2 \|\nabla q_w^k\|^2 &\leq -\gamma\left(\nabla q_y^k, \nabla q_w^k\right) - \left(\mathbf{v} \cdot \nabla q_y^k, q_w^k\right) + \left(y - y_d, q_w^k\right) \\ &= H_1 + H_3 + H_3, \end{aligned} \tag{B.99}$$

where

$$\begin{aligned} H_1 &\leq \gamma\sigma\|\nabla q_w^k\|^2 + \gamma\, C_1(\sigma)\|\nabla q_y^k\|^2, \\ H_2 &\leq \sigma\|\nabla q_w^k\|^2 + C_2(\sigma)\ \|\mathbf{v}\|_{\boldsymbol{\mathcal{D}}}^2 \|\nabla q_y^k\|^2, \\ H_3 &\leq \sigma\|\nabla q_w^k\|^2 + C_3(\sigma)\ \|y - y_d\|^2. \end{aligned}$$

Inserting the estimates of H_1, H_2, H_3 in (B.99), integrating in $(t,T)\,, with 0 \leq t < T$ and using σ small enough, we infer

$$\|\nabla q_y^k(t)\|^2 + \int_t^T \|\nabla q_w^k\|^2\, ds \leq C_2\left[1 + \|\mathbf{v}\|_{\mathbf{L}^\infty(\boldsymbol{\mathcal{D}})}^2\right] \int_t^T \|\nabla q_y^k\|^2\, ds + C_1 \int_t^T \|y - y_d\|^2\, ds.$$

which implies, applying Gronwall's lemma and the estimate (2.26) established in Theorem (2.3),

$$\|\nabla q_y^k(t)\|^2 \leq C(\mathbf{u})\,, \tag{B.100}$$

$$\|q_w^k\|_{L^2(H^1)}^2 \leq C(\mathbf{u})\,. \tag{B.101}$$

With $\theta^k = -\Delta q_y^k$ in (B.97), we realize

$$\|\Delta q_y^k\|_{L^2\left(L_0^2\right)} leq C(\mathbf{u})\,,$$

and then, see [17] and [40],

$$\|q_y^k\|_{L^2(H^2)} \leq C(\mathbf{u}) . \tag{B.102}$$

From (B.100)-(B.102), given the sequences $\{q_y^k\}_{k\in\mathbb{N}}, \{q_w^k\}_{k\in\mathbb{N}}$, we can extract a subsequence (labelled by an index m), such that

$$q_y^m \overset{*}{\rightharpoonup} q_y, \qquad \text{in } L^\infty(H_0) \tag{B.103}$$

$$q_y^m \rightharpoonup q_y, \qquad \text{in } L^2\left(H^2\right) \tag{B.104}$$

$$q_y^m(0) \rightharpoonup q_{y0}, \qquad \text{in } H_0 \tag{B.105}$$

$$q_w^m \rightharpoonup q_w, \qquad \text{in } L^2(H_0), \tag{B.106}$$

where q_y, q_{y0}, q_w satisfy the estimate (2.55) established in Lemma 2.10. Furthermore,

$$\left.\frac{\partial q_y^k}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0, \quad \forall k \in \mathbb{N}.$$

So, using (B.104) above, we can claim that also (2.54) is satisfied. Given $\eta^m \in \mathcal{C}^\infty\left([0,T]; \tilde{V}_m\right)$, applying integration by parts in time, it holds

$$-\int_0^T \left(q_{yt}^k, \eta^k\right) \, dt = \int_0^T \left(\eta_t^k, q_y^k\right) \, dt + \left(q_y^k(0), \eta^k(0)\right), \tag{B.107}$$

So, from the results established above, we can say that $\left(q_y^m, q_w^m\right)$ satisfies, for all m,

$$\begin{aligned}\int_0^T \left[\left(\eta_t^m, q_y^m\right) - \varepsilon^2\left(\nabla q_w^m, \nabla\eta^m\right) + \left(q_w^m - \mathbf{v}\cdot\nabla q_y^m + y - y_d, \eta^m\right)\right] \, dt \\ + \left(q_y^m(0), \eta^m(0)\right) = \frac{1}{\delta}\int_0^T \left(\beta_\delta'(y)\, q_w^m, \eta^m\right) \, dt,\end{aligned} \tag{B.108}$$

$$\int_0^T \left[\ \left(q_w^m, \theta^m\right) + \gamma\left(\nabla q_y^m, \nabla\theta^m\right)\ \right] \, dt = 0, \tag{B.109}$$

for all $\eta^m, \theta^m \in \mathcal{C}^\infty\left([0,T]; \tilde{V}_m\right)$. Given $\eta, \theta \in \mathcal{C}_c^\infty([0,T]; H_0)$, we assume in (B.108), (B.109), $\eta^m = \tilde{P}^m\eta, \theta^m = \tilde{P}^m\theta$, where $\tilde{P}^m$, is the projection operator defined in (B.94). Thus, as $m \to +\infty$, we have

$$\begin{aligned}\int_0^T \left[(\eta_t, q_y) - \varepsilon^2(\nabla q_w, \nabla\eta) + (q_w, \eta) - (\mathbf{v}\cdot\nabla q_y, \eta) + (y - y_d, \eta)\right] \, dt \\ + (q_{y0}, \eta(0)) = \frac{1}{\delta}\int_0^T \left(\beta_\delta'(y)\, q_w, \eta\right) \, dt,\end{aligned} \tag{B.110}$$

$$\int_0^T \left[\ (q_w, \theta) + \gamma(\nabla q_y, \nabla\theta)\ \right] \, dt = 0, \tag{B.111}$$

for all $\eta, \theta \in \mathcal{C}^\infty([0,T];H_0)$. Indeed, the convergence of the linear terms in (B.107), (B.108) to the corresponding terms in (B.110), (B.112) is straightforward. Concerning the nonlinear terms, using the strong convergence of η^m to η in $L^2(H_0)$, the boundedness of $\mathbf{v}$ in $L^\infty(\boldsymbol{\mathcal{D}})$, the weak convergence of q_y^m to q_y in $L^2(H_0)$, the weak convergence of q_w^m to q_w in $L^2(H_0)$ and $0 \le \beta'_\delta \le 1$, we get

$$\begin{aligned}
&\Big|\int_0^T \left(\mathbf{v}\cdot\nabla q_y^m, \eta^m\right)\,dt - \int_0^T \left(\mathbf{v}\cdot\nabla q_y, \eta\right)\,dt\Big| \\
&\le \Big|\int_0^T \left(\mathbf{v}\cdot\nabla q_y^m, \eta^m - \eta\right)\,dt\Big| + \Big|\int_0^T \left(\mathbf{v}\cdot\nabla\left[q_y^m - q_y\right], \eta\right)\,dt\Big| \to 0,
\end{aligned}$$

$$\begin{aligned}
&\Big|\int_0^T \left(\beta'_\delta(y)\,q_w^m, \eta^m\right)\,dt - \int_0^T \left(\beta'_\delta(y)\,q_w, \eta\right)\,dt\Big| \le \\
&\le \Big|\int_0^T \left(\beta'_\delta(y)\,q_w^m, \eta^m - \eta\right)\,dt - \int_0^T \left(\beta'_\delta(y)\left[q_w^m - q_w\right], \eta\right)\,dt\Big| \to 0,
\end{aligned}$$

as $m \to +\infty$. From (B.110), (B.112), noting that the following embedding are dense

$$\mathcal{C}^\infty([0,T];H_0) \hookrightarrow L^2(H_0), \quad \mathcal{C}^\infty([0,T];H_0) \hookrightarrow W_0,$$

we derive that q_y, q_w, q_{y0} satisfy the adjoint equations (2.45c), (2.45d) for all $\eta \in W_0$, $\theta \in L^2(H_0)$. Moreover, from equation (2.45c), we conclude that also the estimate (2.55) holds for $\frac{1}{\delta}\beta'_\delta(y)$. Finally, in order to prove $\mathbf{q_v} \in H^1(\boldsymbol{\mathcal{S}}) \cap L^\infty(\boldsymbol{\mathcal{D}})$, we consider a Galerkin's approximation of the adjoint equation (2.45a) which is analogous to the one used for Stokes equation in the proof of Lemma 2.3. In this way we have that $\mathbf{q}_\mathbf{v}^k, q_y^k$ satisfy

$$-\left(\mathbf{q}_{\mathbf{v}t}^k, \boldsymbol{\psi}^k\right) + \left(\nabla\mathbf{q}_\mathbf{v}^k, \nabla\boldsymbol{\psi}^k\right) - \left(y, \nabla q_y^k\cdot\boldsymbol{\psi}^k\right)\,dt = 0, \tag{B.112}$$

$$\mathbf{q}_\mathbf{v}^k(T) = 0, \tag{B.113}$$

for all k. Substituting $\boldsymbol{\psi}^k = -\mathbf{q}_{\mathbf{v}t}^k$ in (B.112), setting $t = s$ and integrating in (t,T), with $0 \le t < T$, we get

$$\int_t^T \left[\|\mathbf{q}_{\mathbf{v}t}^k\|^2 - \left(\nabla\mathbf{q}_\mathbf{v}^k, \nabla\mathbf{q}_{\mathbf{v}t}^k\right)\right]\,ds = \int_t^T \left(y, \nabla q_y^k\cdot\mathbf{q}_{\mathbf{v}t}^k\right)\,ds,$$

which implies, using (B.113),

$$\begin{aligned}
\int_t^T \|\mathbf{q}_{\mathbf{v}t}^k\|^2 ds + \frac{1}{2}\|\nabla\mathbf{q}_\mathbf{v}^k(t)\|^2 &\le \int_t^T \|y\|_{L^4}\|\nabla q_y^k\|_{L^4}\|\mathbf{q}_{\mathbf{v}t}^k\|\,ds, \\
&\le C\int_t^T \|y\|_{H_0}\|q_y^k\|_{H^2}\|\mathbf{q}_{\mathbf{v}t}^k\|\,ds, \\
&\le \sigma\int_t^T \|\mathbf{q}_{\mathbf{v}t}^k\|^2 ds + C(\sigma)\,\|y\|_{L^\infty(H_0)}\int_t^T \|q_y^k\|_{H^2}^2 ds.
\end{aligned} \tag{B.114}$$

