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On the Navier–Stokes equation perturbed by rough transport noise

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Abstract. We consider the Navier–Stokes system in two and three space dimensions perturbed by transport noise and subject to periodic boundary conditions. The noise arises from perturbing the advecting velocity field by space–time-dependent noise that is smooth in space and rough in time. We study the system within the framework of rough path theory and, in particular, the recently developed theory of unbounded rough drivers. We introduce an intrinsic notion of a weak solution of the Navier–Stokes system, establish suitable a priori estimates and prove existence. In two dimensions, we prove that the solution is unique and stable with respect to the driving noise.

1. Introduction

The theory of rough paths, introduced by Terry Lyons in his seminal work [1], can be briefly described as an extension of the classical theory of controlled differential equations that is robust enough to allow for a pathwise (i.e., deterministic) treatment of stochastic differential equations (SDEs). Since its introduction, the theory of ordinary and partial differential equations driven by rough signals has progressed substantially. We refer the reader to the works of Friz et al. [2,3], Gubinelli et al. [4–6], Gubinelli et al. [7], Hairer [8] for a sample of the literature on the growing subject. In spite of these exciting developments, many PDE methods have not yet found their rough path analogues. For instance, until recently, it was not known how to construct (weak) solutions to rough partial differential equations (RPDEs) using energy methods (or variational methods).

The first results on energy methods for RPDEs were established in [9–11]. In [9], the foundation of the theory of unbounded rough drivers was established and then used to derive the well-posedness of a linear transport equation driven by a rough path in the Sobolev scale. Expanding upon the scope of the theory, the authors of [10] developed a rough version of Gronwall's lemma and proved the well-posedness of nonlinear scalar conservation laws with rough flux. In the framework of unbounded rough drivers, one can define an intrinsic notion of a weak solution of an RPDE that is equivalent to the

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usual definition if the driving path is smooth in time. Additionally, one can obtain an energy estimate of the solution. Prior to the development of the theory of unbounded rough drivers and rough Gronwall lemma, these problems remained open. In particular, how to study the well-posedness of the Navier–Stokes system with rough transport noise was out of reach. Most recently, the theory of unbounded rough drivers has been applied to prove the existence, uniqueness and stability of two classes of equations: (1) linear parabolic PDEs with a bounded and measurable diffusion coefficient driven by rough paths [11] and (2) reflected rough differential equations [12].

The aim of our efforts is to study the Navier–Stokes system subject to rough transport noise. We study the system of equations that govern the evolution of the velocity field $u : \mathbf{R}_+ \times \mathbf{T}^d \to \mathbf{R}^d$ and the pressure $p : \mathbf{R}_+ \times \mathbf{T}^d \to \mathbf{R}$ of an incompressible viscous fluid on the d-dimensional torus $\mathbf{T}^d$ perturbed by transport-type noise:

$$\begin{aligned} \partial_t u + (u - \dot{a}) \cdot \nabla u + \nabla p &= \nu \Delta u, \\ \nabla \cdot u &= 0, \\ u(0) &= u_0 \in L^2(\mathbf{T}^d; \mathbf{R}^d), \end{aligned} \tag{1.1}$$

where $\nu > 0$ is the viscosity coefficient and $\dot{a}$ is the (formal) derivative in time of a function $a = a_t(x) : \mathbf{R}_+ \times \mathbf{T}^d \to \mathbf{R}^d$ that is divergence free in space and has finite p-variation in time for some $p \in [2, 3)$. For example, $\dot{a}$ may represent noise that is white in time and colored in space. Such noise is a formal time derivative of an $L^2(\mathbf{T}^d)$-valued Wiener process. However, one of the main advantages of the theory of rough paths is that drivers that are not necessarily martingales or of finite variation can be considered, which is in direct contrast to the classical semimartingale theory. Consequently, $\dot{a}$ may represent the time derivative of a more general spatially dependent Gaussian or Markov process, such as a fractional Brownian motion, $B^H := (B^{H,1}, \dots, B^{H,K})$ with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$, coupled with a family of vector fields $\sigma = (\sigma_1, \dots, \sigma_K) : \mathbf{T}^d \to \mathbf{R}^{K \times d}$; that is, for $(t, x) \in \mathbf{R}_+ \times \mathbf{T}^d$,

$$a_t(x) = \sum_{k=1}^{K} \sigma_k(x) B_t^{H,k}.$$

Even in the case of the unperturbed Navier–Stokes system, it is unknown whether there exists global smooth solutions, and so we study the perturbed system integrated in time and tested against a smooth test function in space. In particular, it is necessary to make sense of the time integral $\int_0^t (\dot{a}_s \cdot \nabla) u_s \, ds$ as a spatial distribution. Testing this integral against a smooth function $\phi : \mathbf{T}^d \to \mathbf{R}^d$, we get

$$\int_0^t (\dot{a}_s \cdot \nabla) u_s \, \mathrm{d}s(\phi) = -\int_0^t u_s((\dot{a}_s \cdot \nabla)\phi) \, \mathrm{d}s, \tag{1.2}$$

where we have used the divergence-free assumption $\nabla \cdot \dot{a} = 0$. However, the time integral is not a priori well defined since we expect the solution u to inherit the same regularity in time as a (i.e., p-variation). Indeed, L.C. Young's theorem in [13] says that a Riemann–Stieltjes integral $\int f dg$ exists if there are p and q with $p^{-1}+q^{-1} > 1$, such that f is of p-variation and g is of q-variation. Furthermore, a counterexample is given for the case $p^{-1}+q^{-1} = 1$, and hence the theorem of Young cannot be used to define (1.2), unless a has p-variation in time for $p \in [1, 2)$.

The rough path theory of Lyons [1] enables us to define the integral (1.2), provided that we possess additional information about the driving path, namely its iterated integrand. The idea is to iterate the equation for u into the noise integral (1.2) enough times so that the remainder is regular enough in time to be negligible. In the case of transport noise, this iteration leads to an iteration of the spatial derivative. For simplicity, let us explain how this iteration works for the pure-transport equation

$$\partial_t u = (\dot{a} \cdot \nabla)u. \tag{1.3}$$

Integrating (1.3) in time, testing against a smooth function $\phi : \mathbf{T}^d \to \mathbf{R}^d$, and then iterating Eq. (1.3) into itself yields

$$\begin{aligned} u_t(\phi) &= u_s(\phi) - \int_s^t u_r((\dot{a}_r \cdot \nabla)\phi)\, \mathrm{d}r \\ &= u_s(\phi) - u_s\left(\int_s^t (\dot{a}_r \cdot \nabla)\phi\, \mathrm{d}r\right) + \int_s^t \int_s^{r_1} u_{r_2}\left((\dot{a}_{r_2} \cdot \nabla)(\dot{a}_{r_1} \cdot \nabla)\phi\right) \mathrm{d}r_2\, \mathrm{d}r_1 \\ &= u_s(\phi) - u_s\left(\int_s^t (\dot{a}_r \cdot \nabla)\phi\, \mathrm{d}r\right) + u_s\left(\int_s^t \int_s^{r_1} (\dot{a}_{r_2} \cdot \nabla)(\dot{a}_{r_1} \cdot \nabla)\phi\, \mathrm{d}r_2\, \mathrm{d}r_1\right) \\ &\quad - \int_s^t \int_s^{r_1} \int_s^{r_2} u_{r_3}\left((\dot{a}_{r_3} \cdot \nabla)(\dot{a}_{r_2} \cdot \nabla)(\dot{a}_{r_1} \cdot \nabla)\phi\right) \mathrm{d}r_3\, \mathrm{d}r_2\, \mathrm{d}r_1, \end{aligned} \tag{1.4}$$

where we have used the divergence-free assumption $\nabla \cdot \dot{a} = 0$. If we define the operators

$$A^1_{st}\phi = \int_s^t (\dot{a}_r \cdot \nabla)\mathrm{d}r\phi \quad \text{and} \quad A^2_{st}\phi = \int_s^t \int_s^{r_1} (\dot{a}_{r_2} \cdot \nabla)(\dot{a}_{r_1} \cdot \nabla)\mathrm{d}r_2\mathrm{d}r_1\phi, \tag{1.5}$$

and let $\delta u_{st} = u_t - u_s$, then solving the transport equation (1.3) corresponds to finding a map $t \mapsto u_t$ such that $u^\natural$ defined by

$$u^\natural_{st}(\phi) := \delta u_{st}(\phi) - u_s\left(\left[A^{1,*}_{st} + A^{2,*}_{st}\right]\phi\right) \tag{1.6}$$

is of order $o(|t-s|)$, and hence is negligible. That is, the expansion $[A^1_{st} + A^2_{st}]u_s$ tested against ϕ provides a good local approximation of the time integral (1.2), which is uniquely defined by the sewing lemma (see Lemma B.1). Notice that if a is smooth in time and space, then (1.6) is an equivalent formulation of the transport equation (1.3). Because the time singularities in (1.5) are smoothed out by averaging over time, the equation (1.6) does not contain any time derivatives, and hence the formulation is well-suited for irregular drivers. Under certain conditions, the pair $\mathbf{A} = (A^1, A^2)$ defines an *unbounded rough driver* as defined in [9] and in Sect. 2.4 below.

In order to show that the remainder $u^\natural$ is of order $o(|t-s|)$, we shall regard it as a distribution of third order with respect to the space variable; note that three derivatives are taken in (1.4). One of the key aspects of the theory of unbounded rough drivers is the process by which one obtains a priori estimates of the remainder $u^\natural$. (See Sect. 3.) The technique involves obtaining estimates of $\delta u^\natural_{s\theta t} := u^\natural_{st} - u^\natural_{s\theta} - u^\natural_{\theta t}$, interpolating between time and space regularity of various terms, and applying the sewing lemma (i.e., Lemma B.1). This is yet another example of the trade-off between time and space regularity pertinent to many PDE problems. Notice that if a is α-Hölder continuous (essentially equivalent to α^{-1}-variation) with respect to the time variable and the solution u has the same regularity in time, then the first two terms on the right-hand-side of (1.6) are proportional to $|t-s|^\alpha$ and the last term on the right-hand side can be bounded by $|t-s|^{2\alpha}$. Thus, in the case $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, there has to be a cancelation between the terms on the right-hand side to guarantee that $u^\natural$ is of order $o(|t-s|)$. On the other hand, the right-hand side of (1.6) is a distribution of second order with respect to the space variable. Accordingly, the necessary improvement of time regularity can be obtained at the cost of loss of space regularity, that is, considering $u^\natural$ rather as a distribution of third order.

In this paper, we assume that the noise term a can be factorized as follows:

$$a_t(x) = \sigma_k(x) z_t^k = \sum_{k=1}^{K} \sigma_k(x) z_t^k, \tag{1.7}$$

where we adopt the convention of summation over repeated indices $k \in \{1, \dots, K\}$ here and below. We also assume that for all $k \in \{1, \dots, K\}$, the vector fields $\sigma_k : \mathbf{T}^d \to \mathbf{R}^d$ are bounded, divergence free, and twice differentiable with bounded first and second derivatives. The driving signal z is assumed to be a $\mathbf{R}^K$-valued path of finite p-variation for some $p \in [2, 3)$ that can be lifted to a geometric rough path $\mathbf{Z} = (Z, \mathbb{Z})$. The first component of $\mathbf{Z}$ is the increment of z (i.e., $Z_{st} = z_t - z_s$) and the second component is the so-called Lévy's area, which plays the role of the iterated integral $\mathbb{Z}_{st} =: \int_s^t \int_s^r dz_{r_1} \otimes dz_r$. In the smooth setting, the iterated integral can be defined as a Riemann integral, whereas in the rough setting, it has to be given as an input datum; the two-index map $\mathbb{Z}_{st}$ is assumed to satisfy Chen's relation

$$\delta \mathbb{Z}_{s\theta t} := \mathbb{Z}_{st} - \mathbb{Z}_{s\theta} - \mathbb{Z}_{\theta t} = Z_{s\theta} \otimes Z_{\theta t}, \quad s \le \theta \le t,$$

and to be two times as regular in time as the path z. For instance, if z is a Wiener process, then an iterated integral can be constructed using the Stratonovich stochastic integration. Nevertheless, many other important stochastic processes give rise to (two-step) rough paths. For more details, we refer the reader to Sect. 2.3 and the literature mentioned therein.

The motivation for a perturbation of the form $-\dot{a} \cdot \nabla u$ comes from the modeling of a turbulent flow of a viscous fluid. In the Lagrangian formulation, an incompressible fluids evolution is traditionally specified in terms of the flow map of particles initially at X:

$$\dot{\eta}_t(X) = u_t(\eta_t(X)), \quad \eta_0(X) = X \in \mathbf{T}^d, \quad \nabla \cdot u = 0.$$

If we assume the associated fluid flow map is a composition of a mean flow depending on slow time t and a rapidly fluctuating flow with fast timescales $\epsilon^{-1}t$, $\epsilon \ll 1$, then provided that the fast dynamics are sufficiently chaotic, on timescales of order ϵ^{-2}, the averaged slow dynamics are described by the SDE [14]

$$\begin{aligned} d\bar{\eta}_t(X) &= \bar{u}_t(\bar{\eta}_t(X))dt - \sigma_k(\bar{\eta}(X,t)) \circ dw_t^k, \quad \bar{\eta}_0(X) = X \in \mathbf{T}^d, \quad \nabla \cdot \bar{u} = 0, \\ \nabla \cdot \sigma_k &= 0, \end{aligned} \tag{1.8}$$

where $w := \{w^k\}_{k=1}^{\infty}$ is a sequence of independent Brownian motions and the stochastic integral is understood in the Stratonovich sense. The flow dynamics given by (1.8) encompasses models of stochastic passive scalar turbulence that were originally proposed by Kraichnan [15] and further developed in [16,17] and other works. In [18–21], it was shown that the system of equations governing the resolved scale velocity field $\bar{u}$ and pressure p and $\{q_k\}_{k=1}^{\infty}$ is a stochastic version of the Navier–Stokes system with transport noise:

$$d\bar{u} + (\bar{u}dt - \sigma_k \circ dw_t^k) \cdot \nabla \bar{u} + \nabla p dt + \nabla q_k \circ dw_t^k = \nu \Delta \bar{u} dt. \tag{1.9}$$

The existence and uniqueness of solutions of (1.9) has been well-studied [19–22]. In [21], the authors proved the existence of global weak-probabilistic solutions (i.e., martingale solutions) of a general class of stochastic Navier–Stokes equations on the whole space, which included (1.9). Moreover, in dimension two, the uniqueness of the global strong probabilistic solution was established in [21] as well. The existence of strong global solutions for the stochastic Navier–Stokes system (1.9) in three dimensions is still an open problem.

In this paper, we develop a (rough) pathwise solution theory for (1.1), which, in particular, offers a pathwise interpretation of (1.9) for $k \in \{1, \ldots, K\}$. We establish the existence of weak solutions in two and three space dimensions (see Theorem 2.13) by establishing energy estimates, including the recovery of the pressure. (See Sect. 4.1.2.) To prove existence, we use Galerkin approximation combined with a suitable mollification of the driving signal, uniform energy estimates of the solution, and the remainder terms and a compactness argument. In addition, in two space dimensions and for constant vector fields σ_k, we prove uniqueness and pathwise stability with respect to the given driver and initial datum via a tensorization argument (see Theorem 2.14 and Corollary 2.15). This result implies a Wong–Zakai approximation theorem for the Wiener driven SPDE (1.9). To the best of our knowledge, this is the first Wong–Zakai-type result for the Navier–Stokes system (1.9). There are a substantial number of Wong–Zakai results for infinite dimensional stochastic evolution equations in various settings. We mention only the work [23] of Chueshov and Millet in which the authors derive a Wong–Zakai result and support theorem for a general class of stochastic 2D hydrodynamical systems, including 2D stochastic Navier–Stokes. However, the diffusion coefficients in [23] are assumed to have linear growth on $L^2(\mathbf{T}^2; \mathbf{R}^2)$, and hence do not cover transport noise. We do note, however, that in [24], Chueshov and Millet establish a large deviation result for stochastic 2D hydrodynamical systems that does hold true for transport noise.

Our approach relies on a suitable formulation of the system (1.1) that is similar to the formulation of the pure-transport equation (1.6) discussed above. However, due to the structure of (1.1) and the fact that a solution is the pairing of a velocity field and pressure (u, p), the formulation is more subtle. In fact, we present two equivalent (rough) formulations of (1.1) in Sect. 2.5.

Let P be the Helmholtz–Leray projection and $Q = I - P$ (see Sect. 2.1 for more details). Applying P and Q separately to (1.1), we obtain the system of coupled equations

$$\begin{aligned}\partial_t u + P[(u \cdot \nabla)u] &= \nu \Delta u + P[(\dot{a} \cdot \nabla)u]\\ Q[(u \cdot \nabla)u] + \nabla p &= Q[(\dot{a} \cdot \nabla)u].\end{aligned}$$

We can then perform an iteration of the equation for u in the time integral of $P[\dot{a} \cdot \nabla u]$ and $Q[\dot{a} \cdot \nabla u]$ like we illustrated above for the pure-transport equation (1.6). After doing so, we obtain a coupled system of equations for the velocity field and pressure for which the associated unbounded rough drivers are intertwined and a version of the so-called Chen's relation holds true. [See (2.17) and Definition 2.7.] We derive a second equivalent formulation by summing the coupled equations from the first formulation. This second formulation is a single equation for the velocity field in which a modified Chen's relation holds (see (2.21) and Definition 2.11). An alternative way to arrive at the second formulation is by iterating (1.1) and using that $\nabla p = Q[(\dot{a} \cdot \nabla)u] - Q[(u \cdot \nabla)u]$.

The presentation of this paper is organized as follows. In Sect. 2, we define our notion of solution and state our main results. In Sect. 3, we derive a priori estimates of remainder terms, which are used in Sect. 4 to prove our main results. Several auxiliary results that are used to prove the main results are presented in appendix.

2. Mathematical framework and main results

2.1. Notation and definitions

We begin by fixing the notation that we use throughout the paper.

We shall write $a \lesssim b$ if there exists a positive constant C such that $a \leq b$. If the constant C depends only on the parameters $p_1, \ldots, p_n$, we shall also write $C = C(p_1, \ldots, p_n)$ and $\lesssim_{p_1,\ldots,p_n}$.

Let $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. For a given $d \in \mathbf{N}$, let $\mathbf{T}^d = \mathbf{R}^d/(2\pi\mathbf{Z})^d$ be the d-dimensional flat torus and denote by dx the unormalized Lebesgue measure on $\mathbf{T}^d$. As usual, we blur the distinction between periodic functions and functions defined on the torus $\mathbf{T}^d$. For a given Banach space V with norm $|\cdot|_V$, we denote by $\mathcal{B}(V)$ the Borel sigma-algebra of V and by V^* the continuous dual of V. For given Banach spaces V_1 and

V_2, we denote by $\mathcal{L}(V_1, V_2)$ the space of continuous linear operators from V_1 to V_2 with the operator norm denoted by $|\cdot|_{\mathcal{L}(V_1,V_2)}$.

For a given sigma-finite measured space $(X, \mathcal{X}, \mu)$, separable Banach space V with norm $|\cdot|_V$, and $p \in [1, \infty]$, we denote by $L^p(X; V)$ the Banach space of all μ-equivalence classes of strongly measurable functions $f : X \to V$ such that

$$|f|_{L^p(X;V)} := \left(\int_X |f|_V^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} < \infty,$$

equipped with the norm $|\cdot|_{L^p(X;V)}$. We denote by $L^\infty(X; V)$ the Banach space of all μ-equivalence classes of strongly measurable functions $f : X \to V$ such that

$$|f|_{L^\infty(X;V)} := \operatorname{esssup}_X |f|_V := \inf\{a \in \mathbf{R} : \mu(|f|_V^{-1}((a, \infty)) = 0)\} < \infty,$$

where $|f|_V^{-1}((a, \infty))$ denotes the preimage of the set (a, ∞) under the map $|f|_V : X \to \mathbf{R}$, equipped with the norm $|\cdot|_{L^\infty(X;V)}$. It is well known that if $V = H$ is a Hilbert space with inner product $(\cdot,\cdot)_H$, then $L^2(X; H)$ is a Hilbert space equipped with the inner product

$$(f, g)_{L^2(X;H)} = \int_X (f, g)_H \,\mathrm{d}\mu, \quad f, g \in L^2(X; H).$$

For a given Hilbert space H, we let $L^2_T H = L^2([0, T]; H)$ and $L^\infty_T H = L^\infty([0, T]; H)$. Moreover, let $\mathbf{L}^2 = L^2(\mathbf{T}^d; \mathbf{R}^d)$.

For a given Hilbert space V, and real number $T > 0$, we let $C_T H = C([0, T]; H)$ denote the Banach space of continuous functions from $[0, T]$ to H, endowed with the supremum norm in time.

For a given $n \in \mathbf{Z}^d$, let $e_n : \mathbf{T}^d \to \mathbf{C}$ be defined by $e_n(x) = (2\pi)^{-\frac{d}{2}} e^{in\cdot x}$. It is well known that $\{e_n\}_{n\in\mathbf{Z}^d}$ is an orthonormal system of $L^2(\mathbf{T}^d; \mathbf{C})$, and hence for all $f, g \in \mathbf{L}^2$,

$$f = \sum_{n\in\mathbf{Z}^d} \hat{f}_n e_n, \quad (f, g)_{\mathbf{L}^2} = \sum_{n\in\mathbf{Z}^d} \hat{f}_n \cdot \overline{\hat{g}_n},$$

where for each $n \in \mathbf{Z}^d$,

$$\hat{f}^i_n = \int_{\mathbf{T}^d} f^i(x) e_{-n}(x)\, dx, \quad i \in \{1, \ldots, d\}.$$

Let $\mathcal{S}$ be the Fréchet space of infinitely differentiable periodic complex-valued functions with the usual set of semi-norms. Let $\mathcal{S}'$ be the continuous dual space of $\mathcal{S}$ endowed with the weak-star topology. For a given $\Lambda \in \mathcal{S}'$ and test function $\phi \in \mathcal{S}$, we denote by $\Lambda(\phi)$ the value of a distribution Λ at $\phi \in \mathcal{S}$. Since $e_n \in \mathcal{S}$, for a given $f \in \mathcal{S}'$ and $n \in \mathbf{Z}^d$, we define $\hat{f}_n = f(e_n)$. It is well known that $f = \sum_{n\in\mathbf{Z}^d} \hat{f}_n e_n$, where convergence holds in $\mathcal{S}$ if $f \in \mathcal{S}$ and in $\mathcal{S}'$ if $f \in \mathcal{S}'$. This extends trivially to the set $\mathbf{S}' = (\mathcal{S}')^d$ of continuous linear functions from $\mathbf{S} = (\mathcal{S})^d$ to $\mathbf{C}$ endowed with the weak-star topology.

