On Weak Solutions of Stochastic Differential Equations II.

Martina Hofmanová, Jan Seidler

Abstract: In the first part of this paper a new method of proving existence of weak solutions to stochastic differential equations with continuous coefficients having at most linear growth was developed. In this second part we show that the same method may be used even if the linear growth hypothesis is replaced with a suitable Lyapunov condition.

Keywords: stochastic differential equations, weak solutions, fractional integrals

Mathematics Subject Classification: 60H10

Let us consider a stochastic differential equation

$$\mathrm{d}X = b(t,X)\,\mathrm{d}t + \sigma(t,X)\,\mathrm{d}W, \quad X(0) \overset{\mathscr{D}}{\sim} \nu, \tag{1}$$

where $b\colon [0,T]\times\mathbb{R}^m \longrightarrow \mathbb{R}^m$, $\sigma\colon [0,T]\times\mathbb{R}^m \longrightarrow \mathbb{M}_{m\times n}$ are Borel functions and ν is a Borel probability measure on $\mathbb{R}^m$. (In what follows, we shall denote by $\mathbb{M}_{m\times n}$ the space of all m-by-n matrices over $\mathbb{R}$ endowed with the Hilbert-Schmidt norm $\|A\| = (\operatorname{Tr} AA^*)^{1/2}$.)

If the coefficients b and σ are continuous in the second variable and satisfy a linear growth hypothesis

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^m} \frac{\|b(t,x)\| + \|\sigma(t,x)\|}{1+\|x\|} < \infty, \tag{2}$$

then there exists a weak solution to (1) by a theorem established by A. V. Skorokhod some fifty years ago. All proofs of his result that we know have a common basic structure: (1) is approximated with equations having a solution, then tightness of laws of solutions to these approximating equations is shown and finally cluster points of the set of laws are identified as weak solutions to (1). In the first part of our paper [HS] we proposed a new, fairly elementary, version of this argument. In [HS] tightness is proved by means of compactness properties of fractional integrals, while the identification procedure uses results on preservation of the local martingale property under convergence in law, avoiding thus both Skorokhod's theorem on almost surely converging realizations of converging laws and results on integral representation of martingales with absolutely continuous quadratic variation, see [HS] for more details and references.

This research was supported by the GA ČR Grant no. P201/10/0752 and the GA UK Grant no. 556712.

Address correspondence to Jan Seidler, ÚTIA AV ČR, Pod vodárenskou věží 4, 182 08 Praha 8, Praha 8, Czech Republic; E-mail: `seidler@utia.cas.cz`

The purpose of the present paper, which may be viewed as a short addendum to [HS], is to show that the new method may be used even if (2) is relaxed to existence of a suitable Lyapunov function. Namely, we shall prove the following result.

Theorem 1. *Assume that a hypothesis*

(A) *$b(r,\cdot)$ and $\sigma(r,\cdot)$ are continuous on $\mathbb{R}^m$ for any $r \in [0,T]$ and both functions b, σ are locally bounded on $[0,T]\times\mathbb{R}^m$, i.e.*

$$\sup_{r\in[0,T]}\ \sup_{\|z\|\le L}\big\{\|b(r,z)\|\vee\|\sigma(r,z)\|\big\}<\infty$$

for all $L \ge 0$,

is satisfied and a function $V \in \mathscr{C}^2(\mathbb{R}^m)$ may be found such that

(L1) *there exists an increasing function $\kappa\colon \mathbb{R}_+ \longrightarrow\ ]0,\infty[$ such that*

$$\lim_{r\to\infty}\kappa(r)=+\infty$$

and $V(x) \ge \kappa(\|x\|)$ for all $x \in \mathbb{R}^m$,

(L2) *there exists $\gamma \ge 0$ such that*

$$\big\langle b(t,x),DV(x)\big\rangle+\frac{1}{2}\operatorname{Tr}\big(\sigma(t,x)^*D^2V(x)\sigma(t,x)\big)\le\gamma V(x)$$

for all $(t,x) \in [0,T]\times\mathbb{R}^m$.

