Bivariate representation and conjugacy class zeta functions associated to unipotent group schemes

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vorgelegt von
Paula Macedo Lins de Araujo
betreut durch
Prof. Dr. Christopher Voll

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To my family.

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Abstract

The main topic of this doctoral thesis is zeta functions of groups. Let $\mathbf{G}$ be a unipotent group scheme defined over the ring of integers $\mathcal{O}$ of a number field. The group $\mathbf{G}(\mathcal{O})$ of $\mathcal{O}$-rational points is a finitely generated torsion-free nilpotent group. We introduce two bivariate zeta functions related to groups of the form $\mathbf{G}(\mathcal{O})$: firstly the bivariate representation zeta function of $\mathbf{G}(\mathcal{O})$, which enumerates the isomorphism classes of irreducible complex representations of finite dimensions of its congruence quotients, and secondly the bivariate conjugacy class zeta function of $\mathbf{G}(\mathcal{O})$, which enumerates the conjugacy classes of each size of its congruence quotients.

These zeta functions might be used as tools for understanding another (univariate) zeta functions, as they both specialise to class number zeta functions, which enumerate class numbers of the congruence quotients. Additionally, in case of nilpotency class two, bivariate representation zeta functions specialise to twist representation zeta functions, which are zeta functions enumerating the irreducible complex characters of finite dimensions up to tensoring by one-dimensional characters.

We show that bivariate representation and bivariate conjugacy class zeta functions satisfy Euler decompositions and that almost all of their Euler factors are rational and satisfy functional equations. We also prove that they converge on some domains of $\mathbb{C}^2$ and, furthermore, their maximal domains of convergence and meromorphic continuation are independent of the number field $\mathcal{O}$ considered, up to finitely many local factors.

We provide formulae for the bivariate zeta functions of three infinite families of groups of nilpotency class 2 of the form $\mathbf{G}(\mathcal{O})$ which generalise the Heisenberg group of 3×3-unitriangular matrices over $\mathcal{O}$. As an application, we establish formulae for the joint distributions of three statistics on finite hyperoctahedral groups.

Key words and phrases: Group theory, zeta functions, finitely generated nilpotent groups, conjugacy classes, irreducible complex characters, p-adic integration, signed permutation statistics, hyperoctahedral group.

2000 Mathematics Subject Classification: 11M32, 20D15, 20F18, 20E45, 20F69, 32D15, 05E15.

Zusammenfassung

Zetafunktionen von Gruppen sind das Hauptthema dieser Doktorarbeit. Sei $\mathbf{G}$ ein unipotentes Gruppenschema, welches über einen Ganzheitsring $\mathcal{O}$ eines Zahlkörpers definiert ist. Die Gruppe $\mathbf{G}(\mathcal{O})$ der $\mathcal{O}$-rationalen Punkte ist eine endlich erzeugte torsionsfreie nilpotente Gruppe. Wir stellen zwei bivariate Zeta-Funktionen von Gruppen der Form $\mathbf{G}(\mathcal{O})$ vor: erstens die bivariate Darstellungszetafunktion von $\mathbf{G}(\mathcal{O})$, welche die Isomorphieklassen aller endlichdimensionalen irreduziblen komplexen Darstellungen von Kongruenzquotienten von $\mathbf{G}(\mathcal{O})$ kodiert, und zweitens die bivariate Konjugations-klassezetafunktion von $\mathbf{G}(\mathcal{O})$, die die Konjugations-klassen jeder endlichen Größe von Kongruenzquotienten von $\mathbf{G}(\mathcal{O})$ kodiert.

Diese bivariaten Zetafunktionen können benutzen werden, um andere (univariate) Zetafunktionen zu verstehen, denn beide spezialisieren sich zu Klassenzahlzetafunktionen, welche Klassenzahlen von Kongruenzquotienten kodieren. Außerdem spezialisieren sich bivariate Darstellungszetafunktionen von Gruppen des Nilpotenzgrades 2 zu *twist* Darstellungszetafunktionen, welche alle endlich dimensionalen irreduziblen komplexen Darstellungen bis auf Tensorierung mit eindimensionale Darstellungen kodieren.

Wir zeigen, dass bivariate Darstellungs- und Konjugations-klassezetafunktionen Euler-Zerlegungen besitzen, und dass ihre lokalen Faktoren rationale Funktionen sind, welche Funktionalgleichungen genügen. Wir zeigen auch, dass sie jeweils auf einem Gebiet von $\mathbb{C}^2$ konvergieren. Außerdem sind ihre maximalen Konvergenz- und Meromorphiebereiche bis auf endliche viele lokale Faktoren unabhängig von $\mathcal{O}$.

Wir bestimmen explizite Formeln für beide bivariaten Zetafunktionen von drei unendlichen Familien von nilpotenten Gruppen $\mathbf{G}(\mathcal{O})$ des Nilpotenzgrades 2, welche die Heisenberg-Gruppe von 3×3-unipotenten Dreiecksmatrizen über $\mathcal{O}$ verallgemeinern. Als Anwendung ermitteln wir Formeln für die gemeinsamen Verteilungen von drei Statistiken auf endlichen Hyperoktaedergruppen.

Schlüsselwörter: Gruppentheorie, Zetafunktionen, endlich erzeugte nilpotente Gruppen, Konjugations-klassen, irreduzible komplexe Charaktere, p-adische Integration, signierte Permutationsstatistiken, Hyperoktaedergruppen.

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Chapter 1

Introduction and summary of main results

Euler was the first to provide a solution to the Basel problem, which consists of obtaining the precise value of the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

This was done by defining and studying the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \tag{1.0.1}$$

which at the time was regarded as a real function. Euler gave a formula for $\zeta(2m)$ for all $m \in \mathbb{N}$ in terms of Bernoulli numbers. In particular, the solution for the Basel problem is $\zeta(2) = \frac{\pi^2}{6}$.

Riemann extended the zeta function of Euler to a complex variable function. This allowed to analytically continue $\zeta(s)$ around the pole $s = 1$ and to obtain a meromorphic continuation to the whole complex plane. This complex function is known as the *Riemann zeta function*, and has been intensively investigated in the last years, mostly because of the famous Riemann hypothesis and its relation with the distribution of prime numbers; see [21, Section 12 and Theorem 12.3]. The series (1.0.1) is known to converge when the real part $\mathrm{Re}(s)$ of s is larger than 1, and to diverge if $\mathrm{Re}(s) \leq 1$. Euler proved that $\zeta(s)$ satisfies the following decomposition:

$$\zeta(s) = \prod_{p \text{ prime}} \zeta_p(s), \tag{1.0.2}$$

where $\zeta_p(s)$—called *local factors* of $\zeta(s)$—are defined analogously to the Riemann zeta function, but instead of considering all natural numbers, we consider powers of the prime p:

$$\zeta_p(s) = \sum_{i=0}^{\infty} p^{-is} = \frac{1}{1 - p^{-s}}.$$

The decomposition (1.0.2) is known as the *Euler decomposition* of the Riemann zeta function $\zeta(s)$, and it provides a proof for the existence of infinitely many primes, since the harmonic series $\zeta(1)$ diverges.

The definition of the Riemann zeta function was extended by Dirichlet by attaching a coefficient a_n to each term of the sum (1.0.1): the *Dirichlet series* associated to a complex sequence $(a_n)_n$ with $n \in \mathbb{N}$ is

$$D((a_n)_n, s) := \sum_{n=1}^{\infty} a_n n^{-s},$$

where s is a complex variable. This generating function has a right half plane of $\mathbb{C}$ as maximal domain of convergence, possibly empty; see for instance [1, Theorem 11.8] or [18, Theorem 1]. The infimum of all $c \in \mathbb{R}$ such that $D((a_n)_n, s)$ converges on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > c\}$ is called the *abscissa of convergence* of this Dirichlet series—see [1, Theorem 11.9]—and denoted by α. If $D((a_n)_n, s)$ diverges on the whole of $\mathbb{C}$, then $\alpha = -\infty$.

If the sequence $(a_n)_n$ is bounded by an integer polynomial in n, then $(a_n)_n$ is said to have *polynomial growth* and the Dirichlet series associated to $(a_n)_n$ converges for $s \in \mathbb{C}$ with sufficiently large real part $\mathrm{Re}(s)$, that is $\alpha > -\infty$.

Dirichlet series serve algebraic purposes by attaching sequences $(a_n)_n$ encoding some data of algebraic objects. Dedekind, for instance, defined the Dirichlet series associated to the data $(\gamma_n(K))_n$ of a number field K with ring of integers $\mathcal{O}$ given by $\gamma_n(K) := |\{I \trianglelefteq \mathcal{O} \mid |\mathcal{O} : I| = n\}|$:

$$\zeta_K(s) = \sum_{n=1}^{\infty} \gamma_n(K) n^{-s} = \sum_{\mathfrak{a}} |\mathcal{O} : \mathfrak{a}|^{-s}, \tag{1.0.3}$$

where the second sum is over all nonzero ideals $\mathfrak{a}$ of $\mathcal{O}$. This generating function is called *Dedekind zeta function* and satisfies the following *Euler decomposition*:

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - |\mathcal{O} : \mathfrak{p}|^{-s}}, \tag{1.0.4}$$

where $\mathfrak{p}$ ranges over the nonzero prime ideals of $\mathcal{O}$. Decomposition (1.0.4) reflects the unique factorisation of ideals in $\mathcal{O}$. In particular, the Dedekind zeta function $\zeta_{\mathbb{Q}}(s)$ of the rational numbers coincides with the Riemann zeta function.

Zeta functions were introduced as tools in asymptotic group theory by Grunewald, Segal and Smith in [17], where they considered the following data of a torsion-free finitely generated nilpotent group G (or $\mathcal{T}$-*group* for short):

$$\begin{aligned} a_n^{\leq}(G) &= |\{H \leq G \mid |G : H| = n\}|, \\ a_n^{\trianglelefteq}(G) &= |\{H \trianglelefteq G \mid |G : H| = n\}|, \\ a_n^{\wedge}(G) &= |\{H \leq G \mid |G : H| = n,\ \widehat{H} \cong \widehat{G}\}|, \end{aligned}$$

where $\widehat{G}$ denotes the profinite completion of G. The numbers $a_n^{\leq}(G)$, $a_n^{\trianglelefteq}(G)$, and $a_n^{\wedge}(G)$ are finite for all $n \in \mathbb{N}$, since finitely generated groups have only finitely many subgroups of each index, cf. [31, Corollary 1.1.2]. The *subgroup zeta function*, the *normal zeta function*, and the *profinite zeta function* of a $\mathcal{T}$-group G are

$$\zeta_G^{\leq}(s) = \sum_{n=1}^{\infty} a_n^{\leq}(G) n^{-s},\ \ \zeta_G^{\trianglelefteq}(s) = \sum_{n=1}^{\infty} a_n^{\trianglelefteq}(G) n^{-s},\ \ \zeta_G^{\wedge}(s) = \sum_{n=1}^{\infty} a_n^{\wedge}(G) n^{-s}, \tag{1.0.5}$$

respectively. In particular, $\zeta_{\mathbb{Z}}^{\leq}(s) = \zeta_{\mathbb{Z}}^{\trianglelefteq}(s) = \zeta(s)$.

These generating functions are Dirichlet series (1.0.3) and hence they converge for sufficiently large $\mathrm{Re}(s)$, as long as the associated sequences are bounded by polynomials.

Zeta functions are expected to satisfy some arithmetic and analytic properties. Among the arithmetic properties, they should possess Euler decompositions whose local factors are rational functions. Rationality here means the following: let $\zeta^*(s)$ denote a zeta function associated to some data (of a group G, for example) with Euler decomposition $\prod_{p \text{ prime}} \zeta^*_p(s)$. We say that the local factor $\zeta^*_p(s)$ of this zeta function at p is *rational* in p^{-s} if there exists a rational function $W_p(X) \in \mathbb{Q}[X]$ such that $\zeta^*_p(s) = W_p(p^{-s})$.

The zeta functions (1.0.5) satisfy Euler decompositions and their local factors are rational functions; see [17, Proposition 4 and Theorem 1].

If the $\mathcal{T}$-group G has nilpotency class 2, one says that G is a $\mathcal{T}_2$-*group*. The subgroup and normal subgroup zeta functions of free $\mathcal{T}_2$-groups are uniform; see [17, Theorem 2].

One may ask what sort of information some data $a_n(G)$ of a group G can provide about the group and its algebraic features. Some families of groups are characterised by their subgroup growth, that is, they are characterised by how fast the corresponding sequence $(a^{\leq}_n(G))_n$ grows. One example is that arithmetic groups in characteristic zero have the congruence subgroup property if and only if the sequence $(\sum_{i=1}^n a^{\leq}_i(G))_n$ grows strictly less than $n^{\log(n)}$, that is, if there is a constant a such that $\sum_{i=1}^n a^{\leq}_i(G) < n^{a\log(n)}$ for all $n \in \mathbb{N}$; see [31, Theorem 7.1]. Furthermore, the Polynomial Subgroup Growth Theorem [31, Theorem 5.1] asserts that a finitely generated residually finite group G is virtually soluble of finite rank if and only if the sequence $(a_n(G))_n$ is bounded by a polynomial. In particular, a finitely generated residually finite group G is virtually soluble of finite rank if and only if its subgroup zeta function converges somewhere.

1.1 Zeta functions related to representations and conjugacy classes of groups

In finite group theory, character degrees, irreducible representations and conjugacy classes are considerably well studied; see for instance [20]. In order to investigate them in the context of infinite groups, one may investigate zeta functions concerning the distributions of representations of each dimension and conjugacy classes of each size. In the following, we discuss and define zeta functions which encode information about representations and conjugacy classes of groups.

1.1.1 Representation zeta functions

Given a group G, denote by $\mathrm{Rep}(G)$ the set of its isomorphism classes of complex irreducible representations. Set

$$r_n(G) = |\{[\rho] \in \mathrm{Rep}(G) \mid \dim(\rho) = n\}|,$$

where $[\rho]$ is the isomorphism class of the representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$. If G is a topological group, then we only consider continuous representations. If $r_n(G)$ is finite for each $n \in \mathbb{N}$, the group G is said to be *representation rigid*, and one can study the sequence $(r_n(G))_n$ through the Dirichlet series associated to it. Throughout, denote by s a complex variable.

Definition 1.1.1. *The* representation zeta function *of a representation rigid group G is*

$$\zeta_G^{\mathrm{irr}}(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s}.$$

Representation zeta functions of rigid groups are investigated, for instance, in [2, 3, 4, 16, 22, 25, 30]. For a short introduction to representation growth and representation zeta functions, see [23].

A $\mathcal{T}$-group G has infinitely many one-dimensional complex irreducible representations, that is, G is not representation rigid. Hrushovski and Martin introduced in the first version of the paper [19] the Dirichlet series associated to the numbers $\widetilde{r}_n(G)$ of n-dimensional irreducible complex characters of G up to tensoring by one-dimensional characters. The equivalence classes on the set of irreducible complex representations of G under this equivalence relation are called *twist-isoclasses*, and two elements of the same twist-isoclass are said to be *twist-equivalent* to each other. For a $\mathcal{T}$-group G, the numbers $\widetilde{r}_n(G)$ are all finite, see [29, Theorem 6.6], hence one can define a Dirichlet series encoding this data. The following zeta function was defined in the first version of [19].

Definition 1.1.2. *The* twist representation zeta function *of a $\mathcal{T}$-group G is*

$$\zeta_G^{\widetilde{\mathrm{irr}}}(s) = \sum_{n=1}^{\infty} \widetilde{r}_n(G) n^{-s}.$$

This zeta function converges on a (nonempty) complex half-plane, see [48, Lemma 2.1], and has *Euler decomposition*

$$\zeta_G^{\widetilde{\mathrm{irr}}}(s) = \prod_{p \text{ prime}} \zeta_{G,p}^{\widetilde{\mathrm{irr}}}(s), \tag{1.1.1}$$

where $\zeta_{G,p}^{\widetilde{\mathrm{irr}}}(s) = \sum_{i=0}^{\infty} \widetilde{r}_{p^i}(G) p^{-is}$; see [52, Section 4.1]. Twist representation zeta functions of $\mathcal{T}$-groups are studied, for instance, in [11, 12, 19, 37, 46, 48, 49, 51]. For an introduction see [52]. The local factors in (1.1.1) are rational functions in p^{-s}, according to [19, Theorem 1.5]. Moreover, almost all local factors satisfy functional equations under inversion of p; see [51, Theorem D].

Let K be a number field and $\mathcal{O}$ its ring of integers. Let $\mathbf{G}$ be a unipotent group scheme over $\mathcal{O}$. The group $\mathbf{G}(\mathcal{O})$ is a $\mathcal{T}$-group; see [48, Section 2.1.1]. Twist representation zeta functions of groups $\mathbf{G}(\mathcal{O})$ associated to nilpotent Lie lattices were studied in [48]. Stasinski and Voll observe that, since unipotent groups have the Congruence Subgroup Property and the strong approximation property, the twist representation zeta functions of groups of the form $\mathbf{G}(\mathcal{O})$ satisfy the *Euler decomposition*

$$\zeta_{\mathbf{G}(\mathcal{O})}^{\widetilde{\mathrm{irr}}}(s) = \prod_{\mathfrak{p}} \zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}^{\widetilde{\mathrm{irr}}}(s),$$

where $\mathfrak{p}$ ranges over the nonzero prime ideals of $\mathcal{O}$ and the completion of $\mathcal{O}$ at the nonzero prime ideal $\mathfrak{p}$ is denoted by $\mathcal{O}_{\mathfrak{p}}$. This Euler decomposition refines (1.1.1).

We want to study the distribution of irreducible complex representations of some groups of the form $\mathbf{G}(\mathcal{O})$. However, instead of considering the numbers of irreducible complex representations of $\mathbf{G}(\mathcal{O})$ up to some equivalence relation—such as $\widetilde{r}_n(\mathbf{G}(\mathcal{O}))$—, we consider the distributions of the irreducible complex representation of congruence quotients $\mathbf{G}(\mathcal{O}/I)$ of $\mathbf{G}(\mathcal{O})$, where I is a nonzero ideal of $\mathcal{O}$. Since the groups $\mathbf{G}(\mathcal{O}/I)$ are finite, their representation zeta functions (1.1.1) are well defined. Our idea is to define a zeta function associated to $\mathbf{G}(\mathcal{O})$ in two variables which encode the irreducible complex representations of the quotients $\mathbf{G}(\mathcal{O}/I)$: one of the variables keeps track of the level quotient and the other one counts the relevant data.

Definition 1.1.3. *The* bivariate representation zeta function *of* $\mathbf{G}(\mathcal{O})$ *is*

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O})}(s_1, s_2) = \sum_{(0)\neq I \trianglelefteq \mathcal{O}} \zeta^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O}/I)}(s_1)|\mathcal{O} : I|^{-s_2},$$

where s_1 *and* s_2 *are complex variables.*

This generating function is a double Dirichlet series; see Section 2.5. In Proposition 2.5.5, we show that it converges if s_1,$s_2 \in \mathbb{C}$ have sufficiently large real parts. However, the maximal domain of convergence of $\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$ may not be of the form $\{(s_1, s_2) \in \mathbb{C}^2 \mid \mathrm{Re}(s_1) > \alpha_1,\ \mathrm{Re}(s_2) > \alpha_2\}$.

In Proposition 3.1.1, we show the *Euler decomposition*

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O})}(s_1, s_2) = \prod_{\mathfrak{p}} \mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s_1, s_2), \tag{1.1.2}$$

where $\mathfrak{p}$ ranges over the nonzero prime ideals of $\mathcal{O}$. When considering a fixed prime ideal $\mathfrak{p}$, we write simply $\mathcal{O}_{\mathfrak{p}} = \mathfrak{o}$ and $\mathbf{G}_N := \mathbf{G}(\mathfrak{o}/\mathfrak{p}^N)$. With this notation, the *local factor* at $\mathfrak{p}$ is given by

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s_1, s_2) = \mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}(s_1, s_2) = \sum_{N=0}^{\infty} \zeta^{\mathrm{irr}}_{\mathbf{G}_N}(s_1)|\mathfrak{o} : \mathfrak{p}|^{-Ns_2}. \tag{1.1.3}$$

In certain cases, one can study twist representation zeta functions through bivariate representation zeta functions, as we now explain. A nilpotent $\mathcal{O}$-Lie lattice Λ is a free and finitely generated $\mathcal{O}$-module Λ together with an antisymmetric bi-additive form $[\ ,\ ]$ which satisfies the Jacobi identity. Let $\mathbf{G}_\Lambda$ be a unipotent group scheme obtained from a nilpotent $\mathcal{O}$-Lie lattice Λ in the sense of [48, Section 2.1.2]; see Section 2.1. If $\mathbf{G}_\Lambda(\mathcal{O})$ is a $\mathcal{T}_2$-group, the twist representation zeta function $\mathbf{G}_\Lambda(\mathfrak{o})$ can be obtained from its bivariate representation zeta function via the following specialisation, given in Proposition 3.3.1:

$$(1 - q^{r-s_2})\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}_\Lambda(\mathfrak{o})}(s_1, s_2)\ |_{\substack{s_1 \to s-2 \\ s_2 \to r}} = \widetilde{\zeta^{\mathrm{irr}}_{\mathbf{G}_\Lambda(\mathfrak{o})}}(s), \tag{1.1.4}$$

where r is a constant depending on Λ, provided both the left-hand side and the right-hand side converge.

However, no such specialisation is expected to hold in general. In Example 3.3.2, we exhibit a $\mathcal{T}$-group of nilpotency class 3 whose bivariate representation zeta function does not specialise to its twist representation zeta function.

We are mostly interested in studying bivariate representation zeta functions of $\mathcal{T}$-groups of the form $\mathbf{G}_\Lambda(\mathcal{O})$. Stasinski and Voll showed in [48, Theorem A] that almost all local factors of twist representation zeta functions of such groups are rational. More precisely, they proved that there are $t \in \mathbb{N}$ and a rational function $R(X_1, \ldots, X_t, Y)$ such that almost all local factors $\widetilde{\zeta^{\mathrm{irr}}_{\mathbf{G}_\Lambda(\mathfrak{o})}}(s)$ are given by

$$\widetilde{\zeta^{\mathrm{irr}}_{\mathbf{G}_\Lambda(\mathfrak{o})}}(s) = R(\lambda_1, \ldots, \lambda_t, q^{-s}),$$

where $q = |\mathcal{O} : \mathfrak{p}|$ and $\lambda_1, \ldots, \lambda_t$ are algebraic integers depending on the prime $\mathfrak{p}$. More than that, the local factors are *uniform under base extension*: if $\mathfrak{O}$ is a finite extension of $\mathfrak{o}$ with degree of inertia $f = f(\mathfrak{O}, \mathfrak{o})$, then

$$\widetilde{\zeta^{\mathrm{irr}}_{\mathbf{G}_\Lambda(\mathfrak{O})}}(s) = R(\lambda_1^f, \ldots, \lambda_t^f, q^{-fs}); \tag{1.1.5}$$

see [48, Theorem A]. This property is not shared with some other zeta functions of groups: Let $\mathbf{H} = \langle x_1, x_2, y \mid [x_1, x_2] - z \rangle$ be the *Heisenberg group scheme*, so that the group $\mathbf{H}(\mathcal{O})$ is the *Heisenberg group* of upper uni-triangular 3×3-matrices over $\mathcal{O}$. [17, Theorem 3] assures that the local factors $\zeta^{\triangleleft}_{\mathbf{H}(\mathcal{O}_\mathfrak{p})}(s)$ are given by rational functions depending not only on the prime ideal $\mathfrak{p}$ of $\mathcal{O}$ but also on the degree of the finite extension $|K : \mathbb{Q}|$.

Stasinski and Voll also showed that $\widetilde{\zeta^{\mathrm{irr}}_{\mathbf{G}_\Lambda(\mathfrak{o})}}$ satisfies the following local functional equations:

$$\widetilde{\zeta^{\mathrm{irr}}_{\mathbf{G}_\Lambda(\mathfrak{o})}}(s) \mid_{\substack{q \to q^{-1} \\ \lambda_i \to \lambda_i^{-1}}} = q^d \widetilde{\zeta^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}}(s),$$

where $d = \dim(\Lambda' \otimes_{\mathcal{O}} K)$, with $\Lambda' = [\Lambda, \Lambda]$; see [48, Theorem A].

We may wonder whether (almost all) local factors of bivariate representation zeta functions of the groups $\mathbf{G}(\mathcal{O})$ are described by a rational function, and whether these local factors behave uniformly under base extensions and satisfy local functional equations under inversion of parameters. In fact, as we shall see in Section 1.2.1, our first main result Theorem 1 establishes these features for $\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$.

1.1.2 Conjugacy class zeta functions

Conjugacy classes and their sizes reflect properties of groups; see [8] for a survey. One may study the distribution of the conjugacy class sizes of a group G through the sequence of numbers

$$c_n(G) = \{\text{conjugacy classes of } G \text{ of cardinality } n\}.$$

If all numbers $c_n(G)$ are finite, we define the following Dirichlet series.

Definition 1.1.4. *The* conjugacy class zeta function *of the group G is*

$$\zeta^{\mathrm{cc}}_G(s) = \sum_{n=1}^{\infty} c_n(G) n^{-s}.$$

Let again $\mathbf{G}$ be a unipotent group scheme over $\mathcal{O}$. As for the numbers $r_n(\mathbf{G}(\mathcal{O}))$, the numbers $c_n(\mathbf{G}(\mathcal{O}))$ are not all finite. For instance, any free abelian group has infinitely many conjugacy classes of cardinality 1. Analogously to the representation case, we overcome the fact that $c_n(\mathbf{G}(\mathcal{O}))$ may be infinite by considering the finite numbers $c_n(\mathbf{G}(\mathcal{O}/I))$, where I is a nonzero ideal of $\mathcal{O}$, and then attaching them to a double Dirichlet series. This way we obtain a two-variable generating function such that one of the variables keeps track of the level quotient and the other one keeps track of the sizes of the conjugacy classes of these quotients.

Definition 1.1.5. *The* bivariate conjugacy class zeta function *of $\mathbf{G}(\mathcal{O})$ is*

$$\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O})}(s_1, s_2) = \sum_{(0) \neq I \trianglelefteq \mathcal{O}} \zeta^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O}/I)}(s_1) |\mathcal{O} : I|^{-s_2},$$

where s_1 and s_2 are complex variables.

Similarly to bivariate representation zeta functions of groups $\mathbf{G}(\mathcal{O})$, their bivariate conjugacy class zeta functions converge for s_1, $s_2 \in \mathbb{C}$ with sufficiently large real parts—see Proposition 2.5.5—and satisfy the *Euler decomposition*

$$\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O})}(s_1, s_2) = \prod_{\mathfrak{p}} \mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s_1, s_2), \tag{1.1.6}$$

see Proposition 3.1.2, where $\mathfrak{p}$ ranges over the nonzero prime ideals of $\mathcal{O}$ and the *local factors* are

$$\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s_1, s_2) = \mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathfrak{o})}(s_1, s_2) = \sum_{N=0}^{\infty} \zeta^{\mathrm{cc}}_{\mathbf{G}_N}(s_1) |\mathfrak{o} : \mathfrak{p}|^{-Ns_2}. \tag{1.1.7}$$

Almost all of these local factors are rational, behave uniformly under finite base extension, and satisfy functional equations, as we shall see in Theorem 1.

1.1.3 Class number zeta function

The total number of conjugacy classes of a group G is called its *class number* and is denoted by $\mathrm{k}(G)$. Let $\mathrm{Irr}(G)$ be the set of irreducible complex characters of G. If G is a finite group, then $\mathrm{k}(G) = |\mathrm{Irr}(G)| = |\mathrm{Rep}(G)|$. In particular, $\mathrm{k}(G) = \zeta^{\mathrm{cc}}_G(0) = \zeta^{\mathrm{irr}}_G(0)$.

For $\mathcal{T}$-groups of the form $\mathbf{G}(\mathcal{O})$, where $\mathbf{G}$ is a unipotent group scheme, one may define the following generating function.

Definition 1.1.6. *The* class number zeta function *of the $\mathcal{T}$-group $\mathbf{G}(\mathcal{O})$ is*

$$\zeta^{\mathrm{k}}_{\mathbf{G}(\mathcal{O})}(s) = \sum_{(0) \neq I \trianglelefteq \mathcal{O}} \mathrm{k}(\mathbf{G}(\mathcal{O}/I)) |\mathcal{O} : I|^{-s}.$$

The term 'conjugacy class zeta function' is sometimes used for what we call 'class number zeta function'; see for instance [5, 38, 39, 41].

Let $G \leq \mathrm{GL}_m$ be a $\mathbb{Z}$-defined algebraic subgroup which has the strong approximation property. For each $n \in \mathbb{N}$, consider the congruence subgroup $G^n(\mathfrak{o}) = \ker(G(\mathfrak{o}) \to G(\mathfrak{o}/\mathfrak{p}^n))$ of $G(\mathfrak{o})$ and the congruence quotient $G(\mathfrak{o}, n) := G(\mathfrak{o})/G^n(\mathfrak{o}) \cong G(\mathfrak{o}/\mathfrak{p}^n)$. In [5, Lemma 8.1], Berman, Derakhshan, Onn, and Paajanen defined the class number zeta function of groups $G(\mathfrak{o})$, where $\mathfrak{o}$ is the valuation ring of a non-Archimedean local field, by

$$\zeta^{\mathrm{k}}_{G(\mathfrak{o})}(s) = \sum_{n=0}^{\infty} \mathrm{k}(G(\mathfrak{o}, n)) q^{-ns},$$

and show that this zeta function satisfies an Euler decomposition. The proof methods also apply to groups of the form $\mathbf{G}(\mathcal{O})$, since unipotent groups have the strong approximation property; see [35, Lemma 5.5]. This means that the class number zeta functions of groups of the form $\mathbf{G}(\mathcal{O})$ admit Euler decompositions of the form

$$\zeta^{\mathrm{k}}_{\mathbf{G}(\mathcal{O})}(s) = \prod_{\mathfrak{p}} \zeta^{\mathrm{k}}_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s),$$

where $\mathfrak{p}$ ranges over the nonzero prime ideals of $\mathcal{O}$ and the *local factors* are

$$\zeta^{\mathrm{k}}_{\mathbf{G}(\mathfrak{o})}(s) = \sum_{N=0}^{\infty} \mathrm{k}(\mathbf{G}_N) q^{-Ns}.$$

Local class number zeta functions of Chevalley groups $G(\mathfrak{o})$, where $\mathfrak{o}$ is the valuation ring of a non-Archimedean local field of any (sufficiently large) characteristic, are rational functions; see [5, Theorem C]. Moreover, these zeta functions only depend on the size of the residue field of $\mathfrak{o}$.

We study class number zeta functions of groups of the form $\mathbf{G}(\mathcal{O})$ via the following specialisations of the bivariate zeta functions of Definitions 1.1.3 and 1.1.5

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O})}(0,s) = \mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O})}(0,s) = \zeta^{\mathrm{k}}_{\mathbf{G}(\mathcal{O})}(s). \tag{1.1.8}$$

In particular, by showing convergence and Euler decompositions for the bivariate zeta functions and rationality and functional equations for their local factors, we obtain analogous results for class numbers zeta functions via specialisation (1.1.8).

Let $\mathcal{K}$ be a non-Archimedean local field of characteristic zero with compact discrete valuation ring $\mathfrak{O}$. Let $\mathfrak{P}$ be the maximal ideal of $\mathfrak{O}$ and $q = |\mathfrak{O}/\mathfrak{P}|$. Given $G \leq \mathrm{GL}_n(\mathfrak{O})$, denote by G_n the image of G under $\mathrm{GL}_d(\mathfrak{O}) \to \mathrm{GL}_d(\mathfrak{O}/\mathfrak{P}^n)$. Rossmann [38, 39] studied class number zeta functions

$$\zeta^{\mathrm{k}}_{G(\mathfrak{O})}(s) := \sum_{n=0}^{\infty} \mathrm{k}(G_n) q^{-ns}$$

via specialisations of ask zeta functions, which are zeta functions encoding the average sizes of the kernels of modules of matrices over $\mathfrak{O}$. He showed the class number zeta functions of such groups are rational and satisfy functional equations; see [38, Theorem 1.4 and Theorem 4.18].

We conclude Section 1.1 with a simple example.

Example 1.1.7. Let $\mathbf{G}(\mathcal{O})$ be the free abelian torsion-free group $\mathcal{O}^m$, and let $\mathfrak{p}$ be a nonzero prime ideal of $\mathcal{O}$ with $q = |\mathcal{O} : \mathfrak{p}|$. Then, for $N \in \mathbb{N}_0$, it holds that

$$r_{q^i}(\mathbf{G}_N) = c_{q^i}(\mathbf{G}_N) = \begin{cases} q^{mN}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, for $* \in \{\mathrm{irr}, \mathrm{cc}\}$,

$$\mathcal{Z}^{*}_{\mathbf{G}(\mathfrak{o})}(s_1, s_2) = \mathcal{Z}^{*}_{\mathfrak{o}^m}(s_1, s_2) = \sum_{N=0}^{\infty} q^{N(m-s_2)} = \frac{1}{1-q^{m-s_2}}.$$

Consequently, $\mathcal{Z}^{*}_{\mathcal{O}^m}(s_1, s_2) = \zeta_K(s_2 - m)$, where $\zeta_K(s)$ denotes the Dedekind zeta function of the number field K. Moreover, these zeta functions converge on $\{(s_1, s_2) \in \mathbb{C}^2 \mid \mathrm{Re}(s_2) > 1 + m\}$ and admit meromorphic continuation to the whole of $\mathbb{C}^2$; see Section 2.6.

We see that the local factor at $\mathfrak{p}$ is rational in q and q^{-s_2} and satisfies the functional equation

$$\mathcal{Z}^{*}_{\mathfrak{o}^m}(s_1, s_2)\mid_{q \to q^{-1}} = -q^{m-s_2} \mathcal{Z}^{*}_{\mathfrak{o}^m}(s_1, s_2).$$

Specialisation (1.1.8) shows that $\zeta^{\mathrm{k}}_{\mathfrak{o}^m}(s) = \zeta_K(s-m)$. △

1.2 Main results

1.2.1 Arithmetic properties

Our first main result concerns uniform rationality and functional equations of local factors of bivariate representation and bivariate conjugacy class zeta functions of $\mathcal{T}$-groups $\mathbf{G}(\mathcal{O}) = \mathbf{G}_\Lambda(\mathcal{O})$ obtained from nilpotent Lie lattices; see Section 2.1.

Theorem 1. *For each $* \in \{\mathrm{irr}, \mathrm{cc}\}$, there exist a positive integer t^* and a rational function $R^*(X_1, \ldots, X_{t^*}, Y_1, Y_2)$ in $\mathbb{Q}(X_1, \ldots, X_{t^*}, Y_1, Y_2)$ such that, for all but finitely many nonzero prime ideals $\mathfrak{p}$ of $\mathcal{O}$, there exist algebraic integers $\lambda_1^*(\mathfrak{p}), \ldots, \lambda_{t^*}^*(\mathfrak{p})$ for which the following holds. For any finite extension $\mathfrak{O}$ of $\mathfrak{o} := \mathcal{O}_\mathfrak{p}$ with relative degree of inertia $f = f(\mathfrak{O}, \mathfrak{o})$,*

$$\mathcal{Z}^*_{\mathbf{G}(\mathfrak{O})}(s_1, s_2) = R^*(\lambda_1^*(\mathfrak{p})^f, \ldots, \lambda_{t^*}^*(\mathfrak{p})^f, q^{-fs_1}, q^{-fs_2}),$$

where $q = |\mathcal{O} : \mathfrak{p}|$. Moreover, these local factors satisfy the following functional equation:

$$\mathcal{Z}^*_{\mathbf{G}(\mathfrak{O})}(s_1, s_2) \mid_{\substack{q \to q^{-1} \\ \lambda_j^*(\mathfrak{p}) \to \lambda_j^*(\mathfrak{p})^{-1}}} = -q^{f(h-s_2)} \mathcal{Z}^*_{\mathbf{G}(\mathfrak{O})}(s_1, s_2),$$

where $h = \dim_K(\Lambda \otimes K)$.

The algebraic integers $\lambda_k^*(\mathfrak{p})$ are explained in Remark 3.4.5.

The statement of Theorem 1 is analogous to [48, Theorem A], and its proof relies on the methods of [3, 48, 51]; see Section 3.4. The main tools used in the proof of Theorem 1 are the Kirillov orbit method, the Lazard correspondence and $\mathfrak{p}$-adic integration.

As mentioned in Section 1.1.3, a consequence of Theorem 1 is that the local factors of the class number zeta function of $\mathbf{G}(\mathcal{O})$ are rational in $\lambda_i(\mathfrak{p})$, q, and q^{-s} and behave uniformly under base extension. Moreover, for a finite extension $\mathfrak{O}$ of $\mathfrak{o}$ with relative degree of inertia $f = f(\mathfrak{O}, \mathfrak{o})$, the local factors satisfy the functional equation

$$\zeta^{\mathrm{k}}_{\mathbf{G}(\mathfrak{O})}(s) \mid_{\substack{q \to q^{-1} \\ \lambda_j^*(\mathfrak{p}) \to \lambda_j^*(\mathfrak{p})^{-1}}} = -q^{f(h-s)} \zeta^{\mathrm{k}}_{\mathbf{G}(\mathfrak{O})}(s).$$

Rossmann proved independently in [38, Corollary 4.10 and Theorem 4.15], via specialisation of the ask zeta function—cf. [38, Definition 1.3]—, rationality and functional equations for local factors of class number zeta functions of such groups under mild assumptions on the group $\mathbf{G}(\mathfrak{o})$ and the characteristic p of $\mathfrak{o}/\mathfrak{p}$.

1.2.2 Examples: Groups of type F, G, and H

We provide explicit formulae for the bivariate representation and the bivariate conjugacy class zeta functions of three infinite families of $\mathcal{T}_2$-groups. Consequently, we obtain explicit formulae for their twist representation and class number zeta functions. The local factors of these zeta functions are also expressed in terms of sums over finite hyperoctahedral groups, which provides formulae for joint distributions of three statistics on such groups.

Definition 1.2.1. *For $n \in \mathbb{N}$ and $\delta \in \{0,1\}$, consider the nilpotent $\mathbb{Z}$-Lie lattices*

$$\begin{aligned}
\mathcal{F}_{n,\delta} &= \langle x_k, y_{ij} \mid [x_i, x_j] - y_{ij}, 1 \leq k \leq 2n+\delta, 1 \leq i < j \leq 2n+\delta \rangle,\\
\mathcal{G}_n &= \langle x_k, y_{ij} \mid [x_i, x_{n+j}] - y_{ij}, 1 \leq k \leq 2n, 1 \leq i, j \leq n \rangle,\\
\mathcal{H}_n &= \langle x_k, y_{ij} \mid [x_i, x_{n+j}] - y_{ij}, [x_j, x_{n+i}] - y_{ij},\ 1 \leq k \leq 2n, 1 \leq i \leq j \leq n \rangle.
\end{aligned}$$

By convention, relations that do not follow from the given ones are trivial.

Let Λ be one of the $\mathbb{Z}$-Lie lattices of Definition 1.2.1. We consider the unipotent group scheme $\mathbf{G}_\Lambda$ associated to Λ obtained by the construction of [48, Section 2.4], see Section 2.1. Following [48], these unipotent group schemes are denoted by $F_{n,\delta}$, G_n, and H_n, and groups of the form $F_{n,\delta}(\mathcal{O})$, $G_n(\mathcal{O})$, and $H_n(\mathcal{O})$ are called groups of type F, G, and H, respectively.

The unipotent group schemes $F_{n,\delta}$, G_n, and H_n provide different generalisations of the Heisenberg group scheme $\mathbf{H} = \langle x_1, x_2, y \mid [x_1, x_2] - z \rangle$. The interest in such $\mathbb{Z}$-Lie lattices arises from their very construction. Roughly speaking, their defining relations reflect the reduced, irreducible, prehomogeneous vector spaces of complex $n \times n$ antisymmetric matrices, complex $n \times n$-matrices and complex $n \times n$ symmetric matrices, respectively,—here, the relative invariants are given respectively by Pf, det and det, where $\mathrm{Pf}(X)$ denotes the Pfaffian of an antisymmetric matrix X. We refer the reader to [48, Section 6] for details.

Bivariate conjugacy class and class number zeta functions

Theorem 2. *Let $n \in \mathbb{N}$, and $\delta \in \{0,1\}$. Then, for each nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$ with $q = |\mathcal{O} : \mathfrak{p}|$,*

$$\mathcal{Z}^{\mathrm{cc}}_{F_{n,\delta}(\mathfrak{o})}(s_1, s_2) = \frac{1 - q^{\binom{2n+\delta-1}{2} - (2n+\delta-1)s_1 - s_2}}{(1 - q^{\binom{2n+\delta}{2} - s_2})(1 - q^{\binom{2n+\delta}{2} + 1 - (2n+\delta-1)s_1 - s_2})}.$$

Write $q^{-s_1} = T_1$ and $q^{-s_2} = T_2$. For $n \geq 2$,

$$\mathcal{Z}^{\mathrm{cc}}_{G_n(\mathfrak{o})}(s_1, s_2) =$$

$$\frac{(1 - q^{2\binom{n}{2}} T_1^n T_2)(1 - q^{2\binom{n}{2}+1} T_1^{2n-1} T_2) + q^{n^2} T_1^n T_2 (1 - q^{-n})(1 - q^{-(n-1)} T_1^{n-1})}{(1 - q^{n^2} T_2)(1 - q^{n^2} T_1^n T_2)(1 - q^{n^2+1} T_1^{2n-1} T_2)},$$

$$\mathcal{Z}^{\mathrm{cc}}_{H_n(\mathfrak{o})}(s_1, s_2) =$$

$$\frac{(1 - q^{\binom{n}{2}} T_1^n T_2)(1 - q^{\binom{n}{2}+2} T_1^{2n-1} T_2) + q^{\binom{n+1}{2}} T_1^n T_2 (1 - q^{-n+1})(1 - q^{-(n-1)} T_1^{n-1})}{(1 - q^{\binom{n+1}{2}} T_2)(1 - q^{\binom{n+1}{2}+1} T_1^n T_2)(1 - q^{\binom{n+1}{2}+1} T_1^{2n-1} T_2)}.$$

Denote by $\mathrm{Spec}(\mathcal{O})$ the set of prime ideals of $\mathcal{O}$. Specialisation (1.1.8) yields the following.

Corollary 1.2.2. *For all $n \geq 1$ and $\delta \in \{0,1\}$,*

$$\zeta^{\mathrm{k}}_{F_{n,\delta}(\mathcal{O})}(s) = \frac{\zeta_K(s - \binom{2n+\delta}{2} - 1)\zeta_K(s - \binom{2n+\delta}{2})}{\zeta_K(s - \binom{2n+\delta-1}{2})}, \tag{1.2.1}$$

where $\zeta_K(s)$ is the Dedekind zeta function of the number field $K = \mathrm{Frac}(\mathcal{O})$.

Furthermore, for $n \geq 2$,

$$\zeta^{\mathrm{k}}_{G_n(\mathcal{O})}(s) = \prod_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O})\setminus\{(0)\}} \frac{(1-q_{\mathfrak{p}}^{2\binom{n}{2}-s})(1-q_{\mathfrak{p}}^{2\binom{n}{2}+1-s}) + q_{\mathfrak{p}}^{n^2-s}(1-q_{\mathfrak{p}}^{-n})(1-q_{\mathfrak{p}}^{-n+1})}{(1-q_{\mathfrak{p}}^{n^2-s})^2(1-q_{\mathfrak{p}}^{n^2+1-s})},$$

$$\zeta^{\mathrm{k}}_{H_n(\mathcal{O})}(s) = \prod_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O})\setminus\{(0)\}} \frac{(1-q_{\mathfrak{p}}^{\binom{n}{2}-s})(1-q_{\mathfrak{p}}^{\binom{n}{2}+2-s}) + q_{\mathfrak{p}}^{\binom{n+1}{2}-s}(1-q_{\mathfrak{p}}^{-n+1})^2}{(1-q_{\mathfrak{p}}^{\binom{n+1}{2}-s})(1-q_{\mathfrak{p}}^{\binom{n+1}{2}+1-s})^2},$$

where $q_{\mathfrak{p}} = |\mathcal{O}:\mathfrak{p}|$, *for all* $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}) \setminus \{(0)\}$.

In particular, all local factors of the bivariate conjugacy class zeta functions of groups of type F, G, and H are rational in $q_{\mathfrak{p}}$, $q_{\mathfrak{p}}^{-s_1}$, and $q_{\mathfrak{p}}^{-s_2}$, whilst all local factors of their class number zeta functions are rational in $q_{\mathfrak{p}}$ and $q_{\mathfrak{p}}^{-s}$. Moreover, all local factors of both types of zeta functions satisfy functional equations. This (slightly) generalises Theorem 1 for these groups.

Formula (1.2.1) was shown independently in [38]; it is a consequence of [38, Proposition 5.11 and Proposition 6.4]; see Remarks 3.2.13 and 5.1.4.

Bivariate representation and twist representation zeta functions

To state our next result, we introduce some notation.

Let X, Y denote indeterminates in the field $\mathbb{Q}(X,Y)$. Given $n \in \mathbb{N}$, set $(\underline{n})_X = 1 - X^n$ and $(\underline{n})_X! = (\underline{n})_X(\underline{n-1})_X \dots (\underline{1})_X$. For $a, b \in \mathbb{N}_0$ such that $a \geq b$, the X-binomial coefficient of a over b is

$$\binom{a}{b}_X = \frac{(\underline{a})_X}{(\underline{b})_X(\underline{a-b})_X} \in \mathbb{Z}[X].$$

Given $n \in \mathbb{N}$, write $[n] = \{1,\dots,n\}$ and $[n]_0 = [n] \cup \{0\}$. Given a subset $\{i_1,\dots,i_l\} \subset \mathbb{N}$, we write $\{i_1,\dots,i_l\}_<$ meaning that $i_1 < i_2 < \dots < i_l$. For $I = \{i_1,\dots,i_l\}_< \subseteq [n-1]_0$, set $\mu_j := i_{j+1} - i_j$ for all $j \in [l]_0$, where $i_0 = 0$, $i_{l+1} = n$, and define

$$\binom{n}{I}_X = \binom{n}{i_l}_X \binom{i_l}{i_{l-1}}_X \cdots \binom{i_2}{i_1}_X.$$

The Y-Pochhammer symbol is defined as

$$(X;Y)_n = \prod_{i=0}^{n-1}(1 - XY^i).$$

Theorem 3. *Let* $\mathbf{G} \in \{F_{n,\delta}, G_n, H_n\}$ *for some* $n \in \mathbb{N}$ *and* $\delta \in \{0,1\}$. *Then, for each nonzero prime ideal* $\mathfrak{p}$ *of* $\mathcal{O}$ *with* $q = |\mathcal{O}:\mathfrak{p}|$,

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}(s_1,s_2) = \frac{1}{1-q^{\bar{a}(\mathbf{G},n)-s_2}} \sum_{I\subseteq[n-1]_0} f_{\mathbf{G},I}(q^{-1}) \prod_{i\in I} \frac{q^{\bar{a}(\mathbf{G},i)-(n-i)s_1-s_2}}{1-q^{\bar{a}(\mathbf{G},i)-(n-i)s_1-s_2}},$$

where $f_{\mathbf{G},I}(X)$ *and* $\bar{a}(\mathbf{G},i)$, *for all* $I = \{i_1,\dots,i_l\}_< \subseteq [n-1]_0$ *and for all* $i \in [n]_0$, *are defined as in Table 1.1.*

The numbers $\bar{a}(\mathbf{G},i)$ are slight modifications of the numbers $a(\mathbf{G},i)$ given in [48, Theorem C], namely $\bar{a}(F_{n,\delta},i) = a(F_{n,\delta},i)+2i+\delta$ and $\bar{a}(\mathbf{G},i) = a(\mathbf{G},i)+2i$, for $\mathbf{G}(\mathcal{O})$ of type G and H.

Since groups of type F, G, and H are $\mathcal{T}_2$-groups, we may obtain formulae for their twist zeta functions via (1.1.4). The constant r appearing in (1.1.4) in this

$\mathbf{G}$	$f_{\mathbf{G},I}(X)$	$\bar{a}(\mathbf{G},i)$
$F_{n,\delta}$	$\binom{n}{I}_{X^2}(X^{2(i_1+\delta)+1};X^2)_{n-i_1}$	$\binom{2n+\delta}{2}-\binom{2i+\delta}{2}+2i+\delta$
G_n	$\binom{n}{I}_X(X^{i_1+1};X)_{n-i_1}$	n^2-i^2+2i
H_n	$\left(\prod_{j=1}^{l}(X^2;X^2)^{-1}_{\lfloor\mu_j/2\rfloor}\right)(X^{i_1+1};X)_{n-i_1}$	$\binom{n+1}{2}-\binom{i+1}{2}+2i$

Table 1.1: Numerical data associated to $\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O})}$ for $\mathbf{G}\in\{\mathcal{F}_{n,\delta},\mathcal{G}_n,\mathcal{H}_n\}$

case is $r=\bar{a}(\mathbf{G},n)$, that is, $r=2n+\delta$ if $\Lambda=\mathcal{F}_{n,\delta}$ and $r=2n$ if $\Lambda\in\{\mathcal{G}_n,\mathcal{H}_n\}$. Then

$$\widetilde{\zeta^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}}(s)=\sum_{I\subseteq[n-1]_0}f_{\mathbf{G},I}(q^{-1})\prod_{i\in I}\frac{q^{\bar{a}(\mathbf{G},i)-(n-i)(s-2)-r}}{1-q^{\bar{a}(\mathbf{G},i)-(n-i)(s-2)-r}}$$
$$=\sum_{I\subseteq[n-1]_0}f_{\mathbf{G},I}(q^{-1})\prod_{i\in I}\frac{q^{a(\mathbf{G},i)-(n-i)s}}{1-q^{a(\mathbf{G},i)-(n-i)s}},$$

which agrees with [48, Theorem C].