For a given $\alpha \in \mathbf{R}$, we denote by $\mathbf{W}^{\alpha,2}$ the Hilbert space

$$\mathbf{W}^{\alpha,2} = (I-\Delta)^{-\frac{\alpha}{2}}\mathbf{L}^2 = \left\{ f \in \mathbf{S}' : (I-\Delta)^{\frac{\alpha}{2}} f \in \mathbf{L}^2 \right\}$$

with inner product

$$(f,g)_\alpha = \left((I-\Delta)^{\frac{\alpha}{2}} f, (I-\Delta)^{\frac{\alpha}{2}} g \right)_{\mathbf{L}^2} = \sum_{n\in\mathbf{Z}^d} (1+|n|^2)^\alpha \, \hat{f}_n \cdot \overline{\hat{g}_n}, \quad f,g \in \mathbf{W}^{\alpha,2}$$

and induced norm $|\cdot|_\alpha$. For notational simplicity, when $m=0$ we omit the index in the inner product, i.e., $(\cdot,\cdot) := (\cdot,\cdot)_0$. Moreover, for any $u \in \mathbf{W}^{1,2}$, we write $|\nabla u|_0^2 = \sum_{i=1}^d |D_i u|_0^2$. It is easy to see that $\mathbf{W}^{\alpha,2} \subset \mathbf{W}^{\beta,2}$ for $\alpha,\beta \in \mathbf{R}$ with $\alpha > \beta$ and that $\mathbf{S}$ is dense in $\mathbf{W}^{\alpha,2}$ for all $\alpha \in \mathbf{R}$. It can be shown that for all $\alpha,\beta \in \mathbf{R}$, the map $i_{\alpha-\beta,\alpha+\beta} : \mathbf{W}^{\alpha-\beta,2} \to (\mathbf{W}^{\alpha+\beta,2})^*$ defined by

$$i_{\alpha-\beta,\alpha+\beta}(g)(f) = \langle g, f\rangle_{\alpha-\beta,\alpha+\beta} := ((I-\Delta)^{\frac{-\beta}{2}} g, (I-\Delta)^{\frac{\beta}{2}} f)_\alpha,$$

for all $f \in \mathbf{W}^{\alpha+\beta,2}$ and $g \in \mathbf{W}^{\alpha-\beta,2}$, is an isometric isomorphism.

Let

$$\mathbf{H}^0 = \left\{ f \in \mathbf{W}^{0,2} : \ \nabla\cdot f = 0 \right\} = \left\{ f \in \mathbf{W}^{0,2} : \hat{f}_n \cdot n = 0, \ \forall n \in \mathbf{Z}^d \right\}.$$

We define $P : \mathbf{S}' \to \mathbf{S}'$ by

$$Pf = \sum_{n\in\mathbf{Z}^d} \left(\hat{f}_n - \frac{n\cdot \hat{f}_n}{|n|^2} n \right) e_n, \quad f \in \mathbf{L}^2,$$

and let $Q = I - P$. It follows that P is a projection of $\mathbf{L}^2$ onto $\mathbf{H}^0 = P\mathbf{L}^2$ and that $\mathbf{L}^2$ possesses the orthogonal decomposition

$$\mathbf{L}^2 = P\mathbf{L}^2 \oplus Q\mathbf{L}^2.$$

Moreover, it is clear that $P, Q \in \mathcal{L}(\mathbf{W}^{\alpha,2}, \mathbf{W}^{\alpha,2})$ and that P and Q have operator norm less than or equal to one for all $\alpha \in \mathbf{R}$. We set

$$\mathbf{H}^\alpha = P\mathbf{W}^{\alpha,2} \ \ \& \ \ \mathbf{H}^\alpha_\perp = Q\mathbf{W}^{\alpha,2}.$$

It can be shown that for all $\alpha \in \mathbf{R}$ (see Lemma 3.7 in [25]),

$$\mathbf{W}^{\alpha,2} = \mathbf{H}^\alpha \oplus \mathbf{H}^\alpha_\perp,$$

where

$$\langle f, g\rangle_{-\alpha,\alpha} = 0, \quad \forall g \in \mathbf{H}^\alpha_\perp, \quad \forall f \in \mathbf{H}^{-\alpha}, \tag{2.1}$$

and

$$\mathbf{H}^\alpha = \left\{ f \in \mathbf{W}^{\alpha,2} : \ \nabla\cdot f = 0 \right\},$$

$$\mathbf{H}^\alpha_\perp = \{g \in \mathbf{W}^{\alpha,2} : \langle f, g\rangle_{-\alpha,\alpha} = 0, \quad \forall f \in \mathbf{H}^{-\alpha}\}.$$

Using (2.1), one can check that $i_{-\alpha,\alpha} : \mathbf{H}^{-\alpha} \to (\mathbf{H}^\alpha)^*$ and $i_{-\alpha,\alpha} : \mathbf{H}^{-\alpha}_\perp \to (\mathbf{H}^\alpha_\perp)^*$ are isometric isomorphisms for all $\alpha \in \mathbf{R}$.

For each vector $n \in \mathbf{Z}^d - \{0\}$, there exists $d-1$ vectors $\{m_1(n), \dots, m_{d-1}(n)\} \subseteq \mathbf{R}^d$ that are of unit length and orthogonal to n in $\mathbf{R}^d$. Denoting by $\mathbf{e}_j$, $j \in \{1, \dots, d\}$, the standard basis of $\mathbf{R}^d$, it follows that

$$\left\{\mathbf{f}_{0,j} = \mathbf{e}_j(2\pi)^{-\frac{d}{2}} : \ j \in \{1, \dots, d\}\right\} \cup \left\{\mathbf{f}_{n,j} = m_j(n)e_n : \ n \in \mathbf{Z}^d - \{0\}, \right.$$
$$\left. j \in \{1, \dots, d-1\}\right\}$$

is an orthonormal basis of $\{u \in L^2(\mathbf{T}^d; \mathbf{C}^d) : \nabla \cdot u = 0\}$. In dimension two, the unit vector $|n|^{-1}n^\perp = |n|^{-1}[n_2, -n_1]^T$ is orthogonal to $n = [n_1, n_2]^T \in \mathbf{Z}^2 - \{0\}$, and hence

$$\left\{\mathbf{f}_{0,1} = [1, 0]^T(2\pi)^{-\frac{d}{2}}, \ \mathbf{f}_{0,2} = [0, 1]^T(2\pi)^{-\frac{d}{2}}\right\} \cup \left\{\mathbf{f}_{1,n} = |n|^{-1}n^\perp e_n : \ n \in \mathbf{Z}^2 - \{0\}\right\}$$

is an orthonormal basis of $\{u \in L^2(\mathbf{T}^2; \mathbf{C}^2) : \nabla \cdot u = 0\}$.

For a given $n \in \mathbf{Z}^d - \{0\}$ and $j \in \{1, \dots, d-1\}$, let

$$\mathbf{w}^{\sin}_{j,n}(x) := \sqrt{2}(2\pi)^{-\frac{d}{2}} m_j(n) \sin(n \cdot x), \quad \mathbf{w}^{\cos}_{j,n}(x) := \sqrt{2}(2\pi)^{-\frac{d}{2}} m_j(n) \cos(n \cdot x).$$

It follows that

$$\left\{(2\pi)^{-\frac{d}{2}}\mathbf{e}_j : \ j \in \{1, \dots, d\}\right\} \cup \left\{\mathbf{w}^{\sin}_{j,n}(x), \ \mathbf{w}^{\cos}_{j,n}(x) : \ n \in \mathbf{Z}^d - \{0\}, \ n_1 > 0, \right.$$
$$\left. j \in \{1, \dots, d-1\}\right\}$$

is an orthonormal basis of $\mathbf{H}^0$ and an orthogonal basis of $\mathbf{H}^1$. We reindex this basis by $\{h_n\}_{n=1}^\infty$. It is clear that $\mathbf{w}^{\sin}_{j,n}$ and $\mathbf{w}^{\cos}_{j,n}$ are eigenfunctions of the Stokes operator $A = -P\Delta$ on $\mathbf{H}^0$ with corresponding eigenvalues $|n|^2$. Thus, there exist a sequence $\{\lambda_n\}_{n=1}^\infty$ of nonnegative numbers such that $Ah_n = \lambda_n h_n$, for all $n \in \mathbf{N}$.

The following considerations shall enlighten the construction of the unbounded rough drivers associated with (1.1) (see Sect. 2.5). Let $\sigma : \mathbf{T}^d \to \mathbf{R}^d$ be twice differentiable and divergence free. Moreover, assume that the derivatives of σ up to order two are bounded uniformly by a constant N_0. Let $\mathcal{A}^1 = \sigma \cdot \nabla = \sum_{i=1}^d \sigma^i D_i$ and $\mathcal{A}^2 = (\sigma \cdot \nabla)(\sigma \cdot \nabla)$. It follows that there is a constant $N = N(d, N_0, \alpha)$ such that

$$|\mathcal{A}^1|_{\mathcal{L}(\mathbf{W}^{\alpha+1,2}, \mathbf{W}^{\alpha,2})} \leq N, \ \forall \alpha \in [0, 2], \quad |\mathcal{A}^2 f|_{\mathcal{L}(\mathbf{W}^{\alpha+2,2}, \mathbf{W}^{\alpha,2})} \leq N, \ \forall \alpha \in [0, 1].$$

We refer the reader to [26] for the estimates in the fractional norms; the estimates given in [26] are on the whole space, but can easily be adapted to the periodic setting. Since $P \in \mathcal{L}(\mathbf{W}^{\alpha,2}, \mathbf{H}^\alpha)$ and $Q \in \mathcal{L}(\mathbf{W}^{\alpha,2}, \mathbf{H}^\alpha_\perp)$ for all $\alpha \in \mathbf{R}$, both of which have operator norm bounded by 1, we have

$$|P\mathcal{A}^1|_{\mathcal{L}(\mathbf{H}^{\alpha+1}, \mathbf{H}^\alpha)} \leq N, \quad |Q\mathcal{A}^1|_{\mathcal{L}(\mathbf{H}^{\alpha+1}_\perp, \mathbf{H}^\alpha_\perp)} \leq N, \ \forall \alpha \in [0, 2], \tag{2.2}$$

and

$$|P\mathcal{A}^2|_{\mathcal{L}(\mathbf{H}^{\alpha+2},\mathbf{H}^{\alpha})} \leq N, \quad |Q\mathcal{A}^2|_{\mathcal{L}(\mathbf{H}_{\perp}^{\alpha+2},\mathbf{H}_{\perp}^{\alpha})} \leq N, \ \forall \alpha \in [0,1], \tag{2.3}$$

and hence $(P\mathcal{A}^1)^* \in \mathcal{L}((\mathbf{H}^{\alpha})^*, (\mathbf{H}^{\alpha+1})^*)$ and $(Q\mathcal{A}^1)^* \in \mathcal{L}((\mathbf{H}_{\perp}^{\alpha})^*, (\mathbf{H}_{\perp}^{\alpha+1})^*)$ for $\alpha \in [0,2]$ and $(P\mathcal{A}^2)^* \in \mathcal{L}((\mathbf{H}^{\alpha})^*, (\mathbf{H}^{\alpha+2})^*)$ and $(Q\mathcal{A}^2)^* \in \mathcal{L}((\mathbf{H}_{\perp}^{\alpha})^*, (\mathbf{H}_{\perp}^{\alpha+2})^*)$ for $\alpha \in [0,1]$. Making use of the divergence-free property of $\sigma_k, k \in \{1, \dots, K\}$, we find

$$\left(\left(-P\mathcal{A}^1\right) f, g\right) = \left(f, P\mathcal{A}^1 g\right), \quad \forall f, g \in \mathbf{S} \cap \mathbf{H}^0,$$

and

$$\left(\left(-Q\mathcal{A}^1\right) f, g\right) = \left(f, Q\mathcal{A}^1 g\right), \quad \forall f, g \in \mathbf{S} \cap \mathbf{H}_{\perp}^0,$$

which implies that $(-P\mathcal{A}^1)^* = P\mathcal{A}^1$ and $(-Q\mathcal{A}^1)^* = Q\mathcal{A}^1$. Thus, owing to the characterization of the duality between $\mathbf{W}^{\alpha,2}$ and $\mathbf{W}^{-\alpha,2}$ through the $\mathbf{L}^2$ inner product, we have

$$P\mathcal{A}^1 \in \mathcal{L}\left(\mathbf{H}^{-\alpha}, \mathbf{H}^{-(\alpha+1)}\right), \quad Q\mathcal{A}^1 \in \mathcal{L}\left(\mathbf{H}_{\perp}^{-\alpha}, \mathbf{H}_{\perp}^{-(\alpha+1)}\right),$$
$$P\mathcal{A}^2 \in \mathcal{L}\left(\mathbf{H}^{-\alpha}, \mathbf{H}^{-(\alpha+2)}\right), \quad Q\mathcal{A}^2 \in \mathcal{L}\left(\mathbf{H}_{\perp}^{-\alpha}, \mathbf{H}_{\perp}^{-(\alpha+2)}\right).$$

In order to analyze the convective term, we employ the classical notation and bounds. Owing to Lemma 2.1 in [27] adapted to fractional norms (see [28]), the trilinear form

$$b(u, v, w) = \int_{\mathbf{T}^d} ((u \cdot \nabla)v) \cdot w \, dx = \sum_{i,j=1}^{d} \int_{\mathbf{T}^d} u^i D_i v^j w^j \, dx$$

is continuous on $\mathbf{W}^{\alpha_1,2} \times \mathbf{W}^{\alpha_2+1,2} \times \mathbf{W}^{\alpha_3,2}$ if $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}_+$ satisfy

$$\alpha_1 + \alpha_2 + \alpha_3 \geq \frac{d}{2}, \quad \text{if} \quad \alpha_i \neq \frac{d}{2} \ \text{ for all } i \in \{1,2,3\},$$
$$\alpha_1 + \alpha_2 + \alpha_3 > \frac{d}{2}, \quad \text{if } \alpha_i = \frac{d}{2} \ \text{ for some } i \in \{1,2,3\};$$

that is,

$$|b(u, v, w)| \lesssim_{\alpha_1,\alpha_2,\alpha_3,d} |u|_{\alpha_1} |v|_{\alpha_2+1} |w|_{\alpha_3}. \tag{2.4}$$

In the case $d = 2$, by virtue of the Gagliardo–Nirenberg interpolation inequality $|\phi|_{L^4(\mathbf{T}^2,\mathbf{R}^2)} \lesssim |\phi|_0^{\frac{1}{2}} |\phi|_1^{\frac{1}{2}}$, we have

$$|b(u, v, w)| \lesssim |u|_0^{\frac{1}{2}} |u|_1^{\frac{1}{2}} |v|_1 |w|_0^{\frac{1}{2}} |w|_1^{\frac{1}{2}}, \quad \forall u, v, w \in \mathbf{W}^{1,2}, \tag{2.5}$$

which plays an important role in the uniqueness proof. (See Theorem 4.3.) Moreover, for all $u \in \mathbf{H}^{\alpha_1}$ and $(v, w) \in \mathbf{W}^{\alpha_2+1,2} \times \mathbf{W}^{\alpha_3,2}$ such that $\alpha_1, \alpha_2, \alpha_3$ satisfy (2.4), we have

$$b(u, v, w) = -b(u, w, v) \quad \text{and} \quad b(u, v, v) = 0. \tag{2.6}$$

For α_1, α_2, and α_3 that satisfy (2.4) and any given $(u, v) \in \mathbf{W}^{\alpha_1,2} \times \mathbf{W}^{\alpha_2+1,2}$, we define $B(u, v) \in \mathbf{W}^{-\alpha_3,2}$ by

$$\langle B(u, v), w\rangle_{-\alpha_3,\alpha_3} = b(u, v, w), \quad \forall w \in \mathbf{W}^{\alpha_3,2}.$$

Similarly, we define $B_P = PB$ and $B_Q = QB$ and note that

$$B_P := PB : \mathbf{W}^{\alpha_1,2} \times \mathbf{W}^{\alpha_2+1,2} \to \mathbf{H}^{-\alpha_3}, \quad B_Q := QB : \mathbf{W}^{\alpha_1,2} \times \mathbf{W}^{\alpha_2+1,2} \to \mathbf{H}_\perp^{-\alpha_3},$$

for α_1, α_2, and α_3 that satisfy (2.4). We set

$$B(u) = B(u, u), \quad B_P(u) := B_P(u, u), \text{ and } B_Q(u) := B_Q(u, u).$$

2.2. Smoothing operators

As in [9], we will need a family of smoothing operators $(J^\eta)_{\eta\in(0,1]}$ acting on the scale of spaces $(\mathbf{W}^{\alpha,2})_{\alpha\in\mathbf{R}}$; that is, we require a family $(J^\eta)_{\eta\in(0,1]}$ such that for all $\alpha \in \mathbf{R}$ and $\beta \in \mathbf{R}_+$,

$$|(I - J^\eta)f|_\alpha \lesssim \eta^\beta|f|_{\alpha+\beta} \qquad \text{and} \qquad |J^\eta f|_{\alpha+\beta} \lesssim \eta^{-\beta}|f|_\alpha. \tag{2.7}$$

We construct these operators from the frequency cutoff operator $S_N : \mathbf{S}' \to \mathbf{S}$ defined by

$$S_N f = \sum_{|n|<N} \hat{f}_n e_n.$$

It follows that for all $\alpha \in \mathbf{R}$ and $\beta \in \mathbf{R}_+$,

$$|f - S_N f|_\alpha^2 = \sum_{|n|\geq N} \left(1 + |n|^2\right)^\alpha |\hat{f}_n|^2 \leq N^{-2\beta} \sum_{|n|\geq N} \left(1 + |n|^2\right)^{\alpha+\beta} |\hat{f}_n|^2 \leq N^{-2\beta}|f|_{\alpha+\beta}^2$$

and

$$|S_N f|_{\alpha+\beta}^2 = \sum_{|n|<N} \left(1 + |n|^2\right)^{\alpha+\beta} |\hat{f}_n|^2 \leq (1 + N^2)^\beta \sum_{|n|\geq N} \left(1 + |n|^2\right)^\alpha |\hat{f}_n|^2 \lesssim N^{2\beta}|f|_\alpha^2.$$

We define $J^\eta := S_{\lfloor \eta^{-1}\rfloor}$. It is then clear that J^η is a smoothing operator on $\mathbf{W}^{\alpha,2}$ and that it leaves the subspaces $\mathbf{H}^\alpha$ and $\mathbf{H}_\perp^\alpha$ invariant.

2.3. Rough paths

For a given interval I, we define $\Delta_I := \{(s, t) \in I^2 : s \leq t\}$ and $\Delta_I^{(2)} := \{(s, \theta, t) \in I^3 : s \leq \theta \leq t\}$. For a given $T > 0$, we let $\Delta_T := \Delta_{[0,T]}$ and $\Delta_T^{(2)} = \Delta_{[0,T]}^{(2)}$ Let $\mathcal{P}(I)$ denote the set of all partitions of an interval I and let E be a Banach space with norm $|\cdot|_E$. A function $g : \Delta_I \to E$ is said to have finite p-variation for some $p > 0$ on I if

$$|g|_{p-\mathrm{var};I;E} := \sup_{(t_i)\in\mathcal{P}(I)} \left(\sum_i |g_{t_i t_{i+1}}|_E^p\right)^{\frac{1}{p}} < \infty,$$

and we denote by $C_2^{p-\mathrm{var}}(I;E)$ the set of all continuous functions with finite p-variation on I equipped with the semi-norm $|\cdot|_{p-\mathrm{var};I;E}$. In this section, we drop the dependence of norms on the space E when convenient. We denote by $C^{p-\mathrm{var}}(I;E)$ the set of all paths $z: I \to E$ such that $\delta z \in C_2^{p-\mathrm{var}}(I;E)$, where $\delta z_{st} := z_t - z_s$.

For a given interval I, a two-index map $\omega : \Delta_I \to [0,\infty)$ is called superadditive if for all $(s,\theta,t) \in \Delta_I^{(2)}$,

$$\omega(s,\theta) + \omega(\theta,t) \le \omega(s,t).$$

A two-index map $\omega : \Delta_I \to [0,\infty)$ is called a control if it is superadditive, continuous on Δ_I and for all $s \in I$, $\omega(s,s) = 0$.

If for a given $p > 0$, $g \in C_2^{p-\mathrm{var}}(I;E)$, then it can be shown that the 2-index map $\omega_g : \Delta_I \to [0,\infty)$ defined by

$$\omega_g(s,t) = |g|^p_{p-\mathrm{var};[s,t]}$$

is a control (see, e.g., Proposition 5.8 in [29]). It is clear that $|g_{st}| \le \omega_g(s,t)^{\frac{1}{p}}$ for all $(s,t) \in \Delta_I$. If ω is a control such that $|g_{st}| \le \omega(s,t)^{\frac{1}{p}}$, then using superadditivity of the control, we have

$$\sum_i |g_{t_i t_{i+1}}|^p \le \sum_i \omega(t_i,t_{i+1}) \le \omega(s,t),$$

for any partition $(t_i) \in \mathcal{P}([s,t])$. Taking supremum over all partitions yields $\omega_g(s,t) \le \omega(s,t)$. Thus, we could equivalently define a semi-norm on $C_2^{p-\mathrm{var}}(I;E)$ by

$$|g|_{p-var;[s,t]} = \inf\{\omega(s,t)^{\frac{1}{p}} : |g_{uv}| \le \omega(u,v)^{\frac{1}{p}} \text{ for all } (u,v) \in \Delta_{[s,t]}\}.$$

We shall need a local version of the p-variation spaces, for which we restrict the mesh size of the partition by a control.

