(In)Finite Time Dynamical Systems with Homoclinic Structures – Discretization and Approximation –

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Chapter 1

Introduction

Dynamics of the heart beat, turbulences in flows, bursting phenomena, the movement of a flag in the wind and the formation of congestion on the autobahn have one similarity: chaotic behavior. Chaotic phenomena, such as the three-body problem and turbulences are known for a long time. One of the first mathematicians who realized the existence of such irregular dynamics was Poincaré at the end of the 19th century. He examined the consequences of the existence of homoclinic points for the geometrical structure of stable and unstable manifolds. These manifolds contain solutions that converge toward a fixed point ξ in forward and backward time, respectively. Homoclinic structures of autonomous systems are for example solutions which converge in both time directions toward one hyperbolic fixed point ξ. Solutions of this kind are called homoclinic orbits $x(\cdot)$. Each homoclinic point $x(t)$ lies in the intersection of the stable and unstable manifold w.r.t. ξ. Poincaré showed that they produce complex dynamical structures. For more details and a historical overview we refer to [109], [4] and references therein. In 1935 Birkhoff confirmed this complex dynamical structure near homoclinic points by proving that these points are the limit of periodic orbits [24], [25]. All these results are theoretical in nature and the question whether transversal homoclinic points actually exist for real-life problems remained open. In the early 60s the mathematician and meteorologist Edward N. Lorenz discovered chaos in a relatively simple numerical model of a weather forecast. He proved that even small variations of the initial data lead to quite different solutions after a short period of time – the so called butterfly effect – see [96] . Increasingly powerful computers enabled extended numerical computations, which helped with the formulation of scientific problems and the identification of regularities in chaotic motions. This motivated scientists to study Poincaré's and Birkhoff's theoretical achievements about homoclinic points and to continue with their investigation. Smale constructed a geometrical structure – the so called Horseshoe map [2] – which showed the existence of homoclinic orbits and illustrated the theoretical results. Further, he in the West, cf. [121], and Shil'nikov in the East, cf. [119], proved independently that in autonomous discrete time systems the dynamic near a homoclinic point is chaotic. As Kovačiv̌ and Wiggins [89, Introduction] stated

"In fact, it is not an exaggeration to claim that in virtually every manifestation of chaotic behavior thus far, some type of homoclinic behavior is lurking in the background."

extensive studies of homoclinic orbits are essential in the field of dynamical systems. For some significant results and a historical overview we refer to [65], [49], [97] and [103]. These results and increasingly powerful computers enable numerical calculations. One way to implement numerical computations is the discretization of continuous systems, which leads to the question:

Do homoclinic orbits persist under discretization? (Q)

Around the 70s-80s it was proved that one-step methods reflect the long time behavior of differential equations [125], [31], [86], [18] and [17]. For autonomous systems the entire homoclinic orbit lies in the intersection of the stable and unstable manifold. For continuous systems this means that the stable and unstable manifold intersect tangentially. Thus, every homoclinic point of a continuous autonomous system is tangential. Fiedler and Scheurle [50] observed that under discretization with a one-step method the manifolds generically split (with an exponentially small splitting angle w.r.t. the used step size), which implies that for discretized systems there may exist transversal homoclinic orbits. Zou and Beyn [135], [137] proved that the discretization of an autonomous continuous system with a transversal homoclinic orbit induces a closed loop of homoclinic orbits, where most of these trajectories are transversal.

One part of this thesis is the analysis of the question (Q) for nonautonomous continuous systems.

The study of nonautonomous systems, in particular of nonautonomous homoclinic orbits [126], [73], is motivated by the fact that most of the systems modeling a realistic phenomena are nonautonomous, e.g. bacterial growth and tumor drug treatment [87, Section 1.2]. Further, the autonomous setup is not a special case of the nonautonomous situation. Time independent solutions generally do not exist in nonautonomous systems. Furthermore, the manifolds depend on time and are called fiber bundles. Thus, for nonautonomous systems a meaningful definition of homoclinic orbits requires convergence in both time directions toward one reference trajectory. The stable and unstable fiber bundles generally intersect transversally, which means that they only have isolated points of a homoclinic orbit in common for each time. This is a contrast to the autonomous setup, where the entire orbit lies in the intersection, see Figure 1.1. Thus, for nonautonomous continuous systems two different kinds of homoclinic orbits exist, transversal and tangential, see Figure 1.2.

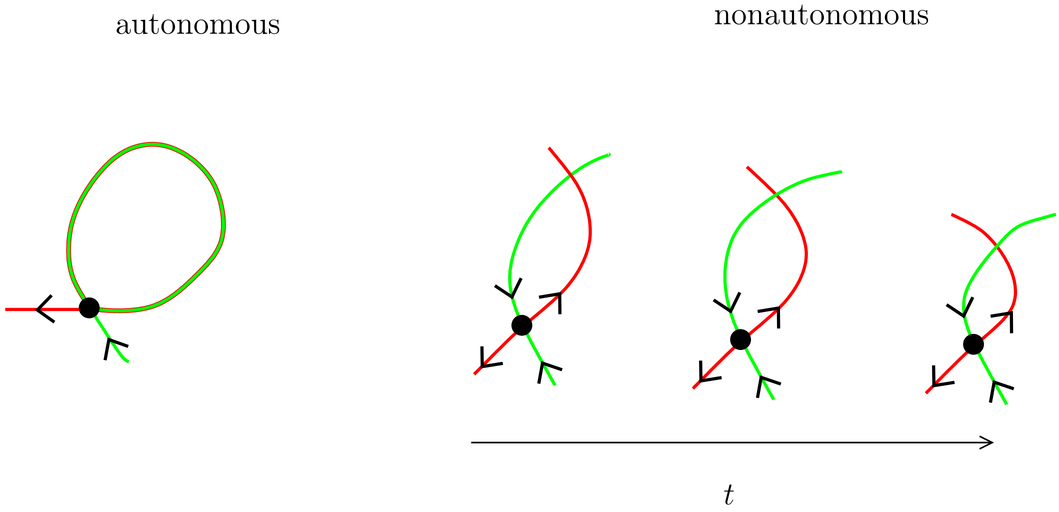

Figure 1.1: Stable (green) and unstable manifolds (autonomous, left) and fiber bundles (nonautonomous, right).

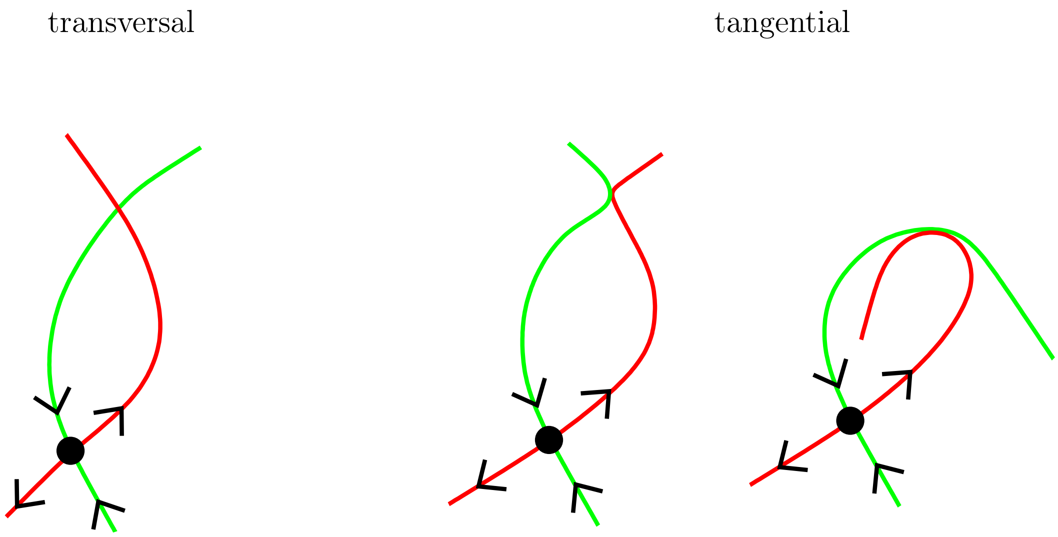

Figure 1.2: Stable (green) and unstable fiber bundles intersecting transversally (left) and tangentially (middle,right).

We discretize nonautonomous ODEs with transversal homoclinic orbits using a one-step method. Under certain conditions we prove in Theorem 7.3.6 that by using a sufficiently small step size the discretized system has a transversal homoclinic orbit as well. Further, we prove that both trajectories lie in a sufficiently small neighborhood.

In this thesis we also analyze the question (Q) for finite time continuous systems.

The theory of finite time dynamical systems is completely different from the theory of infinite time dynamical systems, since the classical asymptotic concepts do not apply to the finite time case. However, the study of these systems is important for at least three reasons. First, modeling observations or collections of data over a finite time interval result in finite time dynamical systems. On top of that, one is interested in the transient behavior of solutions, which is often quite different than the long time behavior. Finally, numerical approximations are only given for finite time intervals. About $10-20$ years ago the first studies in the finite time dynamical system theory dealt amongst other topics with the development of a proper notion of hyperbolicity [59], [62], [16], [43], [15], [12], [13], [45], [14]. Several nonequivalent definitions of finite time hyperbolic systems exist. They can roughly be separated into at least two classes. The first one is based on the concept of exponential dichotomies. We will call such systems M-hyperbolic [43, Definition 1], [14, Definition 1.2], since they require monotonic growth and decay of solutions. The second one is based on the dynamical pattern of the given system. This kind of hyperbolicity is often called D-hyperbolicity and its definition for ODEs is given in [15], [45] and in [43]. For these two classes it holds that a D-hyperbolic system is also an M-hyperbolic system. This was first proved by Haller [62] for three-dimensional continuous systems and extended by Berger et al. [14], [13, Theorem 7] for continuous systems with arbitrary dimensions. Another proof which is based on the fiber bundles of a linearization is given in [43, Theorem 21]. The analysis of discrete finite time systems is not as well-developed as the analysis of continuous finite time systems and by far not as advanced as the analysis of infinite time systems. This motivated us to develop an adequate concept for finite time systems, in particular for discrete finite time systems.

In this thesis we additionally introduce a definition of D-hyperbolicity for discrete systems and prove in Theorem 5.4.2 that a discrete D-hyperbolic system is also M-hyperbolic. Inspired by the study of homoclinic orbits in infinite time systems we develop an approach for finite time homoclinic orbits. In particular, we present an adequate analogon of infinite time fiber bundles for finite time systems that enables a definition of finite time homoclinic orbits. We call a finite time orbit $x(\cdot)$ ε-homoclinic, $\varepsilon > 0$, toward a finite time hyperbolic reference trajectory $\xi(\cdot)$ if

(1) $x(\cdot)$ lies in the intersection of the stable and unstable finite time fiber bundle w.r.t. $\xi(\cdot)$ and if

(2) both endpoints of $x(\cdot)$ each lie in an ε-ball around the corresponding endpoint of $\xi(\cdot)$.

This means that we need a notion of the finite time stable and unstable fiber bundles such that their intersection is not always empty. For finite time stable and unstable fiber bundles there exist various nonequivalent notions and to our knowledge for non of them the fibers intersect. Some authors call the stable and unstable fiber bundles area of attraction and area of repulsion and they are often defined via decay conditions. One way to define finite time fiber bundles, cf. [45], [52], [83], is based on the M-hyperbolic concept, i.e. the (un)stable finite time fiber bundle contains all points, whose orbits satisfy

(D1) monotonically decrease (increase) for all times in the finite time interval.

These fiber bundles do not intersect and, thus, for our purpose to study ε-homoclinic orbits they are not appropriate. Hence, we introduce two alternative notions of finite time fiber bundles. First we additionally require condition (2), i.e. the (un)stable finite time fiber bundle contains all points x, whose orbits $x(\cdot)$ satisfy

(D2) monotonically decrease (increase) for all time in the finite time interval and the endpoint in forward (backward) time of $x(\cdot)$ lies in an ε-ball around the corresponding endpoint of $\xi(\cdot)$.

These fiber bundles still do not intersect, but at least orbits in the fiber bundles satisfy (2). Based on the concept of infinite time fiber bundles we introduce a third notion of finite time fiber bundles. For infinite time autonomous [120, Theorem III.7] and nonautonomous [111, Corollary 4.6.11] systems the (un)stable fiber of a hyperbolic trajectory locally consist of those points, whose orbits stay for all positive times in a sufficiently small neighborhood of ξ and converges toward ξ. This means that our (un)stable finite time fiber bundle contains all points x, whose orbits $x(\cdot)$ satisfy

(D3) the endpoint in forward (backward) time of $x(\cdot)$ lies in a ε-ball around the corresponding endpoint of $\xi(\cdot)$ and monotonically increase in backward (forward) time until the orbit leaves the ε-ball (or until the orbit is not defined anymore).

For this notion of finite time fiber bundles a definition of ε-homoclinic orbits is reasonable, i.e. (1) and (2) may be satisfied for a solution. Finally we analyze whether homoclinic orbits persist under discretization. We prove in Theorem 7.3.6 under certain conditions that the discretization of a finite time system with an ε-homoclinic orbit has a $(2Ch^d + \varepsilon)$-homoclinic orbit, where $C > 0$, h is the step size of the applied one-step method of order d.

In summary, this means that for autonomous and nonautonomous systems for both infinite and finite time the answer of the question (Q) under certain conditions is:

Homoclinic orbits persist under discretization.

To verify whether a homoclinic point is transversal or tangential an analysis of the stable and unstable fiber bundles essential. For finite time systems it is well known, cf. [105, Proposition 5.4], [76, Theorem 9], [124, Theorem 4.2] and [70, Theorem 3.5], that the stable and unstable subspace of the linearization locally approximate the stable and unstable fibers. This inspires the study of the stable and unstable set of linear finite time hyperbolic systems, which are actually cones. For finite time systems Karrasch [83] proved, roughly speaking, that the (un)stable cone of the linearization locally approximates the (un)stable finite time fiber bundle of the original system, with the notion (D1) of finite time fiber bundles. These results imply a promising approach to approximate stable and unstable fiber bundles. Numerical techniques to determine the stable and unstable fiber bundles exist for autonomous infinite time systems [23], [33], [42], [44], [46], [48], [92], for nonautonomous continuous [75], [95], [26], [34] and discrete infinite time systems [112], [114], [74] as well as for finite time systems [59], [55]. Further, approximation results of homoclinic orbits in discrete and continuous autonomous systems are presented in [22] and [19], respectively, and for nonautonomous systems we refer to [70] and [112]. For autonomous infinite time systems the choice of proposed techniques is quite vast. They range from numerical continuation and boundary value problems through Taylor expansions and the parametrization method to fixed point iterations and set orientated methods. Some generalizations of these techniques apply to nonautonomous infinite time systems as well. However, for this kind of systems the literature is quite sparse, in particular for noninvertible systems.

In this thesis we introduce an algorithm which approximates fiber bundles of nonautonomous discrete infinite time systems [Section 6.7]. This algorithm is a generalization of the search circle algorithm in [48]. We develop numerical tools to approximate all ε-homoclinic orbits of a D-hyperbolic system, i.e. the intersection of the stable and unstable finite time fiber bundle. Note that in contrast to infinite time systems stable and unstable fiber bundles in finite time are fat objects. Thus, the intersection of the stable and unstable finite time fibers include more than one ε-homoclinic trajectory. We call the union of all ε-homoclinic trajectories an ε-homoclinic tube. Further we determine the width of the stable and unstable cone and establish upper bounds for the width of the stable and unstable cone as well as of the ε-homoclinic tube. Additionally, we present a more detailed proof of the local approximation Theorem presented in [83] adapted for notion (D3) of finite time fiber bundles. This means the stable and unstable cones of the linearization provide information about the stable and unstable cone, which motivates to study linear finite time systems.

For linear infinite nonautonomous time systems

$$\dot{x} = A(t)x,\ x_{n+1} = A(n)x_n,\ x \in \mathbb{R}^k,\ k \in \mathbb{N},\ t \in \mathbb{R},\ n \in \mathbb{Z}$$

it is well known that a study of the eigenvalues of the matrix $A(\cdot)$ does not help to prove hyperbolicity, see [37, p. 30] for a continuous time example due to Vinograd and for a discrete time example we refer to [47, Example 4.17]. Similar results exist for finite time systems. Autonomous continuous systems (2.6) are M-hyperbolic if the eigenvalues of A do not lie on the imaginary axis and autonomous discrete systems (2.6) are M-hyperbolic if the eigenvalues of A do not have absolute value 1. For 2-dimensional finite time systems Haller [59] presented conditions on the spectral data of A that ensure hyperbolicity of finite time systems. Nevertheless, conditions relying on spectral data have their pitfalls, shown in [12, Section 2]. For 3-dimensional finite time systems the eigenvalues of $A(\cdot)$ do not provide any information about the dynamical properties of the finite time system, for more details see [12, Section 2] and [60, Theorem 1]. Berger [12, Section 2] said about this problem

"Plausible though this may be, it is actually not true."

Therefore, we never use the eigenvalues of $A(\cdot)$ to determine hyperbolicity of a dynamical systems.

Detailed Outline of This Thesis

In **Chapter 2** the notion for this thesis is set. In Section 2.2 we introduce dynamical systems and abbreviate infinite time systems as ift-systems and finite time systems as ft-systems.

In **every chapter** we start with the study of ift-systems and continue with an analogously study of ft-systems.

Chapter 3 starts with the definition of hyperbolicity for continuous and discrete time systems on an infinite time interval. By analyzing the hyperbolicity conditions of an ift-system in Section 3.2 we get a reasonable definition of hyperbolicity for finite time systems, the so called M-hyperbolicity. In Section 3.3 we point out some important differences and similarities between hyperbolic ift-systems and M-hyperbolic ft-systems. For example the uniqueness of the invariant family of projectors of a hyperbolic ift-system and the nonuniqueness for M-hyperbolic ft-systems. The definition of an infinite time exponential dichotomy is independent of the choice of norm, whereas the definition of an finite time exponential dichotomy depends on the norm. We prove for every autonomous hyperbolic ift-system the existence of a proper norm (Lyapunov norm) such that the system is M-hyperbolic on each compact interval. In Section 3.4 we present various Roughness-Theorems, which guarantee the preservation of hyperbolicity under sufficiently small additive perturbations, for both ift- and ft-systems.

In **Chapter 4** we study stable and unstable sets of linear systems. For linear ift-systems these sets are subspaces and for linear ft-systems they are cones. We start this chapter with the definition of the stable and unstable subspaces ${}^{\mathbb{T}}V_{s,u}(\cdot)$ of an hyperbolic ift-system, which have the representation

$$^{\mathbb{T}}V_s(t_0) = \mathcal{R}(P(t_0)), \ ^{\mathbb{T}}V_u(t_0) = \mathcal{N}(P(t_0)),$$

where $t_0 \in \mathbb{T}$, $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$ and $P : \mathbb{T} \to \mathbb{R}^{k\times k}$ is the unique invariant family of projectors of the hyperbolic ift-system. Then, in Section 4.1, we derive a definition for the stable and unstable cone of an ft-system from the definition of the subspaces. In Section 4.2 we prove that the stable and unstable cones ${}^{\mathbb{I}}V_{s,u}(\cdot)$ of an M-hyperbolic ft-system satisfy

$$ {}^{\mathbb{I}}V_s(t_0) = \bigcup_{P(t_0)\in\mathcal{P}_{t_0}} \mathcal{R}(P(t_0)) \text{ and } {}^{\mathbb{I}}V_u(t_0) = \bigcup_{P(t_0)\in\mathcal{P}_{t_0}} \mathcal{N}(P(t_0)) $$

for all $t_0 \in \mathbb{I}$, $\mathbb{I} \subset \mathbb{T}$ a compact interval, where

$$ \begin{aligned} \mathcal{P}_{t_0} :=\{&P(t_0)|P:\mathbb{I}\to\mathbb{R}^{k\times k} \text{ is an invariant family of projectors such} \\ &\text{that the the given ft-system is M-hyperbolic w.r.t. these family}\}. \end{aligned} $$

This means, that the uniquely determined cones can be described by the union of the nonunique families of projectors.

An explicit representation of the stable and unstable subspaces and cones of a linear (ft-)hyperbolic system is of great interest for plotting and is established in **Chapter 5**. For D-hyperbolic systems we are able to find an explicit representation. The definitions of D-hyperbolic systems are based on a Γ-norm, $\|\cdot\|_\Gamma = \sqrt{\langle\cdot,\Gamma\cdot\rangle}$, where $\Gamma \in \mathbb{R}^{k\times k}$ is a positive definite symmetric matrix and $\langle\cdot,\cdot\rangle$ denotes the standard inner product. In Section 5.1 we analyze various types of autonomous ft-systems to find cases where a matrix Γ exists such that the given system is M-hyperbolic w.r.t. $\|\cdot\|_\Gamma$.

In Section 5.2 we define D-hyperbolic systems. Additional to the Definition for continuous ft-systems as in [15], [45] and [43] we give a Definition for discrete ft-systems. The main ingredients of the D-hyperbolicity definitions are the Γ-strain acceleration tensor $M_\Gamma(\cdot)$, the Γ-strain tensor $S_\Gamma(\cdot)$ and zero Γ-strain set $Z_\Gamma(\cdot)$. These tensors describe the numerical pattern of a given continuous system [15], [12, Proposition 2]. We deduce similar properties for discrete systems and present the results in Section 5.2.

In Section 5.3 we develop an explicit representation of stable and unstable cones of discrete D-hyperbolic systems. We state this and an explicit representation of the cones of continuous systems, which is given in [43, Proposition 19]. In Section 5.4 we prove that every D-hyperbolic system is also M-hyperbolic.

We conclude this chapter with various examples of 2- and 3-dimensional finite time systems ranging from autonomous and nonautonomous systems to continuous, discrete invertible and discrete noninvertible systems. Plots of the stable and unstable cones are shown in Section 5.5-5.7. The fact that these cones are fat objects raises the question how wide these cones are. This is a new question in the theory of finite time dynamical systems. In Section 5.6 and 5.7 we analyze the width of stable and unstable cones of invertible ft-systems in 2- and 3-dimensional spaces, respectively. We prove that the width of stable cones decays in backward time while the width of unstable cones decays in forward time. The decay depends on the eigenvalues of the Γ-strain tensor whereas the width at the boundary times depends on the relation between the eigenvalues. Further, we present upper bounds of the width, for which calculation the solution operator is not needed.

In **Chapter 6** we introduce three alternative notions of finite time fiber bundles (D1), (D2) and (D3). The first notion of fiber bundles (D1), that we present in Section 6.1, describes the monotonically stable and unstable ft-fiber bundles. This notion is based on the M-hyperbolicity concept and is also discussed in [45], [52] and [83]. In Section 6.2 and Section 6.3 we develop the other two notions of finite time fiber bundles, the monotonically ε-stable and unstable ft-fiber bundles (D2) and the ε-stable and unstable ft-fiber bundles (D3). In Section 6.4 we analyze the characteristics of the three introduced ft-fiber bundles. In particular, we study their invariance properties and we verify that only the ε-stable and ε-unstable ft-fibers may intersect. These ones are abbreviate as ft-fiber bundles. In Section 6.6 we show roughly speaking that the stable and unstable cone of the linearization locally approximate the stable and unstable ft-fiber bundles. More precisely, we prove this property for the boundaries. We conclude this chapter with a new approach to approximate the stable and unstable fiber bundles. We present an algorithm which applies to both, invertible and noninvertible ift-systems, and is a generalization of the search circle algorithm in [48]. For two examples, one infinite, one finite, we plot the stable and unstable fibers. We calculate the infinite system with the developed algorithm and the finite time systems per iteration.

Chapter 7 contains the study of homoclinic trajectories. We introduce an adequate notion for ε-homoclinic trajectories (finite time) by requiring the conditions (1) and (2). Further, we define the ε-homoclinic tube, which is the union of all ε-homoclinic trajectories of an ft-systems. The two purposes of this chapter are the development of an approach to approximate the tube and the discretization of dynamical systems with homoclinic orbits. In Section 7.1 we develop a boundary value problem which provides the boundary of the ε-homoclinic tube. In Section 7.3 we discretize infinite and finite time dynamical systems with a one-step method. We prove that under certain conditions and for sufficiently small step sizes h the discretization of an ift-system with a transversal homoclinic orbits has a transversal homoclinic orbit as well. Further, both homoclinic orbits lie in an Ch^d-neighborhood, where $C > 0$ and d is the order of the applied one-step method. A similar result holds for ft-systems. If an ft-system with an ε- homoclinic orbits is discretized by a one-step method of order d and step size h then the discretized system has a $(2Ch^d + \varepsilon)$-homoclinic orbit. To obtain this achievements we discretize in Section 7.2 continuous systems, using the h-flow. This has no practical relevance from a numerical point of view but helps to derive error estimates of one-step methods in the last section.

We conclude this thesis with **Chapter 8**, where we present three applications. All for infinite time systems. Note that we studied finite time systems and approximated the ε-homoclinic tube in Section 7.1. In this chapter we construct a 2-dimensional example with an explicitly known homoclinic orbit. Further, we compare orbits of a one-step method with the exact ones and numerically verify our error estimates. For illustrating transversality of the computed orbits we look at the corresponding stable and unstable fiber bundles of the one-step discretization. We calculate the fibers with our algorithm from Section 6.7. The second application is a periodic autonomous ODE and the third one a nonautonomous model from mathematical biology.

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Chapter 2

Basic Concepts

In this chapter the basis for this thesis is set. The first section establishes a general notation used throughout this dissertation. Further, some basic definitions are given. In subsection 2.2 we introduce dynamical systems.

Notations and Basic Definitions

In this section we first define a few symbols. Then we introduce some notations and finally define terms, which are needed in the following.

I	identity matrix
$\mathbb{R}$	real numbers
$\mathbb{R}_{\geq 0}$	real numbers ≥ 0
$\mathbb{R}_{>0}$	real numbers > 0
$\mathbb{Z}$	integral numbers
$\mathbb{N}$	natural numbers without 0
$\mathcal{S}^1$	unit sphere in $\mathbb{R}^k$
$\bar{U}$	the closure of $U \subset \mathbb{R}^k$
∂U	the boundary of $U \subset \mathbb{R}^k$
$\mathcal{C}^j$	set of j–times continuous differentiable systems

In the following we introduce various notations that are used throughout this thesis. Set $r\mathbb{Z} := \{\ldots, -3r, -2r, -r, 0, r, 2r, 3r, \ldots\}$.

We shorten discrete one sided bounded intervals by

$$\mathbb{Z}_N^+ := [N, \infty) \cap \mathbb{Z},\ \mathbb{Z}_N^- := (-\infty, N] \cap \mathbb{Z} \text{ for all } N \in \mathbb{Z}.$$

To have a uniform notation for discrete and continuous compact time intervals we will write for $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$

$$[t_-, t_+]_\mathbb{T} := \begin{cases} [t_-, t_+], & \text{for } \mathbb{T} = \mathbb{R}, \\ [t_-, t_+] \cap \mathbb{Z}, & \text{for } \mathbb{T} = \mathbb{Z}. \end{cases}$$

Analogously, we define $(t_-, t_+]_\mathbb{T}$, $[t_-, t_+)_\mathbb{T}$ and $(t_-, t_+)_\mathbb{T}$.

For a compact time interval $\mathbb{I} = [t_-, t_+]_\mathbb{T}$ we always assume $t_+ > t_-$.
For a compact interval $\mathbb{I} := [n_-, n_+]_\mathbb{Z}$ and $j \in \mathbb{N}$ we define

$$\mathbb{I}_j := [t_-, t_+ - j]_\mathbb{Z},\ {}_j\mathbb{I} := [t_- + j, t_+]_\mathbb{Z}.$$

Let $\varepsilon > 0$ and $x \in \mathbb{R}^k$ then the open and closed ε-ball around x are denoted by

$$B_\varepsilon(x) := \{y \in \mathbb{R}^k |\, \|x - y\|_2 < \varepsilon\},$$
$$B_\varepsilon[x] := \{y \in \mathbb{R}^k |\, \|x - y\|_2 \leq \varepsilon\},$$

where $\| \cdot \|_2$ is the euclidean norm.

To shorten the notation of partial derivatives, we use upper and lower indices, i.e. for $i, j \in \mathbb{N}$

$$(\varphi_n)_{x,h}^{(i,j)}(x, h) := \frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial h^j}\varphi_n(x, h),$$
$$(\varphi_n)_x(x, h) := (\varphi_n)_{x,h}^{(1,0)}(x, h) = \frac{\partial}{\partial x}\varphi_n(x, h).$$

$\Gamma \in \mathbb{R}^{k\times k}$ denotes a positive definite ($\Gamma > 0$) and symmetric ($\Gamma = \Gamma^T$) matrix. The induced Γ-norm is defined by $\| \cdot \|_\Gamma = \sqrt{\langle \cdot, \Gamma \cdot \rangle}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product.

Definition 2.1.1. *A matrix $A \in \mathbb{R}^{k\times k}$ is **degenerate** if 0 is an eigenvalue. Otherwise we say A is **nondegenerate**. We call A **positive definite** ($A > 0$) if $\langle \xi, A\xi \rangle > 0$ holds for all $\xi \in \mathbb{R}^k \setminus \{0\}$ and say A is **negative definite** ($A < 0$) if $\langle \xi, A\xi \rangle < 0$ is true for all $\xi \in \mathbb{R}^k \setminus \{0\}$. If A is not positive and not negative definite and $\langle \xi, A\xi \rangle \neq 0$ for at least one $\xi \in \mathbb{R}^k$ than A is **indefinite**.*

Remark 2.1.2. *In this paper we also call positive semi-definite and negative semi-definite matrices indefinite.*

Definition 2.1.3. *A matrix $P : \mathbb{R}^k \to \mathbb{R}^k$ is called a **projector** if $P \circ P = P$.*

Remark 2.1.4. *We denote the kernel of a matrix P by $\mathcal{N}(P)$ and the range of the matrix by $\mathcal{R}(P)$. The linear case of a vector $v \in \mathbb{R}^k$ is given by $\mathcal{L}(v) := \{\lambda v \in \mathbb{R}^k | \lambda \in \mathbb{R}\}$. And for the dimension of a subspace $U \subset \mathbb{R}^k$ we write $\dim(U)$.*

Definition 2.1.5. *Let $A \subset \mathbb{R}^k$ and $U_A \subset A$ be a subspace. We say that U_A is a subspace of maximal dimension if no subspace $\tilde{U}_A \subset A$ exists with $\dim(\tilde{U}_A) > \dim(U_A)$. If U_A is a subspace of **maximal dimension** then we say A is of dimension $\dim(U_A)$.*

Lemma 2.1.6. *Let $A, B \subset \mathbb{R}^k$ be two sets with $A \cap B = \{0\}$. If there exist subspaces $U_A \subset A$, $U_B \subset B$ with $U_A + U_B = \mathbb{R}^k$ then U_A, U_B are subspaces of maximal dimension.*

Proof. Assume w.l.o.g. there exists a subspace $\tilde{U}_A \subset A$ with $\dim(\tilde{U}_A) > \dim(U_A)$. Then by $U_A \cap U_B \subset A \cap B = \{0\}$ and the dimension formula we obtain

$$\begin{aligned} k &= \dim(U_A + U_B) = \dim(U_A) + \dim(U_B) - \dim(U_A \cap U_B) \\ &= \dim(U_A) + \dim(U_B) < \dim(\tilde{U}_A) + \dim(U_B) = k + \dim(\tilde{U}_A \cap U_B). \end{aligned}$$

This implies $0 < \dim(\tilde{U}_A \cap U_B)$ which is a contradiction to $\tilde{U}_A \cap U_B \subset A \cap B = \{0\}$. Thus U_A is a subspace of maximal dimension. □

Definition 2.1.7. *A function $f : \mathbb{I} \to \mathbb{R}$ is called* ***increasing*** *on $\mathbb{I}$ if for all $t, s \in \mathbb{I}$ with $t \geq s$ we have $f(t) \geq f(s)$. If $f(t) \leq f(s)$ for all $t, s \in \mathbb{I}$ with $t \geq s$ the function is* ***decreasing*** *on $\mathbb{I}$. When we write $f(t)$ is increasing (decreasing) for $t \in \mathbb{I}$ we mean that $f : \mathbb{I} \to \mathbb{R}$ is increasing (decreasing) on $\mathbb{I}$.*

Dynamical Systems

Dynamical systems can be categorized into the following pairs of classes: finite-dimensional and infinite-dimensional systems, continuous and discrete systems, invertible and noninvertible systems as well as autonomous and nonautonomous systems. In this thesis we restrict the study of dynamical systems to finite-dimensional systems. Systems of this kind are generated by e.g. ordinary differential equations, ordinary differential inequalities, ordinary difference equations and ordinary difference inequalities. For a deeper discussion of dynamical systems we refer to [99] and [100]. A historical overview of the field of differential equations and its developments is presented in [78, Section 11.1] and [101, Introduction]. Many problems, such as oscillating circuits, population dynamics, diagnosis of diseases, ocean eddies, tornados and discovery of art forgery, in different fields ranging from physics and biology to geology and sociology can be represented by a differential or difference equation, see [134], [79], [129], [118] and [63]. Thus, the theory of dynamical systems provides powerful tools to analyze such problems.

In this section we introduce dynamical systems that are generated by ordinary differential and ordinary difference equations. Further, we define and analyze some properties of invariant families of projectors, which play a decisive role for hyperbolic systems. Roughly speaking they provide all solutions that decay or grow at certain rates.

For the definition of an autonomous dynamical system and the differences between autonomous and nonautonomous systems we refer to [90, Section 1.1] and [88, Definition 2.1 ff.]. Here, we introduce nonautonomous dynamical systems on the Banach space $\mathbb{R}^k$, $k \in \mathbb{N}$, which include autonomous systems as well, see [88, Definition 2.1].

Definition 2.2.1. *A (nonautonomous)* ***dynamical system*** *is a triple*

$$(\mathbb{R}^k, \mathbb{T}, \varphi),$$

where $\mathbb{T}$ is a time set and $\varphi : X \times \mathbb{T} \times \mathbb{T} \to X$ is a function with the properties

- $\varphi(x, t, t) = x$ *for all* $t \in \mathbb{T}, x \in \mathbb{R}^k$,
- $\varphi(\varphi(x, s, r), t, s) = \varphi(x, t, r)$ *for all* $t, s, r \in \mathbb{T}, t \geq s \geq r, x \in \mathbb{R}^k$.

*A dynamical system is **invertible** if the function $\varphi(\cdot, t, s)$ is invertible for all $t, s \in \mathbb{T}$. It is an **infinite time** system if $\mathbb{T}$ is infinite and a **finite time** system if $\mathbb{T}$ is finite. If $\mathbb{T} \subset \mathbb{R}$ is an interval then the dynamical system is called a **continuous** system and if $\mathbb{T} \subset \mathbb{Z}$ is a discrete time interval – in the following just called an interval – then the system is called a **discrete** system.*

For some results we have to distinguish between infinite time systems and finite time systems. We abbreviate infinite time systems as **ift-systems** and finite time systems as **ft-systems**.

To set a general notion consider a dynamical system

$$(\mathbb{R}^k, \mathbb{I}, \varphi) \text{ with } \varphi \in \mathcal{C}^1(\mathbb{R}^k \times \mathbb{I} \times \mathbb{I}, \mathbb{R}^k) \tag{2.1}$$

where $\mathbb{I}$ denotes an interval. In continuous time, $\mathbb{I} \subset \mathbb{R}$, a differential equation

$$\dot{x}(t) = f(x(t), t), \ t \in \mathbb{I} \tag{2.2}$$

generates such a dynamical system (2.1) and for $f \in \mathcal{C}^{1,0}(\mathbb{R}^k \times \mathbb{I}, \mathbb{R}^k)$ solutions of (2.2) with an initial value $x(t_0) = x_0$ locally exist and are unique (Picard-Lindelöf Theorem [3, Theorem 8.14]). This leads to an invertible solution operator φ. The infinitesimal generator is

$$f(x_0, t) = \lim_{h \to 0, h \in \mathbb{R} \setminus \{0\}} \frac{\varphi(x_0, t+h, t) - x_0}{h}, \ t, t+h \in \mathbb{I}$$

for an initial value $x_0 \in \mathbb{R}^k$. If $\mathbb{I} \subset \mathbb{Z}$ is a discrete time set a difference equation

$$x(n+1) = f(x(n), n), \ n \in \mathbb{I} \tag{2.3}$$

generates a dynamical system (2.1). In contrast to (2.2), the solution operator φ of (2.3) is generally not invertible. If the system is invertible then the solution operator is invertible and satisfies

$$\varphi(u, n, m) := \begin{cases} f(f(\cdots f(u, m), \cdots n-2), n-1), & \text{for } n > m, \\ u, & \text{for } n = m, \\ f^{-1}(f^{-1}(\cdots f^{-1}(u, m-1), \cdots n-1), n), & \text{for } n < m, \end{cases}$$

see [10]. To define hyperbolicity of a solution $\xi(t) = \varphi(\xi(s), t, s), t, s \in \mathbb{I}, t \geq s$ of (2.2) or (2.3) we need the linearization, the variational equation, of (2.2) along $\xi(\cdot)$

$$\dot{u}(t) = f_x(\xi(t), t)u(t) =: A(t)u(t), \ t \in \mathbb{I} \tag{2.4}$$

respectively of (2.3),

$$u(n+1) = f_x(\xi(n), n)u(n) =: A(n)u(n), \;\; n \in \mathbb{I}. \tag{2.5}$$

These equations generate a linear dynamical system

$$(\mathbb{R}^k, \mathbb{I}, \Phi) \text{ with } \Phi(t,s) = \varphi_x(\xi(s), t, s), \;\; t, s \in \mathbb{I}, t \geq s.$$

If φ is the solution operator of (2.2) or (2.3) then Φ is the solution operator of the variational equation (2.4) or (2.5), respectively. In general, we denote a linear dynamical system by

$$(\mathbb{R}^k, \mathbb{I}, \Phi) \text{ with } \Phi \in \mathcal{C}^1(\mathbb{I} \times \mathbb{I}, \mathbb{R}^{k \times k}), \tag{2.6}$$

which is generated by a linear differential equation

$$\dot{u}(t) = A(t)u(t), \;\; t \in \mathbb{I} \tag{2.7}$$

or a linear difference equation

$$u(n+1) = A(n)u(n), \;\; n \in \mathbb{I}_1, \tag{2.8}$$

where $\mathbb{I}_1 := [n_-, n_+ - 1]_{\mathbb{Z}}$ if $\mathbb{I} = [n_-, n_+]_{\mathbb{Z}}$. In the following we assume that

$$f \in \mathcal{C}^{(1,0)}(\mathbb{R}^k \times \mathbb{I}, \mathbb{R}^k), \;\; A \in \mathcal{C}^0(\mathbb{I}, \mathbb{R}^{k \times k})$$

for f and A of the equations (2.2), (2.3), (2.7) and (2.8).

Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$ and $\mathbb{I} \subset \mathbb{T}$ be an interval.

Definition 2.2.2. *A **trajectory** of a dynamical system* (2.1) *is a function $x : \mathbb{I} \to \mathbb{R}^k$ satisfying $x(t) = \varphi(x(s), t, s)$ for all $s, t \in \mathbb{I}$, $t \geq s$.*

For the manageability, it is sometimes helpful to use a linear notation even if the function is actually not linear. Let $X \subset \mathbb{R}^k$, $t, s \in \mathbb{I}$, $t \geq s$ then we define

$$\tilde{\varphi}(t,s)X := \{\varphi(x,t,s) | x \in X\}$$

and identify $\tilde{\varphi}$ with φ. For hyperbolic dynamical systems, which we introduce in Chapter 3, the definition of an invariant family of projectors is required, see e.g. in [115, Definition 4.2]. This family yields roughly speaking an invariant family of subspaces, which contains solutions that decay with a certain rate and an invariant family of subspaces, which contains solutions that grow with a certain rate.

Definition 2.2.3. *We call a **family of projectors** $\boldsymbol{P} : \mathbb{I} \to \mathbb{R}^{k \times k}$ of the dynamical system* (2.6) ***invariant** if the projectors fulfill*

$$\Phi(t,s)P(s) = P(t)\Phi(t,s)$$

for all $t, s \in \mathbb{I}$ with $t \geq s$.

Basic properties of invariant families of projectors, that we need in the following, are summarized in [132, Proposition 6.82] and in the next lemma.

Lemma 2.2.4. *Let $\mathbb{I}$ be a interval and $P : \mathbb{I} \to \mathbb{R}^{k \times k}$ be an invariant family of projectors, then*

$$\Phi(t,s)\mathcal{R}(P(s)) \subset \mathcal{R}(P(t)),\ \Phi(t,s)\mathcal{N}(P(s)) \subset \mathcal{N}(P(t)) \text{ for all } t,s \in \mathbb{I},\ t \geq s.$$

Proof. Let $t, s \in \mathbb{I}$, $t \geq s$ and $\xi \in \mathcal{R}(P(s))$. Then we have $P(s)\xi = \xi$ and we obtain by the invariance of the projectors

$$\Phi(t,s)\xi = \Phi(t,s)P(s)\xi = P(t)\Phi(t,s)\xi.$$

This implies $\Phi(t,s)\xi \in \mathcal{R}(P(t))$. Thus, $\Phi(t,s)\mathcal{R}(P(s)) \subset \mathcal{R}(P(t))$ is satisfied. Let $\nu \in \mathcal{N}(P(s))$, then we have $P(s)\nu = 0$ and by the invariance of the projectors we get

$$0 = \Phi(t,s)P(s)\nu = P(t)\Phi(t,s)\nu.$$

This implies $\Phi(t,s)\nu \in \mathcal{N}(P(t))$. Thus, $\Phi(t,s)\mathcal{N}(P(s)) \subset \mathcal{N}(P(t))$ is satisfied. □

For invertible systems we can easily construct an invariant family of projectors. For two subsets $X_s, X_u \subset \mathbb{R}^k$ such that $X_s \oplus X_u = \mathbb{R}^k$ we can define an invariant family of projectors $P : \mathbb{I} \to \mathbb{R}^k$ by

$$\begin{aligned} \mathcal{N}(P(t)) &:= \Phi(t,t_0)X_u, \\ \mathcal{R}(P(t)) &:= \Phi(t,t_0)X_s \end{aligned}$$

for all $t \in \mathbb{I}$ and a fixed $t_0 \in \mathbb{I}$. For noninvertible systems this is not as simple as for invertible systems. The following Lemma yields a construction of a family of projectors such that this family is invariant.

Lemma 2.2.5. *Let $\mathbb{I} = [t_-, t_+]_{\mathbb{Z}}$ and $\Phi(\cdot,\cdot)$ be the solution operator of* (2.6). *Let $X_s, X_u \subset \mathbb{R}^k$ be two subspaces such that*

$$X_s \oplus X_u = \mathbb{R}^k \text{ and } \dim(\Phi(t,t_-)X_u) = \dim(X_u)$$

for all $t \in \mathbb{I}$. Then the family of projectors $P : \mathbb{I} \to \mathbb{R}^k$, recursively defined by

$$\begin{aligned} \mathcal{N}(P(t)) &:= \Phi(t,t_-)X_u \text{ for } t \in \mathbb{I}, \\ \mathcal{R}(P(t)) &:= \begin{cases} X_s, & \text{for } t = t_-, \\ \Phi(t,t-1)\mathcal{R}(P(t-1)) + W_s(t), & \text{for } t \in \mathbb{I}, t \neq t_-, \end{cases} \end{aligned}$$

is invariant, where $W_s(t) \subset \mathbb{R}^k$ for $t \in \mathbb{I}_1$ are subspaces such that

$$(\Phi(t,t-1)\mathcal{R}(P(t-1)) + W_s(t)) \oplus \Phi(t,t_-)X_u = \mathbb{R}^k.$$

Proof. The invariance of X_u yields that the family of projectors $P : \mathbb{I} \to \mathbb{R}^k$ is well defined. Let $t \in {}_1\mathbb{I}$ and $x \in \mathbb{R}^k$. Then there exist $x_s \in \mathcal{R}(P(t-1)), x_u \in \mathcal{N}(P(t-1))$ such that $x = x_s + x_u$ and

$$P(t-1)x = P(t-1)x_s + P(t-1)x_u = x_s + 0 = x_s \tag{2.9}$$

hold. We define $y_{s,u} := \Phi(t, t-1)x_{s,u}$. For $x_u \in \mathcal{N}(P(t-1)) = \Phi(t-1, t_-)X_u$ there exists a $x_u^u \in X_u$ such that $x_u = \Phi(t-1, t_-)x_u^u$. For y_u we obtain

$$\begin{aligned} y_u &= \Phi(t, t-1)x_u = \Phi(t, t-1)\Phi(t-1, t_-)x_u^u \\ &= \Phi(t, t_-)x_u^u \in \Phi(t, t_-)X_u = \mathcal{N}(P(t)). \end{aligned}$$

Thus

$$P(t)y_u = 0 \tag{2.10}$$

holds. The definition of the ranges $\mathcal{R}(P(\cdot))$ yields

$$y_s = \Phi(t, t-1)x_s \in \Phi(t, t-1)\mathcal{R}(P(t-1)) \subset \mathcal{R}(P(t)).$$

This leads to

$$P(t)y_s = y_s. \tag{2.11}$$

By equation (2.9)-(2.11) we get

$$\begin{aligned} P(t)\Phi(t, t-1)x = P(t)\Phi(t, t-1)x_s + P(t)\Phi(t, t-1)x_u = P(t)y_s + P(t)y_u \\ = y_s = \Phi(t, t-1)x_s = \Phi(t, t-1)P(t-1)x. \end{aligned}$$

Inductively we obtain $P(t)\Phi(t, s) = \Phi(t, s)P(s)$ for all $t, s \in \mathbb{I}$ with $t \geq s$. Hence, the family of projectors is invariant. □

Chapter 3

Hyperbolicity

An important tool to characterize structural stability of a dynamical system is hyperbolicity, see e.g. [103], [27]. In the early eighties Mañe [98] proved that stable systems generated by a $\mathcal{C}^1$ diffeomorphism must be hyperbolic. About fifteen years later Hayashi [64] showed the same statement for $\mathcal{C}^1$ flows.

We start by defining hyperbolicity for continuous and discrete time systems on an infinite time interval. By analyzing the hyperbolicity conditions of an ift-system we get a reasonable definition of hyperbolicity for finite time systems. In Section 3.3 we point out some important differences and similarities between hyperbolic ift-systems and hyperbolic ft-systems. In the last Section 3.4 we present different Roughness-Theorems, which guarantee the preservation of hyperbolicity under sufficiently small additive perturbations.

Infinite Time Hyperbolicity

In the following we define hyperbolicity using the notion of exponential dichotomies, which has been developed for continuous ift-systems by e.g. Coppel [40] and Palmer [104, Chapter 2]. For discrete invertible ift-systems an exponential dichotomy, see [105], is similarly defined as in the continuous time case. If a dynamical ift-systems is noninvertible, we may assume for an invariant family of projectors $P : \mathbb{I} \to \mathbb{R}^k$ the **regularity condition**

$$\Phi(t,s)|_{\mathcal{N}(P(s))} : \mathcal{N}(P(s)) \to \mathcal{N}(P(t)) \text{ is invertible for all } t, s \in \mathbb{I}, t \geq s,$$

where Φ denotes the solution operator. The definition of an exponential dichotomy for noninvertible systems in [36, page 549], [21, Definition 17], [66, Definition 7.6.1], [10, Definition 2.2] and [80, Definition 4.3] is based on this regular condition, whereas the definition in [81, Definition 2.1.2] does not require the condition. If an exponential dichotomy for noninvertible ift-systems is additionally defined with the regularity condition, in contrast to without, then the statements about hyperbolic invertible ift-systems are fundamentally transferable. In this dissertation we study noninvertible systems as well. Accordingly, we use [66, Definition 7.6.1] as a basis.

Definition 3.1.1. *Let* $\mathbb{I}$ *be an infinite time interval. The dynamical system* (2.6) *has an* ***exponential dichotomy*** *on* $\mathbb{I}$ *if there exist constants* $K, \alpha, \beta > 0$ *and an invariant family of projectors* $P : \mathbb{I} \to \mathbb{R}^{k\times k}$ *such that the following holds*

$$\Phi(t,s)|_{\mathcal{N}(P(s))} : \mathcal{N}(P(s)) \to \mathcal{N}(P(t)) \text{ is invertible for all } t,s \in \mathbb{I}, t \geq s, \quad (3.1)$$

and the estimates

$$\|\Phi(t,s)P(s)\| \leq Ke^{-\alpha(t-s)}, \quad \|\Phi(s,t)(\mathrm{I} - P(t))\| \leq Ke^{-\beta(t-s)} \quad (3.2)$$

are satisfied for all $t, s \in \mathbb{I}$, $t \geq s$, *where* $\Phi(s,t)$ *denotes the inverse of* $\Phi(t,s)|_{\mathcal{N}(P(s))}$. *The corresponding* ***data*** *are* $(K, \alpha, \beta, P(\cdot))$ *and we call such a system* ***hyperbolic****.*

The constant α is often called the **decay rate** and β the **growth rate**. If we are not interested in the exact decay or growth rate then the data are presented by $(K, \bar{\alpha}, P(\cdot))$, where $\bar{\alpha} := \min\{\alpha, \beta\}$. On the other hand the data may alternatively have the form $(K, \alpha, \beta, P(\cdot), Q(\cdot))$, where $P(\cdot)$ denotes the family of stable projectors, which fulfills $\|\Phi(t,s)P(s)\| \leq Ke^{-\alpha(t-s)}$ and $Q(\cdot) := \mathrm{I} - P(\cdot)$ denotes the family of unstable projectors, which fulfills $\|\Phi(s,t)Q(t)\| \leq Ke^{-\beta(t-s)}$ for $t, s \in \mathbb{I}, t \geq s$.

Definition 3.1.2. *Let* $(\mathbb{I} \subset \mathbb{R})/(\mathbb{I} \subset \mathbb{Z})$ *be an infinite interval then the linear equation* (2.7) /(2.8) *is called* ***hyperbolic*** *if the corresponding dynamical system* (2.6) *has an exponential dichotomy in the sense of Definition 3.1.1. A trajectory* $\xi(\cdot)$ *of system* (2.1) *generated by equation* (2.2)/(2.3) *is* ***hyperbolic*** *if the corresponding variational equation* (2.4)/(2.5) *is hyperbolic.*

In the next section we derive a reasonable definition of hyperbolicity for finite time systems. This definition uses a vector-norm, not a matrix norm, and we will see that finite time hyperbolicity depends on the chosen norm. By the equivalence of two norms in $\mathbb{R}^{k\times k}$ we immediately observe that Definition 3.1.1 is independent of the choice of the norm, i.e. that the hyperbolicity estimates in (3.2) do not depend on the chosen norm. Using an induced matrix norm in (3.2), yields that Definition 3.1.1 can be rewritten as follows.

Lemma 3.1.3. *Let* $\mathbb{I}$ *be an infinite time interval and* $P : \mathbb{I} \to \mathbb{R}^{k\times k}$ *an invariant family of projectors of* (2.6) *satisfying* (3.1). *Let* $\alpha, \beta > 0$. *Then there exists a constant* $K > 0$ *such that* (3.2) *is satisfied with an induced matrix norm* $\|\cdot\|_M$ *if and only if there exist constants* $C, C' > 0$ *such that the following estimates hold for all* $t, s \in \mathbb{I}$, $t \geq s$

$$\|\Phi(t,s)\xi\| \leq Ce^{-\alpha(t-s)}\|\xi\| \qquad \text{for all } \xi \in \mathcal{R}(P(s)), \quad (3.3)$$

$$\|\Phi(s,t)\xi\| \leq Ce^{-\beta(t-s)}\|\xi\| \qquad \text{for all } \xi \in \mathcal{N}(P(t)), \quad (3.4)$$

$$\|P(s)\|_M \leq C', \quad (3.5)$$

where $\|\cdot\|$ *is the corresponding vector norm to* $\|\cdot\|_M$.

Proof. Assume that the linear dynamical system (2.6) has an exponential dichotomy on $\mathbb{I}$ with data $(K, \alpha, \beta, P(\cdot))$. Let $s \in \mathbb{I}$ and $\xi \in \mathcal{R}(P(s))$. Let the estimates in (3.2) be satisfied. Then for every $t \geq s$ we obtain

$$\|\Phi(t,s)\xi\| = \|\Phi(t,s)P(s)\xi\| \leq \|\Phi(t,s)P(s)\|_M \|\xi\| \leq Ke^{-\alpha(t-s)}\|\xi\|$$

and analogously, for $\xi \in \mathcal{N}(P(s))$, the estimate (3.4) is satisfied. The boundedness of the projectors (3.5) follows directly from (3.2) with $t = s$. Conversely, let (3.3)-(3.5) be true then we get for an induced matrix norm and for all $t, s \in \mathbb{I}$, $t \geq s$

$$\begin{aligned}\|\Phi(t,s)P(s)\|_M &= \sup_{\xi\in\mathbb{R}^k} \frac{\|\Phi(t,s)P(s)\xi\|}{\|\xi\|} = \|P(s)\|_M \sup_{\xi\in\mathbb{R}^k} \frac{\|\Phi(t,s)P(s)\xi\|}{\|P(s)\|_M\|\xi\|} \\ &\leq \|P(s)\|_M \sup_{\xi\in\mathbb{R}^k} \frac{\|\Phi(t,s)P(s)\xi\|}{\|P(s)\xi\|} \leq \|P(s)\|_M \sup_{\xi\in\mathcal{R}(P(s))} \frac{\|\Phi(t,s)\xi\|}{\|\xi\|} \\ &\leq \|P(s)\|_M Ce^{-\alpha(t-s)}.\end{aligned}$$

With equation (3.5) the first estimate in (3.2) holds. Analogously, we can conclude the second estimate in (3.2). □

The estimates (3.3)-(3.5) are well known for defining hyperbolic dynamical systems. For example in [10, Definition 2.2] we find an exponential dichotomy definition, which is similar to the estimates (3.3)-(3.5). Kalkbrenner used in [81, pp.6-7] a similar notation for the hyperbolicity definition of noninvertible systems.

Finite Time Hyperbolicity (M-Hyperbolicity)

In the literature several nonequivalent definitions of finite time hyperbolic systems exist. They can roughly be separated into at least two classes. The first one is based on the concept of exponential dichotomies – we will call such systems M-hyperbolic – and the second one is based on the dynamical pattern of the given system. This kind of hyperbolicity we call D-hyperbolicity. We will define and study both classes. In this chapter we are focused on M-hyperbolic systems and in Chapter 5 we introduce and discuss D-hyperbolic systems. For continuous systems we find a general definition of M-hyperbolicity in [43, Definition 1] and [14, Definition 1.2]. In [16, Definition 2] and [13, Definition 1] the same definition of M-hyperbolicity is presented using a special type of norm, which is also used for defining D-hyperbolicity in [15], [45] and in [43]. We study this special type of norm – the so called Γ-norm – in Section 5.1. Karrasch [82, Definition 3.3] used growth rates to define hyperbolicity of invertible systems and proved in [82, Lemma 3.5] that his definition is equivalent to the D-hyperbolicity definition. For proving whether a system is M-hyperbolic it would be convenient to check concrete conditions on the spectral data of $A(t)$ of (2.4)/(2.5). For ift-systems it is well known that a study of the eigenvalues of the matrix $A(\cdot)$ does not help to prove hyperbolicity, see [37, p. 30] for a continuous time example due to Vinograd and for a discrete time example we refer to [47, Example 4.17]. Similar results exist for ft-systems. Autonomous continuous ft-systems (2.6) are M-hyperbolic if the eigenvalues of A do not lie on the imaginary axis and autonomous discrete ft-systems (2.6) are M-hyperbolic if the eigenvalues of A do not have absolute value 1. For 2-dimensional ft-systems Haller presented in [59] conditions on the spectral data of A that ensure hyperbolicity of ft-systems. Nevertheless, conditions relying on spectral data have their pitfalls, shown in [12, Section 2]. For 3-dimensional ft-systems the eigenvalues of $A(\cdot)$ do not provide any information about the dynamical properties of the ft-system, for more details see [12, Section 2] and [60, Theorem 1]. An alternative is to prove D-hyperbolicity of an ft-system. This might be easier than proving M-hyperbolicity, since the definition of D-hyperbolicity is based on the dynamical pattern of the ft-system. In Section 5.4 we prove that every D-hyperbolic system is also M-hyperbolic.

In this section we only introduce M-hyperbolic systems and we derive the definition (Definition 3.2.3) from the definition of a hyperbolic ift-system (Lemma 3.1.3).

If $\mathbb{I}$ is a compact interval then we observe that the estimate (3.5) is always satisfied for a sufficiently large constant $C' > 0$. Further, the estimates (3.3) and (3.4) are true for every invariant family of projectors $P : \mathbb{I} \to \mathbb{R}^{k\times k}$ satisfying (3.1) by choosing the constant C sufficiently large. To avoid this we need to fix C. The fixing of C causes a dependence on the norm of the hyperbolicity estimates (3.3) and (3.4).

Definition 3.2.1. *Fix $K \in [1,\infty)$ and let $\mathbb{I}$ be a compact interval. Then the dynamical system* (2.6) *is **K-hyperbolic** (on $\mathbb{I}$ and w.r.t the norm $\|\cdot\|$) if there exists an invariant family of projectors $P : \mathbb{I} \to \mathbb{R}^{k\times k}$, which fulfills* (3.1)*, together with exponential rates $\alpha, \beta > 0$ such that for all $t, s \in \mathbb{I}$, $t \geq s$*

$$\begin{aligned} \|\Phi(t,s)\xi\| &\leq Ke^{-\alpha(t-s)}\|\xi\| \quad \text{for all } \xi \in \mathcal{R}(P(s)), \\ \|\Phi(s,t)\xi\| &\leq Ke^{-\beta(t-s)}\|\xi\| \quad \text{for all } \xi \in \mathcal{N}(P(t)) \end{aligned} \tag{3.6}$$

are satisfied, where $\Phi(s,t)$ denotes the inverse of $\Phi(t,s)|_{\mathcal{N}(P(s))}$.

This definition has its pros and cons. In the next example we show one disadvantage, which motivates the search for another definition of finite time hyperbolicity. In this example we prove that the nonhyperbolic ift-system generated by the zero matrix is K-hyperbolic for any $K > 1$ and on any compact interval.

Example 3.2.2. *Consider the linear differential equation*

$$\dot{x} = A(t)x, \ \text{with } x \in \mathbb{R}^2 \text{ and } A(t) = 0 \in \mathbb{R}^{2\times 2} \tag{3.7}$$

for all $t \in \mathbb{I} = [t_-, t_+] \neq \emptyset$*. Then the solution operator* Φ *of* (3.7) *is the identity for all times. Define* $P(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *for all* $t \in \mathbb{I}$*. Fix* $K > 1$ *and define*

$$\alpha := \beta := \frac{\log(K)}{t_+ - t_-} > 0.$$

Choose an arbitrary $s \in \mathbb{I}$ *and a norm* $\|\cdot\|$ *on* $\mathbb{R}^2$*. Then we get for all* $\xi \in \mathcal{R}(P(s))$ *and* $t \in \mathbb{I}$ *with* $t \geq s$

$$Ke^{-\alpha(t-s)}\|\xi\| = Ke^{-\log(K)\frac{(t-s)}{t_+-t_-}}\|\xi\| \geq Ke^{-\log(K)}\|\xi\| = \|\xi\| = \|\Phi(t,s)\xi\|$$

and for all $\xi \in \mathcal{N}(P(s))$ *and* $s \leq t \in \mathbb{I}$

$$Ke^{-\beta(t-s)}\|\xi\| = Ke^{-\log(K)\frac{(t-s)}{t_+-t_-}}\|\xi\| \geq Ke^{-\log(K)}\|\xi\| = \|\xi\| = \|\Phi(s,t)\xi\|.$$

The estimates in (3.6) *are satisfied for an arbitrary, but fixed constant* $K > 1$*. Thus, our example is* K*-hyperbolic with* $K > 1$ *on any compact interval.*

This should not be the case, at least not for the trivial nonhyperbolic system. Hence, the only constant for which the system is not K-hyperbolic is $K = 1$. We choose 1-hyperbolicity as an adequate notion of hyperbolicity for ft-systems.

Solutions of hyperbolic ift-systems, which lie on the stable or unstable manifold of a fixed point x, converge toward the fixed point, but in general not strictly monotone. We require this monotonicity for solutions of ft-systems by setting $K = 1$. To point out this essential difference we call hyperbolic ft-systems M-hyperbolic. We define hyperbolicity for the finite time context as in [16].

Definition 3.2.3. *Let* $\mathbb{I}$ *be a compact interval. The dynamical system* (2.6) *has an* ***ft-exponential dichotomy*** *(finite time) (on* $\mathbb{I}$ *and w.r.t. the norm* $\|\cdot\|$*) if there exist an invariant family of projectors* $P : \mathbb{I} \to \mathbb{R}^{k\times k}$ *and exponential rates* $\alpha, \beta > 0$ *with following properties. The solution operator*

$$\Phi(t,s)|_{\mathcal{N}(P(s))} : \mathcal{N}(P(s)) \to \mathcal{N}(P(t)) \text{ is invertible for all } t,s \in \mathbb{I}, t \geq s \quad (3.8)$$

and the estimates

$$\|\Phi(t,s)\xi\| \leq e^{-\alpha(t-s)}\|\xi\| \qquad \text{for all } \xi \in \mathcal{R}(P(s)), \quad (3.9)$$

$$\|\Phi(s,t)\xi\| \leq e^{-\beta(t-s)}\|\xi\| \qquad \text{for all } \xi \in \mathcal{N}(P(t)) \quad (3.10)$$

are satisfied for all $t, s \in \mathbb{I}$*,* $t \geq s$*, where* $\Phi(s,t)$ *denotes the inverse of* $\Phi(t,s)|_{\mathcal{N}(P(s))}$*. The corresponding* ***data*** *are* $(\alpha, \beta, P(\cdot))$ *and we call such a system* ***M-hyperbolic*** *(monotonically hyperbolic on* $\mathbb{I}$ *w.r.t.* $\|\cdot\|$*).*

Definition 3.2.4. *Let* $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$ *and* $\mathbb{I} \subset \mathbb{T}$ *be a compact interval. The linear equation* (2.7)/(2.8) *is called* ***M-hyperbolic*** *(monotonically hyperbolic on* $\mathbb{I}$ *w.r.t. the chosen norm) if the corresponding dynamical system* (2.6) *has an ft-exponential dichotomy in the sense of Definition 3.2.3. A trajectory* $\xi(\cdot)$ *of system* (2.1) *generated by equation* (2.2)/(2.3) *is* ***M-hyperbolic*** *(on* $\mathbb{I}$ *w.r.t. the chosen norm) if the corresponding variational equation* (2.4)/(2.5) *is M-hyperbolic (on* $\mathbb{I}$ *w.r.t. the chosen norm).*

It is easy to see that 1-hyperbolicity ($K = 1$) is equivalent to M-hyperbolicity. We already mentioned one serious disadvantage of K-hyperbolic systems for $K > 1$, i.e. that the trivial nonhyperbolic ift-system $\dot{x} = 0$ is K-hyperbolic on every finite time interval for $K > 1$. Thus, why should we be interested in K-hyperbolic ($K > 1$) systems? Not all hyperbolic systems on an infinite time interval $\mathbb{I}$ are M-hyperbolic on a finite time subinterval $\mathbb{J} \subset \mathbb{I}$. At least for each finite time interval $\mathbb{J} \subset \mathbb{I}$ there exists a constant $K > 1$ such that the given hyperbolic system is K-hyperbolic on $\mathbb{J}$.

Note that we required for M-hyperbolicity that $-\alpha < 0 < \beta$. A analysis of the consequences if we require $-\alpha < C < \beta$ for a constant $C \in \mathbb{R}$ (it is possible that $0 < -\alpha$ or $\beta < 0$) instead of $-\alpha < 0 < \beta$ might lead to new results but is beyond the scope of this thesis. For the infinite time case there exist studies concerning the spectral splitting described above, cf. [8] and [9].

To prove whether a system is M-hyperbolic we have to verify the estimates (3.9) and (3.10). We show equivalent inequalities as well as invariant properties of the kernel and range of an invariant family of projectors which satisfies (3.8), (3.9) and (3.10).

Lemma 3.2.5. *Let system* (2.6) *be M-hyperbolic on a compact interval* $\mathbb{I}$ *with an invariant family of projectors* $P : \mathbb{I} \to \mathbb{R}^{k \times k}$*. Then*

$$\Phi(t,s)\mathcal{R}(P(s)) \subset \mathcal{R}(P(t)) \text{ for all } t, s \in \mathbb{I},\ t \geq s, \tag{3.11}$$

$$\Phi(t,s)\mathcal{N}(P(s)) = \mathcal{N}(P(t)) \text{ for all } t, s \in \mathbb{I}. \tag{3.12}$$

If the solution operator is invertible we have for all $t, s \in \mathbb{I}$

$$\Phi(t,s)\mathcal{R}(P(s)) = \mathcal{R}(P(t)). \tag{3.13}$$

Proof. Let $t, s \in \mathbb{I}$ and $t \geq s$. By Lemma 2.2.4 we get (3.11) and

$$\Phi(t,s)\mathcal{N}(P(s)) \subset \mathcal{N}(P(t)).$$

The M-hyperbolicity yields that $\Phi(t,s)|_{\mathcal{N}(P(s))} : \mathcal{N}(P(s)) \to \mathcal{N}(P(t))$ is invertible for all $t, s \in \mathbb{I},\ t \geq s$. Then the dimension of both sets satisfies $\dim(\mathcal{N}(P(s))) = \dim(\mathcal{N}(P(t)))$ for all $t, s \in \mathbb{I}$. Since we already showed that the left-hand side of (3.12) is a subset of the right-hand side we get equality in (3.12). Assuming that the solution operator is invertible we obtain that

$$\Phi(t,s) : \mathcal{R}(P(s)) \oplus \mathcal{N}(P(s)) = \mathbb{R}^k \to \mathbb{R}^k = \mathcal{R}(P(t)) \oplus \mathcal{N}(P(t))$$

is invertible for all $t, s \in \mathbb{I},\ t \geq s$. By (3.8) we get the invertibility of

$$\Phi(t,s)|_{\mathcal{R}(P(s))} : \mathcal{R}(P(s)) \to \mathcal{R}(P(t))$$

for all $t, s \in \mathbb{I},\ t \geq s$. Analogous to (3.12) it follows that (3.13) holds. □

In the M-hyperbolic definition, see Definition 3.2.3, we only require the solution operator restricted to the kernel of an invariant family of projectors to be invertible. Which statements can be established under this weak regularity assumption about the solution operator of a linear system if it is not invertible, i.e. if the kernel of the operator is not the zero set?

We will see that the kernel of the solution operator is a subspace of the stable cone, see for M-hyperbolic systems Lemma 3.2.6 and Lemma 4.2.4 and for D-hyperbolic systems see Lemma 5.3.3. In Lemma 3.2.6 we prove that the kernel is a subspace of the range of every invariant family of projectors, which satisfies (3.8), (3.9) and (3.10). Lemma 4.2.4 yields that the range of every invariant family of projectors, which satisfies (3.8), (3.9) and (3.10), is a subset of the stable cone. Note that for continuous systems the kernel of the projectors equals the zero-set, since the solution operator is invertible. Thus, the following lemma is always true in the continuous setting.

Lemma 3.2.6. *Let $\mathbb{I} = [n_-, n_+]_{\mathbb{Z}}$ and (2.8) be M-hyperbolic with solution operator $\Phi(\cdot,\cdot)$. Then for all $n_0 \in \mathbb{I}$ and $P(n_0) \in \mathcal{P}_{n_0} := \{\tilde{P}(n_0) | \tilde{P} : \mathbb{I} \to \mathbb{R}^{k\times k}$ is an invariant family of projectors such that*
(3.8), (3.9) and (3.10) are satisfied with some constants $\alpha, \beta > 0\}$ we have

$$\mathcal{N}(\Phi(n_+, n_0)) \subset \mathcal{R}(P(n_0)).$$

Proof. Fix $n_0 \in \mathbb{I}$. Let $x \in \mathcal{N}(\Phi(n_+, n_0))$ and $P(n_0) \in \mathcal{P}_{n_0}$. Then there exist by [132, Proposition 6.82] an $x_s \in \mathcal{R}(P(n_0))$ and an $x_u \in \mathcal{N}(P(n_0))$ with $x = x_s + x_u$. Thus,

$$\begin{aligned}\Phi(n_+, n_0)x_u &= \Phi(n_+, n_0)(x - x_s) = \Phi(n_+, n_0)x - \Phi(n_+, n_0)x_s \\ &= 0 - \Phi(n_+, n_0)x_s = \Phi(n_+, n_0)(-x_s).\end{aligned}$$

By (3.11) and (3.12) we get

$$\begin{aligned}\mathcal{R}(P(n_+)) &\supset \Phi(n_+, n_0)\mathcal{R}(P(n_0)) \\ &\ni \Phi(n_+, n_0)(-x_s) \\ &= \Phi(n_+, n_0)x_u \in \Phi(n_+, n_0)\mathcal{N}(P(n_0)) = \mathcal{N}(P(n_+)).\end{aligned}$$

This implies that $\Phi(n_+, n_0)x_u \in \mathcal{R}(P(n_+)) \cap \mathcal{N}(P(n_+)) = \{0\}$, see [132, Proposition 6.82]. The invertibility of $\Phi(n_+, n_0)|_{\mathcal{N}(P(n_0))}$ yields $x_u = 0$ and we obtain $x = x_s \in \mathcal{R}(P(n_0))$. Hence, the proof is complete. □

For invertible systems Lemma 3.2.7 gives equivalent estimates to the M-hyperbolic estimates (3.9) and (3.10).

Lemma 3.2.7. *Let $\mathbb{I}$ be a compact interval and $P : \mathbb{I} \to \mathbb{R}^{k\times k}$ an invariant family of projectors of system* (2.6) *satisfying* (3.8) *and let $\alpha, \beta > 0$. Then the family P and the constant $\beta > 0$ satisfy the estimate* (3.10) *if and only if*

$$\|\Phi(t,s)\xi\| \geq e^{\beta(t-s)}\|\xi\| \text{ for all } t,s \in \mathbb{I}, t \geq s \text{ and } \xi \in \mathcal{N}(P(s)) \tag{3.14}$$

is true. If (2.6) *is invertible, then the estimate* (3.9) *is equivalent to*

$$\|\Phi(s,t)\xi\| \geq e^{\alpha(t-s)}\|\xi\| \text{ for all } t,s \in \mathbb{I}, t \geq s \text{ and } \xi \in \mathcal{R}(P(s)). \tag{3.15}$$

Proof. Let $t, s \in \mathbb{I}$ with $t \geq s$ and $\xi \in \mathcal{N}(P(s))$. Then we find by the invariance of the kernel of the family of projectors and by the invertibility of $\Phi(t,s)|_{\mathcal{N}(P(s))}$ a $\nu \in \mathcal{N}(P(t))$ with $\xi = \Phi(s,t)\nu$. By (3.10) we get

$$\|\Phi(s,t)\nu\| \leq e^{-\beta(t-s)}\|\nu\|.$$

Multiplying with $e^{\beta(t-s)} \geq 1$ we obtain

$$\|\Phi(t,s)\xi\| \geq e^{\beta(t-s)}\|\xi\|.$$

Thus, estimate (3.10) is equivalent to (3.14). Similarly, the equivalence of (3.9) and (3.15) follows for invertible systems. □

To obtain equivalent statements of (3.9) and (3.10) we introduce the strain tensor. This tensor is also used in continuum mechanics to describe the rate of deformation of a body of a continuum medium (solid, liquid or gas) locally at a certain time. For more details and physical interpretations we refer to [93, p. 46-57], [123]. The names strain-rate tensor, rate-of-strain tensor or rate-of-deformation tensor or just strain tensor denote the same tensor.

The next definition and statements for continuous time systems originate from [45], [15], [43], [62] and for the two dimensional case from [61]. In addition we introduce similar concepts for discrete time systems.

Let $\Gamma \in \mathbb{R}^{k\times k}$ be a positive definite symmetric matrix ($\Gamma^T = \Gamma > 0$). Then define $\|\cdot\|_\Gamma := \sqrt{\langle\cdot,\cdot\rangle_\Gamma}$, $\langle\cdot,\cdot\rangle_\Gamma := \langle\cdot,\Gamma\cdot\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the standard inner product. Note that this norm – called Γ-norm – is differentiable.

Definition 3.2.8. *Let $\mathbb{T} \in \{\mathbb{R},\mathbb{Z}\}$, $\mathbb{I} \subset \mathbb{T}$ a compact interval and let $\Gamma^T = \Gamma > 0$. Then the symmetric matrix*

$$S_\Gamma(t) := \begin{cases} \frac{1}{2}[\Gamma A(t) + A(t)^T\Gamma], & \text{for } \mathbb{T} = \mathbb{R},\ t \in \mathbb{I}, \\ A(t)^T\Gamma A(t) - \Gamma, & \text{for } \mathbb{T} = \mathbb{Z},\ t \in \mathbb{I}_1 \end{cases}$$

is called the ***Γ-strain tensor*** *of equation* (2.7)/(2.8)*. The set*

$$Z_\Gamma(t) := \{\xi \in \mathbb{R}^k | \langle\xi, S_\Gamma(t)\xi\rangle = 0\}$$

defined for all $t \in \begin{cases} \mathbb{I}, & \text{for } \mathbb{T} = \mathbb{R}, \\ \mathbb{I}_1, & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}$ *is called the* ***zero Γ-strain set*** *of equation* (2.7)/(2.8)*.*

For $\Gamma = \mathrm{I}$ we write $S(\cdot)$ and $Z(\cdot)$ instead of $S_{\mathrm{I}}(\cdot)$ and $Z_{\mathrm{I}}(\cdot)$.

In continuous time the Γ-strain tensor describes the instantaneous change of $\frac{1}{2}\|\xi(\cdot)\|_\Gamma^2$, where $\xi(\cdot)$ is a solution of (2.7), i.e. for $t \in \mathbb{I}$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|_\Gamma^2 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle\xi(t),\Gamma\xi(t)\rangle = \langle\tfrac{1}{2}\dot{\xi}(t),\Gamma\xi(t)\rangle + \langle\xi(t),\tfrac{1}{2}\Gamma\dot{\xi}(t)\rangle \\ &= \langle\tfrac{1}{2}A(t)\xi(t),\Gamma\xi(t)\rangle + \langle\xi(t),\tfrac{1}{2}\Gamma A(t)\xi(t)\rangle \\ &= \langle\xi(t),\tfrac{1}{2}(A(t)^T\Gamma + \Gamma A(t))\xi(t)\rangle \\ &= \langle\xi(t),S_\Gamma(t)\xi(t)\rangle.\end{aligned} \tag{3.16}$$

If $\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|_\Gamma \gtrless 0$ for all $t \in \mathbb{I}$ then $\xi(\cdot)$ is strictly increasing or strictly decreasing w.r.t. the Γ-norm, respectively. For discrete systems we can use the Γ-strain tensor to describe the change of the length of two subsequent solution points, i.e. let $\xi(\cdot)$ be a solution of (2.8) then for $n \in \mathbb{I}_1$

$$\begin{aligned}\|\xi(n+1)\|_\Gamma^2 - \|\xi(n)\|_\Gamma^2 &= \langle\xi(n+1),\Gamma\xi(n+1)\rangle - \langle\xi(n),\Gamma\xi(n)\rangle \\ &= \langle A(n)\xi(n),\Gamma A(n)\xi(n)\rangle - \langle\xi(n),\Gamma\xi(n)\rangle \\ &= \langle\xi(n),[A(n)^T\Gamma A(n) - \Gamma]\xi(n)\rangle = \langle\xi(n),S_\Gamma(n)\xi(n)\rangle.\end{aligned} \tag{3.17}$$

By (3.16)/(3.17) all nontrivial solutions $\xi(\cdot)$ of (2.7)/(2.8) are strictly increasing (strictly decreasing) w.r.t. the Γ-norm if the Γ-strain tensor $S_\Gamma(t)$ is positive (negative) definite for all $t \in \begin{cases}\mathbb{I}, & \text{for } \mathbb{T} = \mathbb{R},\\ \mathbb{I}_1, & \text{for } \mathbb{T} = \mathbb{Z}.\end{cases}$
Thus, the zero Γ-strain set $Z_\Gamma(t)$ is a nontrivial cone if and only if $S_\Gamma(t)$ is indefinite.

Studying equation (3.16) we see by the symmetry of $\langle\cdot,\cdot\rangle_\Gamma$ that every solution $\xi(\cdot)$ of (2.7) satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|_\Gamma^2 &= 2\left(\langle\tfrac{1}{2}A(t)\xi(t),\Gamma\xi(t)\rangle + \langle\xi(t),\tfrac{1}{2}\Gamma A(t)\xi(t)\rangle\right) \\ &= 2\left(\frac{1}{2}\langle A(t)\xi(t),\xi(t)\rangle_\Gamma + \frac{1}{2}\langle\xi(t),A(t)\xi(t)\rangle_\Gamma\right) \\ &= \langle A(t)\xi(t),\xi(t)\rangle_\Gamma + \langle A(t)\xi(t),\xi(t)\rangle_\Gamma \\ &= 2\langle A(t)\xi(t),\xi(t)\rangle\end{aligned} \tag{3.18}$$

for each $t \in \mathbb{I}$.

For proving the Roughness-Theorem 3.4.11 in Section 3.4 we introduce equivalent statements of (3.9) and (3.10) using the Γ-strain tensor. The equivalent statements, which we examine in Lemma 3.2.9, originate for continuous time from [12, Proposition 2] and similar estimates are proved in [16, Lemma 9]. We additionally present and prove a similar statement for discrete time systems.

Lemma 3.2.9. *Let* $\mathbb{T} \in \{\mathbb{R},\mathbb{Z}\}$, $t_\pm \in \mathbb{T}$ *and* $\mathbb{I} = [t_-,t_+]_\mathbb{T}$. *An invariant family of projectors* $P : \mathbb{I} \to \mathbb{R}^{k\times k}$ *of* (2.6) *and constants* $\alpha,\beta > 0$ *satisfy estimates*

(3.9) and (3.10) w.r.t. $\|\cdot\|_\Gamma$ *if and only if the family* $P(\cdot)$ *and the constants*

$$\tilde{\alpha} = \begin{cases} \alpha, & \text{for } \mathbb{T} = \mathbb{R}, \\ 1 - e^{-2\alpha}, & \text{for } \mathbb{T} = \mathbb{Z} \end{cases} > 0, \ \tilde{\beta} = \begin{cases} \beta, & \text{for } \mathbb{T} = \mathbb{R}, \\ e^{2\beta} - 1, & \text{for } \mathbb{T} = \mathbb{Z} \end{cases} > 0 \tag{3.19}$$

satisfy for all $t, s \in \mathbb{I}$, $t \geq s$ *and* $y \in \mathbb{R}^k$

$$\langle \Phi(t,s)P(s)y, S_\Gamma(t)\Phi(t,s)P(s)y \rangle \leq -\tilde{\alpha} \|\Phi(t,s)P(s)y\|_\Gamma^2 \tag{3.20}$$

as well as

$$\langle \Phi(t,s)Q(s)y, S_\Gamma(t)\Phi(t,s)Q(s)y \rangle \geq \tilde{\beta} \|\Phi(t,s)Q(s)y\|_\Gamma^2 , \tag{3.21}$$

where $Q(s) := \mathrm{I} - P(s)$ *for all* $s \in \mathbb{I}$.

Proof. Let $y \in \mathbb{R}^k$ and $s \in \mathbb{I}$. Define $\eta := P(s)y$ and $\mu := Q(s)y$. For $\mathbb{T} = \mathbb{R}$ first assume that (3.9) and (3.10) are satisfied. Then we have for all $t, \tilde{t} \in \mathbb{I}$, $t \geq \tilde{t} \geq s$

$$\begin{aligned} \|\Phi(t,s)\eta\|_\Gamma &= \left\|\Phi(t,\tilde{t})\Phi(\tilde{t},s)\eta\right\|_\Gamma \\ &\leq e^{-\alpha(t-\tilde{t})} \left\|\Phi(\tilde{t},s)\eta\right\|_\Gamma = e^{-\alpha(t-s)+\alpha(\tilde{t}-s)} \left\|\Phi(\tilde{t},s)\eta\right\|_\Gamma , \end{aligned}$$

which is equivalent to

$$e^{\alpha(t-s)} \|\Phi(t,s)\eta\|_\Gamma \leq e^{\alpha(\tilde{t}-s)} \left\|\Phi(\tilde{t},s)\eta\right\|_\Gamma .$$

Thus, $e^{\alpha(t-s)} \|\Phi(t,s)\eta\|_\Gamma$ is decreasing, i.e. $\frac{\mathrm{d}}{\mathrm{d}t} e^{\alpha(t-s)} \|\Phi(t,s)\eta\|_\Gamma \leq 0$. Analogously, we get by (3.14) that

$$e^{-\beta(t-s)} \|\Phi(t,s)\mu\|_\Gamma$$

is increasing. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{-\beta(t-s)} \|\Phi(t,s)\mu\|_\Gamma \geq 0$$

is satisfied. Definition 3.2.8 and equation (3.16) yield

$$\begin{aligned}
&\langle \Phi(t,s)\eta, S_\Gamma(t)\Phi(t,s)\eta\rangle \\
&\qquad +\alpha \left\|\Phi(t,s)\eta\right\|_\Gamma^2 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Phi(t,s)\eta\right\|_\Gamma^2 + \alpha\left\|\Phi(t,s)\eta\right\|_\Gamma^2 \\
&\qquad = \frac{1}{2}e^{-2\alpha(t-s)}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{2\alpha(t-s)}\left\|\Phi(t,s)\eta\right\|_\Gamma^2\right) \\
&\qquad = \frac{1}{2}e^{-2\alpha(t-s)}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\alpha(t-s)}\left\|\Phi(t,s)\eta\right\|_\Gamma\right)^2 \\
&\qquad = \frac{1}{2}e^{-2\alpha(t-s)}2e^{\alpha(t-s)}\left\|\Phi(t,s)\eta\right\|_\Gamma\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\alpha(t-s)}\left\|\Phi(t,s)\eta\right\|_\Gamma\right) \\
&\qquad = e^{-\alpha(t-s)}\left\|\Phi(t,s)\eta\right\|_\Gamma\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\alpha(t-s)}\left\|\Phi(t,s)\eta\right\|_\Gamma\right) \\
&\qquad \leq 0, \\
&\langle \Phi(t,s)\mu, S_\Gamma(t)\Phi(t,s)\mu\rangle \\
&\qquad -\beta\left\|\Phi(t,s)\mu\right\|_\Gamma^2 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Phi(t,s)\mu\right\|_\Gamma^2 - \beta\left\|\Phi(t,s)\mu\right\|_\Gamma^2 \\
&\qquad = \frac{1}{2}e^{2\beta(t-s)}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-2\beta(t-s)}\left\|\Phi(t,s)\mu\right\|_\Gamma^2\right) \\
&\qquad = e^{\beta(t-s)}\left\|\Phi(t,s)\eta\right\|_\Gamma\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\beta(t-s)}\left\|\Phi(t,s)\eta\right\|_\Gamma\right) \\
&\qquad \geq 0.
\end{aligned}$$

Thus, the estimates (3.20) and (3.21) are fulfilled.

Reversely, assume that (3.20) and (3.21) are true. Let $\mathbb{T} = \mathbb{R}$ and $t \in \mathbb{I}$, $t \geq s$. Then we have by (3.19) $\tilde{\alpha} = \alpha$ and, thus,

$$\begin{aligned}
0 &\geq \langle \Phi(t,s)\eta, S_\Gamma(t)\Phi(t,s)\eta\rangle + \tilde{\alpha}\left\|\Phi(t,s)\eta\right\|_\Gamma^2 \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Phi(t,s)\eta\right\|_\Gamma^2 + \alpha\left\|\Phi(t,s)\eta\right\|_\Gamma^2 \\
&= \frac{1}{2}e^{-2\alpha(t-s)}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{2\alpha(t-s)}\left\|\Phi(t,s)\eta\right\|_\Gamma^2\right).
\end{aligned}$$

The positivity of the exponential function yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{2\alpha(t-s)}\left\|\Phi(t,s)\eta\right\|_\Gamma^2\right) \leq 0.$$

This implies that

$$e^{2\alpha(t-s)}\left\|\Phi(t,s)\eta\right\|_\Gamma^2 \leq \left\|\eta\right\|_\Gamma^2$$

is satisfied. Hence, estimate (3.9) is fulfilled. Further, with $\tilde{\beta} = \beta$

$$\begin{aligned}
0 &\leq \langle \Phi(t,s)\mu, S_\Gamma(t)\Phi(t,s)\mu\rangle - \tilde{\beta}\left\|\Phi(t,s)\mu\right\|_\Gamma^2 \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Phi(t,s)\mu\right\|_\Gamma^2 - \beta\left\|\Phi(t,s)\mu\right\|_\Gamma^2 \\
&= \frac{1}{2}e^{2\beta(t-s)}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-2\beta(t-s)}\left\|\Phi(t,s)\mu\right\|_\Gamma^2\right)
\end{aligned}$$

follows, which yields an equivalent statement of estimate (3.14)

$$e^{-2\beta(t-s)} \left\|\Phi(t,s)\mu\right\|_\Gamma^2 \geq \left\|\mu\right\|_\Gamma^2 .$$

Thus, by Lemma 3.2.7 estimate (3.10) is fulfilled.

For $\mathbb{T} = \mathbb{Z}$ let $t \in \mathbb{I}_2$, $t \geq s$. By (3.19) we have $\tilde{\alpha} = -\left(e^{-2\alpha} - 1\right)$ and together with (3.17) we obtain that the next five inequalities are equivalent

$$\begin{aligned}
\langle \Phi(t,s)\eta, S_\Gamma(t)\Phi(t,s)\eta\rangle &\leq -\tilde{\alpha} \left\|\Phi(t,s)\eta\right\|_\Gamma^2 ,\\
\left\|\Phi(t+1,s)\eta\right\|_\Gamma^2 - \left\|\Phi(t,s)\eta\right\|_\Gamma^2 &\leq -\tilde{\alpha} \left\|\Phi(t,s)\eta\right\|_\Gamma^2 ,\\
\left\|\Phi(t+1,s)\eta\right\|_\Gamma^2 - \left\|\Phi(t,s)\eta\right\|_\Gamma^2 &\leq \left(e^{-2\alpha} - 1\right) \left\|\Phi(t,s)\eta\right\|_\Gamma^2 ,\\
\left\|\Phi(t+1,s)\eta\right\|_\Gamma^2 &\leq e^{-2\alpha} \left\|\Phi(t,s)\eta\right\|_\Gamma^2 ,\\
\left\|\Phi(t+1,s)\eta\right\|_\Gamma &\leq e^{-\alpha} \left\|\Phi(t,s)\eta\right\|_\Gamma .
\end{aligned}$$

Hence, (3.9) follows. We get by (3.19) $\beta = \left(e^{2\beta} - 1\right)$ and the equivalence of the following statements

$$\begin{aligned}
\langle \Phi(t,s)\eta, S_\Gamma(t)\Phi(t,s)\eta\rangle &\geq \tilde{\beta} \left\|\Phi(t,s)\eta\right\|_\Gamma^2 ,\\
\left\|\Phi(t+1,s)\mu\right\|_\Gamma^2 - \left\|\Phi(t,s)\mu\right\|_\Gamma^2 &\geq \tilde{\beta} \left\|\Phi(t,s)\eta\right\|_\Gamma^2 ,\\
\left\|\Phi(t+1,s)\mu\right\|_\Gamma^2 - \left\|\Phi(t,s)\mu\right\|_\Gamma^2 &\geq \left(e^{2\beta} - 1\right) \left\|\Phi(t,s)\eta\right\|_\Gamma^2 ,\\
\left\|\Phi(t+1,s)\mu\right\|_\Gamma^2 &\geq e^{2\beta} \left\|\Phi(t,s)\eta\right\|_\Gamma^2 ,\\
\left\|\Phi(t+1,s)\mu\right\|_\Gamma &\geq e^{\beta} \left\|\Phi(t,s)\eta\right\|_\Gamma .
\end{aligned}$$

This yields (3.14) and Lemma 3.2.7 implies (3.10). Thus, (3.9) and (3.10) are equivalent to (3.20) and (3.21). □

A similar statement for continuous systems that is established in [16, Lemma 9] uses the notation of growth rates. For the Roughness-Theorem 3.4.5 in Section 3.4 we need the theory of growth rates as well. Hence, we give a short introduction and point out some of the relations between M-hyperbolic systems and their growth rates. In addition to [82], [43] and [16] we define the extremal growth rates also for noninvertible systems.

For $\mathbb{I} = [n_-, n_+]_\mathbb{Z}$ we define the following function

$$\begin{aligned}
{}_\Phi\mathcal{T}_{\ker} &: \mathbb{R}^k \times \mathbb{I} \to \mathbb{I} \cup \{n_+ + 1\},\\
&(\xi, n_0) \mapsto \mathring{n} := \begin{cases} \min \operatorname{Ker}_{n_0}^\xi, & \text{if } \operatorname{Ker}_{n_0}^\xi \neq \emptyset,\\ n_+ + 1, & \text{otherwise} \end{cases}
\end{aligned}$$

with $\operatorname{Ker}_{n_0}^\xi := \{n \in [n_0, n_+]_\mathbb{Z} | \Phi(n, n_0)\xi = 0\}$. This function provides the earliest time, if it exists, at which the given vector lies in the kernel of the solution operator. For an invertible system ${}_\Phi\mathcal{T}_{\ker}(x) = n_+ + 1$ for all $x \in \mathbb{R}^k$. Thus, our definition of growth rates is for invertible systems equivalent to [82, Definition 2.11].

Definition 3.2.10. *Let Φ be the solution operator of (2.6) for $\mathbb{I} = [t_-, t_+]_\mathbb{T}$, $t_\pm \in \mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$. Let $i \in \{0, \dots, k\}$ and $X \in \mathrm{Gr}_i^k = \mathrm{Gr}(i, \mathbb{R}^k)$, where Gr_i^k denotes the Grassmann manifold, which contains all i-dimensional subspaces of $\mathbb{R}^k$. We define the* ***upper and lower growth rate*** *of X under Φ by*

$$\overline{\lambda}(X, \Phi) := \begin{cases} \sup\limits_{x \in X, \|x\|=1} \overline{\lambda}(x, \Phi), & i \neq 0 \\ -\infty, & i = 0, \end{cases}$$

$$\underline{\lambda}(X, \Phi) := \begin{cases} \inf\limits_{x \in X, \|x\|=1} \underline{\lambda}(x, \Phi), & i \neq 0 \\ \infty, & i = 0 \end{cases}$$

with

$$\overline{\lambda}(x, \Phi) := \begin{cases} \sup\limits_{\substack{t,s \in \mathbb{I}, t \neq s, \\ t,s < {}_\Phi\mathcal{T}_{\mathrm{ker}}(x,t_-)}} \left\{ \frac{\ln(\|\Phi(t,t_-)x\|) - \ln(\|\Phi(s,t_-)x\|)}{t-s} \right\}, & \textit{if case (i)}, \\ -\infty, & \textit{otherwise}, \end{cases}$$

$$\underline{\lambda}(x, \Phi) := \begin{cases} -\infty, & \textit{if } {}_\Phi\mathcal{T}_{\mathrm{ker}}(x, t_-) \in \mathbb{I}, \\ \inf\limits_{t,s \in \mathbb{I}, t \neq s} \left\{ \frac{\ln(\|\Phi(t,t_-)x\|) - \ln(\|\Phi(s,t_-)x\|)}{t-s} \right\}, & \textit{otherwise}, \end{cases}$$

where case (i) is

$${}_\Phi\mathcal{T}_{\mathrm{ker}}(x, t_-) > \begin{cases} t_-, & \textit{for } \mathbb{T} = \mathbb{R}, \\ t_- + 1, & \textit{for } \mathbb{T} = \mathbb{Z} \end{cases}.$$

Further, we define the ***minimal upper and maximal lower i-growth rate*** *of Φ*

$$\overline{\lambda}_i(\Phi) := \begin{cases} \min\limits_{X \in \mathrm{Gr}_i^k} \{\overline{\lambda}(X, \Phi)\}, & i \neq 0 \\ -\infty, & i = 0, \end{cases}$$

$$\underline{\lambda}_i(\Phi) := \begin{cases} \max\limits_{X \in \mathrm{Gr}_i^k} \{\underline{\lambda}(X, \Phi)\}, & i \neq 0 \\ \infty, & i = 0. \end{cases}$$

For a definition of Grassmann manifolds and for the proof of its manifold properties we refer to [94, p.22]. The Grassmann manifold Gr_i^k is the set of all i-dimensional subspaces of $\mathbb{R}^k$. Thus, it is obvious that for any projector $P \in \mathbb{R}^{k \times k}$ with $\dim(\mathcal{R}(P)) =: i$ and $\dim(\mathcal{N}(P)) = k - i =: r$

$$\begin{aligned} \overline{\lambda}_i(\Phi) &\leq \overline{\lambda}(\mathcal{R}(P), \Phi), \\ \underline{\lambda}_r(\Phi) &\geq \underline{\lambda}(\mathcal{N}(P), \Phi) \end{aligned} \tag{3.22}$$

holds. The following relations between M-hyperbolic invertible systems and their growth rates can be found in [82, Lemma 3.3] and in [16, Lemma 12]. For invertible systems Lemma 3.2.11 implies equivalent statements to the exponential dichotomy estimates (3.9) and (3.10) and Lemma 3.2.12 yields equivalent conditions about growth rates to the M-hyperbolicity conditions (3.8), (3.9) and (3.10).

Lemma 3.2.11. *Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$ and $\mathbb{I} = [t_-, t_+]_\mathbb{T}$. Let $P : \mathbb{I} \to \mathbb{R}^{k\times k}$ be a family of invariant projectors of* (2.6) *satisfying* (3.8). *Set $\ell := \dim(\mathcal{R}(P(t_-)))$ and $r := k - \ell$. If the family $P(\cdot)$ and constants $\alpha, \beta > 0$ satisfy* (3.9) *and* (3.10) *then they also satisfy*

$$\overline{\lambda}_\ell(\mathcal{R}(P(t_-)), \Phi) \leq -\alpha \text{ and } \underline{\lambda}_r(\mathcal{N}(P(t_-)), \Phi) \geq \beta. \tag{3.23}$$

If the system (2.6) *is invertible then the statements* (3.9) *and* (3.10) *are equivalent to* (3.23).

Proof. Let (3.9) and (3.10) be satisfied for $\alpha, \beta > 0$ and $P(\cdot)$. For $x \in \mathcal{R}(P(t_-))$ Lemma 2.2.4 yields $\Phi(s, t_-)x \in \mathcal{R}(P(s))$ for all $s \in \mathbb{I}$. Thus, we have

$$\|\Phi(t, t_-)x\| = \|\Phi(t, s)\Phi(s, t_-)x\| \leq e^{\alpha(t-s)} \|\Phi(s, t_-)x\|$$

for all $t, s \in \mathbb{I}$, $t \geq s$. Setting $\bar{t} := {}_\Phi\mathcal{T}_{\mathrm{ker}}(x, t_-)$ we get by Definition 3.2.10

$$\begin{aligned}\sup_{\substack{t,s\in\mathbb{I},t\neq s,\\ t,s<\bar{t}}} \left\{\frac{\ln(\|\Phi(t,t_-)x\|) - \ln(\|\Phi(s,t_-)x\|)}{t-s}\right\} &= \sup_{\substack{t,s\in\mathbb{I},t\neq s,\\ t,s<\bar{t}}} \left\{\frac{\ln\left(\frac{\|\Phi(t,t_-)x\|}{\|\Phi(s,t_-)x\|}\right)}{t-s}\right\} \\ &\leq \sup_{\substack{t,s\in\mathbb{I},t\neq s,\\ t,s<\bar{t}}} \left\{\frac{\ln(e^{-\alpha(t-s)})}{t-s}\right\} \\ &= \sup_{\substack{t,s\in\mathbb{I},t\neq s,\\ t,s<\bar{t}}} \left\{\frac{-\alpha(t-s)}{t-s}\right\} \\ &= -\alpha < 0.\end{aligned}$$

For the growth rate follows

$$\begin{aligned}\overline{\lambda}(\mathcal{R}(P(t_-)), \Phi) = &\sup_{\substack{x\in\mathcal{R}(P(t_-)),\|x\|=1,\\ t,s\in\mathbb{I},t\neq s,\\ t,s<{}_\Phi\mathcal{T}_{\mathrm{ker}}(x,t_-)}} \left\{\frac{\ln(\|\Phi(t,t_-)x\|) - \ln(\|\Phi(s,t_-)x\|)}{t-s}\right\} \\ \leq &-\alpha.\end{aligned}$$

Lemma 3.2.7 yields $\|\Phi(t, s)\xi\| \geq e^{\beta(t-s)} \|\xi\|$ for all $\xi \in \mathcal{N}(P(s))$ and $t, s \in \mathbb{I}$, $t \geq s$. Using Lemma 3.2.5 we get for $x \in \mathcal{N}(P(t_-))$

$$\|\Phi(t, t_-)x\| = \|\Phi(t, s)\Phi(s, t_-)x\| \geq e^{\beta(t-s)} \|\Phi(s, t_-)x\|$$

for all $t, s, \in \mathbb{I}$, $t \geq s$. This estimate implies

$$\begin{aligned}\underline{\lambda}(\mathcal{N}(P(t_-)), \Phi) &= \inf_{\substack{x \in \mathcal{N}(P(t_-)), \|x\|=1, \\ t,s \in \mathbb{I}, t \neq s}} \left\{ \frac{\ln(\|\Phi(t,t_-)x\|) - \ln(\|\Phi(s,t_-)x\|)}{t-s} \right\} \\ &\geq \inf_{t,s \in \mathbb{I}, t \neq s} \left\{ \frac{\beta(t-s)}{t-s} \right\} \\ &\geq \beta.\end{aligned}$$

Conversely, assume that Φ is invertible and (3.23) holds. For any fixed

$$x \in \mathcal{R}(P(t_-)) \setminus \{0\}, \; y \in \mathcal{N}(P(t_-)) \setminus \{0\}$$

set $\mathring{t} := {}_{\Phi}\mathcal{T}_{\ker}(x, t_-)$. By the Definition 3.2.10 of the growth rates we obtain

$$\ln(\|\Phi(t,t_-)x\|) - \ln(\|\Phi(s,t_-)x\|) \leq \overline{\lambda}_\ell(\mathcal{R}(P(t_-)), \Phi)(t-s)$$

for all $t, s \in \mathbb{I}$, $\mathring{t} \geq t > s$ and

$$\ln(\|\Phi(t,t_-)y\|) - \ln(\|\Phi(s,t_-)y\|) \geq \underline{\lambda}_r(\mathcal{N}(P(t_-)), \Phi)(t-s)$$

for all $t, s \in \mathbb{I}$, $t > s$. After transformation and insertion of (3.22) the inequalities

$$\|\Phi(t,t_-)x\| \leq e^{\overline{\lambda}_\ell(\Phi)(t-s)} \|\Phi(s,t_-)x\| \text{ for all } t, s \in \mathbb{I}, \mathring{t} \geq t \geq s, \tag{3.24}$$

$$\|\Phi(t,t_-)y\| \geq e^{\underline{\lambda}_r(\Phi)(t-s)} \|\Phi(s,t_-)y\| \text{ for all } t, s \in \mathbb{I}, t \geq s \tag{3.25}$$

hold. The estimate (3.24) is true even for all $t, s \in \mathbb{I}$, $t \geq s$. By the invertibility of Φ (3.9) follows. Lemma 3.2.7 and the invertibility of Φ yield the equivalence of (3.25) and (3.10). Thus, (3.9) and (3.10) are satisfied. □

With Lemma 3.2.11 we prove an alternative characterization of M-hyperbolicity using growth rates.

Lemma 3.2.12. *Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$ and $\mathbb{I} = [t_-, t_+]_{\mathbb{T}}$. Let (2.6) be invertible and $r \in \{0, \dots, k\}$. Then the following statements are equivalent.*

(a) *System (2.6) is M-hyperbolic (w.r.t. $\|\cdot\|$) on $\mathbb{I}$ with an invariant family of projectors $P : \mathbb{I} \to \mathbb{R}^{k \times k}$ with $\dim(\mathcal{R}(P(t_-))) = r$.*

(b) *The growth rates of (2.6) satisfy*

$$\overline{\lambda}_r(\Phi) < 0, \tag{3.26}$$

$$\underline{\lambda}_\ell(\Phi) > 0, \tag{3.27}$$

with $\ell := k - r$.

If (2.6) is noninvertible then (b) follows from (a).

Proof. For (2.6) let condition (a) be satisfied and denote the dichotomy data by $(\alpha, \beta, P(\cdot))$. We have $\mathcal{R}(P(t_-)) \in \mathrm{Gr}_r^k$, $\mathcal{N}(P(t_-)) \in \mathrm{Gr}_\ell^k$ with $\ell := k - r$. Lemma 3.2.11 and the estimates (3.22) yield

$$\begin{aligned}\overline{\lambda}_r(\Phi) &\leq \overline{\lambda}(\mathcal{R}(P(t_-)), \Phi) = -\alpha < 0,\\ \underline{\lambda}_\ell(\Phi) &\geq \underline{\lambda}(\mathcal{N}(P(t_-)), \Phi) \geq \beta > 0.\end{aligned}$$

Consider that (2.6) is invertible and, conversely, assume (b). By [82, Remark 2.18] and [43, Remark 9] there exist $X_s \in \mathrm{Gr}_r^k$ and $X_u \in \mathrm{Gr}_\ell^k$ such that

$$\overline{\lambda}_r(\Phi) = \overline{\lambda}(X_s, \Phi),\ \underline{\lambda}_\ell(\Phi) = \underline{\lambda}(X_u, \Phi).$$

For any fixed $x \in X_s \setminus \{0\}$, $y \in X_u \setminus \{0\}$ set $\mathring{t} := {}_\Phi\mathcal{T}_{\mathrm{ker}}(x, t_-)$. By Definition 3.2.10 we obtain

$$\begin{aligned}\ln(\|\Phi(t,t_-)x\|) - \ln(\|\Phi(s,t_-)x\|) &\leq \overline{\lambda}_r(\Phi)(t-s) \text{ for all } t, s \in \mathbb{I}, \mathring{t} \geq t > s,\\ \ln(\|\Phi(t,t_-)y\|) - \ln(\|\Phi(s,t_-)y\|) &\geq \underline{\lambda}_\ell(\Phi)(t-s) \text{ for all } t, s \in \mathbb{I}, t > s.\end{aligned}$$

After transformation

$$\begin{aligned}\|\Phi(t,t_-)x\| &\leq e^{\overline{\lambda}_r(\Phi)(t-s)} \|\Phi(s,t_-)x\| \text{ for all } t, s \in \mathbb{I}, t > s,\\ \|\Phi(t,t_-)y\| &\geq e^{\underline{\lambda}_\ell(\Phi)(t-s)} \|\Phi(s,t_-)y\| \text{ for all } t, s \in \mathbb{I}, t > s\end{aligned}$$

are satisfied. By the definition of the extremal growth rates we have $\Phi(t, t_-)x \neq 0$ for all $x \in X_u \setminus \{0\}$ and $t \in \mathbb{I}$. This implies that

$$\Phi(t, t_-)|_{X_u} : X_u \to \Phi(t, t_-)X_u$$

is injective, thus, bijective. This implies that $\Phi(\cdot, t_-)X_u$ is invariant. The subspaces $X_{s,u}$ satisfy by definition

$$X_s \oplus X_u = \mathbb{R}^k.$$

The invertibility of the system yields

$$\Phi(t, t_-) : X_s \oplus X_u = \mathbb{R}^k \to \mathbb{R}^k = \Phi(t, t_-)X_s + \Phi(t, t_-)X_u$$

for all $t \in \mathbb{I}$. Then $\Phi(\cdot, t_-)X_s$ is invariant, since $\Phi(\cdot, t_-)X_u$ is invariant. Thus, we can define a family of projectors $P : \mathbb{I} \to \mathbb{R}^k$ by $\mathcal{R}(P(s)) := \Phi(s, t_-)X_s$, $\mathcal{N}(P(s)) := \Phi(s, t_-)X_u$ for all $s \in \mathbb{I}$. This family of projectors is invariant and we get that

$$\begin{aligned}\|\Phi(t,s)\xi\| &\leq e^{\overline{\lambda}_r(\Phi)(t-s)} \|\xi\|,\\ \|\Phi(t,s)\mu\| &\geq e^{\underline{\lambda}_\ell(\Phi)(t-s)} \|\mu\|\end{aligned}$$

is satisfied for all $\xi \in \mathcal{R}(P(s))$, $\mu \in \mathcal{N}(P(s))$ and $t, s \in \mathbb{I}$, $t \geq s$. Thus, the hyperbolic estimates (3.9) and (3.14) are true and, hence, by Lemma 3.2.7 system (2.6) is M-hyperbolic with exponential rates $\alpha := -\overline{\lambda}_r(\Phi) > 0$ and $\beta := \underline{\lambda}_\ell(\Phi) > 0$. □

Finite and Infinite Time Hyperbolic Systems: Differences and Similarities

In this section we study differences and similarities between infinite time hyperbolic systems (Definition 3.1.1, Lemma 3.1.3) and finite time M–hyperbolic systems (Definition 3.2.3).

One difference is that the invariant family of projectors of an M–hyperbolic (finite time hyperbolic) system is not unique, see Example 3.3.1, whereas the family of projectors of an hyperbolic ift–system is unique.

For autonomous ft–systems there exists at most one autonomous projector such that the system is M–hyperbolic w.r.t. that projector. Still, there may exist more than one nonautonomous family of projectors. The autonomous projector is the unique projector of the associated hyperbolic ift–system, see Theorem 3.3.3. Example 3.3.4 shows that an ft–system can be M–hyperbolic although it is not M-hyperbolic w.r.t. the unique projector of the associated ift–system. Further, we obtain that every autonomous system which is hyperbolic on infinite time is also M–hyperbolic w.r.t. a proper norm (Lyapunov norm) on every finite time interval, cf. Theorem 3.3.2.

In the literature the uniqueness of the family of projectors of a hyperbolic ift–system is well known. For continuous time see e.g. [104, p.227], for invertible systems in discrete time we refer to [105, Proposition 2.3] and for noninvertible systems in discrete time see [66, Theorem 7.6.5]. The following example of a finite time differential equation shows that a system can be M–hyperbolic with different invariant families of projectors. This implies that the invariant family of projectors $P : \mathbb{I} \to \mathbb{R}^k$ according to Definition 3.2.3 is generally not unique. The example is a modified and elaborated version of the second example in [16, Example 4].