Then there exists a weak solution to (1)*.*

(By DV and D^2V we denote the first and second Fréchet derivative of V, respectively.) The assumption (L2) is the well known Khas'minskii's condition for non-explosion (see [K], Theorem 3.5, where equations with locally Lipschitz continuous coefficients are considered), however, we do not work with local solutions and construct global solutions directly. To prove Theorem 1 we approximate coefficients b and σ with bounded continuous functions. Essentially, we mimick the proof of tightness of the laws of solutions to approximating equations from [HS], however, in absence of (2) we do not have uniform moment estimates for approximating processes X_k at our disposal, instead, we have to resort to a well known trick from stability theory and show, roughly speaking, that $(e^{-\gamma t}V(X_k(t)))$ are supermartingales. As a consequence, the proof is less straightforward than the corresponding one in [HS]. Once tightness is proved, the identification procedure from [HS] may be applied without any change, since it does not depend on any particular form of approximations. More precisely, in [HS], Remark 3.2, we proved:

Proposition 2. *Let the assumption* (A) *be satisfied. Let there exist Borel functions $b_k\colon [0,T]\times\mathbb{R}^m \longrightarrow \mathbb{R}^m$ and $\sigma_k\colon [0,T]\times\mathbb{R}^m \longrightarrow \mathbb{M}_{m\times n}$, $k \ge 1$, such that*

1° *$b_k(r,\cdot)$, $\sigma_k(r,\cdot)$ are continuous on $\mathbb{R}^m$ for any $r \in [0,T]$ and $k \ge 1$,* 2° $b_k(r,\cdot) \to b(r,\cdot)$, $\sigma_k(r,\cdot) \to \sigma(r,\cdot)$ *locally uniformly on* $\mathbb{R}^m$ *as* $k \to \infty$ *for any* $r \in [0,T]$,

3° *the functions* b_k, σ_k *are locally bounded on* $[0,T] \times \mathbb{R}^m$ *uniformly in* $k \ge 1$, *that is*

$$\sup_{k\ge 1} \sup_{r\in[0,T]} \sup_{\|z\|\le L} \big\{\|b_k(r,z)\| \vee \|\sigma_k(r,z)\|\big\} < \infty$$

for each $L \ge 1$.

Suppose that for any $k \ge 1$ *there exists a weak solution* $((\Omega_k, \mathscr{F}^k, (\mathscr{F}^k_t), \boldsymbol{P}_k), W_k, X_k)$ *to the problem*

$$\mathrm{d}X = b_k(t,X)\,\mathrm{d}t + \sigma_k(t,X)\,\mathrm{d}W, \quad X(0) \overset{\mathscr{D}}{\sim} \nu. \tag{3}$$

If $\{\boldsymbol{P}_k \circ X_k^{-1};\ k \ge 1\}$ *is a tight set of probability measures on* $\mathscr{C}([0,T];\mathbb{R}^m)$ *then there exists a weak solution to* (1).

Before proceeding to the proof of Theorem 1, we shall recall some definitions and give a few illustrative examples. First, a weak solution to (1) is a triple $((G,\mathscr{G},(\mathscr{G}_t),\boldsymbol{Q}),W,X)$, where $(G,\mathscr{G},(\mathscr{G}_t),\boldsymbol{Q})$ is a stochastic basis with a filtration $(\mathscr{G}_t)$ that satisfies the usual conditions, W is an n-dimensional $(\mathscr{G}_t)$-Wiener process and X is an $\mathbb{R}^m$-valued $(\mathscr{G}_t)$-progressively measurable process such that $\boldsymbol{Q} \circ X(0)^{-1} = \nu$ and

$$X(t) = X(0) + \int_0^t b(r,X(r))\,\mathrm{d}r + \int_0^t \sigma(r,X(r))\,\mathrm{d}W(r)$$

for all $t \in [0,T]$ $\boldsymbol{Q}$-almost surely. In the proof we use the Riemann-Liouville (or fractional integral) operator: if $q \in\ ]1,\infty]$, $\alpha \in\ ]\frac{1}{q},1]$ and $f \in L^q([0,T];\mathbb{R}^m)$, a function $R_\alpha f : [0,T] \longrightarrow \mathbb{R}^m$ is defined by

$$\big(R_\alpha f\big)(t) = \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad 0 \le t \le T.$$

