STRATIFIED EXPONENTIAL INTEGRATOR FOR MODULATED NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. We consider a d-dimensional nonlinear Schrödinger equation with dispersion modulated by a (formal) derivative of a α-Hölder continuous time-dependent function. We derive an exponential integrator based on a stratified Monte-Carlo approximation and show that it converges with order $\alpha + 1/2$. The theoretical results are underlined by numerical simulations where the modulation is given by a trajectory of a fractional Brownian motion with various Hurst parameters $H \in (0,1)$.

1. INTRODUCTION

We study the modulated nonlinear Schrödinger equation of the form

$$(1) \quad i\partial_t u(t,x) + \Delta u(t,x)\partial_t W(t) = |u(t,x)|^2 u(t,x), \quad u(0,x) = u_0(x), \quad (t,x) \in \mathbb{R} \times \mathbb{T}^d,$$

where $W : \mathbb{R} \to \mathbb{R}$ is an arbitrary α-Hölder continuous function of time, $\alpha \in (0,1]$, independent of the space variable. For practical implementation issues we impose periodic boundary conditions, i.e, $x \in \mathbb{T}^d = [0, 2\pi]^d$, $d \in \mathbb{N}$. For notational simplicity we work with cubic nonlinearities. However, the generalization to polynomial nonlinearities $f(u) = |u|^{2p}u$ is straightforward.

Classical semilinear Schrödinger equations with $W(t) = t$ are nowadays extensively studied analytically as well as numerically. In this context, splitting methods (where the right-hand side is split into the kinetic and nonlinear part, respectively) as well as exponential integrators (including Lawson-type Runge–Kutta methods) contribute particularly attractive classes of integration schemes. For an extensive overview on splitting and exponential integration methods we refer to [15, 16, 17, 26], and for their rigorous convergence analysis in the context of semilinear Schrödinger equations we refer to [3, 5, 8, 11, 13, 14, 24] and the references therein.

The situation of a modulated dispersion $\partial_t W(t)\Delta$ is in contrast much more involved as the modulation W is generally only continuous in time. Due to this loss of smoothness classical integration schemes suffer from a severe order reduction as the convergence rate in time heavily depends on the smoothness of W. For an illustration of this phenomenon see also Section 4.

The modulated nonlinear Schrödinger equation under various assumptions on the modulation W has been object of interest in a number of works, see for instance [1, 6, 9, 10, 18, 19, 21, 28] and the references therein. The equation stems as a model describing propagation of light waves in optical dispersion-managed fibers. Numerical analysis of this model in the case of W being a Brownian motion can be found in [2, 7, 25]. In [2], a semi-discrete Crank-Nicolson scheme is analyzed and proved that the strong order of convergence in probability in H^σ, $\sigma > d/2$, is equal to one for initial conditions in H^σ. In [7] an explicit exponential integrator is proposed and it is shown that it is mean-square order 1 in H^σ for initial conditions in $H^{\sigma+4}$.

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In the present paper we put forward an exponential integrator for (1) based on a stratified Monte-Carlo approximation. More precisely, we consider the mild formulation of (1) and approximate the convolution integral appearing on the right hand side by means of a randomized Riemann sum. The key idea is to choose the randomization in such a way, that the associated error is a (discreet time) martingale, which permits to apply the so-called Burkholder-Davis-Gundy inequality (see Theorem 3.4). Remarkably, this allows to gain half in the convergence rate in expectation. More precisely, we establish the following bound for the difference of the exact solution $u(t_n)$ and the numerical solution u^n valid for initial conditions in $H^{\sigma+2}$ (see Theorem 3.1 for details):

$$\left(\mathbb{E}_\xi \max_{n=0,\ldots,N} \|u(t_n) - u^n\|_\sigma^2 \right)^{1/2} \leq c(\tau + \tau^{\alpha+\frac{1}{2}}),$$

where $\mathbb{E}_\xi$ is the expected value associated to the randomization of the numerical scheme and α is the exponent of Hölder continuity of W.

We stress that the expectation above is taken only with respect to the randomization of the scheme, whereas the modulation W is deterministic. In other words, in the case of a random modulation, such as the Brownian motion treated in [2, 7], our result provides a *pathwise* error estimates leading to a *pathwise* convergence analysis. In addition, it applies to a wide range of possible modulations, deterministic or random, and in particular to any other stochastic process with Hölder continuous trajectories.

Recall that in both works [2, 7], a stochastic approach was used and it was only possible to obtain the order of convergence 1 in probability which reads as

$$\lim_{C \to \infty} \mathbb{P}_W \big(\|u(t_n) - u^n\|_\sigma > C\tau \big) = 0,$$

for all $n = 0, \ldots, N$ uniformly with respect to τ. Here we write $\mathbb{P}_W$ to denote the probability measure on the probability space where W is defined (in contrast to the notation $\mathbb{E}_\xi$ above which only concerns the artificial randomness introduced in our scheme). It was conjectured in [2] that a stronger result should hold, namely,

$$\lim_{C \to \infty} \mathbb{P}_W \Big(\max_{n=0,\ldots,N} \|u(t_n) - u^n\|_\sigma > C\tau \Big) = 0,$$

uniformly with respect to τ. However, the proof would require a very tedious computations which were not presented. Our approach allows to overcome this challenge and the convergence analysis is rather elegant and simple.

Our exponential integrator is derived in Section 2 and Section 3 is devoted to the the error analysis. In Section 4 we present numerical experiments for $W = W_H$ being a trajectory of a fractional Brownian motion with various values of the Hurst parameter $H \in (0, 1)$. The case of $H = 1/2$ corresponds to the Brownian motion and, in general, W^H is a zero-mean Gaussian process with covariance

$$\mathbb{E}\left[W_H(t)W_H(s)\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right).$$

It is known that W_H has almost surely α-Hölder continuous trajectories for $\alpha < H$. Our experiments confirm the theoretically obtained order of convergence also for low Hurst parameters.

For completeness, note that various randomized numerical schemes in various settings have already appeared in the literature. Let us particularly mention [12, 20, 22, 23] which inspired our research and where further references can be found. To the best of our knowledge such a method was not yet applied in the context of dispersive equations.

