
AN EXPONENTIAL-TYPE INTEGRATOR FOR THE KdV EQUATION

by

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Abstract. — We introduce an exponential-type time-integrator for the KdV equation and prove its first-order convergence in H^1 for initial data in H^3 without imposing any CFL condition.

We consider the Korteweg-de Vries (KdV) equation

$$(1) \quad \partial_t u(t, x) + \partial_x^3 u(t, x) + \frac{1}{2} \partial_x (u(t, x))^2 = 0, \quad u(0, x) = u_0(x), \quad (t, x) \in \mathbb{R} \times \mathbb{T},$$

where for practical implementation issues we impose periodic boundary conditions. For local-wellposedness results of the periodic KdV equation in low regularity spaces we refer to [1, 5, 19].

In the context of the numerical time integration of (non)linear partial differential equations splitting methods as well as exponential integrators contribute attractive classes of integration methods. We refer to [7, 8, 9, 18] for an extensive overview, and in particular to [3, 4, 16] for the analysis of splitting methods for Schrödinger(-Poisson) equations. In recent years, splitting as well as exponential integration schemes (including Lawson type Runge-Kutta methods [15]) have also gained a lot of attention in the context of the numerical integration of the KdV equation, see for instance [10, 11, 12, 13, 14, 20] and the references therein. We also refer to [2] for a splitting approach for the Kadomtsev-Petviashvili equation.

In particular, a distinguished convergence result was obtained in [11, 10]. In the latter it was proven that the Strang splitting, where the right-hand side of the KdV equation is split into the linear and Burgers part, respectively, is second-order convergent in H^r for initial data in H^{r+5} for $r \geq 1$ without imposing any CFL condition assuming that the Burgers part is solved exactly.

Here we derive a first-order exponential-type time-integrator for the KdV equation (1) based on Duhamel's formula

$$(2) \quad u(t) = e^{-\partial_x^3 t} u_0 + \frac{1}{2} \int_0^t e^{-\partial_x^3(t-s)} \partial_x (u(s))^2 ds$$

Key words and phrases. — KdV equation – exponential-type time integrator – convergence.

looking at the "twisted variable" $v(t) = e^{\partial_x^3 t} u(t)$. This idea of "twisting" the variable is widely used in the analysis of partial differential equations in low regularity spaces (see, for instance [1, 5, 19] for the periodic KdV equation) and also well known in the context of numerical analysis, see [15] for the introduction of Lawson type Runge-Kutta methods. However, instead of approximating the appearing integral with a Runge-Kutta method (see for instance [13]) we use the key relation

$$(3) \quad k_1^3 + k_2^3 - (k_1 + k_2)^3 = -3(k_1 + k_2)k_1k_2$$

which allows us to overcome the *loss of derivative* by integrating the *stiff parts* (i.e., the terms involving ∂_x^3) exactly.

The derived exponential-type integrator is unconditionally stable and we will in particular show its first-order convergence in H^1 for initial data in H^3 without imposing any CFL condition. A key tool in our convergence analysis is a variant of [10, Lemma 3.1].

1. An exponential-type integrator

To illustrate the idea we first consider initial values with zero mean. In Remark 1.2 we point out the generalization to general initial values.

Assumption 1.1. — Assume that the zero-mode of the initial value is zero, i.e., $\hat{u}_0(0) = (2\pi)^{-1} \int_{\mathbb{T}} u(0, x) dx = 0$. Note that the conservation of mass then implies that $\hat{u}_0(t) = 0$.

We will derive a scheme for the "twisted" variable $v(t) = e^{\partial_x^3 t} u(t)$. With this transformation at hand the equation in v reads

$$(4) \quad v(t) = v_0 + \frac{1}{2} \int_0^t e^{s\partial_x^3} \partial_x (e^{-\partial_x^3 s} v(s))^2 ds.$$

For a small time-step τ we iterate Duhamel's formula and approximate the exact solution as follows

$$(5) \quad v(t_n + \tau) \approx v(t_n) + \frac{1}{2} \int_0^\tau e^{(t_n+s)\partial_x^3} \partial_x (e^{-\partial_x^3 (t_n+s)} v(t_n))^2 ds.$$

The key relation (3) now allows us the following integration technique (cf. [1, 5, 19]): We have

$$\begin{aligned}
& \int_0^\tau e^{(t_n+s)\partial_x^3} \partial_x (e^{-\partial_x^3(t_n+s)} v(t_n))^2 ds \\
&= \sum_{k_1, k_2} \int_0^\tau e^{-i(t_n+s)((k_1+k_2)^3 - k_1^3 - k_2^3)} i(k_1 + k_2) \hat{v}_{k_1}(t_n) \hat{v}_{k_2}(t_n) e^{i(k_1+k_2)x} ds \\
&= \sum_{k_1, k_2} \frac{e^{-i(t_n+\tau)((k_1+k_2)^3 - k_1^3 - k_2^3)} - e^{-it_n((k_1+k_2)^3 - k_1^3 - k_2^3)}}{-i((k_1 + k_2)^3 - k_1^3 - k_2^3)} i(k_1 + k_2) \hat{v}_{k_1}(t_n) \hat{v}_{k_2}(t_n) e^{i(k_1+k_2)x} \\
&= \sum_{k_1, k_2} \left(e^{-i(t_n+\tau)((k_1+k_2)^3 - k_1^3 - k_2^3)} - e^{-it_n((k_1+k_2)^3 - k_1^3 - k_2^3)} \right) \frac{1}{-3k_1 k_2} \hat{v}_{k_1}(t_n) \hat{v}_{k_2}(t_n) e^{i(k_1+k_2)x} \\
&= \frac{1}{3} e^{\partial_x^3(t_n+\tau)} \left(e^{-\partial_x^3(t_n+\tau)} \partial_x^{-1} v^n \right)^2 - \frac{1}{3} e^{\partial_x^3 t_n} \left(e^{-\partial_x^3 t_n} \partial_x^{-1} v^n \right)^2.
\end{aligned}$$

Together with the approximation in (5) this yields that

$$(6) \quad v^{n+1} = v^n + \frac{1}{6} e^{\partial_x^3(t_n+\tau)} \left(e^{-\partial_x^3(t_n+\tau)} \partial_x^{-1} v^n \right)^2 - \frac{1}{6} e^{\partial_x^3 t_n} \left(e^{-\partial_x^3 t_n} \partial_x^{-1} v^n \right)^2, \quad \hat{v}_0^{n+1} = 0.$$

In order to obtain an approximation to the original solution $u(t_n)$ of the KdV equation (1) at time $t_n = n\tau$ we then "twist" the variable back again by setting $u^n = e^{-\partial_x^3 t_n} v^n$. This yields the following exponential-type integrator for the KdV equation (1)

$$(7) \quad u^{n+1} = e^{-\tau \partial_x^3} u^n + \frac{1}{6} \left(e^{-\tau \partial_x^3} \partial_x^{-1} u^n \right)^2 - \frac{1}{6} e^{-\tau \partial_x^3} \left(\partial_x^{-1} u^n \right)^2, \quad \hat{u}_0^{n+1} = 0.$$

For sufficiently smooth solutions the numerical scheme (7) is first-order convergent (without any CFL-type condition), see Corollary 2.8 below for the precise convergence result.

