Integrability in Random Two-Matrix Models under Finite-Rank Perturbations

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Abstract

In Quantum Chromodynamics low energy spectral properties of the Dirac operator can be described by random matrix ensembles. In time-series analysis strong statistical fluctuations coincide with eigenvalue statistics of random matrices. These two completely different fields share the same type of random matrix ensembles: chiral symmetric random matrices.
The analysis of two random-matrix models of this type is presented: the product of two coupled Wishart matrices and the sum of two independent Wishart matrices. Here, we expose the integrability of these models and compute quantities being of interest in Quantum Chromodynamics and in time-series analysis, respectively.

Principal Publications

[1] G. Akemann, T. Checinski and M. Kieburg, *Spectral correlation functions of the sum of two independent complex Wishart matrices with unequal covariances*, J. Phys. A: Math. Theor. **49**(31) 315201, 33pp (2016) [arXiv:1509.03466 [math-ph]]

[2] G. Akemann, T. Checinski, D.-Z. Liu and E. Strahov, *Finite rank perturbations in products of coupled random matrices: From one correlated to two Wishart ensembles*, Ann. Inst. Henri Poincaré, Prob. Stat. **55**(1), pp. 441-479 (2019) [arXiv:1704.05224 [math-ph]]

Contents

1 Introduction

Random Matrix Theory (RMT) is a statistical method and a mathematical tool to characterise spectra which arise in a multitude of research topics and applications. Spectra to which RMT is applied can belong to chaotic or to systems with strong and dominant spectral fluctuations. They appear in a variety of areas such as: physics [3], finance [4], medicine [5], sociology [6] and engineering [7]. RMT provides analytical predictions to quantities which can experimentally be measured and it classifies spectra and unifies systems which a priori have nothing in common. RMT has become an indispensable tool in many research fields. To highlight the most well-known: In quantum mechanics we ask for the spectrum of the Hamilton operator, which for complex systems is characterised by a large number of degrees of freedom as for instance in many body systems. In nuclear physics the approach to the heavy-nuclei-Hamilton-operator was very successful in describing the statistics of energy levels measured in experiments, where heavy nuclei were excited by thermal neutrons [8]. In molecular physics, parts of spectra of acetylene or of nitrogen dioxide arising from excitations of vibrational and electronic degrees of freedom can be characterised in their frequency distribution, spacing or rigidity by RMT [9, 10]. In Quantum Chromodynamics, the random matrix appraoch serves an analytical solution in the deepest infrared limit of the eigenvalue statistics of the Dirac operator, which in various thermodynamical regimes are unknown [11, 12]. Recently, another application of RMT was found, in the spatial distribution of a non-interacting Fermion gas at zero temperature due to similar anaytical structures. It turns out that it coincides with the macroscopic eigenvalue distribution of a particular random matrix model [13–15].

In disciplines beyond physics, spectra appear to which RMT provides analytical predictions. The random matrix approach to the analysis of statistical fluctuations in time series especially provided a diversity of application fields, to which many of those listed above belong. In [4] price fluctuations on the asset marker "S&P 500" from 1991-1996 with daily changes were compared to RMT prediction in the global eigenvalue spectrum corresponding to the covariance matrix. Here, statistical fluctuations can be separated from system specific signals to identify the market factor indicating a trend for all observed prices and to estimate the correlations between the price movements. In [5] human EEG signals were studied, which typically differ strongly in time, with respect to individual test persons and given circumstances stimulating the test persons. It was shown that universal features emerge in the recorded data that are coinciding with RMT predictions like the spectral density of covariance matrices evaluated for small time intervals, the level spacing distribution as well as the number variance, where these observations may help to understand synchronisation and stimulation of brain activities in different regions. In [6] reports of criminal offenses throughout ten years within the City of Philadelphia were studied with respect to different neighbourhoods, different types of the recorded crime events and the times of events. For this large data set, RMT provides models to separate noise from significant signals and to uncover relations between neighbourhoods as well as to verify their

underlying dynamics.
Furthermore, we find applications in telecommunications, in which progressive scattering in wireless transmission has to be included [16]. In addition, RMT influences mathematics as well, free probability is one example, where non-commutative random variables are studied, and random matrices are their finite dimensional realisation [17,18]. A further prominent example can be found in number theory, where for example the statistics of the zeros of the Riemann zeta function at the critical line $z = 1/2$ can be provided by RMT [19].
The detailed description of all areas of application of RMT is a great challenge and would go beyond the scope of this thesis. We recommend the comprehensive collection in [3] and a very detailed review of the history of RMT in [20]. The random matrix models being studied in the present thesis are very particularly chosen. They belong to the so-called chiral Gaussian Unitary Ensembles, which are of special interest in Quantum Chromodynamics (QCD) and time-series analysis. These two fields of application are discussed in more detail in Chap. 2, which is meant to be a part of the introduction to the thesis.
The most important concept of RMT is the global symmetry requirement. Although the very first random matrix was proposed in 1928 by Wishart [21], the field of RMT became popular in the 1950s and 1960s for its application in nuclear physics due to global symmetry considerations. Wigner suggested in 1951 to model a Hamilton operator with the help of a random matrix to characterise the generic spectral statistics of heavy nuclei [22]. Obviously, one symmetry of the Hamilton operator is self-adjointness ensuring real eigenvalues. Moreover, further physical symmetries might occur like time-reversal invariance and rotational symmetry with reference to the spin. These symmetries translate to a self-adjoint random matrix by the choice of the algebraic number field for the matrix entries. We denote with $\beta = 1$ time-reversal invariant systems with rotational symmetry or time-reversal systems with integer spin and broken rotational symmetry, then the random matrix must be chosen to be real and symmetric. If the time-reversal symmetry is broken, the random matrix is complex and the self-adjoint property is translated to a hermitian matrix, which we denote by $\beta = 2$. For systems with preserved time-reversal symmetry but with broken rotational symmetry or systems with preserved time-reversal symmetry and half-integer spin, the entries of the random matrix need to be chosen real quaternions and the self-adjointness translates to self-dual hermitian, which is denoted by $\beta = 4$. All entries of the so composed random matrix were taken to be independent and all real independent degrees of freedom were Gaussian distributed. These three ensembles are called: Gaussian Orthogonal Ensemble ($\beta = 1$), Gaussian Unitary Ensemble ($\beta = 2$) and Gaussian Symplectic Ensemble ($\beta = 4$), where the middle part of these names (GβE) indicates the set of similarity transformations keeping the ensemble invariant.
The mathematical formulation of a self-adjoint matrix of size $N \times N$, say H, can be equally presented for all three β values. A self-adjoint matrix,

$$H^\dagger = H\,, \tag{1.1}$$

can be decomposed as

$$H = U\Lambda U^\dagger\,, \quad \text{with } U^\dagger U = \mathbb{1}_N\,, \tag{1.2}$$

where $\mathbb{1}_N$ is the unit matrix of dimensions $N \times N$. Thereby, Λ is a diagonal matrix containing the eigenvalues of H. The matrix U belongs to the orthogonal, unitary or symplectic group, respectively to according β values, and its columns are composed by the eigenvectors of H. The operation $(\ldots)^\dagger$ for the value $\beta = 2$ translates to the transposition and the complex conjugation of the present matrix. The statistics of eigenvectors of random matrices are as important as the eigenvalues. However, for our purposes we focus on the eigenvalues only. The classification with the help of the β index was introduced by Dyson [23, 24]. The Gaussian distribution was a choice of convenience simplifying computations due to the integrability of this particular distribution.
The macroscopic eigenvalue density for all of the GβE for infinite matrix dimensions, $N \to \infty$, is given by a semicircle symmetric with respect to the y-axis, called Wigner semicircle [25]. This situation might not appar often in physics as the typical energy spectra are not bounded from above. The applicability of RMT was facilitated by a quantification of levels in spectra through their relative behaviour to each other. One key quantity is the level spacing distribution describing the probability of finding two consecutive levels in a certain distance. Although the entries of H were chosen statistically independent, the eigenvalues of such a random matrix are not [26]. This correlation depends on Dyson's β index. The first mathematical description of these correlations was provided by a conjecture, known as the Wigner surmise [25] which we recall from [20, Eq. (3.50)],

$$P_\beta(s) = a_\beta s^\beta \, \mathrm{e}^{-b_\beta s^2} \,, \tag{1.3}$$

and illustrate it in Fig. 1.1 together with the Poisson distribution, which is an exponential function.

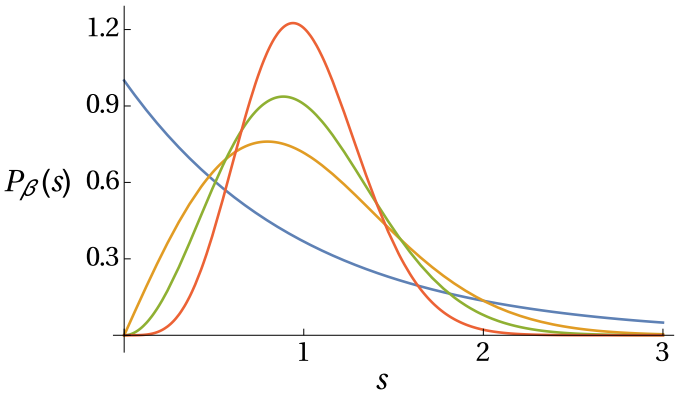

Figure 1.1: Typical level spacing distributions: Poisson distribution (blue) and Wigner surmise labelled by the Dyson-β index: $\beta = 1$ (orange), $\beta = 2$ (green) and $\beta = 4$ (red).

The two constants a_β and b_β are given by the normalisation and a normalised first moment.
The Wigner surmise shows that the eigenvalues of random matrix ensembles for $\beta = 1$, 2 and 4 are repulsively correlated. The most probable distance between two eigenvalues, which can be read off from the maximum in the Wigner surmise, increases by increasing the β value. Simultaneously, from the behaviour near the origin, i.e. $s \ll 1$, we observe a vanishing probability of finding two levels very close to each other with a linear, quadratic or quartic slope, respectively. Level repulsion is absent for the Poisson distribution reflecting statistically independent levels. The measured heavy nuclei levels

agree perfectly with the $\beta = 1$ random matrix prediction [8,27–31]. The same level repulsion could also be observed later on in other systems, such as molecular and atomic spectroscopy [10, 32, 33] and in microwave experiments in the 1990s [34–37]. Systems with broken time-reversal symmetry exhibiting $\beta = 2$ level repulsion could be arranged in microwave experiments in the 1990, too [38, 39], where an external magnetic field was exploited for the symmetry breaking.