In the following let $\sigma > d/2$. We denote by $\|\cdot\|_\sigma$ the standard $H^\sigma = H^\sigma(\mathbb{T}^d)$ Sobolev norm, where we in particular exploit the well-known bilinear estimate

$$(2) \qquad \|fg\|_\sigma \leq c_\sigma \|f\|_\sigma \|g\|_\sigma$$

which holds for some constant $c_\sigma > 0$.

2. Derivation of the numerical scheme

It is possible to make sense of (1) using the associated mild formulation. To this end, let us set

$$S(t) = S^W(t) = e^{iW(t)\partial_x^2}, \quad t \in \mathbb{R},$$

as well as

$$U(t, r) = S(t)S(r)^{-1} = e^{i[W(t)-W(r)]\partial_x^2}$$

such that in particular $U(t, 0) = S(t)$. Note that due to the presence of the modulation W, the problem (1) is not time-homogeneous. Consequently, the linear part of (1) generates an evolution system given by the family of operators $U(t, r)$, $r, t \in \mathbb{R}$, which are (generally) not functions of the difference $t - r$ as it would be case in the classical setting, that is, $W(t) = t$. Intuitively, $U(t, r)$ describes the evolution of the linear part of (1) from time r to time t. Note that since the modulation W does not depend on the space variable x, the above operators are Fourier multipliers given by

$$\mathcal{F}[S(t)f](k) = e^{-iW(t)k^2}\hat{f}_k, \qquad \mathcal{F}[U(t, r)f](k) = e^{-i[W(t)-W(r)]k^2}\hat{f}_k$$

for $f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}_k e^{ik \cdot x} \in L^2$.

In the construction and analysis of our numerical scheme we will in particular exploit that the operators $U(t, r)$ and $S(t)$ are linear isometries in H^σ for all $r, t \in \mathbb{R}$:

Lemma 2.1. *For all $f \in H^\sigma$ and $r, t \in \mathbb{R}$ we have that*

$$(3) \qquad \|U(t, r)f\|_\sigma = \|f\|_\sigma, \quad \|S(t)f\|_\sigma = \|f\|_\sigma.$$

Proof. The assertion follows by the definition of the H^σ norm together with the relation

$$e^{iW(t)\partial_x^2}f(x) = \sum_{k \in \mathbb{Z}^d} e^{-iW(t)|k|^2}\hat{f}_k e^{ik \cdot x}$$

which holds for all $f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}_k e^{ik \cdot x} \in L^2$. □

Accordingly, a solution to (1) satisfies

$$(4) \qquad u(t) = U(t, 0)u_0 - i \int_0^t U(t, r)[|u(r)|^2 u(r)]dr.$$

A well-posedness theory of (1) under rather general assumptions on W was developed in [6]. Remarkably, it was shown that under certain hypothesis of "irregularity" (the so-called (ρ, γ)-irregularity, see [6, Definition 1.2]) of the perturbation W, (1) can be solved in the usual scale of H^s spaces. Without going into details, let us only point out that the theory of [6] covers a wide class of modulations W and in particular also the fractional Brownian motion. The main well-posedness result [6, Theorem 1.8] then states that the modulated NLS (1) in $d = 1$ possesses a global solution in H^s, for $s \geq 0$, and uniqueness holds in a smaller class.

Throughout the remainder of the paper, we assume that u is a unique solution to (1) which belongs to $C([0, T]; H^{\sigma+2})$ for $\sigma > d/2$.

Using the flow property $U(t, \xi) = U(t, r)U(r, \xi)$ valid for all $\xi \leq r \leq t$, we deduce that

$$u(t) = U(t, r)u(r) - i \int_r^t U(t, \xi)[|u(\xi)|^2 u(\xi)]d\xi.$$

Let us now split $[0, T]$ into equidistant partition $t_0, \ldots, t_N$, with the mesh size $\tau = \frac{T}{N}$ which yields that

$$(5) \qquad u(t_{n+1}) = U(t_n + \tau, t_n)u(t_n) - i \int_0^\tau U(t_n + \tau, t_n + r) \left[|u(t_n + r)|^2 u(t_n + r) \right] dr.$$

In order to numerically approximate the above time integral, we first use the approximation

$$u(t_n + r) \approx U(t_n + r, t_n)u(t_n)$$

which is of order r in H^σ for solutions $u \in H^\sigma$:

Lemma 2.2. *Let u be a mild solution to* (1). *Then for all $0 \leq t \leq T$ we have*

$$\|u(t + r) - U(t + r, t)u(t)\|_\sigma \leq cr,$$

where the constant c depends on $\sup_{0 \leq t \leq T} \|u(t)\|_\sigma$, but can be chosen uniformly in r.

Proof. Note that the mild formulation (5) yields that

$$u(t_n + r) = U(t_n + r, t_n)u(t_n) - i \int_0^r U(t_n + r, t_n + \xi) \left[|u(t_n + \xi)|^2 u(t_n + \xi) \right] d\xi.$$

Together with the bilinear estimate (2) this implies that

$$\|u(t_n + r) - U(t_n + r, t)u(t_n)\|_\sigma \leq c \int_0^r \|u(t_n + \xi)\|_\sigma^3 d\xi$$

which yields the assertion. $\qquad \square$

The above Lemma allows us to define the approximation

$$(6)$$
$$u_*(t_{n+1}) := U(t_n + \tau, t_n)u(t_n) - i \int_0^\tau U(t_n + \tau, t_n + r)f\left(U(t_n + r, t_n)u(t_n)\right) dr, \quad f(v) = |v|^2 v,$$

which approximates the exact solution $u(t_{n+1})$ with order τ^2:

Corollary 2.3. *Let u be a mild solution to* (1). *Then the following approximation holds*

$$(7) \qquad\qquad\qquad \|u(t_{n+1}) - u_*(t_{n+1})\|_\sigma \leq c\tau^2$$

for some constant c depending on $\sup_{0 \leq t \leq T} \|u(t)\|_\sigma$, but which is independent of τ.