Then, from (B.114), assuming σ small enough, we can write

$$\int_t^T \|\mathbf{q}_{\mathbf{v}t}^k\|^2 ds + \|\nabla \mathbf{q}_{\mathbf{v}}^k(t)\|^2 \leq C\ \|y\|_{L^\infty(H_0)} \int_t^T \|q_y^k\|_{H^2}^2 ds,$$

for all $0 \leq t < T$. Therefore, using the previous results, we derive

$$\|\mathbf{q}_{\mathbf{v}}^k\|_{L^\infty(\boldsymbol{\mathcal{D}})} \leq C(\mathbf{u}), \tag{B.115}$$

$$\|\mathbf{q}_{\mathbf{v}t}^k\|_{L^2(\boldsymbol{\mathcal{S}})} \leq C(\mathbf{u}). \tag{B.116}$$

So, considering the sequence $\{\mathbf{q}_{\mathbf{v}}^k\}_k$, we can extract a subsequence (labelled by an index m), such that

$$\mathbf{q}_{\mathbf{v}}^m \overset{*}{\rightharpoonup} \mathbf{q}_{\mathbf{v}}, \qquad \text{in } L^\infty(\boldsymbol{\mathcal{D}}),$$
$$\mathbf{q}_{\mathbf{v}t}^m \rightharpoonup \mathbf{q}_{\mathbf{v}t}, \qquad \text{in } L^2(\boldsymbol{\mathcal{S}}).$$

Hence, $\mathbf{q}_{\mathbf{v}} \in H^1(\boldsymbol{\mathcal{S}}) \cap L^\infty(\boldsymbol{\mathcal{D}})$. □

B.2. Proofs of Chapter 3

Proof of Lemma 3.5

Proof. With $\boldsymbol{\psi} = k d_t \mathbf{V}^n$ in the discrete state equations (3.10), using the equality

$$r(r-s) = \frac{1}{2}\left(r^2 - s^2\right) + \frac{1}{2}(r-s)^2, \tag{B.117}$$

we have

$$k\ \|d_t\mathbf{V}^n\|^2 + \frac{\nu}{2}\|\nabla\mathbf{V}^n\|^2 - \frac{\nu}{2}\|\nabla\mathbf{V}^{n-1}\|^2 + \frac{\nu}{2}\|\nabla\mathbf{V}^n - \nabla\mathbf{V}^{n-1}\|^2 = \int_{t_{n-1}}^{t_n} (\boldsymbol{\mathcal{U}}, d_t\mathbf{V}^n)\ dt. \tag{B.118}$$

By Young's inequality (A.13) with $\sigma = 1$, we get

$$\int_{t_{n-1}}^{t_n} (\boldsymbol{\mathcal{U}}, d_t\mathbf{V}^n)\ dt \leq \frac{1}{2}\int_{t_{n-1}}^{t_n} \|d_t\mathbf{V}^n\|^2 dt + \frac{1}{2}\int_{t_{n-1}}^{t_n} \|\boldsymbol{\mathcal{U}}\|^2\ dt,$$

and therefore, from (B.118), we can write

$$\frac{k}{2}\|d_t\mathbf{V}^n\|^2 + \frac{\nu}{2}\|\nabla\mathbf{V}^n\|^2 - \frac{\nu}{2}\|\nabla\mathbf{V}^{n-1}\|^2 + \frac{\nu}{2}\|\nabla\mathbf{V}^n - \nabla\mathbf{V}^{n-1}\|^2 \leq \frac{1}{2}\int_{t_{n-1}}^{t_n} \|\boldsymbol{\mathcal{U}}\|^2\ dt. \tag{B.119}$$

From (B.119), setting $n = i$ and summing up over the index $i = 1, \ldots, n$, with $1 < n \leq N$, we derive

$$\frac{1}{2}\sum_{i=1}^n k\ \|d_t\mathbf{V}^i\|^2 + \frac{\nu}{2}\|\nabla\mathbf{V}^i\|^2 + \frac{\nu}{2}\sum_{i=1}^n \|\nabla\mathbf{V}^i - \nabla\mathbf{V}^{i-1}\|^2 \leq \frac{1}{2}\int_0^{t_n} \|\boldsymbol{\mathcal{U}}\|^2\ dt + \frac{\nu}{2}\|\nabla\mathbf{v}_{0,h}\|^2,$$

which implies the results (3.22), (3.23) and (3.24). Rewriting the first state equation (3.10a) in the following way

$$k\ (P^n, \nabla\cdot\boldsymbol{\psi}) = \left(\mathbf{V}^n - \mathbf{V}^{n-1}, \boldsymbol{\psi}\right) + k\nu\left(\nabla\mathbf{V}^n, \nabla\boldsymbol{\psi}\right) - \int_{t_{n-1}}^{t_n} (\boldsymbol{\mathcal{U}}, \boldsymbol{\psi})\ dt,$$

setting $n = i$ and summing up over the index $i = 1, \ldots, n$, with $1 < n \leq N$, we note

$$\left(\sum_{i=1}^{n} k\ P^i, \nabla\cdot\boldsymbol{\psi}\right) = \left(\mathbf{V}^n - \mathbf{V}^0, \boldsymbol{\psi}\right) + \nu\sum_{i=1}^{n} k\ \left(\nabla\mathbf{V}^i, \nabla\boldsymbol{\psi}\right) - \int_0^{t_n} (\boldsymbol{\mathcal{U}}, \boldsymbol{\psi})\ dt$$

$$\leq C\ \|\nabla\boldsymbol{\psi}\| \left[\|\mathbf{V}^n - \mathbf{v}_{0,h}\| + \nu\sum_{i=1}^{n} k\ \|\nabla\mathbf{V}^i\| + \int_0^{t_n} \|\boldsymbol{\mathcal{U}}\|\ dt\right]. \tag{B.120}$$

Using the inf-sup relation (A.26), from (B.120), we infer

$$C_1\left\|\sum_{i=1}^{n} k\ P^n\right\| \leq \sup_{\boldsymbol{\psi}\in\mathbf{V}_h} \frac{\left(\sum_{i=1}^{n} k\ P^i, \nabla\cdot\boldsymbol{\psi}\right)}{\|\nabla\boldsymbol{\psi}\|}$$
$$\leq C_2\left[\|\mathbf{V}^n - \mathbf{v}_{0,h}\| + \nu\sum_{i=1}^{n} k\ \|\nabla\mathbf{V}^i\| + \sum_{i=1}^{n} k\ \|\mathbf{U}^i\|\right],$$

which implies the result (3.26). By the definition (A.40) of the projection operator $\mathbf{A}^h$, we realize that

$$\left(\tilde{\Delta}_h\mathbf{V}^n, \mathbf{T}^h\tilde{\Delta}_h\mathbf{V}^n\right) = \|\mathbf{T}^h\tilde{\Delta}_h\mathbf{V}^n\|^2 = \|\mathbf{A}^h\mathbf{V}^n\|^2,$$

and, following [46],

$$\|\tilde{\Delta}_h\mathbf{V}^n\| \leq C\ \|\mathbf{A}^h\mathbf{V}^n\| \tag{B.121}$$

So, substituting $k\boldsymbol{\psi} = \mathbf{A}^h\mathbf{V}^n$ in the state equations (3.10), we have

$$\left(\mathbf{V}^n - \mathbf{V}^{n-1}, \mathbf{T}^h\tilde{\Delta}_h\mathbf{V}^n\right) + k\nu\|\mathbf{T}^h\tilde{\Delta}_h\mathbf{V}^n\|^2 - \int_{t_{n-1}}^{t_n} \left(\boldsymbol{\mathcal{U}}^n, \mathbf{T}^h\tilde{\Delta}_h\mathbf{V}^n\right)\ dt = 0,$$

which implies, by Cauchy-Schwarz and Young's inequalities,

$$k\ \nu\ \|\mathbf{A}^h\mathbf{V}^n\|^2 = -k\ \left(d_t\mathbf{V}^n, \mathbf{A}^h\mathbf{V}^n\right) + \int_{t_{n-1}}^{t_n} \left(\boldsymbol{\mathcal{U}}, \mathbf{A}^h\mathbf{V}^n\right)\ dt \tag{B.122}$$
$$\leq \frac{1}{2\sigma}\left[k\|d_t\mathbf{V}^n\|^2 + \int_{t_{n-1}}^{t_n} \|\boldsymbol{\mathcal{U}}\|^2\ dt\right] + k\ \frac{\sigma}{2}\|\mathbf{A}^h\mathbf{V}^n\|^2.$$

In (B.122), summing up over the index $i = 1, \ldots, n$, with $1 < n \leq N$ and assuming σ small enough, we get

$$\sum_{i=1}^{n} k\ \|\mathbf{A}^h\mathbf{V}^i\|^2 \leq C\sum_{i=1}^{n} k\ \|d_t\mathbf{V}^n\|^2 + \int_0^{t_n} \|\boldsymbol{\mathcal{U}}\|^2 dt.$$