Remark 1.2. — If $\hat{u}_0(0) = \alpha \neq 0$ we set $\tilde{u} := u - \alpha$ and look at the modified KdV equation in \tilde{u} , i.e.,

$$(8) \quad \partial_t \tilde{u} + \partial_x^3 \tilde{u} + \alpha \partial_x \tilde{u} + \frac{1}{2} \partial_x (\tilde{u})^2 = 0.$$

Note that the solution $\tilde{u}(t)$ of the modified KdV equation (8) satisfies $\hat{\tilde{u}}_0(t) = 0$ for all t as by the conservation of mass we have that $\hat{u}_0(t) \equiv \hat{u}_0(0) = \alpha$. Thus, we can proceed as above: We look at the twisted variable $\tilde{v}(t) = e^{(\partial_x^3 + \alpha \partial_x)t} \tilde{u}(t)$ and carry out an approximation as above, i.e.,

$$\tilde{v}(t_n + \tau) \approx \tilde{v}(t_n) + \frac{1}{2} \int_0^\tau e^{(t_n+s)(\partial_x^3 + \alpha \partial_x)} \partial_x \left(e^{-(t_n+s)(\partial_x^3 + \alpha \partial_x)} \tilde{v}(t_n) \right)^2 ds.$$

The relation

$$-(k_1 + k_2)^3 + \alpha(k_1 + k_2) + k_1^3 + k_2^3 - \alpha k_1 - \alpha k_2 = -(k_1 + k_2)^3 + k_1^3 + k_2^3$$

then allows us to derive similarly to above an exponential-type integration scheme

$$(9) \quad \begin{aligned} \tilde{v}^{n+1} &= \tilde{v}^n + \frac{1}{6} e^{(\partial_x^3 + \alpha \partial_x)(t_n + \tau)} \left(e^{-(\partial_x^3 + \alpha \partial_x)(t_n + \tau)} \partial_x^{-1} \tilde{v}^n \right)^2 - \frac{1}{6} e^{(\partial_x^3 + \alpha \partial_x)t_n} \left(e^{-(\partial_x^3 + \alpha \partial_x)t_n} \partial_x^{-1} \tilde{v}^n \right)^2 \\ \hat{v}_0^{n+1} &= 0. \end{aligned}$$

Finally, by setting $u^n = e^{-(\partial_x^3 + \alpha \partial_x)t_n} \tilde{v}^n + \alpha$ we then obtain an approximation to the exact solution $u(t_n)$ of the KdV equation (1) (with non-zero zero-mode) at time $t_n = n\tau$.

2. Error analysis

For simplicity we carry out the error analysis for initial values satisfying Assumption 1.1. Furthermore, in the following we denote by $\langle \cdot, \cdot \rangle$ the L^2 scalar product, i.e., $\langle f, g \rangle = \int_{\mathbb{T}} f g dx$ and by $\| \cdot \|_{L^2}$ the corresponding L^2 norm.

In order to obtain a convergence result in H^1 we follow the strategy presented in [16, 10]: We first prove convergence order of one half of the numerical scheme (6) in H^2 for solutions in H^3 , see Section 2.1 Theorem 2.6. This yields essential a priori bounds on the numerical solution in H^2 and allows us to prove first-order convergence globally in H^1 , see Theorem 2.7 in Section 2.2. The latter in particular implies first-order convergence of the exponential-type integration scheme (7) towards the KdV solution (1), see Corollary 2.8 below for the precise convergence result.

2.1. Error analysis in H^2 . — We commence with the error analysis of the numerical scheme (6) in H^2 . In Section 2.1.1 we carry out the stability analysis in H^2 . In Section 2.1.2 we show that the method is consistent of order one half in H^2 for solutions in H^3 .

2.1.1. Stability analysis. — Set

$$(10) \quad \Phi_t^\tau(v) := v + \frac{1}{6} e^{\partial_x^3(t+\tau)} \left(e^{-\partial_x^3(t+\tau)} \partial_x^{-1} v \right)^2 - \frac{1}{6} e^{\partial_x^3 t} \left(e^{-\partial_x^3 t} \partial_x^{-1} v \right)^2, \quad (\widehat{\Phi_t^\tau(v)})_0 := 0.$$

such that for all k we have $v^{k+1} = \Phi_{t_k}^\tau(v^k)$. The following stability result holds for the numerical flow Φ_t^τ :

Lemma 2.1. — *Let $f \in H^2$ and $g \in H^3$. Then, for all $t \in \mathbb{R}$ we have*

$$\left\| \partial_x^2(\Phi_t^\tau(f) - \Phi_t^\tau(g)) \right\|_{L^2} \leq \exp(\tau L) \left\| \partial_x^2(f - g) \right\|_{L^2},$$

where L depends on $\left\| \partial_x^2 f \right\|_{L^2}$ and $\left\| \partial_x^3 g \right\|_{L^2}$.

Proof. — Note that

$$\begin{aligned}
& \|\partial_x^2(\Phi_t^\tau(f) - \Phi_t^\tau(g))\|_{L^2}^2 = \|\partial_x^2(f - g)\|_{L^2}^2 \\
& + \frac{1}{3} \langle \partial_x^2 e^{\partial_x^3(t+\tau)} \left[\left(e^{-\partial_x^3(t+\tau)} \partial_x^{-1} f \right)^2 - \left(e^{-\partial_x^3(t+\tau)} \partial_x^{-1} g \right)^2 \right], \partial_x^2(f - g) \rangle \\
& - \frac{1}{3} \langle \partial_x^2 e^{\partial_x^3 t} \left[\left(e^{-\partial_x^3 t} \partial_x^{-1} f \right)^2 - \left(e^{-\partial_x^3 t} \partial_x^{-1} g \right)^2 \right], \partial_x^2(f - g) \rangle \\
& + \frac{1}{6^2} \|\partial_x^2 e^{\partial_x^3(t+\tau)} \left[\left(e^{-\partial_x^3(t+\tau)} \partial_x^{-1} f \right)^2 - \left(e^{-\partial_x^3(t+\tau)} \partial_x^{-1} g \right)^2 \right] \\
& - \partial_x^2 e^{\partial_x^3 t} \left[\left(e^{-\partial_x^3 t} \partial_x^{-1} f \right)^2 - \left(e^{-\partial_x^3 t} \partial_x^{-1} g \right)^2 \right]\|_{L^2}^2 \\
& =: \|\partial_x^2(f - g)\|_{L^2}^2 + \frac{1}{3} I_1 + \frac{1}{6^2} I_2.
\end{aligned}$$

Lemma 2.3 and Lemma 2.4 below allow us the following bounds on I_1 and I_2 : We have

$$(11) \quad |I_1 + I_2| \leq \tau L \|\partial_x^2(f - g)\|_{L^2}^2,$$

where L depends on $\|\partial_x^2 f\|_{L^2}$ and $\|\partial_x^3 g\|_{L^2}$. Hence,

$$\|\partial_x^2(\Phi_t^\tau(f) - \Phi_t^\tau(g))\|_{L^2}^2 \leq (1 + \tau L) \|\partial_x^2(f - g)\|_{L^2}^2$$

which yields the assertion. \square

In the rest of Section 2.1.1 we will show the essential bound (11). We start with a useful Lemma.

