# Lyapunov Exponents in the Spectral Theory of Primitive Inflation Systems

Dissertation zur Erlangung des akademischen Grades eines
Doktors der Mathematik (Dr. math.)

vorgelegt von

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April 2019

Gedruckt auf alterungsbeständigem Papier °°ISO 9706

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Datum der mündlichen Prüfung: 04 Juni 2019

# Contents

# Abgrenzung des eigenen Beitrags gemäß §10(2) der Promotionsordnung

**(1)** Most of the material in Chapter 2, as well as Sections 3.3, 5.1, and 5.2.2, are part of the author's joint work with Michael Baake and Franz Gähler [BGäM18], which has been submitted to *Communications in Mathematical Physics*. The main results in Section 5.3 are also included in the paper, but the complete derivation appears first in this work.

**(2)** The contents of Section 3.2 and Section 3.5.1 are included in the author's work [Man17a], which was published in *Journal of Mathematical Physics* in 2017.

**(3)** The entire discussion of the non-Pisot family of inflations in Section 4.2.1 is included in the author's joint work with Michael Baake and Uwe Grimm [BGrM18] that was published in *Letters in Mathematical Physics* in 2018.

**(4)** Section 3.2.2 was a part of a joint work with Michael Baake and Michael Coons [BCM17], which is to appear in the *Proceedings of the Jonathan M. Borwein Commemorative Conference*, and is currently in press.

# Acknowledgements

I am eternally indebted to my supervisor and adviser, Michael Baake, for believing in me and giving me a chance to work in his group. His passion for his craft is contagious and without his suggestions, support, and patience this would not have come to reality.

I also thank my other adviser, Franz Gähler, for sharing his expertise on a lot of things and for always being there to guide me when I feel lost.

I would like to thank the current and past members of our group (in alphabetical order): Frederic Alberti, Dirk Frettlöh, Philipp Gohlke, Christian Huck, Tobias Jakobi, Dan Rust, Timo Spindeler, Venta Terauds, and Peter Zeiner, for fruitful discussions and interesting conversations over lunch, without which my stay in Bielefeld would have been boring.

I also like to thank Herr Stephan Merkes and Frau Britta Heitbreder for enthusiastically answering my queries about administrative matters.

I extend my gratitude to the German Research Foundation (DFG) for funding my studies through CRC 701 and CRC 1283.

I am also grateful to Michael Coons and Uwe Grimm, for hosting me as a visiting researcher at the University of Newcastle in NSW, Australia, and at the Open University in Milton Keynes, UK, respectively. This apart from the collaboration and the enlightening chats in conferences and during their visits to Bielefeld.

Credits are also due to Paolo Bugarin and Eden Miro, who paved the way for me to be able to do my PhD in Bielefeld, which is by far the most life-changing decision I made.

My general view of the academic life was influenced by people who I met in conferences, or those who visited Bielefeld. In particular, I heavily benefited from discussions with David Damanik, Natalie Frank, Jeff Lagarias, and Nicolae Strungaru. I would like to specially mention Scott Balchin, who helped me with the numerics regarding Froyland's algorithm and coded a program that computes the Fourier matrix of constant-length substitutions.

To Dan and Mima, no words can ever amount to all the good memories we shared and the tough days we endured together (but maybe a few pints of Weizen will).

My stay in Bielefeld would not have been complete without my fellow Filipino students, with whom I have shared the gloominess of winter. To Cleo, Jay, Joseph, and Sheila: *sa wakas*.

To the Filipino community in Bielefeld and Gütersloh, thank you for being my home away from home, for always being thoughtful, and for always welcoming me to your homes like one of your own.

I also owe a lot to my colleagues and friends in the university: the pubquiz team, Alex, Li, Marie, Omar, and Eduard. Same goes for my playmates and trainers, for volleyball games which proved to be my sanity for the last three years.

To the people I met and visited while I was travelling, here's to the places waiting for us for our next escapades. Special thanks to Charles, Charlotte, Cheska, Emilie, Erika, and Quinn, for hosting me during my trips.

It goes without saying that I share this to my old friends, who knew my plight from the very beginning: Terents, Block V, Gabay, OBB, thanks for rooting for me from miles away.

To my previous colleagues in the Ateneo Math department, thank you for all the support.

To my previous mentors and teachers, thank you for sharing your wisdom and for the encouragement.

To my huge and wonderful family, thank you for being my source of joy and for always being there to push me to carry on.

To Nikko and Pau: *Doktor na si kuya.*

Lastly, to Mama, Papa, and Lola: I would not be where I am today without you. This is all yours.

*Para kina Cristina at Nelson,*
*na mananatiling mas makabuluhan*
*sa lahat ng kinaya,*
*kinakaya,*
*at makakaya kong isulat*

# Introduction

## X-ray Diffraction and Aperiodic Order

The landmark discovery of X-ray diffraction by von Laue and his collaborators in 1912 [FKvL12] revolutionised the state of non-contact characterisation methods of materials. In that work, it was confirmed that X-rays have wavelengths which are compatible with atomic spacing in solids which allows one to consider solids as diffraction gratings—something that with visible light is not possible due to its longer wavelength. It has long been believed that only materials possessing translational symmetry exhibit regularity in the Fourier regime, i.e., a diffraction pattern with isolated points of high intensity signifying a certain degree of order in the material. Such rigidity imposes limitations on allowable geometries, namely only structures with rotational symmetry of order $d \in \{1, 2, 3, 4, 6\}$ are compatible with having a lattice structure; see [Cox61, Sec. 4.5]. Hence, only these structures are expected to exhibit sharp peaks (known as Bragg peaks) when subjected to a diffraction experiment. For a long time, this has been accepted as an equivalence.

On the mathematical side, a first paradigm shift from purely periodic structures stemmed from works of Bohl [Boh93] and Esclangon [Esc04], which initiated further work towards a reasonable generalisation of Fourier theory. The notion of an almost periodic function is usually attributed to Bohr for pioneering a systematic approach towards an extension of periodic concepts within the realm of uniformly continuous functions [Boh47]. His ideas were further extended by various mathematicians to accommodate larger classes; see [Bes54].

In the periodic case, the far-field Fraunhofer picture of the diffraction is known to be the Fourier transform of a finite obstacle, which can be modelled as a finite measure. Once one leaves the periodic setting, working with infinite/unbounded objects is inevitable. Notions of Fourier transformability for unbounded measures had already seen reasonable progress by the early '70s; see [AdL74, BF75]. Moreover, in [Mey72], some connections to number theory were pointed out and the cut-and-project scheme as a method of generating point sets with nice properties was introduced.

Alongside these developments in harmonic analysis was a proliferation of important results on non-periodic tilings. The undecidability of the domino problem was established by Berger in 1966 [Ber66], which meant a tiling of the plane via a finite set of decorated tiles need not be periodic. Within a decade, Penrose solved a related but geometrically different problem in his monumental discovery of tilings of $\mathbb{R}^2$ by six prototiles having no translational symmetry (and hence are non-crystallographic) [Pen74].

Finding connections between these directions of mathematical research received a huge motivational boost from Schechtman's ground-breaking discovery of a real-world quasicrystal in 1982 [SBGC84]. He found that a particular phase of a quenched $\mathsf{Al}_{86}\mathsf{Mn}_{14}$ alloy which has icosahedral symmetry (and hence is non-crystallographic) displayed sharp Bragg peaks, signifying long-range order, which was exclusively attributed to crystals. This, along with the developments that came after, proved that there is indeed a regime between perfectly ordered structures and totally random ones that merits further investigation (hence the term "aperiodic order"). In particular, this justified the quest for an appropriate generalisation of the mathematical theory of diffraction.

Although there were lots of works which already applied Fourier analysis on aperiodic tilings, it was the work of Dworkin and Hof that set the stage for mathematical diffraction. Dworkin provided a first link between diffraction theory and spectral theory of operators [Dwo93], while it was Hof [Hof95] who rigorously established notions of diffraction theory specific to the aperiodic setting.

Under this formalism, one normally views a vertex set $\varLambda$ of an aperiodic tiling $\mathcal{T}$ as a model for a quasicrystal. One distinguishes different atoms by placing different weights signifying distinct scattering strengths. The non-periodicity of such tilings imply that one must deal with (weighted) unbounded measures to describe atomic positions. A subclass of such tilings can be generated via iterated rules on the corresponding building blocks to obtain bigger blocks consisting of unions of the smaller units. This thesis will revolve around such tilings, which are called *inflation tilings*.

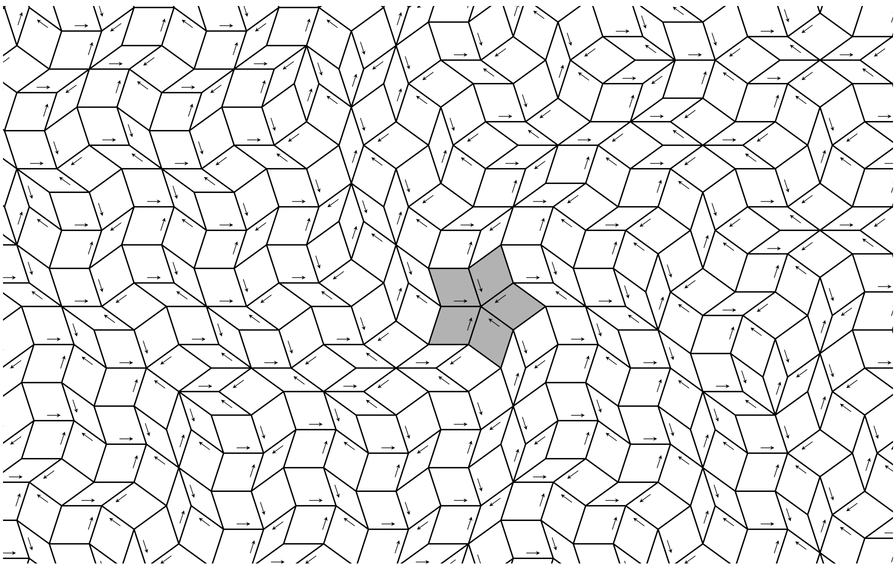


Figure 0.1.: A patch of the Godrèche–Lançon–Billard (GLB) tiling generated by applying the inflation rule in Section 5.3 twice on the shaded vertex star; taken from [BG13] with kind permission.

The main object in diffraction theory is a positive measure on a locally compact Abelian group $G$ (usually taken to be $\mathbb{R}^d$ for explicit examples) called the *diffraction measure* $\widehat{\gamma}$, which has the Lebesgue decomposition

$$\widehat{\gamma} = (\widehat{\gamma})_{\mathsf{pp}} + (\widehat{\gamma})_{\mathsf{ac}} + (\widehat{\gamma})_{\mathsf{sc}},$$

where $(\widehat{\gamma})_{\mathsf{pp}}$ is the *pure point* component and is the analytic analogue of Bragg peaks in a diffraction experiment, $(\widehat{\gamma})_{\mathsf{ac}}$ is the *absolutely continuous* component represented by a locally-integrable function whose non-triviality is usually attributed to a certain level of disorder, and in particular, represents what is called diffuse diffraction, and $(\widehat{\gamma})_{\mathsf{sc}}$ is the *singular continuous* component, which lives on an uncountable set of measure zero and is difficult to detect in experiments.

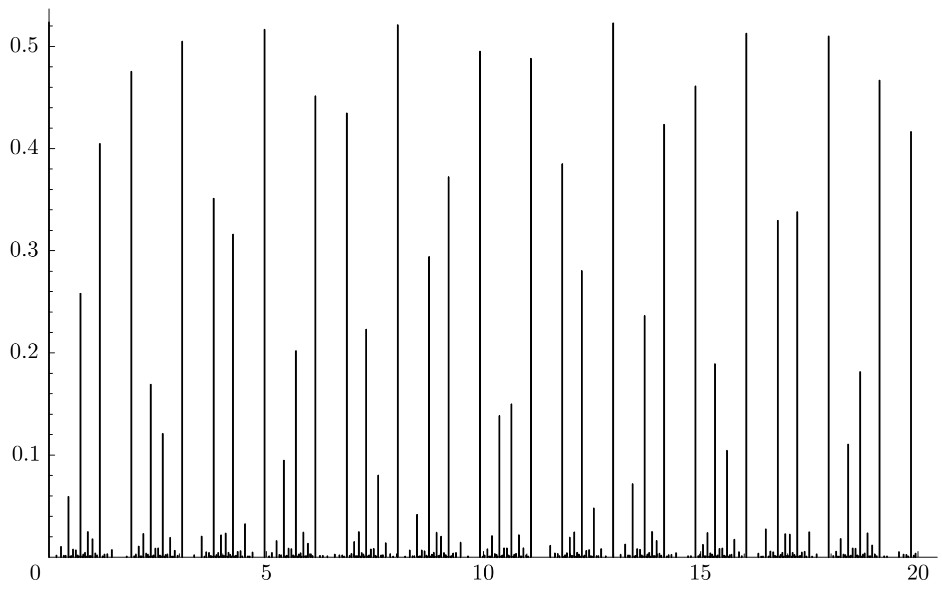



Figure 0.2.: Illustration of the pure point diffraction measure $\widehat{\gamma}$ of the Fibonacci inflation $\varrho_{\mathrm{F}}$; taken from [BG13] with kind permission.

One of the main objectives of mathematical diffraction is to relate the algebraic and geometric properties of the object in question (tiling) to the properties of $\widehat{\gamma}$ (diffraction). This can be summarised into the following questions

**(1)** Given that the configuration of atoms $\varLambda$ satisfies a certain condition $(\mathsf{C})$, what does it imply for $\widehat{\gamma}$?

**(2)** Given that the rule $\varrho$ that generates $\varLambda$ satisfies a certain condition $(\mathsf{C})$, what does it mean for $\widehat{\gamma}$?

### Pure Point Diffraction

Being the fingerprint of long-range order, it is expected that more results are known on the pure point part and on structures which are pure point diffractive, i.e., those for which $\widehat{\gamma} = (\widehat{\gamma})_{\mathsf{pp}}$.

There is a rich literature on the connection of having pure point diffraction to cut-and-project sets (CPS), and to vertex sets being Meyer sets. Moody proved in [Moo02] that a CPS with window having zero-measure boundary has pure point diffraction. It is known from [Mey72] that a tiling vertex set $\varLambda$ is a Meyer set if and only if its a subset of a CPS with compact window. Strungaru showed that weighted Meyer sets with non-trivial pure point diffraction must have relatively dense support for the Bragg peaks [Str05]. For primitive inflation tilings, Sing showed in one dimension [Sin06], which was extended to higher dimensions by Lee and Solomyak [LS08], that pure point diffraction implies that the underlying vertex set is Meyer.

Number-theoretic results are also abundant. For point sets having inflation symmetry, i.e., $\lambda\varLambda \subseteq \varLambda$, Lagarias pointed out that if such a set is Meyer, then $\lambda$ is either Pisot or Salem [Lag99].

It is also well known that for inflation tilings to have non-trivial Bragg peaks, the inflation multiplier $\lambda$ has to be Pisot; see [BT86, BT87, GK97, GL92]. To further this dependence on $\lambda$, Gähler and Klitzing showed that for self-similar tilings, the Bragg spectrum is completely determined by the translation module of the tiling and the inflation factor $\lambda$ [GK97].

### Dynamical Spectrum vs Diffraction Spectrum

Another type of spectrum associated to tilings is the dynamical spectrum which is the spectrum of the unitary Koopman operator associated to the shift $S$ on the hull $\mathbb{X}$; see Appendix A for a brief introduction.

It follows from [Dwo93] that the diffraction spectrum is contained in the dynamical spectrum. In particular, $(\widehat{\gamma})_{\mathsf{pp}}$ is non-trivial if and only if $U_S$ has non-trivial eigenfunctions, which has been immortalised in the literature as the "Dworkin argument", a precise interpretation of which can be found in [BL04] . This further implies that elements of a dynamical system with pure point dynamical spectrum must have pure point diffraction.

Amidst these known results, there are still some standing open questions, one of the biggest of which is the Pisot substitution conjecture.

**Conjecture 0.0.1** (Pisot Substitution Conjecture)**.** *A one-dimensional irreducible substitution $\varrho$ has pure point dynamical spectrum if and only if the eigenvalue $\lambda$ of the substitution matrix is a Pisot (PV) number.*

A few results suggesting the truth of this conjecture include the case with two letters proved by Holander and Solomyak [HS03], and an algorithmic way of deciding whether a given self-affine tiling is pure point via overlap coincidences, which is due to Solomyak [Sol97], and was generalised by Akiyama and Lee [AL11].

A series of independent works also showed the converse of Dworkin's argument, i.e., that under reasonable assumptions, pure point diffraction is equivalent to having pure point dynamical spectrum; see [LMS02, BL04, Gou05]. The question of pure pointedness of the dynamical spectrum for regular CPS was settled by Schlottmann [Sch00].

This equivalence allows one to use techniques in both formalisms to prove specific results. In particular, the Pisot conjecture is proved if one can show that all such systems have pure point diffraction.

In the case where the spectrum contains other types, one does not have this convenient equivalence. However, it was shown in [BLvE15] that, for systems with finite local complexity, one can recover the full dynamical spectrum via the diffraction of suitable factors.

For one-dimensional substitution tilings of constant length, Bartlett has developed an algorithm to determine the corresponding dynamical spectral type [Bar16], continuing previous works of Queffélec in [Que10]. Another classic result is due to Dekking [Dek78], stating that a constant-length substitution of height one is pure point if it admits a coincidence and is partly continuous otherwise, which also holds in higher dimensions; see [Sol97, Fra05].

### Absolutely Continuous Diffraction

Compared to its pure point counterpart, the nature of the continuous component of the diffraction measure remains more mysterious. The Cantor-type structure of the singular continuous component forces one to use multifractal techniques for its description, which is why it is only beginning to be understood in full generality; see [GL90, BG14, BGKS18].

On the other hand, since absolutely continuous diffraction is prevalent in amorphous solids and is seen as a signature of stochasticity, one does not expect to obtain it from deterministic systems. Unfortunately, this is not the case, as there exist completely deterministic systems with absolutely continuous diffraction; see [CGS18, Fra03].

Examples of such systems are rare. In fact, all known deterministic substitutive examples could be derived from the constructions provided in the mentioned references. This strongly suggests that systems with absolutely continuous spectrum satisfy rather restrictive conditions.

It is then natural to ask what these necessary conditions exactly are for $(\widehat{\gamma})_{\mathsf{ac}}$ to be non-trivial and whether, on the contrary, there is a generic sufficient criterion which rules out its existence. Of course, those that imply pure pointedness of the spectrum belong to this set of rules. When one has spectral purity, the Riemann–Lebesgue lemma is useful to detect measures that are not absolutely continuous. For systems which are known *a priori* to have mixed spectra, fewer conditions are known. Recently, Berlinkov and Solomyak provided a necessary criterion in [BS17] for a constant-length substitution to have an absolutely continuous dynamical spectral component. This thesis aims to supplement known criteria, and provide criteria for systems which are not covered by existing ones.

### Main results of this thesis

In this work, we deal primarily with primitive inflation rules seen as generators of tilings, and subsequently, of point sets deemed adequate for diffraction analysis. We harvest the combinatorial-geometric properties of these rules to obtain renormalisation equations satisfied by pair correlation functions, which we then transfer to the Fourier picture. This enables one to dissect each component of the diffraction measure under an appropriate renormalisation scheme. Using tools from the theory of Lyapunov exponents, we make explicit statements regarding $(\widehat{\gamma})_{\mathsf{ac}}$. In particular, we have the following main results:

**(1)** A sufficient criterion that excludes absolutely continuous diffraction, which can be carried out algorithmically for any primitive example (Theorem 2.5.3, Proposition 2.7.7, Theorem 5.1.5)

**(2)** A necessary criterion for general primitive inflation systems to have non-trivial absolutely continuous component (Corollary 2.7.10, Corollary 5.1.6)

**(3)** Spectral analysis of some non-Pisot inflations, which are conjectured to all have purely singular continuous spectra. (Section 4.2.1, Section 5.3).

Moreover, we present the recovery of known singularity results via the method presented in this work, and further point out connections to number-theoretic quantities arising from these objects, such as logarithmic Mahler measures (Proposition 3.2.8, Proposition 5.2.4).

# 1. Prerequisites

## 1.1. Point sets in $\mathbb{R}^d$

Below, we largely follow the monograph [BG13] for notation. For general sets $S_1, S_2 \subset \mathbb{R}^d$, the Minkowski sum (difference) is defined as

$$S_1 \pm S_2 := \{x \pm y \mid x \in S_1, y \in S_2\}.$$

A singleton $\{x\}$ is a set in $\mathbb{R}^d$ comprised of a unique point $x$, and any set $\varLambda$ of the form $\bigcup_{i\in J}\{x_i\}$ where $J$ is countable is called a *point set*. A point set $\varLambda$ is called *discrete* if, for all $x \in \varLambda$, there exists an open neighbourhood $U(x)$ of $x$ such that $\varLambda \cap U(x) = \{x\}$. From this point on, most point sets we deal with are infinite. Assuming the discreteness of $\varLambda$, we get that, for every $x \in \varLambda$, there is an $R(x) > 0$ such that $\mathfrak{B}_{R(x)}(x) \cap \varLambda = \{x\}$, where $\mathfrak{B}_R(x)$ denotes the ball of radius $R$ centred at $x$. If there is a uniform lower bound $R_{\mathrm{p}}$ on $R(x)$, we call $\varLambda$ *uniformly discrete*. If there exists $0 < R_{\mathrm{c}} < \infty$ such that $\varLambda + \overline{\mathfrak{B}_{R_{\mathrm{c}}}(0)} = \mathbb{R}^d$, one calls $\varLambda$ *relatively dense*. The constants $R_{\mathrm{p}}$ and $R_{\mathrm{c}}$ are called the *packing radius* and the *covering radius* of $\varLambda$, respectively.

**Definition 1.1.1.** Point sets that are both uniformly discrete and relatively dense are called *Delone* sets . If $\varLambda$ is relatively dense and $\varLambda - \varLambda$ is uniformly discrete, then $\varLambda$ is called a *Meyer* set.

Every Meyer set is automatically Delone. We refer to [Lag96, Moo97a, Sin06, Str17] for conditions equivalent to the Meyer property.

Consider a discrete point set $\varLambda \subset \mathbb{R}^d$. It is called *locally finite* whenever $K \cap \varLambda$ is at most a finite set, for any compact $K \subset \mathbb{R}^d$. For $x \in \mathbb{R}^d$ and $R > 0$, we call $\mathfrak{P} := \mathfrak{B}_R(x) \cap \varLambda$ a *patch* of $\varLambda$. *Repetitivity* of $\varLambda$ means that, for every patch $\mathfrak{P}$ and for any $y \in \mathbb{R}^d$, there is some $R > 0$ such that $\mathfrak{B}_R(y)$ contains at least one translate of $\mathfrak{P}$. We say that $\varLambda$ has *finite local complexity* or is an FLC set if $\big\{(t + \mathfrak{B}_R(x)) \cap \varLambda \mid t \in \mathbb{R}^d\big\}$ contains at most finitely many patches up to translation, for any $x \in \mathbb{R}^d, R > 0$.

## 1.2. Symbolic dynamics and inflation rules

### 1.2.1. Substitutions

We begin with a finite set $\mathcal{A}_{n_a} = \{a_1, \ldots, a_{n_a}\}$ which we call an *alphabet*, whose elements are called *letters*. Denote by $\mathfrak{F}_{n_a}$ the free group generated by elements of $\mathcal{A}_{n_a}$. A *general substitution rule* $\varrho$ is an endomorphism on $\mathfrak{F}_{n_a}$, i.e., $\varrho(uv) = \varrho(u)\varrho(v)$ and $\varrho(u^{-1}) = \big(\varrho(u)\big)^{-1}$ hold for $u, v \in \mathfrak{F}_{n_a}$. To this rule, one can associate a substitution matrix $M_\varrho$ via the *Abelianisation map* $\vartheta : \mathfrak{F}_{n_a} \mapsto \mathbb{Z}^{n_a}$ which sends an arbitrary element $w$ of $\mathfrak{F}_{n_a}$ to a vector containing the powers of generators $a_i$ in $w$, if elements of $\mathfrak{F}_{n_a}$ are assumed to commute.

In this thesis, we will solely consider such endomorphisms whose images on $a_i$ only contain positive powers, which we will simply call a *substitution*. A finite concatenation of letters $w = \alpha_0\alpha_1 \dots \alpha_{\ell-1}$, where $\alpha_i \in \mathcal{A}_{n_a}$, is called a *word*, whose length $|w|$ is simply the number of letters comprising it. A word $v$ of length $R$ is called a *subword* of $w$, which we denote by $v \triangleleft w$, if there is a $0 \leqslant k \leqslant \ell - R$ such that $v = \alpha_k\alpha_{k+1} \dots \alpha_{k+R-1}$. We denote by $\mathcal{A}_{n_a}^{\ell}$ the set of finite words of length $\ell$ over $\mathcal{A}_{n_a}$, and the set of all finite words (with the empty word $\epsilon$) to be $\mathcal{A}_{n_a}^{*} := \bigcup_{\ell \geqslant 0} \mathcal{A}_{n_a}^{\ell}$.

**Definition 1.2.1.** A *substitution* is a map from a finite alphabet to the set of finite words over it, i.e., $\varrho : \mathcal{A}_{n_a} \to \mathcal{A}_{n_a}^{*}$, with $\varrho(a_i) \neq \epsilon$ for all $i$.

In this work, we formally write a substitution as $\varrho : a_i \mapsto w_i$, where we call the image $w_i$ the *substituted word* of $a_i$. We also adapt the notation $\varrho = (w_1, w_2, \dots, w_{n_a})$ whenever necessary. Note that the endomorphism property of $\varrho$ allows one to extend this to a map from $\mathcal{A}_{n_a}^{*}$ to itself via concatenation of substituted words, i.e., $\varrho(ab) = \varrho(a)\varrho(b)$. This further extends to a map that sends (bi-)infinite words to (bi-)infinite words, which yields a well-defined map on $\mathcal{A}_{n_a}^{\mathbb{N}}$ or $\mathcal{A}_{n_a}^{\mathbb{Z}}$. Powers of $\varrho$, denoted by $\varrho^k$, for some $k \in \mathbb{N}$, are also well defined, and are obtained by applying the rule iteratively on the resulting substituted words.

Through the Abelianisation map $\vartheta$, one constructs the *substitution matrix* $M_{\varrho}$ by counting the number of letters $a_i$ present in $w_j$ and setting it to be the $ij^{\text{th}}$ entry of $M_{\varrho}$. More explicitly,

$$\left(M_{\varrho}\right)_{ij} := \text{card}_{a_i}(\varrho(a_j)).$$

**Definition 1.2.2.** A substitution is called *primitive* if there exists $k \in \mathbb{N}$ such that, for all $1 \leqslant i, j \leqslant n_a$, $a_i$ appears in $\varrho^k(a_j)$.

**Definition 1.2.3.** A non-negative matrix $M$ is *primitive* if there exists $k \in \mathbb{N}$ such that $M^k$ is a strictly positive matrix, i.e., $(M^k)_{ij} > 0$ for all $i, j$.

It is easy to see that $\varrho$ is primitive if and only if $M_{\varrho}$ is a primitive matrix. Unless stated otherwise, our general framework will only concern primitive substitutions.

A finite word $w \in \mathcal{A}_{n_a}^{*}$ is *legal* with respect to $\varrho$, or *$\varrho$-legal*, if $w$ is a subword of a substituted word, i.e., $w \triangleleft \varrho^k(a_i)$, for some $k \in \mathbb{N}, a_i \in \mathcal{A}_{n_a}$. Let $w^{(0)} = a_i|a_j \in \mathcal{A}_{n_a}^2$ be a $\varrho$-legal two-letter subword, where $|$ designates the location of the origin. Fix a power $\varrho^{\ell}$ of the substitution, and consider

$$\lim_{k\to\infty} (\varrho^{\ell})^k(w^{(0)}) := \varrho^{\infty}\big(w^{(0)}\big) = w = \varrho^{\ell}(w).$$

If such a limit exists, we call $w \in \mathcal{A}_{n_a}^{\mathbb{Z}}$ a *bi-infinite fixed point* of $\varrho^{\ell}$ corresponding to the legal seed $w^{(0)}$. For primitive substitutions, the existence of such fixed points is guaranteed by the following result.

**Proposition 1.2.4** ( [BG13, Lem. 4.3]). *Let $\varrho$ be a primitive substitution over $\mathcal{A}_{n_a}, n_a \geqslant 2$. Then, there exists $\ell \in \mathbb{N}$ and $w \in \mathcal{A}_{n_a}^{\mathbb{Z}}$, such that $w$ is a bi-infinite fixed point of $\varrho^{\ell}$, i.e., $\varrho^{\ell}(w) = w$, derived from some legal seed $a_i|a_j$.* □

### 1.2.2. Perron–Frobenius theory

**Theorem 1.2.5** ( [Que10, Thm. 5.4]). *Let $M$ be a primitive matrix. Then, it has a simple real eigenvalue $\lambda_{\mathrm{PF}} \in \mathbb{R}_+$ of maximum modulus. Furthermore, the corresponding left and right eigenvectors, which we denote by $\boldsymbol{L}, \boldsymbol{R}$ consist only of positive entries.* □

We call $\lambda_{\mathrm{PF}}$ the *Perron–Frobenius (PF) eigenvalue*, and $\boldsymbol{L}$ and $\boldsymbol{R}$ the left and right *PF eigenvectors* of $M$, respectively. Several number-theoretic properties of $\lambda_{\mathrm{PF}}$ have remarkable implications to spectral, dynamical, and topological properties of objects derived from $\varrho$. An algebraic integer $\lambda > 1$ is called a *Pisot–Vijayaraghavan (PV) number* if all of its algebraic conjugates $\lambda_1, \ldots, \lambda_{r-1}$ are less than 1 in modulus, i.e., $|\lambda_i| < 1$, for $1 \leqslant i \leqslant r-1$.

An *irreducible* substitution is one whose matrix $M_\varrho$ has irreducible characteristic polynomial, which is equivalent to having all eigenvalues of $M_\varrho$ to be the algebraic conjugates of $\lambda_{\mathrm{PF}}$. A substitution is *Pisot* whenever $\lambda_{\mathrm{PF}}$ of $M_\varrho$ is a PV number, and is *non-Pisot* otherwise. An important class of non-Pisot numbers is the set of *Salem numbers*. An algebraic integer $\lambda > 1$ of degree at least 4 is said to be Salem if all but one of its algebraic conjugates lie on the unit circle.

Due to primitivity, one deduces that each letter appears infinitely often in any infinite substituted word $\varrho^\infty(a_i) = \lim_{n\to\infty} \varrho^n(a_i)$, $a_i \in \mathcal{A}_{n_a}$. The following result provides a quantitative version of the previous statement and how it relates to the right PF eigenvector $\boldsymbol{R}$.

**Proposition 1.2.6** ( [Que10, Prop. 5.8]). *Let $a_i \in \mathcal{A}_{n_a}$. Then,*

$$\lim_{n\to\infty} \frac{\mathrm{card}_{a_j}(\varrho^n(a_i))}{|\varrho^n(a_i)|} = \widetilde{\boldsymbol{R}}_j$$

*where $\widetilde{\boldsymbol{R}} = \boldsymbol{R}/\|\boldsymbol{R}\|_1$.* □

This limit can be interpreted as the *letter frequency* of $a_j$ in $w = \varrho^\infty(a_i)$, which can be extended to bi-infinite words since it is independent of the starting seed $a_i$. Note that this depends solely on $M_\varrho$, and substitutions sharing the same substitution matrix thus yield identical letter frequencies.

**Remark 1.2.7** (Word frequencies via induced substitutions)**.** One can also compute for the frequencies of arbitrary finite $m$-letter legal words of $\varrho$ by working on *induced substitutions* $\varrho^{(m)}$. Such a substitution treats a length-$m$ legal word $av$ as a right-collared word $a|_v$ and maps it to a concatenation of right-collared words which can be obtained from overlapping length-$m$ subwords of $\varrho(av)$. When $\varrho$ is primitive, it is guaranteed that $\varrho^{(m)}$ is also primitive, for any finite $m$; see [BG13, Prop. 4.14]. One can then apply the usual Perron–Frobenius theory to the substitution matrix of $\varrho^{(m)}$ to obtain the frequencies of all $m$-letter legal words, as an analogue of Proposition 1.2.6; see [BG13, Sec. 4.8.3]. ◊

We now give examples of substitution rules, enumerating some of their properties, based on what we have so far. We will encounter these substitutions again in the following chapters.

**Example 1.2.8.**

**(1)** The *Fibonacci substitution* $\varrho_F : a \mapsto ab, b \mapsto a$ has the substitution matrix $M_\varrho = \left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$, with $\lambda_{PF} = \frac{1+\sqrt{5}}{2}$, which is a PV number of degree 2. This makes $\varrho_F$ an irreducible Pisot substitution. Note that, if one considers $\varrho_{Fib}^2$, the bi-infinite word generated by the seeds $a|a$ and $b|a$ are both fixed points. In particular,

$$w = (\varrho_{Fib}^2)^\infty(a|a) = \ldots abaababa|abaababa \ldots = \varrho_{Fib}^2(w).$$

**(2)** The *Thue–Morse substitution* is given by $\varrho_{TM} : a \mapsto ab, b \mapsto ba$. Its substitution matrix has $\{2, 0\}$ as eigenvalues, which makes $\varrho_{TM}$ a Pisot substitution that is not irreducible. Note that $|w_a| = 2 = |w_b|$, which makes it a *constant-length* substitution. One can also check that $\boldsymbol{L} = (1, 1)^T$.

**(3)** The substitution $\varrho_{BNP} : a \mapsto abbb, b \mapsto a$ has the substitution matrix $M_\varrho = \left(\begin{smallmatrix}1&1\\3&0\end{smallmatrix}\right)$, where $\lambda_{PF} = \frac{1+\sqrt{13}}{2}$ and the second eigenvalue satisfies $\left|\frac{1-\sqrt{13}}{2}\right| \approx 1.303 > 1$, which makes $\varrho_{BNP}$ an irreducible non-Pisot substitution. This substitution is systematically treated in [BFGR19].

**(4)** The *Rudin–Shapiro substitution* $\varrho_{RS} : a \mapsto ac, b \mapsto dc, c \mapsto ab, d \mapsto db$ is another example of a constant-length substitution. For $\varrho_{RS}$, $\widetilde{\boldsymbol{R}} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$, which means that every letter is equally frequent in any (bi-)infinite word arising from $\varrho$. ◊

### 1.2.3. The symbolic hull

From Proposition 1.2.4, we know that any primitive substitution gives rise to at least one bi-infinite fixed point $w$. Using $w$, the goal is to construct a subset of $\mathcal{A}_{n_a}^{\mathbb{Z}}$ satisfying some invariance properties.

Let $v \in \mathcal{A}_{n_a}^{\mathbb{Z}}$, with $v_i$ signifying the letter at $i \in \mathbb{Z}$. The (left) *shift operator* $S$ on $\mathcal{A}_{n_a}^{\mathbb{Z}}$ acts via $(Sv)_i := v_{i+1}$. By choosing the sequences we handle to be bi-infinite, we automatically get that $S$ is a homeomorphism and thus invertible. Any closed subset of $\mathcal{A}_{n_a}^{\mathbb{Z}}$ that is $S-$invariant is called a *shift space.*

Now choose any bi-infinite fixed point $w$ of (possibly of a power of) $\varrho$ and construct the space

$$\mathbb{X}(w) = \overline{\{S^i(w) \mid i \in \mathbb{Z}\}},$$

where the closure is taken in the natural product (or local) topology. Note that $\mathbb{X}(w)$ is both $S$ and $\varrho$-invariant; the first follows directly by definition and the second is due to $w$ being a fixed point with a dense orbit in $\mathbb{X}(w)$. We call this shift space the *symbolic hull* of $w$.

From [BG13, Lem. 4.2 and Prop. 4.2], if $\varrho$ is primitive, $\mathbb{X}(w)$ neither depends on the chosen bi-infinite fixed point nor on the power of the substitution that produces such fixed point. Hence, $\varrho$ admits a unique symbolic hull $\mathbb{X} = \mathbb{X}_\varrho := \mathbb{X}(w)$; compare [BG13, Thm. 4.1].

An element $w \in \mathcal{A}_{n_a}^{\mathbb{Z}}$ is called *periodic* if there exists an $r \in \mathbb{Z} \setminus \{0\}$ such that $S^r w = w$ and is *non-periodic* otherwise. We call a substitution *aperiodic* if the hull $\mathbb{X}$ it defines contains no periodic points. We have the following sufficient criterion for aperiodicity for primitive substitutions.

**Theorem 1.2.9** ( [BG13, Thm. 4.6])**.** *Let $\varrho$ be a primitive substitution whose corresponding* PF *eigenvalue $\lambda_{PF}$ is irrational. Then, the corresponding hull (and hence $\varrho$) is aperiodic.*

A substitution $\varrho$ is *locally recognisable* or has the *unique composition property* if, for every $w \in \mathbb{X}_{\varrho}$, one can find a unique $w'$ such that $\varrho(w') = w$. This means that every letter in $w'$ is situated in a unique level-1 substituted word in $w$.

It is well known from a result by Mossé in [Mos92] that aperiodicity is equivalent to local recognisability for primitive substitutions. Balchin and Rust provided an algorithm that determines whether a substitution is locally recognisable, which takes care of the case when $\lambda_{\mathrm{PF}} \in \mathbb{Z}$; see [BR17].

Since we will be more interested in the geometric counterpart $\mathbb{Y}$ of $\mathbb{X}(w)$, we delay discussing further properties of $\mathbb{X}$ and present the corresponding analogues for $\mathbb{Y}$.

### 1.2.4. Inflation systems and the geometric hull

The primitivity of $\varrho$ allows one to associate to it a corresponding *inflation rule*, which we will, by an abuse of notation, also refer to as $\varrho$. Such a rule is constructed by assigning a tile $\mathfrak{t}$ of a certain length to each letter $a_i \in \mathcal{A}_{n_a}$. A natural choice for the tile lengths is given by the left PF eigenvector $\boldsymbol{L}$ of $M_{\varrho}$. This means the tile $\mathfrak{t}_i$ of length $\boldsymbol{L}_i$ is assigned to $a_i$ . Usually, we carry out the assignment such that $\boldsymbol{L}$ is normalised so that the smallest tile has length 1. With this choice, one can construct the inflation rule as follows: under $\varrho$, the associated tile $\mathfrak{t}_j$ to a letter $a_j$ is inflated by a factor of $\lambda_{\mathrm{PF}}$, and is subdivided as a concatenation of constituent tiles according to the arrangement of letters in $\varrho(a_j)$. The image of a tile $\mathfrak{t}_j$, which we denote by $\varrho(\mathfrak{t}_j)$, is called a *supertile*, which is the geometric realisation of a substituted word defined in Section 1.2.1.

The geometric realisation of a bi-infinite fixed point $w$ is then a one-dimensional tiling of $\mathbb{R}$, which we denote by $\mathcal{T}$. Tilings arising from such construction are also called *self-similar*, which mainly alludes to the consistency of the expansion-subdivision scheme with the chosen tile lengths. As an example, the associated inflation rule for the Fibonacci substitution is given in Figure 1.1.

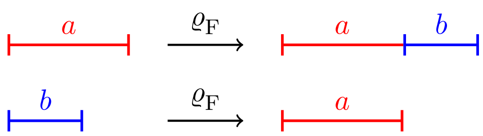



Figure 1.1.: The Fibonacci substitution $\varrho_{\mathrm{F}}$ viewed as an inflation rule.

We then create the geometric analogue of $\mathbb{X}(w)$ as follows: pick a bi-infinite symbolic fixed point $w$ of $\varrho$ and consider its geometric realisation $\mathcal{T}$. To this tiling, we then construct a point set $\varLambda \subset \mathbb{R}$ by choosing the left endpoints of tiles as their markers and colouring these markers depending on the tile type. This coloured point set $\varLambda$ is a suitable object for our diffraction analysis, representing positions of an infinite assembly of $n_a$ types of scatterers.

**Fact 1.2.10.** *Any point set $\varLambda$ arising from a primitive substitution is Delone.* □

**Theorem 1.2.11** ( [Lag99, Thm. 4.1])**.** *Let $\varLambda$ be a Meyer set in $\mathbb{R}^d$ such that, $\lambda\varLambda \subseteq \varLambda$, for some $\lambda > 0$. Then, $\lambda$ is either Pisot or Salem.* □

The following one-dimensional result follows from Theorem 1.2.11 and [Sin06, Sec. 6.3].

**Corollary 1.2.12.** *A point set $\varLambda$ arising from a primitive substitution is Meyer if and only if $\lambda_{\mathrm{PF}}$ is Pisot or Salem.* □

Assuming that one deals with FLC point sets, one can work with the *local topology*. Here, two sets $\varLambda$ and $\varLambda'$ are called $\varepsilon$-close if, for some $t \in \mathfrak{B}_{\varepsilon}(0)$,

$$\varLambda \cap \mathfrak{B}_{1/\varepsilon}(0) = (-t + \varLambda') \cap \mathfrak{B}_{1/\varepsilon}(0), \tag{1.1}$$

which roughly means that these point sets almost agree around a large region containing the origin. From a single point set, we then generate a collection of point sets which satisfy certain dynamical properties. The *geometric hull* $\mathbb{Y}(\varLambda)$ is defined as

$$\mathbb{Y}(\varLambda) = \overline{\{t + \varLambda \mid t \in \mathbb{R}\}},$$

where the closure is taken with respect to the local topology. This, equipped with the continuous $\mathbb{R}$-action via translations, comprises a topological dynamical system $(\mathbb{Y}(\varLambda), \mathbb{R})$.

One notion of equivalence for tilings and point sets is given by mutual local derivability (MLD). For our purposes, we only present here the relevant definitions for point sets in $\mathbb{R}$, but the notions for tilings are completely analogous. A point set $\varLambda$ is *locally derivable* from $\varLambda'$ if there exists a radius $R$ such that whenever

$$(-x + \varLambda) \cap \overline{\mathfrak{B}_R(0)} = (-y + \varLambda) \cap \overline{\mathfrak{B}_R(0)}$$

holds for $x, y \in \mathbb{R}$, one also has

$$(-x + \varLambda') \cap \{0\} = (-y + \varLambda') \cap \{0\}\,.$$

In other words, local derivability allows one to construct a patch of $\varLambda'$ centred at $x_0$ from the structure of a certain patch of $\varLambda$ at the same point. We say that $\varLambda$ and $\varLambda'$ are *mutually locally derivable*, if they are locally derivable from each other. This notion extends to the geometric hulls $\mathbb{Y}(\varLambda)$ and $\mathbb{Y}(\varLambda')$. If $\varLambda$ and $\varLambda'$ are MLD it follows that their hulls are also MLD. A *topological conjugacy* is a homeomorphism between dynamical systems that commutes with the action, which in our case is the translation action by $\mathbb{R}$. Two hulls are MLD if there exists a topological conjugacy between them that is defined locally. We refer to [BG13, Sec. 5.2] for further details.

The hull $\mathbb{Y}(\varLambda)$ (resp. $(\mathbb{Y}(\varLambda), \mathbb{R})$) is called *minimal* if every element $\varLambda' \in \mathbb{Y}(\varLambda)$ has a dense $\mathbb{R}$-orbit, i.e., $\overline{\{t + \varLambda' \mid t \in \mathbb{R}\}} = \mathbb{Y}(\varLambda)$. Recall that a probability measure $\mu$ on a dynamical system $(X, T)$ is *ergodic* if $\mu(\mathfrak{D}) = 0$ or $\mu(\mathfrak{D}) = 1$ holds for every $T-$invariant Borel set $\mathfrak{D}$. Alternatively, we call $T$ an *ergodic transformation* with respect to $\mu$. A system is *uniquely ergodic* if it admits a unique ergodic measure. Further, if it is also minimal, it is called *strictly ergodic*.

Strict ergodicity is known for symbolic hulls arising from primitive substitutions, which is an implication of linear repetitivity; see [Dur00, Len02]. For their geometric counterparts, we have the following result.

**Theorem 1.2.13.** *Let $\varLambda \subset \mathbb{R}$ be a point set which consists of markers from a geometric realisation of a primitive substitution $\varrho$. Then, the geometric hull $\mathbb{Y}(\varLambda)$ it generates is strictly ergodic.*

*Sketch of proof.* The minimality result follows from $\varLambda$ being linearly repetitive; compare [BG13, Prop. 5.3], and the fact that repetitive point sets produce minimal hulls [BG13, Prop. 5.4]. Unique ergodicity follows from Soloymak's result in [Sol97], which generally holds for self-similar tilings $\mathcal{T}$ in $\mathbb{R}^d$, and is transferred to point sets MLD to $\mathcal{T}$. □

**Remark 1.2.14** (Strict ergodicity and patch frequencies)**.** In the symbolic setting, strict ergodicity implies uniform existence and positivity of word frequencies, a result primarily attributed to Oxtoby; see [BG13, Prop. 4.4]. For tilings (also in higher dimensions), uniform existence of patch frequencies under the assumption of unique ergodicity follows from [Sol97, Thm. 3.3]. ◊

**Remark 1.2.15** (Aperiodicity of $\mathbb{Y}$)**.** One can extend the notion of aperiodicity given in Section 1.2.3 to the the geometric hull $\mathbb{Y}(\varLambda)$ by considering $\mathbb{R}$-translates instead of $\mathbb{Z}$-translates. In particular, if $\varLambda$ does not have a non-trivial period $t \in \mathbb{R}$, then the hull it generates is aperiodic; compare [BG13, Prop. 5.5]. ◊

## 1.3. Harmonic analysis and diffraction

### 1.3.1. Fourier transformation of functions

Let the *Schwartz space* $\mathfrak{S}(\mathbb{R}^d)$ be the space of rapidly decaying $C^\infty$−functions on $\mathbb{R}^d$. For $f \in \mathfrak{S}(\mathbb{R}^d)$, the Fourier transform $\mathcal{F} : \mathfrak{S}(\mathbb{R}^d) \to \mathfrak{S}(\mathbb{R}^d)$ is given by

$$\mathcal{F}[f](k) = \widehat{f}(k) := \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} kx} f(x) \,\mathrm{d}x. \tag{1.2}$$

A similar definition holds for $f \in L^1(\mathbb{R}^d)$, with a slight variation that $\widehat{f}$ is no longer necessarily integrable. Continuous (complex) linear functionals $\mathfrak{T}$ on $\mathfrak{S}(\mathbb{R}^d)$ are called *tempered distributions*. The Fourier transform of $\mathfrak{T} \in \mathfrak{S}'(\mathbb{R}^d)$ is given by $\widehat{\mathfrak{T}}(f) := \mathfrak{T}(\widehat{f})$, for test functions $f \in \mathfrak{S}(\mathbb{R}^d)$.

**Example 1.3.1** (Dirac distribution)**.** For a fixed $x \in \mathbb{R}^d$, the corresponding *Dirac distribution* $\delta_x : \mathfrak{S}(\mathbb{R}^d) \to \mathbb{C}$, with $f \mapsto \delta_x(f) := f(x)$, is tempered. Its Fourier transform is defined via

$$\widehat{\delta_x}(f) = \delta_x(\widehat{f}) = \widehat{f}(x) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} xy} f(y) \,\mathrm{d}y := \mathfrak{T}_{h_x}(f),$$

which justifies the convention $\widehat{\delta_x} = h_x = \mathrm{e}^{-2\pi \mathrm{i} xy}$. Here, one identifies $g$ with the functional $\mathfrak{T}_g$ via $\mathfrak{T}_g(f) = \int_{\mathbb{R}^d} g(y) f(y) \,\mathrm{d}y$. ◊

### 1.3.2. Measures

A (complex) *Radon measure* $\mu$ is a continuous linear functional on the space of continuous, compactly supported functions $C_{\mathrm{c}}(G)$, where $G$ is a locally compact Abelian group. By the Riesz–Markov representation theorem, we identify the set of all Radon measures $\mu$ with the set of regular Borel measures on $G$. In this work, we only deal with cases where $G = \mathbb{R}^d$. We denote the set of all measures on $\mathbb{R}^d$ as $\mathcal{M}(\mathbb{R}^d)$.

Given $\mu$, we can construct other measures such as $\widetilde{\mu}$ and $\overline{\mu}$, called its *twist* and *conjugate*, which are defined via their valuation on test functions, i.e., $\widetilde{\mu}(g) = \overline{\mu(\widetilde{g})}$ and $\overline{\mu}(g) = \overline{\mu(\overline{g})}$, where $\widetilde{g}(x) := \overline{g(-x)}$, with $\overline{\,\cdot\,}$ denoting complex conjugation.

For any two functions $f, g \in L^1(\mathbb{R}^d)$, their *convolution* is defined as

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y) g(y) \mathrm{d}y.$$

This definition also works when $f, g \in C_{\mathrm{c}}(\mathbb{R}^d)$.