So, by the result (3.23) and (B.121) above, we derive the estimate (3.25). □

B.3. Proofs of Chapter 4

Proof of Theorem 4.6

Proof. In order to show that $e_{\mathbf{x}}(s(\mathbf{u}),\mathbf{u})$ has a bounded inverse, we need to prove that for all $\mathbf{z}\in\mathbf{Z}$, there exists a unique $\mathbf{d_x}\in\mathbf{X}$, such that

$$e_{\mathbf{x}}(s(\mathbf{u}),\mathbf{u})\,\mathbf{d_x}=\mathbf{z}, \tag{B.123}$$

and

$$\|\mathbf{d_x}\|_{\mathbf{X}}\le C\|\mathbf{z}\|_{\mathbf{Z}}. \tag{B.124}$$

Equation (B.123) is equivalent to find $(\mathbf{d_v},d_y)\in\mathbf{W}_0\times\left[W_0\cap L^\infty(H_0)\cap L^2(H_\Delta)\right]$ which satisfy

$$\begin{aligned}
&\int_0^T \left[\langle \mathbf{d_v}_t,\boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}}+\nu(\nabla\mathbf{d_v},\nabla\boldsymbol{\psi})\right.\\
&\quad + b(\mathbf{d_v},\mathbf{v},\boldsymbol{\psi})+b(\mathbf{v},\mathbf{d_v},\boldsymbol{\psi})\\
&\quad + \rho\left(d_y,\nabla\left[-\varepsilon^2\Delta y-y+y^3\right]\cdot\boldsymbol{\psi}\right)\\
&\quad \left.+ \rho\left(y,\nabla\left[-\varepsilon^2\Delta d_y-d_y+3\,y^2\,d_y\right]\cdot\boldsymbol{\psi}\right)\right]\,dt=\int_0^T\langle\mathbf{z}_1,\boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}}\,dt,
\end{aligned} \tag{B.125}$$

$$\mathbf{d_v}(0)=\mathbf{z}_3\in\boldsymbol{\mathcal{S}}, \tag{B.126}$$

$$\begin{aligned}
&\int_0^T \left[\langle d_{yt},\eta\rangle_{H_0^*,H_0}+\gamma\left(\nabla\left[-\varepsilon^2\Delta d_y-d_y+3y^2d_y\right],\nabla\eta\right)\right.\\
&\quad \left.-(d_y,\mathbf{v}\cdot\nabla\eta)-(y,\mathbf{d_v}\cdot\nabla\eta)\right]\,dt=\int_0^T\langle z_2,\eta\rangle_{H_0^*,H_0}\,dt,
\end{aligned} \tag{B.127}$$

$$d_y(0)=z_4\in H_0, \tag{B.128}$$

for all $\boldsymbol{\psi}\in L^2(\boldsymbol{\mathcal{D}}),\eta\in L^2(H_0)$. We emphasize that $(\mathbf{v},y)$ in (B.125)-(B.128) are solutions of the regularized state equations (4.10), (4.11), for a given $\mathbf{u}\in L^2(\mathbf{L}^2)$. We formulate a Galerkin's approximation of (B.125)-(B.128) applying the same setting used in the proofs of Lemma 2.3 and Lemma 2.7. In this way, for any fixed $k\in\mathbb{N}$, the Galerkin's approximation of (B.125)-(B.128), consists in find $\left(\mathbf{d}_\mathbf{v}^k,d_y^k\right)$, such that

$$\begin{aligned}
&\left(\mathbf{d}_{\mathbf{v}t}^k,\boldsymbol{\psi}^k\right)+\nu\left(\nabla\mathbf{d}_\mathbf{v}^k,\nabla\boldsymbol{\psi}^k\right)\\
&b\left(\mathbf{d}_\mathbf{v}^k,\mathbf{v},\boldsymbol{\psi}^k\right)+b\left(\mathbf{v},\mathbf{d}_\mathbf{v}^k,\boldsymbol{\psi}^k\right)\\
&+\rho\left(d_y^k,\nabla\left[-\varepsilon^2\Delta y-y+y^3\right]\cdot\boldsymbol{\psi}^k\right)\\
&+\rho\left(y,\nabla\left[-\varepsilon^2\Delta d_y^k-d_y^k+3\,y^2\,d_y^k\right]\cdot\boldsymbol{\psi}^k\right)=\langle\mathbf{z}_1,\boldsymbol{\psi}^k\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}},
\end{aligned} \tag{B.129}$$

$$\mathbf{d}_\mathbf{v}^k(0)=\mathbf{P}^k\mathbf{z}_3 \tag{B.130}$$

$$\begin{aligned}
&\left(d_{yt}^k,\eta^k\right)+\gamma\left(\nabla\left[-\varepsilon^2\Delta d_y^k-d_y^k+3y^2d_y^k\right],\nabla\eta^k\right)\\
&\quad-\left(d_y^k,\mathbf{v}\cdot\nabla\eta^k\right)-\left(y,\mathbf{d}_\mathbf{v}^k\cdot\nabla\eta^k\right)=\langle\tilde{z}_2,\eta^k\rangle_{H^{1*},H^1},
\end{aligned} \tag{B.131}$$

$$d_y^k(0)=P^kz_4, \tag{B.132}$$

for all $\boldsymbol{\psi}^k \in \mathbf{W}_k, \eta^k \in V_k$, where

$$\langle \tilde{z}_2, \eta \rangle_{H^{1*},H^1} = \langle z_2, \eta - \frac{1}{|\Omega|}(\eta, 1)\rangle_{H_0^*,H_0}, \quad \forall\, \eta \in H^1.$$

We assume

$$\mathbf{d}_{\mathbf{v}}^k = \sum_{j=1}^{k} b_j(t)\, \boldsymbol{\xi}_j, \quad d_y^k = \sum_{j=1}^{k} c_j(t)\, \phi_j. \tag{B.133}$$

Taking into account that $(\mathbf{v}, y)$ in (B.125)-(B.128) are solutions of the regularized state equations (4.10), (4.11), it can be proved that the linear system associated to (B.129)-(B.132) has a unique solution

$$\mathbf{b}^k(t) = (b_1(t), \ldots, b_k(t))^T, \quad \mathbf{c}^k(t) = (c_1(t), \ldots, c_k(t))^T$$

such that $b_i, c_i \in H^1(0,T)$ for all $i = 1, \ldots, k$. Then $\mathbf{d}_{\mathbf{v}}^k \in H^1(\mathbf{W_k})$, $d_y^k \in H^1(V_k \cap L_0^2)$ solve (B.129)-(B.132) for all $\boldsymbol{\psi}^k \in \mathcal{C}([0,T]; \mathbf{W}_k), \eta^k \in \mathcal{C}([0,T]; V_k)$. Substituting $\eta^k = -\Delta d_y^k$ in (B.131), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla d_y^k\|^2 + \gamma\varepsilon^2\|\nabla\Delta d_y^k\|^2 \\ \le \gamma\left|\left(\nabla d_y^k, \nabla\Delta d_y^k\right)\right| + 3\gamma\left|\left(\nabla\left[y^2\ d_y^k\right], \nabla\Delta d_y^k\right)\right| + \left|\left(d_y^k, \mathbf{v}\cdot\nabla\Delta d_y^k\right)\right| \\ + \left|\left(y, \mathbf{d}_{\mathbf{v}}\cdot\nabla\Delta d_y^k\right)\right| + \left|\langle \tilde{z}_2, \Delta d_y^k\rangle_{H^{1*},H^1}\right| \\ = A_1 + A_2 + A_3 + A_4 + A_5.\end{aligned} \tag{B.134}$$

Using Young's, Holder's, Poincaré's inequalities and the embedding $H^2 \hookrightarrow \mathcal{C}(\bar{\Omega})$, we can write

$$A_1 \le \sigma\|\nabla\Delta d_y^k\|^2 + C_1(\sigma)\,\|\nabla d_y^k\|^2,$$

$$\begin{aligned}A_2 &= 3\gamma\left|\left(2\ y\ d_y^k, \nabla y\cdot\nabla\Delta d_y^k\right) + \left(y^2, \nabla d_y^k\cdot\nabla\Delta d_y^k\right)\right| \\ &\le 3\gamma\left[2\ \|y\|_{\mathcal{C}(\bar{\Omega})}\ \|d_y\|_{L^4}\ \|\nabla y\|_{L^4}\ \|\nabla\Delta d_y^k\| + \|y\|^2_{\mathcal{C}(\bar{\Omega})}\ \|\nabla d_y\|\ \|\nabla\Delta d_y^k\|\right] \\ &\le C\ \|y\|^2_{H^2}\ \|\nabla\Delta d_y^k\|\ \|\nabla d_y\| \le \sigma\|\nabla\Delta d_y^k\|^2 + C_2(\sigma)\ \|y\|^4_{H^2}\ \|\nabla d_y^k\|^2,\end{aligned}$$

$$A_3 \le \|d_y\|\ \|\mathbf{v}\|_{\boldsymbol{\mathcal{C}}(\bar{\Omega})}\ \|\nabla\Delta d_y^k\| \le \sigma\|\nabla\Delta d_y^k\|^2 + C_3(\sigma)\ \|\mathbf{v}\|^2_{\mathbf{H}^2}\|\nabla d_y^k\|^2,$$

$$A_4 \le \|y\|_{\mathcal{C}(\bar{\Omega})}\ \|\mathbf{d}_{\mathbf{v}}^k\|\ \|\nabla\Delta d_y^k\| \le \sigma\|\nabla\Delta d_y^k\|^2 + C_4(\sigma)\ \|y\|^2_{H^2}\ \|\mathbf{d}_{\mathbf{v}}^k\|^2,$$

$$A_5 \le \sigma\|\nabla\Delta d_y^k\|^2 + C_5(\sigma)\,\|\tilde{z}_2\|^2_{H^{1*}}.$$

Hence, inserting the estimates of $A_1, \ldots, A_5$ above in (B.134), we derive

$$\frac{1}{2}\frac{d}{dt}\|\nabla d_y^k\|^2 + \gamma\varepsilon^2\|\nabla\Delta d_y^k\|^2 \tag{B.135}$$

$$\le 5\ \sigma\ \|\nabla\Delta d_y^k\|^2 + \left[C_1(\sigma) + C_2(\sigma)\,\|y\|_{H^2}^4 + C_3(\sigma)\,\|\mathbf{v}\|_{\mathbf{H}^2}^2\right]\|\nabla d_y^k\|^2$$
$$+C_4(\sigma)\,\|y\|_{H^2}^2\ \|\mathbf{d}_{\mathbf{v}}^k\|^2 + C_5(\sigma)\,\|\tilde z_2\|_{H^{1*}}^2.$$