Roughly speaking, the RMT approach to physical systems relies on the composition of an ensemble of Hamilton operators that are randomly distributed but coincide in their symmetry with the underlying physical system. With the help of "good quantum numbers", like spin or parity, the underlying physical Hamilton operator has first to be block-diagonalised. For one of these irreducible blocks an ensemble of random matrices serves as a tool to study generic spectral features. In particular, in the above mentioned situation the local spectral behaviour becomes universal and coincides with RMT results. Thereby, these universal features are reached by averaging over the random matrix ensemble and by taking the limit of infinite matrix dimensions, $N \to \infty$. Here, the limit of infinite matrix dimensions ensures that details of the composed ensemble do not affect local spectral effects, but only the β index is crucial.

The symmetries of the Hamilton random matrix and its irreducibility constraint does not necessarily lead to a possible comparison of the eigenvalues of the random matrix to the spectrum of the physical system. The entire spectrum of a Hamilton operator of a complex system as mentioned above might be large and rich on structure. After the division of the complete spectrum with respect to fixed values of good quantum numbers related to the symmetries discussed above, the studied subspectrum still has to remain sufficiently large for statistical analysis. In addition, for the comparison of the physical system with RMT results, the subspectra have to be zoomed onto the scale of the typical mean level density. This spectral mapping is called *unfolding* [20]. Hence, the correlations can be uncovered and eigenvalue statistics of random matrices may serve as a blueprint. Experimentally, after the unfolding, we are visually faced with sequences of lines. For a demonstration, we recall some typical spectra from Mehta's book [40], which are depicted in Fig. 1.2 and originally taken from Bohigas, Haq and Pandey from 1983 [8]. The differences or similarities between these spectra are a piori not visible. The level spacing distribution quantifies them and infer the global symmetries.

A qualitative understanding of why these spectra exhibit universal features and coincide with random matrix predictions was provided by a conjecture formulated by Bohigas, Giannoni and Schmit in 1984 [30], which refers the studied quantum mechanical system to its classical analog. This conjecture signifies that if the classical analog is a chaotic system then the quantum mechanical counter part shows the same fluctuation properties as predicted by random matrix ensemble. Thereby, the above discussed symmetries play a crucial role for specifying the fluctuation properties. For a detailed discussion about Quantum Chaos and RMT we recommend the book written by Haake: *Quantum Signatures of Chaos*, [41]. The Bohigas-Giannoni-Schmit conjecture is considered as well understood, see [42–44].

The classification of random matrix ensembles has been developed until today. We distinguish between ten Gaussian classes worked out by Altland and Zirnbauer in 1996 [45]. However, we would like to emphasise the chiral Gaussian ensembles (ch$G\beta E$). They were introduced by Verbaarschot in the field of Quantum Chromodynamics [46], to which the random matrix models analysed in this thesis for $\beta = 2$ belong.

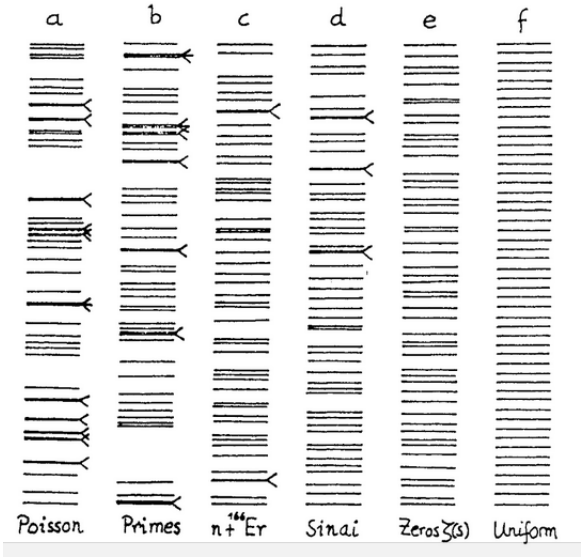

Figure 1.2: Typical spectra on the scale of their typical mean level density, source: Mehta's book [40, Fig. 1.2] and reprinted from [8]: *"Some typical level sequences. From Bohigas et al. (1983). (a) Random levels with no correlation, Poisson statistics. (b) Sequence of prime numbers. (c) Slow neutron resonance levels of the erbium 166 nucleus. (d) Possible eneregy levels of a particle free to move inside the area bounded by* $1/8$ *of a square and a circular arc whose center is the midpoint of the square; i.e., the are specified by the inequalities,* $y \geq 0$, $x \geq y$, $x \leq 1$, *and* $x^2 + y^2 \geq r$ *(Sinai's billiard table). (e) The zeros of the Riemann zeta function on the line* $\mathrm{Re}\, z = 1/2$. *(f) A sequence of equally spaced levels."*.

In RMT, a chiral matrix, say $\mathcal{D}$, is characterised by a block-diagonal form and its anti-hermiticity,

$$\mathcal{D} = \imath \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}, \tag{1.4}$$

which results from the anti-commutation relation with respect to the γ_5-matrix, which we will discuss in the next chapter in depth. The matrix W is assumed to be random and in the simplest case it is composed of independent, Gaussian distributed matrix entries in the respective number field corresponding to the Dyson index β. The dimensions of W can be chosen to $N \times (N + \nu)$. The additional parameter ν is called *number of zero modes* referring to null-eigenvalues of the matrix $W^\dagger W$, which shares its spectrum with $WW^\dagger$. Namely, the spectrum of $\mathcal{D}$ is pure imaginary, there are ν null-eigenvalues and the $2N$ remaining eigenvalues are distributed symmetrically. The distribution of the part of the eigenvalues of $\mathcal{D}$ lying on the positive imaginary axis is equivalent to eigenvalue spectrum of the hermitian random matrix H as

$$H = WW^\dagger, \tag{1.5}$$

which carries the name *Wishart matrix*.

The integrability, which we address in the title, is not referring to integrable physical systems in the sense of integrals of motion nor in terms of the number of conserved quantities: we refer to integrable probabilities, which emerge in the eigenvalue statistics in the studied class of random matrix models. There, all information about the eigenvalues can be derived via an analytical approach yielding an exact and explicit solution as we explain below.

What kind of observables do we study for the eigenvalues of random matrices? In spectral statistics we may ask for global or local properties like the macroscopic or microscopic level densities. As discussed above, the level spacing distribution is one particular observable. Other quantities that are conceivable are distributions of individual eigenvalues and their correlation to the residual part of the spectrum, where extreme eigenvalue statistics are particularly prominent. In the most general sense we ask for correlations among the considered eigenvalues. There are especially two definitions to the so-called k-point correlation function. One of these definitions makes use of Green's functions, which is common in quantum field theories. However, we refer to Dyson's k-point correlation function [24]. For this purpose, we denote the normalised joint probability distribution function for all N eigenvalues λ_1, ..., λ_N of H by P_N. The k-point correlation function is obtained by integrating out $(N-k)$ eigenvalues from P_N,

$$R_k\left(\lambda_1,\ldots,\lambda_k\right) = \frac{N!}{(N-k)!}\int_0^\infty d\lambda_{k+1}\ldots\int_0^\infty d\lambda_N P_N\left(\lambda_1,\ldots,\lambda_N\right). \tag{1.6}$$

This quantity describes the probability of finding one eigenvalue in each of the intervals $[\lambda_j,\lambda_j+d\lambda_j]$ for $j=1,\ \ldots,\ k$. The factor $N!/(N-k)!$ is a combinatorial factor emerging from all possible permutations by the assumption that the eigenvalues are not ordered. The limits of the integrals in Eq. (1.6) already take care of the positive definiteness of the chiral random matrix H as presented in Eq. (1.5). We call a random matrix model integrable if the k-point correlation function for arbitrary $k=1,\ \ldots,\ N$ as defined in Eq. (1.6) can analytically and explicitly be computed as an exact result for finite matrix dimensions, $N<\infty$, in terms of one single function: the kernel.
A well-known result of the general k-point correlation function as defined in Eq. (1.6) is the macroscopic distribution of all eigenvalues, which results from the one-point correlation function as $R_1(\lambda)\sim\rho(\lambda)$ for $N\gg 1$. As mentioned above, for infinite matrix dimensions the global level density for the eigenvalues of GβE forms a semicircle on the real line for all fixed $\beta>0$, the Wigner semicircle [25]. The global level density for ch GβE is restricted to the positive real line and yields in analog consideration a different distribution, which was first derived by Marchenko and Pastur in 1967 [47],

$$\rho_{\mathrm{MP}}(\lambda) = \frac{1}{2\pi}\frac{\sqrt{(\lambda_+-\lambda)(\lambda-\lambda_-)}}{q\lambda}\chi_{[\lambda_-,\lambda_+]}, \quad \text{with } \lambda_\pm = \left(1\pm\sqrt{q}\right)^2. \tag{1.7}$$

The Marchenko-Pastur distribution is illustrated in Fig. 1.3 for different numbers of zero-modes in the ratio $\lim_{N\to\infty} N/(N+\nu)=q\in(0,1]$. Here, the number of zero modes has to scale with the dimension $\nu\to N\nu$ for a deformation of the global level density in the limit to infinite N. The eigenvalues are distributed on the compact support $[\lambda_-,\lambda_+]$, reflected by the indicator function, χ, being unity in the given interval in Eq. (1.7) and zero elsewhere. The Marchenko-Pastur distribution visualises two issues playing central roles in the present thesis. In the application in Quantum Chromodynamics investigations of random matrix models are addressed to the microscopic behaviour of chGUE at the origin for $q=1$. In time-series analysis the macroscopic behaviour of all eigenvalues is of interest.
The content of this thesis is organised as follows: in Chap. 2 we present two applications of chGUE: time-series analysis and Quantum Chromodynamics. By doing so we motivate the choice of the particular random matrix models studied later on in detail: the sum of two independent Wishart matrices and the product of two coupled Wishart matrices. We continue with the eigenvalue representation

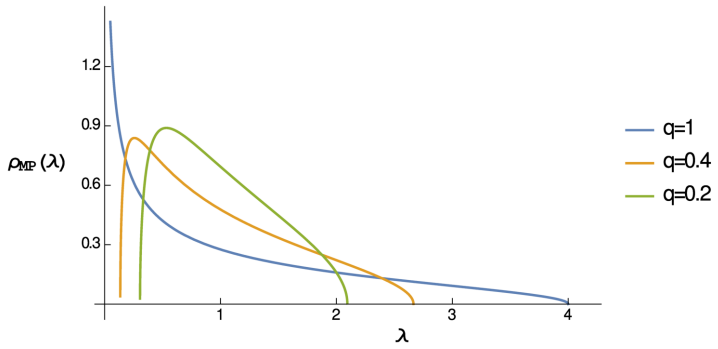

Figure 1.3: Marchenko-Pastur distribution: global level density for the eigenvalues drawn from the chiral Gaussian unitary ensemble for various values of the limiting ratio $\lim_{N\to\infty} N/(N+\nu) = q \in (0,1]$.

of the chosen models in Chap. 3 and discuss the inclusion of external parameters in these models. The discussion in Chap. 4 is addressed to the definition and the integrable structure of determinantal point processes, to which the joint probability distribution functions for the eigenvalues of both models belong. The heart of the thesis is the spectral analysis of these two models presented in Chap. 5 containing the main results published in the principal publications [1] and [2]. The thesis also contains several unpublished results which appear to be new. We close the thesis with a summary and an outlook in Chap. 6.