A measure $\mu$ is called *real* if $\mu = \overline{\mu}$, and a real measure is called *positive* if $\mu(g) \geqslant 0$ for all $g \in C_{\mathrm{c}}(\mathbb{R}^d), g \geqslant 0$. Let us denote the collection of all positive measures on $\mathbb{R}^d$ by $\mathcal{M}^+(\mathbb{R}^d)$. For general measures, we say that $\mu$ is *positive definite* if $\mu(g * \widetilde{g}) \geqslant 0$, for all $g \in C_{\mathrm{c}}(\mathbb{R}^d)$.

To a measure $\mu$, one can also associate its *total variation* $|\mu|$, which is the smallest measure that satisfies $|\mu(g)| \leqslant |\mu|(g)$ for all $g \in C_{\mathrm{c}}(\mathbb{R}^d), g \geqslant 0$. A measure $\mu$ is called *finite* or *bounded* if $|\mu|(\mathbb{R}^d) < \infty$. Otherwise, it is called *unbounded*.

**Remark 1.3.2.** The Dirac distribution $\delta_x$ from Example 1.3.1 also defines a measure, with

$$\delta_x(\mathfrak{D}) = \begin{cases} 1, & \text{if } x \in \mathfrak{D}, \\ 0, & \text{otherwise} \end{cases}$$

for a chosen Borel set $\mathfrak{D} \subset \mathbb{R}^d$. When $\mathfrak{D}$ is countable or finite, the characteristic function $1_{\mathfrak{D}}$ decomposes into $1_{\mathfrak{D}} = \sum_{x \in \mathfrak{D}} \delta_x$, which coincides with the measure $\delta_{\mathfrak{D}}$, and is usually called the *Dirac comb* on $\mathfrak{D}$. ◊

Most measures we deal with in the diffraction theory of inflation systems are unbounded, but still satisfy a certain level of regularity called *translation boundedness*, which will be crucial to our analysis via forbidden growth rates.

**Definition 1.3.3.** A measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ is called *translation bounded* if, for every compact subset $K \subset \mathbb{R}^d$, one has $\sup_{x \in \mathbb{R}^d} |\mu|(x + K) < C_K$, i.e., there exists a constant $C_K$ depending only on $K$ for which $|\mu|(x + K) < C_K$ holds for all translation vectors $x$.

For a given finite measure $\mu$, we define its Fourier transform $\widehat{\mu}$ to be

$$\widehat{\mu}(k) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} k x} \mathrm{d}\mu(x), \tag{1.3}$$

which coincides with the Fourier transform in the distributional sense.

The following lemma can directly be verified using this definition of the Fourier transform and by viewing $\widehat{\mu}$ as a distribution, i.e., $\widehat{\mu}(g) = \mu(\widehat{g})$.

**Lemma 1.3.4.** *For any finite measure $\mu$, the equality $\overline{\widehat{\mu}} = \widehat{\widetilde{\mu}}$ holds.* □

Fourier transformability of unbounded measures is a delicate issue, as there are examples of measures that are transformable as distributions but not as measures; see [Str19, AdL74]. Nevertheless, this is guaranteed for the class of measures we will be working with, which is due to the following results.

**Theorem 1.3.5** (Bochner–Schwartz, [RS80, Thm. IX.10])**.** *Let $\mu$ be a measure that is also a tempered distribution. If $\mu$ is positive definite (or of positive type) on $\mathfrak{S}(\mathbb{R}^d)$, its Fourier transform as a distribution, $\widehat{\mu}$, is a positive measure.* □

We have the following generalisation, which is mostly due to results in [BF75, Ch. I.4]

**Proposition 1.3.6** ( [BG13, Prop. 8.6]). *If $\mu \in \mathcal{M}(\mathbb{R}^d)$ is positive definite, its Fourier transform exists, and is a positive, translation bounded measure on $\mathbb{R}^d$.* □

It will be evident in the following sections that the objects we will be looking at are positive measures, and so we study further characterisations of measures of this type.

### 1.3.3. Decomposition of positive measures

In the ensuing discussion, let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ and $\mathfrak{D}$ an arbitrary Borel set in $\mathbb{R}^d$. The set $P_\mu = \{x : \mu(\{x\}) \neq 0\}$ is called the set of *pure points* of $\mu$. One defines the *pure point component* of $\mu$ to be

$$\mu_{\mathsf{pp}}(\mathfrak{D}) := \sum_{x \in \mathfrak{D} \cap P_\mu} \mu(\{x\}) = \mu(\mathfrak{D} \cap P_\mu).$$

We say that $\mu$ is *atomic* or *pure point* if $\mu(\mathfrak{D}) = \sum_{x \in \mathfrak{D}} \mu(\{x\})$, for all $\mathfrak{D}$.

Next, define $\mu_{\mathsf{c}} := \mu - \mu_{\mathsf{pp}}$ to be the *continuous component* of $\mu$. A measure $\mu$ is absolutely continuous with respect to another measure $\nu$, i.e., $\mu \ll \nu$, if $\nu(\mathfrak{D}) = 0$ implies $\mu(\mathfrak{D}) = 0$. In particular, when one chooses $\nu = \mu_{\mathrm{L}}$ to be Lebesgue measure, we have that $\mu$ is *absolutely continuous* to $\mu_{\mathrm{L}}$ if there exists $h \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that, $\mu = h\mu_{\mathrm{L}}$, i.e.,

$$\mu(g) = \int_{\mathbb{R}^d} g \, \mathrm{d}\mu = \int_{\mathbb{R}^d} g(y)h(y)\,\mathrm{d}y.$$

The locally integrable function $h$ is called the *Radon–Nikodym density* of $\mu$ with respect to $\mu_{\mathrm{L}}$. On the contrary, $\mu$ is said to be *singular* with respect to $\mu_{\mathrm{L}}$ if there is a measurable set $\mathfrak{D}$ with $\mu_{\mathrm{L}}(\mathfrak{D}) = 0$ and $\mu(\mathbb{R}^d \setminus \mathfrak{D}) = 0$. This allows one to write $\mu$ as $\mu = \mu_{\mathsf{ac}} + \mu_{\mathsf{sing}}$, where $\mu_{\mathsf{ac}}$ and $\mu_{\mathsf{sing}} = \mu|_{\mathfrak{D}}$ are its absolutely continuous and singular components, respectively. A singular measure $\mu$ with no pure points is called *singular continuous*, which we denote by $\mu = \mu_{\mathsf{sc}}$.

The mentioned characterisations imply the following result; see [RS80, Thms. 1.13 and 1.14] and [BG13, Thm. 8.3].

**Theorem 1.3.7** (Lebesgue decomposition theorem). *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$. Then, it has a unique decomposition*

$$\mu = \mu_{\mathsf{pp}} + \mu_{\mathsf{ac}} + \mu_{\mathsf{sc}}$$

*with respect to Lebesgue measure $\mu_{\mathrm{L}}$ in $\mathbb{R}^d$.* □

### 1.3.4. Autocorrelation and diffraction measure

Given two finite measures $\mu, \nu$, we define their convolution to be

$$(\mu * \nu)(g) = \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x+y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y). \tag{1.4}$$

Moreover, we have that $\mu * \nu$ is Fourier transformable, with Fourier transform $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$ computable via Eq. (1.3). It turns out that one gets a similar result if one of the measures, say $\nu$, is translation bounded.

**Proposition 1.3.8.** *Let $\mu$ be finite and $\nu$ be translation-bounded and Fourier transformable. Then, $\mu * \nu$ is a translation-bounded, Fourier transformable measure. If $\widehat{\nu}$ is also a measure, then $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$, which is seen as a measure absolutely continuous to $\widehat{\nu}$, with Radon–Nikodym density $\widehat{\mu}$.* □

We will mostly be dealing with unbounded measures, and hence we will be needing an appropriate extension of Proposition 1.3.8 in this setting. To this end, we define the analogue of Eq. (1.4) for unbounded measures.

**Definition 1.3.9.** Let $\mu$ and $\nu$ be unbounded measures in $\mathbb{R}^d$. Their *Eberlein* or *volume-averaged convolution* is defined as

$$\mu \circledast \nu := \lim_{R\to\infty} \frac{\mu_R * \nu_R}{\mathrm{vol}(\mathfrak{B}_R(0))}, \tag{1.5}$$

where $\mu_R$ (resp. $\nu_R$) is the measure $\mu$ (resp. $\nu$) restricted to $\overline{\mathfrak{B}_R(0)}$, provided the limit exists.

**Remark 1.3.10.** Under some mild assumptions on $\mu$ and $\nu$, the sequence of open balls $\{\mathfrak{B}_R(0)\}$ can be replaced by another nested averaging sequence $\mathcal{R} = \{\mathcal{R}_n\}$ so long as it satisfies the van Hove property; compare with [BG13, Def. 2.9]. ◊

In general, the limit in Eq. (1.5) need not exist, but more can be said when $\mu$ and $\nu$ are both translation bounded. To be more specific, we consider $\nu = \widetilde{\mu}$, and the finite approximants of $\mu \circledast \widetilde{\mu}$ given by

$$\gamma_\mu^{(R)} := \frac{\mu_R * \widetilde{\mu}_R}{\mathrm{vol}(\mathfrak{B}_R(0))}$$

which is well defined and positive definite for every $R > 0$. An accumulation point of the sequence $\{\gamma_\mu^{(R)}\}$ is called an *autocorrelation* of $\mu$. If the limit exists, the limit measure $\gamma_\mu$ is called the *natural autocorrelation*.

**Proposition 1.3.11** ( [BG13, Prop. 9.1]). *Let $\mu$ be a translation bounded measure and let $E = \{\gamma_\mu^{(R)}\}$ its family of approximating autocorrelations. Then, $E$ is precompact in the vague topology. Moreover, any accumulation point of this family, of which there is at least one, is translation bounded.* □

We now apply this framework to FLC point sets, which include point sets derived from inflation rules as defined in Section 1.2.4; compare with [BG13, Ex. 9.1] or [Mol13, Rem. 6.3].

**Example 1.3.12.** Let $\varLambda \subset \mathbb{R}^d$ be an FLC point set and consider the weighted Dirac comb $\omega_\varLambda$ constructed on $\varLambda$ by choosing a bounded (generally complex) weight function $W(x)$, i.e., $\omega_\varLambda := \sum_{x\in\varLambda} W(x)\delta_x$. This measure is a translation bounded, pure point measure. To see this, let $C_\omega := \sup\{|W(x)| : x \in \varLambda\} < \infty$. Direct computation then gives

$$\left|\delta_\varLambda\right|(y+K) \leqslant C_\omega \sum_{x\in\varLambda} \delta_x(y+K) \leqslant C_\omega N(K),$$

where $N(K) := \sup\{\mathrm{card}(\varLambda \cap (y+K)) \mid y \in \mathbb{R}^d\}$. Since $\varLambda$ is FLC, it is also locally finite, and hence $N(K) < \infty$, for any compact $K \subset \mathbb{R}^d$, which implies that $C_\omega N(K) < \infty$, implying our claim via Definition 1.3.3.

The autocorrelation resulting from $\omega_\Lambda \circledast \widetilde{\omega_\Lambda}$ is of the form $\gamma_\omega = \sum_{z\in\Lambda-\Lambda} \eta_\omega(z)\delta_z$, where the *autocorrelation coefficients* $\eta_\omega(z)$ can explicitly be written as

$$\eta_\omega(z) = \lim_{R\to\infty} \frac{1}{\mathrm{vol}(\mathfrak{B}_R(0))} \sum_{\substack{x\in\Lambda^{(R)} \\ x-z\in\Lambda}} W(x)\overline{W(x-z)}, \tag{1.6}$$

with $\Lambda^{(R)} := \Lambda \cap \overline{\mathfrak{B}_R(0)}$. ◊

Going back to the general picture, we are now ready to define one of the main objects in this work, which is the diffraction measure.

**Definition 1.3.13.** Let $\mu$ be translation bounded with a well-defined autocorrelation $\gamma_\mu$. The Fourier transform $\widehat{\gamma_\mu}$ is called the *diffraction measure* of $\mu$.

The measure $\gamma_\mu$ is positive definite by construction, and hence Fourier transformable by Proposition 1.3.6. Moreover, $\widehat{\gamma_\mu} \in \mathcal{M}^+(\mathbb{R}^d)$. Invoking Theorem 1.3.7, we get that the diffraction measure splits into

$$\widehat{\gamma_\mu} = \big(\widehat{\gamma_\mu}\big)_{\mathsf{pp}} + \big(\widehat{\gamma_\mu}\big)_{\mathsf{ac}} + \big(\widehat{\gamma_\mu}\big)_{\mathsf{sc}}.$$

One of the major objectives of mathematical diffraction is to understand fundamental implications of properties of $\mu$ to the three components of $\widehat{\gamma_\mu}$.

**Remark 1.3.14.** In an actual X-ray diffraction experiment, the support of the measure $\big(\widehat{\gamma_\mu}\big)_{\mathsf{pp}}$ corresponds to points of high intensities in the diffraction image, which are called *Bragg peaks*. The continuous component describes the *diffuse diffraction* characterised by a noisy background superimposed with the peaks, which usually suggests a certain level of disorder. ◊

## 1.4. Lyapunov exponents

### 1.4.1. Lyapunov exponents for sequences of matrices

In this section, we follow the introduction of general notions and results in the monographs [BP07, Via13].

**Definition 1.4.1.** Given a sequence $\{M_j\}_{j\geqslant 0}$ of matrices in $\mathrm{Mat}(d,\mathbb{C})$, satisfying the condition $\sup_j \|M_j\| < \infty$, one can consider its *Lyapunov exponent* $\chi : \mathbb{C}^d \to \mathbb{R}\cup\{-\infty\}$ defined by

$$\chi(v) = \limsup_{n\to\infty} \frac{1}{n} \log \|M^{(n)} v\|, \tag{1.7}$$

where we have set $M^{(n)} := M_{n-1}M_{n-2}\cdots M_1 M_0$.

Here, we follow the convention that $\log(0) = -\infty$. We also note that $\chi(v)$ does not depend on the norm $\|\cdot\|$ chosen as they are all equivalent. It follows from standard dimension arguments that $\chi(v)$ takes at most $d$ different values: $\chi_1 \geqslant \ldots \geqslant \chi_{d'}$, where $d' \leqslant d$.

From these, one can construct a *filtration* of $\mathbb{C}^d$, i.e., a sequence of subspaces $\{\mathcal{V}^i\}_{i=1}^{d'}$

$$\mathbb{C}^d = \mathcal{V}^1 \supsetneq \mathcal{V}^2 \supsetneq \ldots \supsetneq \mathcal{V}^{d'} \neq \{0\} \tag{1.8}$$

such that $\chi(v) = \chi_i$, for all $v \in \mathcal{V}^i \setminus \mathcal{V}^{i+1}$.

**Remark 1.4.2.** In the case where $\{M_j\}_{j\geqslant 0}$ is made up of a single matrix $M$, the Lyapunov exponents $\chi_i$ are given by $\log|\lambda_i|$, where $\lambda_i$ are the eigenvalues of $M$ and, $\mathcal{V}^i \setminus \mathcal{V}^{i+1}$ are the corresponding (possibly generalised) eigenspaces. When $\{M_j\}_{j\geqslant 0}$ is a convergent sequence with limit $M$, the values of the exponents are also determined by the eigenvalues of $M$. $\Diamond$

A sequence $\{M_j\}_{j\geqslant 0}$ is said to be *forward regular* if

$$\lim_{n\to\infty} \frac{1}{n} \log \left| \det\left(M^{(n)}\right) \right| = \sum_{i=1}^{d} \chi_i', \tag{1.9}$$

provided that the limit exists. Here, $\chi_1' \geqslant \ldots \geqslant \chi_d'$ are the values attained by $\chi$, counted with their multiplicities. Mere existence of the limit does not guarantee forward regularity.

The numbers $\chi_i$ are also related to the singular values of $M^{(n)}$; see [BV17]. Denote by $\mathrm{sing}(M^{(n)})$ the set of singular values $\sigma_1(n) \geqslant \ldots \geqslant \sigma_d(n) \geqslant 0$ of $M^{(n)}$, i.e., the set of eigenvalues of the positive definite matrix $\left(\left(M^{(n)}\right)^\dagger M^{(n)}\right)^{1/2}$. Then, the exponents satisfy

$$\limsup_{n\to\infty} \frac{1}{n} \log \sigma_i(n) \leqslant \chi_i, \ \text{for } 1 < i \leqslant d \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{1}{n} \log \sigma_1(n) = \chi_1.$$

### 1.4.2. Matrix cocycles

One way to generate sequences of matrices is via cocycles. Consider a measure-preserving dynamical system $(X, f, \mu)$ and a measurable matrix-valued map $A : X \to \mathrm{Mat}(d, \mathbb{C})$.

**Definition 1.4.3.** A skew linear map $F : X \times \mathbb{C}^d \to X \times \mathbb{C}^d$ defined by $(x, v) \mapsto \big(f(x), A(x)v\big)$ is called a *linear cocycle over* $f$, where $f$ is the *base dynamics* of the cocycle.

We call $F$ *ergodic* over $(X, f, \mu)$ if $f$ is ergodic. An iteration of this function yields the pair $F^n(x, v) = (f^n(x), A^{(n)}(x)v)$, where the induced fibre action on $\mathbb{C}^d$ is determined by the matrix product

$$A^{(n)}(x) = A(f^{n-1}(x)) \cdot \ldots \cdot A(f(x))A(x).$$

Unless otherwise stated, we assume the base dynamics to be fixed, and we refer to $A^{(n)}(k)$ as the matrix cocycle.

**Example 1.4.4.**

**(1)** Let $\Omega \subset \mathrm{Mat}(d, \mathbb{C})$ be compact. Let $X = \Omega^{\mathbb{Z}}$ with the (left-sided) shift operator $S$ on $X$, with $(Sx)_k = x_{k+1}$, for $\{x_k\}_{k\in\mathbb{Z}} \in \Omega^{\mathbb{Z}}$, and $\mu$ a probability measure on $\Omega$. Consider the locally constant map $A : x \mapsto A(x_0)$. Then, $(S, A)$ defines a cocycle over $X \times \mathbb{C}^d$. Furthermore, $S$ is ergodic with respect to the product measure $\mu^{\mathbb{Z}}$.

**(2)** Let $X = \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, $A : \mathbb{T}^d \to \mathrm{Mat}(d, \mathbb{C})$, and $\widetilde{M}$ be a *toral endomorphism* given by $\widetilde{M} : x \mapsto (Mx) \mod 1$, where $M \in \mathrm{Mat}(d, \mathbb{Z})$. It is well known that $\widetilde{M}$ is ergodic with respect to Lebesgue measure whenever $\det M \neq 0$ and $M$ does not have eigenvalues which are roots of unity [EW11, Cor. 2.20], and is invertible whenever $M \in \mathrm{GL}(d, \mathbb{Z})$, i.e., $\det M = \pm 1$. As in the first example, $(\widetilde{M}, A)$ defines a matrix cocycle. $\Diamond$

For sequences arising from cocycles, more specific versions of Eq. (1.7) and Eq. (1.8) for the Lyapunov exponent $\chi : \mathbb{C}^d \times X \to \mathbb{R} \cup \{-\infty\}$ and the $x$-dependent filtration it defines read

$$\chi(v,x) = \limsup_{n\to\infty} \frac{1}{n} \log \|A^{(n)}(x)v\| \qquad \text{and} \qquad \mathbb{C}^d = \mathcal{V}_x^1 \supsetneq \mathcal{V}_x^2 \supsetneq \ldots \supsetneq \mathcal{V}_x^{d'(x)} \neq \{0\}\,,$$

with $\chi(v,x) = \chi_i(x)$, for all $v \in \mathcal{V}_x^i \setminus \mathcal{V}_x^{i+1}$. We say that $A^{(n)}(x)$ at a given point $x$ is forward regular if the sequence $\{A(f^n(x))\}_{n\geqslant 0}$ is forward regular.

**Lemma 1.4.5.** *Let $v \in \mathbb{C}^d \setminus \{0\}\,, x \in X$. Assuming $A^{(n)}(x)^{-1}$ exists, one has,*

$$\chi_{\min}(x) \leqslant \chi(x,v) \leqslant \chi_{\max}(x),$$

*where*

$$\chi_{\max}(x) = \limsup_{n\to\infty} \frac{1}{n} \log \|A^{(n)}(x)\| \quad \text{and} \quad \chi_{\min}(x) = \liminf_{n\to\infty} \frac{1}{n} \log \|A^{(n)}(x)^{-1}\|^{-1}.$$

*Proof.* Note that the following holds for all non-zero $v$,

$$\|A^{(n)}(x)^{-1}\|^{-1}\|v\| \leqslant \|A^{(n)}(x)v\| \leqslant \|A^{(n)}(x)\|\|v\|.$$

The claim then directly follows by taking the logarithm, and the lim sup and the lim inf of the upper and the lower bound, respectively. □

Define $\phi^+(x) := \max\{0, \phi(x)\}$. The following result on the extremal exponents is due to Furstenberg and Kesten [FK60]; see also [Via13, Thm. 3.12].

**Theorem 1.4.6** (Furstenberg–Kesten)**.** *Let $F : X \times \mathbb{R}^d \to X \times \mathbb{R}^d$ be a matrix cocycle defined by $F(x,v) = (f(x), A(x)v)$, where $A : X \to \mathrm{GL}(d,\mathbb{R})$ is measurable, and $X$ is compact. If $\log^+ \|A^{\pm 1}\| \in L^1(\mu)$, the extremal exponents $\chi_{\min}(x)$ and $\chi_{\max}(x)$ exist as limits for a.e. $x \in X$. Moreover, these functions are $f$-invariant and are $\mu$-integrable.* □

Note that since we assume compactness of $X$ in Theorem 1.4.6, local integrability of $\log^+ \|A^{\pm 1}\|$ is equivalent to integrability; see [Din74, Sec. 15] .

### 1.4.3. Ergodic theorems

The following generalisation of Birkhoff's ergodic theorem for subadditive functions is due to Kingman [Kin73]; compare [Via13, Thm. 3.3].

**Theorem 1.4.7** (Kingman's subadditive ergodic theorem)**.** *Assume $f : X \to X$ to be ergodic with respect to $\mu$. Let $\{\phi_n\}$ be a sequence of functions such that $\phi_1^+$ is $\mu$-integrable and*

$$\phi_{m+n} \leqslant \phi_m + \phi_n \circ f^m \qquad \textit{holds for all } m,n \geqslant 1.$$

*Then,*

$$\lim_{n\to\infty} \frac{1}{n}\phi_n(x) = c = \inf_n \int_X \phi_n(\xi)\,\mathrm{d}\mu(\xi).$$

*for $\mu$-a.e. $x \in X$.* □

For ergodic real-valued cocycles, one has the following central result in the theory of Lyapunov exponents, which is due to Oseledec [Ose68]; see [Via13, Thm. 4.1] and [BP07, Thm. 3.4.3].

**Theorem 1.4.8** (Oseledec's multiplicative ergodic theorem)**.** *Let $f$ be an ergodic transformation of the probability space $(X, \mu)$. Let $A : X \to \mathrm{GL}(d, \mathbb{R})$ be measurable, such that the condition $\log^+ \|A\| \in L^1(\mu)$ holds. Then, for $\mu$-a.e. $x \in X$, the cocycle $A^{(n)}(x)$ is forward regular. Moreover, for these $x$, the Lyapunov exponents $\chi_i(x)$ are constant, i.e., there exist real numbers $\chi_1, \dots, \chi_{d'}$, and a filtration*

$$\mathbb{R}^d = \mathcal{V}_x^1 \supsetneq \mathcal{V}_x^2 \supsetneq \dots \supsetneq \mathcal{V}_x^{d'} \neq \{0\}$$

*such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^{(n)}(x) v_i\| = \chi_i$$

*for all $v_i \in \mathcal{V}_x^i \setminus \mathcal{V}_x^{i+1}$.* □

**Remark 1.4.9** (Exponents from singular values)**.** In the case where Oseledec's theorem holds, the exponents can also be expressed in terms of the singular values $\sigma_1(n) \geqslant \dots \geqslant \sigma_d(n) \geqslant 0$ of $A^{(n)}(x)$, i.e.,

$$\chi_i(x) = \frac{1}{n} \lim_{n\to\infty} \log \sigma_i(n),$$

for a.e. $x \in X$. ♢

**Remark 1.4.10** (cocycles with invertible dynamics)**.** There exists an even stronger notion of regularity, also known as *Lyapunov–Perron regularity.* This requires both the matrix-valued function $A$ and the map $f$ to be invertible so that one can define $A^{(n)}(x)$, for $n < 0$. Under these invertibility assumptions, and that $\log^+ \|A^{-1}\| \in L^1(\mu)$, one gets a two-sided version of Theorem 1.4.8. ♢

Theorem 1.4.6 and Theorem 1.4.8 can easily be extended to complex-valued cocycles since complex matrices could be realised as real maps. The following "realification" scheme is used in [DK14, Sec. 8] and [BHJ03, Sec. 5]. Let $f(x) := f_1(x) + \mathrm{i} f_2(x)$, where $f_1(x)$ and $f_2(x)$ are real-valued functions. We associate to $f$ the matrix

$$f_{\mathbb{R}} := \begin{pmatrix} f_1 & f_2 \\ -f_2 & f_1 \end{pmatrix} \in \mathrm{Mat}(2, \mathbb{R}), \qquad \text{for all } x \in X. \tag{1.10}$$

The *realification* $\mathfrak{R} : \mathrm{Mat}(d, \mathbb{C}) \to \mathrm{Mat}(2d, \mathbb{R})$ is the map that sends $A(x)$ to a real-valued cocycle by sending each entry $A_{\ell j}(x)$ to a $2 \times 2$-block via Eq. (1.10). In other words

$$A(x) = A_1(x) + \mathrm{i} A_2(x) \mapsto \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}.$$

We denote the realification of $A(x)$ as $A_{\mathbb{R}}(x) := \mathfrak{R}(A(x))$. The following results highlight some properties of $A_{\mathbb{R}}(x)$.

**Proposition 1.4.11** ( [DK14, Prop. 8.1])**.** *Let $A$ be a complex matrix cocycle and $A_{\mathbb{R}}$ its realification. Then, the following hold*

**(1)** $\det(A_{\mathbb{R}}) = \left|\det(A)\right|^2$

**(2)** $\mathrm{sing}(A_{\mathbb{R}}) = \mathrm{sing}(A)$, *with each singular value of* $A$ *appearing twice in* $\mathrm{sing}(A_{\mathbb{R}})$

**(3)** $\|A_{\mathbb{R}}\| = \|A\|$. □

**Proposition 1.4.12** ( [DK14, Prop. 8.2])**.** *Any integrable cocycle* $A : \mathbb{T} \to \mathrm{Mat}(d, \mathbb{C})$ *has the same Lyapunov exponents as* $A_{\mathbb{R}}$.

*Proof.* This follows from Proposition 1.4.11 and Remark 1.4.9. □

For the entirety of this work, when we mention that Theorem 1.4.6 or Theorem 1.4.8 holds for specific cocycles, we mean that it holds for their realifications and that the (complex) filtration is derived from the real one.

## 1.5. Polynomials and Mahler measures

**Definition 1.5.1.** Let $p(z) \in \mathbb{C}[z] \setminus \{0\}$. Its *logarithmic Mahler measure* $\mathfrak{m}(p)$ is given by its geometric mean over the unit circle, which formally reads

$$\mathfrak{m}(p) = \int_0^1 \log \left| p(\mathrm{e}^{2\pi \mathrm{i} t}) \right| \mathrm{d}t. \tag{1.11}$$

This interpretation as a mean allows one to extend this definition to polynomials in several variables via

$$\mathfrak{m}(p(z_1, \ldots, z_d)) = \int_{\mathbb{T}^d} \log \left| p(\mathrm{e}^{2\pi \mathrm{i} t_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i} t_d}) \right| \mathrm{d}t_1 \ldots \mathrm{d}t_d.$$

The logarithmic Mahler measure $\mathfrak{m}(\alpha)$ of an algebraic number $\alpha$ is $\mathfrak{m}(p_\alpha)$, where $p_\alpha$ is the monic minimal polynomial of $\alpha$. This notion can also be extended to Laurent polynomials $f \in \mathbb{C}[z^{\pm 1}]$, where we identify $f$ with a polynomial in $p \in \mathbb{C}[z]$, where the two differ by a (multiplied) monomial factor. In the one-dimensional case, Jensen's formula relates this mean to the zeros of $p(z) = c_s \prod_i (z - \alpha_i)$ outside the unit circle; see [Sch95, Prop. 16.1]. This relation explicitly reads

$$\mathfrak{m}(p) = \log |c_s| + \sum_{j=1}^{s} \log(\max \{|\alpha_j|, 1\}). \tag{1.12}$$

In most references, one usually deals with $\mathfrak{M}(p) := \exp(\mathfrak{m}(p))$, which is what is referred to as the *Mahler measure* of $p$. We refer the reader to [Smy08] for a general survey on Mahler measures.

**Lemma 1.5.2.** *Let* $p(z) = c_0 + c_1 z + \ldots + c_s z^s \in \mathbb{C}[z]$, *where* $p$ *is not a monomial. Then,*

$$0 \leqslant \mathfrak{m}(p) < \log \sqrt{\sum_i |c_i|^2}.$$

*Proof.* Since the exponential function is strictly convex, Jensen's inequality is applicable and so we have

$$1 \leqslant \mathfrak{M}(p) < \int_0^1 \left| p(\mathrm{e}^{2\pi \mathrm{i} t}) \right| \mathrm{d}t = \|p\|_1 < \|p\|_2,$$

where both inequalities are strict because $p$ is not a monomial and hence $\left| p(\mathrm{e}^{2\pi \mathrm{i} t}) \right|$ is not constant; see [LL01, Ch. 2.2]. Invoking Parseval's equality, i.e., $\|p\|_2^2 = \sum_i |c_i|^2$, and taking the logarithm of the nested inequality implies the claim. □

When one restricts to polynomials with integer coefficients, one has the following result due to Kronecker.

**Theorem 1.5.3** ( [Kro57]). *Let $p \in \mathbb{Z}[z]$. Then, $\mathfrak{m}(p) = 0$ if and only if $p$ is a product of a monomial and a cyclotomic polynomial.* □

In 1933, D.H. Lehmer asked whether for every $\varepsilon > 0$, there exists a polynomial $p \in \mathbb{Z}[z]$ such that $0 < \mathfrak{m}(p) \leqslant \varepsilon$ [Leh33]. What is currently known as *Lehmer's problem*, conjectures the opposite, i.e., there is a constant $c$ such that, for all $p \in \mathbb{Z}[z]$ with $\mathfrak{m}(p) \neq 0$, one has $\mathfrak{m}(p) \geqslant c$. Evading a general proof, this is a famous long-standing open problem in number theory.

## 1.6. Almost periodic functions and discrepancy analysis

A continuous function $f$ is said to be *Bohr-almost periodic* if for every $\varepsilon > 0$, the set of $\varepsilon$-almost periods

$$\mathcal{AP}_{\varepsilon}(f) := \{t \in \mathbb{R} : \|f - T_t f\|_{\infty} < \varepsilon\}$$

is relatively dense in $\mathbb{R}$. Here, $T_t(f) = f(x-t)$ are the translates of $f$. Bohr-almost periodicity implies boundedness and uniform continuity of $f$.

Define the *Stepanov norm* on $L^1_{\text{loc}}(\mathbb{R})$ to be

$$\|f\|_{\text{S}} := \sup_{x \in \mathbb{R}} \frac{1}{L} \int_x^{x+L} |f(y)| \mathrm{d}y.$$

These norms are equivalent for different $L$, which allows one to fix $L = 1$ unambiguously. A function $f$ is called *Stepanov-almost periodic* if for every $\varepsilon > 0$, the set of $\varepsilon$-almost periods of $f$, this time with respect to $\|\cdot\|_{\text{S}}$, is relatively dense in $\mathbb{R}$. The set of Bohr-almost periodic functions is contained in this class. We refer to [BG13, Sec. 8.2] for a concise introduction on almost periodic functions and to [MS17] for a comprehensive review in relation to almost periodic measures.

For a (Bohr or Stepanov) almost periodic function $f$, its *mean* $\mathbb{M}(f)$ is defined as

$$\mathbb{M}(f) := \lim_{T \to \infty} \frac{1}{2T} \int_{r-T}^{r+T} f(x) \mathrm{d}x,$$

which exists and is independent of $r$; see [Bes54].

A sequence $(x_n)_{n \in \mathbb{N}}$ of real numbers is said to be *uniformly distributed modulo* 1 if, for all $a, b \in \mathbb{R}$, with $0 \leqslant a < b \leqslant 1$ we have

$$\lim_{N \to \infty} \frac{1}{N} \text{card}\big([a,b) \cap \{\langle x_1 \rangle, \ldots, \langle x_n \rangle\}\big) = b - a,$$

where $\langle x \rangle$ denotes the fractional part of $x$.

**Fact 1.6.1** ( [BHL17, Fact 6.2.3]). *Consider $(\alpha^n x)_{n \in \mathbb{N}}$. For a fixed $\alpha \in \mathbb{R}$, $|\alpha| > 1$, this sequence is uniformly distributed mod 1, for a.e. $x \in \mathbb{R}$.* □

Given a sequence $(x_n)_{n \in \mathbb{N}}$, its *discrepancy* is defined as

$$\mathcal{D}_N = \sup_{0 \leqslant a < \leqslant 1} \left| \frac{1}{N} \text{card}\big([a,b) \cap \{\langle x_1 \rangle, \ldots, \langle x_n \rangle\}\big) - (b-a) \right|.$$

Note that being uniformly distributed mod 1 is equivalent to $\mathcal{D}_N \to 0$, as $N \to \infty$.

**Fact 1.6.2** ( [BHL17, Fact 6.2.5]). *Let $\alpha \in \mathbb{R}$, $|\alpha| > 1$. For any fixed $\varepsilon > 0$, the asymptotic behaviour of $\mathcal{D}_N$ for $(\alpha^n x)_{n\in\mathbb{N}}$ is given by*

$$\mathcal{D}_N = \mathcal{O}\left(\frac{(\log(N))^{\frac{3}{2}+\varepsilon}}{\sqrt{N}}\right) \tag{1.13}$$

*for a.e. $x \in \mathbb{R}$.* □

The following generalisations of a theorem by Sobol on averages of (possibly) unbounded functions sampled along uniformly distributed sequences [Sob73] are due to Baake, Haynes, and Lenz; see [BHL17].

**Theorem 1.6.3** ( [BHL17, Thm. 6.4.4]). *Let $\alpha \in \mathbb{R}$ with $|\alpha| > 1$ be given, and let $f$ be Bohr-almost periodic on $\mathbb{R}$. Then, for a.e. $x \in \mathbb{R}$, one has*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\alpha^n x) = \mathbb{M}(f),$$

*where $\mathbb{M}(f)$ is the* mean *of $f$.* □

**Theorem 1.6.4** ( [BHL17, Thm. 6.4.8]). *Let $\alpha \in \mathbb{R}$ with $|\alpha| > 1$ be given, and let $f \in L^1_{\text{loc}}(\mathbb{R})$ be Stepanov almost periodic. Assume that there is a uniformly discrete set $Y \subset \mathbb{R}$ such that $f$, for every $\delta > 0$, is locally Riemann integrable on the complement of $Y + (-\delta, \delta)$. Assume further that there is a $\delta' > 0$ such that, for any $z \in Y$, $f$ is differentiable on the punctured interval $(z - \delta', z + \delta') \setminus \{z\}$ and that, for any $s > 0$,*

$$V_N(s) := \sup_{z\in Y} \left( \int_{z-\delta'}^{z-\frac{1}{N^s}} |f'(x)|\mathrm{d}x + \int_{z+\frac{1}{N^s}}^{z+\delta'} |f'(x)|\mathrm{d}x \right) = \mathcal{O}\big(N^{\frac{s}{2}-r}\big) \tag{1.14}$$

*holds for some $r > 0$ as $N \to \infty$. Then, one has*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\alpha^n x) = \mathbb{M}(f)$$

*for a.e. $x \in \mathbb{R}$.* □

# 2. Renormalisation for Pair Correlations and Absence of Absolutely Continuous Diffraction

In this chapter, we develop a renormalisation scheme satisfied by some ergodic quantities arising from an inflation $\varrho$. We show that this extends to renormalisation schemes satisfied by constituent measures of the autocorrelation $\gamma$ and the diffraction $\widehat{\gamma}$, respectively. The last three sections are dedicated to the main results of this thesis, which are quantitative results relating Lyapunov exponents and absolutely continuous diffraction.

Here, we assume $\varrho$ to be primitive, aperiodic and one-dimensional. A brief remark will be made on how some arguments extend to periodic tilings. Higher-dimensional analogues will be treated in Chapter 5. As described in Section 1.2.4, one can build an inflation dynamical system $(\mathbb{Y}, \mathbb{R})$ from $\varrho$ that is invariant with respect to $\varrho$ regarded as an inflation, and where elements of $\mathbb{Y}$ are translates of geometric realisations of elements of the symbolic hull $\mathbb{X}$.

## 2.1. Fourier matrix and inflation displacement algebra

We now define the main object of study, which is the *Fourier matrix* associated to $\varrho$. Given an inflation $\varrho$ with inflation multiplier $\lambda = \lambda_{\text{PF}}$, we specify the left-most position of a prototile $\mathfrak{t}$ to be its control point. Define the *displacement matrix* $T = (T_{ij})$ by

$$T_{ij} := \{\text{relative positions of } \mathfrak{t}_i \text{ in the supertile } \varrho(\mathfrak{t}_j)\}. \tag{2.1}$$

Entries of this matrix are called *displacement sets* , whose elements are contained in $\alpha\mathbb{Z}[\lambda]$, for some $\alpha \in \mathbb{Q}[\lambda]$. We also define the *total set* $S_T$ to be the union of all displacement sets, i.e., $S_T = \bigcup_{ij} T_{ij}$.

**Definition 2.1.1.** The *Fourier matrix* $B(k)$ is entrywise defined to be

$$B_{ij}(k) := \sum_{t \in T_{ij}} \mathrm{e}^{2\pi \mathrm{i} t k}. \tag{2.2}$$

In a measure-theoretic sense, one can also define it using the Fourier transform for Dirac combs on finite sets, i.e., $B(k)_{ij} := \overline{\widehat{\delta_{T_{ij}}}(k)} = \widehat{\delta_{T_{ij}}}(-k)$.

This matrix is composed of trigonometric polynomials. Moreover, the number of distinct frequencies present in the constituent polynomials is the algebraic degree of $\lambda$. Evaluation at $k = 0$ gives the substitution matrix, i.e., $B(0) = M_{\varrho}$. Furthermore, it satisfies the symmetry relation $B(k) = \overline{B(-k)}$, which enables us to restrict our analysis to $\mathbb{R}_+$. Another way of writing it would be $B(k) = \sum_{t \in S_T} \mathrm{e}^{2\pi \mathrm{i} t k} D_t$, where the $D_t$ are 0-1 matrices given by

$$(D_t)_{ij} = \begin{cases} 1, & \text{if } \varrho(a_j) \text{ contains a tile of type } a_i \text{ at position } t, \\ 0, & \text{otherwise }, \end{cases}$$

which we call *digit matrices*, to be consistent with the notation for constant-length substitutions, but are also referred to as *instruction matrices* in [Bar16, Que10].

**Example 2.1.2** (Square of the Fibonacci)**.** Let $\varrho_{\mathrm{F}}^2$ be the square of the Fibonacci substitution, which reads $\varrho_{\mathrm{F}}^2 : a \mapsto aba, b \mapsto a$. Its realisation as an inflation rule, together with the markers of the tile positions, is given in Figure 2.1.

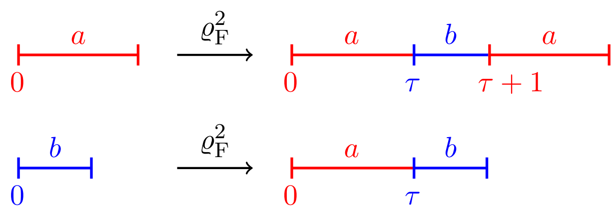



Figure 2.1.: The square of $\varrho_{\mathrm{F}}$ viewed as an inflation rule.

For $\varrho_{\mathrm{F}}^2$, the displacement matrix $T$ and the Fourier matrix $B(k)$ respectively read

$$T = \begin{pmatrix} \{0, \tau+1\} & \{0\} \\ \{\tau\} & \{\tau\} \end{pmatrix} \qquad \text{and} \qquad B(k) = \begin{pmatrix} 1 + \mathrm{e}^{2\pi\mathrm{i}(\tau+1)k} & 1 \\ \mathrm{e}^{2\pi\mathrm{i}\tau k} & \mathrm{e}^{2\pi\mathrm{i}\tau k} \end{pmatrix}.$$

Here, the set of digit matrices that constitute $B(k)$ is given by $\{D_t\} = \{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\}$. ◊

For a given $\varrho$, we define its *inflation displacement algebra (IDA)* $\mathcal{B}$ to be the $\mathbb{C}$-algebra generated by the family $\{B(k) \mid k \in \mathbb{R}\}$, which is a finite-dimensional complex algebra.

**Fact 2.1.3.** *The IDA $\mathcal{B}$ is the same as the $\mathbb{C}$-algebra $\mathcal{B}_D$ generated by the collection of all digit matrices $\{D_x \mid x \in S_T\}$.* □

An algebra is *irreducible* if the only invariant subspaces with respect to the entire set are $\{0\}$ and the full space $\mathbb{C}^{n_a}$. The equivalence $\mathcal{B} = \mathcal{B}_D$ turns out to be very useful when determining the IDA explicitly or when analysing whether it is irreducible or not, because these translate to analogous questions on a finitely-generated algebra.

We also have the following inclusion result between the IDA $\mathcal{B}$ of $\varrho$ and the IDA $\mathcal{B}^{(n)}$ of one of its powers $\varrho^n$.

**Fact 2.1.4.** *Let $B(k)$ be the Fourier matrix of $\varrho$ as defined in Eq. (2.2). Then, for any $n \in \mathbb{N}$, the Fourier matrix of $\varrho^n$ is given by $B^{(n)}(k) = B(k)B^{(n)}(\lambda k) = B(k)B(\lambda k) \cdot \ldots \cdot B(\lambda^{n-1}k)$.* □

**Lemma 2.1.5.** *Let $\varrho$ be a primitive substitution over a finite alphabet with $n_a$ letters and corresponding inflation rule with (fixed) natural tile lengths. If $m, n \in \mathbb{N}$ with $m|n$, one has $\mathcal{B}^{(n)} \subseteq \mathcal{B}^{(m)}$. In particular, $\mathcal{B}^{(n)} \subseteq \mathcal{B}^{(1)} = \mathcal{B}$.* □

From Burnside's theorem, it follows that an IDA $\mathcal{B}$ of dimension $d$ is irreducible if and only if $\mathcal{B} = \mathrm{Mat}(d, \mathbb{C})$. The following criterion guarantees that the IDA of $\varrho$ is irreducible. We provide a sketch of the proof here and refer to [BGäM18, Prop. 3.8] for the full proof.

**Proposition 2.1.6** ( [BGäM18, Prop. 3.8])**.** *Let $\varrho$ be a primitive substitution over a finite alphabet with $n_a \geqslant 2$ letters. If the natural prototile lengths are distinct, the IDA of $\varrho$ is $\mathcal{B} = \mathrm{Mat}(d, \mathbb{C})$ and hence irreducible.*

*Sketch of proof.* Primitivity of $\varrho$ implies that every element of the hull is linearly repetitive, and so every legal patch of length $\ell \geqslant \ell_r$ contains at least one copy of each prototile, for some $\ell_r > 0$. In particular, for some $N \in \mathbb{N}$, all $n$-level supertiles satisfy this property for $n \geqslant N$. When one orders the tiles in descending length, this allows one to recover all elementary matrices $E_{i,1}$, where $1 \leqslant i \leqslant n_a$, as digit matrices. One then proceeds to the second longest supertile, from which one gets $E_{i,2}$ via digit matrices corresponding to $\varrho^n(a_2)$ and, possibly, their linear combinations with the matrices $E_{i,1}$. Continuing this process yields $\mathcal{B}^{(n)} = \mathrm{Mat}(n_a, \mathbb{C})$. Lemma 2.1.5 then implies $\mathcal{B} = \mathrm{Mat}(n_a, \mathbb{C})$, and hence irreducible by Burnside's theorem. □

**Example 2.1.7.**

**(1)** For a bijective constant-length substitution (see Section 3.3 for the definition), the digit matrices $D_t$ are all permutation matrices. It follows from standard representation theory that if the group $G$ generated by the corresponding permutations is the full symmetric group $\Sigma_{n_a}$, the resulting IDA $\mathcal{B} \cong \mathrm{Mat}(n_a - 1, \mathbb{C}) \oplus \mathbb{C}$. One gets the same algebra if one instead has the alternating group $\mathfrak{A}_{n_a}$ because the standard representation $U_{\mathsf{st}}$ remains irreducible when restricted to $\mathfrak{A}_{n_a}$.

When $G$ is Abelian, the matrices $B(k)$ are simultaneously diagonalisable, which yields $\mathcal{B} \cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n \text{ terms}}$.

**(2)** The *period-doubling* substitution $\varrho_{\mathrm{pd}} : 0 \mapsto 01, 1 \mapsto 00$ is also constant-length but is not bijective. The digit matrices given by $\{D_t\} = \{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\}$ generate the three-dimensional algebra

$$\mathcal{B}_{\mathrm{pd}} = \left\{ \begin{pmatrix} c_1 + c_2 & c_1 \\ c_3 & c_2 + c_3 \end{pmatrix} : c_1, c_2, c_3 \in \mathbb{C} \right\}.$$

**(3)** Let $\varrho_{\mathrm{F}}^2$ be as in Example 2.1.2. The IDA $\mathcal{B} = \mathrm{Mat}(2, \mathbb{C})$ by Proposition 2.1.6. In general, this is true for all irreducible substitutions. ◊

A case wherein the inclusion in Lemma 2.1.5 is strict is given in the next example.

**Example 2.1.8.** Consider the alphabet $\mathcal{A}_4 = \{a, b, c, d\}$, and the constant-length substitution $\varrho$, whose first two iterates are given by

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \xmapsto{\varrho} \begin{bmatrix} ad \\ bc \\ da \\ cb \end{bmatrix} \xmapsto{\varrho} \begin{bmatrix} adcb \\ bcda \\ cbad \\ dabc \end{bmatrix}.$$

The corresponding IDA is generated by permutation matrices determined by the columns. In the first iteration, the columns viewed as elements of symmetric group $\Sigma_4$ are $(cd)$ and $(adbc)$, which generate a subgroup isomorphic to the dihedral group $D_4$. This means $\mathcal{B}^{(1)}$ is 6-dimensional, and is isomorphic to $\mathbb{C} \oplus \mathbb{C} \oplus \mathrm{Mat}(2, \mathbb{C})$. Here the restriction of $U_{\mathsf{st}}$ splits as a sum of two irreducible representations, of dimension 2 and 1, respectively. For $\varrho^2$, however, the associated group is isomorphic to Klein's 4-group $C_2 \times C_2$, which is Abelian and hence generates a 4-dimensional, commutative algebra. Here, one generally has that $\mathcal{B}^{(2n)} = \mathcal{B}^{(2)}$ and $\mathcal{B}^{(2n+1)} = \mathcal{B}^{(1)}$ for all $n \in \mathbb{N}$. ◊

The converse of Proposition 2.1.6 does not hold in general, as we shall see next.

**Example 2.1.9.** We consider the return word encoding of a reordered variant of $\varrho_{\mathrm{RS}}$ given by $\varrho_{\mathrm{RS}'} : 0 \mapsto 02, 1 \mapsto 32, 2 \mapsto 01, 3 \mapsto 31$; see [Dur98] for general background on return words and [BR17] for applications to local recognisability. Running the algorithm for the letter 0 yields eight distinct right-collared return words, namely

$$01|_0,\ 02|_0,\ 0131|_0,\ 013132|_0,\ 01313231|_0,\ 02323132|_0,\ 0232313231|_0,$$

which one can use as letters to build an eight-letter substitution on $\mathcal{A}' = \{a, b, \ldots, h\}$ defining a hull MLD to that of $\varrho_{\mathrm{RS}'}$. This is given by $\varrho_{\mathrm{ret}} = (d, ba, g, bca, ha, he, bcfa, bcfe)$, whose substitution matrix $M_\varrho$ reads

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

It has eigenvalues $2, \pm\sqrt{2}, -1, 0$, where the eigenvalue 0 corresponds to a size-4 Jordan block. One can choose the left PF eigenvector to be $\boldsymbol{L} = (2, 2, 4, 4, 6, 8, 8, 10)$ to match the symbolic lengths of the identified return words.