Proof. Taking the difference of the exact solution (5) and the approximation (6) yields with $f(v) = |v|^2 v$ that

$$\|u(t_{n+1}) - u_*(t_{n+1})\|_\sigma \leq \int_0^\tau \left\| U(t_n + \tau, t_n + r) \left[f\left(U(t_n + r, t_n)u(t_n)\right) - f\left(u(t_n + r)\right) \right] \right\|_\sigma dr.$$

The assertion thus follows by Lemma 2.2. $\qquad \square$

Thanks to Corollary 2.3 it remains to find a suitable numerical approximation of

$$u_*(t_{n+1}) = U(t_n + \tau, t_n)u(t_n) - i \int_0^\tau U(t_n + \tau, t_n + r)f\left(U(t_n + r, t_n)u(t_n)\right) dr.$$

To this end, we consider a stratified Monte Carlo approximation of the resulting time integral. More precisely, let $(\xi_n)_{n \in \mathbb{N}_0}$ be a sequence of independent identically distributed random variables having the uniform distribution on $[0, 1]$. We assume that the sequence $(\xi_n)_{n \in \mathbb{N}_0}$ is defined on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we denoted by $\mathbb{E}$ the associated expected value. That is, for every $n \in \mathbb{N}_0$, the mapping $\xi : (\Omega, \mathcal{F}) \to ([0, 1], \mathcal{B}([0, 1]))$ is measurable and for a measurable function $F : \mathbb{R} \to \mathbb{R}$ it holds

$$\mathbb{E}F(\xi_n) = \int_\Omega F(\xi_n(\omega)) \, d\mathbb{P}(\omega) = \int_0^1 F(\xi) \, d\xi,$$

where the last equality follows by a change of variables from the fact that ξ_n is uniformly distributed in $[0, 1]$.

Then we approximate as follows

$$(8) \quad \int_0^\tau U(t_n + \tau, t_n + r) \left[|U(t_n + r, t_n)u(t_n)|^2 U(t_n + r, t_n)u(t_n) \right] dr$$
$$\approx \tau U(t_n + \tau, t_n + \tau\xi_n) \left[|U(t_n + \tau\xi_n, t_n)u(t_n)|^2 U(t_n + \tau\xi_n, t_n)u(t_n) \right].$$

Plugging the above approximation into (6) motivates us to define the *stratified exponential integrator*

$$(9) \quad u^{n+1} = U(t_n + \tau, t_n)u^n - i\tau U(t_n + \tau, t_n + \tau\xi_n) \left[|U(t_n + \tau\xi_n, t_n)u^n|^2 U(t_n + \tau\xi_n, t_n)u^n \right]$$

approximating the solution $u(t)$ of the modulated nonlinear Schrödinger equation (1) at time $t = t_{n+1}$.

3. Convergence analysis

In this section we carry out the convergence analysis of the stratified exponential integrator (15). Our main result reads as follows.

Theorem 3.1. *Let $W \in C^\alpha([0, T])$ for some $\alpha \in (0, 1]$. Then there exits a $\tau_0 > 0$ such that for all $\tau \leq \tau_0$ it holds true that for all $t_N \leq T$*

$$\left(\mathbb{E} \max_{M=0,\dots,N} \left\| u(t_M) - u^M \right\|_\sigma^2 \right)^{1/2} \lesssim \tau + \tau^{\alpha + \frac{1}{2}},$$

where the proportional constant depends on T and $\sup_{0 \leq t \leq T} \|u(t)\|_{\sigma+2}$, but is independent of τ and N.

Remark 3.2 (Classical order of convergence). Let $W \in C^\alpha([0, T])$ for some $\alpha \in (0, 1]$. With classical techniques we can readily derive the classical order of convergence for the stratified exponential integrator (9) approximating solutions of the modulated nonlinear Schrödinger equation (1): Taylor series expansion implies that for all $0 \leq \xi \leq 1$ and $0 \leq r \leq \tau$ it holds that

$$\left\| \left(S(t_n + r) - S(t_n + \xi\tau) \right) f \right\|_\sigma \leq \left| W(t_n + r) - W(t_n + \xi\tau) \right| \|f\|_{\sigma+2} \leq c\tau^\alpha \|f\|_{\sigma+2}.$$

Applying the above estimate in (8) we observe together with Corollary 2.3 that the stratified exponential integrator (9) introduces a local error of order

$$(10) \qquad\qquad\qquad \min\left(\tau^{1+\alpha}, \tau^2 \right) = \tau^{1+\alpha}$$

where the last equality follows as $\alpha \leq 1$. The isometric property in Lemma 2.1 together with the bilinear estimate (2) furthermore allows the stability estimate

$$(11) \qquad\qquad\qquad \|u^{n+1}\|_\sigma \leq \|u^n\|_\sigma + c\tau \|u^n\|_\sigma$$

where the constant c depends on $\|u^n\|_\sigma^2$, but can be chosen independently of τ and ξ_n. Thanks to a Lady Windermere's fan argument (see [15]) we obtain by the local error (10) together with the stability estimate (11) that there exists a $\tau_0 > 0$ such that for all $\tau \le \tau_0$ the global error bound holds

$$(12) \qquad \|u(t_{n+1}) - u^{n+1}\|_\sigma \le c\tau^\alpha \quad \text{with} \quad c = c\left(\sup_{0 \le t \le t_n} \|u(t)\|_{\sigma+2}\right).$$

Remark 3.3 (Apriori bounds). Let $W \in C^\alpha([0,T])$ with $\alpha \in (0,1]$. The classical global error bound (12) in particular implies the apriori boundedness of $\|u^n\|_\sigma$ for all $(\xi_n)_{n \in \mathbb{N}_0}$ in $[0,1]$ and solutions $u \in H^{\sigma+2}$.

In Proposition 3.5 below we will present the essential estimate needed for the proof of the main error result Theorem 3.1. It is based on the following discreet version of the Burkholder-Davis-Gundy inequality (see [4]).

Theorem 3.4. *For each $p \in (0,\infty)$ there exist positive constants c_p and C_p such that for every discrete time $\mathbb{R}^d$-valued martingale $\{X_n; n \in \mathbb{N}_0\}$ and for every $n \in \mathbb{N}_0$ we have*

$$c_p \mathbb{E}\langle X \rangle_n^{p/2} \le \mathbb{E} \max_{k=1,\ldots,n} |X_k|^p \le C_p \mathbb{E}\langle X \rangle_n^{p/2},$$

where $\langle X \rangle_n = |X_0|^2 + \sum_{k=1}^n |X_k - X_{k-1}|^2$ is the quadratic variation of $\{X_n; n \in \mathbb{N}_0\}$.

