WEAK SOLUTIONS FOR A STOCHASTIC MEAN CURVATURE FLOW OF TWO-DIMENSIONAL GRAPHS

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Abstract. We study a stochastically perturbed mean curvature flow for graphs in $\mathbb{R}^3$ over the two-dimensional unit-cube subject to periodic boundary conditions. In particular, we establish the existence of a weak martingale solution. The proof is based on energy methods and therefore presents an alternative to the stochastic viscosity solution approach. To overcome difficulties induced by the degeneracy of the mean curvature operator and the multiplicative gradient noise present in the model we employ a three step approximation scheme together with refined stochastic compactness and martingale identification methods.

1. Introduction

Motion by mean curvature of embedded hypersurfaces in $\mathbb{R}^{N+1}$ is an important prototype of a geometric evolution law and has been intensively studied in the past decades, see for example the surveys [52], [15], [44] or [6]. Mean curvature flow is characterized as a steepest descent evolution for the surface area energy (with respect to an L^2 metric) and constitutes a fundamental relaxation dynamics for many problems where the interface size contributes to the systems energy. In physics it arises for example as an asymptotic reduction of the Allen–Cahn model for the motion of phase boundaries in binary alloys [1].

One of the main difficulties in the mathematical treatment of mean curvature flow is the appearance of topological changes and singularities in finite time, for example by the development of corners and a collapse of parts of the surfaces onto a line in the evolution of a thin dumbbell-shape surface in $\mathbb{R}^3$. Only in particular situations such events are excluded: in the case of initial surfaces given by entire graphs over $\mathbb{R}^N$ classical solutions exist for all times [16]; initally smooth, compact, convex hypersurfaces become round and shrink to a point in finite time [30].

In order to deal with singularity formation and topological changes generalized formulations have been developed. In his pioneering work Brakke [7] employed a geometric measure theory approach to obtain a general global in time existence result. Level set approaches and viscosity solutions were introduced by Evans and Spruck [19, 20, 21, 22] and Chen, Giga, Goto [11]. Evolutions beyond singularity formation and topological changes can also be obtained by De Giorgi's barrier method [5, 4], approximation by the Allen–Cahn equation [18, 31, 3], time-discretization [42, 2] and by elliptic regularization [32]. Several of these approaches have been applied also to more general geometric evolution laws and for perturbations by various forcing terms.

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Stochastic mean curvature flow was proposed in [34] as a refined model incorporating the influence of thermal noise. As a result one may think of a random evolution $(M_t)_{t>0}$ of surfaces in $\mathbb{R}^{N+1}$ given by immersions $\phi_t : M \to \mathbb{R}^{N+1}$, where M is a smooth manifold, and where the increments are given by

$$\mathrm{d}\phi_t(x) = \vec{H}(x,t)\mathrm{d}t + W(\nu(x,t), \phi_t(x), \circ \mathrm{d}t), \quad x \in M, \tag{1.1}$$

where $\vec{H}(x,t)$ denotes the mean curvature vector of M_t in $\phi_t(x)$, $\nu(x,t)$ is the unit normal field on M_t and $W : \mathbb{S}^N \times \mathbb{R}^{N+1} \times \mathbb{R}^+ \to \mathbb{R}^{N+1}$ is a model specific random field with $W(\theta, y, \circ \mathrm{d}t)$ being its Stratonovich differential (here one could even allow for an additional dependence of W on M_t). As an example consider $W(\theta, y, t) = \theta\, \varphi(y)\beta_t$ for $\varphi \in C^\infty(\mathbb{R}^{N+1})$ with a standard real Brownian motion β, inducing the dynamics

$$\mathrm{d}\phi_t(x) = \nu(x,t)\big(\kappa(x,t)\mathrm{d}t + \varphi(\phi_t(x)) \circ \mathrm{d}\beta_t\big), \tag{1.2}$$

where $\kappa(x,t) := \vec{H}(x,t) \cdot \nu(x,t)$ denotes the scalar mean curvature. As in the deterministic case (1.2) can be formulated as a level set equation. Here the evolution of a function $f : \mathbb{R}^{N+1} \times \mathbb{R}^+ \to \mathbb{R}$ is prescribed whose level sets all evolve according to (1.2). This leads to a stochastic partial differential equation (SPDE) of the form

$$\mathrm{d}f(x,t) = |\nabla f|(x,t)\,\mathrm{div}\left(\frac{\nabla f}{|\nabla f|}\right)(x,t)\,\mathrm{d}t + \varphi(x, f(x,t))|\nabla f|(x,t) \circ \mathrm{d}\beta_t. \tag{1.3}$$

We stress that the choice of the Stratonovich differential instead of an Itô term is necessary to retain the geometric meaning of the equation and to make it invariant under reparametrization of the level set function [37].

If we restrict ourselves to random evolutions of graphs, scalar mean curvature, normal vector, and velocity of an evolution $u : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ and the associated graphs are given by

$$\kappa = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right), \quad \nu = \frac{1}{\sqrt{1+|\nabla u(x,t)|^2}}(-\nabla u, 1)^T,$$
$$\mathrm{d}\phi_t \cdot \nu = \frac{1}{\sqrt{1+|\nabla u(x,t)|^2}}\mathrm{d}u.$$

Equation (1.2) then reduces to the SPDE

$$\begin{aligned}\mathrm{d}u(x,t) = {} & \sqrt{1+|\nabla u(x,t)|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)(x,t)\mathrm{d}t \\ & + \sqrt{1+|\nabla u(x,t)|^2}\varphi(x,u(x)) \circ \mathrm{d}\beta_t.\end{aligned} \tag{1.4}$$

Note that we naturally obtain the factor $\sqrt{1+|\nabla u|^2}$ in front of the noise term and that (1.3) reduces to (1.4) for $f(x,y) = y - u(x)$, $(x,y) \in \mathbb{R}^N \times \mathbb{R}$. Vice versa, following the approach of Evans and Spruck [19] one could approximate (1.3) by a problem for rescaled graphs (in $\mathbb{R}^{N+2}$), which leads to an equation similar to (1.4) but with $\sqrt{1+|\nabla u|^2}$ replaced by $\sqrt{\varepsilon^2+|\nabla u|^2}$, $\varepsilon > 0$ a small parameter.

We further observe that the first term on the right-hand side of (1.4) can be rewritten as

$$\sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \left(\mathrm{Id} - \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \otimes \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) : \mathrm{D}^2 u \tag{1.5}$$

and that this term corresponds to a degenerate quasilinear elliptic differential operator of second order in the spatial variable.

Even though we circumvent problems with topological changes by restricting ourselves to graphs, substantial mathematical difficulties are still present in the stochastic case. Most importantly, one has to deal with the multiplicative noise with nonlinear gradient dependence and with the degeneracy in the quasilinear elliptic term, which makes a rigorous treatment challenging. In particular, a general well-posedness theory seems still to be missing. Motivated by the deterministic counterpart of (1.3) Lions and Souganidis introduced a notion of stochastic viscosity solutions [38, 39, 40, 41], but certain technical details of this approach are still being investigated [9, 10, 25]. The model (1.2) with constant $\varphi = \epsilon > 0$ was also studied independently in $N = 1$ by Souganidis and Yip [49] resp. Dirr, Luckhaus and Novaga [14], proving a 'stochastic selection principle' for ϵ tending to zero[1].

Several approaches to construct generalized solutions to other versions of (1.1) can be found in the literature, such as by Yip [51] who selects subsequential limits along tight approximations of a scheme that combines a time-discrete mean curvature flow and a stochastic flow of diffeomorphism of the ambient space. More recently, extending the rigorous analysis of the sharp interface limit of the 1-dimensional stochastic Allen-Cahn equation by Funaki [26] in [47] tightness of solutions for an Allen–Cahn equation perturbed by a stochastic flow was proved. However, both in [51] and in [47] a characterization of the limiting evolution law has not been given. Finally, it was shown in [17] that several variants of (1.1) in dimension $1+1$ can be solved in the variational SPDE framework (see also [27, 23] for refinements resp. numerical analysis), but this approach is not applicable in higher dimensions. For completeness let us also mention that the analysis of associated formal large deviation functionals was started in [35] and remains an active research field to date.

This paper is concerned with equation (1.4), where for simplicity we consider $\varphi = 1$, and graphs over the unit cube in $\mathbb{R}^N$ with periodic boundary condition, that is over the flat torus $\mathbb{T}^N$. This yields the SPDE initial-boundary-value problem

$$\begin{aligned} \mathrm{d}u &= H(\nabla u)\,\mathrm{div}\left(\frac{\nabla u}{H(\nabla u)}\right)\mathrm{d}t + H(\nabla u)\circ \mathrm{d}W, \\ u(0) &= u_0, \qquad t \in (0,T),\ x \in \mathbb{T}^N, \end{aligned} \tag{1.6}$$

where $H(\nabla u) = \sqrt{1 + |\nabla u|^2}$, W is a real-valued Wiener process and $\circ$ denotes the Stratonovich product.

We emphasize that this case is contained in the theory famously announced by Lions and Souganidis in [38, 39, 40, 41]. In this paper, however, our aim is to introduce an alternative approach that is based on energy methods and that yields the existence of weak martingale solutions to (1.6). Even if we consider here a more restrictive setting, we believe that our approach can be extended to more general situations and might be very helpful in problems where a comparison principle and viscosity solution formulations are not available.

The use of energy methods is motivated by the gradient structure of the deterministic mean curvature flow. We prove that also in the case of (1.6) we retain a control over the surface area energy and over the times-space integral of the squared mean curvature, see Proposition 5.1. In the deterministic case, in addition

[1] This means for $\epsilon \to 0$ the level sets the solutions f_t^ϵ to (1.3) converge a.s. to some solution of mean curvature flow even in cases when f_t^0 develops 'fattening', i.e. has zero level sets of positive Lebesgue measure.

one often can prove an L^∞ bound for the gradient (see for example [16]) and consequently the uniform ellipticity of the mean curvature operator. Such a bound is typically obtained from an evolution equation derived for the function $\sqrt{1+|\nabla u|^2}$ and cannot be expected for the stochastic equation (1.6). In contrast, our approach is based on an L^2 bound for ∇u, see Proposition 4.1. These bounds are carefully exploited in a three step approximation and corresponding passages to the limit. Several refined and original tightness and identification arguments together with compensated compactness and Young measures techniques are required, that we believe are of independent interest.

2. Mathematical framework and main results

Our main result is the existence of weak martingale solutions to the Itô form of (1.6) in the case $N = 2$. By a direct calculation one can verify that the Itô-Stratonovich correction corresponding to the stochastic integral in (1.6) is

$$\frac{1}{2}\frac{\nabla u}{H(\nabla u)} \otimes \frac{\nabla u}{H(\nabla u)} : \mathrm{D}^2 u \,\mathrm{d}t$$

and hence, in view of (1.5), equation (1.6) rewrites as

$$\begin{aligned} \mathrm{d}u &= \frac{1}{2}\Delta u \,\mathrm{d}t + \frac{1}{2} H(\nabla u) \operatorname{div}\left(\frac{\nabla u}{H(\nabla u)}\right) \mathrm{d}t + H(\nabla u)\,\mathrm{d}W, \\ u(0) &= u_0, \qquad t \in (0,T),\ x \in \mathbb{T}^N, \end{aligned} \tag{2.1}$$

or equivalently

$$\begin{aligned} \mathrm{d}u &= \Delta u \,\mathrm{d}t - \frac{1}{2}\frac{\nabla u}{H(\nabla u)} \otimes \frac{\nabla u}{H(\nabla u)} : \mathrm{D}^2 u \,\mathrm{d}t + H(\nabla u)\,\mathrm{d}W, \\ u(0) &= u_0, \qquad t \in (0,T),\ x \in \mathbb{T}^N. \end{aligned} \tag{2.2}$$

As we aim at establishing existence of a solution to (2.1) that is weak in both probabilistic and PDEs sense, let us introduce these two notions. From the point of view of the theory of PDEs, we consider solutions that satisfy (2.1) in the sense of distributions and that fulfill a suitable surface area energy inequality. This implies in particular that the mean curvature belongs to L^2 with respect to the surface area measure $H(\nabla u)$.

From the probabilistic point of view, two concepts of solution are typically considered in the theory of stochastic evolution equations, namely, pathwise (or strong) solutions and martingale (or weak) solutions. In the former notion the underlying probability space as well as the driving process is fixed in advance while in the latter case these stochastic elements become part of the solution of the problem. Clearly, existence of a pathwise solution is stronger and implies existence of a martingale solution. In the present work we establish existence of a martingale solution to (2.1). Due to the classical Yamada-Watanabe-type argument (see e.g. [36], [46]), existence of a pathwise solution would then follow if pathwise uniqueness held true, however, uniqueness for (2.1) is out of the scope of the present article. In hand with this issue goes the way how the initial condition is posed: we are given a Borel probability measure on $H^1(\mathbb{T}^N)$, hereafter denoted by Λ, that fulfills some further assumptions specified in Theorem 2.3 and plays the role of an initial law for (2.1), that is, we require that the law of $u(0)$ coincides with Λ.

Definition 2.1. Let Λ be a Borel probability measure on $H^1(\mathbb{T}^N)$. Then

$$\big((\Omega,\mathscr{F},(\mathscr{F}_t),\mathbb{P}),u,W\big)$$

is called a weak martingale solution to (2.1) with the initial law Λ provided

(i) $(\Omega,\mathscr{F},(\mathscr{F}_t),\mathbb{P})$ is a stochastic basis with a complete right-continuous filtration,
(ii) W is a real-valued $(\mathscr{F}_t)$-Wiener process,
(iii) $u\in L^2(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;H^1(\mathbb{T}^N))$,[2]
(iv) the area measure $H(\nabla u)$ belongs to $L^1(\Omega,L^\infty(0,T;L^1(\mathbb{T}^N)))$,
(v) the mean curvature

$$v=\operatorname{div}\left(\frac{\nabla u}{H(\nabla u)}\right)$$

belongs to $L^2\big(\Omega\times[0,T]\times\mathbb{T}^N,H(\nabla u)\,\mathrm{d}\mathbb{P}\otimes\mathrm{d}t\otimes\mathrm{d}x\big)$,
(vi) there exists a $\mathscr{F}_0$-measurable random variable u_0 such that $\Lambda=\mathbb{P}\circ u_0^{-1}$ and for every $\varphi\in C^\infty(\mathbb{T}^N)$ it holds true for a.e. $t\in[0,T]$ a.s.

$$\langle u(t),\varphi\rangle=\langle u_0,\varphi\rangle-\frac12\int_0^t\langle\nabla u,\nabla\varphi\rangle\mathrm{d}s+\frac12\int_0^t\langle H(\nabla u)v,\varphi\rangle\mathrm{d}s+\int_0^t\langle H(\nabla u)\mathrm{d}W,\varphi\rangle.$$

Remark 2.2. According to Definition 2.1(vi), equation (2.1) is satisfied in $H^{-1}(\mathbb{T}^N)$. In particular, the solution u regarded as a class of equivalence in

$$L^2(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;H^1(\mathbb{T}^N))$$

has a representative $\bar u$ with almost surely continuous trajectories in $H^{-1}(\mathbb{T}^N)$ and moreover $\bar u(0)=u_0$.

With this definition at hand we can formulate our main result.

Theorem 2.3. *Assume $N=2$ and that the initial law Λ satisfies*[3]

$$\int_{H^1_x}\|\nabla z\|^2_{L^2_x}\,\mathrm{d}\Lambda(z)<\infty. \tag{2.3}$$

Then there exists a weak martingale solution to (2.1) with the initial law Λ.