Formulae for the twist representation zeta functions of groups of type F, G, and H are given in [48, Theorem B] in terms of Dedekind zeta functions. We remark that the bivariate representation zeta functions of these groups cannot be written in terms of Dedekind zeta functions. For instance, using Theorem 3 one can calculate the bivariate representation zeta function of $F_{2,0}(\mathfrak{o})$. Write $T_1=q^{-s_1}$ and $T_2=q^{-s_2}$. Then, $\mathcal{Z}^{\mathrm{irr}}_{F_{2,0}(\mathfrak{o})}(s_1,s_2)$ equals

$$-\frac{q^7T_1^3T_2^2-q^5T_1^2T_2+q^5T_1T_2-q^4T_1T_2-q^3T_1^2T_2+q^2T_1^2T_2-q^2T_1T_2+1}{(1-q^7T_1T_2)(1-q^6T_1^2T_2)(1-q^4T_2)}.$$

However, it follows from specialisation 1.1.4 and Corollary 1.2.2 that setting $T_1=1$ in this formula will produce a function on q and T_2 which can be written in terms of Dedekind zeta functions.

Sums over finite hyperoctahedral groups

The polynomials $f_{\mathbf{G},I}(X)$ appearing in Table 1.1 can be expressed in terms of distributions of statistics on Weyl groups of type B, also called *hyperoctahedral groups* B_n; see Section 5.3.1. These are the groups of permutations w of the set $[\pm n]_0=\{-n,\dots,n\}$ such that $w(-i)=-w(i)$ for all $i\in[\pm n]_0$.

In Section 5.3.2, we describe the local bivariate representation zeta functions of $\mathbf{G}(\mathcal{O})$ as sums over B_n in terms of statistics on such groups. As the local factors of the bivariate representation and the bivariate conjugacy class zeta functions of $\mathbf{G}(\mathcal{O})$ specialise to the local factors of its class number zeta function, the formulae in terms of statistics on hyperoctahedral groups B_n can be compared with the formulae of Corollary 1.2.2, which leads to formulae for the joint distribution of three functions on Weyl groups of type B; see Propositions 5.3.5 and 5.3.6.

More precisely, the formulae of bivariate representation zeta functions in terms of statistics on hyperoctahedral groups under specialisation (1.1.8) provide a formula of the following form for the class number zeta function of $\mathbf{G}(\mathfrak{o})$:

$$\zeta^{\mathrm{k}}_{\mathbf{G}(\mathfrak{o})}(s)=\frac{\sum_{w\in B_n}\chi_{\mathbf{G}}(w)q^{-\overline{h_{\mathbf{G}}}(w)-\mathrm{des}(w)s}}{\prod_{i=0}^{n}(1-q^{\bar{a}(\mathbf{G},i)-s})};\tag{1.2.2}$$

see Lemma 5.3.4. Here, $\chi_{\mathbf{G}}$ is one of the linear characters $(-1)^{\mathrm{neg}}$ or $(-1)^{\ell}$ of B_n, where $\mathrm{neg}(w)$ denotes the number of negative entries of w, and ℓ is the standard Coxeter length function of B_n. Moreover, the functions $\overline{h_{\mathbf{G}}}$ are sums of statistics on B_n for each $\mathbf{G}$ and $\mathrm{des}(w)$ is the cardinality of the descent set of $w \in B_n$; see Section 5.3.1 for definitions.

The formulae for the bivariate zeta functions given in Theorems 2 and 3 allow us to strengthen Theorem 1 for groups of type F, G, and H by showing that its conclusion holds for *all* local factors:

Theorem 4. *Let* $\mathbf{G} \in \{F_{n,\delta}, G_n, H_n\}$ *and* $* \in \{\mathrm{irr}, \mathrm{cc}\}$. *Then, for every nonzero prime ideal* $\mathfrak{p}$ *of* $\mathcal{O}$ *with* $|\mathcal{O} : \mathfrak{p}| = q$, *the local bivariate zeta function* $\mathcal{Z}^*_{\mathbf{G}(\mathfrak{o})}(s_1, s_2)$ *satisfies the functional equation*

$$\mathcal{Z}^*_{\mathbf{G}(\mathfrak{o})}(s_1, s_2) \mid_{q \to q^{-1}} = -q^{h - s_2} \mathcal{Z}^*_{\mathbf{G}(\mathfrak{o})}(s_1, s_2),$$

where h *is the torsion free rank of* $\Lambda(\mathfrak{o}) = \Lambda \otimes_{\mathfrak{o}} \mathfrak{o}$; *see the exact value of* h *in Table 5.1.*

In fact, Theorem 1 states that almost all local factors satisfy functional equations of such form, whilst Theorems 2 and 3 state that all local factors are given by the same rational functions. We give an alternative proof of Theorem 4 for bivariate representation zeta functions using the descriptions (1.2.2) in Section 5.3.3.

1.2.3 Analytic properties

Having defined and worked with the bivariate zeta functions, it is natural to ask for their domains of convergence. As mentioned in Sections 1.1.1 and 1.1.2, bivariate representation and bivariate conjugacy class zeta functions of groups $\mathbf{G}(\mathcal{O})$, where $\mathbf{G}$ is a unipotent group scheme, converge for $s_1,\ s_2 \in \mathbb{C}$ with sufficiently large real parts. In contrast with the one-variable case, however, the maximal domain of convergence may not be of the form $\{(s_1, s_2) \in \mathbb{C}^2 \mid \mathrm{Re}(s_1) > \alpha_1,\ \mathrm{Re}(s_1) > \alpha_2\}$.

In fact, the formulae of Theorems 2 and 3 show that the domains of convergence of the bivariate zeta functions of groups of type F, G and H are as follows; see Section 2.6.

- $\mathcal{Z}^{\mathrm{cc}}_{F_{n,\delta}(\mathcal{O})}(s_1, s_2)$ converges for
  $$\mathrm{Re}((2n + \delta - 1)s_1 + s_2) > 2 + \binom{2n+\delta}{2} \text{ and } \mathrm{Re}(s_2) > 1 + \binom{2n+\delta}{2},$$
- $\mathcal{Z}^{\mathrm{cc}}_{G_n(\mathcal{O})}(s_1, s_2)$ converges for
  $$\mathrm{Re}((2n-1)s_1+s_2) > 2+n^2,\ \mathrm{Re}(ns_1+s_2) > 1+n^2 \text{ and } \mathrm{Re}(s_2) > 1+n^2,$$
- $\mathcal{Z}^{\mathrm{cc}}_{H_n(\mathcal{O})}(s_1, s_2)$ converges for
  $$\mathrm{Re}((2n - 1)s_1 + s_2) > 2 + \binom{n+1}{2},\ \mathrm{Re}(ns_1 + s_2) > 2 + \binom{n+1}{2} \text{ and } \mathrm{Re}(s_2) > 1 + \binom{n+1}{2},$$
- $\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$ for $\mathbf{G} \in \{F_{n,\delta}, G_n, H_n\}$ converges for
  $$\mathrm{Re}((n - i)s_1 + s_2) > 1 + a(\mathbf{G}, i), \forall i \in [n]_0.$$

These domains are all independent of the ring of integers $\mathcal{O}$ considered. Moreover, it follows easily from the formulae of Theorem 2 that $\mathcal{Z}^{\mathrm{cc}}_{F_{n,\delta}(\mathcal{O})}(s_1, s_2)$ admits meromorphic continuation to the whole $\mathbb{C}^2$, and that $\mathcal{Z}^{\mathrm{cc}}_{G_n(\mathcal{O})}(s_1, s_2)$ and $\mathcal{Z}^{\mathrm{cc}}_{H_n(\mathcal{O})}(s_1, s_2)$ admit meromorphic continuations to open domains which are independent of the ring of integers $\mathcal{O}$. It follows from the formulae of Theorem 3 that bivariate representation zeta functions of groups of type F, G, and H admit meromorphic continuation to

$$\left\{(s_1, s_2) \in \mathbb{C}^2 \mid \mathrm{Re}((n-i)s_1 + s_2) > a(\mathbf{G}, i), \forall i \in [n]_0\right\};$$

see Section 2.6.

These examples raise the question of whether the domains of convergence and meromorphic continuation being independent of $\mathcal{O}$ is a general phenomenon for these bivariate zeta functions. It was previously showed by Dung and Voll in [11, Theorem A] that, for groups of the form $\mathbf{G}_\Lambda(\mathcal{O})$, where $\mathbf{G}_\Lambda$ is a unipotent group scheme associated to a $\mathcal{O}$-Lie lattice Λ, the twist representation zeta functions $\widetilde{\zeta^{\mathrm{irr}}_{\mathbf{G}_\Lambda(\mathcal{O})}}(s)$ converge on some open domain which is independent of $\mathcal{O}$ and admit meromorphic continuations to a larger open domain which is also independent of $\mathcal{O}$.

However, zeta functions of group of the form $\mathbf{G}(\mathcal{O})$, where $\mathbf{G}$ is a unipotent group scheme over $\mathcal{O}$, are not expected in general to have domains of convergence and meromorphy which are independent of $\mathcal{O}$. In fact, the normal zeta function of the Heisenberg Group $\mathbf{H}(\mathcal{O})$ has abscissa of convergence depending on the degree of the extension $|K : \mathbb{Q}|$, see [43, Theorem 1.2] and [44, Theorems 3.2 and 3.8].

Our next main result concerns these properties for the bivariate zeta functions of groups of the form $\mathbf{G}_\Lambda(\mathcal{O})$; we show that for each $* \in \{\mathrm{irr}, \mathrm{cc}\}$ there exists a finite set $\mathcal{Q}^*$ of prime ideals of $\mathcal{O}$ such that the domains of convergence and meromorphic continuation of the bivariate function

$$\mathcal{Z}^{*,\mathcal{Q}^*}_{\mathbf{G}(\mathcal{O})}(s_1, s_2) = \prod_{\mathfrak{p} \notin \mathcal{Q}^*} \mathcal{Z}^*_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s_1, s_2), \tag{1.2.3}$$

are independent of the ring of integers $\mathcal{O}$. This means that, for each finite extension L/K with ring of integers $\mathcal{O}_L$, the domains of convergence and meromorphic continuation of $\mathcal{Z}^*_{\mathbf{G}(\mathcal{O}_L)}(s_1, s_2)$, up to finitely many local factors, are the same as the ones of $\mathcal{Z}^{*,\mathcal{Q}^*}_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$.

Let L/K be a finite extension with ring of integers $\mathcal{O}_L$. In the following, we denote by $\mathcal{O}_{L,\mathfrak{P}}$ the completion of $\mathcal{O}_L$ at the prime ideal $\mathfrak{P}$ of $\mathcal{O}_L$.

Theorem 5. *Denote by $\mathscr{D}^*_{\mathbf{G}(\mathcal{O})}$ the domain of convergence of $\mathcal{Z}^{*,\mathcal{Q}^*}_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$. This function admits meromorphic continuation to an open domain $\mathscr{M}^*_{\mathbf{G}(\mathcal{O})} \supsetneq \mathscr{D}^*_{\mathbf{G}(\mathcal{O})}$. Moreover, for each finite extension L/K with ring of integers $\mathcal{O}_L$, there exists a finite subset $\mathcal{Q}_L \subset \mathrm{Spec}(\mathcal{O}_L)$ such that the bivariate function*

$$\mathcal{Z}^{*,\mathcal{Q}_L}_{\mathbf{G}(\mathcal{O}_L)}(s_1, s_2) = \prod_{\mathfrak{P} \notin \mathcal{Q}_L} \mathcal{Z}^*_{\mathbf{G}(\mathcal{O}_{L,\mathfrak{P}})}(s_1, s_2)$$

satisfies:

1. *The domain of convergence of $\mathcal{Z}^{*,\mathcal{Q}}_{\mathbf{G}(\mathcal{O}_L)}(s_1, s_2)$ coincides with $\mathscr{D}^*_{\mathbf{G}(\mathcal{O})}$ and*

2. *$\mathcal{Z}^{*,\mathcal{Q}}_{\mathbf{G}(\mathcal{O}_L)}(s_1, s_2)$ admits meromorphic continuation to $\mathscr{M}^*_{\mathbf{G}(\mathcal{O})}$.*

In particular, the domains $\mathscr{D}^*_{\mathbf{G}(\mathcal{O})}$ *and* $\mathscr{M}^*_{\mathbf{G}(\mathcal{O})}$ *are independent of* $\mathcal{O}$*. We hence write* $\mathscr{D}^*_{\mathbf{G}} = \mathscr{D}^*_{\mathbf{G}(\mathcal{O})}$ *and* $\mathscr{M}^*_{\mathbf{G}} = \mathscr{M}^*_{\mathbf{G}(\mathcal{O})}$.

1.3 Organisation of chapters

This thesis is composed of the three articles [27, 28, 26]. In the introductory Chapter 2, we give notations and recall results that will be needed. In particular, we explain how to obtain a unipotent group scheme $\mathbf{G}_\Lambda$ from a nilpotent $\mathcal{O}$-Lie lattice Λ, calculate and recall some properties of $\mathfrak{p}$-adic integrals, and recall definitions and analytic properties of complex functions on two variables and double Dirichlet series.

Chapter 3 corresponds to the article [27], which is dedicated to algebraic properties of the bivariate zeta functions of Definitions 1.1.3 and 1.1.5. We start by showing the Euler decompositions presented in (1.1.2) and (1.1.6). Next, we show that almost all local factors of these decompositions can be written as $\mathfrak{p}$-adic integrals. The main tools used are the Kirillov orbit method in the context of bivariate representation zeta functions, and the Lazard correspondence in the context of bivariate conjugacy class zeta functions. These integrals are used in Section 3.3 to prove the specialisation (1.1.4) of local bivariate zeta functions of $\mathcal{T}_2$-groups $\mathbf{G}_\Lambda(\mathcal{O})$ to their twist representation zeta functions, and in Section 3.4 to prove Theorem 1. The latter is proved using the methods of [3, 48, 51], which essentially consist of writing the obtained $\mathfrak{p}$-adic integrals in terms of formulae of Denef type which are uniform under base extensions.

Chapter 4 corresponds to the article [26], which deals with analytic properties of these bivariate zeta functions. In Section 4.1, we use the formulae of local factors in terms of formulae of Denef type given in Chapter 3 to read off their domains of convergence, proving Theorem 5(1). In Section 4.2, we extend these zeta functions meromorphically to open domains which are independent of $\mathcal{O}$, proving Theorem 5(2).

Chapter 5 corresponds to the article [28]; we provide results related to the bivariate zeta functions of groups of type F, G, and H. We calculate in Section 5.1 their bivariate conjugacy class zeta functions and in Section 5.2 their bivariate representation zeta functions. That is, we prove Theorems 2 and 3 in these sections. As an application of these results, we obtain in Section 5.3 formulae for joint distributions of three statistics on finite hyperoctahedral groups and and give an alternative proof for the fact that the bivariate representation zeta functions of these groups satisfy functional equations for all local factors.

Chapter 2

Preliminaries

2.1 Group schemes obtained from nilpotent Lie lattices

Here, we recall from [48, Section 2.1.2] the construction of unipotent group schemes $\mathbf{G}$ associated to nilpotent $\mathcal{O}$-Lie lattices. An $\mathcal{O}$-Lie lattice is a free and finitely generated $\mathcal{O}$-module Λ together with an antisymmetric bi-additive form $[\ ,\ ]$ which satisfies the Jacobi identity.

Let Λ be a nilpotent $\mathcal{O}$-Lie lattice of class c. Fix an $\mathcal{O}$-basis $(x_1, \dots, x_h)$ for Λ. For each $\mathcal{O}$-algebra R, denote by $\Lambda(R)$ the R-module $\Lambda \otimes_{\mathcal{O}} R$ which has basis $(\mathbf{x}_1, \dots, \mathbf{x}_h)$, where $\mathbf{x}_i = x_i \otimes_{\mathcal{O}} 1$.

Suppose that $\Lambda' \subseteq c!\Lambda$, where $\Lambda' = [\Lambda, \Lambda]$ is the derived Lie sublattice. Define a group operation $*$ in $\Lambda(R)$ in terms of Hausdorff series. The obtained group $(\Lambda(R), *)$ is nilpotent of class c and the group operation $*$ is given in terms of polynomials over $\mathcal{O}$ which are independent of the algebra R, when considering coordinates on the basis $(\mathbf{x}_1, \dots, \mathbf{x}_h)$. This process defines a unipotent group scheme $\mathbf{G}_\Lambda$ over $\mathcal{O}$ isomorphic as a scheme to affine h-space over $\mathcal{O}$ which represents the group functor $R \mapsto (\Lambda(R), *)$. The group scheme $\mathbf{G}_\Lambda$ is called the *unipotent group scheme associated to the $\mathcal{O}$-Lie lattice* Λ.

The group $\mathbf{G}_\Lambda(\mathcal{O})$ is a $\mathcal{T}$-group of same nilpotency class c as Λ. If R is a finitely generated pro-p ring, then $\mathbf{G}_\Lambda(R)$ is a finitely generated nilpotent p-group of class c.

For Lie lattices Λ of nilpotency class 2, a different construction of such unipotent group schemes is given in [48, Section 2.4], in which case the hypothesis $\Lambda' \subseteq 2\Lambda$ is not needed. However, if this condition holds, the unipotent group schemes obtained via such construction coincides with the latter ones. We recall briefly this construction.

Assume $c = 2$. Every element ν of $\Lambda(R)$ can be uniquely expressed as $\nu = \sum_{i=1}^{h} a_i \mathbf{x}_i$. Following [48], we adopt multiplicative notation and identify ν with $\underline{\mathbf{x}}^{\underline{a}} = \mathbf{x}_1^{a_1} \dots \mathbf{x}_h^{a_h}$, where $\underline{\mathbf{x}} = (\mathbf{x}_1, \dots, \mathbf{x}_h)$ and $\underline{a} = (a_1, \dots, a_h)$. The group multiplication $*$ in this case is given as follows: for $1 \leq i < j \leq h$,

$$\mathbf{x}_i^{a_i} * \mathbf{x}_j^{a_j} = \mathbf{x}_i^{a_i}\mathbf{x}_j^{a_j}, \qquad \mathbf{x}_j^{a_j} * \mathbf{x}_i^{a_i} = \mathbf{x}_i^{a_i}\mathbf{x}_j^{a_j}\underline{\mathbf{x}}^{a_i a_j \underline{\boldsymbol{\lambda}}_{ij}},$$

where $\underline{\boldsymbol{\lambda}}_{ij} = (\lambda_{ij}^1, \dots, \lambda_{ij}^h)$ is given by $[\mathbf{x}_i, \mathbf{x}_j] = \sum_{k=1}^{h} \lambda_{ij}^k \mathbf{x}_k$. We then extend this operation to the set of all monomials. For each $i \in [h]$, we obtain polynomials $M_i(\underline{\mathbf{X}}, \widetilde{\underline{\mathbf{X}}}) = M_i(X_1, \ldots, X_h, \widetilde{X}_1, \ldots, \widetilde{X}_h)$ and $I_i(\underline{\mathbf{X}}) = I_i(X_1, \ldots, X_h)$ such that

$$\underline{\mathbf{x}}^{\underline{a}} * \underline{\mathbf{x}}^{\underline{a}'} = \underline{\mathbf{x}}^{\underline{a}+\underline{a}'+(M_i(\underline{a},\underline{a}'))_i} \text{ and } (\underline{\mathbf{x}}^{\underline{a}})^{-1} = \underline{\mathbf{x}}^{-\underline{a}+(I_i(\underline{a}))_i}.$$

This process defines a unipotent group scheme $\mathbf{G}_\Lambda$ over $\mathcal{O}$ isomorphic as a scheme to the group functor $R \mapsto (\{\underline{\mathbf{x}}^{\underline{a}} \mid \underline{a} \in R^h\}, *)$.

Example 2.1.1. Let Λ be the Heisenberg Lie lattice given by $\langle x_1, x_2, y \mid [x_1, x_2] - y\rangle$. Given an $\mathcal{O}$-algebra R, set $\mathbf{x}_1 = x_1 \otimes_{\mathcal{O}} 1$, $\mathbf{x}_2 = x_2 \otimes_{\mathcal{O}} 1$, and $\mathbf{x}_3 = y \otimes_{\mathcal{O}} 1$.

The structure constants λ_{ij}^k are such that $\underline{\boldsymbol{\lambda}}_{12} = (0,0,1)$ and $\underline{\boldsymbol{\lambda}}_{13} = \underline{\boldsymbol{\lambda}}_{23} = (0,0,0)$. The group operation $*$ is then given by $\mathbf{x}_2^{a_2} * \mathbf{x}_1^{a_1} = \mathbf{x}_1^{a_1}\mathbf{x}_2^{a_2}\mathbf{x}_3^{a_1 a_2}$, and $\mathbf{x}_j^{a_j} * \mathbf{x}_i^{a_i} = \mathbf{x}_i^{a_i}\mathbf{x}_j^{a_j}$, for all $(i,j) \in \{(1,3),(2,3)\}$. It follows that the polynomials $M_i(\underline{\mathbf{X}}, \widetilde{\underline{\mathbf{X}}})$ and $I_i(\underline{\mathbf{X}})$ vanish everywhere for $i = 1,2$, and $M_3(\underline{\mathbf{X}}) = \widetilde{X}_1 X_2$, $I_3(\underline{\mathbf{X}}) = X_1 X_2$. That is

$$\mathbf{x}_1^{a_1}\mathbf{x}_2^{a_2}\mathbf{x}_3^{a_3} * \mathbf{x}_1^{a_1'}\mathbf{x}_2^{a_2'}\mathbf{x}_3^{a_3'} = \mathbf{x}_1^{a_1+a_1'}\mathbf{x}_2^{a_2+a_2'}\mathbf{x}_3^{a_3+a_3'+a_1' a_2},$$
$$(\mathbf{x}_1^{a_1}\mathbf{x}_2^{a_2}\mathbf{x}_3^{a_3})^{-1} = \mathbf{x}_1^{-a_1}\mathbf{x}_2^{-a_2}\mathbf{x}_3^{-a_3+a_1 a_2}.$$

Denote by $\mathbf{H}$ the unipotent group scheme associated to Λ.

We observe that we cannot define $*$ by means of Hausdorff series, since $\mathbf{H}' \not\subset 2\mathbf{H}$. In fact, we would have

$$\mathbf{x}_1 * \mathbf{x}_2 = \mathbf{x}_1 + \mathbf{x}_2 + \tfrac{1}{2}[\mathbf{x}_1, \mathbf{x}_2] = \mathbf{x}_1 + \mathbf{x}_2 + \tfrac{\mathbf{x}_3}{2},$$

which may not be an element of $\mathbf{H}(R)$. For instance, $\frac{\mathbf{x}_3}{2} \notin \mathbf{H}(\mathbb{Z})$. △

Remark 2.1.2. Let G be a $\mathcal{T}$-group of nilpotency class c and Hirsch length h. Then there exist a $\mathbb{Q}$-Lie algebra $L_G(\mathbb{Q})$ of $\mathbb{Q}$-dimension h and an injective map $\log : G \to L_G(\mathbb{Q})$ such that $\log(G)$ spans $L_G(\mathbb{Q})$ over $\mathbb{Q}$; see [45, Section 6.A]. Moreover, there exists a subgroup H of finite index in G such that $\log(H)$ is a $\mathbb{Z}$-Lie lattice inside the algebra $L_G(\mathbb{Q})$ such that $\log(H)' \subseteq c!\log(H)$, so that H may be regarded as a group of the form $\mathbf{G}(\mathbb{Z})$, where $\mathbf{G}$ is the group scheme obtained from the $\mathbb{Z}$-Lie lattice $\log(H)$.

We may define the bivariate representation and the bivariate conjugacy class zeta functions of G to be the respective bivariate zeta functions of $H = \mathbf{G}(\mathbb{Z})$:

$$\mathcal{Z}_G^*(s_1, s_2) = \mathcal{Z}_{G,H}^*(s_1, s_2) := \mathcal{Z}_{\mathbf{G}(\mathbb{Z})}^*(s_1, s_2), \; * \in \{\mathrm{irr}, \mathrm{cc}\}.$$

If G is such a $\mathcal{T}$-group and $H_1 = \mathbf{G}_1(\mathbb{Z})$ and $H_2 = \mathbf{G}_2(\mathbb{Z})$ are subgroups of G of finite index, then H_1 and H_2 are commensurable and, therefore, they have the same pro-p completion for all but finitely many prime integers p; see [34, Lemma 1.8]. In particular, $\mathcal{Z}_{\mathbf{G}_1(\mathbb{Z}_p)}^*(s_1, s_2) = \mathcal{Z}_{\mathbf{G}_2(\mathbb{Z}_p)}^*(s_1, s_2)$, for all but finitely many primes p, that is, although $\mathcal{Z}_{G,H_1}^*(s_1, s_2)$ and $\mathcal{Z}_{G,H_2}^*(s_1, s_2)$ may not coincide, they are almost the same in the sense that they coincide except for finitely many local factors.

2.2 Some $\mathfrak{p}$-adic integrals

In this section we calculate some $\mathfrak{p}$-adic integrals which will be used. For the rest this section, we fix a nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$ and write $\mathfrak{o} = \mathcal{O}_\mathfrak{p}$. Let q be the cardinality of $\mathcal{O}/\mathfrak{p}$ and p its characteristic.

Given an element $z \in \mathfrak{o}$ satisfying $z \in \mathfrak{p}^e \setminus \mathfrak{p}^{e+1}$ for some $e \in \mathbb{N}_0$, its $\mathfrak{p}$-*adic valuation* is $v_\mathfrak{p}(z) = e$, and its $\mathfrak{p}$-*adic norm* is $|z|_\mathfrak{p} = q^{-v_\mathfrak{p}(z)} = q^{-e}$. Denote by $\|.\|_\mathfrak{p}$ the maximum norm with respect to $|.|_\mathfrak{p}$. For $N \in \mathbb{N}$, we also denote by $v_{\mathfrak{p}}$ the function on $\mathfrak{o}/\mathfrak{p}^N$ given as follows: let $\overline{z}$ be the image of $z \in \mathfrak{o}$ under $\mathfrak{o} \to \mathfrak{o}/\mathfrak{p}^N$ and assume that $z \in \mathfrak{p}^e \setminus \mathfrak{p}^{e+1}$. Then $v_{\mathfrak{p}}(\overline{z}) = e$ if $0 \le e < N$ and, otherwise, $v_{\mathfrak{p}}(\overline{z}) = \infty$. We write $\mathfrak{p}^m$ for the mth ideal power $\mathfrak{p} \cdots \mathfrak{p}$ and $\mathfrak{p}^{(m)}$ is the m-fold Cartesian power $\mathfrak{p} \times \cdots \times \mathfrak{p}$. The valuation $v_{\mathfrak{p}}$ of $\mathfrak{o}$ can be extended to $\mathfrak{o}^n$ by mapping each $\mathbf{z} \in \mathfrak{o}^n$ to $v_{\mathfrak{p}}(\mathbf{z}) = e$ whenever $\mathbf{z} \in \mathfrak{p}^e \mathfrak{o}^n \setminus \mathfrak{p}^{e+1}\mathfrak{o}^n$. The $\mathfrak{p}$-adic norm is $|\mathbf{z}|_{\mathfrak{p}} = q^{-v_{\mathfrak{p}}(\mathbf{z})}$, which coincides with $\|\mathbf{z}\|_{\mathfrak{p}}$. Given $k \in \mathbb{N}$, set

$$W_k^{\mathfrak{o}} := (\mathfrak{o}^k)^* = \{\mathbf{x} \in \mathfrak{o}^k \mid v_{\mathfrak{p}}(\mathbf{x}) = 0\}.$$

From now on, μ denotes the additive Haar measure on $\mathfrak{o}$, normalised so that $\mu(\mathfrak{o}) = 1$. We also denote by μ the product measure on $\mathfrak{o}^n$, for $n \in \mathbb{N}$.

Lemma 2.2.1. *For $r \in \mathbb{C}$ with sufficiently large real part, and for each $k \in \mathbb{N}$,*

$$\int_{w \in \mathfrak{p}^k} |w|_{\mathfrak{p}}^r d\mu = \frac{q^{-k(r+1)}(1-q^{-1})}{1-q^{-k(r+1)}}.$$

Proof. For each $i \in \mathbb{N}$, we see that $\mathfrak{p}^i = \{x \in \mathfrak{o} \mid v_{\mathfrak{p}}(x) \ge i\}$ and $\mathfrak{p}^i \setminus \mathfrak{p}^{i+1} = \{x \in \mathfrak{o} \mid v_{\mathfrak{p}}(x) = i\}$. Thus, the ideal $\mathfrak{p}^k$ is the disjoint union $\mathfrak{p}^k = \bigcup_{i=k}^{\infty} \mathfrak{p}^i \setminus \mathfrak{p}^{i+1}$. It follows that

$$\begin{aligned}\int_{w \in \mathfrak{p}^k} |w|_{\mathfrak{p}}^r d\mu &= \sum_{i=k}^{\infty} \int_{w \in \mathfrak{p}^i \setminus \mathfrak{p}^{i+1}} q^{-ir} d\mu = \sum_{i=k}^{\infty} \mu(\mathfrak{p}^i \setminus \mathfrak{p}^{i+1}) q^{-ir} \\ &= \sum_{i=k}^{\infty} q^{-i(r+1)}(1-q^{-1}) = \frac{q^{-k(r+1)}(1-q^{-1})}{1-q^{-k(r+1)}}.\end{aligned}$$ □

Let $A \subseteq \mathfrak{o}$ and $B \subseteq \mathfrak{o}^n$. In the following, we write

$$\mathcal{K}_{A,B}(r,t) := \int_{(y,\underline{x}) \in A \times B} |y|_{\mathfrak{p}}^r \|x_1, \ldots, x_n, y\|_{\mathfrak{p}}^t d\mu,$$

where r and t are complex variables.

The following lemma is a direct consequence of [38, Lemma 5.8], which assures in particular that, for r, $t \in \mathbb{C}$ with sufficiently large real parts, one has

$$\mathcal{K}_{\mathfrak{o} \times \mathfrak{o}^n}(r,t) = \frac{(1-q^{-1})(1-q^{-r-n-1})}{(1-q^{-r-t-n-1})(1-q^{-t-1})}. \tag{2.2.1}$$

Lemma 2.2.2. *For r, $t \in \mathbb{C}$ with sufficiently large real parts, and for each $n \in \mathbb{N}_0$, the following holds.*

$$\begin{aligned}\mathcal{K}_{\mathfrak{p} \times \mathfrak{o}^n}(r,t) &= \frac{(1-q^{-1})(1-q^{-n}+q^{-s-n}-q^{-r-t-n-1})q^{-r-1}}{(1-q^{-r-t-n-1})(1-q^{-r-1})}, \\ \mathcal{K}_{\mathfrak{p} \times \mathfrak{p}^{(n)}}(r,t) &= \frac{(1-q^{-1})(1-q^{-r-n-1})q^{-r-t-n-1}}{(1-q^{-r-t-n-1})(1-q^{-r-1})}.\end{aligned}$$

Proof. Since $\mathfrak{p} \times \mathfrak{o}^n = \mathfrak{o} \times \mathfrak{o}^n \setminus W_1^{\mathfrak{o}} \times \mathfrak{o}^n$ and $y \in W_1^{\mathfrak{o}}$ implies both $|y|_{\mathfrak{p}} = 1$ and $\|x_1, \ldots, x_n, y\|_{\mathfrak{p}} = 1$, it follows that

$$\mathcal{K}_{\mathfrak{p} \times \mathfrak{o}^n}(r,t) = \mathcal{K}_{\mathfrak{o} \times \mathfrak{o}^n}(r,t) - \mathcal{K}_{W_1^{\mathfrak{o}} \times \mathfrak{o}^n}(r,t) = \mathcal{K}_{\mathfrak{o} \times \mathfrak{o}^n}(r,t) - \mu(W_1^{\mathfrak{o}} \times \mathfrak{o}^n).$$

The first claim then follows from (2.2.1) and the fact that $\mu(W_1^{\mathfrak{o}} \times \mathfrak{o}^n) = 1 - q^{-1}$. Analogously, since $\mathfrak{p} \times \mathfrak{p}^{(n)} = \mathfrak{p} \times \mathfrak{o}^n \setminus \mathfrak{p} \times W_n^{\mathfrak{o}}$,

$$\mathcal{K}_{\mathfrak{p} \times \mathfrak{p}^{(n)}}(r,t) = \mathcal{K}_{\mathfrak{p} \times \mathfrak{o}^n}(r,t) - \mathcal{K}_{\mathfrak{p} \times W_n^{\mathfrak{o}}}(r,t) = \mathcal{K}_{\mathfrak{p} \times \mathfrak{o}^n}(r,t) - (1-q^{-n}) \int_{y \in \mathfrak{p}} |y|_{\mathfrak{p}}^r d\mu.$$

The second claim then follows from the first part and Lemma 2.2.1. □

For the next lemma, consider the vector of variables $\underline{X} = (X_{11}, \dots, X_{2n})$ and the matrix

$$M(\underline{X}) = \begin{bmatrix} X_{11} & X_{12} & \dots & X_{1n} \\ X_{21} & X_{22} & \dots & X_{2n} \end{bmatrix} \in \mathrm{Mat}_{2\times n}(\mathfrak{o}[\underline{X}]).$$

The minors of $M(\underline{X})$ are given by $M_{ij}(\underline{X}) := X_{1i}X_{2j} - X_{1j}X_{2i}$ for $1 \leq i < j \leq n$. Set $\mathbf{M}(\underline{X}) := \{M_{ij}(\underline{X}) \mid 1 \leq i < j \leq n\}$.

Lemma 2.2.3. *For s, $r \in \mathbb{C}$ with sufficiently large real parts, the following holds.*

$$\int_{(y,\underline{x})\in\mathfrak{p}\times W_{2n}^{o}} |y|_{\mathfrak{p}}^{r} \|\mathbf{M}(\underline{x}) \cup \{y\}\|_{\mathfrak{p}}^{s} d\mu =$$

$$\frac{(q^n-1)(1-q^{-1})q^{-r-2n-1}}{(1-q^{-1-r})(1-q^{-r-s-n})}\left((q+1)(1-q^{-r-n})q^{-s} + (q^n-q)(1-q^{-r-s-n})\right).$$

Proof. Let $A_1, \dots, A_k$ be representatives of the classes of $\mathfrak{o}^{2n}/\mathfrak{p}^{(2n)}$ which are different of $\mathfrak{p}^{(2n)}$, that is,

$$\mathfrak{o}^{2n} = \left(\cup_{m=1}^{k} A_m + \mathrm{Mat}_{2\times n}(\mathfrak{p})\right) \cup \mathrm{Mat}_{2\times n}(\mathfrak{p}),$$

where $\mathrm{Mat}_{2\times n}(\mathfrak{p})$ is the set of all $2 \times n$-matrices over $\mathfrak{p}$. In the following, we determine the integrals

$$\mathcal{I}_{A_m}(s,r) := \int_{(y,\underline{x})\in\mathfrak{p}\times(A_m+\mathrm{Mat}_{2\times n}(\mathfrak{p}))} |y|_{\mathfrak{p}}^{r} \|\mathbf{M}(\underline{x}) \cup \{y\}\|_{\mathfrak{p}}^{s} d\mu,$$

because

$$\int_{(y,\underline{x})\in\mathfrak{p}\times W_{2n}^{o}} |y|_{\mathfrak{p}}^{r} \|\mathbf{M}(\underline{x}) \cup \{y\}\|_{\mathfrak{p}}^{s} d\mu = \sum_{m=1}^{k} \mathcal{I}_{A_m}(s,r)$$

If $\underline{x} \in \mathrm{Mat}_{2\times n}(\mathfrak{o})$ and A_m determine the same class modulo $\mathrm{Mat}_{2\times n}(\mathfrak{p})$, then $\mathrm{rk}(\underline{x}) = \mathrm{rk}(A_m)$ modulo $\mathfrak{p}$. We consider the two cases $\mathrm{rk}(A_m) = 1$ and $\mathrm{rk}(A_m) = 2$ modulo $\mathfrak{p}$ separately. For simplicity, assume that $\mathrm{rk}(A_m) = 1$ for $1 \leq m \leq t$, and that $\mathrm{rk}(A_m) = 2$ for $t+1 \leq m \leq k$, for some $t \in [k]_0$.

Case 1: Suppose that $m \in [t]$, that is, $\mathrm{rk}(A_m) = 1$. Then, in particular, $v_{\mathfrak{p}}(M_{ij}(\underline{x})) \geq 1$ for all $1 \leq i < j \leq n$. By making a suitable change of variables, we can consider A_m to be the matrix with $(1,1)$-coordinate 1 and 0 elsewhere. Hence, each $\underline{x} = (x_{ij}) \in A_m + \mathrm{Mat}_{2\times n}(\mathfrak{p})$ is given by $x_{11} = 1 + Q_{11}$ and $x_{ij} = Q_{ij}$ for $(i,j) \neq (1,1)$, where Q_{ij} are suitable elements of $\mathfrak{p}$ for $(i,j) \in [2] \times [n]$. Consequently,

$$M_{ij}(\underline{x}) = \begin{cases} (1+Q_{11})Q_{2i} - Q_{21}Q_{1i}, & \text{for } i = 1 \text{ and } j = 2, \dots, n, \\ Q_{1i}Q_{2j} - Q_{2i}Q_{1j}, & \text{for } 1 < i < j \leq n, \end{cases}$$

so that $\|\mathbf{M}(\underline{x})\|_{\mathfrak{p}} = \|M_{12}(\underline{x}), \dots, M_{1n}(\underline{x})\|_{\mathfrak{p}}$. Therefore

$$\begin{aligned} \mathcal{I}_{A_m}(s,r) &= \int_{(y,\underline{x})\in\mathfrak{p}\times\mathrm{Mat}_{2\times n}(\mathfrak{p})} |y|_{\mathfrak{p}}^{r} \|M_{12}(\underline{x}), \dots, M_{1n}, y\|_{\mathfrak{p}}^{s} d\mu \\ &= \mu(\mathfrak{p}^{(n+1)}) \int_{(y,x_1,\dots,x_{n-1})\in\mathfrak{p}\times\mathfrak{p}^{(n-1)}} |y|_{\mathfrak{p}}^{r} \|x_1, \dots, x_{n-1}, y\|_{\mathfrak{p}}^{s} d\mu \\ &= q^{-n-1} \frac{(1-q^{-1})(1-q^{-r-n})q^{-r-s-n}}{(1-q^{-r-s-n})(1-q^{-r-1})}, \end{aligned}$$

where the domain of integration of the integral in the first equality is justified by the translation invariance of the Haar measure and the last equality is due Lemma 2.2.2.

Case 2: We now assume that $m \in \{t+1, \dots, k\}$, that is, $\mathrm{rk}(A_m) = 2$. In this case, each $\underline{x} \in A_m + \mathrm{Mat}_{2\times n}(\mathfrak{p})$ has rank two modulo $\mathfrak{p}$, which means that at least one of the $M_{ij}(\underline{x})$ has valuation zero. Consequently,

$$\mathcal{I}_{A_m}(s,r) = \int_{(y,\underline{x})\in\mathfrak{p}\times\mathfrak{p}^{2n}} |y|_{\mathfrak{p}}^r d\mu = \frac{q^{-2n-r-1}(1-q^{-1})}{1-q^{-r-1}}.$$

There are $(q+1)(q^n-1)$ matrices of rank 1 and $q(q^n-1)(q^{n-1}-1)$ matrices of rank 2 in $\mathrm{Mat}_{2\times n}(\mathbb{F}_q)$ and, consequently,

$$\begin{aligned}
&\int_{(y,\underline{x})\in\mathfrak{p}\times W_{2n}^{\mathfrak{o}}} |y|_{\mathfrak{p}}^r \|\mathbf{M}(\underline{x}) \cup y\|_{\mathfrak{p}}^s d\mu = \sum_{m=1}^{k} \mathcal{I}_{A_m}(s,r)\\
&= (q+1)(q^n-1)\mathcal{I}_{A_t}(s,r) + q(q^n-1)(q^{n-1}-1)\mathcal{I}_{A_k}(s,r)\\
&= \frac{(q^n-1)(1-q^{-1})q^{-r-2n-1}}{(1-q^{-1-r})(1-q^{-r-s-n})}\left((q+1)(1-q^{-r-n})q^{-s} + (q^n-q)(1-q^{-r-s-n})\right),
\end{aligned}$$

as desired. □

In the following lemma, we show how to write certain $\mathfrak{p}$-adic integrals with domains of integration of the form $\mathfrak{o}\times\mathfrak{o}^n$ in terms of $\mathfrak{p}$-adic integrals with domains of integration of the form $\mathfrak{p}\times W_n^{\mathfrak{o}}$.

In the following lemma we adopt the following notation: $n \in \mathbb{N}$ and $\mathcal{R}(\underline{Y}) = \mathcal{R}(Y_1, \dots, Y_n)$ is a matrix of polynomials $\mathcal{R}(\underline{Y})_{ij} \in \mathfrak{o}[\underline{Y}]$ with $u_{\mathcal{R}} = \max\{\mathrm{rk}_{\mathrm{Frac}(\mathfrak{o})}\mathcal{R}(\mathbf{z}) \mid \mathbf{z} \in \mathfrak{o}^n\}$. Let $F_i(\mathcal{R}(\underline{y}))$ be the set of $i\times i$-minors of $\mathcal{R}(\underline{Y})$.

Lemma 2.2.4. *Let r and t be complex variables. Define*

$$\mathcal{I}(r,t) := \int_{(x,\underline{y})\in\mathfrak{o}\times\mathfrak{o}^n} |x|_{\mathfrak{p}}^r \prod_{i=1}^{u} \frac{\|F_i(\mathcal{R}(\underline{y})) \cup xF_{i-1}(\mathcal{R}(\underline{y}))\|_{\mathfrak{p}}^t}{\|F_{i-1}(\mathcal{R}(\underline{y}))\|_{\mathfrak{p}}^t} d\mu$$

and

$$\mathcal{J}(r,t) := \int_{(x,\underline{y})\in\mathfrak{p}\times W_n^{\mathfrak{o}}} |x|_{\mathfrak{p}}^r \prod_{i=1}^{u} \frac{\|F_i(\mathcal{R}(\underline{y})) \cup xF_{i-1}(\mathcal{R}(\underline{y}))\|_{\mathfrak{p}}^t}{\|F_{i-1}(\mathcal{R}(\underline{y}))\|_{\mathfrak{p}}^t} d\mu.$$

If both $\mathcal{I}(r,t)$ and $\mathcal{J}(r,t)$ converge, then

$$\mathcal{I}(r,t) = \frac{1}{1-q^{-r-ut-n-1}}\left((1-q^{-1}) + \mathcal{J}(r,t)\right). \tag{2.2.2}$$

Proof. Since $\mathfrak{o} = W_1^{\mathfrak{o}} \cup \mathfrak{p}$ and $\mathfrak{o}^n = W_n^{\mathfrak{o}} \cup \mathfrak{p}^{(n)}$, the integral $\mathcal{I}(r,t)$ equals

$$\begin{aligned}
&\int_{(x,\underline{y})\in W_1^{\mathfrak{o}}\times\mathfrak{o}^n} 1\, d\mu + \int_{(x,\underline{y})\in\mathfrak{p}\times\mathfrak{o}^n} |x|_{\mathfrak{p}}^r \prod_{i=1}^{u} \frac{\|F_i(\mathcal{R}(\underline{y})) \cup xF_{i-1}(\mathcal{R}(\underline{y}))\|_{\mathfrak{p}}^t}{\|F_{i-1}(\mathcal{R}(\underline{y}))\|_{\mathfrak{p}}^t} d\mu\\
&= (1-q^{-1}) + \mathcal{J}(r,t) + \int_{(x,\underline{y})\in\mathfrak{p}\times\mathfrak{p}^{(n)}} |x|_{\mathfrak{p}}^r \prod_{i=1}^{u} \frac{\|F_i(\mathcal{R}(\underline{y})) \cup xF_{i-1}(\mathcal{R}(\underline{y}))\|_{\mathfrak{p}}^t}{\|F_{i-1}(\mathcal{R}(\underline{y}))\|_{\mathfrak{p}}^t} d\mu.
\end{aligned}$$

Equality (2.2.2) follows from the change of coordinates $\mathfrak{p}^{(n)} \to \mathfrak{o}^n$ given by $\underline{x} = (x_1, \dots, x_n) \mapsto (x_1/q, \dots, x_n/q)$ and from $\mathfrak{p} \to \mathfrak{o}$ given by $y \mapsto y/q$:

$$\int_{(x,\underline{y})\in\mathfrak{p}\times\mathfrak{p}^{(n)}} |x|_{\mathfrak{p}}^r \prod_{i=1}^{u} \frac{\|F_i(\mathcal{R}(\underline{y})) \cup xF_{i-1}(\mathcal{R}(\underline{y}))\|_{\mathfrak{p}}^t}{\|F_{i-1}(\mathcal{R}(\underline{y}))\|_{\mathfrak{p}}^t} d\mu = q^{-r-ut-n-1}\mathcal{I}(r,t). \qquad \square$$

2.3 Principalisation of ideals

We follow [51, Section 2] here.

Theorem 2.3.1. *[53, Theorem 1.0.1] Let $\mathcal{I}$ be a sheaf of ideals on a smooth algebraic variety defined over a ground field of characteristic zero. There is a* principalisation (Y, h) *of $\mathcal{I}$, that is, a sequence*

$$X = X_0 \overset{h_1}{\longleftarrow} X_1 \overset{h_2}{\longleftarrow} \dots \overset{h_i}{\longleftarrow} X_i \overset{h_{i+1}}{\longleftarrow} \dots \leftarrow X_r = Y$$

of blow-ups $h_i : X_i \to X_{i-1}$ with smooth centres $C_{i-1} \subseteq X_{i-1}$ satisfying:

1. *The exceptional divisor E_i of the induced morphism $h^i = h_1 \circ h_2 \circ \dots \circ h_i : X_i \to X$ has only simple normal crossings and C_i has simple normal crossings with E_i.*
2. *Setting $h = h_1 \circ h_2 \circ \dots \circ h_r$, the total transform $h^*(\mathcal{I})$ is the ideal of a simple normal crossing divisor $\widetilde{E}$ which is a natural linear combination of the irreducible components of the divisor E_r.*

Let R be the valuation ring of a finite extension $\mathbb{K}$ of $\mathbb{Q}_p$, the field of p-adic numbers, and let P be the maximal ideal of R. A principalisation (Y, h) is said to have *good reduction modulo* if $\mathcal{I}$ and (Y, h) are defined over a p-adic field $\mathbb{K}$. By [10, Theorem 2.4], if (Y, h) is a principalisation defined over a number field K with ring of integers $\mathcal{O}$, then (Y, h) has good reduction modulo P for all but finitely many maximal ideals P of $\mathcal{O}$.

Let $l, n, m \in \mathbb{N}$ and fix $I \subseteq [n-1]$ and a finite index set J_κ for each $k \in [l]$. For each $\kappa \in [l]$ and $\iota \in J_\kappa$, let $\mathbf{f}_{\kappa\iota}$ be a finite set of polynomials $f(\underline{Y}) = f(Y_1, \dots, Y_m)$ over a number field K, and let (Y, h) with $h : Y \to \mathbb{A}^m$ be a principalisation of the ideal $\mathcal{I}$ given by

$$\mathcal{I} = \prod_{\kappa=1}^{k} \prod_{\iota \in J_\kappa} (\mathbf{f}_{\kappa\iota}),$$

where $(\mathbf{f}_{\kappa\iota})$ is the ideal generated by the set $\mathbf{f}_{\kappa\iota}$.

Set also $\mathcal{V} = \mathrm{Spec}(K[\underline{Y}]/\mathcal{I})$ and $\mathcal{V}_{\kappa\iota} = \mathrm{Spec}(K[\underline{Y}]/(\mathbf{f}_{\kappa\iota}))$. Let T be a finite set indexing the irreducible components E_u of the pre-image $h^{-1}(\mathcal{V})$. Then there are nonnegative integers N_u and $N_{u\kappa\iota}$ such that

$$h^{-1}(\mathcal{V}) = \sum_{u \in T} N_u E_u \qquad h^{-1}(\mathcal{V}_{\kappa\iota}) = \sum_{k=1}^{l} N_{u\kappa\iota} N_{u\kappa\iota} E_u.$$

Similarly, $\nu_u - 1$ denotes the multiplicity of E_u in the divisor $h^*(dY_1 \wedge \dots \wedge dY_d)$. One calls $(N_{u\kappa\iota}, \nu_u)_{u \in U, \kappa \in [l], \iota \in J_\kappa}$ the *numerical data* of (Y, h).

2.4 Two complex variables

In this section, we recall briefly the meaning of holomorphy and meromorphy for complex functions on two variables. We refer the reader to [13, 14] for further information about functions on several complex variables. We call *domain* a connected open subset of $\mathbb{C}^2$ (with the usual topology).