Example 3.3.1. *Assume $t_-, t_+ \in \mathbb{R}$ with $t_- < 0 < t_+$ and*

$$\dot{x} = A(t)x = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} x,\ x \in \mathbb{R}^2 \textit{ for } t \in \mathbb{I} = [t_-, t_+].$$

The solution operator is $\Phi(t,s) = \operatorname{diag}(e^{t-s}, e^{-2(t-s)})$. Choose the Euclidean–norm. Then the differential equation is M–hyperbolic with the invariant family of projectors $P(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $t \in \mathbb{I}$ and with exponential rates $\bar{\alpha} = 2$, $\bar{\beta} = 1$. If we take $P(0)$ and disturb the stable part towards the unstable direction with $\delta > 0$ we get $P_\delta(0) = \begin{pmatrix} 0 & 0 \\ \delta & 1 \end{pmatrix}$. We see that $P_\delta(0)$ is still a projector and we obtain the invariant family belonging to $P_\delta(0)$ by

$$P_\delta(t) = \Phi(t,0)P_\delta(0)\Phi(0,t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \delta & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \delta e^{-3t} & 1 \end{pmatrix}.$$

In the following we prove that there exist exponential rates such that the given system is M–hyperbolic with this disturbed family of projectors P_δ. Since the *range* $\mathcal{R}(P_\delta(t))$ *does not depend on* δ*, we just need to verify* (3.10)*. For the kernel we get*

$$\mathcal{N}(P_\delta(t)) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| -y = \delta e^{-3t} x \right\}. \tag{3.28}$$

Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{N}(P_\delta(s))$ *with* $\delta \geq 0$ *and assume that* (3.10) *is true using the Euclidean–norm for a* $1 = \bar{\beta} \leq \beta > 0$*. Let* $t, s \in \mathbb{I}$ *with* $t > s$*. Then we have*

$$\begin{aligned} e^{-2(t-s)}x^2 + e^{4(t-s)}y^2 &= \left\| \begin{pmatrix} e^{-(t-s)}x \\ e^{2(t-s)}y \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} e^{-(t-s)} & 0 \\ 0 & e^{2(t-s)} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 \\ &= \left\| \Phi(t,s) \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 \\ &\leq e^{-2\beta(t-s)} \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 = e^{-2\beta(t-s)}(x^2 + y^2). \end{aligned}$$

This equals

$$(e^{-2(t-s)} - e^{-2\beta(t-s)})x^2 \leq (e^{-2\beta(t-s)} - e^{4(t-s)})y^2.$$

Dividing by $(e^{-2\beta(t-s)} - e^{4(t-s)}) < 0$ *and extract the root we get*

$$\left(\frac{e^{-2(t-s)} - e^{-2\beta(t-s)}}{e^{-2\beta(t-s)} - e^{4(t-s)}} \right)^{\frac{1}{2}} |x| \geq |y|.$$

Taking $\beta = 1$ *we obtain* $|y| = 0$ *and, thus, the constant* δ *must be zero. This leads to the same kernel as before. To get more than one family of projectors we have to reduce the exponential rate* β*. Assume* $\beta = 0.5$*. Then*

$$\left(\frac{e^{-2(t-s)} - e^{-(t-s)}}{e^{-(t-s)} - e^{4(t-s)}} \right)^{\frac{1}{2}} |x| = \left(\frac{e^{-(t-s)} - 1}{1 - e^{5(t-s)}} \right)^{\frac{1}{2}} |x| \geq |y|$$

is satisfied for all $t, s \in \mathbb{I}$ *with* $t > s$*. The left-hand side is minimal for* $s = t_-$ *and* $t = t_+$*. Combining this result with the definition of the kernel* (3.28) *we observe that*

$$\left(\frac{e^{-(t_+ - t_-)} - 1}{1 - e^{5(t_+ - t_-)}} \right)^{\frac{1}{2}} e^{3t} \geq \delta \tag{3.29}$$

must be true for all $t \in \mathbb{I}$*. The left-hand side is minimal for* $t = t_-$*. Hence, for all different* δ *such that* (3.29) *is satisfied with* $t = t_-$ *we get another invariant family of projectors, which fulfills estimate* (3.10) *with exponential rate* $\beta = 0.5$*.*

Example 3.3.1 illustrates that by reducing the exponential rates of a hyperbolic ift–system with constant $K = 1$ we generally get more than one invariant family of projectors for the system on every compact interval. In Chapter 4 we prove that the stable and unstable sets are cones which coincide with the union of the kernel and range of all invariant families of projectors. In the following we discuss hyperbolicity of autonomous systems on finite and infinite time intervals. Consider the autonomous system

$$\dot{x}(t) = Ax(t) \text{ respectively } x(t+1) = Ax(t) \tag{3.30}$$

with $A \in \mathbb{R}^{k\times k},\ x \in \mathbb{R}^k$ and $t \in \mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$. Define for all $t \in \mathbb{N}$

$$\begin{aligned} A^{-t} : \mathbb{R}^k &\rightsquigarrow \mathbb{R}^k, \\ x &\mapsto \{y \in \mathbb{R}^k | A^t y = x\}. \end{aligned}$$

The arrow $\rightsquigarrow$ symbolizes that the given map is a set-valued map. For an introduction of set-valued systems we refer to [6]. Further, define for any $t \in \mathbb{T}$

$$B(t) := \begin{cases} e^{At}, & \text{for } \mathbb{T} = \mathbb{R}, \\ A^t, & \text{for } \mathbb{T} = \mathbb{Z}. \end{cases}$$

Denote the solution operator of (3.30) by $\Phi(\cdot,\cdot)$. It satisfies $\Phi(t,s) = B(t-s)$ for all $t, s \in \mathbb{I}$, $t \geq s$.

The next theorem states that for any autonomous hyperbolic ift–system a proper norm (Lyapunov norm) exists such that the given system is M–hyperbolic (w.r.t. this norm) on every compact interval. The definition and properties of a Lyapunov norm can be found in [11], [38, Def. 2.4.9] and [84, Def. 5.3.2].

Theorem 3.3.2. *Let system* (3.30) *have an exponential dichotomy on* $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$ *w.r.t.* $\|\cdot\|$ *with data* $(K, \alpha, \beta, P_s, P_u)$. *Then for any rates* $\tilde{\alpha} \in (0, \alpha)$, $\tilde{\beta} \in (0, \beta)$ *there exists a norm* $\|\cdot\|_L$ *such that system* (3.30) *is M–hyperbolic w.r.t.* $\|\cdot\|_L$ *on every compact interval* $\mathbb{I} \subset \mathbb{T}$ *with projector* P_s.

Proof. Exploiting the dichotomy estimates (3.2), we see that $\|B(t)P_s x\| \leq Ke^{-\alpha t}\|P_s x\|$, $\|B(-t)P_u x\| \leq Ke^{-\beta t}\|P_u x\|$ are bounded for all $t \geq 0$ and $x \in \mathbb{R}^k$. Fix $\tilde{\alpha} \in (0, \alpha)$, $\tilde{\beta} \in (0, \beta)$ and define $\mu := e^{-\tilde{\alpha}} \in (e^{-\alpha}, 1)$, $\gamma := e^{-\tilde{\beta}} \in (e^{-\beta}, 1)$. Choose $a, b \in \mathbb{N}$ sufficiently large such that

$$K\left(\frac{e^{-\alpha}}{\mu}\right)^a < 1,\ K\left(\frac{e^{-\beta}}{\gamma}\right)^b < 1 \tag{3.31}$$

are satisfied. The Lyapunov norm (Lyapunov adapted norm) is defined by

$$\|x\|_L := \begin{cases} \int\limits_0^a \mu^{-r}\|B(r)P_s x\|\,\mathrm{d}r + \int\limits_{-b}^0 \gamma^r \|B(r)P_u x\|\,\mathrm{d}r, & \mathbb{T} = \mathbb{R}, \\ \sum\limits_{r=0}^a \mu^{-r}\|B(r)P_s x\| + \sum\limits_{r=-b}^0 \gamma^r\|B(r)P_u x\|, & \mathbb{T} = \mathbb{Z}. \end{cases}$$

Next we show the ft–exponential dichotomy estimates (3.9) and (3.10). Let $\mathbb{T} = \mathbb{R}$ and $t, s \in \mathbb{T}$ with $t \geq s$. The projectors satisfy

$$P_u P_s = (\mathrm{I} - P_s) P_s = P_s - P_s^2 = P_s - P_s = 0. \tag{3.32}$$

By the invariance of the projector P_s and the solution operator B, by the hyperbolicity estimates (3.2), by equation (3.32) and by estimate (3.31) we have for all $x \in \mathbb{R}^k$

$$\begin{aligned}
&\|\Phi(t,s)P_s x\|_L \\
=& \|B(t-s)P_s x\|_L \\
=& \int_0^a \mu^{-r} \|B(r)P_s B(t-s)P_s x\| \,\mathrm{d}r + \int_{-b}^0 \gamma^r \|B(r)P_u B(t-s)P_s x\| \,\mathrm{d}r \\
=& \int_0^a \mu^{-r} \left\|B(r)B(t-s)P_s^2 x\right\| \mathrm{d}r + \int_{-b}^0 \gamma^r \|B(r)B(t-s)P_u P_s x\| \,\mathrm{d}r \\
=& \int_0^a \mu^{-r} \|B((t-s)+r)P_s x\| \,\mathrm{d}r \\
=& \mu^{(t-s)} \left(\int_{(t-s)}^{a+(t-s)} \mu^{-r} \|B((t-s)+r-(t-s))P_s x\| \,\mathrm{d}r \right) \\
=& \mu^{(t-s)} \left(\int_0^a \mu^{-r} \|B(r)P_s x\| \,\mathrm{d}r \right. \\
& \qquad \left. + \int_a^{a+(t-s)} \mu^{-r} \|B(r)P_s x\| \,\mathrm{d}r - \int_0^{(t-s)} \mu^{-r} \|B(r)P_s x\| \,\mathrm{d}r \right) \\
=& \mu^{(t-s)} \left(\|P_s x\|_L + \int_0^{(t-s)} \left(\mu^{-(r+a)} \|B(r+a)P_s x\| - \mu^{-r} \|B(r)P_s x\| \right) \mathrm{d}r \right) \\
=& \mu^{(t-s)} \left(\|P_s x\|_L + \int_0^{(t-s)} \mu^{-r} \left(\mu^{-a} \|B(a)(P_s B(r)x)\| - \|B(r)P_s x\| \right) \mathrm{d}r \right) \\
\leq& \mu^{(t-s)} \left(\|Px\|_L + \int_0^{(t-s)} \mu^{-r} \left(K \left(\frac{e^{-\alpha}}{\mu} \right)^a \|P_s B(r)x\| - \|P_s B(r)x\| \right) \mathrm{d}r \right) \\
\leq& \mu^{(t-s)} \|P_s x\|_L \\
\leq& e^{-\tilde{\alpha}(t-s)} \|P_s x\|_L .
\end{aligned}$$

For $\mathbb{T} = \mathbb{Z}$ the proof of the estimate follows analogously. For the readers convenience we show the second ft–exponential dichotomy estimate (3.10) for $\mathbb{T} = \mathbb{Z}$. Let $\mathbb{T} = \mathbb{Z}$ and $t, s \in \mathbb{I}$ with $t \geq s$. Then for all $x \in \mathbb{R}^k$ we see similar to the previous calculations that

$$
\begin{aligned}
&\|\Phi(s,t)P_u x\|_L \\
=&\sum_{r=-b}^{0} \gamma^r \|B(-(t-s)+r)P_u x\| \\
=&\sum_{r=-b-(t-s)}^{-(t-s)} \gamma^{r+(t-s)} \|B(r)P_u x\| \\
=&\gamma^{(t-s)} \Bigg(\sum_{r=-b}^{0} \gamma^r \|B(r)P_u x\| \\
&\qquad + \sum_{r=-b-(t-s)}^{-(b+1)} \gamma^r \|B(r)P_u x\| - \sum_{r=-(t-s)+1}^{0} \gamma^r \|B(r)P_u x\| \Bigg) \\
=&\gamma^{(t-s)} \left(\|P_u x\|_L + \sum_{r=-(t-s)+1}^{0} \gamma^r \left(\gamma^{-(b+1)} \|B(r-(b+1))P_u x\| - \|B(r)P_u x\| \right) \right) \\
=&\gamma^{(t-s)} \Bigg(\|P_u x\|_L \\
&\qquad + \sum_{r=-(t-s)+1}^{0} \gamma^r \left(\gamma^{-(b+1)} \|B(-(b+1))P_u B(r)x\| - \|P_u B(r)x\| \right) \Bigg) \\
\leq&\gamma^{(t-s)} \left(\|P_u x\|_L + \sum_{r=-(t-s)+1}^{0} \gamma^r \left(K \left(\frac{e^{-\beta}}{\gamma} \right)^{b+1} - 1 \right) \|P_u B(r)x\| \right) \\
\leq&\gamma^{(t-s)} \|P_u x\|_L \\
\leq&e^{-\tilde{\beta}(t-s)} \|P_u x\|_L ,
\end{aligned}
$$

is satisfied. Analogously, we obtain the same estimate for $\mathbb{T} = \mathbb{R}$. These estimates show that system (3.30) has an exponential dichotomy on $\mathbb{T}$ with $(1, \tilde{\alpha}, \tilde{\beta}, P_s)$ w.r.t. $\|\cdot\|_L$. This implies that (3.30) is M–hyperbolic (w.r.t. $\|\cdot\|_L$) on every compact interval $\mathbb{I} \subset \mathbb{T}$ with rates $\tilde{\alpha}, \tilde{\beta}$ and projector P_s.

□

Next we prove that if an autonomous system is M–hyperbolic with an autonomous projector then the system is also hyperbolic (infinite) with the same projector and exponential rates. This fact and the uniqueness of the infinite time exponential dichotomy projector imply that an autonomous system cannot be M–hyperbolic w.r.t. two different autonomous projectors. Note that for ift-systems the constant K in (3.2) depends on the infimum of the angles between the image and kernel of the invariant family of projectors over time. The estimates (3.9) and (3.10) are only satisfied for ift-systems if the image and kernel of the invariant family of projectors are orthogonal for all times. In [131] this special class of exponential dichotomy of linear ODEs is called a strong dichotomy.

Theorem 3.3.3. *Let system* (3.30) *be M–hyperbolic (w.r.t.* $\|\cdot\|$*) on a compact interval* $\mathbb{I} \subset \mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$ *with an autonomous projector* P *and with exponential rates* $\alpha, \beta > 0$*. Then system* (3.30) *has an exponential dichotomy on* $\mathbb{T}$ *with projector* P *and with rates* α, β*.*

Proof. The M–hyperbolicity yields for $t, s \in \mathbb{I}$ with $t \geq s$ and $x \in \mathbb{R}^k$

$$\|\Phi(t,s)Px\| \leq e^{-\alpha(t-s)} \|Px\| . \tag{3.33}$$

Next, we prove that this estimate is satisfied even for all $t, s \in \mathbb{T}$, $t \geq s$ and $x \in \mathbb{R}^k$. Let $t, s \in \mathbb{T}$ with $t \geq s$ and denote by ℓ the length of the finite interval $\mathbb{I}$. Then there exist $a \in \mathbb{N}_0$ and $b \in \mathbb{R}$ with $0 \leq b < \ell$ such that $t - s = a\ell + b$. With estimate (3.33) and the invariance of P and Φ we obtain for $x \in \mathbb{R}^k$ and all $t, s \in \mathbb{T}$, $t \geq s$

$$\begin{aligned} \|\Phi(t,s)Px\| &= \|B(t-s)Px\| = \|B(a\ell+b)Px\| = \|B(b)P(B(a\ell)x)\| \\ &\leq e^{-\alpha b} \|P(B(a\ell)x)\| = e^{-\alpha b} \|B(\ell)P(B((a-1)\ell)x)\| . \end{aligned}$$

Inductively, we see that

$$\|\Phi(t,s)Px\| \leq e^{-\alpha b} \|B(l)P(B((a-1)l)x)\| \leq e^{-\alpha(b+al)} \|Px\| = e^{-\alpha(t-s)} \|Px\|$$

is satisfied for $t, s \in \mathbb{I}$, $t \geq s$. For the matrix norm we get

$$\begin{aligned} \|\Phi(t,s)P\| &= \sup_{x \in \mathbb{R}^k \setminus \{0\}} \frac{\|\Phi(t,s)Px\|}{\|x\|} \\ &\leq e^{-\alpha(t-s)} \sup_{x \in \mathbb{R}^k \setminus \{0\}} \frac{\|Px\|}{\|x\|} = e^{-\alpha(t-s)} \|P\| = Ke^{-\alpha(t-s)} \end{aligned}$$

with $K := \|P\|$ for all $t, s \in \mathbb{T}$, $t \geq s$. Similarly, it follows that

$$\|\Phi(s,t)(\mathrm{I} - P)\| \leq Ke^{-\beta(t-s)}$$

for all $t, s \in \mathbb{R}$, $t \geq s$ with $K := \|\mathrm{I} - P\|$. This implies that (3.30) has an exponential dichotomy on $\mathbb{T}$ with projector P and exponential rates α and β and constant $K = \max\{\|P\|, \|\mathrm{I} - P\|\}$. □

A hyperbolic ift–system with projector $\bar{P}$ may be M–hyperbolic (w.r.t. $\|\cdot\|$) on a compact interval $\mathbb{I}$ with an invariant family of projectors $P : \mathbb{I} \to \mathbb{R}^k$ but not with $\bar{P}$. The following example illustrates this statement.

Example 3.3.4. *Consider the system*

$$\dot{x} = Ax := \begin{pmatrix} 0.2 & -1 \\ 0 & 0.1 \end{pmatrix} x, \;\; x \in \mathbb{R}^2 \tag{3.34}$$

on $\mathbb{I} \subset \mathbb{R}$. *Then the solution operator of system* (3.34) *is* $\Phi(s+t,s) := e^{At} = \begin{pmatrix} e^{0.2t} & 10(e^{0.1t} - e^{0.2t}) \\ 0 & e^{0.1t} \end{pmatrix}$ *for all* $t, s \in \mathbb{I}$. *On* $\mathbb{I} = \mathbb{R}$ *the system* (3.34) *has an exponential dichotomy with projector* $\bar{P} = 0$. *Let* $\mathbb{I} = [0,1]$ *and use* $\|\cdot\|_2$. *For* $x = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ *we get*

$$\begin{aligned} \|\Phi(0,1)(\mathrm{I} - \bar{P})x\|_2 &= \left\| e^{-A} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} 10(e^{-0.1} - e^{-0.2}) \\ e^{-0.1} \end{pmatrix} \right\|_2 \geq \left\| \begin{pmatrix} 0.86 \\ 0.9 \end{pmatrix} \right\|_2 \\ &> 1 = \|x\|_2 \geq e^{-\beta} \|(\mathrm{I} - \bar{P})x\|_2 \end{aligned}$$

for all $\beta > 0$. *We see that* (3.34) *is not M–hyperbolic (w.r.t.* $\|\cdot\|_2$*) on* $\mathbb{I} = [0,1]$ *with the projector* $\bar{P}$. *Choose* $P(0) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ *and* $P(t) = \Phi(t,0)P(0)\Phi(0,t) = \begin{pmatrix} 0 & 10 - 9e^{0.1t} \\ 0 & 1 \end{pmatrix}$ *for all* $t \in \mathbb{I}$ *and define* $\alpha = 0.01$, *then for all* $t, s \in \mathbb{I}$ *with* $t \geq s$ *the following is true*

$$e^{0.11t} \sqrt{(10 - 9e^{0.1t})^2 + 1} \leq e^{0.11s} \sqrt{(10 - 9e^{0.1s})^2 + 1}. \tag{3.35}$$

Using this setup we obtain for all $t, s \in \mathbb{I}$ *with* $t \geq s$ *and* $x = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T \in \mathbb{R}^2$

$$\begin{aligned} \|\Phi(t,s)P(s)x\|_2 &= \left\| \begin{pmatrix} 10e^{0.1(t-s)} - 9e^{0.2(t-s)}e^{0.1s} \\ e^{0.1(t-s)} \end{pmatrix} \right\|_2 |x_2| \\ &= e^{-0.1s} \left\| \begin{pmatrix} 10e^{0.1t} - 9e^{0.2t} \\ e^{0.1t} \end{pmatrix} \right\|_2 |x_2| \\ &= e^{0.1(t-s)} \left\| \begin{pmatrix} 10 - 9e^{0.1t} \\ 1 \end{pmatrix} \right\|_2 |x_2| = e^{0.1(t-s)} \|P(t)x\|_2 . \end{aligned}$$

We conclude with inequality (3.35) *that*

$$\begin{aligned} \|P(t)x\|_2 = \sqrt{(10 - 9e^{0.1t})^2 + 1}|x_2| &\leq e^{-0.11(t-s)} \sqrt{(10 - 9e^{0.1s})^2 + 1}|x_2| \\ &= e^{-0.11(t-s)} \|P(s)x\|_2 . \end{aligned}$$

Combining these estimates we get for all $t, s \in \mathbb{I}$ *with* $t \geq s$ *and for any* $x \in \mathbb{R}^2$

$$\begin{aligned} \|\Phi(t,s)P(s)x\|_2 &= e^{0.1(t-s)} \|P(t)x\|_2 \leq e^{0.1(t-s)} e^{-0.11(t-s)} \|P(s)x\|_2 \\ &= e^{-0.01(t-s)} \|P(s)x\|_2 . \end{aligned}$$

Let $t, s \in \mathbb{I}$ with $t \geq s$, $x \in \mathbb{R}^2$ and $Q(t) = \mathrm{I} - P(t)$ then

$$\begin{aligned}
&\|\Phi(s,t)Q(t)x\|_2 \\
=&\left\| \begin{pmatrix} e^{-0.2(t-s)} & 10\left(e^{-0.1(t-s)} - e^{-0.2(t-s)}\right) \\ 0 & e^{-0.1(t-s)} \end{pmatrix} \begin{pmatrix} 1 & -10+9e^{0.1t} \\ 0 & 0 \end{pmatrix} x \right\|_2 \\
&= e^{-0.2(t-s)} \left\| \begin{pmatrix} 1 & -10+9e^{0.1t} \\ 0 & 0 \end{pmatrix} x \right\|_2 = e^{-0.2(t-s)} \|Q(t)x\|_2 .
\end{aligned}$$

We see that system (3.34) is M–hyperbolic (w.r.t. $\|\cdot\|_2$) on $\mathbb{I} = [0,1]$ with the invariant family of projectors $P(t) = \begin{pmatrix} 0 & 10-9e^{0.1t} \\ 0 & 1 \end{pmatrix}$.

Perturbation Results

Are (ft-) exponential dichotomies robust under small additive perturbations of the equation? For infinite time systems this is true. Various Roughness-Theorems provide precise bounds on the magnitude of the allowed (additive) perturbation. In the following we list some of these perturbation results. For continuous ift-systems a roughness result, which originates from [40, Chap. 4, Prop. 1], is presented in Theorem 3.4.1. We find a Roughness-Theorem for discrete invertible ift-systems with the content from Theorem 3.4.2 in [105, Proposition 2.10] or [85, Lemma 2.3]. For noninvertible systems we refer to [21, Theo. 19] and [117, Satz 2.1] and we present the statement in Theorem 3.4.3. Some of the perturbation results have been proved for compact time intervals, as well. In this thesis we differentiate between the hyperbolicity terms for ift- and ft-systems. Thus, we present the Roughness-Theorems for ift-systems only for infinite time intervals, but all infinite time Roughness-Theorems are transferable to K-hyperbolic ft-systems. For invertible ft-systems we prove two different roughness results, which guarantee the persistence of M-hyperbolicity. For noninvertible ft-systems the question: “Are M-hyperbolic systems robust under small additive perturbations?” still remains open to our knowledge.

Theorem 3.4.1 (Roughness-Theorem for continuous ift-systems)**.**
Let $\mathbb{I} \in \{[t_-, \infty), (-\infty, t_+], \mathbb{R}\}$ with $t_\pm \in \mathbb{R}$ and assume that the differential equation (2.7) is hyperbolic on $\mathbb{I}$ with data $(K, \alpha, \beta, P(\cdot))$. Let $B(\cdot) : \mathbb{I} \to \mathbb{R}^{k \times k}$ satisfy

$$\delta := \sup_{t \in \mathbb{I}} \|B(t)\| < \frac{\gamma}{4K^2}, \ \textit{with } \gamma = \min\{\alpha, \beta\} \tag{3.36}$$

and denote by $\Psi(\cdot,\cdot)$ the solution operator of

$$\dot{y} = [A(t) + B(t)]y. \tag{3.37}$$

Then (3.37) *is hyperbolic on* $\mathbb{I}$ *with data* $\left(\frac{5K^2}{2}, \gamma - 2K\delta, \gamma - 2K\delta, Q(\cdot)\right)$ *where*

$$\begin{aligned}
\mathcal{N}(Q(t_-)) &:= \mathcal{N}(P(t_-)), && \text{for } \mathbb{I} = [t_-, \infty),\\
\mathcal{R}(Q(t_+)) &:= \mathcal{R}(P(t_+)), && \text{for } \mathbb{I} = (-\infty, t_+],\\
\mathcal{R}(Q(0)) &:= \mathcal{R}(P(0)), \mathcal{N}(Q(0)) := \mathcal{N}(P(0)), && \text{for } \mathbb{I} = \mathbb{R}.
\end{aligned}$$

Theorem 3.4.2 (Roughness-Theorem for discrete invertible ift-systems)**.**
Let $\mathbb{I} \in \{\mathbb{Z}^+_{n_-}, \mathbb{Z}^-_{n_+}, \mathbb{Z}\}, n_\pm \in \mathbb{Z}$ *and assume that the difference equation* (2.8) *is invertible and hyperbolic on* $\mathbb{I}$ *with data* (K, α, P_n).

Then for $0 < \beta < \alpha$ *and every* $E_{\mathbb{I}} \in (\mathbb{R}^{k\times k})^{\mathbb{I}}$ *with*

$$\|E_{\mathbb{I}}\| \le \frac{1}{2} \inf_{n\in\mathbb{I}} \|A_n^{-1}\|^{-1}, \tag{3.38}$$

$$\|E_{\mathbb{I}}\| \le \frac{1}{2} K^{-1} \left(\frac{1}{e^{\beta} - e^{-\alpha}} + \frac{1}{e^{-\beta} - e^{-\alpha}} + \frac{1}{e^{\alpha} - e^{-\beta}} \right)^{-1}, \tag{3.39}$$

the equation

$$u_{n+1} = (A_n + E_n)u_n, \quad n \in \mathbb{I}$$

has an exponential dichotomy on $\mathbb{I}$ *with data* $(2K+1, \beta, Q_n(E_n))$.

Theorem 3.4.3 (Roughness-Theorem for discrete noninvertible ift-systems)**.**

Let $\mathbb{I} \in \{\mathbb{Z}^+_{n_-}, \mathbb{Z}^-_{n_+}, \mathbb{Z}\}, n_\pm \in \mathbb{Z}$ *and assume that the difference equation* (2.8) *is hyperbolic on* $\mathbb{I}$ *with data* (K, α, P_n).

Then for $0 < \beta < \alpha$ *there exists a positive constant* $\gamma = \gamma(K, \alpha, \beta)$ *such that for every* $E_{\mathbb{I}} \in (\mathbb{R}^{k\times k})^{\mathbb{I}}$ *with*

$$\|E_{\mathbb{I}}\| \le \gamma$$

the equation

$$u_{n+1} = (A_n + E_n)u_n, \quad n \in \mathbb{I}_1 := \begin{cases} \mathbb{Z}^+_{n_-}, & \text{for } \mathbb{I} = \mathbb{Z}^+_{n_-},\\ \mathbb{Z}^-_{n_+-1}, & \text{for } \mathbb{I} = \mathbb{Z}^-_{n_+},\\ \mathbb{Z}, & \text{for } \mathbb{I} = \mathbb{Z} \end{cases}$$

has an exponential dichotomy on $\mathbb{I}$ *with data* $(2K, \beta, Q_n(E_n))$.

In the following we study the robustness of M-hyperbolic systems. As we have seen in the infinite time case a small perturbation of a hyperbolic system does not destroy the hyperbolicity. The only drawback is a larger constant and that the new exponential rates are generally smaller. For M-hyperbolicity the system must be “hyperbolic” with constant $K = 1$, so a perturbed M-hyperbolic system is only M-hyperbolic if the constant K does not change. Thus, the infinite time Roughness-Theorems are not transferable to M-hyperbolic systems, whereas all these Roughness-Theorems apply to K-hyperbolic systems.

Theorem 3.4.4 (Roughness-Theorem for cont. K-hyperbolic ft-systems). *Fix $K \in [1,\infty)$. Let equation (2.7) be defined for $t \in \mathbb{I} = [t_-, t_+] \subset \mathbb{R}$ and be K-hyperbolic (on $\mathbb{I}$ w.r.t. $\|\cdot\|$) with an invariant family of projectors $P : \mathbb{I} \to \mathbb{R}^{k\times k}$ and exponential rates $\alpha, \beta > 0$. Let $B : \mathbb{I} \to \mathbb{R}^{k\times k}$ satisfy*

$$\delta := \sup_{t\in\mathbb{I}} \|B(t)\| < \frac{\gamma}{4K^2}, \text{ with } \gamma = \min\{\alpha, \beta\}$$

and denote by $\Psi(\cdot,\cdot)$ the solution operator of

$$\dot{y} = [A(t) + B(t)]\, y. \tag{3.40}$$

Then (3.40) is $\frac{5K^2}{2}$-hyperbolic (on $\mathbb{I}$ w.r.t. $\|\cdot\|$) with an invariant family of projectors $Q : \mathbb{I} \to \mathbb{R}^{k\times k}$ (w.r.t. Ψ), which satisfies $\mathcal{N}(Q(t_-)) = \mathcal{N}(P(t_-))$, and exponential rates $\tilde{\alpha} := \gamma - 2K\delta =: \tilde{\beta}$.

Proof. The expanded system

$$\dot{x} = \tilde{A}(t)x, t \in \mathbb{J} = [t_-, \infty) \tag{3.41}$$

of system (2.7) with

$$\tilde{A}(t) = \begin{cases} A(t), & \text{for } t \in \mathbb{I}, \\ A(t_+), & \text{for } t \in \mathbb{J} \setminus \mathbb{I} \end{cases}$$

has an exponential dichotomy on $\mathbb{J}$ with constant K, exponential rates α, β and the invariant family of projectors $\tilde{P} : \mathbb{J} \to \mathbb{R}^{k\times k}$ with

$$\tilde{P}(s) := \tilde{\Phi}(s, t_-)P(t_-)\tilde{\Phi}(t_-, s) \text{ for all } s \in \mathbb{J}$$

with $\tilde{\Phi}(t,s) := \begin{cases} \Phi(t,s), & \text{for } t, s \in \mathbb{I}, \\ \Phi(t_+, s), & \text{for } t \in \mathbb{J} \setminus \mathbb{I}, s \in \mathbb{I}, \\ \mathrm{Id}, & \text{for } t, s \in \mathbb{J} \setminus \mathbb{I}, \end{cases}$
where $\Phi(t,s)$ denotes the solution operator of (2.7). The Roughness-Theorem 3.4.1 applies to the expanded system (3.41). Thus, for every $\tilde{B} : \mathbb{J} \to \mathbb{R}^{k\times k}$ which fulfills (3.36), the perturbed system $\dot{y} = [\tilde{A}(t)+\tilde{B}(t)]y$ has an exponential dichotomy on $[t_-, \infty)$ with constant $\tilde{K} = \frac{5K^2}{2}$, with an invariant family of projectors $Q^+ : [t_-, \infty) \to \mathbb{R}^{k\times k}$, which satisfies $\mathcal{N}(Q^+(t_-)) = \mathcal{N}(\tilde{P}(t_-)) = \mathcal{N}(P(t_-))$, and with exponential rates $\tilde{\alpha} := \gamma - 2K\delta =: \tilde{\beta}$. Set $B(t) := \tilde{B}(t)$ for $t \in I$ then by Definition 3.2.1 we obtain that (3.40) is $\frac{5K^2}{2}$-hyperbolic with $Q(t) := Q^+(t)$ for $t \in \mathbb{I}$ and with $\tilde{\alpha}, \tilde{\beta}$. □

Analogously, we get statements for discrete time K-hyperbolic systems, which are equivalent to Theorem 3.4.2 and Theorem 3.4.3.

Does there exist for every M-hyperbolic system a sufficiently small perturbation such that the perturbed system is M-hyperbolic, too? Or does every perturbation destroy the M-hyperbolicity? An exponential dichotomy (ift) of an invertible dynamical system persists under sufficiently small perturbations, which we prove in the following. To our knowledge, for noninvertible systems there still does not exist an answer to these questions. A Roughness-Theorem, which uses extremal growth rates, is presented in [82, Theo. 3.13]. This theorem works only for invertible systems. It yields a condition under which a perturbed M-hyperbolic invertible system still remains M-hyperbolic. This condition requires a sufficiently small distance of the solution operators. It does not imply a boundary on the magnitude of the allowed perturbation. However, it is a first indicator that M-hyperbolicity is preserved under sufficiently small perturbations, at least for invertible systems. In [82, Theo. 3.13] the Roughness-Theorem is presented without a proof. For the reader's convenience a proof is stated here in addition.

Theorem 3.4.5 (Roughness-Theorem for invertible ft-systems)**.**
Let $\mathbb{I}$ *be a compact interval. Let system* (2.6) *be invertible and M-hyperbolic (w.r.t.* $\|\cdot\|$*) on* $\mathbb{I}$ *with solution operator* Φ *and with an invariant family of projectors* $P : \mathbb{I} \to \mathbb{R}^{k\times k}$ *with* $\dim(\mathcal{R}(P(t_-))) = r \in \{0, \dots, k\}$*. Then any dynamical system with solution operator* Ψ *is M-hyperbolic (w.r.t.* $\|\cdot\|$*) on* $\mathbb{I}$ *with an invariant family of projectors* $Q : \mathbb{I} \to \mathbb{R}^{k\times k}$ *with* $\dim(\mathcal{R}(Q(t_-))) = r$ *if*

$$\begin{aligned}\tilde{d}_{\mathbb{I}}(\Phi, \Psi) := \sup_{X\in \mathrm{Gr}_1^k} &\left\{\max\left\{|\underline{\lambda}(X, \Phi) - \underline{\lambda}(X, \Psi)|, \left|\overline{\lambda}(X, \Phi) - \overline{\lambda}(X, \Psi)\right|\right\}\right\} \\ &< \min\left\{-\overline{\lambda}_r(\Phi), \underline{\lambda}_{k-r}(\Phi)\right\}. \end{aligned} \tag{3.42}$$

Proof. By Lemma 3.2.12 it suffices to show $\overline{\lambda}_r(\Psi) < 0 < \underline{\lambda}_{k-r}(\Psi)$. By (3.42) we get for $r \neq 0$

$$\begin{aligned}\overline{\lambda}_r(\Psi) &= \min_{X\in \mathrm{Gr}_r^k}\left\{\overline{\lambda}(X,\Psi)\right\} = \min_{X\in \mathrm{Gr}_r^k}\left\{\overline{\lambda}(X,\Psi) - \overline{\lambda}(X,\Phi) + \overline{\lambda}(X,\Phi)\right\} \\ &\leq \inf_{X\in \mathrm{Gr}_r^k}\left\{\left|\overline{\lambda}(X,\Psi) - \overline{\lambda}(X,\Phi)\right|\right\} + \min_{X\in \mathrm{Gr}_r^k}\left\{\overline{\lambda}(X,\Phi)\right\} \\ &\leq \sup_{X\in \mathrm{Gr}_r^k}\left\{\left|\overline{\lambda}(X,\Psi) - \overline{\lambda}(X,\Phi)\right|\right\} + \overline{\lambda}_r(\Phi) \\ &\leq \sup_{X\in \mathrm{Gr}_1^k}\left\{\left|\overline{\lambda}(X,\Psi) - \overline{\lambda}(X,\Phi)\right|\right\} + \overline{\lambda}_r(\Phi) \\ &< -\overline{\lambda}_r(\Phi) + \overline{\lambda}_r(\Phi) = 0\end{aligned}$$

and for $r = 0$ we obtain by definition $\overline{\lambda}_0(\Psi) = -\infty < 0$. For $r \neq k$ we see

$$\begin{aligned}\underline{\lambda}_{k-r}(\Psi) &= \max_{X\in \mathrm{Gr}_{k-r}^k}\{\underline{\lambda}(X,\Psi)\} - \max_{X\in \mathrm{Gr}_{k-r}^k}\{\underline{\lambda}(X,\Phi)\} + \max_{X\in \mathrm{Gr}_{k-r}^k}\{\underline{\lambda}(X,\Phi)\} \\ &\geq -\sup_{X\in \mathrm{Gr}_{k-r}^k}\{|\underline{\lambda}(X,\Psi) - \underline{\lambda}(X,\Phi)|\} + \underline{\lambda}_{k-r}(\Phi) \\ &\geq -\sup_{X\in \mathrm{Gr}_1^k}\{|\underline{\lambda}(X,\Psi) - \underline{\lambda}(X,\Phi)|\} + \underline{\lambda}_{k-r}(\Phi) \\ &> -\underline{\lambda}_{k-r}(\Phi) + \underline{\lambda}_{k-r}(\Phi) = 0\end{aligned}$$

and for $r = k$ we have by definition $\underline{\lambda}_0 = \infty > 0$. $\square$

The last Roughness-Theorem we like to present gives an upper bound on the magnitude of perturbation such that the perturbed M-hyperbolic system is still M-hyperbolic. Therefore, we need various technical calculations. To shorten the actual proof of the Roughness-Theorem 3.4.11 we show the technical details in several lemmas. The theorem originates from Berger [12, Lemma 2], where it is stated for continuous systems only. Consider the differential and difference equation

$$\dot{x}(t) = \tilde{A}(t)x(t), \tag{3.43}$$

$$x(t+1) = \tilde{A}(t)x(t) \tag{3.44}$$

$x \in \mathbb{R}^k$, $\tilde{A} \in \mathcal{C}^0(\mathbb{I}, \mathbb{R}^{k\times k})$, $t \in \mathbb{I}^{\mathbb{R},\mathbb{Z}} := \begin{cases} \mathbb{I}, & \text{for } \mathbb{T} = \mathbb{R}, \\ \mathbb{I}_1, & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}$. Denote their associated solution operator by $\tilde{\Phi}$. For every matrix function $B \in \mathcal{C}^0(\mathbb{I}, \mathbb{R}^{k\times k})$ and induced matrix norm $\|\cdot\|$ define

$$\|B\|_\infty := \max_{t\in\mathbb{I}^{\mathbb{R},\mathbb{Z}}} \|B(t)\|.$$

Lemma 3.4.6. *Let $\mathbb{I} \subset \mathbb{R}$ be a compact interval and $\Gamma = \Gamma^T > 0$. Let Φ be the solution operator of (2.7). Then*

$$e^{-(t-s)\|A\|_{\Gamma,\infty}} \|\mu\|_\Gamma \le \|\Phi(t,s)\mu\|_\Gamma \tag{3.45}$$

is satisfied for all $t, s \in \mathbb{I}$, $t \ge s$ and $\mu \in \mathbb{R}^k$.

Proof. Let $t, s \in \mathbb{I}$, $t \ge s$ and $\mu \in \mathbb{R}^k$. By (3.18) and the Cauchy-Schwarz inequality

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} \|\Phi(t,s)\mu\|_\Gamma^2\right| = |2\langle A(t)\Phi(t,s)\mu, \Phi(t,s)\mu\rangle_\Gamma| \le 2\|A\|_{\Gamma,\infty} \|\Phi(t,s)\mu\|_\Gamma^2.$$

follows. Thus,

$$-2\|A\|_{\Gamma,\infty} \|\Phi(t,s)\mu\|_\Gamma^2 \le \frac{\mathrm{d}}{\mathrm{d}t} \|\Phi(t,s)\mu\|_\Gamma^2$$

and

$$-2\|A\|_{\Gamma,\infty} \left(-\|\Phi(t,s)\mu\|_\Gamma^2\right) \ge \frac{\mathrm{d}}{\mathrm{d}t} \left(-\|\Phi(t,s)\mu\|_\Gamma^2\right)$$

are satisfied. Using [30, Lemma II.4.9], which is similar to Gronwall's Lemma we obtain

$$-\|\Phi(t,s)\mu\|_\Gamma^2 \le \left(-\|\Phi(s,s)\mu\|_\Gamma^2\right) e^{-(t-s)2\|A\|_{\Gamma,\infty}} = -e^{-(t-s)2\|A\|_{\Gamma,\infty}} \|\mu\|_\Gamma^2.$$

Taking the square root and multiplying with -1 we get (3.45). $\square$

Lemma 3.4.7. *Let $\mathbb{I} \subset \mathbb{Z}$ be a compact interval and* (2.8) *be invertible. Further, let $\|\cdot\|$ be any norm in $\mathbb{R}^k$. Then*

$$\left\|A^{-1}\right\|_\infty^{-(t-s)} \|\mu\| \leq \|\Phi(t,s)\mu\| \leq \|A\|_\infty^{(t-s)} \|\mu\| \tag{3.46}$$

is satisfied for all $t, s \in \mathbb{I}$ and $\mu \in \mathbb{R}^k$. If $\left\|\tilde{A} - A\right\|_\infty \leq \frac{1}{2}\|A^{-1}\|_\infty^{-1}$ holds for $\tilde{A}(\cdot)$ of (3.44) *then*

$$\left(2\|A\|_\infty^{(t-s)}\right)^{-1} \|\mu\| \leq \left\|\tilde{\Phi}(t,s)\mu\right\| \leq \left(\frac{1}{2}\|A^{-1}\|_\infty^{-1} + \|A\|_\infty\right)^{(t-s)} \|\mu\|$$

is true for all $t, s \in \mathbb{I}$ and $\mu \in \mathbb{R}^k$.

Proof. For the associated solution operator of (2.8) we have

$$\begin{aligned} \|\Phi(t,s)\| &= \|A(t-1)\cdots A(s)\| \leq \|A(t-1)\|\cdots\|A(s)\| \leq \|A\|_\infty^{(t-s)}, \\ \|\Phi(s,t)\| &= \left\|A^{-1}(s)\cdots A^{-1}(t-1)\right\| \leq \left\|A^{-1}\right\|_\infty^{(t-s)} \end{aligned} \tag{3.47}$$

for all $t, s \in \mathbb{I}$, $t \geq s$. Every invertible matrix B and induced matrix norm $\|\cdot\|$ satisfies

$$\|B\|^{-1} = \|B\|^{-1}\left\|BB^{-1}\right\| \leq \|B\|^{-1}\|B\|\left\|B^{-1}\right\| = \left\|B^{-1}\right\|.$$

Applying this to (3.47) then

$$\left(\left\|A^{-1}\right\|_\infty^{(t-s)}\right)^{-1} \leq \|\Phi(s,t)\|^{-1} \leq \left\|\Phi(s,t)^{-1}\right\| = \|\Phi(t,s)\|$$

follows for all $t, s \in \mathbb{I}$, $t \geq s$. With

$$\left\|A^{-1}\right\|^{-1}\|x\| = \left\|A^{-1}\right\|^{-1}\left\|A^{-1}Ax\right\| \leq \left\|A^{-1}\right\|^{-1}\left\|A^{-1}\right\|\|Ax\| = \|Ax\|$$

(3.46) is proved. Let $\left\|\tilde{A} - A\right\|_\infty \leq \frac{1}{2}\|A^{-1}\|_\infty^{-1}$. Then the Banach-Lemma, cf. [69, Lemma 5.3] yields that $\tilde{A}$ is invertible and satisfies

$$\left\|\tilde{A}^{-1}\right\|_\infty \leq \frac{\|A^{-1}\|_\infty}{1 - \|A^{-1}\|_\infty \left\|\tilde{A} - A\right\|_\infty} \leq \frac{\|A^{-1}\|_\infty}{1 - \frac{1}{2}} = 2\left\|A^{-1}\right\|_\infty.$$

Hence, with (3.46) we obtain for $\tilde{A}$

$$\begin{aligned} \left(2\|A^{-1}\|_\infty^{(t-s)}\right)^{-1}\|\mu\| \leq \left\|\tilde{A}^{-1}\right\|_\infty^{-(t-s)}\|\mu\| \leq \left\|\tilde{\Phi}(t,s)\mu\right\| &\leq \left\|\tilde{A}\right\|^{(t-s)}\|\mu\| \\ &\leq \left(\left\|\tilde{A} - A\right\|_\infty + \|A\|_\infty\right)^{(t-s)}\|\mu\| \\ &\leq \left(\frac{1}{2}\|A^{-1}\|_\infty^{-1} + \|A\|_\infty\right)^{(t-s)}\|\mu\| \end{aligned}$$

for all $t, s \in \mathbb{I}$, $t \geq s$ and $\mu \in \mathbb{R}^k$.

□

Lemma 3.4.8. *Let $\mathbb{I} \subset \mathbb{Z}$ be a compact interval. If for the systems* (2.8) *and* (3.44)

$$\left\|\tilde{A} - A\right\|_\infty \leq 1$$

holds then

$$\left\|\tilde{\Phi}(t,s) - \Phi(t,s)\right\| \leq \left((1 + \|A\|_\infty)^{(t-s)} - \|A\|_\infty^{(t-s)}\right) \left\|\tilde{A} - A\right\|_\infty$$

is satisfied for all $t, s \in \mathbb{I}$, $t \geq s$.

Proof. Fix $s \in \mathbb{I}$. For $t = s$ the statement is trivial. Thus, assume there exists a $T \in \mathbb{I}$ such that the statement is satisfied for all $t \in \mathbb{I}$, $s \leq t < T$. Let $\left\|\tilde{A} - A\right\|_\infty \leq 1$ then

$$\begin{aligned}
&\left\|\tilde{\Phi}(T,s) - \Phi(T,s)\right\| \\
\leq &\left\|\tilde{\Phi}(T,T-1)\left(\tilde{\Phi}(T-1,s) - \Phi(T-1,s)\right)\right\| \\
&+ \left\|\left(\tilde{\Phi}(T,T-1) - \Phi(T,T-1)\right)\Phi(T-1,s)\right\| \\
\leq &\left\|\tilde{A}\right\|_\infty \left((1 + \|A\|_\infty)^{(T-1-s)} - \|A\|_\infty^{(T-1-s)}\right)\left\|\tilde{A} - A\right\|_\infty \\
&+ \left\|\tilde{A} - A\right\|_\infty \|A\|_\infty^{(T-1-s)} \\
\leq &\Big(\left(\left\|\tilde{A} - A\right\|_\infty + \|A\|_\infty\right)\left((1 + \|A\|_\infty)^{(T-1-s)} - \|A\|_\infty^{(T-1-s)}\right) \\
&+ \|A\|_\infty^{(T-1-s)}\Big)\left\|\tilde{A} - A\right\|_\infty \\
\leq &\left((1 + \|A\|_\infty)\left((1 + \|A\|_\infty)^{(T-1-s)} - \|A\|_\infty^{(T-1-s)}\right) + \|A\|_\infty^{(T-1-s)}\right) \\
&\left\|\tilde{A} - A\right\|_\infty \\
= &\left((1 + \|A\|_\infty)^{(T-s)} - \|A\|_\infty^{(T-1-s)} - \|A\|_\infty^{(T-s)} + \|A\|_\infty^{(T-1-s)}\right)\left\|\tilde{A} - A\right\|_\infty \\
= &\left((1 + \|A\|_\infty)^{(T-s)} - \|A\|_\infty^{(T-s)}\right)\left\|\tilde{A} - A\right\|_\infty
\end{aligned}$$

follows. ☐

Lemma 3.4.9. *Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$, $\mathbb{I} = [t_-, t_+]_\mathbb{T}$ and $\Gamma = \Gamma^T > 0$. Let* (2.7)/(2.8) *be invertible and M-hyperbolic w.r.t. $\|\cdot\|_\Gamma$ on $\mathbb{I}$ with data (α, β, P). Fix $s \in \mathbb{I}$ and let $\mu \in \mathbb{R}^k$. Define $\tilde{\eta} := \tilde{\eta}(t) := \tilde{\Phi}(t,s)\mu$, $\eta := \eta(t) = \Phi(t,s)\mu$ for all $t \in \mathbb{I}, t \geq s$. If*

$$\left\|\tilde{A} - A\right\|_{\Gamma,\infty} \leq \delta := \begin{cases} (t_+ - t_-)^{-1}, & \textit{for } \mathbb{T} = \mathbb{R}, \\ \min\left\{1, \frac{1}{2}\left\|A^{-1}\right\|_{\Gamma,\infty}^{-1}\right\}, & \textit{for } \mathbb{T} = \mathbb{Z} \end{cases}$$

holds then

$$\|\tilde{\eta}-\eta\|_\Gamma \left(\|\tilde{\eta}\|_\Gamma + \|\eta\|_\Gamma\right) \leq C \left\|\tilde{A}-A\right\|_{\Gamma,\infty} \left(\|\tilde{\eta}\|_\Gamma^2 + \|\eta\|_\Gamma^2\right), \tag{3.48}$$

is satisfied with

$$C = \begin{cases} 2(t_+ - t_-)e^{1+2(t_+-t_-)\|A\|_{\Gamma,\infty}}, & \text{for } \mathbb{T}=\mathbb{R}, \\ 2\left(\|A^{-1}\|_{\Gamma,\infty} + \|A\|_{\Gamma,\infty}\|A^{-1}\|_{\Gamma,\infty}\right)^{(t_+-t_-)}, & \text{for } \mathbb{T}=\mathbb{Z}. \end{cases}$$

If

$$\left\|\tilde{A}-A\right\|_{\Gamma,\infty} \leq \min\left\{\delta, (2C)^{-1}\right\}$$

holds then

$$\left(\|\tilde{\eta}\|_\Gamma^2 - \|\eta\|_\Gamma^2\right) \leq 4C\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\|\tilde{\eta}\|_\Gamma^2 \tag{3.49}$$

is satisfied.

Proof. The first step is to find an upper bound for $\|\tilde{\eta}-\eta\|_\Gamma$. Let $t, s \in \mathbb{I}$, $t \geq s$ be fixed. The definition of $\tilde{\eta}, \eta$ yields

$$\|\tilde{\eta}-\eta\|_\Gamma = \left\|\left(\tilde{\Phi}(t,s)-\Phi(t,s)\right)\mu\right\|_\Gamma \leq \left\|\tilde{\Phi}(t,s)-\Phi(t,s)\right\|_\Gamma \|\mu\|_\Gamma. \tag{3.50}$$

Let $\mathbb{T}=\mathbb{R}$. Then the variation of constants formula yields

$$\tilde{\Phi}(t,s)-\Phi(t,s) = \int_s^t \Phi(t,\tau)\left(\tilde{A}(\tau)-A(\tau)\right)\tilde{\Phi}(\tau,s)\mathrm{d}\tau, \tag{3.51}$$

since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{\Phi}(t,s)-\Phi(t,s)\right) &= \tilde{A}(t)\tilde{\Phi}(t,s) - A(t)\Phi(t,s) \\ &= A(t)\left(\tilde{\Phi}(t,s)-\Phi(t,s)\right) + \left(\tilde{A}(t)-A(t)\right)\tilde{\Phi}(t,s). \end{aligned}$$

By Lemma 3.4.6 the solution operators Φ and $\tilde{\Phi}$ satisfy

$$\begin{aligned} e^{-(r-z)\|\|A\|_{\Gamma,\infty}}\|\xi\|_\Gamma &\leq \|\Phi(r,z)\xi\|_\Gamma \leq \|\Phi(r,z)\|_\Gamma\|\xi\|_\Gamma, \\ e^{-(r-z)\left\|\tilde{A}\right\|_{\Gamma,\infty}}\|\xi\|_\Gamma &\leq \left\|\tilde{\Phi}(r,z)\xi\right\|_\Gamma \leq \left\|\tilde{\Phi}(r,z)\right\|_\Gamma\|\xi\|_\Gamma \end{aligned}$$

for all $r, z \in \mathbb{I}$, $r \geq z$ and $\xi \in \mathbb{R}^k$. These estimates imply together with (3.51)

$$\begin{aligned}
&\left\|\tilde{\Phi}(t,s) - \Phi(t,s)\right\|_\Gamma \\
\leq & \int_s^t \|\Phi(t,\tau)\|_\Gamma \left\|\tilde{A} - A\right\|_{\Gamma,\infty} \left\|\tilde{\Phi}(\tau,s)\right\|_\Gamma \mathrm{d}\tau \\
\leq & \int_s^t e^{(t-\tau)\|A\|_{\Gamma,\infty}} \left\|\tilde{A} - A\right\|_{\Gamma,\infty} e^{(\tau-s)\|\tilde{A}\|_{\Gamma,\infty}} \mathrm{d}\tau \\
\leq & \left\|\tilde{A} - A\right\|_{\Gamma,\infty} e^{(t-s)\|A\|_{\Gamma,\infty}} \int_s^t e^{(\tau-s)\left(\|\tilde{A}\|_{\Gamma,\infty} - \|A\|_{\Gamma,\infty}\right)} \mathrm{d}\tau \\
\leq & \left\|\tilde{A} - A\right\|_{\Gamma,\infty} e^{(t-s)\|A\|_{\Gamma,\infty}} \int_s^t e^{(\tau-s)\left(\|\tilde{A}-A\|_{\Gamma,\infty} + \|A\|_{\Gamma,\infty} - \|A\|_{\Gamma,\infty}\right)} \mathrm{d}\tau \\
\leq & \left\|\tilde{A} - A\right\|_{\Gamma,\infty} e^{(t-s)\|A\|_{\Gamma,\infty}} \int_s^t e^{(\tau-s)\|\tilde{A}-A\|_{\Gamma,\infty}} \mathrm{d}\tau \\
\leq & e^{(t-s)\|A\|_{\Gamma,\infty}} \left\|\tilde{A} - A\right\|_{\Gamma,\infty} \left(e^{(t-s)\|\tilde{A}-A\|_{\Gamma,\infty}} \frac{1}{\left\|\tilde{A} - A\right\|_{\Gamma,\infty}} - \frac{1}{\left\|\tilde{A} - A\right\|_{\Gamma,\infty}} \right) \\
= & e^{(t-s)\|A\|_{\Gamma,\infty}} \left(e^{(t-s)\|\tilde{A}-A\|_{\Gamma,\infty}} - 1 \right). \qquad (3.52)
\end{aligned}$$

For $0 \leq x \leq 1$ the trivial estimate $e^x - 1 \leq 2x$ is satisfied. If

$$\left\|\tilde{A} - A\right\|_{\Gamma,\infty} \leq (t_+ - t_-)^{-1} =: \delta_1^{\mathbb{R}}$$

holds we get

$$\left(e^{(t-s)\|\tilde{A}-A\|_{\Gamma,\infty}} - 1 \right) \leq 2(t-s) \left\|\tilde{A} - A\right\|_{\Gamma,\infty}.$$

Thus, estimate (3.52) becomes for $\left\|\tilde{A} - A\right\|_{\Gamma,\infty} \leq (t_+ - t_-)^{-1} =: \delta_1^{\mathbb{R}}$

$$\begin{aligned}
\left\|\tilde{\Phi}(t,s) - \Phi(t,s)\right\|_\Gamma \leq & e^{(t-s)\|A\|_{\Gamma,\infty}} \left(e^{(t-s)\|\tilde{A}-A\|_{\Gamma,\infty}} - 1 \right) \\
\leq & 2(t-s) e^{(t-s)\|A\|_{\Gamma,\infty}} \left\|\tilde{A} - A\right\|_{\Gamma,\infty} \\
\leq & C_1^{\mathbb{R}} \left\|\tilde{A} - A\right\|_{\Gamma,\infty} \qquad (3.53)
\end{aligned}$$

with $C_1^{\mathbb{R}} := 2(t_+ - t_-) e^{(t_+ - t_-)\|A\|_{\Gamma,\infty}}$. Let $\mathbb{T} = \mathbb{Z}$. Then Lemma 3.4.8 yields for $\left\|\tilde{A} - A\right\|_{\Gamma,\infty} \leq 1 =: \delta_1^{\mathbb{Z}}$

$$\begin{aligned}
\left\|\tilde{\Phi}(t,s) - \Phi(t,s)\right\|_\Gamma \leq & \left(\left(1 + \|A\|_{\Gamma,\infty}\right)^{(t-s)} - \|A\|_{\Gamma,\infty}^{(t-s)} \right) \left\|\tilde{A} - A\right\|_{\Gamma,\infty} \\
\leq & \left(1 + \|A\|_{\Gamma,\infty}\right)^{(t-s)} \left\|\tilde{A} - A\right\|_{\Gamma,\infty} \\
\leq & C_1^{\mathbb{Z}} \left\|\tilde{A} - A\right\|_{\Gamma,\infty} \qquad (3.54)
\end{aligned}$$

with $C_1^{\mathbb{Z}} := \left(1 + \|A\|_{\Gamma,\infty}\right)^{(t_+ - t_-)}$. Altogether for $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$ let

$$\left\|\tilde{A} - A\right\|_{\Gamma,\infty} \leq \delta_1 := \begin{cases} \delta_1^{\mathbb{R}} = (t_+ - t_-)^{-1}, & \text{for } \mathbb{T} = \mathbb{R}, \\ \delta_1^{\mathbb{Z}} = 1, & \text{for } \mathbb{T} = \mathbb{Z}, \end{cases}$$

then the upper bound (3.50) becomes by (3.53) and (3.54)

$$\|\tilde{\eta} - \eta\|_\Gamma \leq \left\|\tilde{\Phi}(t,s) - \Phi(t,s)\right\|_\Gamma \|\mu\|_\Gamma \leq C_1 \left\|\tilde{A} - A\right\|_{\Gamma,\infty} \|\mu\|_\Gamma$$

with $0 < C_1 := \begin{cases} C_1^{\mathbb{R}} = 2(t_+ - t_-)e^{(t_+ - t_-)\|A\|_{\Gamma,\infty}}, & \text{for } \mathbb{T} = \mathbb{R}, \\ C_1^{\mathbb{Z}} = \left(1 + \|A\|_{\Gamma,\infty}\right)^{(t_+ - t_-)}, & \text{for } \mathbb{T} = \mathbb{Z}. \end{cases}$

Thus, we have

$$\|\tilde{\eta} - \eta\|_\Gamma \left(\|\tilde{\eta}\|_\Gamma + \|\eta\|_\Gamma\right) \leq C_1 \left\|\tilde{A} - A\right\|_{\Gamma,\infty} \left(\|\mu\|_\Gamma \|\tilde{\eta}\|_\Gamma + \|\mu\|_\Gamma \|\eta\|_\Gamma\right). \quad (3.55)$$

Let $\left\|\tilde{A} - A\right\|_{\Gamma,\infty} \leq \delta_2 := \begin{cases} \delta_2^{\mathbb{R}} := \infty, & \text{for } \mathbb{T} = \mathbb{R}, \\ \delta_2^{\mathbb{Z}} := \frac{1}{2}\left\|A^{-1}\right\|_{\Gamma,\infty}^{-1}, & \text{for } \mathbb{T} = \mathbb{Z}. \end{cases}$

Then Lemma 3.4.6 yields for $\mathbb{T} = \mathbb{R}$

$$\begin{aligned} e^{-(t_+ - t_-)\|A\|_{\Gamma,\infty}} \|\mu\|_\Gamma &\leq \|\Phi(t,s)\mu\|_\Gamma = \|\eta\|_\Gamma, \\ e^{-(t_+ - t_-)\|\tilde{A}\|_{\Gamma,\infty}} \|\mu\|_\Gamma &\leq \left\|\tilde{\Phi}(t,s)\mu\right\|_\Gamma = \|\tilde{\eta}\|_\Gamma, \end{aligned}$$

i.e.

$$\begin{aligned} \|\mu\|_\Gamma &\leq \|\eta\|_\Gamma \, e^{(t_+ - t_-)\|A\|_{\Gamma,\infty}}, \\ \|\mu\|_\Gamma &\leq \|\tilde{\eta}\|_\Gamma \, e^{(t_+ - t_-)\|\tilde{A}\|_{\Gamma,\infty}}, \end{aligned} \quad (3.56)$$

and Lemma 3.4.7 yields for $\mathbb{T} = \mathbb{Z}$

$$\begin{aligned} \left\|A^{-1}\right\|_{\Gamma,\infty}^{-(t-s)} \|\mu\|_\Gamma &\leq \|\Phi(t,s)\mu\|_\Gamma = \|\eta\|_\Gamma, \\ \left(2\left\|A^{-1}\right\|_{\Gamma,\infty}^{(t-s)}\right)^{-1} \|\mu\|_\Gamma &\leq \left\|\tilde{\Phi}(t,s)\mu\right\|_\Gamma = \|\tilde{\eta}\|_\Gamma, \end{aligned}$$

i.e.

$$\begin{aligned} \|\mu\|_\Gamma &\leq \|\eta\|_\Gamma \left\|A^{-1}\right\|_{\Gamma,\infty}^{(t-s)}, \\ \|\mu\|_\Gamma &\leq \|\tilde{\eta}\|_\Gamma \, 2\left\|A^{-1}\right\|_{\Gamma,\infty}^{(t-s)}. \end{aligned} \quad (3.57)$$

Let

$$\left\|\tilde{A} - A\right\|_{\Gamma,\infty} \leq \delta := \min\{\delta_1, \delta_2\} = \begin{cases} (t_+ - t_-)^{-1}, & \text{for } \mathbb{T} = \mathbb{R}, \\ \min\left\{1, \frac{1}{2}\left\|A^{-1}\right\|_{\Gamma,\infty}^{-1}\right\}, & \text{for } \mathbb{T} = \mathbb{Z}. \end{cases}$$

Inserting (3.56) and (3.57) in (3.55) we get the first estimate (3.48)

$$\begin{aligned}
&\|\tilde\eta-\eta\|_\Gamma\left(\|\tilde\eta\|_\Gamma+\|\eta\|_\Gamma\right)\\
\le& C_1\left\|\tilde A-A\right\|_{\Gamma,\infty}\left(\|\mu\|_\Gamma\|\tilde\eta\|_\Gamma+\|\mu\|_\Gamma\|\eta\|_\Gamma\right)\\
\le& C_1\left\|\tilde A-A\right\|_{\Gamma,\infty}\begin{cases}\left(e^{(t_+-t_-)\|\tilde A\|_{\Gamma,\infty}}\|\tilde\eta\|_\Gamma^2+e^{(t_+-t_-)\|A\|_{\Gamma,\infty}}\|\eta\|_\Gamma^2\right), & \text{for } \mathbb{T}=\mathbb{R},\\ \left(2\|A^{-1}\|_{\Gamma,\infty}^{(t_+-t_-)}\|\tilde\eta\|_\Gamma^2+\|A^{-1}\|_{\Gamma,\infty}^{(t_+-t_-)}\|\eta\|_\Gamma^2\right), & \text{for } \mathbb{T}=\mathbb{Z}\end{cases}\\
\le& C_1\left\|\tilde A-A\right\|_{\Gamma,\infty}\begin{cases}e^{(t_+-t_-)\left(\|\tilde A-A\|_{\Gamma,\infty}+\|A\|_{\Gamma,\infty}\right)}\left(\|\tilde\eta\|_\Gamma^2+\|\eta\|_\Gamma^2\right), & \text{for } \mathbb{T}=\mathbb{R},\\ 2\|A^{-1}\|_{\Gamma,\infty}^{(t_+-t_-)}\left(\|\tilde\eta\|_\Gamma^2+\|\eta\|_\Gamma^2\right), & \text{for } \mathbb{T}=\mathbb{Z}\end{cases}\\
\le& C\left\|\tilde A-A\right\|_{\Gamma,\infty}\left(\|\tilde\eta\|_\Gamma^2+\|\eta\|_\Gamma^2\right) \qquad (3.58)
\end{aligned}$$

with a constant

$$\begin{aligned}
0<C:=C_1&\begin{cases}e^{1+(t_+-t_-)\|A\|_{\Gamma,\infty}}, & \text{for } \mathbb{T}=\mathbb{R},\\ 2\|A^{-1}\|_{\Gamma,\infty}^{(t_+-t_-)}, & \text{for } \mathbb{T}=\mathbb{Z}.\end{cases}\\
=&\begin{cases}2(t_+-t_-)e^{1+2(t_+-t_-)\|A\|_{\Gamma,\infty}}, & \text{for } \mathbb{T}=\mathbb{R},\\ 2\left(\|A^{-1}\|_{\Gamma,\infty}+\|A\|_{\Gamma,\infty}\|A^{-1}\|_{\Gamma,\infty}\right)^{(t_+-t_-)}, & \text{for } \mathbb{T}=\mathbb{Z}.\end{cases}
\end{aligned}$$

With the triangular inequality, with the third binomial formula and with (3.58) we get the upper bound

$$\begin{aligned}
\left|\|\tilde\eta\|_\Gamma^2-\|\eta\|_\Gamma^2\right| &= \left|\left(\|\tilde\eta\|_\Gamma-\|\eta\|_\Gamma\right)\left(\|\tilde\eta\|_\Gamma+\|\eta\|_\Gamma\right)\right|\\
&\le \left|\|\tilde\eta\|_\Gamma-\|\eta\|_\Gamma\right|\left(\|\tilde\eta\|_\Gamma+\|\eta\|_\Gamma\right)\\
&\le \|\tilde\eta-\eta\|_\Gamma\left(\|\tilde\eta\|_\Gamma+\|\eta\|_\Gamma\right) \qquad (3.59)\\
&\le C\left\|\tilde A-A\right\|_{\Gamma,\infty}\left(\|\tilde\eta\|_\Gamma^2+\|\eta\|_\Gamma^2\right)\\
&\le 2C\left\|\tilde A-A\right\|_{\Gamma,\infty}\|\tilde\eta\|_\Gamma^2+C\left\|\tilde A-A\right\|_{\Gamma,\infty}\left|\|\tilde\eta\|_\Gamma^2-\|\eta\|_\Gamma^2\right|.
\end{aligned}$$

Rearranging the terms in (3.59) yields

$$\left(1-C\left\|\tilde A-A\right\|_{\Gamma,\infty}\right)\left|\|\tilde\eta\|_\Gamma^2-\|\eta\|_\Gamma^2\right|\le 2C\left\|\tilde A-A\right\|_{\Gamma,\infty}\|\tilde\eta\|_\Gamma^2.$$

For $\left\|\tilde A-A\right\|_{\Gamma,\infty}\le\min\left\{\delta,(2C)^{-1}\right\}$ the estimate (3.49) follows by the latter estimate, i.e.

$$\begin{aligned}
\left|\|\tilde\eta\|_\Gamma^2-\|\eta\|_\Gamma^2\right| &= \left|\|\tilde\eta\|_\Gamma^2-\|\eta\|_\Gamma^2\right|\left(1-C\frac{1}{2C}\right)2\\
&\le \left|\|\tilde\eta\|_\Gamma^2-\|\eta\|_\Gamma^2\right|\left(1-C\left\|\tilde A-A\right\|_{\Gamma,\infty}\right)2\\
&\le 4C\left\|\tilde A-A\right\|_{\Gamma,\infty}\|\tilde\eta\|_\Gamma^2.
\end{aligned}$$

□

Lemma 3.4.10. *Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$ and $\mathbb{I} = [t_-, t_+]_\mathbb{T}$. Let $\Gamma = \Gamma^T > 0$ and $\mu \in \mathbb{R}^k$. Fix $s \in \mathbb{I}$. Define $\tilde{\eta} := \tilde{\eta}(t) := \tilde{\Phi}(t,s)\mu$, $\eta := \eta(t) := \Phi(t,s)\mu$ for all $t \in \mathbb{I}$ $t \geq s$. Let*

$$\left\|\tilde{A} - A\right\|_{\Gamma,\infty} \leq \delta := \begin{cases} (t_+ - t_-)^{-1}, & \text{for } \mathbb{T} = \mathbb{R}, \\ \min\left\{1, \frac{1}{2}\left\|A^{-1}\right\|_{\Gamma,\infty}^{-1}\right\}, & \text{for } \mathbb{T} = \mathbb{Z}. \end{cases}$$

Then

$$\left|\langle\tilde{\eta}, \tilde{S}_\Gamma(t)\tilde{\eta}\rangle - \langle\eta, S_\Gamma(t)\eta\rangle\right| \leq \bar{C}\left\|\tilde{A} - A\right\|_{\Gamma,\infty}\left(\|\tilde{\eta}\|_\Gamma^2 + \|\eta\|_\Gamma^2\right) \tag{3.60}$$

is satisfied for all $t \in \mathbb{I}$, $t \geq s$ with

$$\bar{C} := \begin{cases} 1 + \|A\|_{\Gamma,\infty} C, & \text{for } \mathbb{T} = \mathbb{R}, \\ \left(1 + 2\|A\|_{\Gamma,\infty}\right) + \left(\|A\|_{\Gamma,\infty}^2 + 1\right) C, & \text{for } \mathbb{T} = \mathbb{Z} \end{cases} \tag{3.61}$$

and

$$C := \begin{cases} 2(t_+ - t_-)e^{1 + 2(t_+ - t_-)\|A\|_{\Gamma,\infty}}, & \text{for } \mathbb{T} = \mathbb{R}, \\ 2\left(\|A^{-1}\|_{\Gamma,\infty} + \|A\|_{\Gamma,\infty}\|A^{-1}\|_{\Gamma,\infty}\right)^{(t_+ - t_-)}, & \text{for } \mathbb{T} = \mathbb{Z}. \end{cases} \tag{3.62}$$

Proof. For $\mathbb{T} = \mathbb{R}$ we get by equations (3.16) and (3.18), by the symmetry of $\langle\cdot,\cdot\rangle_\Gamma$ and by the Cauchy-Schwarz inequality

$$\begin{aligned}
&\left|\langle\tilde{\eta}, \tilde{S}_\Gamma\tilde{\eta}\rangle - \langle\eta, S_\Gamma\eta\rangle\right| \\
&= \left|\langle\tilde{A}\tilde{\eta}, \tilde{\eta}\rangle_\Gamma - \langle A\eta, \eta\rangle_\Gamma\right| \\
&\leq \left|\langle(\tilde{A} - A)\tilde{\eta}, \tilde{\eta}\rangle_\Gamma\right| + \left|\langle A\tilde{\eta}, \tilde{\eta}\rangle_\Gamma - \langle A\eta, \eta\rangle_\Gamma\right| \\
&\leq \left|\langle(\tilde{A} - A)\tilde{\eta}, \tilde{\eta}\rangle_\Gamma\right| + \left|\langle A(\tilde{\eta} - \eta), \tilde{\eta}\rangle_\Gamma\right| + \left|\langle A\eta, \tilde{\eta}\rangle_\Gamma - \langle A\eta, \eta\rangle_\Gamma\right| \\
&\leq \left|\langle(\tilde{A} - A)\tilde{\eta}, \tilde{\eta}\rangle_\Gamma\right| + \left|\langle A(\tilde{\eta} - \eta), \tilde{\eta}\rangle_\Gamma\right| + \left|\langle A\eta, (\tilde{\eta} - \eta)\rangle_\Gamma\right| \\
&\leq \left\|\tilde{A} - A\right\|_{\Gamma,\infty}\|\tilde{\eta}\|_\Gamma^2 + \|A\|_{\Gamma,\infty}\|\tilde{\eta} - \eta\|_\Gamma\|\tilde{\eta}\|_\Gamma + \|A\|_{\Gamma,\infty}\|\eta\|_\Gamma\|\tilde{\eta} - \eta\|_\Gamma \\
&= \left\|\tilde{A} - A\right\|_{\Gamma,\infty}\|\tilde{\eta}\|_\Gamma^2 + \|A\|_{\Gamma,\infty}\|\tilde{\eta} - \eta\|_\Gamma\left(\|\tilde{\eta}\|_\Gamma + \|\eta\|_\Gamma\right).
\end{aligned} \tag{3.63}$$

For $\mathbb{T}=\mathbb{Z}$ we have by equation (3.17) and the latter arguments

$$\begin{aligned}
&\left|\langle\tilde{\eta},\tilde{S}_\Gamma\tilde{\eta}\rangle-\langle\eta,S_\Gamma\eta\rangle\right|\\
=&\left|\langle\tilde{A}\tilde{\eta},\tilde{A}\tilde{\eta}\rangle_\Gamma+\langle\tilde{\eta},\tilde{\eta}\rangle_\Gamma-\langle A\eta,A\eta\rangle_\Gamma-\langle\eta,\eta\rangle_\Gamma\right|\\
\leq&\left|\langle((\tilde{A}-A)+A)\tilde{\eta},((\tilde{A}-A)+A)\tilde{\eta}\rangle_\Gamma-\langle A\eta,A\eta\rangle_\Gamma\right|\\
&+|\langle\tilde{\eta}-\eta,\tilde{\eta}\rangle_\Gamma|+|\langle\eta,\tilde{\eta}-\eta\rangle_\Gamma|\\
\leq&\left|\langle(\tilde{A}-A)\tilde{\eta},(\tilde{A}-A)\tilde{\eta}\rangle_\Gamma\right|+2\left|\langle(\tilde{A}-A)\tilde{\eta},A\tilde{\eta}\rangle_\Gamma\right|+|\langle A\tilde{\eta},A\tilde{\eta}\rangle_\Gamma-\langle A\eta,A\eta\rangle_\Gamma|\\
&+|\langle\tilde{\eta}-\eta,\tilde{\eta}\rangle_\Gamma|+|\langle\eta,\tilde{\eta}-\eta\rangle_\Gamma|\\
\leq&\left|\langle(\tilde{A}-A)\tilde{\eta},(\tilde{A}-A)\tilde{\eta}\rangle_\Gamma\right|+2\left|\langle(\tilde{A}-A)\tilde{\eta},A\tilde{\eta}\rangle_\Gamma\right|+|\langle A(\tilde{\eta}-\eta),A\tilde{\eta}\rangle_\Gamma|\\
&+|\langle A\eta,A(\tilde{\eta}-\eta)\rangle_\Gamma|+|\langle\tilde{\eta}-\eta,\tilde{\eta}\rangle_\Gamma|+|\langle\eta,\tilde{\eta}-\eta\rangle_\Gamma|\\
\leq&\left\|\tilde{A}-A\right\|_{\Gamma,\infty}^2\|\tilde{\eta}\|_\Gamma^2+2\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\|A\|_{\Gamma,\infty}\|\tilde{\eta}\|_\Gamma^2+\|A\|_{\Gamma,\infty}^2\|\tilde{\eta}-\eta\|_\Gamma\|\tilde{\eta}\|_\Gamma\\
&+\|A\|_{\Gamma,\infty}^2\|\eta\|_\Gamma\|\tilde{\eta}-\eta\|_\Gamma+\|\tilde{\eta}-\eta\|_\Gamma\left(\|\tilde{\eta}\|_\Gamma+\|\eta\|_\Gamma\right)\\
\leq&\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\left(\left\|\tilde{A}-A\right\|_{\Gamma,\infty}+2\|A\|_{\Gamma,\infty}\right)\|\tilde{\eta}\|_\Gamma^2\\
&+\left(\|A\|_{\Gamma,\infty}^2+1\right)\|\tilde{\eta}-\eta\|_\Gamma\left(\|\tilde{\eta}\|_\Gamma+\|\eta\|_\Gamma\right).
\end{aligned}\tag{3.64}$$

Lemma 3.4.9 yields for $\mathbb{T}\in\{\mathbb{R},\mathbb{Z}\}$ the upper bound

$$\|\tilde{\eta}-\eta\|_\Gamma\left(\|\tilde{\eta}\|_\Gamma+\|\eta\|_\Gamma\right)\leq C\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\left(\|\tilde{\eta}\|_\Gamma^2+\|\eta\|_\Gamma^2\right)$$

with C defined in (3.62). By inserting this bound in estimates (3.63) and (3.64) the statement (3.60) follows, i.e.

$$\begin{aligned}
&\left|\langle\tilde{\eta},\tilde{S}_\Gamma\tilde{\eta}\rangle-\langle\eta,S_\Gamma\eta\rangle\right|\\
\leq&\begin{cases}\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\|\tilde{\eta}\|_\Gamma^2+\|A\|_{\Gamma,\infty}\|\tilde{\eta}-\eta\|_\Gamma\left(\|\tilde{\eta}\|_\Gamma+\|\eta\|_\Gamma\right), & \text{for } \mathbb{T}=\mathbb{R}\\ \left\|\tilde{A}-A\right\|_{\Gamma,\infty}\left(\left\|\tilde{A}-A\right\|_{\Gamma,\infty}+2\|A\|_{\Gamma,\infty}\right)\|\tilde{\eta}\|_\Gamma^2 & \\ +\left(\|A\|_{\Gamma,\infty}^2+1\right)\|\tilde{\eta}-\eta\|_\Gamma\left(\|\tilde{\eta}\|_\Gamma+\|\eta\|_\Gamma\right), & \text{for } \mathbb{T}=\mathbb{Z}\end{cases}\\
\leq&\begin{cases}\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\|\tilde{\eta}\|_\Gamma^2+\|A\|_{\Gamma,\infty}C\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\left(\|\tilde{\eta}\|_\Gamma^2+\|\eta\|_\Gamma^2\right), & \text{for } \mathbb{T}=\mathbb{R}\\ \left\|\tilde{A}-A\right\|_{\Gamma,\infty}\left(\left\|\tilde{A}-A\right\|_{\Gamma,\infty}+2\|A\|_{\Gamma,\infty}\right)\|\tilde{\eta}\|_\Gamma^2 & \\ +\left(\|A\|_{\Gamma,\infty}^2+1\right)C\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\left(\|\tilde{\eta}\|_\Gamma^2+\|\eta\|_\Gamma^2\right), & \text{for } \mathbb{T}=\mathbb{Z}\end{cases}\\
\leq&\bar{C}\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\left(\|\tilde{\eta}\|_\Gamma^2+\|\eta\|_\Gamma^2\right)
\end{aligned}$$

with constant $\bar{C}$ defined in (3.61). ☐

Finally, we have all tools at hand to prove the Roughness-Theorem for M-hyperbolic invertible systems. For continuous systems see [12, Lemma 2].

Theorem 3.4.11 (Roughness-Theorem for invertible ft-systems)**.**
Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$*,* $t_\pm \in \mathbb{T}$*,* $\mathbb{I} = [t_-, t_+]_\mathbb{T}$ *and* $\Gamma = \Gamma^T > 0$*. Let* (2.7)/(2.8) *be invertible and M-hyperbolic w.r.t.* $\|\cdot\|_\Gamma$ *on* $\mathbb{I}$ *with data* (α, β, P)*. If*

$$\left\|\tilde{A}(t) - A(t)\right\|_{\Gamma,\infty} < \delta$$

with

$$\begin{aligned} \delta &:= \min\left\{\delta_\mathbb{Z}, (4C_d)^{-1}\frac{1-e^{-\alpha}}{2e^\alpha - e^{-\alpha}}, (4C_d)^{-1}\frac{e^\beta - 1}{e^\beta}\right\}, \\ \delta_\mathbb{Z} &:= \begin{cases} \infty, & \text{for } \mathbb{T} = \mathbb{R}, \\ \min\left\{1, \frac{1}{2}\left\|A^{-1}\right\|_{\Gamma,\infty}^{-1}\right\}, & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}, \\ C_d &:= \max\{C, \bar{C}\} \end{aligned} \tag{3.65}$$

and $C, \bar{C}$ *defined in* (3.62) *and* (3.61) *then* (3.43)/(3.44) *is M-hyperbolic w.r.t.* $\|\cdot\|_\Gamma$ *on* $\mathbb{I}$ *with constants* $\frac{1}{2}\alpha, \frac{1}{2}\beta$*.*

Proof. To show that the system (denoted in the following by g-system) generated by (3.43)/(3.44) is M-hyperbolic with rates $\frac{1}{2}\alpha, \frac{1}{2}\beta$ we apply Lemma 3.2.9. Thus, we have to show that an invariant family of projectors $\tilde{P} : \mathbb{I} \to \mathbb{R}^{k\times k}$ of the g-system exists such that

$$\langle \tilde{\Phi}(t,s)\xi, \tilde{S}_\Gamma(t)\tilde{\Phi}(t,s)\xi\rangle \le -\hat{\alpha}\left\|\tilde{\Phi}(t,s)\xi\right\|_\Gamma^2 \tag{3.66}$$

is satisfied for all $t, s \in \mathbb{I},\ t \ge s$ and all $\xi \in \mathcal{R}(\tilde{P}(s))$ and additionally that

$$\langle \tilde{\Phi}(t,s)\xi, \tilde{S}_\Gamma(t)\tilde{\Phi}(t,s)\xi\rangle \ge \hat{\beta}\left\|\tilde{\Phi}(t,s)\xi\right\|_\Gamma^2 \tag{3.67}$$

holds for all $t, s \in \mathbb{I}$, $t \ge s$ and $\xi \in \mathcal{N}(\tilde{P}(s))$ with

$$\hat{\alpha} := \begin{cases} \frac{1}{2}\alpha, & \text{for } \mathbb{T} = \mathbb{R}, \\ 1 - e^{-\alpha}, & \text{for } \mathbb{T} = \mathbb{Z}, \end{cases}, \quad \hat{\beta} := \begin{cases} \frac{1}{2}\beta, & \text{for } \mathbb{T} = \mathbb{R}, \\ e^\beta - 1, & \text{for } \mathbb{T} = \mathbb{Z}. \end{cases}$$

Let $\xi \in \mathbb{R}^k$ and fix $s \in \mathbb{I}$. Let $t \in \mathbb{I}$, $t \ge s$. Define $\tilde{\eta} := \tilde{\eta}(t) := \tilde{\Phi}(t,s)\xi$ and $\eta := \eta(t) := \Phi(t,s)\xi$. In the following for legibility the dependency of t is not explicitly written down. First, we prove (3.66) for all $\xi \in \mathcal{R}(P(s))$ and (3.67) for all $\xi \in \mathcal{N}(P(s))$ for our fixed s and all $t \in \mathbb{I}$, $t \ge s$. Then we define the family of projectors $\tilde{P}$. From Lemma 3.4.10 we obtain

$$\left|\langle \tilde{\eta}, \tilde{S}_\Gamma\tilde{\eta}\rangle - \langle \eta, S_\Gamma\eta\rangle\right| \le \bar{C}\left\|\tilde{A} - A\right\|_{\Gamma,\infty}\left(\|\tilde{\eta}\|_\Gamma^2 + \|\eta\|_\Gamma^2\right) \tag{3.68}$$

with constant $\bar{C} > 0$ defined in (3.61) for

$$\left\|\tilde{A} - A\right\|_{\Gamma,\infty} \leq \delta_1 := \begin{cases} (t_+ - t_-)^{-1}, & \text{for } \mathbb{T} = \mathbb{R}, \\ \min\left\{1, \frac{1}{2}\left\|A^{-1}\right\|_{\Gamma,\infty}^{-1}\right\}, & \text{for } \mathbb{T} = \mathbb{Z}. \end{cases}$$

Let

$$\left\|\tilde{A} - A\right\|_{\Gamma,\infty} \leq \delta_2 := \min\left\{\delta_1, (2C)^{-1}\right\}, \; C_d := \max\{\bar{C}, C\}$$

with C defined in (3.62). The triangular inequality is used to separate $\langle\tilde{\eta}, \tilde{S}_\Gamma\tilde{\eta}\rangle$ from the rest of (3.68). Then we insert the equivalent M-hyperbolic estimates

$$\langle\eta, S_\Gamma\eta\rangle \leq -\bar{\alpha}\left\|\eta\right\|_\Gamma^2, \; \langle\eta, S_\Gamma\eta\rangle \geq \bar{\beta}\left\|\eta\right\|_\Gamma^2$$

from Lemma 3.2.9 with

$$\bar{\alpha} := \begin{cases} \alpha, & \text{for } \mathbb{T} = \mathbb{R}, \\ 1 - e^{-2\alpha}, & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}, \; \bar{\beta} := \begin{cases} \beta, & \text{for } \mathbb{T} = \mathbb{R}, \\ e^{2\beta} - 1, & \text{for } \mathbb{T} = \mathbb{Z}. \end{cases}$$

Thus, we get with (3.49)

$$\begin{aligned}
&\langle\tilde{\eta}, \tilde{S}_\Gamma, \tilde{\eta}\rangle \\
\leq &\langle\eta, S_\Gamma\eta\rangle + \bar{C}\left\|\tilde{A} - A\right\|_{\Gamma,\infty}\left(\left\|\tilde{\eta}\right\|_\Gamma^2 + \left\|\eta\right\|_\Gamma^2\right) \\
\leq &-\bar{\alpha}\left\|\eta\right\|_\Gamma^2 + \bar{C}\left\|\tilde{A} - A\right\|_{\Gamma,\infty}\left(\left\|\tilde{\eta}\right\|_\Gamma^2 + \left\|\eta\right\|_\Gamma^2\right) \\
= &-\bar{\alpha}\left(\left\|\eta\right\|_\Gamma^2 - \left\|\tilde{\eta}\right\|_\Gamma^2 + \left\|\tilde{\eta}\right\|_\Gamma^2\right) + \bar{C}\left\|\tilde{A} - A\right\|_{\Gamma,\infty}\left(2\left\|\tilde{\eta}\right\|_\Gamma^2 + \left\|\eta\right\|_\Gamma^2 - \left\|\tilde{\eta}\right\|_\Gamma^2\right) \\
\leq &-\bar{\alpha}\left\|\tilde{\eta}\right\|_\Gamma^2 + \bar{\alpha}\left|\left\|\tilde{\eta}\right\|_\Gamma^2 - \left\|\eta\right\|_\Gamma^2\right| \\
&+ 2\bar{C}\left\|\tilde{A} - A\right\|_{\Gamma,\infty}\left\|\tilde{\eta}\right\|_\Gamma^2 + \bar{C}\left\|\tilde{A} - A\right\|_{\Gamma,\infty}\left|\left\|\tilde{\eta}\right\|_\Gamma^2 - \left\|\eta\right\|_\Gamma^2\right| \\
= &-\left(\bar{\alpha} - 2\bar{C}\left\|\tilde{A} - A\right\|_{\Gamma,\infty}\right)\left\|\tilde{\eta}\right\|_\Gamma^2 + \left(\bar{\alpha} + \bar{C}\left\|\tilde{A} - A\right\|_{\Gamma,\infty}\right)\left|\left\|\tilde{\eta}\right\|_\Gamma^2 - \left\|\eta\right\|_\Gamma^2\right| \\
\leq &-\left(\bar{\alpha} - 2\bar{C}\left\|\tilde{A} - A\right\|_{\Gamma,\infty}\right)\left\|\tilde{\eta}\right\|_\Gamma^2 + \left(\bar{\alpha} + \bar{C}\left\|\tilde{A} - A\right\|_{\Gamma,\infty}\right)4C\left\|\tilde{A} - A\right\|_{\Gamma,\infty}\left\|\tilde{\eta}\right\|_\Gamma^2 \\
\leq &-\left(\bar{\alpha} - 2C_d\left(1 + 2\bar{\alpha}\right)\left\|\tilde{A} - A\right\|_{\Gamma,\infty} - 4\bar{C}C\left\|\tilde{A} - A\right\|_{\Gamma,\infty}^2\right)\left\|\tilde{\eta}\right\|_\Gamma^2 \\
\leq &-\left(\bar{\alpha} - 2C_d\left(1 + 2\bar{\alpha}\right)\left\|\tilde{A} - A\right\|_{\Gamma,\infty} - 2\bar{C}\left\|\tilde{A} - A\right\|_{\Gamma,\infty}\right)\left\|\tilde{\eta}\right\|_\Gamma^2 \\
\leq &-\left(\bar{\alpha} - 4C_d\left(1 + \bar{\alpha}\right)\left\|\tilde{A} - A\right\|_{\Gamma,\infty}\right)\left\|\tilde{\eta}\right\|_\Gamma^2 \\
= &-\left(\left(1 - e^{-2\alpha}\right) - 4C_d\left(2 - e^{-2\alpha}\right)\left\|\tilde{A} - A\right\|_{\Gamma,\infty}\right)\left\|\tilde{\eta}\right\|^2
\end{aligned}$$

for all $\mu \in \mathcal{R}(P(s))$. Let $\left\|\tilde{A}-A\right\|_{\Gamma,\infty} \leq \delta_3^1 := \min\left\{\delta_2, (4C_d)^{-1}\frac{e^{-\alpha}-e^{-2\alpha}}{2-e^{-2\alpha}}\right\}$. Then we have

$$\begin{aligned}
&\langle\tilde{\eta}, \tilde{S}_\Gamma\tilde{\eta}\rangle \\
\leq &-\left(\left(1-e^{-2\alpha}\right) - 4C_d\left(2-e^{-2\alpha}\right)\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\right)\|\tilde{\eta}\|_\Gamma^2 \\
\leq &-\left(1-e^{-2\alpha}-\left(e^{-\alpha}-e^{-2\alpha}\right)\right)\|\tilde{\eta}\|_\Gamma^2 \\
= &-\left(1-e^{-\alpha}\right)\|\tilde{\eta}\|_\Gamma^2 \\
= &-\hat{\alpha}\|\tilde{\eta}\|_\Gamma^2 .
\end{aligned}$$

The second estimate (3.67) for our fixed $s \in \mathbb{I}$ results from the same line of argument for $\left\|\tilde{A}-A\right\|_{\Gamma,\infty} \leq \delta_3^2 := \min\left\{\delta_2, (4C_d)^{-1}\frac{e^{2\beta}-e^{\beta}}{e^{2\beta}}\right\}$ and each $\mu \in \mathcal{N}(P(s))$, i.e.

$$\begin{aligned}
&\langle\tilde{\eta}, \tilde{S}_\Gamma\tilde{\eta}\rangle \\
\geq&\langle\eta, S_\Gamma\eta\rangle - \bar{C}\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\left(\|\tilde{\eta}\|_\Gamma^2+\|\eta\|_\Gamma^2\right) \\
\geq&\bar{\beta}\|\eta\|_\Gamma^2 - \bar{C}\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\left(\|\tilde{\eta}\|_\Gamma^2+\|\eta\|_\Gamma^2\right) \\
\geq&\left(\bar{\beta}-2\bar{C}\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\right)\|\tilde{\eta}\|_\Gamma^2 - \left(\bar{\beta}+\bar{C}\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\right)\left|\|\tilde{\eta}\|_\Gamma^2-\|\eta\|_\Gamma^2\right| \\
\geq&\left(\bar{\beta}-2\bar{C}\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\right)\|\tilde{\eta}\|_\Gamma^2 - \left(\bar{\beta}+\bar{C}\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\right)4C\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\|\tilde{\eta}\|_\Gamma^2 \\
\geq&\left(\bar{\beta}-4C_d(1+\bar{\beta})\left\|\tilde{A}-A\right\|_{\Gamma,\infty}\right)\|\tilde{\eta}\|_\Gamma^2 \\
\geq&\left(e^{2\beta}-1-\left(e^{2\beta}-e^{\beta}\right)\right)\|\tilde{\eta}\|_\Gamma^2 \\
=&\left(e^{\beta}-1\right)\|\tilde{\eta}\|_\Gamma^2 \\
=&\hat{\beta}\|\tilde{\eta}\|_\Gamma^2 .
\end{aligned}$$

Before we define the invariant family of projectors $\tilde{P} : \mathbb{I} \to \mathbb{R}^{k\times k}$ fulfilling (3.66) and (3.67) we show

$$\delta := \min\left\{\delta_{\mathbb{Z}}, (4C_d)^{-1}\frac{1-e^{-\alpha}}{2e^{\alpha}-e^{-\alpha}}, (4C_d)^{-1}\frac{e^{\beta}-1}{e^{\beta}}\right\} \leq \min\left\{\delta_2, \delta_3^1, \delta_3^2\right\}, \quad (3.69)$$

where $\delta_{\mathbb{Z}}$ is defined in (3.65). We have

$$\begin{aligned}
\min\left\{\delta_2, \delta_3^1, \delta_3^2\right\} &= \min\left\{\delta_2, (4C_d)^{-1}\frac{e^{-\alpha}-e^{-2\alpha}}{2-e^{-2\alpha}}, (4C_d)^{-1}\frac{e^{2\beta}-e^{\beta}}{e^{2\beta}}\right\} \\
&= \min\left\{\delta_2, (4C_d)^{-1}\frac{1-e^{-\alpha}}{2e^{\alpha}-e^{-\alpha}}, (4C_d)^{-1}\frac{e^{\beta}-1}{e^{\beta}}\right\}.
\end{aligned}$$

For $\delta_2 = \min\{\delta_1, (2C)^{-1}\}$ and $\delta_1 = \begin{cases} (t_+ - t_-)^{-1}, & \text{for } \mathbb{T} = \mathbb{R}, \\ \min\left\{1, \frac{1}{2}\|A^{-1}\|_{\Gamma,\infty}^{-1}\right\}, & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}$

follows with

$$(2C)^{-1} = \begin{cases} \frac{1}{4}(t_+ - t_-)^{-1} e^{-(1+2(t_+-t_-)\|A\|_{\Gamma,\infty})}, & \text{for } \mathbb{T} = \mathbb{R}, \\ \frac{1}{4}\left(1 + \|A\|_{\Gamma,\infty}\right)^{-(t_+-t_-)} \|A^{-1}\|_{\Gamma,\infty}^{-(t_+-t_-)}, & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}$$
$$< \begin{cases} (t_+ - t_-)^{-1}, & \text{for } \mathbb{T} = \mathbb{R}, \\ \frac{1}{2}\|A^{-1}\|_{\Gamma,\infty}^{-(t_+-t_-)}, & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}$$

$$\delta_2 = \begin{cases} (2C)^{-1}, & \text{for } \mathbb{T} = \mathbb{R}, \\ \min\left\{1, \frac{1}{2}\|A^{-1}\|_{\Gamma,\infty}^{-1}(2C)^{-1}\right\}, & \text{for } \mathbb{T} = \mathbb{Z}. \end{cases}$$

With

$$(2C)^{-1} \geq (2C_d)^{-1} \geq (4C_d)^{-1} \geq (4C_d)^{-1}\frac{e^{\beta} - 1}{e^{\beta}}.$$

we obtain (3.69).

Finally, we define the invariant family of projectors $\tilde{P} : \mathbb{I} \to \mathbb{R}^{k\times k}$, which satisfies (3.66) for all $\mu \in \mathcal{R}(\tilde{P}(s))$ and (3.67) for all $\mu \in \mathcal{N}(\tilde{P}(s))$ and all $t \in \mathbb{I}$, $t \geq s$. We already proved that by choosing δ sufficiently small the perturbed equation (3.43)/(3.44) is invertible as well. Thus, the solution operators Φ and $\tilde{\Phi}$ are both invertible. Hence, we define an invariant family of projectors by

$$\tilde{P}(s) := P(s),\ \tilde{P}(t) := \tilde{\Phi}(t,s)\tilde{P}(s)\tilde{\Phi}(s,t),\ t \in \mathbb{I}.$$

Then

$$\langle\tilde{\Phi}(t,s)\mu, \tilde{S}_\Gamma(t)\tilde{\Phi}(t,s)\mu\rangle \leq -\hat{\alpha}\left\|\tilde{\Phi}(t,s)\mu\right\|_\Gamma^2 \text{ for all } \mu \in \mathcal{R}(P(s)) = \mathcal{R}(\tilde{P}(s)),$$
$$\langle\tilde{\Phi}(t,s)\mu, \tilde{S}_\Gamma(t)\tilde{\Phi}(t,s)\mu\rangle \geq \hat{\beta}\left\|\tilde{\Phi}(t,s)\mu\right\|_\Gamma^2 \text{ for all } \mu \in \mathcal{N}(P(s)) = \mathcal{N}(\tilde{P}(s))$$

are satisfied for all $t \in \mathbb{I}$, $t \geq s$. By the invariance of $\tilde{P}$ and by the invertibility of $\tilde{\Phi}$

$$\langle\tilde{\Phi}(t,t_0)\xi, \tilde{S}_\Gamma(t)\tilde{\Phi}(t,t_0)\xi\rangle \leq -\hat{\alpha}\left\|\tilde{\Phi}(t,t_0)\xi\right\|_\Gamma^2 \text{ for all } \xi \in \mathcal{R}(\tilde{P}(t_0),$$
$$\langle\tilde{\Phi}(t,t_0)\xi, \tilde{S}_\Gamma(t)\tilde{\Phi}(t,t_0)\xi\rangle \geq \hat{\beta}\left\|\tilde{\Phi}(t,t_0)\xi\right\|_\Gamma^2 \text{ for all } \xi \in \mathcal{N}(\tilde{P}(t_0)).$$

follows for all $t_0, t \in \mathbb{I}$ with $t \geq t_0$. Thus, by Lemma 3.2.9 the Equation (3.43)/(3.44) is M-hyperbolic with the invariant family of projectors $\tilde{P}$ and rates $\frac{1}{2}\alpha, \frac{1}{2}\beta$. □

Corollary 3.4.12. *Let the assumptions of Theorem 3.4.11 be satisfied. Then for each $s \in \mathbb{I}$ the perturbed equation* (3.43)/(3.44) *is M-hyperbolic with the invariant family of projectors $\tilde{P}(t) := \tilde{\Phi}(t,s)P(s)\tilde{\Phi}(s,t)$, $t \in \mathbb{I}$.*

Chapter 4

Stable and Unstable Subspaces and Cones

For analyzing hyperbolic (i)ft-systems, it is convenient to study their stable and unstable set. These sets are invariant under the solution operator and they comprise the solutions that decay or grow at certain rates. In this chapter we start with linear systems. For linear ift-systems these sets are subspaces and for linear ft-systems they are cones. The stable and unstable subspaces of a linearized system are used to find the stable and unstable manifolds or fiber bundles of the underlying nonlinear system. The Stable Manifold Theorem states, roughly speaking, that the subspaces are tangential to the manifolds. We discuss the nonlinear setup in detail in Section 6.6. There, we present an analogous statement for finite time systems as well.

This chapter starts with the definition of the stable and unstable subspaces of an ift-system and presents an alternative representation using the unique invariant family of projectors. Then we derive a definition for the stable and unstable cone of an ft-system from the definition of the subspaces and prove that the cones have a similar relation to the invariant families of projectors. This means that the uniquely determined cones can be described by the union of the nonunique projectors.

Let system (2.6) be hyperbolic on $\mathbb{I} = \mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$ with the unique invariant family of projectors $P : \mathbb{T} \to \mathbb{R}^k$. Then the **stable and unstable set** of 0 at time $t_0 \in \mathbb{T}$ are defined as

$$\begin{aligned} {}^{\mathbb{T}}V_s(t_0) &:= \left\{ \xi \in \mathbb{R}^k \,\middle|\, \sup_{t \geq t_0} \|\Phi(t, t_0)\xi\| < \infty \right\}, \\ {}^{\mathbb{T}}V_u(t_0) &:= \left\{ \xi \in \mathbb{R}^k \,\middle|\, \sup_{t \leq t_0} \|\Phi(t, t_0)\xi\| < \infty \right\} \end{aligned} \tag{4.1}$$

if the system is invertible. For the definition and for a proof of the following property (4.2) we refer to [104, p.227] for continuous time systems and to [105, Proposition 2.3] for discrete time invertible systems. Analog statments for noninvertible systems are presented in [8, Theorem 2.5]. These sets satisfy

$$^{\mathbb{T}}V_s(t_0) = \mathcal{R}(P(t_0)),\ ^{\mathbb{T}}V_u(t_0) = \mathcal{N}(P(t_0)) \tag{4.2}$$

for all $t_0 \in \mathbb{T}$, which shows that the stable and unstable sets are **subspaces**. This is not true for ft-systems.

For a compact interval $\mathbb{I}$ let system (2.6) be M-hyperbolic. Then an invariant family of projectors $P : \mathbb{I} \to \mathbb{R}^{k\times k}$, which fulfills (3.9) and (3.10) with rates $\alpha, \beta > 0$, is generally not unique. We expect that the stable and unstable t_0-sets $^{\mathbb{I}}V_{s,u}(t_0)$ of the M-hyperbolic system (2.6) satisfy

$$^{\mathbb{I}}V_s(t_0) = \bigcup_{P(t_0)\in\mathcal{P}_{t_0}} \mathcal{R}(P(t_0)) \text{ and } ^{\mathbb{I}}V_u(t_0) = \bigcup_{P(t_0)\in\mathcal{P}_{t_0}} \mathcal{N}(P(t_0)) \tag{4.3}$$

for all $t_0 \in \mathbb{I}$, where

$$\begin{aligned}\mathcal{P}_{t_0} :=\{&P(t_0)|P : \mathbb{I} \to \mathbb{R}^{k\times k} \text{ is an invariant family of projectors, which} \\ &\text{fulfills (3.8), (3.9) and (3.10) with constants } \alpha, \beta > 0\}.\end{aligned}$$

We could use equation (4.3) as a definition for the stable and unstable sets $^{\mathbb{I}}V_{s,u}(\cdot)$. However, we prefer to define the finite time sets similar to (4.1), see Definition 4.1.3/ 4.1.5, and then show in Theorem 4.2.4 that the statements in (4.3) are satisfied.

Stable and Unstable Cones

In [43] we find a definition for the stable and unstable t_0-set $^{\mathbb{I}}V_{s,u}(t_0)$ of a continuous ft-system. We derive an adequate definition for the stable and unstable sets of discrete ft-systems. We will see that the sets $^{\mathbb{I}}V_{s,u}(t_0)$ are double-cones for all $t_0 \in \mathbb{I}$. Therefore, we start with the definition of various types of cones, see [67, Example 1.1.4], [122, Definition 2.1.2] and [28, Definition 2.11 and Exercise 2.12]. Then we define the stable and unstable set and study their properties and we conclude this chapter with the proof of the statements in (4.3).

Definition 4.1.1. *A subset $C \subset \mathbb{R}^k$ is called a* ***cone*** *if $\lambda C \subset C$ for all $\lambda \in \mathbb{R}$, $\lambda \geq 0$. A cone is a* ***convex cone*** *if the cone is convex, i.e. if $C + C \subset C$. We call $C \subset \mathbb{R}^k$ a* ***closed cone*** *if C is closed and an* ***open cone*** *if $C \setminus \{0\}$ is open. Further, we say $D := -C \cup C$ is a* ***(closed, open, convex) double-cone*** *if C is a (closed, open, convex) cone. A cone C is a* ***connected cone*** *if $C \setminus \{0\}$ is connected. Let $D \subset \mathbb{R}^k$ be a double-cone such that $D \setminus \{0\}$ consists of two connected cones $C, -C$, then C is called a* ***half-cone*** *of D and D is called a* ***connected double-cone****. Let $C \subset \mathbb{R}^k$ be a nontrivial connected cone. By defining*

$$\mathcal{X} := \partial C \cap \partial \mathcal{S}^1,$$

where $\mathcal{S}^1 \subset \mathbb{R}^k$ *denotes the unit sphere, we define the* ***width*** $\boldsymbol{d_C}$ *of* C *with the help of the Hausdorff-distance between the connected components of* $\mathcal{X}$*. Let* $\mathcal{X}_i$*,* $1 \leq i \leq n$ *be the connected components of* $\mathcal{X}$ *such that* $\mathcal{X} = \bigcup_{i=1}^n \mathcal{X}_i$*. The width is defined by*

$$d_C := \begin{cases} \max\{\|x-y\|_2 | x, y \in \mathcal{X}\} & , \textit{ if } n = 1; \\ \max\{d_H(\mathcal{X}_i, \mathcal{X}_j) | i, j \in \{1, \ldots, n\}, i \neq j\} & , \textit{ if } n > 1, \end{cases} \tag{4.4}$$

$$\textit{with } d_H(\mathcal{X}_i, \mathcal{X}_j) := \max\{d(x_i, \mathcal{X}_j) | x_i \in \mathcal{X}_i\},$$
$$\textit{and } d(x_i, \mathcal{X}_j) := \min\{\|x_i - x_j\|_2 \, | x_j \in \mathcal{X}_j\}. \tag{4.5}$$

If $D \subset \mathbb{R}^k$ *is a connected double-cone then its width equals the width of its half-cone.*

The ***angle*** $\boldsymbol{\tau_C}$ *of a nontrivial connected cone* $C \subset \mathbb{R}^k$ *with* $-\bar{C} \cup \bar{C} \neq \mathbb{R}^k$ *is defined as*

$$\tau_C := \arccos\left(1 - \frac{(d_C)^2}{2}\right).$$

Remark 4.1.2. *The equations* (4.4)-(4.5) *are well defined. Indeed the Hausdorff-distance* $d_H : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ *and the distance* $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ *are continuous and* $\mathcal{X}$ *is compact (see below) and, hence, they reach their maximum and minimum on* $\mathcal{X} \times \mathcal{X}$*. The set* $\mathcal{X}$ *is compact, since* $\mathcal{X} = \partial C \cap \partial \mathcal{S}^1$ *is bounded by the boundedness of* $\mathcal{S}^1$ *and it is closed since both sets* ∂C *and* $\partial \mathcal{S}^1$ *are closed.*

We should mention that a cone or a double-cone, which is connected, is generally not a connected cone or connected double-cone in the sense of Definition 4.1.1; it must still be connected without 0. For an illustration see the top graphs in Figure 4.1.

We present a few examples of cones in 2-dimensional spaces in Figure 4.1 to see the difference between a connected cone and a cone, which is not a connected cone. In the second row of Figure 4.1 we see on the left a connected cone and on the right a connected double-cone. The pattern part marks one half-cone. The bottom pictures show the angle and width of a (double-)cone in a 2-dimensional space. The boundary of a nonempty open connected double-cone in a 2-dimensional subspace consists of two disjoint subspaces, see middle right part of Figure 4.1. Further, every connected cone in a 2-dimensional subspace is a convex cone. In higher dimensional subspaces this is generally not true, see for example Figure 4.6.

It is easy to see, that the intersection of two (connected, closed, double) cones is also a (connected, closed, double) cone. This is important for the definition of the stable and unstable t_0-sets ${}^{\mathbb{I}}V_{s,u}(t_0)$ of continuous ft-systems as in [43]. This definition is not transferable to the discrete time case, since discrete systems are generally not invertible. We introduce an adequate definition

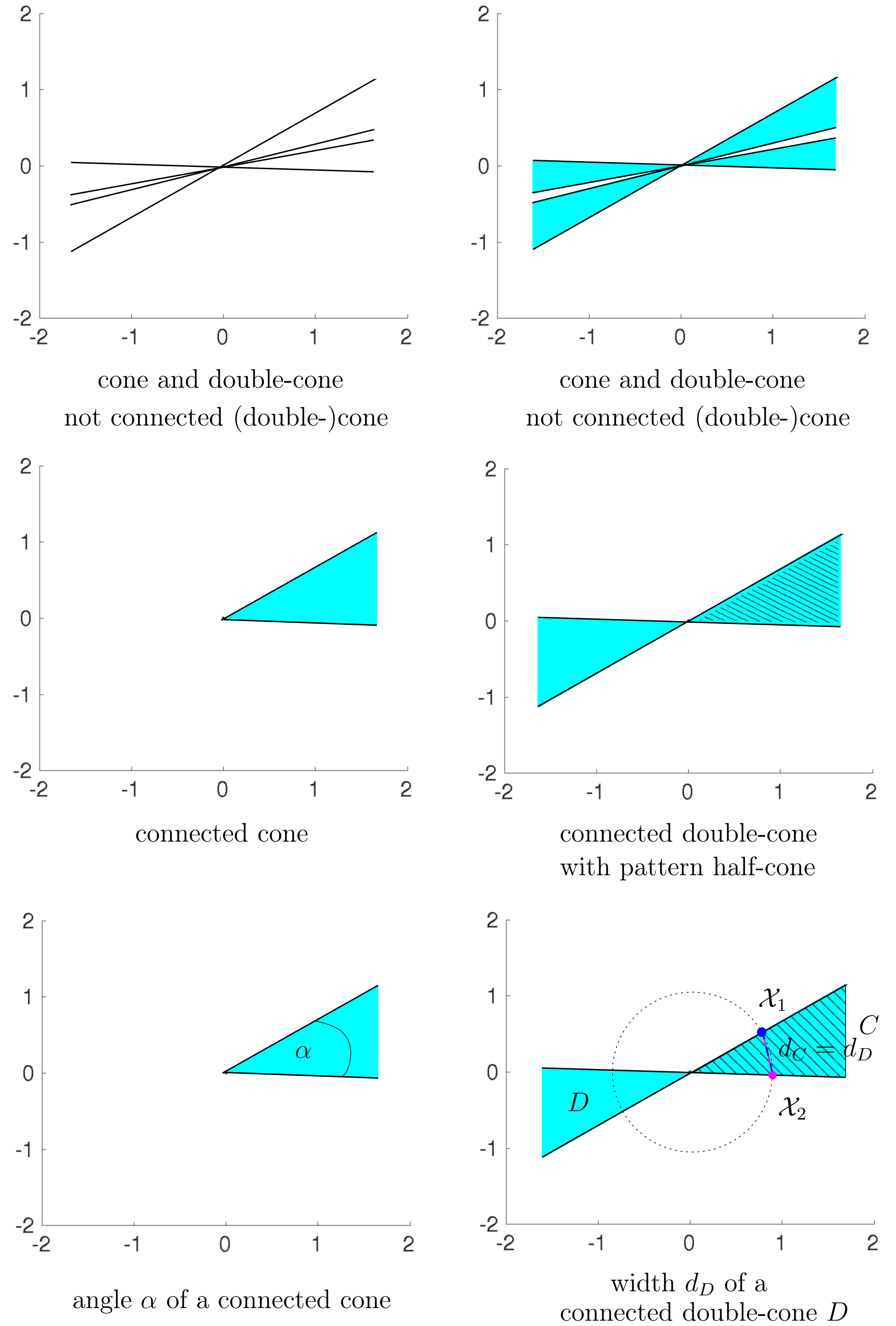


Figure 4.1: Different types of cones and the angle and width of a (double-)cone

for discrete systems in Definition 4.1.5. This definition applies to continuous time systems as well. However, we will define the (un)stable cones of continuous and discrete systems separately, since the definition for continuous systems avoids technicalities that occur for noninvertible systems only.