Our proof relies on a three step approximation scheme. We first regularize the SPDE by adding an uniformly elliptic term $-\eta\Delta^{2K}u$, $K\in\mathbb{N}$ sufficiently large, by adding a term $\varepsilon\Delta u$, and by replacing D^2u in (1.5) by a suitable uniformly bounded truncation $\Theta^R(\mathrm{D}^2u)$. We then obtain a solution of the corresponding regularized equation and pass to the limit first with $R\to\infty$ using a stopping time argument (Theorem 3.1), then in Section 4 with $\eta\to0$, and finally in Section 5 with $\varepsilon\to0$ using the stochastic compactness method.

We stress that the restriction to the spatial dimension $N=2$ is only necessary for the final passage to the limit whereas the results of Sections 3 and 4 are valid for general N. In particular, in Theorem 4.7 we obtain existence of a strong martingale solution to the viscous approximation of (2.1) in any space dimension which might be interesting in its own right.

For the third limit process we have to overcome several substantial difficulties. Our arguments are based on the energy bounds mentioned above. However, we

[2] $\mathcal{P}$ denotes the predictable σ-algebra on $\Omega\times[0,T]$ associated to $(\mathscr{F}_t)_{t\geq0}$

[3] Here and in the sequel, we write L^2_x for $L^2(\mathbb{T}^N)$ and similarly for other spaces.

point out that the only available estimate for higher order derivatives is given by the mean curvature bound and that both the L^2 gradient bound and the bound on area and mean curvature are not available for higher moments. Moreover, our proof of the area and mean curvature bound requires the restriction to $N = 2$.

Therefore, in order to identify the limit of the nonlinear terms, we proceed in several steps. First, in Proposition 5.6, we prove tightness not only for the approximate solutions but also for some (nonlinear) functionals of their gradients. This leads us to the Jakubowski-Skorokhod representation theorem (see [33]), which is valid in a large class of topological spaces that are not necessarily metrizable but retain several important properties of Polish spaces. Next, in Proposition 5.8 we employ compensated compactness and Young measure arguments to deduce a crucial strong convergence property of the gradients in suitable L^p-spaces. Finally, in Subsection 5.3, we employ a refined identification procedure for the stochastic integral. It is based on a general method of constructing martingale solutions in the absence of suitable martingale representation theorems introduced in [45] as well as a method of densely defined martingales from [28] and a local martingale approach from [29].

3. Regularized equation

To begin with, let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration and let W be a real-valued Wiener process relative to $(\mathscr{F}_t)$. In order to prepare the initial data for the first approximation layer, let u_0 be a $(\mathscr{F}_0)$-measurable random variable with the law Λ and for $\varepsilon \in (0,1)$ let u_0^ε be an approximation of u_0 such that for all $p \in [2,4)$

$$\mathbb{E}\|u_0^\varepsilon\|^p_{H^k_x} \leq C_\varepsilon$$

where we fixed $k \in \mathbb{N}$ such that $k > 2 + N/2$. Let the law of u_0^ε on $H^k(\mathbb{T}^N)$ be denoted by Λ^ε. Then according to (2.3) the following estimate holds true uniformly in ε

$$\int_{H^1_x} \|\nabla z\|^2_{L^2_x}\,\mathrm{d}\Lambda^\varepsilon(z) = \mathbb{E}\|\nabla u_0^\varepsilon\|^2_{L^2_x} \leq C \tag{3.1}$$

and $\Lambda^\varepsilon \overset{*}{\rightharpoonup} \Lambda$ in the sense of measures on $H^1(\mathbb{T}^N)$.

As the first step in the proof of existence for (2.1), we consider its equivalent form (2.2) and approximate in the following way

$$\begin{aligned}
&\mathrm{d}u = (1+\varepsilon)\Delta u\,\mathrm{d}t - \frac{1}{2}\frac{(\nabla u)^*}{H(\nabla u)}\mathrm{D}^2 u \frac{\nabla u}{H(\nabla u)}\,\mathrm{d}t - \eta\Delta^{2K} u\,\mathrm{d}t + H(\nabla u)\,\mathrm{d}W,\\
&u(0) = u_0^\varepsilon.
\end{aligned} \tag{3.2}$$

Our aim here is to establish an existence result for ε, η fixed and K sufficiently large.

Theorem 3.1. *Let $k \in \mathbb{N}$ be such that $2 + N/2 < k$ and let $p > 2$. There exists $K \in \mathbb{N}$ such that if $u_0^\varepsilon \in L^p(\Omega; H^k)$ then for any $\varepsilon, \eta \in (0,1)$ there exists $u \in L^p(\Omega \times [0,T], \mathcal{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; H^k)$ that is the unique mild solution to (3.2).*

Proof. In order to guarantee the Lipschitz property of the nonlinear second order term in (3.2), let $R \in \mathbb{N}$ and consider the truncated problem

(3.3)
$$\mathrm{d}u = (1+\varepsilon)\Delta u\,\mathrm{d}t - \frac{1}{2}\frac{(\nabla u)^*}{H(\nabla u)}\Theta^R(\mathrm{D}^2 u)\frac{\nabla u}{H(\nabla u)}\,\mathrm{d}t - \eta\Delta^{2K} u\,\mathrm{d}t + H(\nabla u)\,\mathrm{d}W,$$
$$u(0) = u_0^\varepsilon,$$

where $\Theta^R : \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times N}$ is a truncation, i.e. for $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ we define $\Theta^R(A) = \big(\theta^R(a_{ij})a_{ij}\big)$ where $\theta^R : \mathbb{R} \to [0,1]$ is a smooth truncation satisfying

$$\theta^R(\xi) = \begin{cases} 1, & |\xi| \le R/2 \\ 0, & |\xi| \ge R. \end{cases}$$

Let S denote the semigroup generated by the strongly elliptic differential operator $\eta\Delta^{2K} - (1+\varepsilon)\Delta$. Let $\mathcal{H} = L^p(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;H^k)$ and define the mapping

$$\mathcal{K}u(t) = S(t)u_0^\varepsilon - \frac{1}{2}\int_0^t S(t-s)\frac{(\nabla u)^*}{H(\nabla u)}\Theta^R(\mathrm{D}^2 u)\frac{\nabla u}{H(\nabla u)}\,\mathrm{d}s + \int_0^t S(t-s)H(\nabla u)\,\mathrm{d}W.$$

Then $\mathcal{K}$ maps $\mathcal{H}$ into $\mathcal{H}$ and it is a contraction. Indeed, using the regularization property of the semigroup and Young's inequality for convolutions we obtain for any $u \in \mathcal{H}$ (provided $k < 2K$)

$$\begin{aligned}
&\left\|\int_0^\cdot S(\cdot-s)\frac{(\nabla u)^*}{H(\nabla u)}\Theta^R(\mathrm{D}^2 u)\frac{\nabla u}{H(\nabla u)}\,\mathrm{d}s\right\|_{\mathcal{H}}^p \\
&\le \mathbb{E}\int_0^T\left(\int_0^t\left\|S(t-s)\frac{(\nabla u)^*}{H(\nabla u)}\Theta^R(\mathrm{D}^2 u)\frac{\nabla u}{H(\nabla u)}\right\|_{H^k}\mathrm{d}s\right)^p\mathrm{d}t \\
&\le C\,\mathbb{E}\int_0^T\left(\int_0^t(t-s)^{-k/4K}\left\|\frac{(\nabla u)^*}{H(\nabla u)}\Theta^R(\mathrm{D}^2 u)\frac{\nabla u}{H(\nabla u)}\right\|_{L^2}\mathrm{d}s\right)^p\mathrm{d}t \\
&\le CT^{p(1-k/4K)}\|u\|_{\mathcal{H}}^p,
\end{aligned} \tag{3.4}$$

and similarly for the stochastic term where we apply the Burkholder-Davis-Gundy inequality first (see e.g. [8])

$$\begin{aligned}
\left\|\int_0^\cdot S(\cdot-s)H(\nabla u)\,\mathrm{d}W\right\|_{\mathcal{H}}^p &\le C\int_0^T\mathbb{E}\Big(\int_0^t\big\|S(t-s)H(\nabla u)\big\|_{H^k}^2\mathrm{d}s\Big)^{p/2}\mathrm{d}t \\
&\le C\,\mathbb{E}\int_0^T\left(\int_0^t(t-s)^{-k/2K}\big\|H(\nabla u)\big\|_{L^2}^2\mathrm{d}s\right)^{p/2}\mathrm{d}t \\
&\le CT^{p/2(1-k/2K)}\|u\|_{\mathcal{H}}^p.
\end{aligned}$$

In order to verify the contraction property, we observe that for any $u, v \in \mathcal{H}$

$$\begin{aligned}
&\left\|\frac{(\nabla u)^*}{H(\nabla u)}\Theta^R(\mathrm{D}^2 u)\frac{\nabla u}{H(\nabla u)} - \frac{(\nabla v)^*}{H(\nabla v)}\Theta^R(\mathrm{D}^2 v)\frac{\nabla v}{H(\nabla v)}\right\|_{L^2} \\
&\qquad\le \big\|\mathrm{D}^2 u - \mathrm{D}^2 v\big\|_{L^2} + C_R\big\|\nabla u - \nabla v\big\|_{L^2} \le C_R\|u-v\|_{H^k}
\end{aligned}$$

hence by a similar approach as above

$$\begin{aligned}\Big\| \int_0^{\cdot} S(\cdot - s)\Big(\frac{(\nabla u)^*}{H(\nabla u)} \Theta^R(\mathrm{D}^2 u) \frac{\nabla u}{H(\nabla u)} - \frac{(\nabla v)^*}{H(\nabla v)} \Theta^R(\mathrm{D}^2 v) \frac{\nabla v}{H(\nabla v)} \Big) \,\mathrm{d}s \Big\|_{\mathcal{H}}^p \\ \leq C_R\, \mathbb{E} \int_0^T \Big(\int_0^t (t-s)^{-k/4K} \|u - v\|_{H^k} \mathrm{d}s \Big)^p \mathrm{d}t \leq C_R\, T^{p(1-k/4K)} \|u - v\|_{\mathcal{H}}^p\end{aligned}$$

and

$$\begin{aligned}&\Big\| \int_0^{\cdot} S(\cdot - s)\big(H(\nabla u) - H(\nabla v)\big)\,\mathrm{d}W \Big\|_{\mathcal{H}}^p \\ &\qquad \leq C\, \mathbb{E} \int_0^T \Big(\int_0^t (t-s)^{-k/2K} \big\|H(\nabla u) - H(\nabla v)\big\|_{L^2}^2 \mathrm{d}s \Big)^{p/2} \mathrm{d}t \\ &\qquad \leq C T^{p/2(1-k/2K)} \|u - v\|_{\mathcal{H}}^p.\end{aligned}$$

Therefore, if T is small enough then $\mathcal{K}$ has unique fixed point u in $\mathcal{H}$ which is the mild solution of (3.3). Furthermore, by a standard use of the factorization lemma (see [12]), it has continuous trajectories with values in H^k provided K is large enough, i.e. it belongs to $L^p(\Omega; C([0,T]; H^k))$. Therefore, the condition on T can be easily removed by considering the equation on smaller intervals $[0,\tilde{T}]$, $[\tilde{T}, 2\tilde{T}]$, etc.

As a consequence, for any $R \in \mathbb{N}$ there exists a unique mild solution to (3.3), let it be denoted by u^R. Furthermore, it follows from the Sobolev embedding theorem that

$$\mathbb{E} \sup_{0 \leq t \leq T} \big\|\mathrm{D}^2 u^R\big\|_{L^\infty} \leq C, \tag{3.5}$$

where the constant on the right hand side is independent of R (this can be seen from the fact that the growth estimates for the nonlinear second order term in (3.3) do not depend on R, cf. (3.4)). Hence

$$\tau_R = \inf \Big\{ t > 0; \big\|\mathrm{D}^2 u^R\big\|_{L^\infty} \geq R/2 \Big\}$$

(with the convention $\inf \emptyset = T$) defines an $(\mathscr{F}_t)$-stopping time and u^R is a solution to (3.2) on $[0, \tau_R)$. Besides, due to uniqueness, if $R' > R$ then $\tau_{R'} \geq \tau_R$ and $u^{R'} = u^R$ on $[0, \tau_R)$. Moreover, the blow up cannot occur in a finite time by (3.5) so

$$\tau = \sup_{R \in \mathbb{N}} \tau_R = T \qquad \text{a.s.}$$

and therefore the process u which is uniquely defined by $u := u^R$ on $[0, \tau_R)$ is the unique mild solution to (3.2) on $[0, T]$. □

4. Viscous approximation

Having Theorem 3.1 in hand it is necessary to find sufficient estimates uniform in η in order to justify the passage to the limit as $\eta \to 0$ and obtain a martingale solution to

$$\mathrm{d}u = (1+\varepsilon)\Delta u \,\mathrm{d}t - \frac{1}{2} \frac{(\nabla u)^*}{H(\nabla u)} \mathrm{D}^2 u \frac{\nabla u}{H(\nabla u)} \,\mathrm{d}t + H(\nabla u)\,\mathrm{d}W \tag{4.1}$$

with the initial law Λ. Let us fix $\varepsilon > 0$ and denote by u^η the solution to (3.2) given by Theorem 3.1. Recall that $\mathbb{P} \circ u^\eta(0)^{-1} = \Lambda^\varepsilon$ for all $\eta \in (0,1)$ and the uniform estimate (3.1) holds true.

Proposition 4.1. *For any $p \in [2, 2(1+\varepsilon)]$ it holds true*

$$\mathbb{E}\|\nabla u^\eta(t)\|_{L^2}^p + \frac{p(2(1+\varepsilon)-p)}{2}\mathbb{E}\int_0^t \|\nabla u^\eta\|_{L^2}^{p-2}\|\Delta u^\eta\|_{L^2}^2 \leq \mathbb{E}\|\nabla u_0^\varepsilon\|_{L^2}^p \leq C, \tag{4.2}$$

where the constant C is independent of η and if $p = 2$ then it is also independent of ε.