Lemma 2.2. — *The following estimates hold for $u, v, w \in H^2$*

$$\begin{aligned}
(12) \quad & |\langle \partial_x^2 u, vw - e^{\partial_x^3 \tau} \left[\left(e^{-\partial_x^3 \tau} v \right) \left(e^{-\partial_x^3 \tau} w \right) \right] \rangle| \leq c\tau \|\partial_x^2 u\|_{L^2} \|\partial_x^2 v\|_{L^2} \|\partial_x^2 w\|_{L^2} \\
& |\langle u, (\partial_x v)^2 - e^{\partial_x^3 \tau} \left(e^{-\partial_x^3 \tau} \partial_x v \right)^2 \rangle| \leq c\tau \|\partial_x^2 u\|_{L^2} \|\partial_x^2 v\|_{L^2}^2
\end{aligned}$$

for some constant $c > 0$.

Proof. — The key relation (3) together with the Cauchy-Schwarz inequality allows us the following bound

$$\begin{aligned}
& | \langle \partial_x^2 u, vw - e^{\partial_x^3 \tau} \left[\left(e^{-\partial_x^3 \tau} v \right) \left(e^{-\partial_x^3 \tau} w \right) \right] \rangle | \\
&= \left| \sum_{k_1, k_2} (k_1 + k_2)^2 \hat{u}_{-(k_1+k_2)} \left(1 - e^{-i\tau((k_1+k_2)^3 - k_1^3 - k_2^3)} \right) \hat{v}_{k_1} \hat{w}_{k_2} \right| \\
&= \left| \sum_{k_1, k_2} (k_1 + k_2)^2 \hat{u}_{-(k_1+k_2)} \left(1 - e^{-i\tau 3k_1 k_2 (k_1+k_2)} \right) \hat{v}_{k_1} \hat{w}_{k_2} \right| \\
&\leq 3\tau \sum_{k_1, k_2} |(k_1 + k_2)^2 \hat{u}_{-(k_1+k_2)}| \cdot |(k_1 + k_2) k_1 k_2 \hat{v}_{k_1} \hat{w}_{k_2}| \\
(13) \quad &= 3\tau \sum_{l, k} l^2 |\hat{u}_{-l}| |lk(l-k)| |\hat{v}_k \hat{w}_{l-k}| \\
&\leq 3\tau \sum_{l, k} l^2 |\hat{u}_{-l}| (|k(l-k)| |\hat{v}_k \hat{w}_{l-k}| + |k|^2 |l-k| |\hat{v}_k \hat{w}_{l-k}|) \\
&\leq 3\tau \left(\sum_l l^4 |\hat{u}_l|^2 \right)^{1/2} \left(\sum_l \left(\sum_k |k| |\hat{v}_k| |l-k| |\hat{w}_{l-k}| \right)^2 \right)^{1/2} \\
&\quad + 3\tau \left(\sum_l l^4 |\hat{u}_l|^2 \right)^{1/2} \left(\sum_l \left(\sum_k |k|^2 |\hat{v}_k| |l-k| |\hat{w}_{l-k}| \right)^2 \right)^{1/2} \\
&\leq 3\tau \|\partial_x^2 u\|_{L^2} (\|v^{(1)} * w^{(2)}\|_{l^2} + \|v^{(2)} * w^{(1)}\|_{l^2}),
\end{aligned}$$

where $v^{(j)}(k) := |k|^j |\hat{v}_k|$ and $w^{(j)}(k) := |k|^j |\hat{w}_k|$. By the Young and Cauchy-Schwarz inequality we furthermore obtain that

$$\begin{aligned}
(14) \quad & \|v^{(1)} * w^{(2)}\|_{l^2} + \|v^{(2)} * w^{(1)}\|_{l^2} \leq \|v^{(1)}\|_{l^1} \|w^{(2)}\|_{l^2} + \|w^{(1)}\|_{l^1} \|v^{(2)}\|_{l^2} \leq c \|v^{(2)}\|_{l^2} \|w^{(2)}\|_{l^2} \\
& \leq c \|\partial_x^2 v\|_{L^2} \|\partial_x^2 w\|_{L^2}
\end{aligned}$$

for some constant $c > 0$. Plugging (14) into (13) yields the first assertion.

Similarly we have that

$$\begin{aligned}
& | \langle u, (\partial_x v)^2 - e^{\partial_x^3 \tau} \left(e^{-\partial_x^3 \tau} \partial_x v \right)^2 \rangle | \leq 3\tau \sum_{k_1, k_2} |(k_1 + k_2) \hat{u}_{-(k_1+k_2)}| |k_1 k_2|^2 |\hat{v}_{k_1} \hat{v}_{k_2}| \\
(15) \quad &= 3\tau \sum_{k, l} |l| |\hat{u}_{-l}| k^2 (l-k)^2 |\hat{v}_k \hat{v}_{l-k}| \leq 3\tau \left(\sum_k k^4 |\hat{v}_k|^2 \right)^{1/2} \|u^{(1)} * v^{(2)}\|_{l^2} \\
&\leq c\tau \|\partial_x^2 v\|_{L^2} \|u^{(1)}\|_{l^1} \|v^{(2)}\|_{l^2} \leq c\tau \|\partial_x^2 v\|_{L^2}^2 \|\partial_x^2 u\|_{L^2}
\end{aligned}$$

which yields the second assertion. \square

Lemma 2.3 (Bound on I_1). — We have

$$|I_1| \leq c\tau \left(\|\partial_x^2(f-g)\|_{L^2} + \|\partial_x^3 g\|_{L^2} \right) \|\partial_x^2(f-g)\|_{L^2}^2$$

for some constant $c > 0$.

Proof. — Note that for all $t \in \mathbb{R}$ the following relation holds

$$\langle e^{t\partial_x^3} f, g \rangle = \langle f, e^{-t\partial_x^3} g \rangle.$$

Thus, by setting $(\tilde{f}, \tilde{g}) = e^{-t\partial_x^3}(f, g)$ we obtain that

$$\begin{aligned} I_1 &= \langle \partial_x^2 \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{f} \right)^2 - \partial_x^2 \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{g} \right)^2, e^{-\partial_x^3 \tau} \partial_x^2(\tilde{f} - \tilde{g}) \rangle \\ &\quad - \langle \partial_x^2 \left(\partial_x^{-1} \tilde{f} \right)^2 - \partial_x^2 \left(\partial_x^{-1} \tilde{g} \right)^2, \partial_x^2(\tilde{f} - \tilde{g}) \rangle. \end{aligned}$$