Definition 2.4.1. *Let $U \subseteq \mathbb{C}^2$ be an open set. A continuous function $f : U \to \mathbb{C}$ is* holomorphic *if it is holomorphic in each variable. Equivalently, the function f is holomorphic if it satisfies the system of homogeneous equations $\frac{\partial f}{\partial \overline{z_j}} = 0$, for $j = 1, 2$, where for* $\mathrm{Re}(z_j) = x_j$ *and* $\mathrm{Im}(z_j) = y_j$,

$$\frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Example 2.4.2. Let a and b be nonzero real numbers and $c \in \mathbb{R}$. The function $f : \mathbb{C}^2 \to \mathbb{C}$ given by $f(z_1, z_2) = az_1 + bz_2 + c$ is holomorphic on the whole $\mathbb{C}^2$. Its zero set is

$$V(f) = \{(z_1, z_2) \in \mathbb{C}^2 \mid az_1 + bz_2 = -c\}.$$

In particular, the function $g = \frac{1}{f}$ has set of poles $V(f)$. △

In the one variable case, a function is meromorphic on a certain domain if it is locally the quotient of two holomorphic functions such that the denominator is nonzero. In particular, a meromorphic function may only have finite-order isolated poles. In Example 2.4.2, we see that the rational function g has infinitely many poles and none of them is isolated. However, we shall see that g is a meromorphic function on the whole $\mathbb{C}^2$. This is because meromorphy on several complex variables allows for set of (non-isolated) poles, as long as this set is sufficiently "small". More precisely, we call a subset M of a domain $\Omega \subset \mathbb{C}^2$ *thin* if it is relatively closed on Ω, that is, an intersection of a closed subset with any set, and if for each $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$ there is a neighbourhood $U_{\mathbf{z}}$ of $\mathbf{z}$ and a holomorphic function $f_{\mathbf{z}}$ such that $M \cap U_{\mathbf{z}} \subset V(f_{\mathbf{z}}) = \{\mathbf{x} \in \mathbb{C}^2 \mid f_{\mathbf{z}}(\mathbf{x}) = 0\}$. Particularly, if $f : \Omega \to \mathbb{C}$ is a nonzero holomorphic function, then $V(f) := \{\mathbf{z} \in \mathbb{C}^2 \mid f(\mathbf{z}) = 0\}$ is a thin set.

Definition 2.4.3. *[13, Definition 2.1 of Chap. VI] A* meromorphic function *on a domain $\Omega \subset \mathbb{C}^2$ is a function $f : \Omega \to \mathbb{C}$ such that there exists a thin set $M \subset \Omega$ for which f is holomorphic on $\Omega \setminus M$ and, for each $\mathbf{z}_0 \in \Omega$, there exist a neighbourhood $U_{\mathbf{z}_0}$ of $\mathbf{z}_0$ in Ω and holomorphic functions $g, h : U_{\mathbf{z}_0} \to \mathbb{C}$ with $g \not\equiv 0$ such that $V(h) \subset M$ and*

$$f(\mathbf{z}) = \frac{g(\mathbf{z})}{h(\mathbf{z})}, \text{ for } \mathbf{z} \in U \setminus M.$$

In particular, we see that if $f(\mathbf{z}) = \frac{g(\mathbf{z})}{h(\mathbf{z})}$ with $g, h : \Omega \to \mathbb{C}$ holomorphic and $h \not\equiv 0$, then, since $V(h)$ is thin, f is meromorphic on Ω.

The following result states that the complement of a thin set in a domain is also a domain.

Proposition 2.4.4. *[13, Proposition 1.3 of Chap. VI] Let M be a thin subset of a domain $\Omega \subseteq \mathbb{C}^2$. Then $\Omega \setminus M$ is connected.*

2.5 Double series

In this section, we recall some properties of double series. We refer the reader to [15, Section 7] for further results and definitions on double sequences and double series. For simplicity we write $(a_{m,n}) = (a_{m,n})_{m,n \in \mathbb{N}}$.

We observe that a (single) series $(a_n)_{n \in \mathbb{N}}$ can be regarded as a double series $(a_{m,n})$ by defining $a_{1,n} = a_n$, and $a_{m,n} = 0$, for all $n \in \mathbb{N}$ and $m \in \mathbb{N}_{>1}$. In particular, the results on double series also hold for (single) series. The converse does not hold. For instance, in contrast with single sequences, a convergent double sequence need not be bounded. An example in which this property fails is the double sequence of terms $a_{n,1} = n$ and $a_{n,m} = 1$, for $n \in \mathbb{N}$ and $m \in \mathbb{N}_{>1}$.

However, a double series $\sum\sum_{(m,n)} a_{m,n}$ with nonnegative coefficients is convergent if and only if the double sequence $(A_{n,m})$ of its partial sums

$A_{m,n} := \sum_{k=1}^{m}\sum_{l=1}^{n} a_{k,l}$ is bounded above; [15, Proposition 7.14]. A double sequence $(a_{m,n})$ is monotonically nondecreasing if $a_{m,n} \leq a_{m+1,n}$ and $a_{m,n} \leq a_{m,n+1}$. One defined monotonically increasing, monotonically nonincreasing, and monotonically decreasing similarly. A monotonic double sequence is convergent if and only if it is bounded; see [15, Proposition 7.4].

For the sake of completeness, we show the following Lemmata, which are analogous to similar results on single series.

Lemma 2.5.1. *Let $(a_{m,n})$ be a bounded double sequence and let $\sum\sum_{(m,n)} b_{m,n}$ be an absolutely convergent double series. Then $\sum\sum_{(m,n)} a_{m,n}b_{m,n}$ converges absolutely.*

Proof. There exists $M > 0$ such that $|a_{m,n}| < M$ for all $m, n \in \mathbb{N}$. Since the monotonically non-decreasing double sequence $(\sum_{k=1}^{m}\sum_{l=1}^{n} |b_{k,l}|)_{m,n}$ converges, it is bounded by a positive real number N. Therefore,

$$\sum_{k=1}^{m}\sum_{l=1}^{n} |a_{k,l}b_{k,l}| < M\sum_{k=1}^{m}\sum_{l=1}^{n} |b_{k,l}| < MN.$$

□

Lemma 2.5.2. *A double series $\sum\sum_{(m,n)} a_{m,n}$ converges absolutely if and only if the product $\prod\prod_{(m,n)}(1 + a_{m,n})$ converges absolutely.*

Proof. Denote by $P_{m,n}$ the partial product $\prod_{k=1}^{m}\prod_{l=1}^{n}(1 + |a_{k,l}|)$ and by $S_{m,n}$ the partial sum $\sum_{k=1}^{m}\sum_{l=1}^{n} |a_{k,l}|$. The double sequences $(P_{m,n})$ and $(S_{m,n})$ are positive non-decreasing double sequences and hence they converge if and only if they are bounded. One the one hand, since $1 + x \leq e^x$ for all $x \in \mathbb{R}_{\geq 0}$, it follows that

$$P_{m,n} = \prod_{k=1}^{m}\prod_{l=1}^{n}(1 + |a_{k,l}|) < \prod_{k=1}^{m}\prod_{l=1}^{n} e^{|a_{k,l}|} = e^{s_{m,n}}.$$

On the other hand, it is easy to see that $P_{m,n} \geq 1 + S_{m,n}$. Therefore, $(P_{m,n})$ is bounded if and only if $(S_{m,n})$ is bounded. □

2.5.1 Polynomial growth

It is well known that if a complex sequence $(a_n)_{n\in\mathbb{N}}$ grows at most polynomially, the Dirichlet series $D((a_n)_{n\in\mathbb{N}}, s) := \sum_{n=1}^{\infty} a_n n^{-s}$ converges for $s \in \mathbb{C}$ with sufficiently large real part. We now show that an analogous result holds for double Dirichlet series. We remark that the converse also holds for Dirichlet series.

Definition 2.5.3. *A double sequence $(a_{n,m})_{n,m\in\mathbb{N}}$ of complex numbers is said to have* polynomial growth *if there exist positive real numbers α_1 and α_2 and a constant $C > 0$ such that $|a_{n,m}| < Cn^{\alpha_1}m^{\alpha_2}$ for all $n, m \in \mathbb{N}$.*

Proposition 2.5.4. *If the double sequence $(a_{n,m})_{n,m\in\mathbb{N}}$ has polynomial growth, then there exist α_1, $\alpha_2 \in \mathbb{R}$ such that the double Dirichlet series*

$$D((a_{n,m})_{n,m\in\mathbb{N}}, s_1, s_2) := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n,m}n^{-s_1}m^{-s_2}$$

converges absolutely for $(s_1, s_2) \in \mathbb{C}^2$ satisfying $\mathrm{Re}(s_1) > \alpha_1$ and $\mathrm{Re}(s_2) > \alpha_2$.

Proof. Let $\beta_1,\ \beta_2 \in \mathbb{N}$ and $C > 0$ be such that $|a_{n,m}| < Cn^{\beta_1}m^{\beta_2}$, for all $n, m \in \mathbb{N}$. Then

$$\sum_n \sum_m \left|\frac{a_{n,m}}{n^{s_1}m^{s_2}}\right| \leq C \sum_n \sum_m \frac{1}{n^{\mathrm{Re}(s_1)-\beta_1} m^{\mathrm{Re}(s_2)-\beta_2}}.$$

The relevant statement of Proposition 2.5.4 then follows from the fact that, for $p, q \in \mathbb{R}$, the harmonic double series

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{k^p l^q}$$

converges if and only if $p > 1$ and $q > 1$; see [15, Example 7.10(iii)]. □

For $\mathbf{G}$ a unipotent group scheme over $\mathcal{O}$ and m, n positive integers, write

$$r_{n,m}(\mathbf{G}(\mathcal{O})) = \sum_{\substack{I \trianglelefteq \mathcal{O} \\ |\mathcal{O}:I|=m}} r_n(\mathbf{G}(\mathcal{O}/I)) \text{ and } c_{n,m}(\mathbf{G}(\mathcal{O})) = \sum_{\substack{I \trianglelefteq \mathcal{O} \\ |\mathcal{O}:I|=m}} c_n(\mathbf{G}(\mathcal{O}/I)).$$

The bivariate representation and the bivariate conjugacy class zeta functions of $\mathbf{G}(\mathcal{O})$ are given by the following double Dirichlet series with nonnegative coefficients:

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O})}(s_1, s_2) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} r_{n,m}(\mathbf{G}(\mathcal{O}))n^{-s_1}m^{-s_2},$$

$$\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O})}(s_1, s_2) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{n,m}(\mathbf{G}(\mathcal{O}))n^{-s_1}m^{-s_2}.$$

Proposition 2.5.5. *The bivariate zeta functions* $\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$ *and* $\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$ *converge (at least) on some domain of the form*

$$\{(s_1, s_2) \in \mathbb{C}^2 \mid \mathrm{Re}(s_1) > \alpha_1,\ \ \mathrm{Re}(s_2) > \alpha_2\},$$

for some real constants α_1 *and* α_2*.*

Proof. Let $\gamma_m(\mathcal{O}) := |\{I \trianglelefteq \mathcal{O} \mid |\mathcal{O} : I| = m\}|$. The Dedekind zeta function of the number field K is given by $\zeta_K(s) = \sum_{m=1}^{\infty} \gamma_m m^{-s}$, and is known to converge for $\mathrm{Re}(s) > 1$. In particular, the partial sums $\sum_{m=1}^{M} \gamma_m$ are bounded by $\mathscr{P}(M)$, where $\mathscr{P}(X)$ is a polynomial in $\mathbb{Z}[X]$.

Given $I \trianglelefteq \mathcal{O}$, the finite group $\mathbf{G}(\mathcal{O}/I)$ is a congruence quotient of a torsion-free nilpotent and finitely generated group. Then there exists $\mathscr{Q}(X) \in \mathbb{Z}[X]$ such that, for all $I \trianglelefteq \mathcal{O}$, $|\mathbf{G}(\mathcal{O}/I)| < \mathscr{Q}(m)$, where $m = |\mathcal{O} : I|$.

Given $I \trianglelefteq \mathcal{O}$, the finite group $\mathbf{G}(\mathcal{O}/I)$ has at most $|\mathbf{G}(\mathcal{O}/I)|$ conjugacy classes. Consequently, for each $(n, m) \in \mathbb{N}^2$,

$$c_{n,m}(\mathbf{G}(\mathcal{O})) = \sum_{\substack{(0) \neq I \trianglelefteq \mathcal{O} \\ |\mathcal{O}:I|=m}} c_n(\mathbf{G}(\mathcal{O}/I)) < \mathscr{P}(m)\mathscr{Q}(m).$$

Analogously, $r_{n,m}(\mathbf{G}(\mathcal{O})) < \mathscr{P}(m)\mathscr{Q}(m)$, since $r_n(\mathbf{G}(\mathcal{O}/I)) \leq |\mathbf{G}(\mathcal{O}/I)|$. □

When finite, the abscissa of convergence of a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ gives the precise degree of polynomial growth of the sequence $(\sum_{i=1}^{n} a_i)_n$. However, for double Dirichlet series $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}m^{-s_1}n^{-s_2}$, this might not be the case. For instance, in Example 3.2.15, we show that the bivariate representation zeta function of the Heisenberg group $\mathbf{H}(\mathcal{O})$ is given by

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{H}(\mathfrak{o})}(s_1, s_2) = \frac{1 - q^{-s_1 - s_2}}{(1 - q^{1-s_1-s_2})(1 - q^{2-s_2})}.$$

We see that the maximal domain of convergence of $\mathcal{Z}^{\mathrm{irr}}_{\mathbf{H}(\mathfrak{o})}(s_1, s_2)$ is

$$\mathscr{D}_{\mathbf{H}(\mathcal{O})} := \{(s_1, s_2) \in \mathbb{C}^2 \mid \mathrm{Re}(s_1 + s_2) > 2 \text{ and } \mathrm{Re}(s_2) > 3\}.$$

Then this zeta function converges on

$$\mathscr{D}_{\alpha_1,\alpha_2} := \{(s_1, s_2) \in \mathbb{C}^2 \mid \mathrm{Re}(s_1) > \alpha_1,\ \ \mathrm{Re}(s_2) > \alpha_2\},$$

for many choices of $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, for instance $(-1, 3)$ and $(-2, 4)$. However, we cannot choose the minimum of such pairs, as they are not always comparable.

2.6 Convergence of bivariate Euler products

In this section we recall from [9, Theorem 2.7] the domains of convergence and meromorphy of the Euler products on several variables. In that article, Delabarre deals with Euler products of the form

$$(s_1, \ldots, s_n) \mapsto \prod_{p \text{ prime}} h(p^{-s_1}, \ldots, p^{-s_n}, p^{-c}),$$

for $n > 1$ and a nonzero integral constant c, where $h(X_1, \ldots, X_n, X_{n+1}) \in \mathbb{Z}[X_1, \ldots, X_n, X_{n+1}]$. We observe that Delabarre's main results admit straightforward generalisations to products over prime ideals of $\mathcal{O}$, but we illustrate this just for the case $n = 2$.

For simplicity, denote by P the set of nonzero prime ideals of $\mathcal{O}$. For each $\mathfrak{p} \in P$, denote by $q_{\mathfrak{p}}$ the cardinality of the residue field $\mathcal{O}/\mathfrak{p}$. We are interested in the domains of convergence and meromorphy of the Euler products

$$Z_c(s_1, s_2) = \prod_{\mathfrak{p} \in P} h(q_{\mathfrak{p}}^{-s_1}, q_{\mathfrak{p}}^{-s_2}, q_{\mathfrak{p}}^{-c}),$$

where c is a fixed nonzero integer and $h(X_1, X_2, X_3) \in \mathbb{Z}[X_1, X_2, X_3]$ is a polynomial

$$h(X_1, X_2, X_3) = 1 + \sum_{j=1}^{r} a_j X_1^{\alpha_{1,j}} X_2^{\alpha_{2,j}} X_3^{\alpha_{3,j}},$$

with $a_j \neq 0$ and $\widehat{\boldsymbol{\alpha}}_j = (\alpha_{1,j}, \alpha_{2,j}, \alpha_{2,j}) \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$, for each $j \in [r]$, where for each $n \in \mathbb{N}$, we write $[n] = \{1, \ldots, n\}$. Set $\boldsymbol{\alpha}_j = (\alpha_{1,j}, \alpha_{2,j})$.

The polynomial $h(X_1, X_2, X_3)$ is called *cyclotomic* if there exists a finite set $I \subset \mathbb{N}^{n+1} \setminus \{0\}$ such that

$$h(X_1, X_2, X_3) = \prod_{\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in I} (1 - X_1^{\lambda_1} X_2^{\lambda_2} X_3^{\lambda_3})^{\gamma(\boldsymbol{\lambda})},$$

where the $\gamma(\boldsymbol{\lambda})$ are nonzero positive integers. If h is cyclotomic, then $Z_c(s_1, s_2)$ can be meromorphically continued to the whole $\mathbb{C}^2$. For this reason, from now on, we assume that h is not constant and does not contain cyclotomic factors.

For each $\delta \geq 0$, set

$$W_c(\delta) = \{(s_1, s_2) \in \mathbb{C}^2 \mid \mathrm{Re}(\alpha_{1,j} s_1 + \alpha_{2,j} s_2) > \delta - c\alpha_{j,3},\ \ j \in [r]\}.$$

Proposition 2.6.1. *[9, Theorem 2.7] The product* $(s_1, s_2) \mapsto Z_c(s_1, s_2)$ *converges absolutely in the domain* $W_c(1)$ *and admits meromorphic continuation to* $W_c(0)$.

Set $\hat{h}(X_1, X_2, X_3) = \sum_{j=1}^{r} a_j X_1^{\alpha_{1,j}} X_2^{\alpha_{2,j}} X_3^{\alpha_{3,j}} = h(X_1, X_2, X_3) - 1$. Lemma 2.5.2 then yields that the sum

$$S_c(s_1, s_2) = \sum_{\mathfrak{p} \in P} \hat{h}(q_{\mathfrak{p}}^{-s_1}, q^{-s_2}, q^{-c})$$

converges absolutely in the domain $W_c(1)$.

Remark 2.6.2. Let $Q \subseteq P$ be a finite set of prime ideals of $\mathcal{O}$. Since

$$q_c(s_1, s_2) := \prod_{\mathfrak{p} \in Q} h(q_{\mathfrak{p}}^{-s_1}, q_{\mathfrak{p}}^{-s_2}, q_{\mathfrak{p}}^{-c})$$

is analytic, the infinite product

$$p_c(s_1, s_2) := \prod_{\mathfrak{p} \in P \setminus Q} h(q_{\mathfrak{p}}^{-s_1}, q_{\mathfrak{p}}^{-s_2}, q_{\mathfrak{p}}^{-c}) = \frac{Z_c(s_1, s_2)}{\prod_{\mathfrak{p} \in P} h(q_{\mathfrak{p}}^{-s_1}, q_{\mathfrak{p}}^{-s_2}, q_{\mathfrak{p}}^{-c})}$$

also admits meromorphic continuation to $W_c(0)$. It converges on $W_c(1)$ if the set of zeros $V(p)$ of $q_c(s_1, s_2)$ is not contained in this domain. It follows that Proposition 2.6.1 holds if we consider $Z_c(s_1, s_2)$ as a product over almost all nonzero prime ideals of $\mathcal{O}$, as long as the zeros of the corresponding $h(q_{\mathfrak{p}}^{-s_1}, q_{\mathfrak{p}}^{-s_2}, q_{\mathfrak{p}}^{-c})$ do not lie in $W_c(1)$.

Chapter 3

Arithmetic properties of $\mathcal{Z}^*_{\mathbf{G}(\mathcal{O})}$

This chapter comprises the results of [27], which concerns arithmetic properties of the bivariate zeta functions $\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O})}$ and $\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O})}$.

Firstly, we prove in Section 3.1 that they satisfy Euler decompositions, so that we can relate local and global results. We then write almost all local terms of these bivariate zeta functions in terms of $\mathfrak{p}$-adic integrals in Section 3.2. In particular, this shows that these local factors are rational functions.

In Section 3.3 we prove specialisation (1.1.4), that is, we prove that in case of nilpotency class 2 bivariate representation zeta functions specialise to twist representation zeta functions.

In Section 3.4 we prove Theorem 1.

3.1 Euler decomposition

Most of our main results concern local properties of bivariate representation and bivariate conjugacy class zeta functions. In this section, we show that the corresponding global zeta functions admit Euler decompositions in terms of such local factors, allowing us to relate local and global results.

In the following, $\mathbf{G}$ is a unipotent group scheme defined over $\mathcal{O}$ (not necessarily associated to a nilpotent Lie lattice).

Proposition 3.1.1. *For s_1, $s_2 \in \mathbb{C}$ with sufficiently large real parts,*

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O})}(s_1, s_2) = \prod_{\mathfrak{p}} \mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s_1, s_2),$$

where $\mathfrak{p}$ ranges over all nonzero prime ideals of $\mathcal{O}$.

Proof. For each prime ideal $\mathfrak{p}$ of $\mathcal{O}$, set $q_{\mathfrak{p}} = |\mathcal{O} : \mathfrak{p}|$. Given a nonzero ideal I of $\mathcal{O}$ with prime factorization $I = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$, with $\mathfrak{p}_i \neq \mathfrak{p}_j$ if $i \neq j$, we show that

$$\zeta^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O}/I)}(s) = \prod_{i=1}^{r} \zeta^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O}/\mathfrak{p}_i^{e_i})}(s),$$

so that

$$\begin{aligned}\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O})}(s_1,s_2) &= \sum_{(0)\neq I\lhd\mathcal{O}} \zeta^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O}/I)}(s_1)|\mathcal{O}:I|^{-s_2}\\ &= \sum_{(0)\neq I\lhd\mathcal{O}} \zeta^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O}/\mathfrak{p}_1^{e_1})}(s_1)\cdots\zeta^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O}/\mathfrak{p}_r^{e_r})}(s_1)q_{\mathfrak{p}_1}^{-e_1 s_2}\cdots q_{\mathfrak{p}_r}^{-e_r s_2}\\ &= \prod_{\mathfrak{p}}\sum_{N=0}^{\infty}\zeta^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O}/\mathfrak{p}^N)}(s_1)q_{\mathfrak{p}}^{-Ns_2} = \prod_{\mathfrak{p}}\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s_1,s_2).\end{aligned}$$

Recall that $\mathrm{Irr}(G)$ denotes the set of complex irreducible characters of a group G. For an ideal I as above, since unipotent groups satisfy the strong approximation property—cf. [35, Lemma 5.5]—there is an isomorphism

$$\mathbf{G}(\mathcal{O}/I)\cong\mathbf{G}(\mathcal{O}/\mathfrak{p}_1^{e_1})\times\cdots\times\mathbf{G}(\mathcal{O}/\mathfrak{p}_r^{e_r}). \tag{3.1.1}$$

Hence

$$\mathrm{Irr}(\mathbf{G}(\mathcal{O}/I))\cong\mathrm{Irr}(\mathbf{G}(\mathcal{O}/\mathfrak{p}_1^{e_1}))\times\cdots\times\mathrm{Irr}(\mathbf{G}(\mathcal{O}/\mathfrak{p}_r^{e_r})).$$

For simplicity, write $\mathrm{Irr}_i = \mathrm{Irr}(\mathbf{G}(\mathcal{O}/\mathfrak{p}_i^{e_i}))$. Since $r_n(\mathbf{G}(\mathcal{O}/I)) = |\{\chi\in\mathrm{Irr}(\mathbf{G}(\mathcal{O}/I)) : \chi(1)=n\}|$, it follows that

$$\begin{aligned}\zeta^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O}/I)}(s) &= \sum_{\chi\in\mathrm{Irr}(\mathbf{G}(\mathcal{O}/I))}\chi(1)^{-s} = \sum_{(\chi_1,\ldots,\chi_r)\in\mathrm{Irr}_1\times\cdots\times\mathrm{Irr}_r}\chi_1(1)^{-s}\cdots\chi_r(1)^{-s}\\ &= \prod_{i=1}^{r}\sum_{\chi_i\in\mathrm{Irr}_i}\chi_i(1)^{-s} = \prod_{i=1}^{r}\zeta^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O}/\mathfrak{p}_i^{e_i})}(s).\end{aligned}$$

□

Proposition 3.1.2. *For s_1, $s_2\in\mathbb{C}$ with sufficiently large real parts,*

$$\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O})}(s_1,s_2)=\prod_{\mathfrak{p}}\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s_1,s_2),$$

where $\mathfrak{p}$ ranges over all nonzero prime ideals of $\mathcal{O}$.

Proof. As explained in Proposition 3.1.1, it suffices to show the identity

$$\zeta^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O}/I)}(s)=\prod_{i=1}^{r}\zeta^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O}/\mathfrak{p}^{e_i})}(s),$$

for each nonzero ideal I of $\mathcal{O}$ with prime factorization $I=\mathfrak{p}_1^{e_1}\cdots\mathfrak{p}_r^{e_r}$, with $\mathfrak{p}_i\neq\mathfrak{p}_j$ if $i\neq j$. Because of (3.1.1), each conjugacy class C of $G(\mathcal{O}/I)=\mathbf{G}(\mathcal{O}/\mathfrak{p}_1^{e_1})\times\cdots\times\mathbf{G}(\mathcal{O}/\mathfrak{p}_r^{e_r})$ is of the form $C=C_1\times\cdots\times C_r$, where C_i is a conjugacy class of $\mathbf{G}(\mathcal{O}/\mathfrak{p}_i^{e_i})$, for each $i\in[r]$. Thus

$$c_n(\mathbf{G}(\mathcal{O}/I))=\sum_{\substack{n_1,\ldots,n_r\in\mathbb{N}_0\\ n_1\cdots n_r=n}}c_{n_1}(\mathbf{G}(\mathcal{O}/\mathfrak{p}_1^{e_1}))\cdots c_{n_r}(\mathbf{G}(\mathcal{O}/\mathfrak{p}_r^{e_r})).$$

Again, set $q_{\mathfrak{p}}=|\mathcal{O}:\mathfrak{p}|$ for each nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$. We shall see in Remark 3.2.6 that all conjugacy classes of $\mathbf{G}(\mathcal{O}/\mathfrak{p}_i^{e_i})$ have size a power of $q_{\mathfrak{p}_i}$. Consequently

$$\begin{aligned}\zeta^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O}/I)}(s) &= \sum_{n=1}^{\infty}\sum_{\substack{n_1,\ldots,n_r\in\mathbb{N}_0\\ q_1^{n_1}\ldots q_r^{n_r}=n}}c_{q_{\mathfrak{p}_1}^{n_1}}(\mathbf{G}(\mathcal{O}/\mathfrak{p}_1^{e_1}))\cdots c_{q_{\mathfrak{p}_r}^{n_r}}(\mathbf{G}(\mathcal{O}/\mathfrak{p}_r^{e_r}))(q_{\mathfrak{p}_1}^{n_1}\ldots q_{\mathfrak{p}_r}^{n_r})^{-s}\\ &= \prod_{k=1}^{r}\left(\sum_{n_k=0}^{\infty}c_{q_k^{n_k}}(\mathbf{G}(\mathcal{O}/\mathfrak{p}_k^{e_k}))q_k^{-n_k s}\right)=\prod_{k=1}^{r}\zeta^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O}/\mathfrak{p}_k^{e_k})}(s).\end{aligned}$$

□

3.2 Bivariate zeta functions in terms of $\mathfrak{p}$-adic integrals

Our results rely on the fact that local bivariate representation and local bivariate conjugacy class zeta functions of groups associated to unipotent group schemes can be written in terms of $\mathfrak{p}$-adic integrals. The main goal of this section is to obtain formulae for these local factors in terms of $\mathfrak{p}$-adic integrals. This is done using the methods of [51, Section 2.2], in which Voll shows how Poincaré series encoding elementary divisor types of certain matrices can be written in terms of $\mathfrak{p}$-adic integrals. We recall in Section 3.2.1 these methods and some definitions needed.

Throughout Section 3.2, denote by $\mathbf{G} = \mathbf{G}_\Lambda$ a unipotent group scheme associated to a nilpotent $\mathcal{O}$-Lie lattice Λ. In Section 3.2.2, we show how to rewrite the coefficients of the bivariate representation and the bivariate conjugacy class zeta functions of groups of the form $\mathbf{G}(\mathcal{O})$ in terms of elementary divisor types of certain matrices and use this in Section 3.2.3 to rewrite the mentioned bivariate zeta functions in terms of Poincaré series as the ones of [51, Section 2.2] and hence obtain descriptions of these functions in terms of $\mathfrak{p}$-adic integrals.

For the rest of Section 3.2, fix a nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$, and $\mathfrak{o} = \mathcal{O}_\mathfrak{p}$. Denote by q the cardinality of $\mathcal{O}/\mathfrak{p}$ and by p its characteristic.

3.2.1 Poincaré series and $\mathfrak{p}$-adic integrals

Denote by $\pi \in \mathfrak{o}$ a uniformiser of $\mathfrak{o}$. A matrix $M \in \mathrm{Mat}_{m\times n}(\mathfrak{o}/\mathfrak{p}^N)$ is said to have *elementary divisor type* $(m_1, \dots, m_\epsilon) \in \mathbb{N}_0^\epsilon$ if it is equivalent (by elementary row and column operations) to the to the matrix

$$\begin{bmatrix} \pi^{m_1} & & & \\ & \pi^{m_2} & & \\ & & \ddots & \\ & & & \pi^{m_\epsilon} \\ & & & \end{bmatrix},$$

where ϵ is the rank of M, and $0 \le m_1 \le m_2 \le \cdots \le m_\epsilon \le N$. Write $\nu(M) = (m_1, \dots, m_\epsilon)$ to indicate the elementary divisor type of M.

Given $k, N \in \mathbb{N}$, set

$$W^{\mathfrak{o}}_{k,N} := ((\mathfrak{o}/\mathfrak{p}^N)^k)^* = \{\mathbf{x} \in (\mathfrak{o}/\mathfrak{p}^N)^k \mid v_\mathfrak{p}(\mathbf{x}) = 0\}.$$

For each $k \in \mathbb{N}$, let also $W_{k,0} = (0)^k$, and recall from Section 2.2 the notation $W^{\mathfrak{o}}_k := \{\mathbf{x} \in \mathfrak{o}^k \mid v_\mathfrak{p}(\mathbf{x}) = 0\}$.

Given $n \in \mathbb{N}$ and a matrix $\mathcal{R}(\underline{Y}) = \mathcal{R}(Y_1, \dots, Y_n)$ of polynomials $\mathcal{R}(\underline{Y})_{ij} \in \mathfrak{o}[\underline{Y}]$ with $u_\mathcal{R} = \max\{\mathrm{rk}_{\mathrm{Frac}(\mathfrak{o})}\mathcal{R}(\mathbf{z}) \mid \mathbf{z} \in \mathfrak{o}^n\}$, define for each $\mathbf{m} \in \mathbb{N}_0^{u_\mathcal{R}}$

$$\begin{aligned} \mathfrak{N}^{\mathfrak{o}}_{N,\mathcal{R},\mathbf{m}} &:= \{\mathbf{y} \in W^{\mathfrak{o}}_{n,N} \mid \nu(\mathcal{R}(\mathbf{y})) = \mathbf{m}\} \text{ and} \\ \mathcal{N}^{\mathfrak{o}}_{N,\mathcal{R},\mathbf{m}} &:= |\mathfrak{N}^{\mathfrak{o}}_{N,\mathcal{R},\mathbf{m}}|. \end{aligned} \tag{3.2.1}$$

The number $\mathcal{N}^{\mathfrak{o}}_{N,\mathcal{R},\mathbf{m}}$ is zero unless $\mathbf{m} = (m_1, \dots, m_{u_\mathcal{R}})$ satisfies

$$0 = m_1 \le \cdots \le m_{u_\mathcal{R}} \le N.$$

Let $\underline{r} = (r_1, \ldots, r_{u_{\mathcal{R}}})$ be a vector of variables. Consider the *Poincaré series*

$$\mathcal{P}_{\mathfrak{o},\mathcal{R}}(\underline{r}, t) = \sum_{\substack{N\in\mathbb{N} \\ \mathbf{m}\in\mathbb{N}_0^{u_{\mathcal{R}}}}} \mathcal{N}^{\mathfrak{o}}_{N,\mathcal{R},\mathbf{m}} q^{-tN-\sum_{i=1}^{u_{\mathcal{R}}} r_i m_i}. \tag{3.2.2}$$

In [51, Section 2.2] it is shown that the series (3.2.2) are given in terms of $\mathfrak{p}$-adic integrals of the form:

$$\mathscr{Z}_{\mathfrak{o},\mathcal{R}}(\underline{r}, t) = \frac{1}{1-q^{-1}} \int_{(x,\underline{y})\in\mathfrak{p}\times W_n^{\mathfrak{o}}} |x|^t_{\mathfrak{p}} \prod_{k=1}^{u_{\mathcal{R}}} \frac{\|F_k(\mathcal{R}(\underline{y})) \cup xF_{k-1}(\mathcal{R}(\underline{y}))\|^{r_k}_{\mathfrak{p}}}{\|F_{k-1}(\mathcal{R}(\underline{y}))\|^{r_k}_{\mathfrak{p}}} d\mu, \tag{3.2.3}$$

where μ is the additive Haar measure on $\mathfrak{o}^{n+1}$ normalised so that $\mu(\mathfrak{o}^{n+1}) = 1$, and $F_j(\mathcal{R}(\underline{y}))$ is the set of nonzero $j \times j$-minors of $\mathcal{R}(\underline{y})$.

This is done by decomposing the domain of the integral $\mathscr{Z}_{\mathfrak{o},\mathcal{R}}$ in subdomains where the integrands are constant, as we now explain. Set

$$\Theta_{N,\mathcal{R},\mathbf{m}} := \{(x, \underline{y}) \in \mathfrak{p} \times W_n^{\mathfrak{o}} \mid v_{\mathfrak{p}}(x) = N,\ \nu(\mathcal{R}(\underline{y})) = \mathbf{m}\}.$$

Then, for $(x, \underline{y}) \in \Theta_{N,\mathcal{R},\mathbf{m}}$,

$$|x|^t_{\mathfrak{p}} \prod_{k=1}^{u_{\mathcal{R}}} \frac{\|F_k(\mathcal{R}(\underline{y})) \cup xF_{k-1}(\mathcal{R}(\underline{y}))\|^{r_k}_{\mathfrak{p}}}{\|F_{k-1}(\mathcal{R}(\underline{y}))\|^{r_k}_{\mathfrak{p}}} = q^{-Nt-\sum_{k=1}^{u_{\mathcal{R}}} r_k m_k},$$

and consequently

$$\mathscr{Z}_{\mathfrak{o},\mathcal{R}}(\underline{r}, t) = \frac{1}{1-q^{-1}} \sum_{\substack{N\in\mathbb{N} \\ \mathbf{m}\in\mathbb{N}_0^{u_{\mathcal{R}}}}} |\Theta_{N,\mathcal{R},\mathbf{m}}| q^{-Nt-\sum_{i=1}^{u_{\mathcal{R}}} r_i m_i}.$$

According to [48, Lemma 2.2]

$$\mathcal{N}^{\mathfrak{o}}_{N,\mathcal{R},\mathbf{m}} = (1-q^{-1})^{-1} q^{N(n+1)} \Theta_{N,\mathcal{R},\mathbf{m}},$$

so that

$$\mathcal{P}_{\mathfrak{o},\mathcal{R}}(\underline{r}, t) = \mathscr{Z}_{\mathfrak{o},\mathcal{R}}(\underline{r}, t-n-1). \tag{3.2.4}$$

Suppose now that $M \in \mathrm{Mat}_{n\times n}(\mathfrak{o}/\mathfrak{p}^N)$ is an antisymmetric matrix. Then its elementary divisor type is of the form:

$$\nu(M) = (m_1, m_1, m_2, m_2, \ldots, m_\xi, m_\xi),$$

where 2ξ is the rank of M. For simplicity, we write $\tilde{\nu}(M) = (m_1, m_2, \ldots, m_\xi)$ for the elementary divisor type of the antisymmetric matrix M.

Assume now that $\mathcal{R}(\underline{Y})$ is antisymmetric, in which case $u_{\mathcal{R}}$ is even. For each $\mathbf{m} \in \mathbb{N}_0^{\frac{u_{\mathcal{R}}}{2}}$, we write

$$\widetilde{\mathfrak{N}}^{\mathfrak{o}}_{N,\mathcal{R},\mathbf{m}} := \{\mathbf{y} \in W^{\mathfrak{o}}_{n,N} \mid \tilde{\nu}(\mathcal{R}(\underline{y})) = \mathbf{m}\},$$

and $\mathcal{N}^{\mathfrak{o}}_{N,\mathcal{R},\mathbf{m}} = |\widetilde{\mathfrak{N}}^{\mathfrak{o}}_{N,\mathcal{R},\mathbf{m}}|$. For $\mathcal{R}(\underline{Y})$ antisymmetric, we assume that the vector of variables $\underline{r}$ is of the form $\underline{r} = (r_1, r_1, \ldots, r_{\frac{u_{\mathcal{R}}}{2}}, r_{\frac{u_{\mathcal{R}}}{2}})$ so that

$$\mathcal{P}_{\mathfrak{o},\mathcal{R}}(\underline{r}, t) = \sum_{N\in\mathbb{N},\ \mathbf{m}\in\mathbb{N}_0^{\frac{u_{\mathcal{R}}}{2}}} \mathcal{N}^{\mathfrak{o}}_{N,\mathcal{R},\mathbf{m}} q^{-tN-2\sum_{i=1}^{\frac{u_{\mathcal{R}}}{2}} r_i m_i}. \tag{3.2.5}$$

Given $x \in \mathfrak{o}$ with $v_{\mathfrak{p}}(x) = N$, $\mathbf{y} \in \mathfrak{o}^n$ with $\tilde{\nu}(\mathcal{R}(\mathbf{y})) = \mathbf{m}$, and $k \in [u_{\mathcal{R}}]$, we obtain from [38, Lemma 4.6(i) and (ii)] the following for the antisymmetric matrix $\mathcal{R}(\mathbf{y})$:

$$\frac{\|F_{2k}(\mathcal{R}(\mathbf{y})) \cup xF_{2k-1}(\mathcal{R}(\mathbf{y}))\|_\mathfrak{p}}{\|F_{2k-1}(\mathcal{R}(\mathbf{y}))\|_\mathfrak{p}} = \frac{\|F_{2k-1}(\mathcal{R}(\mathbf{y})) \cup xF_{2(k-1)}(\mathcal{R}(\mathbf{y}))\|_\mathfrak{p}}{\|F_{2(k-1)}(\mathcal{R}(\mathbf{y}))\|_\mathfrak{p}} = q^{-\min(m_k,N)}$$

and

$$\frac{\|F_{2k}(\mathcal{R}(\mathbf{y})) \cup x^2F_{2(k-1)}(\mathcal{R}(\mathbf{y}))\|_\mathfrak{p}}{\|F_{2(k-1)}(\mathcal{R}(\mathbf{y}))\|_\mathfrak{p}} = q^{-2\min(m_k,N)}.$$

Therefore, if $\mathcal{R}(\underline{Y})$ is an antisymmetric matrix, the series (3.2.5) can be described by the $\mathfrak{p}$-adic integral

$$\mathcal{P}_{\mathfrak{o},\mathcal{R}}(\underline{r},t) = \mathscr{Z}_{\mathfrak{o},\mathcal{R}}(\underline{r},t-n-1) =$$
$$\frac{1}{1-q^{-1}}\int_{(x,\underline{y})\in\mathfrak{p}\times W_n^\mathfrak{o}} |x|_\mathfrak{p}^{t-n-1}\prod_{k=1}^{\frac{u_\mathcal{R}}{2}}\frac{\|F_{2k}(\mathcal{R}(\underline{y})) \cup x^2F_{2(k-1)}(\mathcal{R}(\underline{y}))\|_\mathfrak{p}^{r_k}}{\|F_{2(k-1)}(\mathcal{R}(\underline{y}))\|_\mathfrak{p}^{r_k}}d\mu. \tag{3.2.6}$$

3.2.2 The numbers $r_n(\mathbf{G}_N)$ and $c_n(\mathbf{G}_N)$

Recall the notation $\mathbf{G}_N = \mathbf{G}(\mathfrak{o}/\mathfrak{p}^N)$. We now write the local bivariate zeta functions at $\mathfrak{p}$ in terms of sums encoding the elementary divisor types of certain matrices associated to Λ. This is done by rewriting the numbers $r_n(\mathbf{G}_N)$ and $c_n(\mathbf{G}_N)$, for $n \in \mathbb{N}$ and $N \in \mathbb{N}_0$, in terms of the cardinalities $\mathcal{N}^\mathfrak{o}_{N,\mathcal{R},\mathbf{m}}$ of the sets $\mathfrak{N}^\mathfrak{o}_{N,\mathcal{R},\mathbf{m}}$ defined in Section 3.2.1. In each case, $\mathcal{R}$ is one of the two commutator matrices of Λ which we now define.

Let R be either $\mathcal{O}$ or $\mathfrak{o}$. Let N be an R-submodule of some R-module M. The *isolator* of N is the smallest submodule $\iota(N)$ of M containing N such that the R-module quotient $M/\iota(N)$ is torsion-free. The submodule N is said *isolated* if $\iota(N) = N$ or, equivalently, if M/N is torsion free.

Set $\mathfrak{g} = \Lambda(\mathfrak{o}) = \Lambda \otimes_\mathcal{O} \mathfrak{o}$. Let $\mathfrak{g}'$ be the derived Lie sublattice of $\mathfrak{g}$, and let $\mathfrak{z}$ be its centre. According to [48, Lemma 2.5], the centre $\mathfrak{z}$ of $\mathfrak{g}$ is isolated. Consider the following torsion-free $\mathcal{O}$-ranks:

$$h = \mathrm{rk}(\mathfrak{g}), \qquad a = \mathrm{rk}(\mathfrak{g}/\mathfrak{z}), \qquad b = \mathrm{rk}(\mathfrak{g}'), \qquad r = \mathrm{rk}(\mathfrak{g}/\mathfrak{g}'), \qquad z = \mathrm{rk}(\mathfrak{z}).$$

Also $k = \mathrm{rk}(\iota(\mathfrak{g}')/\iota(\mathfrak{g}' \cap \mathfrak{z})) = \mathrm{rk}(\iota(\mathfrak{g}' + \mathfrak{z})/\mathfrak{z})$.

The commutator matrices are defined with respect to a fixed $\mathfrak{o}$-basis $\mathscr{B} = (e_1, \dots, e_h)$ of the $\mathfrak{o}$-Lie lattice $\mathfrak{g}$, satisfying the conditions

$$(e_{a-k+1}, \dots, e_a) \text{ is an } \mathfrak{o}\text{-basis for } \iota(\mathfrak{g}' + \mathfrak{z}),$$
$$(e_{a+1}, \dots, e_{a-k+b}) \text{ is an } \mathfrak{o}\text{-basis for } \iota(\mathfrak{g}' \cap \mathfrak{z}), \text{ and}$$
$$(e_{a+1}, \dots, e_h) \text{ is an } \mathfrak{o}\text{-basis for } \mathfrak{z}.$$

Denote by $\overline{}$ the natural surjection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{z}$. Let $\mathbf{e} = (e_1, \dots, e_a)$. Then $\overline{\mathbf{e}} = (\overline{e_1}, \dots, \overline{e_a})$ is an $\mathfrak{o}$-basis of $\mathfrak{g}/\mathfrak{z}$. The e_i can be chosen so that there are nonnegative integers $c_1, \dots, c_b$ with the property that

$$(\overline{\pi^{c_1}e_{a-k+1}}, \dots, \overline{\pi^{c_k}e_a}) \text{ is an } \mathfrak{o}\text{-basis of } \overline{\mathfrak{g}' + \mathfrak{z}} \text{ and}$$
$$(\pi^{c_{k+1}}e_{a+1}, \dots, \pi^{c_b}e_{a-k+b}) \text{ is an } \mathfrak{o}\text{-basis of } \mathfrak{g}' \cap \mathfrak{z},$$

by the elementary divisor theorem. Fix an $\mathfrak{o}$-basis $\mathbf{f} = (f_1, \dots, f_b)$ for $\mathfrak{g}'$ satisfying

$$(\overline{f_1},\dots,\overline{f_k}) = (\overline{\pi^{c_1}e_{a-k+1}},\dots,\overline{\pi^{c_k}e_a}) \text{ is an } \mathfrak{o}\text{-basis of } \overline{\mathfrak{g}'+\mathfrak{z}} \text{ and}$$
$$(f_{k+1},\dots,f_b) = (\pi^{c_{k+1}}e_{a+1},\dots,\pi^{c_b}e_{a-k+b}) \text{ is an } \mathfrak{o}\text{-basis of } \mathfrak{g}'\cap\mathfrak{z}.$$

For $i,j\in[a]$ and $k\in[b]$, let $\lambda_{ij}^k\in\mathfrak{o}$ be the structure constants satisfying

$$[e_i,e_j] = \sum_{k=1}^{b}\lambda_{ij}^k f_k. \tag{3.2.7}$$

The following matrices were previously defined in [33, Definition 2.1].

Definition 3.2.1. *The* A-commutator *and the* B-commutator matrices *of* $\mathfrak{g} = \Lambda(\mathfrak{o})$ *with respect to* $\mathbf{e}$ *and* $\mathbf{f}$ *are, respectively,*

$$A(X_1,\dots,X_a) = \left(\sum_{j=1}^{a}\lambda_{ij}^k X_j\right)_{ik} \in \mathrm{Mat}_{a\times b}(\mathfrak{o}[\underline{X}]), \textit{ and}$$
$$B(Y_1,\dots,Y_b) = \left(\sum_{k=1}^{b}\lambda_{ij}^k Y_k\right)_{ij} \in \mathrm{Mat}_{a\times a}(\mathfrak{o}[\underline{Y}]),$$

where $\underline{X} = (X_1,\dots,X_a)$ *and* $\underline{Y} = (Y_1,\dots,Y_b)$ *are independent variables.*

Since $\lambda_{ij}^k = -\lambda_{ji}^k$ for all $i,j\in[a]$ and $k\in[b]$, the matrix $B(\mathbf{y})$ is antisymmetric for each $\mathbf{y}\in\mathfrak{o}^b$.

First, we rewrite the numbers $r_n(\mathbf{G}_N)$ in terms of numbers $\mathcal{N}^{\mathfrak{o}}_{N,B,\mathbf{m}}$, and then we describe the numbers $c_n(\mathbf{G}_N)$ in terms of numbers $\mathcal{N}^{\mathfrak{o}}_{N,A,\mathbf{m}}$, where A and B denote the A-commutator and the B-commutator matrices of $\mathfrak{g}$ with respect to $\mathbf{e}$ and $\mathbf{f}$ defined above.

$r_n(\mathbf{G}_N)$ and elementary divisor types of the B-commutator matrix

Given a compact abelian group $\mathfrak{a}$, write $\widehat{\mathfrak{a}} = \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{Z}}(\mathfrak{a},\mathbb{C}^\times)$. Set $\mathfrak{g}_N := \Lambda\otimes_{\mathcal{O}}\mathfrak{o}/\mathfrak{p}^N$, and let $\mathfrak{g}'_N = [\mathfrak{g}_N,\mathfrak{g}_N]$ and $\mathfrak{z}_N = \mathfrak{z}\otimes_{\mathfrak{o}}\mathfrak{o}/\mathfrak{p}^N$.

Given an element w of $\widehat{\mathfrak{g}_N} = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{g}_N,\mathbb{C}^\times)$, define the form

$$B^N_\omega : \mathfrak{g}_N\times\mathfrak{g}_N \to \mathbb{C}^\times,\ (u,v)\mapsto w([u,v]).$$

The radical of B^N_ω is

$$\mathrm{Rad}(B^N_\omega) = \{u\in\mathfrak{g}_N \mid \forall v\in\mathfrak{g}_N : B^N_\omega(u,v)=1\}.$$

If the nilpotency class c of Λ is smaller than the characteristic p of the residue field $\mathfrak{o}/\mathfrak{p}$, then the Kirillov orbit method reduces the problem of enumerating the characters of $\mathbf{G}_N$ to the problem of determining the indices in $\mathfrak{g}_N$ of $\mathrm{Rad}(B^N_\omega)$ for $w\in\mathfrak{g}'_N$. In particular, the statement of the Kirillov orbit method given in [33, Theorem 3.1(1)] asserts that if ω is an element of the coadjoint orbit Ω of $\widehat{\mathfrak{g}_N}$, then the size of this orbit coincides with the index $|\mathfrak{g}:\mathrm{Rad}(B^N_\omega)|$.

The principal congruence quotient $\mathbf{G}_N$ is a finite p-group of nilpotency class c, so that the dimensions of the irreducible complex representations of $\mathbf{G}_N$ are powers of p. Using the Kirillov orbit method [33, Theorem 3.1], O'Brien and Voll show in [33, Section 3.2] that, if $p>c$, then

$$r_{p^i}(\mathbf{G}_N) = \left|\left\{\omega\in\widehat{\mathfrak{g}'_N} \mid |\mathrm{Rad}(B^N_\omega):\mathfrak{z}_N| = p^{-2i}|\mathfrak{g}_N/\mathfrak{z}_N|\right\}\right| \, |\mathfrak{g}_N/\mathfrak{g}'_N|\, p^{-2i}. \tag{3.2.8}$$

Remark 3.2.2. In [48, Section 2.4.2], a Kirillov orbit method formalism is developed for group schemes of nilpotency class $c = 2$ which is valid for *all* primes p. This means that, in case of nilpotency class 2, equation (3.2.8) of $r_n(\mathbf{G}_N)$ holds for all prime ideals $\mathfrak{p}$.

We now relate (3.2.8) to the B-commutator matrix of $\mathfrak{g}$.