Proof. Since mild solution is a weak solution, we consider the function $f(\nabla v) = \|\nabla v\|_{L^2}^p$ and apply similar arguments as in the generalized Itô formula [13, Proposition A.1] to obtain

$$\begin{aligned}
\mathbb{E}\|\nabla u^\eta(t)\|_{L^2}^p &= \mathbb{E}\|\nabla u_0^\varepsilon\|_{L^2}^p + p(1+\varepsilon)\mathbb{E}\int_0^t \|\nabla u^\eta\|_{L^2}^{p-2}\big\langle \nabla u^\eta, \nabla\Delta u^\eta\big\rangle\,\mathrm{d}s \\
&\quad - p\eta\mathbb{E}\int_0^t \|\nabla u^\eta\|_{L^2}^{p-2}\big\langle \nabla u^\eta, \nabla\Delta^{2K} u^\eta\big\rangle\,\mathrm{d}s \\
&\quad - \frac{p}{2}\mathbb{E}\int_0^t \|\nabla u^\eta\|_{L^2}^{p-2}\left\langle \nabla u^\eta, \nabla\left(\frac{(\nabla u^\eta)^*}{H(\nabla u^\eta)}\mathrm{D}^2 u^\eta \frac{\nabla u^\eta}{H(\nabla u^\eta)}\right)\right\rangle\mathrm{d}s \\
&\quad + \frac{p}{2}\mathbb{E}\int_0^t \|\nabla u^\eta\|_{L^2}^{p-2}\big\|\nabla H(\nabla u^\eta)\big\|_{L^2}^2\mathrm{d}s \\
&\quad + \frac{p(p-2)}{2}\mathbb{E}\int_0^t \|\nabla u^\eta\|_{L^2}^{p-4}\big\langle \nabla u^\eta, \nabla H(\nabla u^\eta)\big\rangle^2\mathrm{d}s \\
&= J_1 + \cdots + J_6.
\end{aligned}$$

It holds

$$\begin{aligned}
J_2 + J_3 &\leq -p(1+\varepsilon)\mathbb{E}\int_0^t \|\nabla u^\eta\|_{L^2}^{p-2}\|\Delta u^\eta\|_{L^2}^2\mathrm{d}s, \\
J_4 &\leq \frac{p}{2}\mathbb{E}\int_0^t \|\nabla u^\eta\|_{L^2}^{p-2}\|\Delta u^\eta\|_{L^2}^2\mathrm{d}s, \\
J_5 + J_6 &\leq \frac{p(p-1)}{2}\mathbb{E}\int_0^t \|\nabla u^\eta\|_{L^2}^{p-2}\|\Delta u^\eta\|_{L^2}^2\mathrm{d}s,
\end{aligned}$$

where we used the fact that $\|\Delta u^\eta\|_{L^2} = \|\mathrm{D}^2 u^\eta\|_{L^2}$ due to boundary conditions. Hence the claim follows. ∎

Proposition 4.2. *It holds true*

$$\mathbb{E}\|u^\eta(t)\|_{L^2}^2 \leq C_\varepsilon\big(1 + \mathbb{E}\|u_0^\varepsilon\|_{L^2}^2\big) \leq C,$$

where the constant C is independent of η.

Proof. With regard to Proposition 4.1, the above estimate is a consequence of the Itô formula applied to the function $f(v) = \|v\|_{L^2}^2$:

$$\begin{aligned}\mathbb{E}\|u^\eta(t)\|_{L^2}^2 = \mathbb{E}\|u_0^\varepsilon\|_{L^2}^2 + 2(1+\varepsilon)\mathbb{E}\int_0^t \big\langle u^\eta, \Delta u^\eta\big\rangle \,\mathrm{d}s - 2\eta\mathbb{E}\int_0^t \big\langle u^\eta, \Delta^{2K} u^\eta\big\rangle \,\mathrm{d}s \\ - \mathbb{E}\int_0^t \left\langle u^\eta, \frac{(\nabla u^\eta)^*}{H(\nabla u^\eta)}\mathrm{D}^2 u^\eta \frac{\nabla u^\eta}{H(\nabla u^\eta)}\right\rangle \mathrm{d}s + \mathbb{E}\int_0^t \big\|H(\nabla u^\eta)\big\|_{L^2}^2 \mathrm{d}s \\ = J_1 + \cdots + J_5.\end{aligned}$$

Similar arguments as above imply

$$\begin{aligned}J_2 + J_3 &\leq -2(1+\varepsilon)\mathbb{E}\int_0^t \|\nabla u^\eta\|_{L^2}^2 \mathrm{d}s, \\ J_4 &\leq \frac{1}{2}\mathbb{E}\int_0^t \|u^\eta\|_{L^2}^2 \mathrm{d}s + \frac{1}{2}\mathbb{E}\int_0^t \|\Delta u^\eta\|_{L^2}^2, \\ J_5 &\leq \mathbb{E}\int_0^t \|\nabla u^\eta\|_{L^2}^2 \mathrm{d}s,\end{aligned}$$

hence Proposition 4.1 and the Gronwall lemma completes the proof. □

Proposition 4.3. *Let $p \in (2, 2(1+\varepsilon))$. Then for any $\alpha \in (1/p, 1/2)$ there exists $C_\varepsilon > 0$ such that*

$$\mathbb{E}\|u^\eta\|_{C^{\alpha-1/p}([0,T];H^{2-4K})}^p \leq C_\varepsilon. \tag{4.3}$$

Proof. According to Propositions 4.1 and 4.2, $u^\eta \in L^2(\Omega; L^2(0,T; H^2(\mathbb{T}^N)))$ uniformly in η (not ε). As a consequence,

$$(1+\varepsilon)\Delta u^\eta - \frac{1}{2}\frac{(\nabla u^\eta)^*}{H(\nabla u^\eta)}\mathrm{D}^2 u^\eta \frac{\nabla u^\eta}{H(\nabla u^\eta)} - \eta\Delta^{2K} u^\eta$$

belongs to $L^2(\Omega; L^2(0,T; H^{2-4K}))$ uniformly in η and

$$\mathbb{E}\Big\|u^\eta - \int_0^\cdot H(\nabla u^\eta)\mathrm{d}W\Big\|_{C^{1/2}([0,T];H^{2-4K})} \leq C_\varepsilon.$$

Concerning the stochastic integral, we have by factorization and (4.2)

$$\begin{aligned}\mathbb{E}\Big\|\int_0^\cdot H(\nabla u^\eta)\mathrm{d}W\Big\|_{C^{\alpha-1/p}([0,T];L^2)}^p &\leq C\,\mathbb{E}\Big\|\int_0^\cdot (\cdot - s)^{-\alpha} H(\nabla u^\eta)\mathrm{d}W(s)\Big\|_{L^p(0,T;L^2)}^p \\ &\leq C\int_0^T \mathbb{E}\Big(\int_0^t (t-s)^{-2\alpha}\big(1 + \|\nabla u^\eta\|_{L^2}^2\big)\mathrm{d}s\Big)^{p/2} \mathrm{d}t \\ &\leq C\,T^{p/2(1-2\alpha)}\mathbb{E}\int_0^T \big(1 + \|\nabla u^\eta\|_{L^2}^p\big)\mathrm{d}t \leq C\end{aligned}$$

and the claim follows. □

Now we would like to pass to the limit $\eta \searrow 0$.

4.1. **Compactness.** Let us define the path space $\mathcal{X} = \mathcal{X}_u \times \mathcal{X}_v \times \mathcal{X}_W$, where[4]

$$\mathcal{X}_u = L^2\big(0,T;H^1(\mathbb{T}^N)\big) \cap C\big([0,T];H^{1-4K}(\mathbb{T}^N)\big), \quad \mathcal{X}_v = \big((L^2(0,T;L^2(\mathbb{T}^N)),w\big),$$

$$\mathcal{X}_W = C\big([0,T];\mathbb{R}\big).$$

Let us denote by μ_{u^η} the law of u^η on $\mathcal{X}_u$, $\eta \in (0,1)$, by μ_{v^η} the law of

$$v^\eta := \operatorname{div}\left(\frac{\nabla u^\eta}{H(\nabla u^\eta)}\right)$$

on $\mathcal{X}_v$ and by μ_W the law of W on $\mathcal{X}_W$. Their joint law on $\mathcal{X}$ is then denoted by μ^η.

Proposition 4.4. *The set $\{\mu^\eta;\, \eta \in (0,1)\}$ is tight on $\mathcal{X}$.*

Proof. First, we prove tightness of $\{\mu_{u^\eta};\, \eta \in (0,1)\}$ which follows directly from Proposition 4.1 and 4.3 by making use of the embeddings

$$\begin{aligned}
C^{\alpha-1/p}([0,T];H^{2-4K}(\mathbb{T}^N)) &\hookrightarrow H^\lambda(0,T;H^{2-4K}(\mathbb{T}^N)),\ \lambda<\alpha-1/p,\\
C^{\alpha-1/p}([0,T];H^{2-4K}(\mathbb{T}^N)) &\overset{c}{\hookrightarrow} C([0,T];H^{1-4K}(\mathbb{T}^N)),\\
L^2(0,T;H^2(\mathbb{T}^N)) \cap H^\lambda(0,T;H^{2-4K}(\mathbb{T}^N)) &\overset{c}{\hookrightarrow} L^2(0,T;H^1(\mathbb{T}^N)).
\end{aligned}$$

Here the first embedding follows immediately for the definition of the spaces, the second one is a consequence of the Arzelà-Ascoli theorem and the third one can be found in [24]. For $R>0$ let us define the set

$$B_R = \big\{u \in L^2(0,T;H^2(\mathbb{T}^N)) \cap C^{\alpha-1/p}([0,T];H^{2-4K}(\mathbb{T}^N));\\ \|u\|_{L^2(0,T;H^2)} + \|u\|_{C^{\alpha-1/p}([0,T];H^{2-4K})} \le R\big\}$$

which is thus relatively compact in $\mathcal{X}_u$. Moreover, by Propositions 4.1, 4.2 and 4.3

$$\begin{aligned}
\mu_{u^\eta}\big(B_R^C\big) &\le \mathbb{P}\bigg(\|u^\eta\|_{L^2(0,T;H^2)} > \frac{R}{2}\bigg) + \mathbb{P}\bigg(\|u^\eta\|_{C^{\alpha-1/p}([0,T];H^{2-4K})} > \frac{R}{2}\bigg)\\
&\le \frac{C}{R^2}\mathbb{E}\|u^\eta\|^2_{L^2(0,T;H^2)} + \frac{C}{R^p}\mathbb{E}\|u^\eta\|^p_{C^{\alpha-1/p}([0,T];H^{2-4K})} \le \frac{C}{R^2}
\end{aligned}$$

hence given $\vartheta>0$ there exists $R>0$ such that

$$\mu_{u^\eta}(B_R) \ge 1-\vartheta$$

which yields the claim.

Concerning the tightness of $\{\mu_{v^\eta};\, \eta\in(0,1)\}$ we proceed similarly and make use of the uniform estimate from Proposition 4.1 together with the fact that for $R>0$ the set

$$B_R = \big\{v \in L^2(0,T;L^2(\mathbb{T}^N));\ \|v\|_{L^2(0,T;L^2)} \le R\big\}$$

is relatively compact in $\mathcal{X}_v$.

Since the law μ_W is tight as being a Radon measure on the Polish space $\mathcal{X}_W$, we conclude that also the set of the joint laws $\{\mu^\eta;\, \eta\in(0,1)\}$ is tight and the proof is complete. ∎

[4] If a topological space X is equipped with the weak topology we write (X,w).

The path space $\mathcal{X}$ is not a Polish space and so our compactness argument is based on the Jakubowski-Skorokhod representation theorem instead of the classical Skorokhod representation theorem, see [33]. To be more precise, passing to a weakly convergent subsequence $\mu^n = \mu^{\eta_n}$ (and denoting by μ the limit law) we infer the following result.

Proposition 4.5. *There exists a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ with a sequence of $\mathcal{X}$-valued random variables $(\tilde{u}^n, \tilde{v}^n, \tilde{W}^n)$, $n \in \mathbb{N}$, and $(\tilde{u}, \tilde{v}, \tilde{W})$ such that*

(i) the laws of $(\tilde{u}^n, \tilde{v}^n, \tilde{W}^n)$ and $(\tilde{u}, \tilde{v}, \tilde{W})$ under $\tilde{\mathbb{P}}$ coincide with μ^n and μ, respectively,

(ii) $(\tilde{u}^n, \tilde{v}^n, \tilde{W}^n)$ converges $\tilde{\mathbb{P}}$-almost surely to $(\tilde{u}, \tilde{v}, \tilde{W})$ in the topology of $\mathcal{X}$.

We are immediately able to identify $\tilde{v}^n$, $n \in \mathbb{N}$, and $\tilde{v}$.

Corollary 4.6. *It holds true that*

$$\tilde{v}^n = \operatorname{div}\left(\frac{\nabla \tilde{u}^n}{H(\nabla \tilde{u}^n)}\right) \quad \tilde{\mathbb{P}}\text{-a.s.} \quad \forall n \in \mathbb{N},$$

$$\tilde{v} = \operatorname{div}\left(\frac{\nabla \tilde{u}}{H(\nabla \tilde{u})}\right) \quad \tilde{\mathbb{P}}\text{-a.s.}$$

Proof. According to Proposition 4.1, the mapping

$$\operatorname{supp} \mu_{u^n} \to L^2(0,T; L^2(\mathbb{T}^N)), \; u \mapsto \operatorname{div}\left(\frac{\nabla u}{H(\nabla u)}\right)$$

is well-defined and measurable and hence, the first part of the statement follows directly from the equality of joint laws of (u^n, v^n) and $(\tilde{u}^n, \tilde{v}^n)$. Identification of the limit $\tilde{v}$ follows easily using integration by parts together with the strong convergence of $\nabla \tilde{u}^n$ given by Proposition 4.5. □

Finally, let $(\tilde{\mathscr{F}}_t)$ be the $\tilde{\mathbb{P}}$-augmented canonical filtration of the process $(\tilde{u}, \tilde{W})$, that is

$$\tilde{\mathscr{F}}_t = \sigma\big(\sigma\big(\varrho_t \tilde{u}, \varrho_t \tilde{W}\big) \cup \big\{N \in \tilde{\mathscr{F}};\; \tilde{\mathbb{P}}(N) = 0\big\}\big), \quad t \in [0,T],$$

where ϱ_t is the operator of restriction to the interval $[0,t]$ acting on various path spaces. In particular, if X stands for one of the path spaces $\mathcal{X}_u$ or $\mathcal{X}_W$ and $t \in [0,T]$, we define

$$\varrho_t : X \to X|_{[0,t]}, \quad f \mapsto f|_{[0,t]}.$$

Note that $\tilde{v}$ is also adapted with respect to $(\tilde{\mathscr{F}}_t)$ due to Corollary 4.6.

4.2. **Identification of the limit.** The aim of this subsection is to establish existence of a weak martingale solution to (4.1). Similarly to (2.1) and (2.2), we rewrite (4.1) into a more convenient form given by

$$\mathrm{d}u = \Big(\frac{1}{2} + \varepsilon\Big)\Delta u \,\mathrm{d}t + \frac{1}{2} H(\nabla u) \operatorname{div}\left(\frac{\nabla u}{H(\nabla u)}\right) \mathrm{d}t + H(\nabla u)\,\mathrm{d}W. \tag{4.4}$$

Theorem 4.7. $\big((\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}}), \tilde{W}, \tilde{u}\big)$ *is a strong martingale solution to (4.4) with the initial law Λ^ε, i.e. $\tilde{\mathbb{P}} \circ \tilde{u}(0)^{-1} = \Lambda^\varepsilon$ and*

$$\tilde{u}(t) = \tilde{u}(0) + \Big(\frac{1}{2} + \varepsilon\Big) \int_0^t \Delta \tilde{u} \,\mathrm{d}s + \frac{1}{2} \int_0^t H(\nabla \tilde{u}) \tilde{v} \,\mathrm{d}s + \int_0^t H(\nabla \tilde{u})\,\mathrm{d}W, \tag{4.5}$$

with

$$\tilde v=\operatorname{div}\left(\frac{\nabla\tilde u}{H(\nabla\tilde u)}\right) \tag{4.6}$$

holds true for all $t\in[0,T]$*, almost everywhere in* $(\omega,x)\in\tilde\Omega\times\mathbb{T}^N$.

The proof is based on a new general method of constructing martingale solutions of SPDEs, that does not rely on any kind of martingale representation theorem and therefore holds independent interest especially in situations where these representation theorems are no longer available.