Let us introduce the twisted variable

$$(13) \qquad v(t) := S(t)^{-1}u(t), \qquad v_*(t) := S(t)^{-1}u_*(t).$$

In terms of this new variable, the approximation u_* is given by

$$(14) \qquad v_*(t_{n+1}) = v(t_n) - i\int_0^\tau S(t_n+r)^{-1}\left[|S(t_n+r)v(t_n)|^2 S(t_n+r)v(t_n)\right]dr,$$

whereas the numerical solution $v^n = S(t_n)^{-1}u^n$ satisfies

$$(15) \qquad v^{n+1} := v^n - i\tau S(t_n+\tau\xi_n)^{-1}\left[|S(t_n+\tau\xi_n)v^n|^2 S(t_n+\tau\xi_n)v^n\right].$$

Proposition 3.5. *Let $W \in C^\alpha([0,T])$ for some $\alpha \in (0,1]$. Then it holds true*

$$\mathbb{E} \max_{M=0,\ldots,N} \left\| \sum_{n=0}^M \int_0^\tau S(t_n+r)^{-1}\left[|S(t_n+r)v(t_n)|^2 S(t_n+r)v(t_n)\right]dr \right.$$

$$\left. - \tau \sum_{n=0}^M S(t_n+\tau\xi_n)^{-1}\left[|S(t_n+\tau\xi_n)v(t_n)|^2 S(t_n+\tau\xi_n)v(t_n)\right] \right\|_\sigma^2$$

$$\lesssim \tau^{2\alpha+1} \int_0^T \|v(t)\|_{\sigma+2}^6 dt.$$

Proof. Since $|u|^2 u = u^2 \cdot \bar{u}$, we write

$$\int_0^\tau S(t_n+r)^{-1}\left[|S(t_n+r)v(t_n)|^2 S(t_n+r)v(t_n)\right]dr$$

$$= \int_0^\tau S(t_n+r)^{-1}\left[(S(t_n+r)v(t_n))^2 \overline{S(t_n+r)v(t_n)}\right]dr =: I(\tau).$$

We intend to estimate the error of the stratified Monte Carlo approximation of the above integral $I(\tau)$ introduced above. Namely, in view of the discussion in Section 2, it is approximated by

$$I(\tau) \approx \tau S(t_n + \tau \xi_n)^{-1} \left[|S(t_n + \tau \xi_n)v(t_n)|^2 S(t_n + \tau \xi_n)v(t_n) \right]$$
$$= \tau S(t_n + \tau \xi_n)^{-1} \left[(S(t_n + \tau \xi_n)v(t_n))^2 \overline{S(t_n + \tau \xi_n)v(t_n)} \right] =: J(\tau)$$

As the next step, we observe that

$$I(\tau) = \sum_{\substack{k,k_1,k_2,k_3 \in \mathbb{Z}^d \\ k=-k_1+k_2+k_3}} e^{ik \cdot x} \left(\int_0^\tau e^{iW(t_n+r)[|k|^2+|k_1|^2-|k_2|^2-|k_3|^2]} \hat{v}_{k_2}(t_n) \hat{v}_{k_3}(t_n) \widehat{\overline{v}}_{k_1}(t_n) dr \right)$$

and similarly

$$J(\tau) = \tau \sum_{\substack{k,k_1,k_2,k_3 \in \mathbb{Z}^d \\ k=-k_1+k_2+k_3}} e^{ik \cdot x} \left(e^{iW(t_n+\tau \xi_n)[|k|^2+|k_1|^2-|k_2|^2-|k_3|^2]} \hat{v}_{k_2}(t_n) \hat{v}_{k_3}(t_n) \widehat{\overline{v}}_{k_1}(t_n) \right).$$

For $k \in \mathbb{Z}^d$, let us denote $K := |k|^2 + |k_1|^2 - |k_2|^2 - |k_3|^2$ and $\hat{V}_K(t_n) := \hat{v}_{k_2}(t_n) \hat{v}_{k_3}(t_n) \overline{\hat{v}}_{k_1}(t_n)$ and define the error

$$(16) \qquad E_K^M := \sum_{n=0}^M \left(\int_0^\tau e^{iW(t_n+r)K} - e^{iW(t_n+\tau \xi_n)K} dr \hat{V}_K(t_n) \right), \quad M = 0, \dots, N.$$

Here

$$\tau \sum_{n=0}^M e^{iW(t_n+\tau \xi_n)K}$$

appearing in the second summand of E_K^M can be regarded as a randomized Riemann sum approximation of the integral

$$\int_0^{M\tau} e^{iW(t_n+r)K} dr.$$

Next, we will show that for fixed every K defined above, E_K^M defines a discreet martingale with respect to the parameter $M = 0, \dots, N$ and the filtration $(\mathcal{F}_M)_{M=0,\dots,N}$ given by $\mathcal{F}_M := \sigma(\xi_i; i = 0, \dots, M)$. This will then allow us to apply the Burkholder-Davis-Gundy inequality, Theorem 3.4. As it will be seen below, the martingale property is a consequence of the way the randomization $(\xi_n)_{n \in \mathbb{N}_0}$ was chosen. First, we observe that $E_K^0 = 0$. Since all ξ_n, $n \in \mathbb{N}_0$, are independent and uniformly distributed in the interval $[0, 1]$, it follows that

$$(17) \qquad \begin{aligned} \mathbb{E}\left[\tau e^{iW(t_n+\tau \xi_n)K} \hat{V}_K(t_n) \right] &= \tau \mathbb{E}\left[e^{iW(t_n+\tau \xi_n)K} \right] \hat{V}_K(t_n) \\ &= \tau \int_0^1 e^{iW(t_n+\tau \xi)K} d\xi \hat{V}_K(t_n) = \int_0^\tau e^{iW(t_n+r)K} dr \hat{V}_K(t_n). \end{aligned}$$

As a consequence

$$\begin{aligned} \mathbb{E}\left[E_K^M \right] &= \sum_{n=0}^M \mathbb{E}\left[\int_0^\tau e^{iW(t_n+r)K} - e^{iW(t_n+\tau \xi_n)K} dr \hat{V}_K(t_n) \right] \\ &= \sum_{n=0}^M \left(\int_0^\tau e^{iW(t_n+r)K} dr - \tau \mathbb{E}\left[e^{iW(t_n+\tau \xi_n)K} \right] \right) \hat{V}_K(t_n) = 0. \end{aligned}$$