Using the relation $f^2 - g^2 = (f-g)^2 + 2(f-g)g$ as well as the chain rule yields that

$$\begin{aligned} I_1 &= \langle \partial_x^2 \left(e^{-\partial_x^3 \tau} \partial_x^{-1}(\tilde{f} - \tilde{g}) \right)^2 + 2\partial_x^2 \left(e^{-\partial_x^3 \tau} \partial_x^{-1}(\tilde{f} - \tilde{g}) \right) \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{g} \right), e^{-\partial_x^3 \tau} \partial_x^2(\tilde{f} - \tilde{g}) \rangle \\ &\quad - \langle \partial_x^2 \left(\partial_x^{-1}(\tilde{f} - \tilde{g}) \right)^2 + 2\partial_x^2 \left(\partial_x^{-1}(\tilde{f} - \tilde{g}) \right) \left(\partial_x^{-1} \tilde{g} \right), \partial_x^2(\tilde{f} - \tilde{g}) \rangle \\ &= 2\langle \left(e^{-\partial_x^3 \tau}(\tilde{f} - \tilde{g}) \right)^2 + \left(e^{-\partial_x^3 \tau} \partial_x^{-1}(\tilde{f} - \tilde{g}) \right) \left(e^{-\partial_x^3 \tau} \partial_x(\tilde{f} - \tilde{g}) \right), e^{-\partial_x^3 \tau} \partial_x^2(\tilde{f} - \tilde{g}) \rangle \\ &\quad + 2\langle \left(e^{-\partial_x^3 \tau} \partial_x(\tilde{f} - \tilde{g}) \right) \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{g} \right) + 2 \left(e^{-\partial_x^3 \tau}(\tilde{f} - \tilde{g}) \right) \left(e^{-\partial_x^3 \tau} \tilde{g} \right), e^{-\partial_x^3 \tau} \partial_x^2(\tilde{f} - \tilde{g}) \rangle \\ &\quad + 2\langle \left(e^{-\partial_x^3 \tau} \partial_x^{-1}(\tilde{f} - \tilde{g}) \right) \left(e^{-\partial_x^3 \tau} \partial_x \tilde{g} \right), e^{-\partial_x^3 \tau} \partial_x^2(\tilde{f} - \tilde{g}) \rangle \\ &\quad - 2\langle \left(\partial_x(\tilde{f} - \tilde{g}) \right) \left(\partial_x^{-1}(\tilde{f} - \tilde{g}) \right) + (\tilde{f} - \tilde{g})^2, \partial_x^2(\tilde{f} - \tilde{g}) \rangle \\ &\quad - 2\langle \left(\partial_x^{-1}(\tilde{f} - \tilde{g}) \right) \partial_x \tilde{g} + 2(\tilde{f} - \tilde{g})\tilde{g} + \left(\partial_x(\tilde{f} - \tilde{g}) \right) \left(\partial_x^{-1} \tilde{g} \right), \partial_x^2(\tilde{f} - \tilde{g}) \rangle. \end{aligned}$$

Next we use another key fact namely that

$$\langle vu, \partial_x u \rangle = \frac{1}{2} \langle v, \partial_x(u^2) \rangle = -\frac{1}{2} \langle \partial_x v, u^2 \rangle$$

as well as that

$$\langle u - v, (\partial_x(u - v))^2 \rangle = -\frac{1}{2} \langle (u - v)^2, \partial_x^2(u - v) \rangle.$$

This yields that

$$\begin{aligned}
I_1 &= 2\langle (e^{-\partial_x^3 \tau}(\tilde{f} - \tilde{g}))^2, e^{-\partial_x^3 \tau} \partial_x^2(\tilde{f} - \tilde{g}) \rangle \\
&+ \frac{1}{2}\langle (e^{-\partial_x^3 \tau}(\tilde{f} - \tilde{g}))^2, \partial_x^2 e^{-\partial_x^3 \tau}(\tilde{f} - \tilde{g}) \rangle \\
&- \langle e^{-\partial_x^3 \tau} \tilde{g}, (e^{-\partial_x^3 \tau} \partial_x(\tilde{f} - \tilde{g}))^2 \rangle \\
&+ 4\langle (e^{-\partial_x^3 \tau}(\tilde{f} - \tilde{g})) (e^{-\partial_x^3 \tau} \tilde{g}), e^{-\partial_x^3 \tau} \partial_x^2(\tilde{f} - \tilde{g}) \rangle \\
&+ 2\langle (e^{-\partial_x^3 \tau} \partial_x^{-1}(\tilde{f} - \tilde{g})) (e^{-\partial_x^3 \tau} \partial_x \tilde{g}), e^{-\partial_x^3 \tau} \partial_x^2(\tilde{f} - \tilde{g}) \rangle \\
&- \frac{1}{2}\langle (\tilde{f} - \tilde{g})^2, \partial_x^2(\tilde{f} - \tilde{g}) \rangle \\
&- 2\langle (\tilde{f} - \tilde{g})^2, \partial_x^2(\tilde{f} - \tilde{g}) \rangle \\
&- 2\langle (\partial_x^{-1}(\tilde{f} - \tilde{g})) \partial_x \tilde{g}, \partial_x^2(\tilde{f} - \tilde{g}) \rangle \\
&- 4\langle (\tilde{f} - \tilde{g}) \tilde{g}, \partial_x^2(\tilde{f} - \tilde{g}) \rangle \\
&+ \langle \tilde{g}, (\partial_x(\tilde{f} - \tilde{g}))^2 \rangle.
\end{aligned}$$

Thus, rearranging the terms leads to

$$\begin{aligned}
I_1 &= \frac{5}{2}\langle e^{\partial_x^3 \tau} (e^{-\partial_x^3 \tau}(\tilde{f} - \tilde{g}))^2 - (\tilde{f} - \tilde{g})^2, \partial_x^2(\tilde{f} - \tilde{g}) \rangle \\
&- \langle e^{\partial_x^3 \tau} (e^{-\partial_x^3 \tau} \partial_x(\tilde{f} - \tilde{g}))^2 - (\partial_x(\tilde{f} - \tilde{g}))^2, \tilde{g} \rangle \\
&+ 4\langle e^{\partial_x^3 \tau} (e^{-\partial_x^3 \tau}(\tilde{f} - \tilde{g})) (e^{-\partial_x^3 \tau} \tilde{g}) - (\tilde{f} - \tilde{g})(\tilde{g}), \partial_x^2(\tilde{f} - \tilde{g}) \rangle \\
&+ 2\langle e^{\partial_x^3 \tau} (e^{-\partial_x^3 \tau} \partial_x^{-1}(\tilde{f} - \tilde{g})) (e^{-\partial_x^3 \tau} \partial_x \tilde{g}) - (\partial_x^{-1}(\tilde{f} - \tilde{g})) (\partial_x \tilde{g}), \partial_x^2(\tilde{f} - \tilde{g}) \rangle.
\end{aligned}$$

With the aid of Lemma 2.2 we thus obtain that

$$|I_1| \leq \tau c \left(\|\partial_x^2(f - g)\|_{L^2} + \|\partial_x^3 g\|_{L^2} \right) \|\partial_x^2(f - g)\|_{L^2}^2$$

for some constant $c > 0$. □

Lemma 2.4 (Bound on I_2). — *We have*

$$|I_2| \leq \tau M \|\partial_x^2(f - g)\|_{L^2}^2,$$

where M depends on $\|\partial_x^2 f\|_{L^2}$ and $\|\partial_x^2 g\|_{L^2}$.