First, we show that $\big((\tilde\Omega,\tilde{\mathscr F},(\tilde{\mathscr F}_t),\tilde{\mathbb P}),\tilde W,\tilde u\big)$ is a weak martingale solution to (4.4) with the initial law Λ^ε, i.e. for every $\varphi\in C^\infty(\mathbb{T}^N)$

$$\begin{aligned}\langle\tilde u(t),\varphi\rangle&=\langle\tilde u(0),\varphi\rangle+\Big(\frac12+\varepsilon\Big)\int_0^t\langle\tilde u,\Delta\varphi\rangle\mathrm{d}s+\frac12\int_0^t\langle H(\nabla\tilde u)\tilde v,\varphi\rangle\mathrm{d}s\\&\quad+\int_0^t\langle H(\nabla\tilde u)\mathrm{d}\tilde W,\varphi\rangle,\end{aligned}\tag{4.7}$$

where $\tilde v$ was defined in (4.6).

Towards this end, let us define for all $t\in[0,T]$ and a test function $\varphi\in C^\infty(\mathbb{T}^N)$

$$\begin{aligned}M^n(t)&=\big\langle u^n(t),\varphi\big\rangle-\big\langle u^n_0,\varphi\big\rangle-\Big(\frac12+\varepsilon\Big)\int_0^t\langle u^n,\Delta\varphi\rangle\mathrm{d}s\\&\qquad-\frac12\int_0^t\langle H(\nabla u^n)v^n,\varphi\rangle\mathrm{d}s+\eta_n\int_0^t\big\langle u^n,\Delta^{2K}\varphi\big\rangle\,\mathrm{d}s,\quad n\in\mathbb{N},\\ \tilde M^n(t)&=\big\langle\tilde u^n(t),\varphi\big\rangle-\big\langle\tilde u^n(0),\varphi\big\rangle-\Big(\frac12+\varepsilon\Big)\int_0^t\langle\tilde u^n,\Delta\varphi\rangle\mathrm{d}s\\&\qquad-\frac12\int_0^t\langle H(\nabla\tilde u^n)\tilde v^n,\varphi\rangle\mathrm{d}s+\eta_n\int_0^t\big\langle\tilde u^n,\Delta^{2K}\varphi\big\rangle\,\mathrm{d}s,\quad n\in\mathbb{N},\\ \tilde M(t)&=\big\langle\tilde u(t),\varphi\big\rangle-\big\langle\tilde u(0),\varphi\big\rangle-\Big(\frac12+\varepsilon\Big)\int_0^t\langle\tilde u,\Delta\varphi\rangle\mathrm{d}s-\frac12\int_0^t\langle H(\nabla\tilde u)\tilde v,\varphi\rangle\mathrm{d}s,\end{aligned}$$

we denoted

$$v^n=\operatorname{div}\left(\frac{\nabla u^n}{H(\nabla u^n)}\right),\qquad\tilde v^n=\operatorname{div}\left(\frac{\nabla\tilde u^n}{H(\nabla\tilde u^n)}\right).$$

Hereafter, times $s,t\in[0,T]$, $s\le t$, and a continuous function

$$\gamma:\mathcal{X}_u|_{[0,s]}\times\mathcal{X}_W|_{[0,s]}\longrightarrow[0,1]$$

will be fixed but otherwise arbitrary. The proof is an immediate consequence of the following two lemmas.

Lemma 4.8. *The process* $\tilde W$ *is a* $(\tilde{\mathscr F}_t)$*-Wiener process.*

Proof. Obviously, $\tilde W$ is a Wiener process and is $(\tilde{\mathscr F}_t)$-adapted. According to the Lévy martingale characterization theorem, it remains to show that it is also a $(\tilde{\mathscr F}_t)$-martingale. It holds true

$$\tilde{\mathbb E}\,\gamma\big(\varrho_s\tilde u^n,\varrho_s\tilde W^n\big)\big[\tilde W^n(t)-\tilde W^n(s)\big]=\mathbb{E}\,\gamma\big(\varrho_s u^n,\varrho_s W\big)\big[W(t)-W(s)\big]=0$$

since W is a martingale and the laws of $(\tilde u^n,\tilde W^n)$ and (u^n,W) coincide. Next, the uniform estimate

$$\sup_{n\in\mathbb N}\tilde{\mathbb E}|\tilde W^n(t)|^2=\sup_{n\in\mathbb N}\mathbb{E}|W(t)|^2<\infty$$

and the Vitali convergence theorem yields

$$\tilde{\mathbb{E}}\,\gamma\big(\varrho_s\tilde u,\varrho_s\tilde W\big)\big[\tilde W(t)-\tilde W(s)\big]=0$$

which finishes the proof. ☐

Lemma 4.9. *The processes*

$$\tilde M,\qquad \tilde M^2-\int_0^\cdot\big\langle H(\nabla\tilde u),\varphi\big\rangle^2\,\mathrm{d}r,\qquad \tilde M\tilde W-\int_0^\cdot\big\langle H(\nabla\tilde u),\varphi\big\rangle\,\mathrm{d}r$$

are $(\tilde{\mathscr{F}}_t)$*-martingales.*

Proof. Here, we use the same approach as in the previous lemma. For all $n\in\mathbb{N}$, the process

$$M^n=\int_0^\cdot\big\langle H(\nabla u^n)\,\mathrm{d}W(r),\varphi\big\rangle$$

is a square integrable $(\mathscr{F}_t)$-martingale by (4.2) and therefore

$$(M^n)^2-\int_0^\cdot\big\langle H(\nabla u^n),\varphi\big\rangle^2\,\mathrm{d}r,\qquad M^nW-\int_0^\cdot\big\langle H(\nabla u^n),\varphi\big\rangle\,\mathrm{d}r$$

are $(\mathscr{F}_t)$-martingales. Besides, it follows from the equality of laws that

$$\begin{aligned}&\tilde{\mathbb{E}}\,\gamma\big(\varrho_s\tilde u^n,\varrho_s\tilde W^n\big)\big[\tilde M^n(t)-\tilde M^n(s)\big]\\&\quad=\mathbb{E}\,\gamma\big(\varrho_s u^n,\varrho_s W\big)\big[M^n(t)-M^n(s)\big]=0,\end{aligned}\tag{4.8}$$

$$\begin{aligned}&\tilde{\mathbb{E}}\,\gamma\big(\varrho_s\tilde u^n,\varrho_s\tilde W^n\big)\left[(\tilde M^n)^2(t)-(\tilde M^n)^2(s)-\int_s^t\big\langle H(\nabla\tilde u^n),\varphi\big\rangle^2\,\mathrm{d}r\right]\\&=\mathbb{E}\,\gamma\big(\varrho_s u^n,\varrho_s W\big)\left[(M^n)^2(t)-(M^n)^2(s)-\int_s^t\big\langle H(\nabla u^n),\varphi\big\rangle^2\,\mathrm{d}r\right]=0,\end{aligned}\tag{4.9}$$

$$\begin{aligned}&\tilde{\mathbb{E}}\,\gamma\big(\varrho_s\tilde u^n,\varrho_s\tilde W^n\big)\left[\tilde M^n(t)\tilde W^n(t)-\tilde M^n(s)\tilde W^n(s)-\int_s^t\big\langle H(\nabla\tilde u^n),\varphi\big\rangle\,\mathrm{d}r\right]\\&=\mathbb{E}\,\gamma\big(\varrho_s u^n,\varrho_s W\big)\left[M^n(t)W(t)-M^n(s)W(s)-\int_s^t\big\langle H(\nabla u^n),\varphi\big\rangle\,\mathrm{d}r\right]=0.\end{aligned}\tag{4.10}$$

In order to pass to the limit in (4.8), (4.9) and (4.10), let us first establish the convergence $\tilde M^n(t)\to\tilde M(t)$ a.s. for all $t\in[0,T]$. Let us only make few comments on the mean curvature term. We recall that according to Proposition 4.5 and Corollary 4.6 it holds true that

$$\operatorname{div}\left(\frac{\nabla\tilde u^n}{H(\nabla\tilde u^n)}\right)\rightharpoonup\operatorname{div}\left(\frac{\nabla\tilde u}{H(\nabla\tilde u)}\right)\quad\text{in}\quad L^2(0,T;L^2(\mathbb{T}^N)))\quad\tilde{\mathbb{P}}\text{-a.s.}$$

Moreover,

$$H(\nabla\tilde u^n)\to H(\nabla\tilde u)\quad\text{in}\quad L^2(0,T;L^2(\mathbb{T}^N)))\quad\tilde{\mathbb{P}}\text{-a.s.}$$

and therefore

$$\operatorname{div}\left(\frac{\nabla\tilde u^n}{H(\nabla\tilde u^n)}\right)H(\nabla\tilde u^n)\rightharpoonup\operatorname{div}\left(\frac{\nabla\tilde u}{H(\nabla\tilde u)}\right)H(\nabla\tilde u)$$

in $L^1(0,T;L^1(\mathbb{T}^N)))$ a.s. which yields the desired convergence.

Besides, we observe that according to (4.8), (4.9), (4.10) it follows for every $n\in\mathbb{N}$ that

$$\tilde M^n=\int_0^\cdot\big\langle H(\nabla\tilde u^n),\varphi\big\rangle\,\mathrm{d}\tilde W^n\qquad\tilde{\mathbb{P}}\text{-a.s.}$$

Therefore, the passage to the limit in (4.8) and in the first terms on the left hand side of (4.9) and (4.10) (and the same for the right hand side) can be justified by using the convergence $\tilde{M}^n(t) \to \tilde{M}(t)$ together with the uniform integrability given by Proposition 4.1:

$$\tilde{\mathbb{E}}\big|\tilde{M}^n(t)\big|^p \le C\,\tilde{\mathbb{E}}\bigg(\int_0^t \big\langle H(\nabla\tilde{u}^n),\varphi\big\rangle^2\,\mathrm{d}t\bigg)^{p/2}$$
$$\le C\bigg(1+\tilde{\mathbb{E}}\int_0^T \|\nabla\tilde{u}^n\|_{L^2}^p\,\mathrm{d}t\bigg) \le C < \infty.$$

This estimate also yields the necessary uniform integrability that together with

$$\big\langle H(\nabla\tilde{u}^n),\varphi\big\rangle \to \big\langle H(\nabla\tilde{u}^n),\varphi\big\rangle \quad \text{a.e. } (\omega,r)$$

justifies the passage to the limit in the remaining terms in (4.9) and (4.10) which completes the proof. □

Proof of Theorem 4.7. Once the above lemmas established, we infer that

$$\bigg\langle\!\!\bigg\langle \tilde{M} - \int_0^{\cdot} \big\langle H(\nabla\tilde{u})\,\mathrm{d}\tilde{W},\varphi\big\rangle \bigg\rangle\!\!\bigg\rangle = 0$$

and consequently (4.7) holds true. Moreover, we note that the equation (4.7) is in fact satisfied in a stronger sense: since $\tilde{u} \in L^2(\tilde{\Omega}; L^2(0,T;H^2(\mathbb{T}^N)))$ due to Propositions 4.1 and 4.2 and $H(\nabla\tilde{u})\tilde{v} \in L^1(\tilde{\Omega}; L^1(0,T;L^1(\mathbb{T}^N)))$ which follows from the proof of Lemma 4.9 and therefore (4.5) follows. □

5. Vanishing viscosity limit

The aim of this final section is to study the limit $\varepsilon \to 0$ in (4.4) and complete the proof of Theorem 2.3. Recall that it was proved in Section 4 that for every $\varepsilon \in (0,1)$ there exists

$$\big((\tilde{\Omega}^\varepsilon, \tilde{\mathscr{F}}^\varepsilon, (\tilde{\mathscr{F}}^\varepsilon_t), \tilde{\mathbb{P}}^\varepsilon), \tilde{u}^\varepsilon, \tilde{W}^\varepsilon\big)$$

which is a martingale solution to (4.4) with the initial law Λ^ε. We recall that $\Lambda^\varepsilon \overset{*}{\rightharpoonup} \Lambda$ in the sense of measures on $H^1(\mathbb{T}^N)$. It was shown in [33] that it is enough to consider only one probability space, namely,

$$(\tilde{\Omega}^\varepsilon, \tilde{\mathscr{F}}^\varepsilon, \tilde{\mathbb{P}}^\varepsilon) = \big([0,1], \mathcal{B}([0,1]), \mathcal{L}\big) \qquad \forall \varepsilon \in (0,1)$$

where $\mathcal{L}$ denotes the Lebesgue measure on $[0,1]$. Moreover, we can assume without loss of generality that there exists one common Wiener process W for all ε. Indeed, one could perform the compactness argument of the previous section for all the parameters from any chosen subsequence ε_n at once by redefining

$$\mathcal{X} = \Big(\prod_{n\in\mathbb{N}} \mathcal{X}_u\Big) \times \mathcal{X}_W$$

and proving tightness of the joint laws of $(u^{\eta,\varepsilon_1}, u^{\eta,\varepsilon_2}, \dots, W)$ for $\eta \in (0,1)$. In order to further simplify the notation we also omit the tildas and denote the martingale solution found in Section 4 by

$$\big((\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}), u^\varepsilon, W\big).$$

5.1. **Estimates.** We start with an estimate of the surface area and the mean curvature term. The proof of the following are bounds requires $N = 2$. From now on we therefore restrict ourselves to $N = 2$ and two-dimensional graphs in $\mathbb{R}^3$.

Proposition 5.1. *For any $\varepsilon > 0$ we have the following uniform estimate*

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\int_{\mathbb{T}^2} H(\nabla u^\varepsilon)\,\mathrm{d}x + \frac12\,\mathbb{E}\int_0^T\int_{\mathbb{T}^2}\Big|\operatorname{div}\Big(\frac{\nabla u^\varepsilon}{H(\nabla u^\varepsilon)}\Big)\Big|^2 H(\nabla u^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\ \le C\,\mathbb{E}\int_{\mathbb{T}^2} H(\nabla u^\varepsilon(0))\,\mathrm{d}x + K(\varepsilon),\end{aligned}\tag{5.1}$$

where $K(\varepsilon)\to 0$ as $\varepsilon \to 0$.

Proof. We start with some calculations that hold for any $N \in \mathbb{N}$. In the first step, it is necessary to derive the equation satisfied by ∇u^ε and then apply the Itô formula to the function $p \mapsto \int_{\mathbb{T}^N} H(p)\,\mathrm{d}x$. In order to make the calculation rigorous we make use of the generalized Itô formula as introduced in [13, Appendix A]. To be more precise, we consider

$$\begin{aligned}\mathrm{d}\nabla u^\varepsilon &= \varepsilon\nabla\Delta u^\varepsilon\,\mathrm{d}t + \nabla\Big[H\operatorname{div}\Big(\frac{\nabla u^\varepsilon}{H}\Big)\Big]\mathrm{d}t\\ &\quad+\frac12\nabla\Big[\frac{(\nabla u^\varepsilon)^*}{H}\mathrm{D}^2u^\varepsilon\frac{\nabla u^\varepsilon}{H}\Big]\mathrm{d}t + \nabla H\mathrm{d}W.\end{aligned}\tag{5.2}$$

(For notational simplicity we do not stress the dependence of H on ∇u^ε.) Note that $\nabla u^\varepsilon \in L^2(\Omega; L^2(0,T;H^1(\mathbb{T}^N)))$ according to Proposition 4.1 hence (5.2) can be rewritten as

$$\mathrm{d}\nabla u^\varepsilon = \nabla F(t)\mathrm{d}t + G(t)\mathrm{d}W$$

where $F, G \in L^2(\Omega; L^2(0,T;L^2(\mathbb{T}^N)))$. Besides, $H \in C^2(\mathbb{R}^N)$ has bounded derivatives so the only assumption of [13, Proposition A.1] which is not satisfied is $\nabla u^\varepsilon \in L^2(\Omega; C([0,T];L^2(\mathbb{T}^N)))$. However, following the proof of [13, Proposition A.1], one can easily see that under the boundedness hypothesis for $\mathrm{D}H$ everything works well even without it.