The (easy) properties of $R_\alpha\colon f \longmapsto R_\alpha f$ which we need are summarized in [HS], Lemma 2.2. Finally, by $\mathscr{C}_{1,2}$ we shall denote the set of all $h \in \mathscr{C}^1([0,T]\times\mathbb{R}^m)$ such that $h(t,\cdot) \in \mathscr{C}^2(\mathbb{R}^m)$ for each $t \in [0,T]$ and D_h, $D_x^2 h$ are continuous functions on $[0,T]\times\mathbb{R}^m$, $D_x h(t,x)$ and $D_x^2 h(t,x)$ being the first and second Fréchet derivative of $h(t,\cdot)$ at the point x, respectively.

Example. a) If the coefficients b and σ satisfy (A) and (2) then Theorem 1 is applicable. More generally, assume that

$$2\langle b(t,x),x\rangle + \|\sigma(t,x)\|^2 \le K\big(1+\|x\|^2\big)$$

for some $K < \infty$ and all $t \in [0,T]$, $x \in \mathbb{R}^m$. Then the Lyapunov function $V\colon x \longmapsto 1 + \|x\|^2$ satisfies (L1) and (L2).

b) Suppose that $\sigma\colon [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function bounded on bounded sets and $\sigma(t,\cdot) \in \mathscr{C}(\mathbb{R})$ for each $t \in [0,T]$. Then we may use Theorem 1 with a Lyapunov function $V\colon x \longmapsto \log(e + x^2)$ to deduce that a stochastic differential equation

$$\mathrm{d}X = \sigma(t,X)\,\mathrm{d}W, \quad X_0 \overset{\mathscr{D}}{\sim} \nu$$

has a weak solution. Of course, it is known that explosions cannot occur for one-dimensional stochastic differential equations without drift, irrespective of growth and continuity properties of σ, but a proof based on Lyapunov functions, when available, is much simpler than the one in the general case.

c) Let us consider a stochastic nonlinear oscillator $\ddot{x} + x^{2k+1} = \sigma(x)\dot{w}$, where $k \in \mathbb{N}$ and $\sigma \in \mathscr{C}(\mathbb{R})$, that is rigorously, a system

$$\mathrm{d}X = Y\,\mathrm{d}t, \quad \mathrm{d}Y = -X^{2k+1}\,\mathrm{d}t + \sigma(X)\,\mathrm{d}W. \tag{4}$$

Theorem 1 with a choice

$$V\colon \mathbb{R}^2 \longrightarrow \mathbb{R}, \ \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \log\Bigl(e + \frac{x^{2k+2}}{2k+2} + \frac{y^2}{2}\Bigr)$$

implies that there exists a weak solution of (4) with an arbitrary initial condition ν provided $\sigma^2(x) = O(x^{2k+2})$, $x \to \pm\infty$.

Proof of Theorem 1. For $k \geq 1$, let us define

$$b_k(t,x) = \begin{cases} b(t,x), & 0 \le t \le T, \ \|x\| \le k, \\ b(t,x)\bigl(2 - k^{-1}\|x\|\bigr)^2, & 0 \le t \le T, \ k < \|x\| \le 2k, \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$\sigma_k(t,x) = \begin{cases} \sigma(t,x), & 0 \le t \le T, \ \|x\| \le k, \\ \sigma(t,x)\bigl(2 - k^{-1}\|x\|\bigr), & 0 \le t \le T, \ k < \|x\| \le 2k, \\ 0 & \text{elsewhere.} \end{cases}$$

Obviously, hypotheses 1° and 2° of Proposition 2 are satisfied, moreover $\|b_k\| \le \|b\|$ and $\|\sigma_k\| \le \|\sigma\|$ on $[0,T] \times \mathbb{R}^m$ for all $k \geq 1$ and thus 3° is satisfied as well. The coefficients b_k and σ_k are bounded, so Theorem 0.1 from [HS] implies that there exists a weak solution $((\Omega_k, \mathscr{F}^k, (\mathscr{F}^k_t), \boldsymbol{P}_k), W_k, X_k)$ of (3). Therefore, Theorem 1 will follow from Proposition 2 provided we show that $\{\boldsymbol{P}_k \circ X_k^{-1};\ k \geq 1\}$ is a tight set of measures.