DEFINITION 2.1. Given an interval $I = [a,b]$, a control ϖ and real number $L > 0$, we denote by $C_{2,\varpi,L}^{p-\mathrm{var}}(I;E)$ the space of continuous two-index maps $g : \Delta_I \to E$ for which there exists at least one control ω such that for every $(s,t) \in \Delta_I$ with $\varpi(s,t) \le L$, it holds that $|g_{st}|_E \le \omega(s,t)^{\frac{1}{p}}$. We define a semi-norm on this space by

$$|g|_{p-\mathrm{var},\varpi,L;I} = \inf\Big\{\omega(a,b)^{\frac{1}{p}} : \omega \text{ is a control s.t. } |g_{st}| \\ \le \omega(s,t)^{\frac{1}{p}},\ \forall (s,t) \in \Delta_I \text{ with } \varpi(s,t) \le L\Big\}.$$

REMARK 2.2. By the above analysis, it is clear that we could equivalently define the semi-norm as

$$|g|_{p-\mathrm{var},\varpi,L;I} = \sup_{(t_i)\in\mathcal{P}_{\varpi,L}(I)} \left(\sum_i |g_{t_i t_{i+1}}|^p\right)^{\frac{1}{p}},$$

where $\mathcal{P}_{\varpi,L}(I)$ denotes the family of all partitions of an interval I such that $\varpi(t_i, t_{i+1}) \leq L$ for all neighboring partition points t_i and t_{i+1}. It is clear that

$$C^{p-\mathrm{var}}_{2,\varpi_1,L_1}(I;E) \subset C^{p-\mathrm{var}}_{2,\varpi_2,L_2}(I;E) \tag{2.8}$$

for $\varpi_1 \leq \varpi_2$ and $L_2 \leq L_1$.

REMARK 2.3. Let I be an interval. We could define the local p-variation space for 1-index maps $C^{p-\mathrm{var}}_{\varpi,L}(I;E)$ as above. However, there is no difference between the local and global spaces; that is, $C^{p-\mathrm{var}}_{\varpi,L}(I;E) = C^{p-\mathrm{var}}(I;E)$. Indeed, clearly $C^{p-\mathrm{var}}(I;E) \subset C^{p-\mathrm{var}}_{\varpi,L}(I;E)$. To show $C^{p-\mathrm{var}}_{\varpi,L}(I;E) \subset C^{p-\mathrm{var}}(I;E)$, let ϖ be such there is a partition $(s_j)_{j=1}^J$ of I satisfying $\varpi(s_j, s_{j+1}) \leq L$. Then, for any partition $(t_i) \in \mathcal{P}(I)$, we can always find a refinement $(\tilde{t}_k)$ of (t_i) containing (s_j). It follows from the superadditivity of ϖ that $\varpi(\tilde{t}_k, \tilde{t}_{k+1}) \leq L$. Moreover, either an interval (t_i, t_{i+1}) does not contain any of the $(s_j)_{j=1}^J$ or it contains a set $\{s_{j_1(i)}, \ldots, s_{j_{n(i)}(i)}\}$. In the latter case, we have

$$\delta g_{t_i t_{i+1}} = \delta g_{t_i s_{j_1(i)}} + \sum_{j=j_1(i)}^{j_{n(i)}(i)-1} \delta g_{s_j s_{j+1}} + \delta g_{s_{j_{n(i)}(i)} t_{i+1}}.$$

Thus, for any $g \in C^{p-\mathrm{var}}_{\varpi,L}(I;E)$, we have

$$\sum_{(t_i)\in\mathcal{P}(I)} |\delta g_{t_i t_{i+1}}|^p \lesssim_p \sum_{(\tilde{t}_i)\in\mathcal{P}_{\varpi,L}} |\delta g_{\tilde{t}_i \tilde{t}_{i+1}}|^p \lesssim_p |g|_{p-\mathrm{var},\varpi,L;I},$$

and hence $C^{p-\mathrm{var}}_{\varpi,L}(I;E) = C^{p-\mathrm{var}}(I;E)$.

We now introduce the notion of a rough path. For a thorough introduction to the theory of rough paths, we refer the reader to the monographs [29–31]. For a two-index map $g : \Delta_I \to \mathbf{R}$, we define the second-order increment operator

$$\delta g_{s\theta t} = g_{st} - g_{\theta t} - g_{s\theta}, \quad \forall (s,\theta,t) \in \Delta_I^{(2)}.$$

DEFINITION 2.4. Let $K \in \mathbf{N}$ and $p \in [2,3)$. A continuous p-rough path is a pair

$$\mathbf{Z} = (Z, \mathbb{Z}) \in C_2^{p-\mathrm{var}}\left([0,T];\mathbf{R}^K\right) \times C_2^{\frac{p}{2}-\mathrm{var}}\left([0,T];\mathbf{R}^{K\times K}\right) \tag{2.9}$$

that satisfies the Chen's relation

$$\delta \mathbb{Z}_{s\theta t} = Z_{s\theta} \otimes Z_{\theta t}, \quad \forall (s,\theta,t) \in \Delta_{[0,T]}^{(2)}.$$

A rough path $\mathbf{Z} = (Z, \mathbb{Z})$ is said to be geometric if it can be obtained as the limit in the product topology $C_2^{p-\mathrm{var}}([0,T];\mathbf{R}^K) \times C_2^{\frac{p}{2}-\mathrm{var}}([0,T];\mathbf{R}^{K\times K})$ of a sequence of rough paths $\{(Z^n, \mathbb{Z}^n)\}_{n=1}^\infty$ such that for each $n \in \mathbf{N}$,

$$^{n}_{st} := \delta z^n_{st} \quad \text{and} \quad \mathbb{Z}^n_{st} := \int_s^t \delta z^n_{s\theta} \otimes \mathrm{d} z^n_\theta,$$

for some smooth path $z^n : [0, T] \to \mathbf{R}^K$, where the iterated integral is a Riemann integral. We denote by $C_g^{p-\mathrm{var}}([0, T]; \mathbf{R}^K)$ the set of geometric p-rough paths and endow it with the product topology.

REMARK 2.5. For any continuous p-rough path $\mathbf{Z} = (Z, \mathbb{Z})$, it is clear that we can always find a control ω such that for all $(s, t) \in \Delta_T$,

$$|Z_{st}|^p \leq \omega(s, t) \quad \text{and} \quad |\mathbb{Z}_{st}|^{\frac{p}{2}} \leq \omega(s, t).$$

With abuse of notation, we write $\omega = \omega_Z$. This should compared with (2.10) below.

Throughout this paper, we will only consider geometric rough paths. An advantage of working with geometric rough paths is that a first-order chain rule similar to the one known for smooth paths holds true. We recall that such a chain rule is not true in Itô integration theory, in which only a (second order) Itô formula is available. However, for the Stratonovich integral, a first-order chain rule holds true. Thus, in case of a Brownian motion, a Stratonovich integral should be used for the construction of the iterated integral if one wishes to lift it to a geometric rough path.

2.4. Unbounded rough drivers

Since the rough perturbation in (1.1) is (unbounded) operator valued, it is necessary to generalize the notion of a rough path accordingly. To this end, we define unbounded rough drivers, which can be regarded as operator valued rough paths with values in a suitable space of unbounded operators. In what follows, we call a scale any family $(E^\alpha, |\cdot|_\alpha)_{\alpha\in\mathbf{R}_+}$ of Banach spaces such that $E^{\alpha+\beta}$ is continuously embedded into E^α for $\beta \in \mathbf{R}_+$. For $\alpha \in \mathbf{R}_+$, we denote by $E^{-\alpha}$ the topological dual of E^α, and note that, in general, $E^{-0} \neq E^0$.

DEFINITION 2.6. Let $p \in [2, 3)$ and $T > 0$ be given. A continuous unbounded p-rough driver with respect to the scale $(E^\alpha, |\cdot|_\alpha)_{\alpha\in\mathbf{R}_+}$, is a pair $\mathbf{A} = (A^1, A^2)$ of 2-index maps such that there exists a continuous control ω_A on $[0, T]$ such that for every $(s, t) \in \Delta_T$,

$$\begin{aligned} &|A^1_{st}|^p_{\mathcal{L}(E^{-\alpha}, E^{-(\alpha+1)})} \leq \omega_A(s, t) \ \text{ for } \ \alpha \in [0, 2], \quad |A^2_{st}|^{\frac{p}{2}}_{\mathcal{L}(E^{-\alpha}, E^{-(\alpha+2)})} \\ &\quad \leq \omega_A(s, t) \ \text{ for } \ \alpha \in [0, 1], \end{aligned} \tag{2.10}$$

and Chen's relation holds true,

$$\delta A^1_{s\theta t} = 0, \qquad \delta A^2_{s\theta t} = A^1_{\theta t} A^1_{s\theta}, \quad \forall (s, \theta, t) \in \Delta_T^{(2)}. \tag{2.11}$$

We will show below that Definition 2.6 allows for a formulation of (1.1), (1.7). (See Definitions 2.11 and 2.7.)

2.5. Formulation of the equation

In this section, we derive a rough path formulation of (1.1), (1.7), which will be satisfied by solutions constructed by our main result below, Theorem 2.13. The main ideas of this step were already discussed in Sect. 1 in the simpler setting of the transport equation (1.3).

We fix an arbitrary terminal time $T > 0$ and viscosity $\nu > 0$. Let $d \in \{2, 3\}$. Let $z \in C^{p-\mathrm{var}}([0, T]; \mathbf{R}^K)$ be such that it can be lifted to a continuous geometric p-rough path $\mathbf{Z} = (Z, \mathbb{Z}) \in C_g^{p-\mathrm{var}}([0, T]; \mathbf{R}^K)$ for some $p \in [2, 3)$. For each $k \in \{1, \ldots, K\}$, assume that $\sigma_k : \mathbf{T}^d \to \mathbf{R}^d$ is twice differentiable and divergence free. Moreover, assume that for all $k \in \{1, \ldots, K\}$, σ_k and its derivatives up to order two are bounded uniformly. For given initial condition $u_0 \in \mathbf{H}^0$, we consider the system of Navier–Stokes equations on $(t, x) \in [0, T] \times \mathbf{T}^d$ given by

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u + \nabla p &= \nu \Delta u + (\sigma_k \cdot \nabla) u \, \dot{z}_t^k, \\ \nabla \cdot u &= 0, \\ u(0) &= u_0, \end{aligned} \tag{2.12}$$

where the unknown are the velocity field $u : [0, T] \times \mathbf{T}^d \to \mathbf{R}^d$ and pressure $p : [0, T] \times \mathbf{T}^d \to \mathbf{R}$. Here and below, we use the notation

$$(u \cdot \nabla) u = \sum_{j=1}^{d} u^j \frac{\partial u}{\partial x_j} \quad \text{and} \quad (\sigma_k \cdot \nabla) u \, \dot{z}_t^k = \sum_{k=1}^{K} (\sigma_k \cdot \nabla) u \, \dot{z}_t^k = \sum_{k=1}^{K} \sum_{j=1}^{d} \sigma_k^j \frac{\partial u}{\partial x_j} \dot{z}_t^k.$$

The classical way of studying the Navier–Stokes equation in the variational framework is to decouple the velocity field and the pressure into two equations using the Leray projection P defined in Sect. 2.1. Applying the solenoidal $P : \mathbf{W}^{\alpha,2} \to \mathbf{H}^{\alpha}$ and gradient projection $Q : \mathbf{W}^{\alpha,2} \to \mathbf{H}_{\perp}^{\alpha}$ separately to (2.12) yields

$$\begin{aligned} \partial_t u + P[(u \cdot \nabla) u] &= \nu \Delta u + P[(\sigma_k \cdot \nabla) u] \, \dot{z}_t^k, \\ \nabla p + Q[(u \cdot \nabla) u] &= Q[(\sigma_k \cdot \nabla) u] \, \dot{z}_t^k. \end{aligned} \tag{2.13}$$

We let

$$\pi := \int_0^{\cdot} \nabla p_r \, \mathrm{d}r.$$

As we did for the pure-transport equation (1.3) in the introduction, we integrate the (2.13) over $[s, t]$ and then iterate the equation into itself to obtain

$$\begin{aligned} \delta u_{st} + \int_s^t P[(u_r \cdot \nabla) u_r] \, \mathrm{d}r &= \int_s^t \nu \Delta u_r \mathrm{d}r + \left[A_{st}^{P,1} + A_{st}^{P,2} \right] u_s + u_{st}^{P,\natural}, \\ \delta \pi_{st} + \int_s^t Q[(u_r \cdot \nabla) u_r)] \, \mathrm{d}r &= \left[A_{st}^{Q,1} + A_{st}^{Q,2} \right] u_s + u_{st}^{Q,\natural}, \end{aligned} \tag{2.14}$$

where

$$A_{st}^{P,1}\varphi := P\left[(\sigma_k\cdot\nabla)\varphi\right] Z_{st}^k, \qquad A_{st}^{P,2}\varphi := P\left[(\sigma_k\cdot\nabla)\, P\left[(\sigma_i\cdot\nabla)\varphi\right]\right] \mathbb{Z}_{st}^{i,k},$$
$$A_{st}^{Q,1}\varphi := Q\left[(\sigma_k\cdot\nabla)\varphi\right] Z_{st}^k, \qquad A_{st}^{Q,2}\varphi := Q\left[(\sigma_k\cdot\nabla)\, P\left[(\sigma_i\cdot\nabla)\varphi\right]\right] \mathbb{Z}_{st}^{i,k}.$$

To do this derivation, let us assume that we have a solution $u\in L_T^2\mathbf{H}^1\cap L_T^\infty\mathbf{H}^0$. If we set

$$\mu_\cdot = \int_0^\cdot \left[\nu\Delta u_r - (u_r\cdot\nabla)u_r\right]\mathrm{d}r,$$

then by (2.4) with $\alpha_1=\alpha_3=1$ and $\alpha_2=0$, we have $\mu\in C^{1-\mathrm{var}}([0,T];\mathbf{W}^{-1,2})$. Iterating the first equation of (2.13) into itself gives

$$\begin{aligned}
\delta u_{st} &= P\delta\mu_{st} + \int_s^t P(\sigma_k\cdot\nabla)\left(u_s + P\delta\mu_{sr} + \int_s^r P(\sigma_i\cdot\nabla)u_{r_1}\,\mathrm{d}z_{r_1}^i\right)\mathrm{d}z_r^k\\
&= P\delta\mu_{st} + P(\sigma_k\cdot\nabla)u_s Z_{st}^k + \int_s^t P(\sigma_k\cdot\nabla)\delta\mu_{sr}\,\mathrm{d}z_r^k\\
&\quad + \int_s^t P(\sigma_k\cdot\nabla)\int_s^r P(\sigma_i\cdot\nabla)u_{r_1}\,\mathrm{d}z_{r_1}^i\,\mathrm{d}z_r^k\\
&= P\delta\mu_{st} + P(\sigma\cdot\nabla_k)u_s Z_{st}^k + \int_s^t P(\sigma_k\cdot\nabla)\delta\mu_{sr}\,\mathrm{d}z_r^k +\\
&\quad + \int_s^t P(\sigma_k\cdot\nabla)\int_s^r P(\sigma_i\cdot\nabla)\left(u_s + P\delta\mu_{sr_1} + P\int_s^{r_1}(\sigma_j\cdot\nabla)u_{r_2}\,\mathrm{d}z_{r_2}^j\right)\mathrm{d}z_{r_1}^i\,\mathrm{d}z_r^k\\
&= P\delta\mu_{st} + P(\sigma_k\cdot\nabla)u_s Z_{st}^k + P(\sigma_k\cdot\nabla)P(\sigma_i\cdot\nabla)u_s\mathbb{Z}_{st}^{i,k} + \int_s^t P(\sigma_k\cdot\nabla)\delta\mu_{sr}\,\mathrm{d}z_r^k +\\
&\quad + \int_s^t P(\sigma_k\cdot\nabla)\int_s^r P(\sigma_i\cdot\nabla)\left(P\delta\mu_{sr_1} + P\int_s^{r_1}(\sigma_j\cdot\nabla)u_{r_2}\,dz_{r_2}^j\right)\mathrm{d}z_{r_1}^i\,\mathrm{d}z_r^k\\
&= P\delta\mu_{st} + P[(\sigma_k\cdot\nabla)u_s]Z_{st}^k + P[(\sigma_k\cdot\nabla)P[(\sigma_i\cdot\nabla)u_s]]\mathbb{Z}_{st}^{i,k} + u_{st}^{P,\natural},
\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}
u_{st}^{P,\natural} &:= \int_s^t P(\sigma_k\cdot\nabla)P\delta\mu_{sr}\,\mathrm{d}z_r^k\\
&\quad + \int_s^t P(\sigma_k\cdot\nabla)\int_s^r P(\sigma_i\cdot\nabla)\left(P\delta\mu_{sr_1} + P\int_s^{r_1}(\sigma_j\cdot\nabla)u_{r_2}\,\mathrm{d}z_{r_2}^j\right)\mathrm{d}z_{r_1}^i\,\mathrm{d}z_r^k.
\end{aligned}$$

We expect $u_{st}^{P,\natural}$ be in $C_2^{\frac{p}{3}-\mathrm{var}}([0,T];\mathbf{H}^{-3})$ since $\mu\in C^{1-\mathrm{var}}([0,T];\mathbf{W}^{-1,2})$ and $u\in L_T^\infty\mathbf{H}^0$.

Note that $Q\mu = -\int_0^\cdot Q[(u_r\cdot\nabla)u_r]\mathrm{d}r$. Then, iterating the first equation of (2.13) into second equation, we find

$$\begin{aligned}
\delta\pi_{st} &= Q\delta\mu_{st} + Q\int_s^t(\sigma_k\cdot\nabla)u_r\,\mathrm{d}z_r^k\\
&= Q\delta\mu_{st} + Q\int_s^t(\sigma_k\cdot\nabla)\left(u_s + P\delta\mu_{sr} + P\int_s^r(\sigma_i\cdot\nabla)u_{r_1}\,\mathrm{d}z_{r_1}^i\right)\mathrm{d}z_r^k
\end{aligned}$$

$$= Q\delta\mu_{st} + Q(\sigma_k\cdot\nabla)u_s Z^k_{st} + Q\int_s^t (\sigma_k\cdot\nabla)P\delta\mu_{sr}\,\mathrm{d}z^k_r$$
$$+ Q\int_s^t(\sigma_k\cdot\nabla)P\int_s^r(\sigma_i\cdot\nabla)\left(u_s + P\delta\mu_{sr_1} + P\int_s^{r_1}(\sigma_j\cdot\nabla)u_{r_2}\,\mathrm{d}z^j_{r_2}\right)\mathrm{d}z^i_{r_1}\,\mathrm{d}z^k_r$$
$$= Q\delta\mu_{st} + Q[(\sigma_k\cdot\nabla)u_s]Z^k_{st} + Q[(\sigma_k\cdot\nabla)P[(\sigma_i\cdot\nabla)u_s]]\mathbb{Z}^{i,k}_{st} + u^{Q,\natural}_{st},$$

where

$$u^{Q,\natural}_{st} = Q\int_s^t(\sigma_k\cdot\nabla)P\delta\mu_{sr}\,\mathrm{d}z^k_r$$
$$+ Q\int_s^t(\sigma_k\cdot\nabla)P\int_s^r(\sigma_i\cdot\nabla)\left(P\delta\mu_{sr_1} + P\int_s^{r_1}(\sigma_j\cdot\nabla)u_{r_2}\,\mathrm{d}z^j_{r_2})\,\mathrm{d}z^i_{r_1}\right)\mathrm{d}z^k_r,$$

which is expected to be in $C_2^{\frac{p}{3}-\mathrm{var}}([0,T];\mathbf{H}^{-3}_\perp)$.

Equation (2.14) is to be understood in the sense that we *define* the remainder terms $u^{P,\natural}$ and $u^{Q,\natural}$ from the solution u and π, and have to verify that they are indeed negligible remainders; namely, they are of order $o(|t-s|)$. This will be made precise in Definition 2.7 below.

The pair $\mathbf{A}^P = (A^{P,1}, A^{P,2})$ is an unbounded rough driver (Definition 2.6) on the scale $(\mathbf{H}^\alpha)_{\alpha\in\mathbf{R}_+}$. Indeed, the existence of a control $\omega_{\mathbf{A}^P}$ such that (2.10) holds follows from the discussion in Sect. 2.1 and the fact that $(Z,\mathbb{Z})$ is a p-rough path (Definition 2.4), which also implies Chen's relation (2.11). We note that control $\omega_{\mathbf{A}^P}$ can be chosen to satisfy

$$\omega_{A^P}(s,t) \le C\omega_Z(s,t), \quad \forall (s,t)\in\Delta_T, \tag{2.16}$$

for a constant $C>0$ depending only on d and the bounds on $\sigma = (\sigma_1,\ldots,\sigma_K)$ and its derivatives up to order two.

The pair $\mathbf{A}^Q = (A^{Q,1}, A^{Q,2})$ satisfies (2.10) for the scale $(\mathbf{H}^\alpha_\perp)_{\alpha\in\mathbf{R}_+}$ with a control ω_{A^Q}, which also satisfies the bound (2.16). However, $\mathbf{A}^Q$ is not an unbounded rough driver since it fails to satisfy Chen's relation (2.11). Nevertheless, it satisfies

$$\delta A^{Q,2}_{s\theta t} = A^{Q,1}_{\theta t}A^{P,1}_{s\theta}, \quad \text{for all} \quad (s,\theta,t)\in\Delta^{(2)}_T, \tag{2.17}$$

which is the correct Chen's relation for the system of Eq. (2.13) needed to establish the required time regularity of the remainder $u^{Q,\natural}$ (see Sect. 3 and Lemma 3.5).

We will now define our first notion of solution to (2.12).