Therefore, we have shown that

$$\begin{aligned}\mathrm{d}\int_{\mathbb{T}^N} H\,\mathrm{d}x &= \int_{\mathbb{T}^N}\frac{\nabla u}{H}\nabla\big[\varepsilon\Delta u^\varepsilon\big]\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^N}\frac{\nabla u^\varepsilon}{H}\nabla\Big[H\operatorname{div}\Big(\frac{\nabla u^\varepsilon}{H}\Big)\Big]\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\frac12\int_{\mathbb{T}^N}\frac{\nabla u^\varepsilon}{H}\nabla\Big[\frac{(\nabla u^\varepsilon)^*}{H}\mathrm{D}^2u^\varepsilon\frac{\nabla u^\varepsilon}{H}\Big]\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^N}\frac{\nabla u^\varepsilon}{H}\nabla H\,\mathrm{d}x\,\mathrm{d}W\\ &\quad+\frac12\int_{\mathbb{T}^N}\frac1H\Big(\mathrm{Id}-\frac{\nabla u^\varepsilon}{H}\otimes\frac{\nabla u^\varepsilon}{H}\Big):\Big(\nabla H\otimes\nabla H\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &= J_1+\cdots+J_5.\end{aligned}$$

It follows from the above consideration that the stochastic integral J_4 is a square integrable martingale so has zero expectation. After integration by parts J_2 has a negative sign. For J_1 we have

$$J_1 \le \frac12\int_{\mathbb{T}^N}\Big|\operatorname{div}\Big(\frac{\nabla u^\varepsilon}{H}\Big)\Big|^2 H\,\mathrm{d}x + \frac{\varepsilon^2}{2}\int_{\mathbb{T}^N}|\Delta u^\varepsilon|^2\,\mathrm{d}x.$$

The first term on the right hand side is controlled by J_2 whereas the integral over $\Omega\times[0,T]$ of the second one vanishes as $\varepsilon\to 0$ due to Proposition 4.1. Next, we will show that $J_3 + J_5 = 0$. Indeed,

$$\begin{aligned}
J_3 + J_5 &= -\frac{1}{2}\int_{\mathbb{T}^N} \operatorname{div}\Big(\frac{\nabla u^\varepsilon}{H}\Big)\frac{\nabla u^\varepsilon}{H}\cdot\nabla H \,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \frac{1}{2}\int_{\mathbb{T}^N}\frac{1}{H}\Big(\mathrm{Id} - \frac{\nabla u^\varepsilon}{H}\otimes\frac{\nabla u^\varepsilon}{H}\Big) : \Big(\nabla H\otimes\nabla H\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&= -\frac{1}{2}\int_{\mathbb{T}^N}\frac{1}{H}\Big(\mathrm{Id} - \frac{\nabla u^\varepsilon}{H}\otimes\frac{\nabla u^\varepsilon}{H}\Big) : \mathrm{D}^2 u\,\frac{(\nabla u^\varepsilon)^*}{H}\mathrm{D}^2 u^\varepsilon\frac{\nabla u^\varepsilon}{H}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \frac{1}{2}\int_{\mathbb{T}^N}\frac{1}{H}\mathrm{D}^2 u^\varepsilon\frac{\nabla u^\varepsilon}{H}\Big(\mathrm{Id} - \frac{\nabla u^\varepsilon}{H}\otimes\frac{\nabla u^\varepsilon}{H}\Big)\mathrm{D}^2 u^\varepsilon\frac{\nabla u^\varepsilon}{H}\,\mathrm{d}x\,\mathrm{d}t\\
&= -\frac{1}{2}\int_{\mathbb{T}^N}\frac{1}{H}\Delta u^\varepsilon\frac{(\nabla u^\varepsilon)^*}{H}\mathrm{D}^2 u^\varepsilon\frac{\nabla u^\varepsilon}{H}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \frac{1}{2}\int_{\mathbb{T}^N}\frac{1}{H}\Big(\mathrm{D}^2 u^\varepsilon\frac{\nabla u^\varepsilon}{H}\Big)\cdot\Big(\mathrm{D}^2 u^\varepsilon\frac{\nabla u^\varepsilon}{H}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&= \frac{1}{2}\int_{\mathbb{T}^N}\frac{1}{H}\mathrm{D}^2 u^\varepsilon\frac{\nabla u^\varepsilon}{H}\big(\mathrm{D}^2 u^\varepsilon - \Delta u^\varepsilon\mathrm{Id}\big)\frac{\nabla u^\varepsilon}{H}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

Let $\mathrm{D}^2 u^\varepsilon = \sum_i \lambda_i w_i\otimes w_i$ with $\lambda_i\in\mathbb{R}$, $w_i\in\mathbb{R}^N$, $w_i\cdot w_j = \delta_{ij}$ for all $1\le i,j\le N$. For $v = \sum_i v^i w_i$ we obtain

$$\begin{aligned}
\mathrm{D}^2 u^\varepsilon v\cdot\big(\mathrm{D}^2 u^\varepsilon - \Delta u^\varepsilon\mathrm{Id}\big)v &= \sum_i \lambda_i v^i w_i\cdot\sum_j(\lambda_j v^j w_j - (\sum\lambda_k)v^j w_j)\\
&= \sum_i \lambda_i(-\sum_{k\neq i}\lambda_k)(v^i)^2.
\end{aligned}$$

From now on, we restrict ourselves to $N = 2$. Then the last equation implies

$$\mathrm{D}^2 u^\varepsilon v\cdot\big(\mathrm{D}^2 u^\varepsilon - \Delta u^\varepsilon\mathrm{Id}\big)v = -\lambda_1\lambda_2\sum_i (v^i)^2 \;=\; -(\det\mathrm{D}^2 u)|v|^2$$

and yields

$$J_3 + J_5 = -\frac{1}{2}\int_{\mathbb{T}^2}\frac{|\nabla u^\varepsilon|^2}{H^3}\det(\mathrm{D}^2 u^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t.$$

Next we recall the formula

$$(\operatorname{div} h)|_{\nabla u}\det(\mathrm{D}^2 u) = \operatorname{div}\big(\operatorname{cof}(\mathrm{D}^2 u)\cdot h|_{\nabla u}\big),$$

which can be verified by direct computations. Setting

$$h(z) = |z|^{-2}(1+|z|^2)^{-\frac{1}{2}}\Big(2 - 2\sqrt{1+|z|^2} + |z|^2\Big)z$$

we obtain

$$\operatorname{div} h(z) \;=\; |z|^2(1+|z|^2)^{-\frac{3}{2}},$$

therefore

$$J_3 + J_5 = -\frac{1}{2}\int_{\mathbb{T}^2}\operatorname{div}\big(\operatorname{cof}(\mathrm{D}^2 u)\cdot h|_{\nabla u}\big)\,\mathrm{d}x\,\mathrm{d}t = 0$$

and consequently for every $t\in[0,T]$

$$\begin{aligned}
\mathbb{E}\int_{\mathbb{T}^2} H\big(\nabla u^\varepsilon(t)\big)\,\mathrm{d}x + \frac{1}{2}\mathbb{E}\int_0^t\int_{\mathbb{T}^2}\Big|\operatorname{div}\Big(\frac{\nabla u^\varepsilon}{H}\Big)\Big|^2 H\,\mathrm{d}x\,\mathrm{d}s&\\
\le \mathbb{E}\int_{\mathbb{T}^2} H(\nabla u^\varepsilon(0))\,\mathrm{d}x + K(\varepsilon).&
\end{aligned}$$

In order to obtain (5.1) we proceed similarly, the only difference is in the estimate for the stochastic integral:

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_{\mathbb{T}^2}\mathrm{div}\left(\frac{\nabla u^\varepsilon}{H}\right)H\,\mathrm{d}x\,\mathrm{d}W\right|\\
&\quad\le C\,\mathbb{E}\bigg(\int_0^T\left|\int_{\mathbb{T}^2}\mathrm{div}\left(\frac{\nabla u^\varepsilon}{H}\right)H\,\mathrm{d}x\right|^2\mathrm{d}t\bigg)^{1/2}\\
&\quad\le C\,\mathbb{E}\bigg[\sup_{0\le t\le T}\int_{\mathbb{T}^2}H\,\mathrm{d}x\int_0^T\int_{\mathbb{T}^2}\left|\mathrm{div}\left(\frac{\nabla u^\varepsilon}{H}\right)\right|^2H\,\mathrm{d}x\,\mathrm{d}t\bigg]^{1/2}\\
&\quad\le\frac12\mathbb{E}\sup_{0\le t\le T}\int_{\mathbb{T}^2}H\,\mathrm{d}x+C\,\mathbb{E}\int_0^T\int_{\mathbb{T}^2}\left|\mathrm{div}\left(\frac{\nabla u^\varepsilon}{H}\right)\right|^2H\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le\frac12\mathbb{E}\sup_{0\le t\le T}\int_{\mathbb{T}^2}H\,\mathrm{d}x+C\,\mathbb{E}\int_{\mathbb{T}^2}H(\nabla u^\varepsilon(0))\,\mathrm{d}x+K(\varepsilon)
\end{aligned}$$

which completes the proof. □

As a consequence we deduce an estimate for the L^2-norm of the solution.

Corollary 5.2. *For any $\varepsilon>0$ we have the following uniform estimate*

$$\mathbb{E}\|u^\varepsilon\|_{L^2(0,T;L^2)}\le C.$$

Proof. In the first step we show an estimate for the mean value of u^ε over $\mathbb{T}^2$ and then we apply the Poincaré inequality. Testing (4.4) by $\varphi\equiv 1$ we obtain

$$\mathrm{d}\int_{\mathbb{T}^2}u^\varepsilon\,\mathrm{d}x=\frac12\int_{\mathbb{T}^2}H(\nabla u^\varepsilon)\,\mathrm{div}\left(\frac{\nabla u^\varepsilon}{H(\nabla u^\varepsilon)}\right)\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{T}^2}H(\nabla u^\varepsilon)\,\mathrm{d}x\,\mathrm{d}W.$$

Since the above stochastic integral is a square-integrable martingale, we apply the Burkholder-Davis-Gundy inequality, Proposition 4.1 and Proposition 5.1 and deduce that

$$\mathbb{E}\sup_{0\le t\le T}\left|\int_{\mathbb{T}^2}u^\varepsilon(t)\,\mathrm{d}x\right|\le\mathbb{E}\left|\int_{\mathbb{T}^2}u^\varepsilon(0)\,\mathrm{d}x\right|+C.$$

The Poincaré inequality yields

$$\|u^\varepsilon(t)\|_{L^2(0,T;L^2)}\le C\|\nabla u^\varepsilon(t)\|_{L^2(0,T;L^2)}+\sup_{0\le t\le T}\left|\int_{\mathbb{T}^2}u^\varepsilon(t)\,\mathrm{d}x\right|$$

and the claim follows. □

Finally, we proceed with a uniform estimate for the time derivative of u^ε.

Proposition 5.3. *There exists $s,k>0$ and $p\in[1,\infty)$ such that*

$$\mathbb{E}\|u^\varepsilon\|_{W^{s,2}(0,T;W^{-k,p})}\le C.$$

Proof. In order to estimate the stochastic term, we make use of [24, Lemma 2.1] which gives bounds for fractional time derivatives of a stochastic integrals. We obtain for $s\in[0,1/2)$ that

$$\mathbb{E}\left\|\int_0^\cdot H(\nabla u^\varepsilon)\,\mathrm{d}W\right\|^2_{W^{s,2}(0,T;L^2)}\le C\,\mathbb{E}\int_0^T\|H(\nabla u^\varepsilon)\|^2_{L^2}\mathrm{d}t\le C.$$

Since (u^ε) is bounded in $L^1(\Omega; L^1(0,T; W^{1,2}(\mathbb{T}^2)))$ we deduce that (Δu^ε) is bounded in $L^1(\Omega; L^1(0,T; W^{-1,2}(\mathbb{T}^2)))$ and as a consequence

$$\left(\left(\frac{1}{2}+\varepsilon\right)\int_0^{\cdot} \Delta u^\varepsilon \,\mathrm{d}s\right) \quad \text{is bounded in} \quad L^1(\Omega; W^{1,1}(0,T; W^{-1,2}(\mathbb{T}^2))).$$

Regarding the remaining term, we deduce from (5.1) that

$$\left(H(\nabla u^\varepsilon)\,\mathrm{div}\left(\frac{\nabla u^\varepsilon}{H(\nabla u^\varepsilon)}\right)\right) \quad \text{is bounded in} \quad L^1(\Omega; L^1(0,T; L^1(\mathbb{T}^2)))$$

hence

$$\left(\int_0^{\cdot} H(\nabla u^\varepsilon)\,\mathrm{div}\left(\frac{\nabla u^\varepsilon}{H(\nabla u^\varepsilon)}\right)\mathrm{d}s\right) \quad \text{is bounded in} \quad L^1(\Omega; W^{1,1}(0,T; L^1(\mathbb{T}^2))).$$

Altogether, we obtain that (u^ε) is bounded in $L^1(\Omega; W^{s,2}(0,T; W^{-k,p}))$ where k,p are determined by the Sobolev embedding theorem so that

$$L^1(\mathbb{T}^2) \hookrightarrow W^{-k,p}(\mathbb{T}^2), \qquad W^{-1,2}(\mathbb{T}^2) \hookrightarrow W^{-k,p}(\mathbb{T}^2)$$

and the proof is complete. □

5.2. **Compactness.** Let us define the path space

$$\mathcal{X} = \mathcal{X}_u \times \mathcal{X}_v \times \mathcal{X}_\nu \times \mathcal{X}_V \times \mathcal{X}_I \times \mathcal{X}_{u_0} \times \mathcal{X}_W,$$

where

$$\mathcal{X}_u = L^2\big(0,T; L^2(\mathbb{T}^2)\big), \qquad \mathcal{X}_v = \big(L^2(0,T; L^2(\mathbb{T}^2)), w\big),$$
$$\mathcal{X}_\nu = \big(L^2(0,T; L^2(\mathbb{T}^2)), w\big), \qquad \mathcal{X}_V = \big(L^2(0,T; L^2(\mathbb{T}^2)), w\big),$$
$$\mathcal{X}_I = C\big([0,T]; \mathbb{R}\big), \qquad \mathcal{X}_W = C\big([0,T]; \mathbb{R}\big), \qquad \mathcal{X}_{u_0} = H^1(\mathbb{T}^2).$$

Let us denote by μ_{u^ε} the law of u^ε on $\mathcal{X}_u$, $\eta \in (0,1)$, by μ_v, μ_{ν^ε}, μ_{V^ε} and μ_{I^ε}, respectively, the law of

$$v := \mathrm{div}\left(\frac{\nabla u^\varepsilon}{H(\nabla u^\varepsilon)}\right), \qquad \nu^\varepsilon := \frac{\nabla u^\varepsilon}{H(\nabla u^\varepsilon)},$$

$$V^\varepsilon := \mathrm{div}\left(\frac{\nabla u^\varepsilon}{H(\nabla u^\varepsilon)}\right)\sqrt{H(\nabla u^\varepsilon)}, \qquad I^\varepsilon := \int_0^{\cdot} \|H(\nabla u^\varepsilon)\|_{L^1_x}^{1+\theta}\,\mathrm{d}s$$

on $\mathcal{X}_v$, $\mathcal{X}_\nu$, $\mathcal{X}_V$ and $\mathcal{X}_I$ (for some fixed $\theta \in (0,1)$), respectively, and by μ_W the law of W on $\mathcal{X}_W$. Recall that the law of $u^\varepsilon(0)$ on $\mathcal{X}_{u_0}$ is given by Λ^ε and due to the construction at the beginning of Section 3 we immediately obtain tightness of (Λ^ε). The joint law on $\mathcal{X}$ is then denoted by μ^ε. In order to prove tightness of (μ_ε), we make use of the following compact embedding which can be found in [48, Corollary 5].