With $\boldsymbol{\psi}^k = \mathbf{d}_{\mathbf{v}}^k$ in (B.129), we observe

$$\text{(B.136)}\qquad \frac{1}{2}\frac{d}{dt}\|\mathbf{d}_{\mathbf{v}}^k\|^2 + \nu\|\nabla\mathbf{d}_{\mathbf{v}}^k\|^2$$
$$\le -B\left(\mathbf{d}_{\mathbf{v}}^k, \mathbf{v}, \mathbf{d}_{\mathbf{v}}^k\right) - \rho\left(d_y^k, \nabla\left[-\varepsilon^2\Delta y - y + y^3\right]\cdot\mathbf{d}_{\mathbf{v}}^k\right)$$
$$-\rho\left(y, \nabla\left[-\varepsilon^2\Delta d_y^k - d_y^k + 3\ y^2\ d_y^k\right]\cdot\mathbf{d}_{\mathbf{v}}^k\right) + \langle\mathbf{z}_1, \mathbf{d}_{\mathbf{v}}^k\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}}$$
$$= B_1 + B_2 + B_3 + B_4,$$

where

$$B_1 \le \|\mathbf{d}_{\mathbf{v}}^k\|\ \|\nabla\mathbf{v}\|_{\mathbf{L}^4}\ \|\mathbf{d}_{\mathbf{v}}^k\|_{\mathbf{L}^4} \le \sigma\ \|\nabla\mathbf{d}_{\mathbf{v}}^k\|^2 + C_1(\sigma)\,\|\mathbf{v}\|_{\mathbf{H}^2}^2\ \|\mathbf{d}_{\mathbf{v}}^k\|^2,$$

$$B_2 \le \rho\ \|d_y^k\|_{L^4}\ \left\|\nabla\left[-\varepsilon^2\Delta y - y + y^3\right]\right\|\ \|\mathbf{d}_{\mathbf{v}}^k\|_{L^4} \le$$
$$\le \sigma\|\nabla\mathbf{d}_{\mathbf{v}}^k\|^2 + C_2(\sigma)\left\|\nabla\left[-\varepsilon^2\Delta y - y + y^3\right]\right\|^2\ \|\nabla d_y^k\|^2,$$

$$B_3 \le \rho\ \|y\|_{\mathcal{C}(\bar\Omega)}\ \left\|\nabla\left[-\varepsilon^2\Delta d_y^k - d_y^k + 3\ y^2\ d_y^k\right]\right\|\ \|\mathbf{d}_{\mathbf{v}}^k\|$$
$$\le \rho\ \varepsilon^2\ \|y\|_{\mathcal{C}(\bar\Omega)}\ \|\nabla\Delta d_y^k\|\ \|\mathbf{d}_{\mathbf{v}}^k\| + \rho\ \|y\|_{\mathcal{C}(\bar\Omega)}\ \|\nabla d_y^k\|\ \|\mathbf{d}_{\mathbf{v}}^k\|$$
$$+\,6\ \rho\ \|y\|_{\mathcal{C}(\bar\Omega)}^2\ \|d_y^k\|_{L^4}\ \|\nabla y\|_{L^4}\ \|\mathbf{d}_{\mathbf{v}}^k\| + 3\ \rho\ \|y\|_{\mathcal{C}(\bar\Omega)}^3\ \|\nabla d_y^k\|\ \|\mathbf{d}_{\mathbf{v}}^k\|$$
$$\le \sigma\|\nabla\Delta d_y^k\|^2 + C_4(\sigma)\ \|y\|_{H^2}^2\ \|\mathbf{d}_{\mathbf{v}}^k\|^2 + \|\nabla d_y^k\|^2 + C_2\left[\|y\|_{H^2}^2 + \|y\|_{H^2}^3\right]\ \|\mathbf{d}_{\mathbf{v}}^k\|^2,$$

$$B_4 \le \sigma\ \|\nabla\mathbf{d}_{\mathbf{v}}^k\|^2 + C_3(\sigma)\,\|\mathbf{z}_1\|_{\boldsymbol{\mathcal{D}}^*}^2.$$

Inserting the estimates of $B_1, \ldots, B_4$ in (B.136), we infer

$$\text{(B.137)}\qquad \frac{1}{2}\frac{d}{dt}\|\mathbf{d}_{\mathbf{v}}^k\|^2 + \nu\|\nabla\mathbf{d}_{\mathbf{v}}^k\|^2$$
$$\le 3\ \sigma\ \|\nabla\mathbf{d}_{\mathbf{v}}^k\|^2 + \sigma\|\nabla\Delta d_y^k\|^2$$
$$+\left(1 + C_2(\sigma)\left\|\nabla\left[-\varepsilon^2\Delta y - y + y^3\right]\right\|^2\right)\|\nabla d_y^k\|^2$$
$$+\left(C_1(\sigma)\,\|\mathbf{v}\|_{\mathbf{H}^2}^2 + C_4(\sigma)\ \|y\|_{H^2}^2 + C_2\left[\|y\|_{H^2}^2 + \|y\|_{H^2}^3\right]\right)\|\mathbf{d}_{\mathbf{v}}^k\|^2$$
$$+C_3(\sigma)\,\|\mathbf{z}_1\|_{\boldsymbol{\mathcal{D}}^*}^2.$$

Summing (B.135) and (B.137) and multiplying by two, produces

$$\text{(B.138)}\qquad \frac{d}{dt}\left[\|\mathbf{d}_{\mathbf{v}}^k\|^2 + \|\nabla d_y^k\|^2\right] + 2\ \nu\|\nabla\mathbf{d}_{\mathbf{v}}^k\|^2 + 2\ \gamma\ \varepsilon^2\ \|\nabla\Delta d_y^k\|^2$$
$$\le 6\ \sigma\ \|\nabla\mathbf{d}_{\mathbf{v}}^k\|^2 + 12\ \sigma\ \|\nabla\Delta d_y^k\|^2$$
$$+\left(C_1(\sigma)\,\|y\|_{H^2}^2 + C_2(\sigma)\,\|\mathbf{v}\|_{\mathbf{H}^2}^2 + C_3\left[\|y\|_{H^2}^2 + \|y\|_{H^2}^3\right]\right)\|\mathbf{d}_{\mathbf{v}}^k\|^2$$
$$+\left(C_4(\sigma) + C_5(\sigma)\,\|y\|_{H^2}^4 + C_6(\sigma)\,\|\mathbf{v}\|_{\mathbf{H}^2}^2 + C_7(\sigma)\,\|\nabla\left[-\varepsilon^2\Delta y - y + y^3\right]\|^2\right)\|\nabla d_y^k\|^2$$

$$+C_8(\sigma)\,\|\mathbf{z}_1\|^2_{\mathcal{D}^*} + C_9(\sigma)\,\|\tilde{z}_2\|^2_{H^{1*}}.$$

In (B.138), assuming σ enough small and rearranging, we realize

$$\frac{d}{dt}\Big[\|\mathbf{d}^k_{\mathbf{v}}\|^2 + \|\nabla d^k_y\|^2\Big] + C_1\Big(\|\nabla \mathbf{d}^k_{\mathbf{v}}\|^2 + \|\nabla\Delta d^k_y\|^2\Big) \tag{B.139}$$
$$\leq C_2\ D(\mathbf{v},y)\ \Big[\|\mathbf{d}^k_{\mathbf{v}}\|^2 + \|\nabla d^k_y\|^2\Big] + C_3\ \|\mathbf{z}_1\|^2_{\mathcal{D}^*} + C_4\ \|\tilde{z}_2\|^2_{H^{1*}},$$

where

$$D(\mathbf{v},y) = 1 + \|y\|^2_{H^2} + \|y\|^3_{H^2} + +\|y\|^4_{H^2} + \|\mathbf{v}\|^2_{\mathbf{H}^2} + \Big\|\nabla\left[-\varepsilon^2\Delta y - y + y^3\right]\Big\|^2.$$

Since $y \in L^\infty(H^2)$, $\mathbf{v} \in \mathbf{L}^\infty(\mathbf{H}^2)$, $w := -\varepsilon^2\Delta y - y + y^3 \in L^2(H^1)$, we can integrate (B.139) in the interval $(0,t)$, where $0 < t \leq T$ and applying Gronwall's lemma. In this way, we can claim that there exists a constant $C(\mathbf{v},y)$, dependent on the norms of $\|\mathbf{v}\|$ and $\|y\|$ but independent of k, such that

$$\|\mathbf{d}^k_{\mathbf{v}}(t)\|^2 + \|\nabla d^k_y(t)\|^2 + \|\mathbf{d}^k_{\mathbf{v}}\|^2_{\mathbf{L}^2(0,t;\mathcal{D})} + \|\Delta d^k_y\|^2_{L^2(0,t;H_0)} \tag{B.140}$$
$$\leq C(\mathbf{v},y)\Big[\|\mathbf{z}_1\|^2_{L^2(\mathcal{D}^*)} + \|\tilde{z}_2\|^2_{L^2(H^1)} + \|\mathbf{P}^k\mathbf{z}_3\|^2 + \|\nabla P^k z_4\|^2\Big],$$

for all $t \in (0,T]$. The basis $\{\phi_j\}_{j\in\mathbb{N}}$ used for the Galerkin's approximation, is such that

$$\frac{\partial d^k_y}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0.$$

So, $\Delta d^k_y(t) \in L^2_0$ and then, by Poincaré-Wirtinger's inequality (A.15), we have

$$\|\Delta d^k_y\|_{H^1} \leq C\ \|\nabla\Delta d^k_y\|.$$

Furthermore, following [17] and [40], it holds

$$\|d^k_y\|_{H^2} \leq C\ \|\Delta d^k_y\|.$$

Hence, from (B.140), we get

$$\|\mathbf{d}^k_{\mathbf{v}}\|^2_{L^\infty(\mathcal{S})} + \|d^k_y\|^2_{L^\infty(H_0)} + \|d^k_y\|^2_{L^2(H_\Delta)} \tag{B.141}$$
$$\leq C(\mathbf{u})\ \Big[\|\mathbf{z}_1\|^2_{L^2(\mathcal{D}^*)} + \|\tilde{z}_2\|^2_{L^2(H^{1*})} + \|\mathbf{z}_3\|^2_{\mathcal{S}} + \|z_4\|^2_{H_0}\Big],$$

independently on k. From the Galerkin's approximation (B.129), (B.130), we can write

$$\int_0^T \left(\mathbf{d}^k_{\mathbf{v}t}, \boldsymbol{\psi}^k\right)\ dt = \int_0^T \left[-\nu\left(\nabla\mathbf{d}^k_{\mathbf{v}}, \nabla\boldsymbol{\psi}^k\right) - B\left(\mathbf{d}^k_{\mathbf{v}}, \mathbf{v}, \boldsymbol{\psi}^k\right) - B\left(\mathbf{v}, \mathbf{d}^k_{\mathbf{v}}, \boldsymbol{\psi}^k\right)\right]\ dt$$
$$-\int_0^T \rho\left(d^k_y, \nabla\left[-\varepsilon^2\Delta y - y + y^3\right]\cdot\boldsymbol{\psi}^k\right)\ dt$$
$$+\int_0^T \left[-\rho\left(y, \nabla\left[-\varepsilon^2\Delta d^k_y - d^k_y + 3\ y^2\ d^k_y\right]\cdot\boldsymbol{\psi}^k\right) + \langle\mathbf{z}_1, \boldsymbol{\psi}^k\rangle_{\mathcal{D}^*,\mathcal{D}}\right]\ dt. \tag{B.142}$$