Towards this end, let us define for any $h \in \mathscr{C}_{1,2}$ and $k \geq 1$ a function $L_k h\colon [0,T] \times \mathbb{R}^m \longrightarrow \mathbb{R}$ by

$$(L_k h)(t,x) = \big\langle b_k(t,x), D_x h(t,x) \big\rangle + \frac{1}{2} \operatorname{Tr}\big(\sigma_k(t,x)^* D_x^2 h(t,x) \sigma_k(t,x)\big),$$

$(t,x) \in [0,T] \times \mathbb{R}^m$. The definition of b_k and σ_k and the assumption (L2) imply that

$$L_k V(t,x) \leq \gamma V(x) \quad \text{for all } k \geq 1 \text{ and } (t,x) \in [0,T] \times \mathbb{R}^m.$$

A straightforward calculation shows that if we set $U(t,x) = e^{-\gamma t} V(x)$ then

$$\Big(\frac{\partial U}{\partial t} + L_k U\Big)(t,x) \leq 0 \quad \text{for all } k \geq 1 \text{ and } (t,x) \in [0,T] \times \mathbb{R}^m. \tag{5}$$

Let us fix $k \geq 1$ for a while. From the Itô formula we get

$$\begin{aligned} &U(t \wedge \varrho, X_k(t \wedge \varrho)) - U(s \wedge \varrho, X_k(s \wedge \varrho)) \\ &= \int_{s\wedge\varrho}^{t\wedge\varrho} \Big(\frac{\partial U}{\partial t} + L_k U\Big)(r, X_k(r))\,\mathrm{d}r + \int_{s\wedge\varrho}^{t\wedge\varrho} D_x U(r, X_k(r))^* \sigma_k(r, X_k(r))\,\mathrm{d}W_k(r), \end{aligned}$$

and thus

$$\begin{aligned} &U(t \wedge \varrho, X_k(t \wedge \varrho)) - U(s \wedge \varrho, X_k(s \wedge \varrho)) \\ &\qquad\qquad \leq \int_{s\wedge\varrho}^{t\wedge\varrho} D_x U(r, X_k(r))^* \sigma_k(r, X_k(r))\,\mathrm{d}W_k(r) \end{aligned} \tag{6}$$

by (5), whenever $s, t \in [0,T]$, $s \leq t$ and ϱ is an $[0,T]$-valued $(\mathscr{F}_r^k)$-stopping time.

First, let us choose $s = 0$, $L \geq 0$, and

$$\varrho = \tau_L \equiv \inf\big\{r \geq 0;\ \|X_k(r)\| \geq L\big\}$$

(where we set $\inf \emptyset = T$). Since $U(0,\cdot) = V$ we obtain

$$U(t \wedge \tau_L, X_k(t \wedge \tau_L)) \leq V(X_k(0)) + \int_0^{t\wedge\tau_L} D_x U(r, X_k(r))^* \sigma_k(r, X_k(r))\,\mathrm{d}W_k(r).$$

Let $\chi \subseteq \mathbb{R}^m$ be an arbitrary Borel set such that

$$\int_\chi V(z)\,\mathrm{d}\nu(z) < \infty. \tag{7}$$

(Plainly, any compact set χ satisfies (7).) Denoting by A the set $\{X_k(0) \in \chi\} \in \mathscr{F}_0^k$ we get

$$\begin{aligned} &\mathbf{1}_A U(t \wedge \tau_L, X_k(t \wedge \tau_L)) \\ &\qquad \le \mathbf{1}_A V(X_k(0)) + \int_0^{t\wedge\tau_L} \mathbf{1}_A D_x U(r, X_k(r))^* \sigma_k(r, X_k(r))\,\mathrm{d}W_k(r). \end{aligned}$$