DEFINITION 2.7. A pair of weakly continuous functions $(u,\pi):[0,T]\to\mathbf{H}^0\times\mathbf{H}^{-3}_\perp$ is called a solution of (2.12) if $u\in L^2_T\mathbf{H}^1\cap L^\infty_T\mathbf{H}^0$ and $u^{P,\natural}:\Delta_T\to\mathbf{H}^{-3}$ and $u^{Q,\natural}:\Delta_T\to\mathbf{H}^{-3}_\perp$ defined for all $\phi\in\mathbf{H}^3$, $\psi\in\mathbf{H}^3_\perp$ and $(s,t)\in\Delta_T$ by

$$u^{P,\natural}_{st}(\phi) := \delta u_{st}(\phi) + \int_s^t [\nu(\nabla u_r,\nabla\phi) + B_P(u_r)(\phi)]\,\mathrm{d}r - u_s\left(\left[A^{P,1,*}_{st} + A^{P,2,*}_{st}\right]\phi\right), \tag{2.18}$$

$$u^{Q,\natural}_{st}(\psi) := \delta\pi_{st}(\psi) + \int_s^t B_Q(u_r)(\psi)\,\mathrm{d}r - u_s\left(\left[A^{Q,1,*}_{st} + A^{Q,2,*}_{st}\right]\psi\right), \tag{2.19}$$

satisfy

$$u^{P,\natural} \in C_{2,\varpi,L}^{\frac{p}{3}-\mathrm{var}}([0,T];\mathbf{H}^{-3}) \quad \text{and} \quad u^{Q,\natural} \in C_{2,\varpi,L}^{\frac{p}{3}-\mathrm{var}}([0,T];\mathbf{H}_{\perp}^{-3}), \tag{2.20}$$

for some control ϖ and $L > 0$.

REMARK 2.8. Applying (2.4) with $\alpha_1 = 0, \alpha_2 = 2$, and $\alpha_3 = 0$, we get

$$B_P(u)(\phi) = B_P(u,u)(\phi) = B_P(u,\phi)(u) \lesssim |u|_0^2|\phi|_3,$$

from which it follows that the dr-integral in (2.18) is well defined since $u \in \mathbf{L}_T^\infty\mathbf{H}^0$. One could also obtain an estimate that requires less regularity on ϕ by applying (2.4) with $\alpha_1 = 1, \alpha_2 = 0$, and $\alpha_3 = 1$ to get,

$$|B_P(u)(\phi)| \lesssim |u|_1^2|\phi|_1,$$

from which it follows that the dr-integral in (2.18) is well defined since $u \in \mathbf{L}_T^2\mathbf{H}^1$. However, we must test by $\phi \in \mathbf{H}^3$ to ensure that the remainder term $u^{P,\natural}(\phi)$ has the required time regularity. An analogous argument holds for the B_Q term in (2.20).

REMARK 2.9. In (2.18) and (2.20), we opt for distributional evaluation notation for most terms, and continue to do so throughout the paper. That is,

$$\begin{aligned}
u_{st}^{P,\natural}(\phi) &= \langle u_{st}^{P,\natural},\phi\rangle_{-3,3}, \quad \delta u_{st}(\phi) = (\delta u_{st},\phi)_0, \quad u_s\left(\left[A_{st}^{P,1,*}+A_{st}^{P,2,*}\right]\phi\right)_0 \\
&= \left(u_s,\left[A_{st}^{P,1,*}+A_{st}^{P,2,*}\right]\phi\right)_0, \\
u_{st}^{Q,\natural}(\phi) &= \langle u_{st}^{Q,\natural},\psi\rangle_{-3,3}, \quad u_s\left(\left[A_{st}^{Q,1,*}+A_{st}^{Q,2,*}\right]\psi\right)_0 = \left(u_s,\left[A_{st}^{Q,1,*}+A_{st}^{Q,2,*}\right]\psi\right)_0.
\end{aligned}$$

REMARK 2.10. Due to (2.8), there is no restriction in taking the same ϖ and $L > 0$ for both local variation spaces in (2.20).

We will now discuss an alternative way of formulating the equation. We can arrive at this formulation by performing an iteration directly on (2.12):

$$\begin{aligned}
\delta u_{st} &= \delta\mu_{st} - \delta\pi_{st} + (\sigma_k\cdot\nabla)u_s Z_{st}^k + (\sigma_k\cdot\nabla)(\sigma_i\cdot\nabla)u_s\mathbb{Z}_{st}^{i,k} \\
&\quad + \int_s^t (\sigma_k\cdot\nabla)\delta\mu_{sr}\,\mathrm{d}z_r^k - \int_s^t(\sigma_k\cdot\nabla)\delta\pi_{sr}\,\mathrm{d}z_r^k \\
&\quad + \int_s^t(\sigma_k\cdot\nabla)\int_s^r(\sigma_i\cdot\nabla)\left(\delta\mu_{sr_1} - \delta\pi_{sr_1} + \int_s^{r_1}(\sigma_j\cdot\nabla)u_{r_2}\,\mathrm{d}z_{r_2}^j\right)\mathrm{d}z_{r_1}^i\,\mathrm{d}z_r^k.
\end{aligned}$$

The integral $\int_s^t(\sigma_k\cdot\nabla)\delta\pi_{sr}\,\mathrm{d}z_r^k$ is not regular enough in time for it to be a negligible remainder. Indeed, we expect π to have finite p-variation, so that $\int_0^\cdot(\sigma_k\cdot\nabla)\delta\pi_{sr}\,\mathrm{d}z_r^k$ should only have finite $\frac{p}{2}$-variation. If we define

$$\bar{u}_{st}^{\natural} = \int_s^t(\sigma_k\cdot\nabla)\,\delta\mu_{sr}\,\mathrm{d}z_r^k + \int_s^t(\sigma_k\cdot\nabla)\int_s^r(\sigma_i\cdot\nabla)\Big(\delta\mu_{sr_2} - \delta\pi_{sr_2}$$

$$+\int_s^{r_1} \left(\sigma_j \cdot \nabla\right) u_{r_2}\, \mathrm{d}z^j_{r_2}\Big)\, \mathrm{d}z^i_{r_1}\, \mathrm{d}z^k_r,$$

then we expect $\bar{u}^\natural$ to be in $C_2^{\frac{p}{3}-\mathrm{var}}([0,T];\mathbf{W}^{-3,2})$. Moreover, we have

$$\begin{aligned}\delta u_{st} &= \delta\mu_{st} - \delta\pi_{st} + (\sigma_k\cdot\nabla)u_s Z^k_{st} + (\sigma_k\cdot\nabla)(\sigma_i\cdot\nabla)u_s\mathbb{Z}^{i,k}_{st} + \bar{u}^\natural_{st}\\ &\quad - \int_s^t (\sigma_k\cdot\nabla)\delta\pi_{sr}\,\mathrm{d}z^k_r.\end{aligned}$$

In order to complete the formulation, we use Eq. (2.13) for π to deduce

$$\begin{aligned}-\int_s^t (\sigma_k\cdot\nabla)\,\delta\pi_{sr}\mathrm{d}z^k_r &= -(\sigma_k\cdot\nabla)\,Q\left(\left(\sigma_i\cdot\nabla\right)u_s\right)\mathbb{Z}^{i,k}_{st} + \int_s^t (\sigma_k\cdot\nabla)\,Q\delta\mu_{sr}\mathrm{d}z^k_r\\ &\quad - \int_s^t (\sigma_k\cdot\nabla)\int_s^r Q\left(\sigma_i\cdot\nabla\right)\Big(\delta\mu_{sr_1} - \delta\pi_{sr_1}\\ &\quad + \int_s^{r_1}\left(\sigma_j\cdot\nabla\right)u_{r_1}\mathrm{d}z^j_{r_1}\Big)\mathrm{d}z^i_{r_1}\mathrm{d}z^k_r.\end{aligned}$$

All the terms above except for $(\sigma_k\cdot\nabla)Q\left[(\sigma_i\cdot\nabla)u_s\right]\mathbb{Z}^{i,k}_{st}$ belong to $C_2^{\frac{p}{3}-\mathrm{var}}([0,T];$ $\mathbf{W}^{-3,2})$, and hence we may include them in a new remainder

$$\begin{aligned}u^\natural_{st} &:= \bar{u}^\natural_{st} - \int_s^t (\sigma_k\cdot\nabla)Q\delta\mu_{sr}\,\mathrm{d}z^k_r\\ &\quad - \int_s^t (\sigma_k\cdot\nabla)\int_s^r Q(\sigma_i\cdot\nabla)\left(\delta\mu_{sr_1} + \delta\pi_{sr_1} + \int_s^{r_1}(\sigma_j\cdot\nabla)u_{r_1}\,\mathrm{d}z^j_{r_2}\right)\mathrm{d}z^i_{r_1}\,\mathrm{d}z^k_r.\end{aligned}$$

Combining the above, we obtain

$$\begin{aligned}\delta u_{st} &= \delta\mu_{st} - \delta\pi_{st} + (\sigma_k\cdot\nabla)\,u_s Z^k_{st} + (\sigma_k\cdot\nabla)\,(\sigma_i\cdot\nabla)\,u_s\mathbb{Z}^{i,k}_{st}\\ &\quad - (\sigma_k\cdot\nabla)\,Q\left[\sigma_i\cdot\nabla u_s\right]\mathbb{Z}^{i,k}_{st} + u^\natural_{st}\\ &= \delta\mu_{st} - \delta\pi_{st} + (\sigma_k\cdot\nabla)\,u_s Z^k_{st} + (\sigma_k\cdot\nabla)\,P\left[(\sigma_i\cdot\nabla)\,u_s\right]\mathbb{Z}^{i,k}_{st} + u^\natural_{st}.\end{aligned}$$

Thus, the pair $\mathbf{A} = (A^1, A^2)$ defined by

$$A^1_{st}\varphi = (\sigma_k\cdot\nabla)\varphi Z^k_{st},\quad A^2_{st}\varphi = (\sigma_k\cdot\nabla)P\left[(\sigma_i\cdot\nabla)\varphi\right]\mathbb{Z}^{i,k}_{st}$$

satisfies (2.10) for the scale $(\mathbf{W}^{\alpha,2})_{\alpha\in\mathbf{R}_+}$ with control ω_A. However, this pair does not satisfy Chen's relation (2.11), but does satisfy

$$\delta A^2_{s\theta t} = A^1_{\theta t}PA^1_{s\theta},\quad \forall(s,\theta,t)\in\Delta^{(2)}_T. \tag{2.21}$$

Since $\mathbf{A}^P = P\mathbf{A}$ and $\mathbf{A}^Q = Q\mathbf{A}$, the controls $\omega_{A^P}, \omega_{A^Q}$, and ω_A can be chosen so that

$$\omega_{A^P}(s,t), \omega_{A^Q}(s,t) \le \omega_A(s,t) \le C\omega_Z(s,t),\quad \forall(s,t)\in\Delta_T,$$

where C is a constant depending only on d and the bounds on $\sigma = (\sigma_1,\ldots,\sigma_K)$ and its derivatives up to order two.

Thus, alternatively, we may formulate a solution to (2.12) as follows.

DEFINITION 2.11. A pair of weakly continuous functions $(u, \pi) : [0, T] \to \mathbf{H}^0 \times \mathbf{H}_\perp^{-3}$ is called a solution of (2.12) if $u \in L_T^2\mathbf{H}^1 \cap L_T^\infty\mathbf{H}^0$ and $u^\natural : \Delta_T \to \mathbf{W}^{-3,2}$ defined for all $\phi \in \mathbf{W}^{3,2}$ and $(s,t) \in \Delta_T$ by

$$u_{st}^\natural(\phi) = \delta u_{st}(\phi) + \int_s^t [\nu(\nabla u_r, \nabla\phi) + B(u_r)(\phi)]\,\mathrm{d}r - u_s\left(\left[A_{st}^{1,*} + A_{st}^{2,*}\right]\phi\right) + \delta\pi_{st}(\phi),$$

satisfies $u^\natural \in C_{2,\varpi,L}^{\frac{p}{3}-\text{var}}([0,T];\mathbf{W}^{-3,2})$ for some control ϖ and $L > 0$.

The following lemma says that both formulations were derived in a consistent way and are equivalent.

LEMMA 2.12. *Definitions 2.7 and 2.11 are equivalent.*

Proof. Clearly, $PA_{st}^i = A_{st}^{P,i}$ and $QA_{st}^i = A_{st}^{Q,i}$ for $i \in \{1,2\}$. Moreover, the mapping

$$\begin{aligned} &C_{2,\varpi,L}^{\frac{p}{3}-\text{var}}\left([0,T];\mathbf{W}^{-3,2}\right) \to C_{2,\varpi,L}^{\frac{p}{3}-\text{var}}\left([0,T];\mathbf{H}^{-3}\right) \times C_{2,\varpi,L}^{\frac{p}{3}-\text{var}}\left([0,T];\mathbf{H}_\perp^{-3}\right)\\ &u^\natural \mapsto \left(u^{P,\natural}, u^{Q,\natural}\right) := \left(Pu^\natural, Qu^\natural\right)\end{aligned}$$

is continuous and invertible with inverse

$$\begin{aligned} &C_{2,\varpi_1,L_1}^{\frac{p}{3}-\text{var}}\left([0,T];\mathbf{H}^{-3}\right) \times C_{2,\varpi_2,L_2}^{\frac{p}{3}-\text{var}}\left([0,T];\mathbf{H}_\perp^{-3}\right) \to C_{2,\varpi,L}^{\frac{p}{3}-\text{var}}\left([0,T];\mathbf{W}^{-3,2}\right)\\ &\left(u^{P,\natural}, u^{Q,\natural}\right) \mapsto u^{P,\natural} + u^{Q,\natural}\end{aligned}$$

where $\varpi := \varpi_1 + \varpi_2$ and $L := L_1 \wedge L_2$. The rest of the proof is straightforward. □

In the remainder of the paper, we use Definition 2.7.

2.6. Main results

Let us now formulate our main results.

THEOREM 2.13. *Let $d \in \{2,3\}$. Assume that for each $k \in \{1,\dots,K\}$, $\sigma_k : \mathbf{T}^d \to \mathbf{R}^d$ and its derivatives up to order two are bounded uniformly and that σ_k is divergence free. For a given $u_0 \in \mathbf{H}^0$ and $\mathbf{Z} \in C_g^{p-\text{var}}([0,T];\mathbf{R}^K)$, there exists a solution of (2.12) in the sense of Definition 2.7 satisfying the energy inequality*

$$\sup_{t\in[0,T]} |u_t|_0^2 + \int_0^T |\nabla u_r|_0^2\,\mathrm{d}r \le |u_0|_0^2.$$

Moreover, $u \in C^{p-\text{var}}([0,T];\mathbf{H}^{-1})$ and $\pi \in C^{p-\text{var}}([0,T];\mathbf{H}_\perp^{-3})$.

The proof of this result is presented in Sect. 4.1 as a consequence of the stronger statement in Theorem 4.1. It proceeds in two steps: first (see Sect. 4.1.1), the velocity field is constructed using compactness as a limit of suitable Galerkin approximations combined with an approximation of the driving signal z by smooth paths. Second, the pressure is recovered (see Sect. 4.1.2).

For two space dimensions and constant vector fields, we prove that the solution (u, π) is unique as a consequence of the stronger statement Theorem 4.3. In Sect. 4.2, we prove uniqueness via a tensorization argument, which allows us to estimate the difference of two solutions by the difference of their initial conditions. We remark that one cannot directly use the techniques from [10], since this way of approximating the Dirac-delta violates the divergence-free condition.

THEOREM 2.14. *If $d = 2$ and σ_k is constant function of $x \in \mathbf{T}^d$ for all $k \in \{1, \dots, K\}$, then for a given $u_0 \in \mathbf{H}^0$ and $\mathbf{Z} \in C_g^{p-\text{var}}([0,T]; \mathbf{R}^K)$, there exists a unique solution of (2.12). Moreover, $u \in C_T\mathbf{H}^0$, $\pi \in C^{p-\text{var}}([0,T]; \mathbf{H}_\perp^{-1})$, and*

$$\sup_{t\in[0,T]} |u_t|_0^2 + 2\nu \int_0^T |\nabla u_r|_0^2 \,\mathrm{d}r = |u_0|_0^2.$$

Owing to Theorem 2.14, in dimension two, there exists a solution map Γ that maps every initial condition $u_0 \in \mathbf{H}^0$, family of constant vector fields σ_k, $k \in \{1, \dots, K\}$, and continuous geometric p-rough path $\mathbf{Z} = (Z, \mathbb{Z})$ to a unique solution (u, π) of (2.12). The following stability result is proved in Sect. 4.3.

COROLLARY 2.15. *In dimension two and for constant vector fields σ_k, $k \in \{1, \dots, K\}$, the solution map*

$$\Gamma : \mathbf{H}^0 \times \mathbf{R}^{2\times K} \times C_g^{p-\text{var}}\left([0,T]; \mathbf{R}^K\right) \to L_T^2\mathbf{H}^0 \cap C_T\mathbf{H}_w^0 \times C^{1-\text{var}}\left([0,T]; \mathbf{H}_\perp^{-2}\right)$$
$$(u_0, \sigma, \mathbf{Z}) \mapsto (u, \pi)$$

is continuous.

REMARK 2.16. It is tempting to try to rewrite (1.1) using a flow transformation by following the ideas in [3,32,33]. More specifically, suppose that there is sufficiently regular invertible map $\varphi : [0,T] \times \mathbf{T}^d \to \mathbf{T}^d$ such that

$$\dot{\varphi}_t(x) = \dot{a}_t(\varphi_t(x)), \quad \varphi_0(x) = 0,$$

and let us define $v_t(x) := u_t(\varphi_t(x))$. Differentiating in time, we find

$$\begin{aligned}\partial_t v_t(x) &= \partial_t u_t(\varphi_t(x)) + \dot{a}_t(\varphi_t(x)) \cdot \nabla u_t(\varphi_t(x)) \\ &= \nu \Delta u_t(\varphi_t(x)) - u_t(\varphi_t(x)) \cdot \nabla u_t(\varphi_t(x)) - \nabla p_t(\varphi_t(x)),\end{aligned}$$

which could be rewritten in terms of v using $\nabla v_t(x) = \nabla u_t(\varphi_t(x))\nabla\varphi_t(x)$ provided $\varphi_t(\cdot)$ is a diffeomorphism. If we assume all the driving vector fields are divergence free, then we have $\det(\nabla\varphi_t(x)) = 1$ so that the equation for v is a Navier–Stokes-type equation, including coefficients from a unimodular matrix depending on t and x. This could account for further difficulties, but it seems plausible that one can solve such an equation. The added value of the construction we present in this paper is that it allows for an intrinsic notion of solution to (1.1) and estimates of the corresponding rough integral.

REMARK 2.17. In three dimensions, it is known that the Stratonovich Navier–Stokes equation

$$du + (u \cdot \nabla)u\, dt + \nabla p = \nu \Delta u\, dt + \nabla u \circ dw$$

has a probabilistically weak solution (see, e.g., [19,21,22]). Nevertheless, whether the solution probabilistically strong is still an open question. In other words, it is not known whether the solution to the above equation is adapted to the filtration generated by the Wiener process w. Even though a prime example of a driving rough path in our equation is a Wiener process with its Stratonovich lift and solving rough PDEs corresponds to a non-probabilistic (pathwise) construction of solutions, we still can not answer this question at this point. The reader should note that using the compactness criterion Lemma A.2, we obtain a subsequence of the approximate solutions that a priori depends the randomness variable ω (not a control). The question whether the full sequence converges is very difficult to answer, as it is intimately related to the issue of uniqueness. Indeed, if uniqueness held true in three dimensions, then every subsequence of $\{u^N\}_{N=1}^{\infty}$ would converge to the same limit, and hence the full sequence would converge. As a consequence, the proof of stability in Corollary 2.15 would imply that the solution (u, π) depends continuously on the given data $(u_0, \sigma, \mathbf{Z})$ and is thus adapted to the filtration generated by the Brownian motion.

3. A priori estimates of remainders

In this section, we derive a priori estimates of the remainder terms $u^{P,\natural}$ and $u^{Q,\natural}$ and $|u|_{p-\mathrm{var};[0,T];\mathbf{H}^{-1}}$. Let (u, π) be a solution of (2.13) in the sense of Definition 2.7. For $t \in [0, T]$, we let

$$\mu_t(\phi) = -\int_0^t [\nu(\nabla u_r, \nabla \phi) + B_P(u_r)(\phi)]\, \mathrm{d}r, \quad \phi \in \mathbf{H}^1.$$

It follows that for $(s, t) \in \Delta_T$,

$$\delta u_{st} = \delta \mu_{st} + A_{st}^{P,1} u_s + A_{st}^{P,2} u_s + u_{st}^{P,\natural}, \tag{3.1}$$

where the equality holds in $\mathbf{H}^{-3}$. For all $(s, t) \in \Delta_T$, let

$$\omega_\mu(s, t) = \int_s^t (1 + |u_r|_1)^2\, \mathrm{d}r,$$

where we recall that $|\cdot|_1$ denotes the $\mathbf{H}^1$-norm. Since $u \in L_T^2 \mathbf{H}^1$, ω_μ is a control. Using (2.4) with $\alpha_1 = \alpha_3 = 1$ and $\alpha_2 = 0$, we obtain $|B_P(u_r)|_{-1} \lesssim |u_r|_1^2$, and hence $|\delta \mu_{st}|_{-1} \lesssim \omega_\mu(s, t)$.

We begin with an important lemma which provides an estimate of $u^{P,\natural}$ in terms of the given data. The following result is a special case of [10, Theorem 2.5], but we include a proof for the readers convenience. Let us define the map

$$u_{st}^{\sharp} := \delta u_{st} - A_{st}^{P,1} u_s = \delta \mu_{st} + A_{st}^{P,2} u_s + u_{st}^{P,\natural}. \tag{3.2}$$

The first expression for $u^{\sharp}_{st}$ consists of terms that are less regular in time and more regular in space than the second expression for $u^{\sharp}_{st}$. We use this fact along with the smoothing operators and the sewing lemma (B.1) to estimate the remainder terms.