Lemma 5.4. *Let X, B, Y be Banach spaces such that $X \overset{c}{\hookrightarrow} B \hookrightarrow Y$. If $p, r \in [1,\infty)$ then*

$$L^p(0,T;X) \cap W^{s,r}(0,T;Y) \overset{c}{\hookrightarrow} L^p(0,T;B)$$

provided $s > 0$ if $r \geq p$ and $s > 1/r - 1/p$ if $r \leq p$.

Proposition 5.5. *The set of laws $\{\mu^\varepsilon;\, \varepsilon \in (0,1)\}$ is tight on $\mathcal{X}$.*

Proof. We will show tightness of all the corresponding marginal laws, tightness for the joint laws then follows immediately. Concerning $\{\mu_{u^\varepsilon};\,\varepsilon\in(0,1)\}$, we want to employ the compact embedding

$$L^2(0,T;H^1(\mathbb{T}^2))\cap W^{s,2}(0,T;W^{-k,p}(\mathbb{T}^2))\overset{c}{\hookrightarrow} L^2(0,T;L^2(\mathbb{T}^2))$$

hence for $R>0$ we define the set

$$\begin{aligned}B_R=&\big\{u\in L^2(0,T;H^1(\mathbb{T}^2))\cap W^{s,2}(0,T;W^{-k,p}(\mathbb{T}^2));\\ &\qquad\|u\|_{L^2(0,T;H^1(\mathbb{T}^2))}+\|u\|_{W^{s,2}(0,T;W^{-k,p}(\mathbb{T}^2))}\le R\big\}.\end{aligned}$$

Now, it holds by Chebyshev inequality, Proposition 4.1, Corollary 5.2 and Proposition 5.3

$$\begin{aligned}\mu_{u^\varepsilon}(B_R^c)&\le\mathbb{P}\bigg(\|u^\varepsilon\|_{L^2(0,T;H^1)}>\frac{R}{2}\bigg)+\mathbb{P}\bigg(\|u^\varepsilon\|_{W^{s,2}(0,T;W^{-k,p})}>\frac{R}{2}\bigg)\\ &\le\frac{2}{R}\,\mathbb{E}\|u^\varepsilon\|_{L^2(0,T;H^1)}+\frac{2}{R}\,\mathbb{E}\|u^\varepsilon\|_{W^{s,2}(0,T;W^{-k,p})}\le\frac{C}{R}\end{aligned}$$

which yields tightness of $\{\mu_{u^\varepsilon};\,\varepsilon\in(0,1)\}$.

For $\{\mu_v;\,\varepsilon\in(0,1)\}$, $\{\mu_{\nu^\varepsilon};\,\varepsilon\in(0,1)\}$ and $\{\mu_{V^\varepsilon};\,\varepsilon\in(0,1)\})$ we proceed similarly and make use of the uniform estimate from Proposition 5.1 together with the fact that for $R>0$ the set

$$B_R=\big\{z\in L^2(0,T;L^2(\mathbb{T}^2));\,\|z\|_{L^2(0,T;L^2)}\le R\big\}.$$

is relatively compact in $\big(L^2(0,T;L^2(\mathbb{T}^2)),w\big)$.

Regarding $\{\mu_{I^\varepsilon};\,\varepsilon\in(0,1)\}$ we observe that due to Proposition 4.1

$$\big(\|H(\nabla u^\varepsilon)\|_{L^1_x}^{1+\theta}\big)\quad\text{is bounded in}\quad L^{2/1+\theta}(\Omega;L^{2/1+\theta}(0,T))$$

hence

$$(I^\varepsilon)\quad\text{is bounded in}\quad L^{2/1+\theta}(\Omega;W^{1,2/1+\theta}(0,T))$$

and due to Sobolev imbedding theorem

$$W^{1,2/1+\theta}(0,T)\overset{c}{\hookrightarrow} C([0,T];\mathbb{R}).$$

Therefore, we obtain tightness of $\{\mu_{I^\varepsilon};\,\varepsilon\in(0,1)\}$ on $\mathcal{X}_I$ and the corresponding tightness of μ_W follows by the same reasoning as in Proposition 4.4. ☐

We apply the Jakubowski-Skorokhod representation theorem and obtain a weakly convergent subsequence $\mu^n=\mu^{\varepsilon_n}$ together with a limit law μ such that the following result holds true.

Proposition 5.6. *There exists a probability space $(\tilde\Omega,\tilde{\mathscr{F}},\tilde{\mathbb{P}})$ with a sequence of $\mathcal{X}$-valued random variables $(\tilde u^n,\tilde v^n,\tilde\nu^n,\tilde V^n,\tilde I^n,\tilde W^n,\tilde u_0^n)$, $n\in\mathbb{N}$, and $(\tilde u,\tilde v,\tilde\nu,\tilde V,\tilde I,\tilde W,\tilde u_0)$ such that*

- *(i) the laws of $(\tilde u^n,\tilde v^n,\tilde\nu^n,\tilde V^n,\tilde I^n,\tilde W^n,\tilde u_0^n)$ and $(\tilde u,\tilde v,\tilde\nu,\tilde V,\tilde I,\tilde W,\tilde u_0)$ under $\tilde{\mathbb{P}}$ coincide with μ^n and μ, respectively,*
- *(ii) $(\tilde u^n,\tilde v^n,\tilde\nu^n,\tilde V^n,\tilde I^n,\tilde W^n,\tilde u_0^n)$ converges $\tilde{\mathbb{P}}$-a.s. to $(\tilde u,\tilde v,\tilde\nu,\tilde V,\tilde I,\tilde W,\tilde u_0)$ in the topology of $\mathcal{X}$.*

We are immediately able to identify the approximations $\tilde v^n,\tilde\nu^n,\tilde V^n,\tilde I^n$, $n\in\mathbb{N}$.

Lemma 5.7. *For every $n \in \mathbb{N}$ it holds true a.s.*

$$\tilde{v}^n = \operatorname{div}\left(\frac{\nabla \tilde{u}^n}{H(\nabla \tilde{u}^n)}\right), \qquad \tilde{\nu}^n = \frac{\nabla \tilde{u}^n}{H(\nabla \tilde{u}^n)},$$

$$\tilde{V}^n = \operatorname{div}\left(\frac{\nabla \tilde{u}^n}{H(\nabla \tilde{u}^n)}\right)\sqrt{H(\nabla \tilde{u}^n)}, \qquad \tilde{I}^n = \int_0^{\cdot} \|H(\nabla \tilde{u}^n)\|_{L^1_x}^{1+\theta}\,\mathrm{d}s.$$

Proof. According to our energy estimates and in particular due to Proposition 4.1 and the surface area estimate from Proposition 5.1, the mappings

$$\operatorname{supp}\mu_{u^n} \to L^2(0,T;L^2(\mathbb{T}^2)),\ u \mapsto \operatorname{div}\left(\frac{\nabla u}{H(\nabla u)}\right),$$

$$\operatorname{supp}\mu_{u^n} \to L^2(0,T;L^2(\mathbb{T}^2)),\ u \mapsto \frac{\nabla u}{H(\nabla u)},$$

$$\operatorname{supp}\mu_{u^n} \to L^2(0,T;L^2(\mathbb{T}^2)),\ u \mapsto \operatorname{div}\left(\frac{\nabla u}{H(\nabla u)}\right)\sqrt{H(\nabla u)}$$

and

$$\operatorname{supp}\mu_{u^n} \to C([0,T];\mathbb{R}),\ u \mapsto \int_0^{\cdot} \|H(\nabla u)\|_{L^1_x}^{1+\theta}\,\mathrm{d}s$$

are well-defined and measurable. Therefore, the claim follows directly from the equality of joint laws of $(u^n, v^n, \nu^n, V^n, I^n)$ and $(\tilde{u}^n, \tilde{v}^n, \tilde{\nu}^n, \tilde{V}^n, \tilde{I}^n)$. □

As a consequence of the a.s. convergence $\tilde{u}^n \to \tilde{u}$ in $L^2(0,T;L^2(\mathbb{T}^2))$ and the uniform bound in Proposition 4.1 we deduce that

$$\nabla \tilde{u}^n \rightharpoonup \nabla \tilde{u} \quad \text{in} \quad L^2(\tilde{\Omega}; L^2(0,T;L^2(\mathbb{T}^2))). \tag{5.3}$$

Nevertheless, as our model problem is nonlinear in $\nabla \tilde{u}$ it is crucial to establish the strong convergence in order to be able to pass to the limit.

Proposition 5.8. *For all $p \in [1,2)$, it holds true that*

$$\nabla \tilde{u}^n \to \nabla \tilde{u} \quad \textit{in} \quad L^p(\tilde{\Omega}; L^p(0,T;L^p(\mathbb{T}^2))).$$

Proof. Step 1: Due to the weak convergence (5.3), there exists a Young measure associated to the sequence $(\nabla \tilde{u}^n)$, i.e. there exists $\sigma : \tilde{\Omega} \times [0,T] \times \mathbb{T}^2 \to \mathcal{P}_1(\mathbb{R}^2)$, where $\mathcal{P}_1(\mathbb{R}^2)$ denotes the set of probability measures on $\mathbb{R}^2$, such that for every $B \in C(\mathbb{R}^2)$ with linear growth

$$B(\nabla \tilde{u}^n) \rightharpoonup \bar{B} \quad \text{in} \quad L^2(\tilde{\Omega}; L^2(0,T;L^2(\mathbb{T}^2)))$$

where

$$\bar{B}(t,x) = \langle \sigma_{t,x}, B\rangle \quad \text{a.e.}$$

We refer the reader to [43] for a thorough exposition of the concept of Young measures, the above applied result can be found in [43, Theorem 4.2.1, Corollary 4.2.10]. The desired strong convergence of $\nabla \tilde{u}^n$ will be shown once we prove that for a.e. ω, t, x the Young measure σ is a Dirac mass.

Step 2: In this part of the proof, we show that the following relation holds true a.e.

$$\int_{\mathbb{R}^2} \frac{|p|^2}{\sqrt{1+|p|^2}}\,\mathrm{d}\sigma_{t,x}(p) = \left(\int_{\mathbb{R}^2} p\,\mathrm{d}\sigma_{t,x}(p)\right)\cdot\left(\int_{\mathbb{R}^2} \frac{p}{\sqrt{1+|p|^2}}\,\mathrm{d}\sigma_{t,x}(p)\right). \tag{5.4}$$

Towards this end, we observe that due to Proposition 5.6,

$$\tilde{v}^n \rightharpoonup \tilde{v}, \quad \tilde{\nu}^n \rightharpoonup \tilde{\nu} \quad \text{in} \quad L^2(0,T;L^2(\mathbb{T}^2)) \quad \text{a.s.}$$

Using $|\tilde\nu^n| \leq 1$ and the Vitali convergence Theorem we also deduce that $\tilde\nu \in L^2(\tilde\Omega; L^2(0,T; L^2(\mathbb{T}^2)))$ with

$$\tilde\nu^n \rightharpoonup \tilde\nu \quad \text{in} \quad L^2(\tilde\Omega; L^2(0,T; L^2(\mathbb{T}^2)))$$

Besides, $\tilde\nu^n$ is a continuous and bounded function of $\nabla\tilde u^n$ hence, according to *Step 1*, $\tilde\nu$ is given by

$$\tilde\nu(t,x) = \int_{\mathbb{R}^2} \frac{p}{\sqrt{1+|p|^2}}\,\mathrm{d}\sigma_{t,x}(p).$$

Using integration by parts, it follows easily that $\tilde v = \operatorname{div}\tilde\nu$ almost everywhere. Thus, on the one hand, we employ the Div-Curl Lemma type argument from [22, Theorem 3.1, (3.13)] and obtain

$$\nabla\tilde u^n \cdot \tilde\nu^n \rightharpoonup \nabla u \cdot \tilde\nu \quad \text{in} \quad L^2(0,T; L^2(\mathbb{T}^2)) \quad \text{a.s.}$$

and consequently by the Vitali convergence theorem

$$\nabla\tilde u^n \cdot \tilde\nu^n \rightharpoonup \nabla u \cdot \tilde\nu \quad \text{in} \quad L^2(\tilde\Omega; L^2(0,T; L^2(\mathbb{T}^2))).$$

On the other hand, we deduce from *Step 1* that the weak limit of $\nabla\tilde u^n \cdot \tilde\nu^n$ is also given by

$$\int_{\mathbb{R}^2} \frac{|p|^2}{\sqrt{1+|p|^2}}\,\sigma_{t,x}(p)$$

and (5.4) follows.

Step 3: Next, we will infer from (5.4) that σ reduces to a Dirac mass for a.e. ω, t, x. To simplify the notation, let us denote $f(p) = p$, $g(p) = \frac{p}{\sqrt{1+|p|^2}}$. Then (5.4) reads as

$$\langle\sigma, f\cdot g\rangle = \langle\sigma, f\rangle \cdot \langle\sigma, g\rangle. \tag{5.5}$$

Since $\langle\nu, 1\rangle = 1$, the left hand side of (5.5) can be rewritten as

$$\begin{aligned}\frac12\Big(\int_{\mathbb{R}^2} f(p)\cdot g(p)\mathrm{d}\sigma(p)\int_{\mathbb{R}^2}\mathrm{d}\sigma(q) + \int_{\mathbb{R}^2} f(q)\cdot g(q)\mathrm{d}\sigma(q)\int_{\mathbb{R}^2}\mathrm{d}\sigma(p)\Big)\\ = \frac12\int_{\mathbb{R}^2}\big(f(p)\cdot g(p) + f(q)\cdot g(q)\big)\mathrm{d}\sigma\otimes\sigma(p,q)\end{aligned}$$

whereas for the right hand side, we have

$$\begin{aligned}\frac12\Big(\int_{\mathbb{R}^2} f(p)\mathrm{d}\sigma(p)\cdot\int_{\mathbb{R}^2} g(q)\mathrm{d}\sigma(q) + \int_{\mathbb{R}^2} f(q)\mathrm{d}\sigma(q)\cdot\int_{\mathbb{R}^2} g(p)\mathrm{d}\sigma(p)\Big)\\ = \frac12\int_{\mathbb{R}^2}\big(f(p)\cdot g(q) + f(p)\cdot g(q)\big)\mathrm{d}\sigma\otimes\sigma(p,q).\end{aligned}$$

Thus subtracting the right hand side from the left hand side we deduce that

$$\int_{\mathbb{R}^2}\big(f(p)-f(q)\big)\cdot\big(g(p)-g(q)\big)\mathrm{d}\sigma\otimes\sigma(p,q) = 0.$$

To conclude, it is enough to show that

$$\begin{aligned}&F(p,q) = \big(f(p)-f(q)\big)\cdot\big(g(p)-g(q)\big) > 0 \qquad \forall p,g\in\mathbb{R}^2,\ p\neq q,\\ &F(p,p) = 0.\end{aligned} \tag{5.6}$$

Indeed, as a consequence, the support of σ needs to be a single point hence necessarily $\sigma_{t,x} = \delta_{\nabla\tilde{u}(t,x)}$ a.e. In order to verify (5.6) let us observe that $F(p,q) \geq 0$ holds true if and only if

$$|p|^2\sqrt{1+|q|^2} + |q|^2\sqrt{1+|p|^2} \geq p\cdot q(\sqrt{1+|p|^2} + \sqrt{1+|q|^2}). \tag{5.7}$$

Under the condition that $|p| \neq |q|$, this is further equivalent to

$$|p|^2 + |q|^2 + |p|^2|q|^2 \geq 2p\cdot q + (p\cdot q)^2$$

which leads to

$$|p-q|^2 \geq (p\cdot q)^2 - |p|^2|q|^2.$$

Since the left hand side is always nonnegative and the right hand side always nonpositive due to the Hölder inequality, the inequality is certainly satisfied. Moreover, we see that it becomes equality if and only if $p = q$ which contradicts the assumption that $|p| \neq |q|$. If $|p| = |q|$ we deduce from (5.7) that

$$|p|^2 \geq p\cdot q$$

which holds true due to Hölder inequality. Besides, the equality is achieved if and only if $p = q$.