2 Applications

In the present chapter we propose two fields of application of chiral Gaussian Unitary Ensembles: time-series analysis and Quantum Chromodynamics (QCD). The indicated fields are of interest in modern research and give rise to the study of the two particular random matrix models discussed in this thesis.

2.1 Time-Series Analysis

The application of RMT in time-series analysis is based on the separation of statistical fluctuations from system specific correlations. Time series is an essential issue in many different disciplines and enjoys being a very generic subject. It occurs in a very broad sense by measuring observables at discrete time steps. Questions that can be asked are regarding correlations between these observables and regarding the time evolution of the observables.
In this section we refer to different sources and we strongly reduce the area of time-series analysis to the motivation of our desired random matrix model. In particular, we recommend the broad work representing the state of the art of chiral RMT from 2015 in [48], a big part of which is devoted to covariance matrices.

2.1.1 Covariance Matrix and Fluctuations

We distinguish between *spatial correlations* and *temporal correlations* of a set of observables, which we denote by v_1, ..., v_N. These observables are measured with respect to the time parameter, $v_j = v_j(t)$, at T distinct time steps, t_1, ..., t_T. We arrive at a rectangular table of empirically given data,

$$V = \begin{pmatrix} v_1(t_1) & v_1(t_2) & \dots & v_1(t_T) \\ v_2(t_1) & v_2(t_2) & \dots & v_2(t_T) \\ \vdots & \vdots & \ddots & \vdots \\ v_N(t_1) & v_N(t_2) & \dots & v_N(t_T) \end{pmatrix}. \tag{2.1}$$

The most general question is: what is the correlation between two different observables at different time steps, $v_j(t_k) \leftrightarrow v_{j'}(t_{k'})$ for $j \neq j'$ and $k \neq k'$?
We call spatial the correlation between the two observables v_j and $v_{j'}$, being two rows in the table in Eq. (2.1), in order to highlight the geometrical difference to temporal correlations between two time steps t_k and $t_{k'}$, being between two columns.
Our observables might have different units and might be measured at a different scale. To achieve

dimensionless quantities we apply well known point measures to: *mean* and *variance*,

$$\hat{V} = \begin{pmatrix} \frac{v_1(t_1)-\mu_1}{\sigma_1} & \frac{v_1(t_2)-\mu_1}{\sigma_1} & \cdots & \frac{v_1(t_T)-\mu_1}{\sigma_1} \\ \frac{v_2(t_1)-\mu_2}{\sigma_2} & \frac{v_2(t_2)-\mu_2}{\sigma_2} & \cdots & \frac{v_2(t_T)-\mu_2}{\sigma_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{v_N(t_1)-\mu_2}{\sigma_2} & \frac{v_N(t_2)-\mu_2}{\sigma_2} & \cdots & \frac{v_N(t_T)-\mu_2}{\sigma_2} \end{pmatrix}, \quad \text{with:} \quad \begin{array}{c} \mu_j = \frac{1}{T}\sum_{k=1}^{T} v_j(t_k)\,, \\ \sigma_j^2 = \frac{1}{T}\sum_{k=1}^{T} |v_j(t_k)-\mu_j|^2\,, \\ \text{for all } \; j = 1,\ldots,N\,. \end{array} \tag{2.2}$$

Spatial correlations can be reflected by the *covariances*, which we define with

$$\Sigma_{jj'} = \frac{1}{T}\sum_{k=1}^{T} \hat{V}_{jk}\hat{V}^*_{j'k} = \frac{1}{T}\left(\hat{V}\hat{V}^\dagger\right)_{jj'}\,, \tag{2.3}$$

where we denote with $(\ldots)^*$ complex conjugation. In this way the empirical *covariance matrix*, Σ, is introduced. The rank of the covariance matrix is given by the lower value of the numbers: observables or time steps, $\min(N,T)$. Time-series analysis with less time steps than observables, $T < N$, is very difficult due to zero eigenvalues appearing in the covariance matrix, see for example [49]. For our purposes, the number of time steps is larger than the number of observables. By making use of the number of zero modes introduced in Chap. 1 we set $T = N + \nu$.

The covariance matrix is hermitian, $\Sigma^\dagger = \Sigma$, and positive definite, $\det[\Sigma] > 0$. Due to the zero-mean and unity-variance of $\hat{V}$ presented in Eq. (2.2), the covariance matrix is unity in its diagonal entries and for real valued time series it is valued between $[-1,1]$ in its off-diagonal entries, whereas for complex valued time series the entries of the covariance matrix are valued inside the complex unit circle. A full correlation between two observables is indicated by an entry of the covariance matrix being equal to one and uncorrelated by an entry being equal to zero. Thus, the covariance matrix can be studied with respect to distinguished values of its entries. However, by increasing the number of observables, the covariance matrix is increased and the analysis of its entries has to be reduced to spectral analysis.

There are several extreme scenarios of the resulting covariance matrix Σ, which are affecting its eigenvalues. Let us emphasise the main ones: assuming that all observables are uncorrelated, $\Sigma_{jj'} = \delta_{jj'}$ for all $j,j' = 1, \ldots, N$, then all eigenvalues of Σ are equal to one, $\lambda = 1$. Assuming that all observables are fully correlated, $C_{jj'} = 1$ for all $j,j' = 1, \ldots, N$, then all eigenvalues of Σ are equal to the number of observables, $\lambda = N$. The fully anti-correlated case is also thinkable, but slightly more problematic in its illustration. It is simple to see that a covariance matrix with entries chosen to be negative $C_{jj'} = -1$ for for all pairwise distinct $j \neq j'$ would not satisfy the convergence condition. Thus, we restrict ourselves in this scenario to the idea of the tendency of eigenvalues of a covariance matrix with increased number of anti-correlated observables. Namely, the eigenvalues become smaller than one in the strong anti-correlated regime and, thus, the eigenvalues belonging to anti-correlated observables lye on the interval $\lambda \in (0,1)$.

The eigenvalue distribution of the covariance matrix defined in Eq. (2.3) carries the information of how many observables are correlated with a certain strength. The eigenvectors belonging to certain eigenvalues point out which particular observables are correlated with this particular strength.

The application of RMT in time-series analysis is due to statistical fluctuations in the measured entries of the time series. These fluctuations perturb the measured observables leading to a smear of

the covariance matrix and, thus, true correlations might be hidden.
The main possibility to describe statistical fluctuations is via white noise. A pure white noise in our time series V is composed by assuming that all deviations in the measured observables from their means are described by a Gaussian distributed random variable independently from all other entries. Thus, the entries of the zero-mean and unity-variance time series exchanged by random variables, $\hat{V} \to X$, are distributed by

$$\mathcal{P}(X) = \pi^{-N(N+\nu)} \, \mathrm{e}^{-\mathrm{Tr}\, XX^\dagger}, \quad \text{for} \;\; X \in \mathbb{C}^{N\times(N+\nu)}. \tag{2.4}$$

This ensemble is called complex Wishart-Laguerre ensemble. The influence on the covariance matrix caused by statistical fluctuations motivated John Wishart in 1928 [21] to compose an ensemble of rectangular and Gaussian distributed random matrices with real entries. For spectral analysis of the eigenvalues of the covariance matrix, $XX^\dagger$, the application of the method of orthogonal polynomials is well known [40], where the classical Laguerre polynomials turn out to serve as a mathematical key tool. The global level density of the Wishart-Laguerre ensemble for infinite matrix dimensions, $N \to \infty$ and $N/(N+\nu) \to q$, was discussed in Chap. 1. We recall that the Marchenko-Pastur distribution $\rho_{\mathrm{MP}}(\lambda)$, see Eq. (1.7), is supported on a compact interval, $[\lambda_-, \lambda_+]$, and is sensitive with respect to the parameter q.
Statistical fluctuations in the eigenvalue distribution of the empirically given covariance matrix Σ exhibit a characteristic shape which is of Marchenko-Pastur distribution type. As an example we depict here the comparison of the Marchenko-Pastur distribution and data of stock prises abtracted from [50] in Fig. 2.1. The parameter q serves as a fit parameter deforming the Marchenko-Pastur distribution

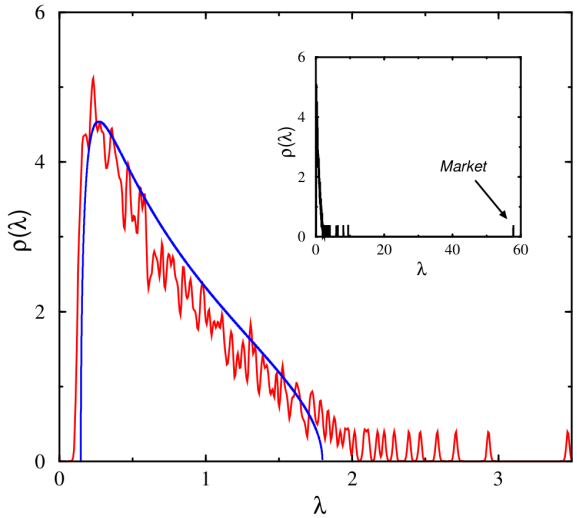

Figure 2.1: Empirical eigenvalue density for 406 stocks from S&P 500 (red curve) compared with the Marchenko-Pastur distribution (blue curve). The presence of one outlier is shown in the inner graph. Abstracted from [50].

as presented in Fig. 1.3. The system specific correlations are represented by eigenvalues lying outside of the bulk called *outliers*, $\lambda \gg \lambda_+$, [4]. This separation of the covariance spectrum is according to the

discussion about the expected behaviour of eigenvalues and the main goal of the application of RMT in time-series analysis.
The comparison of RMT results with empirically measured data has to be improved if the statistical fluctuations become strong with respect to the order of measured observables. System specific correlations might be hidden in the bulk and not separated as outliers. An extension of the Marchenko-Pastur distribution can be done by the so called sample-covariance matrix [51], one-side correlated Wishart ensemble or to what we will refer later as well: *one-epoch model*,

$$\mathcal{P}(X|\Sigma) = \pi^{-N(N+\nu)} \det[\Sigma]^{-(N+\nu)} \, \mathrm{e}^{-\mathrm{Tr}\,\Sigma^{-1}XX^\dagger}, \quad \text{for} \;\; X \in \mathbb{C}^{N\times(N+\nu)}. \tag{2.5}$$

This model can be interpreted as a model for time series driven by statistical fluctuations and coinciding in the first moment in the square form with the empirically measured covariance matrix, see for comparison the Eq. (2.3),

$$\Sigma_{jj'} = \left\langle \frac{1}{N+\nu} \sum_{k=1}^{N+\nu} X_{jk} X^*_{j'k} \right\rangle_{\mathcal{P}(X|\Sigma)}, \tag{2.6}$$

which we will prove explicitly below.
For the application of the one-epoch model an explicit solution for the eigenvalue density of the square form $XX^\dagger$ for finite matrix dimensions is of interest. The standard Wishart-Laguerre ensemble can be solved analytically as well as for its real analogue, $\beta = 1$, meaning that all k-point correlation functions as defined in Eq. (1.6) are accessible, see [52]. This integrability is lost for the real case by introducing the covariance matrix in the one-side correlated Wishart ensemble. However, the analytical approach to the one-point correlation function yielding finite-N expressions was derived in [53] and [54] by making use of supersymmetry techniques.
The joint probability distribution function for the eigenvalues of $XX^\dagger$ implies a group integral, whose solution is required to access all k-point correlation functions, which we will discuss in the next chapter. The derivation of an exact expression for all k-point correlation functions for finite matrix dimensions in the $\beta = 2$ case will be presented in Sec. 5.1.1 as a preliminary consideration.