In addition, by definition of E_K^M we deduce that for every $M = 0, \dots, N$ the random variable E_K^M is measurable with respect to $\mathcal{F}_M$. Hence the stochastic process E_K^M, $M = 0, \dots, N$, is adapted to the filtration $(\mathcal{F}_M)_{M=0,\dots,N}$. To finally verify the martingale property, we let $M \geq m$ and compute the conditional expectation $\mathbb{E}\left[E_K^M | \mathcal{F}_m\right]$. It holds

$$
\begin{aligned}
\mathbb{E}\left[E_K^M | \mathcal{F}_m\right] &= \sum_{n=0}^{M} \mathbb{E}\left[\int_0^\tau e^{iW(t_n+r)K} - e^{iW(t_n+\tau\xi_n)K} dr \hat{V}_K(t_n)\Big| \mathcal{F}_m\right] \\
&= E_K^m + \sum_{n=m+1}^{M} \mathbb{E}\left[\int_0^\tau e^{iW(t_n+r)K} - e^{iW(t_n+\tau\xi_n)K} dr \hat{V}_K(t_n)\Big| \mathcal{F}_m\right] \\
&= E_K^m + \sum_{n=m+1}^{M} \left(\int_0^\tau e^{iW(t_n+r)K} dr - \tau\mathbb{E}\left[e^{iW(t_n+\tau\xi_n)K}\big| \mathcal{F}_m\right]\right) \hat{V}_K(t_n) \\
&= E_K^m + \sum_{n=m+1}^{M} \left(\int_0^\tau e^{iW(t_n+r)K} dr - \tau\mathbb{E}\left[e^{iW(t_n+\tau\xi_n)K}\right]\right) \hat{V}_K(t_n) \\
&= E_K^m,
\end{aligned}
$$

where we used the adaptedness of E_K^m, properties of the conditional expectation, independence of ξ_n as well as (17). Thus E_K^M, $M = 0, \dots, N$, is a martingale with respect to $(\mathcal{F}_M)_{M=0,\dots,N}$.

Hence, we may apply the Parseval identity and the Burkholder-Davis-Gundy inequality, Theorem 3.4, to obtain

$$
\begin{aligned}
\mathbb{E} \max_{M=0,\dots,N} &\left\| \sum_{\substack{k,k_1,k_2,k_3\in\mathbb{Z}^d \\ k=-k_1+k_2+k_3}} e^{ik\cdot x} E_K^M \right\|_\sigma^2 \\
&\leq \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^\sigma \mathbb{E} \max_{M=0,\dots,N} \left| \sum_{n=0}^{M} \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}^d \\ k=-k_1+k_2+k_3}} \int_0^\tau e^{iW(t_n+r)K} - e^{iW(t_n+\tau\xi_n)K} dr \hat{V}_K(t_n) \right|^2 \\
&\lesssim \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^\sigma \mathbb{E} \sum_{n=0}^{N} \left| \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}^d \\ k=-k_1+k_2+k_3}} \int_0^\tau e^{iW(t_n+r)K} - e^{iW(t_n+\tau\xi_n)K} dr \hat{V}_K(t_n) \right|^2 .
\end{aligned}
$$

Next, we have by the Minkowski integral inequality

$$
(18)
\begin{aligned}
&\left(\mathbb{E} \left| \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}^d \\ k=-k_1+k_2+k_3}} \int_0^\tau e^{iW(t_n+r)K} - e^{iW(t_n+\tau\xi_n)K} dr \hat{V}_K(t_n) \right|^2 \right)^{1/2} \\
&\leq \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}^d \\ k=-k_1+k_2+k_3}} \int_0^\tau \left(\mathbb{E} \left| e^{iW(t_n+r)K} - e^{iW(t_n+\tau\xi_n)K} \hat{V}_K(t_n) \right|^2 \right)^{1/2} dr \\
&\leq \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}^d \\ k=-k_1+k_2+k_3}} \int_0^\tau \left(\mathbb{E} \left[\left| (W(t_n+r) - W(t_n+\tau\xi_n))K\hat{V}_K(t_n) \right|^2 \right] \right)^{1/2} dr \\
&\leq \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}^d \\ k=-k_1+k_2+k_3}} \int_0^\tau \left(\mathbb{E} \left[|r-\tau\xi_n|^{2\alpha} \right] \right)^{1/2} dr |K||\hat{V}_K(t_n)| \\
&\lesssim \tau^{\alpha+1} \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}^d \\ k=-k_1+k_2+k_3}} |K||\hat{V}_K(t_n)|,
\end{aligned}
$$

where we used the α-Hölder continuity of W together with

$$
\mathbb{E} \left[|r-\tau\xi_n|^{2\alpha} \right] \leq r^{2\alpha} + \tau^{2\alpha} \lesssim \tau^{2\alpha}.
$$

Therefore the final error is estimated using the bilinear estimate (2) together with the fact that $K = |k|^2 + |k_1|^2 - |k_2|^2 - |k_3|^2 = 2(k-k_2)\cdot(k-k_3)$ as follows

$$
\begin{aligned}
\mathbb{E} \max_{M=0,\ldots,N} &\left\| \sum_{\substack{k,k_1,k_2,k_3\in\mathbb{Z}^d \\ k=-k_1+k_2+k_3}} e^{ik\cdot x} E_K^M \right\|_\sigma^2 \lesssim \tau^{2(\alpha+1)} \sum_{n=0}^N \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^\sigma \left(\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}^d \\ k=-k_1+k_2+k_3}} |K||\hat{V}_K(t_n)| \right)^2 \\
&\lesssim \tau^{2\alpha+1} \int_0^T \|v(t)\|_{H^{\sigma+2}}^6 dt.
\end{aligned}
$$

Consequently the claim follows. $\qquad\square$

Combining Corollary 2.3 with Proposition 3.5, we obtain the estimate for the global error in Theorem 3.1.

Proof of Theorem 3.1. In the following we will derive the bound

$$
(19) \qquad \left(\mathbb{E} \max_{M=0,\ldots,N} \left\| v(t_M) - v^M \right\|_\sigma^2 \right)^{1/2} \lesssim \tau + \tau^{\alpha+\frac{1}{2}}.
$$

The corresponding bound on $u(t_M) - u^M$ then follows by Corollary 2.3 together with the isometric property (3).