Step 4: Since a Young measure being Dirac is equivalent to the convergence in measure we conclude by making use of the a priori estimate from Proposition 4.1. □

As a consequence, we are able to identify the limits $\tilde{V}$ and $\tilde{I}$.

Corollary 5.9. *It holds true a.s.*

$$\tilde{V} = \operatorname{div}\left(\frac{\nabla\tilde{u}}{H(\nabla\tilde{u})}\right)\sqrt{H(\nabla\tilde{u})}, \qquad \tilde{I} = \int_0^{\cdot} \|H(\nabla\tilde{u})\|_{L^1_x}^{1+\theta}\,\mathrm{d}s.$$

Proof. In order to identify the limit of $\tilde{V}^n$, observe that due to Proposition 5.8, for all $q \in [1,\infty)$

$$\frac{\nabla\tilde{u}^n}{H(\nabla\tilde{u}^n)} \to \frac{\nabla\tilde{u}}{H(\nabla\tilde{u})} \quad \text{in} \quad L^q(\tilde{\Omega}; L^q(0,T; L^q(\mathbb{T}^2)))$$

hence according to Proposition 5.1

$$\operatorname{div}\left(\frac{\nabla\tilde{u}^n}{H(\nabla\tilde{u}^n)}\right) \rightharpoonup \operatorname{div}\left(\frac{\nabla\tilde{u}}{H(\nabla\tilde{u})}\right) \quad \text{in} \quad L^2(\tilde{\Omega}; L^2(0,T; L^2(\mathbb{T}^2))).$$

Besides,

$$\sqrt{H(\nabla\tilde{u}^n)} \to \sqrt{H(\nabla\tilde{u})} \quad \text{in} \quad L^2(\tilde{\Omega}; L^2(0,T; L^2(\mathbb{T}^2))) \tag{5.8}$$

and consequently

$$\operatorname{div}\left(\frac{\nabla\tilde{u}^n}{H(\nabla\tilde{u}^n)}\right)\sqrt{H(\nabla\tilde{u}^n)} \rightharpoonup \operatorname{div}\left(\frac{\nabla\tilde{u}}{H(\nabla\tilde{u})}\right)\sqrt{H(\nabla\tilde{u})}$$

in $L^1(\tilde{\Omega}; L^1(0,T; L^1(\mathbb{T}^2)))$ which gives the identification of $\tilde{V}$.

Identification of $\tilde{I}$ follows from the fact that for every $t \in [0,T]$

$$\int_0^t \|H(\nabla\tilde{u}^n)\|_{L^1_x}^{1+\theta}\,\mathrm{d}s \to \int_0^t \|H(\nabla\tilde{u})\|_{L^1_x}^{1+\theta}\,\mathrm{d}s$$

according to Proposition 5.8, Proposition 4.1 and the Vitali convergence theorem. □

5.3. **Identification of the limit.** Let $(\tilde{\mathscr{F}}_t)$ be the $\tilde{\mathbb{P}}$-augmented canonical filtration of the process $(\tilde u, \tilde W, \tilde u_0)$. Note that $\tilde V$ and $\tilde I$ are adapted to $(\tilde{\mathscr{F}}_t)$ as well due to Corollary 5.9. Now everything is prepared to establish the final existence result, which in particular proves the main Theorem 2.3.

Theorem 5.10. $\big((\tilde\Omega, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}}), \tilde u, \tilde W\big)$ *is a weak martingale solution to* (2.1) *with the initial law* Λ. *That is, it satisfies Definition* 2.1 *and in particular for every* $\varphi \in C^\infty(\mathbb{T}^2)$ *it holds true for a.e.* $t\in[0,T]$ *a.s. that*

$$\begin{aligned}\langle \tilde u(t), \varphi\rangle = \langle \tilde u_0, \varphi\rangle + \frac12\int_0^t \langle \tilde u, \Delta\varphi\rangle \mathrm{d}s + \frac12 \int_0^t \langle H(\nabla \tilde u)\tilde v, \varphi\rangle \mathrm{d}s \\ + \int_0^t \langle H(\nabla\tilde u)\mathrm{d}\tilde W, \varphi\rangle,\end{aligned} \tag{5.9}$$

where

$$\tilde v = \operatorname{div}\left(\frac{\nabla \tilde u}{H(\nabla \tilde u)}\right). \tag{5.10}$$

The proof is based on a refined identification limit procedure which in comparison to Subsection 4.2 includes two new ingredients. First, the method of densely defined martingales which was developed in [28] is applied in order to deal with martingales that are only defined for almost all times and no continuity properties are a priori known (see [28, Theorem 4.13, Appendix]). In that case, the corresponding quadratic variations are not well defined and the approach of Subsection 4.2 does not apply directly. Second, the local martingales approach of [29] is invoked to overcome the difficulty in the passage to the limit.

Both issues originate in the lack of uniform moment estimates for ∇u^n. Indeed, on the one hand, we are not able to obtain tightness of (u^n) in any space of continuous (or weakly continuous) functions in time and consequently the passage to the limit in the corresponding martingales can be performed only for a.e. $t\in[0,T]$. On the other hand, we are only able to establish the strong convergence $\nabla\tilde u^n \to \nabla \tilde u$ in $L^p(0,T;L^p(\mathbb{T}^2))$ a.s. for $p\in[1,2)$ and the convergence in $L^2(0,T;L^2(\mathbb{T}^2))$ remains weak, which is not enough to pass to the limit in the quadratic variation. Note that the problem lies in particular in the weak convergence with respect to time rather than space as we consider weak solutions in x anyway.

We claim that as a consequence of Proposition 5.6, it holds true that

$$\tilde u^n \to \tilde u \quad \text{in} \quad L^2(\mathbb{T}^2) \quad \text{in measure} \quad \tilde{\mathbb{P}}\otimes \mathcal{L}_{[0,T]} \tag{5.11}$$

and consequently there exists $\mathcal{D}\subset[0,T]$ of full Lebesgue measure such that (up to subsequence)

$$\tilde u^n(t) \to \tilde u(t) \quad \text{in} \quad L^2(\mathbb{T}^2) \quad \tilde{\mathbb{P}}\text{-a.s.} \quad \forall t\in\mathcal{D}. \tag{5.12}$$

Indeed, (5.11) follows directly from the dominated convergence theorem since for every $\delta\in(0,1)$

$$\tilde{\mathbb{P}}\otimes\mathcal{L}_{[0,T]}\Big(\|\tilde u^n - \tilde u\|_{L^2_x} > \delta\Big) = \tilde{\mathbb{E}}\int_0^T \mathbf{1}_{\{\|\tilde u^n(t)-\tilde u(t)\|_{L^2_x}>\delta\}}\,\mathrm{d}t$$

where for a.e. ω the inner integral converges to 0 due to Proposition 5.6.

Note that $\mathcal{D}$ is dense in $[0,T]$ since it is complement of a set with zero Lebesgue measure. For all $t\in\mathcal{D}$ and a test function $\varphi\in C^\infty(\mathbb{T}^2)$ we define

$$M^n(t)=\big\langle u^n(t),\varphi\big\rangle-\big\langle u^n(0),\varphi\big\rangle-\Big(\frac12+\varepsilon_n\Big)\int_0^t\langle u^n,\Delta\varphi\rangle\mathrm{d}s-\frac12\int_0^t\langle H(\nabla u^n)v^n,\varphi\rangle\mathrm{d}s,$$

$$\tilde M^n(t)=\big\langle \tilde u^n(t),\varphi\big\rangle-\big\langle \tilde u^n_0,\varphi\big\rangle-\Big(\frac12+\varepsilon_n\Big)\int_0^t\langle \tilde u^n,\Delta\varphi\rangle\mathrm{d}s-\frac12\int_0^t\langle H(\nabla \tilde u^n)\tilde v^n,\varphi\rangle\mathrm{d}s,$$

$$\tilde M(t)=\big\langle \tilde u(t),\varphi\big\rangle-\big\langle \tilde u_0,\varphi\big\rangle-\frac12\int_0^t\langle \tilde u,\Delta\varphi\rangle\mathrm{d}s-\frac12\int_0^t\langle H(\nabla \tilde u)\tilde v,\varphi\rangle\mathrm{d}s,$$

and recall that

$$v^n=\operatorname{div}\Big(\frac{\nabla u^n}{H(\nabla u^n)}\Big),\qquad \tilde v^n=\operatorname{div}\Big(\frac{\nabla \tilde u^n}{H(\nabla \tilde u^n)}\Big).$$

Proposition 5.11. *The process $\tilde W$ is a $(\tilde{\mathscr{F}}_t)$-Wiener process, the processes*

$$\tilde M,\qquad \tilde M^2-\int_0^\cdot\big\langle H(\nabla\tilde u),\varphi\big\rangle^2\mathrm{d}r,\qquad \tilde M\tilde W-\int_0^\cdot\big\langle H(\nabla\tilde u),\varphi\big\rangle\mathrm{d}r,$$

indexed by $t\in\mathcal{D}$, are $(\tilde{\mathscr{F}}_t)$-local martingales.

Proof. The first claim follows immediately by the same reasoning as in Lemma 4.8. To prepare the proof of the remaining parts, let $R\in\mathbb{R}^+$ and define

$$\tau_R: C([0,T];\mathbb{R})\to[0,T],\quad f\mapsto\inf\big\{t>0;|f(t)|\ge R\big\}.$$

(with the convention $\inf\emptyset=T$). Then for every I^n, one may use Proposition 4.1 and deduce that $\tau_R(I^n)$ defines an $(\mathscr{F}_t)$-stopping time and the blow up does not occur in a finite time, i.e.

$$\sup_{R\in\mathbb{R}^+}\tau_R(I^n)=T\qquad\text{a.s.}\tag{5.13}$$

The same is valid for the case of $\tilde I^n$ and $\tilde I$. The stopping times $\tau_R(\tilde I)$ will play the role of a localizing sequence for the processes

$$\tilde M,\qquad \tilde M^2-\int_0^\cdot\big\langle H(\nabla\tilde u),\varphi\big\rangle^2\mathrm{d}r,\qquad \tilde M\tilde W-\int_0^\cdot\big\langle H(\nabla\tilde u),\varphi\big\rangle\mathrm{d}r.$$

In particular, we employ $\tau_R(\tilde I^n)$ as a localizing sequence for the approximations

$$\tilde M^n,\qquad (\tilde M^n)^2-\int_0^\cdot\big\langle H(\nabla\tilde u^n),\varphi\big\rangle^2\mathrm{d}r,\qquad \tilde M^n\tilde W^n-\int_0^\cdot\big\langle H(\nabla\tilde u^n),\varphi\big\rangle\mathrm{d}r$$

and pass to the limit. Therefore, it is also necessary to establish the convergence of the stopping times, that is, for a fixed $R\in\mathbb{R}^+$ we need to verify

$$\tau_R(\tilde I^n)\to\tau_R(\tilde I)\qquad\text{a.s.}$$

so it is a question of continuity of $\tau_R(\cdot)$. This is not true in general but due to observations made in [29, Lemma 3.5, Lemma 3.6], there exists a sequence $R_m\to\infty$ such that

$$\tilde{\mathbb{P}}\big(\tau_{R_m}(\cdot)\text{ is continuous at }\tilde I\big)=1\tag{5.14}$$

and in the sequel we only employ R_m from this sequence.

Let us proceed with the proof. We observe that, for all $n \in \mathbb{N}$, the process

$$M^n = \int_0^{\cdot} \big\langle H(\nabla u^n)\,\mathrm{d}W(r), \varphi \big\rangle$$

is a square integrable $(\mathscr{F}_t)$-martingale by (4.2) and therefore

$$(M^n)^2 - \int_0^{\cdot} \big\langle H(\nabla u^n), \varphi \big\rangle^2 \,\mathrm{d}r, \qquad M^n W - \int_0^{\cdot} \big\langle H(\nabla u^n), \varphi \big\rangle \,\mathrm{d}r$$

are $(\mathscr{F}_t)$-martingales. Therefore, as in Lemma 4.9, we obtain for fixed $n \in \mathbb{N}$ from the equality of laws that

$$\begin{aligned} &\tilde{\mathbb{E}}\,\gamma\big(\varrho_s \tilde{u}^n, \varrho_s \tilde{W}^n, \tilde{u}_0^n\big)\big[\tilde{M}^n\big(t \wedge \tau_{R_m}(\tilde{I}^n)\big)\big] \\ &= \tilde{\mathbb{E}}\,\gamma\big(\varrho_s \tilde{u}^n, \varrho_s \tilde{W}^n, \tilde{u}_0^n\big)\big[\tilde{M}^n\big(s \wedge \tau_{R_m}(\tilde{I}^n)\big)\big], \end{aligned} \tag{5.15}$$

(5.16)

$$\begin{aligned} &\tilde{\mathbb{E}}\,\gamma\big(\varrho_s \tilde{u}^n, \varrho_s \tilde{W}^n, \tilde{u}_0^n\big)\Big[(\tilde{M}^n)^2\big(t \wedge \tau_{R_m}(\tilde{I}^n)\big) - \int_{0\wedge\tau_{R_m}(\tilde{I}^n)}^{t\wedge\tau_{R_m}(\tilde{I}^n)} \big\langle H(\nabla \tilde{u}^n), \varphi \big\rangle^2 \,\mathrm{d}r\Big] \\ &= \tilde{\mathbb{E}}\,\gamma\big(\varrho_s \tilde{u}^n, \varrho_s \tilde{W}^n, \tilde{u}_0^n\big)\Big[(\tilde{M}^n)^2\big(s \wedge \tau_{R_m}(\tilde{I}^n)\big) - \int_{0}^{s\wedge\tau_{R_m}(\tilde{I}^n)} \big\langle H(\nabla \tilde{u}^n), \varphi \big\rangle^2 \,\mathrm{d}r\Big], \end{aligned}$$

(5.17)

$$\begin{aligned} &\tilde{\mathbb{E}}\,\gamma\big(\varrho_s \tilde{u}^n, \varrho_s \tilde{W}^n, \tilde{u}_0^n\big)\Big[\tilde{M}^n\tilde{W}^n\big(t \wedge \tau_{R_m}(\tilde{I}^n)\big) - \int_{0}^{t\wedge\tau_{R_m}(\tilde{I}^n)} \big\langle H(\nabla \tilde{u}^n), \varphi \big\rangle \,\mathrm{d}r\Big] \\ &= \tilde{\mathbb{E}}\,\gamma\big(\varrho_s \tilde{u}^n, \varrho_s \tilde{W}^n, \tilde{u}_0^n\big)\Big[\tilde{M}^n\tilde{W}^n\big(s \wedge \tau_{R_m}(\tilde{I}^n)\big) - \int_{0}^{s\wedge\tau_{R_m}(\tilde{I}^n)} \big\langle H(\nabla \tilde{u}^n), \varphi \big\rangle \,\mathrm{d}r\Big], \end{aligned}$$

where $s, t \in [0, T]$, $s \le t$, and

$$\gamma : \mathcal{X}_u|_{[0,s]} \times \mathcal{X}_W|_{[0,s]} \times \mathcal{X}_{u_0} \to [0, 1]$$

is a continuous function.