Proof. — In the following let M denote a constant depending on $\|\partial_x^2 f\|_{L^2}$ and $\|\partial_x^2 g\|_{L^2}$. Setting $(\tilde{f}, \tilde{g}) = e^{-t\partial_x^3}(f, g)$ yields that

$$\begin{aligned}
(16) \quad I_2 &= \langle \partial_x^2 \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{f} \right)^2 - \partial_x^2 \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{g} \right)^2, \partial_x^2 \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{f} \right)^2 - \partial_x^2 \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{g} \right)^2 \rangle \\
&\quad - 2 \langle \partial_x^2 e^{\partial_x^3 \tau} \left[\left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{f} \right)^2 - \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{g} \right)^2 \right], \partial_x^2 \left(\partial_x^{-1} \tilde{f} \right)^2 - \partial_x^2 \left(\partial_x^{-1} \tilde{g} \right)^2 \rangle \\
&\quad + \langle \partial_x^2 \left(\partial_x^{-1} \tilde{f} \right)^2 - \partial_x^2 \left(\partial_x^{-1} \tilde{g} \right)^2, \partial_x^2 \left(\partial_x^{-1} \tilde{f} \right)^2 - \partial_x^2 \left(\partial_x^{-1} \tilde{g} \right)^2 \rangle \\
&= I_2^a + I_2^b
\end{aligned}$$

with

$$\begin{aligned}
I_2^a &= \langle \partial_x^2 \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{f} \right)^2 - \partial_x^2 \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{g} \right)^2 - \partial_x^2 e^{-\partial_x^3 \tau} \left[\left(\partial_x^{-1} \tilde{f} \right)^2 - \left(\partial_x^{-1} \tilde{g} \right)^2 \right], \right. \\
&\quad \left. \partial_x^2 \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{f} \right)^2 - \partial_x^2 \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{g} \right)^2 \right\rangle \\
I_2^b &= - \langle \partial_x^2 e^{\partial_x^3 \tau} \left[\left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{f} \right)^2 - \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{g} \right)^2 \right] - \partial_x^2 \left(\partial_x^{-1} \tilde{f} \right)^2 - \partial_x^2 \left(\partial_x^{-1} \tilde{g} \right)^2, \right. \\
&\quad \left. \partial_x^2 \left(\partial_x^{-1} \tilde{f} \right)^2 - \partial_x^2 \left(\partial_x^{-1} \tilde{g} \right)^2 \right\rangle.
\end{aligned}$$

Similarly to Lemma 2.2 we obtain with $F := \partial_x^2 \left(\partial_x^{-1} \tilde{f} \right)^2 - \partial_x^2 \left(\partial_x^{-1} \tilde{g} \right)^2$ by the key relation (3) using the Cauchy-Schwarz and Young inequality that

$$\begin{aligned}
(17) \quad |I_2^b| &= \left| \sum_{k_1, k_2} \hat{F}_{-(k_1+k_2)} \frac{(k_1+k_2)^2}{k_1 k_2} \left(1 - e^{-i\tau((k_1+k_2)^3 - k_1^3 - k_2^3)} \right) \left(\hat{f}_{k_1} \hat{f}_{k_2} - \hat{g}_{k_1} \hat{g}_{k_2} \right) \right| \\
&= \left| \sum_{k_1, k_2} \hat{F}_{-(k_1+k_2)} \frac{(k_1+k_2)^2}{k_1 k_2} \left(1 - e^{-i\tau 3k_1 k_2 (k_1+k_2)} \right) \left(\hat{f}_{k_1} \hat{f}_{k_2} - \hat{g}_{k_1} \hat{g}_{k_2} \right) \right| \\
&\leq 3\tau \sum_{k_1, k_2} |\hat{F}_{-(k_1+k_2)}| (k_1+k_2) \left| (k_1+k_2)^2 \left(\hat{f}_{k_1} \hat{f}_{k_2} - \hat{g}_{k_1} \hat{g}_{k_2} \right) \right| \\
&\leq 3\tau \sum_{k, l} |l \hat{F}_{-l}| \left((l-k)^2 + 2|(l-k)k| + k^2 \right) \left| (\tilde{f}_k - \tilde{g}_k) \tilde{f}_{l-k} + \tilde{g}_k (\tilde{f}_{l-k} - \tilde{g}_{l-k}) \right| \\
&\leq 6\tau \|\partial_x F\|_{L^2} \left(\sum_{j=0,1} \|(\tilde{f} - \tilde{g})^{(2j)} * \tilde{g}^{(2-2j)}\|_{l^2} + \|(\tilde{f} - \tilde{g})^{(2j)} * \tilde{f}^{(2-2j)}\|_{l^2} \right) \\
&\quad + 6\tau \|\partial_x F\|_{L^2} \left(\|(\tilde{f} - \tilde{g})^{(1)} * \tilde{f}^{(1)}\|_{l^2} + \|(\tilde{f} - \tilde{g})^{(1)} * \tilde{g}^{(1)}\|_{l^2} \right) \\
&\leq c\tau \|\partial_x F\|_{L^2} \|\partial_x^2(f-g)\|_{L^2} (\|\partial_x^2 f\|_{L^2} + \|\partial_x^2 g\|_{L^2}) \\
&\leq \tau M \|\partial_x^2(f-g)\|_{L^2}^2,
\end{aligned}$$

where again we used the notation $\Phi^{(j)}(k) := |k|^j |\hat{\Phi}_k|$. Similarly, we obtain for I_2^a with $F := \partial_x^2 \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \hat{f} \right)^2 - \partial_x^2 \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \hat{g} \right)^2$ that

$$\begin{aligned}
|I_2^a| &= \left| \sum_{k_1, k_2} \hat{F}_{-(k_1+k_2)} \frac{(k_1+k_2)^2}{k_1 k_2} \left(e^{i\tau(k_1^3+k_2^3)} - e^{i\tau(k_1+k_2)^3} \right) \left(\hat{f}_{k_1} \hat{f}_{k_2} - \hat{g}_{k_1} \hat{g}_{k_2} \right) \right| \\
(18) \quad &\leq \sum_{k_1, k_2} \left| \hat{F}_{-(k_1+k_2)} \frac{(k_1+k_2)^2}{k_1 k_2} \right| \left| 1 - e^{-i\tau 3k_1 k_2 (k_1+k_2)} \right| \left| \hat{f}_{k_1} \hat{f}_{k_2} - \hat{g}_{k_1} \hat{g}_{k_2} \right| \\
&\leq \tau M \|\partial_x^2(f-g)\|_{L^2}^2.
\end{aligned}$$

Plugging the bounds (17) and (18) into (16) yields the assertion. \square

2.1.2. Local error analysis. — Let ϕ^t denote the exact flow associated to the reformulated KdV equation (4), i.e., $v(t) = \phi^t(v(0))$. The following local error bound holds for the exponential-type integrator Φ^τ defined in (10) with $v^{k+1} = \Phi_{t_k}^\tau(v^k)$.