For this substitution, one can algorithmically show the irreducibility of the IDA $\mathcal{B}$ via Lemma 2.1.5 and Fact 2.1.3. The digit matrices of $\varrho$ can be flattened into elements of $\mathbb{C}^{64}$. One proceeds by investigating whether, for some $n \geqslant 1$, the span of the digit matrices for $B^{(n)}$ is of dimension 64. Indeed, this is the case for $n = 6$. By Lemma 2.1.5,

$$\operatorname{span}\left\{D_x^{(6)}\right\} = \operatorname{Mat}(8, \mathbb{C}) \subseteq \mathcal{B}_D^{(6)} = \mathcal{B}^{(6)} \subseteq \mathcal{B}^{(1)} = \mathcal{B}.$$

◊

## 2.2. Pair correlation functions

As a starting point for diffraction analysis, we consider a fixed point $\varLambda \in \mathbb{Y}$ generated by a primitive inflation rule $\varrho$, or one of its powers, if necessary. Alternatively, one can view this as choosing a fixed point $\varrho^\infty(a_i|a_j)$ of the substitution and constructing its geometric realisation. As always, we consider the left endpoints of prototiles to be their markers, possibly coloured. Next, we define the *pair correlations* $\nu_{ij}(z)$ to be the relative frequency that a tile of type $i$ is at a distance $z$ to the left of a tile of type $j$ in $\varLambda$, which can formally be expressed as

$$\nu_{ij}(z) = \frac{\operatorname{dens}(\varLambda_i \cap (\varLambda_j - z))}{\operatorname{dens}(\varLambda)} = \lim_{R\to\infty} \frac{\operatorname{card}\big(\varLambda_i^{(R)} \cap (\varLambda_j^{(R)} - z)\big)}{\operatorname{card}\big(\varLambda^{(R)}\big)}. \qquad (2.3)$$

Here, the *density* of a generic point set $\varLambda'$ is given by

$$\mathrm{dens}(\varLambda') = \lim_{R\to\infty} \frac{1}{2R}\mathrm{card}(\varLambda' \cap [-R, R]).$$

These frequencies exist uniformly due to the unique ergodicity of $(\mathbb{Y}, \mathbb{R})$, and are the same for any element $\varLambda \in \mathbb{Y}$ due to minimality.

These correlation functions are non-negative and satisfy $\nu_{ij}(z) = \nu_{ji}(-z)$. Aside from that, $\nu_{ij}(z) > 0$ only whenever $z \in \varLambda_j - \varLambda_i$. Note that this Minkowski difference is the same for any element of the hull, primarily due to minimality, which is equivalent to repetitivity. Seeing $\nu_{ii}(0)$ as the relative frequency of the occurrence of a tile of type $i$ in $\varLambda$, we also have $\sum_{i=1}^{n_a} \nu_{ii}(0) = 1$.

For one-dimensional tilings, we have the additional constraint that $\nu_{ij}(0) = 0$ for all $i \neq j$, which is clear from the choice of the markers. In higher dimensions, this might not be true in general, for example, when the choice of markers does not prohibit two different tiles to be determined by the same point.

These pair correlations and their properties are dealt with for several examples in [BG16], and are extended to the general case in [BGäM18]. In what follows, we state the renormalisation relations satisfied by $\nu_{ij}(z)$.

**Proposition 2.2.1.** *Let $\varLambda \in \mathbb{Y}$ where $\mathbb{Y}$ arises from a primitive, aperiodic substitution over $n_a$ letters, and let the pair correlation functions be defined as in Eq. (2.3). Then, the functions $\nu_{mn}(z)$ exist and are independent of $\varLambda$. Furthermore, they satisfy the system of renormalisation equations given by*

$$\nu_{mn}(z) = \frac{1}{\lambda} \sum_{i,j=1}^{n_a} \sum_{x\in T_{mi}} \sum_{y\in T_{nj}} \nu_{ij}\left(\frac{z+x-y}{\lambda}\right), \tag{2.4}$$

*where $\lambda$ is the inflation multiplier.*

*Proof.* For $\varLambda \in \mathbb{Y}$, fix $R$ and $z_0$ such that, within $(-R+z_0, R+z_0)$, one finds two tiles $\mathfrak{t}_m$ and $\mathfrak{t}_n$ at a distance $z$. Aperiodicity implies local recognisability, which means that every tile is situated in a unique level-1 supertile. For the point set $\varLambda$, this means that markers of type $m$ and $n$ being at a distance $z$ apart correspond to unique markers of type $i$ and $j$ (in another point set $\varLambda'$) inside a window of radius $\frac{R}{\lambda}$ being separated by a distance $\frac{z+x-y}{\lambda}$. Here, $x$ and $y$ are elements of the displacement sets $T_{mi}$ and $T_{nj}$ encoding the location of $\mathfrak{t}_m$ and $\mathfrak{t}_n$ in the respective supertiles $\varrho(\mathfrak{t}_i)$ and $\varrho(\mathfrak{t}_j)$; see Figure 2.2. Since the functions $\nu_{mn}(z)$ are independent of $\varLambda$, one can compare relative frequencies through this renormalisation even if they are associated to different elements of $\mathbb{Y}$. Summing over all supertiles that contain $\mathfrak{t}_m$ and $\mathfrak{t}_n$, all possible relative displacements within each supertile, and renormalising with respect to the averaging diameter given in Eq. (2.3) implies the claim. □

**Remark 2.2.2** (Periodic case)**.** Aperiodicity plays a vital role here since it implies local recognisability. When the hull $\mathbb{Y}$ is periodic, one loses this unique decomposition of $\varLambda$ into supertiles. This could easily be remedied by choosing a fixed decomposition and working out the correlation functions from there. ◊

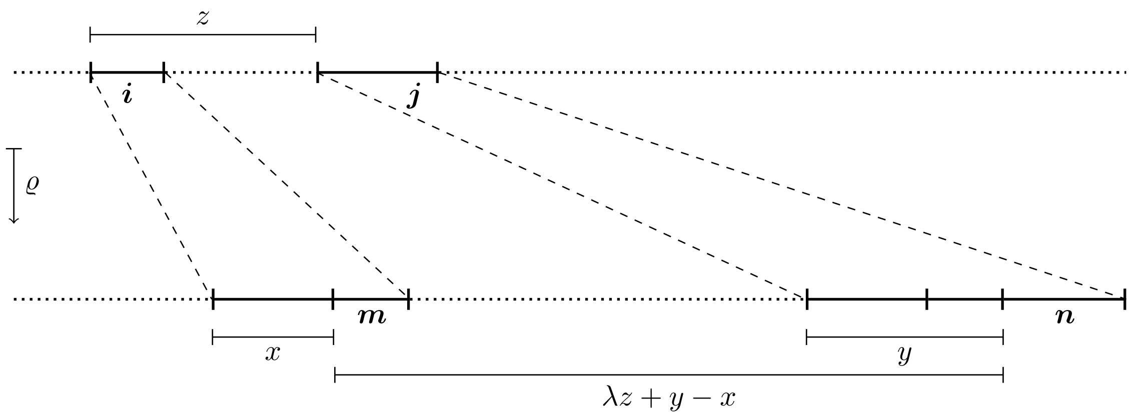



Figure 2.2.: Illustration of the "desubstitution" property which enables the correspondence of pair correlations of supertiles in $\varrho(\varLambda)$ to pair correlations of tiles in $\varLambda$.

**Example 2.2.3** (Fibonacci)**.** One of the examples treated in [BG16] is the Fibonacci substitution $\varrho_{\mathrm{F}}$ as in Example 1.2.8, for which Eq. (2.4) explicitly reads

$$\begin{aligned}
\nu_{aa}(z) &= \tfrac{1}{\tau}\big(\nu_{aa}(\tfrac{z}{\tau}) + \nu_{ab}(\tfrac{z}{\tau}) + \nu_{ba}(\tfrac{z}{\tau}) + \nu_{bb}(\tfrac{z}{\tau})\big),\\
\nu_{ab}(z) &= \tfrac{1}{\tau}\big(\nu_{aa}(\tfrac{z}{\tau} - 1) + \nu_{ba}(\tfrac{z}{\tau} - 1)\big),\\
\nu_{ba}(z) &= \tfrac{1}{\tau}\big(\nu_{aa}(\tfrac{z}{\tau} + 1) + \nu_{ab}(\tfrac{z}{\tau} + 1)\big),\\
\nu_{bb}(z) &= \tfrac{1}{\tau}\big(\nu_{aa}(\tfrac{z}{\tau})\big),
\end{aligned}$$

where $z \in \mathbb{Z}[\tau]$ and $\nu_{ij}(z) \neq 0$ if and only if $z \in \varLambda_j - \varLambda_i$. Here, $\tau = \lambda_{\mathrm{PF}}$ is the golden ratio. ◇

**Remark 2.2.4.** Given any set of complex weights $\boldsymbol{w} = \{W_i\}$, one can rewrite the autocorrelation coefficients $\eta_\omega(z)$ in Eq. (1.6) as

$$\eta_\omega(z) = \mathrm{dens}(\varLambda) \sum_{i,j=1}^{n_a} \overline{W_i}\, \nu_{ij}(z)\, W_j,$$

where $\nu_{ij}(z)$ are the pair correlation functions. ◇

## 2.3. Pair correlation measures and diffraction

From the pair correlation functions $\nu_{mn}$ in the previous section, we can build *pair correlation measures* $\varUpsilon_{mn}$ by treating $\nu_{mn}(z)$ as weights of a Dirac comb on $\varLambda - \varLambda$. We then get pure point measures of the form

$$\varUpsilon_{mn} = \sum_{z \in \varLambda - \varLambda} \nu_{mn}(z)\delta_z.$$

From the non-negativity of the pair correlations, we get that $\varUpsilon_{mn} \geqslant 0$. Moreover, one has $\widetilde{\varUpsilon_{mn}} = \varUpsilon_{nm}$. In particular, $\varUpsilon_{mm}$ is both positive and positive definite.

**Lemma 2.3.1.** *Each pair correlation measure emerges from an Eberlein convolution via*

$$\varUpsilon_{mn} = \frac{\widetilde{\delta_{\varLambda_m}} \circledast \delta_{\varLambda_n}}{\mathrm{dens}(\varLambda)}.$$

*Proof.* It is easy to confirm via test functions that $\widetilde{\delta_{\Lambda_i}} = \delta_{-\Lambda_i}$. The Eberlein convolution can be written as a limit which reads $\widetilde{\delta_{\Lambda_m}} \circledast \delta_{\Lambda_n} = \lim_{R\to\infty}(2R)^{-1}\big(\delta_{-\Lambda_m^{(R)}} * \delta_{\Lambda_n^{(R)}}\big)$, where the convolution of the finite approximants can be simplified to

$$\delta_{-\Lambda_m^{(R)}} * \delta_{\Lambda_n^{(R)}} \; = \sum_{x\in -\Lambda_m^{(R)}, y\in\Lambda_n^{(R)}} \delta_{x+y} \; = \sum_{z\in \Lambda_n^{(R)}-\Lambda_m^{(R)}} \operatorname{card}\big(\Lambda_m^{(R)} \cap (\Lambda_n^{(R)} - z)\big)\delta_z,$$

where the coefficient of $\delta_z$ gives the number of times $z$ is realised as a sum of elements $x \in -\Lambda_m^{(R)}$ and $y \in \Lambda_n^{(R)}$. Expressing $\operatorname{dens}(\Lambda)$ as a limit, and incorporating this to this simplified version of the numerator, proves the claim. □

On the basis of Theorem 1.3.5 and Proposition 1.3.6, one finds the following result.

**Lemma 2.3.2** ( [BGäM18, Lem. 2.3]). *Let $\mu, \nu$ be translation bounded measures such that $\mu \circledast \widetilde{\nu}$ as well as $\mu \circledast \widetilde{\mu}$ and $\nu \circledast \widetilde{\nu}$ exist, all with respect to the same averaging sequence $\mathcal{R}$. Then, $\mu \circledast \widetilde{\nu}$ is a translation bounded and transformable measure, and so is $\widetilde{\mu} \circledast \nu$.* □

**Proposition 2.3.3.** *The pair correlation measures satisfy*

$$\Upsilon_{mn} = \frac{1}{\lambda} \sum_{i,j} \sum_{r\in T_{mi}} \sum_{s\in T_{nj}} \delta_{s-r} * \big(f.\Upsilon_{ij}\big), \tag{2.5}$$

*where $f(x) = \lambda x$ and $(f.\mu)(\mathfrak{D}) = \mu(f^{-1}(\mathfrak{D}))$, for any Borel set $\mathfrak{D} \subset \mathbb{R}$.*

*Proof.* By definition, one can easily verify that $f.\delta_z = \delta_{\lambda z}$, from which we obtain

$$f.\Upsilon_{ij} = \sum_{z\in\Lambda_j-\Lambda_i} \nu_{ij}(z)\delta_{\lambda z} = \sum_{z\in\lambda(\Lambda_j-\Lambda_i)} \nu_{ij}\big(\tfrac{z}{\lambda}\big)\delta_z,$$

where the last equality follows from an appropriate change of variable. Taking its convolution with $\delta_{s-r}$, we get

$$\delta_{s-r} * \big(f.\Upsilon_{ij}\big) \; = \sum_{z\in\lambda(\Lambda_j-\Lambda_i)} \nu_{ij}\big(\tfrac{z}{\lambda}\big)\delta_{z+s-r} \; = \sum_{z\in\lambda(\Lambda_j-\Lambda_i)+s-r} \nu_{ij}\big(\tfrac{z+r-s}{\lambda}\big)\delta_z.$$

Now, we note that the following holds due to the compatibility of the supertile positions, tile displacements and the inflation structure; see Fig. 2.2,

$$\bigcup_{\substack{1\leqslant i,j\leqslant n_a \\ r\in T_{mi}, s\in T_{nj}}} \lambda(\Lambda_j - \Lambda_i) + (s-r) = \Lambda_n - \Lambda_m.$$

With this and Eq. (2.4), the right hand-side of Eq. (2.5) becomes

$$\frac{1}{\lambda}\sum_{i,j}\sum_{r\in T_{mi}}\sum_{s\in T_{nj}}\sum_{z\in\lambda(\Lambda_j-\Lambda_i)+(s-r)} \nu_{ij}\big(\tfrac{z+r-s}{\lambda}\big)\delta_z \; = \sum_{z\in\Lambda_n-\Lambda_m} \nu_{mn}(z)\delta_z = \Upsilon_{mn},$$

which completes the argument. □

Now, let us consider $\Lambda = \bigcup \Lambda_i$ as a weighted point set by choosing a complex weight vector $\boldsymbol{w} = (W_1, \ldots, W_{n_a})$, and putting the weight $W_i$ to any position in $\mathbb{R}$ which is a control point of a tile of type $\mathsf{t}_i$. The resulting weighted comb reads $\omega_\Lambda = \sum_{1\leqslant i\leqslant n_a} W_i\,\delta_{\Lambda_i}$. To this, we associate the natural autocorrelation as $\gamma_{\omega} = \omega_{\Lambda} \circledast \widetilde{\omega}_{\Lambda}$. As explained in Section 1.3.4, for any complex weight vector $\boldsymbol{w}$, $\gamma_{\omega}$ exists and is the same for every element $\Lambda \in \mathbb{Y}$. Invoking Lemma 2.3.1, we can rewrite $\gamma_{\omega}$ in terms of the measures $\varUpsilon_{ij}$ as

$$\gamma_{\omega} = \operatorname{dens}(\Lambda) \sum_{i,j=1}^{n_a} \overline{W}_i \, \varUpsilon_{ij} \, W_j. \tag{2.6}$$

The Fourier transform of each $\varUpsilon_{ij}$ exists due to Lemma 2.3.2. The linearity of the Fourier transform enables us to recover the diffraction measure from Eq. (2.6) as

$$\widehat{\gamma_{\omega}} = \operatorname{dens}(\Lambda) \sum_{i,j=1}^{n_a} \overline{W}_i \, \widehat{\varUpsilon_{ij}} \, W_j. \tag{2.7}$$

Due to an appropriate variant of Lemma 1.3.4 for translation bounded measures, the respective Fourier transforms $\widehat{\varUpsilon_{ij}}$ satisfy

$$\overline{\widehat{\varUpsilon_{mn}}} = \widehat{\widetilde{\varUpsilon_{mn}}} = \widehat{\varUpsilon_{nm}}. \tag{2.8}$$

In particular, the measures $\widehat{\varUpsilon_{mm}}$ are positive and positive definite. The action $f.\mu$ under Fourier transform satisfies

$$\widehat{f.\mu} = \frac{1}{\lambda} f^{-1}.\widehat{\mu}, \tag{2.9}$$

see [BG18, Lem. 2.5]. This, together with the convolution theorem, provides the counterpart of Eq. (2.5), after Fourier transformation, to be

$$\widehat{\varUpsilon_{mn}} = \frac{1}{\lambda^2} \sum_{i,j} \sum_{r \in T_{mi}} \sum_{s \in T_{nj}} \mathrm{e}^{-2\pi \mathrm{i}(s-r)(.)} \big( f^{-1}.\widehat{\varUpsilon_{ij}} \big). \tag{2.10}$$

Now, if we list these measures in lexicographic order, one can deduce from Eq. (2.10) that the resulting vector given by

$$\widehat{\varUpsilon} = \Big( \widehat{\varUpsilon_{11}}, \widehat{\varUpsilon_{12}}, \ldots, \widehat{\varUpsilon_{1n_a}}, \widehat{\varUpsilon_{21}}, \widehat{\varUpsilon_{22}}, \ldots, \widehat{\varUpsilon_{n_a n_a}} \Big)$$

satisfies the vector-valued equation

$$\widehat{\varUpsilon} = \frac{1}{\lambda^2} \boldsymbol{A}(.) (f^{-1}.\widehat{\varUpsilon}) \tag{2.11}$$

with $\boldsymbol{A}(k) = B(k) \otimes \overline{B(k)}$, where $B(k)$ is the Fourier matrix defined in Section 2.1. This scaling relation, together with the decomposition of the diffraction spectrum provided in Eq. (2.7), enables us to analyse the structure of $\widehat{\gamma_{\omega}}$ by analysing $\widehat{\varUpsilon}$.

To continue, consider the decomposition of $\widehat{\varUpsilon_{ij}}$ into its pure point and continuous parts, i.e., $\widehat{\varUpsilon_{ij}} = (\widehat{\varUpsilon_{ij}})_{\mathsf{pp}} + (\widehat{\varUpsilon_{ij}})_{\mathsf{cont}}$. Let $\mathcal{E}_{\mathsf{pp}}$ be the union of all the supporting sets of $(\widehat{\varUpsilon_{ij}})_{\mathsf{pp}}$, for all $i,j$. This is (at most) a countable set, being a finite union of (at most) countable sets. One then obtains a decomposition of the measure vector $\widehat{\varUpsilon} = (\widehat{\varUpsilon})_{\mathsf{pp}} + (\widehat{\varUpsilon})_{\mathsf{cont}}$, where $(\widehat{\varUpsilon})_{\mathsf{pp}}$ is supported on $\mathcal{E}_{\mathsf{pp}}$ and $(\widehat{\varUpsilon})_{\mathsf{cont}}$ on $\mathcal{E}_{\mathsf{cont}} = \mathbb{R} \setminus \mathcal{E}_{\mathsf{pp}}$.

One also has the freedom to choose these supporting sets to be $f$-invariant, which will be crucial for our analysis. To this end, note that the set $\mathcal{E}'_{\mathsf{pp}} := \bigcup_{m \in \mathbb{Z}} f^m(\mathcal{E}_{\mathsf{pp}})$ remains a countable set, and hence is still a null set for $(\widehat{\varUpsilon})_{\mathsf{cont}}$. This set is $f-$invariant and contains the true support of $(\widehat{\varUpsilon})_{\mathsf{pp}}$. Likewise, its $f$-invariant complement $\mathcal{E}'_{\mathsf{cont}} = \mathbb{R} \setminus \mathcal{E}'_{\mathsf{pp}}$ contains a full supporting set of the continuous part, which is immediate since its construction entailed the removal of an at most countable set from $\mathcal{E}'_{\mathsf{cont}}$. This yields a decomposition of $\mathbb{R}$ into the respective supporting sets given by $\mathbb{R} = \mathcal{E}'_{\mathsf{pp}} \,\dot\cup\, \mathcal{E}'_{\mathsf{cont}}$ such that,

$$\big(\widehat{\varUpsilon}\big)_{\mathsf{pp}} = \widehat{\varUpsilon}\big|_{\mathcal{E}'_{\mathsf{pp}}} \quad \text{and} \quad \big(\widehat{\varUpsilon}\big)_{\mathsf{cont}} = \widehat{\varUpsilon}\big|_{\mathcal{E}'_{\mathsf{cont}}}.$$

The continuous component can then be broken down into $\big(\widehat{\varUpsilon}\big)_{\mathsf{cont}} = \big(\widehat{\varUpsilon}\big)_{\mathsf{ac}} + \big(\widehat{\varUpsilon}\big)_{\mathsf{sc}}$, and a similar construction can be employed to ensure that the supports of the respective parts are disjoint and $f$-invariant, which leads to the decomposition

$$\mathbb{R} = \mathcal{E}'_{\mathsf{pp}} \,\dot\cup\, \mathcal{E}'_{\mathsf{ac}} \,\dot\cup\, \mathcal{E}'_{\mathsf{sc}}$$

with $\big(\widehat{\varUpsilon}\big)_{\alpha} = \widehat{\varUpsilon}\big|_{\mathcal{E}'_{\alpha}}$ and $f\big(\mathcal{E}'_{\alpha}\big) = \mathcal{E}'_{\alpha}$ for all $\alpha \in \{\mathsf{pp},\mathsf{ac},\mathsf{sc}\}$.

We take advantage of this decomposition in the next lemma to show that we can carry out the analysis on each spectral component given in Theorem 1.3.7 independently.

**Lemma 2.3.4.** *The renormalisation equation for the vector of measures $\widehat{\varUpsilon}$ in Eq. (2.11) holds individually for each spectral type, i.e.,*

$$(\widehat{\varUpsilon})_{\alpha} = \frac{1}{\lambda^2}\boldsymbol{A}(.)(f^{-1}.\widehat{\varUpsilon})_{\alpha} \tag{2.12}$$

*where $\alpha \in \{\mathsf{pp},\mathsf{ac},\mathsf{sc}\}$.*

*Proof.* Any given measure vector $\boldsymbol{\mu}$ shares the same spectral type with $\boldsymbol{A}(.)\boldsymbol{\mu}$ due to the analytic dependence of $\boldsymbol{A}(k)$ on $k$. In particular, one has $\big(\boldsymbol{A}(.)\boldsymbol{\mu}\big)_{\alpha} = \boldsymbol{A}(.)(\boldsymbol{\mu})_{\alpha}$ holds for each spectral type $\alpha \in \{\mathsf{pp},\mathsf{ac},\mathsf{sc}\}$ . Aside from this, the dilation $f^{-1}$ neither affects the support nor the null sets of the measure as well. The claim then follows by considering restrictions of the measures to their pairwise disjoint $f$-invariant supports. □

**Remark 2.3.5** (Renormalisation of $(\widehat{\varUpsilon})_{\mathsf{pp}}$)**.** It is known from [BG13, Cor. 9.1] that the diffraction $\widehat{\gamma}$ of a locally finite point set $\varLambda$ always possesses a non-trivial Bragg peak at zero. This value is given by

$$\widehat{\gamma}\big(\{0\}\big) = \big(\mathrm{dens}(\varLambda)\big)^2.$$

From Eq. (2.7), one has

$$\big(\widehat{\varUpsilon}_{ij}\big)_{\mathsf{pp}} = \mathrm{dens}(\varLambda) \sum_{k\in\mathcal{E}'_{\mathsf{pp}}} I_{ij}(k)\delta_k,$$

consistent with the definition of the intensities $I_{ij}(k)$ as dimensionless quantities, which at zero satisfy $\sum_{i,j=1}^{n_a} I_{ij}(0) = 1$. Lexicographically ordering $I_{ij}$ yields the intensity vector $\boldsymbol{I}(k)$ associated to $\widehat{\varUpsilon}$. It follows from Lemma 2.3.4 that this vector satisfies the renormalisation

$$\boldsymbol{I}(k) = \lambda^{-2}\boldsymbol{A}(k)\boldsymbol{I}(\lambda k),$$

which describes the behaviour of intensities along orbits of the map $k \mapsto \frac{k}{\lambda}$. For values of $k$ where $\boldsymbol{A}(k)$ is invertible, one also has access to the outward orbit $k \mapsto \lambda k$. In particular, one can derive a variant of a hypothesis by Bombieri and Taylor regarding intensities and how they arise from exponential sums. We refer to [BGäM18, Sec. 3.4] for a complete analysis. ◊

The validity of the central arguments invoked in the remainder of this text relies on the precise implications of Lemma 2.3.4 for the absolutely continuous component $(\widehat{\varUpsilon})_{\mathsf{ac}}$, which we deal with in the next section.

## 2.4. Renormalisation of the Radon–Nikodym density

By definition, the absolutely continuous part $(\widehat{\Upsilon_{ij}})_{\mathsf{ac}}$ of each measure $\widehat{\Upsilon_{ij}}$ is represented by a locally integrable density function, which we denote by $h_{ij}(k) \in L^1_{\text{loc}}(\mathbb{R})$. Following our previous notation, $(\widehat{\Upsilon})_{\mathsf{ac}}$ is to be viewed as a vector of these densities, which we call the *Radon–Nikodym vector* given by $\boldsymbol{h}(k)$. Since our vector entries are now functions, Eq. (2.12) has a simpler formulation.

**Lemma 2.4.1.** *Let $\boldsymbol{h}(k)$ be the Radon–Nikodym vector that defines $(\widehat{\Upsilon})_{\mathsf{ac}}$. Then, it satisfies*

$$\boldsymbol{h}(k) = \frac{1}{\lambda}\boldsymbol{A}(k)\boldsymbol{h}(\lambda k), \tag{2.13}$$

*where this equality holds for Lebesgue-a.e. $k \in \mathbb{R}$.*

*Proof.* For convenience, let us denote $(\widehat{\Upsilon})_{\mathsf{ac}} := \boldsymbol{\xi}$. Pick an arbitrary test function $g \in C_{\mathsf{c}}(\mathbb{R})$. Evaluating $\boldsymbol{\xi}(g)$ yields

$$\boldsymbol{\xi}(g) = \int_{\mathbb{R}} g(k)\mathrm{d}\boldsymbol{\xi}(k) = \int_{\mathbb{R}} g(k)\boldsymbol{h}(k)\mathrm{d}k,$$

where the equality is to be seen as an equivalence between a vector and a vector of integrals involving the same function $g$. On the other hand, given $f(x) = \lambda x$, we have

$$\begin{aligned}\frac{1}{\lambda^2}(\boldsymbol{A}(.)(f^{-1}.\boldsymbol{\xi}))(g) &= \frac{1}{\lambda^2}\int_{\mathbb{R}} g(f^{-1}(k))\boldsymbol{A}(f^{-1}(k))\mathrm{d}\boldsymbol{\xi}(k)\\ &= \frac{1}{\lambda^2}\int_{\mathbb{R}} g\big(\tfrac{k}{\lambda}\big)\boldsymbol{A}\big(\tfrac{k}{\lambda}\big)\boldsymbol{h}(k)\mathrm{d}k = \frac{1}{\lambda}\int_{\mathbb{R}} g(k)A(k)\boldsymbol{h}(\lambda k)\mathrm{d}k,\end{aligned}$$

where the last equality follows from a change of variable that induced the cancellation of $\lambda$ from the denominator. The claim then follows by comparing the associated densities of $\boldsymbol{\xi}$ and $\lambda^{-2}(\boldsymbol{A}(.)(f^{-1}.\boldsymbol{\xi}))$, which we know to be identical from Eq. (2.12) when $\alpha = \mathsf{ac}$. □

At this point, one can already work with Eq. (2.13) and proceed with a growth analysis of the vector $\boldsymbol{h}(k)$ of length $n_a^2$. However, one can still benefit from a dimension reduction that can be harvested from the symmetry properties of the measures $\widehat{\Upsilon_{ij}}$, which obviously also hold for each $h_{ij}(k)$. To be more precise, the relations for $\widehat{\Upsilon_{ij}}$ in Eq. (2.8) and the positivity of $\widehat{\Upsilon_{ii}}$ imply

$$h_{ij}(-k) = h_{ji}(k) = \overline{h_{ij}(k)} \quad \text{and} \quad h_{ii}(k) \geqslant 0 \tag{2.14}$$

for a.e. $k \in \mathbb{R}$, and all $1 \leqslant i, j \leqslant n_a$.

We proceed by constructing the *Radon–Nikodym matrix* $\mathcal{H}(k) = (h_{ij}(k))_{1\leqslant i,j\leqslant n_a}$, which due to Eq. (2.14) is a positive semi-definite Hermitian matrix for a.e. $k$. One can then rewrite Eq. (2.13) as a two-sided renormalisation given by

$$\mathcal{H}(k) = \frac{1}{\lambda}B(k)\mathcal{H}(\lambda k)B^{\dagger}(k). \tag{2.15}$$

By Sylvester's criterion, this matrix decomposes into a sum of positive semi-definite matrices each of rank 1, i.e.,

$$\mathcal{H}(k) = \sum_{\ell=1}^{s}\mathcal{H}_{\ell}(k), \quad \text{where} \quad \mathcal{H}_{\ell}(k) = v_{\ell}(k)v_{\ell}^{\dagger}(k), \tag{2.16}$$

where each $v_\ell(k)$ is a function in $L^2_{\text{loc}}(\mathbb{R})$. With this, the right-hand side of Eq. (2.15) becomes

$$\lambda^{-1} B(k)\mathcal{H}(\lambda k)B^\dagger(k) = \lambda^{-1}\sum_{\ell=1}^{s} B(k)v_\ell(\lambda k)v_\ell^\dagger(\lambda k)B^\dagger(k), \tag{2.17}$$

which allows us to investigate the $n_a$-dimensional iteration

$$v(k) = \frac{1}{\sqrt{\lambda}}B(k)v(\lambda k) \tag{2.18}$$

instead. Whenever $B(k)$ is invertible, we get the outward analogue given by

$$v(\lambda k) = \sqrt{\lambda}B^{-1}(k)v(k) \tag{2.19}$$

which, when iterated, reads

$$v(\lambda^n k) = \lambda^{n/2}B^{-1}(\lambda^{n-1}k)\cdot\ldots\cdot B^{-1}(\lambda k)B^{-1}(k)v(k). \tag{2.20}$$

This equation reveals that the behaviour of $v(k)$ along the sequence $\{\lambda^n k\}_{n\geqslant 0}$ as $k\to\infty$ is completely determined by $\prod_i B^{-1}(\lambda^i k)$. The invertibility of $B(k)$ for a.e. $k\in\mathbb{R}$ is guaranteed whenever $\det B(k)\not\equiv 0$, since $\det B(k)$ is analytic and hence can at most have isolated zeros.

**Remark 2.4.2.** Note that the $v(k)$ we refer to in Eq. (2.20) represents any of the constituent vectors $v_\ell(k)$ in Eq. (2.17). This means that the growth rate of entries of $\mathcal{H}(k)$ under the outward iteration analogue for Eq. (2.15) is bounded from below by the smallest possible growth rate exhibited by $v_\ell(k)$. Roughly speaking, if each $v_\ell(k)$ grows exponentially, so does $h_{ij}(k)$ for all $1\leqslant i,j\leqslant n_a$—something we will explain in more detail in the next section. ♢

We now focus our attention on Eq. (2.20) and explicitly define signatures of exponential growth or decay, which are the corresponding Lyapunov exponents.

## 2.5. Absence of absolutely continuous diffraction

Consider the outward iteration

$$v(\lambda^n k) = \lambda^{n/2}B^{-1}(\lambda^{n-1}k)\cdot\ldots\cdot B^{-1}(\lambda k)B^{-1}(k)v(k).$$

By Lemma 1.4.5, bounds on the exponential asymptotic behaviour of $v(k)$ are determined by the extremal exponents given by

$$\chi_{\max} = \log\sqrt{\lambda} + \limsup_{n\to\infty}\frac{1}{n}\log\|\underbrace{B^{-1}(\lambda^{n-1}k)\cdots B^{-1}(k)}_{(B^{(n)}(k))^{-1}}\|, \tag{2.21}$$

$$\chi_{\min} = \log\sqrt{\lambda} + \liminf_{n\to\infty}\frac{1}{n}\log\|\underbrace{B(k)\cdots B(\lambda^{n-1}k)}_{B^{(n)}(k)}\|^{-1}, \tag{2.22}$$

where one immediately sees that the additional $\log\sqrt{\lambda}$ term is a simple consequence of the pre-factor $\lambda^{n/2}$ in the $n$-th level of the iteration. As we will mainly be interested in the minimal growth rate, we look at $\chi_{\min}$, which can be written as

$$\chi_{\min} = \log\sqrt{\lambda} - \chi^B(k),$$

where

$$\chi^B(k) := \limsup_{n\to\infty} \frac{1}{n} \log \|B^{(n)}(k)\|$$

is the maximal exponent of the matrix cocycle generated by $B(k)$. We refer to $n$-th iterate $B^{(n)}(k)$ of $B(k)$ (which is the Fourier matrix of $\varrho^n$ as stated in Fact 2.1.4) to be the *Fourier cocycle* of $\varrho$. Throughout the text, we refer to $\chi^B$ as the exponent of $B^{(n)}$ and $\chi_{\min}$ to be adjusted exponent with corrective term $\log \sqrt{\lambda}$.

**Remark 2.5.1** (Exponents for the inward iteration)**.** From Remark 1.4.2, the Lyapunov exponents for the inward iteration derived from Eq. (2.18) are given by

$$\chi_i = \log |\lambda_i| - \log \sqrt{\lambda_{\mathrm{PF}}}$$

where $\{\lambda_i\}$ are the eigenvalues of $M_\varrho$. The existence and the general structure of the filtration of $\mathbb{C}^d$ is however rather subtle; see [BFGR19, Sec. 6.5] for an illustrative example. $\Diamond$

**Lemma 2.5.2** ( [BFGR19, Lem. 9.3])**.** *Let $g \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be a non-negative function and let $\lambda > 1$ be fixed. Assume further that there is an interval $[0, a]$, a constant $\delta > 1$, and a measurable function $C$ with $C(x) > 0$ for a.e. $x \in [0, a]$ such that $g(\lambda^m x) \geqslant C(x)\delta^m g(x)$ holds for a.e. $x \in [0, a]$. Then, the absolutely continuous positive measure $g\mu_{\mathrm{L}}$ is translation bounded if and only if $g = 0$ on $[0, a]$ in the Lebesgue sense.*

*Proof.* Suppose $g(x) > 0$ on a subset of $[0, a]$ of positive measure. Due to the almost everywhere positivity of $C(x)$, which implies that $\{x \in [0, a] \mid C(x) = 0 \text{ or } g(x) = 0\}$ is not of full measure, we get

$$c_g := \int_0^a C(x)g(x)\,\mathrm{d}x > 0.$$

Integrating $g$ on a dilated interval, the inequality involving $g(\lambda^m x)$ results in the following estimate:

$$\int_0^{a\lambda^m} g(x)\,\mathrm{d}x \geqslant \delta^m \lambda^m \int_0^a C(x)g(x)\,\mathrm{d}x = c_g(\delta\lambda)^m. \tag{2.23}$$

Since $g\mu_{\mathrm{L}}$ is translation bounded and $g \geqslant 0$, one has

$$\int_0^L g(x)\,\mathrm{d}x = \mathcal{O}(L) \quad \text{as } L \to \infty.$$

This in particular implies that the integral on the left hand-side of Eq. (2.23) is only allowed to grow up to order $\mathcal{O}(\lambda^m)$. This contradicts the previous asymptotic estimate that the growth rate of the integral is at least $\mathcal{O}((\delta\lambda)^m)$, and hence implies the claim. $\square$

**Theorem 2.5.3** (Absence of absolutely continuous diffraction)**.** *Let $\varrho$ be a primitive inflation rule, with inflation multiplier $\lambda$, and corresponding Fourier matrix $B(k)$. Assume further that* $\det B(k) \neq 0$ *for some $k$. If there exists $\varepsilon > 0$ such that $\chi^B(k) \leqslant \log \sqrt{\lambda} - \varepsilon$ for a.e. $k$, the diffraction measure of $\mathbb{Y}$ does not have an absolutely continuous component, i.e., $(\widehat{\gamma_\omega})_{\mathsf{ac}} = 0$, for any choice of weight vector $\boldsymbol{w}$.*

*Proof.* The assumption on $\chi^B(k)$ implies that there exists a $\delta > 0$ such that $\|v(\lambda^n k)\| \approx C_k\, \mathrm{e}^{\delta n k}$ for some positive $C_k$ that depends on $k$. By Lemma 2.5.2, this means $\|v\|^2$ cannot be a Radon–Nikodym density of a translation bounded measure unless $v \equiv 0$ in the Lebesgue sense.

For any choice of weight vector $\boldsymbol{w} \in \mathbb{C}^{n_a}$ , the absolutely continuous portion $(\widehat{\gamma_\omega})_{\mathsf{ac}}$ of the resulting diffraction is a translation bounded measure by Proposition 1.3.6. If $W_j = \delta_{j,m}$, the Radon–Nikodym density of $\widehat{\gamma}_{\mathsf{ac}}$ is the locally integrable function $h_{mm} \geqslant 0$. Note that the densities $h_{mm}(k)$ are the diagonal entries of $\mathcal{H}(k)$. Since a finite sum of translation bounded measures remains translation bounded, $\operatorname{tr}(\mathcal{H}) = \sum_{m=1}^{n_a} h_{mm}$ represents a translation bounded measure.

Due to the Hermiticity of $\mathcal{H}(k)$ and Eq. (2.16), we have

$$\operatorname{tr}(\mathcal{H})(k) = \sum_{m=1}^{n_a} \sum_{\ell=1}^{s} \left|(v_\ell)_m(k)\right|^2.$$

Here, each summand is non-negative, and so cancellation is not possible. Furthermore, if at least one term grows exponentially as $k \to \infty$, the entire sum grows exponentially as well, which violates translation boundedness as explained above. This means $v_\ell(k) = 0$ for Lebesgue-a.e. $k$ and for all $1 \leqslant \ell \leqslant s$, and $\widehat{\varUpsilon}_{\mathsf{ac}} = 0$. This and Eq. (2.7) imply the claim. □

It was shown in [BFGR19] that this $\varepsilon$−condition can be weakened into having $\chi^B(k) < \log \sqrt{\lambda}$ for some subset of an interval $\left[\frac{\varepsilon}{\lambda}, \varepsilon\right]$ of full measure. From this, we obtain the following necessary criterion to have an absolutely continuous component in the diffraction.

**Corollary 2.5.4.** *Let $\varrho$ be a primitive inflation rule, with inflation multiplier $\lambda$, whose Fourier matrix satisfies the non-vanishing determinant condition. If the corresponding diffraction measure $\widehat{\gamma}_{\mathsf{ac}}$ is non-trivial, one has $\chi_{\min} \leqslant 0$.* □

Negative Lyapunov exponents signify that the zero vector is an attractor of the cocycle, which has a strong implication on substitutions which generate Meyer point sets $\varLambda \in \mathbb{Y}$ as follows. We omit the proof here and refer to [BGäM18, Prop. 3.26] instead.

**Proposition 2.5.5.** *Assume that the elements of $\mathbb{Y}$ are Meyer sets, and assume $\widehat{\gamma}_{\mathsf{ac}} \neq 0$. Then, the Radon–Nikodym density $h(k)$ of $\widehat{\gamma}_{\mathsf{ac}}$ does not decay at infinity.* □

**Corollary 2.5.6.** *Assume that the hull $\mathbb{Y}$ contains only Meyer sets and that the diffraction measure $\widehat{\gamma}_{\mathsf{ac}}$ is non-trivial. Then, $\chi(v_\ell, k) = \log \sqrt{\lambda}$ holds for a subset of positive measure in $\mathbb{R}$ and for all $v_\ell$ that constitute $\mathcal{H}(k)$. In particular, this applies whenever the inflation multiplier $\lambda$ of $\varrho$ is a PV-number.* □

As we have seen in Section 1.4, most of the existence results for Lyapunov exponents require that the matrix cocycle takes arguments from a compact manifold, which we do not have here since $B(k)$ is sampled along an orbit in $\mathbb{R}$. In the next section, we will discuss how to remedy this problem by considering a higher-dimensional periodic representation of $B(k)$. With this representation, we will also provide a sufficient criterion to rule out the presence of absolutely continuous components that is verifiable in finite time.

## 2.6. Periodic representations of quasiperiodic functions

Let $p(k)$ be a trigonometric polynomial with finitely many fundamental frequencies, i.e., $p$ is of the form

$$p(k) = \sum_{j \in \mathbb{Z}} c_j \, \mathrm{e}^{2\pi \mathrm{i}\left(a_1^{(j)}\alpha_1 k + a_2^{(j)}\alpha_2 k + \cdots + a_d^{(j)}\alpha_d k\right)} \quad \text{with} \quad a_i^{(j)} \in \mathbb{Z},$$

where $\{\alpha_1, \alpha_2, \dots, \alpha_d\}$ consists of rationally independent real numbers. Such a polynomial an example of a *quasiperiodic function*. This class was first dealt with in [Boh93, Esc04]. It is also the simplest subclass of almost periodic functions. The following fact is mainly due to Bohr, who pointed out that the properties of such polynomials are closer to purely periodic functions than to almost periodic functions; see [Boh47].

**Fact 2.6.1.** *Any quasiperiodic function can be represented as a section of a function on $\mathbb{T}^d$, which is $1-$periodic in each argument. More explicitly, we have*

$$p(k) = \widetilde{p}(x_1, \dots, x_d)|_{x_1=\alpha_1 k,\ x_2=\alpha_2 k,\ \dots,\ x_d=\alpha_d k},$$

*where $\widetilde{p} : \mathbb{T}^d \to \mathbb{C}$.* □

**Lemma 2.6.2.** *Let $B(k)$ be the Fourier matrix of a primitive irreducible inflation $\varrho$, with inflation multiplier $\lambda$ of algebraic degree $d > 1$. Then, $B(k)$ is a quasiperiodic matrix-valued function satisfying*

$$B(k) = \widetilde{B}(x_1, \dots, x_d)|_{x_1=k,\ x_2=\alpha_1 k,\ \dots,\ x_d=\alpha_{d-1} k},$$

*where the numbers $\{1, \alpha_1, \dots, \alpha_{d-1}\}$ represent the (renormalised) tile lengths given by the left PF eigenvector $\boldsymbol{L}$ of $M_\varrho$. Moreover,*

$$B(\lambda k) = \widetilde{B}((x_1, \dots, x_d)M_\varrho)|_{x_1=k,\ x_2=\alpha_1 k,\ \dots,\ x_d=\alpha_{d-1} k}.$$

*Proof.* The frequencies derived from $\boldsymbol{L}$ are all distinct since $M_\varrho$ has an irreducible characteristic polynomial. By construction, every displacement set satisfies $T_{ij} \subset \mathbb{Z}[1, \alpha_1, \dots, \alpha_{d-1}]$, which together with Fact 2.6.1 implies the first claim. The second claim follows from $(1, \alpha_1, \dots, \alpha_{d-1})$ being a left eigenvector of $M_\varrho$ to the eigenvalue $\lambda$. □

From this, we get a representation of the cocycle $B^{(n)}(k)$ as a section of a cocycle over $\mathbb{T}^d$ given by

$$B^{(n)}(k) = \widetilde{B}^{(n)}(x) := \widetilde{B}(x)\widetilde{B}(xM) \cdots \widetilde{B}(xM^{n-1})|_{x_1=k, x_2=\alpha_1 k, \dots, x_d=\alpha_{d-1} k}.$$

**Remark 2.6.3.** One can obtain a different representation by choosing the frequencies to be powers of $\lambda$ instead of entries of $\boldsymbol{L}$. Here, the base dynamics on $\mathbb{T}^d$ will be given by the companion matrix $\mathfrak{C}(p)$ of the minimal polynomial $p_\lambda(z)$ of $\lambda$ instead of $M_\varrho$, whose ergodicity is assured when $\lambda$ is irrational. This is convenient for higher-dimensional examples where the notion of a tile length is no longer available; see Section 5.3. ◊

Working with the new cocycle $\widetilde{B}^{(n)}(x)$ has obvious advantages, as is apparent in the next result.

**Proposition 2.6.4.** *Let $\varrho$ be primitive, aperiodic and irreducible. Then, for a.e. $x \in \mathbb{T}^d$, all Lyapunov exponents for $\widetilde{B}(x)$ exist as limits and are constant. In particular, the exponent $\chi^{\widetilde{B}} := \lim_{n\to\infty} \frac{1}{n} \log \|\widetilde{B}^{(n)}(x)\|$ is constant for a.e. $x$.*

*Proof.* The irreducibility of $M_\varrho$ implies that the PF eigenvalue is irrational. Since $\lambda_{\mathrm{PF}} > 1$, we know that none of its conjugates are roots of unity. By [EW11, Cor. 2.20], $M_\varrho$ defines an ergodic toral endomorphism. The claim then directly follows by an application of Theorem 1.4.8. □

## 2.7. Uniform upper bounds for $\chi^B$

Let $\psi_n(k) := \log \|B^{(n)}(k)\|$. If the norm considered is submultiplicative, this sequence satisfies the following subadditivity relation

$$\psi_{m+n}(k) \leqslant \psi_m(k) + \psi_n(\lambda^m k).$$

Our next goal is the following estimate for a generic cocycle $B^{(n)}(k)$.

**Lemma 2.7.1.** *For any $N \in \mathbb{N}$, and for a.e. $k \in \mathbb{R}$, one has*

$$\chi^B(k) = \limsup_{n\to\infty} \frac{1}{n}\psi_n(k) \leqslant \frac{1}{N}\mathbb{M}(\psi_N),$$

*where $\mathbb{M}(\psi_N) := \int_{\mathbb{T}^d} \log \|\widetilde{B}^{(N)}(x)\| \mathrm{d}x$ and $\widetilde{B}^{(n)}(x)$ is the $d-$dimensional, $1-$periodic representation of $B^{(n)}(k)$.* □

This was proved in particular for a binary non-Pisot substitution in [BFGR19], where one of the crucial elements of the proof is the Bohr almost periodicity of $\psi_n$, which they were able to show by proving that $\|B^{(n)}(k)\|$ is uniformly bounded away from zero for every $n$. The rest of the proof, which we will briefly explain below, relies on some version of Fekete's lemma and the fact that $\{\lambda^j k\}_{j\geqslant 0}$ is uniformly distributed modulo 1 for a.e. $k \in \mathbb{R}$. Fix $N \in \mathbb{N}$ and let $n = mN + r$, with $0 \leqslant r < N$. By inductively invoking the above subadditivity relation, we get

$$\frac{1}{n}\psi_n(k) \leqslant \frac{1}{mN+r}\psi_r(\lambda^{mN}k) + \frac{1}{mN+r}\sum_{\ell=0}^{m-1}\psi_N(\lambda^{\ell N}k). \tag{2.24}$$

The second summand on the right hand-side converges to $\frac{1}{N}\mathbb{M}(\psi_N)$ due to Theorem 1.6.3, while the first converges to zero under the assumption of Bohr almost periodicity.

Here, we relax the Bohr almost periodicity condition on $\psi_n$ by showing that it suffices for $\psi_r$ to be uniformly bounded on almost all orbits $\{\lambda^j k\}_{j\geqslant 0}$, $k \in \mathbb{R}$, for any fixed $r \geqslant 0$. Moreover, we show that this holds whenever $B(k)$ is the Fourier matrix of a primitive $\varrho$, for which $\det(B(k))$ does not vanish for all $k$.

Before we continue, we mention the following well-known bounds for norms of matrices.

**Fact 2.7.2.** *Assuming that $A$ is invertible, $\|AB\| \geqslant \|B\|/\|A^{-1}\|$. In particular, $\|A\| \geqslant C/\|A^{-1}\|$ for some constant $C$.* □

**Fact 2.7.3.** *Let $B(k)$ be the Fourier matrix of $\varrho$ with inflation factor $\lambda$. Then, $\|B(k)\|_{\mathrm{F}} < \infty$ for all $k$, where $\|\cdot\|_{\mathrm{F}}$ is the Frobenius norm. Furthermore, if one considers the adjugate matrix $B^{\mathrm{ad}}$, we have $\|B^{\mathrm{ad}}(k)\|_{\mathrm{F}} < D < \infty$ for all $k$, where $D$ is a suitable constant. This follows from the fact that entries of $B^{\mathrm{ad}}$ are minors of $B$, and hence are trigonometric polynomials as well.* □

We also obtain a trivial global upper bound for $\|B^{(n)}(k)\|_{\mathrm{F}}$ as follows.