2.1.2 Time Evolution in Time-Series Modelling

The approach to time-series analysis described above is intended to study spatial correlations in time series. Thereby, the covariance matrix Σ is assumed to be stationary in the point measures: mean and variance applied in Eq. (2.2). This stationarity reflects reproducibility of each of the observable $v_j(t_k)$, for all j, at every time step, for all k. Thus, no time correlations are included. For our purposes, time dependence in time series is crucial.
Temporal correlation may exist in time series and might describe time dependencies in time series. In an analogous way to the one-side correlated Wishart ensemble in Eq. (2.5), we introduce the doubly-correlated Wishart ensemble, implying non-trivial temporal correlations,

$$\mathcal{P}(X|\Sigma,\Pi) = \pi^{-N(N+\nu)} \det[\Sigma\otimes\Pi]^{-1} \, \mathrm{e}^{-\mathrm{Tr}\,\Sigma^{-1}X\Pi^{-1}X^\dagger}, \quad \text{for} \;\; X \in \mathbb{C}^{N\times(N+\nu)}, \tag{2.7}$$

where $\otimes$ denotes the Kronecker product and Π is a $(N+\nu)\times(N+\nu)$ complex valued and positive definite matrix. This ensemble is to interpret from the view of reproducibility of measured time series. Let us recall that the observables, $\{v_j\}_{j=1,\ldots,N}$, have been measured at $N+\nu$ time steps, t_1, …, $t_{N+\nu}$, which has led us to the composition presented in Eq. (2.1). We now add the assumption that the entire measurement of all observables at all time steps can be be reproduced, which for experiments is reasonable. Consequently, we have more than one realisation of our time series, which we denote with the upper index, $v_j^{(r)}(t)$ with $r=1$, …, R. Then, the most general spatio-temporal correlations of covariance type are given by averaging over all realisations as

$$\left\langle v_j^{(r)}(t_k)\left(v_{j'}^{(r)}(t_{k'})\right)^*\right\rangle_{(R)} = \Sigma_{jj',kk'}\,, \tag{2.8}$$

which may be performed by making use of point measures for averaging over realisations. The correlations in the so obtained $\Sigma_{jj',kk'}$ encode time evolution.
In order to include these correlations into the modelling of white noise via random matrix ensembles, simplifications have to be made. The factorisation of spatial and temporal correlations, $\Sigma_{jj',kk'}\sim\Sigma_{jj'}\otimes\Pi_{kk'}$ motivates the doubly-correlated Wishart ensemble. Factorising but non-trivial spatio-temporal correlations were considered in economics [55], in sociology [6] and telecommunications [16]. Analytically this ensemble was discussed first in [56] and recently, an exact solution to the one-point correlation function was derived with the help of the supersymmetry technique in [57].
For the doubly-correlated Wishart ensemble the first moment condition compared to Eq. (2.6) can now be formulated in both external fixed matrices: Σ and Π,

$$\Sigma_{jj'} = \left\langle \frac{1}{N+\nu}\sum_{k=1}^{N+\nu} X_{jk}X^*_{j'k}\right\rangle_{\mathcal{P}(X|\Sigma,\Pi)} \quad\text{and}\quad \Pi_{kk'} = \left\langle \frac{1}{N}\sum_{j=1}^{N} X_{jk}X^*_{jk'}\right\rangle_{\mathcal{P}(X|\Sigma,\Pi)}\,, \tag{2.9}$$

which will be shown below. The one-side correlated Wishart ensemble from Eq. (2.5) can thus be seen as the limit of stationary observables from the doubly-correlated Wishart ensemble from Eq. (2.7), $\Pi\to\mathbb{1}_{N+\nu}$.
The natural spectral analysis regarding temporal correlations would focus on the random matrix $X^\dagger X$. However, the matrix $X^\dagger X$ shares all non-zero eigenvalues with $XX^\dagger$, which can be seen on the well known Silvester's determinant identity $x^\nu\det\left[x\mathbb{1}_N - XX^\dagger\right] = \det\left[x\mathbb{1}_{N+\nu} - X^\dagger X\right]$ [58]. Thus, in the analysis of time evolution in Wishart ensembles modelling time series, spectral analysis of $X^\dagger X$ do not yield new insights.
Recently, a block form of the temporal-covariance matrix was introduced to study *temporal cross-correlations* in [59, 60], which can be embedded in the doubly correlated Wishart ensemble by setting

$$\Sigma = \mathbb{1}_N \quad\text{and}\quad \Pi = \begin{pmatrix} \mathbb{1}_{N_1} & \Omega \\ \Omega^\dagger & \mathbb{1}_{N_2}\end{pmatrix}, \tag{2.10}$$

with $N_1+N_2 = N+\nu$. Here, the random matrix X is divided in two sectors of length N_1 and N_2 of the form $\begin{pmatrix}X_1 & X_2\end{pmatrix}$ representing time evolution between the time steps N_1 and N_1+1. Questions regarding cross-correlations are related to the product matrix $X_1^\dagger X_2$. In this model, the temporal correlations are non-trivial, whereas the correlations between the observables are set to be trivial. Due to statistical dependence of the two matrices X_1 and X_2 so-called time-lagged correlations can

be modeled.
The *two-epoch model*, which will be studied in this thesis in detail, possesses a similar block structure comprising two random matrices: X_1 of dimensions $N \times (N+\nu_1)$ and X_2 of dimensions $N \times (N+\nu_2)$ arranged analogously to $Y = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$. Kumar introduced in [61] time dependent spatial correlations, which can be embedded into the doubly-correlated Wishart ensemble by setting

$$\Sigma = \Sigma_A \chi_{[1,N+\nu_1]} + \Sigma_B \chi_{[N+\nu_1+1,N+\nu_1+N+\nu_2]} \quad \text{and} \quad \Pi = \mathbb{1}_{N+\nu}\,, \tag{2.11}$$

where χ denotes the indicator function being identity in the indicated interval and zero elsewhere. Here, spatial correlations are assumed to be time dependent, $\Sigma = \Sigma(t)$, and spatial cross-correlations can be introduced with respect to the random matrix $YY^\dagger = X_1 X_1^\dagger + X_2 X_2^\dagger$, for unequal covariances $\Sigma_A \neq \Sigma_B$. The visualisation of differences in the spectra between $YY^\dagger$ from the two-epoch model and $XX^\dagger$ from the one-epoch model for finite matrix dimensions is one of the main goals in this thesis in making use of the integrability of these two models.

2.1.3 First-Moment Condition

The models introduced above: one-side correlated Wishart ensemble in Eq. (2.5) and the doubly-correlated Wishart ensemble in Eq. (2.7) satisfy first moment conditions with respect to the empirical covariance matrices, see Eqs. (2.6) and (2.9). The two-epoch model, which will be discussed in detail, satisfies the first moment condition with respect to the covariance matrix presented in Eq. (2.11).
We would like to present here a simple calculus yielding the first moment conditions for complex covariance matrices.
Let us consider a self-adjoint, complex valued and positive definite $N \times N$-size matrix Σ, which we call covariance matrix. With $\Sigma = U^\dagger \operatorname{diag}(\sigma_1, \ldots, \sigma_N) U$ we denote its eigenvalue decomposition, where all N eigenvalues are pairwise distinct and positive. There is one unique unitary matrix for this diagonalisation with $U^{-1} = U^\dagger$ and $\det[U] = 1$. The derivative with respect to an eigenvalue, $\partial/\partial\sigma_j$, acting on the determinant of the covariance matrix yields the multiplicative inverse of this eigenvalue,

$$\frac{\partial}{\partial \sigma_j} \det[\Sigma] = \frac{1}{\sigma_j} \det[\Sigma]\,, \tag{2.12}$$

which can be seen on the product of all eigenvalues, $\det[\Sigma] = \prod_{j=1}^N \sigma_j$, as a representation for the determinant. It is an important property for our derivation, as determinants of this form arise in the normalisation of our ensembles. The one-side correlated as well as the doubly correlated Wishart ensembles are normalised through the complex generalisation of the Gauss integral,

$$\pi\sigma = \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy\, \mathrm{e}^{-\sigma^{-1}(x^2+y^2)} = \int_{\mathbb{C}} [dX]\, \mathrm{e}^{-\sigma^{-1} X X^*}\,, \tag{2.13}$$

which by extending to $X \in \mathbb{C}^{N\times(N+\nu)}$ yields the determinants in front of the densities: in Eq. (2.5) it is a factor of $\det[\Sigma]^{-(N+\nu)}$ for the one-side correlated Wishart ensemble and in Eq. (2.7) it is $\det[\Sigma \otimes \Pi]^{-1}$ for the doubly correlated Wishart ensemble. Let us stick on the former first moment

condition, which reads by making use of the covariance matrix Σ^{-1} as a source for the first moment:

$$\left\langle \left(XX^\dagger \right)_{jj'} \right\rangle_{\mathcal{P}(X|\Sigma)} = \det\left[\Sigma^{-1}\right]^{N+\nu} \left(-\frac{\partial}{\partial \Sigma^{-1}_{j'j}} \right) \det\left[\Sigma^{-1}\right]^{-(N+\nu)} . \tag{2.14}$$