for all $\boldsymbol{\psi}^k \in \mathcal{C}_c^\infty\left((0,T);\mathbf{W}_k\right)$. So, given $\boldsymbol{\psi} \in \mathcal{C}_c^\infty\left((0,T);\boldsymbol{\mathcal{D}}\right)$, we set $\boldsymbol{\psi}^k = \mathbf{P}^k\boldsymbol{\psi}$ in (B.142). In this way, we derive

$$
\begin{aligned}
\text{(B.143)}\qquad &\int_0^T \left(\mathbf{d}_{\mathbf{v}t}^k, \boldsymbol{\psi}\right)\, dt = \int_0^T \left(\mathbf{d}_{\mathbf{v}t}^k, \mathbf{P}^k\boldsymbol{\psi}\right)\, dt \\
&= \int_0^T \left[-\nu\left(\nabla \mathbf{d}_{\mathbf{v}}^k, \nabla \mathbf{P}^k\boldsymbol{\psi}\right) - B\left(\mathbf{d}_{\mathbf{v}}^k, \mathbf{v}, \mathbf{P}^k\boldsymbol{\psi}\right) - B\left(\mathbf{v}, \mathbf{d}_{\mathbf{v}}^k, \mathbf{P}^k\boldsymbol{\psi}\right)\right]\, dt \\
&- \int_0^T \rho\left(d_y^k, \nabla\left[-\varepsilon^2\Delta y - y + y^3\right]\cdot \mathbf{P}^k\boldsymbol{\psi}\right)\, dt + \\
&+ \int_0^T \left[-\rho\left(y, \nabla\left[-\varepsilon^2\Delta d_y^k - d_y^k + 3\, y^2\, d_y^k\right]\cdot \mathbf{P}^k\boldsymbol{\psi}\right) + \langle \mathbf{z}_1, \mathbf{P}^k\boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}}\right]\, dt = \\
&= D_1 + D_2 + D_3 + D_4 + D_5 + D_6.
\end{aligned}
$$

Using the properties of the projection operator $\mathbf{P}^k$, we note that

$$D_1 \leq \nu \int_0^T \|\nabla \mathbf{d}_{\mathbf{v}}^k\|\ \|\nabla \mathbf{P}^k\boldsymbol{\psi}\|\ dt \leq C\ \|\mathbf{d}_{\mathbf{v}}^k\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})}\ \|\boldsymbol{\psi}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})},$$

$$D_{2,3} \leq \int_0^T \|\mathbf{d}_{\mathbf{v}}^k\|\ \|\nabla \mathbf{v}\|_{\mathbf{L}^4}\ \|\mathbf{P}^k\boldsymbol{\psi}\|_{\mathbf{L}^4}\ dt \leq C\ \|\mathbf{v}\|_{\mathbf{L}^\infty(\mathbf{H}^2)}\ \|\mathbf{d}_{\mathbf{v}}^k\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})}\ \|\boldsymbol{\psi}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})},$$

$$
\begin{aligned}
D_4 &\leq \rho \int_0^T \|d_y^k\|_{L^4}\ \left\|\nabla\left[-\varepsilon^2\Delta y - y + y^3\right]\right\|\ \|\mathbf{P}^k\boldsymbol{\psi}\|_{\mathbf{L}^4}\ dt \\
&\leq C\ \|d_y^k\|_{L^\infty(H_0)}\ \left\| -\varepsilon^2\Delta y - y + y^3\right\|_{L^2(H^1)}\ \|\boldsymbol{\psi}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})},
\end{aligned}
$$

$$
\begin{aligned}
D_5 &\leq \rho \int_0^T \|y\|_{\mathcal{C}(\bar{\Omega})}\ \left\|\nabla\left[-\varepsilon^2\Delta d_y^k - d_y^k + 3\, y^2\, d_y^k\right]\right\|\ \|\mathbf{P}^k\boldsymbol{\psi}\|_{\mathbf{L}^4}\ dt \\
&\leq C\ \|y\|_{L^\infty(H^2)}\ \left(\|\Delta d_y^k\|_{L^2(H^1)} + \|d_y^k\|_{L^2(H_0)} + \|y\|^2_{L^\infty(H^2)}\ \|d_y^k\|_{L^2(H_0)}\right)\|\boldsymbol{\psi}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})},
\end{aligned}
$$

$$D_6 \leq \int_0^T \|\mathbf{z}_1\|_{\boldsymbol{\mathcal{D}}^*}\|\mathbf{P}^k\boldsymbol{\psi}\|_{\boldsymbol{\mathcal{D}}}\ dt \leq \|\mathbf{z}_1\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}}^*)}\|\boldsymbol{\psi}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})}.$$

Inserting the estimates of $D_1, \ldots, D_6$ in (B.143), using (B.141), we infer

$$\left|\int_0^T \left(\mathbf{d}_{\mathbf{v}t}^k, \boldsymbol{\psi}\right)\ dt\right| \leq C\left(\mathbf{u}\right)\left[\|\mathbf{z}_1\|^2_{L^2(\boldsymbol{\mathcal{D}}^*)} + \|\tilde{z}_2\|^2_{L^2(H^{1*})} + \|\mathbf{z}_3\|^2_{\boldsymbol{\mathcal{S}}} + \|z_4\|^2_{H_0}\right]^{\frac{1}{2}}\ \|\boldsymbol{\psi}\|_{L^2(\boldsymbol{\mathcal{D}})},$$

for all $\boldsymbol{\psi} \in \mathcal{C}_c^\infty\left((0,T);\boldsymbol{\mathcal{D}}\right)$. So, from the dense embedding (A.8), we realize that

$$\text{(B.144)}\qquad \|\mathbf{d}_{\mathbf{v}t}^k\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}}^*)} \leq C\left(\mathbf{u}\right)\left[\|\mathbf{z}_1\|^2_{L^2(\boldsymbol{\mathcal{D}}^*)} + \|\tilde{z}_2\|^2_{L^2(H^{1*})} + \|\mathbf{z}_3\|^2_{\boldsymbol{\mathcal{S}}} + \|z_4\|^2_{H_0}\right]^{\frac{1}{2}}.$$

From the Galerkin's approximation (B.131), we have

$$\int_0^T \left(d_{yt}^k, \eta^k\right)\ dt = -\gamma \int_0^T \left(\nabla\left[-\varepsilon^2\Delta d_y^k - d_y^k + 3y^2 d_y^k\right], \nabla\eta^k\right)\ dt$$

$$+\int_0^T \left[\left(d_y^k, \mathbf{v}\cdot\nabla\eta^k\right) + \left(y, \mathbf{d}_{\mathbf{v}}^k\cdot\nabla\eta^k\right) + \langle\tilde{z}_2, \eta^k\rangle_{H^{1*},H^1}\right]\ dt, \tag{B.145}$$

for all $\eta^k \in \mathcal{C}_c^\infty((0,T);V_k)$. So, given $\eta \in \mathcal{C}_c^\infty((0,T);H^1)$, we set in (B.145) $\eta^k = P^k\eta$. We get

$$\begin{aligned}\int_0^T &\left(d_{yt}^k, P^k\eta\right)\ dt \\ =&-\gamma\int_0^T \left(\nabla\left[-\varepsilon^2\Delta d_y^k - d_y^k + 3y^2 d_y^k\right], \nabla P^k\eta\right)\ dt \\ &+\int_0^T \left[\left(d_y^k, \mathbf{v}\cdot\nabla P^k\eta\right) + \left(y, \mathbf{d}_{\mathbf{v}}^k\cdot\nabla P^k\eta\right) + \langle\tilde{z}_2, P^k\eta\rangle_{H^{1*},H^1}\right]\ dt = \\ =&\, E_1 + E_2 + E_3 + E_4.\end{aligned} \tag{B.146}$$

Using the properties of the projection operator P^k, we derive

$$E_1 \le C\left(\|\Delta d_y^k\|_{L^2(H^1)} + \|d_y^k\|_{L^2(H_0)} + \|y\|^2_{L^\infty(H^2)}\ \|d_y^k\|_{L^2(H_0)}\right)\|\eta\|_{L^2(H^1)},$$

$$E_2 \le \int_0^T \|d_y^k\|\ \|\mathbf{v}\|_{\boldsymbol{\mathcal{C}}(\bar\Omega)}\ \|\nabla P^k\eta\|\ dt \le C\ \|\mathbf{v}\|_{\mathbf{L}^\infty(\mathbf{H}^2)}\ \|d_y^k\|_{L^2(H_0)}\ \|\eta\|_{L^2(H^1)},$$

$$E_3 \le \int_0^T \|y\|_{\mathcal{C}(\bar\Omega)}\ \|\mathbf{d}_{\mathbf{v}}^k\|\ \|\nabla P^k\eta\|\ dt \le C\ \|y\|_{L^\infty(H^2)}\ \|\mathbf{d}_{\mathbf{v}}^k\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})}\ \|\eta\|_{L^2(H^1)},$$

$$E_4 \le \int_0^T \|\tilde{z}_2\|_{H^{1*}}\ \|P^k\eta\|_{H^1}\ dt \le \|\tilde{z}_2\|_{L^2(H^{1*})}\ \|\eta\|_{L^2(H^1)}.$$

Inserting the estimates of $E_1,\dots,E_4$ above in (B.146) and using (B.141), we can write

$$\|d_{yt}^k\|_{L^2(H^{1*})} \le C(\mathbf{u})\left[\|\mathbf{z}_1\|^2_{L^2(\boldsymbol{\mathcal{D}}^*)} + \|\tilde{z}_2\|^2_{L^2(H^{1*})} + \|\mathbf{z}_3\|^2_{\boldsymbol{\mathcal{S}}} + \|z_4\|^2_{H_0}\right]^{\frac{1}{2}}. \tag{B.147}$$