Definition 4.1.3. *Consider equation* (2.7) *on* $\mathbb{I} = [t_-, t_+]$. *Let* $t_0 \in \mathbb{I}$ *and* $\|\cdot\|$ *be any norm in* $\mathbb{R}^k$. *We define the two cones*

$$
\begin{aligned}
{}^{\mathbb{I}}V_s^+(t_0) &:= \left\{\xi \in \mathbb{R}^k \middle| \exists \alpha > 0 : \|\Phi(t,t_0)\xi\| e^{\alpha t} \text{ is decreasing for } t \in [t_0, t_+]\right\}, \\
{}^{\mathbb{I}}V_s^-(t_0) &:= \left\{\xi \in \mathbb{R}^k \middle| \exists \alpha > 0 : \|\Phi(t,t_0)\xi\| e^{\alpha t} \text{ is decreasing for } t \in [t_-, t_0]\right\},
\end{aligned}
$$

and we denote their intersection ${}^{\mathbb{I}}V_s(t_0) := {}^{\mathbb{I}}V_s^+(t_0) \cap {}^{\mathbb{I}}V_s^-(t_0)$ *as the* ***stable*** ***t_0-cone*** *w.r.t.* $\|\cdot\|$. *Similarly, we define the* ***unstable t_0-cone*** *w.r.t.* $\|\cdot\|$ *by* ${}^{\mathbb{I}}V_u(t_0) := {}^{\mathbb{I}}V_u^+(t_0) \cap {}^{\mathbb{I}}V_u^-(t_0)$, *where the two cones are defined as*

$$
\begin{aligned}
{}^{\mathbb{I}}V_u^+(t_0) &:= \left\{\xi \in \mathbb{R}^k \middle| \exists \beta > 0 : \|\Phi(t,t_0)\xi\| e^{-\beta t} \text{ is increasing for } t \in [t_0, t_+]\right\}, \\
{}^{\mathbb{I}}V_u^-(t_0) &:= \left\{\xi \in \mathbb{R}^k \middle| \exists \beta > 0 : \|\Phi(t,t_0)\xi\| e^{-\beta t} \text{ is increasing for } t \in [t_-, t_0]\right\}.
\end{aligned}
$$

For noninvertible (discrete) systems we can not define ${}^{\mathbb{I}}V_{s,u}^-$ as above, since $\Phi(t, t_0)$ may not be defined for all $t \in [t_-, t_0]$. Therefore, we first define two maps for $\mathbb{I} = [n_-, n_+]_{\mathbb{Z}}$

$$
\begin{aligned}
{}_{\Phi}\mathcal{T}_{\min} &: \mathbb{R}^k \times \mathbb{I} \to \mathbb{I}, \\
&\quad (\xi, n_0) \mapsto \bar{n} := \min\left\{n \in [n_-, n_0]_{\mathbb{Z}} \middle| \exists x \in \mathbb{R}^k : \Phi(n_0, n)x = \xi\right\}, \\
{}_{\Phi}\mathcal{T}_{\text{pre}} &: \mathbb{R}^k \times \mathbb{I} \rightsquigarrow \mathbb{R}^k, \\
&\quad (\xi, n_0) \mapsto \left\{\mu \in \mathbb{R}^k \middle| \Phi(n_0, \bar{n})\mu = \xi \text{ with } \bar{n} := {}_{\Phi}\mathcal{T}_{\min}(\xi, n_0)\right\}.
\end{aligned}
$$

The arrow $\rightsquigarrow$ indicates that the given map is a set-valued map, see [6]. The first function provides the earliest time at which a preimage of a vector ξ under Φ still exists. The second function yields all preimages to that time (earliest defined time). Figure 4.2 illustrates an example, where the images of the two functions for a pictured vector ξ are marked.

We should mention that in contrast to ${}_{\Phi}\mathcal{T}_{\text{pre}}$ the function ${}_{\Phi}\mathcal{T}_{\text{ker}}$ is not a set-valued function. The image of ${}_{\Phi}\mathcal{T}_{\text{ker}}(\xi, n_0)$ is the earliest time n at which $\Phi(n, n_0)$ maps ξ to 0.

Basic properties of these three functions are presented in the next lemma.

Lemma 4.1.4. *Let* $\mathbb{I} = [n_-, n_+]_{\mathbb{Z}}$ *and* Φ *be the solution operator associated with* (2.8).

(a) Then

$$
{}_{\Phi}\mathcal{T}_{\min}(\xi, n_0) \geq {}_{\Phi}\mathcal{T}_{\min}(\Phi(n_1, n_0)\xi, n_1)
$$

is satisfied for all $\xi \in \mathbb{R}^k$ *and* $n_1, n_0 \in \mathbb{I}$, $n_1 > n_0$.

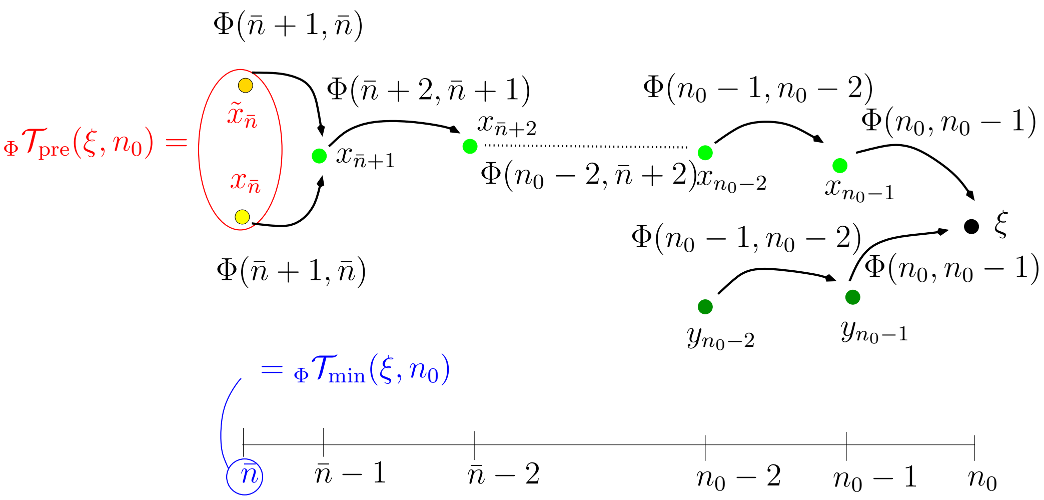


Figure 4.2: An illustration of the two functions ${}_{\Phi}\mathcal{T}_{\text{pre}}(\xi,n_0)$, ${}_{\Phi}\mathcal{T}_{\min}(\xi,n_0)$ for an example vector ξ at time $n_0 \in \mathbb{I}$.

(b) *If*

$$ {}_{\Phi}\mathcal{T}_{\min}(\xi,n_0) = {}_{\Phi}\mathcal{T}_{\min}(\Phi(n_1,n_0)\xi,n_1)$$

holds for all $\xi \in \mathbb{R}^k$ *and* $n_1,n_0 \in \mathbb{I}$, $n_1 > n_0$ *then*

$$ {}_{\Phi}\mathcal{T}_{\text{pre}}(\xi,n_0) \subset {}_{\Phi}\mathcal{T}_{\text{pre}}(\Phi(n_1,n_0)\xi,n_1).$$

(c) *For all*

$$\bar\xi \in \left\{\xi \in \mathbb{R}^k \setminus \{0\} \middle| \exists \beta > 0 : \|\Phi(n,n_-)\xi\| e^{-\beta n} \text{ is increasing for all } n \in \mathbb{I}\right\}$$

we have

$$ {}_{\Phi}\mathcal{T}_{\ker}(\bar\xi,n_-) \notin \mathbb{I}.$$

Proof. Fix $n_1,n_0 \in \mathbb{I}$, $n_1 > n_0$ and let $\xi \in \mathbb{R}^k$. For (a) let $\bar n_0 := {}_{\Phi}\mathcal{T}_{\min}(\xi,n_0)$ then there exists $\bar x_0 \in \mathbb{R}^k$ such that $\Phi(n_0,\bar n_0)\bar x_0 = \xi$ holds. It follows by $\Phi(n_1,\bar n_0)\bar x_0 = \Phi(n_1,n_0)\xi$ that

$$\begin{aligned} {}_{\Phi}\mathcal{T}_{\min}(\Phi(n_1,n_0)\xi,n_1) &= \min\left\{n \in [n_-,n_1]_{\mathbb{Z}} \middle| \exists x \in \mathbb{R}^k : \Phi(n_1,n)x = \Phi(n_1,n_0)\xi\right\} \\ &\le \bar n_0. \end{aligned}$$

For (b) let

$$ {}_{\Phi}\mathcal{T}_{\min}(\xi,n_0) = {}_{\Phi}\mathcal{T}_{\min}(\Phi(n_1,n_0)\xi,n_1)$$

then

$$\begin{aligned} & {}_{\Phi}\mathcal{T}_{\text{pre}}(\xi,n_0) \\ &= \left\{\mu \in \mathbb{R}^k \middle| \Phi(n_0,\bar n)\mu = \xi \text{ with } \bar n := {}_{\Phi}\mathcal{T}_{\min}(\xi,n_0)\right\} \\ &= \left\{\mu \in \mathbb{R}^k \middle| \Phi(n_0,\bar n)\mu = \xi \text{ with } \bar n := {}_{\Phi}\mathcal{T}_{\min}(\Phi(n_1,n_0)\xi,n_1)\right\} \\ &\subset \left\{\mu \in \mathbb{R}^k \middle| \Phi(n_1,n_0)\Phi(n_0,\bar n)\mu = \Phi(n_1,n_0)\xi \text{ with } \bar n := {}_{\Phi}\mathcal{T}_{\min}(\Phi(n_1,n_0)\xi,n_1)\right\} \\ &= {}_{\Phi}\mathcal{T}_{\text{pre}}(\Phi(n_1,n_0)\xi,n_1). \end{aligned}$$

For (c) let

$$\bar{\xi} \in \left\{ \xi \in \mathbb{R}^k \setminus \{0\} \middle| \exists \beta > 0 : \|\Phi(n, n_-)\xi\| e^{-\beta n} \text{ is increasing for all } n \in \mathbb{I} \right\}.$$

Assume there exists an $\hat{n} \in \mathbb{I}$ with

$$\hat{n} = {}_{\Phi}\mathcal{T}_{\ker}(\bar{\xi}, n_-).$$

Then

$$0 = \left\|\Phi(\hat{n}, n_-)\bar{\xi}\right\| e^{-\beta\hat{n}} \geq \left\|\bar{\xi}\right\| e^{-\beta n_-} > 0.$$

This is a contradiction, thus, the assumption is wrong, i.e. ${}_{\Phi}\mathcal{T}_{\ker}(\bar{\xi}, n_-) \notin \mathbb{I}$. □

With these functions we are able to define the stable and unstable cone for noninvertible systems. The main difference to the invertible systems is the definition of ${}^{\mathbb{I}}V_s^-(\cdot)$. In Figure 4.3 we illustrate differences between solutions of invertible and noninvertible systems. Assume that Φ is the solution operator of an invertible system and Ψ is the solution operator of a noninvertible system. Let $\xi \in \mathbb{R}^k$, which is pictured in black in Figure 4.3. The invertible system has an invertible solution operator and yields for any previous time $t \in [n_-, n_0]$ one preimage of ξ under $\Phi(t, n_0)$, marked in blue.

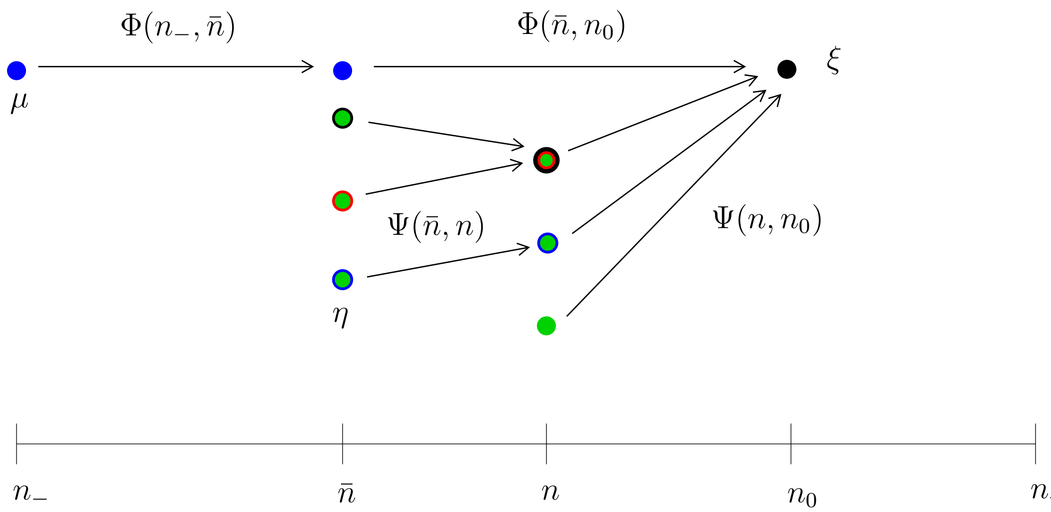


Figure 4.3: Example that illustrates all preimages of a vector ξ for an invertible (blue) and for a noninvertible system (green).

Definition 4.1.3 implies $\xi \in {}^{\mathbb{I}}V_s^-(n_0)$ if

$$\|\Phi(t, n_0)\xi\| \, e^{\alpha t} \text{ is decreasing for all } t \in [n_-, n_0]_{\mathbb{Z}} \text{ and any } \alpha > 0.$$

This is equivalent to $\xi \in {}^{\mathbb{I}}V_s^-(n_0)$ if

$$\|\Phi(t, n_-)\mu\| \, e^{\alpha t} \text{ is decreasing for all } t \in [n_-, n_0]_{\mathbb{Z}} \text{ and any } \alpha > 0,$$

where $\mu \in {}_\Phi\mathcal{T}_{\text{pre}}(\xi, n_0)$ and $n_- = {}_\Phi\mathcal{T}_{\text{pre}}(\xi, n_0)$. This condition does not use the inverse of the solution operator. Thus, it is an ansatz for noninvertible systems.

If a system is noninvertible then the preimages of ξ under $\Psi(n_0, \cdot)$ may not be unique and there may not exist a preimage of ξ to every previous time $t \in [n_-, n_0]_\mathbb{Z}$. All preimages of ξ are plotted in Figure 4.3 in green. The function ${}_\Psi\mathcal{T}_{\min}(\cdot, n_0)$ determines for ξ the earliest time at which a preimage of ξ under Ψ exists. In Figure 4.3 this time is denoted by $\bar{n}$. It is possible that more than one preimage to time $\bar{n}$ exists. Each of these preimages result in a "maximal long" solution, marked in Figure 4.3 by a black, red and blue circle. We want that the vector ξ lies in ${}^\mathbb{I}V_s^-(n_0)$ if ξ is part of a maximal long solution which is "decreasing". More precisely, $\xi \in {}^\mathbb{I}V_s^-(n_0)$ if there exists a $\eta \in {}_\Psi\mathcal{T}_{\text{pre}}(\xi, n_0)$ such that

$$\|\Psi(t, \bar{n})\eta\| \, e^{-\alpha t} \text{ is decreasing for all } t \in [\bar{n}, n_0]_\mathbb{Z}.$$

With this background we define the almost stable and unstable cones of a discrete system. Note that the following definition is analog to Definition 4.1.3 for invertible systems, i.e. the almost stable cone coincides with the stable cone.

Definition 4.1.5. *Consider equation* (2.8) *on* $\mathbb{I} = [n_-, n_+]_\mathbb{Z}$. *Let* $n_0 \in \mathbb{I}$ *and* $\|\cdot\|$ *be any norm in* $\mathbb{R}^k$. *We define the two cones*

$$\begin{aligned}
{}^\mathbb{I}V_s^+(n_0) &:= \big\{\xi \in \mathbb{R}^k \big| \exists \alpha > 0 : \|\Phi(n, n_0)\xi\| \, e^{\alpha n} \textit{ is decreasing for } n \in [n_0, n_+]_\mathbb{Z}\big\}, \\
{}^\mathbb{I}V_s^-(n_0) &:= \big\{\xi \in \mathbb{R}^k \big| \exists \mu_{\bar{n}} \in {}_\Phi\mathcal{T}_{\text{pre}}(\xi, n_0), \bar{n} := {}_\Phi\mathcal{T}_{\min}(\xi, n_0), \\
&\qquad \exists \alpha > 0 : \|\Phi(n, \bar{n})\mu_{\bar{n}}\| \, e^{\alpha n} \textit{ is decreasing for } n \in [\bar{n}, n_0]_\mathbb{Z}\big\},
\end{aligned}$$

and we denote their intersection ${}^\mathbb{I}\bar{V}_s(n_0) := {}^\mathbb{I}V_s^+(n_0) \cap {}^\mathbb{I}V_s^-(n_0)$ *as the* ***almost stable*** $\boldsymbol{n_0}$***-cone*** *w.r.t.* $\|\cdot\|$. *Similarly, we define the* ***unstable*** $\boldsymbol{n_0}$***-cone*** *w.r.t.* $\|\cdot\|$ *by* ${}^\mathbb{I}V_u(n_0) := {}^\mathbb{I}V_u^+(n_0) \cap {}^\mathbb{I}V_u^-(n_0)$, *where the two cones are defined as*

$$\begin{aligned}
{}^\mathbb{I}V_u^+(n_0) &:= \big\{\xi \in \mathbb{R}^k \big| \exists \beta > 0 : \|\Phi(n, n_0)\xi\| \, e^{-\beta n} \textit{ is increasing for } n \in [n_0, n_+]_\mathbb{Z}\big\}, \\
{}^\mathbb{I}V_u^-(n_0) &:= \big\{\xi \in \mathbb{R}^k \big| {}_\Phi\mathcal{T}_{\min}(\xi, n_0) = n_-, \exists \mu_{n_-} \in {}_\Phi\mathcal{T}_{\text{pre}}(\xi, n_0), \\
&\qquad \exists \beta > 0 : \big\|\Phi(n, n_-)\mu_{n_-}\big\| \, e^{-\beta n} \textit{ is increasing for } n \in [n_-, n_0]_\mathbb{Z}\big\}.
\end{aligned}$$

Remark 4.1.6. *All sets in the above definitions are actually cones, even double-cones.*

Indeed, let $t_0 \in \mathbb{I}$ *and* $\xi \in {}^\mathbb{I}V_s^+(t_0)$. *Then there exists* $\alpha > 0$ *such that*

$$\|\Phi(t, t_0)\xi\| e^{\alpha t} \tag{4.6}$$

is decreasing for $t \in [t_0, t_+]_\mathbb{T}$. *Further, for all* $\lambda \in \mathbb{R}$

$$\|\Phi(t, t_0)\lambda\xi\| e^{\alpha t} = |\lambda| \|\Phi(t, t_0)\xi\| e^{\alpha t} \tag{4.7}$$

is decreasing for $t \in [t_0, t_+]_{\mathbb{T}}$. Analogously, we can show the same statement for ${}^{\mathbb{I}}V_s^-(t_0)$ and ${}^{\mathbb{I}}V_u^{\pm}(t_0)$. Hence, all t_0-sets ${}^{\mathbb{I}}V_{s,u}^{\pm}(t_0)$ are cones. Thus, ${}^{\mathbb{I}}V_s(t_0)$, ${}^{\mathbb{I}}\bar{V}_s(t_0)$ and ${}^{\mathbb{I}}V_u(t_0)$, as the intersections of ${}^{\mathbb{I}}V_s^-(t_0)$ and ${}^{\mathbb{I}}V_u^{\pm}(t_0)$, are cones as well. They are double-cones, since equation (4.7) holds for all $\lambda \in \mathbb{R}$.

Before we define the stable cone for noninvertible systems, we study the almost stable cone.

Remark 4.1.7. *For noninvertible systems (2.6) with $\mathbb{I} = [n_-, n_+]_{\mathbb{Z}}$ it is possible that:*

a) The almost stable cone ${}^{\mathbb{I}}\bar{V}_s(\cdot)$ is not forward invariant, i.e.

$$\Phi(n,m){}^{\mathbb{I}}\bar{V}_s(m) \not\subset {}^{\mathbb{I}}\bar{V}_s(n) \text{ for any } n, m \in \mathbb{I}, n > m.$$

b) $\dim({}^{\mathbb{I}}\bar{V}_s(n_+) \cap {}^{\mathbb{I}}V_u(n_+)) > 0.$

The next two examples prove these statements.

Example 4.1.8. *Consider the noninvertible system*

$$x(n+1) = Bx(n), \;\; n \in \mathbb{I} = [0,2]_{\mathbb{Z}}, \;\; B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.8}$$

We show the existence of a vector $x \in {}^{\mathbb{I}}\bar{V}_s(n_+ - 1) = {}^{\mathbb{I}}\bar{V}_s(1)$ such that $Bx \notin {}^{\mathbb{I}}\bar{V}_s(n_+) = {}^{\mathbb{I}}\bar{V}_s(2)$. Hence, we prove a).

Figure 4.4 shows all preimages of $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ under B. We see that $\bar{x} := \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

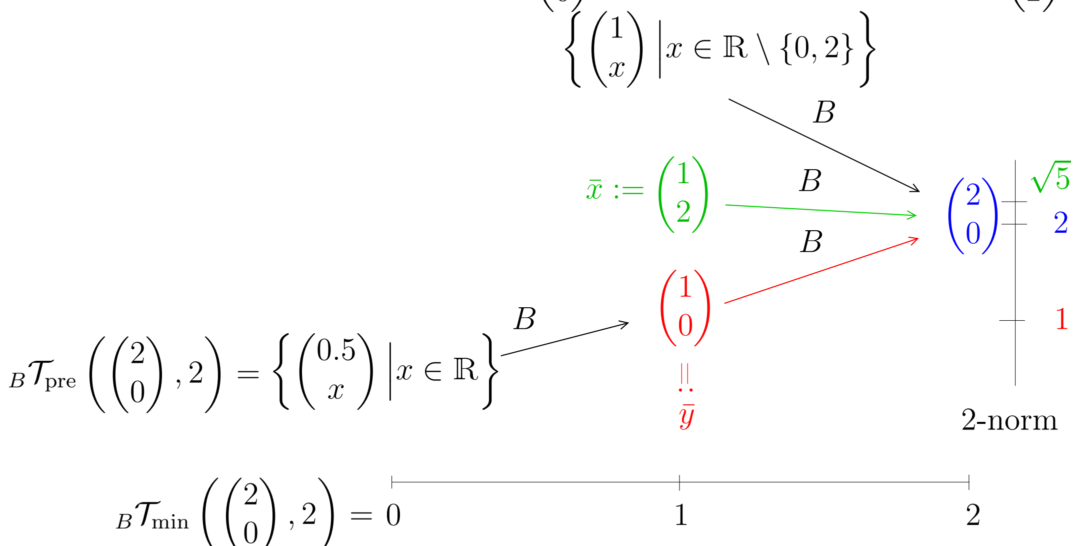


Figure 4.4: Preimages of $(2,0)^T$ under B of equation (4.8).

has no preimage and its norm is decreasing in forward time, i.e. $\|\bar{x}\| \geq \|B\bar{x}\|$. This implies $\bar{x} \in {}^{\mathbb{I}}\bar{V}_s(1)$.

Every preimage of $B\bar{x}$ *in* ${}_B\mathcal{T}_{\text{pre}}(B\bar{x}, 2)$ *under* B^2 *maps under* B *to* $\bar{y} := \begin{pmatrix}1\\0\end{pmatrix}$. *The vector* $\bar{y}$ *maps to* $B\bar{x}$ *and, thus, its norm is not decreasing in forward time* ($\|\bar{y}\| < \|B\bar{x}\|$). *This yields* $B\bar{x} \notin {}^{\mathbb{I}}\bar{V}_s(2)$ *and a) is proved.*

Example 4.1.9. *To show statement b) we consider the system*

$$x(n+1) = B(n)x(n),\ n \in \mathbb{I} = [0,2]_{\mathbb{Z}},\ B(n) = \begin{cases} \begin{pmatrix}2 & 0\\0 & 0.5\end{pmatrix} & , \textit{for } n = 0, \\ \begin{pmatrix}2 & 0\\0 & 0\end{pmatrix} & , \textit{for } n = 1. \end{cases} \tag{4.9}$$

We prove that $x := \begin{pmatrix}2\\0\end{pmatrix}$ *satisfies* $x \in \left({}^{\mathbb{I}}\bar{V}_s(2) \cap {}^{\mathbb{I}}V_u(2)\right) = \left({}^{\mathbb{I}}\bar{V}_s(n_+) \cap {}^{\mathbb{I}}V_u(n_+)\right)$, *hence, we prove b).*

Figure 4.5 shows different preimages of x. $\begin{pmatrix}0.5\\0\end{pmatrix}$ *is a preimage at time*

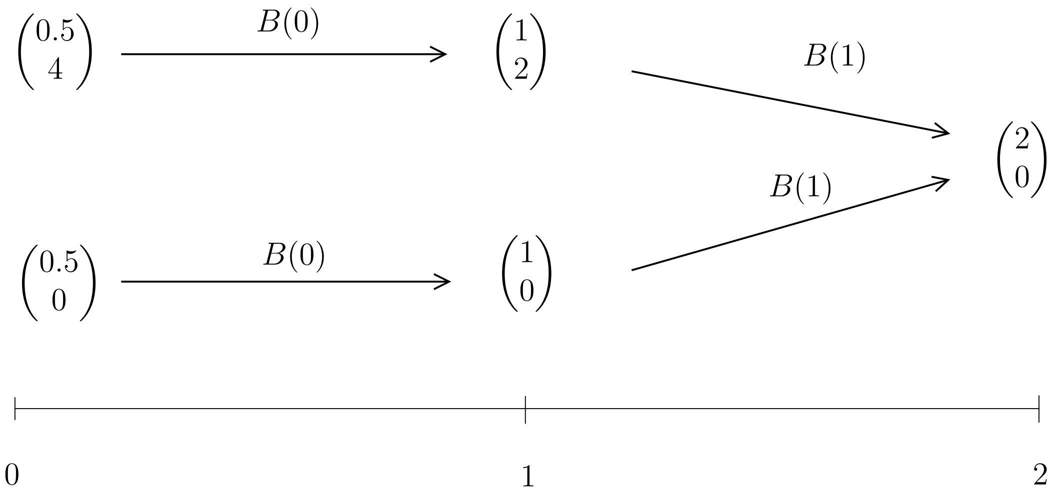


Figure 4.5: Preimages of $(2,0)^T$ under $B(\cdot)$ of equation (4.9).

0 *and its norm is increasing in forward time. Thus,* $x \in {}^{\mathbb{I}}\bar{V}_u(2)$. $\begin{pmatrix}0.5\\4\end{pmatrix}$ *is another preimage of* x *at time* 0 *and its norm is decreasing in forward time. Thus,* $x \in {}^{\mathbb{I}}\bar{V}_s(2)$. *This yields* $x \in \left({}^{\mathbb{I}}\bar{V}_s(2) \cap {}^{\mathbb{I}}V_u(2)\right)$.

The stable and unstable subspace of an hyperbolic ift-system are invariant due to (4.2) and the invariance of the family of projectors, i.e.

$$\Phi(n,m){}^{\mathbb{T}}V_s(m) = {}^{\mathbb{T}}V_s(n) \text{ holds for all } n, m \in \mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}, n \geq m.$$

For continuous ft-systems the stable and unstable cones are invariant as well. This is shown in Lemma 4.2.1.

For noninvertible systems the stable cone should be at least forward invariant. Thus, we need according to example (4.1.8) a modification of the almost stable cone.

Further, only 0 lies in the intersection of the stable and unstable subspace of an ift-system at each time. For noninvertible systems example 4.1.9 shows that this is not the case for the "last" time. Thus, we do not define the stable cone of noninvertible systems for the "last" time.

Definition 4.1.10. *Consider* (2.8) *on* $\mathbb{I} = [n_-, n_+]_{\mathbb{Z}}$. *We define the* ***stable*** $(\boldsymbol{n_+ - 1})$***-cone*** *by*

$$^{\mathbb{I}}V_s(n_+ - 1) := {}^{\mathbb{I}}\bar{V}_s(n_+ - 1).$$

Then we recursively define the ***stable*** $\boldsymbol{n_0}$***-cones***, $n_0 \in \mathbb{I}_2$ *by*

$$^{\mathbb{I}}V_s(n_0) := \left\{ \xi \in {}^{\mathbb{I}}\bar{V}_s(n_0) \middle| \Phi(n_0 + 1, n_0)\xi \in {}^{\mathbb{I}}V_s(n_0 + 1) \right\}.$$

If equation (2.8) *is invertible. Then the cone*

$$^{\mathbb{I}}V_s(n_+) := {}^{\mathbb{I}}\bar{V}_s(n_+)$$

is called ***stable*** $\boldsymbol{n_+}$***-cone***.

This definition yields a forward invariant stable cone. For a proof see Lemma 4.2.1. Note, that the almost stable and stable cones coincide for invertible discrete ft-systems, i.e.

$$^{\mathbb{I}}V_s(n_0) = {}^{\mathbb{I}}\bar{V}_s(n_0) \text{ for all } n_0 \in \mathbb{I}.$$

Characteristics of the (Almost) Stable and Unstable Cone

We rewrite the almost stable and unstable cones of an ft-system such that they are not an intersection as in Definition 4.1.5. Further, we prove some characteristics, e.g. invariant properties, and we show that our cones are equivalent to the cones Karrasch defined in [83, Definition 4.4.]. This helps to finally prove statement (4.3).

For $\mathbb{I} = [n_-, n_+]_{\mathbb{Z}}$ and $n_0 \in \mathbb{I}$ we get with the monotony of e

$$e^{\alpha n} > e^{\alpha m}, \text{ for all } \alpha > 0, \; n, m \in \mathbb{I}, n > m$$

the equivalent representation

$$\begin{aligned}
{}^{\mathbb{I}}\bar{V}_s(n_0) =& \big\{\xi \in \mathbb{R}^k \big| \exists \mu_{\bar{n}} \in {}_{\Phi}\mathcal{T}_{\text{pre}}(\xi, n_0) \text{ with } \bar{n} := {}_{\Phi}\mathcal{T}_{\min}(\xi, n_0) \text{ and} \\
& \exists \alpha > 0 : \|\Phi(n, \bar{n})\mu_{\bar{n}}\| e^{\alpha n} \text{ is decreasing for } n \in [\bar{n}, n_+]_{\mathbb{Z}}\big\} \\
=& \big\{\xi \in \mathbb{R}^k \big| \exists \mu_{\bar{n}} \in {}_{\Phi}\mathcal{T}_{\text{pre}}(\xi, n_0) \text{ with } \bar{n} := {}_{\Phi}\mathcal{T}_{\min}(\xi, n_0) \text{ and} \\
& \exists \alpha > 0 : \|\Phi(n, \bar{n})\mu_{\bar{n}}\| e^{\alpha n} \leq \|\Phi(m, \bar{n})\mu_{\bar{n}}\| e^{\alpha m} \\
& \text{for all } n, m \in [\bar{n}, n_+]_{\mathbb{Z}}, n \geq m\big\} \\
=& \big\{\xi \in \mathbb{R}^k \big| \exists \mu_{\bar{n}} \in {}_{\Phi}\mathcal{T}_{\text{pre}}(\xi, n_0) \text{ with } \bar{n} := {}_{\Phi}\mathcal{T}_{\min}(\xi, n_0) \\
& \text{and } \mathring{n} := {}_{\Phi}\mathcal{T}_{\ker}(\xi, n_0) : \|\Phi(n, \bar{n})\mu_{\bar{n}}\| < \|\Phi(m, \bar{n})\mu_{\bar{n}}\| \\
& \text{for all } n, m \in [\bar{n}, \mathring{n}]_{\mathbb{Z}}, n > m\big\} \cup \{0\}
\end{aligned} \tag{4.10}$$

of the almost stable n_0-cone of (2.8). Similarly, the unstable n_0-cone of (2.8) is

$$\begin{aligned}
{}^{\mathbb{I}}V_u(n_0) =& \big\{\xi \in \mathbb{R}^k \big| {}_{\Phi}\mathcal{T}_{\min}(\xi, n_0) = n_-, \exists \mu_{n_-} \in {}_{\Phi}\mathcal{T}_{\text{pre}}(\xi, n_0), \\
& \exists \beta > 0 : \big\|\Phi(n, n_-)\mu_{n_-}\big\| e^{-\beta n} \text{ is increasing for } n \in \mathbb{I}\big\} \\
=& \big\{\xi \in \mathbb{R}^k \big| {}_{\Phi}\mathcal{T}_{\min}(\xi, n_0) = n_-, \exists \mu_{n_-} \in {}_{\Phi}\mathcal{T}_{\text{pre}}(\xi, n_0), \exists \beta > 0 : \\
& \big\|\Phi(n, n_-)\mu_{n_-}\big\| e^{-\beta n} \geq \big\|\Phi(m, n_-)\mu_{n_-}\big\| e^{-\beta m} \\
& \text{for all } n, m \in \mathbb{I}, n \geq m\big\} \\
=& \big\{\xi \in \mathbb{R}^k \big| {}_{\Phi}\mathcal{T}_{\min}(\xi, n_0) = n_- \exists \mu_{n_-} \in {}_{\Phi}\mathcal{T}_{\text{pre}}(\xi, n_0) : \\
& \big\|\Phi(n, n_-)\mu_{n_-}\big\| > \big\|\Phi(m, n_-)\mu_{n_-}\big\| \text{ for all } n, m \in \mathbb{I}, n > m\big\} \\
& \cup \{0\}.
\end{aligned} \tag{4.11}$$

For invertible systems (2.8) we obtain

$$\begin{aligned}
{}^{\mathbb{I}}V_s(n_0) =& {}^{\mathbb{I}}\bar{V}_s(n_0) \\
=& \big\{\xi \in \mathbb{R}^k \big| \, \|\Phi(n, n_0)\xi\| < \|\Phi(m, n_0)\xi\| \text{ for all } n, m \in \mathbb{I}, n > m\big\} \\
& \cup \{0\},
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
{}^{\mathbb{I}}V_u(n_0) =& \big\{\xi \in \mathbb{R}^k \big| \, \|\Phi(n, n_0)\xi\| > \|\Phi(m, n_0)\xi\| \text{ for all } n, m \in \mathbb{I}, n > m\big\} \\
& \cup \{0\}
\end{aligned} \tag{4.13}$$

for all $n_0 \in \mathbb{I}$. Note, that the cones of (2.7) have the same characterization.

The following lemma summarizes some basic properties of the almost stable, of the stable and of the unstable cone. The statements for the continuous time systems can be found in [43]. For $\mathbb{T} = \mathbb{R}$ we define the sets

$${}^{\mathbb{I}}\bar{V}_s(t) := {}^{\mathbb{I}}V_s(t), \ t \in \mathbb{I}.$$

Lemma 4.2.1. *Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$ and $\mathbb{I} = [t_-, t_+]_{\mathbb{T}}$. Then the stable and unstable cones of system* (2.6) *satisfy*

$${}^{\mathbb{I}}V_u(t_-) = {}^{\mathbb{I}}V_u^+(t_-), \ {}^{\mathbb{I}}V_u(t_+) = {}^{\mathbb{I}}V_u^-(t_+), {}^{\mathbb{I}}\bar{V}_s(t_-) = {}^{\mathbb{I}}V_s^+(t_-), \tag{4.14}$$

and for invertible systems

$$^{\mathbb{I}}V_s(t_+) = {}^{\mathbb{I}}V_s^-(t_+)$$

holds. The sets $^{\mathbb{I}}V_s^+$, $^{\mathbb{I}}V_u^+$ *are forward invariant and the sets* $^{\mathbb{I}}V_s^-$, $^{\mathbb{I}}V_u^-$ *are backward invariant for invertible systems, i.e. for all* $t_0, t_1, t_2 \in \mathbb{I}$ *with* $t_1 \geq t_0 \geq t_2$

$$\Phi(t_1, t_0)^{\mathbb{I}}V_{s,u}^+(t_0) \subset {}^{\mathbb{I}}V_{s,u}^+(t_1) \tag{4.15}$$

holds and we have

$$\Phi(t_2, t_0)^{\mathbb{I}}V_{s,u}^-(t_0) \subset {}^{\mathbb{I}}V_{s,u}^-(t_2), \text{ if (2.6) is invertible.} \tag{4.16}$$

The unstable cone is invariant and the stable cone is at least forward invariant, i.e. for every $t_0, t_1 \in \mathbb{I}$, $t_1 \geq t_0$ *the equation*

$$\Phi(t_1, t_0)^{\mathbb{I}}V_u(t_0) = {}^{\mathbb{I}}V_u(t_1) \tag{4.17}$$

holds and we obtain

$$\Phi(t_1, t_0)^{\mathbb{I}}V_s(t_0) \begin{cases} = {}^{\mathbb{I}}V_s(t_1) & , \text{ if (2.6) is invertible,} \\ \subset {}^{\mathbb{I}}V_s(t_1) & , \text{ otherwise.} \end{cases} \tag{4.18}$$

The almost stable cone is forward invariant w.r.t. t_-, *i.e. for every* $t \in \mathbb{I}$ *we have*

$$\Phi(t, t_-)^{\mathbb{I}}\bar{V}_s(t_-) \subset {}^{\mathbb{I}}\bar{V}_s(t). \tag{4.19}$$

Further it satisfies

$$^{\mathbb{I}}\bar{V}_s(t_-) = {}^{\mathbb{I}}V_s(t_-). \tag{4.20}$$

Their intersection satisfies, if $t_+ \neq t_-$,

$$^{\mathbb{I}}V_u(t) \cap {}^{\mathbb{I}}V_s(t) = \{0\}, \tag{4.21}$$

$$^{\mathbb{I}}V_u(t) \cap {}^{\mathbb{I}}\bar{V}_s(t) = \{0\} \tag{4.22}$$

for all $t \in \begin{cases} \mathbb{I} & , \text{ if (2.6) is invertible,} \\ \mathbb{I}_1 & , \text{ otherwise.} \end{cases}$.

Proof. The equations in (4.14) are true since $^{\mathbb{I}}V_{s,u}^+(t_+) = \mathbb{R}^k = {}^{\mathbb{I}}V_{s,u}^-(t_-)$ holds by the definition of the cones. To show (4.15) let $t_0, t_1 \in \mathbb{I}$ with $t_1 \geq t_0$ and $\xi \in \Phi(t_1, t_0)^{\mathbb{I}}V_s^+(t_0)$. Then there exists a $\mu \in {}^{\mathbb{I}}V_s^+(t_0)$ such that $\xi = \Phi(t_1, t_0)\mu$. Further by the definition of $^{\mathbb{I}}V_s^+(t_0)$ there exists an $\alpha > 0$ such that

$$\|\Phi(t, t_0)\mu\| e^{\alpha t} = \|\Phi(t, t_1)\Phi(t_1, t_0)\mu\| e^{\alpha t} = \|\Phi(t, t_1)\xi\| e^{\alpha t}$$

is decreasing for all $t \in [t_1, t_+]_{\mathbb{T}} \subset [t_0, t_+]_{\mathbb{T}}$. This implies $\xi \in {}^{\mathbb{I}}V_s^+(t_1)$. Analogously, we obtain

$$\Phi(t_1, t_0)^{\mathbb{I}}V_u^+(t_0) \subset {}^{\mathbb{I}}V_u^+(t_1).$$

Let system (2.6) be invertible and let $t_0, t_2 \in \mathbb{I}$, $t_0 \geq t_2$ then

$$\begin{aligned}\Phi(t_2, t_0)^{\mathbb{I}}V_s^-(t_0) =& \{\Phi(t_2, t_0)\xi \in \mathbb{R}^k | \exists \mu \in \mathbb{R}^k : \Phi(t_0, t_-)\mu = \xi, \exists \alpha > 0 : \\ & \quad \|\Phi(t, t_-)\mu\|\, e^{\alpha t} \text{ is decreasing for all } t \in [t_-, t_0]\} \\ \subset& \{\tilde{\xi} \in \mathbb{R}^k | \exists \mu \in \mathbb{R}^k : \Phi(t_2, t_-)\mu = \tilde{\xi}, \exists \alpha > 0 : \\ & \quad \|\Phi(t, t_-)\mu\|\, e^{\alpha t} \text{ is decreasing for all } t \in [t_-, t_2] \subset [t_-, t_0]\} \\ =& {}^{\mathbb{I}}V_s^-(t_2).\end{aligned}$$

The relation (4.16) for the "unstable" cone ${}^{\mathbb{I}}V_u^-$ can be proved in the same way. To show (4.17) let $t_0, t_1 \in \mathbb{I}$, $t_1 \geq t_0$. Then by Lemma 4.1.4 (a)

$$\begin{aligned}\Phi(t_1, t_0)^{\mathbb{I}}V_u(t_0) =& \Big\{\Phi(t_1, t_0)\xi \in \mathbb{R}^k \Big| {}_{\Phi}\mathcal{T}_{\min}(\xi, t_0) = n_-, \exists \mu_{t_-} \in {}_{\Phi}\mathcal{T}_{\text{pre}}(\xi, t_0), \\ & \quad \exists \beta > 0 : \|\Phi(t, t_-)\mu_{t_-}\| e^{-\beta t} \text{ is increasing for all } t \in \mathbb{I}\Big\} \\ =& \Big\{\tilde{\xi} \in \mathbb{R}^k \Big| {}_{\Phi}\mathcal{T}_{\min}(\tilde{\xi}, t_1) = n_-, \exists \mu_{t_-} \in {}_{\Phi}\mathcal{T}_{\text{pre}}(\tilde{\xi}, t_1), \\ & \quad \exists \beta > 0 : \|\Phi(t, t_-)\mu_{t_-}\| e^{-\beta t} \text{ is increasing for all } t \in \mathbb{I}\Big\} \\ =& {}^{\mathbb{I}}V_u(t_1)\end{aligned}$$

holds. Next we show (4.18). If (2.6) is invertible we obtain for all $t_1, t_0 \in \mathbb{I}$, $t_1 \geq t_0$

$$\begin{aligned}\Phi(t_1, t_0)^{\mathbb{I}}V_s(t_0) &= \big\{\Phi(t_1, t_0)\xi \in \mathbb{R}^k \big| \exists \alpha > 0 : \|\Phi(t, t_-)\xi\|\, e^{\alpha n} \text{ is decreasing} \\ & \qquad \text{for all } n \in \mathbb{I},\ \mu \in {}_{\Phi}\mathcal{T}_{\text{pre}}(\xi, t_0)\big\} \\ &= \big\{\xi \in \mathbb{R}^k \big| \exists \alpha > 0 : \|\Phi(n, n_-)\mu\|\, e^{\alpha n} \text{ is decreasing} \\ & \qquad \text{for all } n \in \mathbb{I},\ \mu \in {}_{\Phi}\mathcal{T}_{\text{pre}}(\Phi(t_1, t_0)\xi, t_1)\big\} \\ &= {}^{\mathbb{I}}V_s(t_1).\end{aligned}$$

If (2.6) is not invertible we get directly by Definition 4.1.10

$$\Phi(t_0 + 1, t_0)^{\mathbb{I}}V_s(t_0) \subset {}^{\mathbb{I}}V_s(t_0 + 1) \text{ for } t_0 \in \mathbb{I}_1.$$

Hence, we inductively obtain

$$\Phi(t_1, t_0)^{\mathbb{I}}V_s(t_0) \subset {}^{\mathbb{I}}V_s(t_1) \text{ for all } t_1, t_0 \in \mathbb{I},\ t_1 \geq t_0.$$

Equation (4.14) and (4.15) yield for all $t \in \mathbb{I}$

$$\Phi(t, t_-)^{\mathbb{I}}\bar{V}(t_-) = \Phi(t, t_-)^{\mathbb{I}}V_s^+(t_-) \subset {}^{\mathbb{I}}V_s^+(t). \tag{4.23}$$

For every $\mu \in {}^{\mathbb{I}}\bar{V}_s(t_-)$ there exists an $\alpha > 0$ such that $\|\Phi(t, t_-)\mu\| e^{\alpha t}$ is decreasing for all $t \in \mathbb{I}$. Since ${}_{\Phi}\mathcal{T}_{\min}(\xi, t) = t_-$ holds for all $t \in \mathbb{I}$ and all $\xi \in \Phi(t, t_-){}^{\mathbb{I}}\bar{V}_s(t_-)$ we have

$$\Phi(t, t_-){}^{\mathbb{I}}\bar{V}_s(t_-) \subset {}^{\mathbb{I}}V_s^-(t) \tag{4.24}$$

for all $t \in \mathbb{I}$. Equation (4.23) and (4.24) leads to (4.19).

By (4.14) and (4.19) we obtain

$$\Phi(n_+ - 1, n_-){}^{\mathbb{I}}\bar{V}_s(n_-) \subset {}^{\mathbb{I}}\bar{V}_s(n_+ - 1) = {}^{\mathbb{I}}V_s(n_+ - 1).$$

Thus, ${}^{\mathbb{I}}\bar{V}_s(n_-) = {}^{\mathbb{I}}V_s(n_-)$.

The statement (4.21) follows if (4.22) holds. Therefore, let $t_0 \in \mathbb{I}$, $t_0 < t_+$ and $\xi \in {}^{\mathbb{I}}\bar{V}_s(t_0) \cap {}^{\mathbb{I}}V_u(t_0) \subset {}^{\mathbb{I}}V_s^+(t_0) \cap {}^{\mathbb{I}}V_u^+(t_0)$. By the definition of ${}^{\mathbb{I}}V_s^+(t_0)$ and ${}^{\mathbb{I}}V_u^+(t_0)$ we get that $\alpha, \beta > 0$ exist such that $\|\Phi(t, t_0)\xi\| e^{\alpha t}$ is decreasing and $\|\Phi(t, t_0)\xi\| e^{-\beta t}$ is increasing for $t \in [t_0, t_+]_{\mathbb{T}}$. The properties of an exponential function lead to $\|\Phi(t, t_0)\xi\| = 0$ for all $t \in [t_0, t_+]_{\mathbb{T}}$ and, hence, $\xi = 0$.

Let system (2.6) be invertible. For $t_0 = t_+$ ($t_0 > t_-$) we get that $\xi \in {}^{\mathbb{I}}V_s^-(t_0) \cap {}^{\mathbb{I}}V_u^-(t_0)$ has exactly one preimage for all $t \in [t_-, t_0]_{\mathbb{T}}$. Let $\xi_- \in \mathbb{R}^k$ such that $\xi = \Phi(t_0, t_-)\xi_-$. Then by $\xi \in {}^{\mathbb{I}}V_s^-(t_0)$ there exists an $\alpha > 0$ such that $\|\Phi(t, t_-)\xi_-\| e^{\alpha t}$ is decreasing for all $t \in [t_-, t_0]_{\mathbb{T}}$ and by $\xi \in {}^{\mathbb{I}}V_u^-(t_0)$ there exists a $\beta > 0$ such that $\|\Phi(t, t_-)\xi_-\| e^{-\beta t}$ is increasing for all $t \in [t_-, t_0]_{\mathbb{T}}$. This leads to $\xi_- = 0$ and, thus, $\xi = 0$. □

Another characteristic of the stable and unstable cone is that except for 2-dimensional dynamical systems they are generally not convex double-cones. For an illustration we consider the 3-dimensional discrete equation

$$u(n+1) = A(n)u(n), n \in [1, 6]_{\mathbb{Z}} \tag{4.25}$$

where $A(n) = D(n+1)BD(-n)$ with

$$B = \begin{pmatrix} 0.8 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 1.2 \end{pmatrix}, D(n) = \begin{pmatrix} \cos(n\varphi) & 0 & -\sin(n\varphi) \\ \sin(n\varphi) & 0 & \cos(n\varphi) \\ 0 & 1 & 0 \end{pmatrix} \text{ with } \varphi = \frac{\pi}{3}$$

for all $n \in \mathbb{Z}$. Denote the solution operator by $\Phi(\cdot, \cdot)$. The intersection of the 1-stable cone with the Euclidean unit-ball is illustrated in Figure 4.6.

We observe that the stable cone is not a convex double-cone, i.e. the half-cones are not convex cones. The intersection in Figure 4.6 is approximated using the MATLAB-command `isosurface`. The input of the function `isosurface` is a value table

$$g_1(x) = \max_{\ell \in [1,5]_{\mathbb{Z}}} \{\|\Phi(\ell + 1, 1)x\| - \|\Phi(\ell, 1)x\|\}$$

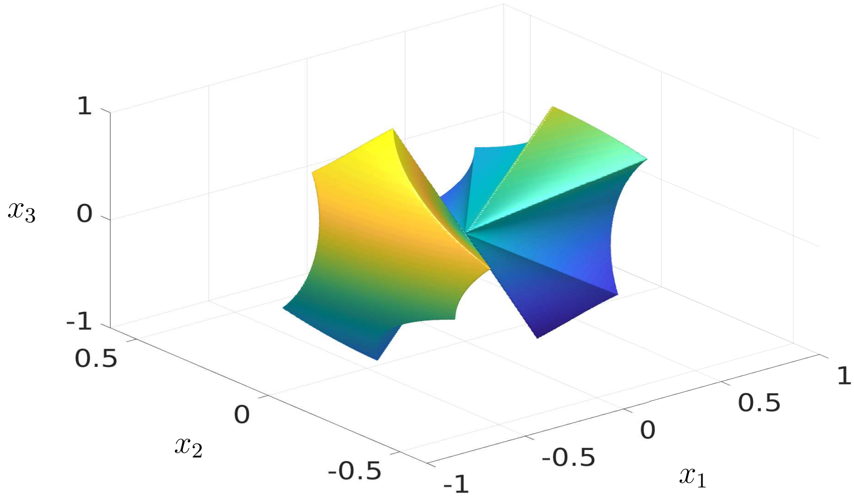


Figure 4.6: 1-stable cone of (4.25) intersected by the Euclidean unit-ball.

for x on the cuboid $[-1.1, 1.1]^3$ discretized with a $500 \times 500 \times 500$ grid. Further cones which are not convex double-cones are presented in [55, Section 3.3]. Note that the stable cones plotted and calculated there are not equivalent to our stable cones. The stable cone defined in [55] coincides with our cone ${}^{\mathbb{I}}V_s^+$.

Before we prove equation (4.3) we show that Karraschs definition of the (un)stable cone, see [83, Definition 4.4], is equivalent to our Definition 4.1.3. The definition in [83] is similar to the equation (4.3), which we want to establish.

Lemma 4.2.2. *Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$ and $\mathbb{I} = [t_-, t_+]_{\mathbb{T}}$ then the Definition 4.1.3/4.1.5 of the unstable cone and almost stable cone is equivalent to the Definition 4.4 of [83], i.e. the following holds*

$$\begin{aligned}
&{}^{\mathbb{I}}V_s(t_-) := \left\{\xi \in \mathbb{R}^k \middle| \exists \alpha > 0 : \|\Phi(t, t_-)\xi\| \, e^{\alpha t} \text{ is decreasing for all } t \in \mathbb{I}\right\} \\
=&V_s(\Phi) := \left\{\xi \in \mathbb{R}^k \middle| \exists X \in \mathrm{Gr}(1, \mathbb{R}^k) : \xi \in X,\ \overline{\lambda}(X, \Phi) < 0\right\} \cup \{0\}, \\
&{}^{\mathbb{I}}V_u(t_-) := \left\{\xi \in \mathbb{R}^k \middle| \exists \beta > 0 : \|\Phi(t, t_-)\xi\| \, e^{-\beta t} \text{ is increasing for all } t \in \mathbb{I}\right\} \\
=&V_u(\Phi) := \left\{\xi \in \mathbb{R}^k \middle| \exists X \in \mathrm{Gr}(1, \mathbb{R}^k) : \xi \in X,\ \underline{\lambda}(X, \Phi) > 0\right\} \cup \{0\},
\end{aligned}$$

with $\overline{\lambda}$ and $\underline{\lambda}$ defined as in (3.26), (3.27).

Proof. We prove ${}^{\mathbb{I}}V_s(t_-) = V_s(\Phi)$ and ${}^{\mathbb{I}}V_u(t_-) = V_u(\Phi)$ similarly follows with Lemma 4.1.4 (c). We begin by introducing a map which gives the extremal growth rate of a single vector. Therefore, we need the sets

$$\begin{aligned}
(\mathbb{I} \times \mathbb{I} \times \mathbb{R}^k)_{\neq}^{(t_-)} &:= \left\{(t, s, \xi) \in (\mathbb{I} \times \mathbb{I} \times \mathbb{R}^k) \middle| t \neq s,\ t, s < {}_{\Phi}\mathcal{T}_{\mathrm{ker}}(\xi, t_-)\right\}, \\
(\mathbb{I} \times \mathbb{I})_{\neq}^{(\xi, t_-)} &:= \left\{(t, s) \in (\mathbb{I} \times \mathbb{I}) \middle| (t, s, \xi) \in (\mathbb{I} \times \mathbb{I} \times \mathbb{R})_{\neq}^{t_-}\right\}.
\end{aligned}$$

The map is well defined by

$$\Delta : \left(\mathbb{I} \times \mathbb{I} \times \mathbb{R}^k\right)^{t_-}_{\neq} \to \mathbb{R}, \qquad (4.26)$$
$$(t,s,\xi) \mapsto \frac{\ln(\|\Phi(t,t_-)\xi\|) - \ln(\|\Phi(s,t_-)\xi\|)}{t-s}.$$

It is easily seen that for all $(t,s,\xi) \in \left(\mathbb{I} \times \mathbb{I} \times \mathbb{R}^k\right)^{t_-}_{\neq}$ and $\lambda \in \mathbb{R}$

$$\Delta(t,s,\xi) = \Delta(s,t,\xi),\ \Delta(t,s,\xi) = \Delta(t,s,\lambda\xi) \qquad (4.27)$$

hold. For $\xi \in \mathbb{R}^k$ define the set

$$X_\xi := \left\{\lambda\xi \in \mathbb{R}^k | \lambda \in \mathbb{R}\right\}.$$

Obviously

$$X_\xi \in \mathrm{Gr}(1,\mathbb{R}^k) \qquad (4.28)$$

for all $\xi \in \mathbb{R}^k \setminus \{0\}$, since $\mathrm{Gr}(1,\mathbb{R}^k)$ is the set of all 1-dimensional subspaces of $\mathbb{R}^k$. By (4.20), (4.26), (4.27) and (4.28) the statement holds, i.e.

$$\begin{aligned}
{}^{\mathbb{I}}V_s(t_-) &= {}^{\mathbb{I}}\bar{V}_s(t_-) \\
&= \left\{\xi \in \mathbb{R}^k \middle| \exists \alpha > 0 : \|\Phi(t,t_-)\xi\|\, e^{\alpha t} \text{ is decreasing for all } t \in \mathbb{I}\right\} \\
&= \Big\{\xi \in \mathbb{R}^k \Big| \exists \alpha > 0 : \|\Phi(t,t_-)\xi\|\, e^{\alpha t} \text{ is decreasing} \\
&\qquad\qquad \text{for all } t \in [t_-, {}_{\Phi}\mathcal{T}_{\ker}(\xi,t_-)]\Big\} \\
&= \Big\{\xi \in \mathbb{R}^k \setminus \{0\} \Big| \exists \alpha > 0 : \|\Phi(t,t_-)\xi\|\, e^{\alpha t} \leq \|\Phi(s,t_-)\xi\|\, e^{\alpha s} \\
&\qquad\qquad \text{for all } (t,s) \in (\mathbb{I} \times \mathbb{I})^{(\xi,t_-)}_{\neq}, t > s\Big\} \cup \{0\} \\
&= \Big\{\xi \in \mathbb{R}^k \setminus \{0\} \Big| \exists \alpha > 0 : \ln(\|\Phi(t,t_-)\xi\|) + \alpha t \leq \ln(\|\Phi(s,t_-)\xi\|) + \alpha s \\
&\qquad\qquad \text{for all } (t,s) \in (\mathbb{I} \times \mathbb{I})^{(\xi,t_-)}_{\neq}, t > s\Big\} \cup \{0\} \\
&= \Big\{\xi \in \mathbb{R}^k \setminus \{0\} \Big| \exists \alpha > 0 : \frac{\ln(\|\Phi(t,t_-)\xi\|) - \ln(\|\Phi(s,t_-)\xi\|)}{t-s} \leq -\alpha \\
&\qquad\qquad \text{for all } (t,s) \in (\mathbb{I} \times \mathbb{I})^{(\xi,t_-)}_{\neq}, t > s\Big\} \cup \{0\} \\
&= \Big\{\xi \in \mathbb{R}^k \setminus \{0\} \Big| \exists \alpha > 0 : \Delta(t,s,\xi) \leq -\alpha \\
&\qquad\qquad \text{for all } (t,s) \in (\mathbb{I} \times \mathbb{I})^{(\xi,t_-)}_{\neq}, t > s\Big\} \cup \{0\} \\
&= \left\{\xi \in \mathbb{R}^k \setminus \{0\} \middle| \Delta(t,s,\xi) < 0 \text{ for all } (t,s) \in (\mathbb{I} \times \mathbb{I})^{(\xi,t_-)}_{\neq}, t > s\right\} \cup \{0\} \\
&= \left\{\xi \in \mathbb{R}^k \setminus \{0\} \middle| \Delta(t,s,\xi) < 0 \text{ for all } (t,s) \in (\mathbb{I} \times \mathbb{I})^{(\xi,t_-)}_{\neq}\right\} \cup \{0\} \\
&= \left\{\xi \in \mathbb{R}^k \setminus \{0\} \middle| \sup_{(t,s) \in (\mathbb{I} \times \mathbb{I})^{(\xi,t_-)}_{\neq}} \Delta(t,s,\xi) < 0\right\} \cup \{0\}
\end{aligned}$$

$$= \left\{ \xi \in \mathbb{R}^k \setminus \{0\} \middle| \sup_{\substack{x \in X_\xi, \|x\|=1 \\ (t,s) \in (\mathbb{I}\times\mathbb{I})_{\neq}^{(\xi,t_-)}}} \Delta(t,s,x) < 0 \right\} \cup \{0\}$$
$$= \left\{\xi \in \mathbb{R}^k \setminus \{0\} \middle| \overline{\lambda}(X_\xi, \Phi) < 0\right\} \cup \{0\}$$
$$= \left\{\xi \in \mathbb{R}^k \middle| \exists X \in \mathrm{Gr}(1,\mathbb{R}^k) : \xi \in X,\ \overline{\lambda}(X,\Phi) < 0\right\} \cup \{0\}$$
$$= V_s(\Phi).$$

□

The next lemma provides a condition, which guarantees that a system is M-hyperbolic. It also shows a part of the statement (4.3). The whole statement (4.3) is proved in Theorem 4.2.4. For a similar result of the following in continuous time we refer to [43, Theorem 14] and [83, Proposition 4.6].

Lemma 4.2.3. *Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$ and system* (2.6) *be defined on the compact interval $\mathbb{I} = [t_-, t_+]_\mathbb{T}$. Let* $\tilde{\mathbb{I}} := \begin{cases} \mathbb{I} & , \text{ if (2.6) is invertible,} \\ \mathbb{I}_1 & , \text{ otherwise.} \end{cases}$
Then $P : \mathbb{I} \to \mathbb{R}^{k\times k}$ is an invariant family of projectors with

$$\mathcal{R}(P(t)) \subset {}^{\mathbb{I}}V_s(t) \text{ for all } t \in \tilde{\mathbb{I}}, \tag{4.29}$$
$$\mathcal{N}(P(t)) \subset {}^{\mathbb{I}}V_u(t) \text{ for all } t \in \mathbb{I} \tag{4.30}$$

if and only if system (2.6) *is M-hyperbolic on $\mathbb{I}$ with this family of projectors.*

Proof. First we show that M-hyperbolicity follows if (4.29) and (4.30) hold. We start with the proof of (3.9).

Let $s \in \tilde{\mathbb{I}}$ and $\xi \in \mathcal{R}(P(s)) \subset {}^{\mathbb{I}}V_s(s) \subset {}^{\mathbb{I}}V_s^+(s)$. Then there exists an $\alpha > 0$ such that $\|\Phi(t,s)\xi\|\, e^{\alpha t}$ is decreasing for $t \in [s, t_+]_\mathbb{T}$. This leads to

$$\|\Phi(t,s)\xi\|\, e^{\alpha t} \le \|\Phi(s,s)\xi\|\, e^{\alpha s} \tag{4.31}$$

for all $t \ge s$. Even if (2.6) is not invertible estimate (4.31) is satisfied for $t \ge t_+ =: s$. Thus, we get the equivalent M-hyperbolic estimate

$$\|\Phi(t,s)\xi\| \le e^{-\alpha(t-s)}\, \|\xi\| \text{ for all } t,s \in \mathbb{I}, t \ge s \text{ and all } \xi \in \mathcal{R}(P(s)). \tag{4.32}$$

Next we prove that

$$\Phi(t,s)|_{\mathcal{N}(P(s))} : \mathcal{N}(P(s)) \to \mathcal{N}(P(t)) \tag{4.33}$$

is invertible for all $t, s \in \mathbb{I}$, $t \ge s$. First we show injectivity. Therefore, let $t, s \in \mathbb{I}$, $t \ge s$ and $\xi_{1,2} \in \mathcal{N}(P(s))$ with $\Phi(t,s)\xi_1 = \Phi(t,s)\xi_2$. Then

$$\Phi(t,s)(\xi_1 - \xi_2) = 0, \text{ i.e. } {}_\Phi\mathcal{T}_{\ker}(\xi_1 - \xi_2, s) \in \mathbb{I}.$$

The subspace property of $\mathcal{N}(P(s))$ yields $\xi_1 - \xi_2 \in \mathcal{N}(P(s))$. For all $\xi \in \mathcal{N}(P(s)) \setminus \{0\}$ Lemma 4.1.4 implies ${}_\Phi\mathcal{T}_{\ker}(\xi, s) \notin \mathbb{I}$. Thus, $\xi_1 - \xi_2 = 0$. This proves that (4.33) is injective.

Invertibility follows if $\mathcal{N}(P(t))$ has the same dimension for all $t \in \mathbb{I}$. Let $t, s \in \mathbb{I}, t \geq s$ and $\mathcal{N}(P(r)) \subset {}^{\mathbb{I}}V_u(r)$ for all $r \in \mathbb{I}$. Lemma 2.1.6 and

$$\mathcal{N}(P(r)) \oplus \mathcal{R}(P(r)) = \mathbb{R}^k$$

imply that $\mathcal{N}(P(r))$, $r \in \mathbb{I}$ is a subspace of maximal dimension in ${}^{\mathbb{I}}V_u(r)$. Equation (4.17) yields the existence of a subspace $U(s) \subset {}^{\mathbb{I}}V_u(s)$ such that $\Phi(t, s)U(s) = \mathcal{N}(P(t))$. This leads to

$$\dim(\mathcal{N}(P(s))) \geq \dim(U(s)) \geq \dim(\mathcal{N}(P(t))). \tag{4.34}$$

From the injectivity of (4.33) and with (4.34) follows

$$\dim(\mathcal{N}(P(t))) = \dim(\mathcal{N}(P(s))).$$

Thus, (4.33) is invertible.

It remains to prove the second M-hyperbolic estimate (3.10). Let $s \in \mathbb{I}$ and $\xi \in \mathcal{N}(P(s)) \subset {}^{\mathbb{I}}V_u(s) \subset {}^{\mathbb{I}}V_u^-(s)$ then there exists a $\beta > 0$ and a unique $\mu \in \mathbb{R}^k$ with $\Phi(s, t_-)\mu = \xi$ such that $\|\Phi(t, t_-)\mu\| \, e^{-\beta t}$ is increasing for $t \in [t_-, s]_{\mathbb{T}}$. This leads to

$$\|\Phi(t, t_-)\mu\| \, e^{-\beta t} \leq \|\Phi(s, t_-)\mu\| \, e^{-\beta s}$$

for all $t \leq s$. Denote the inverse of $\Phi(s, t)|_{\mathcal{N}(P(s))}$ by $\Phi(t, s)$ then we get the equivalent M-hyperbolic statement

$$\|\Phi(t, s)\xi\| \leq e^{\beta(t-s)} \, \|\xi\| \text{ for all } \xi \in \mathcal{N}(P(s)),\ t \leq s. \tag{4.35}$$

The equations (4.32), (4.35) and the invertibility of (4.33) prove that (2.6) is M-hyperbolic on $\mathbb{I}$ with $P(\cdot)$.

Conversely, let (2.6) be M-hyperbolic on $\mathbb{I}$ with the invariant family of projectors $P(\cdot)$ and dichotomy rates $\alpha, \beta > 0$. Let $s \in \mathbb{I}$ and $\mu(s) \in \mathcal{N}(P(s))$ then we have

$$\|\Phi(t, s)\mu(s)\| \leq e^{\beta(t-s)}\|\mu(s)\| \text{ for all } t \in \mathbb{I}, t \leq s$$

which implies

$$\|\Phi(t, s)\mu(s)\|e^{-\beta t} \leq \|\mu(s)\|e^{-\beta s}. \tag{4.36}$$

By the invertibility of $\Phi(t, s)|_{\mathcal{N}(P(s))}$ there exists a $\mu(t_-) \in \mathcal{N}(P(t_-))$ such that $\Phi(s, t_-)\mu(t_-) = \mu(s)$. Together with (4.36) we obtain for all $t \in \mathbb{I}$, $t \leq s$

$$\begin{aligned}\|\Phi(t, t_-)\mu(t_-)\| \, e^{-\beta t} &= \|\Phi(t, s)\Phi(s, t_-)\mu(t_-)\| \, e^{-\beta t} = \|\Phi(t, s)\mu(s)\| \, e^{-\beta t} \\ &\leq \|\mu(s)\| \, e^{-\beta s} = \|\Phi(s, t_-)\mu(t_-)\| \, e^{-\beta s},\end{aligned}$$

i.e. $\|\Phi(t,t_-)\mu(t_-)\|\, e^{-\beta t}$ is increasing for all $t \in \mathbb{I}$. Hence, $\mu(s) \in {}^{\mathbb{I}}V_u(s)$, see (4.11). Thus, $\mathcal{N}(P(s)) \subset {}^{\mathbb{I}}V_u(s)$.

Next we prove $\mathcal{R}(P(t)) \subset {}^{\mathbb{I}}V_s(t)$ for all $t \in \tilde{\mathbb{I}}$.

We have

$$\|\Phi(t,s)\xi(s)\| \le e^{-\alpha(t-s)}\|\xi(s)\| \text{ for all } t,s \in \mathbb{I}, t \ge s \text{ and } \xi(s) \in \mathcal{R}(P(s)),$$

which implies

$$\|\Phi(t,s)\xi(s)\|e^{\alpha t} \le \|\Phi(s,s)\xi(s)\|e^{\alpha s}.$$

For $s \in \tilde{\mathbb{I}} \subset \mathbb{I}$ and $\xi(s) \in \mathcal{R}(P(s))$ it follows that $\|\Phi(t,s)\xi\|e^{\alpha t}$ is decreasing for all $t \in [s,t_+]_{\mathbb{T}}$. Hence, $\xi(s) \in {}^{\mathbb{I}}V_s^+(s)$ for all $s \in \tilde{\mathbb{I}}$ and $\xi(s) \in \mathcal{R}(P(s))$, i.e.

$$\mathcal{R}(P(s)) \subset {}^{\mathbb{I}}V_s^+(s), \text{ for all } s \in \tilde{\mathbb{I}}. \tag{4.37}$$

Combined with (4.14) we have

$$\mathcal{R}(P(t_-)) \subset {}^{\mathbb{I}}V_s^+(t_-) = {}^{\mathbb{I}}\bar{V}_s(t_-) \subset {}^{\mathbb{I}}V_s^-(t_-).$$

By induction we show $\mathcal{R}(P(t)) \subset {}^{\mathbb{I}}V_s^-(t)$ for all $t \in \tilde{\mathbb{I}}$. Fix $s \in \tilde{\mathbb{I}}$. Assume

$$\mathcal{R}(P(t_0)) \subset {}^{\mathbb{I}}V_s^-(t_0) \text{ for all } t_0 < s.$$

Let $\xi \in \mathcal{R}(P(s))$ and $\bar{s} := {}_{\Phi}\mathcal{T}_{\min}(\xi,s)$.

For $\bar{s} = s$ we obtain $\xi \in {}^{\mathbb{I}}V_s^-(s)$. This implies with (4.37) $\mathcal{R}(P(s)) \subset {}^{\mathbb{I}}\bar{V}_s(s)$.

For $\bar{s} < s$, let $\tilde{\xi} \in {}_{\Phi}\mathcal{T}_{\mathrm{pre}}(\xi,s)$. The invariance of the family of projectors leads to

$$\xi = P(s)\xi = P(s)\Phi(s,\bar{s})\tilde{\xi} = \Phi(s,\bar{s})P(\bar{s})\tilde{\xi}.$$

This yields $\bar{\xi} := P(\bar{s})\tilde{\xi} \in \mathcal{R}(P(\bar{s}))$ and

$$\bar{\xi} \in {}_{\Phi}\mathcal{T}_{\mathrm{pre}}(\xi,s). \tag{4.38}$$

Our assumption and $\bar{s} < s$ imply $\bar{\xi} \in \mathcal{R}(P(\bar{s})) \subset {}^{\mathbb{I}}V_s^-(\bar{s})$. By Definition 4.1.5 we get with $\bar{\xi} \subset {}^{\mathbb{I}}V_s^-(\bar{s})$ and (4.38)

$$\xi \in {}^{\mathbb{I}}V_s^-(s). \tag{4.39}$$

Statements (4.39) and (4.37) prove $\mathcal{R}(P(t)) \subset {}^{\mathbb{I}}\bar{V}_s(t)$ for all $t \in \tilde{\mathbb{I}}$. By Definition 4.1.10 we directly obtain

$$\mathcal{R}(P(t_+-1)) \subset {}^{\mathbb{I}}\bar{V}_s(t_+-1) = {}^{\mathbb{I}}V_s(t_+-1).$$

Additionally,

$$\Phi(t_+-1,t_+-2)\mathcal{R}(P(t_+-2)) \subset \mathcal{R}(P(t_+-1)) \subset {}^{\mathbb{I}}V_s(t_+-1)$$

holds and yields

$$\mathcal{R}(P(t_+ - 2)) \subset {}^{\mathbb{I}}V_s(t_+ - 2).$$

Inductively the statement $\mathcal{R}(P(t)) \subset {}^{\mathbb{I}}V_s(t)$ follows for all $t \in \mathbb{I}_1$. If (2.6) is invertible we directly obtain by (4.18)

$$\mathcal{R}(P(t_+)) = \Phi(t_+, t_+ - 1)\mathcal{R}(P(t_+ - 1)) \subset \Phi(t_+, t_+ - 1)^{\mathbb{I}}V_s(t_+ - 1) = {}^{\mathbb{I}}V_s(t_+).$$

□

Finally, we are able to prove that the stable cone is the union of the range whereas the unstable cone is the union of the kernel of all invariant projectors, which satisfy (3.8), (3.9) and (3.10), i.e. (4.3). This is the analog statement to the infinite time case (4.2), where the invariant family is unique. The nonuniqueness of the invariant family of projectors implies that the finite time (un)stable set is not a subspace as in the infinite time case. The (un)stable set is a cone, a union of subspaces.

Theorem 4.2.4. *Let* $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$ *and* $\mathbb{I} = [t_-, t_+]_{\mathbb{T}}$. *Let system* (2.6) *be M-hyperbolic and* $\tilde{\mathbb{I}} := \begin{cases} \mathbb{I} & , \text{ if (2.6) is invertible,} \\ \mathbb{I}_1 & , \text{ otherwise.} \end{cases}$
Then we have

$$\bigcup_{\tilde{P}(t_0) \in \mathcal{P}_{t_0}} \mathcal{R}(\tilde{P}(t_0)) = {}^{\mathbb{I}}V_s(t_0) \text{ for all } t_0 \in \tilde{\mathbb{I}},$$
$$\text{and} \quad \bigcup_{\tilde{P}(t_0) \in \mathcal{P}_{t_0}} \mathcal{N}(\tilde{P}(t_0)) = {}^{\mathbb{I}}V_u(t_0) \text{ for all } t_0 \in \mathbb{I},$$

where $\mathcal{P}_{t_0} := \{\tilde{P}(t_0) | \tilde{P} : \mathbb{I} \to \mathbb{R}^{k \times k}$ *is an invariant family of projectors, which fulfills* (3.8), (3.9) *and* (3.10) *with constants* $\alpha, \beta > 0\}$.

Proof. Let $t_0 \in \mathbb{I}$ and $\tilde{P}(t_0) \in \mathcal{P}_{t_0}$ then system (2.6) is M-hyperbolic with the invariant family of projectors $\tilde{P} : \mathbb{I} \to \mathbb{R}^{k \times k}$ belonging to $\tilde{P}(t_0)$. By Lemma 4.2.3 we get

$$\mathcal{R}(\tilde{P}(t)) \subset {}^{\mathbb{I}}V_s(t) \text{ for all } t \in \tilde{\mathbb{I}}$$
$$\text{and } \mathcal{N}(\tilde{P}(t)) \subset {}^{\mathbb{I}}V_u(t) \text{ for all } t \in \tilde{\mathbb{I}}.$$

Conversely, fix $\bar{t} \in \tilde{\mathbb{I}}$ and let

$$\bar{\xi}(\bar{t}) \in {}^{\mathbb{I}}V_s(\bar{t}).$$

Next we construct a family of projectors $\tilde{P} : \mathbb{I} \to \mathbb{R}^{k \times k}$ with $\bar{\xi}(\bar{t}) \in \mathcal{R}(\tilde{P}(\bar{t}))$. Then we prove $\tilde{P}(t) \in \mathcal{P}_t$ for all $t \in \mathbb{I}$.

Since system (2.6) is M-hyperbolic with an invariant family of projectors $P : \mathbb{I} \to \mathbb{R}^{k\times k}$ we obtain by Lemma 4.2.3

$$\begin{aligned} \mathcal{R}(P(t_0)) \subset {}^{\mathbb{I}}V_s(t_0) \text{ for all } t_0 \in \tilde{\mathbb{I}} \\ \text{and } \mathcal{N}(P(t_0)) \subset {}^{\mathbb{I}}V_u(t_0) \text{ for all } t_0 \in \mathbb{I}. \end{aligned} \tag{4.40}$$

This implies that there exists a subspace

$$S_{\bar{\xi}}(\bar{t}) \subset {}^{\mathbb{I}}V_s(\bar{t})$$

of dimension $\dim(\mathcal{R}(P(\bar{t})))$ with $\bar{\xi}(\bar{t}) \in S_{\bar{\xi}}(\bar{t})$. Every $\xi \in S_{\bar{\xi}}(\bar{t})$ satisfies either ${}_{\Phi}\mathcal{T}_{\text{pre}}(\xi, \bar{t}) = \bar{t}$ or there exists a $\mu \in {}^{\mathbb{I}}V_s(\bar{t}-1)$ with $\xi = \Phi(\bar{t}, \bar{t}-1)\mu$. This implies inductively by the forward invariance of Φ and ${}^{\mathbb{I}}V_s$ (equation (4.18)) the existence of subspaces $S_{\bar{\xi}}(t)$, $t \in [t_-, \bar{t})_{\mathbb{T}}$ with

$$S_{\bar{\xi}}(t) \subset {}^{\mathbb{I}}V_s(t) \tag{4.41}$$

and $\dim(S_{\bar{\xi}}(t)) = \dim(\mathcal{R}(P(t)))$ such that

$$\Phi(t,s)S_{\bar{\xi}}(s) \subset S_{\bar{\xi}}(t) \text{ for all } t, s \in [t_-, \bar{t}]_{\mathbb{T}}, t > s. \tag{4.42}$$

Additionally, by the latter arguments there exits subspaces $S_{\bar{\xi}}(t)$, $t \in (\bar{t}, t_+]_{\mathbb{T}}$ of dimension $\dim(\mathcal{R}(P(t)))$ such that

$$(4.41) \text{ is satisfied for all } t \in \tilde{\mathbb{I}} \tag{4.43}$$

$$\text{and } (4.42) \text{ is satisfied for all } t, s \in [\bar{t}, t_+]_{\mathbb{T}}, t \geq s. \tag{4.44}$$

Define the family of projectors $\tilde{P} : \mathbb{I} \to \mathbb{R}^{k\times k}$ recursively by $\mathcal{N}(\tilde{P}(t)) := \mathcal{N}(P(t))$ and

$$\mathcal{R}(\tilde{P}(t)) := \begin{cases} \Phi(t, t_-)S_{\bar{\xi}}(t_-) & \text{, for (i)} \\ \begin{cases} S_{\bar{\xi}}(t_-) & \text{, if } t = t_-, \\ \Phi(t, t-1)\mathcal{R}(\tilde{P}(t-1)) \oplus W(t) & \text{, if } t > t_- \end{cases} & \text{, for (ii)} \end{cases}$$

with (i) for invertible systems and with (ii) for noninvertible systems, where $W(t) \subset \begin{cases} {}^{\mathbb{I}}V_s(t) & \text{, for } t \in [t_- + 1, t_+ - 1]_{\mathbb{Z}}, \\ \mathbb{R}^k & \text{, for } t = t_+. \end{cases}$
such that

$$\Phi(t, t-1)\mathcal{R}(\tilde{P}(t-1)) \oplus W(t) = S_{\bar{\xi}}(t), \tag{4.45}$$

holds for all $t \in \mathbb{I}$. This family is well defined by equation (4.21) and by the equations (4.40)-(4.44). Further, we have

$$\bar{\xi}(\bar{t}) \in \mathcal{R}(\tilde{P}(\bar{t})),$$

$$\Phi(t, t-1)\mathcal{R}(\tilde{P}(t-1)) \oplus W(t) \oplus \mathcal{N}(P(t)) = S_{\bar{\xi}}(t) \oplus \mathcal{N}(P(t)) = \mathbb{R}^k,$$

It remains to show, that $\tilde{P} : \mathbb{I} \to \mathbb{R}^{k\times k}$ satisfies $\tilde{P}(t) \in \mathcal{P}_t$ for all $t \in \mathbb{I}$.

For invertible systems we obtain that the family of projectors is invariant by definition. By (4.40), (4.41) and (4.18) the relations $\mathcal{N}(\tilde{P}(t)) = \mathcal{N}(P(t)) \subset {}^{\mathbb{I}}V_u(t)$ and $\mathcal{R}(\tilde{P}(t)) = \Phi(t,t_0)S_{\bar{\xi}}(t_-) \subset \Phi(t,t_0)^{\mathbb{I}}V_s(t_0) = {}^{\mathbb{I}}V_s(t)$ hold. Finally, Lemma 4.2.3 leads to $\tilde{P}(t) \in \mathcal{P}_t$ for all $t \in \mathbb{I}$.

For noninvertible systems Lemma 2.2.5 yields that the family of projectors $\tilde{P} : \mathbb{I} \to \mathbb{R}^{k\times k}$ is invariant, since

$$\Phi(t,s)\mathcal{N}(\tilde{P}(s)) = \Phi(t,s)\mathcal{N}(P(s)) = \mathcal{N}(P(t)) = \mathcal{N}(\tilde{P}(t))$$

holds by Lemma 3.2.5 for all $t,s \in \mathbb{I}$, $t \geq s$. Equations (4.40), (4.41) and (4.45) yield $\mathcal{N}(\tilde{P}(t)) = \mathcal{N}(P(t)) \subset {}^{\mathbb{I}}V_u(t)$ and $\mathcal{R}(\tilde{P}(t)) = S_{\bar{\xi}}(t) \subset {}^{\mathbb{I}}V_s(t)$ for all $t \in \mathbb{I}$. Lemma 4.2.3 implies $\tilde{P}(t) \in \mathcal{P}_t$ for all $t \in \mathbb{I}$.

The inclusion

$$\bigcup_{\tilde{P} \in \mathcal{P}_{t_0}} \mathcal{N}(\tilde{P}) \supset {}^{\mathbb{I}}V_u(t_0),$$

for any $t_0 \in \mathbb{I}$, can be shown just as the statement $\bigcup_{\tilde{P} \in \mathcal{P}_{t_0}} \mathcal{R}(\tilde{P}) = {}^{\mathbb{I}}V_s(t_0)$. However, we will prove it for the readers convenience. Fix $\bar{t} \in \mathbb{I}$ and let

$$\bar{\mu}(\bar{t}) \in {}^{\mathbb{I}}V_u(\bar{t}).$$

Next we construct a family of projectors $\hat{P} : \mathbb{I} \to \mathbb{R}^{k\times k}$ with $\bar{\mu}(\bar{t}) \in \mathcal{N}(\hat{P}(\bar{t}))$. Then we show that $\hat{P}(t) \in \mathcal{P}_t$ for all $t \in \mathbb{I}$.

By (4.40) there exists a subspace $U_{\bar{\mu}}(\bar{t}) \subset {}^{\mathbb{I}}V_u(\bar{t})$ of dimension $\dim(\mathcal{N}(P(\bar{t})))$ with $\bar{\mu}(\bar{t}) \in U_{\bar{\mu}}(\bar{t})$. Further, by the invariance of Φ and ${}^{\mathbb{I}}V_u$ (Equation (4.17)) there exist subspaces $U_{\bar{\mu}}(t)$ for all $t \in \mathbb{I}$ with

$$U_{\bar{\mu}}(t) \subset {}^{\mathbb{I}}V_u(t) \tag{4.46}$$

and dimension $\dim(\mathcal{N}(P(t)))$ such that

$$\Phi(t,s)U_{\bar{\mu}}(s) = U_{\bar{\mu}}(t) \text{ for all } t,s \in \mathbb{I},\ t \geq s.$$

Define the family of projectors $\hat{P} : \mathbb{I} \to \mathbb{R}^{k\times k}$ by

$$\mathcal{N}(\hat{P}(t)) := U_{\bar{\mu}}(t),$$
$$\mathcal{R}(\hat{P}(t)) := \mathcal{R}(P(t))$$

for all $t \in \mathbb{I}$. With equation (4.46), (4.40) and (4.21) we get

$$\mathcal{N}(\hat{P}(t)) \oplus \mathcal{R}(\hat{P}(t)) = U_{\bar{\mu}} \oplus \mathcal{R}(P(t)) = \mathbb{R}^k$$

for all $t \in \mathbb{I}$, since $\dim(\mathcal{R}(P(t))) = \dim(\mathcal{N}(P(t))) = k$. Thus, the family is well defined. Further, we have

$$\bar{\mu}(\bar{t}) \in \mathcal{N}(\hat{P}(\bar{t})).$$

The family of projectors is invariant by definition and by (4.40), (4.46) we obtain $\mathcal{N}(\hat{P}(t)) = U_{\bar{\mu}}(t) \subset {}^{\mathbb{I}}V_u(t)$ and $\mathcal{R}(\hat{P}(t)) = \mathcal{R}(P(t)) \subset {}^{\mathbb{I}}V_s(t)$ for all $t \in \mathbb{I}$. Finally, Lemma 4.2.3 leads to $\hat{P}(t) \in \mathcal{P}(t)$ for all $t \in \mathbb{I}$. □

This theorem and Corollary 3.4.12 state that the stable cone of an M-hyperbolic system is a subset of the stable cone of an sufficiently small perturbed system.

Chapter 5

Explicit Representations of (Un)Stable Subspaces and Cones

An explicit representation of the stable and unstable subspaces and cones of a linear (ft-)hyperbolic system is of great interest for nemerical approximation. We start this chapter with the study of ift-systems and their (un)stable subspaces. Then we move on with ft-systems. For a subset of M-hyperbolic systems we are able to find an explicit representation. This kind of systems are D-hyperbolic systems, which we define in Section 5.2.

In infinite time the invariant family of projectors $P : \mathbb{I} \to \mathbb{R}^{k\times k}$, $\mathbb{I} \in \{\mathbb{R}, \mathbb{Z}\}$ of an exponential dichotomy is unique and satisfies

$$^{\mathbb{T}}V_s(t) = \mathcal{R}(P(t)),\ ^{\mathbb{T}}V_u(t) = \mathcal{N}(P(t)).$$

To picture the stable and unstable subspaces we "only" need to find the unique family of projectors P. For invertible systems in discrete time

$$x(n+1) = A(n)x(n),\ n \in \mathbb{Z} \tag{5.1}$$

we get with the help of the Green's function $G(\cdot,\cdot)$, see [105], that $u(n) = G(n, N+1)r$ for $N \in \mathbb{Z}$ is the unique bounded solution of

$$x(n+1) = A(n)x(n) + \delta_{n,N}r,\ n \in \mathbb{Z}, \tag{5.2}$$

where $\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{else} \end{cases}$ is the Kronecker delta. By [71, Theorem 2.1 and Corollary 1] we have the following.

Corollary 5.0.1. *Let the inhomogeneous equation* (5.2) *possess for all* $N \in \mathbb{Z}$ *and* $r \in \mathbb{R}^k$ *a unique bounded solution fulfilling*

$$\|u(n)\| \le \begin{cases} Ke^{-\alpha(n-N-1)}\|r\|, & \text{for } n \ge N+1, \\ Ke^{-\beta(N+1-n)}\|r\|, & \text{for } n \le N. \end{cases}$$

Fix $N \in \mathbb{Z}$ and let $(u^i(n))_{n\in\mathbb{Z}}$ be the unique bounded solution of (5.2) *for $r = e_i$, $i = 1, \ldots, k$, where e_i is the i-th unit vector. Then* (5.1) *possesses an exponential dichotomy on $\mathbb{Z}$ with projector*

$$P(N+1) = (u^1(N+1), u^2(N+1), \ldots, u^k(N+1)).$$

This means, with the help of the unique solution of (5.2) we get the unique projector of (5.1). For ft-systems we observed that the family of projectors is not unique. Hence, we cannot just approximate the projectors to get the stable and unstable cones. Doan, Palmer and Siegmund found an explicit form of the stable and unstable cone if the system is D-hyperbolic, see [43, Proposition 19]. In Section 5.4 we show that all D-hyperbolic systems, which we will introduce in Section 5.2, are also M-hyperbolic w.r.t. the same norm. To verify whether a system is D-hyperbolic we analyze the dynamical characteristics of solutions by using a $\|\cdot\|_\Gamma := \sqrt{\langle\cdot, \Gamma\cdot\rangle}$ norm, where $\Gamma \in \mathbb{R}^{k\times k}$ is a positive definite, symmetric matrix and $\langle\cdot,\cdot\rangle$ denotes the standard inner product. Before we introduce further tensors in Section 5.2, which we need to define hyperbolicity with the help of the dynamical pattern as in [61], [15], [45], [43], we study autonomous systems and analyze whether a matrix Γ exists such that the system is M-hyperbolic with respect to $\|\cdot\|_\Gamma$. After we derived the explicit form of the cones we consider various examples to get an idea how stable and unstable cones are formed. In general the cones are not subspaces. Therefore, it is interesting to know how wide these cones are, i.e. how large the angle of the cone is. In Section 5.6 and 5.7 we present different statements about the width and the angle of a cone.

Γ-Norm and M-Hyperbolicity w.r.t. the Γ-Norm

First we analyze properties of the $\|\cdot\|_\Gamma := \sqrt{\langle\cdot, \Gamma\cdot\rangle}$ norm, where $\Gamma \in \mathbb{R}^{k\times k}$ is a positive definite, symmetric matrix and $\langle\cdot,\cdot\rangle$ denotes the standard inner product. We study different types of autonomous systems to find cases where a matrix Γ exits such that the given system is M-hyperbolic with respect to $\|\cdot\|_\Gamma$.

Theorem 5.1.1. *Let $A \in \mathbb{R}^{k\times k}$ be diagonalizable, $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$ and the system*

$$\begin{cases} \dot{x} = Ax, & \text{for } \mathbb{T} = \mathbb{R}, \\ x(t+1) = Ax(t), & \text{for } \mathbb{T} = \mathbb{Z}, \end{cases} \quad x \in \mathbb{R}^k, t \in \mathbb{T} \tag{5.3}$$

be hyperbolic on $\mathbb{T}$. Then there exists a positive definite symmetric matrix $\Gamma \in \mathbb{R}^{k\times k}$ such that equation (5.3) *is M-hyperbolic on every finite time interval with respect to the Γ-norm.*

Proof. The given autonomous system (5.3) is hyperbolic. Denote by α, β the exponential rates and by P_A the invariant projector. A is diagonalizable, hence, a matrix $S \in \mathrm{Gl}(\mathbb{R}^{k\times k})$ exists such that $S^{-1}AS =: D$ is an diagonal matrix. We define $P_D := S^{-1}P_AS$. Let Φ_A denote the solution operator of (5.3) then $\Phi_D := S^{-1}\Phi_AS$ is the solution operator of

$$\begin{cases} \dot{y} = Dy, & \text{for } \mathbb{T} = \mathbb{R}, \\ y(t+1) = Dy(t), & \text{for } \mathbb{T} = \mathbb{Z}, \end{cases} \quad y \in \mathbb{R}^k,\ t \in \mathbb{T}. \tag{5.4}$$

By definition of P_D and Φ_D as well as by the invariance of Φ_A and the projector P_A the equations

$$\begin{aligned} P_D\Phi_D &= S^{-1}P_ASS^{-1}\Phi_AS = S^{-1}P_A\Phi_AS \\ &\qquad\qquad = S^{-1}\Phi_AP_AS = S^{-1}\Phi_ASS^{-1}P_AS = \Phi_DP_D, \\ P_DP_D &= S^{-1}P_ASS^{-1}P_AS = S^{-1}P_A^2S = S^{-1}P_AS = P_D \end{aligned}$$

hold. This shows that P_D is an invariant projector of (5.4). The relation $0 = P_Dx = S^{-1}P_ASx$, $x \in \mathbb{R}^k$ yields

$$\mathcal{N}(P_D) = S^{-1}\mathcal{N}(P_A).$$

This leads to

$$\Phi_D\mathcal{N}(P_D) = S^{-1}\Phi_ASS^{-1}\mathcal{N}(P_A) = S^{-1}\Phi_A\mathcal{N}(P_A) = S^{-1}\mathcal{N}(P_A) = \mathcal{N}(P_D),$$

since Φ_A satisfies (3.8). Thus, Φ_D satisfies (3.8) and the inverse of $\Phi_D^t|_{\mathcal{N}(P_D)}$, $t \in \mathbb{T}_0^+$ exists. We denote it by Φ_D^{-t}, $t \in \mathbb{T}_0^+$. Let the exponential rates $\alpha, \beta > 0$ satisfy

$$\alpha \le \begin{cases} -\max\limits_{i\in\{1,\cdots,k\}}\{(\Phi_D)_{i,i} | (\Phi_D)_{i,i} < 0\}, & \text{for } \mathbb{T} = \mathbb{R}, \\ -\ln(\max\limits_{i\in\{1,\cdots,k\}}\{|(\Phi_D)_{i,i}| \,|\, |(\Phi_D)_{i,i}| < 1\}), & \text{for } \mathbb{T} = \mathbb{Z}, \end{cases},$$

$$\beta \le \begin{cases} \min\limits_{i\in\{1,\cdots,k\}}\{|(\Phi_D)_{i,i}| \,|\, (\Phi_D)_{i,i} > 0\}, & \text{for } \mathbb{T} = \mathbb{R}, \\ \ln(\min\limits_{i\in\{1,\cdots,k\}}\{|(\Phi_D)_{i,i}| \,|\, |(\Phi_D)_{i,i}| > 1\}), & \text{for } \mathbb{T} = \mathbb{Z}. \end{cases}$$

If they do not, reduce them until they do. It is easy to see that

$$\left\|\Phi_D^tP_D\right\|_2 \le e^{-\alpha t} \text{ for all } t \in \mathbb{T}_0^+, \tag{5.5}$$

$$\left\|\Phi_D^t(\mathrm{I} - P_D)\right\|_2 \le e^{\beta t} \text{ for all } t \in \mathbb{T}_0^- \tag{5.6}$$

holds, for more details see [107, p. 91] and [32, p. 386]. The equation $P_Dx = S^{-1}P_ASx$, $x \in \mathbb{R}^k$ yields

$$\mathcal{R}(P_D) = S^{-1}\mathcal{R}(P_A).$$

Define $\Gamma := (S^{-1})^TS^{-1}$ then Γ is symmetric and we see by

$$\langle\xi, \Gamma\xi\rangle = \langle\xi, (S^{-1})^TS^{-1}\xi\rangle = \langle S^{-1}\xi, S^{-1}\xi\rangle = \left\|S^{-1}\xi\right\|_{\mathrm{I}}^2 > 0$$

for all $\xi \in \mathbb{R}^k \setminus \{0\}$ that Γ is positive definite. The Γ-norm and the identity norm are related as follows

$$\begin{aligned}
\max_{0\neq y\in\mathcal{R}(P_D)} \frac{\|\Phi_D y\|_{\mathrm{I}}}{\|y\|_{\mathrm{I}}} &= \max_{0\neq y\in\mathcal{R}(P_D)} \frac{\langle S^{-1}\Phi_A S y, S^{-1}\Phi_A S y\rangle}{\langle y, y\rangle} \\
&= \max_{0\neq y\in\mathcal{R}(P_D)} \frac{\langle \Phi_A S y, \Gamma\Phi_A S y\rangle}{\langle y, y\rangle} = \max_{0\neq x\in\mathcal{R}(P_A)} \frac{\langle \Phi_A x, \Gamma\Phi_A x\rangle}{\langle S^{-1}x, S^{-1}x\rangle} \\
&= \max_{0\neq x\in\mathcal{R}(P_A)} \frac{\langle \Phi_A x, \Gamma\Phi_A x\rangle}{\langle x, \Gamma x\rangle} = \max_{0\neq x\in\mathcal{R}(P_A)} \frac{\|\Phi_A x\|_\Gamma}{\|x\|_\Gamma}.
\end{aligned}$$

Together with (5.5) and (5.6) we observe that

$$\begin{aligned}
\left\|\Phi_A^t x\right\|_\Gamma &\leq e^{-\alpha t}\,\|x\|_\Gamma \text{ for all } x\in\mathcal{R}(P_A),\ t\in\mathbb{T}_0^+, \\
\left\|\Phi_A^t x\right\|_\Gamma &\leq e^{\beta t}\,\|x\|_\Gamma \text{ for all } x\in\mathcal{N}(P_A),\ t\in\mathbb{T}_0^-
\end{aligned}$$

and consequently (5.3) is M-hyperbolic on every finite time interval with respect to the Γ-norm. □

Does a matrix Γ exist for nondiagonalizable matrices? And if it does, what does this matrix Γ look like? In this thesis we will not answer these questions. However, we will study a Jordan-block matrix in Theorem 5.1.4 with eigenvalue $\lambda < 0$, which generates a hyperbolic system on $\mathbb{R}$ with the identity projector and exponential rate $-\lambda$. We will prove that no positive definite symmetric matrix $\Gamma \in \mathbb{R}^{k\times k}$ exists such that the system generated by the Jordan-block matrix is M-hyperbolic w.r.t. the Γ-norm on a compact interval with the unique infinite time projector (identity projector) and exponential rate $-\lambda$. The question if there exists a matrix Γ such that the system is M-hyperbolic if we reduce the exponential rate $-\lambda$ still remains open.

In the proof of Theorem 5.1.1 we see which matrix Γ we can chose to get M-hyperbolicity w.r.t. $\|\cdot\|_\Gamma$.

Remark 5.1.2. *A possible positive definite symmetric matrix* Γ *such that equation* (5.3) *is M-hyperbolic (w.r.t.* $\|\cdot\|_\Gamma$*) is* $\Gamma = (S^{-1})^T S^{-1}$, *where* $S \in \mathrm{Gl}(\mathbb{R}^{k\times k})$ *such that* $S^{-1}AS$ *is a diagonal matrix.*

Additionally, in the prove of Theorem 5.1.1 a relation between the (un)stable subspace of an autonomous ift-system and the (un)stable cone of the same autonomous ft-system is shown.

Corollary 5.1.3. *Let* $A \in \mathbb{R}^{k\times k}$ *be diagonalizable and system* (5.3) *be hyperbolic on* $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$. *Then by Lemma 4.2.4 the stable (unstable) subspace of the ift-system* (5.3), *which do not depend on the the chosen norm, lies inside the stable (unstable) cone w.r.t.* $\|\cdot\|_\Gamma$ *of any ft-system, which is defined by* A.

Next we analyze a Jordan-block matrix with eigenvalue $\lambda < 0$. In Theorem 3.3.2 we proved that a Lyapunov norm exists such that the given system is M-hyperbolic on every compact interval with the identity projector and a rate $\alpha \leq -\lambda$. We will see that this Lyapunov norm is not generated by a positive definite symmetric matrix Γ, at least not for the exponential rate $-\lambda$.

Theorem 5.1.4. *Let $\mathbb{I} = [t_-, t_+]$, $t_- \neq t_+$ and $A \in \mathbb{R}^{k\times k}$ be a Jordan-block w.r.t. the eigenvalue $\lambda < 0$. Then for every positive definite symmetric $\Gamma \in \mathbb{R}^{k\times k}$ the system*

$$\dot{x} = Ax, \ x \in \mathbb{R}^k \tag{5.7}$$

is not M-hyperbolic (w.r.t. $\|\cdot\|_\Gamma$) with the identity projector and exponential rate $-\lambda$.

Proof. The solution operator of (5.7) satisfies for $s, t+s \in \mathbb{I}$

$$\Phi(t+s,s) := e^{At} = e^{\lambda t} \begin{pmatrix} 1 & t & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & t \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} =: e^{\lambda t}\tilde{A}(t).$$

Assume that equation (5.7) is M-hyperbolic (w.r.t. $\|\cdot\|_\Gamma$) on $\mathbb{I} = [t_-, t_+]$ with the identity projector P, exponential rate $\alpha = -\lambda$ and Γ positive definite and symmetric. Then we have

$$e^{\lambda t}\left\|\tilde{A}(t)x\right\|_\Gamma = \|\Phi(t+s,s)x\|_\Gamma \le e^{-\alpha t}\|x\|_\Gamma = e^{\lambda t}\|x\|_\Gamma \tag{5.8}$$

for all $x \in \mathbb{R}^k = \mathcal{R}(P)$ and $t+s, s \in \mathbb{I}$ with $t \ge 0$. Let $x \in \mathbb{R}^k$ and $\tilde{x} := \begin{pmatrix} x_2 & \cdots & x_k & 0 \end{pmatrix}^T$ then the equation

$$\tilde{A}x = x + t\tilde{x}$$

holds. (5.8) and the symmetry of Γ leads for $t = t_+ - t_- > 0$ to

$$\begin{aligned} 0 &\ge \left\|\tilde{A}x\right\|_\Gamma - e^{-(\lambda-\lambda)t}\|x\|_\Gamma \\ &= \langle \tilde{A}(t)x, \Gamma\tilde{A}(t)x\rangle - \langle x, \Gamma x\rangle \\ &= \langle x + t\tilde{x}, \Gamma(x+t\tilde{x})\rangle - \langle x, \Gamma x\rangle \\ &= \langle x, \Gamma x\rangle + 2t\langle x, \Gamma\tilde{x}\rangle + t^2\langle \tilde{x}, \Gamma\tilde{x}\rangle - \langle x, \Gamma x\rangle \\ &= 2t\langle x, \Gamma\tilde{x}\rangle + t^2\langle \tilde{x}, \Gamma\tilde{x}\rangle. \end{aligned}$$

Take $x = \begin{pmatrix} x_1 & 1 & 0 & \cdots & 0 \end{pmatrix}^T$ with $x_1 > 0$ then

$$\begin{aligned} 0 &\ge 2t(\Gamma_{11}x_1 + \Gamma_{21}) + t^2\Gamma_{11} \\ &= \Gamma_{11}(2tx_1 + t^2) + \Gamma_{21}2t \\ &= 2t\left(\Gamma_{11}x_1 + \frac{t}{2}\Gamma_{11} + \Gamma_{21}\right). \end{aligned} \tag{5.9}$$

By rearranging (5.9) we get

$$-\Gamma_{21} \geq \Gamma_{11}x_1 + \frac{t}{2}\Gamma_{11}. \tag{5.10}$$

Since Γ is positive definite the estimate $0 < e_i^T \Gamma e_i = \Gamma_{ii}$ is satisfied for all $i \in \{1, \dots, k\}$, where e_i denotes the unit vector. This means that the right hand side of (5.10) is unbounded for all $x_1 > 0$. Therefore no Γ_{21} exists such that (5.10) is satisfied for all $x_1 > 0$. Thus no positive definite symmetric matrix Γ exists such that system (5.7) is M-hyperbolic with the identity projector and exponential rate $-\lambda$. □

D-Hyperbolicity

We introduce the Γ-strain acceleration tensor $M_\Gamma(\cdot)$ in addition to the Γ-strain tensor $S_\Gamma(\cdot)$ and zero Γ-strain set $Z_\Gamma(\cdot)$, which are defined in Definition 3.2.8. These tensors are the main ingredients of the D-hyperbolicity definition. For D-hyperbolic systems we can state an explicit representation of the stable and unstable cone, which we do in Section 5.3. Therefore, we analyze these tensors and study their definiteness. The Γ-strain acceleration tensor is called the Cotter-Rivlin rate in continuum mechanics, see [35, Subsection 4.3.13.]. All of the following definitions and statements for continuous time systems originate from [45], [15], [43] and for the two dimensional continuous time case we refer to [61]. In addition, we introduce similar concepts for discrete time systems.

For system (2.7) we impose the following assumption.

(A0) Let the matrix function A of (2.7) satisfy $A \in \mathcal{C}^1(\mathbb{I}, \mathbb{R}^{k\times k})$.

We define the Γ-strain acceleration tensor $M_\Gamma(\cdot)$ and show why the tensor has the given form. Additionally, we present a relation between the definiteness of this tensor and the dynamical characteristics of solutions.

Definition 5.2.1. *Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $\mathbb{I} \subset \mathbb{T}$ be a compact interval and let $\Gamma = \Gamma^T > 0$. For $\mathbb{T} = \mathbb{R}$ assume* **(A0)**. *For every $t \in \begin{cases} \mathbb{I}, & \text{for } \mathbb{T} = \mathbb{R}, \\ \mathbb{I}_2, & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}$ the matrix*

$$M_\Gamma(t) := \begin{cases} \dot{S}_\Gamma(t) + S_\Gamma(t)A(t) + A(t)^T S_\Gamma(t), & \text{for } \mathbb{T} = \mathbb{R}, \\ A(t)^T S_\Gamma(t+1)A(t) - S_\Gamma(t), & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}$$

is called the Γ***-strain acceleration tensor*** *of* (2.7)/(2.8). *Denote by $M_{Z_\Gamma}(t)$ the restriction of $\xi \mapsto \langle \xi, M_\Gamma(t)\xi\rangle$ to $Z_\Gamma(t)$. We call $M_{Z_\Gamma}(t)$ negative/positive definite if it attains only negative/positive values for all $\xi \in Z_\Gamma(t) \setminus \{0\}$ and indefinite if it attains both negative and positive values on $Z_\Gamma(t)$. For $\Gamma = \mathrm{I}$ we write $M(\cdot)$ instead of $M_\mathrm{I}(\cdot)$.*

The dynamical properties of solutions $\xi(\cdot)$ of the continuous system (2.7) intersecting the set $Z_\Gamma(t_0)$, $t_0 \in \mathbb{I}$ depend on the sign of the second derivative of $t \mapsto \|\xi(t)\|^2$ at $t = t_0$. The sign characterizes whether the solution $\xi(\cdot)$ crosses transversally from a region with increasing norm to a region with decreasing norm or vice versa, for more details see Lemma 5.2.4. With equation (3.16) and assumption **(A0)** we get the identity

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|\xi(t)\|_\Gamma^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \xi(t), S_\Gamma(t)\xi(t)\rangle \\ &= \langle \dot\xi(t), S_\Gamma(t)\xi(t)\rangle + \langle \xi(t), \dot S_\Gamma(t)\xi(t) + S_\Gamma(t)\dot\xi(t)\rangle \\ &= \langle \xi(t), (\dot S_\Gamma(t) + S_\Gamma(t)A(t) + A(t)^T S_\Gamma(t))\xi(t)\rangle \\ &= \langle \xi(t), M_\Gamma(t)\xi(t)\rangle. \end{aligned} \tag{5.11}$$

Hence, the dynamical properties of solutions of (2.7) intersecting the set $Z_\Gamma(t_0)$, $t_0 \in \mathbb{I}$ depend on the definiteness of the matrix $M_{Z_\Gamma}(t_0)$. Continuous solutions which pass from a region with increasing (decreasing) norm to a region with decreasing (increasing) norm must intersect the set $Z_\Gamma(\cdot)$. Thus, the definiteness of $M_{Z_\Gamma}(t_0)$, $t_0 \in \mathbb{I}$ determines if a solution can leave or enter a region. More details are presented in Lemma 5.2.2.