As $\mathbf{1}_A \mathbf{1}_{[0,\tau_L[}(\cdot) D_x U(\cdot, X_k(\cdot))^* \sigma_k(\cdot, X_k(\cdot))$ is bounded on $[0,T]\times\Omega_k$ due to continuity of $D_x U$, local boundedness of σ_k and the definition of τ_L, we have

$$\begin{aligned} \boldsymbol{E}_k \mathbf{1}_A U(t\wedge\tau_L, X_k(t\wedge\tau_L)) &\le \boldsymbol{E}_k \mathbf{1}_A V(X_k(0)) = \boldsymbol{E}_k \mathbf{1}_\chi(X_k(0)) V(X_k(0)) \\ &= \int_\chi V(z)\,\mathrm{d}\nu(z); \end{aligned}$$

the right-hand side is independent of $L \ge 0$. Clearly, $\{\tau_L = T\} \uparrow \Omega_k$ $\boldsymbol{P}_k$-almost surely as $L \to \infty$, since X_k has continuous trajectories, so

$$\boldsymbol{E}_k \mathbf{1}_A U(t, X_k(t)) \le \int_\chi V(z)\,\mathrm{d}\nu(z) < \infty$$

by the Fatou lemma.

In particular, if $s,t \in [0,T]$, $s \le t$, then the conditional expectation

$$\boldsymbol{E}_k \big(\mathbf{1}_A U(t, X_k(t)) \,\big|\, \mathscr{F}_s^k\big)$$

is well defined. Using (6) with the stopping time τ_L, replacing the Fatou lemma with its version for conditional expectations but otherwise proceeding as above we arrive at an estimate

$$\boldsymbol{E}_k \big(\mathbf{1}_A U(t, X_k(t)) \,\big|\, \mathscr{F}_s^k\big) \le \mathbf{1}_A U(s, X_k(s)), \quad 0 \le s \le t \le T.$$

Consequently, $\big(\mathbf{1}_A U(t, X_k(t)),\, 0 \le t \le T\big)$ is a nonnegative continuous supermartingale. The maximal inequality for supermartingales implies

$$\begin{aligned} \boldsymbol{P}_k\Big\{ \sup_{0\le t\le T} \mathbf{1}_\chi(X_k(0)) U(t, X_k(t)) > \lambda\Big\} &\le \frac{1}{\lambda} \boldsymbol{E}_k \mathbf{1}_\chi(X_k(0)) V(X_k(0)) \\ &= \frac{1}{\lambda}\int_\chi V(z)\,\mathrm{d}\nu(z), \end{aligned}$$

hence, by the definition of U,

$$\boldsymbol{P}_k\Big\{ \sup_{0\le t\le T} \mathbf{1}_\chi(X_k(0)) V(X_k(t)) > \lambda\Big\} \le \frac{e^{\gamma T}}{\lambda} \int_\chi V\,\mathrm{d}\nu$$

for all $\lambda > 0$; the estimate is uniform in $k \geq 1$. From the assumption (L1) we deduce that

$$\boldsymbol{P}_k\big\{\sup_{0\leq t\leq T} \mathbf{1}_\chi(X_k(0))\|X_k(t)\| > \lambda\big\} \leq \frac{e^{\gamma T}}{\kappa(\lambda)}\int_\chi V\,\mathrm{d}\nu \tag{8}$$

holds for all $\lambda > 0$ and $k \geq 1$.

Now the proof of tightness of $\{\boldsymbol{P}_k \circ X_k^{-1};\ k \geq 1\}$ can be completed essentially in the same manner as in the proof of Proposition 2.1 in [HS]. Let an arbitrary $\varepsilon > 0$ be given, we want to find a relatively compact set $K \subseteq \mathscr{C}([0,T];\mathbb{R}^m)$ so that

$$\sup_{k\geq 1} \boldsymbol{P}_k\big\{X_k \notin K\big\} \leq \varepsilon. \tag{9}$$

Let us take an arbitrary $p \in \,]2,\infty[$ and $\alpha \in \,]\frac{1}{p},\frac{1}{2}[$ and recall that X_k has a representation (see e.g. [HS], Lemma 2.5)

$$X_k(t) = X_k(0) + \big[R_1 b_k(\cdot, X_k(\cdot))\big](t) + \frac{\sin\pi\alpha}{\pi}(R_\alpha Z_k)(t), \quad 0 \leq t \leq T,$$

where

$$Z_k(t) = \int_0^t (t-s)^{-\alpha}\sigma_k(s, X_k(s))\,\mathrm{d}W_k(s), \quad 0 \leq t \leq T.$$