It remains to compute the differentiation with respect to the $j'j$ entry of the inverse covariance matrix applied on its determinant.
The differentiation operator $\partial/\partial\Sigma^{-1}_{j'j}$ is a derivative with respect to the $j'j$-entry of the inverse covariance matrix,

$$\frac{\partial}{\partial \Sigma^{-1}_{j'j}} \Sigma^{-1}_{k'k} = \delta_{j'k'}\delta_{jk} . \tag{2.15}$$

However, it can be interpreted as a matrix entry by itself,

$$\frac{\partial}{\partial \Sigma^{-1}_{j'j}} = \left(\frac{\partial}{\partial \Sigma^{-1}} \right)_{jj'} , \tag{2.16}$$

which defines a differential-operator matrix $\partial/\partial\Sigma^{-1}$. Note that the order of our indices has to change, which can be seen by investigating the equation $\left(\partial/\partial\Sigma^{-1}\right)\Sigma^{-1} = \mathbb{1}_N$ from a sum over k' and j' in Eq. (2.15). We now would like to apply the eigenequation from the beginning of our computation, see Eq. (2.12), on the differentiation left to perform in Eq. (2.14). Here, the diagonalisation of the differential-operator matrix $\partial/\partial\Sigma^{-1}$ has to be introduced.
The transformation of the inverse covariane matrix under similarity transformation belonging to the unitary group $U(N)$ can be investigated on the defining property, $\Sigma^{-1}\Sigma = \mathbb{1}_N$. It is obvious that our inverse covariance matrix can be diagonalised as $\Sigma^{-1} = U^\dagger \operatorname{diag}\left(\sigma_1^{-1},\ldots,\sigma_N^{-1}\right) U$ with the same unitary matrix diagonalising the covariance matrix. The differential-operator matrix has the same transformation property, $\partial/\partial\Sigma^{-1} = U^\dagger \operatorname{diag}\left(\partial/\partial\sigma_1^{-1},\ldots,\partial/\partial\sigma_N^{-1}\right) U$.
It is now a simple matter to apply the eigenequation presented in the beginning in Eq. (2.12) on the matrix form, $\operatorname{diag}\left(\partial/\partial\sigma_1^{-1},\ldots,\partial/\partial\sigma_N^{-1}\right)\det\left[\Sigma^{-1}\right] = \operatorname{diag}\left(\sigma_1,\ldots,\sigma_N\right)\det\left[\Sigma^{-1}\right]$, yielding

$$\frac{\partial}{\partial \Sigma^{-1}} \det\left[\Sigma^{-1}\right] = \Sigma \det\left[\Sigma^{-1}\right] , \tag{2.17}$$

which by applying on the remaining differentiation in Eq. (2.14) finishes the proof of our claim. The first moment conditions for the doubly-correlated Wishart ensemble can be proven analogously.

2.2 Quantum Chromodynamics

QCD is a part of the Standard Model which describes elementary particles [62]. The Standard Model is, due to its success in experimental findings and corresponding predictions, one of the most celebrated achievements in modern physics. In the basic classification we distinguish particles interacting among each other and particles carrying the interactions. There are three interactions included in the standard model: the electromagnetic interaction transmitted by photons, the weak interaction transmitted by so-called Z and $W^\pm$ bosons and the strong interaction transmitted by eight gluons. The latter one couples with particles called quarks, of which we find six in nature, called: *up*, *down*, *charm*, *strange*, *top* and *bottom*. The strong interaction is the subject of QCD.

The application of RMT in QCD is a very sophisticated issue and we only can reduce this topic to the central ideas motivating our random matrix model. We constrain ourselves to selected considerations of global symmetries in the QCD partition function and local spectral properties. We refer to the lecture notes from 2015 [12], which are much more detailed and go beyond the description here, as well as we recommend the wide overview from 2000 [63].

2.2.1 QCD Partition Function and Global Symmetries

There are two global symmetries to which we refer in the application of RMT in QCD: chiral symmetry and anti-hermiticity. These two symmetries are associated with the so-called Dirac operator, of which spectral properties are of interest.
Quarks are fermions with spin 1/2. Thus, the mathematical description of their dynamics is based on the Dirac operator, $\mathcal{D}$. The Dirac operator can be formulated in space-time representation as a first-order partial differential operator,

$$\mathcal{D} = \gamma_\mu \left(\partial_\mu + \imath g_s A_\mu\right), \tag{2.18}$$

where we sum over the Greek letter $\mu = 1,\ldots,4$, indicating four space-time dimensions. This sum represents the geometry of the space-time. For our purposes, the Euclidean metric has to be chosen such that the first-order partial differentiation with respect to the time parameter posses the same sign as the ones with respect to the space parameters,

$$\gamma_\mu \partial_\mu = \gamma_1 \frac{\partial}{\partial x_1} + \gamma_2 \frac{\partial}{\partial x_2} + \gamma_3 \frac{\partial}{\partial x_3} + \gamma_4 \frac{\partial}{\partial t}, \tag{2.19}$$

in natural units. The so-called γ-matrices respect the existence of anti-particles and satisfy the Clifford algebra, which is related to the fermionic nature of quarks, given by anti-commutation relations,

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}, \quad \text{for } \mu,\nu = 1,\ldots,4. \tag{2.20}$$

A standard representation of the γ-matrices as 4×4 matrices is composed essentially by the Pauli matrices,

$$\gamma_k = \begin{pmatrix} 0 & \imath\sigma_k \\ -\imath\sigma_k & 0 \end{pmatrix}, \quad \text{for } k = 1,2,3, \quad \text{and } \gamma_4 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}. \tag{2.21}$$

The strong interaction is carried by gluons, which are bosonic particles. They are introduced by the principle of minimal coupling in the composition of the Dirac operator in Eq. (2.18) through the strong coupling constant g_s. We denote the gluon fields by A^a, where the number of gluons, counted by a, depends on the group characterising the interaction. The gauge fields A can be represented in terms of generators, T^a, of the suitable gauge group $SU(N_c)$, and read in their components

$$A_\mu = \sum_{a=1}^{N_c^2-1} A_\mu^a T^a. \tag{2.22}$$

The so-called *number of colours* N_c depends on the appropriate Lie group $SU(N_c)$, where we have in nature: $N_c = 3$. The generators, T^a, satisfy the corresponding Lie algebra, which implies non-trivial

commutation relations for arbitrary number of colours. The action determining the dynamics of non-Abelian gauge fields is called the Yang-Mills action [64], which we denote by $S_{\mathrm{YM}}[A]$ and we will not specify in more detail.
From this construction, both symmetries can be introduced: the Dirac operator is anti-hermitian,

$$\mathcal{D}^\dagger = -\mathcal{D}\,. \tag{2.23}$$

This symmetry is given due to hermiticity of the γ-matrices in Euclidean space, anti-hermiticity of partial derivatives as quantum mechanical operators representing momentum and energy if multiplied by the imaginary number $\imath$, and the hermiticity of generators of $SU(N_c)$ encoded in the gluon fields A^a.
The Dirac operator obeys the chiral symmetry,

$$0 = \mathcal{D}\gamma_5 + \gamma_5\mathcal{D}\,. \tag{2.24}$$

The chiral symmetry was emphasised in the introductory chapter in Eq. (1.5). Here, this symmetry is represented by an anti-commutation relation to the γ_5 matrix which is composed by the generators of the above-mentioned Clifford algebra,

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \tag{2.25}$$

where its representation after the last equality is due to the standard representation of the γ-matrices as given in Eq. (2.21).
From these two symmetries we obtain information about the spectrum of the Dirac operator. The anti-hermiticity implies pure imaginary eigenvalues of $\mathcal{D}$, whereas the chiral symmetry has as a consequence an axial symmetry in the spectrum of $\mathcal{D}$ with respect to the origin. This can be illustrated in a reformulation of the Dirac operator to

$$\mathcal{D} = \imath \begin{pmatrix} 0 & \mathcal{W} \\ \mathcal{W}^\dagger & 0 \end{pmatrix}. \tag{2.26}$$

In this schematic representation of the Dirac operator, the differential operator as well as the coupling on gluons via $SU(N_c)$-group properties are taken into account in $\mathcal{W}$. The imaginary element $\imath$ in front is due to the anti-hermiticity, which is preserved by the hermiticity of the remaining part. The block structure is due to the chiral symmetry. By making use of the standard formula of a determinant comprising square-block structure, $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det[ad-bc]$, applied on the characteristic polynomial,

$$\det[\lambda\mathbb{1}_4 - \mathcal{D}] = \det\left[\lambda^2\mathbb{1}_2 + \mathcal{W}\mathcal{W}^\dagger\right], \tag{2.27}$$

we obtain the connection to the chiral symmetry presented in Eq. (1.5). The spectrum of $\mathcal{D}$ is up to null-eigenvalues, purely imaginary, as the eigenvalues of $\mathcal{W}\mathcal{W}^\dagger$ are always real and positive. Moreover, its eigenvalues come up in pairs because of the square form in λ on the right hand side in Eq. (2.27).
We now would like to turn to the formulation of the partition function. With the help of the Dirac operator, the fermionic part of the QCD-Lagrangian density can be composed. Therefore, we denote

with N_f the number of quarks, called *flavours*, and with $\psi_j(x)$ the corresponding wave function for $j = 1, \ldots, N_f$. The Lagrangian density reads

$$\mathcal{L}_{\mathrm{D}}\left[\psi_1(x), \ldots, \psi_{N_f}(x)\right] = \sum_{j=1}^{N_f} \bar{\psi}_j(x)\left(\mathcal{D} + m_j \mathbb{1}_4\right)\psi_j(x)\,, \tag{2.28}$$

where m_j denotes the mass of the quark of flavour j and in addition we use the abbreviation $\bar{\psi}_j(x) = \psi_j(x)^\dagger \gamma_4$, where x indicates the space-time dimensions. The partition function reads

$$\mathcal{Z}_{\mathrm{QCD}}^{(N_f)}\left(m_1, \ldots, m_{N_f}\right) = \int d[A]\, d[\psi]\, \mathrm{e}^{-S_{\mathrm{YM}}[A]}\, \mathrm{e}^{\int d^4x \mathcal{L}_{\mathrm{D}}\left[\psi_1(x^\mu), \ldots, \psi_{N_f}(x^\mu)\right]}\,, \tag{2.29}$$

where all possible gauge fields are integrated through $d[A]$ and $d[\psi]$ as Feynman path integrals. The integration over the fermionic fields can be performed in a formal sense. We namely have, with the Grassmann integral [65],

$$\prod_{j=1}^{N_f} \det\left[\mathcal{D} + m_j \mathbb{1}_4\right] = \int d[\psi]\, d\left[\bar{\psi}\right] \exp\left[\int d^4x \sum_{j=1}^{N_f} \bar{\psi}_j(x)\left(\mathcal{D} + m_j \mathbb{1}_4\right)\psi_j(x)\right]. \tag{2.30}$$