Solutions ξ of the discrete system (2.8) can jump from a region with increasing norm to a region with decreasing norm or vice versa without an intermediate stop at $Z_\Gamma(\cdot)$. The positive definiteness of $M_{Z_\Gamma}(n)$ for all $n \in \mathbb{I}$ does not prevent solutions from jumping from a region with increasing norm to a region with decreasing norm. It ensures that a solution has increasing norm after leaving the set $Z_\Gamma(n_0)$, $n_0 \in \mathbb{I}$. This can be seen by the following relation of S_Γ and M_Γ. With equation (3.17) we get for all $n \in \mathbb{I}_2$

$$\begin{aligned} &\big(\big(\|\xi(n+2)\|_\Gamma^2 - \|\xi(n+1)\|_\Gamma^2\big) - \big(\|\xi(n+1)\|_\Gamma^2 - \|\xi(n)\|_\Gamma^2\big)\big) \\ &= \langle \xi(n+1), S_\Gamma(n+1)\xi(n+1)\rangle - \langle \xi(n), S_\Gamma(n)\xi(n)\rangle \\ &= \langle A(n)\xi(n), S_\Gamma(n+1)A(n)\xi(n)\rangle - \langle \xi(n), S_\Gamma(n)\xi(n)\rangle \\ &= \langle \xi(n), [A(n)^T S_\Gamma(n+1)A(n) - S_\Gamma(n)]\xi(n)\rangle \\ &= \langle \xi(n), M_\Gamma(n)\xi(n)\rangle. \end{aligned} \tag{5.12}$$

For a more detailed statement about the dynamical properties of solutions of (2.8) we need to analyze the definiteness of $M_\Gamma(n)$ for all $n \in \mathbb{I}$. In Lemma 5.2.3 and 5.2.5 we present some of them. We start with some properties of the tensor for continuous systems.

Lemma 5.2.2. *Assume* **(A0)**. *Let $\mathbb{I} \subset \mathbb{R}$ be a compact interval, $\Gamma = \Gamma^T > 0$ and let $M_{Z_\Gamma}(t)$ of* (2.7) *be positive definite for all $t \in \mathbb{I}$. Fix $\xi \in \mathbb{R}^k \setminus \{0\}$ and $t_0 \in \mathbb{I}$ with $\langle \xi, S_\Gamma(t_0)\xi\rangle > 0$ then we have*

$$\langle \Phi(t,t_0)\xi, S_\Gamma(t)\Phi(t,t_0)\xi\rangle > 0 \textit{ for all } t \in \mathbb{I}, t > t_0.$$

Let $\xi \in \mathbb{R}^k$ and $t_0 \in \mathbb{I}$ with $\langle \xi, S_\Gamma(t_0)\xi\rangle < 0$ then we have

$$\langle \Phi(t,t_0)\xi, S_\Gamma(t)\Phi(t,t_0)\xi\rangle < 0 \text{ for all } t \in \mathbb{I}, t < t_0.$$

Proof. Let $\xi \in \mathbb{R}^k$ and $t_0 \in \mathbb{I}$ with $\langle \xi, S_\Gamma(t_0)\xi\rangle > 0$. Assume that there exists a $\bar{t} \in \mathbb{I}$, $\bar{t} > t_0$ such that

$$\langle \Phi(t,t_0)\xi, S_\Gamma(t)\Phi(t,t_0)\xi\rangle \begin{cases} = 0, & \text{for } t = \bar{t}, \\ > 0, & \text{for } \bar{t} > t \geq t_0. \end{cases} \tag{5.13}$$

Then $\xi(\bar{t}) := \Phi(\bar{t},t_0)\xi \in Z_\Gamma(\bar{t})$ and by the positive definiteness of $M_{Z_\Gamma}(\bar{t})$ and (5.11) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi(\bar{t}), S_\Gamma(\bar{t})\xi(\bar{t})\rangle = \langle \xi(\bar{t}), M_\Gamma(\bar{t})\xi(\bar{t})\rangle > 0.$$

This implies with $\langle \Phi(\bar{t},t_0)\xi, S_\Gamma(\bar{t})\Phi(\bar{t},t_0)\xi\rangle = 0$ that a $t_1 \in \mathbb{I}$, $t_0 \leq t_1 < \bar{t}$ exists with

$$\langle \Phi(t_1,t_0)\xi, S_\Gamma(t_1)\Phi(t_1,t_0)\xi\rangle < 0$$

which is a contradiction to (5.13). Analogously, we get the second claim ("<"). □

For discrete systems we introduce the corresponding statement.

Lemma 5.2.3. *Let $\mathbb{I} \subset \mathbb{Z}$ be a compact interval, $\Gamma = \Gamma^T > 0$ and let $M_\Gamma(n)$ of (2.8) be positive definite for all $n \in \mathbb{I}_2$. Fix $\xi \in \mathbb{R}^k \setminus \{0\}$ and $n_0 \in \mathbb{I}_2$ with $\langle \xi, S_\Gamma(n_0)\xi\rangle > 0$ then we have for all $n \in \mathbb{I}_1$, $n \geq n_0$ with $\Phi(n,n_0)\xi \neq 0$*

$$\langle \Phi(n,n_0)\xi, S_\Gamma(n)\Phi(n,n_0)\xi\rangle > 0.$$

Let $\xi \in \mathbb{R}^k \setminus \{0\}$ and $n_0 \in \mathbb{I}_1$ with $\langle \xi, S_\Gamma(n_0)\xi\rangle < 0$ and let $\bar{n} \in \mathbb{I}$, $\bar{n} < n_0$ and $\bar{\xi} \in \mathbb{R}^k$ with $\Phi(n_0,\bar{n})\bar{\xi} = \xi$ then we have

$$\langle \Phi(n,\bar{n})\bar{\xi}, S_\Gamma(n)\Phi(n,\bar{n})\bar{\xi}\rangle < 0 \text{ for all } n \in [\bar{n}, n_0 - 1]_{\mathbb{Z}}.$$

Proof. Fix $\xi \in \mathbb{R}^k \setminus \{0\}$. Define $\xi(n) := \Phi(n,n_0)\xi$ for all $n \in \mathbb{I}, n \geq n_0$. Then by the positive definiteness of $M_\Gamma(n_1)$ for all $n_1 \in \mathbb{I}_2$ and equation (5.12) we have for all $n \in \mathbb{I}_2$ with $\xi(n+1) \neq 0$

$$\begin{aligned} 0 &< \langle \xi(n), M_\Gamma(n)\xi(n)\rangle \\ &= \langle \xi(n+1), S_\Gamma(n+1)\xi(n+1)\rangle - \langle \xi(n), S_\Gamma(n)\xi(n)\rangle. \end{aligned} \tag{5.14}$$

With $\langle \xi, S_\Gamma(n_0)\xi\rangle > 0$ we get inductively for all $n \in \mathbb{I}_2$, $n > n_0$ with $\xi(n+1) \neq 0 \neq \xi(n)$

$$\langle \xi(n+1), S_\Gamma(n+1)\xi(n+1)\rangle > \langle \xi(n), S_\Gamma(n)\xi(n)\rangle > \langle \xi(n_0), S_\Gamma(n_0)\xi(n_0)\rangle > 0.$$

The same proof works analogously for the second claim ("<"). □

The characteristics of solutions of the continuous system (2.7), which intersect the zero Γ-strain set $Z_\Gamma(\cdot)$, are summarized in the next lemma.

Lemma 5.2.4. *Assume* **(A0)**. *Let* $\mathbb{I} \subset \mathbb{R}$ *be a compact interval and* $\Gamma = \Gamma^T > 0$. *Assume* $\bar{\xi}(\cdot)$ *is a solution of* (2.7) *with* $\bar{\xi}(t_0) \in Z_\Gamma(t_0)$ *for a* $t_0 \in \mathbb{I}$ *and assume that the* Γ*-strain acceleration tensor fulfills*

$$\langle \bar{\xi}(t_0), M_\Gamma(t_0)\bar{\xi}(t_0)\rangle > 0.$$

Then there exist $t_1, t_2 \in \mathbb{I}$ *with* $t_1 > t_0 > t_2$ *such that*

$$\langle \bar{\xi}(t), S_\Gamma(t)\bar{\xi}(t)\rangle \begin{cases} > 0, & \text{for } t_1 > t > t_0 & (5.15)\\ = 0, & \text{for } t = t_0 & (5.16)\\ < 0, & \text{for } t_0 > t > t_2. & (5.17)\end{cases}$$

Further, if the Γ*-strain acceleration tensor* $M_{Z_\Gamma}(t)$ *is positive definite for all* $t \in \mathbb{I}$ *then equation* (5.15) *holds for all* $t \in \mathbb{I}$ *with* $t > t_0$ *and equation* (5.17) *for all* $t \in \mathbb{I}$ *with* $t < t_0$. *If* $M_{Z_\Gamma}(t)$ *is negative definite for all* $t \in \mathbb{I}$ *then equation* (5.17) *holds for all* $t \in \mathbb{I}$ *with* $t > t_0$ *and equation* (5.15) *for all* $t \in \mathbb{I}$ *with* $t < t_0$.

Proof. Let $\bar{\xi}(t_0) \in Z_\Gamma(t_0)$ and $\langle \bar{\xi}(t_0), M_\Gamma(t_0)\bar{\xi}(t_0)\rangle > 0$. Then we have $\langle \bar{\xi}(t_0), S_\Gamma(t_0)\bar{\xi}(t_0)\rangle = 0$ and with (5.11) we obtain $\frac{\mathrm{d}}{\mathrm{d}t}\langle \bar{\xi}(t_0), S_\Gamma(t_0)\bar{\xi}(t_0)\rangle > 0$. Thus, there exist $t_1, t_2 \in \mathbb{I}$ with $t_1 > t_0 > t_2$ such that for all $t_1 > t > t_0$ equation (5.15) holds and for all $t_2 < t < t_0$ equation (5.17).

Let now $M_{Z_\Gamma}(t)$ be positive definite for all $t \in \mathbb{I}$. By Lemma 5.2.2 and with the statements (5.15)-(5.17) we conclude (5.15) for all $t \in \mathbb{I}$ with $t > t_0$ and (5.17) for all $t \in \mathbb{I}$ with $t < t_0$. The statement for M_{Z_Γ} negative definite follows analogously. □

For the discrete system (2.8) the characteristics of solutions which intersect the zero Γ-stain set $Z_\Gamma(\cdot)$ depend on the definiteness of M_Γ.

Lemma 5.2.5. *Let* $\mathbb{I} = [n_-, n_+]_\mathbb{Z}$ *and* $\Gamma = \Gamma^T > 0$. *Assume* $\bar{\xi}(n_0) \in Z_\Gamma(n_0)$ *for an* $n_0 \in \mathbb{I}_2$. *Let* ${}_\Phi\mathcal{T}_{\min}(\bar{\xi}(n_0), n_0) =: \bar{n}$ *and* $\bar{\xi}(\bar{n}) \in {}_\Phi\mathcal{T}_{\mathrm{pre}}(\bar{\xi}(n_0), n_0)$. *Set* $\bar{\xi}(n) := \Phi(n, \bar{n})\bar{\xi}(\bar{n})$ *for all* $n \in [\bar{n}, n_+]_\mathbb{Z}$. *Let* $M_\Gamma(n)$ *be positive definite for all* $n \in \mathbb{I}_2$ *then we have*

$$\langle \bar{\xi}(n), S_\Gamma(n)\bar{\xi}(n)\rangle \begin{cases} \begin{cases} > 0, & \text{if } \bar{\xi}(n) \neq 0,\\ = 0, & \text{if } \bar{\xi}(n) = 0,\end{cases} & \text{for } n \in \mathbb{I},\ n_+ > n > n_0 \quad (5.18)\\ = 0, & \text{for } n = n_0 \quad (5.19)\\ \begin{cases} < 0, & \text{if } \bar{\xi}(n) \neq 0,\\ = 0, & \text{if } \bar{\xi}(n) = 0\end{cases} & \text{for } n \in \mathbb{I},\ \bar{n} \le n < n_0. \quad (5.20)\end{cases}$$

Let $M_\Gamma(n)$ *be negative definite for all* $n \in \mathbb{I}$, $n \in \mathbb{I}_2$ *then we have that* (5.18) *holds for all* $\bar{n} \le n < n_0$, *equation* (5.19) *for* $n = n_0$ *and equation* (5.20) *for all* $n \in \mathbb{I}$, $n_+ > n > n_0$.

Proof. Let $M_\Gamma(n)$ be positive definite for all $n \in \mathbb{I}_2$. Fix $n_0 \in \mathbb{I}_2$ and let $\bar{\xi}(n_0) \in Z_\Gamma(n_0) \setminus \{0\}$. Then we get equation (5.19) by the definition of the zero Γ-strain set. The positive definiteness of $M_\Gamma(n_0)$ and (5.19) yield

$$\begin{aligned} 0 &< \langle \bar{\xi}(n_0), M_\Gamma(n_0)\bar{\xi}(n_0)\rangle \\ &= \langle \bar{\xi}(n_0+1), S_\Gamma(n_0+1)\bar{\xi}(n_0+1)\rangle - \langle \bar{\xi}(n_0), S_\Gamma(n_0)\bar{\xi}(n_0)\rangle \\ &= \langle \bar{\xi}(n_0+1), S_\Gamma(n_0+1)\bar{\xi}(n_0+1)\rangle \end{aligned}$$

and the positive definiteness of $M_\Gamma(n_0-1)$ implies for $n_0 - 1 \in \mathbb{I}$

$$\begin{aligned} 0 &< \langle \bar{\xi}(n_0-1), M_\Gamma(n_0-1)\bar{\xi}(n_0-1)\rangle \\ &= \langle \bar{\xi}(n_0), S_\Gamma(n_0)\bar{\xi}(n_0)\rangle - \langle \bar{\xi}(n_0-1), S_\Gamma(n_0-1)\bar{\xi}(n_0-1)\rangle \\ &= -\langle \bar{\xi}(n_0-1), S_\Gamma(n_0-1)\bar{\xi}(n_0-1)\rangle. \end{aligned}$$

With Lemma 5.2.3 we obtain (5.18) and (5.20). The statement for negative definite M_Γ follows analogously. □

As in [15, Definition 2.4], [43, Definition 17] and [61] the whole space can be separated into different regions. Every region contains points of a special type. In fact, the type of a point is determined by the definiteness of the Γ-strain tensor $S_\Gamma(\cdot)$ and of the Γ-strain acceleration tensor $M_\Gamma(\cdot)$ of the linearization at this point. The dynamical characteristics of a solution is defined by the regions it passes. They can be grouped in different classes, i.e. if a solution stays in a region of strictly increasing/decreasing norm (repelling/attracting), or if it crosses just once from a region with strictly increasing norm into a region with strictly decreasing norm (quasihyperbolic) or vise versa (hyperbolic), or if it crosses the regions several times (elliptic). Systems are classified by the different types of solutions, which exist for the given system. Important for us are systems that only have repelling, attracting and hyperbolic solutions. Therefore, $S_\Gamma(t)$, $t \in \begin{cases} \mathbb{I}, & \text{for } \mathbb{I} \subset \mathbb{R}, \\ \mathbb{I}_1, & \text{for } \mathbb{I} \subset \mathbb{Z} \end{cases}$ needs to be indefinite, otherwise the system only has attracting or repelling solutions. Additionally, it needs to be nondegenerate (no eigenvalue is 0) to exclude a region with constant norm. As we have seen in Lemma 5.2.2 to Lemma 5.2.5 that $\begin{cases} M_{Z_\Gamma}(t), t \in \mathbb{I}, & \text{for } \mathbb{I} \subset \mathbb{R}, \\ M_\Gamma(t), t \in \mathbb{I}_2, & \text{for } \mathbb{I} \subset \mathbb{Z} \end{cases}$ must be positive definite to get hyperbolic solutions. This type of system is called D-hyperbolic, since it is defined by the notion of the dynamical pattern. We summarize the latter thoughts for a continuous linear dynamical system in a proper definition, which is also presented in [43, Definition 17] [15, Definition 2.4].

Definition 5.2.6. *Assume* **(A0)**. *Let* $\mathbb{I} \subset \mathbb{R}$ *be a compact interval and* $\Gamma = \Gamma^T > 0$. *System* (2.7) *at time* $t \in \mathbb{I}$ *is called*

- ***attracting*** *if* $S_\Gamma(t)$ *is negative definite,*

- ***repelling*** *if* $S_\Gamma(t)$ *is positive definite,*
- ***quasihyperbolic*** *if* $S_\Gamma(t)$ *is indefinite and nondegenerate and* $M_{Z_\Gamma}(t)$ *is negative definite,*
- ***hyperbolic*** *if* $S_\Gamma(t)$ *is indefinite and nondegenerate and* $M_{Z_\Gamma}(t)$ *is positive definite,*
- ***elliptic*** *if* $S_\Gamma(t)$ *is indefinite and nondegenerate* $M_{Z_\Gamma}(t)$ *is indefinite,*
- ***degenerate*** *in the other cases.*

System (2.7) *is called attracting/repelling etc. (on* $\mathbb{I}$*) if it is attracting/repelling etc. for all* $t \in \mathbb{I}$.

Definition 5.2.7. *Assume* **(A0)**. *Let* $\mathbb{I} \subset \mathbb{R}$ *be a compact interval and* $\Gamma = \Gamma^T > 0$. *System* (2.7) *is called* ***D-hyperbolic*** *(dynamical hyperbolic on* $\mathbb{I}$ *w.r.t.* $\|\cdot\|_\Gamma$*) if it is hyperbolic in the sense of Definition 5.2.6.*

Note that if $\|\cdot\|_{\Gamma(t)} := \sqrt{\langle \cdot, \Gamma(t) \cdot \rangle}$ is allowed to depend on time $t \in \mathbb{I}$ then the $\Gamma(t)$-strain tensor equals $S_\Gamma(t) = \frac{1}{2}\left[A(t)\Gamma(t) + \Gamma(t)A^T(t) + \dot{\Gamma}(t)\right]$. Is means, we find for every solution $\bar{x}(\cdot)$ a family of matrices $\Gamma(\cdot)$ such that $\bar{x}(\cdot)$ lies in the attracting or repelling region of the linearized system. A discussion of a time depending norm is beyond the scope of this thesis.

For discrete time systems the D-hyperbolic definition is slightly different. We have one more condition. Surely, there exist adequate definitions of attracting, repelling, etc. discrete ft-systems. However, we do not introduce them in this thesis.

Definition 5.2.8. *Let* $\mathbb{I} \subset \mathbb{Z}$ *be a compact interval and* $\Gamma = \Gamma^T > 0$. *System* (2.8) *is called* ***D-hyperbolic*** *(on* $\mathbb{I}$ *w.r.t.* $\|\cdot\|_\Gamma$*) if* $S_\Gamma(n)$ *is indefinite and nondegenerate for all* $n \in \mathbb{I}_1$*, if* $M_\Gamma(n)$ *is positive definite for all* $n \in \mathbb{I}_2$ *and if* $S_\Gamma(n)$ *has for all* $n \in \mathbb{I}_1$ *the same number of positive eigenvalues* $\lambda > 0$.

The additional condition is

$$S_\Gamma(n) \text{ has for all } n \in \mathbb{I}_1 \text{ the same number of positive eigenvalues } \lambda > 0. \tag{5.21}$$

For continuous D-hyperbolic systems this is always true, since we require that $S_\Gamma(t)$ is nondegenerate for all $t \in \mathbb{I}$. Hence, we do not need to mention this condition explicitly for the continuous time case. However, is it necessary for discrete systems to have an additional condition? Yes, it is. We want that every D-hyperbolic system is an M-hyperbolic system as well. For continuous systems this statement is proved in Theorem 5.4.2. For a reference see [43, Corollary 22]. To obtain this relation for discrete systems the additional condition is important. We first illustrate by an example that we need an additional condition. Then we state why it has to be (5.21).

In the following example we construct a system for which $S_\Gamma(n)$ is indefinite and nondegenerate for all $n \in \mathbb{I}_1$ and for which $M_\Gamma(n)$ is positive definite for all $n \in \mathbb{I}_2$. Note that the additional condition (5.21) is not satisfied. Then we prove that this system is not M-hyperbolic.

Example 5.2.9. *Assume $u(n+1) = A(n)u(n)$ is a finite time system on $\mathbb{I} = \{0,1,2\}$ with $A(0) = \operatorname{diag}\left(\frac{1}{2},\frac{1}{2},2\right)$ and $A(1) = \operatorname{diag}\left(\frac{1}{2},2,2\right)$. Set $\Gamma = \mathrm{I}$. We see that*

$$\begin{aligned} S(0) &= A(0)^T A(0) - \mathrm{I} = \operatorname{diag}\left(-\frac{3}{4},-\frac{3}{4},3\right), \\ S(1) &= A(1)^T A(1) - \mathrm{I} = \operatorname{diag}\left(-\frac{3}{4},3,3\right) \end{aligned}$$

are nondegenerate and indefinite and we get

$$\begin{aligned} M(0) = A(0)^T S(1) A(0) - S(0) &= \operatorname{diag}\left(-\frac{3}{16},\frac{3}{4},12\right) - \operatorname{diag}\left(-\frac{3}{4},-\frac{3}{4},3\right) \\ &= \operatorname{diag}\left(\frac{9}{16},\frac{6}{4},9\right), \end{aligned}$$

which is positive definite. Assume that our system is M-hyperbolic. Then the following must be true for an $\alpha,\beta > 0$ and for a projector $P(1) \in \mathbb{R}^{3\times 3}$

$$\left\|\operatorname{diag}\left(\frac{1}{2},2,2\right)\xi\right\| = \|A(1)\xi\| \le e^{-\alpha}\|\xi\| \text{ for all } \xi \in \mathcal{R}(P(1)), \tag{5.22}$$

$$\left\|\operatorname{diag}\left(2,2,\frac{1}{2}\right)\xi\right\| = \left\|A(0)^{-1}\xi\right\| \le e^{-\beta}\|\xi\| \text{ for all } \xi \in \mathcal{N}(P(1)). \tag{5.23}$$

Estimate (5.22) forces $\mathcal{R}(P(1)) \in \left\{\mathcal{L}(\xi)\middle|\xi \in \mathbb{R}^3 : \xi_1 = 1, \xi_2^2+\xi_3^2 < \frac{1}{4}\right\}$ and estimate (5.23) forces $\mathcal{N}(P(1)) \in \left\{\mathcal{L}(\xi)\middle|\xi \in \mathbb{R}^3 : \xi_3 = 1, \xi_1^2+\xi_2^2 < \frac{1}{4}\right\}$.

By this choices of range and kernel no projector $P(1)$ exists such that $\mathcal{R}(P(1)) + \mathcal{N}(P(1)) = \mathbb{R}^3$. Hence, our system is not M-hyperbolic.

This proves that we need to add another condition to the D-hyperbolic definition. However, why do we require that the number of positive eigenvalues of $S_\Gamma(n)$ is the same for all $n \in \mathbb{I}_1$? In the next section we see that the stable and unstable cone of an M-hyperbolic system depends on $S_\Gamma(\cdot)$. More precisely, the dimension of the (un)stable cone depends on the number of negative (positive) eigenvalues of $S_\Gamma(\cdot)$. If the number of positive eigenvalues changes for different times the dimension of the cones changes. Then by (4.3) also the dimension of the range of "the projectors" changes. This is not allowed for an invariant family of projectors. Thus the number of positive eigenvalues of $S_\Gamma(n)$ need to stay the same for all $n \in \mathbb{I}_1$.

An Explicit Representation of (Un)Stable Cones

The following propositions imply explicit forms of the stable and unstable cones for D-hyperbolic systems. This enables the calculation and illustration of the cones of D-hyperbolic systems. We introduce the explicit representation of ${}^{\mathbb{I}}V_{u,s}^{\pm}$ and their boundaries for the continuous case in Proposition 5.3.1, which originates from [43, Proposition 19]. Then we find an explicit representation of the stable and unstable cones ${}^{\mathbb{I}}V_{s,u}$ and their boundaries. This conclusion is stated in Corollary 5.3.2. Further we develop related results for discrete time systems. These achievements are summarized in Proposition 5.3.4 and Corollary 5.3.5.

Proposition 5.3.1. *Let* $\mathbb{I} = [t_-, t_+]$ *and* $\Gamma = \Gamma^T > 0$. *Assume* **(A0)** *and assume that* (2.7) *is D-hyperbolic on* $\mathbb{I}$. *Then the following statements hold:*

(i) *For each* $t_0 \in [t_-, t_+)$

$$ {}^{\mathbb{I}}V_u^+(t_0) = \{\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(t_0)\xi\rangle > 0\} \cup \{0\} $$

is a connected double-cone and the boundary of ${}^{\mathbb{I}}V_u^+(t_0)$ *satisfies*

$$ \partial {}^{\mathbb{I}}V_u^+(t_0) \subset \{\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(t_0)\xi\rangle = 0\} = Z_\Gamma(t_0). $$

(ii) *For each* $t_0 \in (t_-, t_+]$

$$ {}^{\mathbb{I}}V_s^-(t_0) = \{\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(t_0)\xi\rangle < 0\} \cup \{0\} $$

is a connected double-cone and the boundary of ${}^{\mathbb{I}}V_s^-(t_0)$ *satisfies*

$$ \partial {}^{\mathbb{I}}V_s^-(t_0) \subset \{\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(t_0)\xi\rangle = 0\} = Z_\Gamma(t_0). $$

(iii) *For each* $t_0 \in \mathbb{I}$ *and any* $\xi \in Z_\Gamma(t_0)$ *we get*

$$ \Phi(t, t_0)\xi \in \begin{cases} {}^{\mathbb{I}}V_u^+(t), & \text{for all } t > t_0 \\ {}^{\mathbb{I}}V_s^-(t), & \text{for all } t < t_0. \end{cases} $$

For the proof we refer to [43, Proposition 19]. The stable and unstable cone ${}^{\mathbb{I}}V_{s,u}$ can be characterized with the help of Proposition 5.3.1 and the equations (4.14), (4.18).

Corollary 5.3.2. *Under the assumptions of Proposition 5.3.1 we obtain that*

$$ {}^{\mathbb{I}}V_s(t_0) = \{\xi \in \mathbb{R}^k | \langle \Phi(t_+, t_0)\xi, S_\Gamma(t_+)\Phi(t_+, t_0)\xi\rangle < 0\} \cup \{0\}, \quad (5.24) $$
$$ {}^{\mathbb{I}}V_u(t_0) = \{\Phi(t_0, t_-)\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(t_-)\xi\rangle > 0\} \cup \{0\} \quad (5.25) $$

are open connected double-cones for all $t_0 \in \mathbb{I}$. *Their boundaries satisfy*

$$ \partial {}^{\mathbb{I}}V_s(t_0) = \Phi(t_0, t_+)\partial {}^{\mathbb{I}}V_s(t_+) \subset \Phi(t_0, t_+)Z_\Gamma(t_+), $$
$$ \partial {}^{\mathbb{I}}V_u(t_0) = \Phi(t_0, t_-)\partial {}^{\mathbb{I}}V_u(t_-) \subset \Phi(t_0, t_-)Z_\Gamma(t_-). $$

Proof. Proposition 5.3.1 and Lemma 4.2.1 imply

$$\begin{aligned}{}^{\mathbb{I}}V_s(t_0) &= \Phi(t_0,t_+)^{\mathbb{I}}V_s(t_+) = \Phi(t_0,t_+)^{\mathbb{I}}V_s^-(t_+)\\ &= \{\Phi(t_0,t_+)\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(t_+)\xi\rangle < 0\} \cup \{0\}\\ &= \{\xi \in \mathbb{R}^k | \langle \Phi(t_+,t_0)\xi, S_\Gamma(t_+)\Phi(t_+,t_0)\xi\rangle < 0\} \cup \{0\}\end{aligned}$$

and

$$\begin{aligned}{}^{\mathbb{I}}V_u(t_0) =&\Phi(t_0,t_-)^{\mathbb{I}}V_u^+(t_-)\\ =&\{\Phi(t_0,t_-)\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(t_-)\xi\rangle > 0\} \cup \{0\}.\end{aligned}$$

By Proposition 5.3.1 the sets ${}^{\mathbb{I}}V_u^+(t_-)$, ${}^{\mathbb{I}}V_s^-(t_+)$ are open connected double-cones. Thus, the continuity of the linear function $\Phi(\cdot,\cdot)$ and of $\langle\cdot,\cdot\rangle$ yield by [29, p. 109, Proposition 4] that the stable and unstable t_0-cones ${}^{\mathbb{I}}V_{s,u}(t_0)$, $t_0 \in \mathbb{I}$, are open connected double-cones. The boundaries of ${}^{\mathbb{I}}V_s(t_+)$ and ${}^{\mathbb{I}}V_u(t_-)$ satisfy

$$\begin{aligned}\partial^{\mathbb{I}}V_s(t_+) = \partial^{\mathbb{I}}V_s^-(t_+) \subset \{\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(t_+)\xi\rangle = 0\} = Z_\Gamma(t_+),\\ \partial^{\mathbb{I}}V_u(t_-) = \partial^{\mathbb{I}}V_u^+(t_-) \subset \{\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(t_-)\xi\rangle = 0\} = Z_\Gamma(t_-).\end{aligned}$$

With the invariance of the cones ${}^{\mathbb{I}}V_s(\cdot)$, ${}^{\mathbb{I}}V_u(\cdot)$ (Lemma 4.2.1) and the continuity of $\Phi(\cdot,\cdot)$ we get by [128, Lemma 6.4] that the boundaries $\partial^{\mathbb{I}}V_s(\cdot)$, $\partial^{\mathbb{I}}V_u(\cdot)$ are invariant. Hence, we obtain for every $t_0 \in \mathbb{I}$

$$\begin{aligned}\partial^{\mathbb{I}}V_s(t_0) = \Phi(t_0,t_+)\partial^{\mathbb{I}}V_s(t_+) \subset \Phi(t_0,t_+)Z_\Gamma(t_+),\\ \partial^{\mathbb{I}}V_u(t_0) = \Phi(t_0,t_-)\partial^{\mathbb{I}}V_u(t_-) \subset \Phi(t_0,t_-)Z_\Gamma(t_-).\end{aligned}$$

□

In Lemma 5.7.2 we prove that even

$$\begin{aligned}\partial^{\mathbb{I}}V_s(t_0) = \Phi(t_0,t_+)Z_\Gamma(t_+),\\ \partial^{\mathbb{I}}V_u(t_0) = \Phi(t_0,t_-)Z_\Gamma(t_-)\end{aligned}$$

hold for all $t_0 \in \mathbb{I}$.

We develop similar statements for the stable and unstable cones of discrete systems. The proof of the following proposition for invertible discrete systems is similar to the one of Proposition 5.3.1. The proof of the statements about the cones of a noninvertible system is more involved and requires a few more steps. Therefore, we first show that for noninvertible D-hyperbolic systems the kernel of the solution operator is a subset of the stable cone. Note, that we already proved this statement for M-hyperbolic systems in Lemma 3.2.6 and Lemma 4.2.4.

Lemma 5.3.3. *Let* $\mathbb{I} = [n_-,n_+]_{\mathbb{Z}}$ *and* $\Gamma = \Gamma^T > 0$. *Assume that* (2.8) *is D-hyperbolic on* $\mathbb{I}$ *(w.r.t.* $\|\cdot\|_\Gamma$*). Then for every* $n_0 \in \mathbb{I}_1$ *we have*

$$\mathcal{N}(\Phi(n_+,n_0)) \subset {}^{\mathbb{I}}V_s(n_0).$$

Proof. Let $n_0 \in \mathbb{I}_1$ and $\xi \in \mathcal{N}(\Phi(n_+, n_0))$ with $\bar{n} := {}_\Phi\mathcal{T}_{\min}(\xi, n_0)$ and $\mathring{n} := {}_\Phi\mathcal{T}_{\ker}(\xi, n_0)$ then $\Phi(n, n_0)\xi \neq 0$ for all $n_0 \leq n < \mathring{n}$. By Lemma 5.2.5 we get for all $\mu \in {}_\Phi\mathcal{T}_{\text{pre}}(\xi, n_0)$

$$\langle \Phi(n, \bar{n})\mu, S_\Gamma(n)\Phi(n, \bar{n})\mu \rangle \begin{cases} = 0, & \text{for } n \geq \mathring{n}, \\ < 0, & \text{for } \bar{n} \leq n < \mathring{n}. \end{cases}$$

This implies with (3.17) for all $\bar{n} \leq n < \mathring{n}$

$$0 > \langle \Phi(n, \bar{n})\mu, S_\Gamma(n)\Phi(n, \bar{n})\mu \rangle = \|\Phi(n+1, \bar{n})\mu\|_\Gamma^2 - \|\Phi(n, \bar{n})\mu\|_\Gamma^2$$

which is equivalent to

$$\|\Phi(n, \bar{n})\mu\|_\Gamma < \|\Phi(m, \bar{n})\mu\|_\Gamma$$

for all $n, m \in [\bar{n}, \mathring{n}]_\mathbb{Z}$, $n > m$. This leads with (4.10) to $\xi \in {}^\mathbb{I}\bar{V}_s(n_0)$. With $\Phi(n_0 + 1, n_0)\xi = 0 \in {}^\mathbb{I}V_s(n_0 + 1)$ follows directly that $\xi \in {}^\mathbb{I}V_s(n_0)$. ☐

Proposition 5.3.4. *Let* $\mathbb{I} = [n_-, n_+]_\mathbb{Z}$ *and* $\Gamma = \Gamma^T > 0$. *Assume that* (2.8) *is D-hyperbolic on* $\mathbb{I}$ *(w.r.t.* $\|\cdot\|_\Gamma$*). Then the following statements hold:*

(i) *For each* $n_0 \in {}_1\mathbb{I}$

$$\begin{aligned} {}^\mathbb{I}V_s^-(n_0) = \big\{\xi \in \mathbb{R}^k | {}_\Phi\mathcal{T}_{\min}(\xi, n_0) = n_0 \vee \big({}_\Phi\mathcal{T}_{\min}(\xi, n_0) =: \bar{n} < n_0 \\ \wedge \exists \bar{\mu} \in {}_\Phi\mathcal{T}_{\text{pre}}(\xi, n_0) : \langle \mu, S_\Gamma(n_0 - 1)\mu \rangle < 0 \\ \textit{for } \mu := \Phi(n_0 - 1, \bar{n})\bar{\mu}\big)\big\} \cup \{0\} \end{aligned} \tag{5.26}$$

is a connected double-cone.

(ii) *For each* $n_0 \in \mathbb{I}_1$

$$L(n_0) := \big\{\xi \in \mathbb{R}^k | \langle \Phi(n_+ - 1, n_0)\xi, S_\Gamma(n_+ - 1)\Phi(n_+ - 1, n_0)\xi \rangle < 0\big\}$$

is an open connected double-cone and

$$\begin{aligned} {}^\mathbb{I}\bar{V}_s(n_0) = {}& \big\{\xi \in \mathbb{R}^k | \langle \Phi(n_+ - 1, n_0)\xi, S_\Gamma(n_+ - 1)\Phi(n_+ - 1, n_0)\xi \rangle < 0\big\} \\ & \cup \mathcal{N}(\Phi(n_+, n_0)), \end{aligned} \tag{5.27}$$

$${}^\mathbb{I}V_s(n_0) = \big\{\xi \in L(n_0) | \Phi(n_0 + 1, n_0)\xi \in {}^\mathbb{I}V_s(n_0 + 1)\big\} \cup \mathcal{N}(\Phi(n_+, n_0)). \tag{5.28}$$

(iii) *For each* $n_0 \in \mathbb{I}_1$

$${}^\mathbb{I}V_u^+(n_0) = \big\{\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(n_0)\xi \rangle > 0\big\} \cup \{0\} \tag{5.29}$$

is an open connected double-cone and the boundary of ${}^\mathbb{I}V_u^+(n_0)$ *satisfies*

$$\partial^\mathbb{I}V_u^+(n_0) \subset \big\{\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(n_0)\xi \rangle = 0\big\}. \tag{5.30}$$

Proof. We start with the proof of **(i)** and first show **(5.26)**. Fix $n_0 \in {}_1\mathbb{I}$. By definition of ${}^{\mathbb{I}}V_s^-(n_0)$ we have $0 \in {}^{\mathbb{I}}V_s^-(n_0)$. Let $\xi \in {}^{\mathbb{I}}V_s^-(n_0) \setminus \{0\}$ with ${}_\Phi\mathcal{T}_{\min}(\xi, n_0) =: \bar{n} < n_0$ then there exist an $\alpha > 0$ and a $\bar{\mu} \in \mathbb{R}^k$ with $\Phi(n_0, \bar{n})\bar{\mu} = \xi$ such that $\|\Phi(n, \bar{n})\bar{\mu}\|_\Gamma e^{\alpha n}$ is decreasing for $n \in [\bar{n}, n_0]$. This means that

$$\|\xi\|_\Gamma^2 e^{2\alpha(n_0)} - \|\mu\|_\Gamma^2 e^{2\alpha(n_0-1)} \leq 0$$

is satisfied for $\mu := \Phi(n_0 - 1, \bar{n})\bar{\mu}$. Dividing by $e^{2\alpha(n_0-1)}$ we get with equation (3.17)

$$0 \geq e^{2\alpha} \|\xi\|_\Gamma^2 - \|\mu\|_\Gamma^2 > \|\xi\|_\Gamma^2 - \|\mu\|_\Gamma^2 = \langle \mu, S_\Gamma(n_0 - 1)\mu \rangle.$$

Conversely, let $\xi \in \mathbb{R}^k$. If ${}_\Phi\mathcal{T}_{\min}(\xi, n_0) = n_0$ then we directly see that $\xi \in {}^{\mathbb{I}}V_s^-(n_0)$. Suppose ${}_\Phi\mathcal{T}_{\min}(\xi, n_0) =: \bar{n} < n_0$. Let $\bar{\mu} \in {}_\Phi\mathcal{T}_{\mathrm{pre}}(\xi, n_0)$ such that the estimate $\langle \mu, S_\Gamma(n_0 - 1)\mu \rangle < 0$ is true for $\mu = \Phi(n_0 - 1, \bar{n})\bar{\mu}$. Then Lemma 5.2.3 leads for all $\hat{n} \in [\bar{n}, n_0 - 1]_\mathbb{Z}$ and $\hat{\mu} \in \mathbb{R}^k$ with $\Phi(n_0 - 1, \hat{n})\hat{\mu} = \mu$ to

$$\langle \Phi(n, \hat{n})\hat{\mu}, S_\Gamma(n)\Phi(n, \hat{n})\hat{\mu} \rangle < 0 \text{ for all } n \in [\hat{n}, n_0 - 1]_\mathbb{Z}. \tag{5.31}$$

Defining

$$a := a_\Gamma(\bar{n}) := \max_{n \in [\bar{n}, n_0-1]_\mathbb{Z}} \left\{ \frac{\langle \Phi(n, \bar{n})\bar{\mu}, S_\Gamma(n)\Phi(n, \bar{n})\bar{\mu} \rangle}{\|\Phi(n+1, \bar{n})\bar{\mu}\|_\Gamma^2} \right\}$$

implies for all $n \in [\bar{n}, n_0 - 1]_\mathbb{Z}$

$$a - \frac{\langle \Phi(n, \bar{n})\bar{\mu}, S_\Gamma(n)\Phi(n, \bar{n})\bar{\mu} \rangle}{\|\Phi(n+1, \bar{n})\bar{\mu}\|_\Gamma^2} \geq 0 \tag{5.32}$$

and we obtain by (5.31) that $a < 0$. Further define

$$\alpha := \alpha_\Gamma(\bar{n}) := \frac{1}{2} \ln\left(1 - a_\Gamma(\bar{n})\right) > 0$$

then

$$-a = e^{2\alpha} - 1. \tag{5.33}$$

For all $n \in [\bar{n}, n_0 - 1]_\mathbb{Z}$ we get with (3.17), with (5.33) and with (5.32)

$$\begin{aligned}
&\|\Phi(n+1, \bar{n})\bar{\mu}\|_\Gamma^2 e^{2\alpha(n+1)} - \|\Phi(n, \bar{n})\bar{\mu}\|_\Gamma^2 e^{2\alpha n} \\
=&e^{2\alpha n} \left(\left(e^{2\alpha} - 1\right) \|\Phi(n+1, \bar{n})\bar{\mu}\|_\Gamma^2 + \|\Phi(n+1, \bar{n})\bar{\mu}\|_\Gamma^2 - \|\Phi(n, \bar{n})\bar{\mu}\|_\Gamma^2 \right) \\
=&e^{2\alpha n} \left(\left(e^{2\alpha} - 1\right) \|\Phi(n+1, \bar{n})\bar{\mu}\|_\Gamma^2 + \langle \Phi(n, \bar{n})\bar{\mu}, S_\Gamma(n)\Phi(n, \bar{n})\bar{\mu} \rangle \right) \\
=&e^{2\alpha n} \|\Phi(n+1, \bar{n})\bar{\mu}\|_\Gamma^2 \left(-a + \frac{\langle \Phi(n, \bar{n})\bar{\mu}, S_\Gamma(n)\Phi(n, \bar{n})\bar{\mu} \rangle}{\|\Phi(n+1, n)\bar{\mu}\|_\Gamma^2} \right) \leq 0.
\end{aligned}$$

This means $\|\Phi(n,\bar{n})\bar{\mu}\|_\Gamma e^{\alpha n}$ is decreasing for $n \in [\bar{n}, n_0]_\mathbb{Z}$. Hence, $\xi \in {}^{\mathbb{I}}V_s^-(n_0)$ and (5.26) is shown.

Before we prove that the cone is a connected double-cone we show the statements (5.27) and (5.29).

For the proof of statement **(5.27)** let $\xi \in {}^{\mathbb{I}}\bar{V}_s(n_0) \setminus \mathcal{N}(\Phi(n_+, n_0))$ and define $\bar{n} := {}_\Phi\mathcal{T}_{\min}(\xi, n_0)$. Then there exists (see equation (4.10)) a $\mu \in \mathbb{R}^k$ such that $\Phi(n_0, \bar{n})\mu = \xi$ and

$$\|\Phi(n,\bar{n})\mu\|_\Gamma < \|\Phi(m,\bar{n})\mu\|_\Gamma \text{ for all } n, m \in [\bar{n}, n_+]_\mathbb{Z}, n > m.$$

With equation (3.17)

$$0 > \|\Phi(n+1,\bar{n})\mu\|_\Gamma^2 - \|\Phi(n,\bar{n})\mu\|_\Gamma^2 = \langle \Phi(n,\bar{n})\mu, S_\Gamma(n)\Phi(n,\bar{n})\mu\rangle$$

follows for all $n \in [\bar{n}, n_+ - 1]_\mathbb{Z}$. For $n = n_+ - 1$ we obtain

$$\begin{aligned} 0 &> \langle \Phi(n_+ - 1, \bar{n})\mu, S_\Gamma(n_+ - 1)\Phi(n_+ - 1, \bar{n}\mu\rangle \\ &= \langle \Phi(n_+ - 1, n_0)\Phi(n_0,\bar{n})\mu, S_\Gamma(n_+ - 1)\Phi(n_+ - 1, n_0)\Phi(n_0,\bar{n})\mu\rangle \\ &= \langle \Phi(n_+ - 1, n_0)\xi, S_\Gamma(n_+ - 1)\Phi(n_+ - 1, n_0)\xi\rangle. \end{aligned}$$

Conversely, let $\xi \in \mathbb{R}^k \setminus \mathcal{N}(\Phi(n_+, n_0))$ with

$$\langle \Phi(n_+ - 1, n_0)\xi, S_\Gamma(n_+ - 1)\Phi(n_+ - 1, n_0)\xi\rangle < 0$$

and define $\bar{n} = {}_\Phi\mathcal{T}_{\min}(\xi, n_0)$. Then Lemma 5.2.3 and equation (3.17) yield

$$0 > \langle \Phi(n,\bar{n})\mu, S_\Gamma(n)\Phi(n,\bar{n})\mu\rangle = \|\Phi(n+1,\bar{n})\mu\|_\Gamma^2 - \|\Phi(n,\bar{n})\mu\|_\Gamma^2$$

for all $\mu \in \mathbb{R}^k$ with $\Phi(n_0,\bar{n})\mu = \xi$ and all $n \in [\bar{n}, n_+ - 1]_\mathbb{Z}$. This means that

$$0 > \|\Phi(n,\bar{n})\mu\|_\Gamma - \|\Phi(m,\bar{n})\mu\|_\Gamma \text{ for all } n, m \in [\bar{n}, n_+]_\mathbb{Z},\ n > m$$

is satisfied and it implies $\xi \in {}^{\mathbb{I}}\bar{V}_s(n_0)$. By Lemma 5.3.3 we have

$$\mathcal{N}(\Phi(n_+, n_0)) \subset {}^{\mathbb{I}}V_s(n_0) \subset {}^{\mathbb{I}}\bar{V}_s(n_0). \tag{5.34}$$

Thus, equation (5.27) is shown. Equation (5.28) directly follows from (5.27) and (5.34).

For the proof of statement **(5.29)** fix $n_0 \in \mathbb{I}_1$ and let $\xi \in {}^{\mathbb{I}}V_u^+(n_0) \setminus \{0\}$. Define $\xi(n) := \Phi(n, n_0)\xi$ for all $n \in [n_0, n_+]_\mathbb{Z}$. Then there exists a $\beta > 0$ such that $\|\xi(n)\|_\Gamma e^{-\beta n}$ is increasing for $n \in [n_0, n_+]_\mathbb{Z}$, hence

$$\|\xi(n+1)\|_\Gamma^2 e^{-2\beta(n+1)} - \|\xi(n)\|_\Gamma^2 e^{-2\beta n} \geq 0$$

is satisfied for all $n \in [n_0, n_+ - 1]_\mathbb{Z}$. Dividing by $e^{-2\beta n}$ implies with (3.17)

$$0 \leq \|\xi(n+1)\|_\Gamma^2 e^{-2\beta} - \|\xi(n)\|_\Gamma^2 < \|\xi(n+1)\|_\Gamma^2 - \|\xi(n)\|_\Gamma^2 = \langle \xi(n), S_\Gamma(n)\xi(n)\rangle$$

for all $n \in [n_0, n_+ - 1]_{\mathbb{Z}}$.

Conversely, let $\xi \in \mathbb{R}^k$ with $\langle \xi, S_\Gamma(n_0)\xi \rangle > 0$. Define $\xi(n) := \Phi(n, n_0)\xi$ for all $n \in [n_0, n_+]_{\mathbb{Z}}$. By Lemma 5.2.3 we obtain

$$\langle \xi(n), S_\Gamma(n)\xi(n) \rangle > 0 \text{ for all } n \in \mathbb{I}_1,\ n \geq n_0. \tag{5.35}$$

Defining

$$b_\Gamma := \min_{n \in [n_0, n_+ - 1]_{\mathbb{Z}}} \left\{ \frac{\langle \xi(n), S_\Gamma(n)\xi(n) \rangle}{\|\xi(n)\|_\Gamma^2} \right\}.$$

we see by (5.35) that $b_\Gamma > 0$. It follows

$$-b_\Gamma + \frac{\langle \xi(n), S_\Gamma(n)\xi(n) \rangle}{\|\xi(n)\|_\Gamma^2} \geq 0 \tag{5.36}$$

for all $n \in [n_0, n_+ - 1]_{\mathbb{Z}}$. Further define $\beta := \beta_\Gamma := \frac{1}{2}\ln(1 + b_\Gamma) > 0$ then

$$-b_\Gamma = 1 - e^{2\beta}. \tag{5.37}$$

For all $n \in [n_0, n_+ - 1]_{\mathbb{Z}}$ we get with (3.17), with (5.36) and with (5.37)

$$\begin{aligned}
&\|\xi(n+1)\|_\Gamma^2 e^{-2\beta(n+1)} - \|\xi(n)\|_\Gamma^2 e^{-2\beta n} \\
=&e^{-2\beta(n+1)} \left((1 - e^{2\beta}) \|\xi(n)\|_\Gamma^2 + \|\xi(n+1)\|_\Gamma^2 - \|\xi(n)\|_\Gamma^2 \right) \\
=&e^{-2\beta(n+1)} \left((1 - e^{2\beta}) \|\xi(n)\|_\Gamma^2 + \langle \xi(n), S_\Gamma(n)\xi(n) \rangle \right) \\
=&e^{-2\beta(n+1)} \|\xi(n)\|_\Gamma^2 \left(-b_\Gamma + \frac{\langle \xi(n), S_\Gamma(n)\xi(n) \rangle}{\|\xi(n)\|_\Gamma^2} \right) \geq 0.
\end{aligned}$$

This means that $\|\xi(n)\|_\Gamma e^{-\beta n}$ is increasing for $n \in [n_0, n_+]_{\mathbb{Z}}$, hence $\xi \in {}^{\mathbb{I}}V_u^+(n_0)$. Thus (5.29) follows.

Next we finish the proof of statement **(i)** by proving that ${}^{\mathbb{I}}V_s^-(n_0)$ is a connected double-cone for all $n_0 \in {}_1\mathbb{I}$.

Fix $n_0 \in {}_1\mathbb{I}$. If the set ${}^{\mathbb{I}}V_s^-(n_0) \setminus \{0\}$ is connected then the half-cones of ${}^{\mathbb{I}}V_s(n_0)$ are connected, i.e. ${}^{\mathbb{I}}V_s^-(n_0)$ is a connected double-cone. Assume that ${}^{\mathbb{I}}V_s^-(n_0) \setminus \{0\}$ is not connected. We need to show that the half-cones of ${}^{\mathbb{I}}V_s^-(n_0)$ are connected. Denote by C one half-cone of ${}^{\mathbb{I}}V_s^-(n_0)$. Then every $x \in {}^{\mathbb{I}}V_s^-(n_0) = C \cup (-C)$ satisfies one of the cases

$$\begin{cases} x \in C, & \text{case (a)}, \\ -x \in C, & \text{case (b)}. \end{cases}$$

For every $x \in C \setminus \{0\}$ and $y \in (-C) \setminus \{0\}$ we have

$$\overline{xy} \not\subset {}^{\mathbb{I}}V_s^-(n_0), \tag{5.38}$$

where $\overline{xy}$ denotes the line between x and y, since ${}^{\mathbb{I}}V_s^-(n_0)\setminus\{0\}$ is not connected. Let $x \in C \setminus \{0\}$ and $y \in {}^{\mathbb{I}}V_s^-(n_0) \setminus \{0\}$. To prove that C is connected we show that one of the two cases is true

$$\begin{cases} \overline{xy} \subset {}^{\mathbb{I}}V_s^-(n_0) \setminus \{0\}, & \text{case (1)}, \\ \overline{x(-y)} \subset {}^{\mathbb{I}}V_s^-(n_0) \setminus \{0\}, & \text{case (2)}. \end{cases} \tag{5.39}$$

This is sufficient, since the two cases above lead by (5.38) to

$$\begin{cases} \overline{xy} \subset C, & \text{in case (1)}, \\ \overline{x(-y)} \subset C, & \text{in case (2)}. \end{cases} \tag{5.40}$$

The cases (5.40) state that for every two points $x, y \in C$ there exists a path between these two points which lies in the cone C, i.e. C is connected.

The cases in (5.39) are true, if

$$\begin{cases} tx + (1-t)y \in {}^{\mathbb{I}}V_s^-(n_0) \setminus \{0\}, & \text{for case (1)}, \\ tx - (1-t)y \in {}^{\mathbb{I}}V_s^-(n_0) \setminus \{0\}, & \text{for case (2)}. \end{cases} \tag{5.41}$$

is satisfied for all $t \in [0,1]$.

In the following we show (5.41) first for ${}_\Phi\mathcal{T}_{\min}(x, n_0) = n_0$ and then for ${}_\Phi\mathcal{T}_{\min}(x, n_0) < n_0$. Fix $t \in (0,1)$. For

$${}_\Phi\mathcal{T}_{\min}(x, n_0) = n_0 \tag{5.42}$$

suppose both cases in (5.41) are not true then there exist $\mu_{1,2} \in \mathbb{R}^k$ with

$$\begin{aligned} tx + (1-t)y &= \Phi(n_0, n_0 - 1)\mu_1, \\ tx - (1-t)y &= \Phi(n_0, n_0 - 1)\mu_2, \end{aligned}$$

i.e. ${}_\Phi\mathcal{T}_{\min}(tx \pm (1-t)y, n_0) < n_0$. We get

$$tx = \frac{1}{2}(tx + (1-t)y + tx - (1-t)y) = \Phi(n_0, n_0 - 1)\frac{1}{2}(\mu_1 + \mu_2),$$

which is a contradiction to (5.42). Thus, one of the cases in (5.41) is true.

Assume

$${}_\Phi\mathcal{T}_{\min}(x, n_0) < n_0.$$

Then there exists $\mu_1 \in \mathbb{R}^k$ with

$$\Phi(n_0, n_0 - 1)\mu_1 = x \text{ such that } \langle \mu_1, S_\Gamma(n_0 - 1)\mu_1 \rangle < 0. \tag{5.43}$$

If ${}_\Phi\mathcal{T}_{\min}(tx + (1-t)y, n_0) = n_0$ then the first case of (5.41) is true. So suppose there exists a $\mu \in \mathbb{R}^k$ with

$$tx + (1-t)y = \Phi(n_0, n_0 - 1)\mu,$$

i.e. ${}_{\Phi}\mathcal{T}_{\min}(tx + (1-t)y, n_0) < n_0$. By (5.43) we obtain $(1-t)y = \Phi(n_0, n_0 - 1)(\mu - t\mu_1)$. This implies

$$ {}_{\Phi}\mathcal{T}_{\min}(y, n_0) < n_0, $$

which induces with $y \in {}^{\mathbb{I}}V_s^-(n_0) \setminus \{0\}$ that there exists $\mu_2 \in \mathbb{R}^k$ with

$$ \Phi(n_0, n_0 - 1)y = \mu_2 \text{ such that } \langle \mu_2, S_\Gamma(n_0 - 1)\mu_2 \rangle < 0. \tag{5.44} $$

Define

$$ a := t(1-t)\left(\langle \mu_1, S_\Gamma(n_0 - 1)\mu_2 \rangle + \langle \mu_2, S_\Gamma(n_0 - 1)\mu_1 \rangle\right). \tag{5.45} $$

We have with (5.43) and (5.44)

$$ \begin{aligned} &\begin{cases} \langle t\mu_1 + (1-t)\mu_2, S_\Gamma(n_0 - 1)(t\mu_1 + (1-t)\mu_2)\rangle, & a \le 0, \\ \langle t\mu_1 - (1-t)\mu_2, S_\Gamma(n_0 - 1)(t\mu_1 - (1-t)\mu_2)\rangle, & a > 0, \end{cases} \\ =&\begin{cases} t^2\langle \mu_1, S_\Gamma(n_0 - 1)\mu_1\rangle + (1-t)^2\langle \mu_2, S_\Gamma(n_0 - 1)\mu_2\rangle + a, & a \le 0, \\ t^2\langle \mu_1, S_\Gamma(n_0 - 1)\mu_1\rangle + (1-t)^2\langle \mu_2, S_\Gamma(n_0 - 1)\mu_2\rangle - a, & a > 0, \end{cases} \\ <&0. \end{aligned} \tag{5.46} $$

This implies one of the cases in (5.41) with $tx \pm (1-t)y = \Phi(n_0, n_0 - 1)(t\mu_1 \pm (1-t)\mu_2)$. Thus ${}^{\mathbb{I}}V_s^-(n_0)$ is a connected double-cone for every $n_0 \in {}_1\mathbb{I}$. Now we complete the proof of statements **(ii)** and **(iii)** by showing that

$$ L(n_0) := \left\{\xi \in \mathbb{R}^k | \langle \Phi(n_+ - 1, n_0)\xi, S_\Gamma(n_+ - 1)\Phi(n_+ - 1, n_0)\xi\rangle < 0\right\} $$

is an open connected double-cone for all $n_0 \in \mathbb{I}_1$, since the proof that ${}^{\mathbb{I}}V_u^+(n_0)$ is an open connected double-cone for all $n_0 \in \mathbb{I}_1$ follows analogously to the one of $L(n_0)$.

Fix $\bar{n} \in \mathbb{I}_1$ and let $x, y \in L(\bar{n})$. Define $\mu_1 := \Phi(n_+ - 1, \bar{n})x$ and $\mu_2 := \Phi(n_+ - 1, \bar{n})y$ and a as in (5.45) with the setting $n_0 := n_+$. Then as in (5.46) we get

$$ \begin{aligned} &\langle \Phi(n_+ - 1, \bar{n})(tx \pm (1-t)y), S_\Gamma(n_+ - 1)\Phi(n_+ - 1, \bar{n})(tx \pm (1-t)y)\rangle \\ =&\langle t\mu_1 + (1-t)\mu_2, S_\Gamma(n_+ - 1)(t\mu_1 + (1-t)\mu_2)\rangle < 0, \end{aligned} $$

for $+$ or $-$, i.e. $L(\bar{n})$ is a connected double-cone. That the cone is open follows by the continuity of $\Phi(\cdot,\cdot)$ and $\langle \cdot,\cdot\rangle$, more precisely there exists an $\varepsilon > 0$ such that for all $\xi \in B_\varepsilon(x)$

$$ \langle \Phi(n_+ - 1, \bar{n})\xi, S_\Gamma(n_+ - 1)\Phi(n_+ - 1, \bar{n})\xi\rangle < 0 $$

holds, i.e. $L(n_0)$ is an open connected double-cone for all $n_0 \in \mathbb{I}_1$.

The relation **(5.30)** holds directly, since the sets

$$ \{\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(n_0)\xi\rangle > 0\} \text{ and } \{\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(n_0)\xi\rangle < 0\} $$

are open. □

For the stable and unstable cones of a discrete D-hyperbolic system we get an explicit representation by Lemma 4.2.1 and Proposition 5.3.4. The representation of the boundaries can be shown as in the continuous case (Corollary 5.3.2). First we define for a subset $U \subset \mathbb{R}^k$ and $n, m \in \mathbb{I}$ with $n \geq m$

$$\Phi_{\text{pre}}(n,m)U := \left\{\xi \in \mathbb{R}^k | \Phi(n,m)\xi \in U\right\}.$$

Corollary 5.3.5. *Under the assumptions of Proposition 5.3.4 we obtain that the almost stable n_+-cone is an open connected double-cone with the representation*

$$\begin{aligned}{}^{\mathbb{I}}\bar{V}_s(n_+) =& \mathcal{R}(\Phi(n_+, n_+ - 1))^C \\ & \cup \{\xi \in \mathbb{R}^k |_{\Phi}\mathcal{T}_{\min}(\xi, n_+) =: \bar{n} < n_+ \wedge \exists \bar{\mu} \in {}_{\Phi}\mathcal{T}_{\text{pre}}(\xi, n_+) : \\ & \quad \langle \mu, S_\Gamma(n_+ - 1)\mu\rangle < 0 \text{ for } \mu := \Phi(n_+ - 1, \bar{n})\bar{\mu}\} \cup \{0\}.\end{aligned}$$

For all $n_0 \in \mathbb{I}_1$ we get that the almost stable n_0-cone is a double-cone with the representation

$${}^{\mathbb{I}}\bar{V}_s(n_0) = L(n_0)\dot{\cup}\mathcal{N}(\Phi(n_+ - 1, n_0))$$

and additionally that the stable n_0-cone satisfies

$${}^{\mathbb{I}}V_s(n_0) = \left\{\xi \in L(n_0) \middle| \Phi(n_0 + 1, n_0)\xi \in {}^{\mathbb{I}}V_s(n_0 + 1)\right\} \dot{\cup}\mathcal{N}(\Phi(n_+ - 1, n_0)). \tag{5.47}$$

For all $n_0 \in \mathbb{I}$ the unstable cone is an open connected double-cone, which can be represented by

$${}^{\mathbb{I}}V_u(n_0) = \left\{\Phi(n_0, n_-)\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(n_-)\xi\rangle > 0\right\} \cup \{0\}. \tag{5.48}$$

The boundary of the unstable cone satisfies for all $n_0 \in \mathbb{I}$

$$\partial^{\mathbb{I}}V_u(n_0) = \Phi(n_0, n_-)\partial^{\mathbb{I}}V_u(n_-) \subset \Phi(n_0, n_-)Z_\Gamma(n_-), \tag{5.49}$$

whereas the boundary of the almost stable cone satisfies for $n_0 \in \mathbb{I}_1$

$$\partial^{\mathbb{I}}\bar{V}_s(n_0) \subset \Phi_{\text{pre}}(n_+ - 1, n_0)Z_\Gamma(n_+ - 1).$$

Let (2.8) be invertible then the stable cones ${}^{\mathbb{I}}V_s(n)$ are open connected double-cones for all $n \in \mathbb{I}$ and

$${}^{\mathbb{I}}V_s(n_+) = \Phi(n_+, n_+ - 1)L(n_+ - 1) \cup \{0\}, \tag{5.50}$$

$${}^{\mathbb{I}}V_s(n_0) = {}^{\mathbb{I}}\bar{V}_s(n_0) = L(n_0) \cup \{0\}, \tag{5.51}$$

$$\partial^{\mathbb{I}}V_s(n_0) \subset \Phi(n_0, n_+ - 1)Z_\Gamma(n_+ - 1) \tag{5.52}$$

hold for all $n_0 \in \mathbb{I}$.

Proof. By Proposition 5.3.4 the almost stable n_+-cone is an open connected double-cone, which can also be represented as

$$\begin{aligned}
{}^{\mathbb{I}}\bar{V}_s(n_+) =& {}^{\mathbb{I}}V_s^-(n_+) \\
=& \big\{\xi \in \mathbb{R}^k | {}_{\Phi}\mathcal{T}_{\min}(\xi, n_0) = n_0 \vee \big({}_{\Phi}\mathcal{T}_{\min}(\xi, n_0) =: \bar{n} < n_0 \\
& \qquad \wedge \exists \bar{\mu} \in {}_{\Phi}\mathcal{T}_{\text{pre}}(\xi, n_0) : \langle \mu, S_\Gamma(n_0 - 1)\mu \rangle < 0 \\
& \qquad \text{for } \mu := \Phi(n_0 - 1, \bar{n}) = \bar{\mu}\big)\big\} \cup \{0\} \\
=& \{\xi \in \mathbb{R}^k | {}_{\Phi}\mathcal{T}_{\min}(\xi, n_+) = n_+\} \\
& \cup \{\xi \in \mathbb{R}^k | {}_{\Phi}\mathcal{T}_{\min}(\xi, n_+) =: \bar{n} < n_+ \wedge \exists \bar{\mu} \in {}_{\Phi}\mathcal{T}_{\text{pre}}(\xi, n_+) : \\
& \quad \langle \mu, S_\Gamma(n_+ - 1)\mu \rangle < 0 \text{ for } \mu := \Phi(n_+ - 1, \bar{n})\bar{\mu}\} \cup \{0\}, \\
=& \mathcal{R}(\Phi(n_+, n_+ - 1))^C \\
& \cup \{\xi \in \mathbb{R}^k | {}_{\Phi}\mathcal{T}_{\min}(\xi, n_+) =: \bar{n} < n_+ \wedge \exists \bar{\mu} \in {}_{\Phi}\mathcal{T}_{\text{pre}}(\xi, n_+) : \\
& \quad \langle \mu, S_\Gamma(n_+ - 1)\mu \rangle < 0 \text{ for } \mu := \Phi(n_+ - 1, \bar{n})\bar{\mu}\} \cup \{0\}.
\end{aligned}$$

For all $n_0 \in \mathbb{I}_1$ we see by Proposition 5.3.4 and Lemma 5.2.5 that the almost n_0-stable cone is a double-cone with the representation

$$\begin{aligned}
{}^{\mathbb{I}}\bar{V}_s(n_0) =& \big\{\xi \in \mathbb{R}^k | \langle \Phi(n_+ - 1, n_0)\xi, S_\Gamma(n_+ - 1)\Phi(n_+ - 1, n_0)\xi \rangle < 0\big\} \\
& \cup \mathcal{N}(\Phi(n_+, n_0)) \\
=& \big\{\xi \in \mathbb{R}^k | \langle \Phi(n_+ - 1, n_0)\xi, S_\Gamma(n_+ - 1)\Phi(n_+ - 1, n_0)\xi \rangle < 0\big\} \\
& \dot{\cup} \mathcal{N}(\Phi(n_+ - 1, n_0)) \\
=& L(n_0) \dot{\cup} \mathcal{N}(\Phi(n_+ - 1, n_0))
\end{aligned}$$

Indeed, for $n_0 \in \mathbb{I}_1$ and $\xi \in \mathcal{N}(\Phi(n_+, n_0))$ we have

$$\langle \Phi(n_+, n_+ - 1)\Phi(n_+ - 1, n_0)\xi, S_\Gamma(n_+)\Phi(n_+, n_+ - 1)\Phi(n_+ - 1, n_0)\xi \rangle = 0$$

and Lemma 5.2.5 yields

$$\langle \Phi(n_+ - 1, n_0)\xi, S_\Gamma(n_+ - 1)\Phi(n_+ - 1, n_0)\xi \rangle < 0$$

if $\Phi(n_+ - 1, n_0)\xi \neq 0$, i.e. $\xi \notin \mathcal{N}(\Phi(n_+ - 1, n_0))$. Recursively, we obtain the representation (5.47) of the stable n_0-cone for all $n_0 \in \mathbb{I}_1$.

For all $n_0 \in \mathbb{I}$ the unstable cone is an open connected double-cone and (5.48) holds, since ${}^{\mathbb{I}}V_u(\cdot)$ is invariant and ${}^{\mathbb{I}}V_u^-$ an open connected double-cone by Proposition 5.3.4. The boundary of ${}^{\mathbb{I}}V_u(\cdot)$ fulfills (5.49) by the latter arguments. For the boundary of the almost stable cone we have for $n_0 \in \mathbb{I}_1$

$$\begin{aligned}
\partial {}^{\mathbb{I}}\bar{V}_s(n_0) \subset& \{\xi \in \mathbb{R}^k | \langle \Phi(n_+ - 1, n_0)\xi, S_\Gamma(n_+ - 1)\Phi(n_+ - 1, n_0)\xi \rangle = 0\} \\
=& \{\xi \in \mathbb{R}^k | \Phi(n_+ - 1, n_0)\xi \in Z_\Gamma(n_+ - 1)\} \\
=& \Phi_{\text{pre}}(n_+ - 1, n_0) Z_\Gamma(n_+ - 1).
\end{aligned}$$

Let (2.8) be invertible then we obtain

$$\begin{aligned}{}^{\mathbb{I}}V_s(n_+) &= \left\{\xi \in \mathbb{R}^k \middle| \langle \mu, S_\Gamma(n_+ - 1)\mu\rangle < 0 \text{ for } \mu := \Phi(n_+ - 1, n_+)\xi\right\} \cup \{0\} \\ &= \Phi(n_+, n_+ - 1)L(n_+ - 1) \cup \{0\}\end{aligned}$$

and for all $n_0 \in \mathbb{I}_1$ we have

$${}^{\mathbb{I}}\bar{V}_s(n_0) = L(n_0) \cup \{0\}.$$

Recursively, this yields

$${}^{\mathbb{I}}V_s(n_0) = {}^{\mathbb{I}}\bar{V}_s(n_0) = L(n_0) \cup \{0\}$$

for all $n_0 \in \mathbb{I}_1$ and thus the boundary of the stable cone satisfies (5.52). By Proposition 5.3.4 the set $L(n_+ - 1)$ is a open connected double-cone. The continuity of $\Phi(\cdot, \cdot)$ implies by (5.50) and (5.51) that the stable cones ${}^{\mathbb{I}}V_s(n)$ are open connected double-cones for all $n \in \mathbb{I}$. □

In contrast to invertible systems the stable cone of a noninvertible system is usually not an open cone. Indeed, we proved for all $n_0 \in \mathbb{I}_1$ that ${}^{\mathbb{I}}V_s(n_0)$ is the disjunct union of a subset of an open connected double-cone and a closed double-cone, cf. (5.47).

The explicit form of ${}^{\mathbb{I}}\bar{V}_s(n_+)$ turns out to be quite complicated. Roughly speaking we start with points at the end time n_+ then find special points to a previous time and finally go back to the time n_+. Is this necessary? What happens if we start at the "first" time and then just go forward in time to n_+. We introduce the sets

$$\begin{aligned}A :=& \left\{\xi \in \mathbb{R}^k \middle| {}_\Phi\mathcal{T}_{\min}(\xi, n_+) = \bar{n} < n_+ \wedge \exists \bar{\mu} \in \mathbb{R}^k \text{ with } \bar{\mu} \in {}_\Phi\mathcal{T}_{\text{pre}}(\xi, n_+) : \right. \\ & \qquad \left. \langle \mu, S_\Gamma(n_+ - 1)\mu\rangle < 0 \text{ for } \mu = \Phi(n_+ - 1, \bar{n})\bar{\mu}\right\} \\ =& {}^{\mathbb{I}}V_s(n_+) \setminus \{\xi \in \mathbb{R}^k | {}_\Phi\mathcal{T}_{\min}(\xi, n_+) = n_+\} \\ B :=& \bigcup_{n \in \mathbb{I}_1} \left\{\Phi(n_+, n)\xi \in \mathbb{R}^k \middle| {}_\Phi\mathcal{T}_{\min}(\xi, n) = n, \right. \\ & \qquad \left. \langle \mu, S_\Gamma(n_+ - 1)\mu\rangle < 0 \text{ for } \mu = \Phi(n_+ - 1, n)\xi\right\}.\end{aligned}$$

In Figure 5.1 we illustrate for an example that the sets A and B may be different for noninvertible systems. To find all points in A we start with all point at the end time n_+, marked in green on the left panel. Then we take all preimages to the "minimal" time each, i.e. to time ${}_\Phi\mathcal{T}_{\min}(\cdot, n_+)$. These points are marked in the left half of Figure 5.1 by a red circle. Then we map the points with Φ to time $n_+ - 1$, marked by a red square. Every point μ at time $n_+ - 1$, which satisfies $\langle \mu, S_\Gamma(n_+ - 1)\mu\rangle < 0$ is plotted in blue. Thus, every green point, which is connected to a blue point with a red square lies in A. Hence, the two green points encircled in black lie in A. The right panel of Figure 5.1 shows how we find all points in the set B. Therefore, we start by marking all vectors ξ, which satisfy ${}_\Phi\mathcal{T}_{\min}(\xi, n) = n$ for an $n \in [n_-, n_+ - 1]_{\mathbb{Z}}$.

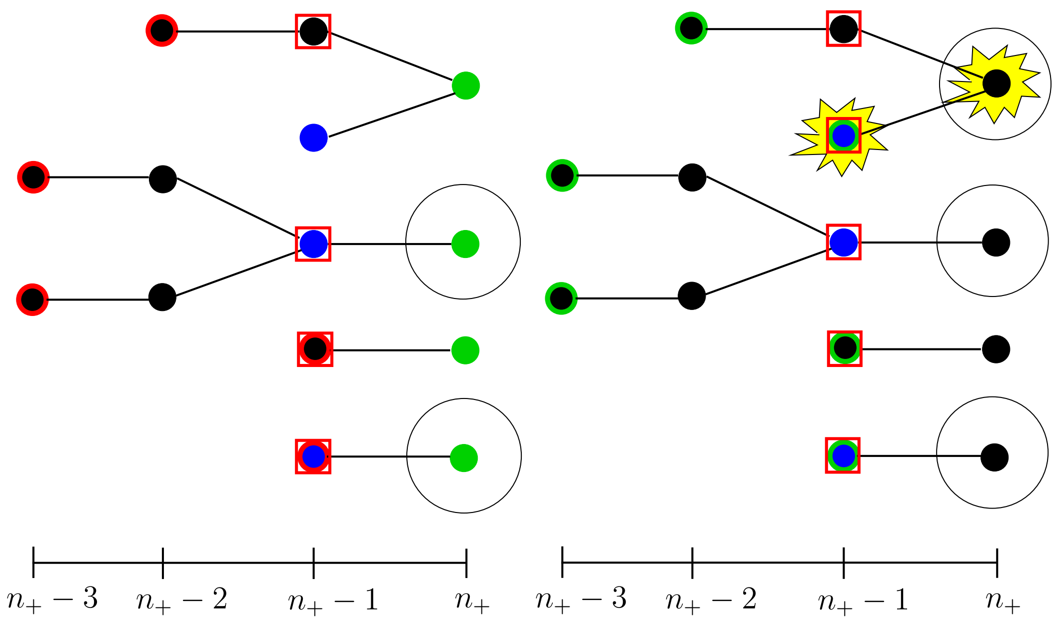


Figure 5.1: Example to illustrate that the sets A and B are not necessary the same sets.

These points are encircled in green. The next step is to map these points with Φ to time $n_+ - 1$, marked in the right half of Figure 5.1 by a red square. Again, all boundary points, which are connected to a blue point marked by a red square lie in B. Hence, the tree points encircled in black lie in B. We see that there exists one more point in the set B, highlighted in yellow, then in set A. This point is not part of a "maximal long" solution with decreasing norm. Thus, the point marked in yellow is not part of ${}^{\mathbb{I}}V_s(n_+)$. This implies that the representation of the stable cone can not be simplified with the help of set B.

M-Hyperbolicity and D-Hyperbolicity

Before we study various examples in Section 5.5 to see how stable and unstable cones can look like we prove that every D-hyperbolic system is also M-hyperbolic. To obtain this result implied by Theorem 5.4.2 we first show some properties of the dimensions of stable and unstable cones. Related results for the continuous time case can be found in [43, Theorem 14, Theorem 21 and Remark 16].

Lemma 5.4.1. *Let* $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{I}$, $\mathbb{I} = [t_-, t_+]_\mathbb{T}$ *and* $\Gamma = \Gamma^T > 0$. *Assume that equation* (2.8) *is D-hyperbolic* ($\mathbb{T} = \mathbb{Z}$) *resp. assume* **(A0)** *and* *that equation* (2.7) *is D-hyperbolic* ($\mathbb{T} = \mathbb{R}$). *Then there exist subspaces*

$$\{0\} \neq \bar{U} \subset {}^{\mathbb{I}}V_u(t_-) \text{ and } \{0\} \neq \bar{S} \subset \begin{cases} {}^{\mathbb{I}}V_s(t_+), & \text{for } \mathbb{T} = \mathbb{R}, \\ {}^{\mathbb{I}}V_s(t_+ - 1), & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}$$

such that

$$\dim(\bar{U}) + \dim(\bar{S}) = k$$

holds. Further, there exist subspaces

$$\{0\} \neq U(t) \subset {}^{\mathbb{I}}V_u(t) \text{ and } \{0\} \neq S(t) \subset {}^{\mathbb{I}}V_s(t)$$

such that

$$U(t) \oplus S(t) = \mathbb{R}^k \text{ and } \dim(U(t)) = \dim(\bar{U}),\ \dim(S(t)) = \dim(\bar{S})$$

hold for all $t \in \mathbb{I}$.

If (2.7) *is not invertible then there exist subspaces*

$$S'(t), N(t) \subset {}^{\mathbb{I}}V_s(t) \text{ with } N(t) := \mathcal{N}(\Phi(t_+ - 1, t)) \neq \{0\}$$

such that

$$S(t) = S'(t) \oplus N(t),\ t \in \mathbb{I}_1.$$

Proof. By Definition 3.2.8 $S_\Gamma(t)$ is symmetric for all $t \in \mathbb{I}$. This yields that all eigenvalues lie in $\mathbb{R}$, see [5, p. 692]. The D-hyperbolicity of (2.7)/(2.8) implies that $S_\Gamma(t)$ is nondegenerate for all $t \in \mathbb{I}$, i.e. no eigenvalue of $S_\Gamma(t)$ equals 0. Additionally, the number of positive eigenvalues ($0 < d < k$) is constant in time. The definition of D-hyperbolicity yields this directly for the discrete case, whereas the same follows for the continuous time case by the continuity of $S_\Gamma(\cdot)$ and that $S_\Gamma(t)$ is nondegenerate for all $t \in \mathbb{I}$. Define

$$\bar{t}_+ := \begin{cases} t_+, & \text{for } \mathbb{T} = \mathbb{R}, \\ t_+ - 1, & \text{for } \mathbb{T} = \mathbb{Z}. \end{cases}$$

Let $\mu_1, \dots, \mu_d$ denote the positive eigenvalues of $S_\Gamma(\bar{t}_+)$ and $\lambda_1, \dots, \lambda_d$ of $S_\Gamma(t_-)$. Further let $\mu_{d+1}, \dots, \mu_k$ denote the negative eigenvalues of $S_\Gamma(\bar{t}_+)$ and $\lambda_{d+1}, \dots, \lambda_k$ of $S_\Gamma(t_-)$. Due to the symmetry of $S_\Gamma(t)$ for all $t \in \mathbb{I}$ there exist orthogonal matrices $Q, R \in \mathbb{R}^{k \times k}$ such that

$$S_\Gamma(t_-) = Q^T \operatorname{diag}(\lambda_1, \dots, \lambda_k) Q,$$
$$S_\Gamma(\bar{t}_+) = R^T \operatorname{diag}(\mu_1, \dots, \mu_k) R$$

With Proposition 5.3.1/5.3.4 we obtain

$$\begin{aligned} {}^{\mathbb{I}}V_u(t_-) = {}^{\mathbb{I}}V_u^+(t_-) &= \left\{\xi \in \mathbb{R}^k \middle| \langle \xi, Q^T \operatorname{diag}(\lambda_1, \dots, \lambda_k) Q\xi \rangle > 0\right\} \cup \{0\} \\ &= \left\{Q^T\xi \middle| \xi \in \mathbb{R}^k : \langle \xi, \operatorname{diag}(\lambda_1, \dots, \lambda_k)\xi \rangle > 0\right\} \cup \{0\}, \end{aligned}$$

for $\mathbb{T}=\mathbb{R}$ we get

$$^{\mathbb{I}}V_s(\bar{t}_+) = {}^{\mathbb{I}}V_s^-(t_+) = \left\{R^T\xi \middle| \xi \in \mathbb{R}^k : \langle \xi, \operatorname{diag}(\mu_1,\dots,\mu_k)\xi\rangle < 0\right\} \cup \{0\}$$

and for $\mathbb{T}=\mathbb{Z}$ we have

$$\begin{aligned} ^{\mathbb{I}}V_s(\bar{t}_+) = {}^{\mathbb{I}}\bar{V}_s(t_+-1) &= \left\{\xi \in \mathbb{R}^k \middle| \langle R\xi, \operatorname{diag}(\mu_1,\dots,\mu_k)R\xi\rangle < 0\right\} \cup \{0\} \\ &= \left\{R^T\xi \middle| \xi \in \mathbb{R}^k : \langle \xi, \operatorname{diag}(\mu_1,\dots,\mu_k)\xi\rangle < 0\right\} \cup \{0\}. \end{aligned}$$

By the orthogonality of Q, R and Lemma 4.2.1 it follows that

$$\bar{U} := \left\{Q^T(\xi_1,\dots,\xi_d,0,\dots,0)^T \middle| \xi_1,\dots,\xi_d \in \mathbb{R}\right\} \subset {}^{\mathbb{I}}V_u(t_-)$$

is a d-dimensional subspace of $^{\mathbb{I}}V_u(t_-)$ and that

$$\bar{S} := \left\{R^T(0,\dots,0,\xi_{d+1},\dots,\xi_k)^T \middle| \xi_{d+1},\dots,\xi_k \in \mathbb{R}\right\} \subset {}^{\mathbb{I}}V_s(\bar{t}_+)$$

is a $(k-d)$-dimensional subspace of $^{\mathbb{I}}V_s(\bar{t}_+)$. For the dimensions we obtain

$$\dim(\bar{U}) + \dim(\bar{S}) = d + (k-d) = k.$$

For invertible systems, Lemma 4.2.1 yields

$$\begin{aligned} U(t) &:= \Phi(t,t_-)\bar{U} \subset \Phi(t,t_-)^{\mathbb{I}}V_u(t_-) = {}^{\mathbb{I}}V_u(t), \\ S(t) &:= \Phi(t,t_+)\bar{S} \subset \Phi(t,t_+)^{\mathbb{I}}V_s(t_+) = {}^{\mathbb{I}}V_s(t) \end{aligned}$$

for every $t \in \mathbb{I}$. Hence, we see

$$\{0\} \subset U(t) \cap S(t) \subset {}^{\mathbb{I}}V_u(t) \cap {}^{\mathbb{I}}V_s(t) = \{0\}.$$

Since $\Phi(\cdot,\cdot)$ is nonsingular for invertible systems the dimension of $U(t)$ and $S(t)$ equals the dimension of $\bar{U}$ and $\bar{S}$, respectively. Combining these results we get

$$\begin{aligned} \dim(U(t)+S(t)) &= \dim(U(t)) + \dim(S(t)) - \dim(U(t)\cap S(t)) \\ &= \dim(\bar{U}) + \dim(\bar{S}) - \dim(\{0\}) = k. \end{aligned}$$

This leads to $S(t) \oplus U(t) = \mathbb{R}^k$ for every $t \in \mathbb{I}$.

If (2.6) is noninvertible we find for all $t \in \mathbb{I}_2$ a subspace

$$\begin{aligned} S'(t) =& \{\bar{\xi} \in \mathbb{R}^k | \Phi(t_+-1,t)\bar{\xi} := R^T(0,\dots,0,\xi_{d+1},\dots,\xi_k)^T, \\ & \xi_i \in \mathbb{R} \text{ for } i \in \{d+1,\dots,k\} \text{ and not all } \xi_i = 0\} \cup \{0\} \\ \subset& ^{\mathbb{I}}\bar{V}_s(t) \end{aligned}$$

with

$$\dim(S'(t)) = k - d - \dim(\mathcal{N}(\Phi(t_+-1,t))), \tag{5.53}$$

$$S'(t) \cap \mathcal{N}(\Phi(t_+-1,t)) = \{0\}. \tag{5.54}$$

We observe that

$$\Phi(t+1,t)S'(t) \subset S'(t+1) \text{ and } \Phi(t_+-1,t_+-2)S'(t_+-2) \subset \bar{S}$$

for all $t \in \mathbb{I}_3$. This implies

$$S'(t) \subset {}^{\mathbb{I}}V_s(t)$$

for all $t \in \mathbb{I}_1$. Lemma 4.2.1 leads to

$$U(t) := \Phi(t,t_-)\bar{U} \subset \Phi(t,t_-)^{\mathbb{I}}V_u(t_-) = {}^{\mathbb{I}}V_u(t)$$

and by Lemma 5.3.3 and (4.21) we obtain

$$U(t) \cap (S'(t) \oplus N(\Phi(t_+-1,t)) \subset {}^{\mathbb{I}}V_u(t) \cap {}^{\mathbb{I}}V_s(t) = \{0\} \tag{5.55}$$

for all $t \in \mathbb{I}_1$ and further we get

$$\begin{aligned} d = \dim(\bar{U}) &= \dim(\Phi(t,t_-)\bar{U}) + \dim(\mathcal{N}(\Phi(t,t_-)) \cap \bar{U}) \\ &= \dim(U(t)) + 0 = \dim(U(t)) \end{aligned} \tag{5.56}$$

Combining (5.53) and (5.56) we have for every $t \in \mathbb{I}_1$

$$\begin{aligned} &\dim(S'(t)) + \dim(\mathcal{N}(\Phi(t_+-1,t))) + \dim(U(t)) \\ =&k - d - \dim(\mathcal{N}(\Phi(t_+-1,t))) + \dim(\mathcal{N}(\Phi(t_+-1,t))) + d = k. \end{aligned}$$

This together with (5.54), (5.55) implies for all $t \in \mathbb{I}_1$

$$U(t) \oplus S'(t) \oplus N(t) = \mathbb{R}^k,$$

where $N(t) = \mathcal{N}(\Phi(t_+-1,t))$. For t_+ and noninvertible systems we have ${}^{\mathbb{I}}V_s(t_+) = \mathbb{R}^k$. Thus, there exist a subspace $S \subset {}^{\mathbb{I}}V_s(t_+)$ such that $U(t_+) \oplus S = \mathbb{R}^k$ holds. □

Finally we show that every D-hyperbolic system is also M-hyperbolic. This result can be found in [43, Corollary 22] for continuous ft-systems. Doan, Palmer and Siegmund additionally proved in [43, Theorem 21] that every continuous dynamical system is M-hyperbolic if the system is either attracting, repelling or quasihyperbolic, see Proposition 5.4.3.

Theorem 5.4.2. *Let* $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$ *and* $\mathbb{I} = [t_-, t_+]_{\mathbb{T}}$. *Let* $\Gamma = \Gamma^T > 0$ *and assume for the continuous case* **(A0)**. *Let system* (2.6) *generated by* (2.7)/(2.8) *be D-hyperbolic w.r.t. the* Γ*-norm. Then system* (2.6) *is M-hyperbolic w.r.t. the* Γ*-norm.*

Proof. By Lemma 5.4.1 there exist subspaces $S(t) \subset {}^{\mathbb{I}}V_s(t)$ and $U(t) \subset {}^{\mathbb{I}}V_u(t)$ for all $t \in \mathbb{I}$ with $S(t) \oplus U(t) = \mathbb{R}^k$. Define the family of projectors by

$$\mathcal{R}(P(t)) := \begin{cases} \Phi(t,t_-)S(t_-), & \text{for } \mathbb{T} = \mathbb{R}, \\ \begin{cases} S(t_-), & \text{for } t = t_-, \\ \Phi(t,t-1)\mathcal{R}(P(t-1)) + W(t), & \text{for } t > t_- \end{cases}, & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}$$
$$\subset {}^{\mathbb{I}}V_s(t),$$

where $W(t) \subset {}^{\mathbb{I}}V_s(t)$ such that

$$\dim(\mathcal{R}(P(t))) = \dim(S(t))$$

and by

$$\mathcal{N}(P(t)) := \Phi(t,t_-)U(t_-) \subset \Phi(t,t_-)^{\mathbb{I}}V_u(t_-) = {}^{\mathbb{I}}V_u(t)$$

for all $t \in \mathbb{I}$. Then Lemma 2.2.5 yields that this family is invariant, since $\dim(\mathcal{N}(P(t))) = \dim(U(t_-)) = k - \dim(S(t_-)) = k - \dim(\mathcal{R}(P(t))$ for all $t \in \mathbb{I}$. Thus, by Lemma 4.2.3 system (2.6) is M-hyperbolic. □

Corollary 5.4.3. *Let* $\mathbb{I} = [t_-, t_+]$, $\Gamma = \Gamma^T > 0$. *Assume* **(A0)**. *If system* (2.4) *is repelling, attracting, hyperbolic or quasihyperbolic according to Definition 5.2.6 then it is also M-hyperbolic.*

Proof. If $S_\Gamma(t)$ is positive (negative) definite, then we directly obtain by (3.16) that there exists an $\alpha > 0$ ($\beta > 0$) such that $\|\Phi(t,s)\xi\|_\Gamma \leq e^{\alpha(t-s)}\|\xi\|_\Gamma$ ($\|\Phi(t,s)\xi\|_\Gamma \leq e^{-\beta(t-s)}\|\xi\|_\Gamma$) is satisfied for all $\xi \in \mathbb{R}^k$, $t,s \in \mathbb{I}$ with $t \geq s$ ($t \leq s$), since $\mathbb{I}$ is a compact interval. Thus, system (2.4) is M-hyperbolic.

Otherwise if (2.4) is hyperbolic then Theorem 5.4.2 implies the M-hyperbolicity of (2.4).

For quasihyperbolic systems analog statements as the ones in Proposition 5.3.1 exist, see [43, Proposition 20], and the M-hyperbolicity follows similar to Theorem 5.4.2. For a proof we refer to [43, Theorem 21]. □

Examples of 2-Dimensional D-Hyperbolic Systems

In this section we study different types of two-dimensional D-hyperbolic ft-systems and plot their stable and unstable cones. The stable and unstable subspaces of the extended ift-system on $\mathbb{R}$ or $\mathbb{Z}$ are pictured for comparison in addition. We start with continuous autonomous time systems and move on with a nonautonomous continuous ft-system.

For discrete systems it is possible that the solution operator is not invertible. Therefore, we plot the cones of two invertible and of one noninvertible discrete autonomous time systems as well as of one nonautonomous noninvertible ft-system. In the next section we analyze the width of the cones. Meanwhile, we have an eye on the decay of the boundary distance in the following examples.

Example 5.5.1. *Consider the system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =: A \begin{pmatrix} x \\ y \end{pmatrix} \tag{5.57}$$

for $t \in \mathbb{I} = [0, \pi]$ and let Γ be the identity. We see that $A \in \mathcal{C}^1(\mathbb{I}, \mathbb{R}^{2\times 2})$ and we obtain that the symmetric strain tensor, defined in Definition 3.2.8, $S = S(t) = \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}$ is indefinite and nondegenerate for each $t \in \mathbb{I}$. The matrix $M(t) = SA + AS = 2A^2 = \begin{pmatrix} 2 & 0 \\ 0 & 32 \end{pmatrix}$, defined in Definition 5.2.1, is positive definite. Hence, the system is D-hyperbolic and Proposition 5.3.1 applies. The solution operator of (5.57) *is*

$$\Phi(t, s) = \begin{pmatrix} e^{t-s} & 0 \\ 0 & e^{-4(t-s)} \end{pmatrix}.$$

Solving $\langle \begin{pmatrix} x \\ y \end{pmatrix}, S \begin{pmatrix} x \\ y \end{pmatrix} \rangle = 0$ we get $|x| = 2|y|$. This leads for every $t_0 \in \mathbb{I}$ to

$$^{\mathbb{I}}V_s(t_0) = \left\{ \begin{pmatrix} e^{t_0-\pi} & 0 \\ 0 & e^{-4(t_0-\pi)} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| |x| < 2|y| \right\} \cup \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\},$$

$$^{\mathbb{I}}V_u(t_0) = \left\{ \begin{pmatrix} e^{t_0} & 0 \\ 0 & e^{-4(t_0)} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| |x| > 2|y| \right\} \cup \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

These cones are illustrated in Figure 5.2. In the left chart the green dashed lines show parts of the boundary of the stable cone, while the red dashed lines represent a part of the boundary of the unstable cone. The green (red) lines display all points on the boundary of the stable (unstable) cone with norm 0.5. *Since the cones are symmetric it suffices to study one half-cone each. They are plotted in the right diagram of Figure 5.2 (green, red). For comparison the stable and unstable subspaces $^{\mathbb{R}}V_s$ (blue) and $^{\mathbb{R}}V_u$ (yellow) of the ift-system* (5.57) *for $\mathbb{I} = \mathbb{R}$ are additionally presented in the right figure.*

We see that the stable subspace $^{\mathbb{R}}V_s$ lies inside the stable cone $^{\mathbb{I}}V_s$ and $^{\mathbb{R}}V_u$ is a subset of $^{\mathbb{I}}V_u$. This underlines the theory, since A is diagonal, cf. Corollary 5.1.3.

Further, we notice that the width of the boundary of the stable cone decreases in backward time while the width of the boundary of the unstable cone decreases in forward time.

This implies the questions:

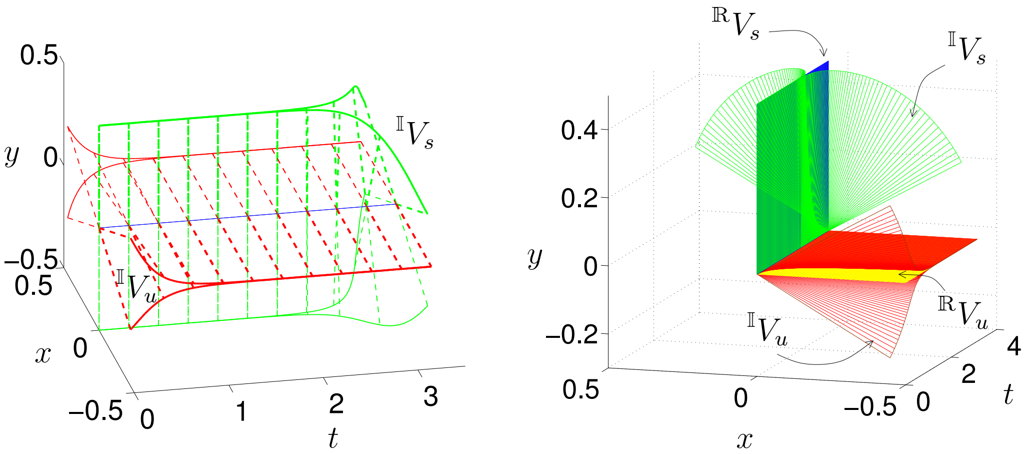


Figure 5.2: Stable (green) and unstable (red) cone of the ft-system (5.57). In the right graph only one half of each cone is plotted. In addition, we displayed in the right figure one half of each stable (blue) and unstable (yellow) subspace of the ift-system (5.57) on $\mathbb{I} = \mathbb{R}$.

- ***How fast decreases the width of the boundaries of*** $^{\mathbb{I}}\boldsymbol{V_s}$ ***and of*** $^{\mathbb{I}}\boldsymbol{V_u}$***?***
- ***How does the decay of the width of the boundaries depend on the system?***

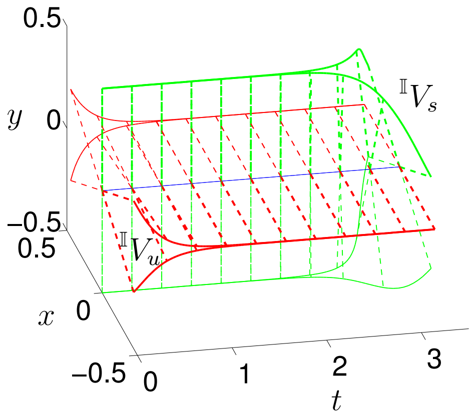
For a first answer we compare the cones of system (5.57) *with the ones of*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{5.58}$$

$t \in \mathbb{I} = [0, \pi]$. *The D-hyperbolicity of system* (5.58) *(w.r.t. the identity norm) follows straight away, since the generating matrices of* (5.58) *and* (5.57) *only differ by the factor* 2. *In Figure 5.3 we plotted in the left the cones of system* (5.57) *and in the right the cones of system* (5.58).

For an exact comparison of the decay of the width of the boundaries we create a table, where we note the angles of the cones at different times. Therefore, denote by $\tau^1_{s,u}(t)$ *the angle of the stable and unstable* t*-cone of system* (5.57) *and by* $\tau^2_{s,u}(t)$ *the angle of the* t*-cones of system* (5.58). *In Table 5.1 the angles of the cones of system* (5.57) *and* (5.58) *are presented for* 8 *different times.*

The stable t_+*-cone* $^{\mathbb{I}}V_s(t_+)$ *as well as the unstable* t_-*-cone* $^{\mathbb{I}}V_u(t_-)$ *are the same for both systems. However, we see in Figure 5.3 that the width of the cones of system* (5.58) *decays faster than the width of the cones of system* (5.57). *Table 5.1 yields that the angle of the (un)stable* t*-cones of system* (5.58) *decays in backward (forward) time twice as fast as the angle of the (un)stable* t*-cones of system* (5.57). *By definition of the angle the same properties hold for the width.*



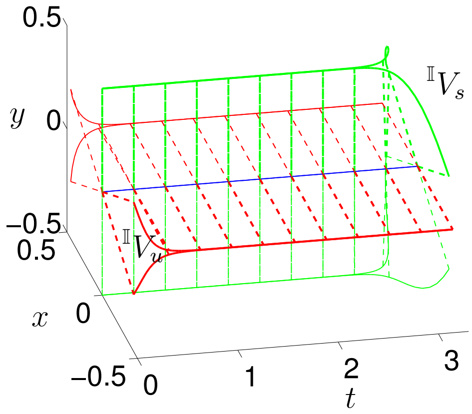


Figure 5.3: Stable (green) and unstable (red) cone of system (5.57) (left) and of system (5.58) (right).

	$t_- = 0$	$t = \frac{\pi}{40}$	$t = \frac{2\pi}{40}$	$t = \frac{4\pi}{40}$	$t = \frac{36\pi}{40}$	$t = \frac{38\pi}{40}$	$t = \frac{39\pi}{40}$	$t_+ = \pi$
τ_s^1	$1.9a$	$2.8a$	$4.2a$	$9.2a$	0.25π	0.47π	0.59π	0.70π
τ_s^2	0	0	0	0	0.055π	0.25π	0.47	0.70π
τ_u^1	0.30π	0.21π	0.14π	0.070π	$2.3a$	$1.0a$	$7.1a$	$4.8a$
τ_u^2	0.30π	0.14π	0.070π	0.014π	0	0	0	0

Table 5.1: Angle of the stable and unstable t-cones of system (5.57) ($\tau_{s,u}^1(t)$) and of (5.58) ($\tau_{s,u}^2(t)$) rounded to mantissa size two, where $a = \pi 10^{-7}$.

The matrix functions defining systems (5.57) and (5.58) differ by the factor 2. Hence, their eigenvalues differ by the factor 2 as well. Thus, we expect that the decay of the width of the stable and unstable cones depends on the eigenvalues of the matrix defining the autonomous system. The width of $^{\mathbb{I}}V_s(t_+)$ and the width of $^{\mathbb{I}}V_u(t_-)$ just depends on the relations of the eigenvalues, since the angles and thus the width of each of this cones is the same for both systems (5.57) and (5.58).

Form these observations we deduce our first conjectures that turn out to be wrong in general:

- **The decay of the width of the cones depends on the eigenvalues of the matrix defining the autonomous system.**
- **The width of $^{\mathbb{I}}V_s(t_+)$ and the width of $^{\mathbb{I}}V_u(t_-)$ just depend on the relations of these eigenvalues.**

In Section 5.6 we analyze the behavior of the width of the cones. We prove in Lemma 5.6.5 that the angle of the cones of system (5.58) decays twice as fast as the angle of the cones of system (5.57) in the right time direction. Further we show that the decay is d-times faster if the defining matrix differ by the factor d. Before we start with the theoretical part, we have a look at a few more examples. The next one shows that the stable and unstable subspaces of an ift-system do not have to lie in the exact center of the cones of a shortened ft-system.

Example 5.5.2. *Consider the system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & q \\ 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =: A \begin{pmatrix} x \\ y \end{pmatrix}, \; q \in \mathbb{N}_0 \tag{5.59}$$

for $t \in \mathbb{I} = [t_-, t_+]$. Let Γ be the identity. Then the symmetric strain tensor

$$S = S(t) = \frac{1}{2}\left(\begin{pmatrix} 1 & q \\ 0 & -4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ q & -4 \end{pmatrix} \right) = \begin{pmatrix} 1 & \frac{q}{2} \\ \frac{q}{2} & -4 \end{pmatrix}$$

has the eigenvalues $\lambda_1 := \frac{-3+\sqrt{25+q^2}}{2} > 0$ and $\lambda_2 := \frac{-3-\sqrt{25+q^2}}{2} < 0$ and is indefinite and nondegenerate for every $t \in \mathbb{I}$ and $q \in \mathbb{N}_0$. The strain acceleration tensor is

$$M = M(t) = SA + A^T S = \begin{pmatrix} 2 & -\frac{q}{2} \\ -\frac{q}{2} & 32 + q^2 \end{pmatrix}.$$

The eigenvalues are $\frac{34+q^2 \pm \sqrt{(34+q^2)^2 - 256 - 7q^2}}{2} > 0$ and the matrix M is positive definite for all $t \in \mathbb{I}$ and $q \in \mathbb{N}_0$. This means that system (5.59) is D-hyperbolic on $\mathbb{I}$. The matrix A has $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as an eigenvector to the eigenvalue 1 and $\begin{pmatrix} q \\ -5 \end{pmatrix}$ as an eigenvector to -4. Then the solution operator is given by

$$\Phi(t,s) = \begin{pmatrix} 1 & q \\ 0 & -5 \end{pmatrix} \begin{pmatrix} e^{t-s} & 0 \\ 0 & e^{-4(t-s)} \end{pmatrix} \begin{pmatrix} 1 & \frac{q}{5} \\ 0 & -\frac{1}{5} \end{pmatrix} = \begin{pmatrix} e^{t-s} & \frac{q}{5}(e^{t-s} - e^{-4(t-s)}) \\ 0 & e^{-4(t-s)} \end{pmatrix}.$$

Next we solve $\langle \begin{pmatrix} x \\ y \end{pmatrix}, S \begin{pmatrix} x \\ y \end{pmatrix} \rangle = 0$, i.e.

$$0 = x^2 + qxy - 4y^2 = x^2 + qxy + \frac{q^2}{4}y^2 - \frac{q^2}{4}y^2 - 4y^2 = \left(x + \frac{q}{2}y\right)^2 - \frac{16+q^2}{4}y^2.$$

Bringing $\frac{16+q^2}{4}y^2$ to the other side, taking the square root and solving the equation for x we get $x = \frac{-q \pm \sqrt{16+q^2}}{2}y$. This leads by (5.24), (5.25) to

$$^{\mathbb{I}}V_s(t_0) = \left\{ \begin{pmatrix} e^{t_0-1} & \frac{q}{5}(e^{t_0-1} - e^{-4(t_0-1)}) \\ 0 & e^{-4(t_0-1)} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| c_1 x < y < c_2 x \right\} \cup \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\},$$

$$^{\mathbb{I}}V_u(t_0) = \left\{ \begin{pmatrix} e^{t_0} & \frac{q}{5}(e^{t_0} - e^{-4(t_0)}) \\ 0 & e^{-4(t_0)} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| c_2 x < y < c_1 x \right\} \cup \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

for every $t_0 \in \mathbb{I}$ with $c_1 := \frac{2}{-q+\sqrt{16+q^2}}$ and $c_2 := \frac{2}{-q-\sqrt{16+q^2}}$.

We have two purposes. We like to show that the stable and unstable subspaces do not lie in the exact center of the stable and unstable cone, respectively. Additionally, we are interested in the decay of the width of the boundaries.

For the first aim let $\mathbb{I} = [0, 1]$ *and* $q = 1$*. In Figure 5.4 one half-cone of the stable cone* ${}^{\mathbb{I}}V_s$ *(green) and unstable cone* ${}^{\mathbb{I}}V_u$ *(red) are illustrated by their boundaries up to a size of norm* 1*. The stable subspace* ${}^{\mathbb{R}}V_s$ *of the ift-system* (5.59) *for* $\mathbb{I} = \mathbb{R}$ *is colored in blue while the unstable subspace* ${}^{\mathbb{R}}V_u$ *is yellow. In the right part of Figure 5.4 the cones and subspaces are projected to the* $x-y$*-plane. Thus, we see where the subspaces of* (5.59) *are located inside the cones.*

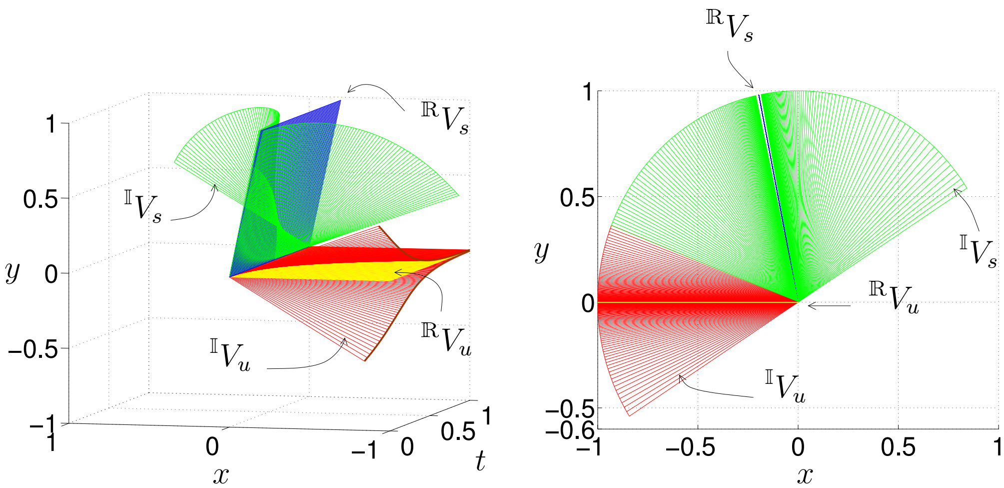


Figure 5.4: One half of the stable ${}^{\mathbb{I}}V_s$ (green) and unstable ${}^{\mathbb{I}}V_u$ (red) cone of system (5.59) with $q = 1$ together with one half of the stable ${}^{\mathbb{R}}V_s$ (blue) and unstable ${}^{\mathbb{R}}V_u$ (yellow) subspace of the same system for $\mathbb{I} = \mathbb{R}$ are illustrated. The right picture is a projection to the $x - y$-plane of the left figure.

In contrast to the first example (5.57) *(Figure 5.2) the subspaces of the expended ift-system are not located in the exact center of the cones. This can easily be seen, since they are not orthogonal towards each other.*

For analyzing the decay of the width of the cones of system (5.59) *for different* q *let* $\mathbb{I} = [0, \pi]$ *as in example* (5.57)*. Denote by* $\tau^q_{s,u}(t)$ *the angle of the* t*-cones of system* (5.59)*, where* $q \in \{0, 1, 2, 3, 4, 5, 6, 10\}$*. Note that system* (5.59) *with* $q = 0$ *equals system* (5.57)*. In Table 5.2 the angles* $\tau^q_{s,u}(\cdot)$ *are each presented for* 6 *different times and are rounded to mantissa size two. Denote by* λ_{approx} *an approximation of* λ *from*

$$\tau^q_{s,u}(t_0)e^{\lambda(t_1-t_0)} \geq \tau^q_{s,u}(t_1), \; t_0, t_1 \in \mathbb{I}, t_0 \leq t_1.$$

Therefore, we approximate $\ln(\tau^q_{s,u}(t)) = \lambda t + d$ *with the MATLAB-function* `polyfit` *for* t *from* 0 *in* 200 *steps up to* π*. In the last column the approximated growth or decay rate* λ_{approx} *is notated with mantissa size four.*

	$t_- = 0$	$t = \frac{\pi}{20}$	$t = \frac{2\pi}{20}$	$t = \frac{18\pi}{20}$	$t = \frac{19\pi}{20}$	$t_+ = \pi$	λ_{approx}
τ_u^0	0.295π	0.143π	0.0659π	$2.31a$	$1.05a$	$0.481a$	-4.996
τ_u^1	0.300π	0.143π	0.0658π	$2.29a\pi$	$1.04a$	$0.472a$	-4.999
τ_u^2	0.312π	0.145π	0.0650π	$2.22a$	$1.01a$	$0.457a$	-5.007
τ_u^3	0.328π	0.145π	0.0633π	$2.12a$	$0.965a$	$0.437a$	-5.018
τ_u^4	0.345π	0.143π	0.0607π	$1.99a$	$0.906a$	$0.408a$	-5.031
τ_u^5	0.361π	0.140π	0.0575π	$1.84a$	$0.840a$	$0.376a$	-5.043
τ_u^6	0.375π	0.135π	0.0540π	$1.70a$	$0.775a$	$0.352a$	-5.054
τ_u^{10}	0.414π	0.111π	0.0410π	$1.24a$	$0.563a$	$0.251a$	-5.088
τ_s^0	$1.92a$	$4.21a$	$9.23a$	0.251π	0.471π	0.705π	4.951
τ_s^1	$1.90a$	$4.17a$	$9.15a$	0.245π	0.461π	0.700π	4.948
τ_s^2	$1.85a$	$4.06a$	$8.90a$	0.229π	0.436π	0.688π	4.940
τ_s^3	$1.76a$	$3.87a$	$8.48a$	0.209π	0.402π	0.672π	4.932
τ_s^4	$1.66a$	$3.63a$	$7.96a$	0.188π	0.366π	0.655π	4.924
τ_s^5	$1.54a$	$3.37a$	$7.39a$	0.168π	0.332π	0.640π	4.919
τ_s^6	$1.42a$	$3.11a$	$6.82a$	0.151π	0.301π	0.626π	4.915
τ_s^{10}	$1.03a$	$2.27a$	$4.97a$	0.104π	0.210π	0.587π	4.912

Table 5.2: Angles of the stable and unstable cones τ_u^q of system (5.59) on $\mathbb{I} = [0, \pi]$ for $q \in \{0, 1, 2, 3, 4, 5, 6, 10\}$ and approximated growth and decay rate λ_{approx}.