Lemma 2.5. — *Let $v(t_k + t) = \phi^t(v(t_k)) \in H^3$ for $0 \leq t \leq \tau$. Then*

$$\|\partial_x^2(\phi^\tau(v(t_k)) - \Phi_{t_k}^\tau(v(t_k)))\|_{L^2} \leq c\tau^{3/2},$$

where c depends on $\sup_{0 \leq t \leq \tau} \|\phi^t(v(t_k))\|_{H^3}$.

Proof. — As $e^{t\partial_x^3}$ is a linear isometry in H^r for all $t \in \mathbb{R}$ the iteration of Duhamel's formula (4) yields that

$$\begin{aligned}
(19) \quad \|\phi^\tau(v(t_k)) - \Phi_{t_k}^\tau(v(t_k))\|_{H^2} &\leq \int_0^\tau \left\| \left(e^{-\partial_x^3(t_k+s)} \phi^s(v(t_k)) \right)^2 - \left(e^{-\partial_x^3(t_k+s)} v(t_k) \right)^2 \right\|_{H^3} ds \\
&\leq \tau c_1 \sup_{0 \leq t \leq \tau} \|\phi^t(v(t_k)) - v(t_k)\|_{H^3},
\end{aligned}$$

where c_1 depends on $\sup_{0 \leq t \leq \tau} \|\phi^t(v(t_k))\|_{H^3}$. Duhamel's formula (4) and integration by parts furthermore yields that

$$\begin{aligned}
\|\phi^t(v(t_j)) - v(t_j)\|_{H^3} &\leq \left\| \int_0^t e^{(t_j+s)\partial_x^3} \partial_x \left(e^{-\partial_x^3(t_j+s)} v(t_j+s) \right)^2 ds \right\|_{H^3} \\
&\leq \left\| \sum_{k_1, k_2} \frac{1}{k_1 k_2} e^{-3it_j k_1 k_2 (k_1+k_2)} \left(e^{-3itk_1 k_2 (k_1+k_2)} - 1 \right) \hat{v}_{k_1}(t_j+t) \hat{v}_{k_2}(t_j+t) e^{i(k_1+k_2)x} \right\|_{H^3} \\
(20) \quad &+ \left\| \sum_{k_1, k_2} \frac{1}{k_1 k_2} e^{-3it_j k_1 k_2 (k_1+k_2)} \left(\hat{v}_{k_1}(t_j+t) \hat{v}_{k_2}(t_j+t) - \hat{v}_{k_1}(t_j) \hat{v}_{k_2}(t_j) \right) e^{i(k_1+k_2)x} \right\|_{H^3} \\
&+ \left\| \int_0^t \sum_{k_1, k_2} e^{-3i(t_j+s)k_1 k_2 (k_1+k_2)} \frac{1}{k_1 k_2} \frac{d}{ds} \left(\hat{v}_{k_1}(t_j+s) \hat{v}_{k_2}(t_j+s) \right) ds e^{i(k_1+k_2)x} \right\|_{H^3} \\
&\leq ct^{1/2} \sup_{k_1, k_2 \in \mathbb{Z} \neq 0} \frac{|k_1+k_2|^{1/2}}{|k_1 k_2|^{1/2}} \|v(t_j+t)\|_{H^3}^2 + ct \sup_{k_1, k_2 \in \mathbb{Z} \neq 0} \frac{|k_1|+|k_2|}{|k_1 k_2|} \sup_{0 \leq s \leq t} \|v(t_j+s)\|_{H^3}^3.
\end{aligned}$$

Plugging (20) into (19) yields the assertion. \square

2.1.3. Global error bound. — The stability analysis in Section 2.1.1 and local error analysis in Section 2.1.2 allows us the following global error bound in H^2 .

Theorem 2.6. — *Let the solution of (4) satisfy $v(t) \in H^3$ for $t \leq T$. Then there exists a $\tau_0 > 0$ such that for all $\tau \leq \tau_0$ and $t_n \leq T$ we have*

$$\|v(t_n) - v^n\|_{H^2} \leq c\tau^{1/2},$$

where c depends on $\sup_{0 \leq t \leq t_n} \|v(t)\|_{H^3}$ and t_n , but can be chosen independently of τ .

Proof. — The triangular inequality yields that

$$(21) \quad \begin{aligned} \|v(t_{k+1}) - v^{k+1}\|_{H^2} &= \|\phi^\tau(v(t_k)) - \Phi_{t_k}^\tau(v^k)\|_{H^2} \\ &\leq \|\phi^\tau(v(t_k)) - \Phi_{t_k}^\tau(v(t_k))\|_{H^2} + \|\Phi_{t_k}^\tau(v(t_k)) - \Phi_{t_k}^\tau(v^k)\|_{H^2}. \end{aligned}$$

Thus, iterating the estimate (21) we obtain with the aid of Lemma 2.1 (with $g = v(t_k) \in H^3$) and Lemma 2.5 that as long as $v^k \in H^2$ (for $0 \leq k \leq n$) we have that

$$\begin{aligned} \|v(t_{n+1}) - v^{n+1}\|_{H^2} &\leq c\tau^{3/2} + e^{\tau L} \|v(t_n) - v^n\|_{H^2} \leq c\tau^{3/2} + e^{\tau L} \left(c\tau^{3/2} + e^{\tau L} \|v(t_{n-1}) - v^{n-1}\|_{H^2} \right) \\ &\leq c\tau^{3/2} \sum_{k=0}^n e^{t_k L} \leq c\tau^{1/2} t_n e^{t_n L}, \end{aligned}$$

where c depends on $\sup_{0 \leq t \leq t_{n+1}} \|v(t)\|_{H^3}$, L depends on $\sup_{0 \leq k \leq n} \|v(t_k)\|_{H^3}$ as well as on $\sup_{0 \leq k \leq n} \|v^k\|_{H^2}$ and we have used the fact that $\hat{v}_0(t_n) \equiv \hat{v}_0^n$. The assertion then follows by a bootstrap, respectively, "Lady Windermere's fan" argument, see, for example [3, 6, 10, 16]. \square

2.2. Error analysis in H^1 . — The error analysis in H^2 of the numerical scheme (6) given in Section 2.1 yields a priori bounds on the numerical solution in H^2 for solutions in H^3 . This allows us to derive the following first-order convergence bound in H^1 .

Theorem 2.7. — *Let the solution of (4) satisfy $v(t) \in H^3$ for $t \leq T$. Then there exists a $\tau_0 > 0$ such that for all $\tau \leq \tau_0$ and $t_n \leq T$ we have*

$$\|v(t_n) - v^n\|_{H^1} \leq c\tau,$$

where c depends on $\sup_{0 \leq t \leq t_n} \|v(t)\|_{H^3}$ and t_n , but can be chosen independently of τ .