LEMMA 3.1. *Assume that* (u,π) *solves* (2.12) *according to Definition* 2.7. *For* $(s,t)\in\Delta_T$ *such that* $\varpi(s,t)\le L$, *let* $\omega_{P,\natural}(s,t):=|u^{P,\natural}|^{\frac{p}{3}}_{\frac{p}{3}-var;[s,t];\mathbf{H}^{-3}}$. *Then there is a constant* $\tilde{L}>0$, *depending only on* p *and* d, *such that for all* $(s,t)\in\Delta_T$ *with* $\varpi(s,t)\le L$ *and* $\omega_A(s,t)\le\tilde{L}$,

$$\omega_{P,\natural}(s,t)\lesssim_p |u|^{\frac{p}{3}}_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)+\omega_\mu(s,t)^{\frac{p}{3}}\left(\omega_A(s,t)^{\frac{1}{3}}+\omega_A(s,t)^{\frac{2}{3}}\right) \tag{3.3}$$

and

$$\omega_{P,\natural}(s,t)\lesssim_p |u|^{\frac{p}{3}}_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)+\left(1+|u|_{L^\infty_T\mathbf{H}^0}\right)^{\frac{2p}{3}}(t-s)^{\frac{p}{3}}\,\omega_A(s,t)^{\frac{1}{12}}. \tag{3.4}$$

Proof. Recall that the second-order increment operator δ is defined on two index maps $g:\Delta^{(2)}_T\to\mathbf{R}$ by $\delta g_{s\theta t}:=g_{st}-g_{\theta t}-g_{s\theta}$ for all $(s,\theta,t)\in\Delta^{(2)}_T$. It is easy to see that for a one-index map f, we have $\delta(\delta f)_{s\theta t}=0$. Applying δ to (2.18), we find that for all $\phi\in\mathbf{H}^3$ and $(s,\theta,t)\in\Delta^{(2)}_T$,

$$\delta u^{P,\natural}_{s\theta t}(\phi)=\delta u_{s\theta}\left(A^{P,2,*}_{\theta t}\phi\right)+u^{\sharp}_{s\theta}\left(A^{P,1,*}_{\theta t}\phi\right),$$

where $u^{\sharp}_{s\theta}$ is defined in (3.2). We decompose $\delta u^{P,\natural}_{s\theta t}(\phi)$ into a smooth (in space) and non-smooth part using the smoothing operator J^η to get

$$u^{P,\natural}_{s\theta t}(\phi)=\delta u^{P,\natural}_{s\theta t}\left(J^\eta\phi\right)+\delta u^{P,\natural}_{s\theta t}\left(\left(I-J^\eta\right)\phi\right),$$

for some $\eta\in(0,1]$ that will be specified later. We will now proceed to analyze term by term. To estimate the non-smooth part, we use (2.7) and that $u^{\sharp}_{s\theta}=\delta u_{s\theta}-A^{P,1}_{s\theta}u_s$ to obtain

$$\begin{aligned}\left|\delta u^{P,\natural}_{s\theta t}((I-J^\eta)\phi)\right| &\le |u|_{L^\infty_T\mathbf{H}^0}\left(\left|A^{P,1*}_{\theta t}((I-J^\eta)\phi)\right|_0+\left|A^{P,1,*}_{s\theta}A^{P,1*}_{\theta t}((I-J^\eta)\phi)\right|_0\right.\\ &\quad\left.+\left|A^{2*}_{\theta t}((I-J^\eta)\phi)\right|_0\right)\\ &\lesssim |u|_{L^\infty_T\mathbf{H}^0}\left(\omega_A(s,t)^{\frac{1}{p}}|(I-J^\eta)\phi|_1+\omega_A(s,t)^{\frac{2}{p}}|(I-J^\eta)\phi|_2\right)\\ &\lesssim |u|_{L^\infty_T\mathbf{H}^0}\left(\omega_A(s,t)^{\frac{1}{p}}\eta^2+\omega_A(s,t)^{\frac{2}{p}}\eta\right)|\phi|_3.\end{aligned}$$

In order to estimate the smooth part, we use the form $u^{\sharp}_{st}=\delta\mu_{s\theta}+A^{P,2}_{s\theta}u_s+u^{P,\natural}_{s\theta}$ to get

$$\begin{aligned}\delta u^{P,\natural}_{s\theta t}\left(J^\eta\phi\right) &= \delta\mu_{s\theta}\left(A^{P,1,*}_{\theta t}J^\eta\phi\right) + u_s\left(A^{P,2,*}_{s\theta}A^{P,1,*}_{\theta t}J^\eta\phi\right) + u^{P,\natural}_{s\theta}\left(A^{P,1,*}_{\theta t}J^\eta\phi\right)\\ &\quad + \delta\mu_{s\theta}\left(A^{P,2,*}_{\theta t}J^\eta\phi\right) + u_s\left(A^{P,1,*}_{s\theta}A^{P,2,*}_{\theta t}J^\eta\phi\right) + u_s\left(A^{P,2,*}_{s\theta}A^{P,2,*}_{\theta t}J^\eta\phi\right)\\ &\quad + u^{P,\natural}_{s\theta}\left(A^{P,2,*}_{\theta t}J^\eta\phi\right),\end{aligned}$$

Estimating each term and using (2.7), for all $(s,\theta,t)\in\Delta^{(2)}_T$ such that $\varpi(s,t)\le L$, we find

$$\begin{aligned}|\delta u^{P,\natural}_{s\theta t}(J^\eta\phi)| &\lesssim \omega_\mu(s,t)\omega_A(s,t)^{\frac{1}{p}}|J^\eta\phi|_2 + |u|_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}}|J^\eta\phi|_3\\ &\quad + \omega_{P,\natural}(s,t)^{\frac{3}{p}}\omega_A(s,t)^{\frac{1}{p}}|J^\eta\phi|_4\\ &\quad + \omega_\mu(s,t)\omega_A(s,t)^{\frac{2}{p}}|J^\eta\phi|_3 + |u|_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}}|J^\eta\phi|_3\\ &\quad + |u|_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)^{\frac{4}{p}}|J^\eta\phi|_4\\ &\quad + \omega_{P,\natural}(s,t)^{\frac{3}{p}}\omega_A(s,t)^{\frac{2}{p}}|J^\eta\phi|_5\\ &\lesssim \Big(\omega_\mu(s,t)\omega_A(s,t)^{\frac{1}{p}} + |u|_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}} + \omega_{P,\natural}(s,t)^{\frac{3}{p}}\omega_A(s,t)^{\frac{1}{p}}\eta^{-1}\\ &\quad + \omega_\mu(s,t)\omega_A(s,t)^{\frac{2}{p}} + |u|_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}} + |u|_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)^{\frac{4}{p}}\eta^{-1}\\ &\quad + \omega_{P,\natural}(s,t)^{\frac{3}{p}}\omega_A(s,t)^{\frac{2}{p}}\eta^{-2}\Big)|\phi|_3. \qquad (3.5)\end{aligned}$$

Setting $\eta = \omega_A(s,t)^{\frac{1}{p}}\lambda$ for some constant $\lambda > 0$ to be determined later, we have

$$\begin{aligned}|\delta u^{P,\natural}_{s\theta t}|_{-3} &\lesssim |u|_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}}(\lambda^{-1}+1+\lambda+\lambda^2) + \omega_\mu(s,t)\omega_A(s,t)^{\frac{1}{p}}\\ &\quad + \omega_\mu(s,t)\omega_A(s,t)^{\frac{2}{p}} + \omega_{P,\natural}(s,t)^{\frac{3}{p}}(\lambda^{-1}+\lambda^{-2})\\ &\lesssim_p \Big(|u|^{\frac{p}{3}}_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)(\lambda^{-1}+1+\lambda+\lambda^2)^{\frac{p}{3}} + \omega_\mu(s,t)^{\frac{p}{3}}\omega_A(s,t)^{\frac{1}{3}}\\ &\quad + \omega_\mu(s,t)^{\frac{p}{3}}\omega_A(s,t)^{\frac{2}{3}} + \omega_{P,\natural}(s,t)(\lambda^{-1}+\lambda^{-2})^{\frac{p}{3}}\Big)^{\frac{3}{p}}.\end{aligned}$$

Applying Lemma B.1, we get

$$\begin{aligned}|u^{P,\natural}_{st}|^{\frac{p}{3}}_{-3} &\lesssim_p |u|^{\frac{p}{3}}_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)\left(\lambda^{-1}+1+\lambda+\lambda^2\right)^{\frac{p}{3}} + \omega_\mu(s,t)^{\frac{p}{3}}\omega_A(s,t)^{\frac{1}{3}}.\\ &\quad + \omega_\mu(s,t)^{\frac{p}{3}}\omega_A(s,t)^{\frac{2}{3}} + \omega_{P,\natural}(s,t)\left(\lambda^{-1}+\lambda^{-2}\right)^{\frac{p}{3}}.\end{aligned}$$

Since $\omega_{P,\natural} = |u^{P,\natural}|^{\frac{p}{3}}_{\frac{p}{3}-var;[s,t];\mathbf{H}^{-3}}$ is equal to the infimum over all controls satisfying $|u^{P,\natural}_{st}|_{-3}\le\omega_{P,\natural}(s,t)^{\frac{3}{p}}$ (see (2.7)), there is a constant $C = C(p,d)$ such that

$$\omega_{P,\natural}(s,t) \le C\left(|u|_{L_T^\infty \mathbf{H}^0}^{\frac{p}{3}}\omega_A(s,t)(\lambda^{-1}+1+\lambda+\lambda^2)^{\frac{p}{3}} + \omega_\mu(s,t)^{\frac{p}{3}}\omega_A(s,t)^{\frac{1}{3}} \right.$$
$$\left. + \omega_\mu(s,t)^{\frac{p}{3}}\omega_A(s,t)^{\frac{2}{3}} + \omega_{P,\natural}(s,t)(\lambda^{-1}+\lambda^{-2})^{\frac{p}{3}}\right).$$

Choosing λ such that $C(\lambda^{-1}+\lambda^{-2})^{\frac{p}{3}} \le \frac{1}{2}$ and $\tilde{L} > 0$ such that $\eta = \omega_A(s,t)^{\frac{1}{p}}\lambda \le \tilde{L}\lambda \le 1$, we obtain (3.3).

The proof of (3.4) replaces the bound $\delta\mu_{st}(\phi) \lesssim \omega_\mu(s,t)|\phi|_1$ with the bound

$$|\delta\mu_{st}(\phi)| \le \int_s^t (\nu|(u_r, \Delta\phi)| + |B_P(u_r)(\phi)|)\,\mathrm{d}r \lesssim \int_s^t \left(|u_r|_0|\phi|_2 + |u_r|_0^2|\phi|_{3-\epsilon}\right)\mathrm{d}r$$
$$\lesssim (t-s)(1+|u|_{L_T^\infty \mathbf{H}^0})^2|\phi|_{3-\epsilon},$$

where we have used the antisymmetric property of B_P and (2.4) with $\alpha_1 = \alpha_3 = 0$ and $\alpha_2 = 3-\epsilon$ for any $\epsilon < \frac{1}{2}$. We note that this is only possible when $d \le 3$. The rest of the proof is similar to the proof of (3.3). Indeed, in (3.5), the term $\omega_\mu(s,t)\omega_A(s,t)^{\frac{1}{p}}$ is replaced with $(1+|u|_{L_T^\infty \mathbf{H}^0})^2(t-s)\omega_A(s,t)^{\frac{1}{p}}\eta^{-1+\epsilon}$ and the term $\omega_\mu(s,t)\omega_A(s,t)^{\frac{2}{p}}$ is replaced with $(1+|u|_{L_T^\infty \mathbf{H}^0})^2(t-s)\omega_A(s,t)^{\frac{2}{p}}\eta^{-2+\epsilon}$. Moreover, we still take $\eta = \omega_A(s,t)^{\frac{1}{p}}\lambda$ and for simplicity let $\epsilon = \frac{1}{4}$. □

REMARK 3.2. We use the estimate (3.4) in the proof of existence, since it is allows us to obtain a bound independent of the Galerkin approximation.

LEMMA 3.3. *Assume that (u,π) is a solution to (2.12). Then $u \in C^{p-\mathrm{var}}([0,T];\mathbf{H}^{-1})$ and there is a constant $\tilde{L} > 0$, depending only on p and d, such that for all $(s,t) \in \Delta_T$ with $\varpi(s,t) \le L$, $\omega_A(s,t) \le \tilde{L}$, and $\omega_{P,\natural}(s,t) \le \tilde{L}$, it holds that*

$$\omega_u(s,t) \lesssim_p (1+|u|_{L_T^\infty \mathbf{H}^0})^p(\omega_{P,\natural}(s,t) + \omega_\mu(s,t)^p + \omega_A(s,t)),$$

where $\omega_u(s,t) := |u|^p_{p-\mathrm{var};[s,t];\mathbf{H}^{-1}}$.

Proof. For all $\eta \in (0,1]$, $(s,t) \in \Delta_T$ and $\phi \in \mathbf{H}^1$, we have

$$\delta u_{st}(\phi) = \delta u_{st}(J^\eta\phi) + \delta u_{st}((I-J^\eta)\phi).$$

Applying (2.7), we find

$$|\delta u_{st}((I-J^\eta)\phi)| \le 2|u|_{L_T^\infty \mathbf{H}^0}|(I-J^\eta)\phi|_0 \lesssim \eta|u|_{L_T^\infty \mathbf{H}^0}|\phi|_1.$$

In order to estimate the smooth part, we expand δu_{st} using (3.1) and then apply (2.7) to get

$$|\delta u_{st}(J^\eta\phi)| \le |u_{st}^{P,\natural}(J^\eta\phi)| + |\delta\mu_{st}(J^\eta\phi)| + |u_s(A_{st}^{P,1,*}J^\eta\phi)| + |u_s(A_{st}^{P,2,*}J^\eta\phi)|$$
$$\lesssim \omega_{P,\natural}(s,t)^{\frac{3}{p}}|J^\eta\phi|_3 + \omega_\mu(s,t)|J^\eta\phi|_1 + |u|_{L_T^\infty \mathbf{H}^0}\omega_A(s,t)^{\frac{1}{p}}|J^\eta\phi|_1$$

$$
\begin{aligned}
&+ |u|_{L_T^\infty \mathbf{H}^0} \omega_A(s,t)^{\frac{2}{p}} |J^\eta \phi|_2 \\
&\lesssim \Big(\omega_{P,\natural}(s,t)^{\frac{3}{p}} \eta^{-2} + \omega_\mu(s,t) + |u|_{L_T^\infty \mathbf{H}^0} \omega_A(s,t)^{\frac{1}{p}} \\
&\quad + |u|_{L_T^\infty \mathbf{H}^0} \omega_A(s,t)^{\frac{2}{p}} \eta^{-1} \Big) |\phi|_1,
\end{aligned}
$$

for all $(s,t) \in \Delta_T$ such that $\varpi(s,t) \le L$. Setting $\eta = \omega_{P,\natural}(s,t)^{\frac{1}{p}} + \omega_A(s,t)^{\frac{1}{p}}$ and choosing $\tilde{L} > 0$ such that $\eta \in (0,1]$, we get

$$
|\delta u_{st}|_{-1} \lesssim_p \left(1 + |u|_{L_T^\infty \mathbf{H}^0}\right) \left(\omega_{P,\natural}(s,t) + \omega_\mu(s,t)^p + \omega_A(s,t)\right)^{\frac{1}{p}},
$$

which proves the claim. □

The following lemma shows that the solution u is controlled by $A^{P,1}$.

LEMMA 3.4. *Assume that (u,π) is a solution of (2.12). Then $u^\sharp \in C_2^{\frac{p}{2}-\mathrm{var}}([0,T]; \mathbf{H}^{-2})$ and there is a constant $\tilde{L} > 0$, depending only on p and d, such that for all $(s,t) \in \Delta_T$ with $\varpi(s,t) \le L$, $\omega_A(s,t) \le \tilde{L}$, and $\omega_{P,\natural}(s,t) \le \tilde{L}$, it holds that*

$$
\omega_\sharp(s,t) \lesssim_p \left(1 + |u|_{L_T^\infty \mathbf{H}^0}\right)^{\frac{p}{2}} \left(\omega_{P,\natural}(s,t) + \omega_\mu(s,t)^{\frac{p}{2}} + \omega_A(s,t)\right),
$$

where $\omega_\sharp(s,t) := |u^\sharp|^{\frac{p}{2}}_{\frac{p}{2}-\mathrm{var};[s,t];\mathbf{H}^{-2}}$.

Proof. For all $\eta \in (0,1]$, $(s,t) \in \Delta_T$ and $\phi \in \mathbf{H}^2$, we have

$$
u^\sharp_{st}(\phi) = u^\sharp_{st}(J^\eta \phi) + u^\sharp_{st}\left((I - J^\eta)\phi\right).
$$

We recall that we have two formulas for $u^\sharp$:

$$
u^\sharp_{st} = \delta u_{st} - A^{P,1}_{st} u_s = \delta \mu_{st} + A^{P,2}_{st} u_s + u^{P,\natural}_{st}.
$$

As explained above, we employ the first formula to estimate the non-smooth part and the second one to estimate the smooth part. Applying (2.7), we find

$$
\begin{aligned}
|u^{\sharp}_{st}\left(\left(I-J^{\eta}\right)\phi\right)| &\leq |\delta u_{st}\left(\left(I-J^{\eta}\right)\phi\right)| + |u_s\left(A^{P,1,*}_{st}\left(I-J^{\eta}\right)\phi\right)| \\
&\leq |u|_{L^\infty_T\mathbf{H}^0}|\left(I-J^{\eta}\right)\phi|_0 + |u|_{L^\infty_T\mathbf{H}^0}\omega_A\left(s,t\right)^{\frac{1}{p}}|\left(I-J^{\eta}\right)\phi|_1 \\
&\lesssim \left(\eta^2|u|_{L^\infty_T\mathbf{H}^0} + \eta|u|_{L^\infty_T\mathbf{H}^0}\omega_A\left(s,t\right)^{\frac{1}{p}}\right)|\phi|_2.
\end{aligned}
$$

In order to estimate the non-smooth part, we apply (2.7) to obtain

$$
\begin{aligned}
|u^{\sharp}_{st}\left(J^{\eta}\phi\right)| &\leq |u^{P,\natural}_{st}\left(J^{\eta}\phi\right)| + |\delta\mu_{st}\left(J^{\eta}\phi\right)| + |u_s\left(A^{P,2,*}_{st}J^{\eta}\phi\right)| \\
&\lesssim \omega_{P,\natural}\left(s,t\right)^{\frac{3}{p}}|J^{\eta}\phi|_3 + \omega_{\mu}\left(s,t\right)|J^{\eta}\phi|_1 + |u|_{L^\infty_T\mathbf{H}^0}\omega_A\left(s,t\right)^{\frac{2}{p}}|J^{\eta}\phi|_2 \\
&\leq \left(\omega_{P,\natural}\left(s,t\right)^{\frac{3}{p}}\eta^{-1} + \omega_{\mu}\left(s,t\right) + |u|_{L^\infty_T\mathbf{H}^0}\omega_A\left(s,t\right)^{\frac{2}{p}}\right)|\phi|_2,
\end{aligned}
$$

for all $(s,t)\in\Delta_T$ with $\varpi(s,t)\leq L$. Setting $\eta=\omega_{P,\natural}(s,t)^{\frac{1}{p}}+\omega_A(s,t)^{\frac{1}{p}}$ and choosing $\tilde{L}>0$ such that $\eta\in(0,1]$, we find

$$
|u^{\sharp}_{st}|_{-2} \lesssim_p (1+|u|_{L^\infty_T\mathbf{H}^0})\left(\omega_{P,\natural}(s,t)+\omega_{\mu}(s,t)^{\frac{p}{2}}+\omega_A(s,t)\right)^{\frac{2}{p}},
$$

which proves the claim. □

We now derive estimates for $\omega_{Q,\natural}$. The computation in the proof of the lemma show why (2.17) is the correct Chen's relation for this system.

LEMMA 3.5. *Assume that* (u,π) *solves* (2.12)*. For* $(s,t)\in\Delta_T$ *such that* $\varpi(s,t)\leq L$*, let* $\omega_{Q,\natural}(s,t):=|u^{Q,\natural}|^{\frac{p}{3}}_{\frac{p}{3}-var;[s,t];\mathbf{H}^{-3}_{\perp}}$*. Then there is a constant* $\tilde{L}>0$*, depending only on* p *and* d*, such that for all* $(s,t)\in\Delta_T$ *with* $\varpi(s,t)\leq L$ *and* $\omega_A(s,t)\leq\tilde{L}$*,*

$$
\omega_{Q,\natural}\left(s,t\right) \lesssim_p |u|^{\frac{p}{3}}_{L^\infty_T\mathbf{H}^0}\omega_A\left(s,t\right)+\omega_{\mu}\left(s,t\right)^{\frac{p}{3}}\omega_A\left(s,t\right)^{\frac{1}{3}}+\omega_{P,\natural}\left(s,t\right)+\omega_u\left(s,t\right)^{\frac{1}{3}}\omega_A\left(s,t\right)^{\frac{2}{3}}. \tag{3.6}
$$

Proof. Applying δ to (2.18), we find that for all $\psi\in\mathbf{H}^3_{\perp}$ and $(s,\theta,t)\in\Delta^{(2)}_T$,

$$
\begin{aligned}
\delta u^{Q,\natural}_{s\theta t}\left(\psi\right) &= u^{Q,\natural}_{st}\left(\psi\right) - u^{Q,\natural}_{s\theta}\left(\psi\right) - u^{Q,\natural}_{\theta t}\left(\psi\right) \\
&= \delta u_{s\theta}\left(A^{Q,1,*}_{\theta t}\psi\right) + \delta u_{s\theta}\left(A^{Q,2,*}_{\theta t}\psi\right) - u_s\left(A^{P,1,*}_{\theta t}A^{Q,1,*}_{\theta t}\psi\right) \\
&= u^{\sharp}_{s\theta}\left(A^{Q,1,*}_{\theta t}\psi\right) + \delta u_{s\theta}\left(A^{Q,2,*}_{\theta t}\psi\right),
\end{aligned}
$$

where we have used (2.17) in the second equality. Using Lemma 3.3, it is easy to see that the last term satisfies (3.6), so we focus on the first term.