**Lemma 2.7.4.** *Let $B(k)$ be a Fourier matrix of $\varrho$, $M_\varrho = B(0)$. Then,*

$$\|B^{(n)}(k)\|_{\mathrm{F}} \leqslant \|M_\varrho\|_{\mathrm{F}}^n.$$

*Proof.* From the submultiplicativity of $\|\cdot\|_{\mathrm{F}}$, we get $\|B^{(n)}(k)\|_{\mathrm{F}} \leqslant \prod_{i=0}^{n-1} \|B(\lambda^i k)\|_{\mathrm{F}}$, for all $k \in \mathbb{R}$. For any entry of $B(\lambda^i k)$, we have $\left|B(\lambda^i k)_{\ell j}\right|^2 \leqslant (M_{\ell j})^2$, which is clear from the fact that $|B(k)_{\ell j}|^2$ attains its maximum when $k = 0$. We then have

$$\prod_{i=0}^{n-1} \|B(\lambda^i k)\|_{\mathrm{F}} \leqslant \|M_\varrho\|_{\mathrm{F}}^n,$$

from which the claim directly follows. □

The points of singularities of our cocycle, i.e., where $B^{(n)}(k)$ is non-invertible, satisfy a certain regularity because the entries of $B(k)$ are all trigonometric polynomials. More formally, we have the following.

**Fact 2.7.5.** *If* $\det B(k)$ *is not identically zero, the zero set* $\mathcal{Z}$ *of* $\det B(k)$ *is an at most countable set. Moreover, from the analyticity and quasi-periodicity of* $\det B(k)$*, we know that* $\mathcal{Z}$ *is finite or is uniformly discrete. In particular, this holds whenever* $\varrho$ *is irreducible.* □

From the structure of the cocycle $B^{(n)}(k)$, the zero set of $\det B^{(n)}(k)$ is given by $\bigcup_{\ell=0}^{n-1} \lambda^{-\ell}\mathcal{Z}$, which obviously is still a null set. Now, let $\mathcal{Z}' = \bigcup_{\ell=0}^{\infty} \lambda^{-\ell}\mathcal{Z}$. We are then interested in the uniform boundedness of the norm of any $n$-th level cocycle on subsets of $\mathbb{R} \setminus \mathcal{Z}'$ of full measure. We will outline how lower bounds of the norm of $B^{(n)}(k)$ depend on the bounds of the determinant $\det B(k)$ along orbits.

When $B(k)$ is invertible, its inverse is $B(k)^{-1} = (\det B(k))^{-1} B^{\mathrm{ad}}(k)$, which gives the upper bound $\|B(k)^{-1}\| \leqslant |\det B(k)|^{-1}\, D$, where $D$ is the constant from Fact 2.7.3. This yields

$$\frac{1}{\|B(k)^{-1}\|} \geqslant \frac{|\det B(k)|}{D}. \tag{2.25}$$

Since we have $B^{(n+1)}(k) = B(k)B^{(n)}(\lambda k)$, we obtain the following lower bound

$$\|B^{(n+1)}(k)\| \geqslant \frac{\|B^{(n)}(k)\|}{\|B(k)^{-1}\|} \geqslant \frac{|\det B(k)|\, \|B^{(n)}(\lambda k)\|}{D}.$$

Iterating this process, we get

$$\|B^{(n+1)}(k)\| \geqslant \frac{C \prod_{i=0}^{n} \left|\det B(\lambda^i k)\right|}{D^{n+1}} \geqslant \frac{\widetilde{C}(n+1;k)^{n+1} C}{D^{n+1}} := \delta_{n+1}(k), \tag{2.26}$$

where $\widetilde{C}(n+1;k) := \min_{0 \le i \le n} \left\{\left|\det B(\lambda^i k)\right|\right\}$.

If we can show that, for a generic substitution $\varrho$, $\delta_{n+1}(k) > 0$ for all $n$ for a.e. $k$, then the first summand in Eq (2.24) really does converge to zero and hence Lemma 2.7.1 holds for $\varrho$. This condition is equivalent to requiring the set $\left\{k \in \mathbb{R} : \inf_{i\in\mathbb{N}} \left|\det B(\lambda^i k)\right| = 0\right\}$ to be a null set.

A sufficient condition would be for the Birkhoff-type averages of $\det B(k)$ to satisfy

$$\lim_{n\to\infty} \frac{1}{n} \log\left|\det B^{(n)}(k)\right| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\left|\det B(\lambda^i k)\right| > -\infty, \tag{2.27}$$

for a.e. $k \in \mathbb{R}$.

**Proposition 2.7.6.** *Let $\varrho$ be a primitive inflation rule, with Fourier matrix $B(k)$ and multiplier $\lambda$. Assume $\det(B(k)) \not\equiv 0$. Then, for a.e. $k \in \mathbb{R}$, one has*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log|\det B(\lambda^i k)| = \mathbb{M}(\log|\det B(k)|).$$

*In particular, Eq. (2.27) holds for a.e. $k \in \mathbb{R}$.*

*Proof.* Since $\det B(k)$ is a trigonometric polynomial, $f(k) = \log|\det B(k)|$ is, at the very least, Stepanov almost periodic. From Fact 2.7.5, the set of zeros $\mathcal{Z}$ is uniformly discrete, which means that for every $\delta > 0$, $f(k)$ is locally Riemmann-integrable outside $\mathcal{Z} + (-\delta, \delta)$ and that there is a $\delta'$ such that $f$ is differentiable on $(z - \delta', z + \delta') \setminus \{z\}$ with $z \in \mathcal{Z}$. Here, one can choose $\delta'$ to be any number $0 < \delta' < R_{\mathrm{p}}(\mathcal{Z})$, where $R_{\mathrm{p}}(\mathcal{Z})$ is the packing radius of $\mathcal{Z}$. In view of Theorem 1.6.3, it suffices to check that Eq. (1.14) holds, i.e., for all $s > 0$, and for some $r > 0$,

$$V_N(s) = \sup_{z\in Y} \left( \int_{z-\delta'}^{z-\frac{1}{N^s}} |f'(x)| \mathrm{d}x + \int_{z+\frac{1}{N^s}}^{z+\delta'} |f'(x)| \mathrm{d}x \right) = \mathcal{O}\big(N^{\frac{s}{2}-r}\big)$$

as $N \to \infty$. Because the singularities of $f$ are logarithmic in nature, we know that the sum of these integrals grows like $\mathcal{O}(\log(N^s))$, which is much less than $\mathcal{O}\big(N^{\frac{s}{2}-r}\big)$. In conjunction with the formula for the discrepancy given in Eq. (1.13), by choosing $s = 1 + \varepsilon$ one obtains

$$\mathcal{D}_N V_N = \mathcal{O}\left( \frac{(\log(N))^{\frac{5}{2}+\varepsilon}}{\sqrt{N}} \right) \xrightarrow{N\to\infty} 0,$$

which describes the deviation of the partial averages from the mean, which is induced by the singularities; compare [BFGR19, Prop. 6.7]. When $\lambda \in \mathbb{Z}$, one has

$$\mathbb{M}(\log|\det B(k)|) = \mathfrak{m}(p) \geqslant 0,$$

for some $p(z) \in \mathbb{Z}[z]$, where $p(\mathrm{e}^{2\pi i k}) = \det B(k)$. Otherwise, $\mathbb{M}(\log|\det B(k)|)$ can be written as an $n_a$−dimensional logarithmic Mahler measure; compare Remark 2.7.8, which is still non-negative, thus completing the proof. □

As a corollary, we get the following result.

**Proposition 2.7.7.** *Let $\varrho$ be primitive with $\det(B(k)) \not\equiv 0$. Then, the Lyapunov exponent $\chi^B(k)$ satisfies the bound given in Lemma 2.7.1 for a.e. $k \in \mathbb{R}$.* □

**Remark 2.7.8.** One can choose the norm used for $\mathbb{M}(\psi_N)$ to be $\|\cdot\|_{\mathrm{F}}$, from which one gets that the upper bound satisfies $\frac{1}{N}\mathbb{M}(\psi_N) = \frac{1}{2}\mathfrak{m}(p^{(N)})$, for some $p^{(N)} \in \mathbb{Z}[x_1, \dots, x_d]$. In particular, one has

$$\frac{1}{N}\mathbb{M}(\log\|B^{(N)}(.)\|_{\mathrm{F}}^2) = \frac{1}{N} \int_{\mathbb{T}^d} \log\Big( \sum_{i,j=1}^{n_a} |p_{ij}^{(N)}(x)|^2 \Big) \mathrm{d}x, \tag{2.28}$$

with $x = (x_1, \dots, x_d) \in \mathbb{T}^d$, and where each $p_{ij}^{(N)}$ is a 1-periodic trigonometric polynomial. ◊

One would expect that $\chi_{\min} \geqslant 0$ holds in more generality other than the cases covered by Proposition 2.5.5. With the global upper bound provided in Lemma 2.7.1, we get the following general non-negativity result.

**Theorem 2.7.9.** *Let $\varrho$ be a primitive inflation rule, with multiplier $\lambda$ and Fourier matrix $B(k)$ satisfying $\det(B(k)) \not\equiv 0$. Then, for a.e. $k \in \mathbb{R}$, one has $\chi_{\min} \geqslant 0$ or, equivalently, $\chi^B(k) \leqslant \log\sqrt{\lambda}$.*

*Proof.* Working with the Frobenius norm, it follows that Eq. (2.28) holds. Applying Jensen's inequality to $\mathbb{M}(\log\|B^{(N)}(.)\|_{\mathrm{F}}^2)$ yields

$$\exp\big(\mathbb{M}(\log\|B^{(N)}(.)\|_{\mathrm{F}}^2)\big) \leqslant \int_{\mathbb{T}^d} \sum_{i,j=1}^{n_a} \big|p_{ij}^{(N)}(x)\big|^2 \mathrm{d}x = \sum_{i,j} \|p_{ij}^{(N)}\|_2^2 = \sum_{i,j} (M_\varrho^N)_{ij},$$

where $M_\varrho$ is the substitution matrix of $\varrho$. The last equality follows from Parseval's equality, and from the fact that the coefficients of $p_{ij}^{(N)}$ are either 0 or 1. The latter holds because of the nature of the control points, i.e., only a single tile can occupy a given tile position for each supertile. One then gets from $\|p_{ij}^{(N)}\|_2^2$ to $\big(M_\varrho^N\big)_{ij}$ via $B^{(N)}(0) = M_\varrho^N$.

From the primitivity of $\varrho$, one is assured of the asymptotic behaviour $\big(M_\varrho^N\big)_{ij} \sim C\lambda^N \boldsymbol{L}_i \boldsymbol{R}_j$ for all $i,j$ as $N \to \infty$, where $\boldsymbol{L}$ and $\boldsymbol{R}$ are the left and right PF eigenvectors of $M_\varrho$, and for some constant $C > 0$. Here, $\boldsymbol{L}$ and $\boldsymbol{R}$ are normalised such that $\sum_i \boldsymbol{R}_i = \sum_i \boldsymbol{L}_i \boldsymbol{R}_i = 1$. This gives

$$\frac{1}{N}\mathbb{M}(\log\|B^{(N)}(.)\|_{\mathrm{F}}^2) \leqslant \frac{1}{N}\log(C'\lambda^N) = \log(\lambda) + \frac{1}{N}\log(C')$$

for some $C' > 0$ and for large enough $N$. Together with Proposition 2.7.7, this implies

$$\chi^B(k) \leqslant \liminf_{N\to\infty} \frac{1}{2N}\mathbb{M}(\log\|B^{(N)}(.)\|_{\mathrm{F}}^2) \leqslant \frac{1}{2}\log(\lambda) = \log\sqrt{\lambda}$$

for a.e. $k \in \mathbb{R}$. □

In accordance with Corollary 2.5.4, we obtain the following generalisation of Corollary 2.5.6.

**Corollary 2.7.10** (Necessary criterion for absolutely continuous diffraction)**.** *Let $\varrho$ be a primitive inflation rule with multiplier $\lambda$ and Fourier matrix $B(k)$, where one has $\det(B(k)) \not\equiv 0$. If the diffraction measure of $\mathbb{Y}$ comprises a non-trivial absolutely continuous component, one has $\chi^B(k) = \log\sqrt{\lambda}$, or equivalently, $\chi_{\min}(k) = 0$, for a set of $k \in \mathbb{R}$ of positive measure.* □

**Remark 2.7.11.** The insufficiency of the criterion provided in Corollary 2.7.10 is clear since $\chi_{\min}$ only measures the minimum possible exponential growth rate of a vector in $\mathbb{R}^{n_a}$ after being subjected to $\lambda^{n/2}B^{(n)}(k)$. It could happen that $\chi_{\min} = 0$ and some component $v^{(\ell)}$ of $\mathcal{H}(k)$ is in some subspace $\mathcal{V}^i$ with $\chi_i > 0$, which implies that $h(k) = 0$; compare the proof of Theorem 2.5.3. Moreover, these exponents do not detect sub-exponential growth, which may still be present, and which may still impede $h(k)$ from being translation bounded. ◊

# 3. Constant-Length Case

In this chapter, we restrict our attention to the constant-length case, where the symbolic and the geometric pictures coincide. This leads us to tacitly use "substitution" instead of "inflation" since both formalisms lead to the same combinatorial and dynamical quantities—something that is not true outside this class. Under some extra assumptions, one has explicit bounds for the Lyapunov exponents discussed in the previous chapter for this subclass. We begin with the summary of the case when $\lambda_{\text{PF}} \in \mathbb{N}$ before discussing general properties of constant-length substitutions.

## 3.1. Integer inflation multiplier: arguments in common

We begin with a primitive substitution $\varrho$ over $n_a$ letters with $\lambda_{\text{PF}} \in \mathbb{N}$. It follows directly that $\boldsymbol{L} \in \mathbb{Q}^{n_a}$, which can be normalised so that $|\mathfrak{t}_j| \in \mathbb{N}$ for all $\mathfrak{t}_j$. The position of prototiles $\mathfrak{t}_i$ in the supertiles $\varrho(\mathfrak{t}_j)$ are then all integers, which means that every entry $T_{ij}$ of the displacement matrix defined in Eq. (2.1) is a finite subset of $\mathbb{Z}$. This implies 1-periodicity of $B(k)$. Recall that we are interested in a matrix cocycle sampled along the orbit $\{\lambda^n k\}_{n \geqslant 0}$, with $k \in \mathbb{R}$.

**Fact 3.1.1** ( [Rén57]). *Consider the map $f : [0,1) \mapsto [0,1)$ defined by $k \mapsto \lambda k \mod 1$. Then, whenever $\lambda \in \mathbb{Z}$, $f$ is ergodic with respect to Lebesgue measure $\mu_{\text{L}}$.* □

Let $\langle k \rangle := k \mod 1 \in [0,1)$, $\lfloor k \rfloor := k - \langle k \rangle$ denote the fractional and integral part of $k \in \mathbb{R}$, respectively.

**Lemma 3.1.2.** *For $\lambda \in \mathbb{Z}, k \in \mathbb{R}, n \in \mathbb{N}$, $f^n \langle k \rangle = \langle \lambda^n k \rangle$, where $f^n(k) = (\underbrace{f \circ \ldots \circ f}_{n \text{ times}})(k)$.*

*Proof.* Multiplying $k$ by $\lambda$, and taking the fractional parts, we get

$$\langle \lambda k \rangle = \underbrace{\langle \lambda \lfloor k \rfloor \rangle}_{0} + \langle \lambda \langle k \rangle \rangle , \tag{3.1}$$

where the first term vanishes since such a product always lies in $\mathbb{Z}$. This yields $f(\langle k \rangle) = \langle \lambda k \rangle$. Iterating this process proves the claim for any $n \in \mathbb{N}$. □

Due to the periodicity of $B(k)$ and Lemma 3.1.2, we then have $B(\lambda^n k) = B(f^n(\langle k \rangle))$, which with Fact 3.1.1 means our matrix cocycle $B^{(n)}(k)$ defined in Fact 2.1.4 is a product of matrices sampled along an orbit of an ergodic transformation $f$ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, which is a compact manifold.

Invoking Theorem 1.4.8, we obtain an existence result for the Lyapunov exponents of $B^{(n)}(k)$.

**Proposition 3.1.3.** *Let $\varrho$ be primitive, with $\lambda_{\text{PF}} \in \mathbb{N}$. Then, all Lyapunov exponents associated to $B^{(n)}(k)$ almost surely exist as limits and are constant for a.e. $k \in \mathbb{R}$. In particular, $\chi_{\min}(k)$ and $\chi_{\max}(k)$ are constant for a.e. $k \in \mathbb{R}$.* □

In this setting, we simply refer to the a.e. values of $\chi_{\min}(k)$ and $\chi_{\max}(k)$ as $\chi_{\min}$ and $\chi_{\max}$. Equivalently, $\chi^B(k) = \chi^B \in \mathbb{R}$ for a.e. $k$. From the ergodicity of $f$, we get the following sharper version of Lemma 2.7.1 via Theorem 1.4.7.

**Lemma 3.1.4.** *Let $\varrho$ be primitive, with Fourier matrix $B(k)$ and multiplier $\lambda_{\mathrm{PF}} \in \mathbb{N}$. Assume that $\det(B(k)) \not\equiv 0$. Then, for a.e. $k \in \mathbb{R}$, one has*

$$\chi^B(k) = \lim_{n\to\infty} \frac{1}{n}\psi_n(k) = \inf_N \frac{1}{N}\mathbb{M}(\psi_N) = \chi^B,$$

*where $\psi_N = \log \|B^{(N)}(k)\|$.* □

**Remark 3.1.5.** This almost everywhere constancy result in Proposition 3.1.3 also strengthens the requirement for the presence of absolutely continuous diffraction in Corollary 2.7.10, that is, $\chi^B(k) = \log \sqrt{\lambda}$ for a.e. $k \in \mathbb{R}$. ◊

**Definition 3.1.6.** A primitive substitution $\varrho$ with $|w_i| = \lambda_{\mathrm{PF}} := L \in \mathbb{N}$, for all substituted words $w_i$ is called a *constant-length substitution*. Equivalently, this means $\boldsymbol{L} = (1, 1, \dots, 1)$.

Let $\mathcal{A}_{n_a} = \{a_1, \dots, a_{n_a}\}$ and $\varrho$ be a constant-length substitution on $\mathcal{A}_{n_a}$. We denote the $m$th *column* of $\varrho$ by

$$\mathcal{C}_m := \begin{bmatrix} (w_1)_m \\ \vdots \\ (w_{n_a})_m \end{bmatrix}, \tag{3.2}$$

where $(w_i)_m$ is the $m$th letter of the word $w_i = \varrho(a_i)$. We follow the convention of indexing the columns starting with 0; compare [BG13, Ch. 4]. If at $m \in \{0, 1, \dots, L-1\}$ the column $\mathcal{C}_m$ is constant, i.e., $(w_i)_m = w_j$ for all $1 \leqslant i \leqslant n_a$, $\varrho$ is said to have a *coincidence* at $m$. Similarly, $\varrho$ is *bijective* at $m$ if the map $\kappa_m : a_i \mapsto (w_i)_m$ is a bijection on $\mathcal{A}_{n_a}$. A *bijective substitution* is that for which $\kappa_m$ is bijective for all $m \in \{0, 1, \dots, L-1\}$.

**Definition 3.1.7.** Let $w$ be an infinite fixed point of a constant-length substitution $\varrho$ of length $L$. The *height* $h(\varrho)$ of $\varrho$ is defined as

$$h(\varrho) := \max\left\{n \geqslant 1 : (n, L) = 1, n \text{ divides } \gcd\{\ell : w_\ell = w_0\}\right\}.$$

When $h(\varrho) \geqslant 2$, one can obtain a substitution $\varrho'$ on legal words of the form $w_0 v$, with $|w_0 v| = h(\varrho)$. This new substitution satisfies $h(\varrho') = 1$ and is called the *pure base* of $\varrho$. If $\varrho$ has coincidence, it is already its own pure base. Further details can be found in [Dek78, Que10].

**Example 3.1.8** ( [Que10, Ex. 6.2])**.** Consider the substitution $\varrho : 0 \mapsto 010, 1 \mapsto 102, 2 \mapsto 201$, and the infinite fixed point arising from 0 given by

$$w = \varrho^\infty(0) = 010102010102010201\dots.$$

It is easy to see that $w_\ell = w_0 = 0$, for all $\ell \in 2\mathbb{N}_0$, and hence $h(\varrho) = 2$. The two-letter words appearing on these positions are $\{01, 02\}$, from which one gets $\varrho' : a \mapsto aab, b \mapsto aba$ via the identification $a \,\widehat{=}\, 01$ and $b \,\widehat{=}\, 02$. ◊

A simple criterion that describes the dynamical spectral type of constant-length substitutions is given by the following result due to Dekking.

**Theorem 3.1.9** ( [Dek78, Thm. 7]). *A substitution dynamical system* $(\mathbb{X}, \mathbb{Z})$ *of constant length has pure point dynamical spectrum if and only if the pure base of the substitution* $\varrho$ *that generates it has a coincidence. Otherwise, its spectrum is partly continuous.* □

## 3.2. Binary constant-length case

For a constant-length substitution $\varrho$ on a binary alphabet $\mathcal{A}_2 = \{0, 1\}$, coincidences are of the form $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$. A binary substitution is bijective if there are no coincidences.

**Example 3.2.1.** For the substitution

$$\varrho := \begin{cases} 0 \mapsto \boxed{0}\,1\;0\,\boxed{1}\,0 \\ 1 \mapsto \boxed{0}\,0\;1\,\boxed{1}\,0 \end{cases}$$

one has $\mathcal{C}_0 = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ and $\mathcal{C}_3 = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, which are both coincidences. ♢

Since there are only four possible column types in the binary case, it is possible to express the entries of the Fourier matrices in terms of trigonometric polynomials associated to these column types. First, we construct the sets

$$C_0 = \left\{ i \mid \mathcal{C}_i = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] \right\}, \qquad C_1 = \left\{ i \mid \mathcal{C}_i = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] \right\},$$
$$P_0 = \left\{ i \mid \mathcal{C}_i = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \right\}, \qquad P_1 = \left\{ i \mid \mathcal{C}_i = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \right\},$$

where $C_0, C_1$ are the positions corresponding to coincidences and $P_0, P_1$ are the bijective positions. These sets are disjoint, and their union is $S_T = \{0, 1, \ldots, L-1\}$. From these, we define the polynomials

$$s_0(z) = \sum_{t \in C_0} z^t, \qquad s_1(z) = \sum_{t \in C_1} z^t, \qquad q(z) = \sum_{t \in P_0} z^t, \qquad r(z) = \sum_{t \in P_1} z^t, \tag{3.3}$$

where $s_0 + s_1 + q + r = \Phi_L = 1 + z + \ldots + z^{L-1}$ with $z = \mathrm{e}^{2\pi \mathrm{i} k}$.

### 3.2.1. Positivity of Lyapunov exponents

**Lemma 3.2.2.** *The Fourier matrix of* $\varrho$ *can be constructed from the polynomials* $s_0, s_1, q$ *and* $r$ *as*

$$B(k) = \begin{pmatrix} (s_0 + q)(z) & (s_0 + r)(z) \\ (s_1 + r)(z) & (s_1 + q)(z) \end{pmatrix}, \qquad \textit{with } \det B(k) = \Phi_L \cdot (q - r).$$

*Proof.* This is immediate from how the polynomials are constructed. □

Bearing in mind that Eq. (1.9) holds for a.e. $k$, one gets an expression for $\chi_{\min}(k) + \chi_{\max}(k)$ as

$$\log(L) - \frac{1}{n} \sum_{m=0}^{n-1} \log |\det B(L^m k)| \xrightarrow[\text{a.e. } k]{n \to \infty} \log(L) - \underbrace{\int_0^1 \log |\Phi_L| \, dk}_{=0} - \underbrace{\int_0^1 \log |q - r| \, dk}_{=\mathfrak{m}(q-r)}, \tag{3.4}$$

where $\mathfrak{m}(q - r)$ is the logarithmic Mahler measure of the polynomial $q - r$. This convergence follows from Birkhoff's ergodic theorem, as $\log |\det B(k)| \in L^1_{\mathrm{loc}}(\mathbb{T}) = L^1(\mathbb{T})$. An alternative route to show convergence would be via a version of Theorem 1.6.3 for 1-periodic locally integrable functions, see [BHL17]. Theorem 1.5.3 implies that the first integral is zero since $\Phi_L$ is cyclotomic.

Using this knowledge about the sum, we now compute both exponents in the following result.

**Proposition 3.2.3.** *The pointwise Lyapunov exponents of an aperiodic, binary constant-length substitution $\varrho$, for a.e. $k \in \mathbb{R}$, are given by*

$$\chi_{\max} = \log \sqrt{L} \quad \text{and}$$
$$\chi_{\min} = \log \sqrt{L} - \mathfrak{m}\,(q-r)\,.$$

*In particular, if $q-r$ is non-reciprocal (i.e. $w_0$ is neither palindromic nor antipalidromic), then $\chi_{\min} \leqslant \log\sqrt{L} - \log(\lambda_{\mathrm{p}})$, where $\lambda_{\mathrm{p}}$ is the plastic number, i.e., the real root of $q(z) = z^3 - z - 1$.*

*Proof.* In the constant-length case, all exponents a.s. exist as limits and hence, one can rewrite $\chi_{\max}$ in Eq. (2.21) as

$$\log\sqrt{L} - \lim_{n\to\infty} \frac{1}{n} \log \left\| B^{\mathrm{ad}}(L^{n-1}k) \cdot \ldots \cdot B^{\mathrm{ad}}(k) \right\| - \frac{1}{n} \sum_{m=0}^{n-1} \log \left| \det \left( B\left(L^m k\right)\right) \right| ,$$

where the last term converges as $n \to \infty$ for a.e. $k$ to $\mathfrak{m}\,(q-r)$ by Birkhoff's ergodic theorem. The extremal exponents then read

$$\chi_{\min}(k) = \log\sqrt{L} - \lim_{n\to\infty} \frac{1}{n} \log \left\| B^{(n)}(k) \right\|$$
$$\chi_{\max}(k) = \log(L) - \chi_{\min}(k) - \mathfrak{m}\,(q-r)\,,$$

where we have used that, for $2\times 2$ matrices, $\|A\|_{\mathrm{F}} = \|A^{\mathrm{ad}}\|_{\mathrm{F}}$.

One can easily check that $\mathbb{C}\big((1,-1)^{\mathrm{T}}\big)$ is invariant with respect to $B(k)$. Comparing $\chi_{\min}$ with the value we have for the iteration on this subspace, we get, for a.e. $k$,

$$\chi_{\min}\,(k) \;\leqslant\; \log\sqrt{L} - \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \log\left( |q-r| \left(z^{L^m}\right)\right) \;=\; \log\sqrt{L} - \mathfrak{m}\,(q-r) \;\leqslant\; \log\sqrt{L},$$

noting that $\log\sqrt{L} - \mathfrak{m}\,(q-r)$ is the value of one of the exponents. From the previous inequality, this cannot be $\chi_{\max}$ when $\mathfrak{m}\,(q-r) > 0$ since $\chi_{\max}\,(k) \geqslant \log\sqrt{L}$. Hence, in general, it must be the smaller of the two. The first claim then follows Eq. (3.4) from forward regularity; see Theorem 1.4.8 and Eq. (1.9). Equality of the two exponents holds when $q-r$ satisfies the conditions of Theorem 1.5.3, which will be tackled in detail in Theorem 3.4.9 for the bijective case.

The upper bound for $\chi_{\min}$ is due to a result by Smyth; see [Smy08, Sec. 5]. □

**Corollary 3.2.4.** *Let $\varrho$ be as in the previous proposition. Then, the Lyapunov exponents $\chi_{\min}$ and $\chi_{\max}$ are both positive.*

*Proof.* When $q-r$ is a monomial, $\mathfrak{m}(q-r) = 0 < \log\sqrt{L}$, for all $L \geqslant 2$. Assume now that $q-r$ is not a monomial. From the disjointness and the completeness of the columns, we get

$$\|q-r\|_2 = \sqrt{L - \mathrm{card}\,(C_0 \cup C_1)} \leqslant \sqrt{L},$$

where equality holds when $\varrho$ is bijective. Lemma 1.5.2 then implies that $\mathfrak{m}(q-r) < \log\sqrt{\lambda}$, from which the claim follows. $\square$

**Corollary 3.2.5.** *Any aperiodic, binary constant-length substitution $\varrho$ has a diffraction $\widehat{\gamma}$ which is singular relative to Lebesgue measure.* $\square$

**Example 3.2.6.** For $\varrho_{\text{TM}}$ and $\varrho_{\text{pd}}$, we have that

$$(q-r)_{\text{TM}} = 1 - \mathrm{e}^{2\pi i k} \qquad \text{and} \qquad (q-r)_{\text{pd}} = -\,\mathrm{e}^{2\pi i k},$$

and hence $\mathfrak{m}\,(q-r) = 0$ for both cases, implying that $\chi_{\min} = \chi_{\max} = \log\sqrt{2}$. $\Diamond$

**Remark 3.2.7.** We point out that the Lyapunov spectrum is not an IDA invariant. Consider the substitution $\varrho : a \mapsto abbab, b \mapsto baaba$, with corresponding Fourier matrix

$$B(k) = \begin{pmatrix} \mathrm{e}^{6\pi i k} & \mathrm{e}^{2\pi i k} + \mathrm{e}^{4\pi i k} + \mathrm{e}^{8\pi i k} \\ \mathrm{e}^{2\pi i k} + \mathrm{e}^{4\pi i k} + \mathrm{e}^{8\pi i k} & \mathrm{e}^{6\pi i k} \end{pmatrix}.$$

Here, $(q-r)\,(z) = z\left(-1 - z + z^2 - z^3\right)$, where $z = \mathrm{e}^{2\pi i k}$. Note that its IDA is isomorphic to that of the Thue–Morse substitution since it is bijective. One can explicitly compute that the Lyapunov exponents for the outward iteration for this substitution are

$$\begin{aligned} \chi_{\max} &= \log\sqrt{5} \approx 0.8047 \quad \text{and} \\ \chi_{\min} &= \log\sqrt{5} - 2\log\,(1.3562) \approx 0.1953, \end{aligned}$$

with $\mathfrak{m}(-1 - z + z^2 - z^3) = 2\log\,(1.3562...)$. $\Diamond$

### 3.2.2. From polynomials to substitutions

A *Borwein* polynomial $p(z) \in \mathbb{Z}[z]$ is a polynomial whose coefficients lie inside $\{-1, 0, 1\}$. Note that, due to the substitutive structure of $\varrho$, the polynomial $q - r$ that determines $\chi^B$ in Proposition 3.2.3 is always Borwein.

It is known that any integer polynomial with $\mathfrak{m}(p) < \log(2)$ must divide a height-1 polynomial; see [Pat72, Boy80]. This makes the set of Borwein polynomials an interesting subclass for Lehmer's problem in $\mathbb{Z}[z]$; see [Smy08].

In what follows, we show that, indeed, given a polynomial $p$, one can construct $\varrho$ having $\mathfrak{m}(p)$ as its Lyapunov exponent.

**Proposition 3.2.8.** *Let $p(z) = \sum_{m=0}^{L-1} c_m z^m \in \mathbb{Z}[z]$ be a Borwein polynomial of degree $L-1$ and $c_0 \neq 0$. Then, there exists at least one primitive binary constant-length substitution $\varrho$ of length $L$ such that, for a.e. $k \in \mathbb{R}$,*

$$\chi^B(k) = \mathfrak{m}(p),$$

*where $\chi^B$ is the Lyapunov exponent of the Fourier cocycle $B^{(n)}$ and $\mathfrak{m}(p)$ is the logarithmic Mahler measure of $p$.*

*Sketch of proof.* Since the construction is best dealt with in concrete cases, we only layout the general technique here and defer the discussion of some subtleties to the examples.

Note that only the bijective columns figure in $\chi^B(k) = \mathfrak{m}(q-r)$, where $q, r$ are determined by positions of columns of type $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, respectively. If there are any coincidences, their positions correspond to zero coefficients in $q-r$. Starting with a polynomial $p(z) = \sum_{m=0}^{L-1} c_m z^m$, the construction of $\varrho$ can then be governed by the column-wise rule

$$\mathcal{C}_m = \begin{cases} \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], & \text{if } c_m = 1, \\ \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], & \text{if } c_m = -1, \\ \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] \text{ or } \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], & \text{if } c_m = 0. \end{cases} \tag{3.5}$$

Via this assignment, one can construct the substituted words $\varrho(a_i)$ by looking at the column concatenation $\mathcal{C}_0\mathcal{C}_1 \dots \mathcal{C}_{L-1}$. □

The substitutions arising from this scheme need not be unique when one factors in the identity

$$\mathfrak{m}(p) = \mathfrak{m}(-p). \tag{3.6}$$

Moreover, some of them might even be non-primitive. We comment on this plurality and illustrate how to circumvent non-primitivity via some examples.

**Example 3.2.9** (Littlewood polynomials)**.** Polynomials for which $c_m \in \{-1, 1\}$ are called *Littlewood polynomials*. In this case, the substitutions $\varrho$ one gets are all bijective. Due to the identity in Eq. (3.6), inversion of columns of a substitution $\varrho$ yields another substitution that admits the same logarithmic Mahler measure as $\chi^B$. As an example, the polynomial $p(z) = -1 - z + z^2 - z^3 + z^4$ gives rise to the primitive substitutions

$$\varrho_p : \begin{cases} 0 \mapsto 11010, \\ 1 \mapsto 00101, \end{cases} \qquad \text{and} \qquad \varrho_{-p} : \begin{cases} 0 \mapsto 00101, \\ 1 \mapsto 11010, \end{cases}$$

with associated Fourier matrices

$$B_p(k) = \begin{pmatrix} \mathrm{e}^{4\pi i k} + \mathrm{e}^{8\pi i k} & 1 + \mathrm{e}^{2\pi i k}\,\mathrm{e}^{6\pi i k} \\ 1 + \mathrm{e}^{2\pi i k}\,\mathrm{e}^{6\pi i k} & \mathrm{e}^{4\pi i k} + \mathrm{e}^{8\pi i k} \end{pmatrix} \qquad \text{and} \qquad B_{-p}(k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B_p(k).$$

Both generated cocycles have the exponent $\chi^B = \mathfrak{m}(p) \approx 0.656256$. ◊

**Example 3.2.10** (Newman polynomials)**.** We next deal with the class of $\{0, 1\}$-polynomials, known as *Newman polynomials*. For this class, one has $r = 0$ in the formula for $B(k)$ given in Lemma 3.2.2, which yields

$$B(k) = \begin{pmatrix} s_0 + q & s_0 \\ s_1 & s_1 + q \end{pmatrix}.$$

If either $s_0$ or $s_1$ is zero, i.e., there are no columns of type $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, the resulting substitution fails to be primitive due to the triangular structure of $B(k)$ (and hence of $M_\varrho = B(0)$). Suppose only one coincidence type is present. One can still obtain a primitive substitution by invoking Eq. (3.6), which induces an inversion of bijective columns from $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. Concretely, given $p(z) = 1 + z^2$, the construction in Proposition 3.2.8 gives

$$\varrho_p : \begin{cases} 0 \mapsto 000, \\ 1 \mapsto 101, \end{cases} \qquad \text{and} \qquad \varrho_{p'} : \begin{cases} 0 \mapsto 010, \\ 1 \mapsto 111, \end{cases}$$

which are both non-primitive. However,

$$\varrho_{-p} : \begin{cases} 0 \mapsto 101, \\ 1 \mapsto 000, \end{cases} \quad \text{and} \quad \varrho_{-p'} : \begin{cases} 0 \mapsto 111, \\ 1 \mapsto 010, \end{cases}$$

are both primitive and aperiodic, and have exponents $\chi^{B} = \mathfrak{m}(p)$. ♢

**Example 3.2.11** (Borwein polynomials)**.** When all coefficients $\{-1, 0, 1\}$ are present in $p(z)$, all standard choices for $\varrho$ via Eq. 3.5 are primitive. Moreover, since there are two column choices for each zero coefficient, one gets $2^n$ distinct substitutions, where $n$ is the number of zero coefficients. One still has, on top of this freedom, Eq. (3.6), which allows one to work with $p$ or $-p$.

As a fitting example, we consider Lehmer's polynomial $\ell_{\mathrm{L}}(z)$ from [Leh33] given by

$$\ell_{\mathrm{L}}(z) = 1 + z - z^3 - z^4 - z^5 - z^6 - z^7 + z^9 + z^{10},$$

which is irreducible and whose root $\lambda$ of maximum modulus is a Salem number. So far, this is known to have the smallest positive logarithmic Mahler measure among integer polynomials, which is given by $\mathfrak{m}(\ell_{\mathrm{L}}) \approx \log(1.176281)$. Here,

$$\varrho_{\ell_{\mathrm{L}}} : \begin{cases} 0 \mapsto 00111111000, \\ 1 \mapsto 11100000011, \end{cases}$$

is one of the substitutions that correspond to $\ell_{\mathrm{L}}(z)$. ♢

**Remark 3.2.12.** For all polynomial subtypes, one obviously can add arbitrary number of coincidences at the tail end of the column expansion $\mathcal{C}_0\mathcal{C}_1 \ldots \mathcal{C}_{L-1}$ and still get the same polynomial $q - r$. This results into substitutions of length greater than $L$. ♢

Proposition 3.2.8 leads to the following dynamical analogue of Lehmer's problem in Section 1.5.

**Question 3.2.13** (Lehmer's problem for substitutions)**.** Does there exist a primitive, binary constant-length substitution $\varrho$ with Lyapunov exponent $0 < \chi^{B} < \mathfrak{m}\big(\ell_{\mathrm{L}}\big) \approx \log(1.17628)$?

**Remark 3.2.14.** Cases when the resulting substitution is periodic are dealt with in [BCM17]. There, it was shown [BCM17, Thm. 3] that for all periodic cases, $\chi^{B}(k) = 0$ for a.e. $k \in \mathbb{R}$. ♢

## 3.3. Abelian bijective case

It is natural to ask whether the conditions in Theorem 2.5.3 can be confirmed on a larger scale. This is the case for a specific class, which we elaborate here. In this section, we assume $\varrho$ to be an aperiodic, primitive, bijective substitution of length $L$ on $\mathcal{A}_{n_a}$, with corresponding Fourier matrix $B(k)$ and associated IDA $\mathcal{B}$.

Due to Fact 2.1.3, we make no distinction between $\mathcal{B}$ and the algebra generated by the digit matrices $\{D_t \mid t \in S_T\}$. In this setting, this is exactly the algebra generated by the matrix representation of the permutations $\{g_0, g_1, \dots, g_{L-1}\}$, where $g_\ell$ is the $\ell$-th column $\mathcal{C}_\ell$ of $\varrho$ given in Eq. (3.2) viewed as an element of the symmetric group $\Sigma_{n_a}$ on $n_a$ letters. In other words,

$$\mathcal{B} = \big\langle \{D_t \mid t \in S_T\} \big\rangle = \big\langle \varPhi(G) \big\rangle,$$

where $G = \langle g_0, g_1, \dots, g_{L-1} \rangle$ and $\varPhi$ is the canonical representation via permutation matrices. From $P^{\mathrm{T}} = P^{-1}$ one has $D_t = \big(\varPhi(g_t)\big)^{\mathrm{T}} = \varPhi(g_t^{-1})$.

We call $G$ a *generating subgroup* for the algebra $\mathcal{B}$, where it is understood that $G$ is a subgroup of $\Sigma_{n_a}$. The primitivity condition on $\varrho$ translates to a condition on its generating subgroup as follows; compare [Que10, Lem. 8.1].

**Lemma 3.3.1.** *Any generating subgroup $G$ for $\mathcal{B}$ must be a transitive subgroup of $\Sigma_{n_a}$.*

*Proof.* Assume to the contrary that $G$ is not transitive. Then, there are $a_i, a_j \in \mathcal{A}_{n_a}$ such that $\sigma(a_i) = a_j$ cannot hold for *any* $\sigma \in G$. Consequently, the representation matrices will be 0 in position $i, j$, as well as all of their linear combinations, and hence all elements of $\mathcal{B}$ by Fact 2.1.3.

Now, this implies that $a_j$ can never appear in any substituted word $\varrho^n(a_i)$ with $n \in \mathbb{N}$. This contradicts the primitivity of $\varrho$, and our claim follows. □

The following property of Abelian subgroups of $\Sigma_{n_a}$ is well known; see [Sco64, Cor. 10.3.3 and Thm. 10.3.4].

**Fact 3.3.2.** *Any transitive Abelian subgroup of $\Sigma_{n_a}$ must be of order $n_a$. So, if $G$ is an Abelian subgroup of $\Sigma_{n_a}$ that is generating for the IDA $\mathcal{B}$ of $\varrho$, it must be of order $n_a$.* □

Bijective substitutions have a rich structure due to the algebraic properties of their columns. These can be exploited to shed light on the spectral measures of the associated dynamical system; see [Bar16, Que10]. When the generating group is Abelian, the measures generating the spectral measure of maximal type $\sigma_{\max}$ can be represented as Riesz products of polynomials arising from the characters $\rho \in \widehat{G}$, evaluated on the columns of $\varrho$; see Appendix A.

The following important result (actually also its higher-dimensional analogue) was outlined in [Que10], and was formally proved in [Bar16]. For binary block substitutions, it also follows from [Fra05, Fra18], and it was shown in [BG14] by a different method.

**Theorem 3.3.3** ( [Bar16, Thm. 4.19])**.** *Any primitive, bijective constant-length substitution that is aperiodic and Abelian has purely singular dynamical spectrum.* □

In what follows, we prove that Theorem 2.5.3 holds for this class. Note that we impose no assumptions on the length or the height of $\varrho$.

**Theorem 3.3.4.** *Let $\varrho$ be a primitive, aperiodic, bijective substitution whose IDA $\mathcal{B}$ is Abelian. Then, for a.e. $k \in \mathbb{R}$, all Lyapunov exponents of $\varrho$ are positive. Moreover, the corresponding diffraction $\widehat{\gamma}$ is singular relative to Lebesgue measure.*

*Proof.* By assumption, the generating subgroup $G$ is Abelian, and all digit matrices $D_t$ commute with one another. Being permutation matrices, they are simultaneously diagonalisable by a unitary matrix $U$. In this case, the diagonal entries of $U D_t U^{-1}$ are values of characters of $G$, written as $\rho_i(g)$. Note that the $\rho_i$ coincide with the irreducible representations of $G$ since the latter is Abelian.

In the diagonal form of $\Phi(g)$, all possible values that occur are roots of unity, so $|\rho_i(g)| = 1$ for all $1 \leqslant i \leqslant n_a$ and $g \in G$. If $\varrho$ has length $L$, the eigenvalues of $B(k)$ are then of the form

$$\beta_j(k) = \sum_{m=0}^{L-1} \overline{\rho_j\left(g_m\right)} u^m, \tag{3.7}$$

which is a polynomial in $u = \mathrm{e}^{2\pi \mathrm{i} k}$ of degree $L-1$, whose coefficients all lie on the unit circle.

From Proposition 3.1.3, we know that the Lyapunov exponents exist and are constant for a.e. $k$, which we are able to explicitly compute from Eq. (3.7). They can be computed for each invariant subspace, where one obtains

$$\chi_j = \log \sqrt{L} - \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} \log \left| \beta_j\left(L^\ell k\right) \right| \underset{k\in\mathbb{R}}{\overset{\text{a.e.}}{=}} \log \sqrt{L} - \int_0^1 \log \left|\beta_j\left(k\right)\right| \mathrm{d}k > 0, \tag{3.8}$$

where the integral is strictly less than $\log \sqrt{L}$ since

$$\exp\left(\int_0^1 \log \left|\beta_j\left(k\right)\right| \mathrm{d}k\right) < \int_0^1 \left|\beta_j\left(k\right)\right| \mathrm{d}k < \left\|\beta_j\left(k\right)\right\|_2 = \sqrt{L}.$$

Here, the first estimate follows from Jensen's inequality and is strict, as is the second since $\beta_j\left(k\right)$ is not a monomial. The last step follows from Parseval's identity, compare with the proof of Lemma 1.5.2, which completes the requirements of Theorem 2.5.3 and thus finishes the proof. □

**Remark 3.3.5.** It is known that the diffraction spectrum is related to the dynamical spectrum of $(\mathbb{X}, \mathbb{Z}, \mu)$; see Appendix A. In particular, for substitutions of constant length, the spectral measure of maximal type $\sigma_{\max}$ is absolutely continuous to the diffraction measure $\widehat{\gamma}$; see Proposition A.0.6. This extends the singularity result in Theorem 3.3.4 to the entire dynamical spectrum, thus giving yet another proof of Theorem 3.3.3. ♢

**Example 3.3.6.** Consider $\varrho^2$ in Example 2.1.8, which is a substitution on $\mathcal{A}_4$, with associated generating subgroup $G = C_2 \times C_2$. The Fourier matrix reads

$$B(k) = \begin{pmatrix} 1 & u^3 & u^2 & u \\ u^3 & 1 & u & u^2 \\ u^2 & u & 1 & u^3 \\ u & u^2 & u^3 & 1 \end{pmatrix}$$

where $u = \mathrm{e}^{2\pi \mathrm{i} k}$, while its corresponding eigenvalues are given by

$$\beta_1 = 1 - u - u^2 + u^3, \qquad \beta_3 = 1 - u + u^2 - u^3,$$
$$\beta_2 = 1 + u - u^2 - u^3, \qquad \beta_4 = 1 + u + u^2 + u^3,$$

with corresponding eigenvectors that are $k$-independent. These four polynomials are products of cyclotomic polynomials, and hence $\mathfrak{m}(\beta_j) = 0$ for $1 \leqslant j \leqslant 4$. This results in a degenerate Lyapunov spectrum for a.e. $k \in \mathbb{R}$, and hence $\chi^B = 0 < \log \sqrt{L} = \log(2)$. ♢

Determining which substitutions have the same Lyapunov exponents is generally a difficult task, especially since the equality of logarithmic Mahler measures, which only depend on roots outside the unit circle, does not imply that they come from the same polynomial. However, as we shall see in the next result, a certain dichotomy gives rise to families of substitutions that share the same Lyapunov spectrum, prior to adding $\log \sqrt{L}$.

**Corollary 3.3.7.** *Consider the constant-length substitution $\varrho = (w_1, w_2, \dots, w_{n_a})$, where one has $|w_i| = L$ for all $i$, and assume that the columns are either bijective or constant. Suppose further that the group $G'$ generated by the bijective columns is Abelian (but not necessarily transitive in $\Sigma_{n_a}$). Then, the Lyapunov exponents associated to $\varrho$ are strictly positive.*

*Proof.* From the premise, the Fourier matrix of $\varrho$ can be decomposed into

$$B(k) = B_{\mathsf{b}}(k) + B_{\mathsf{c}}(k),$$

where $B_{\mathsf{b}}(k)$, $B_{\mathsf{c}}(k)$ are generated by the bijective and coincident columns, respectively. This gives a partition of the positions $\{0, \dots, L-1\} = S_{\mathsf{b}} \cup S_{\mathsf{c}}$. The idea of the proof now is to show that all but one eigenvalue of $B_{\mathsf{b}}(k)$ (and their corresponding eigenvectors) are essentially inherited by $B(k)$. We begin by illustrating how this works for cases when $G'$ is transitive and describe what changes in the case when it is not.

It follows from the proof of Theorem 3.3.4 that $B_{\mathsf{b}}(k)$ has $n_a$ eigenvectors that do not depend on $k$. Moreover, $(n_a - 1)$ of these eigenvectors have a component sum equal to zero, being generators of the invariant subspace corresponding to $U_{\mathsf{st}}$. The remaining eigenvector is given by $v_{n_a} = (1, 1, \dots, 1)^{\mathrm{T}}$.

Consider any eigenvector $v$ of $B_{\mathsf{b}}(k)$, with eigenvalue $\beta(k)$, with zero component sum. Observe that we can write $B_{\mathsf{c}}(k)$ as

$$B_{\mathsf{c}}(k) = \sum_{z \in S_c} \mathrm{e}^{2\pi \mathrm{i} k z} R_{a(z)}, \qquad 1 \leqslant a(z) \leqslant n_a$$

where the matrix $R_m$ is 1 in the $m$-th row and zero elsewhere. Consequently, $R_m v = 0$ for all $1 \leqslant m \leqslant n_a$, which implies $B_{\mathsf{c}}(k)\, v = 0$ and that $v$ is also an eigenvector of $B(k)$, with the same eigenvalue $\beta(k)$.