We see that the angle of the unstable t_--cones increases for greater q and that the angle of the stable t_+-cone decreases. Further, we see that the approximated decay rate is increasing in absolute value and that the approximated growth rate is decreasing when q increases.

This example shows that our conjecture that the decay of the width of the cones depends on the eigenvalues of the defining matrix (A) is generally not correct, otherwise the approximated decay rates would have been the same for all q. Thus, we conjecture that the decay of the width depends on the eigenvalues of the symmetric part $(S := \frac{1}{2}(A + A^T))$ of the matrix (A) defining the autonomous system, since they are quite close to the eigenvalues of the defining matrix (A), but they differ a little bit for different q. Having a look at the first column of Table 5.2 we see that the angle of ${}^{\mathbb{I}}V_u(t_-)$ varies for $q \in \{0, 1, 2, 3, 4, 5, 6, 10\}$. Our first idea about these angles needs to be improved as well.

Our revised conjectures are:

- **The decay of the width depends on the eigenvalues of the symmetric part $(S := \frac{1}{2}(A + A^T))$ of the matrix (A) defining the autonomous system.**
- **The width of ${}^{\mathbb{I}}\boldsymbol{V_u}(\boldsymbol{t_-})$ and the width of ${}^{\mathbb{I}}\boldsymbol{V_s}(\boldsymbol{t_+})$ depend on the**

relation of the eigenvalues of the symmetric part.

In Lemma 5.6.1 we prove this conjectures. Next, we study an nonautonomous ft-system. We construct a nonautonomous ft-system by rotating the autonomous system (5.57). We expect that the decay of the width stays the same, since a rotation leads to a similar strain tensor and thus, the eigenvalues are invariant under this transformation.

Example 5.5.3. *Let* $D = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$ *and* $A = \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}$*. Further, let* $u(t)$ *be a solution of equation* (5.57) *for* $t \in \mathbb{I} = [0, \pi]$*. Then* $v(t) = D(t)u(t)$ *is a solution of the nonautonomous system*

$$\begin{aligned} \dot{v}(t) &= \dot{D}(t)u(t) + D(t)\dot{u}(t) = [\dot{D}(t) + D(t)A]u(t) = [\dot{D}(t) + D(t)A]D^T(t)v(t) \\ &= \begin{pmatrix} \cos^2(t) - 4\sin^2(t) & 5\sin(t)\cos(t) - 1 \\ 5\sin(t)\cos(t) + 1 & \sin^2(t) - 4\cos^2(t) \end{pmatrix} v(t) =: \tilde{A}(t)v(t) \end{aligned} \tag{5.60}$$

for $t \in \mathbb{I}$*. Let* $t, s \in \mathbb{I}$ *then the solution operator is*

$$\begin{aligned} \tilde{\Phi}(t,s) &= D(t)e^{At}e^{A(-s)}D(s)^T \\ &= \begin{pmatrix} \mathfrak{c}(t)\mathfrak{c}(s)e^{t-s} + \mathfrak{s}(t)\mathfrak{s}(s)e^{-4(t-s)} & \mathfrak{c}(t)\mathfrak{s}(s)e^{t-s} - \mathfrak{s}(t)\mathfrak{c}(s)e^{-4(t-s)} \\ \mathfrak{s}(t)\mathfrak{c}(s)e^{t-s} - \mathfrak{c}(t)\mathfrak{s}(s)e^{-4(t-s)} & \mathfrak{s}(t)\mathfrak{s}(s)e^{t-s} + \mathfrak{c}(t)\mathfrak{c}(s)e^{-4(t-s)} \end{pmatrix} \end{aligned}$$

with $\mathfrak{c} = \cos$ *and* $\mathfrak{s} = \sin$*. For the strain tensor we get for every* $t \in \mathbb{I}$

$$\begin{aligned} \tilde{S}(t) &= \tfrac{1}{2}[\tilde{A}(t) + \tilde{A}^T(t)] = \tfrac{1}{2}[\dot{D}(t) + D(t)A]D^T(t) + D(t)[\dot{D}^T(t) + A^T(t)D^T(t)]] \\ &= \tfrac{1}{2}[\dot{D}(t)D^T(t) + D(t)\dot{D}^T(t) + D(t)[A + A^T]D^T(t)] \\ &= D(t)[\tfrac{1}{2}[A + A^T]D^T(t) = D(t)SD^T(t) \\ &= \begin{pmatrix} \cos^2(t) - 4\sin^2(t) & 5\sin(t)\cos(t) \\ 5\sin(t)\cos(t) & \sin^2(t) - 4\cos^2(t) \end{pmatrix}, \end{aligned}$$

since for every differentiable $D^T = D^{-1}$ *holds*

$$\dot{D}(t)D^T(t) + D(t)\dot{D}^T(t) = 0. \tag{5.61}$$

Thus $\tilde{S}(t)$ *and* S *are similar for all* $t \in \mathbb{I}$*. This means the eigenvalues are the same and we obtain that* $\tilde{S}(t)$ *is indefinite and nondegenerate for all* $t \in \mathbb{I}$*. With equation* (5.61) *the strain acceleration tensor satisfies for every* $t \in \tilde{\mathbb{I}}$

$$\begin{aligned} \tilde{M}(t) &= \dot{\tilde{S}}(t) + \tilde{S}(t)\tilde{A}(t) + \tilde{A}^T(t)\tilde{S}(t) \\ &= \dot{D}(t)SD^T(t) + D(t)S\dot{D}^T(t) + D(t)SD^T(t)\dot{D}(t)D^T(t) + D(t)SAD^T(t) \\ &\qquad + D(t)\dot{D}^T(t)D(t)SD^T(t) + D(t)A^TSD^T(t) \\ &= \dot{D}(t)SD^T(t) + D(t)S\dot{D}^T(t) - D(t)S\dot{D}^T(t) - \dot{D}(t)SD^T(t) \\ &\quad + D(t)[SA + A^TS]D^T(t) \\ &= D(t)MD^T(t). \end{aligned}$$

This establishes that $\tilde{M}(t)$ is positive definite for every $t \in \mathbb{I}$. Thus, the rotated system (5.60) *is still D-hyperbolic w.r.t. the identity norm. The stable cone ${}^{\mathbb{I}}V_s$ and the unstable cone ${}^{\mathbb{I}}V_u$ of system* (5.60) *are plotted in Figure 5.5. The right panel shows one half-cone of each cone together with one half of each stable ${}^{\mathbb{R}}V_s$ and unstable subspace ${}^{\mathbb{R}}V_u$ of the extended ift-system* (5.60) *on $\mathbb{I} = \mathbb{R}$. The boundaries and the subspaces for all times $t \in \mathbb{I} = [0, \pi]$ are plotted up to a vector-length of* 0.5.

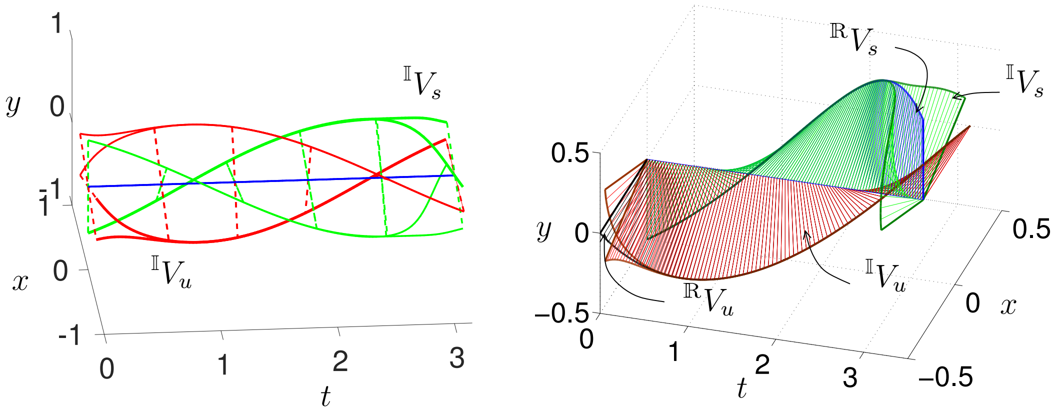


Figure 5.5: Stable (green) and unstable (red) cone of system (5.60). In the right panel one half-cone of each cone is plotted and together with one half of the stable (blue) and unstable (black) subspace of (5.60) for $\mathbb{I} = \mathbb{R}$.

We observe that the cones of this system rotate around the $(0, 0, t)$-axis and the subspaces are still included in the exact center. For the original and rotated systems the angle and the position of the stable t_+-cone and unstable t_--cone are identical. As well, the decay of the angles stays the same only the position of the t-cones ($t \neq t_+$, $t \neq t_-$) differs.

As conjectured the decay of the angles does not change by rotating the system, which is a nonautonomous similarity transformation. Next, we study discrete time systems and start with the invertible case.

Example 5.5.4. *Consider the systems*

$$\begin{pmatrix} x(n+1) \\ y(n+1) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x(n) \\ y(n) \end{pmatrix} =: A \begin{pmatrix} x(n) \\ y(n) \end{pmatrix}, \tag{5.62}$$

$$\begin{pmatrix} x(n+1) \\ y(n+1) \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 0.25 \end{pmatrix} \begin{pmatrix} x(n) \\ y(n) \end{pmatrix} =: A^2 \begin{pmatrix} x(n) \\ y(n) \end{pmatrix} \tag{5.63}$$

for all $n \in \mathbb{I} = [0,8]_{\mathbb{Z}}$ *and let* Γ *be the identity. Then the symmetric strain tensor*

$$S = S(n) = \begin{cases} \begin{pmatrix} 3 & 0 \\ 0 & -\frac{3}{4} \end{pmatrix}, & \textit{of system } (5.62), \\ \begin{pmatrix} 15 & 0 \\ 0 & -\frac{15}{16} \end{pmatrix}, & \textit{of system } (5.63) \end{cases}$$

is indefinite and nondegenerate for all $n \in \mathbb{I}$. *The matrix*

$$M(n) = A^T S A - S = \begin{cases} \begin{pmatrix} 9 & 0 \\ 0 & \frac{9}{16} \end{pmatrix}, & \textit{of system } (5.62), \\ \begin{pmatrix} 15^2 & 0 \\ 0 & \left(\frac{15}{16}\right)^2 \end{pmatrix}, & \textit{of system } (5.63) \end{cases}$$

is positive definite. Thus, the given systems (5.62) *and* (5.63) *are D-hyperbolic and Corollary 5.3.5 applies. The solution operator is*

$$\Phi(n,m) = A^{n-m} = \begin{cases} \begin{pmatrix} 2^{n-m} & 0 \\ 0 & \left(\frac{1}{2}\right)^{n-m} \end{pmatrix}, & \textit{of system } (5.62) \\ \begin{pmatrix} 4^{n-m} & 0 \\ 0 & \left(\frac{1}{4}\right)^{n-m} \end{pmatrix}, & \textit{of system } (5.63). \end{cases}$$

Solving $\langle \begin{pmatrix} x \\ y \end{pmatrix}, S \begin{pmatrix} x \\ y \end{pmatrix} \rangle = 0$ *we get*

$$|x| = \begin{cases} \frac{1}{2}|y|, & \textit{for system } (5.62), \\ \frac{1}{4}|y|, & \textit{for system } (5.63). \end{cases}$$

This leads for every $n_0 \in \mathbb{I}$ to

$$
\begin{aligned}
&{}^{\mathbb{I}}V_s(n_0)\\
&= \begin{cases} \{\xi \in \mathbb{R}^2 \big| \langle A^{8-1-n_0}\xi, SA^{8-1-n_0}\xi\rangle < 0\} \cup \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}, & \text{for } n_0 \in [0,7]_{\mathbb{Z}},\\ \{A\xi \in \mathbb{R}^2 \big| \langle \xi, S\xi\rangle < 0\} \cup \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}, & \text{for } n_0 = 8,\end{cases}\\
&= \{A^{n_0-7}\xi \in \mathbb{R}^2 \big| \langle \xi, S\xi\rangle < 0\} \cup \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}\\
&= \begin{cases} \left\{\begin{pmatrix}2^{n_0-7} & 0\\ 0 & 0.5^{n_0-7}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2 \middle| |x| < \frac{1}{2}|y|\right\} \cup \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}, & \text{for } (5.62),\\ \left\{\begin{pmatrix}4^{n_0-7} & 0\\ 0 & 0.25^{n_0-7}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2 \middle| |x| < \frac{1}{4}|y|\right\} \cup \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}, & \text{for } (5.63),\end{cases}\\
&{}^{\mathbb{I}}V_u(n_0)\\
&= \begin{cases} \left\{\begin{pmatrix}2^{n_0} & 0\\ 0 & 0.5^{n_0}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2 \middle| |x| > \frac{1}{2}|y|\right\} \cup \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}, & \text{for } (5.62),\\ \left\{\begin{pmatrix}4^{n_0} & 0\\ 0 & 0.25^{n_0}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2 \middle| |x| > \frac{1}{4}|y|\right\} \cup \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}, & \text{for } (5.63).\end{cases}
\end{aligned}
$$

In the Figure 5.6 these cones are illustrated. For a clearer picture we only plotted one half-cone each. In the left part of Figure 5.6 the cones of system (5.62) *are presented and in the right part we plotted the cones of system* (5.63) *for comparison. The green (red) lines show the boundaries of the (un)stable cone at each time* $0, 1, 2, \ldots, 8$ *up to a vector length of* 0.5*. The dashed lines illustrate the decrease of the angle between the boundaries. We observe that*

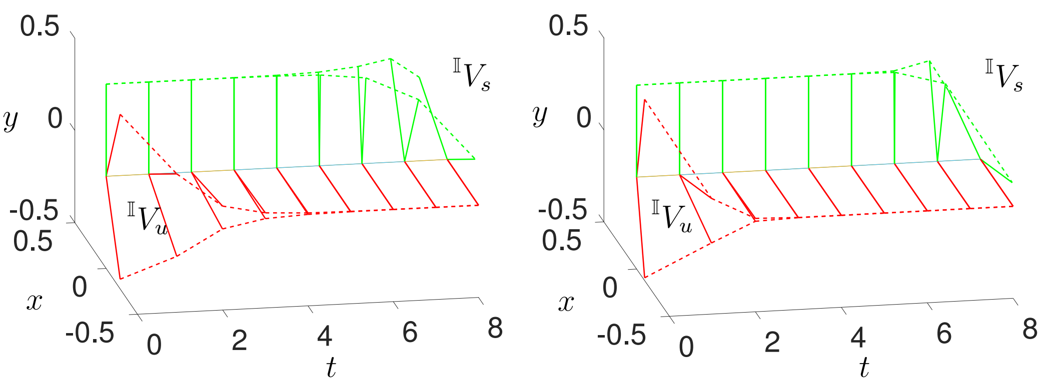


Figure 5.6: One of the stable ${}^{\mathbb{I}}V_s$ (green) and one of the unstable ${}^{\mathbb{I}}V_u$ (red) half-cone of system (5.62) in the left and of system (5.63) on the right.

the decrease of the angle of the cones of system (5.63) *is faster then the one* *of the angle of the cones of system* (5.62)*. We expect that the decrease of the right cones is twice as fast as of the left cones. For accurate data we calculate the angles for all times and present them with mantissa size three in Table 5.3. There,* $\tau^1_{s,u}$ *denotes the angle of the stable and unstable cone of system* (5.62) *and* $\tau^2_{s,u}$ *of system* (5.63)*. Further, we denote by* λ_{approx} *an approximation of* λ *from*

$$\tau^i_{s,u}(n_0)e^{\lambda(n_1-n_0)} \geq \tau^i_{s,u}(n_1), n_0, n_1 \in \mathbb{I}, n_0 \leq n_1, \ i \in \{1,2\}.$$

Therefore, we approximate $\ln(\tau^1_{s,u}(n)) = \lambda n + d$ *with the MATLAB-function* `polyfit` *for* $n \in [0,8]_{\mathbb{Z}}$ *and* $\ln(\tau^2_s(n)) = \lambda n + d$ *for* $n \in [1,8]_{\mathbb{Z}}$ *and* $\ln(\tau^2_u(n)) = \lambda n + d$ *for* $n \in [0,7]_{\mathbb{Z}}$*. We see that the angles of both systems are different for*

t	τ^1_s	τ^2_s	τ^1_u	τ^2_u
0	$1.94 \cdot 10^{-5}$	0	$7.05 \cdot 10^{-1}$	$8.44 \cdot 10^{-1}$
1	$7.77 \cdot 10^{-5}$	$9.49 \cdot 10^{-9}$	$2.95 \cdot 10^{-1}$	$1.56 \cdot 10^{-1}$
2	$3.11 \cdot 10^{-4}$	$1.52 \cdot 10^{-7}$	$7.92 \cdot 10^{-2}$	$9.95 \cdot 10^{-3}$
3	$1.24 \cdot 10^{-3}$	$2.43 \cdot 10^{-6}$	$1.99 \cdot 10^{-2}$	$6.22 \cdot 10^{-4}$
4	$4.97 \cdot 10^{-3}$	$3.89 \cdot 10^{-5}$	$4.97 \cdot 10^{-3}$	$3.89 \cdot 10^{-5}$
5	$1.99 \cdot 10^{-2}$	$6.22 \cdot 10^{-4}$	$1.24 \cdot 10^{-3}$	$2.43 \cdot 10^{-6}$
6	$7.92 \cdot 10^{-2}$	$9.95 \cdot 10^{-3}$	$3.11 \cdot 10^{-4}$	$1.52 \cdot 10^{-7}$
7	$2.95 \cdot 10^{-1}$	$1.56 \cdot 10^{-1}$	$7.77 \cdot 10^{-5}$	$9.49 \cdot 10^{-9}$
8	$7.05 \cdot 10^{-1}$	$8.44 \cdot 10^{-1}$	$1.94 \cdot 10^{-5}$	0
λ_{approx}	1.34	2.68	-1.34	-2.68

Table 5.3: The angle of the stable and unstable cone of system (5.62) ($\tau^1_{s,u}$) and of system (5.63) ($\tau^2_{s,u}$).

all times, but the decay and growth rate of system (5.63) *is twice the decay and growth rate of system* (5.62)*.*

In Lemma 5.6.1 we show, that the angles of the stable and unstable cone depend on S_Γ. For continuous systems we obtain by (3.16) that $\langle x(\cdot), S_\Gamma(\cdot)x(\cdot)\rangle$ is equivalent to the gradient of the solution $x(\cdot) \in \mathbb{R}^k$. Therefore, it does not make a difference whether we take one or two infinitesimal time steps. On the contrary $\langle x(\cdot), S_\Gamma(\cdot)x(\cdot)\rangle$ displays for discrete systems the change of the length of $x(\cdot)$ in one step and obviously it does play a role if we take one or two steps.

For an illustration we show this behavior for a solution $x(\cdot)$ pictured in Figure 5.7 of a dynamical system defined on $\mathbb{I} = [1,9]$. We see that $\|x(2)\| > \|x(1)\|$, whereas $\|x(n+2)\| \leq \|x(n)\|$ for all $n \in \mathbb{I}$. This means that $x(1) \in {}^{\mathbb{I}}V^2_s(1)$ and $x(1) \notin {}^{\mathbb{I}}V^1_s(1)$, where ${}^{\mathbb{I}}V^{1,2}_s(1)$ denotes the stable cone of the "one step/two steps moving" system. Thus ${}^{\mathbb{I}}V^2_s(1) \subset {}^{\mathbb{I}}V^1_s(1)$. Generally, the angle of the stable cone increases if the number of steps increases. This explains why the angle of the stable cone at time $n = 8$ of system (5.62) is smaller than the corresponding angle of system (5.63).

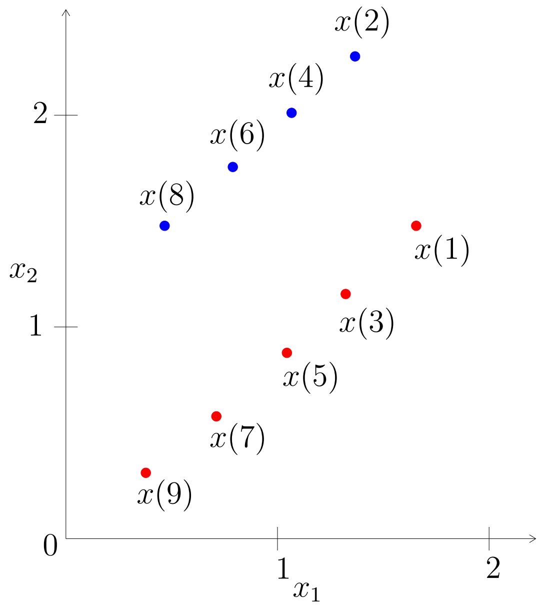


Figure 5.7: An example for illustrating that the angle of a systems changes if the system goes two steps instead of one step at the same time.

Finally, we turn our attention to noninvertible discrete systems.

Example 5.5.5. *Consider the system*

$$\begin{pmatrix} x(n+1) \\ y(n+1) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(n) \\ y(n) \end{pmatrix} =: B \begin{pmatrix} x(n) \\ y(n) \end{pmatrix} \tag{5.64}$$

for $n \in \mathbb{I} = [0,3]_{\mathbb{Z}}$ *and let* Γ *be the identity. Then the symmetric strain tensor* $S = S(n) = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ *of system* (5.64) *is indefinite and nondegenerate for all* $n \in \mathbb{I}$*. The matrix* $M = M(n) = B^T SB - S = \begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix}$ *is positive definite. Thus, the given system* (5.62) *is D-hyperbolic and Corollary 5.3.5 applies. The solution operator is for* $m, n \in \mathbb{I}$ *with* $n \geq m$ *given by*

$$\Phi(n,m) = B^{n-m} = \begin{pmatrix} 2^{n-m} & 0 \\ 0 & 0 \end{pmatrix}.$$

Solving $\langle \begin{pmatrix} x \\ y \end{pmatrix}, S \begin{pmatrix} x \\ y \end{pmatrix} \rangle = 0$ *we get* $\sqrt{3}|x| = |y|$*. Hence, we have*

$$\{\xi \in \mathbb{R}^2 | \langle \xi, S\xi \rangle < 0\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| \sqrt{3}|x| < |y| \right\},$$

$$\{\xi \in \mathbb{R}^2 | \langle \xi, S\xi \rangle > 0\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| \sqrt{3}|x| > |y| \right\},$$

$$\{\xi \in \mathbb{R}^2 | \langle B^i\xi, SB^i\xi \rangle < 0\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| \sqrt{3}|2^i x| < 0 \right\} = \emptyset$$

for all $i \in \mathbb{Z}_+$ *and*

$$\begin{aligned}
&\Big\{\Phi(n_+, n_+ - 1)\xi \in \mathbb{R}^k \Big| \exists \mu \in \mathbb{R}^k, \Phi(n_+, n_+ - 1)\xi = \Phi(n_+, \bar{n})\mu, \\
&\qquad\qquad \bar{n} = {}_\Phi\mathcal{T}_{\min}(\xi, n_+ - 1) : \langle \xi, S\xi \rangle < 0\Big\} \\
=&\left\{ \begin{pmatrix} 2x \\ 0 \end{pmatrix} \in \mathbb{R}^2 \middle| \exists \mu \in {}_\Phi\mathcal{T}_{\text{pre}}\left(\begin{pmatrix} 2x \\ 0 \end{pmatrix}, n_+ - 1 \right) = \mathcal{L}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right), \right. \\
&\qquad\qquad \left. x \in \mathbb{R} : \sqrt{3}|x| < 0 \right\} \\
=&\emptyset.
\end{aligned}$$

The range $\mathcal{R}(\Phi(n_+, n_+ - 1)) = \mathcal{R}(B) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| y = 0 \right\}$ *yields for all* $n_0 \in \mathbb{I}_1$

$$\begin{aligned}
\mathcal{R}(\Phi(n_+, n_+ - 1))^C &= \left\{ \xi \in \mathbb{R}^2 \middle| {}_\Phi\mathcal{T}_{\min}(\xi, n_+) = n_+ \right\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| y \neq 0 \right\}, \\
\mathcal{N}(\Phi(n_+, n_0)) = \mathcal{N}(B^{n_+ - n_0}) &= \mathcal{N}\left(\begin{pmatrix} 2^{n_+ - n_0} & 0 \\ 0 & 0 \end{pmatrix} \right) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x = 0 \right\}.
\end{aligned}$$

By Corollary 5.3.5 and the above statements the cones satisfy for all $n_0 \in \mathbb{I}_2 = [n_-, n_+ - 2]_\mathbb{Z}$*,* $n_1 \in \mathbb{I}$

$$\begin{aligned}
{}^\mathbb{I}\bar{V}_s(n_+) &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| y \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}, \\
{}^\mathbb{I}V_s(n_+) &= \mathbb{R}^k, \\
{}^\mathbb{I}V_s(n_+ - 1) &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| \sqrt{3}|x| < |y| \right\} \cup \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x = 0 \right\} \\
&= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| \sqrt{3}|x| < |y| \right\} \cup \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}, \\
{}^\mathbb{I}V_s(n_0) &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x = 0 \right\}, \\
{}^\mathbb{I}V_u(n_1) &= \left\{ B^{n_1 - n_-} \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| \sqrt{3}|x| > |y| \right\} \cup \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.
\end{aligned}$$

In Figure 5.8 these cones are illustrated. The fixed point 0 *is marked for each time by a black ball.*

Estimates for the Width of (Un)Stable Cones in 2-Dimensional Systems

For cones, the width and its change in time are of interest. For linear systems the width depends on the eigenvalues of the symmetric part of the defining

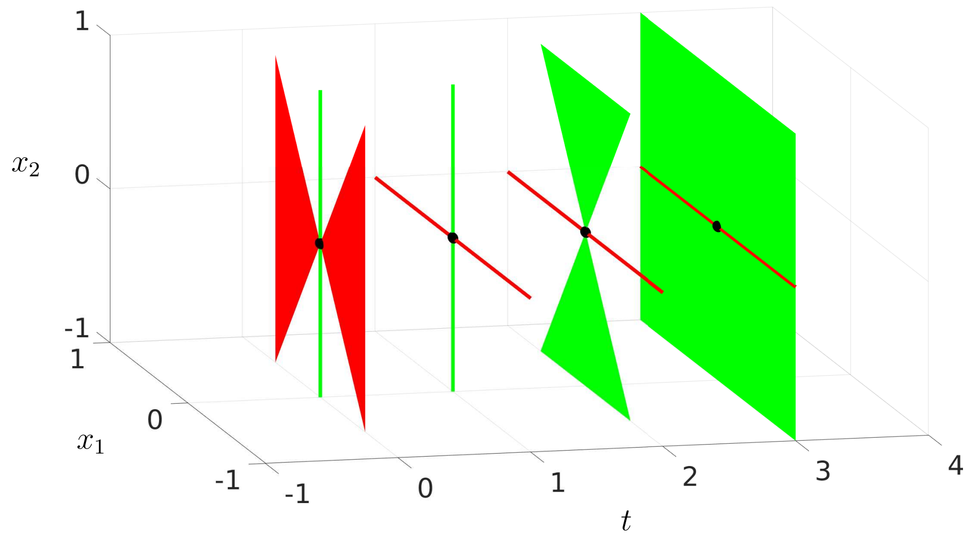


Figure 5.8: The stable (green) and unstable (red) cones of system (5.64).

matrix. The width of stable cones is decreasing in backward time in relation to the negative eigenvalues, while the width of unstable cones is decreasing in forward time in relation to the positive eigenvalues. In this section we concentrate the study on 2-dimensional systems. In Section 5.7 we present and prove estimates of the width of the (un)stable cone in 3 or higher dimensional systems (finite dimensional). The angle and the width of a cone is only defined for connected cones. In this thesis we restrict the study of the width of (un)stable cones to invertible systems. In Corollary 5.3.2 and Corollary 5.3.5 we proved that these cones are open connected double-cones. Note that the results of the unstable cones in discrete invertible M-hyperbolic systems also apply to noninvertible M-hyperbolic (regular) systems. The following lemma characterizes the zero strain set of a linear system, which leads with Corollary 5.3.2 and Corollary 5.3.5 to the boundaries of the stable and unstable cones. A similar statement for continuous systems can be found in [45, Proposition 29]. Additionally, this lemma yields the width of the stable and unstable cones for invertible systems.

First make the assumptions:

(A1) Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$, $\mathbb{I} = [t_-, t_+]_\mathbb{T}$ and $\Gamma = \Gamma^T > 0$. Assume

$$\begin{cases} \textbf{(A0)} \text{ and that system (2.6), generated by (2.7),} & \text{for } \mathbb{T} = \mathbb{R}, \\ \text{that the system (2.6), generated by (2.8),} & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}$$

is D-hyperbolic on $\mathbb{I}$ w.r.t. $\|\cdot\|_\Gamma$.

(A2) Let $\tilde{\mathbb{I}} := \begin{cases} \mathbb{I}, & \text{for } \mathbb{T} = \mathbb{R}, \\ \mathbb{I}_1, & \text{for } \mathbb{T} = \mathbb{Z}, \end{cases}$ $\bar{t}_+ := \bar{t} := \begin{cases} t_+, & \text{for } \mathbb{T} = \mathbb{R}, \\ t_+ - 1, & \text{for } \mathbb{T} = \mathbb{Z}, \end{cases}$

and $\bar{t}_- := t_-$.

(A3) Assume **(A2)** and let $k = 2$. Let $\lambda_1(t) > 0 > \lambda_2(t)$ be the eigenvalues of $S_\Gamma(t)$ and $U(t) = (v_1(t)\ v_2(t))$ be an orthogonal matrix, where $v_i(t)$ are eigenvectors to $\lambda_i(t)$ for $i \in \{1, 2\}$, for all $t \in \tilde{\mathbb{I}}$.

By Definition 3.2.8 the matrix $S_\Gamma(t)$ is symmetric for all $t \in \tilde{\mathbb{I}}$. Thus, the assumption that for $S_\Gamma(t)$ and all $t \in \tilde{\mathbb{I}}$ an orthogonal basis of eigenvalues exists is trivial.

Lemma 5.6.1. *Let $k = 2$. Assume* **(A1)** *and* **(A3)**. *Then the zero Γ-strain set is for every $t \in \tilde{\mathbb{I}}$*

$$\begin{aligned} Z_\Gamma(t) &= \mathcal{L}\left(U(t)\begin{pmatrix}\sqrt{|\lambda_2(t)|}\\ \sqrt{\lambda_1(t)}\end{pmatrix}\right) \cup \mathcal{L}\left(U(t)\begin{pmatrix}-\sqrt{|\lambda_2(t)|}\\ \sqrt{\lambda_1(t)}\end{pmatrix}\right) \\ &= \left\{x \in \mathbb{R}^2 \middle| \exists \lambda \in \mathbb{R} : x = \lambda(\sqrt{|\lambda_2(t)|}v_1(t) \pm \sqrt{\lambda_1(t)}v_2(t))\right\}. \end{aligned} \tag{5.65}$$

Let (2.6) be invertible. Then the width $d_s(t_+)$ of the stable t_+-cone ${}^{\mathbb{I}}V_s(t_+)$ for continuous systems is

$$d_s(t_+) = \sqrt{\frac{4|\lambda_2(t_+)|}{|\lambda_2(t_+)| + \lambda_1(t_+)}} \quad (\mathbb{T} = \mathbb{R}),$$

the width $d_s(t_+ - 1)$ of the stable $t_+ - 1$-cone ${}^{\mathbb{I}}V_s(t_+ - 1)$ for discrete systems is

$$d_s(t_+ - 1) = \sqrt{\frac{4|\lambda_2(t_+ - 1)|}{|\lambda_2(t_+ - 1)| + \lambda_1(t_+ - 1)}} \quad (\mathbb{T} = \mathbb{Z}), \tag{5.66}$$

and the width $d_u(t_-)$ of the unstable t_--cone ${}^{\mathbb{I}}V_u(t_-)$ is

$$d_u(t_-) = \sqrt{\frac{4\lambda_1(t_-)}{|\lambda_2(t_-)| + \lambda_1(t_-)}} \quad (\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}). \tag{5.67}$$

The width $d_s(t)$ of the stable t-cone is for $t \in \mathbb{I}$

$$d_s(t) = \left\| \frac{\Phi(t,\bar{t})U(\bar{t})\begin{pmatrix}\sqrt{|\bar{\lambda}_2|}\\ \sqrt{\bar{\lambda}_1}\end{pmatrix}}{\left\|\Phi(t,\bar{t})U(\bar{t})\begin{pmatrix}\sqrt{|\bar{\lambda}_2|}\\ \sqrt{\bar{\lambda}_1}\end{pmatrix}\right\|_2} - \frac{\Phi(t,\bar{t})U(\bar{t})\begin{pmatrix}-\sqrt{|\bar{\lambda}_2|}\\ \sqrt{\bar{\lambda}_1}\end{pmatrix}}{\left\|\Phi(t,\bar{t})U(\bar{t})\begin{pmatrix}-\sqrt{|\bar{\lambda}_2|}\\ \sqrt{\bar{\lambda}_1}\end{pmatrix}\right\|_2} \right\|_2, \tag{5.68}$$

where $\bar{\lambda}_{1,2} := \lambda_{1,2}(\bar{t})$.

The width $d_u(t)$ of the unstable t-cone is for $t \in \mathbb{I}$

$$d_u(t) = \left\| \frac{\Phi(t,t_-)U(t_-)\begin{pmatrix}\sqrt{|\lambda_2^-|}\\ \sqrt{\lambda_1^-}\end{pmatrix}}{\left\|\Phi(t,t_-)U(t_-)\begin{pmatrix}\sqrt{|\lambda_2^-|}\\ \sqrt{\lambda_1^-}\end{pmatrix}\right\|_2} - \frac{\Phi(t,t_-)U(t_-)\begin{pmatrix}\sqrt{|\lambda_2^-|}\\ -\sqrt{\lambda_1^-}\end{pmatrix}}{\left\|\Phi(t,t_-)U(t_-)\begin{pmatrix}\sqrt{|\lambda_2^-|}\\ -\sqrt{\lambda_1^-}\end{pmatrix}\right\|_2} \right\|_2 , \tag{5.69}$$

where $\lambda_{1,2}^- := \lambda_{1,2}(t_-)$.

Proof. For clarity we do not explicitly mention the dependency on t. Let $\xi \in Z_\Gamma \subset \mathbb{R}^2$. Then there exist constants $a, b \in \mathbb{R}$ such that $\xi = av_1 + bv_2$ holds. Since U is an orthogonal matrix we have $\langle v_1, v_2\rangle = 0$, $\langle v_i, v_i\rangle = 1$ for $i \in \{1,2\}$ and

$$\begin{aligned} 0 = \langle \xi, S_\Gamma \xi\rangle &= \langle av_1 + bv_2, S_\Gamma av_1 + S_\Gamma bv_2\rangle \\ &= \langle av_1, S_\Gamma av_1\rangle + \langle av_1, S_\Gamma bv_2\rangle + \langle bv_2, S_\Gamma av_1\rangle + \langle bv_2, S_\Gamma bv_2\rangle \\ &= a^2\langle v_1, \lambda_1 v_1\rangle + ab\langle v_1, \lambda_2 v_2\rangle + ab\langle v_2, \lambda_1 v_1\rangle + b^2\langle v_2, \lambda_2 v_2\rangle \\ &= a^2\lambda_1 + b^2\lambda_2 \end{aligned}$$

follows. This leads to $a = \pm\sqrt{|\lambda_2|}c$ and $b = \pm\sqrt{\lambda_1}c$, where $c \in \mathbb{R}_0^+$. Thus, by Definition 3.2.8 equation (5.65) is true.

Let system (2.6) be invertible. By Corollary 5.3.2/5.3.5 the sets ${}^{\mathbb{I}}V_s(t)$ and ${}^{\mathbb{I}}V_u(t)$ are connected double-cones for all $t \in \mathbb{I}$ and by Lemma 5.4.1 they are nontrivial. It follows that the half-cones of ${}^{\mathbb{I}}V_s(t)$ and ${}^{\mathbb{I}}V_u(t)$, $t \in \mathbb{I}$ are nontrivial connected cones and the width of all cones is defined, see Definition 4.1.1. To calculate them we need for each cone two linear independent vectors on the boundary of one half-cone. The sums of the pairwise linear independent vectors

$$\begin{aligned} \tilde{v}_1 &:= \sqrt{|\lambda_2|}v_1 + \sqrt{\lambda_1}v_2 \in Z_\Gamma, \\ \tilde{v}_2 &:= -\sqrt{|\lambda_2|}v_1 + \sqrt{\lambda_1}v_2 \in Z_\Gamma, \\ \tilde{v}_3 &:= \sqrt{|\lambda_2|}v_1 - \sqrt{\lambda_1}v_2 \in Z_\Gamma \end{aligned}$$

are $\tilde{v}_1 + \tilde{v}_2 = 2\sqrt{\lambda_1}v_2 =: cv_2$ and $\tilde{v}_1 + \tilde{v}_3 = 2\sqrt{|\lambda_2|}v_1 =: dv_1$. Additionally,

$$\begin{aligned} \langle cv_2, S_\Gamma cv_2\rangle &= c^2\langle v_2, \lambda_2 v_2\rangle = c^2\lambda_2 < 0, \\ \langle dv_1, S_\Gamma dv_1\rangle &= d^2\langle v_1, \lambda_1 v_1\rangle = d^2\lambda_1 > 0 \end{aligned}$$

hold. This means

$$\begin{aligned} \tilde{v}_1(t_+) + \tilde{v}_2(t_+) &\in {}^{\mathbb{I}}V_s(t_+) \text{ for } \mathbb{T} = \mathbb{R}, \\ \tilde{v}_1(t_+ - 1) + \tilde{v}_2(t_+ - 1) &\in {}^{\mathbb{I}}V_s(t_+ - 1) \text{ for } \mathbb{T} = \mathbb{Z}, \\ \tilde{v}_1(t_-) + \tilde{v}_3(t_-) &\in {}^{\mathbb{I}}V_u(t_-) \text{ for } \mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}, \end{aligned}$$

i.e. $\tilde{v}_1(\bar{t})$ and $\tilde{v}_2(\bar{t})$ as well as $\tilde{v}_1(t_-)$ and $\tilde{v}_3(t_-)$ are linear independent vectors on the boundary of a half cone of ${}^{\mathbb{I}}V_s(\bar{t})$ respectively ${}^{\mathbb{I}}V_u(t_-)$. By Definition 4.1.1, by the invertibility of Φ and by the invariance of the cones the width $d_{s,u}(t)$ of ${}^{\mathbb{I}}V_{s,u}(t)$ satisfies (5.68) and (5.69) for all $t \in \mathbb{I}$. Further, the width $d_s(t_+)$ of ${}^{\mathbb{I}}V_s(t_+)$ for $\mathbb{T} = \mathbb{R}$ fulfills

$$d_s(t_+) = \left\| \frac{U(t_+)\begin{pmatrix} \sqrt{|\lambda_2(t_+)|} \\ \sqrt{\lambda_1(t_+)} \end{pmatrix}}{\left\| U(t_+)\begin{pmatrix} \sqrt{|\lambda_2(t_+)|} \\ \sqrt{\lambda_1(t_+)} \end{pmatrix} \right\|_2} - \frac{U(t_+)\begin{pmatrix} -\sqrt{|\lambda_2(t_+)|} \\ \sqrt{\lambda_1(t_+)} \end{pmatrix}}{\left\| U(t_+)\begin{pmatrix} -\sqrt{|\lambda_2(t_+)|} \\ \sqrt{\lambda_1(t_+)} \end{pmatrix} \right\|_2} \right\|_2 .$$

Note that $\|\cdot\|_2$ and the standard inner product $\langle \cdot, \cdot \rangle$ are invariant under orthogonal transformations and that equation $\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} -x \\ y \end{pmatrix} \right\|_2$ holds for every $x, y \in \mathbb{R}$. Thus, we get

$$\begin{aligned} d_s(t_+) &= \left\| \frac{\begin{pmatrix} \sqrt{|\lambda_2(t_+)|} \\ \sqrt{\lambda_1(t_+)} \end{pmatrix}}{\left\| \begin{pmatrix} \sqrt{|\lambda_2(t_+)|} \\ \sqrt{\lambda_1(t_+)} \end{pmatrix} \right\|_2} - \frac{\begin{pmatrix} -\sqrt{|\lambda_2(t_+)|} \\ \sqrt{\lambda_1(t_+)} \end{pmatrix}}{\left\| \begin{pmatrix} \sqrt{|\lambda_2(t_+)|} \\ \sqrt{\lambda_1(t_+)} \end{pmatrix} \right\|_2} \right\|_2 = \left\| \frac{\begin{pmatrix} 2\sqrt{|\lambda_2(t_+)|} \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} \sqrt{|\lambda_2(t_+)|} \\ \sqrt{\lambda_1(t_+)} \end{pmatrix} \right\|_2} \right\|_2 \\ &= \sqrt{\frac{4|\lambda_2(t_+)|}{|\lambda_2(t_+)| + \lambda_1(t_+)}}. \end{aligned}$$

Analogously, we obtain (5.66) and (5.67). □

Lemma 5.6.1 characterizes the width of the stable and unstable t-cones with the help of the solution operator. Additionally, we derive by the following lemma upper bounds of the width of the t-cones, even if we do not have an exact representation of the solution operator.

Lemma 5.6.2. *Let $k = 2$ and assume* **(A1)** *and* **(A3)**. *Let the given system be invertible. Further let $w_{1,2}(\bar{t}) \in \partial^{\mathbb{I}}V_s(\bar{t})$ be linear independent with*

$$w_1(\bar{t}) + w_2(\bar{t}) \in {}^{\mathbb{I}}V_s(\bar{t})$$

and let $w_{1,2}(t_-) \in \partial^{\mathbb{I}}V_u(t_-)$ be linear independent with

$$w_1(t_-) + w_2(t_-) \in {}^{\mathbb{I}}V_u(t_-).$$

Further let $v_s(\bar{t}_\pm) \in {}^{\mathbb{I}}V_s(\bar{t}_\pm)$, $v_u(\bar{t}_\pm) \in {}^{\mathbb{I}}V_u(\bar{t}_\pm)$ with $\|v_s(\bar{t}_\pm)\|_2 = 1 = \|v_u(\bar{t}_\pm)\|_2$ and with corresponding exponential rates $\alpha(\bar{t}_\pm), \beta(\bar{t}_\pm) > 0$, see Definition 4.1.3 and Definition 4.1.5, such that

$$w_{1,2}(\bar{t}_\pm) = v_s(\bar{t}_\pm) + c_{1,2}(\bar{t}_\pm) v_u(\bar{t}_\pm)$$

holds with constants $c_1(\bar{t}_\pm), c_2(\bar{t}_\pm) \in \mathbb{R}$. *Then the width* $d_s(t)$ *of the stable* t*-cone* ${}^{\mathbb{I}}V_s(t)$ *and the width* $d_u(t)$ *of the unstable* t*-cone* ${}^{\mathbb{I}}V_u(t)$ *satisfy with* $\delta(\bar{t}_\pm) := \alpha(\bar{t}_\pm) + \beta(\bar{t}_\pm)$

$$d_s(t) \le C|c_1(\bar{t}) - c_2(\bar{t})|e^{-\delta(\bar{t})(\bar{t}-t)} \text{ for all } t \in \tilde{\mathbb{I}}, \tag{5.70}$$

$$d_u(t) \le C|c_1(t_-) - c_2(t_-)|e^{-\delta(t_-)(t-t_-)} \text{ for all } t \in \mathbb{I}, \tag{5.71}$$

where $C > 0$ *is a constant.*

Proof. For clarity of presentation we leave out the dependency on $\bar{t}$. By Lemma 5.4.2 system (2.6) is M-hyperbolic. Further, by Lemma 4.2.4 and Definition 3.2.3 the estimates (3.10) and (3.15) are satisfied for v_s, v_u, more precisely $v_{s,u}$ and their corresponding exponential rates $\alpha, \beta > 0$ fulfill

$$\|\Phi(t,\bar{t})v_s\|_\Gamma \ge e^{\alpha(\bar{t}-t)} \|v_s\|_\Gamma \ge C_1 e^{\alpha(\bar{t}-t)} \|v_s\|_2 = C_1 e^{\alpha(\bar{t}-t)}, \tag{5.72}$$

$$\|\Phi(t,\bar{t})v_u\|_\Gamma \le e^{-\beta(\bar{t}-t)} \|v_u\|_\Gamma \le C_2 e^{-\beta(\bar{t}-t)} \|v_u\|_2 = C_2 e^{-\beta(\bar{t}-t)} \tag{5.73}$$

for all $t \in \tilde{\mathbb{I}}$ and for constants $C_{1,2} > 0$, since all norms in $\mathbb{R}^k$ are equivalent and $\|v_{s,u}\|_2 = 1$ by the above conditions. Fix $t \in \tilde{\mathbb{I}}$ and define

$$d_1(t) := \|\Phi(t,\bar{t})(v_s + c_1 v_u)\|_2\,,$$
$$d_2(t) := \|\Phi(t,\bar{t})(v_s + c_2 v_u)\|_2\,.$$

Then the width $d_s(t)$ of the stable t-cone satisfies by Definition 4.1.1

$$\begin{aligned} d_s(t) &:= \left\| \Phi(t,\bar{t})(v_s + c_1 v_u)\frac{1}{d_1(t)} - \Phi(t,\bar{t})(v_s + c_2 v_u)\frac{1}{d_2(t)} \right\|_2 \\ &\le \left\| \Phi(t,\bar{t})(v_s + c_1 v_u)\frac{1}{\|\Phi(t,\bar{t})v_s\|_2} - \Phi(t,\bar{t})(v_s + c_2 v_u)\frac{1}{\|\Phi(t,\bar{t})v_s\|_2} \right\|_2 \\ &= \left\| (c_1 - c_2)\Phi(t,\bar{t})v_u \frac{1}{\|\Phi(t,\bar{t})v_s\|_2} \right\|_2 \\ &\le \frac{C_2}{C_1}|c_1 - c_2| \frac{\|\Phi(t,\bar{t})v_u\|_\Gamma}{\|\Phi(t,\bar{t})v_s\|_\Gamma}. \end{aligned}$$

The estimates (5.72), (5.73) and $\delta := \alpha + \beta$ lead to

$$d_s(t) \le \left(\frac{C_2}{C_1}\right)^2 |c_1 - c_2| e^{-\delta(\bar{t}-t)}.$$

Analogously, we obtain the approximation of the width $d_u(t)$, $t \in \mathbb{I}$ of the unstable t-cone ${}^{\mathbb{I}}V_u(t)$ with the help of $v_{s,u}(t_-)$. □

Remark 5.6.3. *If all assumptions of Lemma 5.6.2 are satisfied and* Γ *is the identity we get that* (5.70) *and* (5.71) *are satisfied with* $C = 1$, *i.e. the width* $d_s(t)$ *and* $d_u(t)$ *fulfills*

$$d_s(t) \le |c_1(\bar{t}) - c_2(\bar{t})|e^{-\delta(\bar{t})(\bar{t}-t)} \text{ for all } t \in \tilde{\mathbb{I}},$$
$$d_u(t) \le |c_1(t_-) - c_2(t_-)|e^{-\delta(t_-)(t-t_-)} \text{ for all } t \in \mathbb{I}.$$

If Γ is the identity we find a relation between the width $d_{s,u}(\bar{t}_\pm)$ of the (un)stable $\bar{t}_\pm$-cone ${}^{\mathbb{I}}V_{s,u}(\bar{t}_\pm)$ and the width $d_{s,u}(t)$ of the (un)stable t-cone ${}^{\mathbb{I}}V_{s,u}(t)$ for all $t \in \tilde{\mathbb{I}}$ resp. $t \in \mathbb{I}$. More precisely, the smaller the width of the stable and unstable $\bar{t}_\pm$-cones the smaller the width of the other stable and unstable cones.

Lemma 5.6.4. *Let $k = 2$ and assume* **(A1)** *and* **(A3)** *for $\Gamma = \mathrm{I}$. Let* (2.6) *be invertible. Then*

$$v_1(t) \in {}^{\mathbb{I}}V_u(t),\ v_2(t) \in {}^{\mathbb{I}}V_s(t) \tag{5.74}$$

for all $t \in \tilde{\mathbb{I}}$.
Let $v_1(\bar{t}) \in {}^{\mathbb{I}}V_u(\bar{t})$, $v_2(t_-) \in {}^{\mathbb{I}}V_s(t_-)$. Denote the corresponding exponential rates of $v_2(\bar{t}_\pm) \in {}^{\mathbb{I}}V_s(\bar{t}_\pm)$ by $\tilde{\alpha}(\bar{t}_\pm)$ and of $v_1(\bar{t}_\pm) \in {}^{\mathbb{I}}V_u(\bar{t}_\pm)$ by $\tilde{\beta}(\bar{t}_\pm) > 0$. Then the width $d_s(t)$ and $d_u(t)$ of the stable and unstable t-cones ${}^{\mathbb{I}}V_s(t)$ and ${}^{\mathbb{I}}V_u(t)$ fulfill

$$\begin{aligned} d_s(t) &\leq \sqrt{1 + \frac{|\lambda_2(\bar{t})|}{\lambda_1(\bar{t})}} e^{-\tilde{\delta}(\bar{t})(\bar{t}-t)} d_s(\bar{t}) \text{ for all } t \in \tilde{\mathbb{I}}, \\ d_u(t) &\leq \sqrt{1 + \frac{|\lambda_2(t_-)|}{\lambda_1(t_-)}} e^{-\tilde{\delta}(t_-)(t-t_-)} d_u(t_-) \text{ for all } t \in \mathbb{I} \end{aligned}$$

with $\tilde{\delta}(t_\pm) = \tilde{\alpha}(t_\pm) + \tilde{\beta}(t_\pm)$ and $d_{s,u}(\bar{t}_\pm)$ from Lemma 5.6.1.

Proof. For the eigenvectors $v_{1,2}(t)$ of the Γ-strain tensor $S_\Gamma(t)$, $t \in \tilde{\mathbb{I}}$, we have by the orthonormality of $v_{1,2}$

$$\begin{aligned} \langle v_1(t), S_\Gamma(t) v_1(t)\rangle &= \langle v_1(t), \lambda_1(t) v_1(t)\rangle = \lambda_1(t) > 0, \\ \langle v_2(t), S_\Gamma(t) v_2(t)\rangle &= \langle v_2(t), \lambda_2(t) v_2(t)\rangle = \lambda_2(t) < 0. \end{aligned}$$

Corollary 5.3.2 yields $v_1(t) \in {}^{\mathbb{I}}V_u(t)$ and $v_2(t) \in {}^{\mathbb{I}}V_s(t)$ for all $t \in \tilde{\mathbb{I}}$. Define the linear independent vectors

$$\begin{aligned} w_1(\bar{t}_\pm) &:= v_2(\bar{t}_\pm) + \frac{\sqrt{|\lambda_2(\bar{t}_\pm)|}}{\sqrt{\lambda_1(\bar{t}_\pm)}} v_1(\bar{t}_\pm), \\ w_2(\bar{t}_\pm) &:= v_2(\bar{t}_\pm) - \frac{\sqrt{|\lambda_2(\bar{t}_\pm)|}}{\sqrt{\lambda_1(\bar{t}_\pm)}} v_1(\bar{t}_\pm). \end{aligned}$$

Then equation (5.65) implies

$$\begin{aligned} w_{1,2}(\bar{t}) &\in Z(\bar{t}) = \partial^{\mathbb{I}} V_s(\bar{t}), \\ w_{1,2}(t_-) &\in Z(t_-) = \partial^{\mathbb{I}} V_u(t_-) \end{aligned}$$

by Corollary 5.3.2 and Corollary 5.3.5 and by the fact that exactly two linear independent vectors lie in $Z_\Gamma(\bar{t}_\pm)$. Further, we have by (5.74)

$$w_1(\bar{t}) + w_2(\bar{t}) = 2v_1(\bar{t}) \in {}^{\mathbb{I}}V_s(\bar{t}),$$
$$w_1(t_-) + w_2(t_-) = 2v_1(t_-) \in {}^{\mathbb{I}}V_u(t_-).$$

Lemma 5.6.2 yields that the width satisfies for all $t \in \tilde{\mathbb{I}}$

$$d_s(t) \leq \frac{2\sqrt{|\lambda_2(\bar{t})|}}{\sqrt{\lambda_1(\bar{t})}} e^{-\tilde{\delta}(\bar{t})|\bar{t}-t|} = \sqrt{1 + \frac{|\lambda_2(\bar{t})}{\lambda_1(\bar{t})}} \sqrt{\frac{4|\lambda_2(\bar{t})|}{|\lambda_2(\bar{t})| + \lambda_1}} e^{-\tilde{\delta}(\bar{t})|\bar{t}-t|}$$

where $\tilde{\delta}(\bar{t}) := \tilde{\alpha}(\bar{t}) + \tilde{\beta}(\bar{t})$. Lemma 5.6.1 implies that

$$d_s(\bar{t}) = \sqrt{\frac{4|\lambda_2(\bar{t})|}{|\lambda_2(\bar{t})| + \lambda_1}}.$$

Thus, we obtain

$$d_s(t) = \sqrt{1 + \frac{|\lambda_2(\bar{t})|}{\lambda_1(\bar{t})}} e^{-\tilde{\delta}(\bar{t})|\bar{t}-t|} d_s(\bar{t}).$$

The approximation for the unstable cone follows analogously. □

In Example 5.5.1 we saw numerically that the decay of the angle of the stable and unstable cone of (5.58) is two times as fast as the decay of the angle of the stable and unstable cone of (5.57). This statement is proved in the following.

Lemma 5.6.5. *Let $t_\pm \in \mathbb{R}$ and $\mathbb{I} = [t_-, t_+]$. Assume that*

$$\dot{x}(t) = Ax(t), t \in \mathbb{I} \tag{5.75}$$

is D-hyperbolic on $\mathbb{I} = [t_-, t_+]$ with invertible matrix $A \in \mathbb{R}^{2\times 2}$ and denote by $d^1_{s,u}(t)$ the width of the stable/unstable t-cone. Further, denote the width of the stable/unstable t-cone of

$$\dot{x}(t) = \mu Ax(t), t \in \mathbb{I}, \mu \in \mathbb{R}_{>0} \tag{5.76}$$

by $d^\mu_{s,u}(t)$. Then we have for all $r \in [0, \frac{t_+ - t_-}{\mu}]$

$$d^\mu_s(t_+ - r) = d^1_s(t_+ - \mu r),$$
$$d^\mu_u(t_- + r) = d^1_u(t_- + \mu r).$$

Proof. In this proof we skip the dependency on t. Let λ be an eigenvalue of $S = A$ and v a corresponding eigenvector then we have

$$S_\mu v = \mu Av = \mu\lambda v,$$

where S_μ is the strain tensor of (5.76). Thus, v is an eigenvector of S and S_μ. This implies that the matrix U with orthonormal eigenvectors of S as columns and the matrix U_μ with orthonormal eigenvectors of S_μ as columns satisfy $U = U_\mu$. Additionally, we obtain if and only if λ_1 and λ_2 are the eigenvalues of S then $\mu\lambda_1$ and $\mu\lambda_2$ are the eigenvalues of S_μ. Denote the solution operator of system (5.75) by $\Phi(\cdot,\cdot)$ and the solution operator of system (5.76) by $\Phi_\mu(\cdot,\cdot)$. Fix $r \in [0, \frac{t_+ - t_-}{\mu}]_{\mathbb{T}}$. Then we find the following relation between the two operators

$$\Phi_\mu(t_+ - r, t_+) = e^{\mu A(t_+ - r - t_+)} = e^{A(t_+ - \mu r - t_+)} = \Phi(t_+ - \mu r, t_+).$$

By Lemma 5.6.1 we have for the width of the stable cones

$$\begin{aligned}
&d_s^\mu(t_+ - r)\\
=&\left\| \frac{\Phi_\mu(t_+ - r, t_+)U_\mu \begin{pmatrix} \sqrt{|\mu\lambda_2|} \\ \sqrt{\mu\lambda_1} \end{pmatrix}}{\left\| \Phi_\mu(t_+ - r, t_+)U_\mu \begin{pmatrix} \sqrt{|\mu\lambda_2|} \\ \sqrt{\mu\lambda_1} \end{pmatrix} \right\|_2} - \frac{\Phi_\mu(t_+ - r, t_+)U_\mu \begin{pmatrix} -\sqrt{|\mu\lambda_2|} \\ \sqrt{\mu\lambda_1} \end{pmatrix}}{\left\| \Phi_\mu(t_+ - r, t_+)U_\mu \begin{pmatrix} -\sqrt{|\mu\lambda_2|} \\ \sqrt{\mu\lambda_1} \end{pmatrix} \right\|_2} \right\|_2\\
=&\left\| \frac{\Phi(t_+ - \mu r, t_+)U\sqrt{\mu} \begin{pmatrix} \sqrt{|\lambda_2|} \\ \sqrt{\lambda_1} \end{pmatrix}}{\left\| \Phi(t_+ - \mu r, t_+)U\sqrt{\mu} \begin{pmatrix} \sqrt{|\lambda_2|} \\ \sqrt{\lambda_1} \end{pmatrix} \right\|_2} - \frac{\Phi(t_+ - \mu r, t_+)U\sqrt{\mu} \begin{pmatrix} -\sqrt{|\lambda_2|} \\ \sqrt{\lambda_1} \end{pmatrix}}{\left\| \Phi(t_+ - \mu r, t_+)U\sqrt{\mu} \begin{pmatrix} -\sqrt{|\lambda_2|} \\ \sqrt{\lambda_1} \end{pmatrix} \right\|_2} \right\|_2\\
=&\left\| \frac{\Phi(t_+ - \mu r, t_+)U \begin{pmatrix} \sqrt{|\lambda_2|} \\ \sqrt{\lambda_1} \end{pmatrix}}{\left\| \Phi(t_+ - \mu r, t_+)U \begin{pmatrix} \sqrt{|\lambda_2|} \\ \sqrt{\lambda_1} \end{pmatrix} \right\|_2} - \frac{\Phi(t_+ - \mu r, t_+)U \begin{pmatrix} -\sqrt{|\lambda_2|} \\ \sqrt{\lambda_1} \end{pmatrix}}{\left\| \Phi(t_+ - \mu r, t_+)U \begin{pmatrix} -\sqrt{|\lambda_2|} \\ \sqrt{\lambda_1} \end{pmatrix} \right\|_2} \right\|_2\\
=&d_s^1(t_+ - \mu r).
\end{aligned}$$

□

The defining matrices of system (5.57) and (5.58) are $A = \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}$ and $A^2 = \begin{pmatrix} 2 & 0 \\ 0 & -8 \end{pmatrix}$, respectively. For the angle $\tau_s^1(\cdot)$ of the stable cone of system (5.57) and for the angle $\tau_s^2(\cdot)$ of the stable cone of system (5.58) we obtain the relation

$$\begin{aligned}
\tau_s^2(t_+ - r) &= \arccos\left(\frac{2 - \left(d_s^2(t_+ - r)\right)^2}{2} \right)\\
&= \arccos\left(\frac{2 - \left(d_s^1(t_+ - 2r)\right)^2}{2} \right) = \tau_s^1(t_+ - 2r)
\end{aligned}$$

for $r \in [0, \frac{t_+ - t_-}{2}]$. Thus, the angle of the stable cone of system (5.58) decreases two times as fast as the angle of the stable cone of system (5.57).

Next we check how close the upper bounds of the width given by Lemma 5.6.4 are compared to the exact width of the stable cones of (5.57) and (5.58).

Example 5.6.6. *Consider systems* (5.57) *and* (5.58) *for* $t \in \mathbb{I} = [0, \pi]$. *Let* Γ *be the identity. Then we get by Lemma 5.6.1 for both systems that the width* $d_s = d_s(t_+)$ *of the stable* t_+*-cone satisfies*

$$d_s = \begin{cases} \frac{2\sqrt{4}}{\sqrt{1+4}} = \frac{4}{\sqrt{5}}, & \text{for system (5.57)}, \\ \frac{2\sqrt{8}}{\sqrt{2+8}} = \sqrt{\frac{4\times 8}{10}} = \sqrt{\frac{16}{5}} = \frac{4}{\sqrt{5}}, & \text{for system (5.58)}. \end{cases}$$

Next we verify the assumptions of Lemma 5.6.4. We prove that the eigenvectors of the strain tensor lie inside the infinite time subspaces, i.e. the eigenvectors lie in the stable cone with the infinite time exponential rates. The stable subspace of system (5.57) *and* (5.58) *for* $\mathbb{I} = \mathbb{R}$ *is* $\mathcal{L}\left(\begin{pmatrix}0\\1\end{pmatrix}\right)$ *and the unstable subspace is* $\mathcal{L}\left(\begin{pmatrix}1\\0\end{pmatrix}\right)$. *The eigenvectors of*

$$S = \begin{cases} \begin{pmatrix}1 & 0\\ 0 & -4\end{pmatrix}, & \text{of system (5.57)}, \\ \begin{pmatrix}2 & 0\\ 0 & -8\end{pmatrix}, & \text{of system (5.58)} \end{cases}$$

are $\begin{pmatrix}0\\1\end{pmatrix}$ *to the negative eigenvalue and* $\begin{pmatrix}1\\0\end{pmatrix}$ *to the positive eigenvalue. Thus, the eigenvectors of* S *of system* (5.57) *fulfill the estimates* (3.9) *and* (3.10) *with the exponential rates* $\alpha_1 := 1$ *and* $\beta_1 := 4$, *while the eigenvectors of* S *of* (5.58) *fulfill* (3.9) *and* (3.10) *with* $\alpha_2 := 2$ *and* $\beta_2 := 8$. *Lemma 5.6.4 applies and yields that the width* $d_s^1(t)$ *of the stable* t*-cone of system* (5.57) *satisfies*

$$d_s^1(t) \le \sqrt{1 + \frac{4}{1} e^{-(4+1)(\pi - t)}} \frac{4}{\sqrt{5}} = 4e^{-5(\pi - t)},$$

and the width $d_s^2(t)$ *of the stable* t*-cone of system* (5.58) *satisfies*

$$d_s^2(t) \le \sqrt{1 + \frac{8}{2} e^{-(8+2)(\pi - t)}} \frac{4}{\sqrt{5}} = 4\left(e^{-5(\pi - t)}\right)^2.$$

With this approximated width ($d_{\text{approx}}(\cdot)$) *we find approximations* $\partial^{\mathbb{I}} V_s^{\text{approx}}$ *of the boundaries* $\partial^{\mathbb{I}} V_s$. *For our systems* (5.57) *and* (5.58) *we have*

$$\partial^{\mathbb{I}} V_s(t) = \left\{ \lambda \begin{pmatrix} \frac{d_s(t)}{2} \\ x_2 \end{pmatrix} \middle| \lambda \in \mathbb{R}, (d_s(t))^2 + x_2^2 \right\},$$

$$\partial^{\mathbb{I}} V_s^{\text{approx}}(t) := \left\{ \lambda \begin{pmatrix} \frac{d_{\text{approx}}(t)}{2} \\ x_2 \end{pmatrix} \middle| \lambda \in \mathbb{R}, (d_{\text{approx}}(t))^2 + x_2^2 = 1 \right\}.$$

In Figure 5.9 we plot the boundaries $\partial^{\mathbb{I}} V_s$ *of the stable cone and their approximation* $\partial^{\mathbb{I}} V_s^{\text{approx}}$ *at norm* 1, *projected onto the* $t - x_1$ *plane.*

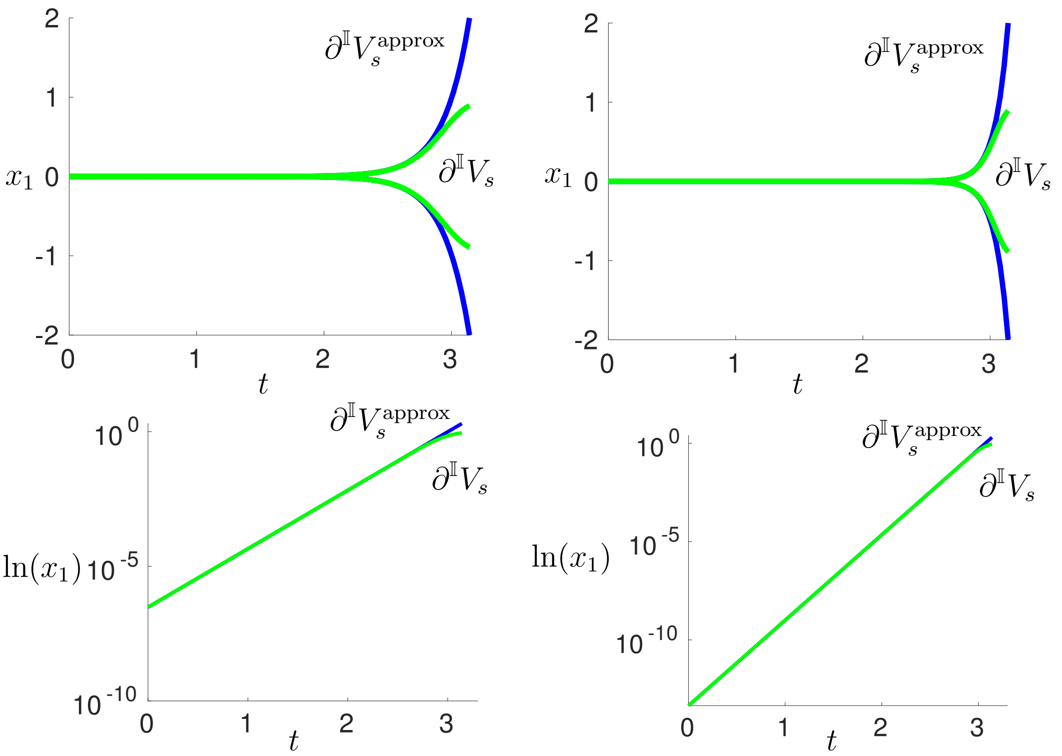


Figure 5.9: In the top the boundaries $\partial^{\mathbb{I}}V_s$ of the stable half-cone (green) and their approximation $\partial^{\mathbb{I}}V_s^{\text{approx}}$ (blue) at norm 1 are projected onto the $t - x_1$ plane. The bottom parts show the decay of the width of the boundaries and of the width of their approximation by a logarithmic scale in backward time. On the left panel we illustrate the stable cone for system (5.57) and on the right panel for system (5.58).

These are shown on the left for system (5.57) *and on the right for system* (5.58). *The approximation* $\partial^{\mathbb{I}}V_s^{\text{approx}}$ *looks quite good for both systems except for times close to* π*, where the width of the cones is much smaller than the approximated width. In the bottom part of Figure 5.9 we see by a logarithmic scale that the width of the cones as well as the distant of the exact boundaries and their approximation decays towards* 0 *in backward time.*

(Un)Stable Cones in 3- or Higher Dimensional D-Hyperbolic Systems

In this section we study two different autonomous 3 dimensional systems. We calculate and plot the stable and unstable cones and will see that the width of the cones decreases in one time direction each. Finally we finish this chapter with an analysis of the width of the stable and unstable cone in three or higher finite dimensional D-hyperbolic systems, where we prove the decrease and obtain bounds for the width.

For this purpose the boundaries of the cones are of relevance. Corollary 5.3.2 and Corollary 5.3.5 motivate to invest in the study of the zero Γ-strain set $Z_\Gamma(\cdot)$. Analogously to Lemma 5.6.1, we obtain the following characterization of the zero Γ-strain set.

Lemma 5.7.1. *Assume* **(A1)** *and* **(A2)***. Then the zero* Γ*-strain set of* (2.7)/(2.8) *is for* $k \geq 2$ *and* $t \in \tilde{\mathbb{I}}$

$$Z_\Gamma(t) = \{U(t)x | x \in \mathbb{R}^k : \sum_{i=1}^{k} x_i^2 \lambda_i(t) = 0\},$$

where $\lambda_i(t)$ *are the eigenvalues of* $S_\Gamma(t)$*,* $v_i(t)$ *the associated eigenvectors such that* $U(t) = (v_1(t) \cdots v_k(t))$ *is orthogonal.*

The next Lemma implies a explicit representation of the boundaries of the stable and unstable cone.

Lemma 5.7.2. *Assume* **(A1)** *and* **(A2)***. Then*

$$Z_\Gamma(\bar{t}) = \partial^{\mathbb{I}} V_s(\bar{t}), \quad Z_\Gamma(t_-) = \partial^{\mathbb{I}} V_u(t_-), \tag{5.77}$$

$$\partial^{\mathbb{I}} V_u(t_-) = \Phi(t_0, t_-) Z_\Gamma(t_-), \ t_0 \in \mathbb{I}. \tag{5.78}$$

Additionally, if (2.6) *is invertible we obtain*

$$\partial^{\mathbb{I}} V_s(t_0) = \Phi(t_0, \bar{t}) Z_\Gamma(\bar{t}) \tag{5.79}$$

for all $t_0 \in \tilde{\mathbb{I}}$.

Proof. Let $\lambda_1(t)$ be the eigenvalues of $S_\Gamma(t)$ and $v_i(t)$ the associated eigenvectors such that $U(t) = (v_1(t) \cdots v_k(t))$ is orthogonal for all $t \in \tilde{\mathbb{I}}$. Let $t \in \tilde{\mathbb{I}}$ and $x \in Z_\Gamma(t) \setminus \{0\}$. Then there exist a_i, $i \in \{1, \ldots, k\}$ such that

$$x = \sum_{i=1}^{k} a_i v_i(t)$$

and by Lemma 5.7.1 we have

$$0 = \langle x, S_\Gamma(t)x \rangle = \sum_{i=1}^{k} a_i^2 \lambda_i(t).$$

Since $x \neq 0$ there exists $j \in \{1, \ldots, k\}$ with $a_j \neq 0$. W.l.o.g. let $\lambda_j > 0$. Thus, for every neighborhood U of x there exists an $\varepsilon > 0$ such hat

$$y_\pm := \sum_{i=1, i \neq j}^{k} a_i v_i(t) + (a_j \pm \varepsilon) v_j(t) \in U.$$

W.l.o.g. let $a_j > 0$. Then for $\varepsilon < a_j$ we get

$$\begin{aligned}
\langle y_\pm, S_\Gamma(t) y_\pm \rangle &= \sum_{i=1, i\neq j}^{k} a_i^2 \lambda_i(t) + (a_j \pm \varepsilon)^2 \lambda_j(t) \\
&= \sum_{i=1}^{k} a_i^2 \lambda_i(t) + (\pm 2\varepsilon a_j + \varepsilon^2)\lambda_j(t) \\
&= (\pm 2\varepsilon a_j + \varepsilon^2)\lambda_j(t) \\
\langle y_+, S_\Gamma(t) y_+ \rangle &= (2\varepsilon a_j + \varepsilon^2)\lambda_j > 0, \\
\langle y_-, S_\Gamma(t) y_- \rangle &= (-2\varepsilon a_j + \varepsilon^2)\lambda_j < (-2\varepsilon^2 + \varepsilon^2)\lambda_j(t) = -\varepsilon^2 \lambda_j(t) < 0.
\end{aligned}$$

By Corollary 5.3.2 and 5.3.5 it follows that

$$y_- \in {}^{\mathbb{I}}V_s(\bar{t}), \ y_+ \in {}^{\mathbb{I}}V_u(t_-)$$

and, thus,

$$x \in \partial^{\mathbb{I}} V_s(\bar{t}), \ x \in \partial^{\mathbb{I}} V_u(t_-).$$

This proves (5.77). The invariance of $\partial^{\mathbb{I}} V_u$ and $\partial^{\mathbb{I}} V_s$ leads to (5.78) and (5.79). □

With this information we are able to plot the boundaries of the (un)stable cones of 3 dimensional systems.

Example 5.7.3. *Consider the system*

$$\dot{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix} x =: Ax \tag{5.80}$$

for $\mathbb{I} = [0, 3]$. *Let* Γ *be the identity. It is easily seen that this is a D-hyperbolic system with solution operator*

$$\Phi(t, s) = \begin{pmatrix} e^{t-s} & 0 & 0 \\ 0 & e^{-2(t-s)} & 0 \\ 0 & 0 & e^{-4(t-s)} \end{pmatrix}.$$

By calculating

$$\begin{aligned}
\Phi(t_0, 3)Z(3) &= \{\xi \in \mathbb{R}^3 | \langle \Phi(3, t_0)\xi, A\Phi(3, t_0)\xi \rangle = 0\}, \\
\Phi(t_0, 0)Z(0) &= \{\xi \in \mathbb{R}^3 | \langle \Phi(0, t_0)\xi, A\Phi(0, t_0)\xi \rangle = 0\}
\end{aligned}$$

for all $t_0 \in \mathbb{I}$ *we obtain the boundaries of* ${}^{\mathbb{I}}V_s(t_0)$ *and* ${}^{\mathbb{I}}V_u(t_0)$, *respectively. The eigenvalues of* A *are* $1, -2$ *and* -4. *Hence, the stable subspace of the infinite system generated by the associated eigenvectors* $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ *must lie* *inside* ${}^{\mathbb{I}}V_s(t_0)$ *for all* $t_0 \in \mathbb{I}$*, cf. Corollary 5.1.3. This leads to the fact, that the stable cone* ${}^{\mathbb{I}}V_s(t_0)$ *is infinite in the two directions of the two eigenvectors, see Figure 5.11. Analogously, we see that the unstable cone* ${}^{\mathbb{I}}V_u(t_0)$ *is infinite in the direction of the eigenvector* $\begin{pmatrix}1 & 0 & 0\end{pmatrix}^T$ *belonging to the eigenvalue* 1*. Figure 5.10 shows the boundaries* $\partial^{\mathbb{I}}V_u(t_0)$ *of system* (5.80) *for different times* t_0 *and the unstable subspace of the infinite system. The lightest red cone illustrates the boundary* $\partial^{\mathbb{I}}V_u(0)$*. The darkest red cone represents the boundary of the unstable cone at time* 0.8*. We see that this cone is close to the unstable subspace, marked in black. The other cones from light to dark red illustrate the boundaries of* ${}^{\mathbb{I}}V_u(t_0)$ *at* $t_0 = 0.2$*,* $t_0 = 0.4$ *and* $t_0 = 0.6$*, respectively. In the right panel of Figure 5.10 the cones are projected to the* $x_2 - x_3$*-plane. We see that they are elliptic, which is caused by the different negative eigenvalues of* S *(*-2 *and* -4*).*

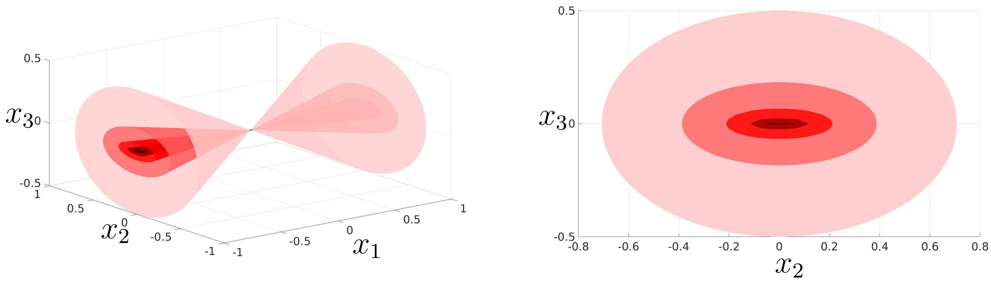


Figure 5.10: The boundaries of the unstable cone ${}^{\mathbb{I}}V_u$ (light to dark red) at time 0 up to 0.8 in 0.2 steps and the infinite subspace (black).

Analogously, Figure 5.11 shows that the boundaries of ${}^{\mathbb{I}}V_s(t_0)$ *converge in backward time (*$t_0 \searrow t_-$*) to the stable subspace of the infinite system, marked in light green-back. The boundary of* ${}^{\mathbb{I}}V_s(3)$ *is plotted in the darkest green, while the boundary of* ${}^{\mathbb{I}}V_s(2.2)$ *is illustrated in the lightest green. The three cones between, from dark to light, represent the boundaries of* ${}^{\mathbb{I}}V_s(t_0)$ *at* $t_0 = 2.8$*,* $t_0 = 2.6$ *and* $t_0 = 2.4$*.*

Since the stable and unstable subspace are generally not orthogonal towards each other we study a second example.

Example 5.7.4. *Consider the modified system*

$$\dot{x} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix} x =: \tilde{A}x \tag{5.81}$$

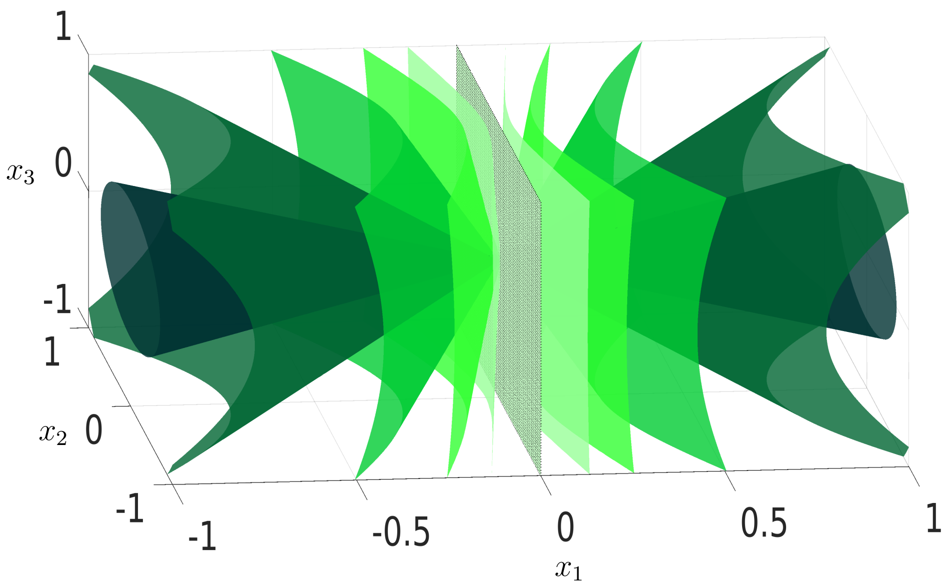


Figure 5.11: The boundaries of the stable cones ${}^{\mathbb{I}}V_s$ (dark to light green) at time 3 down to 2.2 in -0.2 steps and the infinite subspace (light green-black).

for $\mathbb{I} = [0, 3]$ *and let* Γ *be the identity. This system is also D-hyperbolic. The eigenvectors of* $\tilde{A}$ *are* $\begin{pmatrix}1 & 0 & 0\end{pmatrix}^T$, $\begin{pmatrix}1 & -3 & 0\end{pmatrix}^T$ *and* $\begin{pmatrix}0 & 0 & 1\end{pmatrix}^T$ *corresponding to the eigenvalues* $1, -2$ *and* -4*, respectively. To see the difference between the stable cones of system* (5.80) *and* (5.81) *we illustrate in Figure 5.12 the cones of* (5.81) *at the same times (*$3, 2.8, 2.6, 2.4$ *and* 2.2*) and in the same colors as in Figure 5.11.*

Lemma 5.7.6 ensures that the boundary of the stable and unstable cone converge against the stable and unstable subspace of the infinite system if the subspace lies inside the cone. More precisely, we show that the width of the cones decreases exponentially fast in one time direction. The width of the stable cone decreases in backward time whereas the width of the unstable cone decreases in forward time. First we introduce exponential rates of subspaces.

Lemma 5.7.5. *Let* $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$*,* $t_\pm \in \mathbb{I}$*,* $\mathbb{I} = [t_-, t_+]_{\mathbb{T}}$ *be a compact interval and let* $\Gamma = \Gamma^T > 0$*. Assume*

$$\begin{cases} \textbf{(A0)} \text{ and that } (2.6), \text{ generated by } (2.7), & \text{for } \mathbb{T} = \mathbb{R}, \\ \text{that systems } (2.6), \text{ generated by } (2.8), & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}$$

is M-hyperbolic on $\mathbb{I}$ *w.r.t.* $\|\cdot\|_\Gamma$ *and invertible. Let* $S \subset {}^{\mathbb{I}}V_s(t_+)$ *and* $U \subset {}^{\mathbb{I}}V_u(t_+)$ *be subspaces. Then there exist exponential rates* $\alpha, \beta > 0$ *such that*

$$\begin{aligned} \|\Phi(t_0, t_+)v_s\|_\Gamma &\geq e^{\alpha(t_+ - t_0)} \|v_s\|_\Gamma, \\ \|\Phi(t_0, t_+)v_u\|_\Gamma &\leq e^{-\beta(t_+ - t_0)} \|v_u\|_\Gamma \end{aligned} \tag{5.82}$$

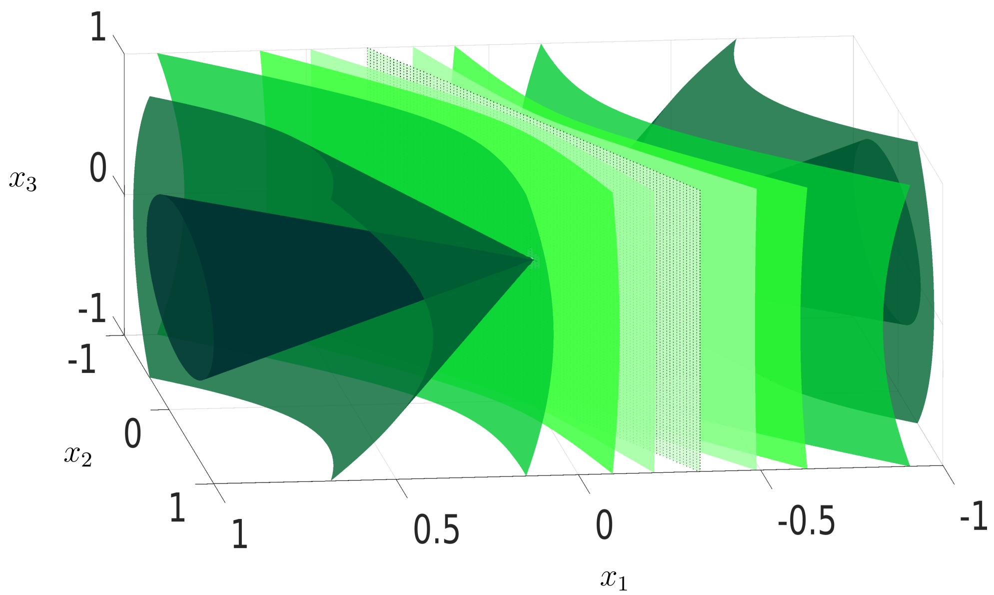


Figure 5.12: The boundaries of the stable cones ${}^{\mathbb{I}}V_s$ (dark to light green) at time 3 down to 2.2 in -0.2 steps and the infinite subspace (light green-black).

hold for all $t_0 \in \mathbb{I}$ *and* $v_s \in S$, $v_u \in U$.

Proof. By Theorem 4.2.4 we find a family of projectors $P : \mathbb{I} \to \mathbb{R}^{k\times k}$ with $S \subset \mathcal{R}(P(t_+))$ and $U \subset \mathcal{N}(P(t_+))$ such that (2.6) is M-hyperbolic on $\mathbb{I}$ with this family of projectors. Hence, there exist exponential rates $\alpha, \beta > 0$ such that

$$\begin{aligned} \|\Phi(t_0,t_+)v_s\|_\Gamma &\geq e^{\alpha(t_+-t_0)}\|v_s\|_\Gamma\,, \\ \|\Phi(t_0,t_+)v_u\|_\Gamma &\leq e^{-\beta(t_+-t_0)}\|v_u\|_\Gamma \end{aligned}$$

hold for all $t_0 \in \mathbb{I}$ and $v_s \in S \subset \mathcal{R}(P(t_+))$, $v_u \in U \subset \mathcal{N}(P(t_+))$. $\square$

Note, that for every D-hyperbolic invertible system the same holds, i.e. for every subspace of the cones there exists an exponential rate in the sense of (5.82).

Lemma 5.7.6. *Assume* **(A1)** *and that system* (2.6) *with* $k = 3$ *is invertible. Let* $S \subset {}^{\mathbb{I}}V_s(t_+)$ *be a subspace of maximal dimension* $r \in \{1,2\}$ *with exponential rate* α *and let* $v_s^1, \dots, v_s^r \in S$ *be a basis. Further let* $v_u^1, \dots, v_u^{k-r} \in {}^{\mathbb{I}}V_u(t_+)$ *be a basis of a subspace* $U \subset {}^{\mathbb{I}}V_u(t_+)$ *with exponential rate* β *which satisfies* $U \oplus S = \mathbb{R}^k$. *Then the distance of the stable cone* ${}^{\mathbb{I}}V_s(t_0)$, $t_0 \in \mathbb{I}$, *satisfies*

$$d_s(t_0) \leq \bar{C}e^{-\delta(t_+-t_0)} \tag{5.83}$$

with $\delta = \alpha + \beta$.

Proof. For clarity we do not always mention the dependency on t_+ explicitly. By Corollary 5.3.2/ 5.3.5 one of the two cases

$$\begin{cases} {}^{\mathbb{I}}V_s(t_+) \text{ is a connected cone,} & \text{case (i)} \\ {}^{\mathbb{I}}V_s(t_+) \text{ has two connected half-cones,} & \text{case (ii)} \end{cases}$$

is true. Set

$${}^{\mathbb{I}}_{C}V_s(t_+) := \begin{cases} {}^{\mathbb{I}}V_s(t_+), & \text{case (i)} \\ \text{half-cone of } {}^{\mathbb{I}}V_s(t_+), & \text{case (ii)} \end{cases}$$

and denote by $\mathcal{X}_i$, $i \in \{1, \ldots, \ell\}$ the connected components of $\partial^{\mathbb{I}}_{C}V_s(t_+) \cap \partial\mathcal{S}^1$, where $\mathcal{S}^1$ is the unit sphere in $\mathbb{R}^k$. Note that the subspace $S \subset {}^{\mathbb{I}}V_s(t_+)$ satisfies

$$\dim(S) \begin{cases} = 1, & \text{if } \ell = 1, \\ = 2, & \text{if } \ell = 2. \end{cases} \tag{5.84}$$

Let

$$\bar{x} \in \partial^{\mathbb{I}}_{C}V_s(t_+) \cap \partial\mathcal{S}^1$$

then we find $\lambda_i, \mu_j \in \mathbb{R}$, $i \in \{1, \ldots, r\}$, $j \in \{1, \ldots, k-r\}$ such that

$$\bar{x} = \sum_{i=1}^{r} \lambda_i v_s^i + \sum_{j=1}^{k-r} \mu_j v_u^j.$$

Define $v_s := \sum_{i=1}^{r} \lambda_i v_s^i \in S \subset {}^{\mathbb{I}}V_s(t_+)$ and $v_u := \sum_{j=1}^{k-r} \mu_j v_u^j \in U \subset {}^{\mathbb{I}}V_u(t_+)$. Since S has the exponential rate α the estimate

$$\|\Phi(t_0, t_+)v_s\|_\Gamma \geq e^{\alpha(t_+ - t_0)} \|v_s\|_\Gamma \tag{5.85}$$

holds by (3.15) for all $t_0 \in \mathbb{I}$. For v_u we get by (3.10) for every $t_0 \in \mathbb{I}$

$$\|\Phi(t_0, t_+)v_u\|_\Gamma \leq e^{-\beta(t_+ - t_0)} \|v_u\|_\Gamma \,. \tag{5.86}$$

First we prove for $\ell = 1$ and every $\bar{x}, \bar{y} \in \mathcal{X}_1$ that there exists a constant $\bar{C} > 0$ such that

$$\left\| \frac{\Phi(t_0, t_+)\bar{x}}{\|\Phi(t_0, t_+)\bar{x}\|_2} - \frac{\Phi(t_0, t_+)\bar{y}}{\|\Phi(t_0, t_+)\bar{y}\|_2} \right\|_2 \leq \bar{C} e^{-\delta(t_+ - t_0)} \tag{5.87}$$

holds for all $t_0 \in \mathbb{I}$, where $\delta := \alpha + \beta$. Let $\ell = 1$. Then we obtain by (5.84)

$$\bar{x} = \lambda_1 v_s^1 + \sum_{j=1}^{k-1} \mu_j v_u^j$$

with $v_s = \lambda_1 v_s^1$ and $v_u = \sum_{j=1}^{k-1} \mu_j v_u^j$. For every $\bar{y} \in \partial_C^{\mathbb{I}} V_s(t_+) \cap \partial\mathcal{S}^1$ there exist $\tilde{\lambda}_1, \tilde{\mu}_j \in \mathbb{R}^k$, $j \in \{1, \ldots, k-1\}$ such that

$$\bar{y} = \tilde{\lambda}_1 v_s^1 + \sum_{j=1}^{k-1} \tilde{\mu}_j v_u^j.$$

Set $y = \frac{\lambda_1}{\tilde{\lambda}_1}\bar{y} = v_s + \frac{\lambda_1}{\tilde{\lambda}_1}\sum_{j=1}^{k-1} \tilde{\mu}_j v_u^j =: v_s + \tilde{v}_u$. Let $t_0 \in \mathbb{I}$. Then we get for the distance between the normed $\Phi(t_0, t_+)\bar{x} \in \partial_C^{\mathbb{I}} V_s(t_0)$ and the normed $\Phi(t_0, t_+)\bar{y} \in \partial_C^{\mathbb{I}} V_s(t_0)$

$$\begin{aligned}
\left\| \frac{\Phi(t_0,t_+)\bar{x}}{\|\Phi(t_0,t_+)\bar{x}\|_2} - \frac{\Phi(t_0,t_+)\bar{y}}{\|\Phi(t_0,t_+)\bar{y}\|_2} \right\|_2 &= \left\| \frac{\Phi(t_0,t_+)\bar{x}}{\|\Phi(t_0,t_+)\bar{x}\|_2} - \frac{\frac{\tilde{\lambda}_1}{\lambda_1}\Phi(t_0,t_+)y}{\frac{\tilde{\lambda}_1}{\lambda_1}\|\Phi(t_0,t_+)y\|_2} \right\|_2 \\
&\leq \left\| \frac{\Phi(t_0,t_+)\bar{x}}{\|\Phi(t_0,t_+)v_s\|_2} - \frac{\Phi(t_0,t_+)y}{\|\Phi(t_0,t_+)v_s\|_2} \right\|_2 \\
&= \frac{\|\Phi(t_0,t_+)(v_u - \tilde{v}_u)\|_2}{\|\Phi(t_0,t_+)v_s\|_2}.
\end{aligned}$$

Using (5.85) and (5.86) and the equivalence of $\|\cdot\|_2$ and $\|\cdot\|_\Gamma$ then there exist constants $C > 0$, $C(\bar{x}, \bar{y}) > 0$ such that the distance satisfies

$$\begin{aligned}
\left\| \frac{\Phi(t_0,t_+)\bar{x}}{\|\Phi(t_0,t_+)\bar{x}\|_2} - \frac{\Phi(t_0,t_+)\bar{y}}{\|\Phi(t_0,t_+)\bar{y}\|_2} \right\|_2 &\leq C \frac{\|\Phi(t_0,t_+)(v_u - \tilde{v}_u)\|_\Gamma}{\|\Phi(t_0,t_+)v_s\|_\Gamma} \\
&\leq C \frac{\|(v_u - \tilde{v}_u)\|_\Gamma \, e^{-\beta(t_+ - t_0)}}{\|v_s\|_\Gamma \, e^{\alpha(t_+ - t_0)}} \\
&\leq C(\bar{x}, \bar{y}) e^{-\delta(t_+ - t_0)} \\
&\leq \bar{C} e^{-\delta(t_+ - t_0)}
\end{aligned}$$

with $\delta = \alpha + \beta$ and $\bar{C} := \max\left\{C(x,y) \middle| x, y \in (\partial_C^{\mathbb{I}} V_s(t_+) \cap \partial S^1)\right\}$ (exists since $\partial_C^{\mathbb{I}} V_s(t_+) \cap \partial S^1$ is compact). This shows estimate (5.87) for $\ell = 1$.

Next we prove for $\ell = 2$ that we find for every $\bar{x} \in \mathcal{X}_i$, $i \in \{1, 2\}$ a $\bar{y} \in \mathcal{X}_j$, $j \neq i$ and a $\bar{C} > 0$ such that (5.87) holds for all $t_0 \in \mathbb{I}$. Let $\ell = 2$. First we construct a special $\bar{y} \in \mathbb{R}^k$ and show that $\bar{y} \in \partial_C^{\mathbb{I}} V_s(t_+) \cap \partial S^1$. Then we prove that $\bar{x}$ and $\bar{y}$ lie in different connecting components of $\partial_C^{\mathbb{I}} V_s(t_+) \cap \partial\mathcal{S}^1$.

Equation (5.24) yields that for $v_s \in {}^{\mathbb{I}}V_s(t_+)$ a $s > 0$ exists such that

$$\langle v_s, S_\Gamma v_s \rangle = -s.$$

By (5.25) we obtain $\langle \Phi(t_-, t_+)v_u, S_\Gamma(t_+)\Phi(t_-, t_+)v_u \rangle > 0$ and with Lemma 5.2.2 $\langle v_u, S_\Gamma(t_+)v_u \rangle > 0$ is satisfied. This means there exists a $u > 0$ such that

$$\langle v_u, S_\Gamma(t_+)v_u \rangle = u.$$

The symmetry of $\langle \cdot, \cdot \rangle$ and S_Γ leads since $\bar{x} \in \partial_C^{\mathbb{I}} V_s(t_+) \subset \partial^{\mathbb{I}} V_s(t_+)$ to

$$\begin{aligned} 0 = \langle \bar{x}, S_\Gamma \bar{x} \rangle &= \langle v_s, S_\Gamma v_s \rangle + 2\langle v_s, S_\Gamma v_u \rangle + \langle v_u, S_\Gamma v_u \rangle \\ &= -s + 2\langle v_s, S_\Gamma v_u \rangle + u. \end{aligned} \tag{5.88}$$

For $y := y(\lambda) := \lambda \left(v_s + \frac{-s}{u} v_u \right)$, $\lambda \in \mathbb{R}$ we see using (5.88)

$$\begin{aligned} \langle y, S_\Gamma y \rangle &= \lambda^2 \langle v_s, S_\Gamma v_s \rangle + \lambda^2 2 \frac{-s}{u} \langle v_s, S_\Gamma v_u \rangle + \lambda^2 \left(\frac{-s}{u} \right)^2 \\ &= \lambda^2 \left(-s - \frac{s}{u} 2 \langle v_s, S_\Gamma v_u \rangle + \frac{s^2}{u^2} u \right) \\ &= \lambda^2 \left(-s - \frac{s}{u}(s-u) + \frac{s^2}{u} \right) \\ &= 0. \end{aligned}$$

This implies $y \in \partial^{\mathbb{I}} V_s(t_+)$. Set $\bar{\lambda} \in \mathbb{R}$ such that $\bar{y} := y(\bar{\lambda}) \in \partial_C^{\mathbb{I}} V_s(t_+)$ and $\|\bar{y}\| = 1$, i.e. $\bar{y} \in \partial_C^{\mathbb{I}} V_s(t_+) \cap \partial \mathcal{S}^1$. Next we show, that $\bar{x}$ and $\bar{y}$ lie in different connecting components. Since $v_u \notin S$ and $\frac{s}{u} > 0$ we observe that the vectors $\bar{x} = v_s + v_u$ and $\bar{y} = v_s - \frac{s}{u} v_u$ lie on different sides of the two dimensional subspace S, see (5.84). This means every path between $\bar{x}$ and $\bar{y}$ goes through S. Further, we have $S \cap \partial^{\mathbb{I}} V_s(t_+) \subset {}^{\mathbb{I}}V_s(t_+) \cap \partial^{\mathbb{I}} V_s(t_+) = \{0\}$, since ${}^{\mathbb{I}}V_s(t_+) \setminus \{0\}$ is by Corollary 5.3.2/ 5.3.5 open. Thus, there does not exist a path between $\bar{x}$ and $\bar{y}$ which lies in $\partial_C^{\mathbb{I}} V_s(t_+) \cap \partial \mathcal{S}^1 \subset \partial^{\mathbb{I}} V_s(t_+) \setminus \{0\}$. Hence, $\bar{x}$ and $\bar{y}$ lie in different connecting components.

Next we show estimate (5.87). Fix $t_0 \in \mathbb{I}$. Then we get for the distance between the normed $\Phi(t_0, t_+)\bar{x}$ and $\Phi(t_0, t_+)\bar{y}$ as in the case $\ell = 1$

$$\begin{aligned} \left\| \frac{\Phi(t_0,t_+)\bar{x}}{\|\Phi(t_0,t_+)\bar{x}\|_2} - \frac{\Phi(t_0,t_+)\bar{y}}{\|\Phi(t_0,t_+)\bar{y}\|_2} \right\|_2 &= \left\| \frac{\Phi(t_0,t_+)\bar{x}}{\|\Phi(t_0,t_+)\bar{x}\|_2} - \frac{\bar{\lambda}\Phi(t_0,t_+)\bar{y}(1)}{\bar{\lambda}\,\|\Phi(t_0,t_+)\bar{y}(1)\|_2} \right\|_2 \\ &\le \left\| \frac{\Phi(t_0,t_+)\bar{x}}{\|\Phi(t_0,t_+)v_s\|_2} - \frac{\Phi(t_0,t_+)\bar{y}(1)}{\|\Phi(t_0,t_+)v_s\|_2} \right\|_2 \\ &\le \frac{\left\|\Phi(t_0,t_+)(v_s + v_u - (v_s - \frac{s}{u} v_u))\right\|_2}{\|\Phi(t_0,t_+)v_s\|_2} \\ &= \frac{(1+\frac{s}{u})\,\|\Phi(t_0,t_+)v_u\|_2}{\|\Phi(t_0,t_+)v_s\|_2} \\ &\le C \frac{(1+\frac{s}{u})\,\|\Phi(t_0,t_+)v_u\|_\Gamma}{\|\Phi(t_0,t_+)v_s\|_\Gamma} \\ &\le C(1+\frac{s}{u}) \frac{\|v_u\|_\Gamma\, e^{-\beta(t_+-t_0)}}{\|v_s\|_\Gamma\, e^{\alpha(t_+-t_0)}} \\ &\le C(\bar{x}) e^{-\delta(t_+-t_0)} \\ &\le \bar{C} e^{-\delta(t_+-t_0)} \end{aligned}$$

for constants $C, C(\bar{x}) > 0$ with $\bar{C} := \max\left\{C(x)\middle|x \in (\partial^{\mathbb{I}}V_s(t_+) \cap \partial S^1)\right\}$ and $\delta = \alpha + \beta$.

Finally we proof (5.83). Note that by Definition 4.1.1 the distance of ${}^{\mathbb{I}}V_s(t_+)$ equals the distance of ${}^{\mathbb{I}}_{C}V_s(t_+)$ for all $t_0 \in \mathbb{I}$. Fix $t_0 \in \mathbb{I}$ and denote by $\mathcal{X}_i(t)$, $i \in \{1, 2\}$ the connecting components of $\partial^{\mathbb{I}}_{C}V_s(t) \cap \partial\mathcal{S}^1$ for all $t \in \mathbb{I}$. For $\ell = 1$ the distance of ${}^{\mathbb{I}}V_s(t_+)$ satisfies by (5.87) and the invertibility of $\Phi(\cdot, \cdot)$

$$\begin{aligned} d_s(t) &= \max\{\|x - y\|_2\,|x, y \in \mathcal{X}_1(t_0)\} \\ &\leq \max\{\bar{C}e^{-\delta(t_+ - t_0)}|x, y \in \mathcal{X}_1(t_0)\} = \bar{C}e^{-\delta(t_+ - t_0)}. \end{aligned}$$

Let $\ell = 2$. Then we obtain by (5.87)

$$\begin{aligned} d_H(\mathcal{X}_1(t_0), \mathcal{X}_2(t_0)) &= \max\{\min\{\|x_1 - x_2\|_2\,|x_2 \in \mathcal{X}_2(t_0)\}|x_1 \in \mathcal{X}_1(t_0)\} \\ &\leq \max\{\|\bar{x} - \bar{y}(\bar{x})\|_2\,|\bar{x} \in \mathcal{X}_1(t_0) \text{ and } \bar{y}(\bar{x}) \text{ as defined before}\} \\ &\leq \max\{\bar{C}e^{-\delta(t_+ - t_0)}|\bar{x} \in \mathcal{X}_1(t_0)\} \\ &= \bar{C}e^{-\delta(t_+ - t_0)} \end{aligned}$$

and directly

$$d_H(\mathcal{X}_2(t_0), \mathcal{X}_1(t_0)) \leq \bar{C}e^{-\delta(t_+ - t_0)}.$$

Thus, the distance of ${}^{\mathbb{I}}V_s(t_+)$ satisfies

$$d_s(t_0) = \max\{d_H(\mathcal{X}_1(t_0), \mathcal{X}_2(t_0)), d_H(\mathcal{X}_2(t_0), \mathcal{X}_1(t_0))\} \leq \bar{C}e^{-\delta(t_+ - t_0)}.$$

□

Chapter 6

Fiber Bundles in Finite and Infinite Time

In dynamical systems, stable and unstable manifolds of a hyperbolic trajectory are important sources for understanding underlying dynamics. Fiber bundles are the nonautonomous equivalent of hyperbolic manifolds in the autonomous case. Their study is important to understand the local behavior of nonlinear systems. In many areas of science and engineering invariant manifolds help formulating problems and finding special solutions. In [133] the benefit of invariant manifolds theory for various examples is discussed.

In this chapter we study finite and infinite time fiber bundles. We present three different ways to define finite time fiber bundles (ft-fiber bundles) and analyze the advantage and disadvantage between these concepts. To study homoclinic trajectories, which is our purpose later, the stable and unstable fiber bundle need to intersect. We analyze this property for the three concepts and find in this way an adequate analogon to the infinite time fiber bundles. In the last section we prove that roughly speaking the (un)stable cone of a linearization locally approximates the (un)stable (ft-)fiber bundles of the original system. This enables the numerical computation of the fiber bundles, which will be focused on in Section 6.7. We start this chapter by defining the invariant infinite time fiber bundles. The definition of invariant fiber bundles for ODE models can be found in [113]. Note that invariant fiber bundles of difference equations are similarly defined, see [7].

Definition 6.0.1. ***Stable and unstable global fiber bundles*** *of a trajectory* $\bar{x}(\cdot)$ *of equation* (2.2)/(2.3) *are defined as*

$$W_{s,u}^{\bar{x}} := \{(x,t) \in \mathbb{R}^k \times \mathbb{R} : \lim_{s \to \pm\infty} \|\varphi(x,s,t) - \bar{x}(s)\| = 0\} \ (\textit{cont.})$$

$$W_u^{\bar{x}} := \{(x_0,t_0)\mathbb{R}^k \times \mathbb{Z} : \exists \textit{ solution } x : \mathbb{Z} \to \mathbb{R}^k : x(t_0) = x \quad \lim_{t \to -\infty} \|x(t) - \bar{x}(t)\| = 0\} \ (\textit{disk.})$$

and ***global t-fibers*** *are* $W_{s,u}^{\bar{x}}(t) := \{x \in \mathbb{R}^k : (x,t) \in W_{s,u}^{\bar{x}}\}$.

Local fiber bundles *w.r.t. a neighborhood* $U(\cdot) \subset \mathbb{R}^k$ *of* $\bar{x}(\cdot)$ *are defined as*

$$
\begin{aligned}
{}_UW_s^{\bar{x}} &:= \{(x,t) \in W_s^{\bar{x}} : \varphi(x,s,t) \in U(s) \quad \forall s \geq t\}, \\
{}_UW_u^{\bar{x}} &:= \{(x,t) \in W_u^{\bar{x}} : \varphi(x,s,t) \in U(s) \quad \forall s \leq t\}
\end{aligned}
$$

and ***local t-fibers*** *are* ${}_UW_{s,u}^{\bar{x}}(t) := \{x \in \mathbb{R}^k : (x,t) \in {}_UW_{s,u}^{\bar{x}}\}$.

Monotonically (Un)Stable Ft-Fiber Bundles

In the literature we find, for example, in [45, Definition 35] a definition of stable and unstable fiber bundles for planar nonautonomous finite time differential equations, which requires a strictly monotonic convergence of solutions in forward or backward time towards the reference trajectory. This definition can be extended to higher dimensional differential equations. Hence, we call these sets monotonically (un)stable ft-fiber bundles. Note, that these sets do not have a manifold structure. However, they are some kind of analogon to the infinite time fiber bundles and, hence, we call them ft-fiber bundles. Thus, if we speak of ft-fiber bundles we do not mean that they satisfy respective topological structure, we only want to point out that they have analog properties as the infinite time fiber bundles.

In Section 6.2 and 6.3 we introduce two more sets, the monotonically ε-(un)stable ft-fiber bundle and respectively the ε-(un)stable ft-fiber bundle. We point out the differences between these concepts and analyze which one of the three definitions is the best suited analogon to the infinite time fiber bundle for our purpose, the study of homoclinic trajectories.

Definition 6.1.1. *Let* $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ *be a solution of* (2.2) *with* $\mathbb{I}$ *a compact interval and* $t_0 \in \mathbb{I}$. *Then the set*

$$
{}^{\mathbb{I}}M_s^{\bar{x}}(t_0) := \left\{ x_0 \in \mathbb{R}^k \Big| \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(x_0,t,t_0) - \bar{x}\| < 0 \text{ for all } t \in \mathbb{I} \right\} \cup \{\bar{x}(t_0)\}
$$

is called the ***monotonically stable $\mathbf{t_0}$-ft-fiber*** *of* $\bar{x}$ *on* $\mathbb{I}$ *w.r.t. the chosen norm and*

$$
{}^{\mathbb{I}}M_u^{\bar{x}}(t_0) := \left\{ x_0 \in \mathbb{R}^k \Big| \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(x_0,t,t_0) - \bar{x}\| > 0 \text{ for all } t \in \mathbb{I} \right\} \cup \{\bar{x}(t_0)\}
$$

is called the ***monotonically unstable $\mathbf{t_0}$-ft-fiber*** *of* $\bar{x}$ *on* $\mathbb{I}$ *w.r.t. the chosen norm. Further the sets*

$$
{}^{\mathbb{I}}M_{s,u}^{\bar{x}} := \left\{ (x_0,t_0) \in \mathbb{R}^k \times \mathbb{I} \middle| x_0 \in {}^{\mathbb{I}}M_{s,u}^{\bar{x}}(t_0) \right\}
$$

are called the ***monotonically stable and unstable ft-fiber bundles*** *of* $\bar{x}$ *on* $\mathbb{I}$ *w.r.t. the chosen norm.*

Next we show that for linear systems the stable and unstable cone, defined for a differentiable norm as in (4.12) and (4.13), coincide with the fiber bundles, defined as above, united with zero.

Lemma 6.1.2. *Let $\|\cdot\|$ be an arbitrary differentiable norm in $\mathbb{R}^k$, $\mathbb{I} \subset \mathbb{R}$ a compact interval and $t_0 \in \mathbb{I}$. Further let ${}^{\mathbb{I}}V_s(t_0)$, ${}^{\mathbb{I}}V_u(t_0)$ be the stable and unstable t_0-cone w.r.t. the chosen norm $\|\cdot\|$ of an M-hyperbolic system on $\mathbb{I}$ with solution operator $\Phi(\cdot,\cdot)$ and exponential rates $\alpha, \beta > 0$. Then we have*

$$ {}^{\mathbb{I}}V_s(t_0) = \left\{\xi \in \mathbb{R}^k \middle| \tfrac{\mathrm{d}}{\mathrm{d}t}\|\Phi(t,t_0)\xi\| < 0 \text{ for all } t \in \mathbb{I}\right\} \cup \{0\} = {}^{\mathbb{I}}M_s^0(t_0), \quad (6.1)$$

$$ {}^{\mathbb{I}}V_u(t_0) = \left\{\xi \in \mathbb{R}^k \middle| \tfrac{\mathrm{d}}{\mathrm{d}t}\|\Phi(t,t_0)\xi\| > 0 \text{ for all } t \in \mathbb{I}\right\} \cup \{0\} = {}^{\mathbb{I}}M_u^0(t_0), \quad (6.2)$$

where ${}^{\mathbb{I}}M_{s,u}(t_0)$ are the monotonically stable and unstable ft-fiber bundles of the given system.