Tensoring the $\mathfrak{o}$-bases $\mathbf{e}$ and $\mathbf{f}$ with $\mathfrak{o}/\mathfrak{p}^N$ yields ordered sets $\mathbf{e}_N = (e_{1,N}, \dots, e_{a,N})$ and $\mathbf{f}_N = (f_{1,N}, \dots, f_{b,N})$ such that $(\overline{e_{1,N}}, \dots, \overline{e_{a,N}})$ is an $\mathfrak{o}/\mathfrak{p}^N$-basis for $\mathfrak{z}_N$ and $\mathbf{f}_N$ is an $\mathfrak{o}/\mathfrak{p}^N$-basis for $\mathfrak{g}'_N$ as $\mathfrak{o}/\mathfrak{p}^N$-modules.

Using similar arguments as the ones of [33, Section 2], we define the following coordinate systems:

$$\phi_N : \mathfrak{g}_N/\mathfrak{z}_N \to (\mathfrak{o}/\mathfrak{p}^N)^a, \qquad \overline{x} = \sum_{j=1}^{a} x_j \overline{e_{j,N}} \mapsto \mathbf{x} = (x_1, \dots, x_a),$$

$$\psi_N : \widehat{\mathfrak{g}'_N} \to (\mathfrak{o}/\mathfrak{p}^N)^b, \qquad \omega = \sum_{j=1}^{b} y_j f^{\vee}_{j,N} \mapsto \mathbf{y} = (y_1, \dots, y_b),$$

where, for $N \in \mathbb{N}_0$, $\mathbf{f}^{\vee}_N = (f^{\vee}_{1,N}, \dots, f^{\vee}_{b,N})$ is the $\mathfrak{o}/\mathfrak{p}$-dual lattice for $\widehat{\mathfrak{g}'_N} = \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}'_N, \mathbb{C}^{\times})$. We notice that $\mathfrak{g}_1/\mathfrak{z}_1$ and $\mathfrak{g}'_1$ are regarded as $\mathfrak{o}/\mathfrak{p}$-vector spaces in the construction of [33, Section 2]. In the coordinate systems above, we regard $\mathfrak{g}_N/\mathfrak{z}_N$ and $\mathfrak{g}'_N$ as $\mathfrak{o}/\mathfrak{p}^N$-modules for all $N \in \mathbb{N}$.

Lemma 3.2.3. *Given $\overline{x} \in \mathfrak{g}_N/\mathfrak{z}_N$ and $\omega \in \widehat{\mathfrak{g}'_N}$ with $\phi_N(\overline{x}) = \mathbf{x} = (x_1, \dots, x_a)$ and $\psi_N(\omega) = \mathbf{y} = (y_1, \dots, y_b)$, it holds that*

$$\overline{x} \in \mathrm{Rad}(B^N_{\omega})/\mathfrak{z}_N \text{ if and only if } B(\mathbf{y})\mathbf{x}^{\mathrm{tr}} = 0,$$

where $\mathbf{x}^{\mathrm{tr}}$ is the transpose of $\mathbf{x}$, regarded as a $1 \times a$-matrix.

Proof. Here, denote by $\overline{}$ the natural surjection $\mathfrak{g}_N \to \mathfrak{g}_N/\mathfrak{z}_N$. An element $\overline{x} \in \mathfrak{g}_N/\mathfrak{z}_N$ belongs to $\mathrm{Rad}(B^N_{\omega})/\mathfrak{z}$ exactly when $\omega[x, v] = 1$, for all $v \in \mathfrak{g}_N$. Fix $x \in \mathfrak{g}_N$ such that $\phi_N(\overline{x}) = \mathbf{x} = (x_1, \dots, x_a) \in (\mathfrak{o}/\mathfrak{p}^N)^a$. Then

$$[e_{i,N}, x] = \left[e_{i,N}, \sum_{j=1}^{a} x_j e_{j,N} \right] = \sum_{j=1}^{a} x_j [e_{i,N}, e_{j,N}] = \sum_{j=1}^{a} \sum_{l=1}^{b} \lambda^l_{ij} x_j f_{l,N}. \quad (3.2.9)$$

Since $\psi(\omega) = \mathbf{y} = (y_1, \dots, y_b)$, for each $i \in [a]$

$$\omega([e_i, x]) = \prod_{k=1}^{b} \left(f^{\vee}_{k,N} \left(\sum_{j=1}^{a} \sum_{l=1}^{b} \lambda^l_{ij} x_j f_{l,N} \right) \right)^{y_k} = \prod_{k=1}^{b} \left(f^{\vee}_{k,N} \left(f_{k,N} \right) \right)^{y_k \sum_{j=1}^{a} \lambda^k_{ij} x_j}.$$

This expression equals 1 exactly when $\sum_{k=1}^{b} \sum_{j=1}^{a} \lambda^k_{ij} x_j y_k = 0$. Now, by definition, $\sum_{k=1}^{b} \lambda^k_{ij} y_k = B(\mathbf{y})_{ij}$, where $B(\mathbf{y})$ is the A-commutator matrix of Definition 3.2.1 evaluated at $\mathbf{y}$. Consequently, $\overline{x} \in \mathrm{Rad}(B^N_{\omega})/\mathfrak{z}_N$ if and only if $\sum_{j=1}^{a} B(\mathbf{y})_{ij} x_j = 0$, for all $j \in [a]$, that is, $B(\mathbf{y})\mathbf{x}^{\mathrm{tr}} = 0$. ☐

Fix an elementary divisor type $\tilde{\nu}(B(\mathbf{y})) = (m_1, \dots, m_{u_B}) \in [N]_0^{u_B}$, where

$$2u_B = \max\{\mathrm{rk}_{\mathrm{Frac}(\mathfrak{o})} B(\mathbf{z}) \mid \mathbf{z} \in \mathfrak{o}^b\}.$$

Since $B(\mathbf{y})$ is similar to the matrix $\mathrm{Diag}(\pi^{m_1}, \pi^{m_1}, \dots, \pi^{m_{u_B}}, \pi^{m_{u_B}}, \mathbf{0}_{a-2u_B})$, where $\mathbf{0}_{a-2u_B} = (0, \dots, 0) \in \mathbb{Z}^{a-2u_B}$, the system $B(\mathbf{y})\mathbf{x}^{\mathrm{tr}} = 0$ in $\mathfrak{o}/\mathfrak{p}^N$ is equivalent to

$$\begin{cases} x_1 \equiv x_2 \equiv 0 \mod \mathfrak{p}^{N-m_1}, \\ x_3 \equiv x_4 \equiv 0 \mod \mathfrak{p}^{N-m_2}, \\ \vdots \\ x_{2u_B-1} \equiv x_{2u_B} \equiv 0 \mod \mathfrak{p}^{N-m_{u_B}}. \end{cases}$$

In particular, for $2u_B < a$, the elements $x_{2u_B+1}, \dots, x_a$ are arbitrary elements of $\mathfrak{o}/\mathfrak{p}^N$. Moreover

$$|\{x \in \mathfrak{o}/\mathfrak{p}^N \mid x \equiv 0 \mod \mathfrak{p}^{N-m_j}\}| = q^{m_j}.$$

Hence, the number of solutions of $B(\mathbf{y})\mathbf{x}^{\mathrm{tr}} = 0$ in $\mathfrak{o}/\mathfrak{p}^N$ is $q^{2(m_1+\dots+m_{u_B})+(a-2u_B)N}$. In other words, Lemma 3.2.3 assures that

$$|\mathrm{Rad}(B_\omega^N)/\mathfrak{z}_N| = q^{2(m_1+\dots+m_{u_B})+(a-2u_B)N},$$

when $B(\mathbf{y})$ has elementary divisor type $(m_1, \dots, m_{u_B})$. In particular, $\mathrm{Rad}(B_\omega^N)/\mathfrak{z}_N$ satisfies

$$|\mathrm{Rad}(B_\omega^N) : \mathfrak{z}_N| = q^{-2i}|\mathfrak{g}_N/\mathfrak{z}_N| = q^{aN-2i}$$

exactly when $B(\mathbf{y})$ has elementary divisor type $(m_1, \dots m_{u_B})$ satisfying $\sum_{j=1}^{u_B} m_j = u_B N - i$. Consequently, expression (3.2.8) can be rewritten as follows, for $r = \mathrm{rk}(\mathfrak{g}/\mathfrak{g}') = h - b$.

$$r_{q^i}(\mathbf{G}_N) = \sum_{\mathbf{m} \in \mathcal{D}_B^N} |\{\mathbf{y} \in (\mathfrak{o}/\mathfrak{p}^N)^b \mid \tilde{\nu}(B(\mathbf{y})) = \mathbf{m}\}| q^{rN-2i}, \tag{3.2.10}$$

where

$$\mathcal{D}_B^N := \left\{ \mathbf{m} = (m_1, \dots, m_{u_B}) \in \mathbb{N}_0^{u_B} \;\middle|\; m_1 \le \dots \le m_{u_B} \le N,\ \sum_{i=1}^{u_B} m_i = u_B N - i \right\}.$$

Remark 3.2.4. The numbers $r_n(\mathbf{G}_N)$ are zero whenever n is not a power of q. In fact, as explained above, for each $w \in \widehat{\mathfrak{g}'_N}$ with $\psi_N(w) = \mathbf{y}$ and $\widetilde{\nu}(B(\mathbf{y})) = (m_1, \dots, m_{u_B})$, one has that

$$|\mathrm{Rad}(B_\omega^N)/\mathfrak{z}_N| = q^{2M+(a-2u_B)N}, \text{ where } M = m_1 + \dots + m_{u_B}.$$

Moreover, $|\mathfrak{g}_N/\mathfrak{z}_N| = q^{aN}$, so that equality $|\mathrm{Rad}(B_\omega^N) : \mathfrak{z}_N| = p^{-2i}|\mathfrak{g}_N/\mathfrak{z}_N|$ is satisfied if and only if $p^i = q^{u_B - M}$.

For a matrix $\mathcal{R}(\underline{Y}) = \mathcal{R}(Y_1, \dots, Y_n)$ of polynomials as the one at the beginning of Section 3.2 and for $\mathbf{m} = (m_1, \dots, m_\epsilon) \in \mathbb{N}_0^\epsilon$, define

$$\mathfrak{W}^{\mathfrak{o}}_{N,\mathcal{R},\mathbf{m}} := \{\mathbf{y} \in (\mathfrak{o}/\mathfrak{p}^N)^n \mid \nu(\mathcal{R}(\mathbf{y})) = \mathbf{m}\}.$$

Expression (3.2.10) is written in terms of cardinalities of such sets, which are related to the cardinalities $\mathcal{N}^{\mathfrak{o}}_{N,\mathcal{R},\mathbf{m}}$ of the sets $\mathfrak{N}^{\mathfrak{o}}_{N,\mathcal{R},\mathbf{m}}$ as follows. Write $\mathbf{m} - m = (m_1 - m, \dots, m_\epsilon - m)$, for all $m \in \mathbb{N}_0$. If $\mathcal{R}(\mathbf{y})$ is such that $v_{\mathfrak{p}}(\mathbf{y}) = v_{\mathfrak{p}}(\mathcal{R}(\mathbf{y}))$, for all $\mathbf{y} \in \mathfrak{o}^n$, then

$$|\mathfrak{W}^{\mathfrak{o}}_{N,\mathcal{R},\mathbf{m}}| = \mathcal{N}^{\mathfrak{o}}_{N-m_1,\mathcal{R},\mathbf{m}-m_1}. \tag{3.2.11}$$

Indeed, the map $\mathfrak{N}^{\mathfrak{o}}_{N-m_1,\mathcal{R},\mathbf{m}-m_1} \to \mathfrak{W}^{\mathfrak{o}}_{N,\mathcal{R},\mathbf{m}}$ given by $\tilde{y} \mapsto \pi^{m_1}\tilde{y}$ is a bijection.

Equality (3.2.11) applied to (3.2.10) yields the following result.

Lemma 3.2.5. *Suppose that either $c = 2$ or $c < p$. For each $i \in \mathbb{N}_0$ and $N \in \mathbb{N}_0$,*

$$r_{q^i}(\mathbf{G}_N) = \sum_{(m_1,\dots,m_{u_B}) \in \mathcal{D}_B^N} \mathcal{N}^{\mathfrak{o}}_{N-m_1,B,(0,m_2-m_1,\dots,m_{u_B}-m_1)} q^{rN-2i}.$$

$c_n(\mathbf{G}_N)$ and elementary divisor types of the A-commutator matrix

We now show an analogous result to Lemma 3.2.5 for the numbers $c_n(\mathbf{G}_N)$.

For $x \in \mathfrak{g}_N/\mathfrak{z}_N$, the adjoint homomorphism $\mathrm{ad}_x : \mathfrak{g}_N/\mathfrak{z}_N \to \widehat{\mathfrak{g}_N}$ is given by $\mathrm{ad}_x(z) = [z,x]$, for all $z \in \mathfrak{g}_N/\mathfrak{z}_N$. Let $\mathrm{ad}_x^\star : \widehat{\mathfrak{g}'_N} \to \widehat{\mathfrak{g}_N/\mathfrak{z}_N}$ be the map $\omega \mapsto \omega \circ \mathrm{ad}_x$. Since the principal congruence quotient $\mathbf{G}_N$ is a finite p-group, the sizes of its conjugacy classes are powers of p and, according to [33, Section 3], for $c < p$,

$$c_{p^i}(\mathbf{G}_N) = \left| \left\{ x \in \mathfrak{g}_N/\mathfrak{z}_N \mid |\mathrm{Ker}(\mathrm{ad}_x^\star)| = p^{-i}|\widehat{\mathfrak{g}'_N}| \right\} \right| \, |\mathfrak{z}_N| \, p^{-i}. \tag{3.2.12}$$

This formula reflects the fact that the Lazard correspondence induces an order-preserving correspondence between subgroups of $\mathbf{G}_N$ and sublattices of $\mathfrak{g}_N$, and maps normal subgroups to ideals. Moreover, centralizers of elements of $\mathbf{G}_N$ correspond to centralizers of elements of $\mathfrak{g}_N$ under the Lazard correspondence.

Remark 3.2.6. The cardinalities of $\mathfrak{g}'_N$ and $\mathfrak{g}_N/\mathfrak{z}_N$ are powers of q, and hence so is the cardinality of $\mathrm{Ker}(\mathrm{ad}_x^\star)$. In particular, the equality $|\mathrm{Ker}(\mathrm{ad}_x^\star)| = p^{-i}|\widehat{\mathfrak{g}'_N}|$ can only be satisfied if p^i is a power of q. That is, the number $c_n(\mathbf{G}_N)$ is zero unless n is a power of q.

Lemma 3.2.7. *Given $x \in \mathfrak{g}_N/\mathfrak{z}_N$ and $w \in \widehat{\mathfrak{g}'_N}$ with $\phi_N(x) = \mathbf{x} = (x_1, \ldots, x_a)$ and $\psi_N(w) = \mathbf{y} = (y_1, \ldots, y_b)$,*

$$w \in \mathrm{Ker}(\mathrm{ad}_x^\star) \text{ if and only if } A(\mathbf{x})\mathbf{y}^{\mathrm{tr}} = 0.$$

Proof. An element $w \in \widehat{\mathfrak{g}'_N}$ belongs to $\mathrm{Ker}(\mathrm{ad}_x^\star)$ exactly when $w[x,v] = 1$ for all $v \in \mathfrak{g}_N/\mathfrak{z}_N$. Expressing these conditions in coordinates, just as in Lemma 3.2.3, we see that the expression on the statement of this lemma holds. □

Fix an elementary divisor type $\nu(A(\mathbf{x})) = (m_1, \ldots, m_{u_A})$, where

$$u_A := \max\{\mathrm{rk}_{\mathrm{Frac}(\mathfrak{o})} A(\mathbf{z}) \mid \mathbf{z} \in \mathfrak{o}^a\}.$$

As in the representation case, we show that the system $A(\mathbf{x})\mathbf{y}^{\mathrm{tr}} = 0$ in $\mathfrak{o}/\mathfrak{p}^N$ has $q^{m_1+m_2+\cdots+m_{u_A}+(b-u_A)N}$ solutions in $\mathfrak{o}/\mathfrak{p}^N$. For $z = \mathrm{rk}(\mathfrak{z}) = h - a$, this yields

$$c_{q^i}(\mathbf{G}_N) = \sum_{\mathbf{m} \in \mathcal{D}_A^N} |\{\mathbf{x} \in (\mathfrak{o}/\mathfrak{p}^N)^a \mid \nu(A(\mathbf{x})) = \mathbf{m}\}| q^{zN-i}, \tag{3.2.13}$$

where

$$\mathcal{D}_A^N := \left\{ \mathbf{m} = (m_1, \ldots, m_{u_A}) \in \mathbb{N}_0^{u_A} \mid m_1 \leq \cdots \leq m_{u_A} \leq N, \; \sum_{i=1}^{u_A} m_i = u_A N - i \right\}.$$

Equality (3.2.11) applied to (3.2.13) gives the following lemma.

Lemma 3.2.8. *Suppose that $p > c$. For each $i \in \mathbb{N}_0$ and $N \in \mathbb{N}_0$,*

$$c_{q^i}(\mathbf{G}_N) = \sum_{(m_1, \ldots, m_{u_A}) \in \mathcal{D}_A^N} \mathcal{N}^{\mathfrak{o}}_{N-m_1, A, (0, m_2-m_1, \ldots, m_{u_A}-m_1)} q^{zN-i}.$$

3.2.3 Bivariate zeta functions as $\mathfrak{p}$-adic integrals

We now write almost all local factors of the bivariate zeta functions of $\mathbf{G}(\mathcal{O})$ in terms of Poincaré series such as (3.2.2) and hence obtain formulae for them in terms of $\mathfrak{p}$-adic integrals. This is done using the descriptions of the numbers $r_n(\mathbf{G}_N)$ of Lemma 3.2.5 and the descriptions of the numbers $c_n(\mathbf{G}_N)$ of

Lemma 3.2.8. Therefore, the finitely many local terms which are not written in terms of $\mathfrak{p}$-adic integrals are the ones whose corresponding prime ideals do not satisfy the assumptions of these Lemmata. That is, for the bivariate representation zeta functions, we write all local factors in terms of $\mathfrak{p}$-adic integrals in case of nilpotency class $c = 2$ and, in case of nilpotency class $c > 2$, we exclude the local terms at prime ideals $\mathfrak{p}$ with residue field cardinality $p \leq c$. For the bivariate conjugacy class zeta functions the exception is given by local factors at prime ideals $\mathfrak{p}$ with residue field cardinality $p \leq c$.

Recall from Section 3.2.2 that the dimensions of irreducible complex representations as well as the sizes of the conjugacy classes of $\mathbf{G}_N$ are powers of q, allowing us to write the local terms of the (univariate) representation and conjugacy class zeta functions of the principal congruence quotient $\mathbf{G}_N$ as

$$\zeta^{\mathrm{irr}}_{\mathbf{G}_N}(s) = \sum_{i=0}^{\infty} r_{q^i}(\mathbf{G}_N) q^{-is} \quad \text{and} \quad \zeta^{\mathrm{cc}}_{\mathbf{G}_N}(s) = \sum_{i=0}^{\infty} c_{q^i}(\mathbf{G}_N) q^{-is}.$$

These sums are finite, since $\mathbf{G}_N$ is a finite group. Consequently, the local factors of the bivariate zeta functions are given by

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}(s_1, s_2) = \sum_{N=0}^{\infty} \sum_{i=0}^{\infty} r_{q^i}(\mathbf{G}_N) q^{-is_1 - Ns_2} \text{ and}$$

$$\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathfrak{o})}(s_1, s_2) = \sum_{N=0}^{\infty} \sum_{i=0}^{\infty} c_{q^i}(\mathbf{G}_N) q^{-is_1 - Ns_2}.$$

The expressions for $r_n(\mathbf{G}_N)$ and $c_n(\mathbf{G}_N)$ given in Lemmata 3.2.5 and 3.2.8 yield, respectively,

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}(s_1, s_2) = \tag{3.2.14}$$
$$\sum_{N=0}^{\infty} \sum_{i=0}^{\infty} \sum_{(m_1,\ldots,m_{u_B}) \in \mathcal{D}^N_B} \mathcal{N}^{\mathfrak{o}}_{N-m_1, B, (0, m_2-m_1, \ldots, m_{u_B}-m_1)} q^{(r-s_2)N-(2+s_1)i},$$

$$\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathfrak{o})}(s_1, s_2) = \tag{3.2.15}$$
$$\sum_{N=0}^{\infty} \sum_{i=0}^{\infty} \sum_{(m_1,\ldots,m_{u_A}) \in \mathcal{D}^N_A} \mathcal{N}^{\mathfrak{o}}_{N-m_1, A, (0, m_2-m_1, \ldots, m_{u_A}-m_1)} q^{(z-s_2)N-(1+s_1)i}.$$

We now show how to rewrite these sums as Poincaré series of the form (3.2.2). In preparation for this, we need two lemmata.

Lemma 3.2.9. *Let s be a complex variable, $(a_m)_{m \in \mathbb{N}_0}$ a sequence of real numbers, and let $q \in \mathbb{Z}_{\geq 2}$. The following holds, provided both series converge.*

$$\sum_{N=1}^{\infty} \sum_{m=0}^{N-1} a_m q^{-sN} = \frac{q^{-s}}{1 - q^{-s}} \left(\sum_{N=0}^{\infty} a_N q^{-sN} \right).$$

Proof. In fact,

$$\begin{aligned}(1 - q^{-s}) \sum_{N=1}^{\infty} \sum_{m=0}^{N-1} a_m q^{-sN} &= \sum_{N=1}^{\infty} \sum_{m=0}^{N-1} a_m q^{-sN} - \sum_{N=1}^{\infty} \sum_{m=0}^{N-1} a_m q^{-s(N+1)} \\ &= a_0 q^{-s} + \sum_{N=1}^{\infty} \sum_{m=0}^{N} a_m q^{-s(N+1)} - \sum_{N=1}^{\infty} \sum_{m=0}^{N-1} a_m q^{-s(N+1)}\end{aligned}$$

$$= a_0 q^{-s} + \sum_{N=1}^{\infty} a_N q^{-s(N+1)} = q^{-s} \sum_{N=0}^{\infty} a_N q^{-sN}. \qquad \square$$

Lemma 3.2.10. *Let s and t be complex variables and $q \in \mathbb{Z}$. Let also $\mathcal{R}(\underline{Y}) = \mathcal{R}(Y_1, \ldots, Y_n)$ be a matrix of polynomials $\mathcal{R}(\underline{Y})_{ij} \in \mathfrak{o}[\underline{Y}]$ with $u = \max\{\mathrm{rk}_{\mathrm{Frac}(\mathfrak{o})} \mathcal{R}(\mathbf{z}) \mid \mathbf{z} \in \mathfrak{o}^n\}$. The following holds, provided both series converge.*

$$\sum_{N=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\substack{0 \le m_1 \le \cdots \le m_u \le N \\ \sum_{j=1}^{u} m_j = uN - i}} \mathcal{N}^{\mathfrak{o}}_{N-m_1, \mathcal{R}, (0, m_2 - m_1, \ldots, m_u - m_1)} q^{-sN - ti} \tag{3.2.16}$$

$$= \frac{1}{1 - q^{-s}} \left(1 + \sum_{N=1}^{\infty} \sum_{(m_1, \ldots, m_u) \in \mathbb{N}_0^u} \mathcal{N}^{\mathfrak{o}}_{N, \mathcal{R}, (m_1, m_2, \ldots, m_u)} q^{-(s+ut)N + t \sum_{j=1}^{u} m_j} \right).$$

Proof. Let $\mathbf{m} = (m_1, \ldots, m_u)$ and recall the notation $\mathbf{m} - m = (m_1 - m, \ldots, m_u - m)$, for $m \in \mathbb{N}_0$.

As $\mathcal{N}^{\mathfrak{o}}_{N, \mathcal{R}, \mathbf{m}} = 0$ unless $0 = m_1 \le m_2 \le \cdots \le m_u \le N$, in which case $0 \le \sum_{j=1}^{u} m_j \le uN$, the condition $\sum_{j=1}^{u} m_j = uN - i$ implies that the only values of i which are relevant for the sum (3.2.16) are $0 \le i \le uN$. Hence, the expression on the left-hand side of (3.2.16) can be rewritten as

$$1 + \sum_{N=1}^{\infty} \sum_{i=0}^{uN} \sum_{\substack{0 \le m_1 \le \cdots \le m_u \le N \\ \sum_{j=1}^{u} m_j = uN - i}} \mathcal{N}^{\mathfrak{o}}_{N-m_1, \mathcal{R}, \mathbf{m} - m_1} q^{-sN - ti}. \tag{3.2.17}$$

Restricting the summation in (3.2.17) to $m_1 = 0$ leads to

$$\sum_{N=1}^{\infty} \sum_{\substack{0 \le m_2 \le \cdots \le m_u \le N \\ \sum_{j=2}^{u} m_j \le (u-1)N}} \mathcal{N}^{\mathfrak{o}}_{N, \mathcal{R}, (0, m_2, \ldots, m_u)} q^{-sN - t(uN - \sum_{j=2}^{u} m_j)}.$$

The fact that $\mathcal{N}^{\mathfrak{o}}_{N, \mathcal{R}, \mathbf{m}} = 0$ unless $0 = m_1 \le m_2 \le \cdots \le m_u \le N$ allows us to rewrite this as

$$\sum_{N=1}^{\infty} \sum_{\mathbf{m} \in \mathbb{N}_0^u} \mathcal{N}^{\mathfrak{o}}_{N, \mathcal{R}, \mathbf{m}} q^{-(s+ut)N + t \sum_{j=1}^{u} m_j} =: \mathcal{S}(s, t).$$

Our goal now is to write the part of the summation in (3.2.17) with $m_1 > 0$ in terms of $\mathcal{S}(s, t)$. Set $\mathbf{m}' = (0, m'_2, \ldots, m'_u)$.

$$\sum_{N=1}^{\infty} \sum_{i=0}^{uN} \sum_{\substack{0 < m_1 \le \cdots \le m_u \le N \\ \sum_{j=1}^{u} m_j = uN - i}} \mathcal{N}^{\mathfrak{o}}_{N-m_1, \mathcal{R}, \mathbf{m} - m_1} q^{-sN - ti}$$

$$= \sum_{N=1}^{\infty} \sum_{m_1=1}^{N} \sum_{\substack{m_1 \le m_2 \le \cdots \le m_u \le N \\ \sum_{j=2}^{u} m_j \le uN - m_1}} \mathcal{N}^{\mathfrak{o}}_{N-m_1, \mathcal{R}, \mathbf{m} - m_1} q^{-sN - t(uN - \sum_{j=1}^{u} m_j)}$$

$$= \sum_{N=1}^{\infty} \sum_{m_1=1}^{N} \sum_{\substack{0 \le m'_2 \le \cdots \le m'_u \le N - m_1 \\ \sum_{j=2}^{u} m'_j \le u(N - m_1)}} \mathcal{N}^{\mathfrak{o}}_{N-m_1, \mathcal{R}, \mathbf{m}'} q^{-(s+ut)N + t\left(\left(\sum_{j=2}^{u} m'_j\right) + u m_1\right)} \tag{3.2.18}$$

$$= \sum_{N=1}^{\infty} q^{-sN} \sum_{m=0}^{N-1} \sum_{\substack{0\leq m_2'\leq\cdots\leq m_u'\leq m \\ \sum_{j=2}^{u} m_j'\leq um}} \mathcal{N}^{\mathfrak{o}}_{m,\mathcal{R},\mathbf{m}'} q^{t\left(\left(\sum_{j=2}^{u} m_j'\right)-um\right)}$$

$$= \sum_{N=1}^{\infty} q^{-sN} \sum_{m=0}^{N-1} \sum_{\mathbf{m}\in\mathbb{N}_0^u} \mathcal{N}^{\mathfrak{o}}_{m,\mathcal{R},\mathbf{m}} q^{t\left(\left(\sum_{j=1}^{u} m_j\right)-um\right)}. \tag{3.2.19}$$

Apply Lemma 3.2.9 to (3.2.19) by setting

$$a_m := \sum_{\mathbf{m}\in\mathbb{N}_0^u} \mathcal{N}^{\mathfrak{o}}_{m,\mathcal{R},\mathbf{m}} q^{t\left(\left(\sum_{j=1}^{u} m_j\right)-um\right)}.$$

This gives

$$\sum_{N=1}^{\infty}\sum_{i=1}^{uN} \sum_{\substack{0<m_1\leq\cdots\leq m_u\leq N \\ \sum_{j=1}^{u} m_j=uN-i}} \mathcal{N}^{\mathfrak{o}}_{N-m_1,\mathcal{R},\mathbf{m}-m_1} q^{-sN-ti}$$

$$= \frac{q^{-s}}{1-q^{-s}}\left(1+\sum_{N=1}^{\infty}\sum_{\mathbf{m}\in\mathbb{N}_0^u} \mathcal{N}^{\mathfrak{o}}_{N,\mathcal{R},\mathbf{m}} q^{-(s+ut)N+t\sum_{j=1}^{u} m_j}\right)$$

$$= \frac{q^{-s}}{1-q^{-s}}\left(1+\mathcal{S}(s,t)\right).$$

Combining the expressions for the parts of the sum with $m_1 = 0$ and $m_1 > 0$ yields

$$\sum_{N=0}^{\infty}\sum_{i=0}^{\infty} \sum_{\substack{0\leq m_1\leq\cdots\leq m_u\leq N \\ \sum_{j=1}^{u} m_j=uN-i}} \mathcal{N}^{\mathfrak{o}}_{N,\mathcal{R},(m_1,m_2,\dots,m_u)} q^{-sN-ti}$$

$$= 1+\mathcal{S}(s,t)+\frac{q^{-s}}{1-q^{-s}}\left(1+\mathcal{S}(s,t)\right) = \frac{1}{1-q^{-s}}\left(1+\mathcal{S}(s,t)\right). \qquad \square$$

By applying Lemma 3.2.10 to (3.2.14) and (3.2.15), we obtain the following.

Proposition 3.2.11. *The bivariate zeta functions of* $\mathbf{G}(\mathfrak{o})$ *can be described by*

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}(s_1,s_2) = \tag{3.2.20}$$

$$\frac{1}{1-q^{r-s_2}}\left(1+\sum_{N=1}^{\infty}\sum_{(m_1,\dots,m_{u_B})\in\mathbb{N}_0^{u_B}} \mathcal{N}^{\mathfrak{o}}_{N,B,\mathbf{m}} q^{-N(u_B s_1+s_2+2u_B-r)-2\sum_{j=1}^{u_B} m_j \frac{(-s_1-2)}{2}}\right),$$

$$\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathfrak{o})}(s_1,s_2) = \tag{3.2.21}$$

$$\frac{1}{1-q^{z-s_2}}\left(1+\sum_{N=1}^{\infty}\sum_{(m_1,\dots,m_{u_A})\in\mathbb{N}_0^{u_A}} \mathcal{N}^{\mathfrak{o}}_{N,A,\mathbf{m}} q^{-N(u_A s_1+s_2+u_A-z)-\sum_{j=1}^{u_A} m_j(-s_1-1)}\right).$$

Expression (3.2.20) is of the form (3.2.5) with $t = u_B s_1 + s_2 + 2u_B - r$ and $r_k = \frac{-s_1-2}{2}$ for each $k \in [u_B]$, whilst (3.2.21) is (3.2.2) with $t = u_A s_1 + s_2 + u_A - z$ and $r_k = -s_1 - 1$ for each $k \in [u_A]$. Therefore these choices of t and $\underline{r}$ applied to (3.2.6) and to (3.2.4) yields the following. Recall that $a+z = \mathrm{rk}(\mathfrak{g}/\mathfrak{z})+\mathrm{rk}(\mathfrak{z}) = \mathrm{rk}(\mathfrak{g}) = h$ and that $b + r = \mathrm{rk}(\mathfrak{g}') + \mathrm{rk}(\mathfrak{g}/\mathfrak{g}') = h$. In the following, we write $\mathscr{Z}_{\mathfrak{o},\mathcal{R}}(r,t)$ meaning $\mathscr{Z}_{\mathfrak{o},\mathcal{R}}(r,t\mathbf{1}_{u_\mathcal{R}})$, where $\mathbf{1}_{u_\mathcal{R}} = (1,\dots,1) \in \mathbb{N}_0^{u_\mathcal{R}}$ and $\mathcal{R}$ is either the A-commutator matrix or the B-commutator matrix of $\mathfrak{g}$.

Proposition 3.2.12. *The bivariate zeta functions of* $\mathbf{G}(\mathfrak{o})$ *can be described by*

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}(s_1,s_2)=\frac{1}{1-q^{r-s_2}}\left(1+\mathscr{Z}_{\mathfrak{o},B}\left(\tfrac{-s_1-2}{2},u_Bs_1+s_2+2u_B-h-1\right)\right), \tag{3.2.22}$$

$$\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathfrak{o})}(s_1,s_2)=\frac{1}{1-q^{z-s_2}}\left(1+\mathscr{Z}_{\mathfrak{o},A}\left(-s_1-1,u_As_1+s_2+u_A-h-1\right)\right), \tag{3.2.23}$$

as long as either $c=2$ *or* $p>c$ *for* (3.2.22)*, and as long as* $p>c$ *for* (3.2.23)*.*

In particular, specialisation (1.1.8) provides the following formulae for the class number zeta function of $\mathbf{G}(\mathfrak{o})$:

$$\zeta^{\mathrm{k}}_{\mathbf{G}(\mathfrak{o})}(s)=\frac{1}{1-q^{z-s}}\left(1+\mathscr{Z}_{\mathfrak{o},A}\left(-1,s+u_A-h-1\right)\right), \tag{3.2.24}$$

$$=\frac{1}{1-q^{r-s}}\left(1+\mathscr{Z}_{\mathfrak{o},B}\left(-1,s+2u_B-h-1\right)\right). \tag{3.2.25}$$

Remark 3.2.13. Recall the $\mathbb{Z}$-Lie lattice $\mathcal{F}_{n,\delta}$ of Definition 1.2.1, and that for $\mathfrak{g}=\mathcal{F}_{n,\delta}\otimes_{\mathcal{O}}\mathfrak{o}$ with centre $\mathfrak{z}$ it holds that $a=\mathrm{rk}(\mathfrak{g}/\mathfrak{z})=2n+\delta$. The B-commutator matrix of $\mathcal{F}_{n,\delta}$ is the generic $a\times a$-antisymmetric matrix. In particular, we see that $\mathfrak{so}_a(\mathfrak{o})=\{B_{\mathcal{F}_{n,\delta}}(\underline{x})\mid \underline{x}\in\mathfrak{o}^b\}$, where $\mathfrak{so}_a(\mathfrak{o})$ is the orthogonal Lie algebra of $a\times a$-matrices M over $\mathfrak{o}$ satisfying $M+M^{\mathrm{Tr}}=0$.

In particular, formula (1.2.1) for the class number zeta function of $F_{n,\delta}(\mathfrak{o})$ also follows from [38, Proposition 5.11], which gives a formula for the ask zeta function $\mathrm{Z}^{\mathrm{ask}}_{\mathfrak{so}_d(\mathfrak{o})}(T)$ of the orthogonal Lie algebra $\mathfrak{so}_d(\mathfrak{o})$, $d\in\mathbb{N}$; see [38, Definition 1.3].

In fact, when comparing the p-adic integral [38, (4.3)] with (3.2.25), we see that $\zeta^{\mathrm{k}}_{F_{n,\delta}(\mathfrak{o})}(s)=\mathrm{Z}^{\mathrm{ask}}_{\mathfrak{so}_a(\mathfrak{o})}(q^{-s+\binom{a}{2}})$, and hence [38, Proposition 5.11] shows (1.2.1).

Remark 3.2.14. Formula (3.2.24) coincides with the $\mathfrak{p}$-adic integral obtained from the $\mathfrak{p}$-adic integral [38, (4.3)] together with the specialisation given in [38, Theorem 1.7].

In fact, for each $x\in\mathfrak{g}$, let $\mathrm{ad}_x:\mathfrak{g}\to\mathfrak{g}'$ be the adjoint homomorphism $\mathrm{ad}_x(z)=[z,x]$, for all $z\in\mathfrak{g}$. As in Section 3.2.2, let $\mathscr{B}=(e_1,\dots,e_h)$ be a basis of $\mathfrak{g}$ with the properties described there; we use the notation that was set up in this context. For each $x\in\mathfrak{g}$, we can write $x=\sum_{i=1}^h x_ie_i$, for some $x_i\in\mathfrak{o}$. Let $\mathbf{x}=(x_1,\dots,x_h)\in\mathfrak{o}^h$. The $b\times h$-matrix representing ad_x is such that its submatrix composed of its first a columns is the transpose $A(\mathbf{x})^{\mathrm{tr}}$ of the A-commutator matrix of Λ and the remaining columns have only zero entries.

The integrals there are taken over $\mathfrak{o}\times\mathfrak{o}^a$ instead of $\mathfrak{p}\times W^{\mathfrak{o}}_a$ as in (3.2.24). The fact that they coincide is due Lemma 2.2.4.

We conclude this section with an example.

Example 3.2.15. Let $\mathbf{H}(\mathcal{O})$ be the Heisenberg group over $\mathcal{O}$. The unipotent group scheme $\mathbf{H}$ is obtained from the $\mathbb{Z}$-Lie lattice

$$\Lambda=\langle x_1,x_2,y\mid [x_1,x_2]-y\rangle.$$

The commutator matrices of $\mathfrak{g}=\Lambda(\mathfrak{o})$ with respect to the ordered sets $\mathbf{e}=$

(x_1, x_2) and $\mathbf{f} = (y)$ are

$$A(X_1, X_2) = \begin{bmatrix} X_2 \\ -X_1 \end{bmatrix} \text{ and } B(Y) = \begin{bmatrix} 0 & Y \\ -Y & 0 \end{bmatrix}.$$

The A-commutator matrix has rank 1 and the B-commutator matrix has rank 2 over the respective fields of rational functions, that is, $u_A = u_B = 1$. Moreover, $h = \mathrm{rk}(\mathfrak{g}) = 3$, and

$$F_1(A(X_1, X_2)) = \{-X_1, X_2\}, \qquad F_2(B(Y)) = \{Y^2\}.$$

Thus, if $(x_1, x_2) \in W_2^{\mathfrak{o}}$, that is, $v_{\mathfrak{p}}(x_1, x_2) = 0$, then $\|F_1(A(x_1, x_2))\|_{\mathfrak{p}} = 1$. Also, if $y \in W_1^{\mathfrak{o}}$, then $v_{\mathfrak{p}}(y^2) = 0$, which gives $\|F_2(B(y))\|_{\mathfrak{p}} = 1$.

It follows from Proposition 3.2.12 and Lemma 2.2.1 that

$$\begin{aligned}
\mathcal{Z}^{\mathrm{irr}}_{\mathbf{H}(\mathfrak{o})}(s_1, s_2) &= \frac{1}{1 - q^{2-s_2}} \left(1 + (1 - q^{-1})^{-1} \int_{(w,y)\in\mathfrak{p}\times W_1^{\mathfrak{o}}} |w|_{\mathfrak{p}}^{s_1+s_2-2} d\mu\right) \\
&= \frac{1 - q^{-s_1-s_2}}{(1 - q^{1-s_1-s_2})(1 - q^{2-s_2})}, \\
\mathcal{Z}^{\mathrm{cc}}_{\mathbf{H}(\mathfrak{o})}(s_1, s_2) &= \frac{1}{1 - q^{1-s_2}} \left(1 + (1 - q^{-1})^{-1} \int_{(w,x_1,x_2)\in\mathfrak{p}\times W_2^{\mathfrak{o}}} |w|_{\mathfrak{p}}^{s_1+s_2-3} d\mu\right) \\
&= \frac{1 - q^{-s_1-s_2}}{(1 - q^{1-s_2})(1 - q^{2-s_1-s_2})}.
\end{aligned}$$

△

3.3 Twist representation zeta functions

In this section, we assume that $\mathbf{G}$ is a unipotent group scheme of nilpotency class 2 associated to a $\mathcal{O}$-Lie lattice Λ without the assumption $\Lambda' \subseteq 2\Lambda$, constructed as explained in Section 2.1. We provide a univariate specialisation of the bivariate representation zeta function of $\mathbf{G}(\mathfrak{o})$ which results in the twist representation zeta function of this group.

According to [48, Corollary 2.11 and Proposition 2.18], the twist representation zeta function of $\mathbf{G}(\mathfrak{o})$ may be written as

$$\widetilde{\zeta^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}}(s) = 1 + \mathscr{Z}_{\mathfrak{o},B}(-s/2, u_B s - b - 1),$$

where $b = \mathrm{rk}(\mathfrak{g}')$, $2u_B = \max\{\mathrm{rk}_{\mathrm{Frac}(\mathfrak{o})} B(\mathbf{z}) \mid \mathbf{z} \in \mathfrak{o}^b\}$ and $\mathscr{Z}_{\mathfrak{o},B}(\mathbf{r}, t)$ is the integral $\mathscr{Z}_{\mathfrak{o},\mathcal{R}}(\mathbf{r}, t)$ given in (3.2.6) when $\mathcal{R}(\underline{Y})$ is the B-commutator matrix $B(\underline{Y})$. Recall that $r = \mathrm{rk}(\mathfrak{g}/\mathfrak{g}') = h - b$. We have shown in Proposition 3.2.12 that

$$(1 - q^{r-s_2})\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}(s_1, s_2) = 1 + \mathscr{Z}_{\mathfrak{o},B}\left((-2 - s_1)/2, u_B s_1 + s_2 + 2u_B - h - 1\right).$$

Comparing the expressions for $\widetilde{\zeta^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}}(s)$ and $(1 - q^{r-s_2})\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}(s_1, s_2)$, we obtain the desired specialisation.

Proposition 3.3.1. *If $\mathbf{G}(\mathfrak{o})$ has nilpotency class 2 and s_1, $s_2 \in \mathbb{C}$ have sufficiently large real parts, then*

$$(1 - q^{r-s_2})\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}(s_1, s_2) \mid_{\substack{s_1 \to s-2 \\ s_2 \to r}} = \widetilde{\zeta^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}}(s).$$

As mentioned in Section 1.1.1, no such specialisation is expected to hold for $\mathbf{G}(\mathcal{O})$ with nilpotency class $c \geq 3$. In fact, when counting isomorphism classes instead of twist-isoclasses of irreducible complex representations of $\mathbf{G}(\mathfrak{o})$, we overcount the ones which are twist-equivalent. As explained in [48, Section 2.2.1], one may count the representations belonging to coadjoint orbits of $\mathfrak{g}$ to obtain the number of representations of $\mathbf{G}(\mathfrak{o})$ up to twist. However, by doing so, for each co-adjoint orbit $\Omega \subset \hat{\mathfrak{g}}$ and $\psi \in \Omega$, one overcounts the number of representations which are twist equivalent to ψ. This number is given by the index of $G_{\psi,2} = \{\omega \in \mathfrak{g} \mid \psi([\omega, \mathfrak{g}]) = 1\}$ in $\mathfrak{g}$. It is clear that this index is 1 in nilpotency class $c = 2$, but might be larger otherwise.

In the following example, we exhibit a group $\mathbf{G}(\mathcal{O})$ of nilpotency class $c = 3$ whose bivariate representation zeta function does not specialise to its twist representation zeta function.

Example 3.3.2. Consider the free nilpotent $\mathbb{Z}$-Lie lattice on 2 generators of class 3:

$$\mathfrak{f}_{3,2} = \langle x_1, x_2, y, z_1, z_2 \mid [x_1, x_2] - y,\ [y, x_1] - z_1,\ [y, x_2] - z_2 \rangle,$$

and relations that do not follow from the given ones are trivial. Let $\mathfrak{F}_{3,2}$ be the unipotent group scheme obtained from $\mathfrak{f}_{3,2}$, and denote by $\mathfrak{z}_{3,2}$ and by $\mathfrak{f}'_{3,2}$ the centre and the derived Lie lattice of $\mathfrak{f}_{3,2}$, respectively.

The B-commutator matrix of $\mathfrak{f}_{3,2}$ with respect to $\mathbf{e} = (y, x_1, x_2)$ and $\mathbf{f} = (z_1, z_2, y)$ is

$$B(Y_1, Y_2, Y_3) = \begin{bmatrix} 0 & Y_1 & Y_2 \\ -Y_1 & 0 & Y_3 \\ -Y_2 & -Y_3 & 0 \end{bmatrix}.$$

Thus, $u_B = 1$, $F_0(B(\underline{Y})) = \{1\}$, and $F_2(B(\underline{Y})) \supseteq \{Y_1^2, Y_2^2, Y_3^2\}$. By Proposition 3.2.12 and Lemma 2.2.1,

$$\begin{aligned}\mathcal{Z}^{\mathrm{irr}}_{\mathfrak{F}_{2,3}(\mathfrak{o})}(s_1, s_2) &= \frac{1}{1 - q^{2-s_2}} \left(1 + (1 - q^{-1})^{-1} \int_{(w, y_1, y_2, y_3) \in \mathfrak{p} \times W_3^{\mathfrak{o}}} |w|_{\mathfrak{p}}^{s_1 + s_2 - 4} d\mu \right) \\ &= \frac{1 - q^{-s_1 - s_2}}{(1 - q^{2-s_2})(1 - q^{3 - s_1 - s_2})}. \end{aligned} \quad (3.3.1)$$

In [37, Table 1], Rossmann provides the following formula for the twist representation zeta function of $\mathfrak{f}_{3,2}$—denoted by $L_{5,9}$ in [37]—, provided q is sufficiently large, by implementing his methods in Zeta [40]:

$$\widetilde{\zeta^{\mathrm{irr}}_{\mathfrak{F}_{3,2}(\mathfrak{o})}}(s) = \frac{(1 - q^{-s})^2}{(1 - q^{1-s})(1 - q^{2-s})}. \quad (3.3.2)$$

Comparing (3.3.1) and (3.3.2), we see that there is no specialisation of the form (1.1.4) for the bivariate representation zeta function of $\mathfrak{F}_{3,2}(\mathfrak{o})$ in terms of its twist representation zeta function.

For completeness, we now calculate the bivariate conjugacy class and the class number zeta functions of $\mathfrak{F}_{3,2}(\mathfrak{o})$. The A-commutator matrix of $\mathfrak{f}_{3,2}$ with respect to $\mathbf{e}$ and $\mathbf{f}$ is

$$A(X_1, X_2, X_3) = \begin{bmatrix} X_2 & X_3 & 0 \\ -X_1 & 0 & X_3 \\ 0 & -X_1 & -X_2 \end{bmatrix}.$$

Thus, $u_A = 2$, $F_0(A(\underline{X})) = \{1\}$, $F_1(A(\underline{X})) = \{-X_1, \pm X_2, X_3\}$, and

$F_2(A(\underline{X})) \supseteq \{X_1^2, -X_2^2, X_3^2\}$. Hence

$$\mathcal{Z}^{\mathrm{cc}}_{\mathfrak{F}_{2,3}(\mathfrak{o})}(s_1,s_2) = \frac{1}{1-q^{2-s_2}}\left(1+(1-q^{-1})^{-1}\int_{(w,x_1,x_2,x_3)\in\mathfrak{p}\times W_3^{\mathfrak{o}}} |w|_{\mathfrak{p}}^{2s_1+s_2-4}d\mu\right)$$
$$= \frac{1-q^{-2s_1-s_2}}{(1-q^{2-s_2})(1-q^{3-2s_1-s_2})}.$$

Specialisation (1.1.8) yields

$$\zeta^{\mathrm{k}}_{\mathfrak{F}_{2,3}(\mathfrak{o})}(s) = \frac{1-q^{-s}}{(1-q^{2-s})(1-q^{3-s})}.$$

This formula agrees with the one given in [38, Section 9.3, Table 1]. △

3.4 Local functional equations—proof of Theorem 1

In this section, we prove Theorem 1.

A formula of Denef type is a finite sum of the form

$$\sum_{i=1}^{m} |\overline{V_i}(\mathfrak{o}/\mathfrak{p})| W_i(q, q^{-s_1}, q^{-s_2}), \tag{3.4.1}$$

where $|V_i(\mathfrak{o}/\mathfrak{p})|$ denotes the number $\mathfrak{o}/\mathfrak{p}$-rational points of reductions modulo $\mathfrak{p}$ of a suitable algebraic variety V_i defined over $\mathcal{O}$ and $W_i(X,Y,Z)$ is a rational function.

Formulae of Denef type are used in [51, Section 2] to show that functions defined in terms of certain $\mathfrak{p}$-adic integrals are rational functions and satisfy functional equations. The expressions given in [51, Section 2] are generalised in [3, Section 4].

As we shall see in Section 3.4.3, the integrals describing the bivariate representation and the bivariate conjugacy class zeta functions of groups of the form $\mathbf{G}(\mathcal{O})$ appearing in Proposition 3.2.12 are special cases of the integrals studied in [51], and hence they fit the framework of [3, 51]. We apply in Section 3.4.3 the methods used in these papers to prove Theorem 1. Firstly, however, we recall from [51] the mentioned family of $\mathfrak{p}$-adic integrals in Section 3.4.1, and then recall from [3, 51] how to write them in terms of formulae of Denef type in Section 3.4.2. In Section 3.4.4, we recall briefly their methods for showing functional equations.

3.4.1 A family of $\mathfrak{p}$-adic integrals

Fix n, d, $l \in \mathbb{N}$, and let $I \subseteq [n-1]$. Define further:

1. J_κ a finite index set, for each $\kappa \in [l]$,
2. $e_{i\kappa\iota} \in \mathbb{Z}_{\geq 0}$, for each $i \in I$, $\kappa \in [l]$ and $\iota \in J_\kappa$,
3. $F_{\kappa\iota}(\underline{Y}) = F_{\kappa\iota}(Y_1, \dots, Y_d)$ a finite set of polynomials over $\mathcal{O}$, for each $\kappa \in [l]$ and $\iota \in J_k$.