The QCD partition function can thus be read as an expectation value over the gluon fields of determinants containing the sum of the Dirac operator and quark masses,

$$\mathcal{Z}_{\mathrm{QCD}}^{(N_f)}\left(m_1, \ldots, m_{N_f}\right) = \int d[A]\, \mathrm{e}^{-S_{\mathrm{YM}}[A]} \prod_{j=1}^{N_f} \det\left[\mathcal{D} + m_j \mathbb{1}_4\right]. \tag{2.31}$$

The quark masses $m_1, \ldots, m_{N_f}$ are breaking both symmetries in the partition function, as the identity matrix is obviously hermitian and commutes with all gamma matrices. Moreover, with the help of the γ_5 matrix, projection operators can be composed as

$$P_\pm = \frac{1}{2}\left(\mathbb{1}_4 \pm \gamma_5\right), \quad P_\pm^2 = P_\pm\,, \quad P_+ P_- = P_- P_+ = 0\,, \quad P_+ + P_- = \mathbb{1}_4\,, \tag{2.32}$$

and, thus, the corresponding Hilbert space can be divided with respect to these two subspaces, $\psi_j = \psi_j^+ + \psi_j^-$, with $\psi_j^\pm = P_\pm \psi_j$. Defining the vector $\psi = \left(\psi_1, \ldots, \psi_{N_f}\right)$, we observe that due to the chiral symmetry of the Dirac operator from Eq. (2.24) the corresponding states preserve the separation in theses chiral subspaces,

$$\bar{\psi}\mathcal{D}\psi = \bar{\psi}^+ \mathcal{D}\psi^+ + \bar{\psi}^- \mathcal{D}\psi^-\,, \tag{2.33}$$

while the mass terms violate this separation $m_j \bar{\psi}_j \psi_j = m_j\left(\bar{\psi}_j^+ \psi_j^- + \bar{\psi}_j^- \psi_j^+\right)$ for all $j = 1, \ldots, N_f$. The symmetry preserved by the Dirac operator in Eq. (2.33) is $\psi^+ \to U^+ \psi^+$ with $U^+ \in U\left(N_f\right)$ and independently $\psi^- \to U^- \psi^-$ with $U^- \in U\left(N_f\right)$. The basic mechanism of breaking the chiral symmetry by the mass term can be conceived by considering the simplifying case of all masses being equal, $m_j = m$ for all $j = 1, \ldots, N_f$,

$$m\bar{\psi}\psi = m\left(\bar{\psi}^+ \psi^- + \bar{\psi}^- \psi^+\right), \tag{2.34}$$

contrary to the Dirac operator part as shown in Eq. (2.33). The mass term formulated in Eq. (2.34) is invariant under the transformations U^+ and U^- only if $U^+ = U^-$. The pattern of breaking the chiral symmetry can be formulated to

$$SU\left(N_f\right) \times SU\left(N_f\right) \to SU\left(N_f\right), \tag{2.35}$$

where the invariance in Eqs.: (2.33) and (2.34) with respect to one degree of freedom as an overall phase, $U\left(N_f\right) = U(1) \times SU\left(N_f\right)$, has been included.
The quark masses can formally be avoided in the QCD partition function in Eq. (2.31) by setting the number of flavours to zero, $N_f = 0$, which is called the *quenched case*. There, the pure gauge field effect on the Dirac spectrum can be studied. The physical interpretation of the quenched case is that the quarks are so heavy that the mass term dominates the determinant and the dynamics described by $\mathcal{D}$ do not contribute. Thus, the determinant can be factorised out of the partition function and only the gauge field part remains.
Our random matrix model is to embed in a further symmetry breaking problem in the QCD partition function. Namely, the introduction of quark chemical potential, μ, is breaking the anti-hermiticity, but preserves the chiral symmetry.
The quark chemical potential can be introduced on the level of the Lagrangian density by adding an additional term in Eq. (2.28) of the form $\mu \sum_{j=1}^{N_f} \psi_j^\dagger(x)\,\psi_j(x) = \mu \sum_{j=1}^{N_f} \bar{\psi}_j \gamma_4(x)\,\psi_j(x)$, where we make use of the identity $\gamma_4\gamma_4 = \mathbb{1}_4$. The chemical potential can thus be directly implied in the Dirac operator in the formulation from Eq. (2.26) through

$$\mathcal{D} = \imath \begin{pmatrix} 0 & \mathcal{W} \\ \mathcal{W}^\dagger & 0 \end{pmatrix} + \mu \begin{pmatrix} 0 & \mathbb{1}_2 \otimes \mathrm{Id} \\ \mathbb{1}_2 \otimes \mathrm{Id} & 0 \end{pmatrix} \tag{2.36}$$

by keeping the formulation of the partition function in Eq. (2.31). The identity element Id in Eq. (2.36) is referring to the space spanned by the generators T^a. The Dirac operator still preserves the chiral symmetry, however, it is anti-hermitian only if the chemical potential vanishes $\mu \to 0$. The application of RMT in QCD on which our random matrix model is based, is built on this block structure of the Dirac operator.
The main structure in the phase diagram of QCD is due to the breaking of the chiral symmetry by the mass term in the fermion determinant in Eq. (2.31), called spontaneous symmetry breaking. This symmetry breaking implies a non-vanishing expectation value of the ground state at low energies called chiral condensate, which can be written as

$$\Sigma = \left|\langle 0|\bar{\psi}\psi|0\rangle\right| = \left|\langle 0|\bar{\psi}^+\psi^+ + \bar{\psi}^-\psi^-|0\rangle\right| \neq 0\,. \tag{2.37}$$

The pattern of this symmetry breaking is the same as described with Eq. (2.35). We distinguish mainly between two phases: the phase of preserved chiral symmetry, $\Sigma = 0$, present at high energies where we find so-called quark-gluon plasma and the phase of broken chiral symmetry, $\Sigma \neq 0$, in which quarks and gluons are confined to compound particles. In certain low-energy regions lattice simulations have tested these ideas from first principles along the temperature axis [66]. The introduction of the chemical potential causes a fundamental problem, called *sign problem*. The main consequence is that the chemical potential, quark or baryonic, can't be included in Monte Carlo simulations on a lattice,

see [67] for a review on this subject.
Another approach to the $\Sigma \neq 0$ area is using effective theories. The chiral perturbation theory serves as an approach by making use of low excitation states, see [68]. In particular, the so-called ϵ-regime in chiral perturbation theory, introduced by Gasser and Leutwyler [69], opened the possibility to approach the chiral transition analytically from the low-energy sector.

2.2.2 Random Matrix Models in QCD

RMT provides an approach to the leading order ϵ-regime as an effective theory based on global symmetries in the computation of spectral properties of the Dirac operator. Due to the Banks-Casher relation [70],

$$\Sigma = \lim_{V\to\infty} \frac{\pi}{V} \rho_{\mathcal{D}}(\lambda), \quad \text{for } \lambda \ll 1, \tag{2.38}$$

where V denotes the volume of the condensate, the chiral condensate Σ can be reached through the computation of the eigenvalue density of the Dirac operator $\rho_{\mathcal{D}}$ for small λ. The next term for the chiral condensate is of the order $\mathcal{O}(|\lambda|)$, which was derived by Stern and Smilga [71] and depends in addition on the pion decay constant. Thus, the local behaviour of the Dirac-operator spectrum near the origin provides an access to test chiral symmetry.
It was suggested by Shuryak and Verbaarschot in 1993 [11] to replace the differential operator $\mathcal{W}$ from the formulation in Eq. (2.26) by a complex valued random matrix W of dimensions $N \times (N+\nu)$. The eigenvalues x_j belong now to a complex Gaussian random matrix $WW^\dagger$. The Yang-Mills action, S_{YM} in Eq. (2.31), was replaced by Gaussian distributions for the entries. The chiral and anti-hermitian symmetries of the Dirac operator are preserved in this model:

$$\mathcal{Z}^{(N_f)}_{\text{Shuryak, Verbaarschot}}\left(m_1,\ldots,m_{N_f}\right) = \int [dW]\, \mathrm{e}^{-N \mathrm{Tr}\, WW^\dagger} \prod_{f=1}^{N_f} \det \begin{bmatrix} m_f \mathbb{1}_N & \imath W \\ \imath W^\dagger & m_f \mathbb{1}_{N+\nu} \end{bmatrix}. \tag{2.39}$$

Their motivation was that spectral fluctuations of the Dirac operator in the local scale near the origin become universal in the limit of infinite matrix dimensions, in the sense that the non-linear interaction from the non-abelian gauge field theory of the exact formulation of QCD can be avoided.
The first inclusion of chemical potential in a random matrix model in the application of QCD was formulated by Stephanov in 1996 [72]. The anti-hermiticity of the Dirac operator is broken by including the chemical potential μ as

$$\mathcal{D}_{\text{Stephanov}} = \imath \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix} + \mu \begin{pmatrix} 0 & \mathbb{1}_{N,\nu} \\ \mathbb{1}^\dagger_{N,\nu} & 0 \end{pmatrix}, \tag{2.40}$$

where we call the extended identity matrix: $\mathbb{1}_{N,\nu} = \begin{pmatrix} \mathbb{1}_N & 0 \end{pmatrix}$. The block structure of the original Dirac operator $\mathcal{D}$ in Euclidean space implying the chemical potential μ presented in Eq. (2.36) is carried over. The corresponding partition function with Gaussian probability weight for the random

matrix W reads

$$\mathcal{Z}^{(N_f)}_{\text{Stephanov}}\left(m_1,\ldots,m_{N_f}\right) = \int [dW]\, \mathrm{e}^{-N\mathrm{Tr}\, WW^\dagger} \times \prod_{j=1}^{N_f} \det\left[\begin{pmatrix} m_j \mathbb{1}_N & \imath W + \mu \mathbb{1}_{N,\nu} \\ \imath W^\dagger + \mu \mathbb{1}^\dagger_{N,\nu} & m_j \mathbb{1}_{N+\nu} \end{pmatrix}\right]. \tag{2.41}$$