The process Z_k is plainly well defined for every $t \in [0,T]$, since σ_k is a bounded function. Let $H \subseteq \mathbb{R}^m$ be a compact set such that $\nu(\mathbb{R}^m \setminus H) = \boldsymbol{P}_k\{X_k(0) \notin H\} < \varepsilon/8$. The set

$$K = \Big\{f \in \mathscr{C}([0,T];\mathbb{R}^m);\ f = x + R_1 v + \frac{\sin\pi\alpha}{\pi}R_\alpha w,\ x \in H,\\ v, w \in L^p(0,T;\mathbb{R}^m),\ |v|_p \vee |w|_p \leq \Lambda\Big\},$$

where by $|\cdot|_p$ the norm of $L^p(0,T;\mathbb{R}^m)$ is denoted, is relatively compact owing to compactness of the operators R_1 and R_α. It remains to show that $\Lambda > 0$ may be found for K to satisfy (9).

From (8) and (L1) we obtain that there exists $\lambda_0 > 0$ such that

$$\sup_{k\geq 1}\boldsymbol{P}_k\big\{\mathbf{1}_H(X_k(0))\sup_{0\leq t\leq T}\|X_k(t)\| > \lambda_0\big\} \leq \frac{e^{\gamma T}}{\kappa(\lambda_0)}\int_H V\,\mathrm{d}\nu < \frac{\varepsilon}{8},$$

therefore the choice of H gives

$$\sup_{k\geq 1}\boldsymbol{P}_k\big\{\sup_{0\leq t\leq T}\|X_k(t)\| > \lambda_0\big\} < \frac{\varepsilon}{4}.$$

Hence if we set

$$B_k = \big\{\omega \in \Omega_k;\ \sup_{0\le t\le T} \|X_k(t,\omega)\| \le \lambda_0\big\},$$

then $\boldsymbol{P}_k(\Omega_k \setminus B_k) < \varepsilon/4$ for all $k \ge 1$.

Obviously,

$$\boldsymbol{P}_k\{X_k \notin K\} \le \boldsymbol{P}_k\{X_k(0) \notin H\} + \boldsymbol{P}_k\big\{|b_k(\cdot, X_k(\cdot))|_p > \Lambda\big\} + \boldsymbol{P}_k\big\{|Z_k|_p > \Lambda\big\}.$$

By the Chebyshev inequality, we get

$$\begin{aligned}\boldsymbol{P}_k\big\{|b_k(\cdot, X_k(\cdot))|_p > \Lambda\big\} &\le \boldsymbol{P}_k(\Omega_k \setminus B_k) + \boldsymbol{P}_k\big\{\omega \in B_k;\ |b_k(\cdot, X_k(\cdot))|_p > \Lambda\big\} \\ &\le \frac{\varepsilon}{4} + \frac{1}{\Lambda^p}\boldsymbol{E}_k \mathbf{1}_{B_k} \int_0^T \|b_k(r, X_k(r))\|^p \,\mathrm{d}r \\ &\le \frac{\varepsilon}{4} + \frac{T}{\Lambda^p} \sup_{\substack{0\le t\le T\\ \|z\|\le\lambda_0}} \|b_k(t,z)\|^p \\ &\le \frac{\varepsilon}{4} + \frac{T}{\Lambda^p} \sup_{\substack{0\le t\le T\\ \|z\|\le\lambda_0}} \|b(t,z)\|^p.\end{aligned}$$

The right-hand side is independent of $k \ge 1$, so there exists $\Lambda_1 > 0$ such that

$$\sup_{k\ge 1} \boldsymbol{P}_k\big\{|b_k(\cdot, X_k(\cdot))|_p > \Lambda\big\} \le \frac{\varepsilon}{3}$$

for all $\Lambda \ge \Lambda_1$. The norm $|Z_k|_p$ may be estimated analogously. Clearly,

$$\begin{aligned}\boldsymbol{P}_k\{|Z_k|_p > \Lambda\} &\le \boldsymbol{P}_k(\Omega_k \setminus B_k) + \boldsymbol{P}_k\{\omega \in B_k;\ |Z_k|_p > \Lambda\} \\ &\le \frac{\varepsilon}{4} + \boldsymbol{P}_k\{\omega \in B_k;\ |Z_k|_p > \Lambda\}.\end{aligned}$$