Proof. — Note that Duhamel's formula (4) implies the first-order consistency bound

$$(22) \quad \begin{aligned} \|\partial_x(\phi^\tau(v(t_k)) - \Phi_{t_k}^\tau(v(t_k)))\|_{L^2} &\leq \int_0^\tau \|\partial_x^2 \left[\left(e^{-\partial_x^3(t_k+s)} \phi^s(v(t_k)) \right)^2 - \left(e^{-\partial_x^2(t_k+s)} v(t_k) \right)^2 \right]\|_{L^2} ds \\ &\leq \tau c_1 \sup_{0 \leq t \leq \tau} \|\phi^t(v(t_k)) - v(t_k)\|_{H^2} \\ &\leq \tau^2 c_1 \sup_{0 \leq t \leq \tau} \|\phi^t(v(t_k))\|_{H^3}, \end{aligned}$$

where c_1 depends on $\sup_{0 \leq t \leq \tau} \|\phi^t(v(t_k))\|_{H^2}$.

Furthermore, as $v(t) \in H^3$ for $t \leq T$ we have the boundedness of the numerical solution in H^2 a priori thanks to Theorem 2.6, i.e., there exists a $\tau_0 > 0$ such that for all $\tau \leq \tau_0$ $v^n \in H^2$ as long as $t_n \leq T$. In particular a stability estimate of type

$$(23) \quad \|\partial_x(\Phi_t^\tau(f) - \Phi_t^\tau(g))\|_{L^2} \leq \exp(\tau L) \|\partial_x(f - g)\|_{L^2}, \quad L = L(\|\partial_x^2 f\|_{L^2}, \|\partial_x^3 g\|_{L^2})$$

is therefore sufficient for our bootstrapping argument in H^1 by choosing $f = v^n \in H^2$ and $g = v(t_n) \in H^3$. The stability bound (23) follows similarly to Lemma 2.1: Note that

$$(24) \quad \begin{aligned} & \|\partial_x(\Phi_t^\tau(f) - \Phi_t^\tau(g))\|_{L^2}^2 = \|\partial_x(f - g)\|_{L^2}^2 \\ & + \frac{1}{3} \langle \partial_x e^{\partial_x^3(t+\tau)} \left[\left(e^{-\partial_x^3(t+\tau)} \partial_x^{-1} f \right)^2 - \left(e^{-\partial_x^3(t+\tau)} \partial_x^{-1} g \right)^2 \right], \partial_x(f - g) \rangle \\ & - \frac{1}{3} \langle \partial_x e^{\partial_x^3 t} \left[\left(e^{-\partial_x^3 t} \partial_x^{-1} f \right)^2 - \left(e^{-\partial_x^3 t} \partial_x^{-1} g \right)^2 \right], \partial_x(f - g) \rangle \\ & + \frac{1}{6^2} \|\partial_x e^{\partial_x^3(t+\tau)} \left[\left(e^{-\partial_x^3(t+\tau)} \partial_x^{-1} f \right)^2 - \left(e^{-\partial_x^3(t+\tau)} \partial_x^{-1} g \right)^2 \right] \\ & - \partial_x e^{\partial_x^3 t} \left[\left(e^{-\partial_x^3 t} \partial_x^{-1} f \right)^2 - \left(e^{-\partial_x^3 t} \partial_x^{-1} g \right)^2 \right]\|_{L^2}^2 \\ & =: \|\partial_x(f - g)\|_{L^2}^2 + \frac{1}{3} I_1 + \frac{1}{6^2} I_2. \end{aligned}$$

Similarly to the proof of Lemma 2.3 we can rewrite I_1 as

$$(25) \quad \begin{aligned} I_1 &= \langle \tilde{f} - \tilde{g}, (\tilde{f} - \tilde{g})^2 - e^{\partial_x^3 \tau} \left(e^{-\partial_x^3 \tau} (\tilde{f} - \tilde{g}) \right)^2 \rangle \\ &+ \langle \tilde{g}, (\tilde{f} - \tilde{g})^2 - e^{\partial_x^3 \tau} \left(e^{-\partial_x^3 \tau} (\tilde{f} - \tilde{g}) \right)^2 \rangle \\ &- 2 \langle \tilde{f} - \tilde{g}, \tilde{g}(\tilde{f} - \tilde{g}) - e^{\partial_x^3 \tau} \left[\left(e^{-\partial_x^3 \tau} \tilde{g} \right) \left(e^{-\partial_x^3 \tau} (\tilde{f} - \tilde{g}) \right) \right] \rangle \\ &- 2 \langle \tilde{f} - \tilde{g}, (\partial_x \tilde{g}) \left(\partial_x^{-1} (\tilde{f} - \tilde{g}) \right) - e^{\partial_x^3 \tau} \left[\left(e^{-\partial_x^3 \tau} (\partial_x \tilde{g}) \right) \left(e^{-\partial_x^3 \tau} \partial_x^{-1} (\tilde{f} - \tilde{g}) \right) \right] \rangle. \end{aligned}$$

As in Lemma 2.2 we obtain by the key relation (3) that

$$I(u, v, w) := |\langle u, vw - e^{\partial_x^3 \tau} \left[\left(e^{-\partial_x^3 \tau} v \right) \left(e^{-\partial_x^3 \tau} w \right) \right] \rangle| \leq 3\tau \sum_{k_1, k_2} |(k_1 + k_2) \hat{u}_{-(k_1+k_2)}| \cdot |k_1 k_2| |\hat{v}_{k_1} \hat{w}_{k_2}|.$$

The Cauchy-Schwarz and Young inequality furthermore yield that

$$(26) \quad \begin{aligned} I(u, v, w) &\leq 3\tau \sum_{k, l} |l| |\hat{u}_{-l}| |(l - k)k| |\hat{v}_k \hat{w}_{l-k}| \leq 3\tau \left(\sum_l |l|^2 |\hat{u}_l|^2 \right)^{1/2} \|v^{(1)} * w^{(1)}\|_{l^2} \\ &\leq c\tau \|\partial_x u\|_{L^2} \min \left(\|v^{(1)}\|_{l^1} \|w^{(1)}\|_{l^2}, \|v^{(1)}\|_{l^2} \|w^{(1)}\|_{l^1} \right) \\ &\leq c\tau \|\partial_x u\|_{L^2} \min \left(\|\partial_x^2 v\|_{L^2} \|\partial_x w\|_{L^2}, \|\partial_x^2 w\|_{L^2} \|\partial_x v\|_{L^2} \right). \end{aligned}$$

The above bound allows us to control the first and last two terms in (25) as long as $f - g, g \in H^2$. Furthermore,

$$(27) \quad \begin{aligned} I(u, v, v) &\leq 3\tau \sum_{k,l} |l| |\hat{u}_{-l}| |(l-k)k| |\hat{v}_k \hat{v}_{l-k}| \leq 3\tau \left(\sum_k |k|^2 |\hat{v}_k|^2 \right)^{1/2} \|u^{(1)} * v^{(1)}\|_{l^2} \\ &\leq 3\tau \|\partial_x v\|_{L^2} \|u^{(1)}\|_{l^1} \|v^{(1)}\|_{l^2} \leq c\tau \|\partial_x v\|_{L^2}^2 \|\partial_x^2 u\|_{L^2}, \end{aligned}$$

which allows us to control the second term in (25) as long as $f - g \in H^1$ and $g \in H^2$.