The random matrix model of Stephanov presented in Eq. (2.41) results in the model by Shuryak and Verbaarschot in Eq. (2.39) by the limit of vanishing chemical potential, $\mu \to 0$. The spectrum of the Dirac operator is for non-vanishing chemical potential affected by the breaking of anti-hermiticity. The eigenvalues of the Dirac operator are for $\mu > 0$ no longer pure imaginary, but they scatter to a two-dimensional picture on the complex plain. Thereby, the extension of the spectrum of the Dirac operator from one dimension to the complex plain has to be studied near the origin.
The RMT approach to the eigenvalue density of the Dirac operator $\rho_{\mathcal{D}}$ at the origin implies the limit of infinite matrix dimensions. Thereby, we distinguish between strong and weak non-hermiticity regimes. In particular, the *weak non-hermiticity* regime, where the chemical potential depends on the matrix dimension as $\mu^2 = \mathcal{O}\left(N^{-1}\right)$.
A further turn in the progress of random matrix models in application to QCD was done by Osborn in 2004 [73]. He introduced a two matrix model by attaching a second complex Wishart matrix on the chemical potential,

$$\mathcal{D}_{\text{Osborn}} = \imath \begin{pmatrix} 0 & W_1 \\ W_1^\dagger & 0 \end{pmatrix} + \mu \begin{pmatrix} 0 & W_2 \\ W_2^\dagger & 0 \end{pmatrix}, \tag{2.42}$$

preserving the hermiticity of the second part of the Dirac operator. The matrices W_1 and W_2 are independent of each other and are of dimensions $N \times (N+\nu)$. For equally distributed entries of both random matrices, the partition function reads:

$$\mathcal{Z}^{(N_f)}_{\text{Osborn}}\left(m_1,\ldots,m_{N_f}\right) = \int [dW_1]\,[dW_2]\, \mathrm{e}^{-N\mathrm{Tr}\left(W_1W_1^\dagger + W_2W_2^\dagger\right)} \times \prod_{j=1}^{N_f} \det\begin{bmatrix} m_j & \imath W_1 + \mu W_2 \\ \imath W_1^\dagger + \mu W_2^\dagger & m_j \end{bmatrix}. \tag{2.43}$$

The eigenvalues of $\mathcal{D}_{\text{Osborn}}$ are now obtained by analogous consideration to Eq. (2.27) through the product $X_1 X_2^\dagger$ with

$$X_1 = \imath W_1 + \mu W_2 \quad \text{and} \quad X_2^\dagger = \imath W_1^\dagger + \mu W_2^\dagger. \tag{2.44}$$

The above-mentioned equations can be seen as a substitution for W_1 and W_2 yielding an equivalent formulation of the partition function in Eq. (2.43),

$$\mathcal{Z}^{(N_f)}_{\text{Osborn}}\left(m_1,\ldots,m_{N_f}\right) = \int [dX_1]\,[dX_2]\, \mathrm{e}^{-N\frac{1+\mu^2}{4\mu^2}\mathrm{Tr}\left(X_1X_1^\dagger + X_2X_2^\dagger\right) - N\frac{1-\mu^2}{4\mu^2}\mathrm{Tr}\left(X_1X_2^\dagger + X_2X_1^\dagger\right)} \times \prod_{j=1}^{N_f} \det\begin{bmatrix} m_j & X_1 \\ X_2^\dagger & m_j \end{bmatrix}. \tag{2.45}$$

The characteristic polynomial for the Dirac operator yields now the product of two coupled Wishart matrices,

$$\det\left[\lambda\mathbb{1}_{2N+\nu}-\mathcal{D}_{\text{Ostborn}}\right]=\det\left[\lambda^2\mathbb{1}_N-X_1X_2^\dagger\right], \tag{2.46}$$

where compared to the characteristic polynomials in Eq. (2.27) a minus sign is regained for vanishing chemical potential, $\mu=0$ in the substitution in Eq. (2.44), where we have $X_2^\dagger=-X_1$. The coupling between these two matrices is parametrised by μ through the term $\left(1-\mu^2\right)/4\mu^2$. For vanishing chemical potential $\mu\to 0$, the one-matrix model of Shuryak and Verbaarschot presented in Eq. (2.39) is reached, whereas if $\mu=1$, the matrices X_1 and X_2 are independent, which represents the maximal non-hermiticity of the Dirac operator. In view of the eigenvalues of the Dirac operator $X_1X_2^\dagger$, the chemical potential $\mu\in(0,1]$ interpolates between one Wishart matrix $WW^\dagger$ and the product of two independent matrices.

2.2.3 Coupled Wishart Matrices with Correlations

Our random matrix model, which is called the product of two coupled Wishart matrices, originates from the model presented in Eq. (2.45). However, it is neither addressed to the singular values of the Dirac operator nor to its eigenvalues. Nonetheless, our analysis is aiming at the singular values of the product $X_1X_2^\dagger$ or equivalently $Y=X_1^\dagger X_2$ appearing in the composition presented above. The first results to singular values of the product of two coupled Wishart matrices have been derived by Akemann and Strahov [74], where the integrable structure of the singular values of $X_1X_2^\dagger$ has been exposed. Moreover, the large-N limit results near the origin were derived by the same authors in [75], where the interpolating property of the chemical potential μ was discussed.
In the course of spectral statistics of the product of two coupled random matrices $X_1X_2^\dagger$, Liu [76] introduced a fixed matrix Ω to parametrise the interpolation between the statistics of independent matrices and coupled matrices. In this work, the issue of finite-rank perturbations as an extension to the model in Eq. (2.45) is introduced, which we will deepen in the following paragraph.
The Gaussian factor of the model created by Osborn under the integral in Eq. (2.45) has been extended by Liu in [76] through the introduction of a fixed matrix Ω parametrising the coupling between X_1 and X_2. We analogously may introduce two further fixed matrices W and Q such that the weight function for X_1 and X_2 reads

$$\mathcal{P}\left(X_1,X_2\right)\propto \mathrm{e}^{-\operatorname{Tr}WX_1^\dagger X_1-\operatorname{Tr}QX_2X_2^\dagger+\operatorname{Tr}\left(\Omega X_1^\dagger X_2+X_2^\dagger X_1\Omega^\dagger\right)}. \tag{2.47}$$

Thereby, the fixed matrices contain the information of the chemical potential and the dependence on the matrix dimension N. In addition, the dimensions of the matrices can be extended to $L\times M$ for X_1 and $M\times M$ for X_2, where $L,M\geq N$. The random matrix model studied by Liu in [76] is given by full degeneracy of W and Q as

$$W\to N\frac{\left(1+\mu^2\right)}{4\mu^2}\mathbb{1}_L\,,\quad Q\to N\frac{\left(1+\mu^2\right)}{4\mu^2}\mathbb{1}_M\,. \tag{2.48}$$

Furthermore, the original model Eq. (2.45) is obtained by setting $L = N$ and by letting the fixed matrix Ω degenerate as

$$\Omega \to N \frac{\left(1-\mu^2\right)}{4\mu^2} \mathbb{1}_N \,. \tag{2.49}$$

In [2] the model presented in Eq. (2.47) is studied in the quenched case $N_f = 0$. Thus, the fermionic determinant does not appear in the corresponding partition function and the formulation of the Dirac operator is left open.
Nonetheless, from the point of view of the unquenched case, the composition of the corresponding Dirac operator, $\mathcal{D}$, is of interest. In particular, a substitution of the form in Eq. (2.44) is needed, from which the dependence on the anti-hermitian part and hermitian part of $\mathcal{D}$ can be pointed out. If we choose that the Dirac operator depends explicitly on the matrix parametrising coupling Ω as

$$\mathcal{D} = \begin{pmatrix} 0 & \Omega X_1 \\ X_2^\dagger & 0 \end{pmatrix}, \tag{2.50}$$

a substitution analogous to Eq. (2.44) can be found, $\Omega X_1 \sim (\imath W_1 + \mu W_2)\,\pi$ and $X_2^\dagger \sim \left(\imath W_1^\dagger + \mu W_2^\dagger\right)\sigma^\dagger$. Here, fixed matrices: π and σ, have again to be taken into account and are related to W and Q as

$$Q = \pi\pi^\dagger, \quad \text{and} \quad W = \Omega^\dagger \sigma^\dagger \sigma\, \Omega \,. \tag{2.51}$$

An equivalent formulation for the partition function to Eq. (2.43), including π and σ, is thus achievable. The role of these fixed matrices, π and σ can be recognised by the analysis of the global symmetries of $\mathcal{D}$ regarding anti-hermiticity and hermiticity. For non-trivial π and σ both symmetries of the Dirac operator in Eq. (2.50) are broken.

3 External Parameters in Matrix Models

In this chapter we discuss the eigenvalue representations of the two random matrix models introduced in Sec. 2.1.2 and in Sec. 2.2.3, which we are the sum of two independent Wishart matrices and the product of two coupled Wishart matrices, respectively. These random matrix models include external fixed matrices, which translate to sets of external parameters in the corresponding eigenvalue representations. By proceeding from the random matrix formulations to eigenvalue representations of our ensembles, unitary group integrals arise, which play a crucial role in their integrability. We present two solvable integral formulas, called the Harish-Chandra/Itzykson–Zuber and the Berezin-Karpelevich integrals. With the help of these integral formulas eigenvalue distribution functions are achievable, which are of determinantal structure ensuring their exact solvability.

3.1 Eigenvalue Representation and Solvable Group Integrals

Here we present the main ideas in the derivation of eigenvalue representations for the two random matrix models introduced in the latter chapter. These random two-matrix models are composed by two rectangular matrices X_1 and X_2 with complex Gaussian distribution. The composite matrix $Y = Y(X_1, X_2)$ is in our consideration again a rectangular matrix with complex entries. We denote the rank of Y by N. The most general way of formulating random two-matrix models unifying the two presented in this thesis might be

$$\mathcal{P}(Y) = \int [dX_1]\,[dX_2]\,\mathcal{P}(X_1, X_2)\,\delta(Y - Y(X_1, X_2))\,, \tag{3.1}$$

where we denote the probability density for X_1 and X_2 by $\mathcal{P}(X_1, X_2)$. For this formulation we make use of the product of Dirac-Delta functions, δ, for all independent real degrees of freedom of the composite matrix Y. The integration has, thus, to be taken over all independent real degrees of freedom, too. The measure $[dX]$ for a rectangular complex matrix, $X \in \mathbb{C}^{N\times M}$, is the flat Lebesgue measure

$$[dX] = \prod_{k=1}^{N}\prod_{l=1}^{M} d\mathrm{Re}\,X_{kl}\, d\mathrm{Im}X_{kl} \tag{3.2}$$

with the decomposition of the k,l entry of X in its real and imaginary components, $X_{kl} = \mathrm{Re}\,X_{kl} + \imath\,\mathrm{Im}\,X_{kl}$. The probability density is normalised to unity with respect to the flat Lebesgue measure as

$$1 = \int [dY]\,\mathcal{P}(Y)\,. \tag{3.3}$$

The singular values of Y are defined by the zeros of the characteristic polynomial of the product of the composite matrix with its self-adjoint,