For each $k \ge 1$ let us define an $(\mathscr{F}_t^k)$-stopping time ζ_k by

$$\zeta_k = \inf\{t \in [0,T];\ \|X_k(t)\| > \lambda_0\},$$

setting again $\inf\emptyset = T$. Using the Chebyshev and Young inequalities and noting that $\zeta_k = T$ on B_k we obtain

$$\begin{aligned}&\boldsymbol{P}_k\{\omega \in B_k; |Z_k|_p > \Lambda\} \\ &\qquad \le \frac{1}{\Lambda^p}\boldsymbol{E}_k \mathbf{1}_{B_k} \int_0^T \|Z_k(s)\|^p \,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Lambda^p} \boldsymbol{E}_k \mathbf{1}_{B_k} \int_0^T \left\| \int_0^s (s-u)^{-\alpha} \sigma_k(u, X_k(u))\, \mathrm{d}W(u) \right\|^p \mathrm{d}s \\
&= \frac{1}{\Lambda^p} \boldsymbol{E}_k \mathbf{1}_{B_k} \int_0^T \left\| \int_0^s (s-u)^{-\alpha} \mathbf{1}_{[0,\zeta_k[}(u) \sigma_k(u, X_k(u))\, \mathrm{d}W(u) \right\|^p \mathrm{d}s \\
&\leq \frac{1}{\Lambda^p} \boldsymbol{E}_k \int_0^T \left\| \int_0^s (s-u)^{-\alpha} \mathbf{1}_{[0,\zeta_k[}(u) \sigma_k(u, X_k(u))\, \mathrm{d}W(u) \right\|^p \mathrm{d}s \\
&\leq \frac{C_p}{\Lambda^p} \boldsymbol{E}_k \int_0^T \left(\int_0^s (s-u)^{-2\alpha} \mathbf{1}_{[0,\zeta_k[}(u) \|\sigma_k(u, X_k(u))\|^2\, \mathrm{d}u \right)^{p/2} \mathrm{d}s \\
&\leq \frac{C_p}{\Lambda^p} \left(\int_0^T u^{-2\alpha}\, \mathrm{d}u \right)^{p/2} \boldsymbol{E}_k \int_0^T \mathbf{1}_{[0,\zeta_k[}(u) \|\sigma_k(u, X_k(u))\|^p\, \mathrm{d}u \\
&\leq \frac{C_p T}{\Lambda^p} \left(\int_0^T u^{-2\alpha}\, \mathrm{d}u \right)^{p/2} \sup_{\substack{0 \leq t \leq T \\ \|z\| \leq \lambda_0}} \|\sigma_k(t, x)\|^p \\
&\leq \frac{C_p T}{\Lambda^p} \left(\int_0^T u^{-2\alpha}\, \mathrm{d}u \right)^{p/2} \sup_{\substack{0 \leq t \leq T \\ \|z\| \leq \lambda_0}} \|\sigma(t, x)\|^p,
\end{aligned}
$$

where C_p is a constant coming from the Burkholder-Gundy-Davis inequality. We see that there exists a constant $\Lambda_2 > 0$ such that

$$
\sup_{k \geq 1} \boldsymbol{P}_k \big\{ |Z_k|_p \geq \Lambda \big\} < \frac{\varepsilon}{3}
$$

for all $\Lambda \geq \Lambda_2$ and hence the proof may be completed easily. Q.E.D.

References

[HS] Hofmanová M., Seidler J. 2012. On weak solutions of stochastic differential equations. *Stochastic Analysis and Applications 30*(1):100-121.

[K] Khasminskii R. 2012. *Stochastic Stability of Differential Equations*, 2nd ed., Springer, Berlin.

M.H., Institute of Information Theory and Automation of the ASCR, Praha, Czech Republic; IRMAR, ENS Cachan Bretagne, Bruz, France; and Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Praha, Czech Republic

J.S., Institute of Information Theory and Automation of the ASCR, Praha, Czech Republic