Considering the sequences $\left\{\mathbf{d}_{\mathbf{v}}^k\right\}_{k\in\mathbb{N}}, \left\{d_y^k\right\}_{k\in\mathbb{N}}$, using the estimates (B.141), (B.144) and (B.147), there exist subsequences (labelled by an index m), such that

$$\mathbf{d}_{\mathbf{v}}^m \rightharpoonup \mathbf{d}_{\mathbf{v}}, \quad \text{in } \mathbf{W}_0, \tag{B.148}$$

$$d_y^m \rightharpoonup d_y, \quad \text{in } W_0, \tag{B.149}$$

$$d_y^m \overset{*}{\rightharpoonup} d_y, \quad \text{in } L^\infty(H_0), \tag{B.150}$$

$$d_y^m \to d_y, \quad \text{in } L^2\left(L_0^2\right), \tag{B.151}$$

$$d_y^m \rightharpoonup d_y, \quad \text{in } L^2(H_\Delta). \tag{B.152}$$

as $m \to +\infty$. Next, we show that $\mathbf{d}_{\mathbf{v}}, d_y$ solve (B.125)-(B.128). It holds

$$\int_0^T \left[\left(\mathbf{d}_{\mathbf{v}t}^m, \boldsymbol{\psi}^m\right) + \nu\left(\nabla\mathbf{d}_{\mathbf{v}}^m, \nabla\boldsymbol{\psi}^m\right)\right. \tag{B.153}$$

$$+ B\left(\mathbf{d}_{\mathbf{v}}^m, \mathbf{v}, \boldsymbol{\psi}^m\right) + B\left(\mathbf{v}, \mathbf{d}_{\mathbf{v}}^m, \boldsymbol{\psi}^m\right)$$
$$+ \rho\left(d_y^m, \nabla\left[-\varepsilon^2 \Delta y - y + y^3\right] \cdot \boldsymbol{\psi}^m\right)$$
$$+ \rho\left(y, \nabla\left[-\varepsilon^2 \Delta d_y^m - d_y^m + 3\, y^2\, d_y^m\right] \cdot \boldsymbol{\psi}^m\right)\Big]\ dt = \int_0^T \langle \mathbf{z}_1, \boldsymbol{\psi}^m\rangle_{\boldsymbol{\mathcal{D}}^*, \boldsymbol{\mathcal{D}}}\, dt,$$

(B.154)
$$\mathbf{d}_{\mathbf{v}}^m(0) = \mathbf{P}^m \mathbf{z}_3,$$

$$\int_0^T \left[\left(d_{yt}^m, \eta^m\right) + \gamma\left(\nabla\left[-\varepsilon^2 \Delta d_y^m - d_y^m + 3y^2 d_y^m\right], \nabla \eta^m\right) + \right.$$

(B.155)
$$\left. - \left(d_y^m, \mathbf{v}\cdot\nabla\eta^m\right) - \left(y, \mathbf{d}_{\mathbf{v}}^m \cdot \nabla \eta^m\right)\right]\ dt = \int_0^T \langle z_2, \eta^m\rangle_{H_0^*, H_0}\, dt,$$

(B.156)
$$d_y^m(0) = P^m z_4,$$

for all m, $\boldsymbol{\psi}^m \in \mathcal{C}_c^\infty\left((0,T); \boldsymbol{\mathcal{D}}\right)$, $\eta^m \in \mathcal{C}_c^\infty\left((0,T); H_0\right)$. So, given $\boldsymbol{\psi} \in L^2(\boldsymbol{\mathcal{D}})$, $\eta \in L^2(H_0)$, we set in (B.153)-(B.156) $\boldsymbol{\psi}^m = \mathbf{P}^m \boldsymbol{\psi}$ and $\eta^m = \tilde{P}^m \eta$, which are such that

(B.157)
$$\|\mathbf{P}^m \boldsymbol{\psi} - \boldsymbol{\psi}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})} \to 0,$$

(B.158)
$$\|\tilde{P}^m \eta - \eta\|_{L^2(H_0)} \to 0,$$

as $m \to +\infty$. Then, using (B.148) and (B.157), we derive

$$\Big|\int_0^T \left(\mathbf{d}_{\mathbf{v}t}^m, \boldsymbol{\psi}^m\right)\ dt - \int_0^T \langle \mathbf{d}_{\mathbf{v}t}, \boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}^*, \boldsymbol{\mathcal{D}}}\ dt\Big| \to 0,$$

$$\Big|\int_0^T \left(\nabla \mathbf{d}_{\mathbf{v}}^m, \nabla \boldsymbol{\psi}^m\right)\ dt - \int_0^T \left(\nabla \mathbf{d}_{\mathbf{v}}, \nabla \boldsymbol{\psi}\right)\ dt\Big| \to 0,$$

from (B.148), (B.157) and the boundedness of $\nabla \mathbf{v}\cdot\boldsymbol{\psi}$ in $\mathbf{L}^2(\mathbf{L}^2)$, we infer

$$\Big|\int_0^T B\left(\mathbf{d}_{\mathbf{v}}^m, \mathbf{v}, \boldsymbol{\psi}^m\right)\ dt - \int_0^T B\left(\mathbf{d}_{\mathbf{v}}, \mathbf{v}, \boldsymbol{\psi}\right)\ dt\Big|$$
$$\Big|\int_0^T B\left(\mathbf{d}_{\mathbf{v}}^m, \mathbf{v}, \boldsymbol{\psi}^m - \boldsymbol{\psi}\right) + B\left(\mathbf{d}_{\mathbf{v}}^m - \mathbf{d}_{\mathbf{v}}, \mathbf{v}, \boldsymbol{\psi}\right)\ dt\Big|$$
$$\le C\ \left[\|\mathbf{d}_{\mathbf{v}}^m\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})}\ \|\mathbf{v}\|_{\mathbf{L}^\infty(\boldsymbol{\mathcal{D}})}\ \|\boldsymbol{\psi}^m - \boldsymbol{\psi}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})}\right]$$
$$+\Big|\int_0^T \left(\int_\Omega \left[\left(\mathbf{d}_{\mathbf{v}}^m - \mathbf{d}_{\mathbf{v}}\right)\cdot\nabla\right]\mathbf{v}\cdot\boldsymbol{\psi}\ dx\right)\ dt\ \Big| \to 0,$$

by (B.148), (B.157) and the boundedness of $\mathbf{v}\cdot\boldsymbol{\psi}$ in $\mathbf{L}^2(\mathbf{L}^2)$, we observe

$$\Big|\int_0^T B\left(\mathbf{v}, \mathbf{d}_{\mathbf{v}}^m, \boldsymbol{\psi}^m\right)\ dt - \int_0^T B\left(\mathbf{v}, \mathbf{d}_{\mathbf{v}}, \boldsymbol{\psi}\right)\ dt\Big|$$
$$\le \Big|\int_0^T B\left(\mathbf{v}, \mathbf{d}_{\mathbf{v}}^m, \boldsymbol{\psi}^m - \boldsymbol{\psi}\right) + B\left(\mathbf{v}, \mathbf{d}_{\mathbf{v}}^m - \mathbf{d}_{\mathbf{v}}, \boldsymbol{\psi}\right)\ dt\ \Big|$$
$$\le C\ \left[\|\mathbf{v}\|_{\mathbf{L}^\infty(\boldsymbol{\mathcal{D}})}\ \|\mathbf{d}_{\mathbf{v}}^m\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})}\ \|\boldsymbol{\psi}^m - \boldsymbol{\psi}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})}\right]$$
$$+\Big|\int_0^T \left(\int_\Omega \left[\mathbf{v}\cdot\nabla\right]\left(\mathbf{d}_{\mathbf{v}}^m - \mathbf{d}_{\mathbf{v}}\right)\cdot\boldsymbol{\psi}\ dx\right)\ dt\ \Big| \to 0,$$

using (B.150), (B.152), (B.157) and the boundedness of $w \cdot \boldsymbol{\psi}$ in $L^2(H^{2*})$, we note

$$
\begin{aligned}
&\Big| \int_0^T \left(d_y^m, \nabla w \cdot \boldsymbol{\psi}^m\right)\, dt - \int_0^T \left(d_y, \nabla w \cdot \boldsymbol{\psi}\right)\, dt \Big| \\
&\leq \Big| \int_0^T \left[\left(d_y^m, \nabla w \cdot \boldsymbol{\psi}^m - \boldsymbol{\psi}\right) + \left(d_y^m - d_y, \nabla w \cdot \boldsymbol{\psi}\right)\right]\, dt \Big| \\
&\leq C\, \|d_y^m\|_{L^\infty(H_0)}\, \|w\|_{L^2(H^1)}\, \|\boldsymbol{\psi}^m - \boldsymbol{\psi}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})} + \Big| \int_0^T \left(d_y^m - d_y, \nabla w \cdot \boldsymbol{\psi}\right)\, dt \Big| \\
&= C\, \|d_y^m\|_{L^\infty(H_0)}\, \|w\|_{L^2(H^1)}\, \|\boldsymbol{\psi}^m - \boldsymbol{\psi}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})} + \Big| \int_0^T \left(w, \nabla \left[d_y^m - d_y\right] \cdot \boldsymbol{\psi}\right)\, dt \Big| \to 0,
\end{aligned}
$$

from (B.149), (B.152), (B.157) and the boundedness of $y \cdot \boldsymbol{\psi}$, $y^2 \cdot \nabla y \cdot \boldsymbol{\psi}$ and $y^3 \cdot \boldsymbol{\psi}$ in $L^2(L^2)$, we realize