As in Theorem 3.3.4, the eigenvalues of $B_{\mathsf{b}}$ could be written in terms of the characters of $G'$ as follows,

$$\beta_j(u) = \sum_{m \in S_{\mathsf{b}}} \overline{\rho_j(g_m)}\, u^m$$

which is always a polynomial in $u = \mathrm{e}^{2\pi \mathrm{i} k}$ of degree at most $L-1$. All of its coefficients have modulus either 0 or 1. Parseval's equation then once again guarantees that the Lyapunov exponents which arise from these eigenvalues are bounded away from $\log \sqrt{L}$. The maximal Lyapunov exponent is then achieved for some $j$, which in turn satisfies

$$\chi^B = \chi_j = \mathfrak{m}(\beta_j) < \log \sqrt{L}.$$

The $(n_a - 1)$ exponents shared by $B$ and $B_{\mathsf{b}}$ clearly satisfy this bound. The idea is to then invoke forward regularity to show that the last exponent is zero, which is done prior to adding $\log \sqrt{L}$. This will confirm that $B$ and $B_{\mathsf{b}}$ indeed share the same set of exponents.

To this end, we note that the $n_a$-th eigenvalue of $B(k)$ is $\beta'_{n_a}(u) = \sum_{m=0}^{L-1} u^m$, which easily follows from the trace formula. By forward regularity, the sum of the exponents under the outward iteration can be derived from Eq. (1.9) as

$$\begin{aligned}
\sum_{m=1}^{n_a} \chi'_m &= -\int_0^1 \log |\det B(k)|\, \mathrm{d}k = -\sum_{m=1}^{n_a} \int_0^1 \log |\beta'_m(u)|\, \mathrm{d}k \\
&= \chi'_1 + \chi'_2 + \dots + \chi'_{n-1} - \mathfrak{m}(\beta'_{n_a}(u)),
\end{aligned}$$

from which it is clear that $\chi'_n = -\mathfrak{m}(\beta'_{n_a}(u)) = 0$, since $\beta'_{n_a}$ is cyclotomic. This completes the proof for the transitive case.

When $G'$ fails to be transitive, we can still use the decomposition $B = B_{\mathsf{b}} + B_{\mathsf{c}}$, where $B_{\mathsf{b}}$ now has to be put into block diagonal form via some elementary matrix operations that partition $\mathcal{A}_{n_a} = \{a_1, \dots, a_{n_a}\}$ into orbits of $G'$. A particularly useful decomposition of $G'$ is $G' \simeq G'_1 \times \cdots \times G'_s$, wherein each subgroup $G'_\ell$ (which can be the trivial subgroup) acts transitively on the $s$ orbits in $\mathcal{A}_{n_a}$. Further, each nontrivial $G'_\ell$ can be written as a finite product of cyclic groups by the fundamental theorem of finite Abelian groups. This also means that the digit matrices afford the splitting

$$D_m = \varPhi(g_m^{-1}) = \bigoplus_{\ell=1}^{s} \overline{\varPhi_\ell(g_m^{(\ell)})}$$

with $g_m = \bigoplus_{\ell=1}^{s} g_m^{(\ell)}$, where $\varPhi_\ell$ is the permutation representation on $G'_\ell$.

With this, we recover the eigenvalues of $B_{\mathsf{b}}$ from each block as

$$\beta_j^{(\ell)}(k) = \sum_{m \in S_{\mathsf{b}}} \overline{\rho_j^{(\ell)}(g_m^{(\ell)})}\, u^m,$$

where $\rho_j^{(\ell)}$ is an irreducible character of $G'_\ell$. An immediate consequence is that $\sum_{m \in S_{\mathsf{b}}} u^m$ has multiplicity $s$ as an eigenvalue of $B_{\mathsf{b}}$ (corresponding to different eigenvectors), since all blocks naturally admit the trivial representation. Note that non-transitivity in conjunction with primitivity of the substitution implies that at least one coincidence must be present, which implies $\mathrm{card}(S_{\mathsf{b}}) < L$.

Similar to the transitive case, any eigenvector of $B_{\mathsf{b}}$ with zero component sum remains an eigenvector of $B$, with the same eigenvalue. All but one copy of the polynomial $\sum_{m \in S_{\mathsf{b}}} u^m$ also remain eigenvalues, but this time with the corresponding eigenvectors being linear combinations of eigenvectors from different blocks. Finally, the uninherited eigenvalue (the one with a $k$-dependent eigenvector) is the cyclotomic polynomial $\sum_{m=0}^{L-1} u^m$, as can be computed from the trace. It is easy to see that the same arguments unambiguously apply as in the transitive case, since the eigenvalues are polynomials in $u$ with coefficients of either zero or unit modulus. □

**Remark 3.3.8.** We stress that the set of constant-length substitutions satisfying the conditions of Corollary 3.3.7 is a subset of substitutions with coincidences. These, by Dekking's criterion given in Theorem 3.1.9, all have pure point spectrum (both diffraction and dynamical). What we have confirmed here using our independent method via Lyapunov exponents is the singularity of the spectrum (which is slightly weaker) for this specific subset. ◊

**Example 3.3.9** ($\mathcal{A}_3$, transitive, $G' \simeq C_3$)**.** Consider the substitution $\varrho_3$, with Fourier matrix $B_3(k)$ given by

$$\varrho_3 : \begin{cases} 0 \mapsto 0022, \\ 1 \mapsto 1002, \\ 2 \mapsto 2012, \end{cases} \qquad B_3(k) = \begin{pmatrix} 1+u & u+u^2 & u \\ 0 & 1 & u^2 \\ u^2+u^3 & u^3 & 1+u^3 \end{pmatrix},$$

where $u = \mathrm{e}^{2\pi \mathrm{i} k}$ as usual. The eigenvalues (and eigenvectors) of $B_3$, which originate from the digit matrices generating the Abelian IDA, are given by

$$\begin{aligned} \beta_1(u) &= 1 + \xi_3^2 u^2 \quad \text{with} \quad v_1 = (\xi_3^2, \xi_3, 1)^{\mathrm{T}} \quad \text{and} \\ \beta_2(u) &= 1 + \xi_3 u^2 \quad \text{with} \quad v_2 = (\xi_3, \xi_3^2, 1)^{\mathrm{T}}, \end{aligned}$$

where $\xi_3 = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$. The third eigenvalue is given by $\beta_3(u) = 1 + u + u^2 + u^3$. All logarithmic Mahler measures of $\beta_1, \beta_2, \beta_3$ are zero since their respective roots all lie on the unit circle. $\Diamond$

**Example 3.3.10** ($\mathcal{A}_4$, non-transitive, $G' \simeq C_2 \times C_2$)**.** The converse of Fact 3.3.2 does not hold in general. There are Abelian subgroups of $\Sigma_{n_a}$ of order $n_a$ which fail to be transitive. In $\Sigma_4$, there are seven subgroups isomorphic to the Klein–4 group $C_2 \times C_2$, only three of which are transitive. Here, we consider a substitution when $G'$ has two disjoint orbits. Consider the substitution $\varrho_V$, alongside with its corresponding Fourier matrix:

$$\varrho_V : \begin{cases} 0 \mapsto 0112, \\ 1 \mapsto 1012, \\ 2 \mapsto 3212, \\ 3 \mapsto 2312, \end{cases} \qquad B(k) = \underbrace{\begin{pmatrix} 1 & u & 0 & 0 \\ u & 1 & 0 & 0 \\ 0 & 0 & u & 1 \\ 0 & 0 & 1 & u \end{pmatrix}}_{B_{\mathsf{b}}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ u^2 & u^2 & u^2 & u^2 \\ u^3 & u^3 & u^3 & u^3 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{B_{\mathsf{c}}}.$$

The eigenvalues of $B$ corresponding to the three $k-$independent eigenvectors of $B_{\mathsf{b}}$ are

$$\begin{aligned} \beta_1(u) &= 1 - u, & v_1 &= (-1, 1, 0, 0)^{\mathrm{T}}, \\ \beta_2(u) &= -1 + u, & v_2 &= (0, 0, -1, 1)^{\mathrm{T}}, \\ \beta_3(u) &= 1 + u, & v_3 &= (-1, -1, 1, 1)^{\mathrm{T}}, \end{aligned}$$

with the last eigenvalue being $\beta_4 = 1 + u + u^2 + u^3$. Here, one sees that $v_3$ is a linear combination of the eigenvectors from the two separate blocks of $B_{\mathsf{b}}$ corresponding to the same eigenvalue $\beta_3$. The positivity of the Lyapunov exponents follows from the same arguments as in our previous examples. $\Diamond$

The following necessary criterion for a primitive substitution of constant length to have an absolutely continuous component in its dynamical spectrum is due to Berlinkov and Solomyak.

**Theorem 3.3.11** ( [BS17, Thm. 1.1])**.** *Let $\varrho$ be a primitive substitution of constant length. Then, if its dynamical spectrum contains an absolutely continuous component, its substitution matrix $M$ must have an eigenvalue of modulus $\sqrt{\lambda_{\mathrm{PF}}}$.* $\square$

One can easily check that the substitution matrix $M_\varrho = B(0)$ of $\varrho_V$ in Example 3.3.10 has eigenvalues $\{4, 2, 0, 0\}$. However, $\varrho_V$ contains coincidences, and hence has pure point spectrum by Dekking's criterion. The absence of absolutely continuous spectral components is also rederived here via the positivity of the Lyapunov exponents.

We provide another example to demonstrate the abundance of such substitutions and comment on how one can systematically construct examples that satisfy the criterion given in Theorem 3.3.11, but do not have absolutely continuous spectrum.

**Example 3.3.12** ($\mathcal{A}_5$, Length 9, $G' \simeq C_4$)**.** Let $\varrho_5$ and its substitution matrix be given by

$$\varrho_5 : \begin{cases} 0 \mapsto 031422300, \\ 1 \mapsto 102422300, \\ 2 \mapsto 213422300, \\ 3 \mapsto 320422300, \\ 4 \mapsto 444422300, \end{cases} \qquad M_{\varrho_5} = \underbrace{\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}}_{B_{\mathsf{b}}(0)} + \underbrace{\begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}}_{B_{\mathsf{c}}(0)}.$$

The eigenvalues of $M$ are $\{9, 3, -1, 1, 1\}$, with 3 again coming from the inherited eigenvalue $1 + u + u^2$ of $B_{ab}$. ♢

In general, one can begin with a (non-primitive) substitution of length $L$ on $n_a$ letters, whose columns are bijective, and with its generating subgroup $G'$ being a non-transitive subgroup of $\Sigma_{n_a}$. From the proof of Corollary 3.3.7, $\beta(k) = 1 + u + \ldots + u^{L-1}$ is always an eigenvalue of $B_{\mathsf{b}}$, and at least one copy of it survives to be an eigenvalue of $B$, which means that $\beta(0) = L$ is an eigenvalue of $M = B(0)$ of the new substitution formed by adding coincidences. One can then choose to add appropriate columns, so that the resulting substitution is primitive, and enough columns, so that it is of length $L^2$. All substitutions emerging from this construction satisfy Theorem 3.3.11, but have pure point spectrum.

**Remark 3.3.13** (Outward and inward filtration are usually unrelated)**.** From Remark 2.5.1, the Lyapunov exponents for the inward iteration are given by

$$\chi_i = \log|\lambda_i| - \log\sqrt{\lambda_{\mathrm{PF}}},$$

where $\{\lambda_i\}$ are the eigenvalues of $M_\varrho$. Furthermore, for Abelian cocycles, $\mathcal{V}_k^i \setminus \mathcal{V}_k^{i+1} = E_i$, where $E_i$ corresponds to the eigenspace of $\lambda_i$, for all $k \in \mathbb{R}$. In general, the arrangement of the subspaces in the inward filtration does not seem to have any implication on the outward filtration. To see this, consider the substitution

$$\varrho : \begin{cases} a \mapsto abb, & c \mapsto cdd, \\ b \mapsto bcc, & d \mapsto daa, \end{cases}$$

with respective inward and outward exponents

| Subspace | $\chi_i^{(\mathrm{in})}$ | $\chi_i^{(\mathrm{out})}$ |
|---|---|---|
| $\langle v_1 \rangle$ ; $v_1 = (1,1,1,1)^{\mathrm{T}}$ | $\log(\sqrt{3})$ | $\log(\sqrt{3})$ |
| $\langle v_2 \rangle$ ; $v_2 = (-1,1,-1,1)^{\mathrm{T}}$ | $-\log(\sqrt{3})$ | $\log(\sqrt{3})$ |
| $\langle v_3 \rangle$ ; $v_3 = (-\mathrm{i},-1,\mathrm{i},1)^{\mathrm{T}}$ | $\log\left(\sqrt{\frac{5}{3}}\right)$ | $\log(\sqrt{3}) - \log(1.44)$ |
| $\langle v_4 \rangle$ ; $v_4 = (\mathrm{i},-1,-\mathrm{i},1)^{\mathrm{T}}$ | $\log\left(\sqrt{\frac{5}{3}}\right)$ | $\log(\sqrt{3}) - \log(1.44)$ |

where one notices that the slowest growing subspace $\langle v_2 \rangle$ with respect to the inward iteration is a subspace of $\mathcal{V}^1_{\mathrm{out}} \setminus \mathcal{V}^2_{\mathrm{out}}$ for the outward one. On the contrary, for the substitution

$$\varrho : \begin{cases} a \mapsto abbb, \\ b \mapsto bccc, \\ c \mapsto caaa, \end{cases}$$

the subspace $V = \langle v_1, v_2 \rangle$, with $v_1 = (\xi_3^2, \xi_3, 1)^{\mathrm{T}}$ and $v_2 = (\xi_3, \xi_3^2, 1)^{\mathrm{T}}$ exhibits the smallest exponential growth both for the outward and inward iterations.

$\Diamond$

## 3.4. Mixed substitutions

**Definition 3.4.1.** A substitution $\varrho$ is called a *global mixture* of $\varrho_1$ and $\varrho_2$ (or is a *globally-mixed substitution*) if $\varrho(a_i) = \varrho_2(\varrho_1(a_i))$ for all $a_i \in \mathcal{A}_{n_a}$. Here we write, $\varrho := \varrho_2 \circ \varrho_1$.

We emphasise that the mixing of the substitutions here is global, as opposed to *local mixing* where for every letter of a finite or infinite word, one has the freedom to choose the substitution rule to apply. Such mixtures are more known as *random substitutions*; see [RS18] and references therein for a general exposition.

**Proposition 3.4.2.** *Let $\varrho_1, \varrho_2$ be two primitive constant-length substitutions on $n_a$ letters, with corresponding inflation factors $\lambda_1, \lambda_2$ and Fourier matrices $B_1(k), B_2(k)$. Consider the (globally) mixed substitution given by $\varrho_{\mathrm{M}} := \varrho_2 \circ \varrho_1$. Then, the Fourier matrix of $\varrho_{\mathrm{M}}$ is given by*

$$B_{\mathrm{M}}(k) = B_2(k) B_1(\lambda_2 k).$$

*Proof.* Let $T^1, T^2, T^{\mathrm{M}}$ be the displacement matrices of the substitutions $\varrho_1, \varrho_2, \varrho_{\mathrm{M}}$, respectively. The Fourier matrix $B_{\mathrm{M}}(k)$ can be computed from $T^{\mathrm{M}}$ by $B_{\mathrm{M}}(k) = \widehat{\overline{\delta_{T^{\mathrm{M}}}}}$. Proving the claim is then equivalent to showing that

$$\delta_{T^{\mathrm{M}}} = \delta_{T^2} * \delta_{\lambda_2 T^1}. \tag{3.9}$$

Consider a specific entry $(\delta_{T^2} * \delta_{\lambda_2 T^1})_{ij}$. This can explicitly be written as

$$\left(\delta_{T^2} * \delta_{\lambda_2 T^1}\right)_{ij} = \sum_{\ell=1}^{n_a} \left(\delta_{T^2_{i\ell}}\right) * \left(\delta_{\lambda_2 T^1_{\ell j}}\right) = \sum_{\ell=1}^{n_a} \delta_{T^2_{i\ell} + \lambda_2 T^1_{\ell j}} = \delta_{\bigcup_\ell T^2_{i\ell} + \lambda_2 T^1_{\ell j}},$$

where we have used the property $\delta_X * \delta_Y = \delta_{X+Y}$, with $X + Y$ being the Minkowski sum of the two sets, and where it is clear that $T^2_{i\ell} + \lambda_2 T^1_{\ell j} = \varnothing$ whenever any of the summands is the empty set. Now fix $i, j, \ell_0$ and assume that $T^2_{i\ell_0}, T^1_{\ell_0 j} \neq \varnothing$. The point measure $\delta_{\lambda_2 T^1_{\ell_0 j}}$ encodes the positions of the inflated tiles of type $\ell_0$ in $\left(\varrho_2 \circ \varrho_1\right)(j)$ prior to subdivision into union of prototiles. The other measure $\delta_{T^2_{i\ell_0}}$ specifies the location of tiles of type $i$ within this $\ell_0-$tile after subdivision. This means that $\delta_{T^2_{i\ell_0} + \lambda_2 T^1_{\ell_0 j}}$ collects all positions of prototiles of type $i$ which live in level-2 supertiles of type $\varrho_2(\ell_0)$. It is then clear that, when one takes the union of all of these positions over all possible tile type $1 \leqslant \ell_0 \leqslant n_a$, one gets all positions of the $i-$tiles in the supertile $(\varrho_2 \circ \varrho_1)(j)$, which proves Eq. (3.9).

To summarise, there is a tile of type $i$ in $\left(\varrho_2 \circ \varrho_1\right)(j)$ at exactly $t = \lambda_2 x + y$ whenever there is a tile of type $\ell$ at $t = x$ in $\varrho_1(j)$ and a tile of type $i$ at $\varrho_2(\ell)$ at $t = y$. This is illustrated in Figure 3.1. The claim now follows from a direct application of the convolution theorem. $\square$

The following corollary is immediate from an inductive argument.

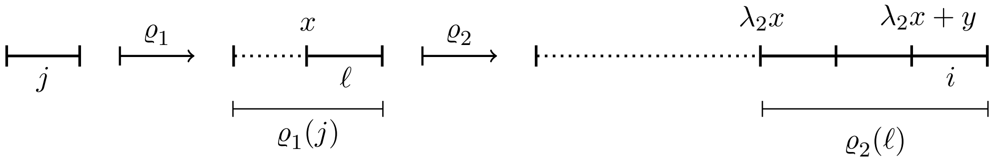



Figure 3.1.: Structure of displacement sets for mixed inflations

**Corollary 3.4.3.** *Let $\varrho_1, \ldots, \varrho_r$ be primitive constant-length substitutions on $n$ letters, with corresponding inflation factors $\lambda_1, \ldots, \lambda_r$. Then,*

$$B(k) = B_r(k) B_{r-1}(\lambda_r k) \cdot \ldots \cdot B_2(\lambda_3 \cdot \ldots \cdot \lambda_r k) B_1(\lambda_2 \cdot \ldots \cdot \lambda_r k)$$

*is the Fourier matrix of $\varrho := \varrho_r \circ \varrho_{r-1} \circ \cdots \circ \varrho_2 \circ \varrho_1$.* □

When the tile lengths are consistent for each substitution, the expand-subdivide scheme proceeds as is, only with a possibly different labelling rule at every step. From this, we get the following generalisation.

**Corollary 3.4.4.** *Corollary 3.4.3 also holds whenever $\varrho_1, \ldots, \varrho_r$ are primitive substitutions which share the same left PF eigenvector $\boldsymbol{L}$.* □

**Remark 3.4.5.** The main motivation for Corollary 3.4.4 is a possible extension of the diffraction formalism to infinite mixtures of substitutions $\varrho = \prod_{i \geqslant 1} \varrho_i$, which are also known as *$S$-adic systems*. The symbolic hulls of some of these systems are known to be strictly ergodic, which is a good starting point in view of pair correlations. We refer to [BD14] for a comprehensive survey of $S$-adic systems. ♢

**Remark 3.4.6.** One cannot expect the same result to hold for all mixed substitutions in general. When mixing substitutions that do not have the same $\boldsymbol{L}$, the natural tile lengths that one gets from the eigenvector of the substitution matrix are totally different from the lengths of the individual substitutions. Aside from this, one also does not expect that the new inflation factor is a simple product of the individual inflation factors. As an example, when one mixes $\varrho_{\mathrm{F}}$ and $\varrho_{\mathrm{TM}}$, one gets

$$\varrho := \varrho_{\mathrm{TM}} \circ \varrho_{\mathrm{F}} : \begin{cases} 0 \mapsto 0110, \\ 1 \mapsto 01, \end{cases}$$

whose inflation factor and natural tile lengths are $\lambda = 3$ and $L_0 = 2, L_1 = 1$. ♢

In what follows, we state known results from number theory about Littlewood polynomials and show how to profit from these and from Corollary 3.4.3 in the bijective binary case.

**Theorem 3.4.7** ( [BC99, Thm. 3.4]). *Let $q(z)$ be a cyclotomic Littlewood polynomial of even degree $L-1$. Then, $q(z)$ can be written as*

$$q(z) = \pm \Phi_{p_1}(\pm z) \Phi_{p_2}(\pm z^{p_1}) \cdot \ldots \cdot \Phi_{p_r}(\pm z^{p_1 p_2 \cdots p_{r-1}})$$

*with $L = p_1 p_2 \cdots p_r$, where all $p_i$ are prime (not necessarily distinct), and $\Phi_{p_i} = \frac{z^{p_i}-1}{z-1}$.* □

**Conjecture 3.4.8** ( [BC99, Conj. 4.1]). *Let $q(z)$ be a cyclotomic Littlewood polynomial of odd degree $L-1$. Then, $q(z)$ affords the same decomposition as in the even case.*

Thangadurai proved Conjecture 3.4.8 in [Tha02] for separable polynomials of degree $2^r p^\ell - 1$. Akhtari and Choi extended this result in [AC08] to all separable polynomials of odd degree and all polynomials of degree $2^r p^\ell - 1$, where $p$ is a prime number.

In Proposition 3.2.8, it was shown that $\mathfrak{m}(q)$ with $q \in \{-1,1\}[z]$ can be recovered as $\chi^B$ of a bijective, binary constant-length substitution $\varrho$. This together with Corollary 3.4.3, Theorem 3.4.7, and Conjecture 3.4.8 gives the following partial classification result.

**Proposition 3.4.9.** *Let $\varrho$ be a bijective, binary constant-length substitution of even length $L$, or of odd length of the form $L = 2^\alpha p^\beta - 1$, where $p$ is an odd prime. Then, the Lyapunov spectrum of $B(k)$ associated to $\varrho$ is degenerate if and only if $\varrho$ can be written as a mixed substitution given by*

$$\varrho := \varrho_\ell \circ \varrho_{\ell-1} \circ \cdots \circ \varrho_2 \circ \varrho_1,$$

*where each $\varrho_i$ is also bijective, of prime length $p_i$ (not necessarily distinct), with $L = p_1 \cdots p_\ell$, such that the corresponding Lyapunov exponent $\chi^B$ for each $\varrho_i$ is also* 0.

*Proof.* Suppose we have a bijective binary substitution $\varrho$ of length $L$, where $L$ satisfies the conditions of the theorem. The corresponding Fourier matrix is of the form

$$B(k) = \begin{pmatrix} q(u) & r(u) \\ r(u) & q(u) \end{pmatrix},$$

where $u = \mathrm{e}^{2\pi \mathrm{i} k}$, and with associated Littlewood polynomial $(q-r)$. Suppose further that the Lyapunov spectrum of $\varrho$ is degenerate, i.e., $\chi^B = 0$. Clearly, $q-r$ must essentially be cyclotomic from Theorem 1.5.3. It follows then from Theorem 3.4.7 and partially proven results on Conjecture 3.4.8 that $q-r$ decomposes into

$$(q-r)(k) = \pm (q-r)_{p_1}(u)(q-r)_{p_2}(u^{p_1}) \cdot \ldots \cdot (q-r)_{p_\ell}(u^{p_1 p_2 \cdots p_{\ell-1}}).$$

One can then choose a corresponding substitution $\varrho_i$ of length $L_i = p_i$ with corresponding polynomial $(q-r)_{p_i}$ via Proposition 3.2.8. This extends to the corresponding Fourier matrices satisfying

$$B(k) = B_1(k) B_2(p_1 k) \cdot \ldots \cdot B_\ell(p_1 p_2 \cdots p_{\ell-1} k),$$

which from Corollary 3.4.3 holds whenever $\varrho = \varrho_1 \circ \varrho_2 \circ \ldots \circ \varrho_\ell$. □

**Remark 3.4.10.** It is possible that some $\varrho_i$ are not primitive, which occurs exactly when $\varrho_i : 0 \mapsto \underbrace{000\ldots0}_{L_i}, 1 \mapsto \underbrace{111\ldots1}_{L_i}$, hence $q - r = q = \frac{z^L - 1}{z-1}$. Aside from this, due to the structure of the constituent polynomials in the decomposition, we know that the corresponding first level substituted words $w_0, w_1$ of $\varrho_i$ are either constant (as in the example above) or alternating. ♢

**Remark 3.4.11.** All bijective binary substitutions in the balanced-weight case (i.e., where weights are chosen to be $W_0 = 1$ and $W_1 = -1$) are suspected to have a singular continuous diffraction measure $\widehat{\gamma}$; see [BGG12, BG14]. For the Thue–Morse measure, an investigation of this measure under the thermodynamic formalism was done in [BGKS18], where the exponent $\chi^B$ is related to the value of the scaling exponent of $\widehat{\gamma}$ on a set of full measure. ♢

## 3.5. Examples with absolutely continuous spectrum

### 3.5.1. Rudin–Shapiro

**Example 3.5.1.** The Rudin–Shapiro substitution first mentioned in Example 1.2.8 is arguably the simplest example of a classical substitution that admits an absolutely continuous component in its diffraction and dynamical spectra. Here, we consider an equivalent substitution on a four-letter alphabet with two letters and their barred counterparts.

Let $\varrho_{\text{RS}} : a \mapsto ab, b \mapsto a\bar{b}, \bar{b} \mapsto \bar{a}b, \bar{a} \mapsto \bar{a}\bar{b}$ be the Rudin–Shapiro substitution, whose Fourier matrix is given by

$$B(k) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \mathrm{e}^{2\pi \mathrm{i}k} & 0 & \mathrm{e}^{2\pi \mathrm{i}k} & 0 \\ 0 & \mathrm{e}^{2\pi \mathrm{i}k} & 0 & \mathrm{e}^{2\pi \mathrm{i}k} \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Since 0 is an eigenvalue of $B(k)$, the matrix fails to be invertible for any $k$, and hence we cannot directly define a cocycle corresponding to the outward iteration from $B$ and test Corollary 2.5.4. However, there exists a $k-$independent matrix that transforms $B(k)$ into block diagonal form, compare with [BG16, Sec. 4.2], which is explicitly given by

$$B'(k) = \left(\begin{array}{cc|cc} 1+\mathrm{e}^{2\pi \mathrm{i}k} & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\boldsymbol{O}$}} \\ -\,\mathrm{e}^{2\pi \mathrm{i}k} & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\boldsymbol{O}$}} & 1 & 1 \\ & & \mathrm{e}^{2\pi \mathrm{i}k} & -\,\mathrm{e}^{2\pi \mathrm{i}k} \end{array}\right),$$

with the matrix $Z_1(k)$ in the upper block getting the zero eigenvalue and the lower block matrix $Z_2(k)$ now being invertible for all $k$ ($\det Z_2(k) = -2\,\mathrm{e}^{2\pi \mathrm{i}k}$). It is also worth noting that this decomposition into invariant subspaces is induced by the bar swap symmetry of the substitution, which is given by $a \longleftrightarrow \bar{a}, b \longleftrightarrow \bar{b}$, which will be tackled in full generality in Section 5.2.3.

Since $Z_2^{-1}(k)$ exists for all $k$, we can consider the cocycle defined by iterating this matrix and compute its corresponding Lyapunov exponent.

Due to the irreducibility of $Z_2(k)$, our technique of computing the exponent for each invariant subspace does not work. However, one notices that $\frac{1}{\sqrt{2}} Z_2\,(k)$ is unitary and so, for any starting vector, $v(k)$ ends up having the same norm as its $n$-th iterate $v\,(2^n k)$ under the cocycle $Z_2^{(n)}(k)$ defined by $Z_2(k)$, which means that both Lyapunov exponents vanish, $\chi_1 = \chi_2 = 0$. ♢

**Remark 3.5.2.** The dynamical system defined by the return word encoding $\varrho_{\text{ret}}$ of $\varrho_{\text{RS}'}$ is topologically conjugate to that of the latter, which implies that it also has an absolutely continuous component in its spectrum. Due to $B(k)$ being singular for all $k$, one cannot use the usual tools to compute exponent bounds. The exponents still exists though, due to Theorem 1.4.7, which this time are allowed to be $-\infty$. The irreducibility of $B(k)$ and the presence of a non-trivial $k$-dependent kernel introduce obstructions in computing the actual exponents. ♢

In what follows, we show that this $k$-independent reducibility is satisfied by other families of substitutive examples that emerge from similar constructions.

### 3.5.2. A nine-letter example

Frank introduced a scheme in [Fra03] by which, given a Hadamard matrix $H$, one can construct substitutions with Lebesgue component in their dynamical spectrum. Subsequently, these systems also display non-trivial absolutely continuous diffraction. Chan, Grimm, and Short generalised this construction to complex Hadamard matrices arising from discrete Fourier transform (DFT) matrices via a modification of Rudin's argument; see [CGS18, Cha18].

One of their examples is on a nine-letter alphabet $\mathcal{A}_9$ is given by $\varrho_9$ defined by the rules

$$\begin{aligned}
&0 \mapsto 012, && \bar{0} \mapsto \bar{0}\bar{1}\bar{2}, && \bar{\bar{0}} \mapsto \bar{\bar{0}}\bar{\bar{1}}\bar{\bar{2}}, \\
&1 \mapsto 0\bar{1}\bar{\bar{2}}, && \bar{1} \mapsto \bar{0}\bar{\bar{1}}2, && \bar{\bar{1}} \mapsto \bar{\bar{0}}1\bar{2}, \\
&2 \mapsto 0\bar{\bar{1}}\bar{2}, && \bar{2} \mapsto \bar{0}1\bar{\bar{2}}, && \bar{\bar{2}} \mapsto \bar{\bar{0}}\bar{1}2,
\end{aligned} \tag{3.10}$$

with 3-cyclic bar symmetry that commutes with $\varrho_9$. This comes from the matrix

$$H_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi_3 & \xi_3^2 \\ 1 & \xi_3^2 & \xi_3 \end{pmatrix},$$

where $\xi_3 = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$. This matrix satisfies $H_3 H_3^{\dagger} = 3\mathbb{I}_3$. One immediately sees that the entries of $H_3$ determine the bar labelling of the images of $0, 1$, and $2$. The images of the barred letters can then be obtained as barred substituted words, with $\bar{\bar{\bar{a}}}_i = a_i$ for all $a_i \in \mathcal{A}_9$.

For the next result, let $C_3 = \langle \sigma \rangle$, with $\sigma = (123)$, and $\varPhi$ be the permutation representation. Moreover, let the matrices $Z_1, Z_2, Z_3$ be given by

$$Z_1 = \begin{pmatrix} 1 & 1 & 1 \\ z & 0 & 0 \\ z^2 & 0 & 0 \end{pmatrix}, \qquad Z_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & z^2 & 0 \end{pmatrix}, \qquad Z_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2 \end{pmatrix}.$$

where $z = \mathrm{e}^{2\pi \mathrm{i} k}$ for $k \in \mathbb{R}$.

**Proposition 3.5.3.** *Let $B(k)$ be the Fourier matrix of $\varrho_9$ given in Eq. (3.10). One has*

**(1)** $B(k) = \varPhi(e) \otimes Z_1 + \varPhi(\sigma) \otimes Z_2 + \varPhi(\sigma^2) \otimes Z_3$.

**(2)** *For all $k \in \mathbb{R}$, $B(k)$ is simultaneously block diagonalisable into*

$$B'(k) = \left(\begin{array}{c|c|c} Z_1' & \boldsymbol{O} & \boldsymbol{O} \\ \hline \boldsymbol{O} & Z_2' & \boldsymbol{O} \\ \hline \boldsymbol{O} & \boldsymbol{O} & Z_3'. \end{array}\right),$$

*where $Z_2'$ and $Z_3'$ are constant multiples of unitary matrices, with multiplier $c = \sqrt{3}$.*

**(3)** *The cocycles defined by the blocks $Z_2'$ and $Z_3'$ have degenerate Lyapunov spectrum*

$$\chi^{Z_2'} = \chi^{Z_3'} = \log(\sqrt{3}) = \log\sqrt{\lambda}.$$

□

In particular, there is a 6-dimensional subspace of $\mathbb{C}$, with $\chi = \log\sqrt{\lambda} - \chi^{B} = 0$, where the absolutely continuous component lives.

We omit the proof here and refer to Section 5.2.3 where we provide a sketch of the general proof for substitutions coming from Frank's construction, which also cover those coming from [CGS18].

### 3.5.3. Globally-mixed examples

**Example 3.5.4.** Consider the following substitutions

$$\varrho_+ : \begin{cases} a \mapsto ab, \\ b \mapsto a\bar{b}, \\ \bar{a} \mapsto \bar{a}\bar{b}, \\ \bar{b} \mapsto \bar{a}b, \end{cases} \qquad \varrho_- : \begin{cases} a \mapsto a\bar{b}, \\ b \mapsto ab, \\ \bar{a} \mapsto \bar{a}b, \\ \bar{b} \mapsto \bar{a}\bar{b}, \end{cases} \qquad \varrho_{-+} = \varrho_- \circ \varrho_+ : \begin{cases} a \mapsto a\bar{b}ab, \\ b \mapsto a\bar{b}\bar{a}\bar{b}, \\ \bar{a} \mapsto \bar{a}b\bar{a}\bar{b}, \\ \bar{b} \mapsto \bar{a}bab, \end{cases}$$

where $\varrho_-$, $\varrho_+$ are variants of the Rudin–Shapiro substitution, whose composition $\varrho_{-+}$ was shown in [CGS18] to have absolutely continuous spectrum. To these substitutions correspond the following set matrices

$$T_- = \begin{pmatrix} \{0\} & \{0\} & \varnothing & \varnothing \\ \varnothing & \{1\} & \{1\} & \varnothing \\ \varnothing & \varnothing & \{0\} & \{0\} \\ \{1\} & \varnothing & \varnothing & \{1\} \end{pmatrix}, \quad T_+ = \begin{pmatrix} \{0\} & \{0\} & \varnothing & \varnothing \\ \{1\} & \varnothing & \varnothing & \{1\} \\ \varnothing & \varnothing & \{0\} & \{0\} \\ \varnothing & \{1\} & \{1\} & \varnothing \end{pmatrix}, \quad T_{-+} = \begin{pmatrix} \{0,2\} & \{0\} & \varnothing & \{2\} \\ \{3\} & \varnothing & \{1\} & \{1,3\} \\ \varnothing & \{2\} & \{0,2\} & \{0\} \\ \{1\} & \{1,3\} & \{3\} & \varnothing \end{pmatrix},$$

whose corresponding measure-valued matrices satisfy the convolution equation

$$\underbrace{\begin{pmatrix} \delta_0 & \delta_0 & 0 & 0 \\ 0 & \delta_1 & \delta_1 & 0 \\ 0 & 0 & \delta_0 & \delta_0 \\ \delta_1 & 0 & 0 & \delta_1 \end{pmatrix}}_{\delta_{T_-}} * \underbrace{\begin{pmatrix} \delta_0 & \delta_0 & 0 & 0 \\ \delta_2 & 0 & 0 & \delta_2 \\ 0 & 0 & \delta_0 & \delta_0 \\ 0 & \delta_2 & \delta_2 & 0 \end{pmatrix}}_{f \cdot \delta_{T_+}} = \underbrace{\begin{pmatrix} \delta_0 + \delta_2 & \delta_0 & 0 & \delta_2 \\ \delta_3 & 0 & \delta_1 & \delta_1 + \delta_3 \\ 0 & \delta_2 & \delta_0 + \delta_2 & \delta_0 \\ \delta_1 & \delta_1 + \delta_3 & \delta_3 & 0 \end{pmatrix}}_{\delta_{T_{-+}}},$$

where $f$ is the dilation $f(x) = 2x$. From this, one obtains the relation between the corresponding Fourier matrices, which explicitly reads

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & u & u & 0 \\ 0 & 0 & 1 & 1 \\ u & 0 & 0 & u \end{pmatrix}}_{B(k)_-} \cdot \underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 \\ u^2 & 0 & 0 & u^2 \\ 0 & 0 & 1 & 1 \\ 0 & u^2 & u^2 & 0 \end{pmatrix}}_{B(2k)_+} = \underbrace{\begin{pmatrix} 1+u^2 & 1 & 0 & u^2 \\ u^3 & 0 & u & u+u^3 \\ 0 & u^2 & 1+u^2 & 1 \\ u & u+u^3 & u^3 & 0 \end{pmatrix}}_{B(k)_{-+}}.$$

This decomposition and the fact that the invariant sectors pair up nicely allow one to show that any finite composition of these two substitutions has at least one Lyapunov exponent $\chi = \log \sqrt{\lambda}$, which is necessary for the presence of an absolutely continuous spectrum. ◊

**Remark 3.5.5.** Note that absolutely continuous spectrum is not always compatible with global mixing. For instance, consider two versions of $\varrho_{\mathrm{RS}}$ and their composition given by

$$\varrho_1 = \begin{cases} a \mapsto ab, \\ b \mapsto ac, \\ c \mapsto db, \\ d \mapsto dc \end{cases} \qquad \varrho_2 = \begin{cases} a \mapsto ab, \\ b \mapsto ad, \\ c \mapsto cd, \\ d \mapsto cb \end{cases} \qquad \varrho_{12} := \varrho_2 \circ \varrho_1 = \begin{cases} a \mapsto abad, \\ b \mapsto abcd, \\ c \mapsto cbad, \\ d \mapsto cbcd. \end{cases} \tag{3.11}$$

Both $\varrho_1$ and $\varrho_2$ have absolutely continuous spectrum, but $\varrho_{12}$ is pure point due to Theorem 3.1.9 since it has coincidences. The eigenvalues of $M_{\varrho_{12}}$ are $\{4, 1, 0, 0\}$, which also implies singularity by Theorem 3.3.11.

Let $B_1(k)$ and $B_2(k)$ be the Fourier matrices of $\varrho_1$ and $\varrho_2$, respectively. It is important to note that the individual eigenspaces of $B_1(k)$ and $B_2(k)$ which correspond to the eigenvalue $\sqrt{2}$ are not equal. On the contrary, the matrices $B(k)_-$ and $B(k)_+$ in Example 3.5.4 share the same eigenspace for $\sqrt{2}$, which makes the existence of an exponent equal to $\log \sqrt{2}$ possible. The resulting Fourier matrix $B(k)$ for $\varrho_{12}$ is non-invertible for all $k$, and $\ker(B(k))$ is $k$-dependent as in Example 2.1.9, which hinders one in computing other exponents aside from $-\infty$ and $\log(2)$, the latter coming from a 1-dimensional invariant subspace. For these two examples with negative infinite Lyapunov exponents, explicit statements on the a.e. rank of $B^{(n)}$ and on block diagonal structures would be desirable; compare [SX17]. $\Diamond$

# 4. Non-Constant-Length Case

Outside the case where $\lambda \in \mathbb{N}$, the resulting Fourier matrix is no longer periodic, which means that one does not automatically have forward regularity. In the first section, we show that this is still true when $\varrho$ is irreducible Pisot. The next three sections are devoted to singularity results that rely on numerical estimates which are guaranteed to hold by Proposition 2.7.7. The last section deals solely with Fibonacci Fourier matrices and a stronger notion of irreducibility.

## 4.1. Existence of exponents for irreducible Pisot substitutions

Let $\varrho$ be a primitive Pisot substitution on $n_a$ letters, with $M_\varrho$ having an irreducible characteristic polynomial. From the irreducibility of $\varrho$, it follows that its inflation multiplier $\lambda$ is of algebraic degree $\deg(\lambda) = n_a$. Recall from Section 2.6 that one can consider $B^{(n)}(k)$ as a section of a 1-periodic, $n_a$-dimensional cocycle, with frequencies $(1, \alpha_1, \ldots, \alpha_{n_a-1})$ given by the (normalised) left PF eigenvector $\boldsymbol{L}$. Denote by $E^u = \mathrm{span}_{\mathbb{R}}\{\boldsymbol{L}\}$ and $E^s$ to be the complement of $E^u$ in $\mathbb{R}^{n_a}$, which is given by the $\mathbb{R}$-span of the other (distinct) eigenvectors $\{v_2, \ldots, v_{n_a}\}$ of $M_\varrho = M$. Together, these subspaces satisfy $\mathbb{R}^{n_a} = E^u \oplus E^s \simeq \mathbb{R} \times \mathbb{R}^{n_a-1}$.

The toral endomorphism $\widetilde{M} : \tilde{x} \mapsto \tilde{x}.M \mod 1$ is ergodic with respect to the Haar measure $\mu_{\mathrm{H}}$ on $\mathbb{T}^{n_a}$, see Example 1.4.4, and commutes with the quotient map $\pi : \mathbb{R}^{n_a} \to \mathbb{T}^{n_a} = \mathbb{R}^{n_a}/\mathbb{Z}^{n_a}$, i.e.,

$$\pi(x.M) = \widetilde{M}(\pi(x)) = \pi(x).M \mod 1, \tag{4.1}$$

for any vector $x \in \mathbb{R}^{n_a}$.

The usual metric $\Delta$ in $\mathbb{R}^{n_a}$ induces a metric $\widetilde{\Delta}$ in $\mathbb{T}^{n_a}$, see [AH94, Ch. 1], which is given by

$$\widetilde{\Delta}(\tilde{x}, \tilde{y}) = \inf_{a,b \in \mathbb{Z}^{n_a}} \{\Delta(\tilde{x}+a, \tilde{y}+b)\} = \inf_{a \in \mathbb{Z}^{n_a}} \{\Delta(x, y+a)\}.$$

Let $\Xi^u := \pi(E^u) \simeq \mathbb{T}$ and $\Xi^s := \pi(E^s) \simeq \mathbb{T}^{n_a-1}$. Note that both projections are invariant with respect to $\widetilde{M}$, and hence are both null sets due to the ergodicity of $\widetilde{M}$.

**Proposition 4.1.1.** *For a.e. $\tilde{x} \in \mathbb{T}^{n_a}$, the orbit $\{\widetilde{M}^n(\tilde{x})\}$ converges to a dense orbit $\{\widetilde{M}^n(\tilde{y})\}$, for some $\tilde{y} \in \Xi^u$.*

*Proof.* Pick a coset representative $x = \tilde{x} + a = (x_1, \ldots, x_{n_a})$ for $\tilde{x} \in \mathbb{T}^{n_a}$, with $a \in \mathbb{Z}^{n_a}$. This vector could be written as $x = c_1 \boldsymbol{L} + x_s$, where $c_1 \in \mathbb{R}$ and $x_s \in E^s$. For $x' = c_1 \boldsymbol{L}$, one has $\tilde{x}' = \pi(x') \in \Xi^u$. Moreover, the distance between the respective iterates of $x$ and $x'$ satisfies

$$\Delta\big(x.M^n, x'.M^n\big) = \|x_s.M^n\| \xrightarrow{n\to\infty} 0.$$

This follows from the Pisot property that all other eigenvalues of $M$ have modulus less than 1, which makes $M$ a contracting map on $E^s$. Note that $c_1 = 0$ means $x + a \in E^s$, implying that $\tilde{x} \in \Xi^s$ and $\widetilde{M}^n(\tilde{x}) \to 0$ as $n \to \infty$. However, one need not worry as $\mu_{\mathrm{H}}(\Xi^s) = 0$.

Since $a$ is chosen arbitrarily, this implies

$$\lim_{n\to\infty} \widetilde{\Delta}(\widetilde{M}(\tilde{x}), \widetilde{M}(\tilde{x}')) = \lim_{n\to\infty} \inf_{a,b\in\mathbb{Z}^{n_a}} \left\{\Delta\big((x+a).M^n, (x'+b).M^n\big)\right\} = 0$$

for a.e. $\tilde{x} \in \mathbb{T}^{n_a}$. □

To continue, define the *induced Lebesgue measure* $\mu_{\mathrm{L}}^{\mathrm{ind}}$ of a subset $V \subset E^u$ via

$$\mu_{\mathrm{L}}^{(u)}(V) := \mu_{\mathrm{L}}(K)$$

whenever $V = \{k.\boldsymbol{L} \mid k \in K \subset \mathbb{R}\}$. Here, $\mu_{\mathrm{L}}$ denotes the standard Lebesgue measure on $\mathbb{R}$. Analogously, for the contracting subspace $E^s$, and for $V = \mathrm{span}_K\{v_2, \dots, v_d\} \subset E^s$, one has $\mu_{\mathrm{L}}^{(s)}(V) := \mu_{\mathrm{L}}(K)$. We endow the subtori $\Xi^s \simeq \mathbb{T}^{n_a-1}$ and $\Xi^u \simeq \mathbb{T}$ with the usual Haar measures $\mu_{\mathrm{H}}^{(s)}$ and $\mu_{\mathrm{H}}^{(u)}$, respectively. The next result is immediate.

**Lemma 4.1.2.** *Let $\pi(V) \subset \Xi^u$, where $V = \mathrm{span}_K\{\boldsymbol{L}\}$. Then, $\mu_{\mathrm{H}}^{(u)}(\pi(V)) > 0$ if and only if $\mu_{\mathrm{L}}^{(u)}(K) > 0$. A similar equivalence holds for $W = \mathrm{span}_K\{v_2, \dots, v_d\} \subset E^s$.* □

Recall from Proposition 2.6.4 that the exponent $\chi^{\widetilde{B}}(\tilde{x})$ for the cocycle defined by $\widetilde{B}$ with base dynamics given by $M$ exists for a.e. $\tilde{x} \in \mathbb{T}^{n_a}$ and is equal to a constant $\chi^{\widetilde{B}}$. Our next goal is to use this existence result and the convergence result in Proposition 4.1.1 to prove almost sure existence of $\chi^B(k)$ for irreducible Pisot substitutions.

**Lemma 4.1.3.** *For all $y = k.\boldsymbol{L} \in E^s$, one has $\chi^B(k) = \chi^{\widetilde{B}}(\tilde{y})$, where $\pi(y) = \tilde{y} \in \mathbb{T}^{n_a}$.*

*Proof.* It follows from the definition of the cocycle $\widetilde{B}$ from Lemma 2.6.2 and 1-periodicity of $\widetilde{B}$ that

$$B^{(n)}(k) = \widetilde{B}^{(n)}(y) = \widetilde{B}^{(n)}(\tilde{y}).$$

The equality of the Lyapunov exponents then follows by considering $\frac{1}{n}\log\|B^{(n)}(k)\|$. □

Next, denote the set of pathological points of $\chi^B(k)$ by $\mathcal{X}$, i.e.,

$$\mathcal{X} := \left\{k \in \mathbb{R} \mid \chi^B(k) \text{ does not exist as a limit or } \chi^B(k) \neq \chi^{\widetilde{B}}\right\}.$$

We show that $\mu_{\mathrm{L}}(\mathcal{X}) = 0$. To this end, let $\mathcal{Y}$ be given as

$$\mathcal{Y} := \{y \in \mathbb{R}^{n_a} \mid y = k.\boldsymbol{L}, k \in \mathcal{X}\} \subset E^u.$$

Let $\widetilde{\mathcal{Y}} := \pi(\mathcal{Y})$ and consider the set $\widetilde{\mathcal{Y}} + \Xi^s$. The next lemma states that the Lyapunov exponent on points in $\widetilde{\mathcal{Y}} + \Xi^s$ are completely determined by the exponent of the component in $\widetilde{\mathcal{Y}}$.

**Lemma 4.1.4.** *For $\mu_{\mathrm{H}}$-a.e. $\tilde{x} \in \widetilde{\mathcal{Y}} + \Xi^s$, one has $\chi^{\widetilde{B}}(\tilde{x}) = \chi^{\widetilde{B}}(\tilde{y})$, where $\tilde{x} = \tilde{y} + \tilde{x}_s$, for some $\tilde{y} \in \widetilde{\mathcal{Y}}$ and $\tilde{x}_s \in \Xi^s$.*

*Proof.* By Proposition 4.1.1, for $\mu_{\mathrm{H}}$-a.e. $\tilde{x} \in \widetilde{\mathcal{Y}}$,

$$\big(\widetilde{M}^n(\tilde{x})\big) \xrightarrow{n\to\infty} \big(\widetilde{M}^n(\tilde{y})\big) = \pi(\lambda^n.y),$$

with $y \in \mathcal{Y}$, where the equality follows from Eq. (4.1). Since $\|\widetilde{B}^{(n)}(.)\|$ displays the same growth asymptotics when it is sampled on converging orbits, the exponential growth behaviour of $\|\widetilde{B}^{(n)}(\tilde{y})\|$ is completely dictated by that of $\|\widetilde{B}^{(n)}(\tilde{x})\|$, from which the equality of the exponents follow. □

We are now ready to prove our main result in this section; compare Lemma 3.1.4.