In order to pass to the limit in (5.15), (5.16) and (5.17), let us first establish the convergence $\tilde{M}^n(t) \to \tilde{M}(t)$ a.s. for all $t \in \mathcal{D}$. Concerning the term $\langle \tilde{u}^n(t), \varphi \rangle$ we conclude immediately due to (5.12). Since convergence of the third term in $\tilde{M}^n(t)$ follows directly from Proposition 5.6, let us proceed with the mean curvature term. We recall that according to Proposition 5.6 and Corollary 5.9 it holds true that

$$\operatorname{div}\left(\frac{\nabla \tilde{u}^n}{H(\nabla \tilde{u}^n)}\right)\sqrt{H(\nabla \tilde{u}^n)} \rightharpoonup \operatorname{div}\left(\frac{\nabla \tilde{u}}{H(\nabla \tilde{u})}\right)\sqrt{H(\nabla \tilde{u})}$$

in $L^2(0, T; L^2(\mathbb{T}^2)))$ almost surely. Moreover, in view of (5.8) we obtain

$$\operatorname{div}\left(\frac{\nabla \tilde{u}^n}{H(\nabla \tilde{u}^n)}\right)H(\nabla \tilde{u}^n) \rightharpoonup \operatorname{div}\left(\frac{\nabla \tilde{u}}{H(\nabla \tilde{u})}\right)H(\nabla \tilde{u})$$

in $L^1(0, T; L^1(\mathbb{T}^2)))$ almost surely. which yields the desired convergence of the corresponding term in $\tilde{M}^n(t)$.

Moreover, we observe that according to (5.15), (5.16), (5.17) and [28, Proposition A.1] it follows for every $n \in \mathbb{N}$ that

$$\tilde{M}^n = \int_0^{\cdot} \big\langle H(\nabla \tilde{u}^n), \varphi \big\rangle \,\mathrm{d}\tilde{W}^n \qquad \forall t \in \mathcal{D} \qquad \tilde{\mathbb{P}}\text{-a.s.}$$

Therefore, the passage to the limit in (5.15) and in the first terms on the left hand side of (5.16) and (5.17) (and the same for the right hand side) can be justified by using the convergence $\tilde{M}^n(t) \to \tilde{M}(t)$ together with the uniform integrability given by

$$\tilde{\mathbb{E}}\big|\tilde{M}^n\big(t\wedge\tau_{R_m}(\tilde{I}^n)\big)\big|^{2+v} \leq C\,\tilde{\mathbb{E}}\bigg(\int_0^{\tau_{R_m}(\tilde{I}^n)}\big\langle H(\nabla\tilde{u}^n),\varphi\big\rangle^2\,\mathrm{d}t\bigg)^{(2+v)/2}$$
$$\leq C\,\tilde{\mathbb{E}}\bigg[\sup_{0\leq t\leq T}\|H(\nabla\tilde{u}^n)\|_{L^1_x}\int_0^{\tau_{R_m}(\tilde{I}^n)}\|H(\nabla\tilde{u}^n)\|_{L^1_x}^{1+\theta}\,\mathrm{d}r\bigg] \leq C_\delta R_m.$$

This estimate also yields the necessary uniform integrability that together with

$$\big\langle H(\nabla\tilde{u}^n),\varphi\big\rangle \to \big\langle H(\nabla\tilde{u}^n),\varphi\big\rangle \quad \text{a.e. } (\omega,r)$$

justifies the passage to the limit in the remaining terms in (5.16) and (5.17). Thus we have shown that $\tilde{M}^2 - \int_0^{\cdot}\big\langle H(\nabla\tilde{u}),\varphi\big\rangle^2\,\mathrm{d}r$ and $\tilde{M}\tilde{W} - \int_0^{\cdot}\big\langle H(\nabla\tilde{u}),\varphi\big\rangle\,\mathrm{d}r$ are densely defined local martingales with respect to $(\tilde{\mathscr{F}}_t)$ and the proof is complete. □

Proof of Theorem 5.10. Having Proposition 5.11 in hand, we apply [28, Proposition A.1] for the stopped processes

$$\tilde{M}\big(\,\cdot\wedge\tau_{R_m}(\tilde{I})\big), \qquad \tilde{M}^2\big(\,\cdot\wedge\tau_{R_m}(\tilde{I})\big) - \int_0^{\cdot\wedge\tau_{R_m}(\tilde{I})}\big\langle H(\nabla\tilde{u}),\varphi\big\rangle^2\,\mathrm{d}r,$$
$$\tilde{M}\tilde{W}\big(\,\cdot\wedge\tau_{R_m}(\tilde{I})\big) - \int_0^{\cdot\wedge\tau_{R_m}(\tilde{I})}\big\langle H(\nabla\tilde{u}),\varphi\big\rangle\,\mathrm{d}r,$$

and deduce that

$$\tilde{M}\big(\,\cdot\wedge\tau_{R_m}(\tilde{I})\big) = \int_0^{\cdot\wedge\tau_{R_m}(\tilde{I})}\big\langle H(\nabla\tilde{u})\,\mathrm{d}\tilde{W},\varphi\big\rangle \qquad \forall t\in\mathcal{D} \quad \tilde{\mathbb{P}}\text{-a.s.}$$

for every $m\in\mathbb{N}$ and consequently (5.9) holds true due to (5.13).

In particular, $\tilde{M}$ can be defined for all $t\in[0,T]$ such that it has a modification which is a continuous $(\tilde{\mathscr{F}}_t)$-local martingale and furthermore, due to Proposition 4.1, it is a $(\tilde{\mathscr{F}}_t)$-martingale. Besides, we observe that (2.1) is satisfied in $H^{-1}(\mathbb{T}^2)$ and, as a consequence, $\tilde{u}$ (as a class of equivalence) has a representative with almost surely continuous trajectories in $H^{-1}(\mathbb{T}^2)$ and hence is measurable with respect to the predictable σ-field $\mathcal{P}$. The continuous embedding $H^1(\mathbb{T}^2)\hookrightarrow H^{-1}(\mathbb{T}^2)$ then implies that $\tilde{u}\in L^2(\tilde{\Omega}\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;H^1(\mathbb{T}^2))$ as required by Definition 2.1. Indeed, any Borel subset of $H^1(\mathbb{T}^2)$ is also Borel in $H^{-1}(\mathbb{T}^2)$ and therefore its preimage under $\tilde{u}$ is predictable. The proof is complete. □

References

[1] S. M. Allen and J. W. Cahn. A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Metall.*, 27:10851095, 1979.

[2] Fred Almgren, Jean E. Taylor, and Lihe Wang. Curvature-driven flows: a variational approach. *SIAM J. Control Optim.*, 31(2):387–438, 1993.

[3] G. Barles, H. M. Soner, and P. E. Souganidis. Front propagation and phase field theory. *SIAM J. Control Optim.*, 31(2):439–469, 1993.

[4] G. Bellettini and M. Novaga. Minimal barriers for geometric evolutions. *J. Differential Equations*, 139(1):76–103, 1997.

[5] G. Bellettini and M. Paolini. Some results on minimal barriers in the sense of De Giorgi applied to driven motion by mean curvature. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5)*, 19:43–67, 1995.

[6] Giovanni Bellettini. *Lecture notes on mean curvature flow, barriers and singular perturbations*, volume 12 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, 2013.

[7] Kenneth A. Brakke. *The motion of a surface by its mean curvature*, volume 20 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1978.

[8] Zdzislaw Brzeźniak. On stochastic convolution in Banach spaces and applications. *Stoch. Stoch. Rep.* 61:245–295, 1997.

[9] Rainer Buckdahn and Jin Ma. Pathwise stochastic Taylor expansions and stochastic viscosity solutions for fully nonlinear stochastic PDEs. *Ann. Probab.*, 30(3):1131–1171, 2002.

[10] Michael Caruana, Peter K. Friz, and Harald Oberhauser. A (rough) pathwise approach to a class of non-linear stochastic partial differential equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(1):27–46, 2011.

[11] Y. G. Chen, Y. Giga, and S. Goto. *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations.* J. Diff. Geom. 33, 749-786, 1991.

[12] G. Da Prato, J. Zabczyk. *Stochastic Equations in Infinite Dimensions.* Encyclopedia Math. Appl., vol. 44, Cambridge University Press, Cambridge, 1992.

[13] A. Debussche, M. Hofmanová, J. Vovelle. Degenerate parabolic stochastic partial differential equations: Quasilinear case. *ArXiv e-prints*, 2013.

[14] Nicolas Dirr, Stephan Luckhaus, and Matteo Novaga. A stochastic selection principle in case of fattening for curvature flow. *Calc. Var. Partial Differential Equations*, 13(4):405–425, 2001.

[15] Klaus Ecker. *Regularity theory for mean curvature flow.* Progress in Nonlinear Differential Equations and their Applications, 57. Birkhäuser Boston Inc., Boston, MA, 2004.

[16] Klaus Ecker and Gerhard Huisken. Interior estimates for hypersurfaces moving by mean curvature. *Invent. Math.*, 105(3):547–569, 1991.

[17] Abdelhadi Es-Sarhir and Max-K. von Renesse. Ergodicity of stochastic curve shortening flow in the plane. *SIAM J. Math. Anal.*, 44(1):224–244, 2012.

[18] Lawrence C. Evans, Halil Mete Soner, and Panagiotis E. Souganidis. Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.*, 45(9):1097–1123, 1992.

[19] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. I. *J. Differential Geom.*, 33(3):635–681, 1991.

[20] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. II. *Trans. Amer. Math. Soc.*, 330(1):321–332, 1992.

[21] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. III. *J. Geom. Anal.*, 2(2):121–150, 1992.

[22] Lawrence C. Evans and Joel Spruck. Motion of level sets by mean curvature. IV. *J. Geom. Anal.*, 5(1):77–114, 1995.

[23] Xiaobing Feng, Yukun Li, and Andreas Prohl. Finite element approximations of the stochastic mean curvature flow of planar curves of graphs. *Stoch. Partial Differ. Equ. Anal. Comput.*, 2(1):54–83, 2014.

[24] F. Flandoli, D. Gątarek. Martingale and stationary solutions for stochastic Navier-Stokes equations. *Probab. Theory Related Fields* 102(3):367-391, 1995.

[25] Peter Friz and Harald Oberhauser. Rough path stability of (semi-)linear SPDEs. *Probab. Theory Related Fields*, 158(1-2):401–434, 2014.

[26] T. Funaki. The scaling limit for a stochastic PDE and the separation of phases. *Probab. Theory Related Fields*, 102(2):221–288, 1995.

[27] B. Gess and M. Röckner. Stochastic variational inequalities and regularity for degenerate stochastic partial differential equations. *ArXiv e-prints*, May 2014.

[28] M. Hofmanová, Degenerate parabolic stochastic partial differential equations, Stoch. Pr. Appl. 123 (12) (2013) 4294-4336.

[29] M. Hofmanová, J. Seidler. On weak solutions of stochastic differential equations. *Stoch. Anal. Appl.* 30(1):100–121, 2012.

[30] Gerhard Huisken. Asymptotic behavior for singularities of the mean curvature flow. *J. Differential Geom.*, 31(1):285–299, 1990.

[31] Tom Ilmanen. Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. *J. Differential Geom.*, 38(2):417–461, 1993.

[32] Tom Ilmanen. Elliptic regularization and partial regularity for motion by mean curvature. *Mem. Amer. Math. Soc.*, 108, 1994.

[33] A. Jakubowski. The almost sure Skorokhod representation for subsequences in nonmetric spaces. *Teor. Veroyatnost. i Primenen* 42(1):209–216, 1997); translation in *Theory Probab. Appl.* 42(1):167–174, 1998.

[34] Kyozi Kawasaki and Takao Ohta. Kinetic drumhead model of interface. I. *Progress of Theoretical Physics*, 67(1):147–163, 1982.

[35] Robert Kohn, Felix Otto, Maria G. Reznikoff, and Eric Vanden-Eijnden. Action minimization and sharp-interface limits for the stochastic Allen-Cahn equation. *Comm. Pure Appl. Math.*, 60(3):393–438, 2007.

[36] I. Gyöngy, N. Krylov. Existence of strong solutions for Itô's stochastic equations via approximations. *Probab. Theory Related Fields* 105(2):143–158, 1996.

[37] P.-L. Lions and P. E. Souganidis. Viscosity solutions of fully nonlinear stochastic partial differential equations. *Sūrikaisekikenkyūsho Kōkyūroku*, (1287):58–65, 2002. Viscosity solutions of differential equations and related topics (Japanese) (Kyoto, 2001).

[38] Pierre-Louis Lions and Panagiotis E. Souganidis. Fully nonlinear stochastic partial differential equations. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(9):1085–1092, 1998.

[39] Pierre-Louis Lions and Panagiotis E. Souganidis. Fully nonlinear stochastic partial differential equations: non-smooth equations and applications. *C. R. Acad. Sci. Paris Sér. I Math.*, 327(8):735–741, 1998.

[40] Pierre-Louis Lions and Panagiotis E. Souganidis. Fully nonlinear stochastic pde with semilinear stochastic dependence. *C. R. Acad. Sci. Paris Sér. I Math.*, 331(8):617–624, 2000.

[41] Pierre-Louis Lions and Panagiotis E. Souganidis. Uniqueness of weak solutions of fully nonlinear stochastic partial differential equations. *C. R. Acad. Sci. Paris Sér. I Math.*, 331(10):783–790, 2000.

[42] Stephan Luckhaus and Thomas Sturzenhecker. Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differential Equations*, 3(2):253–271, 1995.

[43] J. Málek, J. Nečas, M. Rokyta, M. Růžička. *Weak and Measure-valued Solutions to Evolutionary PDEs.* Chapman & Hall, London, Weinheim, New York, 1996.

[44] Carlo Mantegazza. *Lecture notes on mean curvature flow*, volume 290 of *Progress in Mathematics*. Birkhäuser/Springer Basel AG, Basel, 2011.

[45] Z. Brzeźniak, M. Ondreját, Strong solutions to stochastic wave equations with values in Riemannian manifolds, J. Funct. Anal. 253 (2007) 449-481.

[46] C. Prévôt, M. Röckner, A concise course on stochastic partial differential equations, vol. 1905 of Lecture Notes in Math., Springer, Berlin, 2007.

[47] Matthias Röger and Hendrik Weber. Tightness for a stochastic Allen-Cahn equation. *Stoch. Partial Differ. Equ., Anal. Comput.*, 1(1):175–203, 2013.

[48] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl. (4) 146 (1987), 65-96.

[49] P. E. Souganidis and N. K. Yip. Uniqueness of motion by mean curvature perturbed by stochastic noise. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 21(1):1–23, 2004.

[50] Hendrik Weber. Sharp interface limit for invariant measures of a stochastic Allen-Cahn equation. *Comm. Pure Appl. Math.*, 63(8):1071–1109, 2010.

[51] Nung Kwan Yip. Stochastic motion by mean curvature. *Arch. Rational Mech. Anal.*, 144(4):313–355, 1998.

[52] Xi-Ping Zhu. *Lectures on mean curvature flows*, volume 32 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2002.

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