As usual, we split the equality into smooth and non-smooth parts $\psi=J^{\eta}\psi+(I-J^{\eta})\psi$ for $\eta\in(0,1]$ to be determined later. In order to estimate the non-smooth part, we use $u^{\natural}_{s\theta}=\delta u_{s\theta}-A^{P,1}_{s\theta}u_s$ and (2.7) to obtain

$$\begin{aligned}
&\left(\delta u_{s\theta} - A^{P,1}_{s\theta} u_s\right)\left(\left(I - J^{\eta}\right)\psi\right)\\
&= \delta u_{s\theta}\left(A^{Q,1,*}_{\theta t}\left(I - J^{\eta}\right)\psi\right) - u_s\left(A^{P,1,*}_{s\theta} A^{Q,1,*}_{\theta t}\left(I - J^{\eta}\right)\psi\right)\\
&\le 2|u|_{L^\infty_T \mathbf{H}^0}\omega_A(s,t)^{\frac{1}{p}}\left|\left(I - J^{\eta}\right)\psi\right|_1 + |u|_{L^\infty_T \mathbf{H}^0}\omega_A(s,t)^{\frac{2}{p}}\left|\left(I - J^{\eta}\right)\psi\right|_2\\
&\lesssim |u|_{L^\infty_T \mathbf{H}^0}\left(\omega_A(s,t)^{\frac{1}{p}}\eta^2 + \omega_A(s,t)^{\frac{2}{p}}\eta\right)|\psi|_3.
\end{aligned}$$

To estimate the smooth part, we write $u^\sharp_{st} = \delta\mu_{s\theta} + A^{P,2}_{s\theta} u_s + u^{P,\natural}_{s\theta}$ and apply (2.7) to get

$$\begin{aligned}
&\left(\delta u_{s\theta} - A^{P,1}_{s\theta} u_s\right)\left(J^{\eta}\psi\right)\\
&= \delta\mu_{s\theta}\left(A^{Q,1,*}_{\theta t} J^{\eta}\psi\right) + u_s\left(A^{P,2,*}_{s\theta} A^{Q,1,*}_{\theta t} J^{\eta}\psi\right) + u^{P,\natural}_{s\theta}\left(A^{Q,1,*}_{\theta t} J^{\eta}\psi\right)\\
&\lesssim \omega_\mu(s,t)\,\omega_A(s,t)^{\frac{1}{p}}\,|J^{\eta}\psi|_2 + |u|_{L^\infty_T \mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}}\,|J^{\eta}\psi|_3\\
&\quad + \omega_{P,\natural}(s,t)^{\frac{3}{p}}\,\omega_A(s,t)^{\frac{1}{p}}\,|J^{\eta}\psi|_4\\
&\lesssim \left(\omega_\mu(s,t)\,\omega_A(s,t)^{\frac{1}{p}} + |u|_{L^\infty_T \mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}} + \omega_{P,\natural}(s,t)^{\frac{3}{p}}\,\omega_A(s,t)^{\frac{1}{p}}\,\eta^{-1}\right)|\psi|_3.
\end{aligned}$$

Setting $\eta = \omega_A(s,t)^{\frac{1}{p}}$ and choosing $\tilde{L}$ such that $\eta \in (0,1]$, we get

$$\begin{aligned}
|\delta u^{Q,\natural}_{s\theta t}|_{-3} &\lesssim \Big(|u|_{L^\infty_T \mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}} + \omega_\mu(s,t)\omega_A(s,t)^{\frac{1}{p}} + \omega_{P,\natural}(s,t)^{\frac{3}{p}}\\
&\qquad + \omega_u(s,t)^{\frac{1}{p}}\omega_A(s,t)^{\frac{2}{p}}\Big)\\
&\lesssim_p \Big(|u|^{\frac{p}{3}}_{L^\infty_T \mathbf{H}^0}\omega_A(s,t) + \omega_\mu(s,t)^{\frac{p}{3}}\omega_A(s,t)^{\frac{1}{3}} + \omega_{P,\natural}(s,t)\\
&\qquad + \omega_u(s,t)^{\frac{1}{3}}\omega_A(s,t)^{\frac{2}{3}}\Big)^{\frac{3}{p}}.
\end{aligned}$$

Using Lemma B.1, we obtain the first inequality. The proof of the second inequality is similar to the first; see the end of proof of Lemma 3.1. □

By virtue of Lemma 3.5 and (2.19), we see immediately that $\pi \in C^{p-var}([0,T]; \mathbf{H}^{-3}_{\perp})$, although we conjecture that there is better spatial regularity.

4. Proof of the main results

4.1. Existence, proof of Theorem 2.13

4.1.1. Galerkin approximation

We prove the existence of a solution using a Galerkin approximation. Let $\{h_n\}_{n=1}^{\infty}$ be the smooth orthonormal basis of $\mathbf{H}^0$ discussed in Sect. 2.1. Recall that there exist a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of nonnegative numbers such that $-\Delta h_n = \lambda_n h_n$, for all $n \in \mathbf{N}$. For a given $N \in \mathbf{N}$, let $\mathbf{H}_N = \mathrm{span}(\{h_1, \dots, h_N\})$ and $P_N : \mathbf{H}^{-1} \to \mathbf{H}_N$ be defined by

$$P_N v := \sum_{n=1}^{N} (v, h_n) h_n, \quad v \in \mathbf{H}^{-1}.$$

Since $\mathbf{Z} \in C_g^{p-\mathrm{var}}([0,T]; \mathbf{R}^K)$ is a geometric rough path, there is a sequence of $\mathbf{R}^K$-valued smooth paths $\{z^N\}_{N=1}^{\infty}$ such that their canonical lifts $\mathbf{Z}^N = (Z^N, \mathbb{Z}^N)$ converge to $\mathbf{Z}$ in the rough path topology. We assume that

$$|Z_{st}^N| \lesssim \omega_Z(s,t)^{\frac{1}{p}}, \quad |\mathbb{Z}_{st}^N| \lesssim \omega_Z(s,t)^{\frac{2}{p}}, \quad \forall (s,t) \in \Delta_T. \tag{4.1}$$

For convenience, let N_0 denote a constant that bounds $\sigma = (\sigma_1, \dots, \sigma_K)$ and its derivatives up to order two.

We consider the following Nth order Galerkin approximations of (2.12):

$$\partial_t u^N + P_N B_P\left(u^N\right) = \nu P_N \Delta u^N + \sum_{k=1}^{K} P_N P\left[(\sigma_k \cdot \nabla) u^N\right] \dot{z}_t^{N,k}, \tag{4.2}$$

where $u^N(0) = P_N u_0$. If we assume that

$$u_t^N(x) = \sum_{n=1}^{N} c_n^N(t) h_n(x),$$

then plugging in this expansion for $u^N(t,x)$ into (4.2) and testing against h_n we derive an ODE for the coefficients $(c_n^N)_{n=1}^N$:

$$\dot{c}_n^N(t) + \sum_{j,l=1}^{N} B_{j,l,n} c_j^N(t) c_l^N(t) = \nu \lambda_n c_n^N(t) + \sum_{k=1}^{K} \sum_{j=1}^{N} A_{k,j,n} c_j^N(t) \dot{z}_t^{N,k}, \tag{4.3}$$

where $B_{j,l,i} := P_N B_P(h_j, h_l)(h_n)$ and $A_{k,j,n} := ((\sigma_k \cdot \nabla) h_j)(h_n)$. Owing to (2.4) with $\alpha_1 = 1, \alpha_2 = 0$, and $\alpha_3 = 1$, for all j, l and n, we have

$$|B_{j,l,n}| \le |h_j|_1 |h_l|_1 |h_n|_1.$$

Moreover, for all k, j, and i,

$$|A_{k,j,n}| \le |\sigma_k|_{\mathbf{L}^\infty} |h_j|_1 |h_n|_0.$$

Thus, (4.3) has locally Lipschitz coefficients, and so there exists a unique solution $(c_n)_{n=1}^N$ of (4.3) on a time interval $[0, T_N)$, for some $T_N > 0$. Therefore, $u_t^N(x) = \sum_{n=1}^{N} c_n^N(t) h_n(x)$ is a solution of (4.2) on the time interval $[0, T_N)$.

To get a global solution, we derive a global energy estimate of u^N. Testing (4.2) against u^N and using (2.6), the divergence theorem, and that $\nabla \cdot \sigma_k = 0$, for all $k \in \{1, \dots, K\}$, we get

$$\begin{aligned} |u_t^N|_0^2 + 2\nu \int_0^t |\nabla u_s^N|_0^2 \, \mathrm{d}s &= |P_N u_0|_0^2 - 2 \int_0^t P_N B_P \left(u_s^N, u_s^N, u_s^N\right) \mathrm{d}s \\ &\quad + \sum_{k=1}^K \int_0^t \left((\sigma_k \cdot \nabla) u_s^N, u_s^N\right) \dot{z}_s^{N,k} \mathrm{d}s \\ &= |P_N u_0|_0^2 \le |u_0|_0^2, \quad \forall t \in [0, T_N). \end{aligned}$$

It follows that the $\mathbf{L}^2$-norm of u^N is non-increasing in time, and hence that $(c_n)_{n=1}^N$ does not blow up in finite time. Therefore, $u^N \in C_T \mathbf{H}^0 \cap L_T^2 \mathbf{H}^1$ solves (4.2).

Integrating (4.2) over $[s, t]$, and then iterating the equation into the integral against $\dot{z}^N$ as we did in (2.15), we find

$$\delta u_{st}^N = \int_s^t \left(\nu P_N \Delta u_r^N - P_N B_P\left(u_r^N\right)\right) \mathrm{d}r + A_{st}^{N,1} u_s^N + A_{st}^{N,2} u_s^N + u_{st}^{N,\natural}, \tag{4.4}$$

where $\tilde{P}_N := P_N P$,

$$\begin{aligned} A_{st}^{N,1}\phi &:= \tilde{P}_N \left[(\sigma_k \cdot \nabla) \phi\right] Z_{st}^{N,k}, \\ A_{st}^{N,2}\phi &:= \tilde{P}_N \left[(\sigma_k \cdot \nabla) \tilde{P}_N \left[(\sigma_j \cdot \nabla) \phi\right]\right] \mathbb{Z}_{st}^{N,j,k}, \end{aligned}$$

$\mu_t^N := P_N \int_0^t \left(\nu \Delta u_r^N - B_P(u_r^N)\right) \mathrm{d}r$, and

$$\begin{aligned} u_{st}^{N,\natural} &:= \int_s^t \tilde{P}_N \left[(\sigma_k \cdot \nabla) \delta\mu_{sr}^N\right] \dot{z}_r^{N,k} \, \mathrm{d}r \\ &\quad + \tilde{P}_N \int_s^t \int_s^r (\sigma_k \cdot \nabla) \tilde{P}_N \left[(\sigma_i \cdot \nabla) \delta\mu_{sr_1}^N\right] \dot{z}_{r_1}^{N,i} \dot{z}_r^{N,k} \, \mathrm{d}r_1 \, \mathrm{d}r \\ &\quad + \int_s^t \int_s^r \int_s^{r_1} \tilde{P}_N \left[(\sigma_k \cdot \nabla) \tilde{P}_N \left[(\sigma_i \cdot \nabla) \tilde{P}_N \left[(\sigma_j \cdot \nabla) u_{r_2}^N\right]\right]\right] \dot{z}_{r_2}^{N,j} \dot{z}_{r_1}^{N,i} \dot{z}_r^{N,k} \, \mathrm{d}r_2 \, \mathrm{d}r_1 \, \mathrm{d}r, \end{aligned}$$

Owing to (2.2), (2.3), and (4.1), we have that $(A^{N,1}, A^{N,2})$ is uniformly bounded in N as a family of unbounded rough drivers on the scale $(\mathbf{H}^\alpha)_{\alpha \in \mathbf{R}_+}$. That is, there exists a control ω_{A^N} such that (2.10) holds and for all $(s, t) \in \Delta_T$,

$$\omega_{A^N}(s, t) \lesssim_{N_0} \omega_Z(s, t).$$

It is straightforward to check that $u^{N,\natural} \in C_2^{\frac{p}{3}-\text{var}}([0, T]; H_N)$ by estimating term by term; one makes use of (2.2), (2.3), (2.4), and that u^N is smooth in space and z^N is smooth in time. For all $(s, t) \in \Delta_T$, let $\omega_{N,\natural}(s, t) := |u^{N,\natural}|_{\frac{p}{3}-\text{var};[s,t];\mathbf{H}^{-3}}^{\frac{p}{3}}$. Arguing as in Lemma 3.1, we find that there is an $L > 0$ such that for all $(s, t) \in \Delta_T$ with $\omega_Z(s, t) \le L$,

$$\omega_{N,\natural}(s,t) \lesssim_p |u^N|_{L_T^\infty H_N}^{\frac{p}{3}} \omega_{A^N}(s,t) + \left(1+|u^N|_{L_T^\infty H_N}\right)^{\frac{2p}{3}} (t-s)^{\frac{p}{3}} \omega_{A^N}(s,t)^{\frac{1}{12}}$$
$$\lesssim_{p,N_0} |u_0|_0^{\frac{p}{3}} \omega_Z(s,t) + (1+|u_0|_0)^{\frac{2p}{3}} (t-s)^{\frac{p}{3}}. \tag{4.5}$$

THEOREM 4.1. *There exists a subsequence of $\{u^N\}_{N=1}^\infty$ that converges weakly in $L_T^2\mathbf{H}^1$, weak-* in $L_T^\infty\mathbf{H}^0$, and strongly in $L_T^2\mathbf{H}^0 \cap C_T\mathbf{H}^{-1}$ to a solution of (2.18) that is weakly continuous in $\mathbf{H}^0$.*

Proof. Since $\{u^N\}_{N=1}^\infty$ is uniformly bounded in $L_T^2\mathbf{H}^1 \cap L_T^\infty\mathbf{H}^0$, an application of Banach–Alaoglu yields a subsequence, which we will relabel as $\{u^N\}_{n=1}^\infty$, that converges weakly in $L_T^2\mathbf{H}^1$ and weak-* in $L_T^\infty\mathbf{H}^0$. To obtain a further subsequence that converges strongly in $L_T^2\mathbf{H}^0 \cap C_T\mathbf{H}^{-1}$, we need to apply Lemma A.2; that is, we need to show there exists a controls ω and $\bar{\omega}$ and $L, \kappa > 0$ independent of N such that $|\delta u_{st}^N|_{-1} \le \omega(s,t)^\kappa$ for all $(s,t) \in \Delta_T$ with $\bar{\omega}(s,t) \le L$. The proof of this is similar to the proof of Lemma 3.3, except that we need a slightly different bound on the drift term. This bound, in particular, does not yield p-variation of the solution.

Let $\phi \in \mathbf{H}^1$. Decomposing δu_{st}^N into a smooth and non-smooth part using J^η for some $\eta \in (0,1]$, we get

$$\begin{aligned}
|\delta u_{st}^N(\phi)| &\le |\delta u_{st}^N(J^\eta\phi)| + |\delta u_{st}^N((I-J^\eta)\phi)| \\
&\lesssim \omega_{N,\natural}(s,t)^{\frac{3}{p}} |J^\eta\phi|_3 + (t-s)\left(1+|u^N|_{L_T^\infty H_N}\right)^2 |J^\eta\phi|_3 \\
&\quad + |u^N|_{L_T^\infty H_N}\left(\omega_{A^N}(s,t)^{\frac{1}{p}}|\phi|_1 + \omega_{A^N}(s,t)^{\frac{2}{p}}|J^\eta\phi|_2\right) \\
&\quad + |u^N|_{L_T^\infty H_N}|(I-J^\eta)\phi|_0 \\
&\lesssim \eta^{-2}\omega_{N,\natural}(s,t)^{\frac{3}{p}}|\phi|_1 + \eta^{-2}(t-s)(1+|u_0|_0)^2|\phi|_1 \\
&\quad + |u_0|_0\left(\omega_Z(s,t)^{\frac{1}{p}} + \eta^{-1}\omega_Z(s,t)^{\frac{2}{p}}\right)|\phi|_1 + \eta|u_0|_0|\phi|_1.
\end{aligned}$$

Using (4.5) together with $\eta = \omega_Z(s,t)^{\frac{1}{p}} + (t-s)^{\frac{1}{p}}$ and $L > 0$ chosen such that $\eta \in (0,1]$, we find

$$\begin{aligned}
|\delta u_{st}^N|_{-1} &\le (1+|u_0|_0)^2\Big[\left(\omega_Z(s,t)^{\frac{3}{p}} + (t-s)\,\omega_Z(s,t)^{\frac{1}{p}}\right)\eta^{-2} \\
&\quad + (t-s)\,\eta^{-2} + \left(\omega_Z(s,t)^{\frac{1}{p}} + \omega_Z(s,t)^{\frac{2}{p}}\eta^{-1}\right) + \eta\Big] \\
&\lesssim_{N_0} (1+|u_0|_0)^2\left(\omega_Z(s,t)^{\frac{1}{p}} + (t-s)^{1-\frac{2}{p}}\right).
\end{aligned}$$

By Lemma A.2, there is a subsequence of $\{u^N\}_{N=1}^\infty$ which we continue to denote by $\{u^N\}_{N=1}^\infty$ converging strongly to an element u in $C_T\mathbf{H}^{-1} \cap L_T^2\mathbf{H}^0$. Furthermore, owing to Lemma A.3, we know that u is continuous with values in $\mathbf{H}_w^0$ (i.e., $\mathbf{H}^0$ equipped with the weak topology).

Our goal now is to pass to the limit in (4.4) tested against some $\phi \in \mathbf{H}^3$ as N tends to infinity. Clearly,

$$|(A_{st}^{N,1} - A_{st}^{P,1})\phi|_0 = \Big|P_N P\left[\sigma_k \cdot \nabla)\phi\right] Z_{st}^{N,k} - P\left[(\sigma_k \cdot \nabla)\phi\right] Z_{st}^k\Big|_0$$
$$\leq |P_N P\left[(\sigma_k \cdot \nabla)\phi\right](Z_{st}^{N,k} - Z_{st}^k)|_0 + |(I - P_N)P\left[(\sigma_k \cdot \nabla)\phi\right] Z_{st}^k|_0.$$

Making use of (2.2), we get

$$|P_N P\,(\sigma_k \cdot \nabla)\,\phi|_0 |Z_{st}^{N,k} - Z_{st}^k| \lesssim_{N_0} |\phi|_1 |Z_{st}^N - Z_{st}|,$$
$$|(I - P_N)P\,[(\sigma_k \cdot \nabla)\,\phi]\,Z_{st}^k|_0 \leq |I - P_N|_{\mathcal{L}(\mathbf{H}^0,\mathbf{H}^0)}\,|P\,[(\sigma_k \cdot \nabla)\,\phi]|_0\,|Z_{st}| \lesssim_{N_0} |I - P_N|_{\mathcal{L}(\mathbf{H}^0,\mathbf{H}^0)} |\phi|_1 |Z_{st}|.$$

Moreover, we have

$$\left|\left(A_{st}^{N,2} - A_{st}^{P,2}\right)\phi\right|_0 \leq \left|\tilde{P}_N\left[(\sigma_k \cdot \nabla)\tilde{P}_N[(\sigma_j \cdot \nabla)\phi]\right](\mathbb{Z}_{st}^{N,j,k} - \mathbb{Z}_{st})\right|_0$$
$$+ \left|(I - P_N)P\left[(\sigma_k \cdot \nabla)P[(\sigma_j \cdot \nabla)\phi]\right]\mathbb{Z}_{st}^{j,k}\right|_0$$
$$+ \left|\tilde{P}_N\left[(\sigma_k \cdot \nabla)(I - P_N)P[(\sigma_j \cdot \nabla)\phi]\right]\mathbb{Z}_{st}^{j,k}\right|_0.$$

Now, applying (2.3), we find

$$\left|\tilde{P}_N\Big[\,(\sigma_k \cdot \nabla)\,\tilde{P}_N\Big[\Big(\sigma_j \cdot \nabla\Big)\phi\Big]\Big]\left(\mathbb{Z}_{st}^{N,j,k} - \mathbb{Z}_{st}^{j,k}\right)\right|_0 \lesssim_{N_0} |\phi|_2 \left|\mathbb{Z}_{st}^{N,j,k} - \mathbb{Z}_{st}^{j,k}\right|,$$
$$\left|\,(I - P_N)\,P\left[(\sigma_k \cdot \nabla)\,P[(\sigma_j \cdot \nabla)\phi]\right]\mathbb{Z}_{st}^{j,k}\right|_0 \lesssim_{N_0} |I - P_N|_{\mathcal{L}(\mathbf{H}^0,\mathbf{H}^0)} |\phi|_2 \left|\mathbb{Z}_{st}^{j,k}\right|,$$
$$\left|\tilde{P}_N\,(\sigma_k \cdot \nabla)\,(I - P_N)\,P[(\sigma_j \cdot \nabla)\phi])\mathbb{Z}_{st}^{j,k}\right|_0 \lesssim_{N_0} |I - P_N|_{\mathcal{L}(\mathbf{H}^1,\mathbf{H}^1)} |\phi|_2 \left|\mathbb{Z}_{st}^{j,k}\right|.$$

Therefore,

$$A_{st}^{N,i,*}\phi \to A_{st}^{P,i,*}\phi$$

in $\mathbf{H}^0$ for $i \in \{1, 2\}$ as $N \to \infty$, and hence

$$\left|\left(u_s^N, A_{st}^{N,i,*}\phi\right) - \left(u_s, A_{st}^{P,i,*}\phi\right)\right| \leq_{N_0} \left|\left(u_s^N - u_s, A_{st}^{N,i,*}\phi\right) - \left(u_s, (A_{st}^{P,i,*} - A_{st}^{N,i,*})\phi\right)\right|$$
$$\lesssim_{N_0} \left|u_s^N - u_s\right|_{-1} |\phi|_3 + |u_s|_0 \left|\left(A_{st}^{P,i,*} - A_{st}^{N,i,*}\right)\phi\right|_0 \to 0$$

as $N \to \infty$. Finally, using the strong convergence in $L_T^2\mathbf{H}^0$ of $\{u^N\}$ and (2.4), we find

$$\left|\int_s^t \left[B_P(u_r)(\phi) - B_P\left(u_r^N\right)(\phi)\right] \mathrm{d}r\right|$$
$$\leq \left|\int_s^t B_P\left(u_r - u_r^N, u_r\right)(\phi)\,\mathrm{d}r\right| + \left|\int_s^t B_P\left(u_r^N, u_r - u_r^N\right)(\phi)\,\mathrm{d}r\right|$$
$$\lesssim \int_s^t \left|u_r - u_r^N\right|_0 |u_r|_0\,\mathrm{d}r\,|\phi|_3 + \int_s^t \left|u_r - u_r^N\right|_0 \left|u_r^N\right|_0 \mathrm{d}r\,|\phi|_3 \to 0$$

as $N \to \infty$.