**Theorem 4.1.5.** *Let $\varrho$ be a primitive one-dimensional irreducible Pisot substitution. Then, for a.e. $k \in \mathbb{R}$, $\chi^B(k)$ exists as a limit and is equal to the almost everywhere exponent $\chi^{\widetilde{B}}$ of $\widetilde{B}$, i.e.,*

$$\chi^B(k) = \lim_{n\to\infty} \frac{1}{n} \log \|B^{(n)}(k)\| = \inf_N \frac{1}{N} \mathbb{M}(\log \|\widetilde{B}^{(N)}(\cdot)\|) = \chi^{\widetilde{B}}.$$

*Proof.* Fix $0 < \delta < 1$ and $k_0 \in \mathbb{R}$, and consider the set $\widetilde{\mathcal{Y}}' = \pi(\mathcal{X} \cap \mathfrak{B}_\delta(k_0)) \subset \widetilde{\mathcal{Y}}$. Here, $\mathfrak{B}_\delta(k_0)$ is the open ball of radius $\delta$ around $k_0 \in \mathbb{R}$. Now choose an $0 < \varepsilon < 1$ and consider the $\varepsilon$-neighbourhood $U_\varepsilon(0)$ of 0 in $\Xi^s$, i.e.,

$$U_\varepsilon(0) := \pi(\mathrm{span}_{\mathfrak{B}_\varepsilon(0)} \{v_2, \ldots, v_d\}).$$

From this, construct the $\varepsilon$-thickening of $\widetilde{\mathcal{Y}}'$ along $\Xi^s$ given by $\widetilde{\mathcal{Y}}' + U_\varepsilon(0)$; see Figure 4.1. Note that, in general, $E^u$ need not be orthogonal to $E^s$, though it is always transversal.

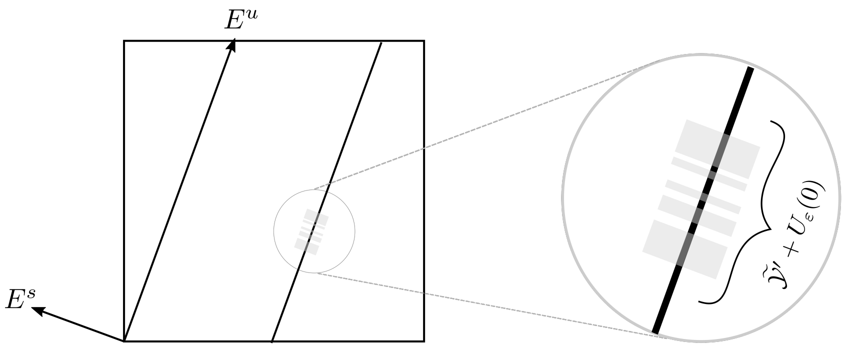



Figure 4.1.: Illustration in $\mathbb{T}^2$ of the $\varepsilon$-thickening of an exceptional subset $\widetilde{\mathcal{Y}}' \subset \Xi^u$.

Let $\mathcal{Z}$ be the set of exceptional points for $\chi^{\widetilde{B}}(\tilde{x})$, i.e.,

$$\mathcal{Z} := \left\{ \tilde{x} \in \mathbb{T}^{n_a} \mid \chi^{\widetilde{B}}(\tilde{x}) \text{ does not exist as a limit or } \chi^{\widetilde{B}}(\tilde{x}) \neq \chi^{\widetilde{B}} \right\}.$$

From Proposition 2.6.4, one has $\mu_{\mathrm{H}}(\mathcal{Z}) = 0$.

From Lemma 4.1.3 and Lemma 4.1.4, for $\mu_{\mathrm{H}}$-a.e. $\tilde{x} \in \widetilde{\mathcal{Y}}' + U_\varepsilon(0)$, one has $\chi^{\widetilde{B}}(\tilde{x}) = \chi^B(k)$, with $k \in \mathcal{X}$. This means that there exists a full-measure subset $\widetilde{\mathcal{Y}}'' \subset \widetilde{\mathcal{Y}}' + U_\varepsilon(0)$, i.e.,

$$\mu_{\mathrm{H}}(\widetilde{\mathcal{Y}}'') = \mu_{\mathrm{H}}(\widetilde{\mathcal{Y}}' + U_\varepsilon(0)),$$

which satisfies $\widetilde{\mathcal{Y}}'' \subset \mathcal{Z}$. Since $\widetilde{\mathcal{Y}}''$ is a subset of a null set in $\mathbb{T}^{n_a}$, one automatically gets $\mu_{\mathrm{H}}(\widetilde{\mathcal{Y}}'') = \mu_{\mathrm{H}}(\widetilde{\mathcal{Y}}' + U_\varepsilon(0)) = 0$.

Alternatively, one can write $\mu_{\mathrm{H}}$ in $\mathbb{T}^{n_a}$ as a product measure, i.e.,

$$\mu_{\mathrm{H}}(X \times Y) = \alpha \cdot \mu_{\mathrm{H}}^{(u)}(X) \cdot \mu_{\mathrm{H}}^{(s)}(Y),$$

for $X \subset \Xi^u$ and $Y \subset \Xi^s$, and for some $\alpha > 0$; compare [Hal74, Sec. 35, Thm. B].

With this, one gets

$$0 = \mu_{\mathrm{H}}(\widetilde{\mathcal{Y}}' + U_\varepsilon(0)) = \alpha \cdot \mu_{\mathrm{H}}^{(u)}(\widetilde{\mathcal{Y}}') \cdot \mu_{\mathrm{H}}^{(s)}(U_\varepsilon(0)).$$

Since $\mu_{\mathrm{H}}^{(s)}(U_\varepsilon(0)) > 0$, this implies $\mu_{\mathrm{H}}^{(u)}(\widetilde{\mathcal{Y}}') = 0$ and thus $\mu_{\mathrm{L}}(\mathcal{X} \cap \mathfrak{B}_\delta(k_0)) = 0$ by Lemma 4.1.2.

This holds for any arbitrary open ball $\mathfrak{B}_\delta(k_0)$ with $0 < \delta < 1$, and since $\mathbb{R}$ is locally compact, one gets $\mu_{\mathrm{L}}(\mathcal{X}) = 0$. □

**Remark 4.1.6.** One can also prove the upper bound estimate in Proposition 2.7.7 for irreducible Pisot substitutions without invoking results in discrepancy analysis using the same arguments in Theorem 4.1.5 by showing that the Birkhoff averages of $\log|\det B(k)|$ can only diverge on a set of measure zero via an almost-everywhere result for $\log|\det \widetilde{B}(x)|$. ♢

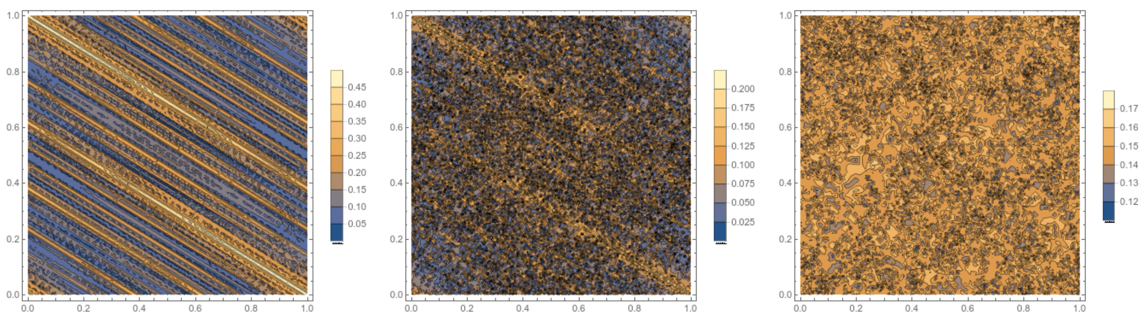
The following result is due to Fan, Saussol and Schmeling.

**Theorem 4.1.7** ( [FSS04, Thm. 2.5]). *Let $\{u_n\} = \{\alpha^n\}_{n\geqslant 0}$, for some Pisot $\alpha > 1$, and $\{f_n\}_{n\geqslant 0}$ be a sequence of subadditive Bohr almost periodic functions such that, for any $n \geqslant 1$, one has $\sup_m \big(f_m(u_n k) - f_m(k)\big) < \infty$, for a.e. $k \in \mathbb{R}$. Then,*

$$\lim_{n\to\infty} \frac{1}{n} f_n(k) = \inf_n \frac{1}{n} \mathbb{M}(f_n)$$

*for a.e. $k \in \mathbb{R}$.* □

**Remark 4.1.8.** For cocycles which do not have singularities, one can use Theorem 4.1.7 to prove almost sure existence. Theorem 4.1.5 is an existence result for a specific class of cocycles which are allowed to have (local) singularities. A numerical illustration of this convergence for $\widetilde{B}$ is given in Figure 4.2. ♢



(a) $N = 10$ (b) $N = 30$ (c) $N = 1000$

Figure 4.2.: Contour plot of $\frac{1}{N}\log\|\widetilde{B}^{(N)}(x,y)\|$ over $[0,1)^2$ for the Fibonacci substitution.

## 4.2. Non-Pisot examples

### 4.2.1. A family of non-Pisot substitutions

In this section, we consider the family of substitutions on $\mathcal{A}_2 = \{0,1\}$ defined by

$$\varrho_m : \quad 0 \mapsto 01^m, 1 \mapsto 0, \quad \text{with } m \in \mathbb{N}. \tag{4.2}$$

Its substitution matrix is $M_m = \left(\begin{smallmatrix} 1 & 1 \\ m & 0 \end{smallmatrix}\right)$, whose eigenvalues are $\lambda_m^{\pm} = \frac{1}{2}(1 \pm \sqrt{4m+1})$, with PF eigenvalue $\lambda_m = \lambda_m^{+}$. The left PF eigenvector $\boldsymbol{L}$ is given by $(\lambda_m, 1)$, which means the associated tiles have lengths $|\mathfrak{t}_0| = \lambda_m$ and $|\mathfrak{t}_1| = 1$. One can easily check that a bi-infinite fixed point $w$ of $\varrho_m^2$ can be obtained from the legal seed $0|0$. This fixed point gives rise to the geometric hull $\mathbb{Y}_m$.

The first member $\varrho_1$ of this family is the classic Fibonacci substitution $\varrho_{\mathrm{F}}$.

**Fact 4.2.1.** *The* PF *eigenvalue satisfies* $\lambda_m \in \mathbb{N}$ *if and only if* $m = \ell(\ell+1)$, *for some* $\ell \in \mathbb{N}$. *In these cases,* $\lambda_m = \ell + 1$. *The resulting geometric hull* $\mathbb{Y}_m$ *is* MLD *to a hull* $\mathbb{Y}'_m$ *of a constant-length substitution given by*

$$\widetilde{\varrho}_m : a \mapsto ab^{\ell}, b \mapsto a^{\ell+1}.$$

*Sketch of proof.* The first statement follows from direct computation. For the second claim, we assume $|\mathsf{t}_a| = |\mathsf{t}_b| = \lambda_m$. The proof relies on elements in $\mathbb{Y}'_m$ being locally recognisable; compare with [BGrM18, Prop. 2.3]. In particular, the symbol 1 appears in blocks of length $\ell(\ell+1)$ in $w_m$, which allows one to define local maps from $w_m = \varrho_m^{\infty}(0|0)$ to $u_m = (\widetilde{\varrho}_m)^{\infty}(a|a)$, and vice versa. These maps extend to any element of the hull, which then implies the claim. □

**Proposition 4.2.2.** *For* $m = 1$ *and all* $m = \ell(\ell+1)$, *where* $\ell \in \mathbb{N}$, *the dynamical system* $(\mathbb{Y}_m, \mathbb{R})$ *has pure point spectrum, both in the diffraction and in the dynamical senses.*

*Proof.* Since each $\widetilde{\varrho}_m$ is constant-length and has a coincidence, Theorem 3.1.9 implies that the symbolic dynamical systems $(\mathbb{X}'_m, \mathbb{Z})$ all have pure point dynamical spectra . One can then view $\mathbb{Y}'_m$ as a suspension of $\mathbb{X}'_m$ with a constant roof function. Since the spectral type is preserved under MLD-equivalence, we also get that $(\mathbb{Y}_m, \mathbb{R})$ has pure point dynamical spectrum. From this, the pure point nature of the diffraction measure $\widehat{\gamma}_m$ is guaranteed by [LMS02, Thm. 3.2]. □

**Remark 4.2.3.** An independent, although weaker result regarding the diffraction spectra of the family of substitutions $\widetilde{\varrho}_m$ can be obtained by invoking Corollary 3.2.4, i.e., $\chi_{\min} > 0$, for each $\varrho_m$, which implies the singularity of $\widehat{\gamma}_m$, see [BGrM18, Sec. 5.3] for details. ◊

When $m \neq 1$ and $m \neq \ell(\ell+1), \ell \in \mathbb{N}$, $\lambda_m$ is a non-Pisot number (and not a unit), which makes $\varrho_m$ a non-Pisot substitution. This has immediate implications on $\mathbb{Y}_m$ due to Corollary 1.2.12.

**Fact 4.2.4.** *Let* $\varrho_m$ *be given as in Eq.* (4.2), *with* $m \neq 1$ *and* $m \neq \ell(\ell+1)$. *Then, no point set* $\varLambda \in \mathbb{Y}_m$ *is a Meyer set. Furthermore, the diffraction* $\widehat{\gamma}_m$ *has no non-trivial pure point component, i.e.,* $\widehat{\gamma}_m = I_0\delta_0 + (\widehat{\gamma}_m)_{\mathsf{cont}}$. □

The goal is then to prove that $(\widehat{\gamma}_m)_{\mathsf{cont}}$ is singular continuous, i.e., $(\widehat{\gamma}_m)_{\mathsf{ac}} = 0$ by showing $\chi^B(k) < \log \sqrt{\lambda_m}$, for a.e. $k$. For readability, we fix $m$ and refer to the diffraction as $\widehat{\gamma}$, and to the PF eigenvalue as $\lambda$.

For a chosen $m \in \mathbb{N}$, the displacement matrix reads

$$T = \begin{pmatrix} \{0\} & \{0\} \\ \varGamma & \varnothing \end{pmatrix}, \quad \text{with } \varGamma := \{\lambda, \lambda+1, \ldots, \lambda+m-1\}.$$

From this, we get the Fourier matrix $B(k)$ as

$$B(k) = D_0 + p(k) D_\lambda, \quad \text{with} \quad p(k) = z^{\lambda}(1 + z + \ldots + z^{m-1})\big|_{z=\mathrm{e}^{2\pi \mathrm{i} k}}$$

and digit matrices $D_0 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $D_\lambda = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, which are independent of $m$. From Proposition 2.1.6, the IDA $\mathcal{B}_m$ is irreducible.

For $m > 1$, the cocycle $B^{(n)}(k)$ is invertible for $k \notin \bigcup_{\ell=0}^{n-1} \lambda^{-\ell} \mathcal{Z}_m$, where $\mathcal{Z}_m = \mathbb{Z}/m$. For $m = 1$, $|\det B(k)| \equiv 1$, for all $k \in \mathbb{R}$, which makes it everywhere invertible. Analogous to forward regular cocycles, Proposition 2.7.6 yields the following result.

**Proposition 4.2.5.** *For a.e. $k \in \mathbb{R}$, one has* $\lim_{n\to\infty} \frac{1}{n} \log \bigl|\det B^{(n)}(k)\bigr| = 0$.

*Proof.* The claim is trivial for $m = 1$. For $m \geqslant 2$, Proposition 2.7.6 implies

$$\frac{1}{n} \log \bigl|\det B^{(n)}(k)\bigr| \;=\; \frac{1}{n} \sum_{\ell=0}^{n-1} \log \bigl|\det B(\lambda^{\ell} k)\bigr| \;\xrightarrow{n\to\infty}\; \mathbb{M}(\log|\det B(\cdot)|)$$

for a.e. $k \in \mathbb{R}$. The mean $\mathbb{M}(\log|\det B(\cdot)|)$ can easily be computed since $\log|\det B(k)|$ is periodic, and is given by

$$\int_0^1 \log|\det B(t)| \mathrm{d}t \;=\; \int_0^1 \log\bigl|1 + z + \ldots + z^{m-1}\bigr|_{z=\mathrm{e}^{2\pi \mathrm{i} t}} \mathrm{d}t \;=\; \mathfrak{m}(1 + z + \ldots + z^{m-1}) = 0,$$

where the last equality follows from Kronecker's lemma. □

**Lemma 4.2.6.** *For a.e. $k \in \mathbb{R}$, one has* $\chi_{\max}(k) + \chi_{\min}(k) = \log(\lambda)$.

*Proof.* For any invertible $B$, one has $B^{-1} = \frac{1}{\det B} B^{\mathrm{ad}}$. The adjoint satisfies $(AB)^{\mathrm{ad}} = B^{\mathrm{ad}} A^{\mathrm{ad}}$. Since we are dealing with $2\times 2$-matrices, $\|B^{\mathrm{ad}}\|_{\mathrm{F}} = \|B\|_{\mathrm{F}}$. The claim then follows from a simple computation involving Eqs. (2.21) and (2.22), and Proposition 4.2.5. □

In view of Section 2.6, for $\lambda \notin \mathbb{N}$, $B(k)$ can be written as

$$B(k) = \widetilde{B}(x,y)\big|_{x=\lambda k,\; y=k}$$

with $\widetilde{B}(x,y) = \begin{pmatrix} 1 & 1 \\ \tilde{p}(x,y) & 0 \end{pmatrix}$ and $\tilde{p}(x,y) = \mathrm{e}^{2\pi \mathrm{i} x}(1 + z + \ldots + z^{m-1})\big|_{z=\mathrm{e}^{2\pi \mathrm{i} y}}$.

This generates the cocycle $\widetilde{B}^{(n)}(x,y)$, whose dynamics is given by $(x,y) \mapsto (x,y)M_m$ on $\mathbb{T}^2$. When $\lambda \in \mathbb{N}$, $B(k)$ is periodic and $B^{(n)}(k)$ is an ergodic cocycle over the map $k \mapsto \lambda k$ in $[0,1)$; see Section 3.1.

Lemma 2.7.1 holds for this family of substitutions since no Fourier matrix $B(k)$ is identically singular. The upper bound

$$2\chi^B(k) \leqslant \frac{1}{N} \mathbb{M}(\log \|\widetilde{B}^{(N)}(\cdot)\|_{\mathrm{F}}^2) \tag{4.3}$$

holds for a.e. $k \in \mathbb{R}$ and for all $N \in \mathbb{N}$. This brings us to a sufficient criterion for positivity of $\chi_{\min}$ for $\varrho_m$. For the following result, let

$$q(z) = 2z^{m-1} + (1 + z + \ldots + z^{m-1})^2.$$

**Fact 4.2.7.** *If $\mathfrak{m}(q) < \log\lambda$, then $\chi^B(k) < \log\sqrt{\lambda}$, for a.e. $k \in \mathbb{R}$.*

*Proof.* Substituting $N = 1$ to the upper bound from Eq. (4.3), we obtain

$$2\chi^B(k) \leqslant \mathbb{M}\bigl(\log\|\widetilde{B}(\cdot)\|_{\mathrm{F}}^2\bigr) = \int_0^1 \log(2 + |p(t)|^2)\mathrm{d}t = \int_0^1 \log|q(z)|_{z=\mathrm{e}^{2\pi i t}} \mathrm{d}t = \mathfrak{m}(q) \tag{4.4}$$

for a.e. $k \in \mathbb{R}$. Here, the validity of $\bar{z} = z^{-1}$ on the unit circle was used to get the second equality. □

**Lemma 4.2.8.** *For any $m \in \mathbb{N}$, the logarithmic Mahler measure of the polynomial $q$ satisfies the inequality* $\mathfrak{m}(q) < \log\sqrt{46} \approx 1.914321$.

*Proof.* Here, we employ an argument from [Clu59, BCJ13] that was also used, in a similar context, in Section 3.2.1. By a simple geometric series calculation, one finds that $q(z) = \frac{r(z)}{(z-1)^2}$ with

$$r(z) \;=\; z^{2m} + 2z^{m+1} - 6z^m + 2z^{m-1} + 1 \;=\; \sum_{\ell=0}^{2m} c_\ell \, z^\ell. \tag{4.5}$$

Consequently, we have $\mathfrak{m}(q) = \mathfrak{m}(r) - \mathfrak{m}\big((z-1)^2\big) = \mathfrak{m}(r)$.

Assume that $m \geqslant 2$, so that the exponents of $r(z)$ in Eq. (4.5) are distinct. Consequently,

$$\|r\|_2^2 \;=\; \sum_{\ell=0}^{2m} |c_\ell|^2 \;=\; 46,$$

so that $\mathfrak{M}(r) < \sqrt{46}$, independently of $m$ by Lemma 1.5.2 . This inequality also holds trivially for $m = 1$, and we get $\mathfrak{m}(q) = \mathfrak{m}(r) < \log \sqrt{46}$ for all $m \in \mathbb{N}$ as claimed. □

With this bound, one has $\mathfrak{m}(q) < \log \lambda_m$ for all $m \geqslant 40$.

**Remark 4.2.9.** A better bound for $\mathfrak{m}(q)$ can be obtained from Eq. (4.4) by noting that one has $p(t) = \frac{1-\mathrm{e}^{2\pi \mathrm{i} m t}}{1-\mathrm{e}^{2\pi \mathrm{i} t}}$, which transforms the first integral to be

$$\mathfrak{m}(q) = \int_0^1 \log\left(2 + \left(\frac{\sin(m\pi t)}{\sin(\pi t)}\right)^2\right) \mathrm{d}t.$$

Here, one observes that $\sin(m\pi t)^2$ oscillates between 0 and 1 for small perturbations of $t$, while $\sin(\pi t)$ remains almost constant. With this, under the integral, one can replace the second term by $\sin(\pi t)^2 = \frac{1}{2}(1 - \cos(2\pi t))$ to get the upper bound

$$\mathfrak{m}(q) \leqslant \int_0^1 \log \frac{3 - 2\cos(2\pi t)}{1 - \cos(2\pi t)} \mathrm{d}t = \mathfrak{m}(z^2 - 3z + 1) + \log(2) = \log(3 + \sqrt{5}) \approx 1.655571.$$

This universal bound is less that $\log(\lambda_m)$ for all $m \geqslant 23$. ◊

The following result is due to Boyd [Boy81] for polynomials in $\ell = 2$ variables, and has been generalised by Lawton [Law83, Thm. 2] to any $\ell \geqslant 2$.

**Theorem 4.2.10.** *Suppose $r(z) \in \mathbb{Z}[z]$ can be written as $r(z) = \tilde{r}(z, z^m)$, with $\tilde{r} \in \mathbb{Z}[z_1, z_2]$. Then,*

$$\lim_{m \to \infty} \mathfrak{m}(\tilde{r}(z, z^m)) = \mathfrak{m}(\tilde{r}(z_1, z_2)),$$

*where $\mathfrak{m}(\tilde{r}(z_1, z_2))$ is a two-dimensional logarithmic Mahler measure.* □

Note that the polynomial $r(z)$ from Eq. (4.5) is of the form required in Theorem 4.2.10, with $\tilde{r}(z_1, z_2) = -z_2(6 - 2(z_1 + z_1^{-1}) - (z_2 + z_2^{-1}))$, and hence has the limit

$$\begin{aligned} \mathfrak{m}(\tilde{r}(z, w)) &= \int_{\mathbb{T}^2} \log(6 - 2\cos(2\pi t_1) - 2\cos(2\pi t_2)) \mathrm{d}t_1 \mathrm{d}t_2 \\ &= 2 \int_0^1 \operatorname{arsinh}(\sqrt{2} \sin(\pi t_2)) \mathrm{d}t_2 \;\approx\; 1.550675. \end{aligned}$$

If one knows that $\mathfrak{m}(q)$ is increasing in $m$, this result automatically implies $\mathfrak{m}(q) < \log(\lambda)$ for all $m \geqslant 18$. Unfortunately, this is yet to be shown.

Table 4.1.: Some relevant values for the quantities in the inequality of Eq. (4.6) for the family $\varrho_m$. The numerical error is less than $10^{-3}$ in all cases listed.

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\log(\lambda)$ | 0.481 | 0.693 | 0.834 | 0.941 | 1.027 | 1.099 | 1.161 | 1.216 | 1.265 | 1.309 |
| $N = N(m)$ | 6 | 4 | 4 | 3 | 3 | 3 | 2 | 2 | 2 | 2 |
| $\frac{1}{N}\mathbb{M}\big(\log \lVert B^{(N)}(.)\rVert_{\mathrm{F}}^2\big)$ | 0.439 | 0.677 | 0.770 | 0.924 | 0.949 | 0.964 | 1.144 | 1.152 | 1.157 | 1.161 |

| $m$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\log(\lambda)$ | 1.349 | 1.386 | 1.421 | 1.453 | 1.483 | 1.511 | 1.538 | 1.563 | 1.587 | 1.609 |
| $N = N(m)$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 |
| $\frac{1}{N}\mathbb{M}\big(\log \lVert B^{(N)}(.)\rVert_{\mathrm{F}}^2\big)$ | 1.164 | 1.166 | 1.168 | 1.169 | 1.170 | 1.171 | 1.172 | 1.546 | 1.547 | 1.547 |

**Lemma 4.2.11.** *For any $m \geqslant 18$ and a.e. $k$, the $\chi_{\min}$ is strictly positive.*

*Proof.* Lemma 4.2.8 and Remark 4.2.9 confirm that Fact 4.2.7 holds for $m \geqslant 23$, which directly implies the claim. For the remaining $m$, one can use Jensen's formula to compute $\mathfrak{m}(q)$ numerically, which can be done up to some reasonable error bound, to check that, indeed, $\mathfrak{m}(q) < \log(\lambda)$ for these cases; see Table 4.1 for some of the values. ☐

In order to establish positivity for $m < 18$, it suffices to find the smallest $N = N(m)$ for which the right hand-side in Eq. (4.3) satisfies

$$\frac{1}{N(m)}\mathbb{M}(\log \lVert \widetilde{B}^{(N(m))}(\cdot)\rVert_{\mathrm{F}}^2) = \frac{1}{N(m)} \int_{[0,1]^2} \log \lVert \widetilde{B}^{(N(m))}(x,y)\rVert_{\mathrm{F}}^2 \,\mathrm{d}x\,\mathrm{d}y \;<\; \log(\lambda). \tag{4.6}$$

To this end, one can calculate the integral in Eq. (4.6) numerically up to a reasonable level of precision, and without ambiguity. The minimal values of $N(m)$ are given in Table 4.1.

Together with Lemma 4.2.11, one has the following result.

**Proposition 4.2.12.** *For any $m \in \mathbb{N}$, and for a.e. $k \in \mathbb{R}$, $\chi_{\min}$ is strictly positive.* ☐

We summarise our results from Proposition 4.2.2, Fact 4.2.4, and Proposition 4.2.12, in conjunction with Theorem 2.5.3 as follows.

**Theorem 4.2.13.** *Consider the inflation tiling, with prototiles of natural length, defined by $\varrho_m$. For $m = 1$ and $m = \ell(\ell+1)$ with $\ell \in \mathbb{N}$, the tiling has pure point diffraction, which can be calculated with the projection method. The corresponding tiling dynamical system $(\mathbb{Y}_m, \mathbb{R})$ has pure point dynamical spectrum.*

*For all remaining cases, the pure point part of the diffraction consists of the trivial Bragg peak at 0, while the remainder of the diffraction is of singular continuous type.* ☐

**Remark 4.2.14.** For the cases where $\lambda$ is non-Pisot, the resulting tiling dynamical system $(\mathbb{Y}, \mathbb{R})$ is weakly mixing due to [Sol97, Thm. 5.1], i.e., there are no non-trivial eigenfunctions. ◊

### 4.2.2. Example with a Salem multiplier

Note that Lemma 2.7.1 also holds for Salem substitutions. Consider the substitution $\varrho_{\mathrm{S}}$ with substitution matrix given by

$$\varrho_{\mathrm{S}} : \begin{cases} a \mapsto b, & c \mapsto cb, \\ b \mapsto d, & d \mapsto acd. \end{cases} \qquad M_\varrho = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Here, $M_\varrho$ is primitive, with characteristic polynomial $p(z) = z^4 - 2z^3 + z^2 - 2z + 1$, and whose two real eigenvalues are $\lambda = \lambda_{\mathrm{PF}} \approx 1.8832$ and $\lambda_2 = \frac{1}{\lambda}$, while the other two satisfy $|\lambda_3| = |\lambda_4| = 1$. The PF eigenvalue $\lambda$ is a Salem number of minimal degree.

Denote by $\ell_{a_i}$ the length associated to $a_i$ and let $\boldsymbol{L}$ be the (normalised) left PF-eigenvector of $M_\varrho$ with $\ell_a = 1$. From the resulting inflation rule, one gets the Fourier matrix $B(k)$ as

$$B(k) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & \mathrm{e}^{2\pi \mathrm{i} \ell_c k} & 0 \\ 0 & 0 & 1 & \mathrm{e}^{2\pi \mathrm{i} k} \\ 0 & 1 & 0 & \mathrm{e}^{2\pi \mathrm{i}(1+\ell_c)k} \end{pmatrix}, \qquad \widetilde{B}(x,y,z,w) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & \mathrm{e}^{2\pi \mathrm{i} z} & 0 \\ 0 & 0 & 1 & \mathrm{e}^{2\pi \mathrm{i} x} \\ 0 & 1 & 0 & \mathrm{e}^{2\pi \mathrm{i}(x+z)} \end{pmatrix},$$

where $B(k) = \widetilde{B}(x,y,z,w)|_{x=k,\ y=\ell_b k,\ z=\ell_c k,\ w=\ell_d k}$. As in Section 4.2.1, the cocycle $\widetilde{B}^{(n)}(x,y,z,w)$ is generated by the base dynamics $(x,y,z,w) \mapsto (x,y,z,w).M_\varrho$ on $\mathbb{T}^4$.

From Lemma 2.7.1, it suffices to find an $N$ for which

$$\frac{1}{2N}\mathbb{M}(\log \|\widetilde{B}^{(N)}(x,y,z,w)\|_{\mathrm{F}}^2) = \frac{1}{2N}\int_{\mathbb{T}^4} \log \|\widetilde{B}^{(N)}(x,y,z,w)\|_{\mathrm{F}}^2 < \log\sqrt{\lambda} \approx 0.316487$$

to be able to confirm the absence of absolutely continuous diffraction, which we carry out numerically as in the previous section. The estimates are given in Table 4.2.

| $N$ | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|
| $\frac{1}{N}\mathbb{M}\big(\log \|B^{(N)}(.)\|_{\mathrm{F}}\big)$ | 0.321 | 0.316 | 0.312 | 0.308 | 0.305 |

Table 4.2.: Numerical upper bounds for $\chi^B$ for the Salem substitution $\varrho_{\mathrm{S}}$. The numerical error is less than $10^{-3}$ in all cases listed.

**Proposition 4.2.15.** *The diffraction measure $\widehat{\gamma}$ of the hull associated to $\varrho_{\mathrm{S}}$ is essentially singular continuous, i.e.,*

$$\widehat{\gamma} = I_0 \delta_0 + \widehat{\gamma}_{\mathsf{sc}}.$$

□

## 4.3. Noble means family

Consider the family of substitutions given by

$$\varrho_{m,j} : \begin{cases} a \mapsto a^j b a^{m-j}, \\ b \mapsto a, \end{cases}$$

where $0 \leqslant j \leqslant m$, which are called *noble means* substitutions. For general notions, we refer to [Mol13, Ch. 2]. For a given $m$, this substitution family has the substitution matrix $\left(\begin{smallmatrix} m & 1 \\ 1 & 0 \end{smallmatrix}\right)$, with corresponding Perron–Frobenius eigenvalue $\lambda_m = \frac{m+\sqrt{m^2+4}}{2}$, both independent of $j$. Moreover, this family falls under the *unimodular Pisot* class. Due to this special structure, the tiles $a$ and $b$ have associated natural lengths $\lambda_m$ and 1 when one considers its geometric realisation as an inflation rule.

For a generic member $\varrho_{m,j}$, the displacement and Fourier matrices are given by

$$T = \begin{pmatrix} \varGamma_m & \{0\} \\ \{j\lambda_m\} & \emptyset \end{pmatrix} \quad \text{and} \quad B(k) = \begin{pmatrix} p(z) + wz^j q(z) & 1 \\ z^j & 0 \end{pmatrix},$$

where $w = \mathrm{e}^{2\pi \mathrm{i} k}$ and $z = w^{\lambda_m}$, and the set $\varGamma_m$ is given by

$$\varGamma_m = \{0, \lambda_m, 2\lambda_m, \ldots, (j-1)\lambda_m, j\lambda_m + 1, (j+1)\lambda_m + 1, \ldots, (m-1)\lambda_m + 1\}.$$

From here, one essentially recovers the polynomials $p$ and $q$ which read

$$p(z) = 1 + z + \cdots + z^{j-1} \quad \text{and}$$
$$q(z) = 1 + \cdots + z^{m-j-1}.$$

**Proposition 4.3.1.** [BG13, Prop. 4.6 and Rem. 4.7] *For a fixed $m$, the symbolic hull $\mathbb{X}_{m,j}$ generated by $\varrho_{m,j}$ is the same for all $0 \leqslant j \leqslant m$. The same is true for the corresponding geometric hulls.* □

Since the diffraction is a property of the hull for strictly ergodic systems, it suffices to carry out the cocycle analysis for $j = m$. Let $\varrho_m := \varrho_{m,m} : a \mapsto a^m b, b \mapsto a$, with the Fourier matrix

$$B(k) = \begin{pmatrix} 1 + z + \ldots + z^{m-1} & 1 \\ z^m & 0 \end{pmatrix}.$$

As in the non-Pisot case in Section 4.2.1, we have

$$\mathbb{M}(\log \|\tilde{B}(x,y)\|_{\mathrm{F}}^2) = \mathfrak{m}(r(z)) < \log \sqrt{46} < \log(\lambda_7),$$

where $r(z) = z^{2m} + 2z^{m+1} - 6z^m + 2z^{m-1} + 1$.

**Proposition 4.3.2.** *Let $\varrho_{m,j}$ be any of the noble means substitutions defined above. Then, both Lyapunov exponents associated to $\varrho_{m,j}$ are positive.* □

Table 4.3.: Numerical upper bounds for $\chi^B$ for the noble means substitutions $\varrho_m$. The numerical error is less than $10^{-3}$ in all cases listed.

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\log(\lambda_m)$ | 0.481 | 0.881 | 1.195 | 1.444 | 1.647 | 1.818 |
| $N = N(m)$ | 6 | 3 | 2 | 2 | 1 | 1 |
| $\frac{1}{N}\mathbb{M}(\log \|B^{(N)}(.)\|_{\mathrm{F}}^2)$ | 0.439 | 0.835 | 1.114 | 1.162 | 1.496 | 1.511 |

**Remark 4.3.3.** It is known that the corresponding hull $\mathbb{Y}$ arises from a cut-and-project scheme, and so it must be pure point diffractive. The complete description of the heights and locations of the Bragg peaks are briefly treated in [Mol13, Sec. 6.2]. The pure point nature of $\widehat{\gamma}$ also follows from a result that all irreducible Pisot substitutions on binary alphabets have pure point dynamical spectrum [HS03], and hence are also pure point diffractive. The random version which displays both pure point and absolutely continuous part is dealt with in [Mol13, Mol14]. ◊

## 4.4. Strong irreducibility for Fibonacci Fourier matrices

Most computations for Lyapunov exponents in the constant-length case in Chapter 3 and for its higher-dimensional counterparts in Section 5.2 rely on the presence of invariant subspaces, where one can easily compute exponents for specific directions or spot unitarity of blocks. Outside this regime, it is natural to ask whether one can profit from irreducibility.

This advantage is known for random matrix cocycles, which is given by Fursternberg's representation of $\chi$, which requires a stronger notion of irreducibility. Let $\mathcal{S}$ be a set of matrices. Let $V = V_1 \cup \cdots \cup V_s$ be a finite union of proper subspaces of $\mathbb{R}^d$. We say that $V$ is *invariant* under $\mathcal{S}$ if $Mv \in V$, for all $v \in V$ and $M \in \mathcal{S}$. A collection $\mathcal{S}$ is called *strongly irreducible* if there exists no such finite union $V$ that is invariant under $\mathcal{S}$. Under the assumption that $\operatorname{supp}(\mu)$ is strongly irreducible, where $\mu$ is the support of the random cocycle, one gets an integral formula for $\chi$; see Appendix B for some details.

For probability measures $\mu$ supported on invertible real matrices, one has the following result.

**Proposition 4.4.1.** [BL85, Prop. 4.3] *Let $\mu$ be a probability measure on $\mathrm{GL}(2,\mathbb{R})$, $G_\mu$ be the group generated by supp$(\mu)$. If $|\det M| = 1$, for all $M \in G_\mu$, and $G_\mu$ is non-compact, strong irreducibility is equivalent to the following condition:*

- *For any $\bar{v} \in \mathbb{RP}^1$, the set $\mathcal{W}_{\bar{v}} = \{M \cdot \bar{v} \mid M \in G_\mu\}$ contains more than two elements.* □

**Remark 4.4.2.** Note that Proposition 4.4.1 can be extended to general subgroup $G$ of $\mathrm{GL}(d,\mathbb{R})$ whose elements are of determinant 1 or $-1$, i.e., if $G$ is non-compact, $G$ is strongly irreducible if and only if $\operatorname{card}(\{M \cdot \bar{v} \mid M \in G\}) > d$, for every $\bar{v} \in \mathbb{RP}^{d-1}$, which is equivalent to $|\mathcal{Q}| > d$, where $\mathcal{Q}$ is the set of directions in $\mathbb{RP}^{d-1}$ which is invariant with respect to $G_\mu$; see Appendix B.2 for the original proof for $d = 2$. ◊

For a moment, we intentionally forget the order defined by our deterministic orbit, and just consider the constituent matrices. Since Furstenberg's formula is stated for real cocycles, we work with the corresponding realification $B_{\mathbb{R}}(k)$. In what follows, we consider the Fibonacci substitution $\varrho : a \mapsto ab, b \mapsto a$, with Fourier matrix $B(k) \in \mathrm{GL}(2,\mathbb{C})$, which in turn implies $B_{\mathbb{R}}(k) \in \mathrm{GL}(4,\mathbb{R})$ for all $k$. We suspect that, in general, this everywhere invertibility holds when $\varrho$ is an automorphism of the free group $\mathfrak{F}_{n_a}$.

**Proposition 4.4.3.** *The group $G_k$ generated by the set of matrices $\{B_{\mathbb{R}}(\tau^n k)\}_{n\in\mathbb{N}_0}$ associated to the Fibonacci substitution is strongly irreducible for a.e. $k \in \mathbb{R}$.*

*Proof.* We employ a strategy similar to what was used in [DSS114] to show the positivity of exponents for Bernoulli–Anderson transfer matrices. The realification $B_{\mathbb{R}}(k)$ of $B(k)$ is given by

$$M_1(k) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \cos(2\pi\tau k) & \sin(2\pi\tau k) & 0 & 0 \\ -\sin(2\pi\tau k) & \cos(2\pi\tau k) & 0 & 0 \end{pmatrix},$$

with $\det B_{\mathbb{R}}(k) = 1$ for all $k \in \mathbb{R}$. We build the corresponding cocycle $B_{\mathbb{R}}^{(n)}(k)$ analogously as in the complex case, and we see that this real cocycle acts on the real projective space $\mathbb{RP}^3$, which we identify with $\mathbb{S}^3/\{-1,1\}$ and is parametrised by the triple $(\phi_1, \phi_2, \phi_3)$, with $0 \leqslant \phi_1, \phi_2 \leqslant \pi$ and $0 \leqslant \phi_3 < 2\pi$.

A generic element of the projective space has the representation

$$\overline{v}(\phi_1, \phi_2, \phi_3) = \big(\cos(\phi_1),\ \sin(\phi_1)\cos(\phi_2),\ \sin(\phi_1)\sin(\phi_2)\cos(\phi_3),\ \sin(\phi_1)\sin(\phi_2)\sin(\phi_3)\big),$$

consistent with the usual spherical coordinates in four dimensions. For an arbitrary but fixed $k$, we consider the group $G_k$ generated by the set $\{B_{\mathbb{R}}(\tau^n k)\}_{n\in\mathbb{N}}$. We note that the matrix

$$M_3(n,m,k) = B_{\mathbb{R}}(\tau^n k) \cdot B_{\mathbb{R}}(\tau^m k)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(2\pi(\tau^m - \tau^n)k) & \sin(2\pi(\tau^m - \tau^n)k) \\ 0 & 0 & -\sin(2\pi(\tau^m - \tau^n)k) & \cos(2\pi(\tau^m - \tau^n)k) \end{pmatrix},$$

is in $G_k$ for all $n, m \in \mathbb{N}$.

In order to exploit the equivalent criterion of strong irreducibility given in Proposition 4.4.1, we first need to show that $G_k$ is non-compact, i.e., that there exists a sequence of matrices with unbounded norms.

To do so, we consider a generic direction $\overline{v}_0 \in \mathbb{RP}^3$, and show that, for a.e chosen $k$, one can find a sequence of matrices $(A_n)_{n\in\mathbb{N}_0} \subset G_k^{\mathbb{N}_0}$ such that $\|\overline{v}_0 A_0 A_1 \dots A_{n-1} A_n\| \to \infty$ as $n \to \infty$. We first note that the matrix $M_3$ changes only the third parameter of an element in $\mathbb{RP}^3$, i.e.,

$$\overline{v}(\phi_1, \phi_2, \phi_3) \cdot M_3(n,m,k) = \overline{v}(\phi_1, \phi_2, \phi_3 + 2\pi(\tau^n - \tau^m)k).$$

Now we let $\overline{v} \in \mathbb{RP}^3$. Explicit calculations give us

$$\|\overline{v}(\phi_1, \phi_2, \phi_3) \cdot M_3(n,m,k)\|^2 = C + D + E,$$

with

$$\begin{aligned} C &= 1 + \cos(\phi_1)^2 + \cos(\phi_2)^2 \sin(\phi_1)^2, \\ D &= \underbrace{\cos(2\pi(\tau^n - \tau^m)k + \phi_3)}_{D_1} \underbrace{\sin(2\phi_1)\sin(\phi_2)}_{D_2}, \\ E &= \sin(\phi_1)^2 \underbrace{\sin(2\pi(\tau^n - \tau^m)k + \phi_3)}_{E_1} \underbrace{\sin(2\phi_2)}_{E_2}. \end{aligned}$$

**Case 1.** $\phi_1, \phi_2 \notin \{0, \frac{\pi}{2}, \pi\}$

From the assumption, none of $\cos(\phi_1), \sin(\phi_1), \cos(\phi_2)$, or $\sin(\phi_2) \neq 0$ are zero, which implies $C > 1$. To show that for any such direction there is a matrix element that is norm-wise expanding, it suffices to confirm that there exists $n, m \in \mathbb{N}$ for which $D, E \geqslant 0$, for any $\phi_3 \in [0, 2\pi]$. This can be achieved by choosing $n, m \in \mathbb{N}$ such that

$$\operatorname{sgn}(D_1)\operatorname{sgn}(D_2) \geqslant 0 \quad \text{and} \quad \operatorname{sgn}(E_1)\operatorname{sgn}(E_2) \geqslant 0,$$

where $\operatorname{sgn}(x) := \frac{x}{|x|}$, for $x \neq 0$, and 0 otherwise. This freedom of choice, which is independent of $\phi_3$, follows from the fact that $((\tau^n - \tau^m)k) \mod 1$ is dense in $[0,1)$ for a.e. $k \in \mathbb{R}$.

**Case 2.** $\phi_1, \phi_2 \in \{0, \pi\}$

When either $\phi_1$ or $\phi_2 \in \{0, \pi\}$, one has $\|\overline{v} \cdot M_3(n, m, k)\|^2 = 2$, for any $m, n, \phi_3$.

**Case 3.** $\phi_1 = \frac{\pi}{2}$ or $\phi_2 = \frac{\pi}{2}$

When exactly one of the two parameters is equal to $\frac{\pi}{2}$, the norm simplifies to a form similar to Case 1, where one can choose $m, n$ so that $\sin(2\phi_1)$ or $\sin(2\phi_2)$ is consistent with the sign of $\sin(2\pi(\tau^n - \tau^m)k + \phi_3)$.

When $\phi_1 = \phi_2 = \frac{\pi}{2}$, the matrix $M_3$ is norm preserving for any $n, m$, and $\phi_3$. However, we note that

$$\overline{v}\left(\frac{\pi}{2}, \frac{\pi}{2}, \phi_3\right) \xrightarrow{M_3(m,n,k)} \overline{v}(2\pi(\tau^n - \tau^m)k + \phi_3, 0, 0),$$

which is already covered by Case 2. This completes our claim on the non-compactness of $G_k$.

Now, we can invoke Proposition 4.4.1 to show strong irreducibility. For this, we need, for every $\overline{v}_0 \in \mathbb{RP}^3$, matrices $A_1, A_2, A_3, A_4$ such that

$$\overline{v}_0 \overset{A_1}{\longmapsto} \overline{v}_1 \overset{A_2}{\longmapsto} \overline{v}_2 \overset{A_3}{\longmapsto} \overline{v}_3 \overset{A_4}{\longmapsto} \overline{v}_4,$$

with $\overline{v}_j$ being distinct, for $0 \leqslant j \leqslant 4$. It is easy to see that the matrix $M_3$ generates more than four directions for any starting element $\overline{v}_0$ as it only changes $\phi_3$, and one can choose suitable powers $n, m$ so that this condition on $|\mathcal{W}_{\overline{v}}|$ is satisfied. □

**Remark 4.4.4.** Using the same techniques, one can prove that for a.e. $x \in \mathbb{T}^2$, the corresponding lifted cocycle $\left\{\widetilde{B}_{\mathbb{R}}(x.M^n)\right\}_{n\in\mathbb{N}_0}$ is also strongly irreducible, where the crucial point is the denseness of $x.M^n$ in $\mathbb{T}^2$ for a.e. $x \in \mathbb{T}^2$, which is due to the base dynamics being ergodic. ♢

Whether one can use this property in our context to compute $\chi^B$ is still unclear. For one, deterministic products constitute a null set in the set of realisations of Bernoulli systems. Moreover, the program presented in Appendix B.3, which numerically estimates the exponent, only works presuming one has a finitely-supported $\mu$, which one forgoes when one considers $\{B_{\mathbb{R}}(\lambda^n k)\}_{n\in\mathbb{N}_0}$. However, results like Theorem 4.1.5 suggest that a link between $\chi^B$ and the random one given by Furstenberg's formula might actually exist.

# 5. Higher-Dimensional Examples

## 5.1. Formulation in higher dimensions

### 5.1.1. Inflation tilings in $\mathbb{R}^d$

Let $\mathcal{P} = \{\mathfrak{t}_1, \dots, \mathfrak{t}_{n_a}\}$ be a finite set of prototiles (up to translation), where each $\mathfrak{t}_i \subset \mathbb{R}^d$ is compact and has non-empty interior, with $\overline{\mathfrak{t}_i^\circ} = \mathfrak{t}_i$. A *stone inflation* $\varrho$ is a rule, together with an expansive linear map $Q$, such that $Q(\mathfrak{t}_i)$ is mapped to a union of non-overlapping translates of elements in $\mathcal{P}$. This means

$$Q(\mathfrak{t}_i) = \bigcup_{j=1}^{n_a} \mathfrak{t}_j + F_{ij},$$

where $F_{ij} \subset \mathbb{R}^d$ are finite sets. This union is the supertile associated to $\mathfrak{t}_i$.

Once again, an incidence (substitution) matrix $M_\varrho$ can be associated to $\varrho$, and its primitivity implies that $\varrho$ is primitive like in the one-dimensional case.

Before we proceed, let us introduce a generalised notion of a local topology, which is called the *local rubber topology*. Under this topology, tilings or point sets are $\varepsilon$-close if they "almost agree" within a large region around the origin. It generalises the local topology since the translation vector $t$ in Eq. 1.1 needed for large patches of two tilings or point sets $\varLambda, \varLambda'$ to coincide is allowed to vary for different points $x \in \varLambda' \cap \mathfrak{B}_{1/\varepsilon}(0)$, provided $t_x \in \mathfrak{B}_\varepsilon(0)$ for all $t_x$; see [BL04, Sec. 4] for a precise formulation.