In the following we denote the error $v(t_n) - v^n$ by e_n, $n = 0, \ldots, N$, and set $f(v) := |v|^2 v$. The error in v reads

$$e_{n+1} := v(t_{n+1}) - v^{n+1} = (v(t_{n+1}) - v_*(t_{n+1})) + (v_*(t_{n+1}) - v^{n+1})$$
$$= (v(t_{n+1}) - v_*(t_{n+1})) + (v(t_n) - v^n)$$
$$- i \int_0^\tau S(t_n + r)^{-1} f\left(S(t_n + r)v(t_n)\right) dr + i\tau S(t_n + \tau\xi_n)^{-1} f\left(S(t_n + \tau\xi_n)v^n\right),$$

where the last equality is obtained by subtracting (15) from (14).

Next we use that

$$f(v) = f(w) + (\overline{v - w})v^2 + \overline{w}(v + w)(v - w).$$

The above relation with $v = S(t_n + \tau\xi_n)v^n$ and $w = S(t_n + \tau\xi_n)v(t_n)$ yields that

$$e_{n+1} = (v(t_{n+1}) - v_*(t_{n+1})) + e_n$$
$$- i\Big[\int_0^\tau S^{-1}(t_n + r)f\left(S(t_n + r)v(t_n)\right) dr$$
$$- \tau S^{-1}(t_n + \tau\xi_n)\Big\{ f\left(S(t_n + \tau\xi_n)v(t_n)\right) + \overline{S(t_n + \tau\xi_n)e_n}v^2 + \overline{w}(v + w)S(t_n + \tau\xi_n)e_n \Big\} \Big]$$
$$= (v(t_{n+1}) - v_*(t_{n+1})) + e_n$$
$$- i\Big[\int_0^\tau S^{-1}(t_n + r)f\left(S(t_n + r)v(t_n)\right) dr - \tau S^{-1}(t_n + \tau\xi_n)f\left(S(t_n + \tau\xi_n)v(t_n)\right) \Big]$$
$$+ \tau\mathcal{B}_n e_n$$
$$= (v(t_{n+1}) - v_*(t_{n+1})) + e_n - iL(v(t_n)) + \tau\mathcal{B}_n e_n,$$

where

$$L(v(t_n)) := \Big[\int_0^\tau S^{-1}(t_n + r)f\left(S(t_n + r)v(t_n)\right) dr - \tau S^{-1}(t_n + \tau\xi_n)f\left(S(t_n + \tau\xi_n)v(t_n)\right) \Big]$$

and $\mathcal{B}_n$ denotes some bounded operator, depending on v^n, $v(t_n)$ but bounded uniformly in ω, τ, n due to Remark 3.3. Iterating the above formula yields

$$e_M = (v(t_M) - v_*(t_M)) + (v(t_{M-1}) - v_*(t_{M-1})) + e_{M-2} - iL(v(t_{M-2})) + \tau\mathcal{B}_{M-2}e_{M-2}$$
$$- iLv(t_{M-1}) + \tau\mathcal{B}_{M-1}e_{M-1}$$
$$= \sum_{k=0}^{M-1} (v(t_{M-k}) - v_*(t_{M-k})) + e_0 - i\sum_{k=1}^M L(v(t_{M-k})) + \tau\sum_{k=1}^M \mathcal{B}_{M-k}e_{M-k}$$
$$= \sum_{n=1}^M (v(t_n) - v_*(t_n)) + e_0 - i\sum_{n=0}^{M-1} L(v(t_n)) + \tau\sum_{n=0}^{M-1} \mathcal{B}_n e_n.$$

Hence we estimate (using Corollary 2.3, definition of the twisted variables (13), the isometric property (3) and the fact that $e_0 = 0$)

$$(20) \qquad \begin{aligned} \mathbb{E} \max_{M=1,\ldots,N} \|e_M\|_\sigma^2 &\lesssim \max_{M=1,\ldots,N} \left\| \sum_{n=1}^M (v(t_n) - v_*(t_n)) \right\|_\sigma^2 \\ &+ \mathbb{E} \max_{M=1,\ldots,N} \left\| \sum_{n=0}^{M-1} L(v(t_n)) \right\|_\sigma^2 + \tau^2 \mathbb{E} \max_{M=1,\ldots,N} \left\| \sum_{n=0}^{M-1} \mathcal{B}_n e_n \right\|_\sigma^2. \end{aligned}$$

Note that the first term on the right hand side does not depend on (ξ_n) and hence is independent of ω. According to Corollary 2.3, definition of the twisted variables (13) and the isometric property (3) it can be estimated by

$$\max_{M=1,\ldots,N} \left\| \sum_{n=1}^{M} (v(t_n) - v_*(t_n)) \right\|_\sigma^2 \leq \left(\sum_{n=1}^{N} \|(v(t_n) - v_*(t_n))\|_\sigma \right)^2 \lesssim \tau^2.$$

The last term on the right hand side of (20) can be estimated as follows

$$\tau^2 \mathbb{E} \max_{M=1,\ldots,N} \left\| \sum_{n=0}^{M-1} \mathcal{B}_n e_n \right\|_\sigma^2 \leq \tau^2 \mathbb{E} \left(\sum_{n=0}^{N-1} \|\mathcal{B}_n e_n\|_\sigma \right)^2 \lesssim \tau^2 \mathbb{E} \left(\sum_{n=0}^{N-1} \|e_n\|_\sigma \right)^2$$

$$\leq \tau^2 N \mathbb{E} \sum_{n=0}^{N-1} \|e_n\|_\sigma^2 \lesssim \tau \sum_{n=0}^{N-1} \mathbb{E} \max_{M=0,\ldots,n} \|e_M\|_\sigma^2.$$

Therefore, in view of Proposition 3.4 we estimate the second term on the right hand side of (20) and obtain

$$\mathbb{E} \max_{M=0,\ldots,N} \|e_M\|_\sigma^2 \lesssim \tau^2 + \tau^{2\alpha+1} + \tau \sum_{n=0}^{N-1} \mathbb{E} \max_{M=0,\ldots,n} \|e_M\|_\sigma^2.$$

Finally, we apply the discreet Gronwall Lemma to deduce

$$(21) \qquad\qquad \mathbb{E} \max_{M=0,\ldots,N} \|e_M\|_\sigma^2 \lesssim \tau^2 + \tau^{2\alpha+1},$$

where the proportional constant depends on T but is independent of τ and N. This implies the estimate (19) as long as $\|v^n\|_\sigma$ is bounded.