Since all of the terms in Eq. (4.4) converge when applied to ϕ, the remainder $u_{st}^{N,\natural}(\phi)$ converges to some limit $u_{st}^{P,\natural}(\phi)$. Owing to the uniform bound (4.5), we have $u^{P,\natural} \in C_{2,\varpi,L}^{\frac{p}{3}-\mathrm{var}}([0,T]; \mathbf{H}^{-3})$ for some control ϖ depending only on ω_Z and $L > 0$ depending only on p, which proves that u is a solution of (2.18). □

4.1.2. Pressure recovery

To finalize the proof of existence, we need to prove that the pressure term π exists and satisfies (2.19). To this end, we first show that we can construct the rough integral

$$I_t = Q\int_0^t (\sigma_k \cdot \nabla)u_r \,\mathrm{d}Z_r^k, \quad I_0 = 0,$$

using the sewing lemma, Lemma B.1. Let $h_{st} = A_{st}^{Q,1}u_s + A_{st}^{Q,2}u_s$ for $(s,t) \in \Delta_T$. It follows that $h \in C_2^{p-\mathrm{var}}([0,T]; \mathbf{H}_\perp^{-2})$. Applying the δ operator to h and using (2.17), for $(s,\theta,t) \in \Delta_T^{(2)}$, we have

$$\begin{aligned}\delta h_{s\theta t} &= \left(\delta A_{s\theta t}^{Q,2}\right)u_s - A_{\theta t}^{Q,1}\delta u_{s\theta} - A_{\theta t}^{Q,2}\delta u_{s\theta}\\ &= A_{\theta t}^{Q,1}A_{s\theta}^{P,1}u_s - A_{\theta t}^{Q,1}\delta u_{s\theta} - A_{\theta t}^{Q,2}\delta u_{s\theta}\\ &= -A_{\theta t}^{Q,1}u_{s\theta}^{\sharp} - A_{\theta t}^{Q,2}\delta u_{s\theta},\end{aligned}$$

where we recall that $u_{st}^{\sharp} = \delta u_{st} - A_{st}^{P,1}u_s$ [see (3.2)]. Owing to Lemma 3.3 and (3.4), which establish the regularity of δu and $u^{\sharp}$, there are controls ω and ϖ and an $L > 0$ such that for all (s,θ,t) with $\varpi(s,t) \le L$, we have

$$|\delta h_{s\theta t}|_{-3} \lesssim_p \left(\omega_A(s,t)^{\frac{1}{3}}\omega_{\sharp}(s,t)^{\frac{2}{3}} + \omega_A(s,t)^{\frac{2}{3}}\omega_u(s,t)^{\frac{1}{3}}\right)^{\frac{3}{p}} =: \omega(s,t)^{\frac{3}{p}}.$$

Therefore, by Lemma B.1, there exists a unique path $I \in C^{p-\mathrm{var}}([0,T]; \mathbf{H}_\perp^{-3})$ and two-index map $I^{\natural} \in C_{2,\varpi,L}^{p-\mathrm{var}}([0,T]; \mathbf{H}_\perp^{-3})$ such that

$$\delta I_{st} = A_{st}^{Q,1}u_s + A_{st}^{Q,2}u_s + I_{st}^{\natural}.$$

and

$$|I_{s\theta t}^{\natural}|_{-3} \lesssim_p \omega(s,t)^{\frac{3}{p}}.$$

We define

$$\pi_t := -\int_0^t B_Q(u_r)\,\mathrm{d}r + I_t,$$

or alternatively using the local approximation

$$\delta\pi_{st} = -\int_s^t B_Q(u_r)\,\mathrm{d}r + A_{st}^{Q,1}u_s + A_{st}^{Q,2}u_s + u_{st}^{Q,\natural},$$

where $u_{st}^{Q,\natural} := I_{st}^{\natural}$. Owing to Lemma 3.5 and (2.19), we have that $\pi \in C^{p-var}([0,T]; \mathbf{H}_\perp^{-3})$.

4.2. Uniqueness in two spatial dimensions, proof of Theorem 2.14

The objective of this section is to prove that the solution (u, π) of (2.12) is unique when $d = 2$ and σ_k is constant function of $x \in \mathbf{T}^2$ for all $k \in \{1, \dots, K\}$. Assume for a moment that all functions are smooth and that we have two solutions of (2.12):

$$\partial_t u_t^i = \nu \Delta u_t^i - P(u_t^i \cdot \nabla) u_t^i + P(\sigma_k \cdot \nabla) u_t^i \dot{z}_t^k, \quad i \in \{1, 2\}.$$

Then $v := u^1 - u^2$ satisfies

$$\partial_t v = \nu \Delta v - (B_P(u^1) - B_P(u^2)) + P(\sigma_k \cdot \nabla) v \dot{z}_t^k,$$

and the chain rule gives for all $x \in \mathbf{T}^2$,

$$\begin{aligned}\frac{1}{2}\partial_t |v(x)|^2 = \nu v(x) \cdot \Delta v(x) - v(x) \cdot (B_P(u^1(x)) - B_P(u^2(x))) \\ + v(x) \cdot (\sigma_k \cdot \nabla) v(x) \dot{z}_t^k.\end{aligned}$$

One could proceed by integrating with respect to x to obtain uniqueness and energy estimates. However, in the rough case, many of our objects are distributions, and so the action of integrating with respect to x is actually applying a distribution to a test function.

Since we do not expect our solution to be regular enough to perform this operation, we shall employ a doubling of the variables trick; that is, we consider $t \mapsto v_t^{\otimes 2}(x, y) := v_t(x) v_t(y)^{\mathrm{T}}$, where T denotes the transpose. This is a well defined operation for any distribution and we get the formula for the square by testing this distribution against an approximation of the Dirac-delta in $x = y$. We remark that one cannot directly use the techniques from [10], since this way of approximating the Dirac-delta violates the divergence-free condition.

Let u^1 and u^2 be solutions of (2.12), as defined by Definition 2.7. For all $\phi \in \mathbf{H}^3$ and $i \in \{1, 2\}$ and $(s, t) \in \Delta_T$, we have

$$\delta u_{st}^i(\phi) = \delta \mu_{st}^i(\phi) + u_s^i\left(\left[A_{st}^{P,1,*} + A_{st}^{P,2,*}\right]\phi\right) + u_{st}^{i;P,\natural}(\phi),$$

where

$$\mu_t^i(\phi) = -\int_0^t \left[\nu\left(\nabla u_r^i, \nabla\phi\right) + B_P(u_r^i)(\phi)\right] \mathrm{d}r.$$

Setting $v = u^1 - u^2$, $v^\natural = u^{1;P,\natural} - u^{2;P,\natural}$ and $\mu_t(\phi) = -\int_0^t [\nu(\nabla v_r, \nabla\phi) + (B_P(u_r^1) - B_P(u_r^2))(\phi)] \,\mathrm{d}r$, we have

$$\delta v_{st}(\phi) = \delta \mu_{st}(\phi) + v_s\left(\left[A_{st}^{P,1,*} + A_{st}^{P,2,*}\right]\phi\right) + v_{st}^\natural.$$

Define

$$\omega_\mu(s, t) = \omega_{\mu^1}(s, t) + \omega_{\mu^2}(s, t),$$

and notice that

$$|\delta\mu_{st}(\phi)|_{-1} \lesssim \omega_\mu(s,t).$$

We denote by $a \hat{\otimes} b$ the symmetrization of the tensor product of two functions $a, b : \mathbf{T}^2 \to \mathbf{R}^2$; that is,

$$a \hat{\otimes} b\,(x,y) := \frac{1}{2}\,(a \otimes b + b \otimes a)\,(x,y) = \frac{1}{2}\big(a\,(x)\,b\,(y)^{\mathrm{T}} + b\,(x)\,a\,(y)^{\mathrm{T}}\big), \quad (x,y) \in \mathbf{T}^2.$$

LEMMA 4.2. *The weakly continuous mapping* $v_t^{\otimes 2} : [0,T] \to \mathbf{H}_x^{-3} \otimes \mathbf{H}_y^{-3}$ *satisfies the equation*

$$\delta v_{st}^{\otimes 2} - 2\int_s^t \left(\nu v_r \hat{\otimes} \Delta v_r - v_r \hat{\otimes} \left(B_P\left(u_r^1\right) - B_P\left(u_r^2\right)\right)\right) \mathrm{d}r = \left(\Gamma_{st}^1 + \Gamma_{st}^2\right) v_s^{\otimes 2} + v_{st}^{\otimes 2,\natural},$$

where

$$\Gamma^1 := A^{P,1} \otimes I + I \otimes A^{P,1}, \quad \Gamma^2 := A^{P,2} \otimes I + I \otimes A^{P,2} + A^{P,1} \otimes A^{P,1},$$

and $v^{\otimes 2,\natural} \in C_{2,\varpi,L}^{\frac{p}{3}-\mathrm{var}}([0,T]; \mathbf{H}_x^{-3} \otimes \mathbf{H}_y^{-3})$, *for a control* ϖ *and* $L > 0$.

Proof. Elementary algebraic manipulations yield

$$\begin{aligned}
\delta v_{st}^{\otimes 2} &= 2v_s \hat{\otimes} \delta v_{st} + \delta v_{st} \otimes \delta v_{st} = 2v_s \hat{\otimes} v_{st}^{\natural} + 2v_s \hat{\otimes} \delta\mu_{st} + 2v_s \hat{\otimes} A_{st}^1 v_s \\
&\quad + 2v_s \hat{\otimes} A_{st}^{P,2} v_s + \left(v_{st}^{\natural} + \delta\mu_{st} + A_{st}^{P,2} v_s\right)^{\otimes 2} + 2\left(v_{st}^{\natural} + \delta\mu_{st} + A_{st}^{P,2} v_s\right) \\
&\quad \hat{\otimes} A_{st}^{P,1} v_s + A_{st}^{P,1} v_s \otimes A_{st}^{P,1} v_s.
\end{aligned}$$

Thus,

$$\delta v_{st}^{\otimes 2} - 2\int_s^t \left[\nu v_r \hat{\otimes} \Delta v_r - v_r \hat{\otimes} \left(B_P(u_r^1) - B_P(u_r^2)\right)\right] \mathrm{d}r = \left(\Gamma_{st}^1 + \Gamma_{st}^2\right) v_s^{\otimes 2} + v_{st}^{\otimes 2,\natural}, \tag{4.6}$$

where

$$\begin{aligned}
v_{st}^{\otimes 2,\natural} &:= -2\int_s^t \delta v_{sr} \hat{\otimes} \left[\nu \Delta v_r + (B_P\left(u_r^1\right) - B_P\left(u_r^2\right)\right] \mathrm{d}r + 2v_{st}^{\natural} \hat{\otimes} v_s \\
&\quad + \left(v_{st}^{\natural} + \delta\mu_{st} + A_{st}^{P,2} v_s\right)^{\otimes 2} + 2\left(v_{st}^{\natural} + \delta\mu_{st} + A_{st}^{P,2} v_s\right) \hat{\otimes} A_{st}^{P,1} v_s \\
&= -2\int_s^t \nu \delta v_{sr} \hat{\otimes} \Delta v_r \mathrm{d}r + 2\int_s^t \delta v_{sr} \hat{\otimes} \left[B_P\left(u_r^1\right) - B_P\left(u_r^2\right)\right] \mathrm{d}r + 2v_{st}^{\natural} \hat{\otimes} v_s \\
&\quad + v_{st}^{\natural} \otimes v_{st}^{\natural} + v_{st}^{\natural} \hat{\otimes} \delta\mu_{st} + v_{st}^{\natural} \hat{\otimes} A_{st}^{P,2} v_s + \delta\mu_{st} \otimes \delta\mu_{st} + \delta\mu_{st} \hat{\otimes} A_{st}^{P,2} v_s \\
&\quad + A_{st}^{P,2} v_s \otimes A_{st}^{P,2} v_s + 2v_{st}^{\natural} \hat{\otimes} A_{st}^{P,1} v_s + \delta\mu_{st} \hat{\otimes} A_{st}^{P,1} v_s + A_{st}^{P,2} v_s \hat{\otimes} A_{st}^{P,1} v_s.
\end{aligned}$$

Estimating $v_{st}^{\otimes 2,\natural}$ term by term and making use of (2.2), (2.3), and (2.4), we find that there is a control ϖ and $L > 0$ such that $v^{\otimes 2,\natural} \in C_{2,\varpi,L}^{\frac{p}{3}-\mathrm{var}}([0,T]; \mathbf{H}_x^{-3} \otimes \mathbf{H}_y^{-3})$. $\square$

Let $\{\mathbf{f}_{0,1}, \mathbf{f}_{0,2}\} \cup \{\mathbf{f}_{1,n}\}_{n\in\mathbf{Z}^2-\{0\}}$ be the orthonormal basis of $\{u \in L^2(\mathbf{T}^2; \mathbf{C}^2) : \nabla \cdot u = 0\}$ described in Sect. 2.1. Define

$$F_N(x, y) := \mathbf{f}_{0,1} \otimes \mathbf{f}_{0,1} + \mathbf{f}_{0,2} \otimes \mathbf{f}_{0,2} + \sum_{|n|<N, n\neq 0} \mathbf{f}_{1,n}(x) \otimes \overline{\mathbf{f}_{1,n}(y)}.$$

It follows that for all $f, g \in \mathbf{H}^0$, $f \otimes g(F_N) \to (f, g)$ as $N \to \infty$, and hence $v^{\otimes 2}(F_N) \to |v|_0^2$ as $N \to \infty$. Since $\nabla \mathbf{f}_{1,n} = in\mathbf{f}_{1,n}$, for all $n \in \mathbf{Z}^2 - \{0\}$, we have $\nabla_x F_N + \nabla_y F_N = 0$.

Motivated by this, we will test Eq. (4.6) F_N and pass to the limit as $N \to \infty$ to derive the equation for the square. Because σ_k is constant, we have

$$\Gamma_{st}^{1,*} F_N = ((\sigma_k \cdot \nabla_x) F_N + (\sigma_k \cdot \nabla_y) F_N) Z_{st}^k = 0$$

and

$$\begin{aligned} \Gamma_{st}^{2,*} F_N &= (\sigma_k \cdot \nabla_x) \left(\sigma_j \cdot \nabla_x\right) F_N \mathbb{Z}_{st}^{j,k} + \left(\sigma_k \cdot \nabla_y\right) \left(\sigma_j \cdot \nabla_y\right) F_N \mathbb{Z}_{st}^{j,k} \\ &\quad + (\sigma_k \cdot \nabla_x) \left(\sigma_j \cdot \nabla_y\right) F_N Z_{st}^j Z_{st}^k \\ &= (\sigma_k \cdot \nabla_x) \left(\sigma_j \cdot \nabla_x\right) F_N \mathbb{Z}_{st}^{j,k} + (\sigma_k \cdot \nabla_x) \left(\sigma_j \cdot \nabla_x\right) F_N \mathbb{Z}_{st}^{k,j} \\ &\quad - (\sigma_k \cdot \nabla_x) \left(\sigma_j \cdot \nabla_x\right) F_N Z_{st}^j Z_{st}^k \\ &= 0, \end{aligned}$$

where we have used $(\sigma_k \cdot \nabla)(\sigma_j \cdot \nabla) = (\sigma_j \cdot \nabla)(\sigma_k \cdot \nabla)$ and $\mathbb{Z}_{st}^{j,k} + \mathbb{Z}_{st}^{k,j} = Z_{st}^j Z_{st}^k$. Applying the divergence theorem, we get

$$\begin{aligned} \int_s^t \nu v_r \otimes \Delta v_r(F_N)\,\mathrm{d}r &= -\int_s^t \nu v_r \otimes \nabla v_r(\nabla_y F_N)\,\mathrm{d}r = \int_s^t \nu v_r \otimes \nabla v_r(\nabla_x F_N)\,\mathrm{d}r \\ &= -\int_s^t \nu \nabla v_r \otimes \nabla v_r(F_N)\,\mathrm{d}r, \end{aligned}$$

and hence that

$$2\int_s^t \nu v_r \hat{\otimes} \Delta v_r(F_N)\,\mathrm{d}r = -2\int_s^t \nu \nabla v_r \otimes \nabla v_r(F_N)\,\mathrm{d}r.$$

Since $v \in L_T^2 \mathbf{H}^1$, we have $\nabla v_r \otimes \nabla v_r(F_N) \to |\nabla v_r|_0^2$ as $N \to \infty$ for almost all $r \in [s, t]$. Owing to the bound $|\nabla v_r \otimes \nabla v_r(F_N)| \leq |\nabla v_r|_0^2$, we apply the dominated convergence theorem to get that

$$\lim_{N\to\infty} 2\int_s^t \nu v_r \hat{\otimes} \Delta v_r(F_N)\,dr = -2\nu \int_s^t |\nabla v_r|_0^2\,dr.$$

Using the divergence theorem again, we find

$$\int_s^t v_r \otimes (u_r^i \cdot \nabla) u_r^i (F_N)\, \mathrm{d}r = -\int_s^t v_r \otimes (u_r^i)^T u_r^i (\nabla_y F_N)\, \mathrm{d}r$$
$$= -\int_s^t \nabla v_r \otimes (u_r^i)^T u_r^i (F_N)\, \mathrm{d}r.$$

Using the interpolation inequality $|(u_r^i)^T u_r^i|_0 \lesssim |u_r^i|_0^{\frac{1}{2}} |u_r^i|_1^{\frac{1}{2}}$, we apply the dominated convergence theorem to get

$$\lim_{N\to\infty} 2\int_s^t v_r \hat{\otimes} B_P(u_r^i)(F_N)\, \mathrm{d}r = 2\int_s^t B_P(u_r^i)(v_r)\, \mathrm{d}r.$$

We are now ready to finish the proof of uniqueness.

THEOREM 4.3. *Let $d = 2$ and assume the vector fields $\sigma_k(x) = \sigma_k$, $k \in \{1, \dots, K\}$, are constant. Suppose that u^1 and u^2 are two solutions of* (2.12) *in the sense of Definition* 2.7. *Then the difference $v = u^1 - u^2$ satisfies*

$$|v_t|_0^2 + 2\int_0^t (B_P(u_r^1) - B_P(u_r^2))(v_r)\, \mathrm{d}r + 2\nu \int_0^t |\nabla v_r|_0^2\, \mathrm{d}r = |v_0|_0^2, \quad \forall t \in [0, T]. \tag{4.7}$$

Furthermore, there is a constant $c = c(\nu, T)$ such that

$$|v_t|_0^2 + \int_0^t |\nabla v_r|_0^2\, \mathrm{d}r \lesssim_{\nu,T} |v_0|_0^2 \exp\left\{c \int_0^t |u_r^1|_0^2 |u_r^1|_1^2\, \mathrm{d}r\right\}, \quad \forall t \in [0, T]. \tag{4.8}$$

Therefore, there exists a unique solution u of (2.12).

REMARK 4.4. The right-hand side of (4.8) is finite. Indeed, we have

$$\int_0^t |u_r^1|_0^2 |u_r^1|_1^2\, \mathrm{d}r \le \sup_{t\in[0,T]} |u_t^1|_0^2 \int_0^T |u_r^1|_1^2\, \mathrm{d}r,$$

which is finite since $u \in L_T^2 \mathbf{H}^1 \cap L_T^\infty \mathbf{H}^0$.

Proof of Theorem 4.3. Testing equation (4.6) against F_N and using that $\Gamma_{st}^{i,*} F_N = 0$ for $i \in \{1, 2\}$, we find

$$\delta v_{st}^{\otimes 2}(F_N) - 2\int_s^t \left(\nu v_r \hat{\otimes} \Delta v_r + v_r \hat{\otimes} \left(B_P\left(u_r^1\right) - B_P\left(u_r^2\right)\right)\right) \mathrm{d}r(F_N) = v_{st}^{\otimes 2,\natural}(F_N).$$

Since the left-hand-side is an increment of a function, the right-hand-side $(s, t) \mapsto v_{st}^{\otimes 2,\natural}(F_N)$ must be as well. By virtue of Lemma 4.2, we know that $v^{\otimes 2,\natural}(F_N)$ has finite $\frac{p}{3}$-variation, which is only possible if $v_{st}^{\otimes 2,\natural}(F_N) = 0$. Thus, for every $N \in \mathbf{N}$,

$$\delta v_{st}^{\otimes 2}(F_N) - 2\int_s^t \left(\nu v_r \hat{\otimes} \Delta v_r - v_r \hat{\otimes} \left(B_P\left(u_r^1\right) - B_P\left(u_r^2\right)\right)\right) \mathrm{d}r(F_N) = 0.$$

Passing to the limit as $N \to \infty$ in the above equality, we get

$$\delta(|v|_0^2)_{st}+2\int_s^t \left(B_P\left(u_r^1\right)-B_P\left(u_r^2\right)\right)(v_r)\,\mathrm{d}r+2\nu\int_s^t |\nabla v_r|_0^2\,\mathrm{d}r=0,\quad \forall t\in[0,T].$$

Moreover, using (2.6), (2.5), and Young's inequality (i.e, $ab \le \epsilon a^{\frac{4}{3}} + c_\epsilon b^4, \forall a, b, \epsilon \ge 0$, where c_ϵ is a constant depending only on ϵ), for every $\epsilon > 0$, we have

$$(B_P(u^1)-B_P(u^2))(v)=-B_P(v,v)(u^1)\le c|v|_1^{\frac{3}{2}}|v|_0^{\frac{1}{2}}|u^1|_0^{\frac{1}{2}}|u^1|_1^{\frac{1}{2}}\le \epsilon|v|_1^2+c_\epsilon|v|_0^2|u^1|_0^2|u^1|_1^2,$$

and hence

$$|v_t|_0^2+2\nu\int_0^t |\nabla v_r|_0^2\,\mathrm{d}r\le |v_0|_0^2+\epsilon\int_0^t |v_r|_1^2\,\mathrm{d}r+c_\epsilon\int_0^t |v_r|_0^2|u_r^1|_0^2|u_r^1|_1^2\,\mathrm{d}r.$$

Choosing ϵ small enough, we find

$$|v_t|_0^2+\int_0^t |\nabla v_r|_0^2\,\mathrm{d}r\lesssim_\nu |v_0|_0^2+\int_0^t |v_r|_0^2(1+|u_r^1|_0^2|u_r^1|_1^2)\,\mathrm{d}r.$$

We then complete the proof by applying Gronwall's lemma. From the uniqueness of the velocity and the pressure recovery in Sect. 4.1.2, we immediately obtain the uniqueness of the associated pressure π. □

4.2.1. Energy equality and continuity

Letting $u^1 = u$ and $u^2 = 0$ in (4.7), where u is the unique solution, we obtain the following corollary.