Let also $\mathcal{W}(\mathfrak{o}) \subseteq \mathfrak{o}^d$ be a union of cosets modulo $\mathfrak{p}^{(d)}$. The following integral is defined and investigated in [51, Section 2]:

$$Z_{\mathcal{W}(\mathfrak{o}),I}(\underline{s}) = \int_{\mathfrak{p}^{|I|}\times\mathcal{W}(\mathfrak{o})} \prod_{\kappa=1}^{l} \left\| \bigcup_{\iota\in J_\kappa} \left(\prod_{i\in I} X_i^{e_{i\kappa\iota}} \right) F_{\kappa\iota}(\underline{Y}) \right\|_{\mathfrak{p}}^{s_\kappa} d\mu, \qquad (3.4.2)$$

where $\underline{s} = (s_1, \ldots, s_l)$ is a vector of complex variables and $\underline{X} = (X_i)_{i\in I}$ and $\underline{Y} = (Y_1, \ldots, Y_d)$ are independent integration variables. The notation $\left(\prod_{i\in I} X_i^{e_{i\kappa\iota}}\right) F_{\kappa\iota}(\underline{Y})$ means the set $\{\left(\prod_{i\in I} X_i^{e_{i\kappa\iota}}\right) f(\underline{Y}) \mid f(\underline{Y}) \in F_{\kappa\iota}(\underline{Y})\}$. The term $d\mu$ denotes the additive Haar measure on $\mathfrak{o}^{|I|+d}$, normalised so that $d\mu(\mathfrak{o}^{|I|+d}) = 1$.

The numbers d, l, n as well as the data I, J_κ, $e_{i\kappa\iota}$, and $F_{\kappa\iota}(\underline{Y})$ will be referred to as the data associated to the integral $Z_{\mathcal{W}(\mathfrak{o}),I}(\underline{s})$.

3.4.2 Formulae of Denef type

We now recall from [51, Section 2] how to write the $\mathfrak{p}$-adic integrals (3.4.2) in terms of formulae of Denef type.

We make the further assumptions on the data associated to the integral $Z_{\mathcal{W}(\mathfrak{o}),I}(\underline{s})$: firstly, $d = n^2$, so that we can identify $\mathfrak{o}^d$ with the set $\mathrm{Mat}_{n\times n}(\mathfrak{o})$ and, secondly, $\mathcal{W}(\mathfrak{o}) = \mathrm{GL}_n(\mathfrak{o})$. Thirdly, the ideals $(F_{\kappa\iota})$ are assumed to be invariant under the natural action of the standard Borel subgroup $B \subseteq \mathrm{GL}_n$ of upper triangular matrices in GL_n, acting on $K[Y_1, \ldots, Y_{n^2}]$ by right matrix multiplication.

Consider the $\mathcal{O}$-ideal

$$\mathcal{I} = \prod_{\kappa=1}^{l} \prod_{\iota\in J_\kappa} (F_{\kappa\iota}(\underline{Y})),$$

and fix a principalisation (Y, h) of $\mathcal{I}$—cf. Section 2.3—with $h : Y \to \mathrm{GL}_d/B$.

Let T be the finite set indexing the irreducible components E_u of the pre-image of h of the subvariety $\mathcal{V}$ of GL_d/B defined by $\mathcal{I}$. Set $|T| = t$. The numerical data associated to (Y, h) is $(N_{u\kappa\iota}, \nu_u)_{u\kappa\iota}$, where $N_{u\kappa\iota}$ denotes the multiplicity of the irreducible component E_u in the pre-image under h of the variety defined by the ideal $(F_{\kappa\iota})$ and $\nu_u - 1$ denotes the multiplicity of E_u in the divisor $h^*(d\mu(\mathbf{y}))$; see Section 2.3.

Definition 3.4.1. *Let* $N \in \mathbb{N}$, $U \subseteq T$, $(d_{\kappa\iota}) \in \mathbb{N}_0^{\prod_{\kappa=1}^{l} J_\kappa}$, *and write* $\mathbf{m} = (m_u)_{u\in U} \in \mathbb{N}^U$ *and* $\boldsymbol{n} = (n_i)_{i\in I} \in \mathbb{N}^I$. *Define*

$$\mathcal{L}(\mathbf{m}, \mathbf{n}) = \sum_{i\in I} n_i + \sum_{u\in U} \nu_u m_u,$$

$$\mathcal{L}_{\kappa\iota}(\mathbf{m}, \mathbf{n}) = \sum_{i\in I} e_{i\kappa\iota} n_i + \sum_{u\in U} N_{u\kappa\iota} m_u, \ \textit{for}\ \kappa \in [l], \iota \in J_\kappa,$$

and

$$\Xi^N_{U,I,(d_{\kappa\iota})}(q, \underline{s}) = \sum_{(\mathbf{m},\mathbf{n})\in\mathbb{N}^U_{\geq N}\times\mathbb{N}^I} q^{-\mathcal{L}(\mathbf{m},\mathbf{n}) - \sum_{\kappa=1}^{l} s_\kappa \min\{\mathcal{L}_{\kappa\iota}(\mathbf{m}) - d_{\kappa\iota} | \iota\in J_\kappa\}}.$$

For the special case $N = 1$ *and* $(d_{\kappa\iota}) = (0)$, *write* $\Xi_{U,I}(q, \underline{s}) := \Xi^1_{U,I,(0)}(q, \underline{s})$.

We remark that (Y, h) has good reduction modulo $\mathfrak{p}$ for all but finitely many prime ideals $\mathfrak{p}$ of $\mathcal{O}$. In case of good reduction, we consider the following numbers of $\mathfrak{o}/\mathfrak{p}$-rational points of reductions modulo $\mathfrak{p}$ of algebraic varieties over $\mathcal{O}$: for each $U \subseteq T$,

$$c_U(\mathfrak{o}/\mathfrak{p}) = |\{a \in Y(\mathfrak{o}/\mathfrak{p}) \mid (a \in E_u(\mathfrak{o}/\mathfrak{p}) \Leftrightarrow u \in U) \text{ and } \overline{h}(a) \in \overline{\mathcal{W}(\mathfrak{o})}\}|,$$

where $^-$ denotes reduction modulo $\mathfrak{p}$.

Proposition 3.4.2. *[51, Theorem 2.2] If (Y,h) has good reduction modulo $\mathfrak{p}$, then*

$$Z_{\mathcal{W}(\mathfrak{o}),I}(\underline{s}) = \frac{(1-q^{-1})^{n+|I|}}{q^{\binom{n}{2}}} \sum_{U \subseteq T} c_{U,I}(\mathfrak{o}/\mathfrak{p})(q-1)^{|U|}\Xi_{U,I}(q,\underline{s}).$$

We denote by Q_1 the finite set of prime ideals $\mathfrak{p}$ such that (Y,h) has bad reduction modulo $\mathfrak{p}$. The local factors given by the prime ideals $\mathfrak{p} \in Q_1$ are the ones excluded in the statement of Theorem 1 (together with the prime ideals $\mathfrak{p}$ satisfying $p \leq c$ when considering $\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O})}$, and the prime ideals $\mathfrak{p}$ satisfying $p \leq c$ for $c \neq 2$ when considering $\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O})}$). However, the integrals $Z_{\mathcal{W}(\mathfrak{o}),I}(\underline{s})$ with $\mathfrak{p} \in Q_1$ can also be written in terms of formulae of Denef type, as follows.

Given $N \in \mathbb{N}$ and $\mathbf{a} \in Y(\mathfrak{o}/\mathfrak{p}^N)$, set

$$Y^N_{\mathbf{a}} = \{\mathbf{y} \in Y(\mathfrak{o}) \mid \mathbf{y} \equiv \mathbf{a} \bmod \mathfrak{p}^N\}.$$

As explained in [3, Section 4.3], for each $\mathfrak{p} \in Q_1$, there exists $N \in \mathbb{N}$ such that the following holds: on the cosets $Y^N_{\mathbf{a}}$, there exist $U(\mathbf{a}) \subseteq T$, $j(\mathbf{a}) \in \mathbb{N}_0$ and $(d_{\kappa\iota}(\mathbf{a})) \in \mathbb{N}_0^{\prod_{\kappa=1}^{l} J_\kappa}$ such that

$$\|F_{\kappa\iota} \circ h\| = q^{-d_{\kappa\iota}(\mathbf{a})} \prod_{u \in U(\mathbf{a})} |\gamma_u|_{\mathfrak{p}}^{N_{u\kappa\iota}} \text{ and}$$

$$h^*(d\mu(\mathbf{y})) = q^{-j(\mathbf{a})} \prod_{u \in U(\mathbf{a})} |\gamma_u|_{\mathfrak{p}}^{\nu_u - 1} d\mu(\underline{\gamma}),$$

where γ_u are the coordinate functions of $\underline{\gamma}$ for each $u \in U(\mathbf{a})$. Denote by $(Y_{\mathfrak{k}}, h_{\mathfrak{k}})$ the principalisation over the field of fractions $\mathfrak{k} = \mathrm{Frac}(\mathfrak{o})$ of $\mathfrak{o}$ obtained from (Y,h) by base extension.

Given $U \subseteq T$, $j \in \mathbb{N}_0$ and $(d_{\kappa\iota}) \in \mathbb{N}_0^{\prod_{\kappa=1}^{l} J_\kappa}$, consider the following number of $\mathfrak{o}/\mathfrak{p}$-rational points of reductions modulo $\mathfrak{p}$ of an algebraic variety over $\mathcal{O}$:

$$c_{U,j,(d_{\kappa\iota})}(\mathfrak{o}/\mathfrak{p}^N) := |\{\mathbf{a} \in Y(\mathfrak{o}/\mathfrak{p}^N) \mid U(\mathbf{a}) = U,\ j(\mathbf{a}) = j,\ (d_{\kappa\iota}(\mathbf{a})) = (d_{\kappa\iota}) \text{ and } \overline{h_{\mathfrak{k}}}(\mathbf{a}) \in \mathrm{GL}_n(\mathfrak{o}/\mathfrak{p}^N)\}|.$$

Proposition 3.4.3. *[3, Corollary 4.2] If (Y,h) has bad reduction modulo $\mathfrak{p}$, there exist $N \in \mathbb{N}$, finite sets $J \subset \mathbb{N}_0$, and $\Delta \subset \mathbb{N}_0^{\prod_{\kappa=1}^{l} J_\kappa}$ such that*

$$Z_{\mathcal{W}(\mathfrak{o}),I}(\underline{s}) = \frac{(1-q^{-1})^{|I|}}{q^{Nn^2}} \sum_{\substack{U \subseteq T, j \in J\\ (d_{\kappa\iota}) \in \Delta}} c_{U,j,(d_{\kappa\iota})}(\mathfrak{o}/\mathfrak{p})(q^N - q^{N-1})^{|U|} q^{-j} \Xi^N_{U,I,(d_{\kappa\iota})}(q,\underline{s}).$$

3.4.3 Proof of Theorem 1

Let L be a finite extension of the field of fractions $K = \mathrm{Frac}(\mathcal{O})$ of $\mathcal{O}$. For a fixed prime ideal $\mathfrak{P}$ of $\mathcal{O}_L$ dividing $\mathfrak{p}$, write $\mathfrak{O}$ for the completion $\mathcal{O}_{L,\mathfrak{P}}$. Denote by $f = f(\mathfrak{O},\mathfrak{o})$ the relative degree of inertia, hence $|\mathfrak{O}/\mathfrak{P}| = q^f$. Set $\mathfrak{g}_L = \Lambda(\mathfrak{O})$, and let $\mathfrak{z}_L$ be its centre and $\mathfrak{g}'_L$ its derived Lie sublattice. Since $\mathcal{O}_L$ is a ring of integers of a number field L, we can choose ordered sets $\mathbf{e}$ and $\mathbf{f}$ as the ones of

Section 3.2.2 such that $\overline{\mathbf{e}}$ and $\mathbf{f}$ are bases of $\mathfrak{g}_L/\mathfrak{z}_L$ and $\mathfrak{g}'_L$, respectively. Define the commutator matrices $A(\underline{X})$ and $B(\underline{Y})$ of $\mathfrak{g}_L$ with respect to $\mathbf{e}$ and $\mathbf{f}$ as in Definition 3.2.1. Recall the $\mathfrak{p}$-adic integrals $\mathscr{Z}_{\mathfrak{O},B}(\underline{r},t)$ and $\mathscr{Z}_{\mathfrak{O},A}(\underline{r},t)$ given in (3.2.6) and (3.2.3), respectively. Consider the following functions:

$$\widetilde{\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{O})}}(s_1,s_2) = 1 + \mathscr{Z}_{\mathfrak{O},B}\left(\tfrac{-s_1-2}{2}, u_B s_1 + s_2 + 2u_B - h - 1\right), \tag{3.4.3}$$

$$\widetilde{\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathfrak{O})}}(s_1,s_2) = 1 + \mathscr{Z}_{\mathfrak{O},A}(-1-s_1, u_A s_1 + s_2 + u_A - h - 1). \tag{3.4.4}$$

We call the functions (3.4.3) and (3.4.4) the *main terms* of the bivariate representation, respectively, of the bivariate conjugacy class zeta functions of $\mathbf{G}(\mathfrak{O})$.

We have shown in Proposition 3.2.12 that

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{O})}(s_1,s_2) = \frac{1}{1-q^{f(r-s_2)}}\widetilde{\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{O})}}(s_1,s_2) \text{ and}$$

$$\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathfrak{O})}(s_1,s_2) = \frac{1}{1-q^{f(z-s_2)}}\widetilde{\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathfrak{O})}}(s_1,s_2).$$

It thus suffices to show the relevant statement of Theorem 1 for the bivariate zeta functions' main terms. In fact, for $x = f(z-s_2)$ and $x = f(r-s_2)$, the term $(1-q^x)^{-1}$ is rational and

$$\frac{1}{1-q^x}\,|_{q\to q^{-1}} = -q^x\frac{1}{1-q^x}.$$

Therefore, we only need to show that the $\mathfrak{p}$-adic integrals appearing in (3.4.3) and (3.4.4) fit the framework of [48, Section 2.3] and [3, Section 4]. In other words, we must show that their integrands are defined over $\mathcal{O}$—hence only their domains of integration vary with the ring $\mathfrak{O}$—and that these $\mathfrak{p}$-adic integrals can be expressed in terms of the integrals given in [51, Section 2.1].

The condition that the integrands of the integrals appearing in (3.4.3) and (3.4.4) are defined over $\mathcal{O}$ is needed since the $\mathfrak{O}$-bases defined in Section 3.2.2 are only defined locally, so that the matrices $A(\underline{X})$ and $B(\underline{Y})$ are also defined locally. We must assure that there exist $\mathfrak{O}$-bases $\mathbf{e}$ and $\mathbf{f}$ as the ones of Section 3.2.2 such that the commutator matrices $A(\underline{X})$ and $B(\underline{Y})$, defined with respect with these $\mathbf{e}$ and $\mathbf{f}$ are defined over $\mathcal{O}$, and hence so are the sets of polynomials $F_j(A(\underline{X}))$ and $F_{2j}(B(\underline{Y}))$.

Since the matrix $B(\underline{Y})$ is the same as the one appearing in the integrands of [48, (2.8)] and $A(\underline{X})$ is obtained in an analogous way, the argument of [48, Section 2.3] also holds in this case. Namely, we choose an $\mathcal{O}$-basis $\mathbf{f}$ for a free finite-index $\mathcal{O}$-submodule of the isolator $i(\Lambda')$ of the derived $\mathcal{O}$-Lie sublattice of Λ; see Section 3.2.2. By [48, Lemma 2.5], $\mathbf{f}$ can be extended to an $\mathcal{O}$-basis $\mathbf{e}$ for a free finite-index $\mathcal{O}$-submodule M of Λ. If the residue characteristic p of $\mathfrak{p}$ does not divide $|\Lambda : M|$ or $|i(\Lambda') : \Lambda'|$, this basis $\mathbf{e}$ may be used to obtain an $\mathfrak{O}$-basis for $\Lambda(\mathfrak{O})$, by tensoring the elements of $\mathbf{e}$ with $\mathfrak{O}$.

Remark 3.4.4. The condition "p does not divide $|i(\Lambda') : \Lambda'|$" is missing in [48], but this omission does not affect the proof of [48, Theorem A], since this condition only excludes a finite number of prime ideals $\mathfrak{p}$. This was first pointed out in [11, Section 3.3].

We now relate $\widetilde{\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{O})}}(s_1,s_2)$ and $\widetilde{\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathfrak{O})}}(s_1,s_2)$ with the general integral (3.4.2).

Set $I = \{1\}$ and write $x_1 = x$. Set also $n = b = \mathrm{rk}(\mathfrak{g}')$, hence $d = b^2$. In addition, we set $l = 2u_B + 1$, and $J_k = \{1, 2\}$ for $k \in [u_B]$, and $J_k = \{1\}$ for $u_B < k \leq 2u_B + 1$. Moreover,

k	j	F_{kj}	e_{1kj}
$\leq u_B$	1	$F_{2k}(B(\underline{y}))$	0
$u_B < k \leq 2u_B$	1	$F_{2(k-1-u_B)}(B(\underline{y}))$	0
$2u_B + 1$	1	$\{1\}$	1
$\leq u_B$	2	$F_{2(k-1)}(B(\underline{y}))$	2

Table 3.1: Data associated to the integral $Z_{\mathrm{GL}_b(\mathfrak{O}),\{1\}}$

We see that, with this set-up, the integral (3.4.2) is given by

$$Z_{\mathrm{GL}_b(\mathfrak{O}),\{1\}}(\underline{s}) = \int_{\mathfrak{P}\times \mathrm{GL}_b(\mathfrak{O})} \|x\|_{\mathfrak{P}}^{s_{2u_B+1}} \cdot \tag{3.4.5}$$

$$\prod_{k=1}^{u_B} \|F_{2k}(B(\underline{Y})) \cup x^2 F_{2(k-1)}(B(\underline{Y}))\|_{\mathfrak{P}}^{s_k} \prod_{k=u_B+1}^{2u_B} \|F_{2(k-1-u_B)}(B(\underline{Y}))\|_{\mathfrak{P}}^{s_k} d\mu.$$

Although the domain of integration of the integral (3.4.5) involves $\mathrm{GL}_b(\mathfrak{O})$, the integrand only depends on the entries of the first column, say, since the B-commutator matrix is defined in b variables. Consequently, we can interpret $W_b^{\mathfrak{O}}$ as the "space of first columns" of $\mathrm{GL}_b(\mathfrak{O})$ and we may consider the domain of integration of (3.4.5) to be $W_b^{\mathfrak{O}}$ as long as we correct the integral by multiplying it by the measure of the remaining entries of matrices of $\mathrm{GL}_b(\mathfrak{O})$. That is, $Z_{\mathrm{GL}_b(\mathfrak{O}),\{1\}}(\underline{s})$ is equal to

$$\left(\prod_{\theta=1}^{b-1}(1-q^{-f\theta})\right)^{-1} \int_{\mathfrak{P}\times W_b^{\mathfrak{O}}} \|x\|_{\mathfrak{P}}^{s_{2u_B+1}} \prod_{k=1}^{u_B} \|F_{2k}(B(\underline{Y})) \cup x^2 F_{2(k-1)}(B(\underline{Y}))\|_{\mathfrak{P}}^{s_k} \cdot$$

$$\prod_{k=u_B+1}^{2u_B} \|F_{2(k-1-u_B)}(B(\underline{Y}))\|_{\mathfrak{P}}^{s_k} d\mu.$$

Let $\mathbf{1}_{u_B} = (1, \ldots, 1) \in \mathbb{Z}^{u_B}$, $\mathbf{0}_{u_B} = (0, \ldots, 0) \in \mathbb{Z}^{u_B}$, and write

$$\mathbf{a}_1^{\mathrm{irr}} = (-\frac{1}{2}\mathbf{1}_{u_B}, \frac{1}{2}\mathbf{1}_{u_B}, u_B), \qquad \mathbf{a}_2^{\mathrm{irr}} = (\mathbf{0}_{u_B}, \mathbf{0}_{u_B}, 1),$$
$$\mathbf{b}^{\mathrm{irr}} = (-\mathbf{1}_{u_B}, \mathbf{1}_{u_B}, 2u_B - h - 1),$$

It follows that

$$\widetilde{\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{O})}}(s_1, s_2) = 1 + \frac{(1-q^{-f})^{-1}}{\prod_{\theta=1}^{b-1}(1-q^{-f\theta})} Z_{\mathrm{GL}_b(\mathfrak{O}),\{1\}}\left(\mathbf{a}_1^{\mathrm{irr}} s_1 + \mathbf{a}_2^{\mathrm{irr}} s_2 + \mathbf{b}^{\mathrm{irr}}\right).$$

Analogously, for $n = a$, $d = a^2$, and $l = 2u_A + 1$, one can find appropriate data J_k, e_{1jk}, and $F_{kj}(\underline{X})$ such that

$$\widetilde{\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathfrak{O})}}(s_1, s_2) = 1 + \frac{(1-q^{-f})^{-1}}{\prod_{\theta=1}^{a-1}(1-q^{-f\theta})} Z_{\mathrm{GL}_a(\mathfrak{O}),\{1\}}(\mathbf{a}_1^{\mathrm{cc}} s_1 + \mathbf{a}_2^{\mathrm{cc}} s_2 + \mathbf{b}^{\mathrm{cc}}),$$

for

$$\mathbf{a}_1^{\mathrm{cc}} = (-\mathbf{1}_{u_A}, \mathbf{1}_{u_A}, u_A), \qquad \mathbf{a}_2^{\mathrm{cc}} = (\mathbf{0}_{u_A}, \mathbf{0}_{u_A}, 1),$$
$$\mathbf{b}^{\mathrm{cc}} = (-\mathbf{1}_{u_A}, \mathbf{1}_{u_A}, u_A - h - 1).$$

Therefore, the bivariate zeta functions fit the framework of [51, Section 2.1] and [3, Section 4], which concludes the proof of Theorem 1; see Section 3.4.4.

3.4.4 Functional equations

We now recall the methods of [3, 51] for showing the existence of functional equations for the integrals (3.4.2). Here, we make the additional assumption that $\mathfrak{p}$ is such that the principalisation (Y,h) given in Section 3.4.2 has good reduction modulo $\mathfrak{p}$.

Consider the normalised integral

$$\widetilde{Z_{\mathcal{W}(\mathfrak{o}),I}}(\underline{s}) = \left((1-q^{-1})^{|I|}\prod_{k=1}^{n}(1-q^{-k})\right)^{-1} Z_{\mathcal{W}(\mathfrak{o}),I}(\underline{s}). \tag{3.4.6}$$

Let $b_U(\mathfrak{o}/\mathfrak{p})$ denote the number of $\mathfrak{o}/\mathfrak{p}$-rational points of the reduction modulo $\mathfrak{p}$ of the smooth projective variety $E_U = \bigcap_{u\in U} E_u$. These numbers are related to the numbers $c_U(\mathfrak{o}/\mathfrak{p})$ defined in Section 3.4.2:

$$c_V(\mathfrak{o}/\mathfrak{p}) = \sum_{V\subseteq U\subseteq T}(-1)^{|U\setminus V|} b_U(\mathfrak{o}/\mathfrak{p}).$$

According to [51, Corollary 2.1], one may write

$$\widetilde{Z_{\mathcal{W}(\mathfrak{o}),I}}(\underline{s}) = \left(\prod_{i=1}^{n}\frac{(q-1)}{q^i-1}\right)\sum_{U\subseteq T} b_U(\mathfrak{o}/\mathfrak{p})\sum_{V\subseteq U}(-1)^{|U\setminus V|}(q-1)^{|V|}\Xi_V(q,\underline{s}). \tag{3.4.7}$$

Let L be a finite extension of $K = \mathrm{Frac}(\mathcal{O})$, and let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_L$. Denote by $\mathfrak{O}$ the completion of the ring of integers $\mathcal{O}_L$ at $\mathfrak{P}$. Let also $\mathfrak{K} = \mathrm{Frac}(\mathfrak{O})$. By base extension, we obtain a principalisation $(Y_{\mathfrak{K}}, h_{\mathfrak{K}})$. All such principalisations have good reduction modulo $\mathfrak{P}$; cf. [10, Proposition 2.3 and Theorem 2.4]. If $\mathfrak{O}|\mathfrak{o}$ has degree of inertia $f = f(\mathfrak{O},\mathfrak{o})$—and hence $|\mathfrak{O}/\mathfrak{P}| = q^f$—then

$$\widetilde{Z_{\mathcal{W}(\mathfrak{O}),I}}(\underline{s}) = \left((1-q^{-f})^{|I|}\prod_{k=1}^{n}(1-q^{-kf})\right)^{-1} Z_{\mathcal{W}(\mathfrak{O}),I}(\underline{s}).$$

According to [51, Section 2.1] and [3, (4.10)], the numbers $b_U(\mathfrak{O}/\mathfrak{P})$ are given by alternating sums of powers of Frobenius eigenvalues:

$$b_U(\mathfrak{O}/\mathfrak{P}) = \sum_{i=0}^{2(\binom{n}{2}-|U|)}(-1)^i\sum_{j=1}^{t_{U,i}}\alpha^f_{U,i,j},$$

where $t_{U,i}$ are nonnegative integers and $\alpha_{U,i,j}$ are nonzero complex numbers satisfying

$$b_U^{-1}(\mathfrak{O}/\mathfrak{P}) := q^{-f(\binom{n}{2}-|U|)}b_U(\mathfrak{O}/\mathfrak{P}) = \sum_{i=0}^{2(\binom{n}{2}-|U|)}(-1)^i\sum_{j=1}^{t_{U,i}}\alpha^{-f}_{U,i,j};$$

see [51, (13)] and [3, (4.11)].

Remark 3.4.5. The numbers $\alpha_{U,i,j}$ are algebraic integers which will be denoted by $\lambda_1(\mathfrak{p}),\ldots,\lambda_t(\mathfrak{p})$. As we have seen in Section 3.4.3, the bivariate zeta functions may be written in terms of integrals $\widetilde{Z_{\mathcal{W}(\mathfrak{O}),I}}(\underline{s})$. The algebraic numbers $\lambda^*_k(\mathfrak{p})$, appearing in the statement of Theorem 1 are given by such $\lambda_k(\mathfrak{p})$.

The effect of inverting q and q^{-s_κ} in $\Xi_U(q,\underline{s})$ is clear, since these are rational functions in these parameters. It then follows that, for each $i \in [n]$

$$\left(\widetilde{Z_{\mathcal{W}(\mathfrak{o}),\varnothing}}(\underline{s}) + (1-q^{-fn})\widetilde{Z_{\mathcal{W}(\mathfrak{o}),\{i\}}}(\underline{s})\right) \Big|_{\substack{q\to q^{-1}\\ \lambda_j\to\lambda_j^{-j}}} = q^{-fn}\left(\widetilde{Z_{\mathcal{W}(\mathfrak{o}),\varnothing}}(\underline{s}) + (1-q^{-fn})\widetilde{Z_{\mathcal{W}(\mathfrak{o}),\{i\}}}(\underline{s})\right). \tag{3.4.8}$$

In Section 3.4.3, we have shown that for each $* \in \{\mathrm{irr}, \mathrm{cc}\}$ there exist vectors $\mathbf{a}_1^*$, $\mathbf{a}_2^*$, and $\mathbf{b}^*$ and suitable $n \in \mathbb{Z}$ such that

$$\begin{aligned}\widetilde{\mathcal{Z}^*_{\mathbf{G}(\mathfrak{O})}}(s_1,s_2) &= 1 + \left((1-q^{-f})\prod_{k=1}^{n-1}(1-q^{-fk})\right)^{-1} Z_{\mathrm{GL}_n(\mathfrak{O}),\{1\}}(\mathbf{a}_1^* s_1 + \mathbf{a}_2^* s_2 + \mathbf{b}^*)\\ &= 1 + (1-q^{-nf})\widetilde{Z_{\mathrm{GL}_n(\mathfrak{O}),\{1\}}}(\mathbf{a}_1^* s_1 + \mathbf{a}_2^* s_2 + \mathbf{b}^*).\end{aligned}$$

Moreover, it is not difficult to see that

$$Z_{\mathrm{GL}_n(\mathfrak{O}),\varnothing}(\mathbf{a}_1^* s_1 + \mathbf{a}_2^* s_2 + \mathbf{b}^*) = 1,$$

so that

$$\begin{aligned}\widetilde{\mathcal{Z}^*_{\mathbf{G}(\mathfrak{O})}}(s_1,s_2) = {} & \widetilde{Z_{\mathrm{GL}_n(\mathfrak{O}),\varnothing}}(\mathbf{a}_1^* s_1 + \mathbf{a}_2^* s_2 + \mathbf{b}^*)\\ &+ (1-q^{-nf})\widetilde{Z_{\mathrm{GL}_n(\mathfrak{O}),\{1\}}}(\mathbf{a}_1^* s_1 + \mathbf{a}_2^* s_2 + \mathbf{b}^*).\end{aligned}$$

Thus, the functional equations for the bivariate zeta function $\mathcal{Z}^*_{\mathbf{G}(\mathfrak{O})}(s_1,s_2)$ follows from (3.4.8).

Chapter 4

Analytic properties of $\mathcal{Z}^*_{\mathbf{G}(\mathcal{O})}$

This chapter comprises the results of [26], which concerns analytic properties of the bivariate zeta functions $\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O})}$ and $\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O})}$. The main goal here is to prove Theorem 5.

Let $\mathbf{G}$ be the unipotent group scheme obtained from the nilpotent $\mathcal{O}$-Lie lattice Λ. In Proposition 2.5.5, we have shown that the bivariate zeta functions $\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$ and $\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$ converge at least on some domains of the form

$$\{(s_1, s_2) \in \mathbb{C}^2 \mid \mathrm{Re}(s_1) > \alpha_1,\ \ \mathrm{Re}(s_2) > \alpha_2\},$$

for some real constants α_1 and α_2. Recall from Section 2.4 that a domain is a connected open subset of $\mathbb{C}^2$ with the usual topology.

In this chapter, we show that the maximal domains of convergence of these zeta functions are independent of the ring of integers $\mathcal{O}$ and that they admit meromorphic continuations to domains which are also independent of $\mathcal{O}$, all this possibly with exception of finitely many local factors. This is done using the formulae of Denef type describing these zeta functions obtained in Section 3.4.

Throughout this chapter, we adopt the notation introduced in Section 3.4.2. In particular, (Y, h) is the principalisation of the $\mathcal{O}$-ideal $\mathcal{I} = \prod_{\kappa=1}^{l} \prod_{\iota \in J_\kappa} (F_{\kappa\iota}(\underline{Y}))$ given in that section.

Recall that Q_1 is the finite set of all nonzero prime ideals $\mathfrak{p}$ such that (Y, h) has bad reduction modulo $\mathfrak{p}$. Let Q_2^{irr} be the finite set of all nonzero prime ideals $\mathfrak{p}$ of $\mathcal{O}$ with residue field of characteristic p satisfying:

1. p divides $|\Lambda : M||\iota(\Lambda') : \Lambda'|$, or
2. for $c \neq 2$, all $p \leq c$.

where $\iota(\Lambda')$ is the isolator of the derived Lie sublattice Λ' and M is the free $\mathcal{O}$-submodule of Λ of finite index described in Section 3.4.3. Moreover, we denote by Q_2^{cc} the finite set of all nonzero prime ideals $\mathfrak{p}$ of $\mathcal{O}$ with residue field of characteristic p satisfying:

1. p divides $|\Lambda : M||\iota(\Lambda') : \Lambda'|$, or
2. $p \leq c$.

For $* \in \{\mathrm{irr}, \mathrm{cc}\}$, denote by Q^* the finite set $Q_1 \cup Q_2^*$ of "bad primes".

Recall that $\widetilde{\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}}(s_1, s_2)$ and $\widetilde{\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(\mathfrak{o})}}(s_1, s_2)$ are the main terms (3.4.3) and (3.4.4) of the bivariate representation, respectively, of the bivariate conjugacy class zeta functions of $\mathbf{G}(\mathfrak{o})$. When determining the domains of convergence and meromorphic continuation of the global bivariate zeta functions, it suffices to determine the respective domains of convergence of the products of the main terms of their local factors. In fact, the products $\prod_{\mathfrak{p}}(1-q^{r-s_1})^{-1}$ and $\prod_{\mathfrak{p}}(1-q^{z-s_1})^{-1}$ converge for $\mathrm{Re}(s_2) > 1-r$ and $\mathrm{Re}(s_2) > 1-z$, respectively, and both admit meromorphic continuation to the whole $\mathbb{C}^2$; cf. Section 2.6.

In Section 4.1, we prove Theorem 5(1): for $* \in \{\mathrm{irr}, \mathrm{cc}\}$, we determine in Section 4.1.1 the domain of convergence $\mathscr{D}^*_{\mathbf{G}(\mathcal{O})}$ of the infinite product

$$\mathscr{G}^*_{\mathbf{G}(\mathcal{O})}(s_1, s_2) := \prod_{\mathfrak{p} \notin Q^*} \widetilde{\mathcal{Z}^*_{\mathbf{G}(\mathfrak{o})}}(s_1, s_2), \tag{4.0.1}$$

and show that $\mathscr{D}^*_{\mathbf{G}(\mathcal{O})}$ is independent of $\mathcal{O}$, that is $\mathscr{D}^*_{\mathbf{G}(\mathcal{O})} = \mathscr{D}^*_{\mathbf{G}}$. In Section 4.1.2, we determine the domains of convergence $\mathscr{C}^*_{\mathfrak{p}}$ of the local factors $\widetilde{\mathcal{Z}^*_{\mathbf{G}(\mathfrak{o})}}(s_1, s_2)$ for $\mathfrak{p} \in Q_1$ and show that they strictly contain the domain $\mathscr{D}^*_{\mathbf{G}}$. Denote by $\mathscr{C}^*_{Q_1}$ the intersection of all $\mathscr{C}^*_{\mathfrak{p}}$ with $\mathfrak{p} \in Q_1$. Since Q_1 is finite, this means that the domain of convergence of the infinite product

$$\mathscr{B}^*_{\mathbf{G}(\mathcal{O})}(s_1, s_2) := \prod_{\mathfrak{p} \notin Q_2^*} \widetilde{\mathcal{Z}^*_{\mathbf{G}(\mathfrak{o})}}(s_1, s_2) \tag{4.0.2}$$

is $\mathscr{D}^*_{\mathbf{G}}$, since $\mathscr{D}^*_{\mathbf{G}} \cap \mathscr{C}^*_{Q_1} = \mathscr{D}^*_{\mathbf{G}}$. The primes belonging to Q_2^* are the primes excluded in Theorem 5.

In Section 4.2 we prove Theorem 5(2): we show in Sections 4.2.2 and 4.2.1 that the product $\mathscr{G}^*_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$ admits meromorphic continuation to a domain $\mathscr{M}_{\mathscr{G}^*}$ which is independent of $\mathcal{O}$. We then show in Section 4.2.3 that $\mathscr{M}^*_{\mathbf{G}} = \mathscr{M}_{\mathscr{G}^*} \cap \mathscr{C}^*_{Q_1}$ is a domain strictly containing $\mathscr{D}^*_{\mathbf{G}}$, and then conclude that $\mathscr{B}^*_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$ admits meromorphic continuation to $\mathscr{M}^*_{\mathbf{G}}$.

4.1 Convergence

4.1.1 Good reduction

Let $* \in \{\mathrm{irr}, \mathrm{cc}\}$. The goal of this section is to determine the maximal domain of convergence $\mathscr{D}^*_{\mathbf{G}(\mathcal{O})}$ of $\mathscr{G}^*_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$, given in (4.0.1)

Recall the integrals $Z_{\mathcal{W}(\mathfrak{o}),I}(\underline{s})$ defined in (3.4.2), and that for $\mathfrak{p} \notin Q^*$ this integral may be written in terms of the functions $\Xi_{U,I}(q, \underline{s})$ for $U \subseteq T$; see Definition 3.4.1 and Proposition 3.4.2. In this chapter, we only consider the case $I = \{1\}$ and thus we write simply $Z_{\mathcal{W}(\mathfrak{o})}(\underline{s})$ for $Z_{\mathcal{W}(\mathfrak{o}),\{1\}}(\underline{s})$ and drop the subscripts I and i appearing in the data associated to $Z_{\mathcal{W}(\mathfrak{o})}(\underline{s})$. Recall that for each $* \in \{\mathrm{irr}, \mathrm{cc}\}$ there are integral vectors $\mathbf{a}_1^*$, $\mathbf{a}_2^*$, $\mathbf{b}^*$, a positive integer $n = n^*$ and suitable data associated to the integral $Z_{\mathrm{GL}_n(\mathfrak{o})}(\underline{s})$ such that

$$\begin{aligned}\widetilde{\mathcal{Z}^*_{\mathbf{G}(\mathfrak{o})}}(s_1, s_2) &= 1 + (1-q^{-1})^{-1} \widetilde{Z_{\mathrm{GL}_n(\mathfrak{o})}}(\mathbf{a}_1^* s_1 + \mathbf{a}_2^* s_2 + \mathbf{b}^*) \\ &= 1 + \frac{(1-q^{-1})^{-1}}{\prod_{\theta=1}^{n-1}(1-q^{-\theta})} Z_{\mathrm{GL}_n(\mathfrak{o})}(\mathbf{a}_1^* s_1 + \mathbf{a}_2^* s_2 + \mathbf{b}^*). \end{aligned} \tag{4.1.1}$$

The functions $\Xi_U(q, \underline{s})$ are rewritten in [11, Section 3.1] in terms of zeta functions of polyhedral cones in a fan. This allows the deduction of formulae for the integrals $Z_{\mathcal{W}(\mathfrak{o})}(\underline{s})$ from which one can read off the domain of convergence. In analogy to [11], we apply this formula to the integrals $Z_{\mathrm{GL}_n(\mathfrak{o})}(\mathbf{a}_1^* s_1 + \mathbf{a}_2^* s_2 + \mathbf{b}^*)$ with $\mathfrak{p} \notin Q^*$ to determine the domain of convergence of $\mathscr{G}^*_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$. We recall from [11, Section 3.1] the notation needed.

Let t be the cardinality of the set T defined in Section 3.4.2. Let $\{R_i\}_{i\in[w]_0}$ be a finite triangulation of $\mathbb{R}^{t+1}_{\geq 0}$ consisting of pairwise disjoint cones R_i such that each of them is a relatively open simple rational polyhedral cone with the property of eliminating the "min-terms" in the exponent of q in $\Xi_U(q, \underline{s})$. Assume that $R_0 = \{0\}$ and that $R_1, \ldots, R_z$ are the one-dimensional cones in this triangulation. For each $j \in [z]$ let $\mathbf{r}_j \in \mathbb{N}_0^{t+1}$ denote the shortest integral vector on the cone R_j. Then $R_j = \mathbb{R}_{>0}\mathbf{r}_j$.

All cones R_i are generated (as semigroups) by one-dimensional cones, so that for each $i \in [w]$ there exists a set $M_i \subseteq [z]$ such that R_i is the direct sum of monoids

$$R_i = \bigoplus_{j\in M_i} \mathbb{R}_{>0}\mathbf{r}_j.$$

Since $R_i = \mathbb{R}_{>0}\mathbf{r}_i$ exactly when $i \in [z]$, it follows that $|M_i| = 1$ if and only if $i \in [z]$. Because the R_j are simple,

$$R_i \cap \mathbb{N}_0^{t+1} = \bigoplus_{j\in M_i} \mathbb{N}\mathbf{r}_j.$$

For $U \subseteq T$, the domain of summation of $\Xi_U(q, \underline{s})$ is

$$\mathscr{C}_U = \{\underline{m} \in \mathbb{N}_0^t \times \mathbb{N} \mid m_u = 0 \text{ if and only if } u \in T\backslash U\}.$$

Denote by W'_U the (unique) subset of $[w]$ such that $\mathscr{C}_U$ is the disjoint union

$$\mathscr{C}_U = \bigcup_{i\in W'_U} R_i \cap \mathbb{N}_0^{t+1},$$

and by W' the union of all W'_U, that is, $W' \subseteq [w]$ is the set of index of cones which do not lie in the boundary component $\mathbb{R}_{\geq 0} \times \{0\}$ of $\mathbb{R}^{t+1}_{\geq 0}$.

Given $i \in W'$, denote by U_i the unique subset $U \subseteq T$ such that $i \in W'_U$, and $c_i(\mathfrak{o}/\mathfrak{p}) := c_{U_i}(\mathfrak{o}/\mathfrak{p})$.

Proposition 4.1.1. *[11, Proposition 3.2] For $\mathfrak{p} \notin Q^*$, there exist $\mathscr{A}_{j\kappa} \in \mathbb{N}$ and $\mathscr{B}_j \in \mathbb{N}_0$ for each $j \in [z]$ and $\kappa \in [l]$ such that*

$$Z_{\mathrm{GL}_n(\mathfrak{o})}(\underline{s}) = \frac{(1-q^{-1})^{n+1}}{q^{\binom{n}{2}}} \sum_{i\in W'} c_i(\mathfrak{o}/\mathfrak{p})(q-1)^{|U_i|} \prod_{j\in M_i} \frac{q^{-(\sum_{\kappa=1}^{l} \mathscr{A}_{j\kappa}s_k + \mathscr{B}_j)}}{1-q^{-(\sum_{\kappa=1}^{l} \mathscr{A}_{j\kappa}s_\kappa + \mathscr{B}_j)}}.$$

Proposition 4.1.1 applied to (4.1.1) gives the following result.

Proposition 4.1.2. *For $\mathfrak{p} \notin Q^*$, there exist A^*_{1j}, A^*_{2j}, $B^*_j \in \mathbb{Q}$, for each $j \in [z]$, such that $\widetilde{\mathcal{Z}^*_{\mathbf{G}(\mathfrak{o})}}(s_1, s_2)$ is given by*

$$1 + \frac{(1-q^{-1})^n q^{-\binom{n}{2}}}{\prod_{\theta=1}^{n-1}(1-q^{-\theta})} \sum_{i\in W'} c_i(\mathfrak{o}/\mathfrak{p})(q-1)^{|U_i|} \prod_{j\in M_i} \frac{q^{-A^*_{1j}s_1 - A^*_{2j}s_2 - B^*_j}}{1-q^{-A^*_{1j}s_1 - A^*_{2j}s_2 - B^*_j}}.$$

Remark 4.1.3. The numbers $\mathscr{A}_{j\kappa}$ of Proposition 4.1.1 are constructed so that $\sum_{j\in M_i} \mathscr{A}_{j\kappa} = 0$ if and only if the cone R_i lies in the boundary component $\mathbb{R}_{\geq 0} \times \{0\}$, that is, if and only if $i \notin W'$. Similar arguments show that A^*_{1j}, A^*_{2j} of Proposition 4.1.2 are such that $\sum_{j\in M_i} A_{1j}$ and $\sum_{j\in M_i} A_{2j}$ are zero if and only if $i \notin W'$. Moreover, all the B^*_j's of Proposition 4.1.2 are nonnegative, this follows from similar arguments as for the $\mathscr{B}_j$; see their construction in [11, Section 3.1] and [11, Remark 3.6].

The numbers $c_i(\mathfrak{o}/\mathfrak{p})$ are all divisible by

$$q^{\binom{n-1}{2}}(1-q^{-1})^{-(n-1)}\prod_{\theta=1}^{n-1}(1-q^{-\theta}),$$

because of the way of construction of the relevant integrals; see [11, Remark 3.5].

Proposition 4.1.2 shows that the poles of the main terms of the bivariate zeta functions are the ones occurring in the terms

$$(1-q^{A^*_{1j}s_1-A^*_{2j}s_2-B^*})^{-1},$$

for $j\in M_i$ and $i\in W'$ such that $(A^*_{1j},A^*_{2j})\neq(0,0)$. Since $(A^*_{1j},A^*_{2j})\neq(0,0)$ exactly when $j\in W'$, it follows that the poles of $\widetilde{\mathcal{Z}^*_{\mathbf{G}(\mathfrak{o})}}(s_1,s_2)$ are unions of sets

$$\mathrm{P}^*_j=\{(s_1,s_2)\mid A^*_{1j}s_1+A^*_{2j}s_2=B^*_j\},\ j\in[z]\cap W'.$$

Consequently, the domain of convergence of $\widetilde{\mathcal{Z}^*_{\mathbf{G}(\mathfrak{o})}}(s_1,s_2)$ is a finite intersection of sets of the following form.

Definition 4.1.4. *For each $\delta\geq 0$ and each $i\in W'\cap[z]$, set*

$$\mathscr{D}_{i,\delta}=\{(s_1,s_2)\in\mathbb{C}^2\mid \mathrm{Re}(A^*_{1i}s_1+A^*_{2i}s_2)>1-B^*_i-\delta\}.$$

Proposition 4.1.2 shows that the generating function $\widetilde{\mathcal{Z}^*_{\mathbf{G}(\mathfrak{o})}}(s_1,s_2)$ converges at least on the domain $\bigcap_{j\in[z]\cap W'}\mathscr{D}_{j,1}$.

We now want to investigate the domain of convergence of the infinite product $\mathscr{G}^*_{\mathbf{G}(\mathcal{O})}(s_1,s_2)$. For each $i\in W$, let

$$\widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2)=\frac{(1-q^{-1})^n q^{-\binom{n}{2}}}{\prod_{\theta=1}^{n-1}(1-q^{-\theta})}c_i(\mathfrak{o}/\mathfrak{p})(q-1)^{|U_i|}\prod_{j\in M_i}\frac{q^{-A^*_{1j}s_1-A^*_{2j}s_2-B^*_j}}{1-q^{-A^*_{1j}s_1-A^*_{2j}s_2-B^*_j}}. \tag{4.1.2}$$

By Proposition 4.1.2,

$$\widetilde{\mathcal{Z}^*_{\mathbf{G}(\mathfrak{o})}}(s_1,s_2)=1+\sum_{i\in W'}\widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2).$$

Thus,

$$\mathscr{G}^*_{\mathbf{G}(\mathcal{O})}(s_1,s_2)=\prod_{\mathfrak{p}\notin Q^*}\left(1+\sum_{i\in W'}\widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2)\right). \tag{4.1.3}$$

We now determine the domain of absolute convergence $\mathscr{D}_i$ of the product $\prod_{\mathfrak{p}\notin Q^*}(1+\widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2))$, that is, of the sum $\sum_{\mathfrak{p}\notin Q^*}\widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2)$; cf. Lemma 2.5.2. Since W' is a finite set, $\mathscr{G}^*_{\mathbf{G}(\mathcal{O})}(s_1,s_2)$ converges absolutely on $\bigcap_{i\in W'}\mathscr{D}_i$.

In preparation for that, we need some notation. In the set-up of Section 3.4.2, T is the finite set of irreducible components E_u of the pre-image under h of the variety defined by $\mathcal{I}$, and $E_U:=\bigcap_{u\in U}E_u$. Denote by d_U the dimension of E_U. For each $U\subseteq T$, it holds that $d_U=\binom{n}{2}-|U|$; see [42, Proposition 4.13]. The family of the irreducible components over K of E_U of maximal dimension d_U is denoted by $\{F_{U,b}\}_{b\in I_U}$, where I_U is a finite set of indices. For $b\in I_U$, denote by $l_{\mathfrak{p}}(F_{U,b})$ the number of irreducible components of $\overline{F_{U,b}}$ over $\mathfrak{o}/\mathfrak{p}^N$ which are absolutely irreducible over an algebraic closure of $\mathfrak{o}/\mathfrak{p}^N$.

We now record a useful consequence of the Lang-Weil estimate.

Lemma 4.1.5. *[42, Proposition 4.9] There exists $C > 0$ such that for all $U \subseteq T$ and $\mathfrak{p} \notin Q^*$,*

$$\left| c_U(\mathfrak{o}/\mathfrak{p}) - \sum_{b \in I_U} l_{\mathfrak{p}}(F_{U,b}) q^{d_U} \right| < C q^{d_U - \frac{1}{2}},$$

and $l_{\mathfrak{p}}(F_{U,b}) > 0$ for a set of prime ideals with positive density. This means in particular that, for any sequence $(r_{\mathfrak{p}})_{\mathfrak{p} \notin Q^*}$ of rational numbers, a sum of the form $\sum_{\mathfrak{p} \notin Q^*} c_U(\mathfrak{o}/\mathfrak{p}) r_{\mathfrak{p}}$ converges absolutely if and only if $\sum_{\mathfrak{p} \notin Q^*} q^{d_U} r_{\mathfrak{p}}$ converges absolutely.

In [42, Proposition 4.9], it is shown that, for each $b \in I_U$, the number $l_{\mathfrak{p}}(F_{U,b})$ is positive for a set of prime ideals of positive density. We remark that the finitely many prime ideals excluded are elements of Q^*; see the proof of [42, Lemma 4.7].

Proposition 4.1.6. *For each $i \in W'$, the domain of absolute convergence of the product $\prod_{\mathfrak{p} \notin Q^*}(1 + \widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1, s_2))$ is*

$$\mathscr{D}_i := \bigcap_{j \in M_i} \mathscr{D}_{j,1} \cap \left\{ (s_1, s_2) \in \mathbb{C}^2 \mid \sum_{j \in M_i} \operatorname{Re}\left(A^*_{1j} s_1 + A^*_{2j} s_2\right) > 1 - \sum_{j \in M_i} B_j \right\}.$$

Proof. If $j \in M_i \cap W'$, then each term $\frac{q^{-A^*_{1j}s_1 - A^*_{2j}s_2 - B^*_j}}{1 - q^{-A^*_{1j}s_1 - A^*_{2j}s_2 - B^*_j}}$ converges absolutely if and only if $(s_1, s_2) \in \mathscr{D}_{j,1}$, for each $j \in M_i$. If $j \in M_i \setminus W'$, the corresponding term $\frac{q^{-A^*_{1j}s_1 - A^*_{2j}s_2 - B^*_j}}{1 - q^{-A^*_{1j}s_1 - A^*_{2j}s_2 - B^*_j}}$ has no poles and converges on the whole $\mathbb{C}^2$.