$$
\begin{aligned}
&\Big| \int_0^T \left(y, \nabla \left[-\varepsilon^2 \Delta d_y^m - d_y^m + 3\, y^2\, d_y^m\right] \cdot \boldsymbol{\psi}^m\right)\, dt \\
&\quad - \int_0^T \left(y, \nabla \left[-\varepsilon^2 \Delta d_y - d_y + 3\, y^2\, d_y\right] \cdot \boldsymbol{\psi}\right)\, dt \Big| \\
&\leq \Big| \int_0^T \left(y, \nabla \left[-\varepsilon^2 \Delta d_y^m - d_y^m + 3\, y^2\, d_y^m\right] \cdot \left[\boldsymbol{\psi}^m - \boldsymbol{\psi}\right]\right)\, dt \\
&\quad + \int_0^T \left(y, \nabla \left[-\varepsilon^2 \Delta \left(d_y^m - d_y\right) - \left(d_y^m - d_y\right) + 3\, y^2\, \left(d_y^m - d_y\right)\right] \cdot \boldsymbol{\psi}\right)\, dt \Big| \\
&\leq C\, \|y\|_{L^\infty(H^2)}\, \left[\|\Delta d_y^m\|_{L^2(H^1)} + \|d_y^m\|_{L^2(H_0)} \left(1 + \|y\|^2_{L^\infty(H^2)}\right)\right] \|\boldsymbol{\psi}^m - \boldsymbol{\psi}\|_{\mathbf{L}^2(\boldsymbol{\mathcal{D}})} \\
&\quad + \varepsilon^2 \Big| \int_0^T \left(y, \nabla \Delta \left[d_y^m - d_y\right] \cdot \boldsymbol{\psi}\right)\, dt \Big| + \Big| \int_0^T \left(y, \nabla \left[d_y^m - d_y\right] \cdot \boldsymbol{\psi}\right)\, dt \Big| \\
&\quad + 6 \Big| \int_0^T \left(y^2\, \left[d_y^m - d_y\right], \nabla y \cdot \boldsymbol{\psi}\right)\, dt \Big| + 3 \Big| \int_0^T \left(y^3, \nabla \left[d_y^m - d_y\right] \cdot \boldsymbol{\psi}\right)\, dt \Big| \to 0,
\end{aligned}
$$

by (B.149) and (B.158), we have

$$
\Big| \int_0^T \left(d_{yt}^m, \eta^m\right)\, dt - \int_0^T \langle d_{yt}, \eta \rangle_{H_0^*, H_0}\, dt \Big| \to 0,
$$

using (B.149), (B.152), (B.158) and the boundedness of $y \cdot \nabla y \cdot \nabla \eta$ in $L^2(H^{1*})$ and $y^2 \cdot \nabla \eta$ in $L^2(L^2)$, we get

$$
\begin{aligned}
&\Big| \int_0^T \left(\nabla \left[-\varepsilon^2 \Delta d_y^m - d_y^m + 3\, y^2\, d_y^m\right], \nabla \eta^m\right)\, dt \\
&\quad - \int_0^T \left(\nabla \left[-\varepsilon^2 \Delta d_y - d_y + 3\, y^2\, d_y\right], \nabla \eta\right)\, dt \Big| \\
&= \Big| \int_0^T \left(\nabla \left[-\varepsilon^2 \Delta d_y^m - d_y^m + 3\, y^2\, d_y^m\right], \nabla \left[\eta^m - \eta\right]\right)\, dt \\
&\quad + \int_0^T \left(\nabla \left[-\varepsilon^2 \Delta \left(d_y^m - d_y\right) - \left(d_y^m - d_y\right) + 3\, y^2\, \left(d_y^m - d_y\right)\right], \nabla \eta\right)\, dt \Big|
\end{aligned}
$$

$$\leq C \left[\|\Delta d_y^m\|_{L^2(H^1)} + \|d_y^m\|_{L^2(H_0)} \left(1 + \|y\|^2_{L^\infty(H^2)}\right)\right] \|\eta^m - \eta\|_{L^2(H_0)}$$
$$+\varepsilon^2 \left| \int_0^T \left(\nabla\Delta \left[d_y^m - d_y\right], \nabla\eta\right) \, dt \right| + \left| \int_0^T \left(\nabla \left[d_y^m - d_y\right], \nabla\eta\right) \, dt \right|$$
$$+6 \left| \int_0^T \left(y \, \left[d_y^m - d_y\right], \nabla y \cdot \nabla\eta\right) \, dt \right| + 3 \left| \int_0^T \left(y^2, \nabla \left[d_y^m - d_y\right] \cdot \nabla\eta\right) \, dt \right| \to 0,$$

from (B.149), (B.151) and (B.158), we can write

$$\left| \int_0^T \left(d_y^m, \mathbf{v} \cdot \nabla\eta^m\right) \, dt - \int_0^T \left(d_y, \mathbf{v} \cdot \nabla\eta\right) \, dt \right|$$
$$= \left| \int_0^T \left(d_y^m, \mathbf{v} \cdot \nabla\left[\eta^m - \eta\right]\right) \, dt + \int_0^T \left(d_y^m - d_y, \mathbf{v} \cdot \nabla\eta\right) \, dt \right|$$
$$\leq \|d_y^m\|_{L^2(H_0)} \, \|\mathbf{v}\|_{\mathbf{L}^\infty(\boldsymbol{\mathcal{D}})} \, \|\eta^m - \eta\|_{L^2(H_0)} + \|d_y^m - d_y\|_{L^2(L^2)} \, \|\mathbf{v}\|_{\mathbf{L}^\infty(\mathbf{H}^2)} \, \|\eta\|_{L^2(H_0)} \to 0,$$

by (B.148), (B.158) and the boundedness of $y \cdot \nabla\eta$ in $L^2\left(\mathbf{L}^2\right)$, we derive

$$\left| \int_0^T \left(y, \mathbf{d}_\mathbf{v}^m \cdot \nabla\eta^m\right) \, dt - \int_0^T \left(y, \mathbf{d}_\mathbf{v} \cdot \nabla\eta\right) \, dt \right|$$
$$= \left| \int_0^T \left(y, \mathbf{d}_\mathbf{v}^m \cdot \nabla\left[\eta^m - \eta\right]\right) \, dt + \int_0^T \left(y, \left[\mathbf{d}_\mathbf{v}^m - \mathbf{d}_\mathbf{v}\right] \cdot \nabla\eta\right) \, dt \right|$$
$$\leq \|y\|_{L^\infty(H^2)} \, \|\mathbf{d}_\mathbf{v}^m\|_{\mathbf{L}^2(\mathbf{L}^2)} \, \|\eta^m - \eta\|_{L^2(H_0)} + \left| \int_0^T \left(y, \left[\mathbf{d}_\mathbf{v}^m - \mathbf{d}_\mathbf{v}\right] \cdot \nabla\eta\right) \, dt \right| \to 0.$$

So, we can claim that $(\mathbf{d}_\mathbf{v}, d_y) \in \mathbf{W}_0 \times \left(W_0 \cap L^\infty(H_0) \cap L^2(H_\Delta)\right)$ satisfies the equations (B.125) and (B.127), for all $\boldsymbol{\psi} \in \mathcal{C}_c^\infty\left((0,T); \boldsymbol{\mathcal{D}}\right)$, $\eta \in \mathcal{C}_c^\infty\left((0,T); H_0\right)$. Then, from the dense embeddings

$$\mathcal{C}_c^\infty\left((0,T); H_0\right) \hookrightarrow L^2\left(H_0\right),$$
$$\mathcal{C}_c^\infty\left((0,T); \boldsymbol{\mathcal{D}}\right) \hookrightarrow L^2\left(\boldsymbol{\mathcal{D}}\right),$$

we infer that $(\mathbf{d}_\mathbf{v}, d_y)$ satisfies the equations (B.125) and (B.127), for all $\boldsymbol{\psi} \in L^2\left(\boldsymbol{\mathcal{D}}\right)$, $\eta \in L^2\left(H_0\right)$. Concerning the initial conditions (B.126), (B.128), considering $\boldsymbol{\psi} = \boldsymbol{\xi}\left(1 - t/T\right), \ \boldsymbol{\xi} \in \boldsymbol{\mathcal{D}}$ and $\eta = \zeta\left(1 - t/T\right), \ \zeta \in H_0$, we note

$$\left(\mathbf{d}_\mathbf{v}^m(0) - \mathbf{d}_\mathbf{v}(0), \boldsymbol{\xi}\right) = \int_0^T \left[-\left(\mathbf{d}_{\mathbf{v}t}^m, \boldsymbol{\psi}\right) + \langle \mathbf{d}_{\mathbf{v}t}, \boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}} - \langle \boldsymbol{\psi}_t, \mathbf{d}_\mathbf{v}^m - \mathbf{d}_\mathbf{v}\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}}\right] \, dt \to 0,$$
$$\left(d_y^m(0) - d_y(0), \zeta\right) = \int_0^T \left[-\left(d_{yt}^m, \eta\right) + \langle d_{yt}, \eta\rangle_{H^{1*},H^1} - \langle \eta_t, d_y^m - d_y\rangle_{H^{1*},H^1}\right] \, dt \to 0,$$

as $m \to +\infty$, for all $\boldsymbol{\xi} \in \boldsymbol{\mathcal{D}}, \ \zeta \in H_0$. Furthermore

$$\mathbf{d}_\mathbf{v}^m(0) = \mathbf{P}^m \mathbf{z}_3 \to \mathbf{z}_3, \qquad \text{in } \boldsymbol{\mathcal{S}},$$
$$d_y^m(0) = P^m z_4 \to z_4, \qquad \text{in } L_0^2.$$

Then, we can conclude $\mathbf{d}_\mathbf{v}(0) = \mathbf{z}_3, \ d_y(0) = z_4$. It remains to show that the solution $(\mathbf{d}_\mathbf{v}, d_y)$ of equations (B.125)-(B.128) is unique. Let us assume that $(\mathbf{d}_{\mathbf{v}1}, d_{y1})$,