Proof. For $\xi \in {}^{\mathbb{I}}V_s(t_0)$ the inequality $\|\Phi(t,t_0)\xi\|e^{\alpha t} \le \|\Phi(s,t_0)\xi\|e^{\alpha s}$ holds for all $t, s \in \mathbb{I}$ with $t \ge s$. Thus, we obtain for all $t \in \mathbb{I}$

$$0 \ge \frac{\mathrm{d}}{\mathrm{d}t}\left[\|\Phi(t,t_0)\xi\|e^{\alpha t}\right] = e^{\alpha t}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi(t,t_0)\xi\| + \|\Phi(t,t_0)\xi\|\alpha e^{\alpha t}. \quad (6.3)$$

The inequality is trivial for $\xi = 0$. For $\xi \ne 0$ we have $\|\Phi(t,t_0)\xi\|\alpha e^{\alpha t} > 0$ for all $t \in \mathbb{I}$. By (6.3) the estimate $\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi(t,t_0)\xi\| < 0$ holds for all $t \in \mathbb{I}$, since $e^{\alpha t} > 0$. This leads to

$$ {}^{\mathbb{I}}V_s(t_0) \subset \{\xi \in \mathbb{R}^k | \tfrac{\mathrm{d}}{\mathrm{d}t}\|\Phi(t,t_0)\xi\| < 0 \text{ for all } t \in \mathbb{I}\} \cup \{0\} = {}^{\mathbb{I}}M_s^0(t_0).$$

Next we show ${}^{\mathbb{I}}M_s^0(t_0) \subset {}^{\mathbb{I}}V_s(t_0)$. By definition of ${}^{\mathbb{I}}V_s(t_0)$ it follows that $0 \in {}^{\mathbb{I}}V_s(t_0)$ for all $t_0 \in \mathbb{I}$. Let $\xi \in {}^{\mathbb{I}}M_s^0(t_0) \setminus \{0\}$ then there exists an $r \in \mathbb{R}$ with $r > 0$ such that

$$\max_{t \in \mathbb{I}} \frac{\mathrm{d}}{\mathrm{d}t}\|\Phi(t,t_0)\xi\| = -r,$$

since $\mathbb{I}$ is a compact interval. Choose α small enough such that

$$\max_{t \in \mathbb{I}} \|\Phi(t,t_0)\xi\|\alpha \le r$$

holds then we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\Phi(t,t_0)\xi\|e^{\alpha t}\right] &= e^{\alpha t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi(t,t_0)\xi\| + \|\Phi(t,t_0)\xi\|\alpha\right) \\
&\le e^{\alpha t}\left(\max_{t \in \mathbb{I}}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi(t,t_0)\xi\| + \max_{t \in \mathbb{I}}\|\Phi(t,t_0)\xi\|\alpha\right) \\
&\le e^{\alpha t}(-r + r) = 0
\end{aligned}$$

for all $t \in \mathbb{I}$. Hence, the inequality

$$\|\Phi(t,t_0)\xi\|e^{\alpha t} \le \|\Phi(s,t_0)\xi\|e^{\alpha s}$$

holds for all $t, s \in \mathbb{I}$ with $t \ge s$. This leads to $\xi \in {}^{\mathbb{I}}V_s(t_0)$ and (6.1) is proved. Analogously, we can show (6.2). □

Remark 6.1.3. *Assume* **(A0)**. *Let* $\Gamma = \Gamma^T > 0$ *and let system* (2.7) *be D-hyperbolic on the compact interval* $\mathbb{I} \subset \mathbb{R}$ *w.r.t. the* Γ*-norm. Denote by* $\Phi(\cdot,\cdot)$ *the solution operator then we have*

$$
\begin{aligned}
{}^{\mathbb{I}}V_s(t_0) &= \left\{\xi \in \mathbb{R}^k | \langle \Phi(t,t_0)\xi, S_\Gamma(t)\Phi(t,t_0)\xi\rangle < 0 \text{ for all } t \in \mathbb{I}\right\} \cup \{0\}\,,\\
{}^{\mathbb{I}}V_u(t_0) &= \left\{\xi \in \mathbb{R}^k | \langle \Phi(t,t_0)\xi, S_\Gamma(t)\Phi(t,t_0)\xi\rangle > 0 \text{ for all } t \in \mathbb{I}\right\} \cup \{0\}\,.
\end{aligned}
$$

Indeed, since a Γ*-norm is differentiable the statements follow from equations* (6.1), (6.2) *and equation* (3.16). *Without using equations* (6.1) *and* (6.2) *we already showed (see equations* (5.24), (5.25)*)*

$$
\begin{aligned}
{}^{\mathbb{I}}V_s(t_0) &= \left\{\xi \in \mathbb{R}^k | \langle \Phi(t_+,t_0)\xi, S_\Gamma(t_+)\Phi(t_+,t_0)\xi\rangle < 0\right\} \cup \{0\},\\
{}^{\mathbb{I}}V_u(t_0) &= \left\{\xi \in \mathbb{R}^k | \langle \Phi(t_-,t_0)\xi, S_\Gamma(t_-)\Phi(t_-,t_0)\xi\rangle > 0\right\} \cup \{0\}.
\end{aligned}
$$

Since our system is D-hyperbolic $M_{Z_\Gamma}(t)$ *is positive definite for all* $t \in \mathbb{I}$ *and we get by Corollary 5.2.2 for all* $\xi_s \in {}^{\mathbb{I}}V_s(t_0) \setminus \{0\}$, $\xi_u \in {}^{\mathbb{I}}V_u(t_0) \setminus \{0\}$ *and* $t \in \mathbb{I} = [t_-, t_+]$

$$
\begin{aligned}
\Phi(t,t_0)\xi_s &= \Phi(t,t_+)\Phi(t_+,t_0)\xi_s \in \{\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(t)\xi\rangle < 0\},\\
\Phi(t,t_0)\xi_u &= \Phi(t,t_-)\Phi(t_-,t_0)\xi_u \in \{\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(t)\xi\rangle > 0\}.
\end{aligned}
$$

In the following we study an example of a solution of an infinite time autonomous hyperbolic dynamical system, which lies on the infinite time stable manifold. We show that a finite part of this solution lies on the finite time monotonically unstable ft-fiber bundle, defined in Definition 6.1.1. One hyperbolic system that has the properties described in the following example is generated by the differential equation

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -u_2 + 0.98u_1 \\ u_1^2 - u_2 \end{pmatrix}. \tag{6.4}
$$

Example 6.1.4. *Consider a hyperbolic autonomous dynamical system on* $\mathbb{R}$, *which has a fixed point* ξ *and a nontrivial solution* $x(\cdot)$ *with* $\lim_{t\to\infty} x(t) = \xi$, *i.e.* $x(t) \in W_s^\xi(t)$. *Denote by* φ *the solution operator. Let* $t_-, t_+, s_-, s_+ \in \mathbb{R}$ *with*

$$
t_+ > t_- > s_+ > s_-,\ t_+ - t_- = s_+ - s_-.
$$

The solution $x(\cdot)$ *curvess as presented in Figure 6.1.*

For $\mathbb{I} = [t_-, t_+]$ *we observe that* $x(t_0) \in {}^{\mathbb{I}}M_s^\xi(t_0)$ *for all* $t_0 \in \mathbb{I}$, *while for* $\mathbb{I} = [s_-, s_+]$ *we get* $x(t_0) \in {}^{\mathbb{I}}M_u^\xi(t_0)$ *for all* $t_0 \in \mathbb{I}$.

Thus, we see that a finite part of an infinite solution can lie on the monotonically unstable ft-fiber although the infinite solution lies on the stable fiber.

To avoid this or at least to reduce the possibility of such a case we introduce the variable $\varepsilon > 0$ and require a addional condition. The new bundles are called monotonically ε-(un)stable ft-fiber bundles and are introduced in the next section.

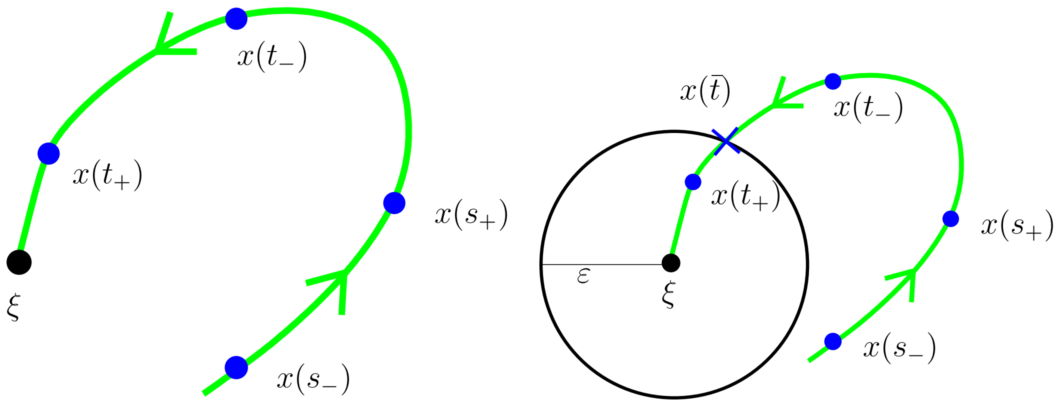


Figure 6.1: A fixed point ξ (black) and a solution $x(\cdot)$ which converges to the fixed point (presented in green). In the right an ε-ball is drawn around ξ and the solution point $x(\bar{t})$ on the border is marked by a blue cross.

Monotonically ε-(Un)Stable Ft-Fiber Bundles

In this section we introduce a second way to define ft-fiber bundles, the so called monotonically ε-(un)stable ft-fiber bundles. We find similarities to the monotonically (un)stable ft-fiber bundles. Additionally to the continuous time case, we define the monotonically ε-(un)stable ft-fibers for discrete time systems as well.

In the following let $B_\varepsilon(y) := \left\{x \in \mathbb{R}^k \middle| \, \|x-y\|_2 < \varepsilon\right\}$ denote the open ε-ball around $y \in \mathbb{R}^k$ with radius $\varepsilon \in \mathbb{R}_{>0}$.

Definition 6.2.1. *Let $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ be a solution of (2.2) and $\varepsilon > 0$. Then the set*

$$ {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_{s} := \Big\{ (x_0, t_0) \in \mathbb{R}^k \times \mathbb{I} \,\Big|\, \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(x_0, t, t_0) - \bar{x}(t)\| < 0 \text{ for all } t \in \mathbb{I}, \\ \varphi(x_0, t_+, t_0) \in B_\varepsilon(\bar{x}(t_+)) \Big\} \cup \{(\bar{x}(t_0), t_0) | t_0 \in \mathbb{I}\} $$

is called the ***monotonically ε-stable ft-fiber bundle*** *of $\bar{x}$ on $\mathbb{I}$ w.r.t. the chosen differentiable norm and*

$$ {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_{u} := \Big\{ (x_0, t_0) \in \mathbb{R}^k \times \mathbb{I} \,\Big|\, \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(x_0, t, t_0) - \bar{x}(t)\| > 0 \text{ for all } t \in \mathbb{I}, \\ \varphi(x_0, t_-, t_0) \in B_\varepsilon(\bar{x}(t_-)) \Big\} \cup \{\bar{x}(t_0), t_0) | t_0 \in \mathbb{I}\} $$

is called the ***monotonically ε-unstable ft-fiber bundle*** *of $\bar{x}$ on $\mathbb{I}$ w.r.t. the chosen differentiable norm.*

$$ {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_{s}(t) := \left\{ x_0 \in \mathbb{R}^k \middle| (x_0, t) \in {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_{s} \right\}, \; {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_{u}(t) := \left\{ x_0 \in \mathbb{R}^k \middle| (x_0, t) \in {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_{u} \right\} $$

are the ***monotonically ε-stable*** *and* ***monotonically ε-unstable ft-t-fibers****.*

The almost stable and unstable cones with the explicit form given in (4.10), (4.11) represent ft-fiber bundles of a linearized system. The construction helps to define the ft-fiber bundles of a nonautonomous nonlinear difference equation. First we need a kernel function for $\mathbb{I} = [n_-, n_+]_\mathbb{Z}$ and a solution $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ of (2.3)

$$
{}_\varphi\mathcal{T}^{\bar{x}}_{\ker} : \mathbb{R}^k \times \mathbb{I} \to \mathbb{I} \cup \{n_+ + 1\},
$$
$$
(\xi, n_0) \mapsto \begin{cases} \min\{n \in [n_0, n_+]_\mathbb{Z} | \varphi(\xi, n, n_0) - \bar{x}(n) = 0\}, & \text{if it exists,} \\ n_+ + 1, & \text{otherwise.} \end{cases}
$$

This function yields the earliest time at which a vector ξ is mapped onto the solution $\bar{x}$, i.e. the earliest time at which ξ lies in the kernel of $\varphi(\xi, \cdot, n_0) - \bar{x}(\cdot)$.

Definition 6.2.2. *Let $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ be a solution of* (2.3) *and $\varepsilon > 0$. Then the set*

$$
\begin{aligned}
{}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s := \Big\{ (\xi, n_0) \in \mathbb{R}^k \times \mathbb{I} \Big| & \exists \mu \in {}_\varphi\mathcal{T}_{\mathrm{pre}}(\xi, n_0) \text{ with } \bar{n} := {}_\varphi\mathcal{T}_{\min}(\xi, n_0) \text{ and} \\
& \mathring{n} := {}_\varphi\mathcal{T}^{\bar{x}}_{\ker}(\mu, \bar{n}) : \|\varphi(\mu, n, \bar{n}) - \bar{x}(n)\| < \|\varphi(\mu, m, \bar{n}) - \bar{x}(m)\| \\
& \text{for all } n, m \in [\bar{n}, \mathring{n}]_\mathbb{Z}, n > m \text{ and } \varphi(\xi, n_+, n_0) \in B_\varepsilon(\bar{x}(n_+)), \Big\}
\end{aligned}
$$

*is called the **monotonically ε-stable ft-fiber bundle** of $\bar{x}$ on $\mathbb{I}$ w.r.t. the chosen norm and*

$$
\begin{aligned}
{}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_u := \Big\{ (\xi, n_0) \in \mathbb{R}^k \times \mathbb{I} \Big| & {}_\varphi\mathcal{T}_{\min}(\xi, n_0) = n_- \text{ and } \exists \mu \in {}_\varphi\mathcal{T}_{\mathrm{pre}}(\xi, n_0) \cap B_\varepsilon(\bar{x}(n_-)) : \\
& \|\varphi(\mu, n, n_-) - \bar{x}(n)\| > \|\varphi(\mu, m, n_-) - \bar{x}(m)\| \text{ for all } n, m \in \mathbb{I}, n > m \Big\} \\
& \cup \{(\bar{x}(t_0), t_0) | t_0 \in \mathbb{I}\}
\end{aligned}
$$

*is called the **monotonically ε-unstable ft-fiber bundle** of $\bar{x}$ on $\mathbb{I}$ w.r.t. the chosen norm.*

$$
{}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s(n) := \left\{ x_0 \in \mathbb{R}^k \middle| (x_0, n) \in {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s \right\}, \; {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_u(n) := \left\{ x_0 \in \mathbb{R}^k \middle| (x_0, n) \in {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_u \right\}
$$

*are called the **monotonically ε-stable** and **ε-unstable ft-n-fibers**.*

The following example shows the main advantage of monotonically ε-stable and ε-unstable ft-fiber bundles over monotonically stable and unstable ft-fiber bundles.

Example 6.2.3. *Consider the setting in Example 6.1.4 and in Figure 6.1. Let $\varepsilon > 0$ and $\mathbb{I} := [r_-, r_+]$. Then a solution $y(\cdot)$ with $y(t) \in {}^{\mathbb{I}}_{\varepsilon}M^{\xi}_s(t)$, $t \in \mathbb{I}$ satisfies $y(r_+) \in B_\varepsilon(\xi)$, i.e. the boundary point $x(r_+)$ of $x(\cdot)$ lies in the ε-ball around the fixed point ξ. A solution $z(\cdot)$ with $z(t) \in {}^{\mathbb{I}}_{\varepsilon}M^{\xi}_u(t)$, $t \in \mathbb{I}$ satisfies $y(r_-) \in B_\varepsilon(\xi)$, i.e. the boundary point $x(r_-)$ of $x(\cdot)$ lies in the ε-ball around the fixed point ξ. For $r_\pm := t_\pm$ the given solution $x(\cdot)$ satisfies $x(t_+) \in B_\varepsilon(\xi)$, cf. the right part* *of Figure 6.1. This leads to* $x(t_0) \in {}^{\mathbb{I}}_{\varepsilon}M_s^{\xi}(t_0)$ *for all* $t_0 \in \mathbb{I}$. *For* $r_\pm := s_\pm$ *the solution* $x(\cdot)$ *shown in the right half of Figure 6.1 satisfies*

$$x(s_-) \notin B_\varepsilon(\xi).$$

Thus, for $\mathbb{I} = [s_-, s_+]$ *we have*

$$x(t_0) \in {}^{\mathbb{I}}M_u^{\xi}(t_0), \quad x(t_0) \notin {}^{\mathbb{I}}_{\varepsilon}M_u^{\xi}(t_0)$$

for all $t_0 \in \mathbb{I}$.

This shows that Definition 6.2.1 prevents or at least reduces the risk that finite parts of a stable infinite time solution lie on the unstable finite time fiber.

A disadvantage of Definition 6.2.1 in contrast to Definition 6.1.1 is that not all finite time parts of a monotonically stable infinite time solution lie in the monotonically ε-stable ft-fiber although they lie on the monotonically stable ft-fiber. This fact illustrates the following example.

Example 6.2.4. *Consider a hyperbolic autonomous dynamical system on* $\mathbb{I} = \mathbb{R}$, *which has a fixed point* ξ *and two nontrivial solutions* $y(\cdot)$, $z(\cdot)$ *which monotonically converges towards the fixed point* ξ *as presented in Figure 6.2, where* $t_-, t_+, s_-, s_+ \in \mathbb{R}$ *with*

$$t_+ > t_- > s_+ > s_-, \; t_+ - t_- = s_+ - s_-.$$

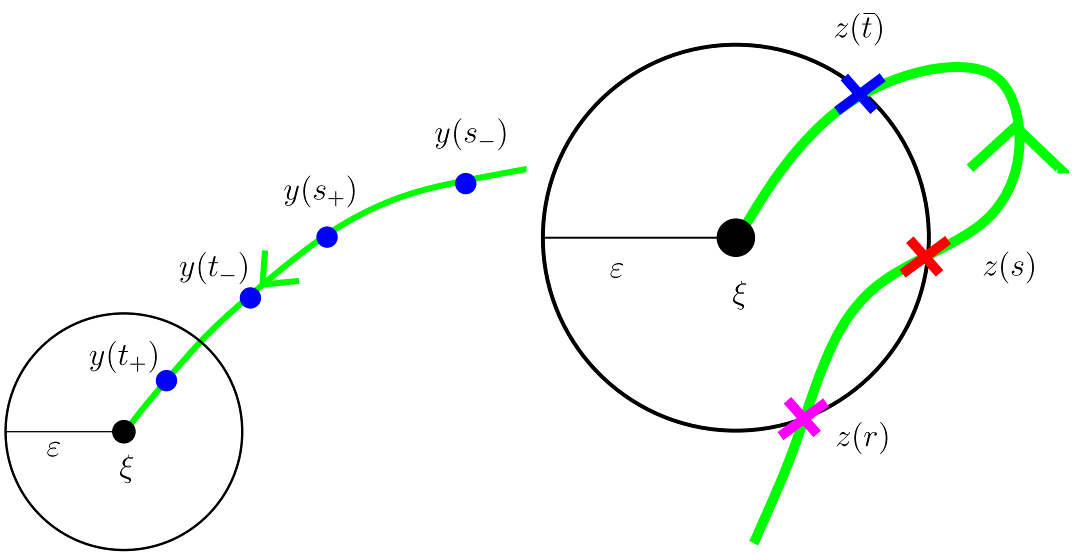


Figure 6.2: A fixed point ξ in an ε-ball (black) and solutions $y(\cdot)$ (left) and $z(\cdot)$ (right) (presented in green) which converges towards the fixed point.

For $\mathbb{I} = [t_-, t_+]$ we have $y(t_0) \in {}^{\mathbb{I}}_{\varepsilon}M^{\xi}_{s}(t_0)$ for all $t_0 \in \mathbb{I}$, while for $\hat{\mathbb{I}} = [s_-, s_+]$ we have

$$y(t_0) \in {}^{\hat{\mathbb{I}}}M^{\xi}_{s}(t_0), \quad y(t_0) \notin {}^{\hat{\mathbb{I}}}_{\varepsilon}M^{\xi}_{s}(t_0)$$

for all $t_0 \in \hat{\mathbb{I}}$.

Thus, Definition 6.2.1 may yields less solutions which are stable in infinite time than Definition 6.1.1. The differential equation (6.4) generates a dynamical system with the required properties of example 6.2.4.

Both ft-fiber bundles (monotonically and monotonically ε) have one disadvantage in common. If a infinite stable solution ends in the ε-ball the finite time parts may not line in the ft-fibers. An illustration is presented the following example.

Example 6.2.5. *Consider the setting of Example 6.2.4. The solution $z(\cdot)$ intersects three times the boundary of the ε-ball. The intersections are marked with an cross at time $\bar{t}$ in blue, at s in red and at r in magenta. For $\mathbb{I} = [s-1, \bar{t}+1]$ we see*

$$z(t_0) \notin {}^{\mathbb{I}}M^{\xi}_{s}(t_0), z(t_0) \notin {}^{\mathbb{I}}_{\varepsilon}M^{\xi}_{s}(t_0)$$

for all $t_0 \in \mathbb{I}$, although $x(t_+) \in B_{\varepsilon}(\xi)$. The finite solution does not monotonically converge towards the fixed point ξ.

We analyze convergence properties of solutions on the continuous stable infinite fiber bundles. Therefore, let $\bar{\xi}$ be a fixed point of a continuous infinite time dynamical system and $\bar{x}(\cdot)$ be a solution with $\bar{x}(t) \in W^{\bar{\xi}}_{s}(t)$ for all $t \in \mathbb{R}$. The solution $\bar{x}$ convergences towards the fixed point $\bar{\xi}$ but in general not monotonically. It satisfies

$$\lim_{t\to\infty} \bar{x}(t) = \bar{\xi}.$$

This means we find a $T \in \mathbb{R}$ such that $\frac{\mathrm{d}}{\mathrm{d}t} \|\bar{x}(t) - \xi\| < 0$ for all $t > T$. An adaption to a finite time interval (if $\mathbb{I}$ is sufficiently large and $T \in \mathbb{I} \setminus \{t_+\}$) leads to the existence of a $T \in \mathbb{I} \setminus \{t_+\}$ such that $\frac{\mathrm{d}}{\mathrm{d}t} \|\bar{x}(t) - \xi\| < 0$ for all $t \in \mathbb{I}$, $t > T$. Thus, we do not require monotonic convergence on the whole interval $\mathbb{I}$. Hence, we should demand monotonic convergence inside an ε-ball around ξ for all forward defined times from the time at which the solution last intersects with the boundary of the ε-ball. Important is that we do not require monotonicity for all times, at which the solution lies inside the ε-ball. This means for the solution $z(\cdot)$ illustrated in the right panel of Figure 6.2 that we require monotonicity for all times $t \in \mathbb{I}$, $t > \bar{t}$ ($x(\bar{t})$ is marked by a blue cross) and not for times $t \in \mathbb{I}$ with $r < t < s$. Ft-fiber bundles with these properties are defined in the next section and are called ε-(un)stable ft-fiber bundles.

ε-(Un)Stable Ft-Fiber Bundles

A third way to define ft-fiber bundles is presented in this section. In contrast to the other two definitions a monotonically decrease or growth of the whole solution is not required for solutions on these ft-fiber bundles, which are called the ε-(un)stable ft-fiber bundles. As mentioned in the last section for the definition we need the last and first time at which a solution intersects with the boundary of the ε-ball.

Therefore, we introduce two functions, which yield the first and last time at which an orbit intersects with the boundary of an ε-ball around a given trajectory. Let $\varepsilon > 0$ then define

$$
{}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon} : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{I} \to \mathbb{I} \cup \{t_+ + 1\},
$$
$$
(\mu, \bar{x}, \bar{t}) \mapsto \begin{cases} t_+ + 1, & \text{if } \varphi(\mu, t_+, \bar{t}) \notin B_\varepsilon(\varphi(\bar{x}, t_+, \bar{t})), \\ \min\left\{\hat{t} \in \mathbb{I} | \hat{t} \geq \bar{t}, \varphi(\mu, t, \bar{t}) \in B_\varepsilon(\varphi(\bar{x}, t, \bar{t})) \forall t \in \mathbb{I}, t > \hat{t}\right\}, & \text{else,} \end{cases}
$$

$$
{}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\max}_{\varepsilon} : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{I} \cup \{t_- - 1\},
$$
$$
(\mu, \bar{x}) \mapsto \begin{cases} t_- - 1, & \text{if } \mu \notin B_\varepsilon(\bar{x}), \\ \max\left\{\hat{t} \in \mathbb{I} | \varphi(\mu, t, t_-) \in B_\varepsilon(\varphi(\bar{x}, t, t_-)) \forall t \in \mathbb{I}, t < \hat{t}\right\}, & \text{else.} \end{cases}
$$

For the solution $z(\cdot)$, illustrated in the right panel of Figure 6.2, the first function yields ${}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(x(t), \xi, t) = \bar{t}$ for all $t \in \mathbb{I}$ if $\bar{t} \in \mathbb{I}$. In the following lemmas we summarize some basic properties, as the symmetry, of these functions.

Lemma 6.3.1. *Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$ and $\mathbb{I} = [t_-, t_+]_{\mathbb{T}}$ be a compact interval. Let $\varepsilon > 0$ and φ be the solution operator of* (2.1) *then*

$$
{}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(x, y, t_0) = {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(y, x, t_0), \ {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\max}_{\varepsilon}(x, y) = {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\max}_{\varepsilon}(y, x)
$$

hold for all $x, y \in \mathbb{R}^k$ and $t_0 \in \mathbb{I}$.

Proof. Let $x, y \in \mathbb{R}^k$ and $t_0 \in \mathbb{I}$. Assume $\varphi(x, t_+, t_0) \notin B_\varepsilon(\varphi(y, t_+, t_0))$ then

$$
\varepsilon \leq \|\varphi(x, t_+, t_0) - \varphi(y, t_+, t_0)\|_2 = \|\varphi(y, t_+, t_0) - \varphi(x, t_+, t_0)\|_2 ,
$$

i.e. $\varphi(y, t_+, t_0) \notin B_\varepsilon(\varphi(x, t_+, t_0))$. Thus, ${}^{\mathbb{I}}_{\varphi}B^{\min}_{\varepsilon}(x, y, t_0) = {}^{\mathbb{I}}_{\varphi}B^{\min}_{\varepsilon}(y, x, t_0)$ holds. Next assume $\varphi(x, t_+, t_0) \in B_\varepsilon(\varphi(y, t_+, t_0))$ then

$$
\begin{aligned}
{}^{\mathbb{I}}_{\varphi}B^{\min}_{\varepsilon}(x, y, t_0) &= \min\left\{\hat{t} \in \mathbb{I} \middle| \hat{t} \geq t_0, \varphi(x, t, t_0) \in B_\varepsilon(\varphi(y, t, t_0)) \forall t \in \mathbb{I}, t > \hat{t}\right\} \\
&= \min\left\{\hat{t} \in \mathbb{I} \middle| \hat{t} \geq t_0, \varphi(y, t, t_0) \in B_\varepsilon(\varphi(x, t, t_0)) \forall t \in \mathbb{I}, t > \hat{t}\right\} \\
&= {}^{\mathbb{I}}_{\varphi}B^{\min}_{\varepsilon}(y, x, t_0)
\end{aligned}
$$

follows. Analogously, we get ${}^{\mathbb{I}}_{\varphi}B^{\max}_{\varepsilon}(x, y) = {}^{\mathbb{I}}_{\varphi}B^{\max}_{\varepsilon}(y, x)$. □

Lemma 6.3.2. *Let* $\mathbb{I}$ *be a compact interval and* $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ *be a solution of* (2.2)/(2.3) *and let* $\varepsilon > 0$. *For each* $x \in \mathbb{R}^k$

$$ {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(x, \bar{x}(t_1), t_1) \leq {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(\varphi(x, t_2, t_1), \bar{x}(t_2), t_2) $$

is satisfied for all $t_1, t_2 \in \mathbb{I}$, $t_2 \geq t_1$.

Proof. Let $x \in \mathbb{R}^k$, $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$ and $t_1, t_2 \in \mathbb{I} := [t_-, t_+]_{\mathbb{T}}$, $t_2 \geq t_1$. If $t_+ + 1 = {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(x, \bar{x}(t_1), t_1) \notin \mathbb{I}$ then

$$ \varphi(\varphi(x, t_2, t_1), t_+, t_2) = \varphi(x, t_+, t_1) \notin B_\varepsilon(\bar{x}(t_+)). $$

Thus $t_+ + 1 = {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(\varphi(x, t_2, t_1), \bar{x}(t_2), t_2) \notin \mathbb{I}$. Assume ${}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(x, \bar{x}(t_1), t_1) \in \mathbb{I}$ then

$$ \begin{aligned} &{}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(x, \bar{x}(t_1), t_1) \\ =& \min\{\hat{t} \in \mathbb{I} \big| \hat{t} \geq t_1, \varphi(x, t, t_1) \in B_\varepsilon(\bar{x}(t)) \text{ for all } t \in \mathbb{I}, t > \hat{t}\} \\ \leq& \{\hat{t} \in \mathbb{I} \big| \hat{t} \geq t_2, \varphi(x, t, t_1) \in B_\varepsilon(\bar{x}(t)) \text{ for all } t \in \mathbb{I}, t > \hat{t}\} \\ =& \{\hat{t} \in \mathbb{I} \big| \hat{t} \geq t_2, \varphi(\varphi(x, t_2, t_1), t, t_2) \in B_\varepsilon(\bar{x}(t)) \text{ for all } t \in \mathbb{I}, t > \hat{t}\} \\ =& {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(\varphi(x, t_2, t_1), \bar{x}(t_2), t_2). \end{aligned} $$

□

With the help of the latter studied functions we are able to define the ε-stable and ε-unstable ft-fiber bundles. We start with the continuous case.

Definition 6.3.3. *Let* $\mathbb{I} = [t_-, t_+]$ *and* $\varepsilon > 0$. *Let* $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ *be a solution of* (2.2). *Then the set*

$$ \begin{aligned} {}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s} := \Bigg\{ & (x_0, t_0) \in \mathbb{R}^k \times \mathbb{I} \Bigg| \hat{t} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(\varphi(x_0, t_-, t_0), \bar{x}(t_-), t_-) \in \mathbb{I} \text{ and} \\ & \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(x_0, t, t_0) - \bar{x}(t)\| < 0 \text{ for all } t \in \mathbb{I}, t > \hat{t} \Bigg\} \cup \{(\bar{x}(t_0), t_0) | t_0 \in \mathbb{I}\} \end{aligned} $$

is called the ***ε-stable ft-fiber bundle*** *of* $\bar{x}$ *on* $\mathbb{I}$ *w.r.t. the chosen differentiable norm. The* ***ε-unstable ft-fiber bundle*** *of* $\bar{x}$ *on* $\mathbb{I}$ *w.r.t. the chosen differentiable norm is defined as*

$$ \begin{aligned} {}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{u} := \Bigg\{ & (x_0, t_0) \in \mathbb{R}^k \times \mathbb{I} \Bigg| \hat{t} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\max}_{\varepsilon}(\varphi(x_0, t_-, t_0), \bar{x}(t_-)) \in \mathbb{I} \text{ and} \\ & \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(x_0, t, t_0) - \bar{x}(t)\| > 0 \text{ for all } t \in \mathbb{I}, t < \hat{t} \Bigg\} \cup \{(\bar{x}(t_0), t_0) | t_0 \in \mathbb{I}\}. \end{aligned} $$

The sets

$$ \begin{aligned} {}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s}(t) &:= \left\{x_0 \in \mathbb{R}^k \big| (x_0, t) \in {}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s}\right\}, \\ {}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{u}(t) &:= \left\{x_0 \in \mathbb{R}^k \big| (x_0, t) \in {}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{u}\right\} \end{aligned} $$

are the ***ε-stable*** *and* ***ε-unstable ft-t-fibers***.

Analogously to the continuous time case, we define the ε-stable and ε-unstable ft-fiber bundles.

Definition 6.3.4. *Let $\mathbb{I} := [n_-, n_+]_{\mathbb{Z}}$ with $n_\pm \in \mathbb{Z}$ and $\varepsilon < 0$. Further, let $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ be a solution of (2.3). Then the set*

$$\begin{aligned} {}^{\mathbb{I}}_{\varepsilon}W_s^{\bar{x}} :=\Big\{ &(\xi, n_0) \in \mathbb{R}^k \times \mathbb{I} \Big| \exists \mu \in {}_{\varphi}\mathcal{T}_{\text{pre}}(\xi, n_0) \text{ with } \bar{n} := {}_{\varphi}\mathcal{T}_{\min}(\xi, n_0), \\ &\mathring{n} := {}_{\varphi}\mathcal{T}_{\ker}^{\bar{x}}(\mu, \bar{n}) \text{ and } \hat{n} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}_{\varepsilon}^{\min}(\mu, \bar{x}(\bar{n}), \bar{n}) \in \mathbb{I} : \\ &\|\varphi(\mu, n, \bar{n}) - \bar{x}(n)\| < \|\varphi(\mu, m, \bar{n}) - \bar{x}(m)\| \\ &\text{for all } n, m \in \mathbb{I}, \mathring{n} \geq n > m > \hat{n} \Big\} \end{aligned}$$

is called the ***ε-stable ft-fiber bundle*** *of $\bar{x}$ on $\mathbb{I}$ w.r.t. the chosen norm. The* ***ε-unstable ft-fiber bundle*** *of $\bar{x}$ on $\mathbb{I}$ w.r.t. the chosen norm is defined as*

$$\begin{aligned} {}^{\mathbb{I}}_{\varepsilon}W_u^{\bar{x}} :=\Big\{ &(\xi, n_0) \in \mathbb{R}^k \times \mathbb{I} \Big| {}_{\varphi}\mathcal{T}_{\min}(\xi, n_0) = n_- \text{ and } \exists \mu \in {}_{\varphi}\mathcal{T}_{\text{pre}}(\xi, n_0) \text{ with} \\ &\hat{n} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}_{\varepsilon}^{\max}(\mu, \bar{x}(n_-)) \in \mathbb{I} : \\ &\|\varphi(\mu, n, n_-) - \bar{x}(n)\| > \|\varphi(\mu, m, n_-) - \bar{x}(m)\| \\ &\text{for all } n, m \in \mathbb{I}, m < n < \hat{n} \Big\} \cup \left\{ (\bar{x}(n), n) \middle| n \in \mathbb{I} \right\}. \end{aligned}$$

The sets

$$\begin{aligned} {}^{\mathbb{I}}_{\varepsilon}W_s^{\bar{x}}(n) &:= \left\{ x \in \mathbb{R}^k | (x, n) \in {}^{\mathbb{I}}_{\varepsilon}W_s^{\bar{x}} \right\}, \\ {}^{\mathbb{I}}_{\varepsilon}W_u^{\bar{x}}(n) &:= \left\{ x \in \mathbb{R}^k | (x, n) \in {}^{\mathbb{I}}_{\varepsilon}W_u^{\bar{x}} \right\} \end{aligned}$$

are called the ***ε-stable*** *and* ***ε-unstable ft-n-fiber****.*

Characteristics of the Ft-Fiber Bundles

In this section we analyze the characteristics of the ft-fiber bundles defined in Definition 6.1.1, Definition 6.2.1, 6.2.2 and Definition 6.3.3, 6.3.4. In particular we study their invariance properties and we verify whether the stable and unstable fibers intersect. These facts enable us to determine an adequate analogon of the infinite time fiber bundles for finite time systems.

Remark 6.4.1. *The ft-fiber bundles ${}^{\mathbb{I}}M_s^{\bar{x}}(t_0)$, ${}^{\mathbb{I}}_{\varepsilon}M_s^{\bar{x}}(t_0)$ and ${}^{\mathbb{I}}_{\varepsilon}W_s^{\bar{x}}(t_0)$ of an invertible system are open sets for all $t \in \mathbb{I}$. The ft-fiber bundles ${}^{\mathbb{I}}M_u^{\bar{x}}(t_0)$, ${}^{\mathbb{I}}_{\varepsilon}M_u^{\bar{x}}(t_0)$ and ${}^{\mathbb{I}}_{\varepsilon}W_u^{\bar{x}}(t_0)$ of any dynamical system are open for all $t \in \mathbb{I}$.*

Indeed, the monotonically ft-fiber bundles ${}^{\mathbb{I}}M_{s,u}^{\bar{x}}(t_0)$ are open for all $t \in \mathbb{I} \subset \mathbb{R}$, see [45, Remark 36].

To prove that ${}^{\mathbb{I}}_{\varepsilon}M_s^{\bar{x}}(t_0)$, $t_0 \in \mathbb{I}$ of an invertible system is open it is sufficient to prove that for each $x_0 \in {}^{\mathbb{I}}_{\varepsilon}M_s^{\bar{x}}(t_0)$ a $\delta > 0$ exists such that

$$\varphi(t_+, t_0)B_\delta(x_0) \subset B_\varepsilon(\bar{x}(t_+)).$$

Let $t_0 \in \mathbb{I}$ and $x_0 \in {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s(t_0)$. By the continuity of φ we find for every $B_\delta(x_0)$ a $\delta_1 > 0$ such that $\varphi(t_+, t_0)B_\delta(x_0) \subset B_{\delta_1}(\varphi(x_0, t_+, t_0))$. Thus, for δ sufficiently small we obtain

$$\varphi(t_+, t_0)B_\delta(x_0) \subset B_{\delta_1}(\varphi(x_0, t_+, t_0)) \subset B_\varepsilon(\bar{x}(t_+)).$$

For noninvertible systems there exist a $t_0 \in \mathbb{I}$ and a $\bar{x}(t_0) \neq \xi \in {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s(t_0)$ with ${}_{\varphi}\mathcal{T}^{\bar{x}}_{\mathrm{ker}}(\xi, t_0) = t_0$. With these points the sets are not open. Since for every t_0 and each $\xi \in {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_u(t_0)$ we have ${}_{\varphi}\mathcal{T}^{\bar{x}}_{\mathrm{ker}}(\xi, t_0) \notin \mathbb{I}$ the set ${}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_u(t_0)$ of any dynamical system is open. Analogously, we obtain that ${}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_u(t_0)$, $t_0 \in \mathbb{I}$ of an invertible system and ${}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_u(t_0)$, $t_0 \in \mathbb{I}$ of any dynamical system are open.

The ft-fibers have invariance properties. For ${}^{\mathbb{I}}M^{\bar{x}}_{s,u}$ we refer to [45, Theorem 37]. The following lemma shows that ${}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_{s,u}(t)$ are "invariant" sets for all $t \in \mathbb{I}$, $\varepsilon > 0$. In Lemma 6.4.3 we prove the same invariance properties for ${}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s,u}$, $\varepsilon > 0$.

Lemma 6.4.2. *Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$ and $\mathbb{I} = [t_-, t_+]_{\mathbb{T}}$. Assume that $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ is a solution of (2.2)/(2.3) and $\varepsilon > 0$. Then the monotonically ε-unstable ft-fiber bundle ${}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_u$ is invariant and the monotonically ε-stable ft-fiber bundle ${}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s$ is at least forward invariant w.r.t. t_-, i.e. we have for all $t_1, t_0 \in \mathbb{I}$, $t_1 > t_0$*

$$ {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_u(t_1) = \varphi(t_1, t_0){}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_u(t_0), \tag{6.5}$$

$$ {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s(t_1) \supset \varphi(t_1, t_-){}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s(t_-) \tag{6.6}$$

and if φ is invertible

$$ {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s(t_1) = \varphi(t_1, t_0){}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s(t_0). \tag{6.7}$$

Proof. We show the invariance for the ε-stable ft-fibers. For $\mathbb{T} = \mathbb{R}$ let $t_0, t_1 \in \mathbb{I}$ and $x(t_0) \in {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s(t_0) \setminus \{\bar{x}(t_0)\}$. Then we have for all $t \in \mathbb{I}$

$$0 > \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(x(t_0), t, t_0) - \bar{x}(t)\| = \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(\varphi(x(t_0), t_1, t_0), t, t_1) - \bar{x}(t)\|.$$

This implies $\varphi(x(t_0), t_1, t_0) \in {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s(t_1)$ and we get

$$\varphi(t_1, t_0){}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s(t_0) \subset {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s(t_1). \tag{6.8}$$

Thus (6.6) is shown. Since φ is invertible and (6.8) holds for all $t_0, t_1 \in \mathbb{I}$ we get (6.7). By the same arguments we obtain Equation (6.5).

For $\mathbb{T} = \mathbb{Z}$ we get for all $t \in \mathbb{I}$

$$\begin{aligned}
&\varphi(t,t_-)^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s(t_-) \\
&= \big\{\varphi(\xi,t,t_-) \in \mathbb{R}^k \big| \xi \in {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s(t_-)\big\} \\
&= \big\{\varphi(\xi,t,t_-) \in \mathbb{R}^k \big| \varphi(\xi,t_+,t_-) \in B_\varepsilon(\bar{x}(t_+)) \text{ and} \\
&\quad \|\varphi(\xi,t,t_-) - \bar{x}(t)\| < \|\varphi(\xi,s,t_-) - \bar{x}(s)\| \\
&\quad \text{for all } t,s \in \mathbb{I}, \mathring{t} \geq t > s \text{ with } \mathring{t} := {}_{\varphi}\mathcal{T}^{\bar{x}}_{\ker}(\xi,t_-)\big\} \\
&\subset \big\{\tilde{\xi} \in \mathbb{R}^k \big| {}_{\varphi}\mathcal{T}_{\min}(\tilde{\xi},t) = t_-, \mathring{t} := {}_{\varphi}\mathcal{T}^{\bar{x}}_{\ker}(\tilde{\xi},t) : \varphi(\tilde{\xi},t_+,t) \in B_\varepsilon(\bar{x}(t_+)), \\
&\quad \exists \xi \in {}_{\varphi}\mathcal{T}_{\mathrm{pre}}(\tilde{\xi},t) : \|\varphi(\xi,t,t_-) - \bar{x}(t)\| < \|\varphi(\xi,s,t_-) - \bar{x}(s)\| \\
&\quad \text{for all } t,s \in \mathbb{I}, \mathring{t} \geq t > s\big\} \\
&\subset {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s(t).
\end{aligned}$$

Further for all $t_1, t_0 \in \mathbb{I}$ with $t_1 > t_0$ we obtain by Lemma 4.1.4

$$\begin{aligned}
&\varphi(t_1,t_0)^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_u(t_0) \\
&= \big\{\varphi(\xi,t_1,t_0) \in \mathbb{R}^k \big| {}_{\varphi}\mathcal{T}_{\min}(\xi,t_0) = t_- \wedge \exists \mu \in {}_{\varphi}\mathcal{T}_{\mathrm{pre}}(\xi,t_0) \cap B_\varepsilon(\bar{x}(t_-)) : \\
&\quad \|\varphi(\mu,t,t_-) - \bar{x}(t)\| > \|\varphi(\mu,s,t_-) - \bar{x}(s)\| \text{ for all } t,s \in \mathbb{I}, t > s\big\} \\
&\quad \cup \{\bar{x}(t_0)\} \\
&= \big\{\tilde{\xi} \in \mathbb{R}^k \big| {}_{\varphi}\mathcal{T}_{\min}(\tilde{\xi},t_1) = t_- \wedge \exists \mu \in {}_{\varphi}\mathcal{T}_{\mathrm{pre}}(\tilde{\xi},t_1) \cap B_\varepsilon(\bar{x}(t_-)) : \\
&\quad \|\varphi(\mu,t,t_-) - \bar{x}(t)\| > \|\varphi(\mu,s,t_-) - \bar{x}(s)\| \text{ for all } t,s \in \mathbb{I}, t > s\big\} \\
&\quad \cup \{\bar{x}(t_0)\} \\
&= {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_u(t_1).
\end{aligned}$$

Let φ be invertible then we have

$$\begin{aligned}
&\varphi(t_1,t_0)^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s(t_0) \\
&= \big\{\varphi(t_1,t_0)\xi \in \mathbb{R}^k \big| \|\varphi(\xi,t,t_0) - \bar{x}(t)\| < \|\varphi(\xi,s,t_0) - \bar{x}(s)\| \\
&\quad \text{for all } t,s \in \mathbb{I}, t > s \text{ and } \varphi(\xi,t_+,t_0) \in B_\varepsilon(\bar{x}(t_+))\big\} \cup \{\bar{x}(t_1)\} \\
&= \Big\{\tilde{\xi} \in \mathbb{R}^k \Big| \Big\|\varphi(\tilde{\xi},t,t_1) - \bar{x}(t)\Big\| < \Big\|\varphi(\tilde{\xi},s,t_1) - \bar{x}(s)\Big\| \\
&\quad \text{for all } t,s \in \mathbb{I}, t > s \text{ and } \varphi(\tilde{\xi},t_+,t_1) \in B_\varepsilon(\bar{x}(t_+))\Big\} \cup \{\bar{x}(t_1)\} \\
&= {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s(t_1).
\end{aligned}$$

□

Note that ${}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_s$ is in general not forward invariant, see Example 4.1.8.

The ft-fiber bundles have the same invariance properties as the monotonically ft-fiber bundles.

Lemma 6.4.3. *Let* $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$ *and* $\mathbb{I} := [t_-, t_+]_\mathbb{T}$ *and let* $\varepsilon > 0$. *Further, let* $x(\cdot)$ *be a trajectory of system (2.2)/(2.3) then the* ε*-unstable ft-t-fibers are invariant under* $\varphi(\cdot, t)$ *for all* $t \in \mathbb{I}$ *and the* ε*-stable ft-*t_-*-fiber is forward invariant, i.e.*

$$\varphi(t, t_-)\,{}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s}(t_-) \subset {}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s}(t) \text{ for all } t \in \mathbb{I},$$
$$\varphi(t_1, t_0)\,{}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{u}(t_0) = {}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{u}(t_1) \text{ for all } t_1, t_0 \in \mathbb{I},\ t_1 \geq t_0.$$

If the system is invertible then the ft-t-fibers are invariant under $\varphi(\cdot, t)$ *for all* $t \in \mathbb{I}$ *,i.e.*

$$\varphi(t_1, t_0)\,{}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s,u}(t_0) = {}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s,u}(t_1) \text{ for all } t_1, t_0 \in \mathbb{I}.$$

Proof. Let $\varepsilon > 0$. Let $\bar{x}(\cdot)$ be a trajectory of (2.3), $t_0 \in \mathbb{I}$. For all $t_1 \in \mathbb{I}$, $t_1 \geq t_0$ we obtain

$$\begin{aligned}
&\varphi(t_1, t_0)\,{}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{u}(t_0)\\
=&\big\{\varphi(x, t_1, t_0) \in \mathbb{R}^k \big|\, {}_{\varphi}\mathcal{T}_{\min}(x, t_0) = t_- \text{ and } \exists \mu \in {}_{\varphi}\mathcal{T}_{\text{pre}}(x, t_0) \text{ with}\\
&\quad \hat{t} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\max}_{\varepsilon}(\mu, \bar{x}(t_-)) \in \mathbb{I} : \|\varphi(\mu, t, t_-) - \bar{x}(t)\| > \|\varphi(\mu, s, t_-) - \bar{x}(s)\|\\
&\quad \text{for all } s, t \in \mathbb{I},\ s < t < \hat{t}\big\} \cup \{\bar{x}(t_1)\}\\
=&\big\{y \in \mathbb{R}^k \big|\, {}_{\varphi}\mathcal{T}_{\min}(y, t_1) = t_- \text{ and } \exists \mu \in {}_{\varphi}\mathcal{T}_{\text{pre}}(y, t_1) \text{ with}\\
&\quad \hat{t} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\max}_{\varepsilon}(\mu, \bar{x}(t_-)) \in \mathbb{I} : \|\varphi(\mu, t, t_-) - \bar{x}(t)\| > \|\varphi(\mu, s, t_-) - \bar{x}(s)\|\\
&\quad \text{for all } s, t \in \mathbb{I},\ s < t < \hat{t}\big\} \cup \{\bar{x}(t_1)\}\\
=&{}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{u}(t_1).
\end{aligned}$$

The forward invariance of the ε-stable ft-t_--fiber is given by

$$\begin{aligned}
&\varphi(t, t_-)\,{}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s}(t_-)\\
:=&\Big\{\varphi(x_-, t, t_-) \in \mathbb{R}^k \Big|\, \mathring{t} := {}_{\varphi}\mathcal{T}^{\bar{x}}_{\ker}(x_-, t_-) \text{ and } \hat{t} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(x_-, \bar{x}(t_-), t_-) \in \mathbb{I} :\\
&\quad \|\varphi(x_-, t_1, t_-) - \bar{x}(t_1)\| < \|\varphi(x_-, t_0, t_-) - \bar{x}(t_0)\|\\
&\quad \text{for all } t_1, t_0 \in \mathbb{I},\ \mathring{t} \geq t_1 > t_0 > \hat{t}\Big\} \cup \{\bar{x}(t)\}\\
=&\Big\{x \in \mathbb{R}^k \Big|\, \exists x_- \in {}_{\varphi}\mathcal{T}_{\text{pre}}(x, t) \text{ with } t_- := {}_{\varphi}\mathcal{T}_{\min}(x, t), \mathring{t} := {}_{\varphi}\mathcal{T}^{\bar{x}}_{\ker}(x_-, t_-) \text{ and}\\
&\quad \hat{t} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(x_-, \bar{x}(t_-), t_-) \in \mathbb{I} :\\
&\quad \|\varphi(x_-, t_1, t_-) - \bar{x}(t_1)\| < \|\varphi(x_-, t_0, t_-) - \bar{x}(t_0)\|\\
&\quad \text{for all } t_1, t_0 \in \mathbb{I},\ \mathring{t} \geq t_1 > t_0 > \hat{t}\Big\} \cup \{\bar{x}(t)\}\\
\subset&\Big\{x \in \mathbb{R}^k \Big|\, \exists \mu \in {}_{\varphi}\mathcal{T}_{\text{pre}}(x, t) \text{ with } \bar{t} := {}_{\varphi}\mathcal{T}_{\min}(x, t), \mathring{t} := {}_{\varphi}\mathcal{T}^{\bar{x}}_{\ker}(\mu, \bar{t}) \text{ and}\\
&\quad \hat{t} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(\mu, \bar{x}(\bar{t}), \bar{t}) \in \mathbb{I} :\\
&\quad \|\varphi(\mu, t_1, \bar{n}) - \bar{x}(t_1)\| < \|\varphi(\mu, t_0, \bar{n}) - \bar{x}(t_0)\|\\
&\quad \text{for all } t_1, t_0 \in \mathbb{I},\ \mathring{t} \geq t_1 > t_0 > \hat{t}\Big\} \cup \{\bar{x}(t)\}\\
=&{}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s}(t).
\end{aligned}$$

Let $\bar{x}(\cdot)$ be a trajectory of (2.2) or of an invertible system (2.3). For $t_1 \in \mathbb{I}$ we have

$$\begin{aligned}
&\varphi(t_1,t_0){}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s}(t_0)\\
:=&\Big\{\varphi(x_0,t_1,t_0)\in\mathbb{R}^k \Big| \hat{t} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(\varphi(x_0,t_-,t_0),\bar{x}(t_-),t_-)\in\mathbb{I} \text{ and}\\
&\|\varphi(x_0,t,t_0)-\bar{x}(t)\| < \|\varphi(x_0,s,t_0)-\bar{x}(s)\| \text{ for all } t,s\in\mathbb{I},\ t>s>\hat{t}\Big\}\\
&\cup\{\bar{x}(t_1)\}\\
=&\Big\{x\in\mathbb{R}^k \Big| \hat{t} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(\varphi(\varphi(x,t_0,t_1),t_-,t_0),\bar{x}(t_-),t_-)\in\mathbb{I} \text{ and}\\
&\|\varphi(\varphi(x,t_0,t_1),t,t_0)-\bar{x}(t)\| < \|\varphi(\varphi(x,t_0,t_1),s,t_0)-\bar{x}(s)\|\\
&\text{for all } t,s\in\mathbb{I},\ t>s>\hat{t}\Big\}\\
&\cup\{\bar{x}(t_1)\}\\
=&\Big\{x\in\mathbb{R}^k \Big| \hat{t} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(\varphi(x,t_-,t_1),\bar{x}(t_-),t_-)\in\mathbb{I} \text{ and}\\
&\|\varphi(x,t,t_1)-\bar{x}(t)\| < \|\varphi(x,s,t_1)-\bar{x}(s)\| \text{ for all } t,s\in\mathbb{I},\ t>s>\hat{t}\Big\}\\
&\cup\{\bar{x}(t_1)\}\\
=&{}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s}(t_1).
\end{aligned}$$

Analogously, we get for the ε-unstable ft-fibers $\varphi(t_1,t_0){}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{u}(t_0) = {}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{u}(t_1)$. □

It is easy to see that the monotonically ε-(un)stable ft-t-fibers are subsets of the ε-(un)stable ft-t-fibers and of the monotonically (un)stable ft-t-fibers, i.e.

$$\begin{aligned}
{}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_{s,u}(t) &\subset {}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s,u}(t) \text{ for all } t\in\mathbb{I},\\
{}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_{s,u}(t) &\subset {}^{\mathbb{I}}M^{\bar{x}}_{s,u}(t) \text{ for all } t\in\mathbb{I}.
\end{aligned} \tag{6.9}$$

In the following we analyze where and when the sets ${}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_{s,u}$ and ${}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s,u}$ coincide. We show in Lemma 6.4.4 that the ft-fibers ${}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s,u}$ and the monotonically ft-fibers ${}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_{s,u}$ coincide in a small neighborhood of the solution $\bar{x}$. For linear D-hyperbolic systems we prove in Lemma 6.4.5 that the (monotonically) ε-(un)stable ft-fibers coincide with the ε-(un)stable ft-fibers and locally coincide with the (un)stable cones.

Lemma 6.4.4. *Let $\mathbb{T}\in\{\mathbb{R},\mathbb{Z}\}$, $t_\pm\in\mathbb{T}$ and $\mathbb{I}=[t_-,t_+]_{\mathbb{T}}$ be a compact interval. Let $\bar{x}(\cdot)$ be a solution of system (2.1) and fix $\varepsilon>0$. Then there exists an $\bar{\varepsilon}>0$ such that*

$${}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_{s,u}(t_-)\cap B_{\bar{\varepsilon}}(\bar{x}(t_-)) = {}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s,u}(t_-)\cap B_{\bar{\varepsilon}}(\bar{x}(t_-)).$$

Proof. By (6.9) it follows that the left-hand side of the claim is a subset of the right-hand side.

For the other inclusion first note that an $\bar{\varepsilon} > 0$ exists such that

$$\varphi(t,t_-)B_{\bar{\varepsilon}}(\bar{x}(t_-)) \subset B_\varepsilon(\bar{x}(t)) \tag{6.10}$$

holds for all $t \in \mathbb{I}$, since φ is continuous, $\mathbb{I}$ is a compact interval and $\bar{x}(t) = \varphi(\bar{x}(t_-),t,t_-)$ for all $t \in \mathbb{I}$. Let $x_0 \in {}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s,u}(t_-) \cap B_{\bar{\varepsilon}}(\bar{x}(t_-))$. Then by equation (6.10) we have

$$\varphi(x_0,t,t_-) \subset B_\varepsilon(\bar{x}(t)), \text{ for all } t \in \mathbb{I}.$$

This implies with $t_- = {}_{\varphi}\mathcal{T}_{\min}(x_0,t_-)$ and all $x \in {}_{\varphi}\mathcal{T}_{\text{pre}}(x_0,t_-)$ that

$${}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(x,\bar{x}(t_-),t_-) = t_-,\ {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\max}_{\varepsilon}(x,\bar{x}(t_-)) = t_+,$$

which means that the solution corresponding to x_0 stays in the ε-ball for all times $t \in \mathbb{I}$. By definition the solutions lies in ${}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s,u}$ and thus it is monotonically increasing/ decreasing for all times $t \in \mathbb{I}$. Hence, we see

$$x_0 \in {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_{s,u}(t_-) \cap B_{\bar{\varepsilon}}(\bar{x}(t_-)).$$

□

Next we show that the ε-(un)stable ft-fiber of a linear D-hyperbolic system locally coincides with the (un)stable cone, see Definition 4.1.3/4.1.5.

Lemma 6.4.5. *Assume* **(A1)**. *Then for every $\varepsilon > 0$, $t_0 \in \mathbb{I}$ and solution $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ of* (2.7)/(2.8) *we get*

$${}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s,u}(t_0) = {}^{\mathbb{I}}_{\varepsilon}M^{\bar{x}}_{s,u}(t_0). \tag{6.11}$$

Further the sets ${}^{\mathbb{I}}_{\varepsilon}W^{0}_{s,u}(t_0)$ and ${}^{\mathbb{I}}\bar{V}_s(t_0)/{}^{\mathbb{I}}V_u(t_0)$ locally coincide, i.e.

$${}^{\mathbb{I}}_{\varepsilon}W^{0}_{s}(t_0) = {}^{\mathbb{I}}\bar{V}_s(t_0) \cap \Phi_{\text{pre}}(t_+,t_0)B_\varepsilon(0), \tag{6.12}$$
$${}^{\mathbb{I}}_{\varepsilon}W^{0}_{u}(t_0) = {}^{\mathbb{I}}V_u(t_0) \cap \Phi(t_0,t_-)B_\varepsilon(0). \tag{6.13}$$

Proof. Let $\varepsilon > 0$ and $t_0 \in \mathbb{I}$. Assume $x_0 \in {}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s}(t_0) \setminus \{\bar{x}(t_0)\}$ and let $t_{\min} := {}_{\Phi}\mathcal{T}_{\min}(x_0,t_0)$ then there exists an $x_{\min} \in {}_{\Phi}\mathcal{T}_{\text{pre}}(y_0,n_0)$ (in the case $\mathbb{T} = \mathbb{R}$ the preimage $x_{\min}$ is unique and exists to time t_-) such that

$$\hat{t} := {}^{\mathbb{I}}_{\Phi}\mathcal{B}_\varepsilon(x_{\min},\bar{x}(t_{\min}),t_{\min}) \in \mathbb{I}$$

and

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\,\|\Phi(t,t_{\min})x_{\min} - \bar{x}(t)\|_\Gamma < 0, & \text{for } \mathbb{T} = \mathbb{R}, \\ \|\Phi(s,t_{\min})x_{\min} - \bar{x}(s)\|_\Gamma < \|\Phi(t,t_{\min})x_{\min} - \bar{x}(t)\|_\Gamma, & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}$$

holds for all $\begin{cases} t \in \mathbb{I},\ t > \hat{t}, & \text{for } \mathbb{T} = \mathbb{R}, \\ t, s \in \mathbb{I},\ \mathring{t} \geq s > t > \hat{t}, & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}$
with $\mathring{t} := {}_{\varphi}\mathcal{T}_{\text{ker}}^{\bar{x}}(x_{\min}, t_{\min})$ (in the case $\mathbb{T} = \mathbb{R}$ we have $\mathring{t} = t_+$). By equation (3.16)/(3.17) we get

$$0 > \langle \Phi(t, t_{\min})x_{\min} - \bar{x}(t), S_\Gamma(t)(\Phi(t, t_{\min})x_{\min} - \bar{x}(t)) \rangle \tag{6.14}$$

for all $t \in \mathbb{I}$, $\begin{cases} t > \hat{t}, & \text{for } \mathbb{T} = \mathbb{R}, \\ \mathring{t} - 1 \geq t > \hat{t}, & \text{for } \mathbb{T} = \mathbb{Z}. \end{cases}$
By the D-hyperbolicity of (2.7)/ (2.8) Lemma 5.2.2/5.2.3 applies and we obtain equation (6.14) for all $t \in [t_{\min}, \hat{t})_{\mathbb{T}}$. This leads to

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} \left\| \Phi(t, t_{\min})x_{\min} - \bar{x}(t) \right\|_\Gamma < 0, & \text{for } \mathbb{T} = \mathbb{R}, \\ \left\| \Phi(s, t_{\min})x_{\min} - \bar{x}(s) \right\|_\Gamma < \left\| \Phi(t, t_{\min})x_{\min} - \bar{x}(t) \right\|_\Gamma, & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}$$

for all $\begin{cases} t \in \mathbb{I},\ t \geq t_{\min}, & \text{for } \mathbb{T} = \mathbb{R}, \\ t, s \in \mathbb{I},\ \mathring{t} \geq s > t \geq t_{\min}, & \text{for } \mathbb{T} = \mathbb{Z}. \end{cases}$
By the definition of the monotonically ε-stable ft-t_0-fiber we get $x_0 \in {}^{\mathbb{I}}_{\varepsilon}M_s^{\bar{x}}(t_0)$ and thus

$${}^{\mathbb{I}}_{\varepsilon}W_s^{\bar{x}}(t_0) \subset {}^{\mathbb{I}}_{\varepsilon}M_s^{\bar{x}}(t_0).$$

With equation (6.9) we get (6.11).

The statement about the unstable sets follows analogously.

Next we note that $\bar{x}(\cdot) := 0$ is a solution of the linear equation (2.7)/(2.8) and as a consequence the left-hald sides of the claims (6.12) and (6.13) are well defined. Using (6.11) and the definition of the monotonically ε-stable ft-t_0-fiber we get for $\mathbb{T} = \mathbb{R}$

$$\begin{aligned} {}^{\mathbb{I}}_{\varepsilon}W_s^{0}(t_0) =& {}^{\mathbb{I}}_{\varepsilon}M_s^{0}(t_0) \\ =& \left\{ x_0 \in \mathbb{R}^k \,\middle|\, \frac{\mathrm{d}}{\mathrm{d}t} \left\| \Phi(t, t_0)x_0 \right\|_\Gamma < 0 \text{ for all } t \in \mathbb{I},\ \Phi(t_+, t_0)x_0 \in B_\varepsilon(0) \right\} \\ & \cup \{0\} \end{aligned}$$

and for $\mathbb{T} = \mathbb{Z}$

$$\begin{aligned} {}^{\mathbb{I}}_{\varepsilon}W_s^{0}(t_0) =& {}^{\mathbb{I}}_{\varepsilon}M_s^{0}(t_0) \\ =& \big\{ \xi \in \mathbb{R}^k \big| \exists x_{\min} \in {}_{\Phi}\mathcal{T}_{\text{pre}}(\xi, t_0) \text{ with } t_{\min} := {}_{\Phi}\mathcal{T}_{\min}(\xi, t_0), \\ & \mathring{t} := {}_{\Phi}\mathcal{T}_{\text{ker}}^{0}(x_{\min}, t_{\min}) : \left\| \Phi(s, t_{\min})x_{\min} \right\|_\Gamma < \left\| \Phi(t, t_{\min})x_{\min} \right\|_\Gamma \\ & \text{for all } s, t \in \mathbb{I}, \mathring{t} \geq s > t \geq t_{\min}, \Phi(t_+, t_0)\xi \in B_\varepsilon(0) \big\}. \end{aligned}$$

Equation (4.12)/ (4.10) prove that the ft-fiber bundles satisfy equation (6.12). The second claim (6.13) follows analogously. □

For every finite time solution $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ of (2.2) or (2.3) and every $\varepsilon > 0$ the intersection of the monotonically (ε-)stable and the (ε-)unstable ft-fiber bundle is empty, i.e.

$$^{\mathbb{I}}M_s^{\bar{x}} \cap {}^{\mathbb{I}}M_u^{\bar{x}} = \emptyset,\ {}^{\mathbb{I}}_{\varepsilon}M_s^{\bar{x}} \cap {}^{\mathbb{I}}_{\varepsilon}M_u^{\bar{x}} = \emptyset \text{ for every solution } \bar{x} : \mathbb{I} \to \mathbb{R}^k,\ \varepsilon > 0.$$

(Un)Stable Ft-Fiber Bundles and Ft-Hyperbolicity

We determine an adequate analogon of the infinite time fiber bundles for finite time systems with the help of the studied characteristics in the previous section. Further we define ft-hyperbolic trajectories and the stable and unstable ft-fiber bundles of the linearization. These fiber bundles roughly speaking locally approximate the fiber bundles of the original system, which we prove in the next section.

Because of the invariance of the monotonically (ε-)unstable ft-fiber bundles and the at least forward invariance of the monotonically (ε-)stable ft-fiber bundles w.r.t. t_- we can not extend the fibers such that they will intersect. Our purpose later is to find homoclinic trajectories, which lie in the intersection of the stable and unstable fibers, see Chapter 7. Hence, we are interested in the intersection of the stable and unstable ft-fibers. Thus, in this thesis we choose the ε-stable and ε-unstable ft-fibers (Definition 6.3.3/6.3.4) as the analogon of the stable and unstable fibers (Definition 6.0.1) for finite time systems.

Next we define stable and unstable ft-fiber bundles of a linearization and ft-hyperbolic solutions. We need the following condition:

(A3) Let $f : \mathbb{R}^k \times \mathbb{I} \to \mathbb{R}^k$ of equation (2.2) be a continuous function with continuous derivatives f_x and $f_x^{(2)}$.

Roughly speaking a solution $\bar{x}$ of a nonlinear system is call ft-hyperbolic if the linearization w.r.t. $\bar{x}$ is D-hyperbolic.

Definition 6.5.1. *Assume* **(A3)**. *Let $\mathbb{I} \subset \mathbb{R}$ be a compact interval and $\Gamma = \Gamma^T > 0$. A solution $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ of* (2.2) *is called **ft-hyperbolic** (finite time hyperbolic on $\mathbb{I}$ w.r.t. $\|\cdot\|_\Gamma$) if for all $t \in \mathbb{I}$ the symmetric part*

$$S_\Gamma^{\bar{x}}(t) := \tfrac{1}{2}(\Gamma f_x(\bar{x}(t), t) + (f_x(\bar{x}(t), t))^T \Gamma)$$

of $f_x(\bar{x}(t), t)$ is indefinite and

$$M_\Gamma^{\bar{x}}(t) := \dot{S}_\Gamma^{\bar{x}}(t) + S_\Gamma^{\bar{x}}(t) f_x(\bar{x}(t), t) + f_x(\bar{x}(t), t)^T S_\Gamma^{\bar{x}}(t)$$

with $\dot{S}_\Gamma^x(t) = (S_\Gamma^x)_t(t) + (S_\Gamma^x)_x(t) f(x(t), t)$ is positive definite for all $t \in \mathbb{I}$.

Definition 6.5.2. *Let $\mathbb{I} \subset \mathbb{Z}$ be a compact interval and $\Gamma = \Gamma^T > 0$. A solution $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ of* (2.3) *is called* ***ft-hyperbolic*** *(finite time hyperbolic on $\mathbb{I}$ w.r.t. $\|\cdot\|_\Gamma$) if for all $n \in \mathbb{I}_1$ the symmetric matrix*

$$S_\Gamma^{\bar{x}}(n) := ((f_n)_x(\bar{x}(n)))^T \, \Gamma (f_n)_x(\bar{x}(n)) - \Gamma$$

is indefinite and

$$M_\Gamma^{\bar{x}}(n) := ((f_n)_x(\bar{x}(n)))^T \, S_\Gamma^{\bar{x}}(n+1)(f_n)_x(\bar{x}(n)) - S_\Gamma^{\bar{x}}(n)$$

is positive definite for all $n \in \mathbb{I}_2$.

If $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ is an ft-hyperbolic solution of (2.2)/(2.3) then the linearized equation (2.4)/(2.5) is D-hyperbolic. Let $\tilde{\mathbb{I}} := \begin{cases} \mathbb{I}_1, & \text{if (2.3) is noninvertible,} \\ \mathbb{I}, & \text{otherwise} \end{cases}$ and let $\bar{t} := \begin{cases} t_+ - 1, & \text{for (2.3),} \\ t_+, & \text{for (2.2).} \end{cases}$
We call

$$\begin{aligned} {}^{\mathbb{I}}V_s^{\bar{x}} := & \Big\{(x_0, t_0) \in \mathbb{R}^k \times \tilde{\mathbb{I}} \Big| \langle \Phi(\bar{t}, t_0)x_0, S_\Gamma^{\bar{x}}(\bar{t})\Phi(\bar{t}, t_0)x_0 \rangle < 0 \Big\} \\ & \cup \{(0, t_0) \in \mathbb{R}^k \times \tilde{\mathbb{I}}\} \end{aligned} \tag{6.15}$$

and

$$\begin{aligned} {}^{\mathbb{I}}V_u^{\bar{x}} := & \big\{(\Phi(t_0, t_-)x_-, t_0) \in \mathbb{R}^k \times \mathbb{I} \big| \langle x_-, S_\Gamma^{\bar{x}}(t_-)x_- \rangle > 0 \big\} \\ & \cup \{(0, t_0) \in \mathbb{R}^k \times \mathbb{I}\} \end{aligned} \tag{6.16}$$

the **stable and unstable ft-fiber bundle of the linearization**. These sets coincide with the stable and unstable cone of the linearized system, see Corollary 5.3.2/ 5.3.5. By Lemma 6.4.5 this is a well-considered definition.

We show in Section 6.6 roughly speaking that the (un)stable ft-fiber bundle of the linearization locally approximates the (un)stable ft-fiber bundle of the original system. For this purpose [83, Theorem 4.13] functions as a basis. We first show that our monotonically ε-(un)stable t_--fiber is equivalent to a modified version of the domains of finite-time attraction/repulsion of [83, Definition 4.9].

Lemma 6.5.3. *Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$, $\mathbb{I} = [t_-, t_+]_\mathbb{T}$, $\varepsilon > 0$ and let $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ be an ft-hyperbolic solution of* (2.2)/ (2.3). *Then the monotonically ε-stable* *and ε-unstable ft-t_--fibers of Definition 6.2.1/ 6.2.2*

$$
{}^{\mathbb{I}}_{\varepsilon}M_s^{\bar{x}}(t_-) = \begin{cases} \big\{\xi \in \mathbb{R}^k \big| \frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(\xi,t,t_-)-\bar{x}(t)\| < 0 \textit{ for all } t \in \mathbb{I}, \\ \varphi(\xi,t_+,t_-) \in B_\varepsilon(\bar{x}(t_+))\big\} \cup \{\bar{x}(t_-)\} & , \textit{ for } \mathbb{T}=\mathbb{R}, \\ \big\{\xi \in \mathbb{R}^k \big| \|\varphi(\xi,t,t_-)-\bar{x}(t)\| < \|\varphi(\xi,s,n_-)-\bar{x}(s)\| \\ \textit{ for all } t,s \in \mathbb{I},\ \mathring{t} \geq t > s \textit{ with } \mathring{t} := {}_{\varphi}\mathcal{T}^{\bar{x}}_{\mathrm{ker}}(\xi,t_-), \\ \varphi(\xi,t_+,t_-) \in B_\varepsilon(\bar{x}(t_+))\big\} & , \textit{ for } \mathbb{T}=\mathbb{Z}, \end{cases}
$$

$$
{}^{\mathbb{I}}_{\varepsilon}M_u^{\bar{x}}(t_-) = \begin{cases} \big\{\xi \in \mathbb{R}^k \big| \frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(\xi,t,t_-)-\bar{x}(t)\| > 0 \textit{ for all } t \in \mathbb{I}, \\ \xi \in B_\varepsilon(\bar{x}(t_-))\big\} \cup \{\bar{x}(t_-)\} & , \textit{ for } \mathbb{T}=\mathbb{R}, \\ \big\{\xi \in \mathbb{R}^k \big| \|\varphi(\xi,t,t_-)-\bar{x}(t)\| > \|\varphi(\xi,s,t_-)-\bar{x}(s)\| \\ \textit{ for all } t,s \in \mathbb{I},\ t > s, \xi \in B_\varepsilon(\bar{x}(t_-))\big\} \cup \{\bar{x}(t_-)\} & , \textit{ for } \mathbb{T}=\mathbb{Z} \end{cases}
$$

are equivalent to

$$
M^s_{\bar{x}} := \left\{ \xi \in \mathbb{R}^k \setminus \{\bar{x}(t_-)\} \,\middle|\, \sup_{(t,s)\in(\mathbb{I}\times\mathbb{I})^{\xi}_{\neq}} \Delta^{\bar{x}}_{\varphi}(t,s,\xi) < 0,\ \varphi(\xi,t_+,t_-) \in B_\varepsilon(\bar{x}(t_+)) \right\}
$$
$$
\cup \{\bar{x}(t_-)\},
$$
$$
M^u_{\bar{x}} := \left\{ \xi \in \mathbb{R}^k \setminus \{\bar{x}(t_-)\} \,\middle|\, \inf_{(t,s)\in(\mathbb{I}\times\mathbb{I})^{\xi}_{\neq}} \Delta^{\bar{x}}_{\varphi}(t,s,\xi) > 0,\ \xi \in B_\varepsilon(\bar{x}(t_-)) \right\}
$$
$$
\cup \{\bar{x}(t_-)\},
$$

respectively with

$$
\begin{aligned} \Delta^{\bar{x}}_{\varphi} : (\mathbb{I}\times\mathbb{I}\times\mathbb{R}^k)_{\neq} &\to \mathbb{R}, \\ (t,s,\xi) &\mapsto \frac{\ln(\|\varphi(\xi,t,t_-)-\bar{x}(t)\|) - \ln(\|\varphi(\xi,s,t_-)-\bar{x}(s)\|)}{t-s}, \\ \left(\mathbb{I}\times\mathbb{I}\times\mathbb{R}^k\right)_{\neq} &:= \left\{(t,s,\xi) \in \mathbb{I}\times\mathbb{I}\times\mathbb{R}^k \middle| t \neq s,\ t,s \leq {}_{\varphi}\mathcal{T}^{\bar{x}}_{\mathrm{ker}}(\xi,t_-)\right\}, \\ (\mathbb{I}\times\mathbb{I})^{\xi}_{\neq} &:= \left\{(t,s) \in \mathbb{I}\times\mathbb{I} | t \neq s,\ t,s \leq {}_{\varphi}\mathcal{T}^{\bar{x}}_{\mathrm{ker}}(\xi,t_-)\right\}. \end{aligned} \tag{6.17}
$$

Proof. Let $\xi \in \mathbb{R}^k \setminus \{\bar{x}(t_-)\}$. Set $\mathring{t} := {}_{\varphi}\mathcal{T}^{\bar{x}}_{\mathrm{ker}}(\xi,t_-)$. Note that $\mathring{t} = t_+$ for $\mathbb{T}=\mathbb{R}$. Then we have

$$
\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(\xi,t,t_-)-\bar{x}(t)\| < 0 \text{ for all } t \in \mathbb{I}, & \text{for } \mathbb{T}=\mathbb{R}, \\ \|\varphi(\xi,t,t_-)-\bar{x}(t_-)\| < \|\varphi(\xi,s,t_-)-\bar{x}(t_-)\| \\ \text{ for all } t,s \in \mathbb{I}, \mathring{t} \geq t > s, & \text{for } \mathbb{T}=\mathbb{Z} \end{cases}
$$
$$
\Leftrightarrow \|\varphi(\xi,t,t_-)-\bar{x}(t)\| - \|\varphi(\xi,s,t_-)-\bar{x}(s)\| < 0 \text{ for all } t,s \in \mathbb{I}, \mathring{t} \geq t > s
$$
$$
\Leftrightarrow \Delta^{\bar{x}}_{\varphi}(t,s,\xi) = \frac{\ln(\|\varphi(\xi,t,t_-)-\bar{x}(t)\|) - \ln(\|\varphi(\xi,s,t_-)-\bar{x}(s)\|)}{t-s} < 0
$$
$$
\text{for all } t,s \in \mathbb{I}, \mathring{t} \geq t > s
$$
$$
\Leftrightarrow \sup_{(t,s)\in(\mathbb{I}\times\mathbb{I})^{\xi}_{\neq}} \Delta^{\bar{x}}_{\varphi}(t,s,\xi) < 0.
$$

Thus, the stable sets are equivalent. The equivalence of the unstable sets follows analogously. □

Local Approximation of (Ft-)Fiber Bundles

In infinite time, $\mathbb{I} \in \{\mathbb{R}, \mathbb{Z}\}$, local stable and unstable fiber bundles ${}_UW^0_{s,u}$ (see Def. 6.0.1) of a hyperbolic fixed point 0 exist and can be represented by a local graph representation. For discrete invertible systems we refer to [70, Theorem 3.2], for noninvertible systems to [58, Theorem and proof 3.8] and [124, Theorem 4.1] and for continuous systems see [76, Theorem 6.5)]. Further references are [110, Theorem 4.9], [112, Theorem 3.2] and [113, Theorem 3.2]. More precisely, if (2.4)/(2.5) has an exponential dichotomy with the family of projectors $P : \mathbb{I} \to \mathbb{R}^k$, there exist maps $s^{\pm} : \mathbb{I} \times U \to \mathbb{R}^k$ with

$$s^+(t,x) = s^+(t, P(t)x) \in \mathcal{N}(P(t)),\ s^-(t,x) = s^-(t, (\mathrm{I} - P(t))x) \in \mathcal{R}(P(t))$$

such that the fiber bundles of (2.2)/(2.3) satisfy

$$\begin{aligned} {}_UW^0_s &= \left\{(t, x + s^+(t,x)) \in \mathbb{I} \times \mathbb{R}^k | x \in \mathcal{R}(P(t)) \cap U\right\}, \\ {}_UW^0_u &= \left\{(t, x + s^-(t,x)) \in \mathbb{I} \times \mathbb{R}^k | x \in \mathcal{N}(P(t)) \cap U\right\}. \end{aligned}$$

With this representation we can show that the stable and unstable subspace of the linearization locally approximate the stable and unstable fibers. In particular, under smoothness assumptions we have

$$T_0W^0_s(t_0) = V^0_s(t_0),\ T_0W^0_u(t_0) = V^0_u(t_0) \tag{6.18}$$

for $t_0 \in \mathbb{I}$ where $T_0W^0_{s,u}(t_0)$ denotes the tangent space of $W^0_{s,u}(t_0)$ at 0, i.e.

$$T_0W^0_{s,u}(t_0) := \left\{\nu \in \mathbb{R}^k | \exists \zeta \in \mathcal{C}^1((-\delta,\delta), {}^{\mathbb{I}}_{\varepsilon}W^0_{s,u}(t_0)) : \zeta(0) = 0, \dot{\zeta}(0) = \nu\right\},$$

see [21, p. 3370]. For a proof and more details we refer to [105, Proposition 5.4], [76, Theorem 9], [124, Theorem and proof 4.2] and [70, Theorem 3.5].

In the following we show for finite time systems roughly speaking that the (un)stable cone of the linearization locally approximates the (un)stable ft-fiber bundle of the original system. More precisely, the boundaries locally approximate each other. This statement proved in Theorem 6.6.1 holds for continuous systems as well as for discrete systems. In particular, we show a result for our ft-fibers, which is analogous to [83, Theorem 4.13]. We use some of the definitions and techniques presented in [83]. From now on let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$ and let $\mathbb{I} \subset \mathbb{T}$ be a compact interval. First we define the **growth rate of a point** by

$$\begin{aligned} \underline{\mu} &: \mathbb{R}^k \setminus \{0\} \times C(\mathbb{I} \times \mathbb{I} \times \mathbb{R}^k, \mathbb{R}^k) \to \mathbb{R},\ (x,\varphi) \mapsto \inf_{(t,s) \in (\mathbb{I}\times\mathbb{I})^x_{\neq}} \Delta_\varphi(t,s,x), \\ \overline{\mu} &: \mathbb{R}^k \setminus \{0\} \times C(\mathbb{I} \times \mathbb{I} \times \mathbb{R}^k, \mathbb{R}^k) \to \mathbb{R},\ (x,\varphi) \mapsto \sup_{(t,s) \in (\mathbb{I}\times\mathbb{I})^x_{\neq}} \Delta_\varphi(t,s,x), \end{aligned}$$

with $\Delta_\varphi := \Delta^0_\varphi$, $(\mathbb{I}\times\mathbb{I})^x_{\neq}$ defined in (6.17).

Then we have by definition of $\overline{\lambda}, \underline{\lambda}$, see equations (3.26) -(3.27), for all $X \in \mathrm{Gr}(1,\mathbb{R}^k)$ and $\bar{x} \in X \setminus \{0\}$

$$\overline{\lambda}(X,\Phi) = \sup_{\substack{x\in X, \|x\|=1,\\ (t,s)\in(\mathbb{I}\times\mathbb{I})^x_{\neq}}} \Delta_\Phi(t,s,x) = \sup_{x\in X,\|x\|=1} \overline{\mu}(x,\Phi) = \overline{\mu}(\bar{x},\Phi), \tag{6.19}$$

$$\underline{\lambda}(X,\Phi) = \inf_{\substack{x\in X, \|x\|=1,\\ (t,s)\in(\mathbb{I}\times\mathbb{I})^x_{\neq}}} \Delta_\Phi(t,s,x) = \inf_{x\in X,\|x\|=1} \underline{\mu}(x,\Phi) = \underline{\mu}(\bar{x},\Phi). \tag{6.20}$$

Let φ be the solution operator of of the differential equation (2.2)/the difference equation (2.3) and Φ be the solution operator of the of the linearization along the zero reference trajectory (2.4)/(2.5). For analyzing the common properties of the solution operator φ and the solution operator Φ we introduce a measure of their approximation as in [83, equation (44a),(44b)]

$$m : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \tag{6.21}$$

$$\nu \mapsto \begin{cases} 0, & \nu = 0,\\ \sup\limits_{x\in B_\nu[0]\setminus\{0\}} \max\{|\underline{\mu}(x,\varphi) - \underline{\mu}(x,\Phi)|, |\overline{\mu}(x,\varphi) - \overline{\mu}(x,\Phi)|\}, & \nu \neq 0, \end{cases}$$

where $B_\nu[0] := \left\{x \in \mathbb{R}^k \middle| \, \|x\| \leq \nu\right\}$ denotes the close ν-ball around 0 with radius ν.

In the following we need that m is continuous at 0. Karrasch yields conditions under which m is continuous at 0 in the discrete and in the continuous time case. For more information we refer to [83, Lemma 4.10 and Lemma 4.11]. Let 0 be a fixed point of (2.2)/(2.3), $\varepsilon, \delta > 0$ and ${}^{\mathbb{I}}_{\varepsilon}W^0_{s,u}$ be the ft-fiber bundles of (2.2)/(2.3) of 0. Then the **stable and unstable tangent sets** at a general point $x \in \partial^{\mathbb{I}}_{\varepsilon}W^0_{s,u}(t)$, $t \in \mathbb{I}$, are defined by

$$T_x\partial^{\mathbb{I}}_{\varepsilon}W^0_{s,u}(t) := \left\{\nu \in \mathbb{R}^k | \exists \zeta \in \mathcal{C}^1((-\delta,\delta), \partial^{\mathbb{I}}_{\varepsilon}W^0_{s,u}(t)) : \zeta(0) = x, \dot{\zeta}(0) = \nu\right\}, \tag{6.22}$$

where $\partial^{\mathbb{I}}_{\varepsilon}W^0_{s,u}$ denotes the boundary of the ft-fiber bundles ${}^{\mathbb{I}}_{\varepsilon}W^0_{s,u}$. An analogous definition of (6.22) can be found in [21, p. 3370]. We show that the tangent set at 0 of the boundary of the ft-fibers of the original system equals the cone boundary of the linearized (along zero) system.

Theorem 6.6.1. *Let* $\varepsilon > 0$, $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$ *and* $\mathbb{I} = [t_-, t_+]_{\mathbb{T}}$ *be a compact interval. Let* $\Gamma = \Gamma^T > 0$ *and* $0 : \mathbb{I} \to \mathbb{R}^k$ *be an ft-hyperbolic solution (w.r.t.* $\|\cdot\|_\Gamma$*) of equation* (2.2)/(2.3)*. Assume* **(A1)***. Let* ${}^{\mathbb{I}}_{\varepsilon}W^0_{s,u}$ *be the ft-fiber bundles of equation* (2.2)/(2.3) *and let* ${}^{\mathbb{I}}V^0_{s,u}$ *be the ft-fiber bundles of the linearization* (2.4)/(2.5)*. If the function* m *defined in* (6.21) *is continuous at* 0 *then the tangent set at* 0 *of the boundary* $\partial^{\mathbb{I}}_{\varepsilon}W^0_{s,u}$ *equals the boundary* $\partial^{\mathbb{I}}V^0_{s,u}$*, i.e. for all* $t \in \mathbb{I}$ *we have*

$$T_0\varphi(t,t_-)\partial^{\mathbb{I}}_{\varepsilon}W^0_{s,u}(t_-) = \Phi(t,t_-)\partial^{\mathbb{I}}V^0_{s,u}(t_-). \tag{6.23}$$

Proof. Lemma 6.4.4 yields the existence of an $\bar{\varepsilon} > 0$ such that

$$ {}^{\mathbb{I}}_{\varepsilon}W_{s,u}(t_-) \cap B_{\bar{\varepsilon}}(0) = {}^{\mathbb{I}}_{\varepsilon}M^0_{s,u}(t_-) \cap B_{\bar{\varepsilon}}(0) $$

and by Lemma 6.5.3 we obtain

$$ {}^{\mathbb{I}}_{\varepsilon}W^0_{s,u}(t_-) \cap B_{\bar{\varepsilon}}(0) = {}^{\mathbb{I}}_{\varepsilon}M_{s,u}(t_-) \cap B_{\bar{\varepsilon}}(0) = M_0^{s,u} \cap B_{\bar{\varepsilon}}(0). $$

This means that we have

$$ {}^{\mathbb{I}}_{\varepsilon}W^0_s(t_-) \cap B_{\bar{\varepsilon}}(0) = \left\{\xi \in B_{\bar{\varepsilon}}(0) \middle| \overline{\mu}(\xi,\varphi) < 0\right\} \cup \{0\}, \tag{6.24} $$
$$ {}^{\mathbb{I}}_{\varepsilon}W^0_u(t_-) \cap B_{\bar{\varepsilon}}(0) = \left\{\xi \in B_{\bar{\varepsilon}}(0) \middle| \underline{\mu}(\xi,\varphi) > 0\right\} \cup \{0\}. \tag{6.25} $$

For the stable and unstable ft-fibers of the linearization, see (6.15) and (6.16), we obtain by Lemma 4.2.2 and (6.19), (6.20)

$$ \begin{aligned} {}^{\mathbb{I}}V^0_s(t_-) &= \left\{\xi \in \mathbb{R}^k \setminus \{0\} \middle| \exists X \in \mathrm{Gr}(1,\mathbb{R}^k) : \xi \in X,\ \overline{\lambda}(X,\Phi) < 0\right\} \cup \{0\}, \\ &= \left\{\xi \in \mathbb{R}^k \setminus \{0\} \middle| \overline{\mu}(x,\Phi) < 0\right\} \cup \{0\}, \end{aligned} \tag{6.26} $$
$$ {}^{\mathbb{I}}V^0_u(t_-) = \left\{\xi \in \mathbb{R}^k \setminus \{0\} \middle| \underline{\mu}(x,\Phi) > 0\right\} \cup \{0\}. \tag{6.27} $$

First we show that for any

$$ \begin{gathered} X, Y \in \mathrm{Gr}(1,\mathbb{R}^k) \text{ with } X \subset {}^{\mathbb{I}}V^0_s(t_-),\ Y \subset {}^{\mathbb{I}}V^0_u(t_-) \text{ there exists} \\ \text{a } \delta > 0 \text{ such that } B_\delta(0) \cap X \subset {}^{\mathbb{I}}_{\varepsilon}W^0_s(t_-),\ B_\delta(0) \cap Y \subset {}^{\mathbb{I}}_{\varepsilon}W^0_u(t_-). \end{gathered} \tag{6.28} $$

Let $X, Y \in \mathrm{Gr}(1,\mathbb{R}^k)$ with $X \subset {}^{\mathbb{I}}V^0_s(t_-)$ and $Y \subset {}^{\mathbb{I}}V^0_u(t_-)$ then by (6.26) and (6.27) as well as by the definition of $\overline{\mu}$ and $\underline{\mu}$ we have for all $x_1, x_2 \in X \setminus \{0\}$, $y_1, y_2 \in Y \setminus \{0\}$

$$ \overline{\mu}(x_1,\Phi) = \overline{\mu}(x_2,\Phi) < 0,\ \underline{\mu}(y_1,\Phi) = \underline{\mu}(y_2,\Phi) > 0. $$

By the continuity of m there exists a $\delta > 0$ such that $m(\nu) < -\overline{\mu}(x,\Phi)$ and $m(\nu) < \underline{\mu}(y,\Phi)$ for each $\nu \in [0,\delta]$ and $x \in X \setminus \{0\}$, $y \in Y \setminus \{0\}$. For all $x \in B_\delta[0] \cap X \setminus \{0\}$, $y \in B_\delta[0] \cap Y \setminus \{0\}$ we have with (6.21)

$$ \begin{aligned} -\overline{\mu}(x,\Phi) &> m(\delta) \geq \left|\overline{\mu}(x,\varphi) - \overline{\mu}(x,\Phi)\right|, \\ \underline{\mu}(y,\Phi) &> m(\delta) \geq \left|\underline{\mu}(y,\Phi) - \underline{\mu}(y,\varphi)\right|. \end{aligned} $$

This implies for all $x \in B_\delta[0] \cap X \setminus \{0\}$, $y \in B_\delta[0] \cap Y \setminus \{0\}$

$$ \overline{\mu}(x,\varphi) < 0 \text{ and } \underline{\mu}(y,\varphi) > 0, $$

which leads by (6.24) and (6.25) for $\delta < \bar{\varepsilon}$ to (6.28).

Define

$$ \begin{aligned} \mathcal{V}_{s,u} &:= \left(\mathbb{R}^k \setminus \overline{{}^{\mathbb{I}}V^0_{s,u}(t_-)}\right) \cup \{0\}, \\ \mathcal{W}_{s,u} &:= \left(\mathbb{R}^k \setminus \overline{{}^{\mathbb{I}}_{\varepsilon}W^0_{s,u}(t_-)}\right) \cup \{0\}. \end{aligned} $$

We prove that for any

$$X, Y \in \mathrm{Gr}(1,\mathbb{R}^k) \text{ with } X \subset \mathcal{V}_s,\ Y \subset \mathcal{V}_u \text{ there exists a } \delta > 0 \\ \text{such that } B_\delta[0] \cap X \subset \mathcal{W}_s,\ B_\delta[0] \cap Y \subset \mathcal{W}_u. \tag{6.29}$$

Therefore, we need the boundary of ${}^{\mathbb{I}}V^0_{s,u}(t_-)$ w.r.t. $\overline{\mu}$ and $\underline{\mu}$. By the D-hyperbolicity of the linearization we can apply Corollary 5.3.2 and 5.3.5 as well as Lemma 5.2.2 and 5.2.3 and we obtain

$$\begin{aligned}
&{}^{\mathbb{I}}V^0_s(t_-)\\
&= \left\{\xi \in \mathbb{R}^k \setminus \{0\} \middle| \overline{\mu}(\xi,\Phi) < 0\right\} \cup \{0\}\\
&= \left\{\xi \in \mathbb{R}^k \middle| \langle \Phi(\bar{t},t_-)\xi, S_\Gamma(\bar{t})\Phi(\bar{t},t_-)\xi\rangle < 0\right\} \cup \mathcal{N}(\Phi(\bar{t},t_-)),\\
&V^a\\
&:= \left\{\xi \in \mathbb{R}^k \setminus \{0\} \middle| \overline{\mu}(\xi,\Phi) > 0\right\} \cup \{0\}\\
&= \left\{\xi \in \mathbb{R}^k \middle| \sup_{t,s\in\mathbb{I},\ t>s} \|\Phi(t,t_-)\xi\|_\Gamma - \|\Phi(s,t_-)\xi\|_\Gamma > 0\right\} \cup \{0\}\\
&= \begin{cases} \left\{\xi \in \mathbb{R}^k \middle| \frac{\mathrm{d}}{\mathrm{d}t}\|\Phi(t,t_-)\xi\|_\Gamma > 0 \text{ for any } t \in \mathbb{I}\right\} & , \text{ for } \mathbb{T} = \mathbb{R},\\ \left\{\xi \in \mathbb{R}^k \middle| \|\Phi(t+1,t_-)\xi\|_\Gamma - \|\Phi(t,t_-)\xi\|_\Gamma > 0 \text{ for any } t \in \mathbb{I}_1\right\}, & \text{ for } \mathbb{T} = \mathbb{Z}, \end{cases}\\
&\quad \cup \{0\}\\
&= \left\{\xi \in \mathbb{R}^k \middle| \langle \Phi(t,t_-)\xi, S_\Gamma(t)\Phi(t,t_-)\xi\rangle > 0 \text{ for any } t \in \tilde{\mathbb{I}}\right\} \cup \{0\}\\
&\supset \left\{\xi \in \mathbb{R}^k \middle| \langle \Phi(\bar{t},t_-)\xi, S_\Gamma(\bar{t})\Phi(\bar{t},t_-)\xi\rangle > 0\right\} \cup \{0\} =: \bar{V}^a.
\end{aligned}$$

Statement (4.19) and Lemma 5.7.2 yield

$$\begin{aligned}
\overline{{}^{\mathbb{I}}V^0_s(t_-)} \subset& \Phi_{\mathrm{pre}}(\bar{t},t_-)\overline{{}^{\mathbb{I}}V^0_s(\bar{t})}\\
=& \Phi_{\mathrm{pre}}(\bar{t},t_-)\left({}^{\mathbb{I}}V^0_s(\bar{t}) \cup Z_\Gamma(\bar{t})\right)\\
=& \left\{\Phi_{\mathrm{pre}}(\bar{t},t_-)\xi \in \mathbb{R}^k \middle| \langle \xi, S_\Gamma(\bar{t})\xi\rangle \le 0\right\}\\
=& \left\{\xi \in \mathbb{R}^k \middle| \langle \Phi(\bar{t},t_-)\xi, S_\Gamma(\bar{t})\Phi(\bar{t},t_-)\xi\rangle \le 0\right\}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\left({}^{\mathbb{I}}V^0_s(t_-) \setminus \{0\}\right) \dot\cup \left(V_a \setminus \{0\}\right) \dot\cup \left\{\xi \in \mathbb{R}^k \middle| \overline{\mu}(\xi,\Phi) = 0\right\}\\
=& \mathbb{R}^k\\
=& \overline{{}^{\mathbb{I}}V^0_s(t_-)} \dot\cup \left(\bar{V}^a \setminus \{0\}\right)\\
\subset& \overline{{}^{\mathbb{I}}V^0_s(t_-)} \cup \left(V^a \setminus \{0\}\right)\\
=& \partial^{\mathbb{I}}V^0_s(t_-) \cup \left({}^{\mathbb{I}}V^0_s(t_-) \setminus \{0\}\right) \cup \left(V^a \setminus \{0\}\right).
\end{aligned}$$

Thus,

$$\left\{\xi \in \mathbb{R}^k \middle| \overline{\mu}(\xi,\Phi) = 0\right\} \subset \partial^{\mathbb{I}}V^0_s(t_-).$$

Analogously, we get

$$\big\{\xi \in \mathbb{R}^k \big| \underline{\mu}(\xi,\Phi) = 0\big\} \subset \partial^{\mathbb{I}} V_u^0(t_-).$$

These statements yield together with (6.24), (6.25), (6.26), (6.27) and the continuity of $\overline{\mu}$ and $\underline{\mu}$

$$\begin{aligned}
\overline{{}^{\mathbb{I}}V_s^0(t_-)} &= \big\{\xi \in \mathbb{R}^k \setminus \{0\} \big| \overline{\mu}(\xi,\Phi) \le 0\big\} \cup \{0\},\\
\overline{{}^{\mathbb{I}}V_u^0(t_-)} &= \big\{\xi \in \mathbb{R}^k \setminus \{0\} \big| \underline{\mu}(\xi,\Phi) \ge 0\big\} \cup \{0\},\\
\overline{{}^{\mathbb{I}}_{\varepsilon}W_s^0(t_-) \cap B_{\bar\varepsilon}(0)} &\subset \{\xi \in B_{\bar\varepsilon}[0] \setminus \{0\} | \overline{\mu}(\xi,\varphi) \le 0\} \cup \{0\},\\
\overline{{}^{\mathbb{I}}_{\varepsilon}W_u^0(t_-) \cap B_{\bar\varepsilon}(0)} &\subset \big\{\xi \in B_{\bar\varepsilon}[0] \setminus \{0\} | \underline{\mu}(\xi,\varphi) \ge 0\big\} \cup \{0\}
\end{aligned}$$

and hence

$$\begin{aligned}
\mathcal{V}_s &= \Big(\mathbb{R}^k \setminus \overline{{}^{\mathbb{I}}V_s^0(t_-)}\Big) \cup \{0\}\\
&= \big\{\xi \in \mathbb{R}^k \setminus \{0\} \big| \overline{\mu}(\xi,\Phi) > 0\big\} \cup \{0\},\\
\mathcal{V}_u &= \Big(\mathbb{R}^k \setminus \overline{{}^{\mathbb{I}}V_u^0(t_-)}\Big) \cup \{0\}\\
&= \big\{\xi \in \mathbb{R}^k \setminus \{0\} \big| \underline{\mu}(\xi,\Phi) < 0\big\} \cup \{0\},\\
\mathcal{W}_s \cap B_{\bar\varepsilon}[0] &:= \Big(\Big(\mathbb{R}^k \setminus \overline{{}^{\mathbb{I}}_{\varepsilon}W_s^0(t_-) \cap B_{\bar\varepsilon}(0)}\Big) \cup \{0\}\Big) \cap B_{\bar\varepsilon}[0]\\
&\supset \{\xi \in B_{\bar\varepsilon}[0] \setminus \{0\} | \overline{\mu}(\xi,\varphi) > 0\} \cup \{0\}. \qquad (6.30)\\
\mathcal{W}_u \cap B_{\bar\varepsilon}[0] &:= \Big(\Big(\mathbb{R}^k \setminus \overline{{}^{\mathbb{I}}_{\varepsilon}W_u^0(t_-) \cap B_{\bar\varepsilon}(0)}\Big) \cup \{0\}\Big) \cap B_{\bar\varepsilon}[0]\\
&\supset \big\{\xi \in B_{\bar\varepsilon}[0] \setminus \{0\} | \underline{\mu}(\xi,\varphi) < 0\big\} \cup \{0\}. \qquad (6.31)
\end{aligned}$$

Let $X, Y \in \mathrm{Gr}(1,\mathbb{R}^k)$ with $X \subset \mathcal{V}_s$, $Y \subset \mathcal{V}_u$. Then analogously to the prove of (6.28) there exists a $\delta > 0$ such that for all $x \in B_\delta[0] \cap X \setminus \{0\}$, $y \in B_\delta[0] \cap Y \setminus \{0\}$ the following holds

$$\begin{aligned}
\overline{\mu}(x,\Phi) &> m(\delta) \ge |\overline{\mu}(x,\Phi) - \overline{\mu}(x,\varphi)|\,,\\
-\underline{\mu}(y,\Phi) &> m(\delta) \ge \big|\underline{\mu}(y,\varphi) - \underline{\mu}(y,\Phi)\big|\,,
\end{aligned}$$

which imply

$$\overline{\mu}(x,\varphi) > 0,\ \underline{\mu}(y,\varphi) < 0.$$

This leads with (6.30) and (6.31) and $\delta < \bar\varepsilon$ to

$$B_\delta[0] \cap X \subset \mathcal{W}_s,\ B_\delta[0] \cap Y \subset \mathcal{W}_u,$$

i.e. (6.29) is proved.

Our next step is to prove

$$\partial^{\mathbb{I}} V_{s,u}^0(t_-) = T_0 \partial_{\varepsilon}^{\mathbb{I}} W_{s,u}^0(t_-). \qquad (6.32)$$

Let $X \in \mathrm{Gr}(1, \mathbb{R}^k)$ with $X \subset \partial^{\mathbb{I}} V^0_{s,u}(t_-)$ and assume

$$X \not\subset T_0 \partial^{\mathbb{I}}_{\varepsilon} W^0_s(t_-).$$

Define

$$\begin{aligned}
T_{s,u} &:= \left\{ X \in \mathrm{Gr}(1, \mathbb{R}^k) \middle| X \subset T_0 \partial^{\mathbb{I}}_{\varepsilon} W^0_{s,u}(t_-) \right\}, \\
T^a_{s,u} &:= \left\{ X \in \mathrm{Gr}(1, \mathbb{R}^k) \middle| \exists \delta > 0 : X \cap B_\delta[0] \subset \mathcal{W}_s \right\}, \\
T^i_{s,u} &:= \left\{ X \in \mathrm{Gr}(1, \mathbb{R}^k) \middle| \exists \delta > 0 : X \cap B_\delta[0] \subset {}^{\mathbb{I}}_{\varepsilon} W^0_{s,u}(t_-) \right\}.
\end{aligned}$$

Then

$$T_{s,u} \dot{\cup} T^a_{s,u} \dot{\cup} T^i_{s,u} = \mathbb{R}^k$$

is satisfied. For X we obtain w.l.o.g.

$$X \subset T^i_{s,u} \text{ with } d_T(X, T^a_{s,u}) = \rho, \tag{6.33}$$

where

$$\begin{aligned}
d_T(X, T^b_{s,u}) &:= \inf \left\{ d(X, Y) | Y \in \mathrm{Gr}(1, \mathbb{R}^k),\ Y \subset T^b_{s,u} \right\}, b \in \{i, a\}, \\
d(X, Y) &:= \inf \left\{ \|x - y\|_2 \middle| x \in X, y \in Y \text{ with } \|x\|_2 = 1 = \|y\|_2 \right\}.
\end{aligned}$$

Let $X_1 \in \mathrm{Gr}(1, \mathbb{R}^k)$ with

$$X_1 \in \mathcal{V}_{s,u} \text{ and } d(X_1, X) < \rho. \tag{6.34}$$

Then by (6.33) we have

$$X_1 \subset T^i_{s,u}. \tag{6.35}$$

Statement (6.34), (6.29) and the definition of $T^{i,a}_{s,u}$ yield

$$X_1 \subset T^a_{s,u},$$

which is a contradiction to (6.35) and ,thus,

$$\partial^{\mathbb{I}} V^0_{s,u}(t_-) \subset T_0 \partial^{\mathbb{I}}_{\varepsilon} W^0_{s,u}(t_-)$$

holds.

Conversely, let $Y \in \mathrm{Gr}(1, \mathbb{R}^k)$ with $Y \subset T_0 \partial^{\mathbb{I}}_{\varepsilon} W^0_{s,u}(t_-)$ and assume $Y \not\subset \partial^{\mathbb{I}} V^0_{s,u}(t_-)$. Then $Y \subset {}^{\mathbb{I}} V_{s,u}(t_-)$ or $Y \subset \mathcal{V}_{s,u}$. The statements (6.28) and (6.29) yield a contradiction. Thus,

$$\partial^{\mathbb{I}} V^0_{s,u}(t_-) \supset T_0 \partial^{\mathbb{I}}_{\varepsilon} W^0_{s,u}(t_-)$$

holds and (6.32) is shown.

Finally, we prove claim (6.23). Let $x \in \Phi(t,t_-)\partial^{\mathbb{I}} V^0_{s,u}(t_-)$ then there exists a

$$y \in \partial^{\mathbb{I}} V^0_{s,u}(t_-) = T_0 \partial^{\mathbb{I}}_{\varepsilon} W^0_{s,u}(t_-) \tag{6.36}$$

with

$$x = \Phi(t,t_-)y. \tag{6.37}$$

Further by (6.36) and (6.22) there exists a function $\zeta \in \mathcal{C}^1((-\delta,\delta), \partial^{\mathbb{I}}_{\varepsilon} W^0_{s,u}(t_-))$ with $\zeta(0) = 0$ and $\dot{\zeta}(0) = y$. The function

$$\varphi(t,t_-)\zeta \in \mathcal{C}^1((-\delta,\delta), \varphi(t,t_-)\partial^{\mathbb{I}}_{\varepsilon} W^0_{s,u}(t_-))$$

satisfies $\varphi(\zeta(0),t,t_-) = \varphi(0,t,t_-) = 0$ and since Φ is the linearization of φ we get with (6.37)

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(\zeta(0),t,t_-) = \Phi(t,t_-)\dot{\zeta}(0) = \Phi(t,t_-)y = x.$$

This implies $x \in T_0\varphi(t,t_-)\partial^{\mathbb{I}}_{\varepsilon} W^0_{s,u}(t_-)$, which leads to

$$\Phi(t,t_-)\partial^{\mathbb{I}} V^0_{s,u}(t_-) \subset T_0\varphi(t,t_-)\partial^{\mathbb{I}}_{\varepsilon} W^0_{s,u}(t_-).$$

Conversely, let $x \in T_0\varphi(t,t_-)\partial^{\mathbb{I}}_{\varepsilon} W^0_{s,u}(t_-)$. Then there exists a function $\zeta \in \mathcal{C}^1((-\delta,\delta), \varphi(t,t_-)\partial^{\mathbb{I}}_{\varepsilon} W^0_{s,u}(t_-))$ with $\zeta(0) = 0$ and $\dot{\zeta}(0) = x$. Since $\varphi \in \mathcal{C}^1(\mathbb{I} \times \mathbb{I} \times \mathbb{R}^k, \mathbb{R}^k)$ there exists a function $\nu \in \mathcal{C}^1((-\delta,\delta), \partial^{\mathbb{I}}_{\varepsilon} W^0_{s,u}(t_-))$ with $\nu(0) = 0$ such that $\zeta(\rho) = \varphi(\nu(\rho),t,t_-)$ holds for all $\rho \in (-\delta,\delta)$. Then $0 = \varphi(\nu(0),t,t_-) = \zeta(0)$ is still true and we get by the definition of a tangent set (6.22) and equation (6.32) that

$$\begin{aligned} x = \dot{\zeta}(0) &= \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\nu(0),t,t_-) \\ &= \Phi(t,t_-)\dot{\nu}(0) \in \Phi(t,t_-)T_0\partial^{\mathbb{I}}_{\varepsilon} W^0_{s,u}(t_-) = \Phi(t,t_-)\partial^{\mathbb{I}} V^0_{s,u}(t_-). \end{aligned}$$

This leads to

$$\Phi(t,t_-)\partial^{\mathbb{I}} V^0_{s,u}(t_-) \supset T_0\varphi(t,t_-)\partial^{\mathbb{I}}_{\varepsilon} W^0_{s,u}(t_-)$$

and our claim (6.23) is shown. □

An Algorithm to Calculate Fiber Bundles

In this section we numerically approximate the fiber bundles we introduced and studied in the last sections. More precisely, we approximate fiber bundles of infinite and finite time systems for both invertible and noninvertible systems. Numerical tools for their computation in invertible infinite time systems are often based on continuation techniques, see for example [106, Subchapter 6.2], [68], [91] or [92]. If the system is noninvertible, stable sets cannot be computed via backward iteration. To avoid this problem, the authors of [48] proposed a refined approach – the so called search circle algorithm – for computing stable sets without applying the inverse mapping for 2-dimensional autonomous infinite time systems.