For $(s_1, s_2) \in \mathscr{D}_{j,1}$, the convergent sequence $((1 - q^{-A^*_{1j}s_1 - A^*_{2j}s_2 - B^*_j})^{-1})$ is a decreasing sequence when q increases, and hence it is bounded. The sequence $\left(\prod_{\theta=1}^{n-1}(1 - q^{-\theta})^{-1}\right)$ is also bounded when q increases. Therefore, to determine where the series $\sum_{\mathfrak{p} \notin Q^*} \widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1, s_2)$ converges absolutely, it suffices to determine the domain of absolute convergence of the series

$$\sum_{\mathfrak{p} \notin Q^*} (1 - q^{-1})^n q^{-\binom{n}{2}} c_i(\mathfrak{o}/\mathfrak{p})(q-1)^{|U_i|} q^{-\sum_{j \in M_i}(A^*_{1j}s_1 + A^*_{2j}s_2 + B^*_j)}.$$

The Lang-Weil estimate of Lemma 4.1.5 guarantees that the series above converges absolutely if and only if so does the following series:

$$\sum_{\mathfrak{p} \notin Q^*} (1 - q^{-1})^n q^{-\binom{n}{2} + d_U} (q-1)^{|U_i|} q^{-\sum_{j \in M_i}(A^*_{1j}s_1 + A^*_{2j}s_2 + B^*_j)},$$

which in turn converges absolutely for $(s_1, s_2) \in \mathbb{C}^2$ satisfying

$$\operatorname{Re}\left(\sum_{j \in M_i} A^*_{1j}s_1 + A^*_{2j}s_2\right) > 1 - \sum_{j \in M_i} B_j + |U_i| - \binom{n}{2} + d_{U_i} = 1 - \sum_{j \in M_i} B^*_j,$$

because of the identity $d_{U_i} = \binom{n}{2} - |U_i|$ and Proposition 2.6.1. It follows that the domain of absolute convergence of the series $\sum_{\mathfrak{p} \notin Q^*} \widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1, s_2)$, and hence of the product $\prod_{\mathfrak{p} \notin Q^*}(1 + \widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1, s_2))$ is $\mathscr{D}_i$, as desired. □

If $i \in W' \cap [z]$, since $M_i = \{i\}$, the set $\mathscr{D}_i$ is given simply by

$$\mathscr{D}_i = \{(s_1, s_2) \in \mathbb{C}^2 \mid \operatorname{Re}(A^*_{1i}s_1 + A^*_{2i}s_2) > 1 - B^*_i\} = \mathscr{D}_{i,0}. \tag{4.1.4}$$

We now show that the domain of absolute convergence of $\mathscr{G}^*_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$ is given by an intersection of such sets.

Corollary 4.1.7. *The product $\mathscr{G}^*_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$ converges on the domain*

$$\mathscr{D}^*_{\mathbf{G}(\mathcal{O})} = \mathscr{D}^*_{\mathbf{G}} := \bigcap_{i \in [z] \cap W'} \mathscr{D}_i, \tag{4.1.5}$$

which is independent of the ring of integers $\mathcal{O}$.

Proof. It is clear that $\mathscr{D}^*_{\mathbf{G}}$ is independent of $\mathcal{O}$, since so are the sets $\mathscr{D}_i$. Proposition 4.1.6 shows that $\mathscr{G}^*_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$ converges absolutely on $\bigcap_{i \in W'} \mathscr{D}_i$. We claim that $\bigcap_{i \in W'} \mathscr{D}_i = \bigcap_{i \in [z] \cap W'} \mathscr{D}_i$.

Let $(s_1, s_2) \in \bigcap_{i \in W' \cap [z]} \mathscr{D}_i$. Given $k \in W'$

$$\begin{aligned}\sum_{j \in M_k} \mathrm{Re}(A^*_{1j} s_1 + A^*_{2j} s_2) &= \sum_{j \in M_k \cap W'} \mathrm{Re}(A^*_{1j} s_1 + A^*_{2j} s_2) \\ &> \sum_{j \in M_k \cap W'} (1 - B^*_j) \geq 1 - \sum_{j \in M_k} B^*_j.\end{aligned}$$

The equality is justified by the fact that $(A^*_{1j}, A^*_{2j}) = (0,0)$ if and only if $j \notin W'$, and the second inequality follows from the fact that $B^*_j \geq 0$ for all $j \in W'$. We have shown that $(s_1, s_2) \in \mathscr{D}_k$ for each $k \in W'$. Therefore, $\bigcap_{i \in [z] \cap W'} \mathscr{D}_i \subseteq \bigcap_{k \in W'} \mathscr{D}_k$. □

4.1.2 Bad reduction

For each $\mathfrak{p} \in Q_1$, denote by $\mathscr{C}^*_{\mathfrak{p}}$ the domain of convergence of the local factor $\mathcal{Z}^*_{\mathbf{G}(\mathfrak{o})}(s_1, s_2)$. We now show that $\mathscr{C}^*_{\mathfrak{p}} \supsetneq \mathscr{D}^*_{\mathbf{G}}$. A consequence is that

$$\mathscr{B}^*_{\mathbf{G}(\mathcal{O})}(s_1, s_2) = \prod_{\mathfrak{p} \notin Q^*_2} \widetilde{\mathcal{Z}^*_{\mathbf{G}(\mathfrak{o})}}(s_1, s_2)$$

converges absolutely on $\mathscr{D}^*_{\mathbf{G}}$ because Q_1 is a finite set.

Recall that the main terms of the bivariate zeta functions are given in (4.1.1) in terms of the $\mathfrak{p}$-adic integrals $Z_{\mathrm{GL}_n(\mathfrak{o})}(\mathbf{a}^*_1 s_1 + \mathbf{a}^*_2 s_2 + \mathbf{b}^*)$ of (3.4.2), where $\mathbf{a}^*_1$, $\mathbf{a}^*_2$ and $\mathbf{b}^*$ are the integral vectors defined in Section 3.4.3. The poles of $Z(\mathbf{a}^*_1 s_1 + \mathbf{a}^*_2 s_2 + \mathbf{b}^*)$, in turn, are the poles of functions $\Xi^N_{U,(d_{\kappa\iota})}(q, \mathbf{a}_1 s_1 + \mathbf{a}_2 s_2 + \mathbf{b})$ of Definition 3.4.1, by Proposition 3.4.3.

The next proposition is analogous to [3, Proposition 4.5] and is proven in the same way.

Proposition 4.1.8. *Given q, $N \in \mathbb{N}$, a family of integers $(d_{\kappa\iota})$ for $\kappa \in [l]$ and $\iota \in J_\kappa$, and $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{b} \in \mathbb{Z}^l$, the set of poles of $\Xi^N_{U,(d_{\kappa\iota})}(q, \mathbf{a}_1 s_1 + \mathbf{a}_2 s_2 + \mathbf{b})$ is independent of q, N and $(d_{\kappa\iota})$, for all $U \subseteq T$.*

Since the function $\Xi^N_{U,(d_{\kappa\iota})}(q, \mathbf{a}_1 s_1 + \mathbf{a}_2 s_2 + \mathbf{b})$ may be rewritten as

$$q^{\sum_{u \in U}(N-1)\nu_u} \Xi_{U,(d^1_{\kappa\iota} + \sum_{u \in U} N_{u\kappa\iota}(N-1))}(q, \mathbf{a}_1 s_1 + \mathbf{a}_2 s_2 + \mathbf{b}),$$

it follows from Proposition 4.1.8 that the functions $\Xi^N_{U,(d_{\kappa\iota})}(q, \mathbf{a}_1 s_1 + \mathbf{a}_2 s_2 + \mathbf{b})$ and $\Xi^1_{U,(0)}(q, \mathbf{a}_1 s_1 + \mathbf{a}_2 s_2 + \mathbf{b}) = \Xi_U(q, \mathbf{a}_1 s_1 + \mathbf{a}_2 s_2 + \mathbf{b})$ have the same poles.

In particular, the function $Z_{\mathrm{GL}_n(\mathfrak{o})}(\mathbf{a}^*_1 s_1 + \mathbf{a}^*_2 s_2 + \mathbf{b}^*)$ converges absolutely on the domain $\bigcap_{i \in [z] \cap W'} \mathscr{D}_{j,1} \supsetneq \mathscr{D}^*_{\mathbf{G}}$ as in the good reduction case. This concludes the proof of Theorem 5(1).

4.2 Meromorphic continuation

We start by showing that the bivariate function $\mathscr{G}^*_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$ admits meromorphic continuation to a domain $\mathscr{M}_{\mathscr{G}^*} \supsetneq \mathscr{D}^*_{\mathbf{G}}$; recall the concept of meromorphy in two complex variables in Definition 2.4.3.

For each $i \in W'$, set $\mathrm{R}_i = \{j \in W' \mid \mathscr{D}_j = \mathscr{D}_i\}$, where $\mathscr{D}_i$ is the set defined in (4.1.4). Set also

$$\mathscr{R} = \left\{ i \in W' \cap [z] \mid \bigcap_{j \in W' \setminus \mathrm{R}_i} \mathscr{D}_j \neq \mathscr{D}^*_{\mathbf{G}} \right\}.$$

In other words, $\mathscr{R}$ is the set of indices i such that the boundary $\partial\mathscr{D}_i$ of $\mathscr{D}_i$ shares infinitely many points with the boundary $\partial\mathscr{D}^*_{\mathbf{G}}$ of $\mathscr{D}^*_{\mathbf{G}}$.

For each $* \in \{\mathrm{irr}, \mathrm{cc}\}$ and $\mathfrak{p} \notin Q^*$, define

$$V^*_{\mathfrak{p}}(s_1, s_2) = \prod_{i \in \mathscr{R}} (1 - c_i(\mathfrak{o}/\mathfrak{p}) q^{-d_{U_i}} q^{-A^*_{1i}s_1 - A^*_{2i}s_2 - B^*_i}).$$

Observe that $\prod_{\mathfrak{p} \notin Q^*} V_{\mathfrak{p}}$ is convergent on $\mathscr{D}^*_{\mathbf{G}}$, since for each $i \in \mathscr{R}$ the sum $\sum_{\mathfrak{p} \notin Q^*} c_i(\mathfrak{o}/\mathfrak{p}) q^{-d_{U_i}} q^{-A^*_{1i}s_1 - A^*_{2i}s_2 - B^*_i}$ converges on $\mathscr{D}_i$. Recall from (4.1.3) that $\mathscr{G}^*_{\mathbf{G}(\mathcal{O})}(s_1, s_2) = \prod_{\mathfrak{p} \notin Q^*}(1 + \sum_{i \in W'} \widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1, s_2))$. Then, for $(s_1, s_2) \in \mathscr{D}^*_{\mathbf{G}}$,

$$\mathscr{G}^*_{\mathbf{G}(\mathcal{O})}(s_1, s_2) = \frac{\prod_{\mathfrak{p} \notin Q^*}(1 + \sum_{i \in W'} \widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1, s_2)) V^*_{\mathfrak{p}}(s_1, s_2)}{\prod_{\mathfrak{p} \notin Q^*} V^*_{\mathfrak{p}}(s_1, s_2)},$$

provided that the numerator on the right-hand side converges. In the following, we show that

(i) The product $\prod_{\mathfrak{p} \notin Q^*}(1 + \sum_{i \in W'} \widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1, s_2)) V^*_{\mathfrak{p}}(s_1, s_2)$ is meromorphic on a domain $\mathscr{M}^1_{\mathscr{G}^*} \supsetneq \mathscr{D}^*_{\mathbf{G}}$ which is independent of $\mathcal{O}$, and

(ii) The product $\prod_{\mathfrak{p} \notin Q^*} V^*_{\mathfrak{p}}(s_1, s_2)$ is meromorphic on a domain $\mathscr{M}^2_{\mathscr{G}^*} \supsetneq \mathscr{D}^*_{\mathbf{G}}$, which is independent of $\mathcal{O}$.

4.2.1 Proof of (i)

We now introduce some convenient notation. The following is a modification of the relations $\equiv$ of [42, Section 4] and $\equiv_\Delta$ of [11, Definition 4.4].

Definition 4.2.1. *Given families $(f_{\mathfrak{p}}(s_1, s_2))_{\mathfrak{p} \notin Q^*}$ and $(g_{\mathfrak{p}}(s_1, s_2))_{\mathfrak{p} \notin Q^*}$ of bivariate complex functions and a domain $\mathcal{D}$, we write*

$$\prod_{\mathfrak{p} \notin Q^*} f_{\mathfrak{p}} \equiv_{\mathcal{D}} \prod_{\mathfrak{p} \notin Q^*} g_{\mathfrak{p}}$$

to indicate that $\sum_{\mathfrak{p} \notin Q^*}(f_{\mathfrak{p}}(s_1, s_2) - g_{\mathfrak{p}}(s_1, s_2))$ is absolutely convergent on $\mathcal{D}$.

The following Lemmata establish convenient properties of $\equiv_{\mathcal{D}}$.

Lemma 4.2.2. *Let $(f_{\mathfrak{p}}(s_1, s_2))$, $(g_{\mathfrak{p}}(s_1, s_2))$, and $(h_{\mathfrak{p}}(s_1, s_2))$ be families of bivariate complex functions indexed by $\mathfrak{p} \notin Q^*$, and let $\mathcal{D}$ and $\mathcal{D}'$ be domains of $\mathbb{C}^2$. If $\prod_{\mathfrak{p} \notin Q^*} f_{\mathfrak{p}} \equiv_{\mathcal{D}} \prod_{\mathfrak{p} \notin Q^*} g_{\mathfrak{p}}$ and $\prod_{\mathfrak{p} \notin Q^*} g_{\mathfrak{p}} \equiv_{\mathcal{D}'} \prod_{\mathfrak{p} \notin Q^*} h_{\mathfrak{p}}$, then*

$$\prod_{\mathfrak{p} \notin Q^*} f_{\mathfrak{p}} \equiv_{\mathcal{D} \cap \mathcal{D}'} \prod_{\mathfrak{p} \notin Q^*} h_{\mathfrak{p}}.$$

In particular, if $\prod_{\mathfrak{p}\notin Q^*} g_\mathfrak{p}(s_1,s_2)$ *converges absolutely on the domain* $\mathcal{D}'$ *and* $\prod_{\mathfrak{p}\notin Q^*} f_\mathfrak{p} \equiv_\mathcal{D} \prod_{\mathfrak{p}\notin Q^*} g_\mathfrak{p}$*, then* $\prod_{\mathfrak{p}\notin Q^*} f_\mathfrak{p}(s_1,s_2)$ *converges absolutely on the domain* $\mathcal{D}\cap\mathcal{D}'$.

Proof. The first claim follows from the fact that

$$\sum_{\mathfrak{p}\notin Q^*} |f_\mathfrak{p}(s_1,s_2)-h_\mathfrak{p}(s_1,s_2)| \le \sum_{\mathfrak{p}\notin Q^*} (|f_\mathfrak{p}(s_1,s_2)-g_\mathfrak{p}(s_1,s_2)|+|g_\mathfrak{p}(s_1,s_2)-h_\mathfrak{p}(s_1,s_2)|).$$

By definition, $\prod_{\mathfrak{p}\notin Q^*} g_\mathfrak{p}(s_1,s_2)$ being absolutely convergent on $\mathcal{D}'$ is equivalent to $\prod_{\mathfrak{p}\notin Q^*} g_\mathfrak{p} \equiv_{\mathcal{D}'} 1$. The second claim then follows from the first part of Lemma 4.2.2. □

Lemma 4.2.3. *Let* $(f_\mathfrak{p}(s_1,s_2))$*,* $(g_\mathfrak{p}(s_1,s_2))$*, and* $(X_\mathfrak{p}(s_1,s_2))$ *be families of bivariate complex functions indexed by* $\mathfrak{p}\notin Q^*$*, and let* $\mathcal{D}$ *and* $\mathcal{D}'$ *be domains of* $\mathbb{C}^2$*. If* $\prod_{\mathfrak{p}\notin Q^*} f_\mathfrak{p} \equiv_\mathcal{D} \prod_{\mathfrak{p}\notin Q^*} g_\mathfrak{p}$ *and* $(X_\mathfrak{p}(s_1,s_2))$ *is bounded on* $\mathcal{D}'$*, then*

$$\prod_{\mathfrak{p}\notin Q^*} f_\mathfrak{p} X_\mathfrak{p} \equiv_{\mathcal{D}\cap\mathcal{D}'} \prod_{\mathfrak{p}\notin Q^*} g_\mathfrak{p} X_\mathfrak{p}.$$

Proof. This is clear, as the partial sums $\sum |f_\mathfrak{p}(s_1,s_2)-g_\mathfrak{p}(s_1,s_2)||X_\mathfrak{p}(s_1,s_2)|$ are bounded on $\mathcal{D}\cap\mathcal{D}'$. □

In the following, we write $\mathscr{D}_{\mathscr{R},\delta} = \bigcap_{i\in\mathscr{R}} \mathscr{D}_{i,\delta}$ for each $\delta>0$.

Proposition 4.2.4. *There exists a domain* $\mathcal{D}_1$ *which is independent of* $\mathcal{O}$ *satisfying the following condition: for each* $\delta>0$ *the intersection* $\mathcal{D}_1\cap\mathscr{D}_{\mathscr{R},\delta}$ *is a domain strictly containing* $\mathscr{D}^*_\mathbf{G}$ *and such that*

$$\prod_{\mathfrak{p}\notin Q^*}\left(1+\sum_{i\in W'}\widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}\right)V^*_\mathfrak{p} \equiv_{\mathcal{D}_1} \prod_{\mathfrak{p}\notin Q^*}\left(1+\sum_{i\in\mathscr{R}}\widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}\right)V^*_\mathfrak{p}. \tag{4.2.1}$$

Proof. The domain $\mathcal{D}'_1 := \bigcap_{i\in W'\setminus\mathscr{R}} \mathscr{D}_i$ strictly contains $\mathscr{D}^*_\mathbf{G}$, by choice of $\mathscr{R}$, and is independent of $\mathcal{O}$, since so are the domains $\mathscr{D}_i$, for all $i\in W'$.

The domain $\mathcal{D}'_1$ has the property that, for each $\delta>0$, the intersection $\mathcal{D}'_1\cap\mathscr{D}_{\mathscr{R},\delta}$ is a domain strictly containing $\mathscr{D}^*_\mathbf{G}$. In fact, if $\mathcal{D}'_1\cap\mathscr{D}_{\mathscr{R},\delta}=\mathscr{D}^*_\mathbf{G}$, then since $\mathscr{D}_{\mathscr{R},\delta}$ is a translation of $\mathscr{D}^*_\mathbf{G}$ which strictly contains it, we must have $\mathcal{D}'_1=\mathscr{D}^*_\mathbf{G}$.

The definition of $\equiv_{\mathcal{D}_1}$ yields

$$\prod_{\mathfrak{p}\notin Q^*}\left(1+\sum_{i\in W'}\widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}\right) \equiv_{\mathcal{D}_1} \prod_{\mathfrak{p}\notin Q^*}\left(1+\sum_{i\in\mathscr{R}}\widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}\right).$$

Since the sequence $(V^*_\mathfrak{p}(s_1,s_2))_{\mathfrak{p}\notin Q^*}$ is positive monotonically non-increasing on $\mathcal{D}_{V_\mathfrak{p}} := \bigcap_{i\in\mathscr{R}}\mathscr{D}_{i,(d_{U_i}+1)}$, Lemma 4.2.3 assures that (4.2.1) holds for $\mathcal{D}_1 := \mathcal{D}'_1\cap\mathscr{D}_{V_\mathfrak{p}}$.

Clearly, given γ, $\gamma'>0$, the intersection $\mathscr{D}_{\mathscr{R},\gamma}\cap\mathscr{D}_{\mathscr{R},\gamma'}$ is $\mathscr{D}_{i,\min\{\gamma,\gamma'\}}$. Thus, for each $\delta>0$, and for $\gamma := \min\{d_{U_i} \mid i\in\mathscr{R}\}$,

$$\mathcal{D}_1\cap\mathscr{D}_{\mathscr{R},\delta} \supseteq \mathcal{D}'_1\cap\mathscr{D}_{\mathscr{R},\min\{\delta,\gamma+1\}} \supsetneq \mathscr{D}^*_\mathbf{G}.$$ □

We now use Lemma 4.2.2 to show that there exists $\delta>0$ such that

$$\prod_{\mathfrak{p}\notin Q^*}\left(1+\sum_{i\in\mathscr{R}}\widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}\right)V^*_\mathfrak{p} \equiv_{\mathscr{D}_{\mathscr{R},\delta}} 1. \tag{4.2.2}$$

Then (4.2.2) and Proposition 4.2.4 together imply

$$\prod_{\mathfrak{p}\notin Q^*}\left(1+\sum_{i\in W'}\widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}\right)V^*_{\mathfrak{p}}\equiv_{\mathscr{M}^1_{\mathscr{G}^*}}1,$$

where $\mathscr{M}^1_{\mathscr{G}^*}:=\mathcal{D}_1\cap\mathscr{D}_{\mathscr{R},\delta}$, which is independent of the ring of integers $\mathcal{O}$. In preparation for this, we need three lemmata. In following, consider the auxiliary functions

$$\overline{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2)=c_i(\mathfrak{o}/\mathfrak{p})q^{-d_{U_i}}q^{-A^*_{1i}s_1-A^*_{2i}s_2-B^*_i},\ i\in\mathscr{R}.$$

Lemma 4.2.5. *There exist $\delta_2>0$ and a domain $\mathcal{D}_2\supseteq\mathscr{D}_{\mathscr{R},\delta_2}$ such that*

$$\prod_{\mathfrak{p}\notin Q^*}V^*_{\mathfrak{p}}\equiv_{\mathcal{D}_2}\prod_{\mathfrak{p}\notin Q^*}\left(1-\sum_{i\in\mathscr{R}}\overline{\mathcal{Z}^*_{i,\mathfrak{p}}}\right).$$

Proof. We first notice that

$$\begin{aligned}&\sum_{\mathfrak{p}\notin Q^*}|V^*_{\mathfrak{p}}(s_1,s_2)-(1-\sum_{i\in\mathscr{R}}\overline{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2))|\\ &=\sum_{\mathfrak{p}\notin Q^*}\left|\prod_{i\in\mathscr{R}}\left(1-\overline{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2)\right)-\left(1-\sum_{i\in\mathscr{R}}\overline{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2)\right)\right|\\ &=\sum_{\mathfrak{p}\notin Q^*}\left|\sum_{l=2}^{|\mathscr{R}|}\sum_{\substack{I\subseteq\mathscr{R}\\|I|=l}}(-1)^l\prod_{i\in I}\overline{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2)\right|.\end{aligned}\tag{4.2.3}$$

By applying successively the Lang-Weil estimate of Lemma 4.1.5 to (4.2.3), we obtain that $\sum_{\mathfrak{p}\notin Q^*}|V^*_{\mathfrak{p}}(s_1,s_2)-(1-\sum_{i\in\mathscr{R}}\overline{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2))|$ converges if and only if the series

$$\sum_{\mathfrak{p}\notin Q^*}\left|\sum_{l=2}^{|\mathscr{R}|}\sum_{\substack{I\subseteq\mathscr{R}\\|I|=l}}(-1)^l q^{-\sum_{i\in I}(A^*_{1i}s_1+A^*_{2i}s_2+B^*_i)}\right|$$

converges, which in turn converges on the domain $\mathcal{D}_2$ defined by

$$\left\{(s_1,s_2)\in\mathbb{C}^2\mid\sum_{i\in I}\mathrm{Re}(A^*_{1i}s_1+A^*_{2i}s_2)>1-\sum_{i\in I}B_i,\ I\subseteq\mathscr{R}\text{ with }|I|\geq 2\right\}.$$

Finally, if $(s_1,s_2)\in\mathscr{D}_{\mathscr{R},\frac12}$, then for each $I\subseteq\mathscr{R}$ with $|I|\geq 2$,

$$\sum_{i\in I}\mathrm{Re}(A^*_{1i}s_1+A^*_{2i}s_2)>\sum_{i\in I}\left(\frac12-B^*_i\right)\geq 1-\sum_{i\in I}B^*_i,$$

that is, $\mathscr{D}_{\mathscr{R},\frac12}\subseteq\mathcal{D}_2$. □

Lemma 4.2.6. *There exist $\delta_3>0$ and a domain $\mathcal{D}_3\supseteq\mathscr{D}_{\mathscr{R},\delta_3}$ such that*

$$\prod_{\mathfrak{p}\notin Q^*}\left(1+\sum_{i\in\mathscr{R}}\widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}\right)\equiv_{\mathcal{D}_3}\prod_{\mathfrak{p}\notin Q^*}\left(1+\sum_{i\in\mathscr{R}}\overline{\mathcal{Z}^*_{i,\mathfrak{p}}}\right).$$

Proof. For each $\mathfrak{p}\notin Q^*$ and $i\in\mathscr{R}$, set

$$\mathcal{S}_{\mathfrak{p},i}(s_1,s_2)=(1-q^{-1})^n q^{-\binom{n}{2}}(q-1)^{|U_i|}q^{d_{U_i}}-(1-q^{-A^*_{1i}s_1-A^*_{2i}s_2-B^*_i})\prod_{\theta=1}^{n-1}(1-q^{-\theta}).$$

For each $i \in \mathscr{R}$ the sequences

$$\left(\prod_{\theta=1}^{n-1}(1-q^{-\theta})^{-1}\right) \quad \text{and} \quad ((1-q^{-A^*_{1i}s_1-A^*_{2i}s_2-B^*_i})^{-1})$$

are positive and monotonically non-increasing for $\mathrm{Re}(A^*_{1i}s_1 + A^*_{2i}s_2) > -B^*_i$ when q increases. Thus, if the series

$$\sum_{\mathfrak{p}\notin Q^*}\sum_{i\in\mathscr{R}}\left|\mathcal{S}_{\mathfrak{p},i}(s_1,s_2)c_i(\mathfrak{o}/\mathfrak{p})q^{-d_{U_i}}q^{-A^*_{1i}s_1-A^*_{2i}s_2-B^*_i}\right|$$

converges absolutely on $\mathcal{D}_3$, then the series

$$\sum_{\mathfrak{p}\notin Q^*}\sum_{i\in\mathscr{R}}\left|\widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2)-\overline{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2)\right|$$
$$=\sum_{\mathfrak{p}\notin Q^*}\sum_{i\in\mathscr{R}}\frac{|\mathcal{S}_{\mathfrak{p},i}(s_1,s_2)c_i(\mathfrak{o}/\mathfrak{p})q^{-d_{U_i}}q^{-A^*_{1i}s_1-A^*_{2i}s_2-B^*_i}|}{|(\prod_{\theta=1}^{n-1}(1-q^{-\theta}))(1-q^{-A^*_{1i}s_1-A^*_{2i}s_2-B^*_i})|}$$

also converges absolutely on $\mathcal{D}_3 \cap \mathscr{D}_{\mathscr{R},1}$.

The claim of Lemma 4.2.6 then follows from the fact that the series

$$\sum_{\mathfrak{p}\notin Q^*}\sum_{i\in\mathscr{R}}\left(c_i(\mathfrak{o}/\mathfrak{p})q^{-2A^*_{1i}s_1-2A^*_{2i}s_2-2B^*_i}\prod_{\theta=1}^{n-1}(1-q^{-\theta})\right)$$

converges absolutely on $\mathscr{D}_{\mathscr{R},\frac{1}{2}}$, because of the Lang-Weil estimate of Lemma 4.1.5 and Proposition 2.6.1. □

Lemma 4.2.7. *The product*

$$\prod_{\mathfrak{p}\notin Q^*}\left(\left(1+\sum_{i\in\mathscr{R}}\overline{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2)\right)\left(1-\sum_{i\in\mathscr{R}}\overline{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2)\right)\right)$$

converges absolutely on the domain $\mathscr{D}_{\mathscr{R},\frac{1}{2}}$.

Proof. Let us show that

$$\prod_{\mathfrak{p}\notin Q^*}\left(1+\sum_{i\in\mathscr{R}}\overline{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2)\right)\left(1-\sum_{i\in\mathscr{R}}\overline{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2)\right)\equiv_{\mathscr{D}_{\mathscr{R},\frac{1}{2}}}1.$$

In fact,

$$\sum_{\mathfrak{p}\notin Q^*}\left|1-(1+\sum_{i\in\mathscr{R}}\overline{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2))(1-\sum_{i\in\mathscr{R}}\overline{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2))\right|$$
$$=\sum_{\mathfrak{p}\notin Q^*}\sum_{i\in\mathscr{R}}\sum_{j\in\mathscr{R}}\left|\overline{\mathcal{Z}^*_{i,\mathfrak{p}}}(s_1,s_2)\overline{\mathcal{Z}^*_{j,\mathfrak{p}}}(s_1,s_2)\right|$$
$$=\sum_{\mathfrak{p}\notin Q^*}\sum_{i\in\mathscr{R}}\sum_{j\in\mathscr{R}}\left|c_i(\mathfrak{o}/\mathfrak{p})c_j(\mathfrak{o}/\mathfrak{p})q^{-d_{U_i}-d_{U_j}}q^{-(A^*_{1i}+A^*_{1j})s_1-(A^*_{2i}+A^*_{2j})s_2-(B^*_i+B^*_j)}\right|,$$

which, by Lemma 4.1.5, converges if and only if the following series converges:

$$\sum_{\mathfrak{p}\notin Q^*}\sum_{i\in\mathscr{R}}\sum_{j\in\mathscr{R}}\left|q^{-(A^*_{1i}+A^*_{1j})s_1-(A^*_{2i}+A^*_{2j})s_2-(B^*_i+B^*_j)}\right|.$$

Proposition 2.6.1 assures that the latter series converges on
$\mathcal{D}_4 := \{(s_1,s_2)\in\mathbb{C}^2 \mid \mathrm{Re}((A^*_{1i}+A^*_{1j})s_1+(A^*_{2i}+A^*_{2j})s_2) > 1-B^*_i-B^*_j,\ i,j\in\mathscr{R}\}$.

In particular, if we choose $i = j$ in $\mathscr{R}$, we see that for each $(s_1, s_2) \in \mathcal{D}_4$,

$$\mathrm{Re}(A^*_{1i}s_1 + A^*_{2i}s_2) > \frac{1 - 2B^*_i}{2} = 1 - B^*_i - \frac{1}{2}.$$

In other words, $\mathcal{D}_4 \subseteq \mathscr{D}_{\mathscr{R},\frac{1}{2}}$. The equality $\mathcal{D}_4 = \mathscr{D}_{\mathscr{R},\frac{1}{2}}$ holds, since $(s_1, s_2) \in \mathscr{D}_{\mathscr{R},\frac{1}{2}}$ implies

$$\mathrm{Re}((A^*_{1i} + A^*_{1j})s_1 + (A^*_{2i} + A^*_{2j})s_2) > \frac{1 - 2B^*_i}{2} + \frac{1 - 2B^*_j}{2} = 1 - B_i - B_j. \quad \square$$

There is $\delta > 0$ such that the domains $\mathcal{D}_2$ and $\mathcal{D}_3$ of Lemmata 4.2.5 and 4.2.6 satisfy

$$\mathcal{D}_2 \cap \mathcal{D}_3 \cap \mathscr{D}_{\mathscr{R},\frac{1}{2}} \supseteq \mathscr{D}_{\mathscr{R},\delta_2} \cap \mathscr{D}_{\mathscr{R},\delta_3} \cap \mathscr{D}_{\mathscr{R},\frac{1}{2}} = \mathscr{D}_{\mathscr{R},\delta}.$$

It then follows from Lemmata 4.2.2, 4.2.5, 4.2.6, and 4.2.7 that

$$\begin{aligned}\prod_{\mathfrak{p}\notin Q^*}(1 + \sum_{i\in\mathscr{R}} \widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}})V^*_{\mathfrak{p}} &\equiv_{\mathscr{D}_{\mathscr{R},\delta}} \prod_{\mathfrak{p}\notin Q^*}(1 + \sum_{i\in\mathscr{R}} \widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}})(1 - \sum_{i\in\mathscr{R}} \overline{\mathcal{Z}^*_{i,\mathfrak{p}}}) \\ &\equiv_{\mathscr{D}_{\mathscr{R},\delta}} \prod_{\mathfrak{p}\notin Q^*}(1 + \sum_{i\in\mathscr{R}} \overline{\mathcal{Z}^*_{i,\mathfrak{p}}})(1 - \sum_{i\in\mathscr{R}} \overline{\mathcal{Z}^*_{i,\mathfrak{p}}}) \equiv_{\mathscr{D}_{\mathscr{R},\delta}} 1,\end{aligned}$$

which confirms (4.2.2).

4.2.2 Proof of (ii)

For $i \in \mathscr{R}$, we define the following functions, which are analogous to the $V_i(s)$ of [11, Section 4.2].

$$V^*_i(s_1, s_2) := \prod_{\mathfrak{p}\notin Q^*}(1 - c_i(\mathfrak{o}/\mathfrak{p})q^{-d_{U_i}}q^{-A^*_{1i}s_1 - A^*_{2i}s_2 - B^*_i}).$$

It suffices to show that each $V^*_i(s_1, s_2)$ admits meromorphic continuation to $\mathscr{D}_{i,\Delta}$, for some $\Delta > 0$. Then, since $\mathscr{R}$ is finite, it will follow that

$$\prod_{i\in\mathscr{R}} V^*_i(s_1, s_2) = \prod_{\mathfrak{p}\notin Q^*} V^*_{\mathfrak{p}}(s_1, s_2)$$

admits meromorphic continuation to $\mathscr{M}^2_{\mathscr{G}^*} := \bigcap_{i\in\mathscr{R}} \mathscr{D}_{i,\Delta}$.

The following proposition is analogous to [42, Lemma 4.6].

Proposition 4.2.8. *For each $i \in W$ and $b \in I_{U_i}$, the function*

$$V_{b,i}(s_1, s_2) = \prod_{\mathfrak{p}\notin Q^*}(1 - l_{\mathfrak{p}}(F_{U_i,b})q^{-A^*_{1i}s_1 - A^*_{2i}s_2 - B^*_i})$$

converges absolutely on $\mathscr{D}_i$. Moreover, there exists $\delta_i > 0$ such that $V_{b,i}(s_1, s_2)$ admits meromorphic continuation to $\mathscr{D}_{i,\delta_i}$.

Proof. For each $i \in \mathscr{R}$ and $b \in I_{U_i}$, the convergence of $V_{b,i}(s_1, s_2)$ follows from the fact pointed out in the proof of [42, Lemma 4.6] that $l_{\mathfrak{p}}(F_{U_i,b})$ is bounded by the number of absolutely irreducible components of $F_{U_i,b}$. Then, for a sufficiently large $C > 0$, the sum $\sum_{\mathfrak{p}\notin Q^*} l_{\mathfrak{p}}(F_{U,b})q^{-A^*_{1i}s_1 - A^*_{2i}s_2 - B^*_i}$ is majored by $C\sum_{\mathfrak{p}\notin Q^*} q^{-A^*_{1i}s_1 - A^*_{2i}s_2 - B^*_i}$, which converges for $\mathrm{Re}(A^*_{1i}s_1 + A^*_{2i}s_2) > 1 - B^*_i$.

Let $L|K$ be a finite Galois extension and S the finite set of prime ideals $\mathfrak{p}$ of $\mathcal{O}$ which are unramified and of the prime ideals $\mathfrak{p}$ such that the reduction of $F_{U_i,b}$ mod $\mathfrak{p}$ is smooth. Denote by $\mathrm{Frob}_{\mathfrak{p}}$ ($\mathfrak{p}$ unramified) the conjugacy class in the Galois group of $L|K$ consisting of Frobenius elements. Given $a_1, a_2, b \in \mathbb{R}$

with $(a_1, a_2) \neq (0,0)$ and a representation ρ of the Galois Group of $L|K$, one can show that the Artin L-function

$$L_{F_{U,b}}(a_1 s_1 + a_2 s_2 + b) = \prod_{\mathfrak{p}} \det(1 - \rho(\mathrm{Frob})_{\mathfrak{p}} q^{-a_1 s_1 - a_2 s_2 - b})^{-1}$$

converges for $\mathrm{Re}(a_1 s_1 + a_2 s_2) > 1 - b$ and admits meromorphic continuation to the whole $\mathbb{C}^2$, the same way that $L_{F_{U,b}}(s)$ does; see [32, Section 10 of Chap.VII]. This is due essentially to the facts that, although we are considering two variables, the function $L_{F_U,b}(a_1 s_1 + a_2 s_2 + b)$ is being taken over values on $\mathbb{C}$ given by the entire function $\omega : \mathbb{C}^2 \to \mathbb{C}$ defined by $\omega(s_1, s_2) = a_1 s_1 + a_2 s_2 + b$.

In particular, the second part of this proposition follows from similar arguments as the ones of [42, Lemma 4.6]. □

Proposition 4.2.8 assures that $\sum_{\mathfrak{p} \notin Q^*} l_{\mathfrak{p}}(F_{U_i,b}) q^{-A^*_{1i} s_1 - A^*_{2i} s_2 - B^*_i}$ converges absolutely on $\mathscr{D}_i$ and admits meromorphic continuation to $\mathscr{D}_{i,\delta_i}$ for some $\delta_i > 0$ and, hence, the sum $\sum_{b \in I_{U_i}} \sum_{\mathfrak{p} \notin Q^*} l_{\mathfrak{p}}(F_{U_i,b}) q^{-A^*_{1i} s_1 - A^*_{2i} s_2 - B^*_i}$ also does, because I_{U_i} is finite.

For each $i \in \mathscr{R}$, define

$$\widetilde{V_i^*}(s_1, s_2) = \prod_{b \in I_{U_i}} \prod_{\mathfrak{p} \notin Q^*} (1 - l_{\mathfrak{p}}(F_{U_i,b}) q^{-A^*_{1i} s_1 - A^*_{2i} s_2 - B^*_i}) = \prod_{b \in I_{U_i}} V_{b,i}(s_1, s_2). \tag{4.2.4}$$

Since I_{U_i} is finite, Proposition 4.2.8 assures that $\widetilde{V_i^*}(s_1, s_2)$ converges on $\mathscr{D}_i$ and admits meromorphic continuation to $\mathscr{D}_{i,\delta_i}$ for some $\delta_i > 0$.

The Lang-Weil estimate of Lemma 4.1.5 gives a positive constant Δ_i such that $V_i^*(s_1, s_2) \equiv_{\mathscr{D}_{i,\Delta_i}} \widetilde{V_i^*}(s_1, s_2)$ for each $i \in \mathscr{R}$. It follows from Lemma 4.2.2 that $V_i^*(s_1, s_2)$ is a meromorphic function on $\mathscr{D}_{i,\min\{\delta_i, \Delta_i\}}$, and therefore $\prod_{\mathfrak{p} \notin Q^*} V_i^*(s_1, s_2)(s_1, s_2)$ is meromorphic on $\mathscr{D}_{\mathscr{R},\Delta} = \bigcap_{i \in \mathscr{R}} \mathscr{D}_{i,\Delta}$, for $\Delta = \min\{\delta_i, \Delta_i \mid i \in \mathscr{R}\}$.

4.2.3 Proof of Theorem 5(2)

It follows from the results of Sections 4.2.1 and 4.2.2 that $\mathscr{G}^*_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$ is meromorphic on the domain $\mathscr{M}^1_{\mathscr{G}^*} \cap \mathscr{M}^2_{\mathscr{G}^*}$, which is independent of $\mathcal{O}$. Moreover, $\mathscr{M}^2_{\mathscr{G}^*} = \mathscr{D}_{\mathscr{R},\Delta}$ for some $\Delta > 0$ and the intersection of $\mathscr{M}^1_{\mathscr{G}^*}$ with a domain of the form $\mathscr{D}_{\mathscr{R},\delta}$ with $\delta > 0$ is a domain strictly containing $\mathscr{D}^*_{\mathbf{G}}$.

In Section 4.1.2, we have shown that, for $\mathfrak{p} \in Q_1$, the domain of convergence $\mathscr{C}^*_{\mathfrak{p}}$ of $\widetilde{\mathcal{Z}^*_{\mathbf{G}(\mathfrak{o})}}(s_1, s_2)$ is a domain of the form $\bigcap_{i \in [z] \cap W'} \mathscr{D}_{i,\delta}$. Denote by $\mathscr{C}^*_{Q_1}$ the intersection of all $\mathscr{C}^*_{\mathfrak{p}}$ with $\mathfrak{p} \in Q_1$.

Since the function

$$\prod_{\mathfrak{p} \notin Q_2^*} \widetilde{\mathcal{Z}^*_{\mathbf{G}(\mathfrak{o})}}(s_1, s_2)$$

is meromorphic on $\mathscr{M}^*_{\mathbf{G}} = \mathscr{M}^*_{\mathbf{G}(\mathcal{O})} := \mathscr{M}^1_{\mathscr{G}^*} \cap \mathscr{M}^2_{\mathscr{G}^*} \cap \mathscr{C}^*_{Q_1}$, it is left to show that $\mathscr{M}^*_{\mathbf{G}}$ is a domain strictly containing $\mathscr{D}^*_{\mathbf{G}}$.

In fact, for each $i \in [z] \cap W'$ the domain $\mathscr{D}_{i,\delta}$ is a translation of the domain $\mathscr{D}_i$. Thus, $\mathscr{R}$ is also the set of all indices $i \in [z] \cap W'$ such that the boundary $\partial \mathscr{D}_{i,\delta}$ shares infinitely many points with the boundary $\partial \left(\bigcap_{i \in [z] \cap W'} \mathscr{D}_{i,\delta} \right)$. In other words, $\bigcap_{i \in [z] \cap W'} \mathscr{D}_{i,\delta} = \bigcap_{i \in \mathscr{R}} \mathscr{D}_{i,\delta} = \mathscr{D}_{\mathscr{R},\delta}$. Therefore, the domains of convergence $\mathscr{C}^*_{\mathfrak{p}}$ for $\mathfrak{p} \in Q_1$ are domains of the form $\mathscr{D}_{\mathscr{R},\delta}$ with $\delta > 0$, and hence $\mathscr{C}^*_{Q_1} = \mathscr{D}_{\mathscr{R},\gamma}$ for some $\gamma > 0$, which concludes the proof of Theorem 5(2).

Chapter 5

Groups of type F, G, and H

This chapter comprises the results of [28], which concerns bivariate zeta functions of groups of type F, G, and H.

Fix $n \in \mathbb{N}$ and $\delta \in \{0,1\}$. Recall the nilpotent $\mathbb{Z}$-Lie lattices of Definition 1.2.1:

$$\mathcal{F}_{n,\delta} = \langle x_k, y_{ij} \mid [x_i, x_j] - y_{ij}, 1 \leq k \leq 2n+\delta, 1 \leq i < j \leq 2n+\delta \rangle,$$

$$\mathcal{G}_n = \langle x_k, y_{ij} \mid [x_i, x_{n+j}] - y_{ij}, 1 \leq k \leq 2n, 1 \leq i, j \leq n \rangle,$$

$$\mathcal{H}_n = \langle x_k, y_{ij} \mid [x_i, x_{n+j}] - y_{ij}, [x_j, x_{n+i}] - y_{ij},\ 1 \leq k \leq 2n, 1 \leq i \leq j \leq n \rangle.$$

By convention, relations that do not follow from the given ones are trivial.

In this chapter, we consider the unipotent group scheme $\mathbf{G} = \mathbf{G}_\Lambda$ associated to one of the $\mathbb{Z}$-Lie lattices $\mathcal{F}_{n,\delta}$, $\mathcal{G}_n$, or $\mathcal{H}_n$ given above, that is, Λ is one of the $\mathbb{Z}$-Lie lattices $\mathcal{F}_{n,\delta}$, $\mathcal{G}_n$, or $\mathcal{H}_n$.

In this chapter, we prove Theorems 2 and 3, in Sections 5.1 and 5.2, respectively. We also give an alternative prove to Theorem 4 for the representation case in Section 5.3.3. In Section 5.3.2, formulae for the joint distribution of three functions on Weyl groups of type B are obtained by writing the local class number zeta functions of $\mathbf{G}(\mathcal{O})$ as sums over finite hyperoctahedral groups in terms of statistics on such groups and then comparing these formulae with the ones given in Corollary 1.2.2.

Fix a prime ideal $\mathfrak{p}$ of $\mathcal{O}$. As in Section 3.2.2, set $\mathfrak{g} = \Lambda(\mathfrak{o}) = \Lambda \otimes_{\mathcal{O}} \mathfrak{o}$ and let $\mathfrak{g}'$ and $\mathfrak{z}$ be the derived Lie sublattice and the centre of $\mathfrak{g}$, respectively. We observe that the numbers h, a, b, r, and z defined in Section 3.2.2 are given as follows in the current context.

Λ	$h = \mathrm{rk}(\mathfrak{g})$	$a = \mathrm{rk}(\mathfrak{g}/\mathfrak{z})$	$b = \mathrm{rk}(\mathfrak{g}') = \mathrm{rk}(\mathfrak{z}) = z$
$\mathcal{F}_{n,\delta}$	$\binom{2n+\delta+1}{2}$	$2n+\delta$	$\binom{2n+\delta}{2}$
$\mathcal{G}_n$	n^2+2n	$2n$	n^2
$\mathcal{H}_n$	$\binom{n+1}{2}+2n$	$2n$	$\binom{n+1}{2}$

Table 5.1: Constants associated to $\mathcal{F}_{n,\delta}$, $\mathcal{G}_n$, and $\mathcal{H}_n$

5.1 Bivariate conjugacy class zeta functions—Proof of Theorem 2

5.1.1 Commutator matrices

Proposition 3.2.12 describes bivariate zeta functions in terms of $\mathfrak{p}$-adic integrals whose integrand is given in terms of minors of commutator matrices. In order to explicitly calculate these integrals, we describe the A-commutator matrix of groups of type F, G, and H. In this chapter we write $A_\Lambda(\underline{X})$ instead of $A(\underline{X})$ for the commutator matrix of $\mathfrak{g} = \Lambda(\mathfrak{o})$.

We determine the ordered sets $\mathbf{e}$ and $\mathbf{f}$ defined in Section 3.2.2 in the context of the Lie lattice $\Lambda \in \{\mathcal{F}_{n,\delta}, \mathcal{G}_n, \mathcal{H}_n\}$. The ordered set $\mathbf{e}$ is given by $\mathbf{e} = (x_1, \dots, x_a)$, where the x_i are the elements appearing in the presentation of Λ of Definition 1.2.1 and the ordering is $x_i > x_{i+1}$ for each $i \in [a-1]$. Then $\overline{\mathbf{e}} = (\overline{e_1}, \dots, \overline{e_a})$ is an $\mathfrak{o}$-basis of $\mathfrak{g}/\mathfrak{z}$, where $^-$ denotes the natural surjection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{z}$.

To determine $\mathbf{f}$, define

$$D_\Lambda = \begin{cases} \{(i,j) \in [2n+\delta]^2 \mid 1 \le i < j \le 2n+\delta\}, & \text{if } \Lambda = \mathcal{F}_{n,\delta}, \\ [n]^2, & \text{if } \Lambda = \mathcal{G}_n, \\ \{(i,j) \in [n]^2 \mid 1 \le i \le j \le n\}, & \text{if } \Lambda = \mathcal{H}_n, \end{cases}$$

and let y_{ij} be the elements appearing in the relations of the presentation of Λ of Definition 1.2.1. Then $\mathbf{f} = (y_{ij})_{(i,j)\in D_\Lambda}$, with ordering given by $y_{ij} > y_{kl}$, whenever either $i < k$ or $i = k$ and $j < l$. For simplicity, we write $\mathbf{f} = (y_{ij})_{(i,j)\in D_\Lambda} = (f_1, \dots, f_b)$ so that $f_1 > \dots > f_b$. The following lemma relates the notations $\mathbf{f} = (f_1, \dots, f_b)$ and $\mathbf{f} = (y_{ij})_{(i,j)\in D_\Lambda}$.