$(\mathbf{d}_{\mathbf{v}2}, d_{y2})$ are two solutions of (B.125)-(B.128). Then, $(\mathbf{h}_\mathbf{v}, h_y) = (\mathbf{d}_{\mathbf{v}2} - \mathbf{d}_{\mathbf{v}1}, d_{y2} - d_{y1})$ satisfies

$$\text{(B.159)} \qquad \int_0^T [\langle \mathbf{h}_{\mathbf{v}t}, \boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}^*,\boldsymbol{\mathcal{D}}} + \nu\left(\nabla \mathbf{h}_\mathbf{v}, \nabla \boldsymbol{\psi}\right) + B\left(\mathbf{h}_\mathbf{v}, \mathbf{v}, \boldsymbol{\psi}\right) + B\left(\mathbf{v}, \mathbf{h}_\mathbf{v}, \boldsymbol{\psi}\right)$$
$$+ \rho\left(h_y, \nabla\left[-\varepsilon^2 \Delta y - y + y^3\right] \cdot \boldsymbol{\psi}\right) + \rho\left(y, \nabla\left[-\varepsilon^2 \Delta h_y - h_y + 3\, y^2\, h_y\right] \cdot \boldsymbol{\psi}\right)\Big]\ dt = 0,$$
$$\text{(B.160)} \qquad \mathbf{h}_\mathbf{v}(0) = 0,$$
$$\text{(B.161)} \qquad \int_0^T \left[\langle h_{yt}, \eta\rangle_{H_0^*,H_0} + \gamma\left(\nabla\left[-\varepsilon^2 \Delta h_y - h_y + 3y^2 h_y\right], \nabla \eta\right)\right.$$
$$\left. - \left(h_y, \mathbf{v} \cdot \nabla \eta\right) - \left(y, \mathbf{h}_\mathbf{v} \cdot \nabla \eta\right)\right]\ dt = 0,$$
$$\text{(B.162)} \qquad h_y\left(0\right) = 0,$$

for all $\boldsymbol{\psi} \in L^2\left(\boldsymbol{\mathcal{D}}\right), \eta \in L^2\left(H_0\right)$. We set $\boldsymbol{\psi} = \chi_{[0,t]}\ \mathbf{h}_\mathbf{v}$ in (B.159) and $\eta = \chi_{[0,t]}\ h_y$, with $0 < t \leq T$, where

$$\chi_{[0,t]}\left(s\right) = \begin{cases} 1 & \text{if} \quad s \in [0,t]\,, \\ 0 & \text{otherwise} \end{cases}$$

Thus, using Young's inequality, we realize

$$\text{(B.163)} \qquad \int_0^t \left[\frac{1}{2}\frac{d}{ds}\|\mathbf{h}_\mathbf{v}\|^2 + \nu\|\nabla \mathbf{h}_\mathbf{v}\|^2\right]\ ds$$
$$= -\int_0^t \Big[B\left(\mathbf{h}_\mathbf{v}, \mathbf{v}, \mathbf{h}_\mathbf{v}\right)$$
$$+\rho\left(h_y, \nabla w \cdot \mathbf{h}_\mathbf{v}\right) + \rho\left(y, \nabla\left[-\varepsilon^2 \Delta h_y - h_y + 3\, y^2\, h_y\right] \cdot \mathbf{h}_\mathbf{v}\right)\Big]\ ds$$
$$\leq \int_0^t \Big[\|\mathbf{h}_\mathbf{v}\|\ \|\nabla \mathbf{v}\|_{\mathbf{L}^4}\ \|\mathbf{h}_\mathbf{v}\|_{\mathbf{L}^4} + \|h_y\|_{L^4}\ \|\nabla w\|_{L^4}\ \|\mathbf{h}_\mathbf{v}\| + \|y\|_{\mathcal{C}(\bar\Omega)}$$
$$\times\Big(\|\nabla \Delta h_y\| + \|\nabla h_y\| + \|y\|_{\mathcal{C}(\bar\Omega)}\ \|h_y\|_{L^4}\ \|\nabla y\|_{L^4} + \|y\|^2_{\mathcal{C}(\bar\Omega)}\ \|\nabla h_y\|\Big)\|\mathbf{h}_\mathbf{v}\|\Big]\ ds$$
$$\leq \int_0^t \Big[\sigma\|\nabla \mathbf{h}_\mathbf{v}\|^2 + \sigma\|\nabla \Delta h_y\|^2 + C_1\ \|\nabla h_y\|^2 +$$
$$+\Big(C_2\left(\sigma\right)\ \|\mathbf{v}\|^2_{\mathbf{H}^2} + C_3\ \|w\|^2_{H^2} + C_4\left(\sigma\right)\ \|y\|^2_{H^2} + C_5\ \|y\|^6_{H^2}\Big)\|\mathbf{h}_\mathbf{v}\|^2\Big]\ ds,$$

$$\text{(B.164)} \qquad \int_0^t \left[\frac{1}{2}\frac{d}{ds}\|\nabla h_y\|^2 + \gamma\ \varepsilon^2\|\nabla \Delta h_y\|^2\right]\ ds$$
$$= -\int_0^t \Big[\gamma\left(\nabla h_y, \nabla \Delta h_y\right) - 6\ \gamma\left(y\ h_y, \nabla y \cdot \nabla \Delta h_y\right) - 3\ \gamma\left(y^2, \nabla h_y \cdot \nabla \Delta h_y\right)$$
$$+ \left(h_y, \mathbf{v} \cdot \nabla \Delta h_y\right) + \left(y, \mathbf{h}_\mathbf{v} \cdot \nabla \Delta h_y\right)\Big]\ ds$$
$$\leq \int_0^t \Big[\gamma\ \|\nabla h_y\|\ \|\nabla \Delta h_y\| + 6\ \gamma\ \|y\|_{\mathcal{C}(\bar\Omega)}\ \|h_y\|_{L^4}\ \|\nabla y\|_{L^4}\ \|\nabla \Delta h_y\|$$
$$+3\ \gamma\ \|y\|^2_{\mathcal{C}(\bar\Omega)}\ \|\nabla h_y\|\ \|\nabla \Delta h_y\| + \|h_y\|\ \|\mathbf{v}\|_{\boldsymbol{\mathcal{C}}(\bar\Omega)}\ \|\nabla \Delta h_y\|$$
$$+\|y\|_{\mathcal{C}(\bar\Omega)}\ \|\mathbf{h}_\mathbf{v}\|\ \|\nabla \Delta h_y\|\Big]\ ds$$

$$\leq \int_0^t \Big[5\ \sigma\ \|\nabla \Delta h_y\|^2 + C_2(\sigma)\ \|y\|_{H^2}^2\ \|\mathbf{h_v}\|^2$$
$$+ \Big(C_3(\sigma) + C_4(\sigma)\ \|y\|_{H^2}^4 + C_5(\sigma)\ \|\mathbf{v}\|_{\mathbf{H}^2}^2 \Big) \|\nabla h_y\|^2 \Big]\ ds,$$

Summing (B.163) and (B.164) and multiplying by two, we have

$$\int_0^t \left[\frac{d}{ds} \Big(\|\mathbf{h_v}\|^2 + \|\nabla h_y\|^2 \Big) + 2\ \nu \|\nabla \mathbf{h_v}\|^2 + 2\ \gamma\ \varepsilon^2 \|\nabla \Delta h_y\|^2 \right] ds \tag{B.165}$$
$$\leq \int_0^t \Big[2\ \sigma\ \|\nabla \mathbf{h_v}\|^2 + 12\ \sigma\ \|\nabla \Delta h_y\|^2$$
$$+ \Big(C_1(\sigma)\ \|\mathbf{v}\|_{\mathbf{H}^2}^2 + C_2\ \|w\|_{H^2}^2 + C_3(\sigma)\ \|y\|_{H^2}^2 + C_4\ \|y\|_{H^2}^6 \Big) \|\mathbf{h_v}\|^2$$
$$+ \Big(C_5 + C_6(\sigma) + C_7(\sigma)\ \|y\|_{H^2}^4 + C_8(\sigma)\ \|\mathbf{v}\|_{\mathbf{H}^2}^2 \Big) \|\nabla h_y\|^2 \Big]\ ds.$$

Choosing in (B.165) $\sigma < \nu$ and $6\ \sigma < \gamma\ \varepsilon^2$, we get

$$\|\mathbf{h_v}(t)\|^2 + \|\nabla h_y(t)\|^2 \leq \|\mathbf{h_v}(0)\|^2 + \|\nabla h_y(0)\|^2$$
$$+ C \int_0^t \Big[1 + \|w\|_{H^2}^2 + \|\mathbf{v}\|_{\mathbf{H}^2}^2 + \|y\|_{H^2}^2 + \|y\|_{H^2}^4 + \|y\|_{H^2}^6 \Big] \Big(\|\mathbf{h_v}\|^2 + \|\nabla h_y\|^2 \Big)\ ds,$$

which implies, applying Gronwall's lemma,

$$\|\mathbf{h_v}(t)\|^2 + \|\nabla h_y(t)\|^2 \leq \Big[\|\mathbf{h_v}(0)\|^2 + \|\nabla h_y(0)\|^2 \Big]$$
$$\times \exp\left(C \int_0^t \Big[1 + \|w\|_{H^2}^2 + \|\mathbf{v}\|_{\mathbf{H}^2}^2 + \|y\|_{H^2}^2 + \|y\|_{H^2}^4 + \|y\|_{H^2}^6 \Big]\ ds \right).$$

Then from the initial conditions (B.160), (B.162), we can claim $\mathbf{h_v} = 0,\ h_y = 0$. So, we have shown that given

$$\mathbf{z} = (\mathbf{z}_1, z_2, \mathbf{z}_3, z_4) \in \mathbf{Z} = L^2(\boldsymbol{\mathcal{D}}^*) \times L^2(H_0^*) \times \boldsymbol{\mathcal{S}} \times H_0,$$

the system of PDEs (B.125)-(B.128) has a unique solution

$$\mathbf{d_x} = (\mathbf{d_v}, d_y) \in \mathbf{X} = \mathbf{W}_0 \times \big[W_0 \cap L^\infty(H_0) \cap L^2(H_\Delta) \big].$$

Furthermore, from the estimates (B.141), (B.144) and (B.147), we have derived

$$\|\mathbf{d_v}\|_{\mathbf{W}_0}^2 + \|d_y\|_{W_0}^2 + \|d_y\|_{L^\infty(H_0)}^2 + \|d_y\|_{L^2(H_\Delta)}^2$$
$$\leq C(\mathbf{u}) \Big[\|\mathbf{z}_1\|_{L^2(\boldsymbol{\mathcal{D}}^*)}^2 + \|z_2\|_{L^2(H_0^*)}^2 + \|\mathbf{z}_3\|_{\boldsymbol{\mathcal{S}}}^2 + \|z_4\|_{H_0}^2 \Big].$$

Hence the equation (B.123) and the estimate (B.124) are satisfied. This conclude the proof. □

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