**Fact 5.1.1.** *The hull of a primitive stone inflation is compact in the local rubber topology, its elements are all locally indistinguishable, and the hull gives rise to a minimal topological dynamical system* $(\mathbb{Y}, \mathbb{R}^d)$*. The latter is strictly ergodic, which implies uniform existence of patch frequencies for every tiling* $\mathcal{T} \in \mathbb{Y}$*.* □

**Remark 5.1.2.** When the tilings $\mathcal{T}$ we consider are FLC, the local rubber topology coincides with the usual local topology in the spirit of Eq. (1.1). It is well known that primitivity implies minimality both in the FLC and the non-FLC case; see [FS14a, Prop. 3.2] and [FS14b, Prop. 3.1]. Unique ergodicity in the non-FLC case requires extra care, but is nevertheless shown in different settings which cover stone inflations; compare [FR14, Prop. 4.5] for almost repetitive Delone sets, [FS14b, Thm. 4.5] for fusion tilings with infinite local complexity, and [LS19, Thm. 4.13] for primitive substitution tilings on finitely many prototiles. ◊

Now, fix a control point for every element of $\mathcal{P}$ such that any tiling $\mathcal{T} \in \mathbb{Y}$ is MLD to the resulting collection of control points, which we call $\varLambda$. Unlike in one dimension, where the left endpoint of tiles proves to be a canonical choice, there might be different possible choices in higher dimensions. And since control points of different tiles might coincide, this time, it is practical to colour each identification point to distinguish different occupants. The choice of the markers would not affect much of our arguments since $\mathbb{Y}$ and the derived set of (weighted) point sets are topologically conjugate.

### 5.1.2. Displacement and Fourier matrix

We now construct the higher-dimensional analogues of notions introduced in Section 2.1.

Denote by $T_{ij}$ the set of control point positions of tiles of type $i$ inside $\varrho(\mathfrak{t}_j)$, relative to the image of the control point of $\mathfrak{t}_j$ under $\varrho$. We also get that $M_\varrho = \mathrm{card}(T)$, where $T = (T_{ij})$ is the set-valued displacement matrix. Moreover, applying $Q$ to $T$ yields the positions of supertiles in level-2 supertiles.

The displacement matrix $T^{(n)}$ for $\varrho^n$ can then be recursively computed to be

$$T_{ij}^{(n)} = \bigcup_{\ell=1}^{n_a} (T_{i\ell} + QT_{\ell j}^{(n-1)}), \tag{5.1}$$

where $+$ denotes the Minkowski sum of two point sets; compare with Eq. (3.9). As in the one-dimensional case, the *Fourier matrix* $B(k)$ of $\varrho$ reads

$$B(k) := \overline{\widehat{\delta_T}}(k) = \widehat{\delta_T}(-k),$$

with $k \in \mathbb{R}^d$. For each $k$, $B(k) \in \mathrm{Mat}(n_a, \mathbb{C})$, with entries of $B$ being trigonometric polynomials in $d$ variables.

**Lemma 5.1.3.** *Let $\varrho$ be a stone inflation on finitely many prototiles in $\mathbb{R}^d$, for the linear expansion $Q$, with Fourier matrix $B(k)$. Then, for $n \in \mathbb{N}$, the Fourier matrix of $B^{(n)}(k)$ is given by*

$$B^{(n)}(k) = B(k)B(Q^{\mathrm{T}}k)\dots B((Q^{\mathrm{T}})^{n-1}k).$$

*Moreover, it satisfies $B^{(1)}(k) = B(k)$ and $B^{(n+1)}(k) = B(k)B^{(n)}(Q^{\mathrm{T}}k)$. Here $Q^{\mathrm{T}}$ is the transpose of $Q$.*

*Proof.* For $n = 2$, Eq. (5.1) and the convolution theorem for Fourier transforms yield

$$\left(B^{(2)}(k)\right)_{ij} = \overline{\widehat{\delta_{T_{ij}^{(2)}}}} = \overline{\widehat{\left(\sum_{\ell=1}^{n_a} \delta_{T_{i\ell}} * \delta_{QT_{\ell j}}\right)}} = \sum_{\ell=1}^{n_a} \sum_{\substack{r\in T_{i\ell}\\ s\in T_{\ell j}}} \mathrm{e}^{2\pi \mathrm{i}k(r+Qs)} = \sum_{\ell=1}^{n_a} \sum_{\substack{r\in T_{i\ell}\\ s\in T_{\ell j}}} \mathrm{e}^{2\pi \mathrm{i}kr}\, \mathrm{e}^{2\pi \mathrm{i}(Q^{\mathrm{T}})k\cdot s},$$

where the last equality follows from the identity $Q^{\mathrm{T}}k \cdot x = k \cdot Qx$. It is then easy to see that this is equal to $\left(B(k)B(Q^{\mathrm{T}}k)\right)_{ij}$ for $1 \leqslant i, j \leqslant n_a$. The validity of the general formula is then obvious by an inductive argument. □

### 5.1.3. Renormalisation relations

As in one dimension, the pair correlation function at $z$ is the relative frequency that markers of type $i$ and $j$ are separated by a vector $z \in \mathbb{R}^d$ from $i$ to $j$, and is given by

$$\nu_{ij}(z) = \frac{\mathrm{dens}(\varLambda_i \cap (\varLambda_j - z))}{\mathrm{dens}(\varLambda)}.$$

This is independent of the choice $\varLambda$ from the hull given a fixed marking system.

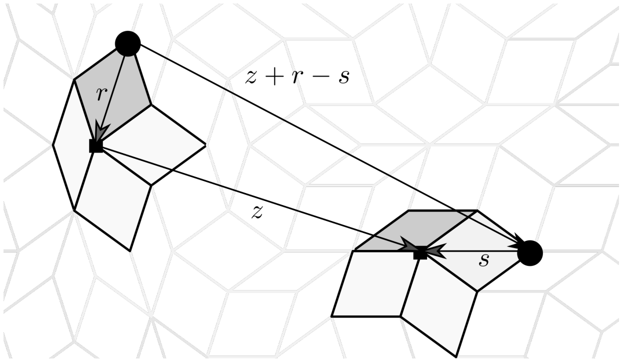



Figure 5.1.: If the two tiles (gray) at distance $z$ have offsets $r$ and $s$ within their covering supertiles, the latter are at a distance $z+r-s$ apart. Here, the distances are always defined via the control points of the tiles.

The higher-dimensional analogue of Eq. (2.4), which was first announced in [Man17b], reads

$$\nu_{ij}(z) = \frac{1}{|\det Q|} \sum_{m,n=1}^{n_a} \sum_{\substack{r\in T_{im}\\ s\in T_{jn}}} \nu_{mn}(Q^{-1}(z+r-s)),$$

the proof of which proceeds in a manner similar to that of Prop. 2.2.1; see Figure 5.1 for a higher-dimensional analogue of Figure 2.2.

Setting $\Upsilon_{ij} = \sum_{z\in\Lambda-\Lambda} \nu_{ij}(z)\delta_z$, we get the corresponding version of Eq. (2.5) for the pair correlation measures

$$\Upsilon_{ij} = \frac{1}{|\det Q|} \sum_{m,n=1}^{n_a} \widetilde{\delta_{T_{im}}} * \delta_{T_{jn}} * (Q.\Upsilon_{mn}). \tag{5.2}$$

It is important to note that, in the non-FLC case, unique ergodicity of the tiling dynamical system $(\mathbb{Y}, \mathbb{R}^d)$ implies that the measures $\Upsilon_{ij}$ are well defined and indeed constitutes a well-defined autocorrelation measure $\gamma$ [BL04, Thm. 5]. Moreover, the finiteness of the prototile set guarantees that each measure $\Upsilon_{ij}$ satisfies the renormalisation relation in Eq. (5.2).

Before we can proceed, we will need the following lemma.

**Lemma 5.1.4** ( [BG18, Lem. 2.5]). *Set $Q^* = (Q^{\mathrm{T}})^{-1}$ and let $\mu \in \mathcal{M}(\mathbb{R}^d)$. Then, one has $\widehat{Q.\mu} = |\det Q|^{-1} Q^*.\widehat{\mu}$.*

*Proof.* For $g \in C_{\mathrm{c}}(\mathbb{R}^d)$, $Q: \mathbb{R}^d \to \mathbb{R}^d$, one has $\widehat{Q.\mu}(g) = \mu(\hat{g}\circ Q)$, where $\hat{g}$ is defined via Eq. (1.2). Here, the right hand-side is given by

$$\mu(\hat{g}\circ Q) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{-2\pi\mathrm{i} Qk\cdot x} g(x)\,\mathrm{d}x\,\mathrm{d}\mu(k).$$

On the other hand, we have $Q^*.\hat{\mu}(g) = \hat{\mu}(g\circ Q^*) = \mu(\widehat{g\circ Q^*})$ with

$$\widehat{g\circ Q^*}(k) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi\mathrm{i}k\cdot x} g(Q^*(x))\,\mathrm{d}x = \big|\det Q^{\mathrm{T}}\big| \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi\mathrm{i}Q^{\mathrm{T}}y\cdot k} g(y)\,\mathrm{d}y,$$

where the last equality follows via the change of variable $y = Q^* x$. The identity $Qk\cdot x = k\cdot Q^{\mathrm{T}}x$ implies the claim, since $g$ was chosen arbitrarily. Eq. (2.9) in one dimension follows as a special case . □

Taking the Fourier transform of Eq. (5.2) yields the corresponding renormalisation for $\widehat{\varUpsilon_{ij}}$, which by Lemma 5.1.4 reads

$$\widehat{\varUpsilon_{ij}} = (\det Q)^{-2} \sum_{m,n=1}^{n_a} B_{im}(.)\overline{B_{jn}(.)}(Q^{*}.\widehat{\varUpsilon_{mn}}).$$

Via the exact arguments from Lemma 2.3.4, this equation also holds independently for each spectral type $\alpha \in \{\mathsf{pp}, \mathsf{ac}, \mathsf{sc}\}$.

In particular, we recover Eq. (2.13) for the Radon–Nikodym vector $\boldsymbol{h}(k)$, which now reads

$$\boldsymbol{h}(k) = \frac{1}{|\det Q|} \boldsymbol{A}(k) \boldsymbol{h}(Q^{\mathrm{T}} k).$$

### 5.1.4. Lyapunov exponents and absolutely continuous diffraction

The dimensional reduction arguments in Sec. 2.4 can also be applied here, which allows to study the cocycle $B^{(n)}(k)$ and the relevant Lyapunov exponents

$$\chi^{B}(k) := \limsup_{n\to\infty} \frac{1}{n} \log \left\| B^{(n)}(k) \right\| \quad \text{and} \quad \chi_{\min}(k) = \log \sqrt{|\det Q|} - \chi^{B}(k). \tag{5.3}$$

The following results are the analogues of Theorem 2.5.3 and Corollary 2.5.4 for higher-dimensional stone inflations.

**Theorem 5.1.5.** *Let $\varrho$ be a primitive stone inflation in $\mathbb{R}^d$, with finite prototile set $\mathcal{P}$ and expansive linear map $Q$. Let $B(k)$ be the corresponding Fourier matrix, with $\det B(k) \neq 0$ for some $k \in \mathbb{R}^d$. If there is an $\varepsilon > 0$ such that*

$$\chi^{B}(k) \leqslant \log \sqrt{|\det Q|} - \varepsilon$$

*holds for a.e. $k \in \mathbb{R}^d$, where $\chi^{B}(k)$ is the maximal exponent defined in Eq. (5.3), the diffraction measure of the system cannot have an absolutely continuous component.* □

**Corollary 5.1.6.** *Let $\varrho$ be a primitive stone inflation satisfying the conditions of Theorem 5.1.5. Then, one has $\chi_{\min}(k) \geqslant 0$ for a.e. $k \in \mathbb{R}^d$. Moreover, if the system displays a non-trivial diffraction of absolutely continuous type, one must have $\chi_{\min}(k) = 0$, i.e., $\chi^{B}(k) = \log \sqrt{|\det Q|}$, for some subset of $\mathbb{R}^d$ of positive measure.* □

Employing the techniques used in Section 4.1, one can show the almost sure existence and almost everywhere constancy of $\chi^{B}(k)$ for stone inflations whose inflation multiplier in each direction is a PV-number.

**Proposition 5.1.7.** *Let $\varrho$ be an $d$-dimensional primitive stone inflation with finitely many prototiles and a diagonal expansion map $Q$. Assume further that all eigenvalues of $Q$ are Pisot (not necessarily irreducible). Then, the Lyapunov exponent $\chi^{B}(k)$ exists as a limit almost surely and is equal to the constant $\chi^{\widetilde{B}}$, where $\widetilde{B}$ is the appropriate higher-dimensional $1$-periodic representation of $B$.*

*Proof.* Let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of $Q$, with corresponding algebraic degrees $r_i$. For each $\lambda_i$, the corresponding companion matrix $\mathfrak{C}_i$ of its minimal polynomial is an integer matrix that defines an ergodic toral endomorphism $\widetilde{\mathfrak{C}}_i : x \mapsto x.\mathfrak{C}_i \mod 1$ on $\mathbb{T}^{r_i}$. The $\sum_i r_i$-dimensional block matrix

$$Q' := \mathrm{diag}(\mathfrak{C}_1, \dots, \mathfrak{C}_m)$$

then defines an ergodic toral endomorphism on $\mathbb{T}^{r_1} \times \cdots \times \mathbb{T}^{r_m}$.

The almost sure existence of the exponent for the cocycle $\widetilde{B} : \prod_i \mathbb{T}^{r_i} \to \mathbb{C}$ with base dynamics $Q'$ is again immediate from Oseledec's theorem; compare Proposition 2.6.4.

For each $1 \leqslant i \leqslant d$, one can show that, for a.e. $x_i \in \mathbb{T}^{r_i}$, one has the orbit convergence

$$\widetilde{\mathfrak{C}}_i^{\,n}(x_i) \xrightarrow{n\to\infty} \pi_i\big((k_i, \lambda_i k_i, \dots, \lambda_i^{r_i - 1} k_i).\mathfrak{C}_i^n\big) = \pi_i(\lambda_i^n k_i)$$

for some $k_i \in \mathbb{R}^{r_i}$. Here, $\pi_i$ is the quotient map on $\mathbb{T}^{r_i}$. Exploiting the block diagonal structure of $\mathfrak{C}$, and proceeding in a similar manner as in Theorem 4.1.5, one can deduce that the exceptional set $\mathcal{X} = \prod_i \mathcal{X}_i$ of $k \in \mathbb{R}^d$ for which $\chi^B(k)$ either does not exist as a limit or is not equal to $\chi^{\widetilde{B}}$ satisfies $\mu_{\mathrm{L}}(\mathcal{X}) = 0$. □

**Remark 5.1.8** (Primitive inflations which are not stone inflations)**.** There exist primitive inflations in $\mathbb{R}^d$ which are not stone inflations, i.e., the level-1 supertiles are not exact inflated versions of the prototiles but are nevertheless finite unions of their translates. To some of these inflations, one can associate stone inflations on prototiles with fractal boundaries (usually obtained via variants of the von Koch curve construction), which generate tiling hulls that are MLD to the original ones; see [BG13, Rem. 6.9 and Rem. 6.11] for examples. For this subclass of inflations, the renormalisation relations in Section 5.1.3 and the consequent criteria involving Lyapunov exponents in this section apply as in the stone inflation case. ◊

## 5.2. Substitutions in $\mathbb{Z}^d$

Generalisations of constant-length substitutions in higher dimensions are called *block substitutions*, which map letters into sequences in $\mathbb{Z}^d$. The corresponding geometric rule is called a *block inflation*, where $\varrho$ maps labelled or coloured unit cubes in $\mathbb{R}^d$ with support $[0,1)^d$ into $d$-dimensional rectangular union of such cubes, whose support is of the form

$$[0, L_1 - 1) \times [0, L_2 - 1) \times \cdots \times [0, L_d - 1)\,.$$

Here, $L_1, \dots, L_d$ are the respective inflation factors in each direction. The associated expansion map is $Q = \mathrm{diag}(L_1, \dots, L_d)$. The vertex set of the tiling arising from such a $\varrho$ can then be viewed as a colouring of $\mathbb{Z}^d$ . We refer to [Bar16, Fra03] for a formal exposition, and to [BG14] for a complete treatment under the renormalisation scheme discussed in this work.

We choose the control points of tiles to be the origin, which in two and three-dimensional examples are the lower left vertex of the cube. Due to this structure, the total set is given by

$$S_T = \{0, \dots, L_1 - 1\} \times \{0, \dots, L_2 - 1\} \times \dots \times \{0, \cdots, L_d - 1\}\,,$$

and the displacement sets $T_{ij}$ consist of elements in $\mathbb{Z}^d$. Employing arguments similar to those used in Section 3.1, one can show that $B(k)$ is periodic, from which the almost sure existence of the Lyapunov exponents is immediate. Notions of bijectivity and coincidences are also extended once one fixes an order on the prototile set $\mathcal{P} = \{\mathfrak{t}_1, \dots, \mathfrak{t}_{n_a}\}$. Define the map $\kappa_m : \mathcal{P} \to \mathcal{P}$ as

$$\mathfrak{t}_i \mapsto (\varrho(\mathfrak{t}_i))_m \quad \text{with} \quad m \in S_T. \tag{5.4}$$

We say that $\varrho$ is a bijection at $m$ if $\kappa_m$ is bijective. If this holds for all $m \in S_T$, one calls $\varrho$ a *bijective inflation*. A coincidence at $m \in S_T$ means that the image of $\kappa_m$ consists of a single tile, i.e., $(\varrho(\mathfrak{t}_i))_m = \mathfrak{t}_j$ for all $1 \leqslant i \leqslant n_a$, for some $1 \leqslant j \leqslant n_a$.

**Example 5.2.1.** The inflation $\varrho$ over a binary alphabet in Figure 5.2 has the Fourier matrix

$$B(k) = \begin{pmatrix} 1 + x^2 + y^2 + x^2y^2 & (1 + x + x^2)(1 + y + y^2) \\ x + y + xy + xy^2 + x^2y & 0 \end{pmatrix}$$

where $k = (k_1, k_2)$ and $x = \mathrm{e}^{2\pi \mathrm{i} k_1}, y = \mathrm{e}^{2\pi \mathrm{i} k_2}$. ◊

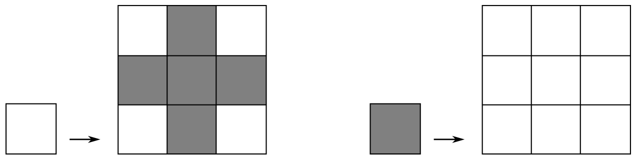


Figure 5.2.: A block substitution in $\mathbb{Z}^2$ with coincidences at $m \in \{(0, 0), (2, 0), (0, 2), (2, 2)\}$.

### 5.2.1. Binary block substitutions

Most of the arguments in Section 3.2 can be extended to the higher-dimensional case, which allows one to prove the following versions of Proposition 3.2.3 and Corollary 3.2.4; compare [BG18, Sec. 7].

**Proposition 5.2.2.** *The pointwise Lyapunov exponents of a primitive aperiodic binary block inflation $\varrho$ on $\mathbb{Z}^d$, for a.e. $k \in \mathbb{R}^d$, are given by*

$$\begin{aligned} \chi_{\max} &= \log \sqrt{L_1 L_2 \dots L_d} \quad \text{and} \\ \chi_{\min} &= \log \sqrt{L_1 L_2 \dots L_d} - \mathfrak{m}(q - r), \end{aligned}$$

*where the polynomials $q(x_1, \dots, x_d)$ and $r(x_1, \dots, x_d)$ are determined by the bijective positions, and $\mathfrak{m}(q - r)$ is a $d$-dimensional logarithmic Mahler measure.* □

**Corollary 5.2.3.** *Let $\varrho$ be as in Proposition 5.2.2. Then, the Lyapunov exponents associated with $\varrho$ are both positive. In particular, this means that the corresponding diffraction measure $\widehat{\gamma}$ is singular with respect to Lebesgue measure.* □

Corollary 5.2.3 provides an alternative proof for the same result in [BG14, Thm. 3].

The correspondence between substitutions and height-1 integer polynomials stated in Proposition 3.2.8 can be extended to any dimension. Here, we adapt the definition of a Borwein polynomial in Section 3.2.2 to the multivariate case.

**Proposition 5.2.4.** *Let $p(x_1,\dots,x_d)\in\mathbb{Z}[x_1,\dots,x_d]$ be a multivariate Borwein polynomial of degree $N\leqslant(L-1)^d$ with nonzero constant term. Then, there exists at least one primitive binary block inflation $\varrho$ with expansion map $Q=\mathrm{diag}(L,\dots,L)$ such that, for a.e. $k\in\mathbb{R}^d$,*

$$\chi^B(k)=\mathfrak{m}(p),$$

*where $\chi^B$ is the Lyapunov exponent of the Fourier cocycle $B^{(n)}$ and $\mathfrak{m}(p)$ is the logarithmic Mahler measure of the multivariate polynomial $p$.* □

This result yields some interesting examples. We refer to [BL12] for a succinct overview and [Boy98] for an extensive survey on multivariate Mahler measures and their relations to $L$-functions of elliptic curves.

**Example 5.2.5.** The exponent $\chi^B$ associated to the inflation rule in Example 5.2.1 is the logarithmic Mahler measure

$$\chi^B\;=\;\mathfrak{m}(x+y+xy+x^2y+xy^2)\;=\;\mathfrak{m}\big(x+\frac{1}{x}+y+\frac{1}{y}+1\big),$$

which is equal to a special $L$-series value given by

$$\mathfrak{m}\big(x+\frac{1}{x}+y+\frac{1}{y}+1\big)\;=\;L'(E_{15},0)\;=\;\frac{15}{4\pi^2}L(E_{15},2).$$

Here, $E_{15}$ is the elliptic curve of conductor 15; see [BL12]. ♢

**Example 5.2.6.** Consider the block substitution $\varrho_1$ in three dimensions given in Figure 5.3.

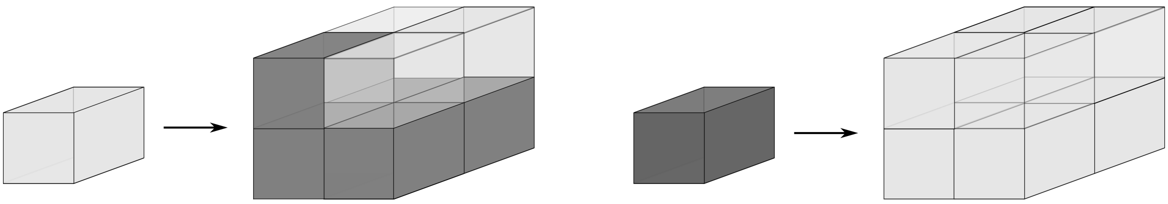


Figure 5.3.: The three-dimensional block substitution $\varrho_1$

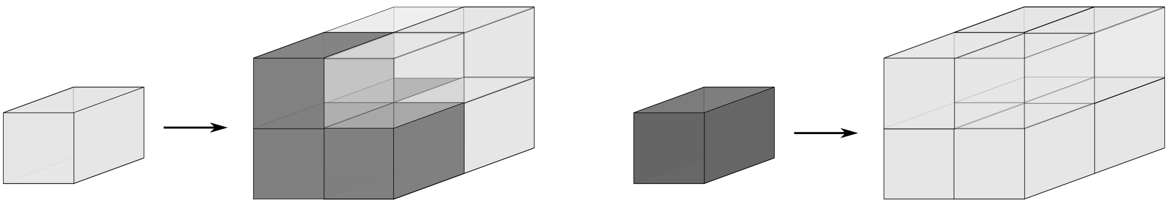


Figure 5.4.: The three-dimensional block substitution $\varrho_2$

Its corresponding Lyapunov exponent is equal to

$$\chi^B=\mathfrak{m}(1+x+y+xy+z),$$

which is conjectured to be equal to $2L'(E_{15},-1)$; see [BL12]. Similarly, for the substitution $\varrho_2$ in Figure 5.4, one has

$$\chi^B=\mathfrak{m}(1+x+y+z)=\frac{7}{2\pi^2}\zeta(3),$$

where $\zeta(s)$ is the Riemann zeta function. ♢

### 5.2.2. Abelian bijective block inflations

Here, we consider an extension of notions in Section 3.3 to higher dimensions. Assume that $\varrho$ is a bijective block inflation. The map $\kappa_m$ from Eq. (5.4) induces a map $m \mapsto \sigma_m$ from $S_T$ to $\Sigma_{n_a}$, where $\sigma_m$ is defined via

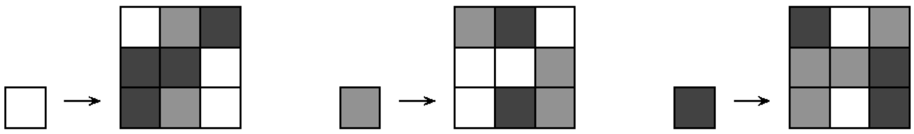
$$\big(\kappa_m(\mathsf{t}_1), \ldots, \kappa_m(\mathsf{t}_{n_a})\big) = \big(\mathsf{t}_{\sigma_m(1)}, \ldots, \mathsf{t}_{\sigma_m(n_a)}\big).$$

As before, the generating subgroup is given by $G = \Big\langle \{\sigma_m\}_{m\in S_T} \Big\rangle \subseteq \Sigma_{n_a}$. We say that $\varrho$ is Abelian if $G$ is Abelian. Automatically, the IDA $\mathcal{B}$ is also Abelian.

The bounding steps in Theorem 3.3.4 generalise conveniently when one considers bijective Abelian inflations in higher dimensions. Together with a higher-dimensional analogue of the argument in Remark 3.3.5, we recover Bartlett's singularity result completely for any dimension in the following result.

**Theorem 5.2.7.** *Let $\varrho$ be a primitive, aperiodic, bijective block inflation in $\mathbb{Z}^d$ whose IDA $\mathcal{B}$ is Abelian. Then, for a.e. $k \in \mathbb{R}^d$, all Lyapunov exponents of $\varrho$ are strictly positive. Consequently, the corresponding diffraction and dynamical spectra are both singular.* □

**Example 5.2.8** (Block substitution on three tiles)**.** Let $\varrho_{\text{2D}}$ be defined by



By inspection, one sees that this is indeed bijective and that $G \simeq C_3$. The eigenvalues of the Fourier matrix $B(k_1, k_2)$ are given by

$$\begin{aligned}
\beta_1(k_1,k_2) &= (1+x+x^2)(1+y+y^2),\\
\beta_2(k_1,k_2) &= (x^2+x^2y+y^2) + \xi_3(1+y+xy+x^2y^2) + \xi_3^2(x+xy^2),\\
\beta_3(k_1,k_2) &= (x^2+x^2y+y^2) + \xi_3^2(1+y+xy+x^2y^2) + \xi_3(x+xy^2),
\end{aligned}$$

where $x = \mathrm{e}^{2\pi \mathrm{i} k_1}, y = \mathrm{e}^{2\pi \mathrm{i} k_2}$ and $\xi_3 = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$. Viewed as polynomials in two variables with complex coefficients, all of them are of height one, and have logarithmic Mahler measures strictly less than $\log(3)$. Note that the same boundedness result holds for block substitutions that are not homotheties, i.e., those with $Q \neq c\mathbb{I}_{n_a}$ for some $c \in \mathbb{N}$, as long as their generating subgroups are Abelian. ◊

### 5.2.3. Examples with absolutely continuous spectrum

As mentioned, a construction from Hadamard matrices of higher-dimensional substitutions with absolutely continuous spectrum is detailed out in [Fra03]. An example found there on an eight-letter alphabet (four letters and their barred counterparts) given in Figure 5.5 is one of the simplest ones in two dimensions.

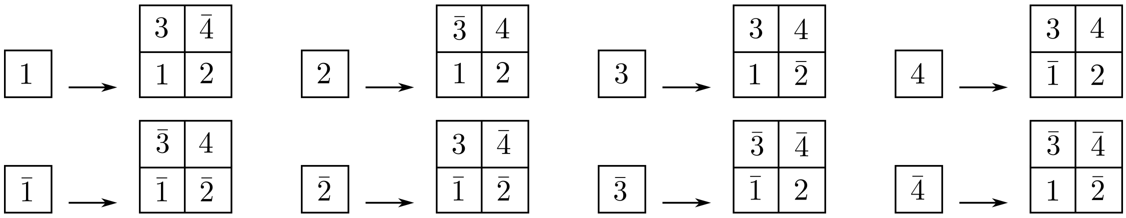



Figure 5.5.: Frank's substitution in $\mathbb{Z}^2$

This substitution arises from the $4 \times 4-$Hadamard matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

and clearly has bar swap symmetry. We assume that each tile's control point is given by its lower left corner. One can then directly construct its Fourier matrix which reads

$$B(k_1, k_2) = \left( \begin{array}{c|c} Z_1(x,y) & Z_2(x,y) \\ \hline Z_2(x,y) & Z_1(x,y) \end{array} \right) \qquad \text{where}$$

$$Z_1(x,y) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ x & x & 0 & x \\ y & 0 & y & y \\ 0 & xy & xy & xy \end{pmatrix} \qquad \text{and} \qquad Z_2(x,y) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & x & 0 \\ 0 & y & 0 & 0 \\ xy & 0 & 0 & 0 \end{pmatrix}.$$

Here, we denote $x = \mathrm{e}^{2\pi \mathrm{i} k_1}$ and $y = \mathrm{e}^{2\pi \mathrm{i} k_2}$ for convenience. Note that $B(0,0)$ is just the substitution matrix $M_\varrho$ with $\lambda_{\mathrm{PF}} = 4$.

The block symmetric, or more aptly, 2-circulant block structure of $B(k_1, k_2)$ allows it to be written as

$$B(k_1, k_2) = \Phi(e) \otimes Z_1 + \Phi(\sigma) \otimes Z_2 = \mathbb{I}_2 \otimes Z_1 + \mathbb{J}_2 \otimes Z_2 \tag{5.5}$$

where $\{e, \sigma\} \simeq C_2$ and $\Phi$ is the permutation representation. Here, $\mathbb{J}_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Since we know *a priori* that this substitution gives rise to tilings exhibiting absolutely continuous diffraction, we also know that the cocycle $B^{(n)}(k_1, k_2)$ has at least one Lyapunov exponent equal to $\log \sqrt{\lambda} = \log 2$, which is what we explicitly confirm next. We begin with lemmata which will aid us in determining the singular values of $B^{(n)}$.

**Lemma 5.2.9.** *Let $F(x,y), G(x,y)$ be matrices consisting of polynomials in $x, y$ such that, for a given row $i$ of $F$ or $G$, $F_{ij} = C^{(1)}_{ij} p_i(x,y), G_{ij} = C^{(2)}_{ij} p_i(x,y)$, where $p$ is a monomial that depends only on $i$ and $C^{(\alpha)}_{ij} \in \{0,1\}$ . Then, $G^\dagger(x,y)F(x,y)$ is a constant integer matrix.*

*Proof.* Computing a given entry of $G^\dagger F$ explicitly, we get

$$\left(G^\dagger F\right)_{ij} = \sum_\ell G^\dagger_{i\ell} F_{\ell j} = \sum_\ell C^{(2)}_{i\ell} C^{(1)}_{\ell j} \overline{p_\ell} p_\ell = \sum_\ell C^{(2)}_{i\ell} C^{(1)}_{\ell j} \in \mathbb{Z},$$

implying the claim. ☐

**Lemma 5.2.10.** *Let $B(k_1, k_2)$ be the same as in Eq. (5.5). Then, $B^\dagger(k_1, k_2)B(k_1, k_2)$ is constant for all $(k_1, k_2) \in \mathbb{R}^2$. Consequently, the singular values of $B(k_1, k_2)$ are equal to those of $B(0, 0) = M_\varrho$.*

*Proof.* Using the form given in Eq. (5.5), we get

$$B^\dagger B = \mathbb{I}_2 \otimes (Z_1^\dagger Z_1 + Z_2^\dagger Z_2) + \mathbb{J}_2 \otimes (Z_2^\dagger Z_1 + Z_1^\dagger Z_2). \tag{5.6}$$

The first claim is then just a consequence of Lemma 5.2.9. The second claim follows by evaluating at $k_1 = k_2 = 0$. □

We now show that we can get the singular values of $M_\varrho$ as the moduli of its eigenvalues by showing that it is normal. This, together with the fact that we know exactly what the invariant subspaces are and which eigenvalues occur in each subspace, will give the complete characterisation of the block diagonal form of $B(x, y)$ with prescribed growth rates.

**Proposition 5.2.11.** *Let $M_\varrho = B(0, 0)$, where $B(k_1, k_2)$ is given in Eq. (5.5). Then, $M_\varrho$ is normal, i.e., $M_\varrho^\dagger M_\varrho = M_\varrho M_\varrho^\dagger$, and the singular values of $M_\varrho$ are the moduli of its eigenvalues.*

*Proof.* Let $Z_1(1, 1) = Y_1$ and $Z_2(1, 1) = Y_2$. Setting $k_1 = k_2 = 0$ in Eq. (5.6) and doing the same for $M_\varrho M_\varrho^\dagger$, we obtain

$$\begin{aligned} M_\varrho^\dagger M_\varrho &= \mathbb{I}_2 \otimes (Y_1^\dagger Y_1 + Y_2^\dagger Y_2) + \mathbb{J}_2 \otimes (Y_2^\dagger Y_1 + Y_1^\dagger Y_2) \\ M_\varrho M_\varrho^\dagger &= \mathbb{I}_2 \otimes (Y_1 Y_1^\dagger + Y_2 Y_2^\dagger) + \mathbb{J}_2 \otimes (Y_2 Y_1^\dagger + Y_1 Y_2^\dagger). \end{aligned}$$

Next, we note that $Y_1 + Y_2 = \begin{pmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \end{pmatrix}$ and $Y_1 - Y_2 = H = \begin{pmatrix} 1&1&1&-1 \\ 1&1&-1&1 \\ 1&-1&1&1 \\ -1&1&1&1 \end{pmatrix}$. Obviously, $Y_1 + Y_2$ is normal, and $Y_1 - Y_2$ is normal because it is a Hadamard matrix. From this, we get the following equations

$$\begin{aligned} (Y_1^\dagger Y_1 + Y_2^\dagger Y_2) + (Y_2^\dagger Y_1 + Y_1^\dagger Y_2) &= (Y_1 Y_1^\dagger + Y_2 Y_2^\dagger) + (Y_2 Y_1^\dagger + Y_1 Y_2^\dagger) \\ (Y_1^\dagger Y_1 + Y_2^\dagger Y_2) - (Y_2^\dagger Y_1 + Y_1^\dagger Y_2) &= (Y_1 Y_1^\dagger + Y_2 Y_2^\dagger) - (Y_2 Y_1^\dagger + Y_1 Y_2^\dagger). \end{aligned}$$

Taking sums and differences of these two equations yields

$$Y_1^\dagger Y_1 + Y_2^\dagger Y_2 = Y_1 Y_1^\dagger + Y_2 Y_2^\dagger \quad \text{and} \quad Y_2^\dagger Y_1 + Y_1^\dagger Y_2 = Y_2 Y_1^\dagger + Y_1 Y_2^\dagger,$$

from which the claim is immediate. □

Now, it is not hard to see that two invariant subspaces of $B(k_1, k_2)$ in $\mathbb{C}^8$ are given by

$$V_+ = (1, 1)^{\mathrm{T}} \otimes \mathbb{C}^4 \quad \text{and} \quad V_- = (1, -1)^{\mathrm{T}} \otimes \mathbb{C}^4.$$

One can then choose a basis of $\mathbb{C}^4$, say the canonical basis $\{e_i\}$, and consider the unitary matrix $S$ composed of column vectors of the form $\frac{1}{\sqrt{2}}(1, 1)^{\mathrm{T}} \otimes e_i$, $\frac{1}{\sqrt{2}}(1, -1)^{\mathrm{T}} \otimes e_i$. This puts $B(k_1, k_2)$ into block diagonal form given by

$$B(k_1, k_2) \cong \left( \begin{array}{c|c} B_1(k_1, k_2) & \boldsymbol{O} \\ \hline \boldsymbol{O} & B_2(k_1, k_2) \end{array} \right),$$

where $B_1, B_2$ correspond to the restriction to the even and odd sectors, respectively.

**Lemma 5.2.12.** *$B_1(0,0)$ has eigenvalues 4 and 0 (the latter with multiplicity 3), while $B_2(0,0)$ has four eigenvalues of modulus 2.*

*Proof.* Choose $v_+ \in V_+$ and assume that it is an eigenvector of $B_1(0,0)$. Then, the corresponding eigenvalues are exactly those of $Y_1 + Y_2$. Since $\mathrm{rank}(Y_1 + Y_2) = 1$, we know that we get 0 of multiplicity 3 as an eigenvalue and 4 is the PF eigenvalue. If we do the same for a vector $v_- \in V_-$, we see that the eigenvalues of $B_2$ must correspond to the eigenvalues of the defining Hadamard matrix $H = Y_1 - Y_2$. Since $H$ is Hadamard, it satisfies $HH^{\mathrm{T}} = H^{\mathrm{T}}H = 4\mathbb{I}_4$, hence all eigenvalues are of modulus 2. In fact, in this case, we have the eigenvalues $\pm 2$, both with multiplicity two [YH82]. In other words, $B_2(0,0) = 2U$, where $U$ is unitary. $\square$

We now collect all of these observations into the following result.

**Proposition 5.2.13.** *Let $k = (k_1, k_2) \in \mathbb{R}^2$ and $B(k)$ be the Fourier matrix from Eq. (5.5). Let $Q = Q^{\mathrm{T}} = \mathrm{diag}(2,2)$ be the inflation map and let the associated cocycle be*

$$B^{(n)} = B(k)B(k.Q)\cdots B(k.Q^{n-1}).$$

*Then, there exists a similarity transformation that decomposes $B^{(n)}$ into*

$$B^{(n)}(k) \cong \left(\begin{array}{c|c} B_1^{(n)}(k) & \boldsymbol{O} \\ \hline \boldsymbol{O} & B_2^{(n)}(k) \end{array}\right),$$

*where $B_2^{(n)} = B_2(k) \cdot \ldots \cdot B_2(k.Q^{n-1}) = 2^n U^{(n)}(x,y)$, with $U^{(n)}$ being unitary for all $n \in \mathbb{N}$ and all $k \in \mathbb{R}^2$.*

*Proof.* The unitarity of $U$ (and hence of $U^{(n)}$) follows from the invariance of the singular values from Lemma 5.2.9, the normality of the substitution matrix in Proposition 5.2.11, and the explicit computations of the eigenvalues in the odd sector in Lemma 5.2.12. $\square$

Since the cocycle $B_2^{(n)}$ can be expressed as a product of $2^n$ and a unitary matrix $U^{(n)}(x,y)$ for any $(k_1, k_2)$, we get the following consequence.

**Corollary 5.2.14.** *The cocycle $B_2^{(n)}$ has a degenerate Lyapunov spectrum, with a single growth exponent equal to $\chi^{B_2} = \log(2)$.* $\square$

For a generic higher-dimensional substitution having the same bar swap symmetry, the results we have here about the invariance of the singular values, normality of $M_\varrho$, and the eigenvalues of $B_2$ at $k = (0, \ldots, 0)$ being of modulus $\sqrt{\lambda_{\mathrm{PF}}}$, could all be confirmed using exactly the same line of reasoning. This leads us to the following general result.

**Theorem 5.2.15.** *Let $\varrho$ be a substitution with bar swap symmetry over $4d$ letters in $\mathbb{Z}^r$ constructed from a Hadamard matrix of size $2d$, with volume inflation factor $\lambda = \ell_1 \cdots \ell_r = 2d$. Then, its Fourier matrix $B(\cdot)$ has a block diagonal form, where the second block which acts on the odd sector $V_- = (1,-1)^T \otimes \mathbb{C}^{2d}$ is a product of a unitary matrix with $\sqrt{\lambda}$. Furthermore, the cocycle induced by this block has degenerate Lyapunov spectrum given by $\chi = \log\sqrt{\lambda}$.* $\square$

## 5.3. Non-Pisot example: The Godrèche–Lançon–Billard tiling

In this section, we investigate the spectral type of the diffraction of a two-dimensional tiling with non-Pisot inflation. The supertiles first appeared in a work by Lançon and Billard [LB88] as a decoration rule on Penrose tilings generated by thick and thin Penrose rhombuses, whose largest internal angles are $\frac{3\pi}{5}$ and $\frac{4\pi}{5}$, respectively. They called such a tiling "binary" because its vertex set can be decorated by two types of atoms (big and small) in such a way that the resulting atomic packing is considerably dense in $\mathbb{R}^2$.

It was in the paper [GL92] by Godrèche and Lançon where the said decoration rule was fully realised as an inflation rule, which generates binary tilings that are spectrally different from Penrose tilings. For instance, no such tiling can have non-trivial Bragg peaks in its diffraction, following arguments by Solomyak [Sol97] for inflation tilings because it has a non-Pisot inflation multiplier, and invoking Dworkin's argument [Dwo93], whereas Penrose tilings are known to have pure point spectra.

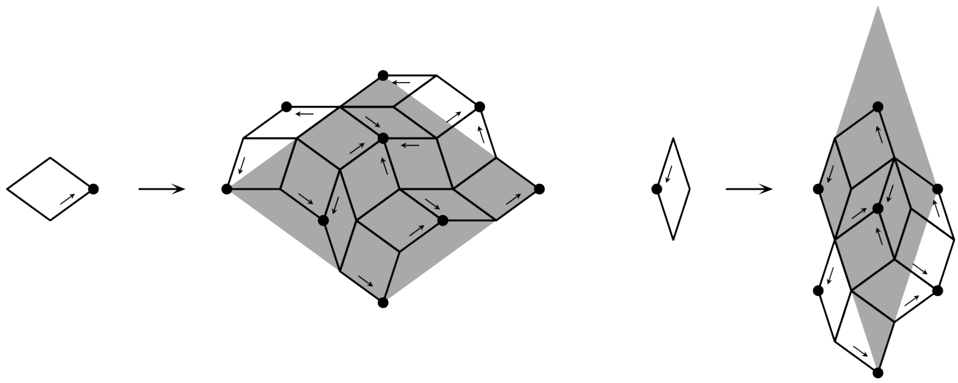


Figure 5.6.: The inflation rule $\varrho_{\text{GLB}}$ for the Godrèche–Lançon–Billard tiling, with the control points for the tiles.

We consider a chiral version $\varrho_{\text{GLB}}$ of the original inflation rule in [GL92], meaning we decorate each tile with an identifying arrow to specify its orientation, but the reflected versions of the chosen prototiles do not appear in the tiling; see [BG13, Sec. 6.5] for a detailed survey on this tiling. From here onwards, we refer to a tiling generated by such a rule a *GLB tiling*. The expansion map $Q$ is given by $Q = \lambda^2\mathbb{I}$, where

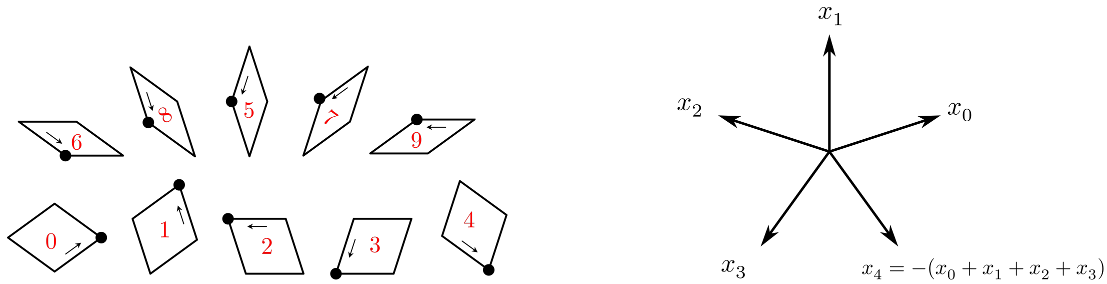
$$\lambda^2 = 4\cos^2\left(\frac{\pi}{10}\right) = \frac{1}{2}(5+\sqrt{5}) \approx 3.618 \tag{5.7}$$

is a non-Pisot number, with minimal polynomial $p_{\lambda^2}(z) = z^2 - 5z + 5$. It must also be noted that $\varrho_{\text{GLB}}$ is not a stone inflation, but a reformulation as such, using tiles with fractal boundaries, is given in [Fra08, BG13]. Nevertheless, the renormalisation scheme still holds since we only have finitely many prototiles. Here, we show that continuous component of the diffraction of any GLB tiling, which is given by $\widehat{\gamma} - I_0\delta_0$, is purely singular continuous.

Let $\mathcal{P} = \{\mathfrak{t}_0, \mathfrak{t}_1, \ldots, \mathfrak{t}_9\}$ be the ordered prototile set of $\varrho_{\text{GLB}}$. We choose the control point of a tile $\mathfrak{t}$ to be the vertex to which its decorating arrow points to. The tiles constituting $\mathcal{P}$, together with their associated control points, are provided in Figure 5.7. The prototiles $\mathfrak{t}_i$ with $0 \leqslant i \leqslant 4$ are the five rotated copies of the thick rhombus, and those with $5 \leqslant i \leqslant 9$ pertain to the thin rhombuses. Note that the prototiles are numbered in such a way that a rotation by $\frac{2\pi}{5}$ cyclically sends a tile to the next prototile of the same geometry (thick or thin), i.e.,

$$R_{\frac{2\pi}{5}} \mathsf{t}_i = \begin{cases} \mathsf{t}_{i+1(\mathrm{mod}5)}, & 0 \leqslant i \leqslant 4, \\ \mathsf{t}_{i+1(\mathrm{mod}5)+5}, & 5 \leqslant i \leqslant 9, \end{cases} \tag{5.8}$$

where $R_\theta$ corresponds to the geometric rotation by an angle $\theta$ in $\mathbb{R}^2$.




(a) Markers and labels for the prototiles

(b) Generating vectors for the set of control points of a GLB tiling

Figure 5.7.: The prototile set for $\varrho_{\mathrm{GLB}}$, together with the vectors that generate the associated set of control points.

**Remark 5.3.1.** The original inflation rule given in [BG13] includes the rotations of each prototile by an angle $\pi$ in its prototile set $\mathcal{P}$, which doubles the size of $\mathcal{P}$ to 20. The rule given in Figure 5.6 is thus chosen to be the square of the original to eliminate these additional tiles. The resulting analysis under the diffraction program remains unaffected as different powers of the same rule yield the same tiling dynamical system $(\mathbb{Y}, \mathbb{R}^2)$. ◊

To proceed with the analysis via Lyapunov exponents, one needs the location in $\mathbb{R}^2$ of the tile markers, which could be cumbersome if one insists with the usual Cartesian coordinates. Instead, we take advantage of the five-fold symmetry of a GLB tiling and work with a different set of generators.

**Fact 5.3.2.** *The corresponding vertex set $\varLambda$ of a GLB tiling is a subset of $\mathbb{Z}[\xi_5]$, where one has $\xi_5 = \mathrm{e}^{\frac{2\pi \mathrm{i}}{5}} \cong (\cos(\frac{2\pi}{5}), \sin(\frac{2\pi}{5}))$.* □

The set of control points $\varLambda'$ also lies in a submodule of $\mathbb{Z}[\xi_5]$ since it is contained in the vertex set. This submodule also contains the set of control points of the level-1 supertiles. For the next result, $\lambda = \sqrt{\frac{1}{2}(5+\sqrt{5})} \approx 1.902$ and the vectors $x_i$ are given in Figure 5.7.

**Fact 5.3.3.** *The set $\varLambda'$ of control points lies in the submodule $\mathbb{Z}[x_0, x_1, x_2, x_3] \subset \mathbb{Z}[\xi_5]$, where $x_0 = \lambda \, \mathrm{e}^{\mathrm{i}\frac{\pi}{10}}$, and $x_i = \xi_5^i x_0$, for $0 \leqslant i \leqslant 3$.* □

The corresponding positions of control points of tiles for $\varrho_{\mathrm{GLB}}(\mathsf{t}_0)$ and $\varrho_{\mathrm{GLB}}(\mathsf{t}_5)$ are given in Figure 5.8. The weighted Dirac comb given by $\omega = \sum_{x \in \varLambda'} W(x) \delta_x$ is then constructed by assigning weights $W(x)$ to each point in $\varLambda'$, namely $\frac{2}{5}$ and $\frac{1}{5}$ for points occupied by thin and thick rhombuses, respectively. Such a weight assignment is employed to ensure that the total weight at every control point adds up to 1.