Lemma 5.1.1. *Let $\omega_\Lambda : D_\Lambda \to [b]$ be the map satisfying $y_{ij} = f_{\omega(i,j)}$. Then*

$$\omega_\Lambda(i,j) = \begin{cases} (i-1)a - \binom{i+1}{2} + j, & \text{if } \Lambda = \mathcal{F}_{n,\delta}, \\ (i-1)n + j, & \text{if } \Lambda = \mathcal{G}_n, \\ (i-1)n - \binom{i}{2} + j, & \text{if } \Lambda = \mathcal{H}_n. \end{cases}$$

Proof. For $\Lambda = \mathcal{F}_{n,\delta}$, the ordering of the y_{ij} is given by the following identities:

$$\omega_{\mathcal{F}_{n,\delta}}(i,j+1) = \omega(i,j) + 1, \qquad 1 \le i < j < a,$$
$$\omega_{\mathcal{F}_{n,\delta}}(i,i+1) = \omega_{\mathcal{F}_{n,\delta}}(i-1,a) + 1, \qquad 1 \le i < a,$$

In particular, for $1 \le i < j < a$,

$$\begin{aligned} \omega_{\mathcal{F}_{n,\delta}}(i,j) &= \omega_{\mathcal{F}_{n,\delta}}(i,j-1) + 1 = \dots = \omega_{\mathcal{F}_{n,\delta}}(i,i+1) + j - i - 1 \\ &= \omega_{\mathcal{F}_{n,\delta}}(i-1,a) + j - i. \end{aligned}$$

Since $\omega_{\mathcal{F}_{n,\delta}}(1,j) = j-1$, we see that $\omega_{\mathcal{F}_{n,\delta}}(2,j) = (a-1) + j - 2$, and thus $\omega_{\mathcal{F}_{n,\delta}}(3,j) = (a-1) + (a-2) + j - 3$. Inductively,

$$\omega_{\mathcal{F}_{n,\delta}}(i,j) = \sum_{k=1}^{i-1}(a-k) + j - i = (i-1)a - \binom{i}{2} + j - i = (i-1)a + \binom{i+1}{2} + j.$$

The other cases follow from similar arguments. □

Let $\underline{X} = (X_1, \dots, X_a)$ be a vector of variables and, for $m \in [a]$, set

$$\mathfrak{C}_{\Lambda,m} = \{\omega_\Lambda(m,j) \mid (m,j) \in D_\Lambda\}.$$

We want to determine the submatrix $A_\Lambda^{(m)}(\underline{X})$ of $A_\Lambda(\underline{X})$ composed by the columns of index in $\mathfrak{C}_{\Lambda,m}$ so that

$$A_{\mathcal{F}_{n,\delta}}(\underline{X}) = \left[\begin{array}{cccc} A^{(1)}_{\mathcal{F}_{n,\delta}}(\underline{X}) & A^{(2)}_{\mathcal{F}_{n,\delta}}(\underline{X}) & \dots & A^{(a-1)}_{\mathcal{F}_{n,\delta}}(\underline{X}) \end{array}\right],$$
$$A_\Lambda(\underline{X}) = \left[\begin{array}{cccc} A^{(1)}_{\Lambda}(\underline{X}) & A^{(2)}_{\Lambda}(\underline{X}) & \dots & A^{(n)}_{\Lambda}(\underline{X}) \end{array}\right],$$

for $\Lambda \in \{\mathcal{G}_n, \mathcal{H}_n\}$. For $\Lambda \in \{\mathcal{F}_{n,\delta}, \mathcal{G}_n, \mathcal{H}_n\}$, the matrices $A_\Lambda(\underline{X})$ all have size $a \times b$. Set

$$\nu_{\Lambda,m} = \begin{cases} (m-1)a - \binom{m+1}{2} + m, & \text{if } \Lambda = \mathcal{F}_{n,\delta} \\ (m-1)n, & \text{if } \Lambda = \mathcal{G}_n \\ (m-1)n - \binom{m}{2} + m - 1, & \text{if } \Lambda = \mathcal{H}_n, \end{cases}$$

so that $\mathfrak{C}_{\Lambda,m} = \{\nu_{\Lambda,m}+1, \dots, \nu_{\Lambda,m}+k_{\Lambda,m}\}$, where

$$k_{\Lambda,m} = \begin{cases} a - m, & \text{if } \Lambda = \mathcal{F}_{n,\delta}, \\ n, & \text{if } \Lambda = \mathcal{G}_n, \\ n - m + 1, & \text{if } \Lambda = \mathcal{H}_n, \end{cases}$$

that is, the jth column of $A_\Lambda^{(m)}(\underline{X})$ is the $(\nu_{\Lambda,m}+j)$th column of $A_\Lambda(\underline{X})$.

The relations of Λ show that, for $(i,j) \in D_\Lambda$ and for $k \in \mathfrak{C}_{\Lambda,m}$, the structure constants involving (i,j) are the ones in the following table:

Λ	structure constants involving (i,j)
$\mathcal{F}_{n,\delta}$	$\lambda_{ij}^k = \begin{cases} 1, & \text{if } k = \omega_{\mathcal{F}_{n,\delta}}(i,j), \\ 0, & \text{otherwise,} \end{cases}$
$\mathcal{G}_n$	$\lambda_{i(n+j)}^k = \begin{cases} 1, & \text{if } k = \omega_{\mathcal{G}_n}(i,j), \\ 0, & \text{otherwise,} \end{cases}$
$\mathcal{H}_n$	$\lambda_{i(n+j)}^k = \lambda_{j(n+i)}^k = \begin{cases} 1, & \text{if } k = \omega_{\mathcal{H}_n}(i,j), \\ 0, & \text{otherwise.} \end{cases}$

Table 5.2: Structure constants for $\mathcal{F}_{n,\delta}$, $\mathcal{G}_n$, and $\mathcal{H}_n$

Since $\mathfrak{C}_{\Lambda,m}$ is composed of all $\omega_\Lambda(m,j)$ with $(m,j) \in D_\Lambda$, it is clear that the indices $k \in \mathfrak{C}_{\Lambda,m}$ of the columns of $A_\Lambda^{(m)}(\underline{X})$ cannot equal $\omega_\Lambda(i,j)$ if $i \neq m$. In particular, $\lambda_{ij}^k = 0$ if $i, j \neq m$. Every $k \in \mathfrak{C}_{\Lambda,m}$ is of the form $k = \nu_{\Lambda,m} + l$, for some $l \in [k_{\Lambda,m}]$. Recall that the (i,l)th entry of $A_\Lambda^{(m)}(\underline{X})$ is the $(i, \nu_{\Lambda,m}+l)$th entry of $A_\Lambda(\underline{X})$, that is,

$$A_\Lambda^{(m)}(\underline{X})_{il} = A_\Lambda(\underline{X})_{i(\nu_{\Lambda,m}+l)}.$$

In the following, we determine $A_\Lambda^{(m)}(\underline{X})$ of each type separately.

A-commutator matrices of groups of type F

For $\Lambda = \mathcal{F}_{n,\delta}$, the index $k = \nu_{\mathcal{F}_{n,\delta},m} + l$ coincides with $\omega_{\mathcal{F}_{n,\delta}}(m,j) = \nu_{\mathcal{F}_{n,\delta},m} + j - m$ if and only if $j = l + m$. It follows that $\lambda_{ij}^k = 1$ if and only if $i = m$ and $j = m + l$. Hence the (m, l)th entry of $A^{(m)}_{\mathcal{F}_{n,\delta}}(\underline{X})$ is

$$A^{(m)}_{\mathcal{F}_{n,\delta}}(\underline{X})_{ml} = \sum_{j=1}^{a} \lambda_{mj}^{\nu_{\mathcal{F}_{n,\delta},m}+l} X_j = X_{m+l},$$

and, for $i \neq m$, its (i, l)th entry is

$$A^{(m)}_{\mathcal{F}_{n,\delta}}(\underline{X})_{il} = -\sum_{j=1}^{a} \lambda_{ji}^{\nu_{\mathcal{F}_{n,\delta},m}+l} X_j = \begin{cases} -X_m, & \text{if } i = m + l, \\ 0, & \text{otherwise.} \end{cases}$$

Given $s, r \in \mathbb{N}$, let $\mathbf{0}_{s\times r}$ be the $(s \times r)$-zero matrix and let $\mathbf{1}_s$ be the $(s \times s)$-identity matrix, both over $\mathfrak{o}[\underline{X}]$. It follows that, for each $m \in [a - 1]$,

$$A^{(m)}_{\mathcal{F}_{n,\delta}}(\underline{X}) = \begin{bmatrix} & \mathbf{0}_{(m-1)\times(2n+\delta-m)} & \\ X_{m+1} & X_{m+2} \quad \ldots & X_{2n+\delta} \\ & -X_m \, \mathbf{1}_{(2n+\delta-m)} & \end{bmatrix} \in \mathrm{Mat}_{(2n+\delta)\times(2n+\delta-m)}(\mathfrak{o}[\underline{X}]).$$

A-commutator matrices of groups of type G

For $\Lambda = \mathcal{G}_n$, the index $k = \nu_{\mathcal{G}_n,m} + l$ coincides with $\omega_{\mathcal{G}_n}(m, j) = \nu_{\mathcal{G}_n,m} + j$ if and only if $j = l$. It follows that $\lambda^k_{i(n+j)} = 1$ if and only if $i = m$ and $j = l$. Hence the (m, l)th entry of $A^{(m)}_{\mathcal{G}_n}(\underline{X})$ is

$$A^{(m)}_{\mathcal{G}_n}(\underline{X})_{ml} = \sum_{j=1}^{a} \lambda_{mj}^{\nu_{\mathcal{G}_n,m}+l} X_j = X_{n+l},$$

and, for $i \neq m$, its (i, l)th entry is

$$A^{(m)}_{\mathcal{G}_n}(\underline{X})_{il} = -\sum_{j=1}^{a} \lambda_{ji}^{\nu_{\mathcal{G}_n,m}+l} X_j = \begin{cases} -X_m, & \text{if } i = n + l, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, for each $m \in [n]$,

$$A^{(m)}_{\mathcal{G}_n}(\underline{X}) = \begin{bmatrix} & \mathbf{0}_{(m-1)\times n} & \\ X_{n+1} & X_{n+2} \quad \ldots & X_{2n} \\ & \mathbf{0}_{(n-m)\times n} & \\ & -X_m \mathbf{1}_n & \end{bmatrix} \in \mathrm{Mat}_{2n\times n}(\mathfrak{o}[\underline{X}]). \tag{5.1.1}$$

A-commutator matrices of groups of type H

For $\Lambda = \mathcal{H}_n$, the index $k = \nu_{\mathcal{H}_n,m} + l$ coincides with $\omega_{\mathcal{H}_n}(m, j) = \nu_{\mathcal{H}_n,m} + j - m + 1$ if and only if $j = m + l - 1$. If follows that $\lambda^k_{i(n+j)} = \lambda^k_{j(n+i)} = 1$ if and only if either $i = m$ and $j = m + l - 1$ or $j = m$ and $i = m + l - 1$. Therefore

$$A^{(m)}_{\mathcal{H}_n}(\underline{X})_{ml} = \sum_{j=1}^{a} \lambda_{mj}^{\nu_{\mathcal{H}_n,m}+l} X_j = X_{n+m+l-1},$$

$$A^{(m)}_{\mathcal{H}_n}(\underline{X})_{(n+m)l} = -\sum_{j=1}^{a} \lambda_{j(n+m)}^{\nu_{\mathcal{H}_n,m}+l} X_j = -X_{m+l-1}.$$

For $i \in [n] \setminus \{m\}$, the (i,l)th entry of $A^{(m)}_{\mathcal{H}_n}(\underline{X})$ is

$$A^{(m)}_{\mathcal{H}_n}(\underline{X})_{il} = \sum_{j=1}^{n} \lambda_{j(n+i)}^{\nu_{\mathcal{H}_n,m+l}} X_{n+j} = \begin{cases} X_{n+m}, & \text{if } i = m+l-1, \\ 0, & \text{otherwise.} \end{cases}$$

For $i = n+t$ with $t \in [n] \setminus \{m\}$, the (i,l)th entry of $A^{(m)}_{\mathcal{H}_n}(\underline{X})$ is

$$A^{(m)}_{\mathcal{H}_n}(\underline{X})_{il} = -\sum_{j=1}^{n} \lambda_{j(n+t)}^{\nu_{\mathcal{H}_n,m+l}} X_{j} = \begin{cases} -X_{m}, & \text{if } t = m+l-1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, for each $m \in [n]$,

$$A^{(m)}_{\mathcal{H}_n}(\underline{X}) = \begin{bmatrix} & \mathbf{0}_{(m-1)\times(n-m+1)} & & \\ X_{n+m} & X_{n+m+1} & \dots & X_{2n} \\ & X_{n+m} & & \\ & & \ddots & \\ & & & X_{n+m} \\ & \mathbf{0}_{(m-1)\times(n-m+1)} & & \\ -X_{m} & -X_{m+1} & \dots & -X_{n} \\ & -X_{m} & & \\ & & \ddots & \\ & & & -X_{m} \end{bmatrix} \in \mathrm{Mat}_{2n\times(n-m+1)}(\mathfrak{o}[\underline{X}]). \tag{5.1.2}$$

Example 5.1.2. We now illustrate the form of each commutator matrix.

$$A_{\mathcal{F}_{2,0}}(\underline{X}) = \begin{bmatrix} X_2 & X_3 & X_4 & & & \\ -X_1 & & & X_3 & X_4 & \\ & -X_1 & & -X_2 & & X_4 \\ & & -X_1 & & -X_2 & -X_3 \end{bmatrix},$$

$$A_{\mathcal{G}_3}(\underline{X}) = \begin{bmatrix} X_4 & X_5 & X_6 & & & & & & \\ & & & X_4 & X_5 & X_6 & & & \\ & & & & & & X_4 & X_5 & X_6 \\ -X_1 & & & -X_2 & & & -X_3 & & \\ & -X_1 & & & -X_2 & & & -X_3 & \\ & & -X_1 & & & -X_2 & & & -X_3 \end{bmatrix},$$

$$A_{\mathcal{H}_3}(\underline{X}) = \begin{bmatrix} X_4 & X_5 & X_6 & & & \\ & X_4 & & X_5 & X_6 & \\ & & X_4 & & X_5 & X_6 \\ -X_1 & -X_2 & -X_3 & & & \\ & -X_1 & & -X_2 & -X_3 & \\ & & -X_1 & & -X_2 & -X_3 \end{bmatrix},$$

where the omitted entries equal zero. △

It is not difficult to see that $A_\Lambda(\underline{X})$ has rank $a-1$ in all cases, that is, $u_A = a-1$.

We now proceed to a detailed analysis of the A-commutator matrix in each individual type.

5.1.2 Conjugacy class zeta functions of groups of type F

Lemma 5.1.3. *For $w \in \mathfrak{p}$ and $\mathbf{x} \in W_a^{\mathfrak{o}}$, that is, for $\mathbf{x} \in \mathfrak{o}^a$ such that $v_{\mathfrak{p}}(\mathbf{x}) = 0$,*

$$\frac{\|F_k(A_{\mathcal{F}_{n,\delta}}(\mathbf{x})) \cup wF_{k-1}(A_{\mathcal{F}_{n,\delta}}(\mathbf{x}))\|_{\mathfrak{p}}}{\|F_{k-1}(A_{\mathcal{F}_{n,\delta}}(\mathbf{x}))\|_{\mathfrak{p}}} = 1, \text{ for all } k \in [a-1]. \tag{5.1.3}$$

Proof. The columns of $A_{\mathcal{F}_{n,\delta}}(\underline{X})$ are of the form

$$\begin{matrix} k\text{th row} \{ \\ \\ j\text{th row} \{ \end{matrix} \begin{bmatrix} X_j \\ \\ -X_k \end{bmatrix}, \text{ for each } j,k \in [a], \tag{5.1.4}$$

where the nondisplayed entries equal 0. Denote column (5.1.4) by $C_{j,k}$. For each $i \in [a]$, consider the $(a \times (a-1))$-submatrix $K_i(\underline{X})$ of $A_{\mathcal{F}_{n,\delta}}(\underline{X})$ composed of the columns $C_{i,1}, \ldots, C_{i,i-1}, C_{i+1,i}, C_{i+2,i}, \ldots, C_{a,i}$ in this order. That is,

$$K_i(\underline{X}) = \begin{bmatrix} \overbrace{X_i}^{C_{i,1}} & & & & & \\ & \ddots & & & & \\ & & \overbrace{X_i}^{C_{i,i-1}} & & & \\ -X_1 & \ldots & -X_{i-1} & \overbrace{X_{i+1}}^{C_{i+1,i}} & & \overbrace{X_a}^{C_{a,i}} \\ & & & -X_i & & \\ & & & & \ddots & \\ & & & & & -X_i \end{bmatrix},$$

Given $\mathbf{x} \in W_a^{\mathfrak{o}}$, it is clear that, for at least one $i_0 \in [a]$, the matrix $K_{i_0}(\mathbf{x})$ has rank $a-1$. That is, for each $k \in [a-1]$, there exists a $(k \times k)$-minor of $K_{i_0}(\mathbf{x})$ which is a unit. Since the $(k \times k)$-minors of $K_{i_0}(\mathbf{x})$ are elements of $F_k(A_{\mathcal{F}_{n,\delta}}(\mathbf{x}))$, expression (5.1.3) follows. □

Lemma 5.1.3, Proposition 3.2.12, and Lemma 2.2.1 yield

$$\begin{aligned} &\mathcal{Z}^{\mathrm{cc}}_{\mathcal{F}_{n,\delta}(\mathfrak{o})}(s_1, s_2) \\ &= \frac{1}{1-q^{\binom{2n+\delta}{2}-s_2}} \left(1 + (1-q^{-1})^{-1} \int_{(w,\underline{x}) \in \mathfrak{p} \times W^{\mathfrak{o}}_{2n+\delta}} |w|_{\mathfrak{p}}^{(2n+\delta-1)s_1+s_2-\binom{2n+\delta}{2}-2} d\mu\right) \\ &= \frac{1-q^{\binom{2n+\delta-1}{2}-(2n+\delta-1)s_1-s_2}}{(1-q^{\binom{2n+\delta}{2}-s_2})(1-q^{\binom{2n+\delta}{2}+1-(2n+\delta-1)s_1-s_2})}, \end{aligned}$$

proving Theorem 2 for groups of type F.

Remark 5.1.4. Formula (1.2.1) for the class number zeta function $\mathcal{F}_{n,\delta}(\mathcal{O})$ reflects the K-minimality of $\Lambda = \mathcal{F}_{n,\delta}$; see [38, Lemma 6.2 and Definition 6.3]. In fact, the proof of Lemma 5.1.3 shows that

$$\frac{\|F_k(A_{\mathcal{F}_{n,\delta}}(\underline{x})) \cup yF_{k-1}(A_{\mathcal{F}_{n,\delta}}(\underline{x}))\|_{\mathfrak{p}}}{\|F_{k-1}(A_{\mathcal{F}_{n,\delta}}(\underline{x}))\|_{\mathfrak{p}}} = \|\underline{x}, y\|_{\mathfrak{p}}.$$

Thus, the formula for the local factors of the class number zeta function of $F_{n,\delta}(\mathcal{O})$ given in Corollary 1.2.2 coincides with the formula of the class number zeta function of $\mathcal{F}_{n,\delta}(\mathfrak{o})$ given by the specialisation of the formula given in [38, Proposition 6.4]; see Remark 3.2.14.

5.1.3 Conjugacy class zeta functions of groups of type G

We first describe the determinant of a square matrix in terms of its 2×2-minors, which will be used to describe the minors of $A_{\mathcal{G}_n}(\underline{X})$. Given a matrix $M=(m_{ij})$, let $\widetilde{M}_{(i,j),(r,s)} = \begin{vmatrix} m_{ij} & m_{is} \\ m_{rj} & m_{rs} \end{vmatrix}$.

Lemma 5.1.5. *Given $t\in\mathbb{N}$, let $G=(g_{ij})_{1\leq i,j\leq 2t}$ and $U=(u_{ij})_{1\leq i,j\leq 2t+1}$ be matrices with $g_{ij}=g(\underline{X})_{ij}$, $u_{ij}=u(\underline{X})_{ij}\in\mathfrak{o}[\underline{X}]$. Let $\mathbf{i}=\{i_1,\dots,i_t\}$, $\mathbf{j}=\{j_1,\dots,j_t\}\subset [2t]$. Then, for suitable $\alpha_{\mathbf{i},\mathbf{j}},\beta_{\mathbf{i},\mathbf{j}}\in\{-1,1\}$,*

$$\det(G) = \sum_{\substack{\mathbf{i}\cup\mathbf{j}=[2t]\\ i_q<j_q,\ \forall q\in[t]}} \alpha_{\mathbf{i},\mathbf{j}}\widetilde{G}_{(1,i_1),(2,j_1)}\widetilde{G}_{(3,i_2),(4,j_2)}\cdots\widetilde{G}_{(2t-1,i_t),(2t,j_t)},$$

$$\det(U) = \sum_{i=1}^{2t+1}\sum_{\substack{\mathbf{i}\cup\mathbf{j}=[2t+1]\setminus\{i\}\\ i_q<j_q,\ \forall q\in[t]}} \beta_{\mathbf{i},\mathbf{j}}u_{1i}\widetilde{U}_{(1,i_1),(2,j_1)}\widetilde{U}_{(3,i_2),(4,j_2)}\cdots\widetilde{U}_{(2t-1,i_t),(2t,j_t)}.$$

Proof. Given two subsets $I,J\subseteq[2t]$ of equal cardinality m, denote by $\widehat{G}_{I,J}$ the determinant of the $(2t-m)\times(2t-m)$-submatrix of G obtained by excluding the rows of indices in I and columns of index in J. The entries of the submatrix $G_{\{1\},\{k\}}=(\tilde{g}_{ij})_{ij}$ obtained from G by excluding its first row and its kth column are given by

$$\tilde{g}_{ij} = \begin{cases} g_{(i+1)j}, & \text{if } j\in[k-1],\\ g_{(i+1)(j+1)}, & \text{if } j\in\{k,\dots,2t-1\}.\end{cases}$$

Consequently,

$$\widehat{G}_{\{1\},\{k\}} = \sum_{j=1}^{k-1}(-1)^{1+j}g_{2j}\widehat{G}_{\{1,2\},\{j,k\}} + \sum_{j=k}^{2t-1}(-1)^{1+j}g_{2(j+1)}\widehat{G}_{\{1,2\},\{k,j+1\}}.$$

It follows that

$$\begin{aligned}\det(G) &= \sum_{k=1}^{2t}(-1)^{1+k}g_{1k}\widehat{G}_{\{1\},\{k\}}\\ &= \sum_{k=1}^{2t}\left(\sum_{j=1}^{k-1}(-1)^{k+j}g_{1k}g_{2j}\widehat{G}_{\{1,2\},\{j,k\}} - \sum_{j=k+1}^{2t}(-1)^{k+j}g_{1k}g_{2j}\widehat{G}_{\{1,2\},\{k,j\}}\right)\\ &= \sum_{m=1}^{2t-1}\sum_{i=m+1}^{2t}(-1)^{i+m-1}(g_{1m}g_{2i}-g_{1i}g_{2m})\widehat{G}_{\{1,2\}\{m,i\}}\end{aligned}$$

$$= \sum_{m=2}^{2t-1} \sum_{i=m+1}^{2t} (-1)^{i+m-1} \widetilde{G}_{(1,m),(2,i)} \widehat{G}_{\{1,2\},\{m,i\}}.$$

The relevant claim of Lemma 5.1.5 for the matrix G follows by induction on t.

The claim for the matrix U follows by the first part, since its determinant is

$$\det(U) = \sum_{i=1}^{2t+1} (-1)^{i+1} u_{1i} \widehat{U}_{\{1\},\{i\}}. \qquad \square$$

Lemma 5.1.6. *For each $r \in [2n]$, the nonzero elements of $F_r(A_{\mathcal{G}_n}(\underline{X}))$ are either of one of the following forms or a sum of these terms.*

$$X_{i_1} \cdots X_{i_\omega} X_{n+j_1} \cdots X_{n+j_\lambda} \text{ or } -X_{i_1} \cdots X_{i_\omega} X_{n+j_1} \cdots X_{n+j_\lambda}.$$

Proof. Lemma 5.1.5 describes each element of $F_k(A_{\mathcal{G}_n}(\underline{X}))$ in terms of sums of products of (2×2)-minors of $A_{\mathcal{G}_n}(\underline{X})$. It then suffices to show that these minors are all either 0 or of the forms X_iX_j or $-X_iX_j$, for some $i, j \in [2n]$. This can be seen from the description of $A_{\mathcal{G}_n}(\underline{X})$ in terms of the blocks (5.1.1). $\square$

The proof of Theorem 2 for groups of type G follows from the following Proposition.

Proposition 5.1.7. *Let $\underline{X} = (X_1, \ldots, X_{2n})$ be a vector of variables. Given $\lambda, \omega \in [n]_0$ such that $0 < \omega + \lambda \leq 2n-1$, for all choices of $i_1, \ldots, i_\omega, j_1, \ldots, j_\lambda \in [n]$, one of*

$$X_{i_1} \cdots X_{i_\omega} X_{n+j_1} \cdots X_{n+j_\lambda} \text{ or } -X_{i_1} \cdots X_{i_\omega} X_{n+j_1} \cdots X_{n+j_\lambda}$$

is an element of $F_{\omega+\lambda}(A_{\mathcal{G}_n}(\underline{X}))$.

In fact, for $x, y \in \mathfrak{o}$, it holds that $\min\{v_\mathfrak{p}(x+y), v_\mathfrak{p}(x), v_\mathfrak{p}(y)\} = \min\{v_\mathfrak{p}(x), v_\mathfrak{p}(y)\}$. Thus, if some term of the form

$$X_{i_1}X_{i_2} \cdots X_{i_\omega} X_{n+j_1} \cdots X_{n+j_\lambda} - X_{k_1}X_{k_2} \cdots X_{k_\omega} X_{n+l_1} \cdots X_{n+l_\lambda}$$

is a minor of the commutator matrix $A_{\mathcal{G}_n}(\underline{X})$, then, assuming the claim in Proposition 5.1.7 holds, both

$$X_{i_1}X_{i_2} \cdots X_{i_\omega} X_{n+j_1} \cdots X_{n+j_\lambda} \text{ and } X_{k_1}X_{k_2} \cdots X_{k_\omega} X_{n+l_1} \cdots X_{n+l_\lambda}$$

are minors of this commutator matrix (up to sign), and hence, when considering these three terms, only the last two will be relevant in order to determine $\|F_r(A_{\mathcal{G}_n}(\underline{X}))\|_\mathfrak{p}$. In this case, we may assume that all elements are of the form given in Proposition 5.1.7 while computing $\|F_r(A_{\mathcal{G}_n}(\underline{X}))\|_\mathfrak{p}$ and $\|F_r(A_{\mathcal{G}_n}(\underline{X})) \cup wF_{r-1}(A_{\mathcal{G}_n}(\underline{X}))\|_\mathfrak{p}$.

Firstly we show Proposition 5.1.7 for the case where both $\{i_1, \ldots, i_\omega\}$ and $\{j_1, \ldots, j_\lambda\}$ have cardinality smaller than n.

Lemma 5.1.8. *Let $\omega, \lambda \in [n]_0$ not both zero and not both n. Given $i_1, \ldots, i_\omega, j_1, \ldots, j_\lambda \in [n]$ such that $|\{i_1, \ldots, i_\omega\}|, |\{j_1, \ldots, j_\lambda\}| < n$, either*

$$X_{i_1} \cdots X_{i_\omega} X_{n+j_1} \cdots X_{n+j_\lambda} \text{ or } -X_{i_1} \cdots X_{i_\omega} X_{n+j_1} \cdots X_{n+j_\lambda}$$

is an element of $F_{\lambda+\omega}(A_{\mathcal{G}_n}(\underline{X}))$.

Proof. For each $(\mathbf{i}, \mathbf{j}) = (i_1, \ldots, i_\omega, j_1, \ldots, j_\lambda)$ as in the assumption of Lemma 5.1.8, we construct explicitly a submatrix of $A_{\mathcal{G}_n}(\underline{X})$ which is, up to reordering of rows and columns, of the form

$$\left[\begin{array}{ccc|ccc} X_{n+j_1} & & & & & \\ & \ddots & & & T(\underline{X}) & \\ & & X_{n+j_\lambda} & & & \\ \hline & & & -X_{i_1} & & \\ & W(\underline{X}) & & & \ddots & \\ & & & & & -X_{i_\omega} \end{array}\right] \tag{5.1.5}$$

where $T(\underline{X}) = (t(\underline{X})_{ij})$ and $W(\underline{X}) = (w(\underline{X})_{ij})$ are such that $t(\underline{X})_{ij} = 0$ and $w(\underline{X})_{ij} = 0$, if $i \leq j$. It is clear that the determinant of this matrix is one of $\pm X_{i_1} \cdots X_{i_\omega} X_{n+j_1} \cdots X_{n+j_\lambda}$.

The main fact we use is that the columns of $A_{\mathcal{G}_n}(\underline{X})$ are of the form

$$\begin{array}{r} i\text{th row}\,\{ \\ \\ (n+j)\text{th row}\,\{ \end{array} \begin{bmatrix} X_{n+j} \\ \\ -X_i \end{bmatrix}, \tag{5.1.6}$$

where the nondisplayed terms equal zero. For each $i, j \in [n]$, there is exactly one column of $A_{\mathcal{G}_n}(\underline{X})$ with X_{n+j} in the ith row, and exactly one column with $-X_j$ in the $(n+i)$th row.

Fix $l_1 \in [n] \setminus \{i_1, \ldots, i_\omega\}$ and let c_1 denote the unique column of $A_{\mathcal{G}_n}(\underline{X})$ with X_{n+j_1} in the l_1th row. Inductively, fix $l_k \in [n] \setminus \{l_1, \ldots, l_{k-1}, i_k, \ldots, i_\omega\}$, for each $k \in [\lambda]$, and let c_k be the unique column of $A_{\mathcal{G}_n}(\underline{X})$ with X_{n+j_k} in the l_kth row.

Analogously, fix $m_1 \in [n] \setminus \{j_1, \ldots, j_\lambda\}$ and let $\mathcal{C}_1$ be the index of the unique column of $A_{\mathcal{G}_n}(\underline{X})$ with $-X_{i_1}$ in the $(n+m_1)$th row, and, inductively, fix $m_q \in [n] \setminus \{m_1, \ldots, m_{q-1}, j_q, \ldots, j_\lambda\}$, for each $q \in [\omega]$, and let $\mathcal{C}_q$ be the index of the unique column of $A_{\mathcal{G}_n}(\underline{X})$ with $-X_{i_q}$ in the $(n+m_q)$th row.

From (5.1.6), one sees that the columns c_k and $\mathcal{C}_q$ are given by

$$\begin{array}{r} l_k\text{th row}\,\{ \\ \\ (n+j_k)\text{ th row}\,\{ \end{array} \overbrace{\begin{bmatrix} X_{n+j_k} \\ \\ -X_{l_k} \end{bmatrix}}^{c_k} \qquad \begin{array}{r} i_q\text{th row}\,\{ \\ \\ (n+m_q)\text{ th row}\,\{ \end{array} \overbrace{\begin{bmatrix} X_{n+m_q} \\ \\ -X_{i_q} \end{bmatrix}}^{\mathcal{C}_q}. \tag{5.1.7}$$

By construction, the indices c_k are all distinct, and so are the indices $\mathcal{C}_q$. If $c_k = \mathcal{C}_q$ for some $k \in [\lambda]$ and some $q \in [\omega]$, then we would obtain $l_k = i_q$. Analogously, the indices $l_1, \ldots, l_\lambda, n+m_1, \ldots, n+m_\omega$ are all distinct.

Consider the matrix $M_{(\mathbf{i},\mathbf{j})}(\underline{X})$ composed of columns c_k and $\mathcal{C}_q$ and of rows l_k and $n+m_q$, for $k \in [\lambda]$ and $q \in [\omega]$. This matrix is of the form (5.1.5) for some matrices $T(\underline{X}) \in \mathrm{Mat}_{\lambda\times\omega}(\mathfrak{o}[\underline{X}])$ and $W(\underline{X}) \in \mathrm{Mat}_{\omega\times\lambda}(\mathfrak{o}[\underline{X}])$. Let us show that, in fact, $t(\underline{X})_{ij} = 0$ and $w(\underline{X})_{ij} = 0$ for $i \leq j$.

The only nonzero entries of $\mathcal{C}_q$ are the ones of indices i_q and $n+m_q$. We chose each l_k so that $l_k \notin \{i_1, \ldots, i_k\}$. Since any of the rows $l_1, \ldots, l_q$ is the i_qth row of $A_{\mathcal{G}_n}(\underline{X})$, it follows that $t(\underline{X})_{iq} = 0$, for all $i \leq q$. Analogously, since the only nonzero entries of c_k are l_k and $n+j_k$ and $m_q \notin \{j_1,\dots,j_q\}$, it follows that $w(\underline{X})_{ik}=0$, for all $i\le k$. □

Proof of Proposition 5.1.7. Lemma 5.1.8 shows the claim of Proposition 5.1.7 for all cases, except for $\omega=n$ and $i_1,\dots,i_n$ all distinct, and for $\lambda=n$ and $j_1,\dots,j_n$ all distinct. Let us show the last case, the other one is analogous.

Assume that $j_1,\dots,j_n$ are all distinct and $\omega\in[n-1]_0$. For $k\in[n]$, we can define l_k as in the proof of Lemma 5.1.8, since $|\{i_1,\dots,i_\omega\}|<n$. We also set c_k as in the proof of Lemma 5.1.8. As $|\{j_1,\dots,j_n\}|=n$, we cannot choose $m_1\in[n]\setminus\{j_1,\dots,j_n\}$. Instead, we consider the rows $n+j_k$, for $k\in[\omega]$. Denote by $\mathcal{C}_q$ the column of $A_{\mathcal{G}_n}(\underline{X})$ with $-X_{i_q}$ in the $(n+j_q)$th row. By construction, the indices c_k, for $k\in[n]$, are all distinct, and so are the indices $\mathcal{C}_q$, for $q\in[\omega]$. The indices c_k and $\mathcal{C}_q$ coincide, for some $k\in[n]$ and $q\in[\omega]$, if and only if $i_q=l_q$. It follows that all c_k and $\mathcal{C}_q$ are distinct. Let $M_{\mathbf{i},\mathbf{j}}(\underline{X})$ be the submatrix of $A_{\mathcal{G}_n}(\underline{X})$ composed by columns c_k and $\mathcal{C}_q$ and of rows l_k and $n+j_q$, for each $k\in[n]$ and $q\in[\lambda]$, where $\mathbf{i}=(i_1,\dots,i_\lambda)$ and $\mathbf{j}=(j_1,\dots,j_n)$.

Then, as in Lemma 5.1.8, $B(\underline{X})$ is of the form (5.1.5), but the matrix $W(T)$ is such that $w(\underline{X})_{ij}=0$ if $i\ne j$. □

In particular, Proposition 5.1.7 shows that, for each $r\in[2n]$ and each $k\in[n]$, either X_r^k or $-X_r^k$ is an element of $F_k(A_{\mathcal{G}_n}(\underline{X}))$. Hence, if $\mathbf{x}\in W_{2n}^{\mathfrak{o}}$, then at least one $(k\times k)$-minor of $A_{\mathcal{G}_n}(\mathbf{x})$ has valuation zero. This gives

$$\frac{\|F_k(A_{\mathcal{G}_n}(\mathbf{x}))\cup wF_{k-1}(A_{\mathcal{G}_n}(\mathbf{x}))\|_{\mathfrak{p}}}{\|F_{k-1}(A_{\mathcal{G}_n}(\mathbf{x}))\|_{\mathfrak{p}}}=1,\ \text{for all } k\in[n]. \tag{5.1.8}$$

For $k\in\{n+1,\dots,2n-1\}$, the elements of $F_k(A_{\mathcal{G}_n}(\underline{X}))$ can be assumed to be of the form

$$X_{i_1}\cdots X_{i_\omega}X_{n+j_1}\cdots X_{n+j_\lambda},$$

where $\omega,\lambda\in[n]_0$ satisfy $\omega+\lambda=k$, and $i_1,\ \dots,\ i_\omega,j_1,\ \dots,\ j_\lambda\in[n]$.

Given $\mathbf{x}\in W_{2n}^{\mathfrak{o}}$, let $M=v_{\mathfrak{p}}(x_1,\dots,x_n)$ and $N=v_{\mathfrak{p}}(x_{n+1},\dots,x_{2n})$. Then

$$\left\|\bigcup_{\substack{\omega+\lambda=k\\0\le\omega,\lambda\le n}}\{X_{i_1}\cdots X_{i_\omega}X_{n+j_1}\cdots X_{n+j_\lambda}\mid i_1,\dots,i_\omega,j_1,\dots,j_\lambda\in[n]\}\right\|_{\mathfrak{p}}$$
$$=q^{-n\min\{M,N\}-(k-n)\max\{M,N\}}.$$

Consequently, for $w\in\mathfrak{p}$,

$$\frac{\|F_k(A_{\mathcal{G}_n}(\mathbf{x}))\cup wF_{k-1}(A_{\mathcal{G}_n}(\mathbf{x}))\|_{\mathfrak{p}}}{\|F_{k-1}(A_{\mathcal{G}_n}(\mathbf{x}))\|_{\mathfrak{p}}}=\begin{cases}\|x_1,\dots,x_n,w\|_{\mathfrak{p}}, & \text{if } 0=N\le M,\\ \|x_{n+1},\dots,x_{2n},w\|_{\mathfrak{p}}, & \text{if } 0=M\le N.\end{cases} \tag{5.1.9}$$

Combining (5.1.8) and (5.1.9) yields

$$\prod_{k=1}^{2n-1}\frac{\|F_k(A_{\mathcal{G}_n}(\mathbf{x}))\cup wF_{k-1}(A_{\mathcal{G}_n}(\mathbf{x}))\|_{\mathfrak{p}}}{\|F_{k-1}(A_{\mathcal{G}_n}(\mathbf{x}))\|_{\mathfrak{p}}}=$$
$$\begin{cases}\|x_1,\dots,x_n,w\|_{\mathfrak{p}}^{n-1}, & \text{if } 0=M\le N,\\ \|x_{n+1},\dots,x_{2n},w\|_{\mathfrak{p}}^{n-1}, & \text{if } 0=N\le M.\end{cases}$$

Consequently, the $\mathfrak{p}$-adic integral given in (3.2.23) in this case is

$$\int_{(w,\underline{x})\in\mathfrak{p}\times W_{2n}^{\mathfrak{o}}} |w|_{\mathfrak{p}}^{(2n-1)s_1+s_2-n^2-2} \prod_{k=1}^{2n-1} \frac{\|F_k(A_{\mathcal{G}_n}(\underline{x}))\cup wF_{k-1}(A_{\mathcal{G}_n}(\underline{x}))\|_{\mathfrak{p}}^{-1-s_1}}{\|F_{k-1}(A_{\mathcal{G}_n}(\underline{x}))\|_{\mathfrak{p}}^{-1-s_1}} d\mu$$
$$= 2\int_{(w,x_1,\ldots,x_{2n})\in\mathfrak{p}\times\mathfrak{p}^n\times W_n^{\mathfrak{o}}} |w|_{\mathfrak{p}}^{(2n-1)s_1+s_2-n^2-2}\|x_1,\ldots,x_n,w\|_{\mathfrak{p}}^{-(n-1)(1+s_1)}d\mu$$
$$+\int_{(w,x_1,\ldots,x_{2n})\in\mathfrak{p}\times W_n^{\mathfrak{o}}\times W_n^{\mathfrak{o}}} |w|_{\mathfrak{p}}^{(2n-1)s_1+s_2-n^2-2}d\mu$$
$$= \left(1-q^{-n}+2q^{-1+(n-1)s_1}-q^{n^2-ns_1-s_2}-q^{n^2-n-ns_1-s_2}\right)\cdot$$
$$\frac{(1-q^{-1})(1-q^{-n})q^{n^2+1-(2n-1)s_1-s_2}}{(1-q^{n^2+1-(2n-1)s_1-s_2})(1-q^{n^2-ns_1-s_2})},$$

where the first and the second integrals of the second equality are calculated in Lemmata 2.2.2 and 2.2.1, respectively.

It follows from Proposition 3.2.12 that

$$\mathcal{Z}^{\mathrm{cc}}_{G_n(\mathfrak{o})}(s_1,s_2) = \frac{1}{1-q^{n^2-s_2}}\left(1+\mathscr{Z}_{\mathfrak{o},A_{\mathcal{G}_n}}\left(-s_1-1,(2n-1)s_1+s_2-n^2-2\right)\right) =$$
$$\frac{(1-q^{2\binom{n}{2}}T_1^nT_2)(1-q^{2\binom{n}{2}+1}T_1^{2n-1}T_2)+q^{n^2}T_1^nT_2(1-q^{-n})(1-q^{-(n-1)}T_1^{n-1})}{(1-q^{n^2}T_2)(1-q^{n^2}T_1^nT_2)(1-q^{n^2+1}T_1^{2n-1}T_2)},$$

where $T_1=q^{-s_1}$ and $T_2=q^{-s_2}$, proving Theorem 2 for groups of type G.

5.1.4 Conjugacy class zeta functions of groups of type H

In this section, we denote by $A(\underline{X})_{ij}$ the $(i,j)th$ coordinate of the commutator matrix $A_{\mathcal{H}_n}(\underline{X})$.

By (5.1.2), each column of $A_{\mathcal{H}_n}(\underline{X})$ is of one of the following forms:

$$\begin{matrix} s\text{th row}\{ \\ \\ (n+s)\text{th row}\{ \end{matrix}\begin{bmatrix} X_{n+s} \\ \\ -X_s \end{bmatrix}, \quad (5.1.10) \qquad \begin{matrix} s\text{th row}\{ \\ r\text{th row}\{ \\ (n+s)\text{th row}\{ \\ (n+r)\text{th row}\{ \end{matrix}\begin{bmatrix} X_{n+r} \\ X_{n+s} \\ -X_r \\ -X_s \end{bmatrix}, \quad (5.1.11)$$

where the nondisplayed entries equal zero. These columns have the following symmetry:

$$A(\underline{X})_{(n+i)k} = \begin{cases} -X_j, & \text{if and only if } A(\underline{X})_{ik}=X_{n+j}, \\ 0, & \text{if and only if } A(\underline{X})_{ik}=0. \end{cases} \tag{5.1.12}$$

For each $s\in[n]$, there is exactly one column of the form (5.1.10), and the columns (5.1.11) occur exactly once for each pair $s<r$ of elements of $[n]$.

Lemma 5.1.9. *For $w\in\mathfrak{p}$, $\mathbf{x}\in W_{2n}^{\mathfrak{o}}$ and $k\in[n]$,*

$$\frac{\|F_k(A_{\mathcal{H}_n}(\mathbf{x}))\cup wF_{k-1}(A_{\mathcal{H}_n}(\mathbf{x}))\|_{\mathfrak{p}}}{\|F_{k-1}(A_{\mathcal{H}_n}(\mathbf{x}))\|_{\mathfrak{p}}}=1.$$

Proof. Fix $m \in [n]$. For each $q \in [m-1]$, denote by $\mathcal{C}_q$ the index of the unique column of $A_{\mathcal{H}_n}(\underline{X})$ which has X_{n+m} in the qth row. Recall that $A^{(m)}_{\mathcal{H}_n}(\underline{X})$ is the submatrix of $A_{\mathcal{H}_n}(\underline{X})$ given in (5.1.2). The submatrix $U_m(\underline{X})$ of $A_{\mathcal{H}_n}(\underline{X})$ composed of columns $\mathcal{C}_1, \ldots, \mathcal{C}_{m-1}$ and the columns of $A^{(m)}_{\mathcal{H}_n}(\underline{X})$ and rows $1, \ldots, n$ is

$$U_m(\underline{X}) = \left[\begin{array}{cccc|cccc} \overbrace{X_{n+m}}^{\mathcal{C}_1} & \overbrace{}^{\mathcal{C}_2} & & \overbrace{}^{\mathcal{C}_{m-1}} & \multicolumn{4}{c}{\overbrace{}^{A^{(m)}_{\mathcal{H}_n}(\underline{X})}} \\ & X_{n+m} & & & & & & \\ & & \ddots & & & & & \\ & & & X_{n+m} & & & & \\ X_{n+1} & X_{n+2} & \ldots & X_{n+m-1} & X_{n+m} & X_{n+m+1} & \ldots & X_{2n} \\ & & & & & X_{n+m} & & \\ & & & & & & \ddots & \\ & & & & & & & X_{n+m} \end{array}\right].$$

Symmetry (5.1.12) implies that the submatrix $L_m(\underline{X})$ of $A_{\mathcal{H}_n}(\underline{X})$ composed of columns $\mathcal{C}_1, \ldots, \mathcal{C}_{m-1}$ and the columns of $A^{(m)}_{\mathcal{H}_n}(\underline{X})$ and rows $n+1, \ldots, 2n$ is

$$L_m(\underline{X}) = \left[\begin{array}{cccc|cccc} \overbrace{X_{n+m}}^{\mathcal{C}_1} & \overbrace{}^{\mathcal{C}_2} & & \overbrace{}^{\mathcal{C}_{m-1}} & \multicolumn{4}{c}{\overbrace{}^{A^{(m)}_{\mathcal{H}_n}(\underline{X})}} \\ & -X_m & & & & & & \\ & & \ddots & & & & & \\ & & & -X_m & & & & \\ -X_1 & -X_2 & \ldots & -X_{m-1} & -X_m & -X_{m+1} & \ldots & -X_n \\ & & & & & -X_m & & \\ & & & & & & \ddots & \\ & & & & & & & -X_m \end{array}\right].$$

If $\mathbf{x} = (x_1, \ldots, x_{2n})$ is such that $(x_{n+1}, \ldots, x_{2n}) \in W^{\mathfrak{o}}_n$, then there exists $m_1 \in [n]$ such that the matrix $U_{m_1}(\mathbf{x})$ has maximal rank n, that is, for each $k \in [n]$, at least one of the $(k \times k)$-minors of $U_{m_1}(\mathbf{x})$ is a unit. Analogously, if $(x_1, \ldots, x_n) \in W^{\mathfrak{o}}_n$, then there exists $m_2 \in [n]$ such that the matrix $L_{m_2}(\mathbf{x})$ has maximal rank n. Since the $(k \times k)$-minors of $U_{m_1}(\mathbf{x})$ and of $L_{m_2}(\mathbf{x})$ are elements of $F_k(A_{\mathcal{H}_n}(\mathbf{x}))$, the result follows. □

In the next lemma, we show that the sets $F_{n+l}(A_{\mathcal{H}_n}(\underline{X}))$, for $l \in [n-1]$, are given in terms of linear combinations of products of (i,j)-minors $M_{ij}(\underline{X}) := X_i X_{n+j} - X_j X_{n+1}$ of the following matrix

$$M(X_1, \ldots, X_{2n}) = \begin{bmatrix} X_1 & X_2 & \ldots & X_n \\ X_{n+1} & X_{n+2} & \ldots & X_{2n} \end{bmatrix} \in \mathrm{Mat}_{2\times n}(\mathfrak{o}[X_1, \ldots, X_{2n}]).$$

Lemma 5.1.10. *Let $k = n+l$, for some $l \in [n-1]$. Then the nonzero elements of $F_k(A_{\mathcal{H}_n}(\underline{X}))$ are sums of terms of the form*

$$X_{f_1} \ldots X_{f_r} M_{i_1 j_1}(\underline{X}) \ldots M_{i_s j_s}(\underline{X}),$$

for $i_1, \ldots, i_s, j_1, \ldots, j_s \in [n]$, and $f_1, \ldots, f_r \in [2n]$, where $r + 2s = k$ and $s \geq l$.

Proof. Lemma 5.1.5 describes each element G of $F_k(A_{\mathcal{H}_n}(\underline{X}))$ in terms of sums of products of minors of the form $\widetilde{G}_{(m_1,n_1),(m_2,n_2)}$. It then suffices to show that these minors are all either 0 or of the forms X_uX_v, $-X_uX_v$ or $M_{ij}(\underline{X})$, for some $u, v \in [2n]$ and $1 \le i < j \le n$.

Since $k = n+l$, there are at least l pairs of rows of G whose indices in $A_{\mathcal{H}_n}(\underline{X})$ are of the form t and $n+t$, for some $t \in [n]$. Denote by λ the exact number of such pairs of rows occurring in G, and assume that, for $m \in \{1, 3, \dots, 2\lambda - 1\}$, the mth and the $(m+1)$th rows of G correspond, respectively, to rows of indices of the form t and $n+t$ in $A_{\mathcal{H}_n}(\underline{X})$, for some $t \in [n]$. In this case,

$$A(\underline{X})_{ij} = 0 \text{ if and only if } A(\underline{X})_{(i+1)j} = 0,$$

for all $i \in \{1, 3, \dots, 2\lambda - 1\}$ and $j \in \left[\binom{n+1}{2}\right]$, because of (5.1.12). Therefore, for $k_1, k_2 \in \left[\binom{n+1}{2}\right]$ distinct and $m \in \{1, 3, \dots, 2\lambda - 1\}$, the minor $\widetilde{G}_{(m,k_1),(m+1,k_2)}$ is either 0 or $M_{ij}(\underline{X})$, for some $1 \le i < j \le n$, as the columns of this minor are either of the form $(0,0)^{\mathrm{T}}$ or $(X_{n+i}, -X_i)^{\mathrm{T}}$, for some $i \in [n]$.

For $i, j \in [n]$ distinct, there is at most one column of $A_{\mathcal{G}_n}(\underline{X})$ whose nonzero rows are the ones of indices in $\{i, j, n+i, n+j\}$, it follows that each of the remaining minors of G are either equal to 0 or of one of the forms X_iX_j or $-X_iX_j$, for some distinct $i, j \in [2n]$. □

Let $\mathbf{x} = (x_1, \dots, x_{2n}) \in W_{2n}^{\mathfrak{o}}$ with $v_{\mathfrak{p}}(x_{f_0}) = 0$, say. Then

$$v_{\mathfrak{p}}(x_{f_0}^r M_{i_1j_1}(\mathbf{x}) \cdots M_{i_sj_s}(\mathbf{x})) \le v_{\mathfrak{p}}(x_{f_1} \cdots x_{f_{r'}} M_{i_1j_1}(\mathbf{x}) \cdots M_{i_sj_s}(\mathbf{x})), \quad (5.1.13)$$

for all $r, r' \in \mathbb{N}$, $f_1, \dots, f_{r'} \in [2n]$ and $i_1, \dots, i_s, j_1, \dots, j_s \in [n]$.