Using the bounds (26) and (27) in (25) yields that

$$(28) \quad |I_1| \leq \tau L \|\partial_x(f - g)\|_{L^2}^2, \quad L = L(\|\partial_x^2(f - g)\|_{L^2}, \|\partial_x^2 g\|_{L^2}).$$

Next we write $I_2 = I_2^a + I_2^b$ with

$$\begin{aligned} I_2^a &= \langle \partial_x \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{f} \right)^2 - \partial_x \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{g} \right)^2 - \partial_x e^{-\partial_x^3 \tau} \left[\left(\partial_x^{-1} \tilde{f} \right)^2 - \left(\partial_x^{-1} \tilde{g} \right)^2 \right], \right. \\ &\quad \left. \partial_x \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{f} \right)^2 - \partial_x \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{g} \right)^2 \right\rangle \\ I_2^b &= - \langle \partial_x e^{\partial_x^3 \tau} \left[\left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{f} \right)^2 - \left(e^{-\partial_x^3 \tau} \partial_x^{-1} \tilde{g} \right)^2 \right] - \partial_x \left(\partial_x^{-1} \tilde{f} \right)^2 - \partial_x \left(\partial_x^{-1} \tilde{g} \right)^2, \right. \\ &\quad \left. \partial_x \left(\partial_x^{-1} \tilde{f} \right)^2 - \partial_x \left(\partial_x^{-1} \tilde{g} \right)^2 \right\rangle. \end{aligned}$$

Note that by the Cauchy-Schwarz and Young inequality we have with $F := \partial_x \left(\partial_x^{-1} \tilde{f} \right)^2 - \partial_x \left(\partial_x^{-1} \tilde{g} \right)^2$ that

$$(29) \quad \begin{aligned} |I_2^b| &= \left| \sum_{k_1, k_2} \hat{F}_{-(k_1+k_2)} \frac{(k_1+k_2)}{k_1 k_2} \left(1 - e^{-i\tau((k_1+k_2)^3 - k_1^3 - k_2^3)} \right) \left(\hat{f}_{k_1} \hat{f}_{k_2} - \hat{g}_{k_1} \hat{g}_{k_2} \right) \right| \\ &= \left| \sum_{k_1, k_2} \hat{F}_{-(k_1+k_2)} \frac{(k_1+k_2)}{k_1 k_2} \left(1 - e^{-i\tau 3k_1 k_2 (k_1+k_2)} \right) \left(\hat{f}_{k_1} \hat{f}_{k_2} - \hat{g}_{k_1} \hat{g}_{k_2} \right) \right| \\ &\leq 3\tau \sum_{k_1, k_2} |\hat{F}_{-(k_1+k_2)}| (k_1+k_2) \left| \left(\hat{f}_{k_1} \hat{f}_{k_2} - \hat{g}_{k_1} \hat{g}_{k_2} \right) \right| \\ &\leq 3\tau \left(\sum_l l^2 |\hat{F}_l|^2 \right)^{1/2} \left(\|(\tilde{f} - \tilde{g})^{(1)} * \tilde{f}^{(0)}\|_{l^2} + \|(\tilde{f} - \tilde{g})^{(0)} * \tilde{f}^{(1)}\|_{l^2} \right) \\ &\quad + 3\tau \left(\sum_l l^2 |\hat{F}_l|^2 \right)^{1/2} \left(\|(\tilde{f} - \tilde{g})^{(1)} * \tilde{g}^{(0)}\|_{l^2} + \|(\tilde{f} - \tilde{g})^{(0)} * \tilde{g}^{(1)}\|_{l^2} \right) \\ &\leq c\tau \|\partial_x F\|_{L^2} \left(\|(\tilde{f} - \tilde{g})^{(1)}\|_{l^2} (\|\tilde{f}^{(0)}\|_{l^1} + \|\tilde{g}^{(0)}\|_{l^1}) + \|(\tilde{f} - \tilde{g})^{(0)}\|_{l^1} (\|\tilde{f}^{(1)}\|_{l^2} + \|\tilde{g}^{(1)}\|_{l^2}) \right) \\ &\leq M\tau \|\partial_x(\tilde{f} - \tilde{g})\|_{L^2}^2, \end{aligned}$$

where M depends on $\|\partial_x f\|_{L^2}$ and $\|\partial_x g\|_{L^2}$. A similar bound holds for I_2^a which implies that

$$(30) \quad |I_2| \leq M\tau \|\partial_x(f - g)\|_{L^2}^2, \quad M = M(\|\partial_x f\|_{L^2}, \|\partial_x g\|_{L^2}).$$

Plugging the bounds (28) as well as (30) into (24) yields the stability estimate (23).

With the aid of the stability estimate (23) and the local error bound (22) the proof then follows the line of argumentation to the proof of Theorem 2.6. \square

Corollary 2.8. — *Let the solution of the KdV equation (1) satisfy $u(t) \in H^3$ for $t \leq T$. Then there exists a $\tau_0 > 0$ such that for all $\tau \leq \tau_0$ and $t_n \leq T$ the exponential-type integration scheme (7) is first-order convergent in H^1 , i.e.,*

$$\|u(t_n) - u^n\|_{H^1} \leq c\tau,$$

where c depends on $\sup_{0 \leq t \leq t_n} \|u(t)\|_{H^3}$ and t_n , but can be chosen independently of τ .

Proof. — The assertion follows from Theorem 2.7 as $e^{t\partial_x^3}$ is a linear isometry in H^1 for all $t \in \mathbb{R}$. \square

3. Numerical experiments

In this section we numerically underline the first-order convergence rate of the numerical scheme (7) approximating the solution $u(t_n)$ of the KdV equation (1) given in Corollary 2.8. For the space discretization we use a Fourier pseudo spectral method, see [17], where we choose the largest Fourier mode $K = 2^{12}$ (i.e., the spatial mesh size $\Delta x = 0.0015$) and integrate up to $T = 2$. The error measured in the corresponding discrete H^1 norm for the initial value $u(0, x) = 2\operatorname{sech}(\frac{1}{2}x)^2 \sin(x)$ is illustrated in Figure 1.

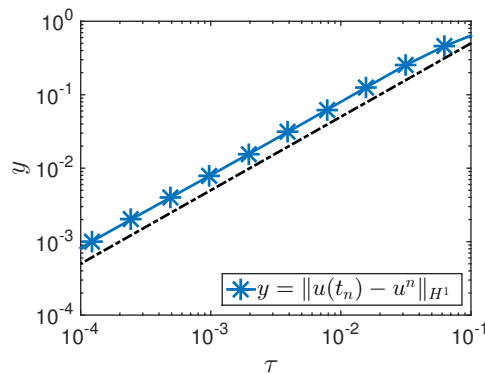


FIGURE 1. Orderplot (double logarithmic). Error in u^n defined in (7) measured in a discrete H^1 norm. The slope of the dashed line is one.

Acknowledgement

K. Schratz gratefully acknowledges financial support by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173.

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