The method that we introduce here generalizes these ideas to the nonautonomous case where 0 is a hyperbolic fixed point. Another algorithm that works for invertible and noninvertible, autonomous and nonautonomous 2 or 3 dimensional systems is the contour algorithm. Hüls introduced this algorithm in [74] for discrete time systems and additionally, applied it to continuous time systems in [75].

Most of these techniques also apply for finite time systems. Before we study algorithms for finite time systems, we present a generalization of the search circle algorithm for infinite time system as in [54, Subsection 5.2].

Consider

$$x(n+1) = f(x(n), n) =: f_n(x(n)), \quad f_n(0) = 0,\ x(n) \in \mathbb{R}^2 \text{ for all } n \in \mathbb{Z}.$$

The algorithm chooses the first point on the tangent space of the stable fiber, which is a good linear approximation of the fiber. This subspace can formally be expressed as the stable subspace of an exponential dichotomy, which the variational equation

$$u_{n+1} = Df_n(0)u_n, \quad n \in \mathbb{Z}$$

possesses due to our hyperbolicity assumption, see (6.18). Indeed, these subspaces are numerically accessible for at least discrete invertible systems, see Corollary 5.0.1.

One step of the algorithm works as follows. Assume we already have an approximation of the $(n+1)$-th fiber, given by the set of points $M^{n+1} := \{p_1^{n+1}, \dots, p_{\ell_{n+1}}^{n+1}\}$ that are marked in blue in Figure 6.3. Further assume that the points $p_1^n, \dots, p_r^n$ on the n-th fiber have also been computed (light blue data in Figure 6.3). We search for the next point p_{r+1}^n on a circular segment γ shown in Figure 6.3. Therefore, its boundary points p_{end}^n and p_{start}^n are mapped by f_n. If the angle α of the circular segment is chosen appropriately, $f_n(p_{\text{start}}^n)$ and $f_n(p_{\text{end}}^n)$ lie on different sides of the $(n+1)$-th fiber and thus, $f_n(\gamma)$ has a common intersection with this fiber. For its approximation, we first detect the neighboring points p_{left}^{n+1}, $p_{\text{right}}^{n+1} \in M^{n+1}$ and then calculate the point of intersection between the line segment $\overline{p_{\text{left}}^{n+1} p_{\text{right}}^{n+1}}$ and $f_n(\gamma)$ using bisection. Its preimage under f_n is the next point p_{r+1}^n on the n-th fiber. In case $f_n(\gamma)$ lies beyond the $(n+1)$-th fiber, the continuation of the n-th fiber stops and we proceed with the $(n-1)$-th fiber. Note that the first fibers that we compute in this way are rather short, but expanding dynamics on the stable fibers – in backward time – lead to an increase of length if n decreases.

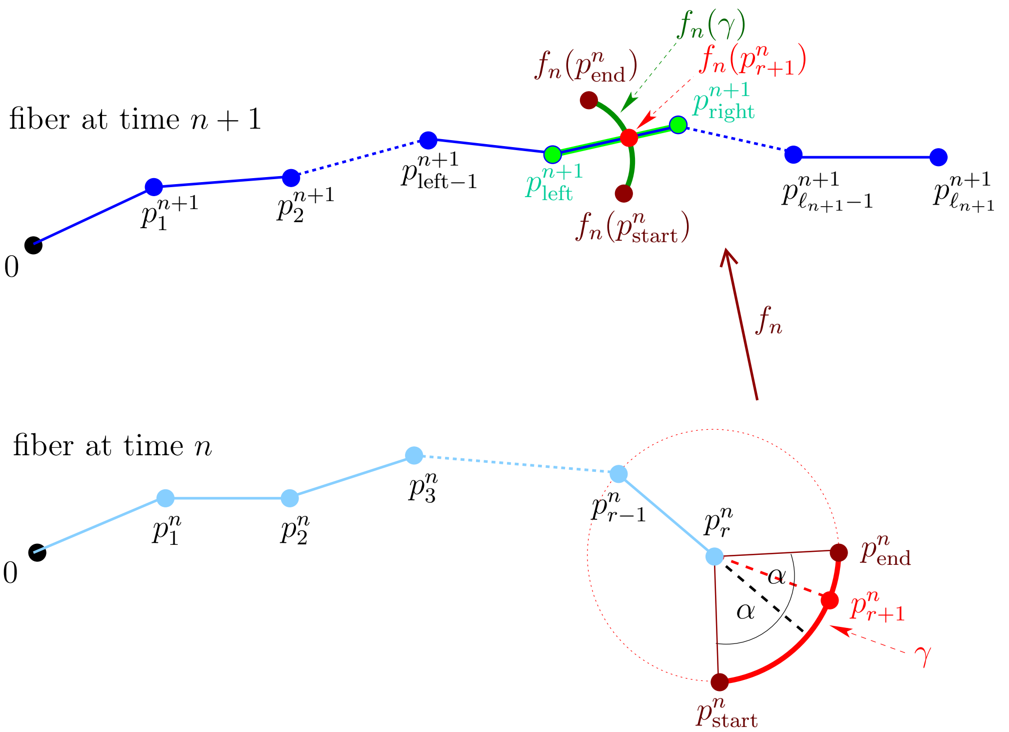


Figure 6.3: Approximation of stable fiber bundles.

Figure 6.4 illustrates this increase of length for our artificial example from Section 8.1. In the left diagram the stable fibers are extremely close to each other. To reveal the differences we rotate the highlighted area of the left picture and show its zoom in the right part of Figure 6.4. The bottom stable fiber (darkest shade of green) in Figure 6.4 is the first computed one and belongs to time $70h$. Since this is the first fiber we approximate we start with a small part of the stable subspace of the linearization. Then by using our algorithm we computed the rest of the plotted fibers; the last computed one is the 65-th fiber at the top (lightest shade of green). Particularly, the shades of green from dark to light show the order, in which stable fibers are calculated by our algorithm.

We finally note that details on the choice of the search angle α and on techniques for step size control have a similar implementation in autonomous systems and can be found in [48].

The computation of unstable fiber bundles is not so involved. We can choose points on the tangent space and iterate them in forward time, jumping in this way from fiber to fiber (neglecting small approximation errors).

In Figure 6.5 the stable and unstable fibers of the trivial solution of the model from Section 8.1 are shown. The stable fibers (green) are computed with the algorithm from above while the unstable fibers (red) are approximated by forward iteration. The diagram visualizes the fiber bundles on the time interval $[-30h, 30h]$.

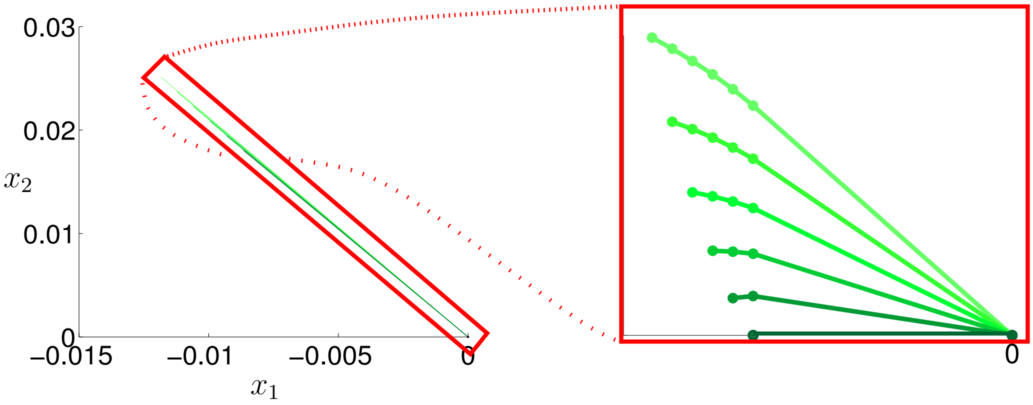


Figure 6.4: Computation of stable fibers for (8.3), (8.4) with $h = 0.04$.

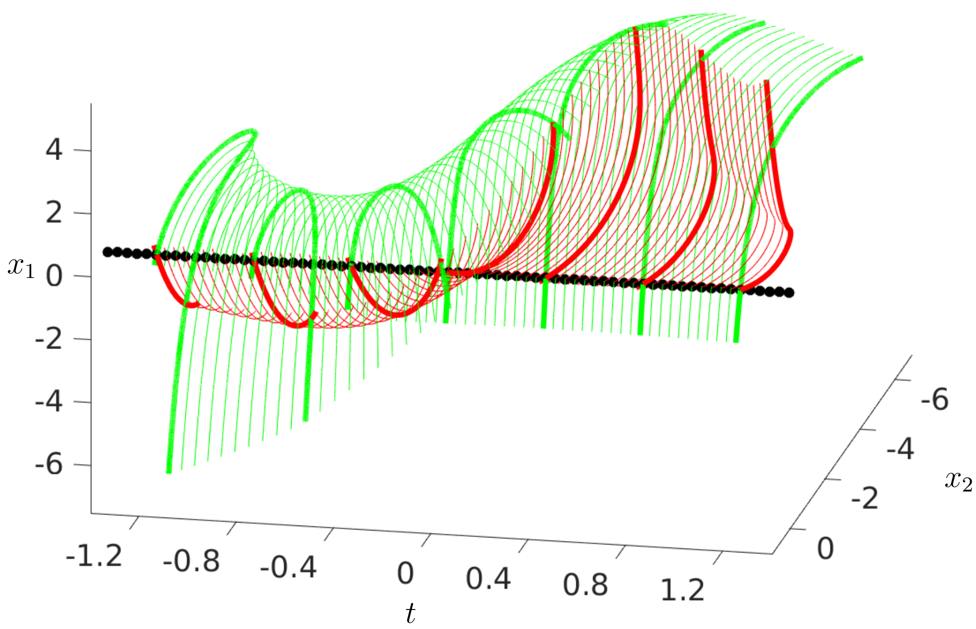


Figure 6.5: Stable and unstable fiber bundles of (8.3), (8.4) with $h = 0.04$.

In the following we approximate 2-dimensional finite time dynamical systems. Analogously to the infinite time case, we compute the stable and unstable fibers via iteration if the given system is invertible. By Theorem 6.6.1 the stable and unstable cones of the linearization locally approximate the stable and unstable fibers. Thus, all points in the cones which are at least ε close to the ft-hyperbolic solution are a good first approximation. By iterating these sets we get approximations of the stable and unstable fibers for all times in the given interval. Hence, the length of the fibers increase in the respective time direction as in the infinite time case. In contrast to infinite time systems we can not extend the time interval such that our algorithm provids sufficiently long fibers. For the infinite time system (8.3) we approximated the fiber bundles on $[-70h, 70h]$ and plotted them only on the time interval $[-30h, 30h]$ for $h = 0.04$. In Figure 6.5 these fibers are illustrated and we see that they are sufficiently long. For comparison we plotted in Figure 6.6 the boundary of the stable and unstable fibers of the finite time system (7.9), which we study in Chapter 7. In the left top picture the increase of the length of the stable fibers is visualized. In the right top part the stable and unstable fibers, approximated by iteration, are presented. As expected, the fibers are not one dimensional as the ones of the infinite time system. At the bottom of Figure 6.6 the stable and unstable fibers are plotted for the times 1.9 (left) and 0.7 (right). It is easily seen that the length of the stable fibers at times close to 2.5 and of the unstable fibers at times close to -2.5 is very short, while in the middle of the interval both fibers, the stable and unstable ones are of a proper length.

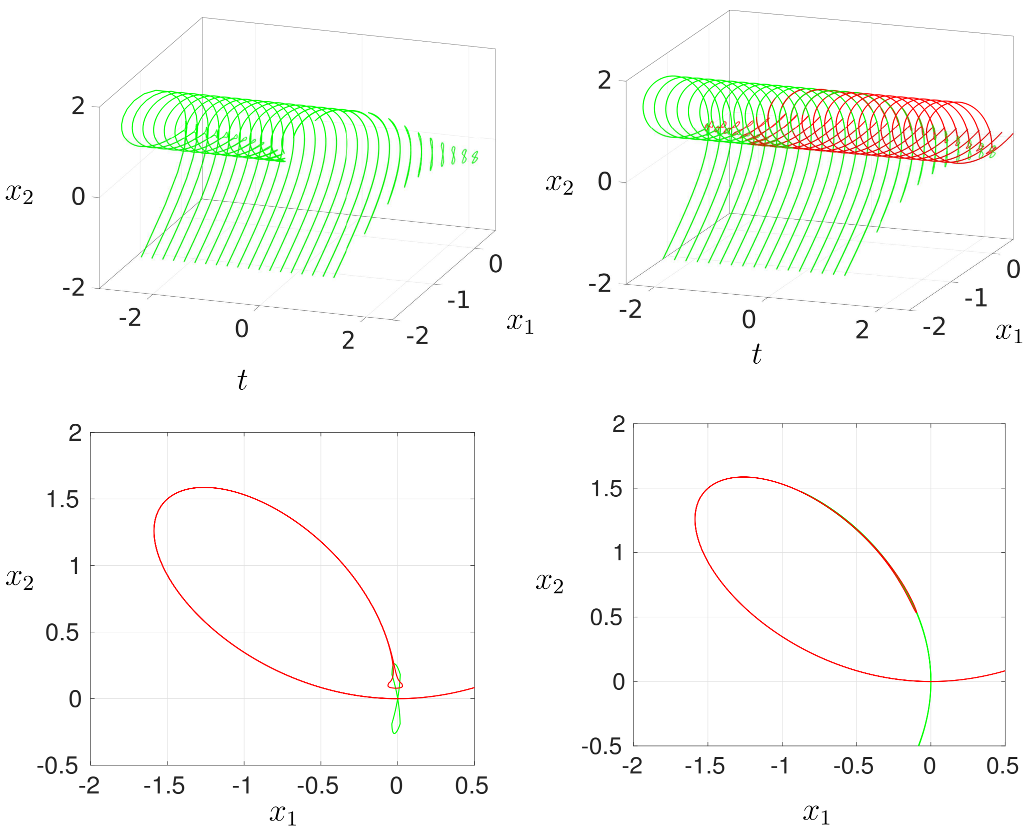


Figure 6.6: Stable (green) and unstable (red) fibers of (7.9) computed via iteration; at the bottom left at time 1.9 and at the bottom right at time 0.7.

This is not the case for all finite time systems. If the interval is to short then the fibers may not grow fast enough. Thus, we may only be able to approximate the fibers in a small neighborhood of the solution. Hence, we probably do not get the information we are interested in, for example, if the stable and unstable fibers intersect. Another drawback of finite time fiber bundles is that they are generally not invariant or at least forward invariant. More precisely, for noninvertible systems they are not invariant. If the system is invertible we already found a way to approximate the fiber bundles. If the system is not invertible we would like to approximate the fibers by a similar algorithm as described before for noninvertible, infinite time systems. The problem is that we need forward invariance of the boundaries for the algorithm to work. Assume the boundary of the stable fibers are forward invariant then, we can apply the algorithm onto each of the two boundary curves. If the boundary of the stable fiber bundle is not forward invariant we get an approximation of the fibers by defining a new forward invariant set, which includes the fibers. Let

$$\begin{aligned} {}^{\mathbb{I}}_{\varepsilon}\hat{W}^{\bar{x}}_{s} := \{(x_0, n_0) \in \mathbb{R}^k \times \mathbb{I} \big| \hat{n} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(x_0, \bar{x}(n_0), n_0) \in \mathbb{I}, \mathring{n} := {}_{\varphi}\mathcal{T}^{\bar{x}}_{\ker}(x_0, n_0) : \\ \|\varphi(x_0, n_2, n_0) - \bar{x}(n_2)\| < \|\varphi(x_0, n_1, n_0) - \bar{x}(n_1)\| \\ \text{for all } n_1, n_2 \in \mathbb{I}, \mathring{n} \geq n_2 > n_1 > \hat{n}\} \end{aligned}$$

be the **forward $\boldsymbol{\varepsilon}$-stable ft-fiber bundle** of a solution $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ of (2.3) for an $\varepsilon > 0$. For this set we easily see that

$${}^{\mathbb{I}}_{\varepsilon}W^{\bar{x}}_{s} \subset {}^{\mathbb{I}}_{\varepsilon}\hat{W}^{\bar{x}}_{s}.$$

Further, these sets are forward invariant, i.e.

$$\varphi(n_2, n_1){}^{\mathbb{I}}_{\varepsilon}\hat{W}^{\bar{x}}_{s}(n_1) \subset {}^{\mathbb{I}}_{\varepsilon}\hat{W}^{\bar{x}}_{s}(n_2)$$

holds for all $n_1, n_2 \in \mathbb{I}$, $n_2 > n_1$ by the following. Let $n_1, n_2 \in \mathbb{I}$, $n_2 > n_1$ and let $x \in {}^{\mathbb{I}}_{\varphi}\hat{W}^{\bar{x}}_{s}(n_1)$ then $\hat{n} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(x, \bar{x}(n_1), n_1) \in \mathbb{I}$ and

$$\|\varphi(x, n, n_1) - \bar{x}(n)\| < \|\varphi(x, m, n_1) - \bar{x}(m)\|$$

holds for all $n, m \in \mathbb{I}$, $\mathring{n} \geq n > m \geq \hat{n}$ with $\mathring{n} := {}_{\varphi}\mathcal{T}^{\bar{x}}_{\ker}(x, n_1)$. Further, by Lemma 6.3.2 we have $\hat{n} \leq \tilde{n} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(\varphi(x, n_2, n_1), \bar{x}(n_2), n_2) \in \mathbb{I}$. Hence,

$$\|\varphi(\varphi(x, n_2, n_1), n, n_2) - \bar{x}(n)\| < \|\varphi(\varphi(x, n_2, n_1), m, n_2) - \bar{x}(m)\|$$

holds for all $\mathring{n} \geq n > m \geq \tilde{n} \geq \hat{n}$. Last we prove that either $\mathring{n} \leq n_2$ or ${}_{\varphi}\mathcal{T}^{\bar{x}}_{\ker}(\varphi(x, n_2, n_1), n_2) = \mathring{n}$ holds. Let $\mathring{n} > n_2$. Then

$$\begin{aligned} n_2 < \mathring{n} &= \min\{n \in [n_1, n_+]_{\mathbb{Z}} \big| \varphi(x, n, n_1) = 0\} \\ &= \min\{n \in [n_2, n_+]_{\mathbb{Z}} \big| \varphi(\varphi(x, n_2, n_1), n, n_2) = 0\} = {}_{\varphi}\mathcal{T}^{\bar{x}}_{\ker}(x, n_2). \end{aligned}$$

Thus, $\varphi(x, n_2, n_1) \in {}^{\mathbb{I}}_{\varepsilon}\hat{W}_s^{\bar{x}}(n_2)$.

Note that Contour techniques also apply to finite time systems. In [55] an algorithm is introduced and applied to various examples of 2 and 3 dimensional systems. The stable fibers considered there coincide with our forward invariant set ${}^{\mathbb{I}}_{\varepsilon}\hat{W}_s^{\bar{x}}$. Thus, we can apply the contour algorithm to approximate the forward ε-stable ft-fiber bundle. The obtained approximation is a superset of the original stable ft-fiber bundle and, thus, an approximation for the original stable ft-fiber bundle.

Chapter 7

(In)Finite Time Homoclinic Trajectories

Systems that describe phenomena with chaotic properties often exhibit homoclinic orbits. Examples for chaotic behavior are electrical circuits [127], lasers in nonlinear optics [41], bursting phenomena in mathematical biology [1] and chemical reactions with chaotic oscillations in the reactant concentrations [56]. Further references to examples are mentioned in [89]. All these examples show homoclinic dynamics and, thus, the analysis of homoclinic orbits is of great interest.

In this chapter we study homoclinic trajectories of continuous and discrete, finite and infinite time systems. These trajectories lie in the intersection of stable and unstable fiber bundles. In finite time the intersection is generally more than a point or curve. Thus, we define the homoclinic tube, which is the union of all homoclinic trajectories. We present a way to numerically approximate this tube and prove that the distant of the boundaries decays to the middle of the finite time interval if the distant remains sufficiently small for all times. Further, we prove in Section 7.3 that homoclinic orbits of continuous time systems induce homoclinic orbits of a system, discretized by a one-step method. Therefore, we analyze the h-flow of a dynamical system in Section 7.2.

Most results for infinite time systems that are presented in this chapter originate from the publication [54].

The following definition of homoclinic trajectories can be found for discrete infinite time systems in [73] and for a similar definition in continuous infinite time we refer to [19].

Definition 7.0.1. *Two bounded* ***trajectories*** $x(\cdot)$ *and* $y(\cdot)$ *of the system* (2.1) *for* $\mathbb{I} := \mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$ *are* ***homoclinic*** *toward each other if*

$$\lim_{\substack{t\to\pm\infty,\\ t\in\mathbb{T}}} \|x(t) - y(t)\| = 0. \tag{7.1}$$

If $y(t) = 0$ for all $t \in \mathbb{T}$ is a (trivial) trajectory (equilibrium) of (2.1) and if $x(\cdot)$ is homoclinic toward $y(\cdot)$ then (7.1) has the form $\lim_{t\to\pm\infty} \|x(t)\| = 0$. Unless stated otherwise, homoclinic always means homoclinic toward the equilibrium 0. A homoclinic trajectory is also called a **homoclinic orbit**. Studying the definition of the stable and unstable fiber bundle (Def. 6.0.1) we observe that equation (7.1) is equivalent to

$$x(t) \in W_s^y(t) \cap W_u^y(t) \text{ for all } t \in \mathbb{T}. \tag{7.2}$$

This means that every homoclinic orbit lies in the intersection of the stable and unstable fiber bundle.

To obtain an adequate definition for "homoclinic" trajectories in finite time we transfer the properties of an infinite time trajectory to the finite time case. The condition (7.1) only works for infinite time trajectories. For finite time systems we cannot take $\lim_{t\to\pm\infty}$. Hence, it is only meaningful to assume that the trajectories are close to each other at the boundary times $t_\pm$. This leads to the condition

$$\|x(t_\pm) - y(t_\pm)\|_2 < \varepsilon, \tag{7.3}$$

for a fixed $\varepsilon > 0$.

To point out the dependency of ε we define ε-homoclinic trajectories for finite time system. The condition (7.3) does not yield (7.2) for our ft-fiber bundles. Thus, in finite time we have to examine (7.2) explicitly. In contrast to infinite time systems, fibers in finite time are fat objects. Hence, the intersection of the stable and unstable ft-fiber bundle normally includes more than one ε-homoclinic trajectory. Therefore, we define the intersection of the stable and unstable ft-fibers as the ε-homoclinic tube. This tube is the union of all ε-homoclinic trajectories.

Definition 7.0.2. *Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$ and $\mathbb{I} = [t_-, t_+]_\mathbb{T}$ be a compact interval with $t_\pm \in \mathbb{T}$. Two different **trajectories** $x(\cdot)$ and $y(\cdot)$ of system (2.1) are **ε-homoclinic towards each other** (form an **ε-homoclinic trajectory pair**) if*

$$x(t) \in {}^{\mathbb{I}}_{\varepsilon}W_s^y(t) \cap {}^{\mathbb{I}}_{\varepsilon}W_u^y(t) \text{ for a } t \in \mathbb{I}. \tag{7.4}$$

*Trajectories which are ε-homoclinic towards the hyperbolic fixed point 0 are called **ε-homoclinic trajectories**.*

*Let $\bar{x}(\cdot)$ be a trajectory. The **ε-homoclinic t-tube***

$${}^{\mathbb{I}}T_\varepsilon^{\bar{x}}(t) := {}^{\mathbb{I}}_{\varepsilon}W_s^{\bar{x}}(t) \cap {}^{\mathbb{I}}_{\varepsilon}W_u^{\bar{x}}(t)$$

*is the set of all trajectories of (2.2) or (2.3) at time $t \in \mathbb{I}$ which are ε-homoclinic towards $\bar{x}(\cdot)$. The **ε-homoclinic tube** is defined by*

$${}^{\mathbb{I}}T_\varepsilon^{\bar{x}} = \left\{(x_0, t_0) \in \mathbb{R}^k \times \mathbb{I} | x_0 \in {}^{\mathbb{I}}T_\varepsilon^{\bar{x}}(t_0)\right\}.$$

Remark 7.0.3.
- *For invertible systems an ε-homoclinic trajectory pair satisfies* (7.4) *for all $t \in \mathbb{I}$, see Corollary 7.0.6.*
- *Caused by the monotonicity of ${}^{\mathbb{I}}_{\varepsilon}W^{y}_{s,u}(\cdot)$ a solution $x(\cdot) \in {}^{\mathbb{I}}_{\varepsilon}W^{y}_{s}(\cdot) \cap {}^{\mathbb{I}}_{\varepsilon}W^{y}_{u}(\cdot)$ has to leave the ε-ball around $y(\cdot)$. This means, the only trivial ε-homoclinic trajectory pair $(x(\cdot), y(\cdot))$ satisfies $y(\cdot) = x(\cdot)$. Indeed, from $\|x(t) - y(t)\|_2 < \varepsilon$ for all $t \in \mathbb{I}$ follows ${}^{\mathbb{I}}_{\varepsilon}W^{y}_{s,u}(t) = {}^{\mathbb{I}}_{\varepsilon}M^{y}_{s,u}(t)$ and thus, the intersection is*

$$
{}^{\mathbb{I}}_{\varepsilon}W^{y}_{s}(t) \cap {}^{\mathbb{I}}_{\varepsilon}W^{y}_{u}(t) = {}^{\mathbb{I}}_{\varepsilon}M^{y}_{s}(t) \cap {}^{\mathbb{I}}_{\varepsilon}M^{y}_{u}(t) = \{y\}.
$$

- *The ε-homoclinic tube may consists of two tubes, since the fibers extent to two sides.*

As mentioned before we want that an ε-homoclinic trajectory pair satisfies (7.3). This is not explicitly required in the Definition 7.0.2. However, (7.3) follows from (7.4).

Lemma 7.0.4. *Let $(x(\cdot),\, y(\cdot))$ form an ε-homoclinic trajectory pair of system* (2.2)/(2.3) *then $\|x(t_\pm) - y(t_\pm)\|_2 < \varepsilon$ is satisfied.*

Proof. By Definition 7.0.2 we have $x(t) \in {}^{\mathbb{I}}_{\varepsilon}W^{y}_{s}(t) \cap {}^{\mathbb{I}}_{\varepsilon}W^{y}_{u}(t)$ for a $t \in \mathbb{I}$. Then the definitions of the t-fibers lead to $x(t_\pm) \in B_\varepsilon(y(t_\pm))$. Thus, we have $\|x(t_\pm) - y(t_\pm)\|_2 < \varepsilon$. □

Lemma 7.0.5. *The definition of an ε-homoclinic trajectory pair is symmetric, i.e. if $(x(\cdot), y(\cdot))$ is an ε-homoclinic trajectory pair then $(y(\cdot), x(\cdot))$ is one as well.*

Proof. Let $(x(\cdot), y(\cdot))$ be an ε-homoclinic trajectory pair. We show that if

$$
x(t_0) \in {}^{\mathbb{I}}_{\varepsilon}W^{y}_{s}(t_0) \cap {}^{\mathbb{I}}_{\varepsilon}W^{y}_{u}(t_0)
$$

for a $t_0 \in \mathbb{I}$ then

$$
y(t_0) \in {}^{\mathbb{I}}_{\varepsilon}W^{x}_{s}(t_0) \cap {}^{\mathbb{I}}_{\varepsilon}W^{x}_{u}(t_0).
$$

Let $x(t_0) \in {}^{\mathbb{I}}_{\varepsilon}W^{y}_{s}(t_0) \setminus \{0\}$ and $\bar{t} = {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(x(t_-), y(t_-), t_-)$. Then we have

$$
\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\|x(t) - y(t)\| < 0 \text{ for all } t \in \mathbb{I}, t > \bar{t}, & \text{in continuous time,} \\ \|x(t) - y(t)\| < \|x(s) - y(s)\| \text{ for all } t, s \in \mathbb{I}, t > s > \bar{t}, & \text{in discrete time.} \end{cases}
$$

By the symmetry of the estimates and the symmetry of ${}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(\cdot,\cdot,t)$ (Lemma 6.3.1) for all $t \in \mathbb{I}$ we have $y(t_0) \in {}^{\mathbb{I}}_{\varepsilon}W^{x}_{s}(t_0)$. Analogously, we can prove that from $x(t_0) \in {}^{\mathbb{I}}_{\varepsilon}W^{y}_{u}(t_0)$ the statement $y(t_0) \in {}^{\mathbb{I}}_{\varepsilon}W^{x}_{u}(t_0)$ follows. □

Corollary 7.0.6. *Let $\varepsilon > 0$. If $x(\cdot)$, $y(\cdot)$ are two different trajectories of an invertible systems* (2.2)/(2.3) *such that there exist $t_0, t_1 \in \mathbb{I}$ with*

$$x(t_0) \in {}^{\mathbb{I}}_{\varepsilon}W^y_s(t_0), \quad x(t_1) \in {}^{\mathbb{I}}_{\varepsilon}W^y_u(t_1)$$

then we get for all $t \in \mathbb{I}$

$$x(t) \in {}^{\mathbb{I}}_{\varepsilon}W^y_s(t) \cap {}^{\mathbb{I}}_{\varepsilon}W^y_u(t).$$

If $x(\cdot)$, $y(\cdot)$ are two different trajectories of a (non)invertible system (2.3) *such that there exists $n_0 \in \mathbb{I}$ with*

$$x(t_-) \in {}^{\mathbb{I}}_{\varepsilon}W^y_s(t_-), \quad x(n_0) \in {}^{\mathbb{I}}_{\varepsilon}W^y_u(n_0)$$

then we get for all $n \in \mathbb{I}$

$$x(n) \in {}^{\mathbb{I}}_{\varepsilon}W^y_s(n) \cap {}^{\mathbb{I}}_{\varepsilon}W^y_u(n).$$

Proof. Let (2.2)/(2.3) be invertible and $x(t_0) \in {}^{\mathbb{I}}_{\varepsilon}W^y_s(t_0)$, $x(t_1) \in {}^{\mathbb{I}}_{\varepsilon}W^y_u(t_1)$ for any $t_0, t_1 \in \mathbb{I}$. Then we have for all $t \in \mathbb{I}$ by the invariance of the fiber bundles (Lemma 6.4.3)

$$\begin{aligned} x(t) &= \varphi(x(t_0), t, t_0) \in \varphi(t, t_0){}^{\mathbb{I}}_{\varepsilon}W^y_s(t_0) = {}^{\mathbb{I}}_{\varepsilon}W^y_s(t), \\ x(t) &= \varphi(x(t_1), t, t_1) \in \varphi(t, t_1){}^{\mathbb{I}}_{\varepsilon}W^y_u(t_1) = {}^{\mathbb{I}}_{\varepsilon}W^y_u(t), \end{aligned}$$

which prove the first statement.

Let (2.2)/(2.3) be noninvertible and $x(t_-) \in {}^{\mathbb{I}}_{\varepsilon}W^y_s(t_-)$, $x(n_0) \in {}^{\mathbb{I}}_{\varepsilon}W^y_u(n_0)$ for any $n_0 \in \mathbb{I}$. Then we have for all $n \in \mathbb{I}$

$$\begin{aligned} x(n) &= \varphi(x(t_-), n, t_-) \in \varphi(n, t_-){}^{\mathbb{I}}_{\varepsilon}W^y_s(t_-) \subset {}^{\mathbb{I}}_{\varepsilon}W^y_s(n), \\ x(n) &= \varphi(x(n_0), n, n_0) \in \varphi(n, n_0){}^{\mathbb{I}}_{\varepsilon}W^y_u(n_0) = {}^{\mathbb{I}}_{\varepsilon}W^y_u(n), \end{aligned}$$

which prove the second statement. □

In the following we need the Banach space of bounded functions which is defined for $i \in \mathbb{N}$ as

$$X^i = \Big\{ u(\cdot) \in \mathcal{C}^i(\mathbb{R}, \mathbb{R}^k) : \|u\|_i = \sum_{j=0}^{i} \sup_{t \in \mathbb{R}} \big\| u^{(j)}(t) \big\|_\infty < \infty \Big\}.$$

For infinite time systems the way the stable and unstable fibers intersect is of interest. If they intersect transversally than homoclinic orbits are preserved under a small perturbation of the system. For tangential intersections homoclinic orbits generally vanish if the system is perturbed. Thus, we analyze whether the intersection is transversal or tangential. First we have a look at the definition of a infinite time transversal orbit and an illustration of the different types of intersections.

Definition 7.0.7. *A homoclinic orbit $\bar{x}(\cdot)$ of* (2.2) *for* $\mathbb{I} = \mathbb{R}$ *is called* ***transversal*** *if the following is true.*

(T) *A function $u(\cdot) \in X^1$ satisfies*

$$\dot{u}(t) = f_x(\bar{x}(t), t)u(t), \ t \in \mathbb{R}$$

if and only if $u(\cdot) = 0$.

Geometrically, transversality means that stable and unstable fiber bundles intersect transversally along the homoclinic orbit, see Theorem 7.0.10. In Figure 7.1 both types of intersection, transversal (top left) and tangential (bottom left) are pictured. For each system we also plotted the fiber bundles of two sufficiently small perturbed systems (second and third coulomb). Note that in nonautonomous continuous ift-systems the intersection of stable and unstable fibers at a fixed time does not necessarily have a whole orbit in common. As mentioned before a transversal intersection is preserved under small perturbation whereas a tangential intersection does typically not survive under perturbation. This can be seen in Figure 7.1. The second and third picture of the first row still have a homoclinic (blue) point whereas the second and third picture of the second row do not have a homoclinic point. For stable and unstable ft-fibers the way of intersection does not play a role for the existence of ε-homoclinic points of a perturbed system. For small perturbations an intersection is always preserved if the fibers are fat sets. In the right panels of Figure 7.1 two different kinds of intersection are plotted, before and after perturbing the system. The fiber bundles are illustrated as fat sets, since this is in general the case. Figure 7.1 shows that the fibers of the perturbed systems (fifths and sixths column) intersect for both types of intersection (blue points).

Studying the last row of Figure 7.1 we claim that

Claim 7.0.8. *It is possible to get a nontrivial ε-homoclinic tube of an autonomous system although the same system for* $\mathbb{I} = \mathbb{R}$ *does not have a nontrivial homoclinic trajectory.*

In Section 7.1 we illustrate this statement by an explicit example and we give numerical tools to approximate an ε-homoclinic tube.

Before we summarize alternative characterizations of transversality in Theorem 7.0.10 we impose for system (2.2) a few assumptions under which we later show that homoclinic orbits of (2.2) induce homoclinic orbits of a discretized system.
From now on assume $\mathbb{I} = \mathbb{R}$ or that $\mathbb{I} = [t_-, t_+] \subset \mathbb{R}$ is a compact interval.

(A4) Let $h > 0$ and if $\mathbb{I}$ is compact such that $\frac{t_-}{h}, \frac{t_+}{h} \in \mathbb{Z}$. Define

$$\mathbb{J} := \mathbb{J}_h := \begin{cases} \mathbb{Z}, & \text{if } \mathbb{I} = \mathbb{R}, \\ [n_-, n_+]_{\mathbb{Z}} := [\frac{t_-}{h}, \frac{t_+}{h}]_{\mathbb{Z}}, & \text{if } \mathbb{I} = [t_-, t_+]. \end{cases} \tag{7.5}$$

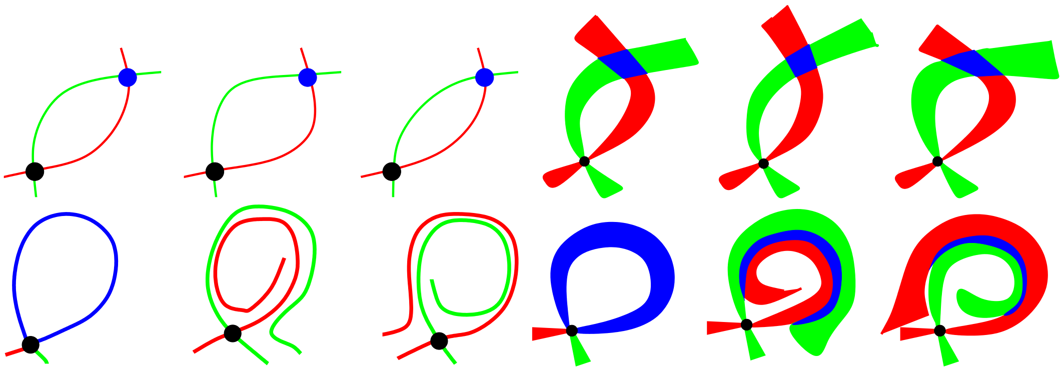


Figure 7.1: Examples of fiber bundles (infinite time left panels, finite time right panels). Two different kinds of intersection, tangential (bottom) and transversal (top), of the stable (green) and unstable (red) fiber are plotted. The fibers are plotted before (1. and 4. column) and after perturbing (2., 3., 5. and 6. column) the system. All homoclinic points (intersections) are marked in blue.

Further set

$$\mathbb{J}_1 := \begin{cases} \mathbb{Z}, & \text{if } \mathbb{J} = \mathbb{Z}, \\ [\frac{t_-}{h}, \frac{t_+}{h} - 1]_{\mathbb{Z}}, & \text{if } \mathbb{J} = [\frac{t_-}{h}, \frac{t_+}{h}]_{\mathbb{Z}}. \end{cases} \tag{7.6}$$

(A5) $f \in \mathcal{C}^1(\mathbb{R}^k \times \mathbb{I}, \mathbb{R}^k)$ satisfies conditions, assuring existence and uniqueness of global solutions of (2.2) as well as the following estimates for the solution operator φ. For any compact set $\mathcal{K} \subset \mathbb{R}^k$ there exist constants $C_1(\mathcal{K}), h_1(\mathcal{K}) > 0$ such that the inequality

$$\|\varphi_x(x, t, s)\| \le C_1(\mathcal{K})$$

holds for all $x \in \mathcal{K}$ and $|t - s| \le h_1(\mathcal{K})$. For $n \in \mathbb{J}_1$ let

$$\varphi_n(x, h) := \varphi(x, (n + 1)h, nh)$$

be $\mathcal{C}^d$ smooth w.r.t. h, $d \ge 1$. Mixed derivatives $(\varphi_n)^{(1,\ell)}_{x,h}$, $\ell \in \{0, \dots, d\}$ exist and satisfy the uniform Lipschitz condition

$$\left\|(\varphi_n)^{(1,d)}_{x,h}(x, \mu_1) - (\varphi_n)^{(1,d)}_{x,h}(x, \mu_2)\right\| \le C_1(\mathcal{K}) \|\mu_1 - \mu_2\|$$

for all $x \in \mathcal{K}$, $0 \le \mu_{1,2} \le h_1(\mathcal{K})$ and $n \in \mathbb{J}_1$.

Further, let $\left\|(\varphi_n)^{(r,1)}_{x,h}(x, h)\right\| \le C_1(\mathcal{K})$ for all $n \in \mathbb{Z}$, $r \in \{0, 1\}$, $x \in \mathcal{K}$ and $0 \le h \le h_1(\mathcal{K})$.

(A6) $0 \in \mathbb{R}^k$ satisfies $f(0, t) = 0$ for all $t \in \mathbb{I}$.

Remark 7.0.9. *If* (2.2) *possesses a (hyperbolic) bounded solution* $\xi(\cdot) \in X^0$, *then the transformed system*

$$\dot{y} = g(y,t), \quad g(y,t) := f(y + \xi(t), t) - f(\xi(t), t)$$

has 0 *as a* t*-independent (hyperbolic) equilibrium. Thus, without loss of generality* **(A6)** *is fulfilled. Note that* **(A5)** *is invariant under this transformation.*

(A7) 0 is **(ft-)hyperbolic**. Denote the data of the corresponding variational equation

$$\dot{x} = f_x(0, \cdot)x$$

by $(K, \beta, P^{s,u}(\cdot))$.

(A8) A nontrivial homoclinic orbit $\bar{x}(\cdot)$ of (2.2) exists.

Next we present equivalent statements to (T) from Defintion 7.0.7.

Theorem 7.0.10. *Let* $\mathbb{I} = \mathbb{R}$. *Assume* **(A5)-(A8)**, *then the following statements are equivalent*

($\mathrm{a_c}$) *The homoclinic orbit* $\bar{x}(\cdot)$ *is transversal in the sense of* (T).

($\mathrm{b_c}$) *The variational equation*

$$\dot{u} = f_x(\bar{x}(\cdot), \cdot)u \tag{7.7}$$

has an exponential dichotomy on $\mathbb{R}$.

($\mathrm{c_c}$) *The linear operator*

$$L(\bar{x}) : X^1 \to X^0, \; L(\bar{x})u(\cdot) := \dot{u}(\cdot) - f_x(\bar{x}(\cdot), \cdot)u(\cdot)$$

is a homeomorphism.

($\mathrm{d_c}$) *The tangent spaces* $T_{\bar{x}(0)}W^0_{s,u}(0)$ *of the fibers* $W^0_{s,u}(0)$ *at the point* $\bar{x}(0)$ *satisfy*

$$T_{\bar{x}(0)}W^0_u(0) \oplus T_{\bar{x}(0)}W^0_s(0) = \mathbb{R}^k.$$

Proof. **($\mathbf{a_c}$)⇒($\mathbf{b_c}$):** Since $\bar{x}(\cdot)$ is a homoclinic orbit w.r.t. the hyperbolic equilibrium 0, the Roughness-Theorem 3.4.1 applies and gives an exponential dichotomy of (7.7) that we can extend to $\mathbb{R}^+$ and $\mathbb{R}^-$. By assuming ($\mathrm{a_c}$), a nontrivial bounded solution cannot exist. Using [104, Prop. 2.1] half sided dichotomies can be combined into a dichotomy on $\mathbb{R}$.

(b_c)⇒(c_c): Assuming (b_c) then $\mathcal{N}(L(\bar{x})) = \{0 \in X^1\}$ directly follows. On the other hand by [39, Chapter V.1] we obtain for each $r \in X^0$ – using Green's function – a unique bounded solution in X^1 of the inhomogeneous equation

$$\dot{u} = f_x(\bar{x}(\cdot), \cdot)u + r(\cdot).$$

Thus $L(\bar{x})$ is injective and surjective.

(c_c)⇒(a_c): The claim immediately follows, since $L(\bar{x})$ is a homeomorphism.

(a_c)⇔ (d_c): For a proof we refer to the end of Section 7.2 where we introduce the discrete equivalent of these statements.

□

We see that every homoclinic orbit which is transversal is also a hyperbolic trajectory and vice versa. Since Theorem 7.0.10 is not applicable to finite time systems we assume

(A9) The homoclinic orbit $\bar{x}(\cdot)$ is (ft-)hyperbolic.

For the infinite time case this is equivalent to $\bar{x}(\cdot)$ is a transversal orbit, as assumed in [54]. Several of the following results hold true for bounded (ft-)hyperbolic trajectories that need not to be homoclinic. For this case, we assume

(A10) Let $\bar{y}(\cdot)$ be a (ft-)hyperbolic bounded trajectory of (2.2). Denote by $(\bar{K}, \bar{\beta}, \bar{Q}^{s,u}(\cdot))$ the (ft-)dichotomy data of the corresponding variational equation

$$\dot{u} = f_x(\bar{y}(\cdot), \cdot)u$$

and let $S^{\bar{y}}(t, s)$ be its solution operator.

Approximation of ε-Homoclinic Tubes and Numerical Tools

In this subsection we concentrate on ε-homoclinic tubes. We prove Claim 7.0.8, i.e.

it is possible to get a nontrivial ε-homoclinic tube of an autonomous system although the same system for $\mathbb{I} = \mathbb{R}$ does not have a nontrivial homoclinic trajectory,

with the help of an explicit example. Further, we find numerical tools to approximate this tubes and a theoretical statement about their width. To actually plot ε-homoclinic tubes it is helpful that they are invariant under the solution operator.

Lemma 7.1.1. *Let $\bar{x}(\cdot)$ be an ft–hyperbolic trajectory of the invertible ft–system (2.2)/ (2.3) and let $\varepsilon > 0$. Then the ε-homoclinic t-tube*

$$^{\mathbb{I}}T_{\varepsilon}^{\bar{x}}(t) = {}^{\mathbb{I}}W_{s}^{\bar{x}}(t) \cap {}^{\mathbb{I}}W_{u}^{\bar{x}}(t)$$

is open and invariant under $\varphi(\cdot, t)$.

Proof. By Lemma 6.4.3 both, the ε-stable $^{\mathbb{I}}W_{s}^{\bar{x}}(t)$ and the ε-unstable $^{\mathbb{I}}W_{u}^{\bar{x}}(t)$ ft-t-fibers are invariant under $\varphi(\cdot, t)$ for all $t \in \mathbb{I}$, i.e. the intersection is invariant. By Definition 6.3.3/ 6.3.4 we see that $^{\mathbb{I}}_{\varepsilon}W_{s,u}^{\bar{x}}(t)$ are open sets for all $t \in \mathbb{I}$. Thus, the ε-homoclinic t-tube is open. □

By the continuity of $\varphi(\cdot, \cdot)$ we get that the boundary of the tube is invariant. To plot the tube (boundary of the tube) it is sufficient to calculate the boundary of $^{\mathbb{I}}T_{\varepsilon}^{\bar{x}}(t_-)$ and plot its image under $\varphi(t, t_-)$ for all $t \in \mathbb{I}$.

To show that Claim 7.0.8 is true we first consider the autonomous system

$$\dot{x} = \begin{pmatrix} 1.6\,(x(1) + x(2)^2) \\ -1.6\,(-x(1)^2 + x(2)) \end{pmatrix} = f(x),\ x \in \mathbb{R}^2 \tag{7.8}$$

on $\mathbb{I} = \mathbb{R}$. For this system a nontrivial homoclinic orbit exists. With the Numlab application for Matlab [102] the manifolds of the hyperbolic fixed point 0 are plotted for the parameters $h := 0.01$ and $l = 300$, where h denote the step size of the classical Runge-Kutta method and l the number of steps. The generated plot is presented in the left of Figure 7.2. The time independency of the system induces that the fibers are autonomous. This means that if the manifolds intersect then they have the whole homoclinic orbit in common, since the fibers are invariant. Hence, the manifolds intersect tangentially and the existing homoclinic orbit is tangential. As illustrated in Figure 7.1 (left bottom panel) the homoclinic orbit vanishes by perturbing the given system (7.8). To prove this we study the perturbed system

$$\dot{x} = \begin{pmatrix} 1.6\,(x(1) + x(2)^2) \\ -1.6016\,(-x(1)^2 + x(2)) \end{pmatrix} = f(x),\ x \in \mathbb{R}^2. \tag{7.9}$$

For comparison we also let Numlab plot the manifolds of the hyperbolic fixed point 0 of the perturbed system (7.9) for the same parameters ($h = 0.01$, $l = 300$). These manifolds are pictured in the right half of Figure 7.2. Note that the manifolds do not intersect. Hence, the system (7.9) does not have a nontrivial homoclinic trajectory.

In the following example we show that on a finite time interval system (7.9) has a nontrivial ε-homoclinic tube. This proves Claim 7.0.8. Further we develop a tool to approximate the tube.

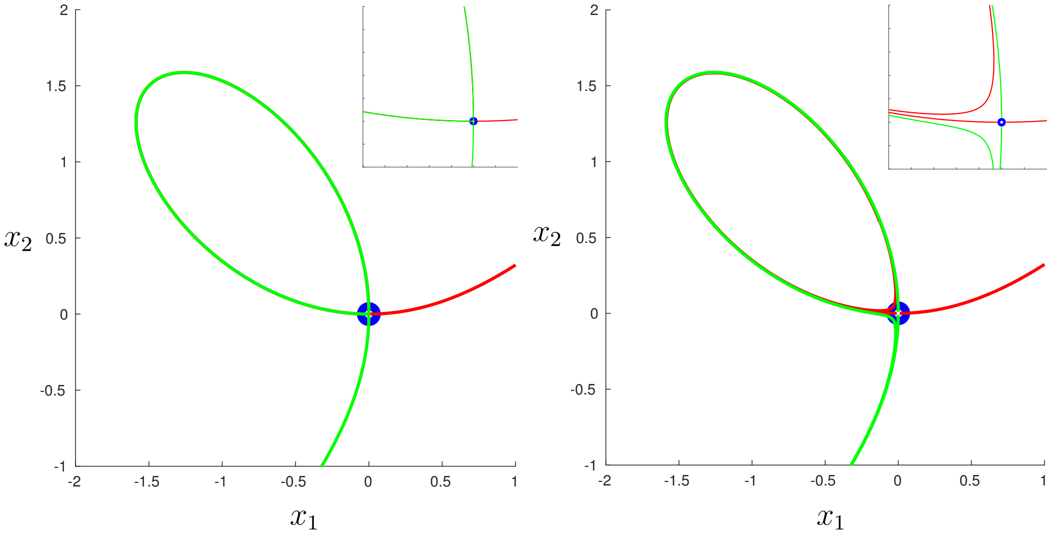


Figure 7.2: The stable and unstable manifolds of the fixed point 0 of system (7.8) (left) and (7.9) (right).

Example 7.1.2. *Consider the finite time autonomous system* (7.9) *on* $\mathbb{I} = [-2.5, 2.5]$*. We see that* 0 *is a fixed point. Our aim is to find all nontrivial* ε*-homoclinic trajectories towards* 0 *for* $\varepsilon := 0.1$ *and prove that at least one* ε*-homoclinic orbit exists. First we start with the approximation of the stable and unstable fiber bundles. Therefore, we compute the stable and unstable cones* $V_{s,u}$ *of the linearization with the help of Lemma 5.6.1. To apply this lemma we show that* 0 *is an ft-hyperbolic solution, i.e.* $f_x(0)$ *is D-hyperbolic w.r.t. the euclidean norm (*$\Gamma = \mathrm{I}$*). The symmetric part of* $f_x(0)$ *is* $f_x(0) = \operatorname{diag}(1.6, -1.6016)$ *itself (indefinite and nondegenerate) and the acceleration tensor is* $M = 2f_x(0)^2$ *(positive definite). Thus,* 0 *is an ft-hyperbolic solution and by Lemma 5.6.1 we have*

$$\partial V_s(2.5) = \partial V_u(-2.5) = \mathcal{L}\left(\begin{pmatrix}\sqrt{1.6016}\\ \sqrt{1.6}\end{pmatrix}\right) \cup \mathcal{L}\left(\begin{pmatrix}-\sqrt{1.6016}\\ \sqrt{1.6}\end{pmatrix}\right).$$

Since these boundaries approximate the boundaries of the fiber bundles locally, see Theorem 6.6.1, we take as a first approximation all points in the cones which are at least ε *close to* 0*.*

These sets are illustrated in Figure 7.3 in the light green ($V_s(2.5) \cap B_\varepsilon(0)$*) and in light red (*$V_u(-2.5) \cap B_\varepsilon(0)$*). To prove which points*

$$x_u(-2.5) \in \partial\left(V_u(-2.5) \cap B_\varepsilon(0)\right)$$

(bright red in the top left picture) satisfy

$$x_u(2.5) = \Phi(2.5, -2.5)x_u(-2.5) \in \overline{V_s(2.5) \cap B_\varepsilon(0)}$$

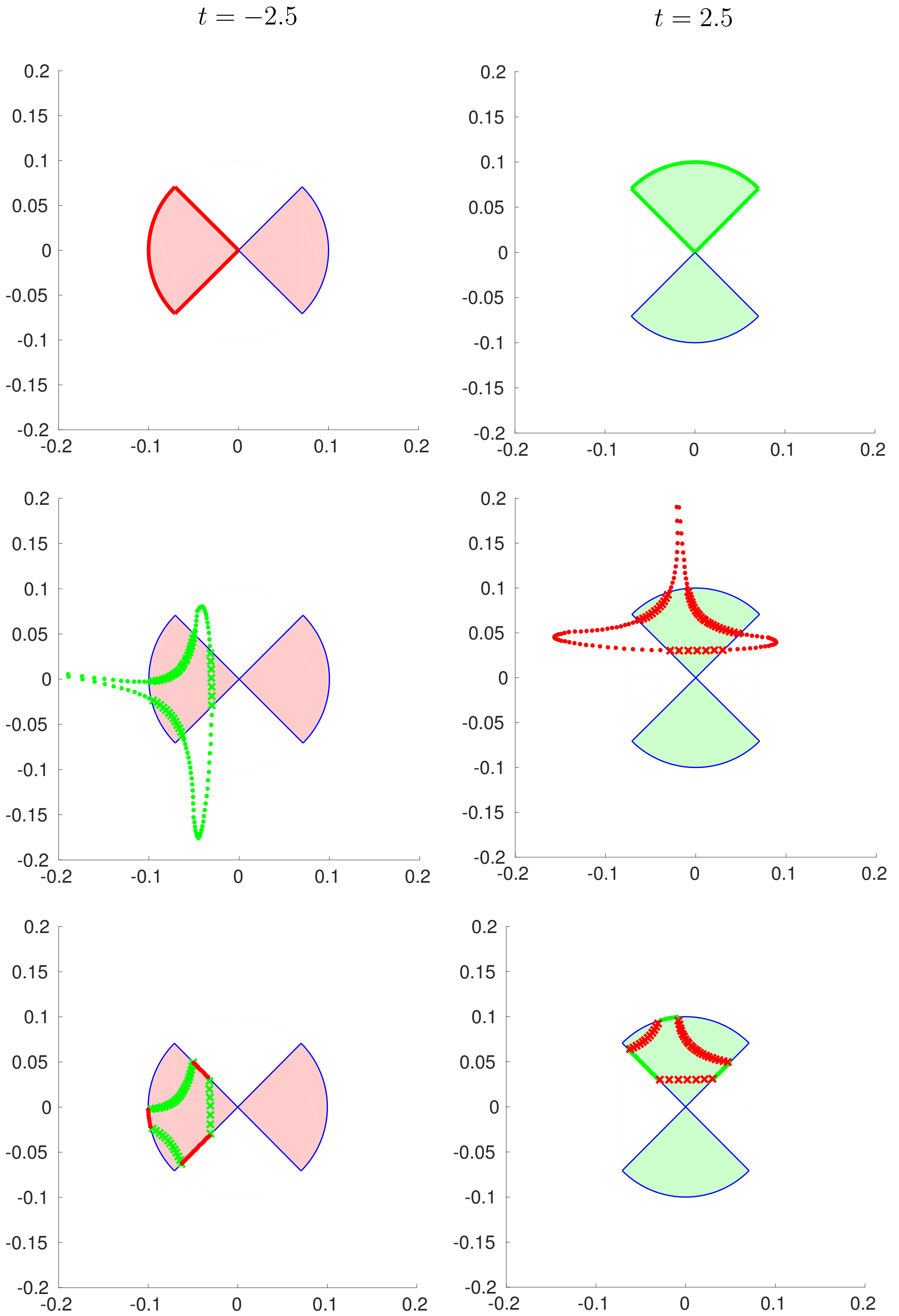


Figure 7.3: Finding all $\varepsilon = 0.1$-homoclinic trajectories to the fixed point 0 of system (7.9).

we need the solution operator $\Phi(\cdot,\cdot)$ *of system* (7.9)*. Since the solution operator is not given explicitly we approximate* $x_u(2.5)$ *with the help of the classical Runge-Kutta method of order 4, see [108, S.172]. This is a discretization method, thus, we need a step size*

$$h := \frac{5}{n},\ n = 500.$$

We calculate $\{\Phi(2.5,-2.5)x|x \in (\partial(V_s(-2.5) \cap B_\varepsilon(0)) \cap G)\}$*, where* G *is a adjusted fine grid. These points are plotted in red in the right middle part of Figure 7.3. The ones which satisfy* $x_u(2.5) \in \overline{V_s(2.5) \cap B_\varepsilon(0)}$ *are marked with an* x *and are also plotted (red) in the panel below. Further, their preimages at time* -2.5 *are plotted (red) in the left bottom part. Next we search for all points*

$$x_u(-2.5) \in V_u(-2.5) \cap B_\varepsilon(0)$$

which satisfy

$$x_u(2.5) = \Phi(2.5,-2.5)x_u(-2.5) \in \partial(V_s(2.5) \cap B_\varepsilon(0)).$$

With time reversal and negative step size for the classical Runge-Kutta method we obtain an approximation of $\{\Phi(-2.5,2.5)x|x \in (\partial(V_s(2.5) \cap B_\varepsilon(0)) \cap G)\}$*, illustrated in green in the left middle part of Figure 7.3. The points* x *which satisfy*

$$x \in V_u(-2.5) \cap B_\varepsilon(0)$$

are marked by an x *and are also plotted (green) in the panel below. Further, their images at time* 2.5 *are shown (green) in the right bottom graphic.* $\Phi(\cdot,-2.5)x_u(-2.5)$ *is an* ε*-homoclinic trajectory towards* 0 *if and only if the point* $x_u(-2.5)$ *lies inside the (by the red and green* x*) marked set in the left bottom panel of Figure 7.3. The boundary points at time* 2.5 *lie inside the marked set in the right bottom panel.*

The boundary of the $\varepsilon = 0.1$*-homoclinic tube w.r.t. the fixed point* 0 *is illustrated in the top of Figure 7.4. To approximate it we started with the boundary (green and red points in the right bottom picture of Figure 7.3) of the* $\varepsilon = 0.1$*-homoclinic* $t = 2.5$*-tube. Then we searched the previous points with the help of the classical Runge-Kutta method and step size* -0.01*. The boundary of all* $\varepsilon = 0.1$*-homoclinic orbit points at time* 2.5 *are the boundary point in green. The magenta boundary points mirror the boundary of the* $\varepsilon = 0.1$*-homoclinic tube at time* -2.5*. For each* 0.5*-time step the points of the boundary are marked in different colors from magenta over yellow to green. The left panel of Figure 7.4 shows a sector of the orbit tube from the top. The whole tube projected to the* x_1*-*x_2*-plane is plotted in the right panel of Figure 7.4.*

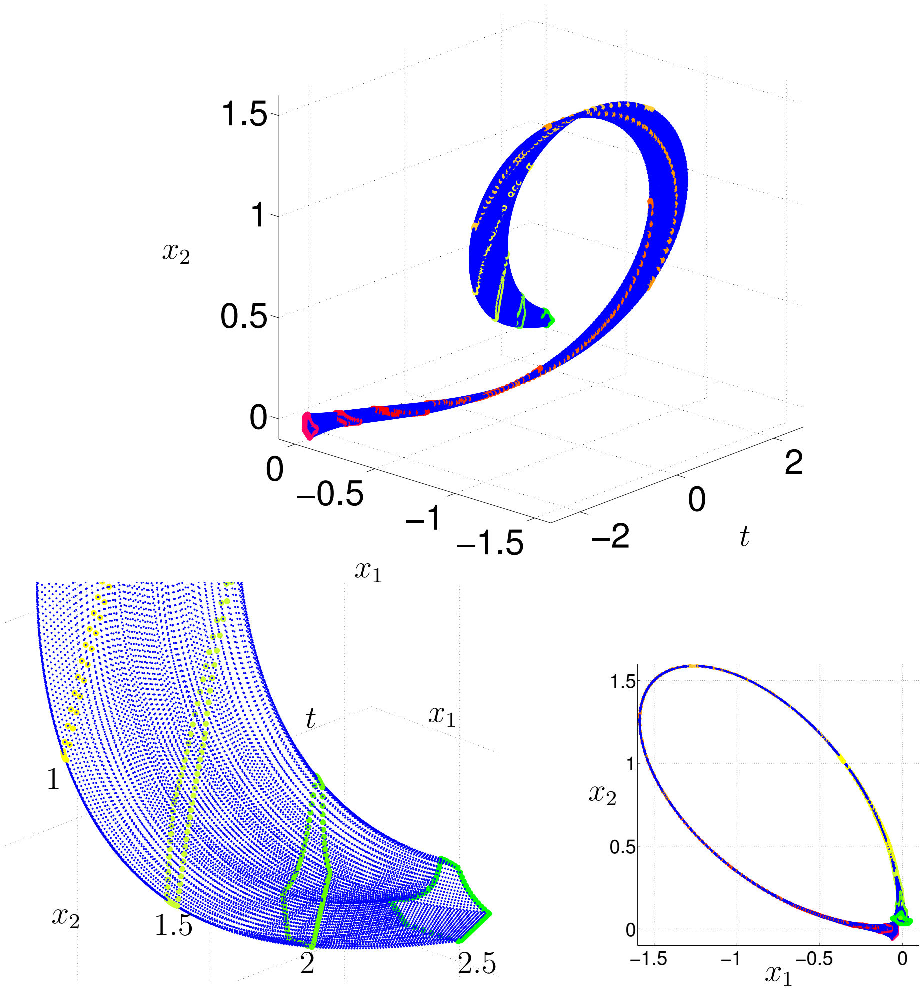


Figure 7.4: All $\varepsilon = 0.1$-homoclinic trajectories w.r.t. the fixed point 0 of system (7.9).

With the method described above we construct a boundary value problem, which enables a direct calculation of the boundary of the tube. This ansatz also applies to noninvertible dynamical systems. Consider

$$\dot{x} = f(x,t), \; x \in \mathbb{R}^k, t \in \mathbb{I} = [t_-, t_+]$$

and assume that 0 is an ft-hyperbolic solution. Let $\Gamma = \Gamma^T > 0$ and let $S_\Gamma(t) = \frac{1}{2}[\Gamma f_x(0,t) + f_x(0,t)^T \Gamma]$ be the symmetric part of $f_x(0,t)$ for all $t \in \mathbb{I}$. Denote by ${}^{\mathbb{I}}V_{s,u}(t)$, $t \in \mathbb{I}$ the stable and unstable t-cones of the linearized system. We discretize this system by applying a one-step method

$$x_{n+1} = \tilde{f}_n(x_n), n \in [0, N]_{\mathbb{Z}}$$

with step size $h = \frac{t_+ - t_-}{N}$ and $N \in \mathbb{N}, N > 0$. First define an operator whose zeros imply trajectories of the one-step method, i.e. $F : \mathbb{R}^{(N+1)k} \to \mathbb{R}^{Nk}$,

$$F\begin{pmatrix} x_0 \\ \vdots \\ x_N \end{pmatrix} := \begin{pmatrix} x_1 - \tilde{f}_0(x_0) \\ \vdots \\ x_N - \tilde{f}_{N-1}(x_{N-1}) \end{pmatrix}.$$

Let $X = \begin{pmatrix} x_0 & \dots & x_N \end{pmatrix}^T$, $\text{fix}_2 \in \mathbb{R}$ be variable and $\text{fix}_1 \in \mathbb{R}$ be a constant. Define the operator $F_1 : \mathbb{R}^{(N+1)k+1} \to \mathbb{R}^{(N+1)k+1}$ by

$$F_1 \begin{pmatrix} X \\ \text{fix}_2 \end{pmatrix} := \begin{pmatrix} F(X) \\ b^1_{\text{proj}}(x_0, x_N, \text{fix}_2) \\ \langle x_0, S_\Gamma(t_-)x_0 \rangle \end{pmatrix},$$

with boundary condition

$$b^1_{\text{proj}} : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k,\ b^1_{\text{proj}}(x, y, \mu) := \begin{pmatrix} Y_s(t_-)^T x - \text{fix}_1 \\ Y_u(t_+)^T y - \mu \end{pmatrix},$$

where $Y_{s,u}(t)$ forms a basis of $(U_{u,s}(t))^\perp$ and $U_{s,u}(t) \subset {}^{\mathbb{I}}V_{s,u}(t)$ is a vector space of maximal dimension for all $t \in \mathbb{I}$. This is an adapted version of the projection boundary condition presented in [72] for the infinite time case. Note that $Y_s(t_-)^T x - \text{fix}_1$ displays the "distance" between x and the ft-hyperbolic solution 0 at time t_-. For an ε-homoclinic orbit the "distance" does not need to be zero. It is variable in the interval $(-\varepsilon, \varepsilon)$, since the boundary point of a homoclinic orbit $x(\cdot)$ satisfies $\|x(t_\pm)\| < \varepsilon$.

Further define the following operator $F_2 : \mathbb{R}^{(N+1)k+1} \to \mathbb{R}^{(N+1)k+1}$ by

$$F_2 \begin{pmatrix} X \\ \text{fix}_2 \end{pmatrix} := \begin{pmatrix} F(X) \\ b^1_{\text{proj}}(x_0, x_N, \text{fix}_2) \\ \|x_0\|_2 - \varepsilon \end{pmatrix}.$$

Let now $\text{fix}_2 \in \mathbb{R}$ be a constant.
Then define the operators $F_{3,4} : \mathbb{R}^{(N+1)k+1} \to \mathbb{R}^{(N+1)k+1}$ by

$$F_3 \begin{pmatrix} X \\ \text{fix}_1 \end{pmatrix} := \begin{pmatrix} F(X) \\ b^2_{\text{proj}}(x_0, x_N, \text{fix}_1) \\ \langle x_N, S_\Gamma(t_+)x_N \rangle \end{pmatrix},\ F_4 \begin{pmatrix} X \\ \text{fix}_1 \end{pmatrix} := \begin{pmatrix} F(X) \\ b^2_{\text{proj}}(x_0, x_N, \text{fix}_1) \\ \|x_N\|_2 - \varepsilon \end{pmatrix},$$

with the boundary condition

$$b^2_{\text{proj}} : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k,\ b^2_{\text{proj}}(x, y, \mu) := \begin{pmatrix} Y_s(t_-)^T x - \mu \\ Y_u(t_+)^T y - \text{fix}_2 \end{pmatrix}.$$

If $\text{fix}_{1,2}$ are fixed, then $b^1_{\text{proj}}(x, y, \text{fix}_2) = b^2_{\text{proj}}(x, y, \text{fix}_1)$ for all $x, y \in \mathbb{R}$. This means that if (x, fix_2) is a zeros of F_1 with fixed fix_1 and if $x(N+1)$ solves the last equation of $F_{3,4}$ then (x, fix_1) is a zero of $F_{3,4}$ with fixed fix_2.

To get the boundary of an ε-homoclinic tube we first need to find a $\text{fix}_1 \in (0, \varepsilon]$ such that $X \in \mathbb{R}^{(N+1)k}$ with $F_1(X, \mu) = 0$ for a $\mu \in \mathbb{R}$ lies on the boundary of the ε-homoclinic tube. Note that the stable cone is a good approximation of the stable fiber in a small neighborhood of 0. Thus, it is sufficient to prove that the boundary point $X(N+1) = x_N$ (at time t_+) lies on the boundary of the intersection of the stable cone and the ε-ball around 0, i.e.

$$\begin{aligned} \langle x_N, S_\Gamma(t_+) x_N \rangle \leq 0,\ \|x_N\|_2 = \varepsilon \\ \text{or } \langle x_N, S_\Gamma(t_+) x_N \rangle = 0,\ \|x_N\|_2 \leq \varepsilon. \end{aligned} \tag{7.10}$$

We change fix_1 sufficiently small and approximate with the Newton method a zero of F_1 with the changed fix_1. If the condition (7.10) is satisfied for the new zero we continue with this procedure until the condition is not satisfied. This leads to a set of trajectories which form one part of the tube boundary. The boundary of the ε-homoclinic tube consists of six segments. The first segment is the one we just calculated. The other five segments are obtained in a similar way. We need to find a zero

$$\begin{aligned} \text{of } F_1 \text{ for } \text{fix}_1 \in [-\varepsilon, 0), \\ \text{of } F_2 \text{ for } \text{fix}_1 \in [-\varepsilon, \varepsilon], \end{aligned}$$

which satisfy condition (7.10). Via continuation w.r.t. fix_1 in the given interval as long as (7.10) is true we get two more curves of the tube boundary. Further we need a zero

$$\begin{aligned} \text{of } F_3 \text{ for } \text{fix}_2 \in [-\varepsilon, 0) \text{ and } \text{fix}_2 \in (0, \varepsilon], \\ \text{of } F_4 \text{ for } \text{fix}_2 \in [-\varepsilon, \varepsilon] \end{aligned}$$

which satisfy

$$\begin{aligned} \langle x_0, S_\Gamma(t_-) x_0 \rangle \geq 0,\ \|x_0\| = \varepsilon \\ \text{or } \langle x_0, S_\Gamma(t_-) x_0 \rangle = 0,\ \|x_0\| \leq \varepsilon. \end{aligned} \tag{7.11}$$

Via continuation this time w.r.t fix_2 as long as (7.11) is true we get the rest of the tube boundary.

We can calculate all six parts of the tube boundary separately or combined. For the first alternative we need for each function $F_{2,4}$ one initial trajectory and for $F_{1,3}$ two, one for a fixed parameter in $[-\varepsilon, 0)$ and one for a fixed parameter in $(0, \varepsilon]$. This means, we need six initial trajectories. Via continuation we obtain six curves, which together form the boundary of the tube. The second alternative requires only one initial trajectory. Figure 7.5 describes a way to obtain the entire boundary with just one initial trajectory. We start with one initial trajectory x, one initial parameter fix_2 and a fixed $\text{fix}_1 \in (0, \varepsilon]$ such that (x, fix_2) is a zero of F_1 which satisfies (7.10). This solution and fixed constant fix_1 are marked in black in Figure 7.5. Via continuation as described above w.r.t. fix_1 as long as (7.10) is satisfied we get a curve with boundary points $(x^\pm, \text{fix}_2^\pm)$. Denote the fixed constants by $\text{fix}_1^\pm$. The boundary points satisfy

$\langle x_N^+, S_\Gamma x_N^+ \rangle = 0$ and $\left\| x_N^- \right\|_2 = \varepsilon$. Then (x^+, fix_1^+) is a zero of F_3 with constant fix_2^+ and (x^-, fix_1^-) is a zero of F_4 with constant fix_2^-. Via continuation w.r.t. fix_2 we get two more parts of the tube boundary (curves) and for each boundary point with fixed constant a initial point for an other F_i function (with fixed constant). Continuing this procedure until the curves reach its initial point, we get the entire boundary via continuation from one initial trajectory. In Figure 7.5 we see how to change fix_1 and fix_2 such that we get a zero of another F_i function. Further it shows which function F_i we have to use.

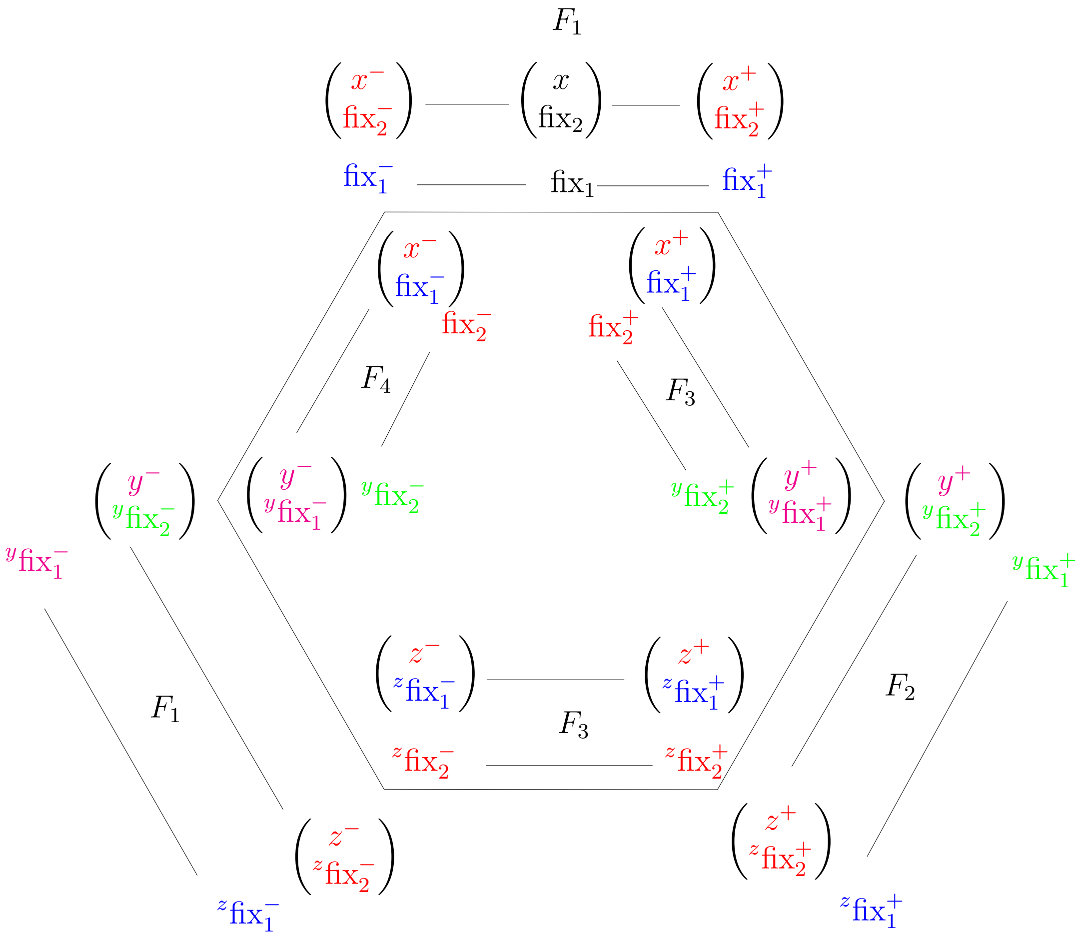


Figure 7.5: Circle of continuation to approximate the boundary of an ε-homoclinic tube with one initial trajectory x.

Finally we apply this method to equation (7.9).

Example 7.1.11. ***(part 2)***

We apply the continuation algorithm from above, which is based on the boundary value problems F_i, and obtain as illustrated in Figure 7.6 the boundary of the tube. We plotted the boundary at time -2.5 and 2.5 to illustrate *where the boundary of the tube and the stable and unstable cone coincide. The boundary value problem requires for the solution points at these times that they lie in the (un)stable cone and an ε-ball.*

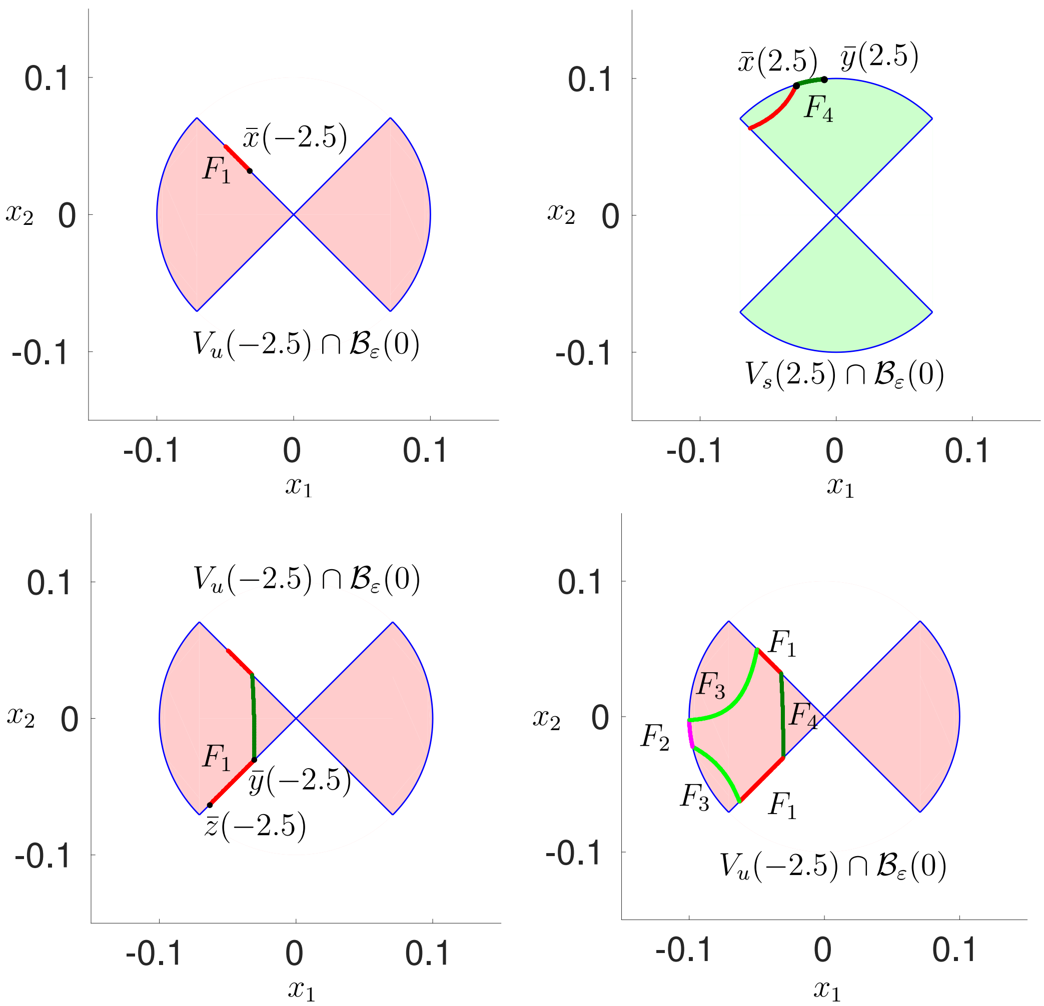


Figure 7.6: Calculation of all points of the ε-homoclinic tube of system (7.9) with the boundary value problem.

Having another look at Figure 7.4 we see in the bottom panel that the distance between the boundaries decreases if the distant to the fixed point increases. To analyze this in more detail we take three points X_1, X_2 and X_3 on the boundary of the ε-tube at time 2.5. These points are marked in black in Figure 7.7.

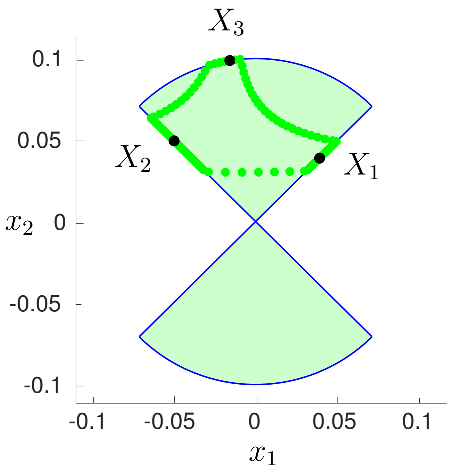


Figure 7.7: Three reference points of the boundary of the ε-homoclinic tube at time 2.5 (marked in black) of system (7.9).

We approximate the corresponding orbits $X_1(\cdot)$, $X_2(\cdot)$ and $X_3(\cdot)$ with the classical Runge-Kutta method and step size $h = -0.01$. First we take the orbit $X_1(\cdot)$ and calculating for each 0.01-time step the smallest distance d to the orbit $X_2(\cdot)$, i.e.

$$d(t_0) := \min_{t\in(\mathbb{I}\cap h\mathbb{Z})} \|X_1(t_0) - X_2(t)\|_2 \, , \; t_0 \in \mathbb{I} \cap 0.01\mathbb{Z}.$$

In Figure 7.8 the blue dashed line shows this distance. The solid blue line gives us the distance between the same orbits approximated with a smaller step size $h = -0.001$. The green lines illustrates the distance of $X_1(\cdot)$ to the orbit $X_3(\cdot)$ and the black lines of $X_2(\cdot)$ to $X_3(\cdot)$. The dashed lines show the distance between the orbits approximated with step size $h = -0.01$ while the solid lines illustrate the distance between the orbits approximated with step size $h = -0.001$. In the right part of Figure 7.8 we computed for each logarithmic distance graph a linear fit to all points from time -2.3 to -1.8 and a second fit to all points from time 1.3 to 1.8.

The linear fits provide approximations of the gradients. We observe that the distance decays exponentially fast to the middle of the interval. In general the distance does not decays towards the the exact middle of the interval. The place (time) depends on the exponential rates. The graphs corresponding to step size $h = -0.001$ (solid lines) decreases until they reach J_1 and the graphs corresponding to step size $h = -0.01$ (dashed lines) only until they reach $J \subset J_1$ (in Figure 7.8 the intervals J and J_1 are coordinated for the black graphs). We expect that the decay does not stop in an interval J of J_1, as the plotted graphs, if we theoretically calculate the distance between the continuous orbits. In Figure 7.9 we illustrate the reason why the decay stops. In the left part we plotted the three orbits $X_1(\cdot)$ (blue), $X_2(\cdot)$ (red) and $X_3(\cdot)$ (green) projected to

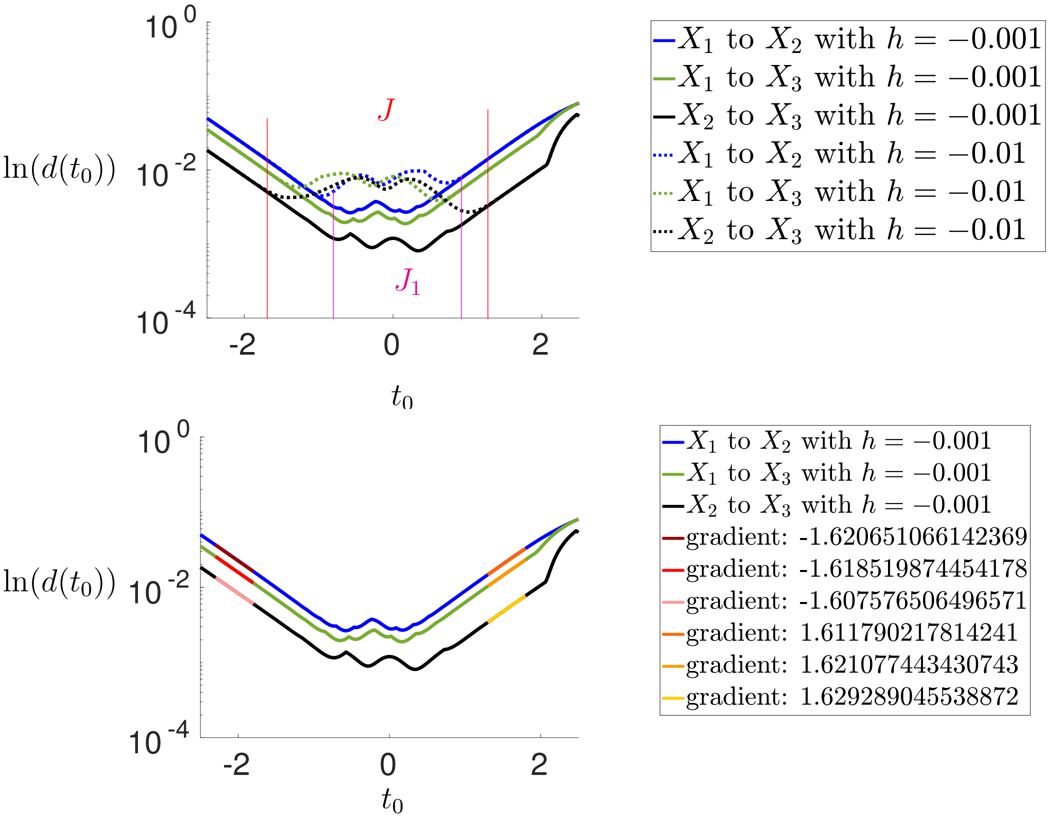


Figure 7.8: The distance of three different trajectories $(\Phi(\cdot, 2.5)Xi,\ i = 1, 2, 3)$ over the time interval.

the x_1-x_2-plane. The right part shows a section of these orbits (points of the orbit for times close to 0). Two distances calculated between the discretized orbits are marked with a black line. The brown lines illustrate the distance between the continuous orbits, which are shorter than the black lines.

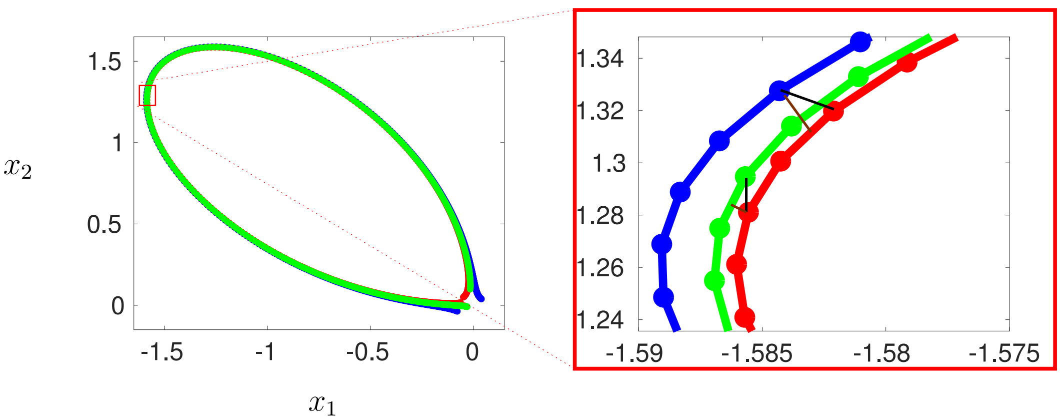


Figure 7.9: Numerical artifacts are causing that the distance does not exponentially decrease for times close to 0.

We expect that the approximation of the distances gets more precise the smaller we choose the step size. Further we await that in the theoretical case the distance decays exponentially fast towards the middle of the given interval $\mathbb{I}$. Beyn, Hüls and Schenke studied in [21] infinite time discrete hyperbolic dynamical systems. They showed that two finite orbit segments in a small neighborhood of a hyperbolic orbit converge exponentially fast towards each other (for t towards some time in the given finite time interval). Inspired by this publication we study a solution $x(\cdot)$ of a continuous finite time dynamical system which generates a K-hyperbolic, $K \in [1, \infty)$, variational equation. We prove that two solutions of the original system in a small neighborhood of the solution $x(\cdot)$ converge exponentially fast towards each other (for t towards some time in the interval).

Theorem 7.1.12. *Let* $\mathbb{I} = [t_-, t_+]$, $K \in [1, \infty)$ *and* $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ *be a solution of equation* (2.2). *Further let*

$$\dot{u}(t) = f_x(\bar{x}(t), t)u(t)$$

be K*-hyperbolic (w.r.t.* $\|\cdot\|$*) on* $\mathbb{I}$ *with* $\alpha, \beta > 0$. *Then there exists an* $\bar{\varepsilon} > 0$, $0 < \tilde{\alpha} \leq \alpha$, $0 < \tilde{\beta} \leq \beta$ *and a constant* $C > 0$ *such that for all* $0 < \varepsilon \leq \bar{\varepsilon}$ *any two solutions* z_1, z_2 *of* (2.2) *with* $\sup_{t \in \mathbb{I}} \|z_{1,2}(t) - \bar{x}(t)\| \leq \varepsilon$ *satisfy*

$$\|z_1(t) - z_2(t)\| \leq 5\varepsilon K^2 C(e^{-\tilde{\alpha}(t-t_-)} + e^{-\tilde{\beta}(t_+-t)})$$

for all $t \in \mathbb{I}$.

Proof. Define $d(t) = z_1(t) - z_2(t)$ for all $t \in \mathbb{I}$. Then $d(\cdot)$ solves

$$\begin{aligned} \dot{d}(t) = \dot{z}_1(t) - \dot{z}_2(t) &= f(z_1(t), t) - f(z_2(t), t) \\ &= \int_0^1 f_x(z_2(t) + sd(t), t)\mathrm{d}s d(t) =: A(t)d(t). \end{aligned}$$

Next we show that

$$\dot{u}(t) = A(t)u(t) \tag{7.12}$$

has an exponential dichotomy on $\mathbb{I}$ with $\tilde{K} = \frac{5K^2}{2}$ and exponential rates $0 < \tilde{\alpha} \leq \alpha$, $0 < \tilde{\beta} \leq \beta$. For that purpose we use Theorem 3.4.4 and need to show that

$$\sup_{t \in \mathbb{I}} \|A(t) - f_x(\bar{x}(t), t)\| < \frac{\min\{\alpha, \beta\}}{4}.$$

The modulus of continuity is given by

$$\begin{aligned}&\omega(f_x, \mathbb{R}^k, \delta)\\ &= \sup\left\{\sup_{t\in\mathbb{I}} \|f_x(x(t),t) - f_x(y(t),t)\| \,\middle|\, x, y : \mathbb{I} \to \mathbb{R}^k : \sup_{t\in\mathbb{I}} \|x(t) - y(t)\| \le \delta\right\}.\end{aligned}$$

Choose $\bar{\varepsilon} > 0$ such that $\omega(f_x, \mathbb{R}^k, \varepsilon) < \frac{\min\{\alpha,\beta\}}{4}$ for all $0 < \varepsilon \le \bar{\varepsilon}$. Then by

$$\begin{aligned}\sup_{t\in\mathbb{I}} \|z_2(t) + sd(t) - \bar{x}(t)\| &\le s \sup_{t\in\mathbb{I}} \|z_1(t) - \bar{x}(t)\| + (1-s) \sup_{t\in\mathbb{I}} \|z_2(t) - \bar{x}(t)\|\\ &\le s\varepsilon + (1-s)\varepsilon = \varepsilon, \;\; s \in [0,1],\end{aligned}$$

we get

$$\begin{aligned}\sup_{t\in\mathbb{I}} \|A(t) - f_x(\bar{x}(t),t)\| &\le \int_0^1 \sup_{t\in\mathbb{I}} \|f_x(z_2(t) + sd(t), t) - f_x(\bar{x}(t),t)\| \,\mathrm{d}s\\ &\le \int_0^1 \omega(f_x, \mathbb{R}^k, \varepsilon)\mathrm{d}s\\ &= \omega(f_x, \mathbb{R}^k, \varepsilon) < \frac{\min\{\alpha,\beta\}}{4}.\end{aligned}$$

This implies an exponential dichotomy (w.r.t. $\|\cdot\|$) of (7.12) on $\mathbb{I}$ with $\tilde{K} = \frac{5K^2}{2}$, $0 < \tilde{\alpha} \le \alpha$, $0 < \tilde{\beta} \le \beta$ and an invariant family of projectors $Q : \mathbb{I} \to \mathbb{R}^{k\times k}$. Denote the solution operator of (7.12) by Ψ, then the solution $d(\cdot)$ of (7.12) satisfies

$$\begin{aligned}d(t) &= Q(t)d(t) + (\mathrm{I} - Q(t))d(t) = Q(t)\Psi(t,t_-)d(t_-) + (\mathrm{I} - Q(t))\Psi(t,t_+)d(t_+)\\ &= \Psi(t,t_-)Q(t_-)d(t_-) + \Psi(t,t_+)(\mathrm{I} - Q(t_+))d(t_+).\end{aligned}$$

Combining this with

$$\begin{aligned}\sup_{t\in\mathbb{I}} \|d(t)\| &= \sup_{t\in\mathbb{I}} \|z_1(t) - \bar{x}(t) + \bar{x}(t) - z_2(t)\|\\ &\le \sup_{t\in\mathbb{I}} \|z_1(t) - \bar{x}(t)\| + \sup_{t\in\mathbb{I}} \|z_2(t) - \bar{x}(t)\| \le \varepsilon + \varepsilon = 2\varepsilon\end{aligned}$$

we obtain

$$\begin{aligned}\|d(t)\| &\le \|\Psi(t,t_-)Q(t_-)d(t_-)\| + \|\Psi(t,t_+)(\mathrm{I} - Q(t_+))d(t_+)\|\\ &\le \frac{5K^2}{2} e^{-\tilde{\alpha}(t-t_-)} \|Q(t_-)d(t_-)\| + \frac{5K^2}{2} e^{-\tilde{\beta}(t_+-t)} \|(\mathrm{I} - Q(t_+))d(t_+)\|\\ &\le \frac{5K^2}{2} \sup_{t\in\mathbb{I}} \|d(t)\| \, C \left(e^{-\tilde{\alpha}(t-t_-)} + e^{-\tilde{\beta}(t_+-t)}\right)\\ &\le 5\varepsilon K^2 C \left(e^{-\tilde{\alpha}(t-t_-)} + e^{-\tilde{\beta}(t_+-t)}\right)\end{aligned}$$

with $C = \max\{\|Q(t_-)\|, \|\mathrm{I} - Q(t_+)\|\}$.

□

Corollary 7.1.13. *Let* $\mathbb{I} = [t_-, t_+]$, $K \in [1, \infty)$ *and* $\bar{x} : \mathbb{I} \to \mathbb{R}^k$ *be a solution of equation* (2.2). *Further let*

$$\dot{u}(t) = f_x(\bar{x}(t), t)u(t)$$

be M-hyperbolic (w.r.t. $\|\cdot\|$*) on* $\mathbb{I}$ *with* $\alpha, \beta > 0$. *Then there exists an* $\bar{\varepsilon} > 0$, $0 < \tilde{\alpha} \leq \alpha$, $0 < \tilde{\beta} \leq \beta$ *and a constant* $C > 0$ *such that for all* $0 < \varepsilon \leq \bar{\varepsilon}$ *any two solutions* z_1, z_2 *of* (2.2) *with* $\sup_{t \in \mathbb{I}} \|z_{1,2}(t) - \bar{x}(t)\| \leq \varepsilon$ *satisfy*

$$\|z_1(t) - z_2(t)\| \leq 2\varepsilon C(e^{-\tilde{\alpha}(t-t_-)} + e^{-\tilde{\beta}(t_+-t)})$$

for all $t \in \mathbb{I}$.

Proof. Using Theorem 3.4.11 instead of Theorem 3.4.4 we obtain the constant 1 instead of $\frac{5K^2}{2}$. □

For K-hyperbolic systems we conclude that the distance of the boundary of the ε-homoclinic tube decreases to the middle of the given interval if the distance stays sufficiently small for all times within the interval. In our example (7.9) this is the case.

Discretization by the h-Flow

In this subsection, we discretize the differential equation (2.2), using the h-flow. From a numerical point of view, this ansatz is of no practical relevance. It is introduced for deriving error estimates of one-step methods in the next subsection.

From now on if not other mentioned assume **(A4)**. Further, let $\varphi(\cdot, t, s)$ be defined for all $t, s \in \mathbb{I}$ (invertible). We consider the ***h*-flow**

$$\varphi_n(x, h) := \varphi(x, (n+1)h, nh),\ n \in \mathbb{J}_1 \tag{7.13}$$

and note that the resulting dynamical system generated by the difference equation

$$x_{n+1} = \varphi_n(x_n, h), n \in \mathbb{J}_1, x_n \in \mathbb{R}^k \tag{7.14}$$

is invertible. An orbit

$$\varphi_{\mathbb{J}}(\cdot, h) := \begin{cases} \varphi_{\mathbb{Z}}(\cdot, h), & \text{for } \mathbb{J} = \mathbb{Z} \\ (\cdot; \varphi_{\mathbb{J}_1}(\cdot, h)), & \text{for } \mathbb{J} \neq \mathbb{Z} \end{cases}$$

that satisfies the boundary condition $b(\cdot, \varphi_{n_+-1}(\cdot, h)) = 0)$ is a zero of the operator

$$\Upsilon : S_{\mathbb{J}} \times \mathbb{R} \to S_{\mathbb{J}},\ (x_{\mathbb{J}}, h) \mapsto \begin{pmatrix} x_{n+1} - \varphi_n(x_n, h), & n \in \mathbb{J}_1 \\ hb(x_{n_-}, x_{n_+}), & \text{if } \mathbb{J} \neq \mathbb{Z} \end{pmatrix}$$

that operates on the Banach space of bounded sequences

$$S_{\mathbb{J}} := \begin{cases} \{x_{\mathbb{Z}} = (x_n)_{n\in\mathbb{Z}} : x_n \in \mathbb{R}^k, \|x_{\mathbb{Z}}\| := \sup_{n\in\mathbb{Z}} \|x_n\| < \infty\}, & \text{if } \mathbb{J} = \mathbb{Z}, \\ \{x_{\mathbb{J}} = (x_n)_{n\in\mathbb{J}} : x_n \in \mathbb{R}^k\} & , \text{ if } \mathbb{J} \neq \mathbb{Z}. \end{cases}$$

Assuming **(A10)**, the discretized orbit

$$\bar{y}_{\mathbb{J}}(h) := (\bar{y}_n(h))_{n\in\mathbb{J}} := (\bar{y}(nh))_{n\in\mathbb{J}} \tag{7.15}$$

is bounded and a zero of Υ, where φ is the solution operator of (2.2), if the boundary condition equals zero. Further, we look at the corresponding variational equation, $b \in \mathcal{C}^1(\mathbb{R}^{2k}, \mathbb{R}^k)$, that we obtain by differentiating Υ w.r.t. the first component

$$\Upsilon_{x_{\mathbb{J}}}(\bar{y}_{\mathbb{J}}(h), h) : S_{\mathbb{J}} \to S_{\mathbb{J}}, \tag{7.16}$$
$$u_{\mathbb{J}} \mapsto \begin{pmatrix} u_{n+1} - (\varphi_n)_x(\bar{y}_n(h), h)u_n, & n \in \mathbb{J}_1 \\ h(D_1 b(\bar{y}_{n_-}, \bar{y}_{n_+})u_{n_-} + D_2 b(\bar{y}_{n_-}, \bar{y}_{n_+})u_{n_+}), & \text{if } \mathbb{J} \neq \mathbb{Z}. \end{pmatrix}$$

Transversality of homoclinic orbits $x_{\mathbb{Z}}(h)$ in discrete time systems is characterized by one of the equivalent properties given in Theorem 7.2.1, see Theorem 7.0.10 for the continuous time case and note that all results applied in the proof have a discrete time counterpart. For the equivalence of (a_Δ) and (d_Δ) we particularly refer to [70, Lemma 3.7] .

Theorem 7.2.1. *Let* $\mathbb{I} = \mathbb{R}$. *Assume* **(A5)-(A8)** *and let* $h > 0$. *Then* $\bar{x}_{\mathbb{Z}}(h)$ *is a homoclinic orbit of* (7.13). *Furthermore, the following statements are equivalent:*

(a_Δ) $u_{n+1} = (\varphi_n)_x(\bar{x}_n(h), h)u_n, \quad u_{\mathbb{Z}} \in S_{\mathbb{Z}} \quad \Leftrightarrow \quad u_{\mathbb{Z}} = 0.$

(b_Δ) *The variational equation*

$$u_{n+1} = (\varphi_n)_x(\bar{x}_n(h), h)u_n, \quad n \in \mathbb{Z}$$

has an exponential dichotomy on $\mathbb{Z}$.

(c_Δ) $\Upsilon_{x_{\mathbb{Z}}}(\bar{x}_{\mathbb{Z}}, h)$ *is a homeomorphism.*

(d_Δ) *The tangent spaces* $T_{\bar{x}_0}W^0_{s,u}(0,h)$ *of the global stable and unstable* 0*-fibers* $W^0_{s,u}(0,h)$ *of the h-flow* $x_{n+1} = \varphi_n(x_n, h)$ *satisfy*

$$T_{\bar{x}_0}W^0_s(0,h) \oplus T_{\bar{x}_0}W^0_u(0,h) = \mathbb{R}^k.$$

As in the continuous time case the statements in Theorem 7.2.1 are not equivalent for finite time systems. We want at least that from statement (b_Δ) statement (c_Δ) follows. Therefore, it is important that the domain of definition and the range of $\Upsilon_{x_{\mathbb{Z}}}$ have the same dimension. This is the reason why we added for finite time systems to the one-step-method a boundary condition. To prove that from (b_Δ) the statement (c_Δ) follows we need assumptions depending on the boundary condition.

(A11) Let the images of the projectors $\bar{Q}^s(t_-)$, $\bar{Q}^u(t_+)$ satisfy $\mathcal{R}(\bar{Q}^s(t_-)) \oplus \mathcal{R}(\bar{Q}^u(t_+)) = \mathbb{R}^k$ and let the boundary condition $b(\cdot,\cdot)$ satisfies $b \in \mathcal{C}^1(\mathbb{R}^{2k}, \mathbb{R}^k)$ and $b := b(\bar{y}_{n_-}, \bar{y}_{n_+}) = 0$. Further, let

$$\begin{aligned} B(\bar{y}_{\mathbb{J}}) :=& h\left(D_1 b + D_2 b \Phi(n_+, n_-)_{|\mathcal{R}(\bar{Q}^s(t_-))} \quad D_1 b \bar{\Phi}(n_-, n_+) + D_2 b_{|\mathcal{R}(\bar{Q}^u(t_+))}\right) : \\ & \mathcal{R}(\bar{Q}^s(n_-)) \oplus \mathcal{R}(\bar{Q}^u(n_+)) \to \mathbb{R}^k \end{aligned}$$

be invertible. For any compact set $\mathcal{K} \subset \mathbb{R}^k$ there exist a constant $C(\mathcal{K}) > 0$ such that

$$\left\| b_x(x_{n_-}, x_{n_+}) - b_x(y_{n_-}, y_{n_+}) \right\| \le C(\mathcal{K}) \left\| x_{\mathbb{J}} - y_{\mathbb{J}} \right\|$$

holds for all $x_n, y_n \in \mathbb{R}^k$, $n \in \mathbb{J}$.

With these assumptions we obtain that for ft-hyperbolic trajectories $y_{\mathbb{J}}$ the operator $\Upsilon_{x_{\mathbb{J}}}(y_{\mathbb{J}}, h)$ is a homeomorphism ("$(\mathrm{b}_\Delta) \to (\mathrm{c}_\Delta)$"). The proof is similar to [73, Theorem 4] and [21, Theorem 6] for infinite autonomous and nonautonomous systems on finite time intervals.

Theorem 7.2.2. *Let $\mathbb{I} = [t_-, t_+]$. Assume* **(A4)**, **(A5)** *and that $\bar{y}(\cdot)$ is a trajectory of* (2.2). *Then $\bar{y}_{\mathbb{J}}(h)$ is a trajectory of* (7.13). *Let the variational equation*

$$u_{n+1} = (\varphi_n)_x)(\bar{y}_n(h), h) u_n, \ n \in \mathbb{J}_1$$

have an ft-exponential dichotomy with the invariant family of projectors $\bar{Q}^{s,u}(hn) = \bar{Q}^{s,u}_n(h)$, $n \in \mathbb{J}$ and solution operator $\Phi(n,m)$, $m, n \in \mathbb{J}$, $n \ge m$. Further assume **(A11)**. *Then $\Upsilon_{x_{\mathbb{J}}}(y_{\mathbb{J}}, h)$ is a homeomorphism.*

Proof. By **(A5)** it is sufficient to show that (7.16) is bijective. Hence, we have to prove that for every $y_{\mathbb{J}_1} \in S_{\mathbb{J}_1}$ and $r \in \mathbb{R}^k$ there exists a unique solution $\bar{u}_{\mathbb{J}}$ of the boundary value problem

$$\Upsilon_{x_{\mathbb{J}}}(\bar{y}_{\mathbb{J}}, h) u_{\mathbb{J}} = \begin{pmatrix} y_{\mathbb{J}_1} \\ r \end{pmatrix}.$$

Denote by $\Phi(m,n)$ the inverse of $\Phi(n,m)|_{\mathcal{R}(\bar{Q}^u_m(h))}$ for all $n, m \in \mathbb{I}$, $n \ge m$. The general solution of the linear equation is given by

$$u_n := \Phi(n, n_-) v_- + \Phi(n, n_+) v_+ + \sum_{m \in \mathbb{J}_1} G(n, m+1) y_m$$

with $v_- \in \mathcal{R}(\bar{Q}^s_{n_-}(h))$ and $v_+ \in \mathcal{R}(\bar{Q}^u_{n_+}(h))$, where $G(\cdot,\cdot)$ is the Green's function, see [105].

To show that the solution is unique we have to prove that v_- and v_+ are unique. We insert the general solution into the boundary condition

$$\begin{aligned} r =& h D_1 b(\bar{y}_{n_-}, \bar{y}_{n_+})[v_- + \Phi(n_-, n_+) v_+ + \sum_{m \in \mathbb{J}_1} G(n_-, m+1) y_m] \\ & + h D_2 b(\bar{y}_{n_-}, \bar{y}_{n_+})[\Phi(n_+, n_-) v_- + v_+ + \sum_{m \in \mathbb{J}_1} G(n_+, m+1) y_m]. \end{aligned}$$

Since $y_{\mathbb{J}_1}$ is fixed and given we define

$$R = hD_1 b(\bar{y}_{n_-}, \bar{y}_{n_+}) \sum_{m\in\mathbb{J}_1} G(n_-, m+1) y_m + hD_2 b(\bar{y}_{n_-}, \bar{y}_{n_+}) \sum_{m\in\mathbb{J}_1} G(n_+, m+1) y_m.$$

Set $b = b(\bar{y}_{n_-}, \bar{y}_{n_+})$. Then we need to solve

$$\begin{aligned} r - R &= h[D_1 b + D_2 b\Phi(n_+, n_-)]v_- + h[D_1 b\Phi(n_-, n_+) + D_2 b]v_+ \\ &= h\begin{pmatrix} D_1 b + D_2 b\Phi(n_+, n_-) & D_1 b\Phi(n_-, n_+) + D_2 b \end{pmatrix} \begin{pmatrix} v_- \\ v_+ \end{pmatrix} \\ &= B(\bar{y}_{\mathbb{J}}) \begin{pmatrix} v_- \\ v_+ \end{pmatrix}. \end{aligned}$$

By **(A11)** the vectors v_- and v_+ are unique. □

(Ft-)Hyperbolicity as well as transversality of a trajectory in continuous time carries over to the discrete trajectory generated by the h-flow.

Lemma 7.2.3. (a) *Assume* **(A4)**, **(A5)** *and* **(A10)**, *then $\bar{y}_{\mathbb{J}}(h)$ is a (ft-)hyperbolic bounded trajectory of $\varphi_n(\cdot, h)$.*

(b) *Let $\mathbb{J} = \mathbb{Z}$. Assume* **(A5)-(A8)**, *then the following statements are equivalent.*

- (b_1) *The continuous time orbit $\bar{x}(\cdot)$ is transversal.*
- (b_2) *There exists an $\hat{h} > 0$ such that $\bar{x}_{\mathbb{Z}}(h)$ is a transversal homoclinic orbit of the h-flow $\varphi_n(\cdot, h)$ for all step sizes $0 < h \le \hat{h}$.*

Proof. Assume **(A4)**, **(A5)** and **(A10)**. Denote by $\Phi(\cdot,\cdot)$ the solution operator of the variational equation

$$u_{n+1} = (\varphi_n)_x(\bar{y}_n(h), h)u_n, \quad n \in \mathbb{J}_1 \tag{7.17}$$

and observe by the following that

$$\Phi(n, m) = S^{\bar{y}}(nh, mh), \quad n, m \in \mathbb{J}. \tag{7.18}$$

Since $S^{\bar{y}}(\cdot,\cdot)$ is the solution operator of $\dot{u} = f_x(\bar{y}(\cdot), \cdot)u$ we get for $s, t \in \mathbb{I}$

$$\frac{\mathrm{d}}{\mathrm{d}t} S^{\bar{y}}(t, s) = f_x(\bar{y}(t), t) S^{\bar{y}}(t, s). \tag{7.19}$$

Further, $\varphi(\bar{y}(s), t, s)$ is a solution of $\dot{x} = f(x, t)$. Inserting and differentiating by x yields with $\varphi(\bar{y}(s), t, s) = \bar{y}(t)$ the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_x(\bar{y}(s), t, s) = f_x(\bar{y}(t), t)\varphi_x(\bar{y}(s), t, s). \tag{7.20}$$

Comparing (7.19) and (7.20) we get by the uniqueness of the solution operator

$$S^{\bar{y}}(t,s) = \varphi_x(\bar{y}(s),t,s). \tag{7.21}$$

Since $\Phi(\cdot,\cdot)$ is the solution operator of (7.17) we get with (7.13), (7.15) and (7.21)

$$\begin{aligned}\Phi(n+1,n) &= (\varphi_n)_x(\bar{y}_n(h),h) \\ &= \varphi_x(\bar{y}(nh),(n+1)h,nh) = S^{\bar{y}}((n+1)h,nh).\end{aligned} \tag{7.22}$$

This implies (7.18) since $\Phi(\cdot,\cdot)$ and $S^{\bar{y}}(\cdot,\cdot)$ are both solution operators. Using the continuous time dichotomy data from **(A10)**, we define

$$Q_n^{s,u}(h) := \bar{Q}^{s,u}(hn),\ \ n \in \mathbb{J}$$

and immediately obtain that (7.17) has an (ft-)exponential dichotomy on $\mathbb{J}$ with data $(\bar{K}, h\bar{\beta}, Q_n^{s,u}(h))$. This completes the proof of (a).