From Figure 5.8, one can calculate the displacement sets $T_{n0}$ and $T_{n5}$ in terms of the basis vectors $\{x_j\}_{0\leqslant j\leqslant 3}$, which are given by

$$\begin{aligned}
&T_{00}=\{0,-x_0,\,x_0+x_1+x_2+x_3,\,x_1+x_2+x_3\} && T_{05}=\{-x_0-2x_1-x_2-2x_3\}\\
&T_{10}=\{x_1+x_2+x_3,\,x_0+x_1+x_2+x_3\} && T_{15}=\{-x_3,-x_1-x_3\}\\
&T_{20}=\{x_1+x_2+x_3\} && T_{25}=\varnothing\\
&T_{30}=\{x_1+2x_2+2x_3\} && T_{35}=\varnothing\\
&T_{40}=\{x_1+x_2+2x_3,\,-x_0+x_3\} && T_{45}=\{-x_0-2x_1-x_2-x_3,\,-x_0-2x_1-x_2-2x_3\}\\
&T_{50}=\{x_1+x_2+2x_3\} && T_{55}=\{0,-x_1,-x_1-x_3\}\\
&T_{60}=\{-x_0,\,x_1+x_2+x_3\} && T_{65}=\varnothing\\
&T_{70}=\varnothing && T_{75}=\{-x_1-x_3\}\\
&T_{80}=\varnothing && T_{85}=\{-x_0-x_1-x_2-2x_3\}\\
&T_{90}=\{x_1+2x_2+x_3,\,x_0+x_1+2x_2+x_3\} && T_{95}=\varnothing.
\end{aligned} \tag{5.9}$$

Define the map $\mathfrak{L}: S_T=\bigcup T_{ij}\to\mathbb{Z}^4$ via $\mathfrak{L}:\sum_{\ell=0}^{3}a_\ell x_\ell\mapsto(a_0,a_1,a_2,a_3)$, which encodes elements of $T_{ij}$ as row vectors in terms of the basis $\{x_\ell\}$. Let $\mathfrak{L}(T_{ij})=\widetilde{T}_{ij}$. From the symmetry relations satisfied by the prototiles given in Eq. (5.8), one can derive the sets $\widetilde{T}_{ij}$ from the fundamental displacement sets $\widetilde{T}_{n0}$ and $\widetilde{T}_{n5}$ as follows.

**Lemma 5.3.4.** *Any set $\widetilde{T}_{ij}$ is related to a unique element of the collection $\left\{\widetilde{T}_{n0},\widetilde{T}_{n5}\right\}$, with $0\leqslant n\leqslant 9$, via*

$$\widetilde{T}_{ij}=\begin{cases}\widetilde{T}_{i-j(\mathrm{mod}5),5\lfloor\frac{j}{5}\rfloor}.\Theta^j, & i\leqslant 4,\\ \widetilde{T}_{i-j+5(\mathrm{mod}10),5\lfloor\frac{j}{5}\rfloor}.\Theta^j, & i>4,\ 0\leqslant(i-j)<5,\\ \widetilde{T}_{i-j(\mathrm{mod}10),5\lfloor\frac{j}{5}\rfloor}.\Theta^j, & \textit{otherwise},\end{cases} \tag{5.10}$$

*where $\lfloor m\rfloor$ is the floor of $m$ and the matrix*

$$\Theta=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-1&-1&-1&-1\end{pmatrix}$$

*represents rotation by $\frac{2\pi}{5}$.* □

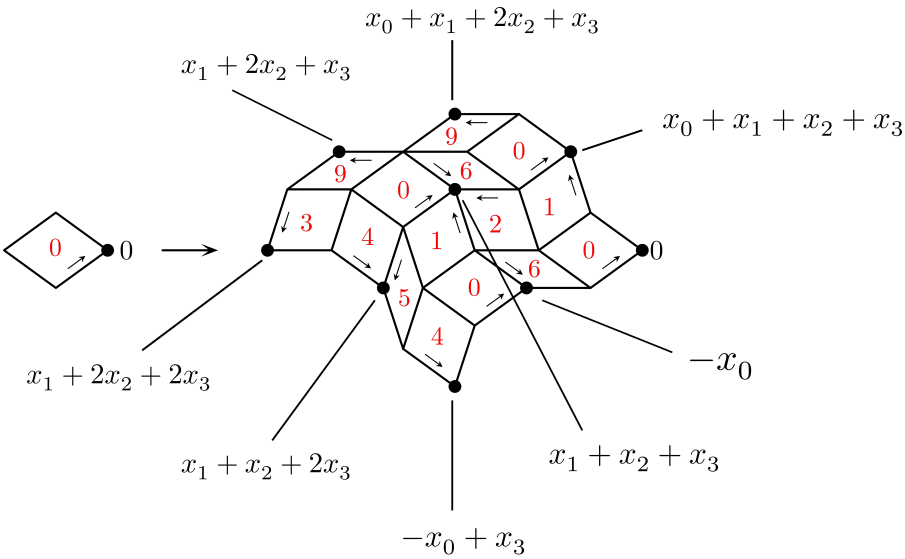
Now, from the sets $T_{ij}$, one has $B(k)$ via $B_{ij}(k)=\sum_{t\in T_{ij}}\mathrm{e}^{2\pi\mathrm{i}t\cdot k}$, where $k=(k_1,k_2)\in\mathbb{R}^2$. Define the matrix-valued function $\widetilde{B}$ on $\mathbb{T}^4$ via

$$\widetilde{B}_{ij}(\tilde{z}):=\sum_{\tilde{t}\in\widetilde{T}_{ij}}\mathrm{e}^{2\pi\mathrm{i}\tilde{t}\cdot\tilde{z}},$$

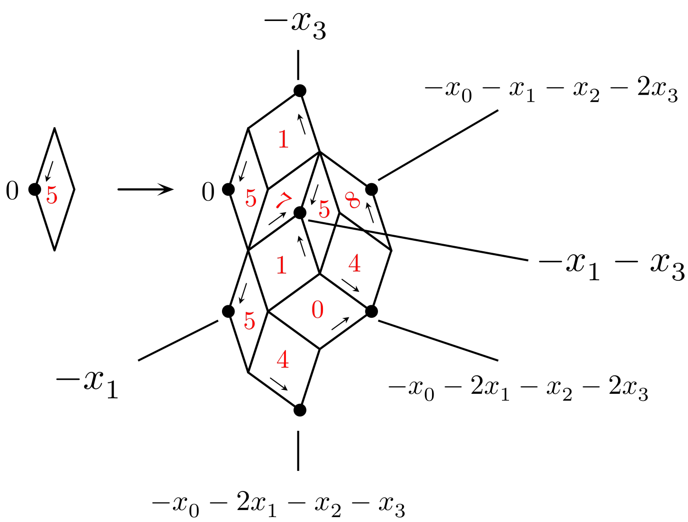
with $\tilde{z}=(\tilde{z}_0,\ldots,\tilde{z}_3)\in\mathbb{T}^4$. One then has the following equality.

**Lemma 5.3.5.** *For any $k\in\mathbb{R}^2$, there exists a $\tilde{z}\in\mathbb{T}^4$ with*

$$B(k)=\widetilde{B}(\tilde{z})|_{\tilde{z}=(\alpha_0,\ldots,\alpha_3)}.$$




(a) The inflation rule for the thick rhombus $\mathfrak{t}_0$, with the position of markers in $\mathbb{R}^2$ in the level-1 supertile.




(b) The inflation rule for the thin rhombus $\mathfrak{t}_5$, with the position of markers in $\mathbb{R}^2$ in the level-1 supertile.

Figure 5.8.: Level-1 supertiles of the GLB tiling

*Proof.* For a fixed $k \in \mathbb{R}^2$, one has

$$B_{ij}(k) = \sum_{t \in T_{ij}} \mathrm{e}^{2\pi \mathrm{i} t \cdot k} = \sum_{t \in T_{ij}} \mathrm{e}^{2\pi \mathrm{i} \sum_{\ell=0}^{3} a_\ell^{(t)} x_\ell \cdot k},$$

where $\mathfrak{L}(t) = \left(a_0^{(t)}, \ldots, a_3^{(t)}\right)$. Let $\mathcal{L} := \mathbb{Z}^4$ and $\{y_\ell\}_{0 \leqslant \ell \leqslant 3}$ be the standard lattice basis for $\mathcal{L}$. Let $\mathcal{L}^* \simeq \mathbb{Z}^4$ be the dual lattice for $\mathcal{L}$, with basis $\{y_\ell^*\}_{0 \leqslant \ell \leqslant 3}$. Since $\operatorname{span}_{\mathbb{R}} \mathcal{L}^* = \mathbb{R}^4$, for every $z \in \mathbb{R}^4$, there exist $\alpha_\ell \in \mathbb{R}$ such that $z = \sum_{\ell=0}^{3} \alpha_\ell y_\ell^*$.

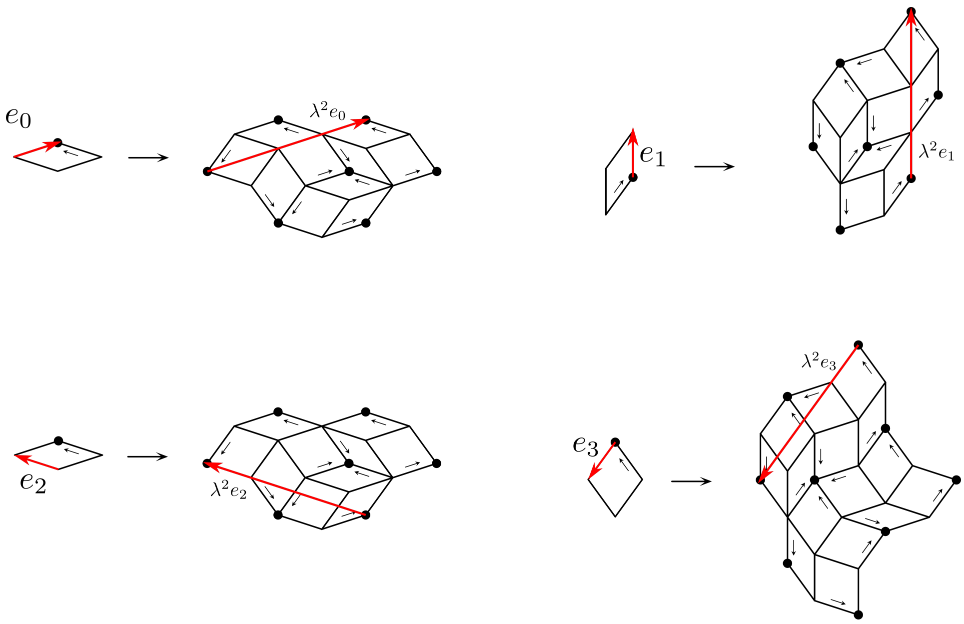



Figure 5.9.: Computing the expansion of the unit vectors $e_j := x_j/\|x_j\|$ by $\lambda^2$ via rotated level-1 supertiles of $\varrho_{\mathrm{GLB}}$

For arbitrary $z \in \mathbb{R}^4$ and $\tilde{t} \in \mathfrak{L}(S_T) \subset \mathbb{Z}^4$ one has

$$\langle \tilde{t} \cdot z \rangle = \left\langle \sum_{\ell,r} a_\ell^{(t)} \alpha_r y_\ell \cdot y_r^* \right\rangle = \left\langle \sum_\ell a_\ell^{(t)} \alpha_\ell \right\rangle = \sum_\ell \left\langle a_\ell^{(t)} \langle \alpha_\ell \rangle \right\rangle,$$

where $\langle k \rangle = k \mod 1$. The last equality follows from Lemma 3.1.2 since $a_\ell^{(t)} \in \mathbb{Z}$ for all $t \in S_T$ and $0 \leqslant \ell \leqslant 3$.

Now choosing $\alpha_\ell = \langle x_\ell \cdot k \rangle$, one obtains

$$\widetilde{B}_{ij}(\tilde{z}) = \sum_{\tilde{t} \in \widetilde{T}_{ij}} \mathrm{e}^{2\pi \mathrm{i} \tilde{t} \cdot \tilde{z}} = \sum_{\tilde{t} \in \widetilde{T}_{ij}} \mathrm{e}^{2\pi \mathrm{i} \sum_{\ell=0}^{3} a_\ell^{(t)} \langle x_\ell \cdot k \rangle} = \sum_{t \in T_{ij}} \mathrm{e}^{2\pi \mathrm{i} t \cdot k} = B_{ij}(k),$$

which finishes the proof. □

With this, one gets a representation for $B(k)$ as a section of the matrix-valued function $\widetilde{B}$ on $\mathbb{T}^4$ via

$$B(k) = \widetilde{B}(\tilde{z})\big|_{\tilde{z}_1 = \langle x_0 \cdot k \rangle,\ \tilde{z}_2 = \langle x_1 \cdot k \rangle,\ \tilde{z}_3 = \langle x_2 \cdot k \rangle,\ \tilde{z}_4 = \langle x_3 \cdot k \rangle}.$$

To find the appropriate base dynamics for $\widetilde{B}$, one needs a linear transformation $Q'$ which satisfies

$$(x_0, x_1, x_2, x_3) Q' = \lambda^2 (x_0, x_1, x_2, x_3).$$

To this end, it suffices to consider the unit vectors $e_j := x_j/\|x_j\|$ and express $\lambda^2 e_j$ as an integer linear combination of the vectors $\{e_\ell\}$. One convenient way to compute this is to look at rotated version of the prototiles of $\varrho_{\mathrm{GLB}}$ and see where these unit vectors get map to in the corresponding level-1 supertiles; see Figure 5.9. This process yields

$$Q' = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 0 & 3 & 1 & -1 \\ -1 & 1 & 3 & 0 \\ -1 & 0 & 1 & 2 \end{pmatrix}.$$

Consequently, for the cocycle $B^{(n)}(k)$ under $Q$, one has

$$B^{(n)}(k) = \widetilde{B}^{(n)}(\tilde{z})\big|_{\tilde{z}_1=\langle x_0 \cdot k\rangle,\, \tilde{z}_2=\langle x_1 \cdot k\rangle,\, \tilde{z}_3=\langle x_2 \cdot k\rangle,\, \tilde{z}_4=\langle x_3 \cdot k\rangle},$$

where $\widetilde{B}^{(n)}(\tilde{z}) = \widetilde{B}(\tilde{z})\widetilde{B}^{(n-1)}(\tilde{z}Q')$ for $\tilde{z} \in \mathbb{T}^4$. The new base dynamics is given by $\widetilde{Q'} : \tilde{z} \mapsto \tilde{z}Q'$.

The cocycle $\widetilde{B}(\tilde{z})$ is invertible for a.e. $\tilde{z} \in \mathbb{T}^4$. To see this, one can consider the average

$$\frac{1}{N}\sum_{i=0}^{N-1} \log|\det \widetilde{B}\big(\tilde{z}(Q')^i\big)| \xrightarrow{N\to\infty} \int_{\mathbb{T}^4} \log|\det \widetilde{B}(\tilde{z})|\mathrm{d}\tilde{z},$$

where convergence follows from Theorem 1.4.7. Then, $\widetilde{B}(\tilde{z})$ is only identically singular when this Birkhoff average is $-\infty$. One can numerically check that this integral is indeed finite. In fact, when it is not $-\infty$, it is equal to $\mathfrak{m}(q)$, for some polynomial $q \in \mathbb{Z}[z_1, z_2, z_3, z_4]$, and hence is non-negative, as in our case here. Due to boundedness properties of $\widetilde{B}$, one can employ the same estimates as in Lemma 2.7.1, i.e., for a.e. $k \in \mathbb{R}^2$ and for all $N \in \mathbb{N}$,

$$2\chi^B(k) \leqslant \frac{1}{N}\mathbb{M}(\log\|\widetilde{B}^{(N)}(.)\|_{\mathrm{F}}^2).$$

As in the one-dimensional examples, this upper bound can be calculated numerically up to a certain level of precision. To rule out the presence of $\widehat{\gamma}_{\mathsf{ac}}$, it suffices to find an $N$ for which the upper bound given in terms of the mean is strictly less than

$$\log(\det Q) = 4\log(\lambda) \approx 2.571862.$$

The relevant numerical values are given in Table 5.1.

| $N$ | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|
| $\frac{1}{N}\mathbb{M}\big(\log\|B^{(N)}(.)\|_{\mathrm{F}}^2\big)$ | 2.571 | 2.517 | 2.474 | 2.440 | 2.411 | 2.387 | 2.367 |

Table 5.1.: Numerical upper bounds for $\chi^B$ for the GLB inflation $\varrho_{\mathrm{GLB}}$. The numerical error is less than 0.005 in all cases listed.

**Proposition 5.3.6.** *For a.e. $k \in \mathbb{R}^2$, the Lyapunov exponents associated to $B(k)$ for the GLB tiling are all strictly positive.* □

Theorem 5.1.5 implies the next result.

**Corollary 5.3.7.** *Given a GLB tiling, with set of control points $\varLambda'$, the measure $\omega = \sum_{x\in\varLambda'} W(x)\delta_x$ with the defined weights $W(x)$ has a singular diffraction with a trivial Bragg peak at zero, but is otherwise singular continuous.* □

More specifically, one has

$$\widehat{\gamma_\omega} = I_0\delta_0 + (\widehat{\gamma_\omega})_{\mathsf{sc}} \quad \text{with} \quad I_0 = \left(\frac{5-\sqrt{5}}{10}\operatorname{dens}(\varLambda)\right)^2,$$

where $\operatorname{dens}(\varLambda) = \frac{1+\sqrt{5}}{5}\lambda \approx 1.231$, under the assumption that we work with rhombuses of unit edge length.

For any GLB tiling, the set of control points $\varLambda'$ and the corresponding vertex set $\varLambda$ are MLD, which means their corresponding diffractions are of the same spectral type. Thus, our analysis via Lyapunov exponents also extends to $\varLambda$ and is given by the following result.

**Corollary 5.3.8.** *Consider the uniform Dirac comb $\delta_\varLambda$, where $\varLambda$ is the vertex set of a GLB tiling. Then, its corresponding diffraction is singular and is given by*

$$\widehat{\gamma} = \operatorname{dens}(\varLambda)^2\delta_0 + (\widehat{\gamma})_{\mathsf{sc}},$$

*where* $\operatorname{dens}(\varLambda) = \frac{1+\sqrt{5}}{5}\lambda$. □

**Remark 5.3.9.** The choice of the generating basis for the control points is not unique. One can also work directly with the fifth roots of unity $\left\{\xi_5^j\right\}$ instead of their rotated and inflated versions which we used in this work. Aside from this, one can also use the vector

$$y = \left(\tfrac{1}{2}k_1,\, \tfrac{1}{2}\lambda k_1,\, \tfrac{1}{2}\lambda^2 k_2,\, \tfrac{1}{2}\lambda^3 k_2\right),$$

which satisfies

$$(1\cdot k,\, \xi_5\cdot k,\, \xi_5^2\cdot k,\, \xi_5^3\cdot k) = yP, \qquad \text{with} \quad P = \begin{pmatrix} 2 & -3 & 2 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

for a different coordinatisation of the control points. For this choice, the base dynamics is given by the matrix $Q'' = \mathfrak{C}_{\lambda^2}^2 \otimes \mathbb{I}_2$, where $\mathfrak{C}_{\lambda^2}$ is the companion matrix of $p_{\lambda^2}(z)$. ♢

# 6. Summary and Outlook

In this work, we systematically established a renormalisation scheme satisfied by the pair correlation functions $\nu_{ij}(z)$ for primitive inflation systems, which we extended to the pair correlation measures $\varUpsilon_{ij}$ and to their Fourier transforms $\widehat{\varUpsilon_{ij}}$. Since $\widehat{\varUpsilon_{ij}}$ fully determines the diffraction $\widehat{\gamma}$, this scheme enables one to test properties of $\widehat{\gamma}$ in accordance with the scaling it satisfies. In particular, we rigorously carried out the implications for the Radon–Nikodym density $h(k)$ of the absolutely continuous component $(\widehat{\gamma})_{\mathsf{ac}}$.

Incorporating the translation boundedness of $(\widehat{\gamma})_{\mathsf{ac}}$ to the picture, one discovers that some growth behaviour under derived scaling prohibits its existence. This growth behaviour is encoded in the corresponding minimal Lyapunov exponent $\chi_{\min}$ of an analytic matrix cocycle $B^{(n)}(k)$, which can be directly constructed given $\varrho$. Synthesised together, one obtains a sufficient criterion for the absence of $(\widehat{\gamma})_{\mathsf{ac}}$, given in Theorem 2.5.3, which can numerically be confirmed for any primitive example satisfying some mild conditions. Moreover, a necessary condition for its existence for general primitive inflations complements that of [Bar16, BS17] in the constant-length case, and provides an entirely new one for the non-constant-length case.

The last three chapters delved into several classes of examples and provided proofs of some existence and positivity results related to the exponent $\chi_{\min}$. Moreover, using our program we confirmed that all known deterministic substitutive examples with nontrivial $(\widehat{\gamma})_{\mathsf{ac}}$ satisfy the necessary condition in Corollary 2.7.10. This leads to an open question which asks whether there exists a substitution $\varrho$ with $\chi_{\min} = 0$ but has $(\widehat{\gamma})_{\mathsf{ac}} \neq 0$.

The analysis of the Radon–Nikodym density $h(k)$ we presented here is rigorous, but is not robust enough to encompass all possible scenarios. Translation boundedness is the root of all the arguments derived from Lyapunov exponents, but this only rules out candidates for densities. Roughly speaking, if all *possible* constituent vectors grow exponentially under $B^{(n)}(k)$, there is no other way but for the *actual* vectors making up $h(k)$ to be zero. However, our method does not prescribe a structure of $h(k)$ if $\chi_{\min} = 0$. The difficulty lies in the fact that we know almost nothing about $h(k)$, expect that it is locally-$L^1$ and has dense support when $\varLambda$ is Meyer. A closer look at $h(k)$ from scratch would be the logical next step.

Our characterisation of inflation systems using their Lyapunov spectra is still far from being complete. For one, aside from numerical estimates, we still do not have a general method to compute the actual exponents when the IDA $\mathcal{B}$ is irreducible. The non-negativity result of $\chi_{\min}$ in Theorem 2.7.9 is quite tempting, as one might hope that pointing out *when* it is zero is an easy task. Unfortunately, this is not the case. From the submultiplicativity of matrix norms, it is intuitive that, under some invertibility assumptions, $N^{-1} \log \| B(k) B(\lambda k) \dots B(\lambda^{N-1} k) \|$ has to be monotonically decreasing in $N$. But whether or why it crosses the threshold value $\log(\sqrt{\lambda})$ in general is totally unclear. A general positivity result is not far-fetched though, as we know almost everywhere constancy results like in the Pisot and the integer multiplier case. We hope to find more sophisticated ways of bounding the exponents, which might lead to proving that irreducible Pisot substitutions do not possess absolutely continuous diffraction—something that might bring us closer to a proof of the Pisot substitution conjecture.

Another possible topic for future work would be a deeper, more involved investigation of the relation between the diffraction and the dynamical spectra. In recent works [BS18a, BS18b] by Bufetov and Solomyak, they deal with the appropriate generalisation of the cocycle $B^{(n)}(k)$ for flows associated to $S$-adic systems, where substitutions are a special subclass. There, they presented a dynamical analogue of the singularity result Theorem 2.5.3 for spectral measures $\sigma_f$ and the flow they generate. We confirm this dynamical singularity result, for constant roof functions, for the binary and the Abelian bijective case using known connections between the diffraction and the dynamical section briefly stated in Appendix A. One might hope that this route is not limited to the constant-length case, and a general equivalence of $\sigma_{\max}$ and some linear combination of spectral measures of lookup functions $\mathbb{1}_{\varrho^n(a)}$ would be desirable.

A remarkable feature of the lifted cocycles $\widetilde{B}^{(n)}(x)$ on $\mathbb{T}^d$ is that the base dynamics (whether $\widetilde{M}$ or $\widetilde{\mathfrak{C}}$) is usually an ergodic toral endomorphism. In [Bac17, KS18], it was proved that for certain class of cocycles with hyperbolic base dynamics, the Lyapunov exponents can be approximated by exponents on periodic orbits. It would be interesting whether these results and known results on periodic points of toral endomorphisms can shed light on the values of the exponents and the approximative behaviour they satisfy.

Beyond the realm of Lyapunov exponents, there are also interesting questions pertaining to diffraction of aperiodic structures in general. In particular, MLD equivalences allow one to construct patches around specific points of a tiling based on sufficient information about patches of another tiling. This construction is local and extends continuously to the hull $\mathbb{Y}$. It would be interesting to know how the pair correlations $\nu_{ij}(z)$ behave under these local derivations. This might be the key to a rigorous description of the effect of local derivations on the spectral components of $\widehat{\gamma}$.

# A. Dynamical Spectrum

This brief exposition on spectral theory of dynamical systems is mainly derived from [Que10] and [Bar16], and the results on the relation of the diffraction and the dynamical spectrum are taken from [BL04] and [BLvE15]. Let $\varrho$ be a one-dimensional primitive substitution $\varrho$, with symbolic hull $\mathbb{X}$. The corresponding substitution dynamical system is given by $(\mathbb{X}, \mathbb{Z}, \mu)$, where the $\mathbb{Z}$-action is induced by the shift map $S : (Sw)_i = w_{i+1}$, for all $w \in \mathbb{X}$. The measure $\mu$ is usually chosen to be the word frequency measure on cylinder sets, which is $S$-invariant, making $(\mathbb{X}, \mathbb{Z}, \mu)$ a measure-theoretic dynamical system (MTDS). With $(\mathbb{X}, \mathbb{Z}, \mu)$ comes the Hilbert space $\mathcal{H} = L^2(\mathbb{X}, \mu)$ with the standard inner product $\langle f|g\rangle := \int_{\mathbb{X}} \overline{f(x)} g(x) \,\mathrm{d}\mu(x)$.

To the shift $S$, one associates the the unitary Koopman operator $U_S : \mathcal{H} \to \mathcal{H}$ defined as

$$f \mapsto U_S f \qquad \text{with} \qquad (U_S f)(x) = f(Sx).$$

**Definition A.0.1.** The *dynamical spectrum* of $(\mathbb{X}, \mathbb{Z}, \mu)$ is the spectrum of $U_S$.

An eigenvalue of $U_S$ is a number $\alpha_i \in \mathbb{C}$ satisfying $U_S f_i = \alpha_i f_i$, for some $f \in L^2(\mathbb{X}, \mu)$. From the unitarity of $U_S$, one has $\alpha_i = \mathrm{e}^{2\pi\mathrm{i}\lambda_i}$, for some $\lambda_i \in \mathbb{R}$, which allows one to see the set of eigenvalues as a subset of $\mathbb{R}$. There exists a subspace $\mathcal{H}_{\mathsf{pp}} = \langle f_1, f_2, \ldots\rangle_{\mathbb{C}} \subset \mathcal{H}$ that is spanned by the eigenfunctions $f_i$. When $f_i$ is continuous, one calls $\alpha_i$ a *topological eigenvalue*. The topological point spectrum is the set of all topological eigenvalues. In the setting chosen above, all eigenvalues are topological.

The MTDS $(\mathbb{X}, \mathbb{Z}, \mu)$ is said to have *pure point dynamical spectrum* when $\mathcal{H}_{\mathsf{pp}} = \mathcal{H}$. When one has this, it is well known that the corresponding geometric hull $\mathbb{Y}$ also has pure point diffraction.

**Theorem A.0.2.** [BL04, Thm. 9] *If $(\mathbb{X}, \mathbb{Z}, \mu)$ is pure point, then the set of eigenvalues* $\mathrm{spec}(U_S)$ *is given by the additive subgroup of $\widehat{\mathbb{Z}} = \mathbb{T}$ generated by the position of the Bragg peaks, i.e.,*

$$\mathrm{spec}(U_S) = L^{\circledast},$$

*where $L^{\circledast} = \big\langle k \in \mathbb{T} \mid \widehat{\gamma}(\{k\}) > 0 \big\rangle$ is the Fourier module.* □

The previous result holds for more general systems with $\mathbb{R}$-actions, where the group is then seen as a subset of $\widehat{\mathbb{R}} = \mathbb{R}$.

For every function $f \in \mathcal{H}$ one can define its *spectral measure* $\sigma_f$ to be the measure induced by the inner product

$$\big\langle f \mid U_S^n f \big\rangle = \int_0^1 \mathrm{e}^{2\pi\mathrm{i}nu} \mathrm{d}\sigma_f(u).$$

In particular, for an eigenfunction $f_i$, one has $\sigma_{f_i} = \delta_{\alpha_i}$. For any nonzero $g \in \mathcal{H}_{\mathsf{pp}}^{\perp}$, $\sigma_g$ is a continuous measure.

A measure $\mu_1$ is said to be *absolutely continuous* with respect to another measure $\mu_2$, which we denote by $\mu_1 \ll \mu_2$, if the null sets of $|\mu_2|$ are also null sets of $|\mu_1|$. Two measures are *equivalent*, i.e., $\mu_1 \sim \mu_2$, if $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$.

**Fact A.0.3** ( [Que10, Thm. 2.3]). *There exists a function* $f_0$ *such that for all* $f \in \mathcal{H}$, *one has* $\sigma_f \ll \sigma_{f_0} := \sigma_{\max}$. *This spectral measure is known as the* spectral measure of maximal type. *The spectral type of* $(\mathbb{X}, \mathbb{Z}, \mu)$ *is the spectral type of the measure* $\sigma_{\max}$. □

**Theorem A.0.4** ( [Bar16, Thm. 4.4], [Que10, Prop. 7.2]). *Let* $\varrho$ *be a primitive aperiodic substitution of constant length. Then, one has the following equivalence*

$$\sigma_{\max} \sim \sum_{a \in \mathcal{A}} \sum_{n \geqslant 1} 2^{-n} \sigma_{\mathbb{1}_{\varrho^n(a)}},$$

*where* $\mathbb{1}_{\varrho^n(a)}$ *is the indicator function of the inflated word* $\varrho^n(a)$ *at 0.* □

Let $\gamma_{\mathbb{X}} = \sum_{m \in \mathbb{Z}} \eta_{\mathbb{X}}(m) \delta_m$ be the autocorrelation of $\mathbb{X}$. It follows from standard arguments that the diffraction measure $\widehat{\gamma}$ of $\mathbb{X}$ generated by a constant-length substitution is $\mathbb{Z}-$periodic in $\mathbb{R}$ [BF75], which allows it to be written as

$$\widehat{\gamma} = \theta_{\mathbb{X}} * \delta_{\mathbb{Z}}.$$

where $\theta_{\mathbb{X}}$ is a measure on $[0,1) \cong \mathbb{T}$, and is called the fundamental diffraction of $\mathbb{X}$. By Bochner's theorem, it is related to $\eta_{\mathbb{X}}$ via

$$\eta_{\mathbb{X}}(m) = \int_0^1 \mathrm{e}^{2\pi i m u} \mathrm{d}\theta_{\mathbb{X}}(u).$$

**Proposition A.0.5** ( [BLvE15, Prop. 2]). *Let* $\mathbb{X}$ *be a uniquely ergodic subshift over a finite alphabet* $\mathcal{A}$. *Let* $W \subset \mathbb{C}$ *be finite and* $g : \mathbb{X} \to W$ *continuous, with spectral measure* $\sigma_g$ *and let* $\mathbb{W}$ *denote the subshift factor. Then, the fundamental diffraction of* $\mathbb{W}$ *satisfies* $\theta_{\mathbb{W}} = \sigma_g$. □

In particular, one has that

$$\sigma_{\mathbb{1}_{\varrho^n(a)}} = \theta_{\mathbb{W}}$$

for some factor $\mathbb{W} \subset \{0,1\}^{\mathbb{Z}}$. Note that the indicator functions on other positions can be realised as a shifted version of that at 0, which allows one, via introduction of appropriate weights for each letter, to construct the full fundamental diffraction $\theta_{\mathbb{X}}$. Since mere shifting does not alter the spectral type, we note that

$$\sigma_{\mathbb{1}_a} \ll \theta_{\mathbb{X}}$$

for all $a \in \mathcal{A}$. Now, when one goes to the indicator functions of supertiles, one gets nothing but an inflated version of $\theta_{\mathbb{X}}$, which gives

$$\sigma_{\mathbb{1}_{\varrho^n(a)}} \ll \widetilde{L}^n . \theta_{\mathbb{X}} \ll \theta_{\mathbb{X}}$$

where $\widetilde{L} : x \mapsto Lx$. These equivalences also hold for arbitrary translates of $\sigma_{\mathbb{1}_{\varrho^n(a)}}$.

**Proposition A.0.6.** *The maximal spectral type of* $(\mathbb{X}, \mathbb{Z}, \mu)$ *is a subset of the spectral types retrievable from the diffraction* $\widehat{\gamma}$. □

# B. Furstenberg's Representation

## B.1. Random cocycles

**Theorem B.1.1** ( [FK83, Thm. 3.5]). *Let $J = \{M_1, \dots, M_r\} \subset \mathrm{GL}(d, \mathbb{R})$ satisfy*

**(1)** $\|M_i\|, \|M_i^{-1}\| < \infty$, *for all* $1 \leqslant i \leqslant r$ *and*

**(2)** $J$ *is strongly irreducible.*

*Consider the random cocycle $M^{(n)}$ on $J^{\mathbb{N}_0}$ given by $M^{(n)} = M_{i_{n-1}} \cdots M_{i_1} M_{i_0}$ where $M_{i_r} \in J$ for all $i_r$. Then, for a.e. sequence of matrices and for all starting directions $\bar{v} \in \mathbb{RP}^{d-1}$, the Lyapunov exponent given by $\chi(v) = \lim_{n\to\infty} \|M^{(n)} v\|$ exists and is independent of $v$.* □

Exactly the same result holds when one replaces the set $\{M_{i_r}\}$ by a sequence of independent and identically distributed random variables $\{Y_i\}_{i\geqslant 0}$ with distribution $\mu$, where supp $(\mu)$ is compact in $\mathrm{GL}(d, \mathbb{R})$. Not only it is known that the limit almost surely exists, but one can also have an explicit formula for this constant value via a space average with respect to a stationary measure, which is given by the following result.

**Theorem B.1.2** (Furstenberg's formula, [Fur63, Thm. 8.5], [Via13, Thm. 6.8]). *Consider a strongly irreducible matrix cocycle $F : X \times \mathbb{R}^d \to X \times \mathbb{R}^d$ that is locally constant similar to Example 1.4.4, No. 1. The largest exponent associated to $F$ is given by*

$$\chi_{\max}^{F} = \int_{X \times \mathbb{RP}^{d-1}} \log \frac{\|B(k)v\|}{\|v\|} \, \mathrm{d}(\mu \times \eta),$$

*where $\eta$ is a stationary measure on $\mathbb{RP}^{d-1}$ with respect to $F$.* □

Here, a measure $\eta$ is said to be *stationary* with respect to $F$ if, for every measurable set $\mathfrak{D} \subset \mathbb{RP}^{d-1}$,

$$\eta(\mathfrak{D}) = \int_X \eta\big(F_x^{-1}(\mathfrak{D})\big) \, \mathrm{d}\mu(x),$$

where $F_x^{-1}(\mathfrak{D}) = \{\bar{v} : \pi_2(F(x, v)) \in \mathfrak{D}\}$ and $\pi_2 : (x, v) \mapsto \bar{v}$. When we are dealing with a random cocycle where $J$ is finite, with associated set of probabilities $\{p_i\}$, the integral in Theorem B.1.2 becomes

$$\chi_{\max}^{\mathrm{rand}} = \sum_{i=1}^{|J|} p_i \int_{\mathbb{RP}^{d-1}} \log \|M_i v\| \mathrm{d}\eta(\bar{v}).$$

for any stationary measure $\eta$.

## B.2. Strong irreducibility

*Proof of Proposition 4.4.1.* It is obvious that card $(\{M \cdot \bar{v} \mid M \in G_\mu\}) > 2$ when $G_\mu$ is strongly irreducible, for all $\bar{v} \in \mathbb{RP}^1$. Hence, it suffices to prove the other direction.

Suppose that $G_\mu$ is not strongly irreducible. We have to show that there exists a direction $\bar{v}$ such that card $(\{M \cdot \bar{v} \mid M \in G_\mu\}) \leqslant 2$.

From the premise, we know that there is a union of subspaces preserved by all matrices $M \in G_\mu$. Let $\mathcal{Q} = \{\bar{v}_1, \ldots, \bar{v}_r\} \subset \mathbb{RP}^1$ be the set of directions preserved by matrices in $G_\mu$. If $|\mathcal{Q}| \leqslant 2$, then for each $\bar{v} \in \mathcal{Q}$, one has card $(\{M \cdot \bar{v} \mid M \in G_\mu\}) \leqslant 2$. We now show that it is impossible for $|\mathcal{Q}| \geqslant 3$. Note that each $M \in G_\mu$ induces a permutation $\iota(M) \in \Sigma_r$ and $\iota$ is a group homomorphism. The kernel of $\iota$ given by

$$\ker(\iota) = \{M \in G_\mu \mid M \cdot \bar{v}_i = \bar{v}_i, i = 1, \ldots, r\}$$

is a closed normal subgroup of $G_\mu$, and $G_\mu/\ker(\iota)$ is finite. This follows from the first isomorphism theorem and that fact that $\iota(G_\mu) \subset \Sigma_r$. Since $G_\mu$ is not compact, $\ker(\iota)$ is not finite. If $r \geqslant 3$, consider three vectors $v_1, v_2, v_3$ with directions $\bar{v}_1, \bar{v}_2, \bar{v}_3$. We can write $v_3 = \alpha v_1 + \beta v_2$ for some $\alpha \neq 0, \beta \neq 0$. For each $M \in \ker(\iota)$, and $\lambda_i \neq 0$, we have $M \cdot v_i = \lambda_i v_i$, for $i = 1, 2, 3$. This yields $\alpha\lambda_3\bar{v}_1 + \beta\lambda_3\bar{v}_2 = \lambda_3\bar{v}_3 = M\bar{x}_3$, which in turn implies

$$M\bar{v}_3 \;=\; \alpha M\bar{v}_1 + \beta M\bar{v}_2 \;=\; \alpha\lambda_1\bar{v}_1 + \beta\lambda_2\bar{v}_2.$$

Hence, $\lambda_1 = \lambda_2 = \lambda_3$, which means $M = \lambda_1\mathbb{I}_2$. But since $|\det M| = 1$, each $M \in \ker(\iota)$ must either be $\mathbb{I}_2$ or $-\mathbb{I}_2$. This contradicts the fact that $\ker(\iota)$ is infinite, and hence $r \leqslant 2$. □

## B.3. Approximation of stationary measures

The difficulty of exploiting the general formula for computing the exponent in Theorem B.1.2 lies in finding a suitable stationary measure as there seems to be no general method of computing it for generic cocycles. However, Froyland and Aihara provided an algorithmic method in [FA00] to circumvent this difficulty, which is given by the following program.

**(1)** Partition $\mathbb{RP}^{d-1}$ into $m$ connected sets $V_1, \ldots, V_m$ of small diameter

**(2)** Choose a representative point $\bar{v}_i \in V_i$ for $1 \leqslant i \leqslant m$, and for each matrix $M_k$ construct the matrix

$$(J_m)_{i\ell}(k) = \begin{cases} 1, & M_k\bar{v}_i \in V_\ell \\ 0, & \text{otherwise,} \end{cases}$$

then combine them to form the matrix $J_m := \sum_k J_m(k)$.

**(3)** Retrieve the left eigenvector of $J_m$ with non-negative entries (i.e., the one corresponding to the eigenvalue 1) and consider its statistically normalised version $j_m$.

**(4)** Construct the point measure $\eta_m = \sum_{i=1}^{m} (j_m)_i \delta_{\bar{v}_i}$. This measure provides an approximation of the stationary measure $\eta$.

**Theorem B.3.1** ( [FA00, Thm. 3.2]). *Assume that the conditions of Theorem B.1.1 are in place. Let* $\{j_m\}_{m=m_0}^{\infty}$ *be the sequence of eigenvectors as constructed above. Then,*

$$\chi_m^{\mathrm{rand}} := \sum_{i=1}^{|J|} p_i \sum_{\ell=1}^{m} (j_m)_\ell \log \|M_i v_\ell\| \xrightarrow{m\to\infty} \chi_{\max}^{\mathrm{rand}}.$$

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# List of Symbols

| | |
|---|---|
| $\ll$ | absolute continuity relation for measures |
| $\Vert \cdot \Vert_{\mathrm{F}}$ | Frobenius norm |
| $\mathbb{1}_{\varrho^n(a)}$ | indicator function of $\varrho^n(a)$ at 0 |
| $\sim$ | equivalence relation for measures |
| $\lhd$ | subword relation |
| $\boldsymbol{A}(k)$ | Kronecker product of $B(k)$ with its conjugate |
| $A_{\mathbb{R}}(x)$ | realification of the complex cocycle $A(x)$ |
| $\mathcal{A}_{n_a}$ | alphabet with $n_a$ letters |
| $\mathcal{A}_{n_a}^{*}$ | set of all finite words over $\mathcal{A}_{n_a}$ |
| $\mathcal{A}_{n_a}^{\mathbb{Z}}$ | set of bi-infinite words over $\mathcal{A}_{n_a}$ |
| $\mathcal{A}_{n_a}^{\ell}$ | set of length-$\ell$ words over $\mathcal{A}_{n_a}$ |
| $\mathcal{A}_{n_a}^{\mathbb{N}}$ | set of one-sided infinite words over $\mathcal{A}_{n_a}$ |
| $\mathcal{AP}_{\varepsilon}(f)$ | set of $\varepsilon$-almost periods of $f$ |
| $\mathfrak{A}_{n_a}$ | alternating group over $n_a$ symbols |
| $B(k)$ | Fourier matrix |
| $\widetilde{B}(x)$ | lifted Fourier matrix on $\mathbb{T}^d$ |
| $\widetilde{B}^{(n)}(x)$ | lifted Fourier cocyle on $\mathbb{T}^d$ |
| $\mathfrak{B}_R(x)$ | ball of radius $R$ centred at $x$ |
| $\mathcal{B}$ | inflation displacement algebra of $\varrho$ |
| $\mathcal{B}_D$ | $\mathbb{C}$-algebra generated by the digit matrices |
| $\mathcal{B}^{(n)}$ | inflation displacement algebra of $\varrho^n$ |
| $C_{\mathrm{c}}(G)$ | space of compactly-supported functions on $G$ |
| $\mathcal{C}_m$ | $m$-th column of $\varrho$ |
| $\mathfrak{C}(p)$ | companion matrix of $p_{\lambda}$ |
| $\chi$ | Lyapunov exponent |
| $\chi^{B}$ | Lyapunov exponent of the cocycle $B^{(n)}(k)$ |
| $\chi^{\widetilde{B}}$ | Lyapunov exponent of the cocycle $\widetilde{B}^{(n)}(x)$ |
| $\chi_{\max}(x)$ | maximum Lyapunov exponent at $x$ |
| $\chi_{\min}(x)$ | minimum Lyapunov exponent at $x$ |
| $\chi_{\max}^{\mathrm{rand}}$ | Lyapunov exponent of a random cocycle $M^{(n)}$ |
| $\mathfrak{D}$ | Borel set in $\mathbb{R}^d$ |
| $D_t$ | Digit matrix at $t$ |
| $\delta_x$ | Dirac distribution at $x$ |
| $\delta_{\mathfrak{D}}$ | Dirac comb on a countable set $\mathfrak{D}$ |

| | |
|---|---|
| $\Delta$ | Euclidean metric in $\mathbb{R}^d$ |
| $\widetilde{\Delta}$ | induced Euclidean metric in $\mathbb{T}^d$ |
| $\mathcal{D}_N$ | $N$-th level discrepancy |
| $E^s$ | stable subspace of $M$ |
| $E^u$ | unstable subspace of $M$ |
| $\epsilon$ | empty word |
| $\mathcal{E}'_\alpha$ | $f$-invariant support of $\big(\widehat{\varUpsilon}\big)_\alpha$ |
| $\widehat{f}$ | Fourier transform of $f \in \mathfrak{S}(\mathbb{R}^d)$ |
| $\mathcal{F}$ | Fourier transformation on $\mathfrak{S}(\mathbb{R}^d)$ |
| $\mathfrak{F}_{n_a}$ | free group over $n_a$ letters |
| $\gamma_\mu$ | natural autocorrelation of $\mu$ |
| $\widehat{\gamma_\mu}$ | diffraction measure of $\mu$ |
| $\mathcal{H}$ | space of $L^2$-functions on $\mathbb{X}$ with respect to $\mu$ |
| $\mathcal{H}_{\mathsf{pp}}$ | $\mathbb{C}$-span of eigenfunctions of $U_S$ |
| $\mathcal{H}(k)$ | Radon–Nikodym matrix of densities $h_{ij}(k)$ |
| $\boldsymbol{h}(k)$ | Radon–Nikodym vector of densities $h_{ij}(k)$ |
| $\boldsymbol{I}(k)$ | vector of intensities $I_{ij}(k)$ |
| $L^1_{\text{loc}}(\mathbb{R}^d)$ | space of locally $L^1$-functions on $\mathbb{R}^d$ |
| $L^2_{\text{loc}}(\mathbb{R})$ | space of locally $L^2$-functions on $\mathbb{R}$ |
| $\boldsymbol{L}$ | left Perron–Frobenius eigenvector of $M$ |
| $\lambda_{\text{PF}}$ | Perron–Frobenius eigenvalue of $M$ |
| $\varLambda$ | point set |
| $M_\varrho$ | substitution matrix of $\varrho$ |
| $\mathfrak{m}(p)$ | logarithmic Mahler measure of $p$ |
| $\mathfrak{M}(p)$ | Mahler measure of $p$ |
| $\mathcal{M}(\mathbb{R}^d)$ | set of measures on $\mathbb{R}^d$ |
| $\mathcal{M}^+(\mathbb{R}^d)$ | set of positive measures on $\mathbb{R}^d$ |
| $\mathbb{M}(f)$ | mean of an almost periodic function $f$ |
| $\mu$ | measure |
| $\mu \circledast \nu$ | Eberlein convolution of $\mu$ and $\nu$ |
| $\widehat{\mu}$ | Fourier transform of $\mu$ |
| $\mu_{\text{H}}$ | Haar measure on $\mathbb{T}^d$ |
| $\mu_{\text{L}}$ | Lebesgue measure in $\mathbb{R}^d$ |
| $\mu_{\mathsf{ac}}$ | absolutely continuous component of $\mu$ |
| $\mu_{\mathsf{c}}$ | continuous component of $\mu$ |
| $\mu_{\text{L}}^{\text{ind}}$ | induced Lebesgue measure on $\Xi^u$ |
| $\mu_{\mathsf{pp}}$ | pure point component of $\mu$ |
| $\mu_{\mathsf{sc}}$ | singular continuous component of $\mu$ |
| $\mu_{\mathsf{sing}}$ | singular component of $\mu$ |

| | |
|---|---|
| $\nu_{ij}(z)$ | pair correlation functions |
| $\omega_{\varLambda}$ | weighted Dirac comb on $\varLambda$ |
| $P_{\mu}$ | set of pure points of $\mu \in \mathcal{M}^{+}(\mathbb{R}^{d})$ |
| $\mathfrak{P}$ | patch of $\varLambda$ |
| $\varPhi$ | permutation representation on $G$ |
| $\Phi_{p_i}$ | the cyclotomic polynomial $1 + z + \ldots + z^{p_i - 1}$ |
| $\mathcal{P}$ | set of prototiles of $\mathcal{T}$ |
| $\mathbb{RP}^{d-1}$ | real projective space of dimension $d$ |
| $\boldsymbol{R}$ | right Perron–Frobenius eigenvector of $M$ |
| $\varrho$ | substitution/inflation rule |
| $\varrho^{(m)}$ | induced substitution on length-$m$ legal words |
| $\varrho(\mathsf{t}_j)$ | level-1 supertile of $\mathsf{t}_j$ |
| $\rho$ | irreducible character of $G$ |
| $S_T$ | total set |
| $S$ | left shift operator |
| $\mathfrak{S}(\mathbb{R}^{d})$ | space of rapidly-decaying $C^{\infty}$-functions on $\mathbb{R}^{d}$ |
| $\sigma_{\max}$ | spectral measure of maximal type |
| $\sigma_f$ | spectral measure associated to $f$ |
| $\Sigma_{n_a}$ | symmetric group over $n_a$ symbols |
| $T_{ij}$ | displacement set |
| $\mathfrak{T}$ | tempered distribution |
| $\vartheta$ | Abelianisation map |
| $\mathsf{t}$ | tile |
| $\mathcal{T}$ | tiling |
| $\theta_{\mathbb{X}}$ | fundamental diffraction of $\mathbb{X}$ |
| $\mathbb{T}^{d}$ | $d$-dimensional torus |
| $U_S$ | Koopman operator associated to $S$ |
| $U_{\mathsf{st}}$ | standard representation |
| $\varUpsilon_{mn}$ | pair correlation measures |
| $\mathcal{V}^{i}$ | $i$-th constituent subspace of a filtration |
| $w^{(0)}$ | legal seed of $w$ |
| $\boldsymbol{w}$ | weight vector |
| $\mathbb{X}$ | symbolic hull of $\varrho$ |
| $\mathbb{X}(w)$ | symbolic hull generated by $w$ |
| $\Xi^{s}$ | projection of $E^{s}$ on $\mathbb{T}^{d}$ |
| $\Xi^{u}$ | projection of $E^{u}$ on $\mathbb{T}^{d}$ |
| $\mathbb{Y}(\varLambda)$ | geometric hull generated by $\varLambda$ |
| $\eta_{\omega}(z)$ | autocorrelation coefficients of $\gamma_{\omega}$ |

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