Set $\mathbf{M}(\mathbf{x}) = \{M_{ij}(\mathbf{x}) \mid 1 \le i < j \le n\}$. If $\|\mathbf{M}(\mathbf{x})\|_{\mathfrak{p}} = \|M_{i_0j_0}(\mathbf{x})\|_{\mathfrak{p}}$, for some i_0, j_0, then

$$\begin{aligned}&\|\{M_{i_1j_1}(\mathbf{x}) \cdots M_{i_lj_l}(\mathbf{x}) \mid 1 \le i_m < j_m \le n,\ m \in [k]\}\|_{\mathfrak{p}} \\ &= \|M_{i_0j_0}(\mathbf{x})\|_{\mathfrak{p}}^l = \|\mathbf{M}(\mathbf{x})\|_{\mathfrak{p}}^l. \end{aligned} \quad (5.1.14)$$

Expressions (5.1.13) and (5.1.14) then assure that, for $m \in [n-1]_0$ and for $i_1, \dots, i_l, j_1, \dots, j_l \in [n]$, and $f_1, \dots, f_m \in [2n]$,

$$v_{\mathfrak{p}}(x_{f_0}^m M_{i_0j_0}(\mathbf{x})^l) \le v_{\mathfrak{p}}(x_{f_1} \dots x_{f_r} M_{i_1j_1}(\mathbf{x}) \cdots M_{i_sj_s}(\mathbf{x})),$$

Lemma 5.1.10 states that the $k \times k$-minors of $A_{\mathcal{H}_n}(\underline{X})$ are of the form

$$X_{f_1} \cdots X_{f_r} M_{i_1j_1}(\underline{X}) \cdots M_{i_sj_s}(\underline{X}),$$

or sums of such terms, where $r + 2s = k$ and $s \ge l$. The maximal value for r occurs when $s = l$.

We now show that, for all $k = n+l$ with $l \in [n-1]$, all terms of the form $X_f^m M_{ij}(\underline{X})^l$ are elements of $F_k(A_{\mathcal{H}_n}(\underline{X}))$, for $k = m + 2l$. This implies in particular that, for $\mathbf{x} \in W_{2n}^{\mathfrak{o}}$ as above, the term $x_{f_0}^m M_{i_0j_0}(\mathbf{x})^l$ is an element of $F_k(A_{\mathcal{H}_n}(\mathbf{x}))$ and, therefore

$$\|F_k(A_{\mathcal{H}_n}(\mathbf{x}))\|_{\mathfrak{p}} = \|x_{f_0}^m M_{i_0j_0}(\mathbf{x})^l\|_{\mathfrak{p}} = \|M_{i_0j_0}(\mathbf{x})\|_{\mathfrak{p}}^l = \|\mathbf{M}(\mathbf{x})\|_{\mathfrak{p}}^l.$$

Assuming this holds, the integrand of (3.2.23) can be simplified as follows.

$$\begin{aligned}&\frac{\|F_{n+l}(A_{\mathcal{H}_n}(\mathbf{x})) \cup wF_{n+l-1}(A_{\mathcal{H}_n}(\mathbf{x}))\|_{\mathfrak{p}}}{\|F_{n+l-1}(A_{\mathcal{H}_n}(\mathbf{x}))\|_{\mathfrak{p}}} \\ &= \frac{\|\{M_{ij}(\mathbf{x})^l \mid 1 \le i < j \le n\} \cup w\{M_{ij}(\mathbf{x})^{l-1} \mid 1 \le i < j \le n\}\|_{\mathfrak{p}}}{\|\{M_{ij}(\mathbf{x})^{l-1} \mid 1 \le i < j \le n\}\|_{\mathfrak{p}}} \\ &= \|\{M_{ij}(\mathbf{x}) \mid 1 \le i < j \le n\} \cup \{w\}\|_{\mathfrak{p}} = \|\mathbf{M}(\mathbf{x}) \cup \{w\}\|_{\mathfrak{p}}. \end{aligned} \quad (5.1.15)$$

Proposition 5.1.11. *Given $l \in [n-1]$, let $k = n+l$ and $m = n-l$. Then, for all $\overline{f} \in [2n]$ and $1 \le i < j \le n$, either $X_{\overline{f}}^m M_{ij}(\underline{X})^l$ or $-X_{\overline{f}}^m M_{ij}(\underline{X})^l$ is an element of $F_k(A_{\mathcal{H}_n}(\underline{X}))$.*

Proof. Let $f \in [n]$ and $1 \leq i < j \leq n$. We show that, up to sign, both $X_f^m M_{ij}(\underline{X})^l$ and $X_{n+f}^m M_{ij}(\underline{X})^l$ lie in $F_k(A_{\mathcal{H}_n}(\underline{X}))$.

First, we show that $X_{n+f}^m M_{ij}(\underline{X})^l \in F_k(A_{\mathcal{H}_n}(\underline{X}))$. We consider the cases $m \geq 3$, $m = 2$ and $m = 1$ separately. In most cases, we do the following: we choose specific indices $r_1, \ldots, r_m, \mathcal{R}_1, \ldots, \mathcal{R}_l$ of rows of $A_{\mathcal{H}_n}(\underline{X})$, and then set c_s, $\mathcal{C}_q^i$, and $\mathcal{C}_q^j$ to be indices of columns of $A_{\mathcal{H}_n}(\underline{X})$ as in the following table.

Index	Unique column of $A_{\mathcal{H}_n}(\underline{X})$ satisfying
c_s	r_sth entry is X_{n+f}
$\mathcal{C}_q^i$	$\mathcal{R}_q$th entry is X_{n+i}
$\mathcal{C}_q^j$	$\mathcal{R}_q$th entry is X_{n+j}

Table 5.3: Indices of columns—proof of Proposition 5.1.11

The choices of r_s and $\mathcal{R}_q$ are made such that the submatrix $\tilde{A}(\underline{X})$ of $A_{\mathcal{H}_n}(\underline{X})$ obtained by its rows of indices $r_1, \ldots, r_m$, $\mathcal{R}_1$, $n + \mathcal{R}_1$, $\ldots$, $\mathcal{R}_l$, $n + \mathcal{R}_l$ and columns $c_1, \ldots, c_m, \mathcal{C}_1^i, \mathcal{C}_1^j, \ldots, \mathcal{C}_l^i, \mathcal{C}_l^j$, in this order, is of the form

$$\left[\begin{array}{cccc|cccccc} X_{n+f} & X_{n+r_2} & \cdots & X_{n+r_m} & & & & & & \\ & X_{n+f} & & & & & * & & & \\ & & \ddots & & & & & & & \\ & & & X_{n+f} & & & & & & \\ \hline & & & & X_{n+i} & X_{n+j} & & & & \\ & & & & -X_i & -X_j & & & & \\ & & 0 & & & & \ddots & & & \\ & & & & & & & & X_{n+i} & X_{n+j} \\ & & & & & & & & -X_i & -X_j \end{array}\right], \quad (5.1.16)$$

which has determinant $X_{n+f}^m M_{ij}(\underline{X})^l$.

Case 1. Assume that $m \geq 3$. First, we consider $f \notin \{i, j\}$. Set $r_1 = f$, $r_2 = i$, $r_3 = j$. Inductively, fix $r_s \in [n] \setminus \{r_1, \ldots, r_{s-1}\}$, for each $s \in \{4, \ldots, m\}$. Fix also $\mathcal{R}_1 \in [n] \setminus \{r_1, \ldots, r_m\}$ and, inductively, $\mathcal{R}_q \in [n] \setminus \{r_1, \ldots, r_m, \mathcal{R}_1, \ldots, \mathcal{R}_{q-1}\}$.

The submatrix $\tilde{A}(\underline{X})$ of $A_{\mathcal{H}_n}(\underline{X})$ described above is of the form (5.1.16).

In fact, column c_1 is of the form (5.1.10) and, for $s \in \{2, \ldots, m\}$, c_s is of the form (5.1.11), so that the only nonzero entries of c_s in $A_{\mathcal{H}_n}(\underline{X})$ are the ones of index f, r_s, $n + f$, and $n + r_s$. Since $r_1 = f$ and $r_s \notin \{r_1, \ldots, r_{s-1}\}$, it follows that the nonzero entries of this column which appear in the submatrix $\tilde{A}(\underline{X})$ are X_{n+r_s} in the row of index $r_1 = f$, and X_{n+f} in the row of index r_s.

Given $q \in [l]$, the only nonzero entries of $\mathcal{C}_q^i$ in $A_{\mathcal{H}_n}(\underline{X})$ are the ones of rows whose index are elements of $\{i, \mathcal{R}_q, n + i, n + \mathcal{R}_q\}$. Since $\mathcal{R}_q \neq i$, it follows that the row $n + i$ is not one of the rows of index $n + \mathcal{R}_t^i$, $t \in [l]$, that is, the only nonzero rows of the form $\mathcal{R}_t^i$ or of the form $n + \mathcal{R}_t^i$ in $\mathcal{C}_q^i$ which appear in $\tilde{A}(\underline{X})$ are the ones with $t = q$. The same argument shows that, the only nonzero entries of $\mathcal{C}_q^j$ of the form $\mathcal{R}_t$ or $n + \mathcal{R}_t$ in $\tilde{A}(\underline{X})$ are the ones with $t = q$.

If $f \in \{i, j\}$, fix $r_1 = f$, $r_2 \in \{i, j\} \setminus \{f\}$, and set inductively $r_s \in [n] \setminus \{r_1, \ldots, r_{s-1}\}$, for each $s \in \{3, \ldots, m\}$. The indices $\mathcal{R}_t$ are chosen as in the former case. The matrix $\tilde{A}(\underline{X})$ is in this case of the form (5.1.16), by similar arguments as the ones for the former case.

Case 2. Assume that $m = 2$, that is, we want to find a minor of the form $X_{n+f}^2 M_{ij}(\underline{X})$. If $f \notin \{i,j\}$, set $r_1 = f$, $r_2 = i$, and $\mathcal{R}_1 = j$. Then fix, inductively, $\mathcal{R}_q \in [n] \setminus \{r_1, r_2, \mathcal{R}_1, \dots, \mathcal{R}_{q-1}\}$.

If $f \in \{i,j\}$, we set $r_1 = f$, $r_2 \in \{i,j\} \setminus \{f\}$ and $\mathcal{R}_q$, for $q \in [l]$, as in the former cases.

These choices give matrices $\tilde{A}(\underline{X})$ of the form (5.1.16).

Case 3. Assume that $m = 1$. If $f \in \{i,j\}$, set $r_1 = f$, $\mathcal{R}_1 \in \{i,j\} \setminus \{f\}$, $\mathcal{R}_2 \in [n] \setminus \{r_1, \mathcal{R}_1\}$, and, inductively, $\mathcal{R}_t \in [n] \setminus \{r_1, \mathcal{R}_1, \dots, \mathcal{R}_{t-1}\}$. The obtained matrix $\tilde{A}(\underline{X})$ is of the desired form.

For $m = 1$ and $f \notin \{i,j\}$, we need a slightly different construction: we set $r_1 = f$, but, in this case, we consider c_1^i and c_1^j, which are the indices of the columns of $A_{\mathcal{H}_n}(\underline{X})$ containing, respectively, X_{n+i} and X_{n+j} in the r_1th row. Then set $\mathcal{R}_1 = i$ and $\mathcal{R}_2 = j$ and, inductively, $\mathcal{R}_q \in [n] \setminus \{r_1, \mathcal{R}_1, \dots, \mathcal{R}_{q-1}\}$, for all $q \in \{3, \dots, l\}$. Denote by $\mathcal{C}_q^i$ and by $\mathcal{C}_q^j$ the index of the columns of $A_{\mathcal{H}_n}(\underline{X})$ containing, respectively, X_{n+i} and X_{n+j} in the $\mathcal{R}_q$th row. There are only $2l-1$ indices $\mathcal{C}_q^j$ and $\mathcal{C}_q^j$ in total, since $\mathcal{C}_1^j = \mathcal{C}_2^i$.

Similar arguments as the ones of the former cases show that the matrix composed of rows r_1, $\mathcal{R}_1$, $n+\mathcal{R}_1$, $\dots$, $\mathcal{R}_l$, $n+\mathcal{R}_l$ and columns c_1^i, c_1^j, $\mathcal{C}_1^i$, $\mathcal{C}_1^j$, $\mathcal{C}_2^j$, $\dots$, $\mathcal{C}_l^i$, $\mathcal{C}_l^j$, in this order, is

$$\left[\begin{array}{cc|cc|c|cc|c|cc}
X_{n+i} & X_{n+j} & 0 & 0 & 0 & 0 & 0 & & 0 & 0 \\ \hline
X_{n+f} & 0 & X_{n+i} & X_{n+j} & 0 & X_{n+\mathcal{R}_3} & 0 & & X_{n+\mathcal{R}_l} & 0 \\
-X_f & 0 & -X_i & -X_j & 0 & -X_{\mathcal{R}_3} & 0 & & -X_{\mathcal{R}_l} & 0 \\ \hline
0 & X_{n+f} & 0 & X_{n+i} & X_{n+j} & 0 & X_{n+\mathcal{R}_3} & & 0 & X_{n+\mathcal{R}_l} \\
0 & -X_f & 0 & -X_i & -X_j & 0 & -X_{\mathcal{R}_3} & & 0 & -X_{\mathcal{R}_l} \\ \hline
0 & 0 & 0 & 0 & 0 & X_{n+i} & X_{n+j} & & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -X_i & -X_j & & 0 & 0 \\ \hline
 & & & & & & & \ddots & & \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & & X_{n+i} & X_{n+j} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & -X_i & -X_j
\end{array}\right].$$

The determinant of such matrix is

$$M_{ij}(\underline{X})^{l-2} \det \left(\left[\begin{array}{cc|cc|c}
X_{n+i} & X_{n+j} & 0 & 0 & 0 \\ \hline
X_{n+f} & 0 & X_{n+i} & X_{n+j} & 0 \\
-X_f & 0 & -X_i & -X_j & 0 \\ \hline
0 & X_{n+f} & 0 & X_{n+i} & X_{n+j} \\
0 & -X_f & 0 & -X_i & -X_j
\end{array}\right] \right) = X_{n+f} M_{ij}(\underline{X})^l.$$

The minors of the form $X_f^m M_{ij}(\underline{X})^l$ (up to sign) are obtained by repeating the constructions above for each case but considering rows $n + r_s$ instead of r_s, for all $s \in [m]$. The determinants of the matrices obtained in this way are of the desired form because of the symmetry of the columns of $A_{\mathcal{H}_n}(\underline{X})$ given by (5.1.12). □

For each $\mathbf{x} \in W_{2n}^{\mathfrak{o}}$, combining (5.1.15) with Lemma 5.1.9, we obtain

$$\prod_{k=1}^{2n-1} \frac{\|F_k(A_{\mathcal{H}_n}(\mathbf{x})) \cup w F_{k-1}(A_{\mathcal{H}_n}(\mathbf{x}))\|_{\mathfrak{p}}}{\|F_{k-1}(A_{\mathcal{H}_n}(\mathbf{x}))\|_{\mathfrak{p}}} = \|\mathbf{M}(\mathbf{x}) \cup \{w\}\|_{\mathfrak{p}}^{n-1}.$$

Thus, for groups of the form $H_n(\mathfrak{o})$, the $\mathfrak{p}$-adic integral appearing in (3.2.23) is

$$\mathcal{J}_{H_n}(s_1,s_2) := \int_{(w,\underline{x})\in\mathfrak{p}\times W_{2n}^{\mathfrak{o}}} |w|_{\mathfrak{p}}^{(2n-1)s_1+s_2-\binom{n+1}{2}-2} \|\{\mathbf{M}(\underline{x})\cup\{w\}\|_{\mathfrak{p}}^{-(n-1)(1+s_1)} d\mu,$$

which is a specialisation of the integral given in Lemma 2.2.3. Combining Lemma 2.2.3 with Proposition 3.2.12 yields

$$\begin{aligned}\mathcal{Z}^{\mathrm{cc}}_{H_n(\mathfrak{o})}(s_1,s_2) &= \frac{1}{1-q^{\binom{n+1}{2}-s_2}}\left(1+(1-q^{-1})^{-1}\mathcal{J}_{H_n}(s_1,s_2)\right)\\ &= ZF_{H_n}(q,q^{-s_1},q^{-s_2}),\end{aligned}$$

where $ZF_{H_n}(q,T_1,T_2)$ is given by

$$\frac{(1-q^{\binom{n}{2}}T_1^nT_2)(1-q^{\binom{n}{2}+2}T_1^{2n-1}T_2)+q^{\binom{n+1}{2}}T_1^nT_2(1-q^{-n+1})(1-q^{-(n-1)}T_1^{n-1})}{(1-q^{\binom{n+1}{2}}T_2)(1-q^{\binom{n+1}{2}+1}T_1^nT_2)(1-q^{\binom{n+1}{2}+1}T_1^{2n-1}T_2)}.$$

This proves Theorem 2 for groups of type H.

5.2 Bivariate representation zeta functions—proof of Theorem 3

Recall that $\mathfrak{g} := \Lambda(\mathfrak{o})$, and the constants a, b, r and z associated to it given in Table 5.1. Consider the B-commutator matrix $B_\Lambda(\underline{Y}) = B(\underline{Y})$ of $\mathfrak{g}$ of Definition 3.2.1 with respect to the $\mathbf{e}$ and $\mathbf{f}$ given in Section 5.1.1.

Recall the numbers $\mathcal{N}^{\mathfrak{o}}_{N,B,\mathbf{m}} = |\{\mathbf{y}\in W^{\mathfrak{o}}_{b,N} \mid \nu(B(\mathbf{y}))=\mathbf{m}\}|$ of Section 3.2.1. Write $\mathbf{m}=(m_1,\dots,m_{u_B})$. Recall from Section 3.2.3 that

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}(s_1,s_2) = \tag{3.2.20}$$
$$(1-q^{r-s_2})\left(1+\sum_{N=1}^{\infty}\sum_{\mathbf{m}\in\mathbb{N}_0^{u_B}}\mathcal{N}^{\mathfrak{o}}_{N,B,\mathbf{m}}q^{-N(u_Bs_1+s_2+2u_B-r)-2\sum_{j=1}^{u_B}m_j\frac{(-s_1-2)}{2}}\right).$$

Given a set $I=\{i_1,\dots,i_l\}_<\subseteq[n-1]_0$, recall that $\mu_j := i_{j+1}-i_j$ for all $j\in[l]_0$, where $i_0=0$, $i_{l+1}=n$. Choose $r_I=(r_i)_{i\in I}\in\mathbb{N}^I$ and let $N=\sum_{i\in I}r_i$.

Following [48, Section 3], define the following sets, which form a partition of $W^{\mathfrak{o}}_{b,N}$:

$$\begin{aligned}\mathrm{N}_{I,r_I}(\mathbf{G}) =\{\mathbf{y}\in W^{\mathfrak{o}}_{b,N} : \nu(B(\mathbf{y})) =(&\underbrace{0,\dots,0}_{\mu_l\text{ terms}},\underbrace{r_{i_l},\dots,r_{i_l}}_{\mu_{l-1}\text{ terms}},\underbrace{r_{i_l}+r_{i_{l-1}},\dots,r_{i_l}+r_{i_{l-1}}}_{\mu_{l-2}\text{ terms}},\dots,\underbrace{N,\dots,N}_{\mu_0\text{ terms}})\}.\end{aligned}$$

For $\Lambda\in\{\mathcal{F}_{n,\delta},\mathcal{G}_n,\mathcal{H}_n\}$, it holds that $\mathfrak{z}=\mathfrak{g}'$, so that

$$r := \mathrm{rk}(\mathfrak{g}/\mathfrak{g}') = a := \mathrm{rk}(\mathfrak{g}/\mathfrak{z}) = \begin{cases}2n+\delta, & \text{if } \mathbf{G}=F_{n,\delta},\\ 2n, & \text{if } \mathbf{G}\in\{G_n,H_n\}.\end{cases}$$

For simplicity, consider $\delta=0$ when $\mathbf{G}\in\{G_n,H_n\}$, so that we can write $a = 2n+\delta$ uniformly.

Using these facts and the equality $\bar{a}(\mathbf{G},n)=2n+\delta$, as in Table 1.1, we rewrite $\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}$ as follows.

$$\begin{aligned}(1-q^{\bar{a}(\mathbf{G},n)-s_2})&\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}(s_1,s_2)\\ &=\sum_{I\subseteq[n-1]_0}\sum_{r_I\in\mathbb{N}^I}|\mathrm{N}_{I,r_I}(\mathbf{G})|q^{-\sum_{i\in I}r_i(ns_1+s_2+2n-r)-\sum_{i\in I}ir_i(-2-s_1)}\\ &=\sum_{I\subseteq[n-1]_0}\sum_{r_I\in\mathbb{N}^I}|\mathrm{N}_{I,r_I}(\mathbf{G})|q^{\sum_{i\in I}r_i(-(n-i)s_1-s_2+2i+\delta)}.\end{aligned}\tag{5.2.1}$$

The cardinalities $|\mathrm{N}_{I,r_I}(\mathbf{G})|$ are described in [48, Proposition 3.4] in terms of the polynomials $f_{\mathbf{G},I}$ and the numbers $\bar{a}(\mathbf{G},i)$ defined in Table 1.1 as follows.

$$|\mathrm{N}_{I,r_I}(\mathbf{G})|=f_{\mathbf{G},I}(q^{-1})q^{\sum_{i\in I}r_i(\bar{a}(\mathbf{G},i)-2i-\delta)}.\tag{5.2.2}$$

Combining (5.2.1) with (5.2.2) yields

$$\begin{aligned}\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}(s_1,s_2)&=\frac{1}{1-q^{\bar{a}(\mathbf{G},n)-s_2}}\sum_{I\subseteq[n-1]_0}\sum_{r_I\in\mathbb{N}^I}f_{\mathbf{G},I}(q^{-1})q^{\sum_{i\in I}r_i(\bar{a}(\mathbf{G},i)-(n-i)s_1-s_2)}\\ &=\frac{1}{1-q^{\bar{a}(\mathbf{G},n)-s_2}}\sum_{I\subseteq[n-1]_0}f_{\mathbf{G},I}(q^{-1})\prod_{i\in I}\frac{q^{\bar{a}(\mathbf{G},i)-(n-i)s_1-s_2}}{1-q^{\bar{a}(\mathbf{G},i)-(n-i)s_1-s_2}}.\end{aligned}$$

This concludes the proof of Theorem 3.

5.3 Hyperoctahedral groups and functional equations

In this section, we relate the formulae of Theorem 3 to statistics on Weyl groups of type B, also called hyperoctahedral groups B_n. Specialisation (1.1.8) then provides formulae for the class number zeta functions of groups of type F, G, and H in terms of such statistics. By comparing these formulae to the ones of Corollary 1.2.2, we obtain formulae for joint distributions of three functions on such Weyl groups.

We also use the descriptions of the bivariate representation zeta functions in terms of Weyl group statistics in Section 5.3.3 in order to prove Theorem 4.

Some required notation regarding hyperoctahedral groups is given in Section 5.3.1.

5.3.1 Hyperoctahedral groups B_n

We briefly recall the definition of the hyperoctahedral groups B_n and some statistics associated to them. For further details about Coxeter groups and hyperoctahedral groups we refer the reader to [6].

The Weyl groups of type B are the groups B_n, for $n\in\mathbb{N}$, of all bijections $w:[\pm n]\to[\pm n]$ with $w(-a)=-w(a)$, for all $a\in[\pm n]$, with operation given by composition. Given an element $w\in B_n$ write $w=[a_1,\dots,a_n]$ to denote $w(i)=a_i$.

Definition 5.3.1. *For $w\in B_n$, the* inversion number*, the* number of negative entries *and the* number of negative sum pairs *of w are defined, respectively, by*

$$\begin{aligned}\mathrm{inv}(w) &= |\{(i,j) \in [n]^2 \mid i < j, w(i) > w(j)\}|,\\ \mathrm{neg}(w) &= |\{i \in [n] \mid w(i) < 0\}|,\\ \mathrm{nsp}(w) &= |\{(i,j) \in [n]^2 : i \neq j, w(i) + w(j) < 0\}|.\end{aligned}$$

Let $s_i = [1, \dots, i-1, i+1, i, \dots, n]$ for $i \in [n-1]$ and $s_0 = [-1, 2, \dots, n]$ be elements of B_n. Then (B_n, S_B) is a Coxeter system, where $S_B = \{s_i\}_{i \in [n-1]_0}$.

In [6, Proposition 8.1.1] it is shown that the *Coxeter length* on B_n with respect to the generating set S_B is given by

$$\ell(w) = \mathrm{inv}(w) + \mathrm{neg}(w) + \mathrm{nsp}(w), \text{ for } w \in B_n.$$

The *right descent* of $w \in B_n$ is the set

$$D(w) = \{s_i \in S_B \mid w(i) > w(i+1)\}.$$

For simplicity, we identify S_B with $[n-1]_0$ in the obvious way, so that $D(w) \subseteq [n-1]_0$. Moreover, for $I \subseteq S_B$, define

$$B_n^I = \{w \in B_n \mid D(w) \subseteq I^{\mathrm{c}} = S_B \setminus I\}.$$

Example 5.3.2. Let $w_0 = [-1, \dots, -n]$ be the longest element of B_n. Then

$$\mathrm{inv}(w_0) = \binom{n}{2}, \quad \mathrm{neg}(w_0) = n, \quad \ell(w_0) = n^2, \quad D(w_0) = S_B. \qquad \triangle$$

Consider $w \in B_n$. The following statistics are used in the present work.

$$L(w) = \frac{1}{2}|\{(i,j) \in [\pm n]_0^2 \mid i < j, w(i) > w(j), i \not\equiv 0 \bmod 2\}|, \tag{5.3.1}$$

$$\begin{aligned}\mathrm{des}(w) &= |D(w)|,\\ \sigma(w) &= \sum_{i \in D(w)} n^2 - i^2,\\ \mathrm{maj}(w) &= \sum_{i \in D(w)} i,\\ \mathrm{rmaj}(w) &= \sum_{i \in D(w)} n - i.\end{aligned}$$

The statistics $\mathrm{des}(w)$, $\mathrm{maj}(w)$, and $\mathrm{rmaj}(w)$ are called the *descent number*, the *major index*, and the *reverse major index* of w, respectively.

5.3.2 Bivariate representation zeta functions and statistics of Weyl groups

The following lemma describes the polynomials $f_{\mathbf{G},I}$ defined in Table 1.1 in terms of statistics on the groups B_n, where $\mathbf{G} \in \{F_{n,\delta}, G_n, H_n\}$.

Lemma 5.3.3. *Let $n \in \mathbb{N}, \delta \in \{0,1\}$ and $I \subseteq [n-1]_0$. Then*

1. *[48, Proposition 4.6]*

$$f_{F_{n,\delta},I}(X) = \sum_{w \in B_n^{I^{\mathrm{c}}}} (-1)^{\mathrm{neg}(w)} X^{(2\ell + (2\delta - 1)\,\mathrm{neg})(w)},$$

$$f_{G_n,I}(X) = \sum_{w \in B_n^{I^{\mathrm{c}}}} (-1)^{\mathrm{neg}(w)} X^{\ell(w)},$$

2. [7, Theorem 5.4]

$$f_{H_n,I}(X) = \sum_{w \in B_n^{I^c}} (-1)^{\ell(w)} X^{L(w)}.$$

Lemma 5.3.4. *Given $n \in \mathbb{N}$, $\delta \in \{0,1\}$, and a prime ideal $\mathfrak{p}$ of $\mathcal{O}$,*

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}(s_1,s_2) = \frac{\sum_{w \in B_n} \chi_{\mathbf{G}}(w) q^{-h_{\mathbf{G}}(w)} \prod_{i \in D(w)} q^{\bar{a}(\mathbf{G},i)-(n-i)s_1-s_2}}{\prod_{i=0}^{n}(1-q^{\bar{a}(\mathbf{G},i)-(n-i)s_1-s_2})},$$

where, for each $w \in B_n$,

$\mathbf{G}$	$\chi_{\mathbf{G}}(w)$	$h_{\mathbf{G}}(w)$
$F_{n,\delta}$	$(-1)^{\mathrm{neg}(w)}$	$2\ell(w) + (2\delta-1)\,\mathrm{neg}(w)$
G_n	$(-1)^{\mathrm{neg}(w)}$	$\ell(w)$
H_n	$(-1)^{\ell(w)}$	$L(w)$

Table 5.4: Statistics associated to $\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}(s_1,s_2)$ for $\mathbf{G} \in \{\mathcal{F}_{n,\delta}, \mathcal{G}_n, \mathcal{H}_n\}$

Proof. Applying Lemma 5.3.3 to the formulae of Theorem 3, one obtains the following expression for $\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathfrak{o})}(s_1,s_2)$:

$$\frac{1}{1-q^{\bar{a}(\mathbf{G},n)-s_2}} \sum_{I \subseteq [n-1]_0} \sum_{w \in B_n^{I^c}} \chi_{\mathbf{G}}(w) q^{-h_{\mathbf{G}}(w)} \prod_{i \in I} \frac{q^{\bar{a}(\mathbf{G},i)-(n-i)s_1-s_2}}{1-q^{\bar{a}(\mathbf{G},i)-(n-i)s_1-s_2}},$$

which can be rewritten as the claimed sum because of [48, Lemma 4.4]. □

Specialisation (1.1.8) applied to Lemma 5.3.4 yields that

$$\begin{aligned}\zeta^{\mathrm{k}}_{\mathbf{G}(\mathfrak{o})}(s_1,s_2) &= \frac{\sum_{w \in B_n} \chi_{\mathbf{G}}(w) q^{-h_{\mathbf{G}}(w)} \prod_{i \in D(w)} q^{\bar{a}(\mathbf{G},i)-s}}{\prod_{i=0}^{n}(1-q^{\bar{a}(\mathbf{G},i)-s})} \\ &= \frac{\sum_{w \in B_n} \chi_{\mathbf{G}}(w) q^{-h_{\mathbf{G}}(w)} q^{\left(\sum_{i \in D(w)} \bar{a}(\mathbf{G},i)\right)-\mathrm{des}(w)s}}{\prod_{i=0}^{n}(1-q^{\bar{a}(\mathbf{G},i)-s})}. \qquad (5.3.2)\end{aligned}$$

Proposition 5.3.5. *For $n \in \mathbb{N}$ and $\delta \in \{0,1\}$, the following holds in $\mathbb{Q}[X,Z]$.*

$$\sum_{w \in B_n} (-1)^{\mathrm{neg}(w)} X^{-(2(\ell-\sigma)+(2\delta-1)\,\mathrm{neg}-(2\delta-3)\,\mathrm{rmaj}-(2n+\delta)\,\mathrm{des})(w)} Z^{\mathrm{des}(w)}$$

$$= \left(1 - X^{\binom{2n+\delta-1}{2}} Z\right) \prod_{i=2}^{n} \left(1 - X^{\binom{2n+\delta}{2}-\binom{2i+\delta}{2}+2i+\delta} Z\right)$$

Proof. On the one hand, since

$$\bar{a}(F_{n,\delta},i) = \binom{2n+\delta}{2} - \binom{2i+\delta}{2} + 2i+\delta = 2(n^2-i^2)+(2\delta-3)(n-i)+2n+\delta,$$

it follows that

$$q^{\left(\sum_{i \in D(w)} \bar{a}(F_{n,\delta},i)\right)-\mathrm{des}(w)s} = q^{(2\sigma+(2\delta-3)\,\mathrm{rmaj}+(2n+\delta-s)\,\mathrm{des})(w)}.$$

Hence

$$\zeta^{\mathrm{k}}_{F_{n,\delta}(\mathfrak{o})}(s) = \frac{\sum_{w \in B_n} (-1)^{\mathrm{neg}(w)} q^{-(2(\ell-\sigma)+(2\delta-1)\,\mathrm{neg}-(2\delta-3)\,\mathrm{rmaj}-(2n+\delta-s)\,\mathrm{des})(w)}}{\prod_{i=0}^{n}(1-q^{\bar{a}(F_{n,\delta},i)-s})}.$$

On the other hand, Corollary 1.2.2 asserts that

$$\zeta^{\mathrm{k}}_{F_{n,\delta}(\mathfrak{o})}(s) = \frac{1-q^{\binom{2n+\delta-1}{2}-s}}{(1-q^{\binom{2n+\delta}{2}+1-s})(1-q^{\binom{2n+\delta}{2}-s})} = \frac{1-q^{\binom{2n+\delta-1}{2}-s}}{\prod_{i=0}^{1}(1-q^{\bar{a}(F_{n,\delta},i)-s})}.$$

Therefore

$$\sum_{w\in B_n}(-1)^{\operatorname{neg}(w)}q^{-(2(\ell-\sigma)+(2\delta-1)\operatorname{neg}-(2\delta-3)\operatorname{rmaj}-(2n+\delta-s)\operatorname{des})(w)}$$

$$=\left(1-q^{\binom{2n+\delta-1}{2}-s}\right)\prod_{i=2}^{n}\left(1-q^{\bar{a}(F_{n,\delta},i)}q^{-s}\right)$$

$$=\left(1-q^{\binom{2n+\delta-1}{2}-s}\right)\prod_{i=2}^{n}\left(1-q^{\binom{2n+\delta}{2}-\binom{2i+\delta}{2}+2i+\delta}q^{-s}\right).$$

The formal identity follows as these formulae hold for all prime powers q and all $s\in\mathbb{C}$ with sufficiently large real part. □

For a geometric interpretation of $\ell-\sigma$, we refer the reader to [50, Section 2]. It can be easily checked that, for $n\geq 2$ and $w\in B_n$,

$$\prod_{i\in D(w)}q^{\bar{a}(G_n,i)-s}=q^{(\sigma+2\operatorname{maj}-s\operatorname{des})(w)},\tag{5.3.3}$$

$$\prod_{i\in D(w)}q^{\bar{a}(H_n,i)-s}=q^{\frac{1}{2}(\sigma-3\operatorname{rmaj})(w)+(2n-s)\operatorname{des}(w)}.\tag{5.3.4}$$

The following proposition follows from (5.3.2), Corollary 1.2.2, equalities (5.3.3) and (5.3.4), and arguments analogous to those given in the proof of Proposition 5.3.5.

Proposition 5.3.6. *For $n\in\mathbb{N}$ and $i\in[n]_0$, set*

$$f_1(n,i)=n^2-i^2+2i,\ \text{ and } f_2(n,i)=\binom{n+1}{2}-\binom{i+1}{2}+2i.$$

Then, the following identities hold in $\mathbb{Q}[X,Z]$.

$$\sum_{w\in B_n}(-1)^{\operatorname{neg}(w)}X^{-(\ell-\sigma-2\operatorname{maj})(w)}Z^{\operatorname{des}(w)}=$$

$$\left((1-X^{2\binom{n}{2}}Z)(1-X^{2\binom{n}{2}+1}Z)+X^{n^2}Z(1-X^{-n})(1-X^{-n+1})\right)\prod_{i=3}^{n}(X^{f_1(n,i)}Z),$$

and

$$\sum_{w\in B_n}(-1)^{\ell(w)}X^{-\frac{1}{2}(2L-\sigma+3\operatorname{rmaj}+4n\operatorname{des})(w)}Z^{\operatorname{des}(w)}=$$

$$\left((1-X^{\binom{n}{2}}Z)(1-X^{\binom{n}{2}+2}Z)+X^{\binom{n+1}{2}}Z(1-X^{-n+1})^2\right)\prod_{i=3}^{n}(X^{f_2(n,i)}Z).$$

Remark 5.3.7. By setting $X=1$ in the equations of Propositions 5.3.5 and 5.3.6, we obtain the equalities

$$\sum_{w\in B_n}(-1)^{\operatorname{neg}(w)}Z^{\operatorname{des}(w)}=\sum_{w\in B_n}(-1)^{\ell(w)}Z^{\operatorname{des}(w)}=(1-Z)^n,$$

which were first proven in [36, Theorem 3.2].

5.3.3 Functional equations—proof of Theorem 4 (representation case)

We recall that the formulae of Proposition 3.2.12 of the local factors of the bivariate representation zeta function of groups of type F, G, and H hold for all nonzero prime ideals $\mathfrak{p}$, since we consider the construction of the unipotent group schemes of class 2 given in [48, Section 2.4]. In particular, the descriptions of the local terms of the bivariate representation zeta functions of groups of type F, G, and H in terms of Weyl statistics given in Lemma 5.3.4 also hold for all nonzero prime ideals. We use Lemma 5.3.4 to show that all local terms of these bivariate zeta functions satisfy functional equations. Recall that, for each $n \in \mathbb{N}$, the longest element of B_n is $w_0 = [-1, -2, \ldots, -n]$.

Theorem 4 follows from the same arguments of the proof of [24, Theorem 2.6] applied to the expressions of Lemma 5.3.4. In fact, although $h_{\mathbf{G}}$ is not one of the statistics $\mathbf{b} \cdot \mathbf{l_L}$ or $\mathbf{b} \cdot \mathbf{l_R}$ defined in [24, Theorem 2.6], it satisfies the equations (2.6) of [24], that is,

$$h_{\mathbf{G}}(ww_0) + h_{\mathbf{G}}(w) = h_{\mathbf{G}}(w_0).$$

In fact, one can easily show that $g \in \{\mathrm{inv}, \mathrm{neg}, \ell\}$ satisfies $g(ww_0) = g(w_0) - g(w)$, for all $w \in B_n$, and the equation $L(ww_0) = L(w_0 w) = L(w_0) - L(w)$ is [47, Corollary 7]. Therefore the conclusion of [24, Theorem 2.6] also holds for the expressions given in Lemma 5.3.4.

Index

List of symbols

Domains

$\mathscr{D}_i$	$\{(s_1, s_2) \in \mathbb{C}^2 \mid \mathrm{Re}(A^*_{1i}s_1 + A^*_{2i}s_2) > 1 - B^*_i\}$, for $i \in W' \cap [z]$, pp. 55
$\mathscr{D}_{i,\delta}$	$\{(s_1, s_2) \in \mathbb{C}^2 \mid \mathrm{Re}(A^*_{1i}s_1 + A^*_{2i}s_2) > 1 - B^*_i - \delta\}$, pp. 54
$\mathscr{D}^*_{\mathbf{G}}$	$\bigcap_{i \in [z] \cap W'} \mathscr{D}_i$, pp. 56
$\mathscr{D}_{\mathscr{R},\delta}$	$\bigcap_{i \in \mathscr{R}} \mathscr{D}_{i,\delta}$, for $\delta > 0$, pp. 58

Fields, groups and rings

K	Number field with ring of integers $\mathcal{O}$, pp. 2
$\mathcal{O}$	Ring of integers of number field K, pp. 2
$\mathfrak{p}$	Prime ideal of ring of integers $\mathcal{O}$, pp. 2
$\mathfrak{p}^m$	mth ideal power $\mathfrak{p} \cdots \mathfrak{p}$, pp. 19
$\mathfrak{p}^{(m)}$	m-fold Cartesian power $\mathfrak{p} \times \cdots \times \mathfrak{p}$, pp. 19
$\mathcal{O}_{\mathfrak{p}} = \mathfrak{o}$	Completion of $\mathcal{O}$ at prime ideal $\mathfrak{p}$, pp. 4
$\mathrm{Spec}(\mathcal{O})$	Set of prime ideals of $\mathcal{O}$, pp. 10
$\mathrm{Frac}(\mathcal{O})$	Field of fractions of $\mathcal{O}$, pp. 46
$\mathrm{Frac}(\mathfrak{o}) = \mathfrak{k}$	Field of fractions of $\mathfrak{o}$, pp. 46
$\mathbb{Q}_p$	Field of p-adic numbers, pp. 22
$\widehat{G}$	Profinite completion of a group G, pp. 2
$\widehat{\mathfrak{a}}$	$\mathrm{Hom}^{\mathrm{cont}}_{\mathbb{Z}}(\mathfrak{a}, \mathbb{C}^\times)$, for a compact abelian group $\mathfrak{a}$, pp. 34
$W^{\mathfrak{o}}_k$	$\{\mathbf{x} \in \mathfrak{o}^k \mid v_{\mathfrak{p}}(\mathbf{x}) = 0\}$, pp. 19
$W^{\mathfrak{o}}_{k,N}$	$\{\mathbf{x} \in (\mathfrak{o}/\mathfrak{p}^N)^k \mid v_{\mathfrak{p}}(\mathbf{x}) = 0\}$, pp. 31

$\mathbf{G}_N$	$\mathbf{G}(\mathfrak{o}/\mathfrak{p}^N)$, pp. 5

Group schemes and Lie lattices

Λ	$\mathcal{O}$-Lie lattice, pp. 17
$\mathbf{G}$	Unipotent group scheme, pp. 4
$\mathbf{G}_\Lambda$	Unipotent group scheme associated to Λ (also denoted by $\mathbf{G}$), pp. 17
$\mathbf{H}$	Heisenberg group scheme, pp. 6
$\mathcal{F}_{n,\delta}$	Nilpotent $\mathbb{Z}$-lattice of Definition 1.2.1, pp. 10
$\mathcal{G}_n$	Nilpotent $\mathbb{Z}$-lattice of Definition 1.2.1, pp. 10
$\mathcal{H}_n$	Nilpotent $\mathbb{Z}$-lattice of Definition 1.2.1, pp. 10
$\mathfrak{g}$	$\Lambda(\mathfrak{o}) := \Lambda \times_{\mathcal{O}} \mathfrak{o}$, pp. 33
$\mathfrak{g}'$	Derived Lie sublattice $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$, pp. 33
$\mathfrak{z}$	Centre of $\mathfrak{g}$, pp. 33

Hyperoctahedral groups and Statistics

B_n	Hyperoctahedral group, pp. 79
$[a_1, \dots, a_n]$	Element w of B_n given by $w(i) = a_i$, for each $i \in [n]$, pp. 79
w_0	Longest element $w_0 = [-1, \dots, -n]$ of B_n, pp. 80
$D(w)$	Right descent $\{s_i \in S_B \mid w(i) > w(i+1)\}$, pp. 80
inv	Inversion number, pp. 79
neg	Number of negative entries, pp. 79
nsp	Number of negative sum pairs, pp. 79
maj	Major index $\mathrm{maj}(w) = \sum_{i \in D(w)} i$, $w \in B_n$, pp. 80
rmaj	Reverse major index $\mathrm{rmaj}(w) = \sum_{i \in D(w)} n - i$, $w \in B_n$, pp. 80
ℓ	Coxeter length of B_n, pp. 80
σ	Statistic on B_n given by $\sigma(w) = \sum_{i \in D(w)} n^2 - i^2$, $w \in B_n$, pp. 80
L	Statistic on B_n given in (5.3.1), pp. 80

Matrices

$\mathbf{0}_n$	$(0,\dots,0)\in\mathbb{Z}^n$
$\mathbf{0}_{s\times r}$	$(s\times r)$-zero matrix
$\mathbf{1}_s$	$(s\times s)$-identity matrix
$\mathrm{Mat}_{a\times b}(R)$	Set of all $a\times b$-matrices over a ring R
M^{tr}	Transpose of the matrix M
$\nu(M)$	Elementary divisor type of the matrix M, pp. 31
$\tilde{\nu}(M)$	Elementary divisor type of an antisymmetric matrix M, pp. 32
$A(\underline{X})$	A-Commutator matrix, pp. 34
$A_\Lambda(\underline{X})$	A-Commutator matrix of $\Lambda\in\{\mathcal{F}_{n,\delta},\mathcal{G}_n\mathcal{H}_n\}$, pp. 64
$B(\underline{X})$	B-Commutator matrix, pp. 34
$F_j(M)$	Set of all $j\times j$-minors of the matrix M, pp. 32

$\mathfrak{p}$-adic integrals and functions

$v_\mathfrak{p}$	$\mathfrak{p}$-adic valuation, pp. 18
$\lvert.\rvert_\mathfrak{p}$	$\mathfrak{p}$-Adic norm, pp. 18
$\lvert\lvert.\rvert\rvert_\mathfrak{p}$	Maximum $\mathfrak{p}$-adic norm, pp. 18
$\mathscr{Z}_{\mathfrak{o},\mathcal{R}}$	p-Adic integral defined in (3.2.3), pp. 32
$\widetilde{\mathcal{Z}^*_{i,\mathfrak{p}}}$	Function given in (4.1.2), pp. 54
$Z_{\mathcal{W}(\mathfrak{o}),I}$	Integral given in (3.4.2), pp. 45

q-Symbols

$(\underline{n})_X$	$1-X^n$, pp. 11
$(\underline{n})_X!$	$(\underline{n})_X(\underline{n-1})_X\cdots(\underline{1})_X$, pp. 11
$\binom{a}{b}_X$	$\frac{(\underline{a})_X}{(\underline{b})_X(\underline{a-b})_X}$, for $a\geq b$, pp. 11
$\binom{n}{I}_X$	$\binom{n}{i_l}_X\binom{i_l}{i_{l-1}}_X\cdots\binom{i_2}{i_1}_X$, where $I=\{i_1,\dots,i_l\}_<$, pp. 11
$(X;Y)_n$	$\prod_{i=0}^{n-1}(1-XY^i)$ (Pochhammer symbol), pp. 11

Sets and cardinalities of sets

$\mathbb{N}$	$\{1, 2, \dots\}$
$\mathbb{N}_0$	$\{0, 1, 2, \dots\}$
$[n]$	$\{1, \dots, n\},\ n \in \mathbb{N}$
$[n]_0$	$\{0, 1, \dots, n\},\ n \in \mathbb{N}$
$[\pm n]_0$	$\{-n, \dots, n\}$, $n \in \mathbb{N}$
$\{i_1, \dots, i_l\}_<$	Set $\{i_1, \dots, i_l\}$ such that $i_1 < \cdots < i_l$, pp. 11
$\mathrm{Rep}(G)$	Set of isomorphism classes of complex irreducible representations of a group G, pp. 3
$\mathrm{Irr}(G)$	Set of isomorphism classes of complex irreducible characters of a group G, pp. 7
$\mathfrak{N}^{\circ}_{N,\mathcal{R},\mathbf{m}}$	$\{\mathbf{y} \in W^{\circ}_{n,N} \mid \nu(\mathcal{R}(\mathbf{y})) = \mathbf{m}\}$, pp. 31
$\mathcal{N}^{\circ}_{N,\mathcal{R},\mathbf{m}}$	Cardinality of $\mathfrak{N}^{\circ}_{N,\mathcal{R},\mathbf{m}}$, pp. 31
$\mathcal{D}^N_A$	$\{(m_1, \dots, m_{u_A}) \in [N]_0^{u_A} \mid m_1 \le \cdots \le m_{u_A},\ \sum_{i=1}^{u_A} m_i = u_A N - i\}$, pp. 37
$\mathcal{D}^N_B$	$\{(m_1, \dots, m_{u_B}) \in [N]_0^{u_B} \mid m_1 \le \cdots \le m_{u_B},\ \sum_{i=1}^{u_B} m_i = u_B N - i\}$, pp. 36
Q_1	Set of prime ideals $\mathfrak{p}$ such that (Y, h) has bad reduction modulo $\mathfrak{p}$, pp. 46
Q^*	Set $Q_1 \cup Q_2^*$ of "bad primes", pp. 51
T	Set indexing the irreducible components of the pre-image of h of the subvariety defined by $\mathcal{I}$, pp. 45
W'	Set of index i of cones R_i which do not lie in the boundary component $\mathbb{R}^t_{\ge 0} \times \{0\}$ of $\mathbb{R}^{t+1}_{\ge 0}$, pp. 53
$\mathscr{R}$	Set of indices $i \in W'$ such that the boundary $\partial\mathscr{D}_i$ of $\mathscr{D}_i$ shares infinitely many points with the boundary $\partial\mathscr{D}^*_{\mathbf{G}}$ of $\mathscr{D}^*_{\mathbf{G}}$, pp. 57

Zeta functions

ζ	Riemann zeta function, pp. 1
ζ_K	Dedekind zeta function of a number field K, pp. 2

ζ_G^{irr}	Representation zeta function of a rigid group G, pp. 4
$\zeta_G^{\widetilde{\mathrm{irr}}}$	Twist representation zeta function of a $\mathcal{T}$-group G, pp. 4
ζ_G^{cc}	Conjugacy class zeta function of a group G, pp. 6
$\zeta_{\mathbf{G}(\mathcal{O})}^{\mathrm{k}}$	Class number zeta function of $\mathbf{G}(\mathcal{O})$, pp. 7
$\mathcal{Z}_{\mathbf{G}(\mathcal{O})}^{\mathrm{irr}}$	Bivariate representation zeta function of $\mathbf{G}(\mathcal{O})$, pp. 5
$\mathcal{Z}_{\mathbf{G}(\mathcal{O})}^{\mathrm{cc}}$	Bivariate conjugacy class zeta function of $\mathbf{G}(\mathcal{O})$, pp. 6
$\mathcal{Z}_{\mathbf{G}(\mathcal{O})}^{*}$	Bivariate representation or bivariate conjugacy class zeta function of $\mathbf{G}(\mathcal{O})$ (i.e. $* \in \{\mathrm{irr}, \mathrm{cc}\}$), pp. 8

Other Functions

$\Xi_{U,I,(d_{\kappa\iota})}^{N}$	Function given in Definition 3.4.1, pp. 45
$\Xi_{U,I}$	Same function as $\Xi_{U,I,(0)}^{1}$, see Definition 3.4.1, pp. 45

Other Symbols

$r_n(G)$	Number of isomorphism classes of n-dimensional representations of a group G, pp. 3
$\widetilde{r}_n(G)$	Number of twist-equivalent n-dimensional representations of a $\mathcal{T}$-group G, pp. 4
$c_n(G)$	Number of conjugacy classes of a group G of size n, pp. 6
$\mathrm{k}(G)$	Class number of a group G, pp. 7
$\mathrm{re}(s)$	Real part of a complex number s
μ	Additive Haar measure on $\mathfrak{o}$ normalised so that $\mu(\mathfrak{o}) = 1$ or the product measure on $\mathfrak{o}^n$, $n \in \mathbb{N}$, pp. 19
$(N_{u\kappa\iota}, \nu_u)$	Numerical data of a principalisation, pp. 22
$\iota(M)$	Isolator of the module M, pp. 33
$\equiv_{\mathcal{D}}$	Relation given in Definition 4.2.2, pp. 57

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