For proving (b) assume **(A5)-(A8)**. If (b_1) holds true, then Theorem 7.0.10 guarantees that (5.1) has an exponential dichotomy on $\mathbb{R}$. An application of (a) combined with the observation that the h-flow preserves homoclinic structures proves (b_2).

To show the implication "(b_2)$\Rightarrow$(b_1)", we assume that (b_1) is not satisfied, i.e. the orbit $\bar{x}(\cdot)$ is not transversal. Then a nontrivial bounded solution $u(\cdot)$ of (5.1) exists. As a consequence, we find an $\tilde{h} > 0$ such that $u_{\mathbb{Z}}(h) = (u(nh))_{n\in\mathbb{Z}} \neq 0$ for all $0 < h \leq \tilde{h}$ and $u_{n+1}(h) = S^{\bar{x}}((n+1)h, nh)u_n(h)$ holds for all $n \in \mathbb{Z}$, where $S^{\bar{x}}(s,t)$ is the solution operator of (5.1). Applying the identity

$$S^{\bar{x}}((n+1)h,nh) = (\varphi_n)_x(\bar{x}_n(h),h)$$

for all $n \in \mathbb{Z}$ (see (7.22)) we obtain

$$u_{n+1}(h) = (\varphi_n)_x(\bar{x}_n,h)u_n(h), \quad n \in \mathbb{Z}.$$

Since $u_{\mathbb{Z}}(h) \neq 0$ for all $h \leq \tilde{h}$ Theorem 7.2.1 (a_Δ) applies and thus $\bar{x}_{\mathbb{Z}}(h)$ is not transversal for all $h \leq \tilde{h}$. This violates condition (b_2). □

Before we are finally able to prove the last equivalence relation of Theorem 7.0.10 we show that the stable and unstable 0-fiber bundles $W^0_{s,u}(0)$ of the continuous infinite time system generated by (2.2) coincide with those $W^0_{s,u}(0,h)$ of the system generated by (7.14) for h sufficiently small. For finite time systems this statement is generally not true. We can not guarantee that a function which is strictly decreasing (increasing) on a discrete interval is also strictly decreasing (increasing) on the continuous interval no matter how small the step size h is. This is not dramatical since Lemma 7.2.4 is only needed to finish the proof of Theorem 7.0.10. Hence, the statement $W^0_{s,u}(0,h) = W^0_{s,u}(0)$ is important for transversal orbits. We showed that we do not have to explicitly distinguish between transversal and tangentil finite time homoclinic orbits.

Lemma 7.2.4. *Assume* **(A4)**, **(A5)** *and* **(A6)** *and* $0 \in \mathbb{I}$. *Let* $\mathbb{I} = [t_-, t_+]$ *and* $\varepsilon > 0$. *Then for every* $h > 0$ *we have*

$$ {}^{\mathbb{I}}_{\varepsilon}W^0_{s,u}(0) \subset {}^{\mathbb{I}}_{\varepsilon}W^0_{s,u}(0,h), $$

where ${}^{\mathbb{I}}_{\varepsilon}W^0_{s,u}(0,h)$ *are the (un)stable* 0*-fiber bundles of* (7.14) *and* ${}^{\mathbb{I}}_{\varepsilon}W^0_{s,u}(0)$ *of* (2.2). *Let* $\mathbb{I} = \mathbb{R}$. *Then there exists a constant* $\hat{h} > 0$ *such that*

$$ W^0_{s,u}(0,h) = W^0_{s,u}(0) $$

holds for all $0 < h < \hat{h}$, *where* $W^0_{s,u}(0,h)$ *are the (un)stable* 0*-fiber bundles of* (7.14) *and* $W^0_{s,u}(0)$ *of* (2.2).

Proof. Let $\mathbb{I} = [t_-, t_+]$ with $0 \in \mathbb{I}$. By the invertibilily of the solution operator we have

$$ \begin{aligned} {}^{\mathbb{I}}_{\varepsilon}W^0_{s}(0) =& \{\xi \in \mathbb{R}^k \big| \hat{t} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(\varphi(\xi, t_-, 0), 0) \in \mathbb{I} : \\ & \qquad \frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(\xi, t, 0)\| \le 0 \text{ for all } t \in \mathbb{I}, t > \hat{t}\} \\ =& \{\xi \in \mathbb{R}^k \big| \hat{t} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(\varphi(\xi, t_-, 0), 0) \in \mathbb{I} : \\ & \qquad \|\varphi(\xi, t, 0)\| \le \|\varphi(\xi, s, 0) \text{ for all } t, s \in \mathbb{I}, t > s > \hat{t}\} \\ \subset& \{\xi \in \mathbb{R}^k \big| \hat{t} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(\varphi(\xi, t_-, 0), 0) \in \mathbb{J} : \\ & \qquad \|\varphi(\xi, th, 0)\| \le \|\varphi(\xi, sh, 0)\| \text{ for all } t, s \in \mathbb{J}, t > s > \hat{t}\} \\ =& {}^{\mathbb{I}}_{\varepsilon}W^0_{s,u}(0,h). \end{aligned} $$

Analogously, the statement holds for the unstable fiber bundles.
Let $\mathbb{I} = \mathbb{R}$. Then $W^0_{s,u}(0) \subset W^0_{s,u}(0,h)$ is true since

$$ \left\{ x \in \mathbb{R}^k \Big| \lim_{\substack{s \to \pm\infty, \\ s \in \mathbb{R}}} \|\varphi(x, s, 0)\| = 0 \right\} \subset \left\{ x \in \mathbb{R}^k \Big| \lim_{\substack{n \to \pm\infty, \\ n \in \mathbb{Z}}} \|\varphi(x, nh, 0)\| = 0 \right\} $$

holds. For proving the other inclusion let $x_0 \in W^0_{s,u}(0,h)$. For every $t \in \mathbb{R}$ we find an $n \in \mathbb{Z}$ such that $nh \le t \le (n+1)h$ holds. With **(A5)**, **(A6)** and the mean value theorem we get

$$ \begin{aligned} & \sup_{nh \le t \le (n+1)h} \|\varphi(x_0, t, 0)\| \\ =& \sup_{nh \le t \le (n+1)h} \|\varphi(\varphi(x_0, nh, 0), t, nh) - \varphi(0, t, nh)\| \\ =& \sup_{nh \le t \le (n+1)h} \Big\| \int_0^1 \varphi_x(s\varphi(x_0, nh, 0), t, nh) \mathrm{d}s \varphi(x_0, nh, 0) \Big\| \\ \le& C_1 \|\varphi(x_0, nh, 0)\|. \end{aligned} \tag{7.23} $$

From $x_0 \in W^0_{s,u}(0,h)$ it follows that $\lim_{\substack{n \to \pm\infty, \\ n \in \mathbb{Z}}} \|\varphi(x_0, nh, 0)\| = 0$ holds. With (7.23) we get $\lim_{\substack{t \to \pm\infty, \\ t \in \mathbb{R}}} \|\varphi(x_0, t, 0)\| = 0$ and consequently $x_0 \in W^0_{s,u}(0)$. ☐

Combining these results, we prove the remaining statements of Theorem 7.0.10.

Proof of Theorem 7.0.10, "(a_c) ⇔ (d_c)". It follows from Lemma 7.2.3 (b) that an $\hat{h} > 0$ exists such that (a_c) ⇔ (a_Δ holds for all $h \leq \hat{h}$). Applying Theorem 7.2.1 we get (a_Δ) ⇔ (d_Δ) and finally, Lemma 7.2.4 yields (d_Δ holds for all $h \leq \hat{h}$) ⇔ (d_c). □

Corollary 7.2.5. *Let $\bar{x}_\mathbb{Z}$ be a transversal homoclinic orbit of the map $\varphi_n(\cdot, h)$. Then $\bar{x}_\mathbb{Z}$ is a* ***regular*** *solution of the operator Υ, i.e. $\Upsilon(\bar{x}_\mathbb{Z}, h) = 0$ and $\Upsilon_{x_\mathbb{Z}}(\bar{x}_\mathbb{Z}, h)$ is a homeomorphism.*

Discretization by a One-Step Method

For a general one-step method, we prove a closeness result for approximate trajectories in our nonautonomous context. From this, we conclude our main theorem - the persistence of homoclinic orbits under one-step discretizations. We still assume **(A4)**.

We consider a general one-step method

$$x_{n+1} = \psi_n(x_n, h), \quad x_n \in \mathbb{R}^k, \quad n \in \mathbb{J}_1 \tag{7.24}$$

with step size $h > 0$. The orbits of (7.24) are zeros of the operator

$$\tilde{\Upsilon} : S_\mathbb{J} \times \mathbb{R} \to S_\mathbb{J}, \quad (x_\mathbb{J}, h) \mapsto \begin{pmatrix} x_{n+1} - \psi_n(x_n, h), & n \in \mathbb{J}_1 \\ hb(x_{n_-}, x_{n_+}), & \text{for } \mathbb{J} = [n_-, n_+]_\mathbb{Z} \end{pmatrix}$$

if their boundary points are zeros of the boundary condition b. We assume consistency of order d as well as smoothness:

(A12) For any compact set $\mathcal{K} \subset \mathbb{R}^k$ there exist constants $C_2(\mathcal{K}), h_2(\mathcal{K}) > 0$ such that the consistency estimate of order $d \in \mathbb{N}$

$$\|\varphi_n(x, h) - \psi_n(x, h)\| \leq C_2(\mathcal{K}) h^{d+1}$$

holds for all $n \in \mathbb{J}_1$, $x \in \mathcal{K}$ and $0 < h \leq h_2(\mathcal{K})$.

(A13) Mixed derivatives of $\psi_n(x, h)$ up to order 3 exist. For any compact set $\mathcal{K} \subset \mathbb{R}^k$ the derivatives are continuous and uniformly bounded by some constant $\tilde{C}(\mathcal{K})$ in $\mathcal{K} \times (0, h_3(\mathcal{K})]$, with $0 < h_3(\mathcal{K})$ sufficiently small. Furthermore, $\psi_n(x, h)$ is $\mathcal{C}^d$ smooth in h and mixed derivatives $(\psi_n)^{(1,d)}_{x,h}(x, h)$ exist and satisfy the uniform Lipschitz estimate

$$\left\| (\psi_n)^{(1,d)}_{x,h}(x, \mu_1) - (\psi_n)^{(1,d)}_{x,h}(x, \mu_2) \right\| \leq C_3(\mathcal{K}) \, |\mu_1 - \mu_2|$$

for all $n \in \mathbb{J}_1$, $x \in \mathcal{K}$ and $0 < \mu_{1,2} \leq h_3(\mathcal{K})$ with a constant $C_3(\mathcal{K}) > 0$.

The following lemma summarizes closeness estimates between the h-flow and an one-step method with step size h. Garay proved in [51] closeness estimates for autonomous infinite time systems which are also satisfied for finite time systems. With our uniformity Assumptions **(A12)** and **(A13)** Garay's approach immediately carries over to the nonautonomous case. For the readers convenience, a sketch of the proof is presented. For more details the reader is refert to [53, Lemma 2.3.1].

Lemma 7.3.1. *Assume* **(A4)**,**(A5)**, **(A12)** *and* **(A13)**. *Then for any compact set $\mathcal{K} \subset \mathbb{R}^k$ there exist constants $\tilde{C}(\mathcal{K}), C_4(\mathcal{K}), h_4(\mathcal{K}) > 0$ such that for all $x \in \mathcal{K}$ and $0 < h \le h_4(\mathcal{K})$ with $h_4(\mathcal{K}) \le h_{1,2,3}(\mathcal{K})$ the following statements hold true:*

(i) $\sup_{n\in\mathbb{J}_1} \|(\varphi_n)_x(x,h) - (\psi_n)_x(x,h)\| \le C_4(\mathcal{K})h^{d+1}$,

(ii) $\psi_n(x,h) = x+h\Delta_n(x,h)$ *for all* $n \in \mathbb{J}_1$, *where* $\Delta_n(x,h) := \int_0^1 (\psi_n)_h(x,sh)\mathrm{d}s$ *has the same smoothness properties as ψ_n except for losing one derivative with respect to h. Further, for $r \in \{0,1,2\}$ the following estimates are true:*

$$\sup_{n\in\mathbb{J}_1} \|(\Delta_n)_x^{(r)}(x,h)\| \le \tilde{C}(\mathcal{K}), \tag{7.25}$$

$$\sup_{n\in\mathbb{J}_1} \|(\psi_n)_x(x,h)^{-1}\| \le \frac{1}{1-h\tilde{C}(\mathcal{K})}. \tag{7.26}$$

Proof. With Taylor's formula at $h = 0$ and **(A13)** we get for all $n \in \mathbb{J}_1$

$$\begin{aligned}&\varphi_n(x,h) - \psi_n(x,h)\\ &= \int_0^1 \left(\tfrac{(1-s)^{d-1}}{(d-1)!}(\varphi_n(x,sh) - \psi_n(x,sh))_h^{(d)} - (\varphi_n(x,0) - \psi_n(x,0))_h^{(d)}\right) \mathrm{d}sh^d\end{aligned}$$

for all $x \in \mathcal{K}$, $0 < h \le h_4(\mathcal{K})$. By differentiating this expression w.r.t. x and using **(A5)** and **(A12)** we get the estimate from (i). The second statement follows immediately from the mean value theorem. The estimate (7.25) is a direct consequence of **(A13)** and with the Banach-Lemma we obtain (7.26) for sufficiently small h. □

Corollary 7.3.2. *Assume* **(A4)** *and* **(A5)**. *Then for any compact set $\mathcal{K} \subset \mathbb{R}^k$ there exist constants $C_1(\mathcal{K}), h_4(\mathcal{K}) > 0$ such that for all $x \in \mathcal{K}$ and $0 < h \le h_4(\mathcal{K})$ with $h_4(\mathcal{K}) \le h_1(\mathcal{K})$*

$$\sup_{n\in\mathbb{J}_1} \left\|(\varphi_n)_x(x,h)^{-1}\right\| \le \frac{1}{1-hC_1(\mathcal{K})}$$

holds.

Now, we have all tools at hand to prove h^d-closeness between orbits of the continuous time system and orbits of the one-step discretization, see [136, Theorem 4.3] for a related result in autonomous systems.

Theorem 7.3.3. *Assume* **(A4)**,**(A5)** and **(A10)-(A13)**. *Then there exist constants $h_5, \delta > 0$ such that for all $0 < h \leq h_5$ the operator $\tilde{\Upsilon}(\cdot, h)$ has a unique zero $\tilde{y}_{\mathbb{J}}(h)$ in a δ-neighborhood of $\bar{y}_{\mathbb{J}}(h)$.*
Furthermore, $\tilde{y}_{\mathbb{J}}(h)$ is a (ft-)hyperbolic bounded trajectory of (7.24) *that satisfies*

$$\sup_{n\in\mathbb{J}} \|\tilde{y}_n(h) - \bar{y}_n(h)\| = \mathcal{O}(h^d). \tag{7.27}$$

Proof. Let $\mathcal{K} \subset \mathbb{R}^k$ be compact and sufficiently large. We prove the statements from above by applying Lemma A.0.1 with the settings: $F = \tilde{\Upsilon}$, $Y = S_{\mathbb{J}}$, $\Lambda = \mathbb{R}^+$, $Z = S_{\mathbb{J}}$ and $\bar{v}_0(h) = \bar{y}_{\mathbb{J}}(h)$, $\delta_1 = \delta$ and $\delta_2 = h_5$.

We verify the assumptions of Lemma A.0.1 and first prove that $\tilde{\Upsilon}(\cdot, h)$ for $0 < h \leq h_5$ is invertible with uniformly bounded inverse. Since the boundary conditions of $\tilde{\Upsilon}_{x_{\mathbb{J}}}(x_{\mathbb{J}}, h)$ and $\Upsilon_{x_{\mathbb{J}}}(x_{\mathbb{J}}, h)$ are the same we get by Lemma 7.3.1 the closeness estimate

$$\begin{aligned} \left\|\tilde{\Upsilon}_{x_{\mathbb{J}}}(x_{\mathbb{J}}, h) - \Upsilon_{x_{\mathbb{J}}}(x_{\mathbb{J}}, h)\right\| &= \sup_{n\in\mathbb{J}_1} \|(\varphi_n)_x(x_n, h) - (\psi_n)_x(x_n, h)\| \\ &\leq C_4(\mathcal{K}) h^{d+1} \end{aligned} \tag{7.28}$$

for all $0 < h \leq h_4(\mathcal{K})$ and $x_{\mathbb{J}} \in \mathcal{K}^{\mathbb{J}}$. Lemma 7.2.3 (a) yields an exponential dichotomy of the generated dynamical system (7.17). The exponential rate is $h\bar{\beta}$ the invariant family of projector is $Q_n^{s,u}(h) := \bar{Q}^{s,u}(hn)$ and for $\mathbb{J} = \mathbb{Z}$ we have a constant $\bar{K}$. Denote the corresponding solution operator by $\Phi(n, m)$, $m, n \in \mathbb{J}_1$. Further, by Theorem 7.2.1 for $\mathbb{I} = \mathbb{Z}$ and by Theorem 7.2.2 for $\mathbb{I} = [n_-, n_+]_{\mathbb{Z}}$ we see that $\Upsilon_{x_{\mathbb{J}}}(\bar{y}_{\mathbb{J}}(h), h)$ is a homeomorphism for all $0 < h \leq h_4(\mathcal{K})$. Then the Banach-Lemma and the estimate (7.28) guarantee the existence of a possible smaller bound $0 < h_5 \leq h_4(\mathcal{K})$ such that $\tilde{\Upsilon}_{x_{\mathbb{J}}}(\bar{y}_{\mathbb{J}}(h), h)$ is a homeomorphism for all $0 < h \leq h_5$. Consequently, we obtain for any $\tilde{r}_{\mathbb{J}}$, $r_{\mathbb{J}} \in S_{\mathbb{J}}$ unique solutions $\tilde{u}_{\mathbb{J}}$, $u_{\mathbb{J}} \in S_{\mathbb{J}}$ of the inhomogeneous equations

$$\tilde{\Upsilon}_{x_{\mathbb{J}}}(\bar{y}_{\mathbb{J}}(h), h)\tilde{u}_{\mathbb{J}} = \tilde{r}_{\mathbb{J}}, \quad \Upsilon_{x_{\mathbb{J}}}(\bar{y}_{\mathbb{J}}(h), h)u_{\mathbb{J}} = r_{\mathbb{J}}. \tag{7.29}$$

Assume $\mathbb{J} = \mathbb{Z}$. By [105, Lemma 2.7] we obtain

$$\|u_{\mathbb{Z}}\| \leq \bar{K}\frac{1 + e^{-h\bar{\beta}}}{1 - e^{-h\bar{\beta}}} \|r_{\mathbb{Z}}\| = \bar{K}\frac{1 + e^{-h\bar{\beta}}}{1 - e^{-h\bar{\beta}}} \|\Upsilon_{x_{\mathbb{Z}}}(\bar{y}_{\mathbb{Z}}(h), h)u_{\mathbb{Z}}\|. \tag{7.30}$$

Assume $\mathbb{J} = [n_-, n_+]_{\mathbb{Z}}$. Then we have for the norm of the Green's function

$$\|G(n, m)\| \leq e^{-h\bar{\beta}|n-m|}$$

for all $n, m \in \mathbb{J}$. For a proof of the estimate see [105]. Let $n \in \mathbb{J}$, $n \neq n_-$ be fixed then we obtain

$$\begin{aligned}
\left\| \sum_{m\in\mathbb{J}_1} G(n,m+1) \right\| &\leq \sum_{m\in\mathbb{J},m\neq n_-} \|G(n,m)\| \leq \sum_{m\in\mathbb{J},m\neq n_-} e^{-h\bar{\beta}|n-m|} \\
&= \sum_{n_-<m\leq n} e^{-h\bar{\beta}|n-m|} + \sum_{n<m\leq n_+} e^{-h\bar{\beta}|n-m|} \\
&= \sum_{i=0}^{n-n_-1} e^{-h\bar{\beta}i} + \sum_{i=1}^{n_+-n} e^{-h\bar{\beta}i} \\
&= \frac{1-(e^{-h\bar{\beta}})^{n-n_-}}{1-e^{-h\bar{\beta}}} + \frac{e^{-h\bar{\beta}}(1-(e^{-h\bar{\beta}})^{n_+-n})}{1-e^{-h\bar{\beta}}} \\
&\leq \frac{1+e^{-h\bar{\beta}}}{1-e^{-h\bar{\beta}}} =: C_G(h)
\end{aligned}$$

and for $n = n_-$ we have

$$\begin{aligned}
\left\| \sum_{m\in\mathbb{J}_1} G(n,m+1) \right\| &\leq \sum_{m\in\mathbb{J},m\neq n_-} e^{-h\bar{\beta}|n-m|} = \sum_{n_-<m\leq n_+} e^{-h\bar{\beta}|n_-m|} = \sum_{i=1}^{n_+-n_-} e^{-h\bar{\beta}i} \\
&= \frac{e^{-h\bar{\beta}}(1-(e^{-h\bar{\beta}})^{n_+-n_-})}{1-e^{-h\bar{\beta}}} \leq \frac{1+e^{-h\bar{\beta}}}{1-e^{-h\bar{\beta}}} = C_G(h).
\end{aligned}$$

Since $u_\mathbb{J}$ is a solution of the second inhomogeneous equation of (7.29) there exist $v_- \in \mathcal{R}(\bar{Q}^s_{n_-}(h))$, $v_+ \in \mathcal{R}(\bar{Q}^u_{n_+}(h))$ such that

$$u_n = \Phi(n,n_-)v_- + \Phi(n,n_+)v_+ + \sum_{m\in\mathbb{J}_1} G(n,m+1)r_m$$

holds for all $n \in \mathbb{J}$. Studying the boundary condition of $\Upsilon_{x_\mathbb{J}}(\bar{y}_\mathbb{J}(h),h)$ and denoting $b := b(\bar{y}_{n_-}, \bar{y}_{n_+})$ we have just as in the proof of Theorem 7.2.1

$$B(\bar{y}_\mathbb{J}) \begin{pmatrix} v_- \\ v_+ \end{pmatrix} = r_{n_+} - h \left(D_1 b \sum_{m\in\mathbb{J}_1} G(n_-,m+1)r_m + D_2 b \sum_{m\in\mathbb{J}_1} G(n_+,m+1)r_m \right).$$

First define some constants

$$C_b := \max\left\{ \|D_1 b\|, \|D_2 b\|, 1 \right\}, \tag{7.31}$$
$$C_{B^{-1}} := \max\left\{ \left\|B^{-1}(\bar{y}_\mathbb{J})\right\|, 1 \right\}. \tag{7.32}$$

Then we get by **(A11)**,$C_G(h) = \frac{1+e^{-h\bar\beta}}{1-e^{-h\bar\beta}} \geq 1$ and $h \leq 1$

$$\begin{aligned}
&\left\|\begin{pmatrix} v_- \\ v_+ \end{pmatrix}\right\| \\
\leq &\left\|B^{-1}(\bar y_{\mathbb{J}})\right\| \left\| r_{n_+} - h\left(D_1 b \sum_{m\in\mathbb{J}_1} G(n_-, m+1) r_m + D_2 b \sum_{m\in\mathbb{J}_1} G(n_+, m+1) r_m\right)\right\| \\
\leq &C_{B^{-1}}\left(\|r_{\mathbb{J}}\| + hC_b\left(\left\|\sum_{m\in\mathbb{J}_1} G(n_-, m+1)\right\| + \left\|\sum_{m\in\mathbb{J}_1} G(n_+, m+1)\right\|\right)\|r_{\mathbb{J}}\|\right) \\
\leq &C_{B^{-1}}\left(1 + 2hC_bC_G(h)\right)\|r_{\mathbb{J}}\| \\
\leq &3C_{B^{-1}}C_bC_G(h)\|r_{\mathbb{J}}\|.
\end{aligned} \tag{7.33}$$

For the solution $u_{\mathbb{J}}$ we obtain by the hyperbolicity and (7.33), (7.31) and (7.32)

$$\begin{aligned}
\|u_n\| &\leq \|\Phi(n, n_-)v_-\| + \|\Phi(n, n_+)v_+\| + \left\|\sum_{m\in\mathbb{J}_1} G(n, m+1)\right\| \|r_{\mathbb{J}}\| \\
&\leq \|v_-\| + \|v_+\| + C_G(h)\|r_{\mathbb{J}}\| \\
&\leq 2\left\|\begin{pmatrix} v_- \\ v_+ \end{pmatrix}\right\| + C_G(h)\|r_{\mathbb{J}}\| \\
&\leq 6C_{B^{-1}}C_bC_G(h)\|r_{\mathbb{J}}\| + C_G(h)\|r_{\mathbb{J}}\| \\
&\leq 7C_{B^{-1}}C_bC_G(h)\|r_{\mathbb{J}}\| \\
&\leq \bar K C_G(h)\left\|\Upsilon_{x_{\mathbb{J}}}(\bar y_{\mathbb{J}}(h), h)u_{\mathbb{J}}\right\|
\end{aligned} \tag{7.34}$$

for a constant $\bar K = 7C_{B^{-1}}C_b$ and all $n \in \mathbb{J}$.

For $\mathbb{J} = \mathbb{Z}$ and $\mathbb{J} = [n_-, n_+]_{\mathbb{Z}}$ we get by an elementary estimate for the exponential (see [53, Lemma 1.3.1.]) and estimate (7.30), (7.34)

$$\|u_{\mathbb{J}}\| \leq \bar K \frac{1+e^{-h\bar\beta}}{1-e^{-h\bar\beta}}\left\|\Upsilon_{x_{\mathbb{J}}}(\bar y_{\mathbb{J}}(h), h)u_{\mathbb{J}}\right\| \leq \frac{1}{\nu h}\left\|\Upsilon_{x_{\mathbb{J}}}(\bar y_{\mathbb{J}}(h), h)u_{\mathbb{J}}\right\| \tag{7.35}$$

with some constant $\nu > 0$ that does neither depend on h nor on $r_{\mathbb{J}}$.

Combining (7.28) with (7.35) for $r_{\mathbb{J}} := \Upsilon_{x_{\mathbb{J}}}(\bar y_{\mathbb{J}}(h), h)\tilde u_{\mathbb{J}}$, the estimate

$$\begin{aligned}
\|\tilde r_{\mathbb{J}}\| &= \left\|\tilde\Upsilon_{x_{\mathbb{J}}}(\bar y_{\mathbb{J}}(h), h)\tilde u_{\mathbb{J}}\right\| \\
&\geq \left\|\Upsilon_{x_{\mathbb{J}}}(\bar y_{\mathbb{J}}(h), h)\tilde u_{\mathbb{J}}\right\| - \left\|(\Upsilon_{x_{\mathbb{J}}}(\bar y_{\mathbb{J}}(h), h) - \tilde\Upsilon_{x_{\mathbb{J}}}(\bar y_{\mathbb{J}}(h), h))\tilde u_{\mathbb{J}}\right\| \\
&\geq \nu h\|\tilde u_{\mathbb{J}}\| - C_4(\mathcal{K})h^{d+1}\|\tilde u_{\mathbb{J}}\| \geq \frac{1}{2}\nu h\left\|\tilde\Upsilon_{x_{\mathbb{J}}}(\bar y_{\mathbb{J}}(h), h)^{-1}\tilde r_{\mathbb{J}}\right\|
\end{aligned} \tag{7.36}$$

holds for all $0 < h \leq h_5$ with a possibly smaller h_5. Since (7.36) holds true for all $\tilde r_{\mathbb{J}} \in S_{\mathbb{J}}$ we conclude

$$\left\|\tilde\Upsilon_{x_{\mathbb{J}}}(\bar y_{\mathbb{J}}(h), h)^{-1}\right\|^{-1} \geq \frac{\nu h}{2}. \tag{7.37}$$

Next, we verify Assumption (A.1) of Lemma A.0.1. Therefore define

$$\sigma(h) := \frac{\nu h}{2} \quad \text{and} \quad \kappa(h) := \frac{\sigma(h)}{2}. \tag{7.38}$$

Lemma 7.3.1, **(A13)** and the mean value theorem yield with $\bar{C(\mathcal{K})} := \sup\left\{C(\mathcal{K}), \tilde{C}(\mathcal{K})\right\}$

$$\begin{aligned}
&\left\|\tilde{\Upsilon}_{x_{\mathbb{J}}}(x_{\mathbb{J}}, h) - \tilde{\Upsilon}_{x_{\mathbb{J}}}(\bar{y}_{\mathbb{J}}(h), h)\right\| && (7.39)\\
\leq &\sup\left\{h \left\|((\Delta_n)_x(\bar{y}_n(h), h) - (\Delta_n)_x(x_n, h))_{n\in\mathbb{J}_1}\right\|, h\left\|b_x(x_{n_-}, x_{n_+}) - b_x(\bar{y}_{n_-}(h), \bar{y}_{n_+}(h))\right\|\right\},\\
\leq &\sup\left\{h\int_0^1 \left\|\left((\Delta_n)_x^{(2)}(\bar{y}_n(h) + s(x_n - \bar{y}_n(h)), h)\right)_{n\in\mathbb{J}_1}\right\| \mathrm{d}s \left\|\bar{y}_{\mathbb{J}}(h) - x_{\mathbb{J}}\right\|, hC(\mathcal{K})\left\|x_{\mathbb{J}} - \bar{y}_{\mathbb{J}}(h)\right\|\right\}\\
\leq &\bar{C(\mathcal{K})}h\left\|\bar{y}_{\mathbb{J}}(h) - x_{\mathbb{J}}\right\| && (7.40)
\end{aligned}$$

for all $x_{\mathbb{J}} \in \mathcal{K}^{\mathbb{J}}$ and $0 < h \leq h_4(\mathcal{K})$. Note that for sufficiently small δ the estimate $\bar{C}(\mathcal{K})h\delta \leq \kappa(h)$ holds. By (7.40)(7.37) and (7.38) Assumption (A.1) is confirmed for all $\|x_{\mathbb{J}} - \bar{y}_{\mathbb{J}}(h)\| \leq \delta$:

$$\begin{aligned}
\left\|\tilde{\Upsilon}_{x_{\mathbb{J}}}(x_{\mathbb{J}}, h) - \tilde{\Upsilon}_{x_{\mathbb{J}}}(\bar{y}_{\mathbb{J}}(h), h)\right\| &\leq \bar{C}(\mathcal{K})h\delta \leq \kappa(h) < \sigma(h) = \frac{\nu h}{2}\\
&\leq \left\|\tilde{\Upsilon}_{x_{\mathbb{J}}}(\bar{y}_{\mathbb{J}}(h), h)^{-1}\right\|^{-1}.
\end{aligned}$$

Assumption (A.2) of Lemma A.0.1 immediately follows from **(A12)** for sufficiently small h_5, $0 < h \leq h_5$:

$$\begin{aligned}
\left\|\tilde{\Upsilon}(\bar{y}_{\mathbb{J}}(h), h)\right\| &= \left\|\tilde{\Upsilon}(\bar{y}_{\mathbb{J}}(h), h) - \Upsilon(\bar{y}_{\mathbb{J}}(h), h)\right\|\\
&= \left\|(\varphi_n(\bar{y}_n(h), h) - \psi_n(\bar{y}_n(h), h))_{n\in\mathbb{J}_1}\right\|\\
&\leq C_2(\mathcal{K})h^{d+1} \leq \frac{\nu h}{4}\delta \leq \frac{\sigma(h)}{2}\delta = (\sigma(h) - \kappa(h))\delta.
\end{aligned}$$

Thus, Lemma A.0.1 applies and guarantees the existence of a unique zero $\tilde{y}_{\mathbb{J}}(h)$ of $\tilde{\Upsilon}(\cdot, h)$ in a δ-neighborhood of $\bar{y}_{\mathbb{J}}(h)$, satisfying the inequality (7.27)

$$\begin{aligned}
\left\|\tilde{y}_{\mathbb{J}}(h) - \bar{y}_{\mathbb{J}}(h)\right\| &\leq (\sigma(h) - \kappa(h))^{-1}\left\|\tilde{\Upsilon}(\bar{y}_{\mathbb{J}}(h), h)\right\|\\
&\leq \tfrac{4}{\nu h}C_2(\mathcal{K})h^{d+1} = \frac{4C_2(\mathcal{K})}{\nu}h^d
\end{aligned} \tag{7.41}$$

for all $0 < h \leq h_5$.

Next, we prove hyperbolicity of this solution.

In order to show that the variational equation, given in terms of the operator $\tilde{\Upsilon}_{x_{\mathbb{J}}}(\tilde{y}_{\mathbb{J}}(h), h)$, has an exponential dichotomy on $\mathbb{J}$, we apply the Roughness-Theorem 3.4.2/3.4.11 with the settings: $A(n) := A_n =: (\varphi_n)_x(\bar{y}_n(h), h)$ and $\tilde{A}(n) := \tilde{A}_n := (\psi_n)_x(\tilde{y}_n(h), h)$ and $E_n := \tilde{A}(n) - A(n) = (\psi_n)_x(\tilde{y}_n(h), h) - (\varphi_n)_x(\bar{y}_n(h), h)$, $n \in \mathbb{J}_1$.

With (7.40) and (7.41) it follows

$$\begin{aligned}\left\|\tilde{\Upsilon}_{x_{\mathbb{J}}}(\tilde{y}_{\mathbb{J}}(h),h)-\tilde{\Upsilon}_{x_{\mathbb{J}}}(\bar{y}_{\mathbb{J}}(h),h)\right\| \leq&\bar{C}(\mathcal{K})h\left\|\tilde{y}_{\mathbb{J}}(h)-\bar{y}_{\mathbb{J}}(h)\right\|\\ \leq&\bar{C}(\mathcal{K})h\frac{4C_3(\mathcal{K})}{\nu}h^d=\hat{C}(\mathcal{K})h^{d+1}\end{aligned}$$

and combining this result with (7.28) we obtain

$$\begin{aligned}\|E_{\mathbb{J}_1}\| &= \left\|\tilde{A}_{\mathbb{J}_1}-A_{\mathbb{J}_1}\right\|\\ &= \sup_{n\in\mathbb{J}_1}\|(\psi_n)_x(\tilde{y}_n(h),h)-(\varphi_n)_x(\bar{y}_n(h),h)\|\\ &= \|\tilde{\Upsilon}_{x_{\mathbb{J}}}(\tilde{y}_{\mathbb{J}}(h),h)-\Upsilon_{x_{\mathbb{J}}}(\bar{y}_{\mathbb{J}}(h),h)\|\\ &\leq \|\tilde{\Upsilon}_{x_{\mathbb{J}}}(\tilde{y}_{\mathbb{J}}(h),h)-\tilde{\Upsilon}_{x_{\mathbb{J}}}(\bar{y}_{\mathbb{J}}(h),h)\|+\|\tilde{\Upsilon}_{x_{\mathbb{J}}}(\bar{y}_{\mathbb{J}}(h),h)-\Upsilon_{x_{\mathbb{J}}}(\bar{y}_{\mathbb{J}}(h),h)\|\\ &\leq (\hat{C}(\mathcal{K})+C_4(\mathcal{K}))h^{d+1}. \end{aligned}\tag{7.42}$$

By Corollary 7.3.2 we observe for h sufficiently small

$$\left\|A_{\mathbb{J}_1}^{-1}\right\|=\sup_{n\in\mathbb{J}_1}\|(\varphi_n)_x(\bar{y}_n(h),h)^{-1}\|\leq\frac{1}{1-hC_1(\mathcal{K})}\tag{7.43}$$

and together with (7.42) this yields the estimate

$$\frac{1}{2}\inf_{n\in\mathbb{J}_1}\|A_n^{-1}\|^{-1}\geq\frac{1}{2}(1-hC_1(\mathcal{K}))\geq(\hat{C}(\mathcal{K})+C_4(\mathcal{K}))h^{d+1}\geq\|E_{\mathbb{J}_1}\|\tag{7.44}$$

for h sufficiently small. This is the first Assumption (3.38) of the Roughness-Theorem 3.4.2. For verifying the second Assumption (3.39) and the condition of the Roughness-Theorem 3.4.11, note that (7.17) has an exponential dichotomy on $\mathbb{J}$ with rate $h\bar{\beta}$ and for $\mathbb{J}=\mathbb{Z}$ with constant $\bar{K}$. By Taylor expanding we observe

$$\left(\frac{1}{e^{\frac{h\bar{\beta}}{2}}-e^{-h\bar{\beta}}}+\frac{1}{e^{-\frac{h\bar{\beta}}{2}}-e^{-h\bar{\beta}}}+\frac{1}{e^{h\bar{\beta}}-e^{-\frac{h\bar{\beta}}{2}}}\right)^{-1}=\frac{3}{10}\bar{\beta}h+\mathcal{O}(h^2),\tag{7.45}$$

$$\left(\frac{1-e^{-h\bar{\beta}}}{2e^{h\bar{\beta}}-e^{-\bar{\beta}}}\right)=\bar{\beta}h+\mathcal{O}(h^2),\tag{7.46}$$

$$\left(\frac{e^{h\bar{\beta}-1}}{e^{h\bar{\beta}}}\right)=\bar{\beta}h+\mathcal{O}(h^2).\tag{7.47}$$

As a consequence

$$\begin{aligned}\|E_{\mathbb{J}_1}\| \leq&(\hat{C}(\mathcal{K})+C_4(\mathcal{K}))h^{d+1}\\ \leq&\frac{1}{2}\bar{K}^{-1}\left(\frac{1}{e^{\frac{h\bar{\beta}}{2}}-e^{-h\bar{\beta}}}+\frac{1}{e^{-\frac{h\bar{\beta}}{2}}-e^{-h\bar{\beta}}}+\frac{1}{e^{h\bar{\beta}}-e^{-\frac{h\bar{\beta}}{2}}}\right)^{-1}\end{aligned}\tag{7.48}$$

holds true for h sufficiently small, i.e. the second assumption (3.39) is satisfied. Minimize $h_5 > 0$ such that (7.43), (7.44) and (7.48) are satisfied for all $0 < h \le h_5$.

Next we prove the condition of the Roughness-Theorem 3.4.11. In the following let $C_0 > 0$ be a generic constant which will increase if necessary. Assumption **(A5)** yields

$$\|A_{\mathbb{J}_1}\| \le C_1(\mathcal{K})$$

for $0 < h \le h_1(\mathcal{K})$ ($h_5 \le h_4(\mathcal{K}) \le h_1(\mathcal{K})$). Together with (7.43) we obtain for the constants C, $\bar{C}$ and C_d from the Roughness-Theorem 3.4.11

$$\begin{aligned} C &= 2\left(\left\|A_{\mathbb{J}_1}^{-1}\right\| + \|A_{\mathbb{J}_1}\|\left\|A_{\mathbb{J}_1}^{-1}\right\|\right)^{(n_+-n_-)} \\ &\le 2\left(\frac{1}{1-hC_1(\mathcal{K})} + \frac{C_1(\mathcal{K})}{1-hC_1(\mathcal{K})}\right)^{(n_+-n_-)} \\ &\le \frac{C_0}{(1-hC_1(\mathcal{K}))^{(n_+-n_-)}} \end{aligned}$$

and

$$\begin{aligned} \bar{C} &= (1 + 2\|A_{\mathbb{J}_1}\|) + \left(\|A_{\mathbb{J}_1}\|^2 + 1\right) C \\ &\le (1 + 2C_1(\mathcal{K})) + \left(C_1(\mathcal{K})^2 + 1\right)\frac{C_0}{(1-hC_1(\mathcal{K}))^{(n_+-n_-)}} \\ &\le C_0\left(1 + \frac{1}{(1-hC_1(\mathcal{K}))^{(n_+-n_-)}}\right) = C_0\frac{(1-hC_1(\mathcal{K}))^{(n_+-n_-)} + 1}{(1-hC_1(\mathcal{K}))^{(n_+-n_-)}} \\ &\le C_0\frac{2}{(1-hC_1(\mathcal{K}))^{(n_+-n_-)}} \le \frac{C_0}{(1-hC_1(\mathcal{K}))^{(n_+-n_-)}} \end{aligned}$$

and

$$C_d = \max\{C, \bar{C}\} \le \frac{C_0}{(1-hC_1(\mathcal{K}))^{(n_+-n_-)}}$$

for all $0 < h \le h_5$. This implies

$$(4C_d)^{-1} \ge \frac{(1-hC_1(\mathcal{K}))^{(n_+-n_-)}}{4C_0} > \frac{1}{8C_0} \tag{7.49}$$

for h sufficiently small. Combined with (7.42), (7.46) and with (7.47)

$$\begin{aligned} \left\|\tilde{A}_{\mathbb{J}_1} - A_{\mathbb{J}_1}\right\| &\le (\hat{C}(\mathcal{K}) + C_4(\mathcal{K}))h^{d+1} \\ &< \min\left\{1, (4C_d)^{-1}\frac{1-e^{-h\bar{\beta}}}{2e^{h\bar{\beta}} - e^{-h\bar{\beta}}}, (4C_d)^{-1}\frac{e^{h\bar{\beta}}-1}{e^{h\bar{\beta}}}\right\} \end{aligned} \tag{7.50}$$

follows for h sufficiently small. Thus, the condition of the Roughness-Theorem 3.4.11 is satisfied. Minimize $h_5 > 0$ such that (7.49) and (7.50) are satisfied for all $0 < h \leq h_5$. Theorem 3.4.2/3.4.11 applies and guarantees that the variational equation, given in terms of $\tilde{\Upsilon}_{x_{\mathbb{J}}}(\tilde{y}_{\mathbb{J}}(h), h)$, has an exponential dichotomy on $\mathbb{J}$ with rate $\frac{h\bar{\beta}}{2}$ and for $\mathbb{J} = \mathbb{Z}$ with constant $2\bar{K} + 1$. Thus, $\tilde{y}_{\mathbb{J}}(h)$ is a (ft-)hyperbolic bounded trajectory of the h-step method $\psi_{\mathbb{J}_1}(\cdot, h)$ for all $0 < h \leq h_5$. □

We exploit this result for analyzing discretized homoclinic orbits. Note that the application of a one-step method turns the equilibrium 0 into a bounded trajectory.

Corollary 7.3.4. *Assume* **(A5)**-**(A7)**, **(A12)**,**(A13)** *and* **(A11)**. *Choose* h_5 *and* δ *as in Theorem 7.3.3. Then there exists a* $C > 0$ *such that for all* $0 < h \leq h_5$ *a unique (ft-)hyperbolic bounded trajectory* $\tilde{\xi}_{\mathbb{J}}(h)$ *of* (7.24) *in a* δ*-neighborhood of the equilibrium* 0 *of* (2.2) *exists which satisfies*

$$\sup_{n \in \mathbb{J}} \left\| \tilde{\xi}_n(h) - 0 \right\| \leq C h^d. \tag{7.51}$$

The variational equation, belonging to $\tilde{\Upsilon}_{x_{\mathbb{Z}}}(\tilde{\xi}_{\mathbb{Z}}(h), h)$, *has an (ft-)exponential dichotomy on* $\mathbb{J}$ *with rate* $\frac{h\beta}{2}$ *and for* $\mathbb{J} = \mathbb{Z}$ *with constant* $2K + 1$.

Further, assume **(A8)** *and* **(A9)**. *Then there exists a unique (ft-)hyperbolic bounded trajectory* $\tilde{x}_{\mathbb{J}}(h)$ *of* (7.24) *in a* δ*-neighborhood of the (ft-)hyperbolic (*ε*-)homoclinic orbit* $\bar{x}_{\mathbb{J}}(h)$ *(*$\varepsilon > 0$*) of the* h*-flow which satisfies*

$$\sup_{n \in \mathbb{J}} \|\tilde{x}_n(h) - \bar{x}_n(h)\| \leq C h^d. \tag{7.52}$$

Furthermore, for $\mathbb{J} = \mathbb{Z}$ *there exists an* $N \in \mathbb{Z}^+$ *such that*

$$\sup_{n \in \mathbb{Z}^{\pm}_{\pm N}} \|\tilde{x}_n(h) - \tilde{\xi}_n(h)\| \leq 3 C h^d \tag{7.53}$$

holds with $\mathbb{Z}^+_N := [N, \infty) \cap \mathbb{Z}$, $\mathbb{Z}^-_{-N} := (-\infty, -N] \cap \mathbb{Z}$. *For* $\mathbb{J} = [n_-, n_+]_{\mathbb{Z}}$

$$\left\| \tilde{x}_{n_\pm}(h) - \tilde{\xi}_{n_\pm}(h) \right\| \leq 2 C h^d + \varepsilon \tag{7.54}$$

is satisfied.

Proof. Fix $0 < h \leq h$. It remains to prove the estimate (7.53) for sufficiently large $|n|$, $n \in \mathbb{Z}$ and (7.54). First consider $\mathbb{J} = \mathbb{Z}$. With (7.51), (7.52) and **(A8)**, see Definition 7.0.1, there exist a $N := N(h) \in \mathbb{Z}^+$ such that

$$\begin{aligned} &\sup_{n \in \mathbb{Z}^{\pm}_{\pm N}} \|\tilde{x}_n(h) - \tilde{\xi}_n(h)\| \\ &\leq \sup_{n \in \mathbb{Z}} \|\tilde{x}_n(h) - \bar{x}_n(h)\| + \sup_{n \in \mathbb{Z}^{\pm}_{\pm N}} \|\bar{x}_n(h) - 0\| + \sup_{n \in \mathbb{Z}} \|0 - \tilde{\xi}_n(h)\| \leq 3 C h^d. \end{aligned}$$

For $\mathbb{J} = [n_-, n_+]_{\mathbb{Z}}$ we obtain by the latter arguments

$$\begin{aligned}\left\|\tilde{x}_{n_\pm}(h) - \tilde{\xi}_{n_\pm}(h)\right\| &\leq \sup_{n\in\mathbb{J}} \|\tilde{x}_n(h) - x_n(h)\| + \|x_{n_\pm}(h) - 0\| + \sup_{n\in\mathbb{J}} \left\|0 - \tilde{\xi}_n(h)\right\| \\ &\leq 2Ch^d + \varepsilon.\end{aligned}$$

□

For $\mathbb{J} = [n_-, n_+]_{\mathbb{Z}}$ the estimate (7.54) implies that $(\tilde{x}_{\mathbb{J}}(h), \tilde{\xi}_{\mathbb{J}}(h))$ forms a $(2Ch^d + \varepsilon)$-homoclinic pair. For the infinite time cases $\mathbb{J} = \mathbb{Z}$ we know, from the previous results, that the tails of the discretized hyperbolic bounded trajectories $\tilde{x}_{\mathbb{Z}}(h)$ and $\tilde{\xi}_{\mathbb{Z}}(h)$ of (7.24) stay in a common small neighborhood. In the following we show that these trajectories are indeed homoclinic toward each other for all $0 < h \leq h_5$. This can be achieved by establishing the identity

$$\lim_{n\to\pm\infty} \left\|\tilde{x}_n(h) - \tilde{\xi}_n(h)\right\| = 0.$$

The next lemma states that if two hyperbolic bounded trajectories stay in a sufficiently small common neighborhood, then they converge towards each other. Note that a related result for the autonomous case can be found in [105, Lemma 5.3].

Lemma 7.3.5. *Assume that $f_n \in \mathcal{C}^1(\mathbb{R}^k, \mathbb{R}^k)$ for $n \in \mathbb{Z}$ and let $\xi_{\mathbb{Z}}$ be a bounded trajectory of the difference equation*

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{Z}. \tag{7.55}$$

Further, assume that $(f_n)_x$ is uniformly Lipschitz with constant L in a neighborhood of $\xi_{\mathbb{Z}}$ for all $n \in \mathbb{Z}$ and that the variational equation

$$u_{n+1} = (f_n)_x(\xi_n)u_n, \quad n \in \mathbb{Z} \tag{7.56}$$

has an exponential dichotomy on $\mathbb{Z}$ with data $(K, \alpha, P_n^{s,u})$. Fix $n_1 \in \mathbb{N}$ and let $x_{\mathbb{Z}}$ be a second bounded trajectory of (7.55), satisfying the following estimates with some constant $0 < \beta < \alpha$:

$$\|(x_n - \xi_n)_{n\in\mathbb{T}}\| \leq L^{-1} \inf_{n\in\mathbb{T}} \|(f_n)_x(\xi_n)^{-1}\|^{-1}, \tag{7.57}$$

$$\|(x_n - \xi_n)_{n\in\mathbb{T}}\| \leq L^{-1}K^{-1} \left(\frac{1}{e^\beta - e^{-\alpha}} + \frac{1}{e^{-\beta} - e^{-\alpha}} + \frac{1}{e^\alpha - e^{-\beta}}\right)^{-1}, \tag{7.58}$$

for $\mathbb{T} \in \{\mathbb{Z}^-_{n_1}, \mathbb{Z}^+_{n_1}\}$.

Then there exists a constant $\tilde{C} > 0$ such that the exponential estimate

$$\|x_n - \xi_n\| \leq \tilde{C} e^{-\beta|n-n_1|} \tag{7.59}$$

holds for all $n \in \mathbb{T}$.

Proof. First we define $z_n := x_n - \xi_n$ for $n \in \mathbb{T}$ and get with the mean value theorem

$$\begin{aligned} z_{n+1} = x_{n+1} - \xi_{n+1} &= f_n(x_n) - f_n(\xi_n) \\ &= f_n(z_n + \xi_n) - f_n(\xi_n) = \int_0^1 (f_n)_x(\xi_n + sz_n)\mathrm{d}s z_n. \end{aligned}$$

By defining $B_n = \int_0^1 (f_n)_x(\xi_n + sz_n)\mathrm{d}s$ we see that $z_{\mathbb{T}}$ is a bounded solution of

$$u_{n+1} = B_n u_n, \quad n \in \mathbb{T}. \tag{7.60}$$

To finish the proof, we show that (7.60) has an exponential dichotomy on $\mathbb{T}$ with data $(2K+1, \beta, Q_n^{s,u})$.

Assume this dichotomy is already known. Then we get $z_n = Q_n^u z_n$ for $n \in \mathbb{T}^- = \mathbb{Z}_{n_1}^-$ and $z_n = Q_n^s z_n$ for $n \in \mathbb{T}^+ = \mathbb{Z}_{n_1}^+$ since $z_{\mathbb{T}}$ is a bounded solution of (7.60). Denote by $\Phi(\cdot,\cdot)$ the corresponding solution operator, then

$$\|z_n\| = \|\Phi(n, n_1) Q_{n_1}^{s,u} z_{n_1}\| \le (2K+1)e^{-\beta|n-n_1|}\|z_{n_1}\|$$

for $n \in \mathbb{T}^\pm$ which completes the proof of (7.59).

For proving an exponential dichotomy of (7.60), we start with (7.56) that already has an exponential dichotomy and apply the Roughness-Theorem 3.4.2. To verify its assumptions, we use the estimate

$$\begin{aligned} \|B_n - (f_n)_x(\xi_n)\| &\le \int_0^1 \|(f_n)_x(\xi_n + sz_n) - (f_n)_x(\xi_n)\|\,\mathrm{d}s \\ &\le L\int_0^1 \|sz_n\|\mathrm{d}s \le \frac{1}{2}L\|z_n\| \text{ for all } n \in \mathbb{T}. \end{aligned}$$

Then Assumption (3.38) of the Roughness-Theorem 3.4.2 directly follows from (7.57) and (3.39) from (7.58). □

Our next step is to show that discretized infinite time homoclinic trajectories converge towards each other. For this task, Lemma 7.3.5 is applied to the one-step method $\psi_{\mathbb{Z}}(\cdot, h)$ and the hyperbolic bounded trajectories $\tilde{\xi}_{\mathbb{Z}}(h)$ and $\tilde{x}_{\mathbb{Z}}(h)$.

From Corollary 7.3.4 we know that the variational equation, belonging to $\tilde{\Upsilon}_{x_{\mathbb{Z}}}(\tilde{\xi}_{\mathbb{Z}}(h), h)$, has an exponential dichotomy on $\mathbb{Z}$ with constant $2K+1$ and dichotomy rate $\frac{h\beta}{2}$. Let $\mathcal{K} \subset \mathbb{R}^k$ be compact and sufficiently large such that $\tilde{\xi}_{\mathbb{Z}}(h), \tilde{x}_{\mathbb{Z}}(h) \in \mathcal{K}^{\mathbb{Z}}$. Then the Lipschitz constant of $(\psi_n)_x$ is $L := h\tilde{C}(\mathcal{K})$, see equation (7.40). The first Assumption (7.57) of Lemma 7.3.5 for $\mathbb{T}_\pm := \mathbb{Z}_{\pm N}^\pm$ follows with (7.26) and (7.53) for h sufficiently small:

$$\begin{aligned} \sup_{n\in\mathbb{T}^\pm} \|\tilde{x}_n(h) - \tilde{\xi}_n(h)\| &\le 3Ch^d \le h^{-1}\tilde{C}(\mathcal{K})^{-1}(1 - h\tilde{C}(\mathcal{K})) \\ &\le L^{-1} \inf_{n\in\mathbb{Z}} \|(\psi_n)_x(\tilde{\xi}_n(h), h)^{-1}\|^{-1}. \end{aligned}$$

The second one (7.58) is fulfilled with $\bar{\beta} := \frac{\beta}{2}$ since

$$\sup_{n\in\mathbb{T}^{\pm}} \|\tilde{x}_n(h) - \tilde{\xi}_n(h)\| \leq 3Ch^d \leq \frac{h^{-1}\tilde{C}(\mathcal{K})^{-1}}{2K+1} 3C(2K+1)\tilde{C}(\mathcal{K})h^2$$
$$\leq \frac{L^{-1}}{2K+1}\left(\frac{1}{e^{\frac{h\bar{\beta}}{2}} - e^{-h\bar{\beta}}} + \frac{1}{e^{-\frac{h\bar{\beta}}{2}} - e^{-h\bar{\beta}}} + \frac{1}{e^{h\bar{\beta}} - e^{-\frac{h\bar{\beta}}{2}}}\right)^{-1}$$

follows from (7.53) and (7.45) for h sufficiently small. Now we apply Lemma 7.3.5 and get a constant $C_5 > 0$ such that

$$\|\tilde{x}_n(h) - \tilde{\xi}_n(h)\| \leq C_5 e^{-\frac{h\beta}{4}|n-N|} \tag{7.61}$$

for all $n \in \mathbb{T}^{\pm}$.

Summarizing these results we have seen that bounded trajectories in continuous time lead to bounded trajectories in discrete time, staying close to each other. This achievement holds for infinite and finite time systems. Furthermore, if the tails of two trajectories of a nonautonomous infinite time system lie for all future (past) times in a sufficiently small neighborhood, then they converge exponentially fast towards each other in forward (backward) time. As a consequence, (ε-)homoclinic orbits induce ($2Ch^d + \varepsilon$-)homoclinic orbits of the system discretized by a one-step method which are close to themselves. We summarize this, our main result, in the following theorem.

Theorem 7.3.6. *Let $\bar{x}(\cdot)$ be a(n) (ε-)homoclinic orbit of the continuous time system* (2.2) *w.r.t. the fixed point* 0 *and assume that our Assumptions* **(A5)**-**(A9)**, **(A12)**,**(A13)** *and* **(A11)** *are satisfied. Then we find constants $\bar{h}, C > 0$ such that two (ft-)hyperbolic bounded trajectories $\tilde{\xi}_{\mathbb{J}}(h)$ and $\tilde{x}_{\mathbb{J}}(h)$ of the one-step approximation* (7.24) *exist which satisfy*

$$\sup_{n\in\mathbb{J}} \|\tilde{\xi}_n(h) - 0\| \leq Ch^d, \quad \sup_{n\in\mathbb{J}} \|\tilde{x}_n(h) - \bar{x}_n(h)\| \leq Ch^d,$$

$$\begin{cases} \lim_{n\to\pm\infty} \left\|\tilde{x}_n(h) - \tilde{\xi}_n(h)\right\| = 0, & \text{for } \mathbb{J} = \mathbb{Z}, \\ \left\|\tilde{x}_{n_\pm}(h) - \tilde{\xi}_{n_\pm}(h)\right\| < 2Ch^d + \varepsilon, & \text{for } \mathbb{J} = [n_-, n_+]_{\mathbb{Z}} \end{cases} \text{for all } 0 < h < \bar{h},$$

i.e. $(\tilde{x}_{\mathbb{J}}(h), \tilde{\xi}_{\mathbb{J}}(h))$ *forms a* (($2Ch^d + \varepsilon$)-)*homoclinic orbit pair of* (7.24).

Chapter 8

Applications

In this last chapter we study various examples, which underline our theoretical results. The infinite time statements originate from the publication [54]. First, we construct a 2-dimensional example with an explicitly known homoclinic orbit. We compare orbits of a one-step method with the exact ones and numerically verify our error estimates for various step sizes.

For illustrating transversality of the computed orbits we plot the corresponding stable and unstable fiber bundles of the one-step discretization. We apply our algorithm from Section 6.7.

Periodic forcing of an autonomous ODE leads to a special class of nonautonomous systems. We construct a model of this type and discuss the underlying autonomous dynamics and their influence on invariant fiber bundles along a homoclinic orbit.

Finally, a 2-dimensional continuous time model from mathematical biology is introduced that is nonautonomous due to time variant environmental influences. For its time discretization we compute a homoclinic orbit as well as invariant fiber bundles.

For finite time situations we already approximated and potted the ε-homoclinic tube for an 2-dimensional systems, cf. Section 7.1.

An Artificial Example with Explicitly Known Homoclinic Orbits

We start with the Hamiltonian system

$$\dot{x} = f(x) = \begin{pmatrix} x_2 \\ x_1^2 - 4 \end{pmatrix},$$

which has the homoclinic orbit

$$\hat{x}(t) = 2(1 - 3\mathrm{sech}^2(t), 6\mathrm{sech}^2(t)\tanh(t))$$

with respect to the fixed point $(2, 0)$, see [20, Section 11.2.2], [57, Section 7.3].

To construct a nonautonomous example, we first shift the fixed point to $(0,0)$. This leads us to the new system

$$\dot{x} = f(x) = \begin{pmatrix} x_2 \\ x_1^2 + 4x_1 \end{pmatrix} \tag{8.1}$$

with corresponding homoclinic orbit

$$\bar{x}(t) = 6(-\text{sech}^2(t), 2\text{sech}^2(t)\tanh(t)). \tag{8.2}$$

Next we add a nonautonomous term as follows

$$\begin{aligned} \dot{x} &= g(x,t) := f(x) + \begin{pmatrix} x_1(x_1 - \bar{x}_1(t)) \\ x_2(x_2 - \bar{x}_2(t)) \end{pmatrix} \\ &= \begin{pmatrix} x_2 + x_1^2 + 6\text{sech}^2(t)x_1 \\ x_1^2 + 4x_1 + x_2^2 - 12\text{sech}^2(t)\tanh(t)x_2 \end{pmatrix}. \end{aligned} \tag{8.3}$$

Obviously, $(0,0)$ is for all $t \in \mathbb{R}$ a fixed point; furthermore, (8.2) is still a homoclinic orbit w.r.t. $(0,0)$ of this new system.

For a one-step discretization, we choose Heun's method with step size h which has order $d = 2$ and obtain the discrete time system

$$x_{n+1} = F_n(x_n) := x_n + \frac{h}{2}\left(g(x_n, t_n) + g(x_n + hg(x_n, t_n), t_{n+1})\right), \quad n \in \mathbb{Z}. \tag{8.4}$$

Tools for the numerical approximation of homoclinic orbits in nonautonomous systems have been proposed in [70, 73]. The key idea lies in introducing boundary value problems to obtain error controlled orbit segments on a finite time interval. More precisely, we compute an orbit segment $(\tilde{x}_{n_-}, \dots, \tilde{x}_{n_+})$ by solving the periodic boundary value problem

$$\begin{pmatrix} x_{n_-+1} - F_{n_-}(x_{n_-}) \\ \vdots \\ x_{n_+} - F_{n_+-1}(x_{n_+-1}) \\ x_{n_-} - x_{n_+} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \tag{8.5}$$

using Newton's method with an appropriate initial guess. For the model (8.4), we start with the exact orbit (8.2). Note that the sparse structure of the derivative allows efficient computations. We solve (8.5) on the time-interval $[-30, 30]$ with the step size $h = 0.03$, i.e. $n_\pm = \pm 1000$. Figure 8.1 shows the orbit with time dependence (right) and without it (left).

The middle and lower diagrams in Figure 8.1 illustrate the homoclinic orbit together with transversally intersecting fibers. In the left middle panel the orbit projected to the x_1-x_2-plain and the fiber bundles at time $20h$ are presented. The right middle panel pictures the fibers at the next time instance $21h$. The lower diagram visualizes transversally intersecting fiber bundles on

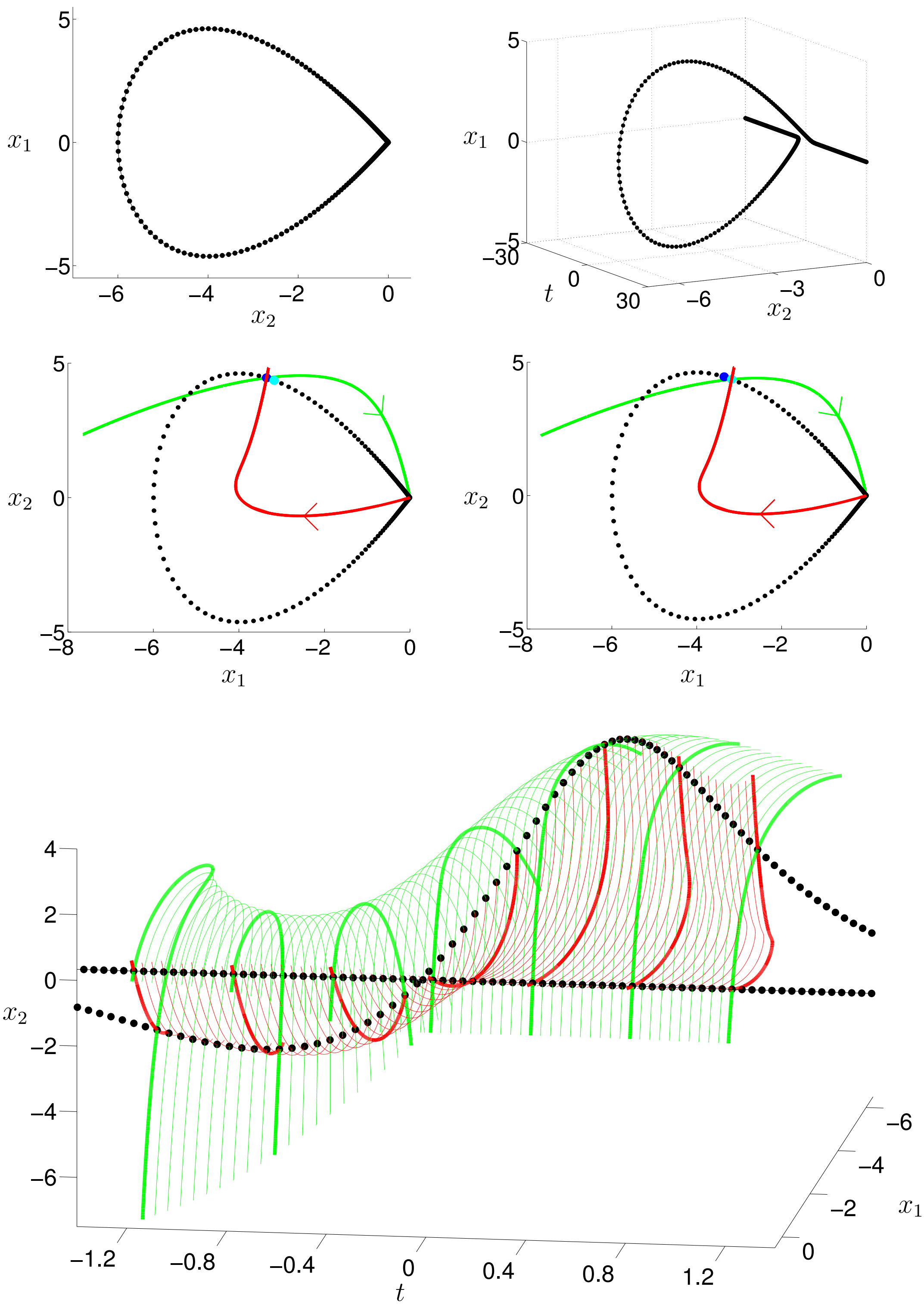


Figure 8.1: Homoclinic orbit and transversally intersecting fiber bundles of (8.3), (8.4) with $h = 0.04$.

the time interval $[-30h, 30h]$. The stable fibers (green) are computed with the algorithm from Section 6.7 while the unstable fibers (red) are approximated by forward iteration.

We conclude that this orbit is truly nonautonomous, since two fibers at different times do not coincide.

Theorem 7.3.6 states that the maximal error $e_{\mathbb{Z}}(h) := \max_{n\in\mathbb{Z}} \|\tilde{x}_n - \bar{x}(hn)\|$ that occurs by approximating the original orbit using an h-step method of order d is less than Ch^d, with some constant $C > 0$. Furthermore, the computation of finite orbit segments by solving (8.5) introduces a second error. A precise analysis of this second error, cf. [73, Theorem 5], reveals that its maxima occur at the boundary points of the finite interval whereas this error decreases exponentially fast towards the midpoint. Thus, we avoid secondary errors by computing a solution of (8.5) on the time-interval $[-30, 30]$ and determine the maximal error $e(h) := \max_{n\in[-\frac{5}{h},\frac{5}{h}]\cap\mathbb{Z}} \|\tilde{x}_n - \bar{x}(hn)\|$ only on the center $[-5, 5]$. Figure 8.2 illustrates the numerical output of this procedure for various step sizes from 0.00005 up to 0.04 in a double logarithmic scale. The slope of the graph represents the exponent d. In Figure 8.2 it is 1.9930 in accordance with Theorem 7.3.6.

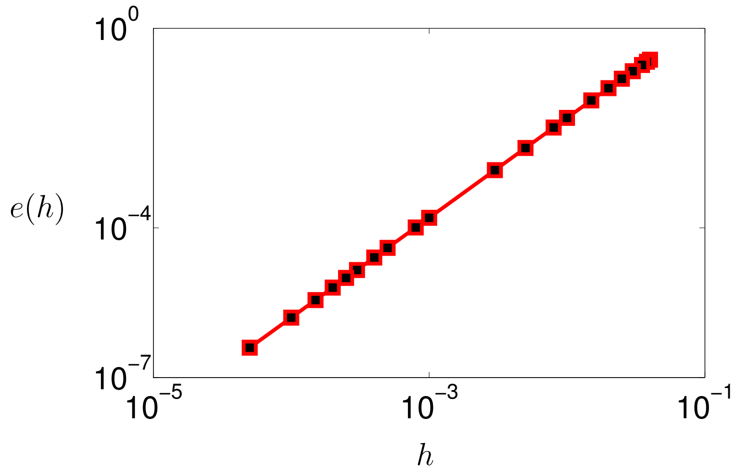


Figure 8.2: Maximal error between exact and numerically approximated orbits.

A Periodic Nonautonomous ODE

In τ-periodic ODEs, stable (and unstable) fibers of a fixed point at times t and $t+\tau$, $t \in \mathbb{R}$ coincide. For an illustration we modify (8.1) to the $\tau = \pi$-periodic model

$$\dot{x} = f(x,t) = \begin{pmatrix} -(1 + 0.3\sin(2t))x_2 \\ x_1^2 + x_1 \end{pmatrix}. \tag{8.6}$$

Discretizing this system with Heun's method and step size $h = \frac{\pi}{30}$ leads to a 30-periodic difference equation of the form

$$x_{n+1} = g_n(x_n), \quad g_n = g_{n+30}, \quad n \in \mathbb{Z}. \tag{8.7}$$

This discrete time system exhibits a homoclinic orbit w.r.t. the fixed point $(-1, 0)$, see Figure 8.3 for an illustration.

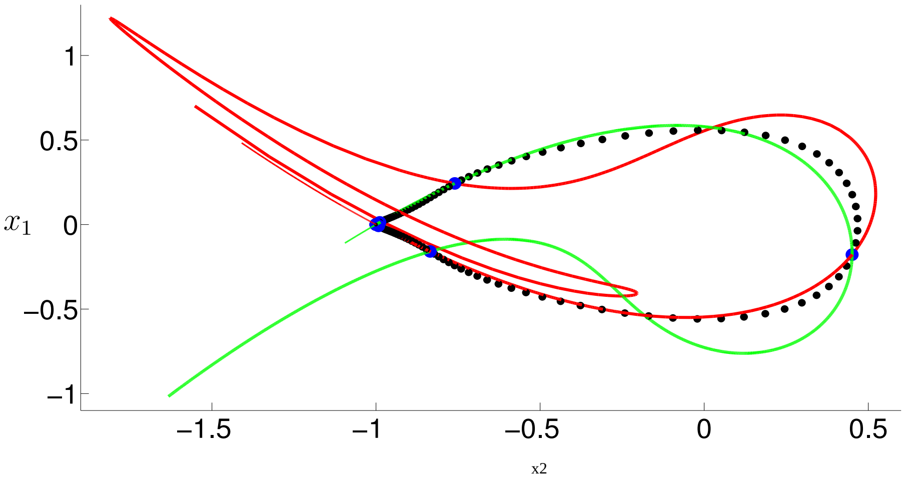


Figure 8.3: Homoclinic orbit segment on the time interval $[-200h, 200h] \cap h\mathbb{Z}$ of (8.6) and the fibers at time $(20 + 30n)h$, $n \in [-7, 6] \cap \mathbb{Z}$, $h = \frac{\pi}{30}$.

We further observe that stable and unstable fibers intersect transversally at every 30th point along the orbit. Stable and unstable fibers at time $20h$ are depicted in Figure 8.3. For their computation, we apply the algorithm from Section 6.7.

Note that alternatively, autonomous tool for computing homoclinic orbits from [22] as well as the search circle algorithm, introduced in [48], are directly applicable to $G_n := g_{n+29} \circ \cdots \circ g_n$, $n \in \mathbb{Z}$ fixed. We do not follow this route, since this problem typically has a worse condition number than the nonautonomous equation (8.7).

An Example from Mathematical Biology

Let us apply our techniques to a more realistic model from mathematical biology. The dynamics of the growth of algea and zooplankton, typically Daphnia, is presented in [116] with the help of a periodically forced predator-prey system. The authors introduce a 2-dimensional ODE

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= 0.5A\left(1 - \frac{A}{10}\right) - 0.4Z\left(\frac{A}{A+0.6}\right) + 0.01(10 - A), \\ \frac{\mathrm{d}Z}{\mathrm{d}t} &= 0.24Z\frac{A}{A+0.6} - 0.15Z - E\frac{Z^2}{Z^2+0.5^2}, \end{aligned} \tag{8.8}$$

where (A) describes the amount of edible algea and (Z) the amount of large herbivorous zooplankton. The growth of zooplankton is influenced by the fish population and some other environmental terms (E). Our first step is to search for homoclinic structures in the autonomous system (8.8), see [116, Figure 6b], which can be found for

$$E := 0.0784372294995495865\ldots$$

Next we add a time-dependent perturbation to E reflecting time dependent environmental influences. Choosing

$$E(t) := 0.0784372294995495865 + \exp(-0.2t^2)$$

(8.8) is a continuous time nonautonomous 2-dimensional ODE of the form $\dot{x} = g(x,t)$ with $x := \begin{pmatrix} A & Z \end{pmatrix}^T$. We start with an analysis of the underlying dynamics by searching for homoclinic structures. To study this we are looking at the discretized system. For the one-step discretization $x_{n+1} = F_n(x_n)$ of the system we take Heun's method (8.4). First we compute a bounded trajectory $\hat{x}_{[n_-,n_+]}$ of (8.4) replacing the fixed point from the autonomous case. For this task, we solve, as in Section 8.1, the periodic boundary value problem (8.5) on the time-interval $[-1750, 1750]$ with step size $h = 0.5$, i.e. $n_\pm = \pm 3500$. Using this bounded solution $\hat{x}_{[n_-,n_+]}$ the transformed system

$$y_{n+1} = G_n(y_n),\ G_n(y_n) := F_n(y_n + \hat{x}_n) - \hat{x}_{n+1},\ n \in [n_-, n_+ - 1] \qquad (8.9)$$

has $(0,0)$ as an n independent fixed point.

To obtain a homoclinic orbit w.r.t. the fixed point $(0,0)$, see Figure 8.4 (left), we solve the periodic boundary value problem (8.5) with $G_n(\cdot)$ instead of $F_n(\cdot)$ and initial value

$$(0,\ldots,0,x_n,G_n(x_n),G_{n+1}G_n(x_n),\ldots,G_{n+249}G_{n+248}\cdots G_n(x_n),0,\ldots,0),$$
$$n = -125,\ x_{-125} = \begin{pmatrix} -0.081 \\ 0.096 \end{pmatrix}.$$

In the top right diagram of Figurer 8.4 the stable (green) and unstable (red) fibers at time 0 are plotted. The stable fiber is approximated with the algorithm introduced in Section 6.7. The fibers intersect each other transversally in a single point. This is also the case for fibers in the time interval $[-30h, 30h]$, see the lower diagram in Figure 8.4. For the original continuous time system, this is a strong evidence for transversal homoclinic trajectories, satisfying **(A9)**.

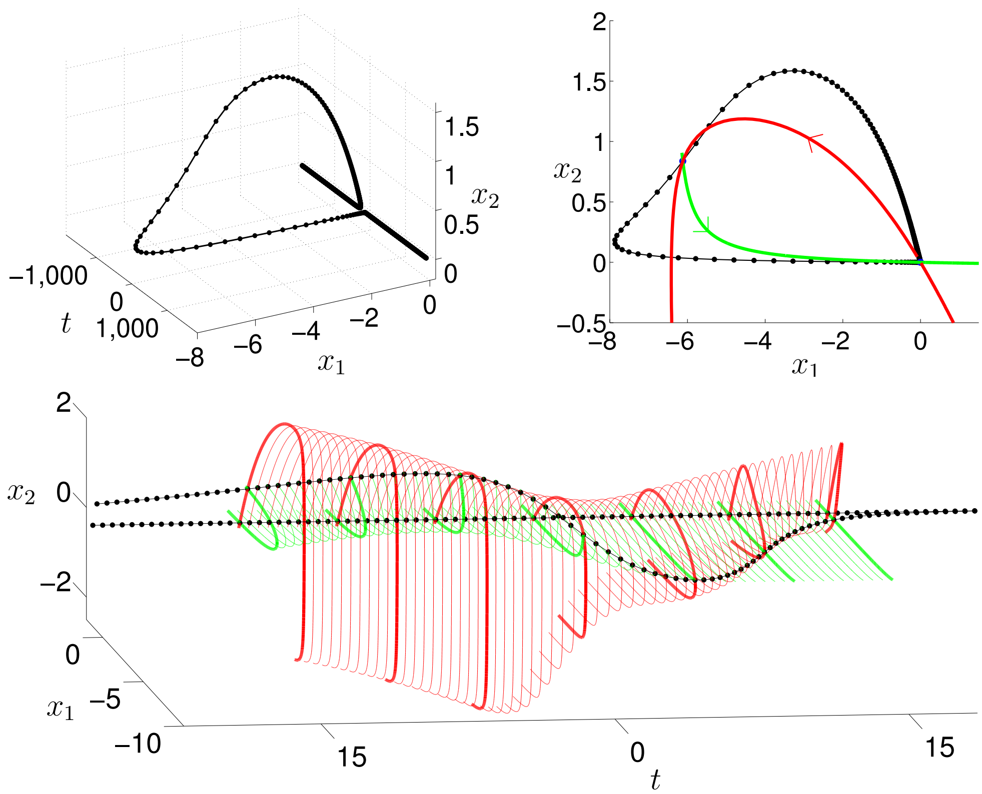


Figure 8.4: Homolinic orbit of (8.9) with $h = 0.5$ (top left) and transversally intersecting fiber bundles (right and bottom).

Appendix A

A Lipschitz Inverse Mapping Theorem

The following quantitative version of the Lipschitz inverse mapping theorem cf. [130, §3 Lemma 1], [77, Appendix C] is essential for proving Theorem 7.3.3.

Lemma A.0.1. *Let $F : Y \times \Lambda \to Z$ be a $\mathcal{C}^\ell$, $\ell \geq 1$ mapping from a Banach space $Y \times \Lambda$ into some Banach space Z. Assume there exists a function $\bar{v}_0 : \Lambda \to Y$ such that $F_v(\bar{v}_0(\varepsilon), \varepsilon)$ are homeomorphisms for all $|\varepsilon| \leq \delta_2$, and there exist some constants $\kappa(\varepsilon) > 0$, $\sigma(\varepsilon) > 0$ such that for all $\|v - \bar{v}_0(\varepsilon)\| \leq \delta_1$ and $|\varepsilon| \leq \delta_2$ we have*

$$\|F_v(v, \varepsilon) - F_v(\bar{v}_0(\varepsilon), \varepsilon)\| \leq \kappa(\varepsilon) < \sigma(\varepsilon) \leq \left\|F_v(\bar{v}_0(\varepsilon), \varepsilon)^{-1}\right\|^{-1}, \tag{A.1}$$

$$\|F(\bar{v}_0(\varepsilon), \varepsilon)\| \leq (\sigma(\varepsilon) - \kappa(\varepsilon))\delta_1. \tag{A.2}$$

Then for any $|\varepsilon| \leq \delta_2$, $F(\cdot, \varepsilon)$ has a unique zero $\tilde{v}(\varepsilon)$ with $\|\tilde{v}(\varepsilon) - \bar{v}_0(\varepsilon)\| \leq \delta_1$ that is $\mathcal{C}^\ell$-smooth w.r.t. ε. The following estimates hold for all $\|v_i - \bar{v}_0(\varepsilon)\| \leq \delta_1$, $i = 1, 2$

$$\|\tilde{v}(\varepsilon) - \bar{v}_0(\varepsilon)\| \leq (\sigma(\varepsilon) - \kappa(\varepsilon))^{-1} \|F(\bar{v}_0(\varepsilon), \varepsilon)\|,$$

$$\|v_1 - v_2\| \leq (\sigma(\varepsilon) - \kappa(\varepsilon))^{-1} \|F(v_1, \varepsilon) - F(v_2, \varepsilon)\|.$$

Appendix B

Assumptions, Functions, Sets

Assumptions

(A0) Let the matrix function A of (2.7) / (2.8) satisfy $A \in \mathcal{C}^1(\mathbb{I}, \mathbb{R}^{k\times k})$.

(A1) Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$, $t_\pm \in \mathbb{T}$, $\mathbb{I} = [t_-, t_+]_\mathbb{T}$ and $\Gamma = \Gamma^T > 0$. Assume

$$\begin{cases} \textbf{(A0)} \text{ and that system (2.6), generated by (2.7),} & \text{for } \mathbb{T} = \mathbb{R}, \\ \text{that the system (2.6), generated by (2.8),} & \text{for } \mathbb{T} = \mathbb{Z} \end{cases}$$

is D-hyperbolic on $\mathbb{I}$ w.r.t. $\|\cdot\|_\Gamma$.

(A2) Let $\tilde{\mathbb{I}} := \begin{cases} \mathbb{I}, & \text{for } \mathbb{T} = \mathbb{R}, \\ \mathbb{I}_1, & \text{for } \mathbb{T} = \mathbb{Z}, \end{cases}$ $\bar{t}_+ := \bar{t} := \begin{cases} t_+, & \text{for } \mathbb{T} = \mathbb{R}, \\ t_+ - 1, & \text{for } \mathbb{T} = \mathbb{Z}, \end{cases}$ and $\bar{t}_- := t_-$.

(A3) Assume **(A2)** and let $k = 2$. Let $\lambda_1(t) > 0 > \lambda_2(t)$ be the eigenvalues of $S_\Gamma(t)$ and $U(t) = (v_1(t)\ v_2(t))$ be an orthogonal matrix, where $v_i(t)$ are eigenvectors to $\lambda_i(t)$ for $i \in \{1, 2\}$, for all $t \in \tilde{\mathbb{I}}$.

(A4) Let $h > 0$ and if $\mathbb{I}$ is compact such that $\frac{t_-}{h}, \frac{t_+}{h} \in \mathbb{Z}$. Define

$$\mathbb{J} := \mathbb{J}_h := \begin{cases} \mathbb{Z}, & \text{if } \mathbb{I} = \mathbb{R}, \\ [n_-, n_+]_\mathbb{Z} := [\frac{t_-}{h}, \frac{t_+}{h}]_\mathbb{Z}, & \text{if } \mathbb{I} = [t_-, t_+]. \end{cases}$$

Further set

$$\mathbb{J}_1 := \begin{cases} \mathbb{Z}, & \text{if } \mathbb{J} = \mathbb{Z}, \\ [\frac{t_-}{h}, \frac{t_+}{h} - 1]_\mathbb{Z}, & \text{if } \mathbb{J} = [\frac{t_-}{h}, \frac{t_+}{h}]_\mathbb{Z}. \end{cases}$$

(A5) $f \in \mathcal{C}^1(\mathbb{R}^k \times \mathbb{I}, \mathbb{R}^k)$ satisfies conditions, assuring existence and uniqueness of global solutions of (2.2) as well as the following estimates for the solution operator φ. For any compact set $\mathcal{K} \subset \mathbb{R}^k$ there exist constants $C_1(\mathcal{K}), h_1(\mathcal{K}) > 0$ such that the inequality

$$\|\varphi_x(x,t,s)\| \leq C_1(\mathcal{K})$$

holds for all $x \in \mathcal{K}$ and $|t-s| \leq h_1(\mathcal{K})$. For $n \in \mathbb{J}_1$ let

$$\varphi_n(x,h) := \varphi(x,(n+1)h, nh)$$

be $\mathcal{C}^d$ smooth w.r.t. h, $d \geq 1$. Mixed derivatives $(\varphi_n)^{(1,\ell)}_{x,h}$, $\ell \in \{0,\dots,d\}$ exist and satisfy the uniform Lipschitz condition

$$\left\|(\varphi_n)^{(1,d)}_{x,h}(x,\mu_1) - (\varphi_n)^{(1,d)}_{x,h}(x,\mu_2)\right\| \leq C_1(\mathcal{K})\,\|\mu_1 - \mu_2\|$$

for all $x \in \mathcal{K}$, $0 \leq \mu_{1,2} \leq h_1(\mathcal{K})$ and $n \in \mathbb{J}_1$.

Further, let $\left\|(\varphi_n)^{(r,1)}_{x,h}(x,h)\right\| \leq C_1(\mathcal{K})$ for all $n \in \mathbb{Z}$, $r \in \{0,1\}$, $x \in \mathcal{K}$ and $0 \leq h \leq h_1(\mathcal{K})$.

(A6) $0 \in \mathbb{R}^k$ satisfies $f(0,t) = 0$ for all $t \in \mathbb{I}$.

(A7) 0 is **(ft-)hyperbolic**. Denote the data of the corresponding variational equation

$$\dot{x} = f_x(0,\cdot)x$$

by $(K, \beta, P^{s,u}(\cdot))$.

(A8) A nontrivial homoclinic orbit $\bar{x}(\cdot)$ of (2.2) exists.

(A9) The homoclinic orbit $\bar{x}(\cdot)$ is (ft-)hyperbolic.

(A10) Let $\bar{y}(\cdot)$ be a (ft-)hyperbolic bounded trajectory of (2.2). Denote by $(\bar{K}, \bar{\beta}, \bar{Q}^{s,u}(\cdot))$ the (ft-)dichotomy data of the corresponding variational equation

$$\dot{u} = f_x(\bar{y}(\cdot),\cdot)u$$

and let $S^{\bar{y}}(t,s)$ be its solution operator.

(A11) Let the images of the projectors $\bar{Q}^s(t_-)$, $\bar{Q}^u(t_+)$ satisfy $\mathcal{R}(\bar{Q}^s(t_-)) \oplus \mathcal{R}(\bar{Q}^u(t_+)) = \mathbb{R}^k$ and let the boundary condition $b(\cdot,\cdot)$ satisfies $b \in \mathcal{C}^1(\mathbb{R}^{2k}, \mathbb{R}^k)$ and $b := b(\bar{y}_{n_-}, \bar{y}_{n_+}) = 0$. Further, let

$$\begin{aligned} B(\bar{y}_{\mathbb{J}}) :=& h\left(D_1 b + D_2 b\Phi(n_+,n_-)_{|\mathcal{R}(\bar{Q}^s(t_-))} \quad D_1 b\bar{\Phi}(n_-,n_+) + D_2 b_{|\mathcal{R}(\bar{Q}^u(t_+))}\right) : \\ & \mathcal{R}(\bar{Q}^s(n_-)) \oplus \mathcal{R}(\bar{Q}^u(n_+)) \to \mathbb{R}^k \end{aligned}$$

be invertible. For any compact set $\mathcal{K} \subset \mathbb{R}^k$ there exist a constant $C(\mathcal{K}) > 0$ such that

$$\left\|b_x(x_{n_-}, x_{n_+}) - b_x(y_{n_-}, y_{n_+})\right\| \leq C(\mathcal{K})\,\|x_{\mathbb{J}} - y_{\mathbb{J}}\|$$

holds for all $x_n, y_n \in \mathcal{K}$, $n \in \mathbb{J}$.

(A12) For any compact set $\mathcal{K} \subset \mathbb{R}^k$ there exist constants $C_2(\mathcal{K}), h_2(\mathcal{K}) > 0$ such that the consistency estimate of order $d \in \mathbb{N}$

$$\|\varphi_n(x,h) - \psi_n(x,h)\| \leq C_2(\mathcal{K}) h^{d+1}$$

holds for all $n \in \mathbb{J}_1$, $x \in \mathcal{K}$ and $0 < h \leq h_2(\mathcal{K})$.

(A13) Mixed derivatives of $\psi_n(x,h)$ up to order 3 exist. For any compact set $\mathcal{K} \subset \mathbb{R}^k$ the derivatives are continuous and uniformly bounded by some constant $\tilde{C}(\mathcal{K})$ in $\mathcal{K} \times (0, h_3(\mathcal{K})]$, with $0 < h_3(\mathcal{K})$ sufficiently small. Furthermore, $\psi_n(x,h)$ is $\mathcal{C}^d$ smooth in h and mixed derivatives $(\psi_n)^{(1,d)}_{x,h}(x,h)$ exist and satisfy the uniform Lipschitz estimate

$$\left\|(\psi_n)^{(1,d)}_{x,h}(x,\mu_1) - (\psi_n)^{(1,d)}_{x,h}(x,\mu_2)\right\| \leq C_3(\mathcal{K}) \|\mu_1 - \mu_2\|$$

for all $n \in \mathbb{J}_1$, $x \in \mathcal{K}$ and $0 < \mu_{1,2} \leq h_3(\mathcal{K})$ with a constant $C_3(\mathcal{K}) > 0$.

Functions

$$
\begin{aligned}
{}_{\Phi}\mathcal{T}_{\ker} &: \mathbb{R}^k \times \mathbb{I} \to \mathbb{I} \cup \{n_+ + 1\} \\
(\xi, n_0) &\mapsto \mathring{n} := \begin{cases} \min \operatorname{Ker}^{\xi}_{n_0}, & \text{if } \operatorname{Ker}^{\xi}_{n_0} \neq \emptyset \\ n_+ + 1, & \text{otherwise} \end{cases} \\
\operatorname{Ker}^{\xi}_{n_0} &:= \{n \in [n_0, n_+]_{\mathbb{Z}} | \Phi(n, n_0)\xi = 0\} \\
{}_{\Phi}\mathcal{T}_{\min} &: \mathbb{R}^k \times \mathbb{I} \to \mathbb{I} \\
(\xi, n_0) &\mapsto \bar{n} := \min\{n \in [n_-, n_0]_{\mathbb{Z}} | \exists x \in \mathbb{R}^k : \Phi(n_0, n)x = \xi\} \\
{}_{\Phi}\mathcal{T}_{\mathrm{pre}} &: \mathbb{R}^k \times \mathbb{I} \rightsquigarrow \mathbb{R}^k \\
(\xi, n_0) &\mapsto \left\{\mu \in \mathbb{R}^k \middle| \Phi(n_0, \bar{n})\mu = \xi \text{ with } \bar{n} := {}_{\Phi}\mathcal{T}_{\min}(\xi, n_0)\right\} \\
\Phi_{\mathrm{pre}}(n, m)U &:= \left\{\xi \in \mathbb{R}^k | \Phi(n, m)\xi \in U\right\} \\
{}_{\varphi}\mathcal{T}^{\bar{x}}_{\ker} &: \mathbb{R}^k, \times \mathbb{I} \to \mathbb{I} \cup \{n_+ + 1\} \\
(\xi, n_0) &\mapsto \begin{cases} \min\{n \in [n_0, n_+]_{\mathbb{Z}} | \varphi(\xi, n, n_0) - \bar{x}(n) = 0\}, & \text{if it exists} \\ n_+ + 1, & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
{}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon} : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{I} \to \mathbb{I} \cup \{t_+ + 1\}
$$

$$
(\mu, \bar{x}, \bar{t}) \mapsto \begin{cases} t_+ + 1, & \text{if } \varphi(\mu, t_+, \bar{t}) \notin B_\varepsilon(\varphi(\bar{x}, t_+, \bar{t})) \\ \min\left\{\hat{t} \in \mathbb{I} | \hat{t} \geq \bar{t}, \varphi(\mu, t, \bar{t}) \in B_\varepsilon(\varphi(\bar{x}, t, \bar{t})) \forall t \in \mathbb{I}, t > \hat{t}\right\}, & \text{else} \end{cases}
$$

$$
{}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\max}_{\varepsilon} : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{I} \cup \{t_- 1\},
$$

$$
(\mu, \bar{x}) \mapsto \begin{cases} t_- 1, & \text{if } \mu \notin B_\varepsilon(\bar{x}) \\ \max\left\{\hat{t} \in \mathbb{I} | \varphi(\mu, t, t_-) \in B_\varepsilon(\varphi(\bar{x}, t, t_-)) \forall t \in \mathbb{I}, t < \hat{t}\right\}, & \text{else.} \end{cases}
$$

Tensor and Their Properties

$$
S_\Gamma(t) := \begin{cases} \frac{1}{2}[\Gamma A(t) + A(t)^T \Gamma], & \text{for } \mathbb{T} = \mathbb{R},\ t \in \mathbb{I}, \\ A(t)^T \Gamma A(t) - \Gamma, & \text{for } \mathbb{T} = \mathbb{Z},\ t \in \mathbb{I}_1, \end{cases}
$$

$$
M_\Gamma(t) := \begin{cases} \dot{S}_\Gamma(t) + S_\Gamma(t)A(t) + A(t)^T S_\Gamma(t), & \text{for } \mathbb{T} = \mathbb{R},\ t \in \mathbb{I}, \\ A(t)^T S_\Gamma(t+1)A(t) - S_\Gamma(t), & \text{for } \mathbb{T} = \mathbb{Z},\ t \in \mathbb{I}_2 \end{cases}
$$

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\xi(t)\|^2_\Gamma = \langle \xi(t), S_\Gamma(t)\xi(t)\rangle \tag{3.16}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t} \|\xi(t)\|^2_\Gamma = 2\langle A(t)\xi(t), \xi(t)\rangle \tag{3.18}
$$

$$
\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} \|\xi(t)\|^2_\Gamma = \langle \xi(t), M_\Gamma(t)\xi(t)\rangle \tag{5.11}
$$

$$\|\xi(n+1)\|_\Gamma^2 - \|\xi(n)\|_\Gamma^2 = \langle \xi(n), S_\Gamma(n)\xi(n)\rangle \tag{3.17}$$

$$\begin{aligned}&\Big(\big(\|\xi(n+2)\|_\Gamma^2 - \|\xi(n+1)\|_\Gamma^2\big) \\ &\quad - \big(\|\xi(n+1)\|_\Gamma^2 - \|\xi(n)\|_\Gamma^2\big)\Big) = \langle \xi(n), M_\Gamma(n)\xi(n)\rangle\end{aligned} \tag{5.12}$$

Sets

$$\begin{aligned} Z_\Gamma(t) &:= \{\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(t)\xi\rangle = 0\} \\ L(n_0) &:= \big\{\xi \in \mathbb{R}^k | \langle \Phi(n_+ - 1, n_0)\xi, S_\Gamma(n_+ - 1)\Phi(n_+ - 1, n_0)\xi\rangle < 0\big\}\end{aligned}$$

Cones Continuous

$$\begin{aligned}
{}^{\mathbb{I}}V_s^+(t_0) &:= \big\{\xi \in \mathbb{R}^k \big| \exists \alpha > 0 : \|\Phi(t,t_0)\xi\| e^{\alpha t} \text{ is decreasing for } t \in [t_0, t_+]\big\} \\
{}^{\mathbb{I}}V_s^-(t_0) &:= \big\{\xi \in \mathbb{R}^k \big| \exists \alpha > 0 : \|\Phi(t,t_0)\xi\| e^{\alpha t} \text{ is decreasing for } t \in [t_-, t_0]\big\} \\
{}^{\mathbb{I}}V_u^+(t_0) &:= \big\{\xi \in \mathbb{R}^k \big| \exists \beta > 0 : \|\Phi(t,t_0)\xi\| e^{-\beta t} \text{ is increasing for } t \in [t_0, t_+]\big\} \\
{}^{\mathbb{I}}V_u^-(t_0) &:= \big\{\xi \in \mathbb{R}^k \big| \exists \beta > 0 : \|\Phi(t,t_0)\xi\| e^{-\beta t} \text{ is increasing for } t \in [t_-, t_0]\big\} \\
{}^{\mathbb{I}}V_s(t_0) &:= {}^{\mathbb{I}}V_s^+(t_0) \cap {}^{\mathbb{I}}V_s^-(t_0) \\
{}^{\mathbb{I}}V_u(t_0) &:= {}^{\mathbb{I}}V_u^+(t_0) \cap {}^{\mathbb{I}}V_u^-(t_0)
\end{aligned}$$

$${}^{\mathbb{I}}V_s(t_0) = \{\xi \in \mathbb{R}^k | \langle \Phi(t_+, t_0)\xi, S_\Gamma(t_+)\Phi(t_+, t_0)\xi\rangle < 0\} \cup \{0\} \tag{5.24}$$

$${}^{\mathbb{I}}V_u(t_0) = \{\Phi(t_0, t_-)\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(t_-)\xi\rangle > 0\} \cup \{0\} \tag{5.25}$$

Cones Discrete

$$\begin{aligned}
{}^{\mathbb{I}}V_s^+(n_0) &:= \big\{\xi \in \mathbb{R}^k \big| \exists \alpha > 0 : \|\Phi(n,n_0)\xi\|\, e^{\alpha n} \text{ is decreasing for } n \in [n_0, n_+]_{\mathbb{Z}}\big\} \\
{}^{\mathbb{I}}V_s^-(n_0) &:= \big\{\xi \in \mathbb{R}^k \big| \exists \mu_{\bar n} \in {}_\Phi\mathcal{T}_{\text{pre}}(\xi, n_0), \bar n := {}_\Phi\mathcal{T}_{\min}(\xi, n_0), \\
&\qquad \exists \alpha > 0 : \|\Phi(n,\bar n)\mu_{\bar n}\|\, e^{\alpha n} \text{ is decreasing for } n \in [\bar n, n_0]_{\mathbb{Z}}\big\} \\
{}^{\mathbb{I}}V_u^+(n_0) &:= \big\{\xi \in \mathbb{R}^k \big| \exists \beta > 0 : \|\Phi(n,n_0)\xi\|\, e^{-\beta n} \text{ is increasing for } n \in [n_0, n_+]_{\mathbb{Z}}\big\} \\
{}^{\mathbb{I}}V_u^-(n_0) &:= \big\{\xi \in \mathbb{R}^k \big| {}_\Phi\mathcal{T}_{\min}(\xi, n_0) = n_-, \exists \mu_{n_-} \in {}_\Phi\mathcal{T}_{\text{pre}}(\xi, n_0), \\
&\qquad \exists \beta > 0 : \big\|\Phi(n, n_-)\mu_{n_-}\big\|\, e^{-\beta n} \text{ is increasing for } n \in [n_-, n_0]_{\mathbb{Z}}\big\} \\
{}^{\mathbb{I}}\bar V_s(t_0) &:= {}^{\mathbb{I}}V_s^+(t_0) \cap {}^{\mathbb{I}}V_s^-(t_0) \\
{}^{\mathbb{I}}V_u(t_0) &:= {}^{\mathbb{I}}V_u^+(t_0) \cap {}^{\mathbb{I}}V_u^-(t_0) \\
{}^{\mathbb{I}}V_s(n_+ - 1) &:= {}^{\mathbb{I}}\bar V_s(n_+ - 1) \\
{}^{\mathbb{I}}V_s(n_0) &:= \Big\{\xi \in {}^{\mathbb{I}}\bar V_s(n_0) \Big| \Phi(n_0 + 1, n_0)\xi \in {}^{\mathbb{I}}V_s(n_0 + 1)\Big\} \ \ (n_0 \in \mathbb{I}_2) \\
{}^{\mathbb{I}}V_s(n_+) &:= {}^{\mathbb{I}}\bar V_s(n_+) \text{ (invertible)}
\end{aligned}$$

$$
\begin{aligned}
{}^{\mathbb{I}}\bar{V}_s(n_+) &= \mathcal{R}(\Phi(n_+, n_+ - 1))^C \\
&\quad \cup \{\xi \in \mathbb{R}^k | {}_{\Phi}\mathcal{T}_{\min}(\xi, n_+) =: \bar{n} < n_+ \wedge \exists \bar{\mu} \in {}_{\Phi}\mathcal{T}_{\mathrm{pre}}(\xi, n_+) : \\
&\qquad \langle \mu, S_\Gamma(n_+ - 1)\mu \rangle < 0 \text{ for } \mu := \Phi(n_+ - 1, \bar{n})\bar{\mu}\} \cup \{0\} \\
{}^{\mathbb{I}}\bar{V}_s(n_0) &= L(n_0) \dot{\cup} \mathcal{N}(\Phi(n_+ - 1, n_0)) \; (n_0 \in \mathbb{I}_1) && (5.47) \\
{}^{\mathbb{I}}V_s(n_0) &= \left\{\xi \in L(n_0) \middle| \Phi(n_0 + 1, n_0)\xi \in {}^{\mathbb{I}}V_s(n_0 + 1)\right\} \; (n_0 \in \mathbb{I}_1) \\
&\quad \dot{\cup} \mathcal{N}(\Phi(n_+ - 1, n_0)) && (5.52) \\
{}^{\mathbb{I}}V_u(t_0) &= \{\Phi(t_0, t_-)\xi \in \mathbb{R}^k | \langle \xi, S_\Gamma(t_-)\xi \rangle > 0\} \cup \{0\} && (5.48) \\
{}^{\mathbb{I}}V_s(n_+) &= \Phi(n_+, n_+ - 1)L(n_+ - 1) \cup \{0\} \text{ (invertible)} && (5.50) \\
{}^{\mathbb{I}}V_s(n_0) &= {}^{\mathbb{I}}\bar{V}_s(n_0) = L(n_0) \cup \{0\} \text{ (invertible)} && (5.51)
\end{aligned}
$$

Fibers Continuous

$$
\begin{aligned}
{}^{\mathbb{I}}M_s^{\bar{x}} &:= \left\{(x_0, t_0) \in \mathbb{R}^k \times \mathbb{I} \middle| \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(x_0, t, t_0) - \bar{x}\| < 0 \text{ for all } t \in \mathbb{I}\right\} \\
&\quad \cup \{(\bar{x}(t_0), t_0) | t_0 \in \mathbb{I}\} \\
{}^{\mathbb{I}}M_u^{\bar{x}} &:= \left\{(x_0, t_0) \in \mathbb{R}^k \times \mathbb{I} \middle| \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(x_0, t, t_0) - \bar{x}\| > 0 \text{ for all } t \in \mathbb{I}\right\} \\
&\quad \cup \{(\bar{x}(t_0), t_0) | t_0 \in \mathbb{I}\} \\
{}^{\mathbb{I}}_{\varepsilon}M_s^{\bar{x}} &:= \Bigg\{(x_0, t_0) \in \mathbb{R}^k \times \mathbb{I} \,\Bigg|\, \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(x_0, t, t_0) - \bar{x}(t)\| < 0 \text{ for all } t \in \mathbb{I}, \\
&\qquad \varphi(x_0, t_+, t_0) \in B_\varepsilon(\bar{x}(t_+))\Bigg\} \cup \{(\bar{x}(t_0), t_0) | t_0 \in \mathbb{I}\} \\
{}^{\mathbb{I}}_{\varepsilon}M_u^{\bar{x}} &:= \Bigg\{(x_0, t_0) \in \mathbb{R}^k \times \mathbb{I} \,\Bigg|\, \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(x_0, t, t_0) - \bar{x}(t)\| > 0 \text{ for all } t \in \mathbb{I}, \\
&\qquad \varphi(x_0, t_-, t_0) \in B_\varepsilon(\bar{x}(t_-))\Bigg\} \cup \{\bar{x}(t_0), t_0) | t_0 \in \mathbb{I}\} \\
{}^{\mathbb{I}}_{\varepsilon}W_s^{\bar{x}} &:= \Bigg\{(x_0, t_0) \in \mathbb{R}^k \times \mathbb{I} \,\Bigg|\, \hat{t} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}_{\varepsilon}^{\min}(\varphi(x_0, t_-, t_0), \bar{x}(t_-), t_-) \in \mathbb{I} \text{ and} \\
&\qquad \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(x_0, t, t_0) - \bar{x}(t)\| < 0 \text{ for all } t \in \mathbb{I}, t > \hat{t}\Bigg\} \cup \{(\bar{x}(t_0), t_0) | t_0 \in \mathbb{I}\} \\
{}^{\mathbb{I}}_{\varepsilon}W_u^{\bar{x}} &:= \Bigg\{(x_0, t_0) \in \mathbb{R}^k \times \mathbb{I} \,\Bigg|\, \hat{t} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}_{\varepsilon}^{\max}(\varphi(x_0, t_-, t_0), \bar{x}(t_-)) \in \mathbb{I} \text{ and} \\
&\qquad \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(x_0, t, t_0) - \bar{x}(t)\| > 0 \text{ for all } t \in \mathbb{I}, t < \hat{t}\Bigg\} \cup \{(\bar{x}(t_0), t_0) | t_0 \in \mathbb{I}\}
\end{aligned}
$$

Fibers Discrete

$$
\begin{aligned}
{}^{\mathbb{I}}_{\varepsilon}M_s^{\bar{x}} :=\Big\{ & (\xi, n_0) \in \mathbb{R}^k \times \mathbb{I} \Big| \exists \mu \in {}_{\varphi}\mathcal{T}_{\mathrm{pre}}(\xi, n_0) \text{ with } \bar{n} := {}_{\varphi}\mathcal{T}_{\min}(\xi, n_0) \text{ and} \\
& \mathring{n} := {}_{\varphi}\mathcal{T}^{\bar{x}}_{\mathrm{ker}}(\mu, \bar{n}) : \|\varphi(\mu, n, \bar{n}) - \bar{x}(n)\| < \|\varphi(\mu, m, \bar{n}) - \bar{x}(m)\| \\
& \text{for all } n, m \in [\bar{n}, \mathring{n}]_{\mathbb{Z}}, n > m \text{ and } \varphi(\xi, n_+, n_0) \in B_\varepsilon(\bar{x}(n_+)), \Big\} \\
{}^{\mathbb{I}}_{\varepsilon}M_u^{\bar{x}} :=\Big\{ & (\xi, n_0) \in \mathbb{R}^k \times \mathbb{I} \Big| {}_{\varphi}\mathcal{T}_{\min}(\xi, n_0) = n_- \text{ and } \exists \mu \in {}_{\varphi}\mathcal{T}_{\mathrm{pre}}(\xi, n_0) \cap B_\varepsilon(\bar{x}(n_-)) : \\
& \|\varphi(\mu, n, n_-) - \bar{x}(n)\| > \|\varphi(\mu, m, n_-) - \bar{x}(m)\| \text{ for all } n, m \in \mathbb{I}, n > m \Big\} \\
& \cup \{(\bar{x}(t_0), t_0) | t_0 \in \mathbb{I}\} \\
{}^{\mathbb{I}}_{\varepsilon}W_s^{\bar{x}} :=\Big\{ & (\xi, n_0) \in \mathbb{R}^k \times \mathbb{I} \Big| \exists \mu \in {}_{\varphi}\mathcal{T}_{\mathrm{pre}}(\xi, n_0) \text{ with } \bar{n} := {}_{\varphi}\mathcal{T}_{\min}(\xi, n_0), \\
& \mathring{n} := {}_{\varphi}\mathcal{T}^{\bar{x}}_{\mathrm{ker}}(\mu, \bar{n}) \text{ and } \hat{n} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\min}_{\varepsilon}(\mu, \bar{x}(\bar{n}), \bar{n}) \in \mathbb{I} : \\
& \|\varphi(\mu, n, \bar{n}) - \bar{x}(n)\| < \|\varphi(\mu, m, \bar{n}) - \bar{x}(m)\| \\
& \text{for all } n, m \in \mathbb{I}, \mathring{n} \geq n > m > \hat{n} \Big\} \\
{}^{\mathbb{I}}_{\varepsilon}W_u^{\bar{x}} :=\Big\{ & (\xi, n_0) \in \mathbb{R}^k \times \mathbb{I} \Big| {}_{\varphi}\mathcal{T}_{\min}(\xi, n_0) = n_- \text{ and } \exists \mu \in {}_{\varphi}\mathcal{T}_{\mathrm{pre}}(\xi, n_0) \text{ with} \\
& \hat{n} := {}^{\mathbb{I}}_{\varphi}\mathcal{B}^{\max}_{\varepsilon}(\mu, \bar{x}(n_-)) \in \mathbb{I} : \\
& \|\varphi(\mu, n, n_-) - \bar{x}(n)\| > \|\varphi(\mu, m, n_-) - \bar{x}(m)\| \\
& \text{for all } n, m \in \mathbb{I}, m < n < \hat{n} \Big\} \cup \Big\{ (\bar{x}(n), n) \Big| n \in \mathbb{I} \Big\} .
\end{aligned}
$$

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