Recall the twisted variables (cf. (13))

$$v(t) = S(t)^{-1} u(t), \quad v^n = S(t)^{-1} u^n$$

such that in particular due to the isometric property (3) we can conclude that

$$\|v(t)\|_\sigma = \|u(t)\|_\sigma, \qquad \|v^n\|_\sigma = \|u^n\|_\sigma$$

which thanks to Remark 3.3 implies the apriori boundedness of $\|v^n\|_\sigma$.

Thanks to Lemma 2.1 and Corollary 2.3 we then in particular obtain that

$$\mathbb{E} \max_{M=0,\ldots,N} \left\| u(t_M) - u^M \right\|_\sigma^2 \lesssim \tau^2 + \tau^{2\alpha+1}$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4. NUMERICAL EXPERIMENTS

In this section we numerically underline the theoretical convergence result of Theorem 3.1. Furthermore, we compare the convergence behavior of our newly derived stratified exponential integrator (9) with a classical Strang splitting and exponential integration scheme. For the latter we refer to [7, 13, 16, 24] and the references therein. The numerical experiments emphasize the favorable error behavior of our newly derived scheme over classical integration methods in the presence of a non-smooth function W. Thereby, the classical exponential integrator reads

$$(22) \qquad\qquad u_E^{n+1} = U(t_n + \tau, t_n) u_E^n - i\tau U(t_n + \tau, t_n) \left(|u_E^n|^2 u_E^n \right)$$

whereas to the modulated Schrödinger equation (1) associated Strang splitting scheme takes the form

$$\psi_-^{n+1/2} = e^{-i\frac{\tau}{2}|\psi^{n+1}|^2}\psi^{n+1}$$

(23)
$$\psi_+^{n+1/2} = e^{i(W(t_{n+1})-W(t_n))\partial_x^2}\psi_-^{n+1/2}$$

$$\psi^{n+1} = e^{-i\frac{\tau}{2}|\psi_+^{n+1/2}|^2}\psi_+^{n+1/2}.$$

In all numerical experiments we choose the initial value

$$u_0(x) = \frac{\cos(x)}{2-\sin(x)}$$

and use a standard Fourier pseudospectral method for the space discretization where we choose the largest Fourier mode $K = 2^7$ (i.e., the spatial mesh size $\Delta x = 0.049$). To simulate the fractional Brownian motion $W_H(t)$ with Hurst parameter H we follow the code in [27, Chapter 12.4.2]. Moreover we consider a smooth example, where $W_\infty(t) = \sin(t)$. In the smooth setting $W_\infty(t) = \sin(t)$ we take as a reference solution the Strang splitting scheme with a very small time step size. In the less regular case of W_H, i.e., for $H = 1/2$, $H = 1/4$ or $H = 1/10$ we take as a reference solution the schemes themselves with a very small time step size.
To compute the error $(\mathbb{E}\max_{M=0,\dots,N}\|e_M\|_1^2)^{1/2}$ (see Theorem 3.1) of our stratified exponential integrator (9) we proceed as follows: We denote by $u^{\mathrm{ref}}(1)$ the reference solution at time $T = 1$ and by $u_{(\xi_i)_i^k}(1)$ the approximation computed with the stratified exponential integration scheme (9), by using the sequence $(\xi_i)_i^k$. By taking $m \in \mathbb{N}$ sequences, we now use the approximation

$$\left(\mathbb{E}\max_{M=0,\dots,N}\|e_M\|_1^2\right)^{\frac{1}{2}} \approx \left(\frac{1}{m}\sum_{k=1}^m \left\|u^{\mathrm{ref}}(1) - u_{(\xi_i)_i^k}(1)\right\|_1^2\right)^{\frac{1}{2}},$$

where the derivative is computed by means of the Fourier transform.
For the Strang splitting and the exponential integrator we compute the classical error in a discrete H^1 norm.

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REFERENCES

[1] F. Kh. Abdullaev, J. C. Bronski, and G. Papanicolaou, *Soliton perturbations and the random Kepler problem.* Physica D. Nonlinear Phenomena 135 (2000), no. 3-4, 369386, DOI 10.1016/S0167-2789(99)00118-9.

[2] R. Belaouar, A. de Bouard, A. Debussche, *Numerical analysis of the nonlinear Schrödinger equation with white noise dispersion.* Stochastic Partial Differential Equations: Analysis and Computations, 3, 1, pp. 103–132 (2015).

[3] C. Besse, B. Bidégaray, S. Descombes, *Order estimates in time of splitting methods for the nonlinear Schrödinger equation.* SIAM J. Numer. Anal. 40:26–40 (2002).

[4] D. L. Burkholder, *Martingale transforms.* Ann. Math. Statist., 37:1494–1504 (1966).

[5] E. Celledoni, D. Cohen, B. Owren, *Symmetric exponential integrators with an application to the cubic Schrödinger equation.* Found. Comput. Math. 8:303–317 (2008).

[6] K. Chouk, M. Gubinelli, *Nonlinear PDEs with modulated dispersion I: Nonlinear Schrödinger equations.* Communications in Partial Differential Equations. 2015.40(11):2047–81.

[7] D. Cohen, G. Dujardin, *Exponential integrators for nonlinear Schrdinger equations with white noise dispersion.* Stochastics and Partial Differential Equations: Analysis and Computations, to appear.
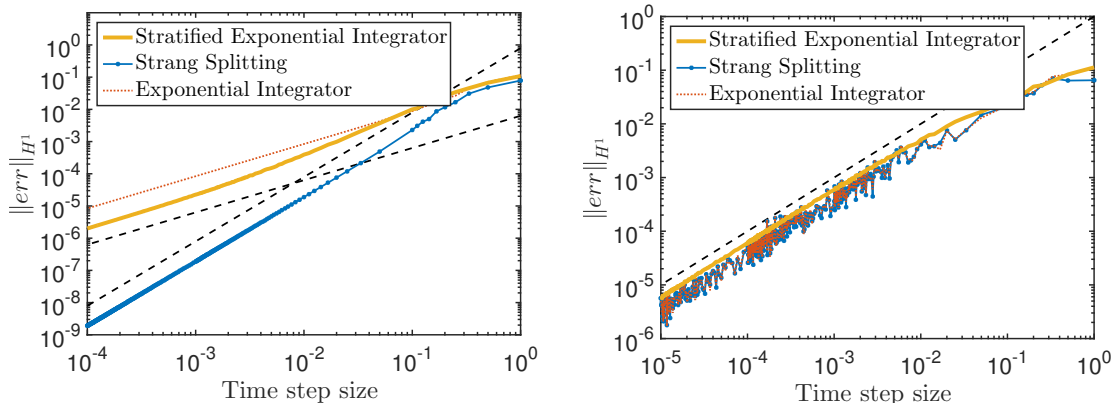
(A) $W_\infty(t) = \sin(t)$. The slope of the dashed lines is one and two, respectively.