COROLLARY 4.5. *Let $d = 2$ and assume the vector fields $\sigma_k(x) = \sigma_k$ are constant for all $k \in \{1, \dots, K\}$. Then the unique solution u of* (2.12) *is in $C_T\mathbf{H}^0$ and satisfies the energy equality:*

$$|u_t|_0^2+2\nu\int_0^t |\nabla u_r|_0^2\,\mathrm{d}r=|u_0|_0^2,\quad \forall t\in[0,T]. \tag{4.9}$$

Proof. We start by showing that u is continuous as a mapping with values in $\mathbf{H}^0$ equipped with the weak topology. It is immediate from (3.1) that $\lim_{s\to t} u_s(\phi) = u_t(\phi)$ for any $\phi \in \mathbf{H}^3$. Moreover, since $\{|u_s|_0\}_{s\in[0,T]}$ is bounded, there exists a subsequence $\{u_{s_n}\}_n \subset \{u_s\}_{s\to t}$ such that $u_{s_n}(\phi)$ has a limit for all $\phi \in \mathbf{H}^3$. Because $\mathbf{H}^3$ is dense in $\mathbf{H}^0$ and weak limits are unique, we must have convergence $\lim_{s\to t} u_s(\phi) = u_t(\phi)$ for all $\phi \in \mathbf{H}^0$. By virtue of the energy equality (4.9), we have that $\lim_{s\to t} |u_s|_0 = |u_t|_0$, which implies strong convergence. □

REMARK 4.6. For constant vector fields σ_k, we have $A_{st}^{Q,i} u_s = 0$ for $i \in \{1, 2\}$, and so (2.19) reduces to the deterministic case; that is,

$$\pi_t=-\int_0^t Q(u_r\cdot\nabla)u_r\,\mathrm{d}r.$$

Applying (2.4) with $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = 1$, we find that $\pi \in C^{1-\mathrm{var}}([0,T];\mathbf{H}_\perp^{-1})$.

4.3. Stability in two spatial dimension, proof of Corollary 2.15

Proof of Corollary 2.15. For $n \in \mathbf{N}$, consider a sequence of initial conditions $\{u_0^n\}_{n=1}^{\infty} \subset \mathbf{H}^0$, constant vector fields $\{\sigma^n\}_{n=1}^{\infty} \subset \mathbf{R}^{d\times K}$ and continuous geometric p-rough paths $\{\mathbf{Z}^n = (Z^n, \mathbb{Z}^n)\}_{n=1}^{\infty} \in C_g^{p-\mathrm{var}}([0,T];\mathbf{R}^K)$. According to Theorem 4.1, there exists a sequence $(u^n, \pi^n)_{n=1}^{\infty}$ of solutions to (2.13) corresponding to the datum $\{(u_0^n, \sigma^n, \mathbf{Z}^n)\}_{n=1}^{\infty}$. Moreover, by virtue of the energy equality (4.9), we have

$$|u_t^n|_0^2 + 2\nu \int_0^t |\nabla u_r^n|^2 \,\mathrm{d}r = |u_0^n|^2, \quad \forall t \in [0,T]. \tag{4.10}$$

Thus, in view of Lemmas 3.1 and 3.3 and Remark 2.3, we obtain

$$|u^n|_{p-\mathrm{var};[0,T];\mathbf{H}^{-1}} \leq c\big(|u_0^n|_0, |\sigma^n|, |Z^n|_{p-\mathrm{var};[0,T]}, |\mathbb{Z}^n|_{\frac{p}{2}-\mathrm{var};[0,T]}\big), \tag{4.11}$$

for some function c that is increasing in its arguments.

Assume now that $u_0^n \to u_0$ in $\mathbf{H}^0$, $\sigma^n \to \sigma$ in $\mathbf{R}^{2\times K}$ and $\mathbf{Z}^n \to \mathbf{Z} = (Z,\mathbb{Z})$ in the rough path topology (2.9) (i.e., $Z^n \to Z$ in $C_2^{p-\mathrm{var}}([0,T];\mathbf{R}^K)$ and $\mathbb{Z}^n \to \mathbb{Z}$ in $C_2^{p-\mathrm{var}}([0,T];\mathbf{R}^{K\times K})$). Then the estimates (4.10) and (4.11) yields a uniform (in n) bound for the sequence $\{u^n\}_{n=1}^{\infty}$ in $L_T^\infty\mathbf{H}^0 \cap L_T^2\mathbf{H}^1 \cap C^{p-\mathrm{var}}([0,T];\mathbf{H}^{-1})$. Hence, due to Lemma A.3, there exists $u \in L_T^\infty\mathbf{H}^0 \cap L_T^2\mathbf{H}^1 \cap C^{p-\mathrm{var}}([0,T];\mathbf{H}^{-1})$ such that, up to a subsequence,

$$u^n \to u \quad \text{in} \quad L_T^2\mathbf{H}^0 \cap C_T\mathbf{H}_w^0$$

as n tends to infinity.

Similar to the proof of Theorem 4.1, we may pass to the limit in the equation and verify that u solves (2.13) with the datum $(u_0, \sigma, \mathbf{Z})$. Since uniqueness holds true for (2.13) in two dimensions with constant vector fields, we deduce that the whole sequence u^n converges to u in $L_T^2\mathbf{H}^0 \cap C_T\mathbf{H}_w^0$.

To see the convergence of π^n, we note that since the vector fields are constant we have $A_{st}^{Q,i}u_s = 0$ for $i \in \{1,2\}$, and hence

$$\pi_t^n = -\int_0^t B_Q(u_r^n)\,\mathrm{d}r.$$

The convergence $\pi^n \to \pi$ in $C^{1-\mathrm{var}}([0,T];\mathbf{H}_\perp^{-2})$ follows since u^n converges to u in $L_T^2\mathbf{H}^0$. Indeed,

$$\begin{aligned}
&\left|\int_s^t B_Q(u_r)(\psi) - B_Q(u_r^n)(\psi)\,\mathrm{d}r\right| \\
&\quad\leq \left|\int_s^t B_Q\left(u_r - u_r^n, u_r\right)(\psi)\,\mathrm{d}r\right| + \left|\int_s^t B_Q\left(u_r^n, u_r - u_r^n\right)(\psi)\,\mathrm{d}r\right| \\
&\quad\lesssim \int_s^t |u_r - u_r^n|_0 |u_r|_1 |\psi|_2 \,\mathrm{d}r + \int_s^t |u_r^n|_1 |\psi|_2 |u_r - u_r^n|_0 \,\mathrm{d}r
\end{aligned}$$

for every $\psi \in \mathbf{H}_\perp^2$, where we have used (2.6) and (2.4) with $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = 2$, as well as $\alpha_1 = 1$, $\alpha_2 = 1$ and $\alpha_3 = 0$. □

A Compact embedding results

The following compact embedding result is comparable to the fractional version of the Aubin-Lions compactness result (see, e.g., [22, Theorem 2.1]). Before we come to the embedding itself, we need to prove a simple lemma.

LEMMA A.1. *If ω is a continuous control, then*

$$\lim_{a\to 0} \sup_{s\in[0,T]} \sup_{t\in[s,s+a]} \omega(s,t) = 0.$$

Proof. Owing to superadditivity, for any $t \in [s, s+a]$, we have $\omega(s,t) \le \omega(s, s+a)$, and hence the claim follows once we show that

$$\lim_{a\to 0} \sup_{s\in[0,T]} \omega(s, s+a) = 0.$$

Suppose, by contradiction, that there exists an $\epsilon > 0$ and a sequence $\{(s_n, a_n)\}_{n=1}^{\infty} \subset [0,T] \times [0,1]$ such that $\lim_{n\to\infty} a_n = 0$ and

$$\omega(s_n, s_n + a_n) > \epsilon, \quad \forall n \in \mathbf{N}.$$

Since $[0,T]$ is compact, there exists an $s \in [0,T]$ and a subsequence $\{(s_{n_k}, a_{n_k})\}_{k=1}^{\infty} \subset \{(s_n, a_n)\}_{n=1}^{\infty}$ converging to $(s, 0)$. By the continuity of the control ω, we have

$$\epsilon \le \lim_{k\to\infty} \omega(s_{n_k}, s_{n_k} + a_{n_k}) = \omega(s,s) = 0,$$

which is a contradiction. □

LEMMA A.2. *Let ω and ϖ be a controls on $[0,T]$ and $L, \kappa > 0$. Let*

$$X = L_T^2 \mathbf{H}^1 \cap \Big\{ g \in C_T \mathbf{H}^{-1} : |\delta g_{st}|_{-1} \le \omega(s,t)^{\kappa},\ \forall (s,t) \in \Delta_T \text{ with } \varpi(s,t) \le L \Big\}$$

be endowed with the norm

$$|g|_X = |g|_{L_T^2 \mathbf{H}^1} + \sup_{t\in[0,T]} |g_t|_{-1} + \sup \left\{ \frac{|\delta g_{st}|_{-1}}{\omega(s,t)^{\kappa}} : (s,t) \in \Delta_T \ \text{ s.t. } \varpi(s,t) \le L \right\}.$$

Then X is compactly embedded into $C_T \mathbf{H}^{-1}$ and $L_T^2 \mathbf{H}^0$.

Proof. For each $a \in (0, L]$ and every $g \in L_T^2 \mathbf{H}^{-1}$, let us define the function $J_a g : [0,T] \to \mathbf{H}^{-1}$ by

$$J_a g_s = \frac{1}{a} \int_s^{s+a} g_t \,\mathrm{d}t = \frac{1}{a} \int_0^a g_{s+t} \,\mathrm{d}t,$$

where we extend g to $\mathbf{R}_+$ by letting $g = g_T$ outside $[0,T]$. Clearly, $s \mapsto J_a g_s$ is continuous from $[0,T]$ into $\mathbf{H}^{-1}$; that is, J_a is a well-defined map from $L_T^2 \mathbf{H}^{-1}$ to $C_T \mathbf{H}^{-1}$. Moreover, using Hölder's inequality, for $i \in \{-1, 1\}$, we find

$$|J_a g_s|_i \le \frac{1}{a} \int_0^a |g_{s+t}|_i \,\mathrm{d}t \le \frac{1}{\sqrt{a}} \left(\int_0^a |g_{s+t}|_i^2 \,\mathrm{d}t \right)^{\frac{1}{2}}, \tag{A.1}$$

which implies

$$\int_0^T |J_a g_s|_i^2 \mathrm{d}s \le \frac{1}{a}\int_0^T \int_0^a |g_{s+t}|_i^2 \,\mathrm{d}t\mathrm{d}s = \int_0^T |g_t|_i^2 \,\mathrm{d}t,$$

and hence $|J_a g|_{L^2_T \mathbf{H}^i} \le |g|_{L^2_T \mathbf{H}^i}$; that is, $J_a : L^2_T \mathbf{H}^i \to \mathbf{H}^i$ is a bounded operator for $i \in \{-1, 1\}$.

Let us show that $J_a g \to g$ in $C_T \mathbf{H}^{-1}$ as $a \to 0$ uniformly with respect to X. For each $s \in [0, T]$, $g \in X$, we have

$$\begin{aligned} |J_a g_s - g_s|_{-1} &= \frac{1}{a}\left|\int_s^{s+a} g_t \,\mathrm{d}t - \int_s^{s+a} g_s \,\mathrm{d}t\right|_{-1} \le \frac{1}{a}\int_s^{s+a} |g_t - g_s|_{-1} \,\mathrm{d}t \\ &\le \frac{1}{a}\int_s^{s+a} \omega(s,t)^{\kappa} \,\mathrm{d}t \le \sup_{t\in[s,s+a]} \omega(s,t)^{\kappa}, \end{aligned}$$

which converges uniformly in s to 0 as $a \to 0$ by Lemma A.1.

Let $\mathcal{G}$ be a bounded subset of $L^2_T \mathbf{H}^1$, with norm bound denoted N_0. Using Hölder's inequality, for all $s, t \in [0, T]$ and $g \in \mathcal{G}$, we obtain

$$|J_a g_t - J_a g_s|_{-1} = \frac{1}{a}\left|\int_{t+a}^{s+a} g_r \,\mathrm{d}r - \int_s^t g_t \,\mathrm{d}t\right|_{-1} \le \frac{2}{a} N_0 \sqrt{|s-\bar{s}|},$$

and hence for a fixed a, $J_a\mathcal{G}$ is uniformly equicontinuous $\mathbf{H}^{-1}$. Owing to (A.1), for each $s \in [0, T]$, we have that $|J_a g_s|_1 \le \frac{1}{\sqrt{a}} N_0$, and hence for a fixed a, $J_a\mathcal{G}$ is pointwise bounded in $\mathbf{H}^1$. Since $\mathbf{H}^1$ is compactly embedded in $\mathbf{H}^{-1}$, for a fixed a, $J_a\mathcal{G}$ is pointwise relatively compact in $\mathbf{H}^{-1}$. Therefore, by the generalized Arzelà–Ascoli theorem $J_a\mathcal{G}$ is relatively compact in $C_T \mathbf{H}^{-1}$.

To conclude the proof, let $\{g^n\}_{n=1}^{\infty}$ be a bounded sequence in X. In particular, by Banach–Alaoglu, there exists a subsequence $\{g^{n_k}\}_{k=1}^{\infty}$ that converges in the weak*-topology of $L^2_T \mathbf{H}^1$ to some $g \in L^2_T \mathbf{H}^1$. We can reduce to the case $g = 0$, and hence the proof of the compact embedding of X in $C_T \mathbf{H}^{-1}$ is complete if we can show that $|g^{n_k}|_{C_T \mathbf{H}^{-1}} \to 0$ as $k \to \infty$.

To this end, for any fixed $a \in [0, L]$, by the above Arzelà–Ascoli argument, $\{J_a g^{n_k}\}_{k=1}^{\infty}$ has a convergent subsequence in $C_T \mathbf{H}^{-1}$, which we also denote by $\{J_a g^{n_k}\}_{k=1}^{\infty}$. We note that this subsequence may depend on a. Combining this with the fact that $g^{n_k} \to 0$ in the weak*-topology of $L^2_T \mathbf{H}^1$, we see that for any $f \otimes \phi \in C_T \otimes \mathbf{H}^1$, we have

$$\lim_{k\to\infty} \int_0^T J_a g_r^{n_k}(\phi) f_r \,\mathrm{d}r = \lim_{k\to\infty} \int_0^T g_r^{n_k}(\phi) J_a^* f_r \,\mathrm{d}r = 0,$$

so that $\lim_{k\to\infty} J_a g^{n_k} = 0$ in $C_T \mathbf{H}^{-1}$. Since all subsequences converges to the same limit, this means the full sequence converges. For any $a \in (0, L]$

$$\begin{aligned} |g^{n_k}|_{C_T\mathbf{H}^{-1}} &\le |J_a g^{n_k}|_{C_T\mathbf{H}^{-1}} + |J_a g^{n_k} - g^{n_k}|_{C_T\mathbf{H}^{-1}} \\ &\le |J_a g^{n_k}|_{C_T\mathbf{H}^{-1}} + \sup_{s\in[0,T]} \sup_{t\in[s,s+a]} \omega(s,t)^\kappa. \end{aligned}$$

Letting $k \to \infty$ first and then $a \to 0$, we find that $|g^{n_k}|_{C_T\mathbf{H}^{-1}} \to 0$ as $k \to \infty$, which shows that X is compactly embedded in $C_T\mathbf{H}^{-1}$.

Let us now show that the set X is compactly embedded in $L_T^2\mathbf{H}^0$. Using Young's inequality, for $h \in \mathbf{H}^1$ and any $\epsilon > 0$,

$$|h|_0^2 = h(h) \le |h|_{-1}|h|_1 \le C_\epsilon |h|_{-1}^2 + \epsilon |h|_1^2$$

for some appropriate constant $C_\epsilon > 0$. Consequently, proceeding with the same sequence above, we find

$$|g^{n_k}|^2_{L_T^2\mathbf{H}^0} \le C_\epsilon |g^{n_k}|^2_{L_T^2\mathbf{H}^{-1}} + \epsilon |g^{n_k}|^2_{L_T^2\mathbf{H}^1} \le C_\epsilon |g^{n_k}|^2_{C_T\mathbf{H}^{-1}} + \epsilon \sup_{n\in\mathbf{N}} |g^n|^2_{L_T^2\mathbf{H}^1}.$$

Letting $k \to \infty$ first, we have

$$\lim_{n\to\infty} |g^n|^2_{L_T^2\mathbf{H}^0} \le \epsilon \sup_{n\in\mathbf{N}} |g^n|^2_{L_T^2\mathbf{H}^1},$$

and then letting $\epsilon \to 0$, we conclude the proof. □

Denote by $C_T\mathbf{H}_w^0$ the space of $\mathbf{H}^0$-valued weakly continuous functions on $[0, T]$.

LEMMA A.3. *Let ω and ϖ be controls on $[0, T]$ and $L, \kappa > 0$. Let*

$$Y = L_T^\infty\mathbf{H}^0 \cap \left\{ g \in C_T\mathbf{H}^{-1} : |\delta g_{st}|_{-1} \le \omega(s,t)^\kappa,\ \forall (s,t) \in \Delta_T \text{ with } \varpi(s,t) \le L \right\},$$

be endowed with the norm

$$|g|_Y = |g|_{L_T^\infty\mathbf{H}^0} + \sup_{t\in[0,T]} |g_t|_{-1} + \sup\left\{ \frac{|\delta g_{st}|_{-1}}{\omega(s,t)^\kappa} : (s,t) \in \Delta_T \ \text{ s.t. } \varpi(s,t) \le L \right\}.$$

Then Y is compactly embedded into $C_T\mathbf{H}_w^0$.

Proof. Let $g \in Y$ be arbitrarily chosen. First, we will show that for all $\varphi \in \mathbf{H}^0$, the mapping

$$t \mapsto \langle g_t, \varphi \rangle \in C_T\mathbf{R}. \tag{A.2}$$

To this end, we observe that since $g \in L_T^\infty\mathbf{H}^0$, it follows that there exists $R > 0$ such that $g_t \in B_R$ for all $t \in [0, T]$, where $B_R \subset \mathbf{H}^0$ is a ball of radius R. Let $\{h_n\}_{n=1}^\infty \subset \mathbf{H}^1$ be a family whose finite linear combinations are dense in $\mathbf{H}^0$. Then

$$\begin{aligned} |\langle g_t, \varphi\rangle - \langle g_s, \varphi\rangle| &\le \left| \left\langle g_t - g_s, \sum_{n\le M} \beta_n h_n \right\rangle \right| + \left| \left\langle g_t - g_s, \varphi - \sum_{n\le M} \beta_n h_n, \right\rangle \right| \\ &\le \left| \left\langle g_t - g_s, \sum_{n\le M} \beta_n h_n, \right\rangle \right| + 2R \left| \varphi - \sum_{n\le M} \beta_n h_n \right|_0 \end{aligned}$$

$$\leq c(M)\omega(s,t)^{\kappa} + 2R\left|\varphi - \sum_{n\leq M}\beta_n h_n\right|_0, \tag{A.3}$$

where the last term can be made small uniformly for all $s, t \in [0, T]$ by taking M large enough and suitable $\{\beta_m\}_{m=1}^{M}$. Hence, (A.2) follows. The compactness of the embedding follows from the generalized Arzelà–Ascoli theorem. Indeed, the ball B_R is relatively weakly compact, and the desired equicontinuity follows from (A.3). □

B Sewing lemma

The following lemma, referred to as the *sewing lemma*, lies at the very foundation of the theory of rough paths. The proof is a straightforward modification of Lemma 2.1 in [10]. See, also, Lemma 4.2 in [31].

LEMMA B.1. *(c.f. Lemma 2.1 in* [10] *and Lemma 4.2 in* [31] *Let I be a subinterval of* $[0, T]$*, E be a Banach space and* $\zeta \in [0, 1)$*. Let ω and ϖ be controls on I and* $L > 0$*. Assume that $h : \Delta_I \to E$ is such that for all $(s, u, t) \in \Delta_I^{(2)}$ with $\varpi(s, t) \leq L$,*

$$|\delta h_{sut}| \leq \omega(s,t)^{\frac{1}{\zeta}}.$$

Then there exists a unique path $\mathcal{I}h : I \to E$ with $\mathcal{I}h_0 = 0$ such that $\Lambda h := h - \delta\mathcal{I}h \in C^{\zeta-\mathrm{var}}_{2,\varpi,L}(I; E)$. Moreover, there exists a universal constant $C_\zeta > 0$ such that for all $(s, t) \in \Delta_I$ with $\varpi(s, t) \leq L$,

$$|(\Lambda h)_{st}| \leq C_\zeta\,\omega(s,t)^{\frac{1}{\zeta}}. \tag{B.1}$$

Furthermore, if $h \in C^{p-\mathrm{var}}_{2,\varpi,L}(I; E)$ for some $p \geq \zeta$, then $\mathcal{I}h \in C^{p-\mathrm{var}}(I; E)$.

The following corollary is immediate since $\mathcal{I}h$ is a path with $\mathcal{I}h_0 = 0$, and hence vanishes if $\mathcal{I}h \in C^{p-\mathrm{var}}(I; E)$ for $p < 1$.

COROLLARY B.2. *Assume the hypothesis of Lemma B.1. If $h \in C^{p-\mathrm{var}}_{2,\varpi,L}(I; E)$ for some $p < 1$, then for all $(s, t) \in \Delta_I$ with $\varpi(s, t) \leq L$,*

$$|h_{st}| \leq C_\zeta\,\omega(s,t)^{\frac{1}{\zeta}}.$$

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