(B) Brownian motion (Hurst parameter $H = 1/2$). The slope of the dashed line is one.

FIGURE 1. Convergence plot of the Strang splitting scheme (23), the classical exponential integrator (22) and the stratified exponential integrator (9) with 100 sequences in the case of a smooth function W (left) and a non-regular, Brownian motion W (right).
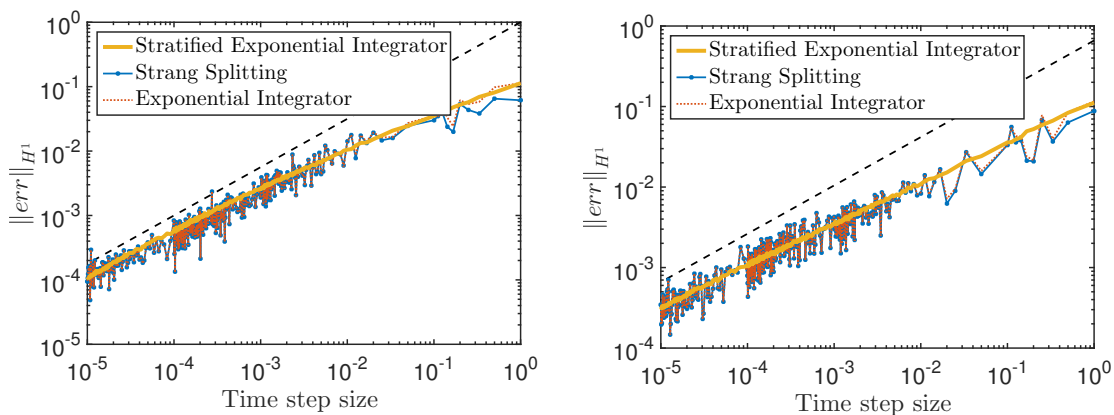
(A) Hurst parameter 1/4. The slope of the dashed line is 3/4.

(B) Hurst parameter 1/10. The slope of the dashed line is 3/5.

FIGURE 2. Convergence plot of the Strang splitting scheme (23), the classical exponential integrator (22) and the stratified exponential integrator (9) with 100 sequences in the case of fractional Brownian motions with different Hurst parameters.

[8] D. Cohen, L. Gauckler, *One-stage exponential integrators for nonlinear Schrödinger equations over long times.* BIT 52:877–903 (2012).

[9] A. de Bouard and A. Debussche, *The nonlinear Schrödinger equation with white noise dispersion.* Journal of Functional Analysis 259 (5) 1300–1321 (2010).

[10] A. Debussche and Y. Tsutsumi, *1D quintic nonlinear Schrödinger equation with white noise dispersion.* Journal de Mathématiques Pures et Appliquées 96 4) 363–376 (2011).

[11] G. Dujardin, *Exponential Runge-Kutta methods for the Schrödinger equation.* Appl. Numer. Math. 59:1839–1857 (2009).

[12] M. Eisenmann, M. Kovács, R. Kruse, S. Larsson, *On a randomized, backward Euler method for nonlinear evolution equations with time-irregular coefficients.* arXiv:1709:01018.

[13] E. Faou, *Geometric Numerical Integration and Schrödinger Equations.* European Math. Soc. Publishing House, Zürich 2012.

[14] L. Gauckler, *Convergence of a split-step Hermite method for the Gross–Pitaevskii equation.* IMA J. Numer. Anal. 31:396–415 (2011).

[15] E. Hairer, C. Lubich, G. Wanner, *Geometric Numerical Integration. Structure-Preserving Algorithms for Ordinary Differential Equations.* Second edition, Springer, Berlin 2006.

[16] M. Hochbruck, A. Ostermann, *Exponential integrators.* Acta Numer. 19:209–286 (2010).

[17] H. Holden, K. H. Karlsen, K.-A. Lie, N. H. Risebro, *Splitting for Partial Differential Equations with Rough Solutions.* European Math. Soc. Publishing House, Zürich 2010.

[18] D. Hundertmark, Y. R. Lee, *Decay estimates and smoothness for solutions of the dispersion managed nonlinear Schrdinger equation.* Communications in Mathematical Physics 286 (3) 851–873 (2009).

[19] D. Hundertmark, Y. R. Lee, *Super-exponential decay of diffraction managed solitons.* Communications inMathematical Physics 309 (1) 1–21 (2012).

[20] A. Jentzen and A. Neuenkirch, *A random Euler scheme for Caratheodory differential equations.* J. Comput. Appl. Math., 224(1):346359, 2009.

[21] M. Kunze, J. Moeser, V. Zharnitsky, *Ground states for the higher-order dispersion managed NLS equation in the absence of average dispersion.* Journal of Differential Equations 209 (1) 77–100 (2005).

[22] R. Kruse and Y. Wu, *Error analysis of randomized Runge-Kutta methods for differential equations with time-irregular coefficients.* Comput. Methods Appl. Math., 17(3):479498, 2017.

[23] R. Kruse and Y. Wu, *A randomized Milstein method for stochastic differential equations with non-differentiable drift coefficients.* arXiv:1706:09964.

[24] C. Lubich, *On splitting methods for Schrödinger–Poisson and cubic nonlinear Schrödinger equations.* Math. Comp. 77:2141–2153 (2008).

[25] R. Marty, *On a splitting scheme for the nonlinear Schrödinger equation in a random medium.* Communications in Mathematical Sciences 4 (4) 679–705(2006).

[26] R.I. McLachlan, G.R.W. Quispel, *Splitting methods*, Acta Numer. 11:341–434 (2002).

[27] V. Schmidt, *Stochastic Geometry, Spatial Statistics and Random Fields*, Models and Algorithms, Springer International Publishing Switzerland, 2015

[28] V. Zharnitsky, E. Grenier, C. K.R.T. Jones, and S. K. Turitsyn, *Stabilizing effects of dispersion management.* Physica D: Nonlinear Phenomena 152-153 794–817 (2001).

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