$$0 = \det\left[y\mathbb{1}_N - YY^\dagger\right] = \prod_{k=1}^{N}(y - y_k)\,, \tag{3.4}$$

where we denote the squared singular values of Y by $y_1, \ldots, y_N$.
As discussed in the introduction, Chap. 1 in Eq. (2.24), the squared singular values of Y or equivalently the eigenvalues of the matrix $H = YY^\dagger$, are real and positive, $y_{k=1,\ldots,N} \in \mathbb{R}_+$. In addition, the matrix $YY^\dagger$ shares non-zero eigenvalues with the matrix $Y^\dagger Y$. The number of zero modes, ν, appears as a dimension of the rectangular matrix $Y \in \mathbb{C}^{N\times(N+\nu)}$. The spectral accordance of these two matrices follows from Sylvester's determinant identity [58]:

$$y^\nu \det\left[y\mathbb{1}_N - YY^\dagger\right] = \det\left[y\mathbb{1}_{N+\nu} - Y^\dagger Y\right]\,, \tag{3.5}$$

where the number of zero modes is always non-negative $\nu \geq 0$. From the definition of singular values and Sylvester's determinant identity presented in Eq. (3.4) and Eq. (3.5) the singular value decomposition can be concluded. By the standard eigenvalue decompositions of the hermitian matrices $YY^\dagger$ and $Y^\dagger Y$ and by taking into account the cyclicity of the determinant operation we obtain immediately the singular value decomposition of Y,

$$Y = V_Y \begin{pmatrix} \Lambda_Y^{\frac{1}{2}} & 0_{N,\kappa} \end{pmatrix} U_Y\,, \quad \text{with } \Lambda_Y = \begin{pmatrix} y_1 & 0 & \ldots & 0 \\ 0 & y_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & y_N \end{pmatrix}, \tag{3.6}$$

where the square root is taken entrywise, $\Lambda_Y^{\frac{1}{2}} = \operatorname{diag}\left(\sqrt{y_1}, \ldots, \sqrt{y_N}\right)$. The two matrices V_Y and U_Y are unitary, their dimensions are according to the presented decomposition and, due to remaining degrees of freedom, their direct product is an element of the unitary coset space $[U(N)\times U(N+\nu)]/U(1)^N$. The unitary matrices V_Y and U_Y comprise eigenvectors of the matrix Y, where we have two sets called left and right eigenvectors. For our purposes, we are interested only in observables that depend solely on singular values.
Let us denote an observable by $\mathcal{O}$. The observable $\mathcal{O}$ is a test function, a Schwartz function on the space $\mathbb{C}^{N\times(N+\nu)}$. The sole dependence on singular values, $\mathcal{O}(Y) = \mathcal{O}(y_1, \ldots, y_N)$, as well as its invariance with respect to permutations among the singular values $\{y_k\}_{k=1}^N$, yields its unitary bi-invariance as

$$\mathcal{O}(Y) = \mathcal{O}(VYU)\,, \quad \text{for all } V \in \mathrm{U}(N) \quad \text{and } U \in \mathrm{U}(N+\nu)\,. \tag{3.7}$$

The condition of unitary bi-invariance ensures the applicability of the diagonalising unitary matrices V_Y and U_Y from the singular value decomposition presented in Eq. (3.6). With the help of the observable $\mathcal{O}$, the expectation value with respect to the probability density $\mathcal{P}(Y)$ for the random matrix Y composed in Eq. (3.1) can be introduced. Taking into account that the observable $\mathcal{O}$ depends solely on the singular values of Y, the joint probability distribution function for the squared

singular values y_1, ..., y_N can be defined via the expectation value as

$$\langle \mathcal{O}(Y) \rangle_Y = \int [dY]\, \mathcal{P}(Y)\, \mathcal{O}(Y) = \int_0^\infty dy_1 \ldots \int_0^\infty dy_N\, P_N(y_1, \ldots, y_N)\, \mathcal{O}(y_1, \ldots, y_N)\,. \tag{3.8}$$

Assuming the unitary bi-invariance of the observable $\mathcal{O}$ is broken, the joint probability distribution function for the singular values $P_N(y_1, \ldots, y_N)$ would vary depending on the eigenspace preferred by the observable, $\mathcal{O}$, in the definition in Eq. (3.8). Thus, the joint probability distribution function $P_N(y_1, \ldots, y_N)$ can only be well defined with the help of an unitary bi-invariant observable as introduced in Eq. (3.7). The joint probability distribution function P_N has obviously to be normalised to unity for probabilistic interpretation. According to the normalisation from Eq. (3.3) we simply have from the above equation that

$$1 = \int_0^\infty dy_1 \ldots \int_0^\infty dy_N\, P_N(y_1, \ldots, y_N)\,. \tag{3.9}$$

The explicit expression for the joint probability distribution function $P_N(y_1, \ldots, y_N)$ for a given random matrix model from $\mathcal{P}(Y)$ is the very first challenge in the question of exact solvability of spectral properties. In this section we want to emphasise the difficulties occurring in its derivation by including external fixed matrices. The unitary bi-invariance as introduced with the aid of our observable in Eq. (3.7) plays a crucial role. Namely, the breaking of the unitary bi-invariance by the probability density $\mathcal{P}(Y)$ causes difficulties in the derivation of the joint probability distribution function P_N.
Let us again underline that the right hand side of the definition of the expectation value in Eq. (3.8) does not depend on eigenvectors of Y, whereas in the matrix formulation it implicitly does, due to the decomposition presented in Eq. (3.6). The role of the flat Lebesgue measure $[dY]$ defined in Eq. (3.2) is crucial. The conjugation of Y by an arbitrary $V \in U(N)$ from the left and an arbitrary $U \in U(N+\nu)$ from the right yields a trivial Jacobian, because the modulus of the determinant of unitary matrices is one, $|\det V| = 1 = |\det U|$,

$$[dY] \to [d(VYU)] = [dY]\,, \quad \text{for } V \in U(N) \quad \text{and } U \in U(N+\nu)\,. \tag{3.10}$$

This property is known from the so-called Haar measures [77, 78]. Nonetheless, choosing the singular value decomposition from Eq. (3.6), the measure $[dY]$ has to be partitioned in its singular value dependent part and the corresponding unitary matrices dependent part, which yields a non-trivial Jacobian [52],

$$[dY] = c \left(\prod_{j=1}^{N} dy_j y_j^\nu \right) \Delta_N(y_1, \ldots, y_N)^2\, d\mu(V_Y)\, d\mu(U_Y)\,. \tag{3.11}$$

We denote with Δ_N the Vandermonde determinant of N elements, which is defined by an ordered product of differences of the implied elements,

$$\Delta_N(y_1, \ldots, y_N) = \prod_{1 \le i < j \le N} (y_j - y_i) = \det \left[y_i^{j-1} \right]_{i,j=1}^{N}\,, \tag{3.12}$$

where the last equality can be seen by elementary raw and column operations. The constant c is singular value independent and $d\mu$ is the appropriate Haar measure with respect to its argument. It is

invariant from the left as well as from the right hand side by a conjugation of a unitary group element according to the given argument. We observe in the decomposition of the measure $[dY]$ presented in Eq. (3.11) that the singular value dependent part factorises from the eigenvector dependent part.
The simplest scenario in the derivation of the joint probability distribution function P_N is if the probability density $\mathcal{P}(Y)$ preserves unitary bi-invariance. Namely then, from the definition in Eq. (3.8), we can write

$$P_N(y_1,\ldots,y_N) = c\left(\prod_{j=1}^{N} dy_j y_j^{\nu}\right)\Delta_N(y_1,\ldots,y_N)^2\,\mathcal{P}\left(\Lambda_Y^{\frac{1}{2}} \quad 0_{N,\kappa}\right), \quad \text{for } \mathcal{P}(Y) = \mathcal{P}(VYU), \quad (3.13)$$

where c can now be computed from the condition given in Eq. (3.9) as an alternative to the integration measure decomposition as introduced in Eq. (3.11).
The factorisation of the eigenvector dependent part from the eigenvalue dependent part in the classical random matrix ensembles, GβE, through the invariance of the probability density under conjugations by unitary matrices was investigated already in the 1960s by Hua [26]. The factorisation of the eigenvectors from the eigenvalues opened the method of orthogonal polynomials which is applied on the pure eigenvalue statistics, investigated especially by Mehta and greatly comprised in his book [40].
The introduction of external fixed matrices into the random matrix model breaks the unitary bi-invariance $\mathcal{P}(Y) \neq \mathcal{P}(VYU)$. The dependence on eigenvectors does not factorise from the probability density $\mathcal{P}(Y)$ under the singular value decomposition as introduced in Eq. (3.6). The joint probability distribution function as defined on the right hand side in Eq. (3.8) can only be achieved by performing integrals over unitary groups which are of the form:

$$\mathcal{I}(y_1,\ldots,y_N) = \int_{[U(N)\times U(N+\nu)]/U(1)^N} d\mu(V_Y)\,d\mu(U_Y)\,\mathcal{P}\left(V_Y\left(\Lambda_Y^{\frac{1}{2}} \quad 0_{N,\kappa}\right)U_Y\right). \quad (3.14)$$

We want to present now the most elementary integral formulas which can be applied to the problem of broken unitary bi-invariance under the integrand as presented in Eq. (3.14). They are called: the Harish-Chandra/Itzykson–Zuber and the Berezin-Karpelevich integrals. For the presentation of the first, we would like to introduce a hermitian external fixed matrix Q of dimensions $N \times N$ with positive definite eigenvalues q_1, …, q_N with the eigenvalue decomposition $Q = V_Q \Lambda_Q V_Q^\dagger$. From here on, we consider the one-side correlated Wishart ensemble as introduced in the context of time-series analysis in Sec. 2.1. Here, we are breaking the unitary bi-invariance from the left in the integral given in Eq. (3.14) with an integrand consisting by an exponential as

$$\int_{[U(N)\times U(N+\nu)]/U(1)^N} d\mu(V_Y)\,d\mu(U_Y)\,\mathrm{e}^{-\mathrm{Tr}\,QV_Y\Lambda_Y V_Y^\dagger} \propto \int_{U(N)} d\mu(V_Y)\,\mathrm{e}^{-\mathrm{Tr}\,\Lambda_Q V_Y \Lambda_Y V_Y^\dagger}. \quad (3.15)$$

In this equation we made use of the Haar measure property discussed in the course of the Eq. (3.10) by a conjugation of the matrix $V_Y \in U(N)$, such that the dependence on V_Q has been absorbed, $V_Y \to V_Q V_Y$. In addition, we factorised the integral over $d\mu(U_Y)$ as the integrand does not depend on $U_Y \in U(N+\nu)/U(1)^N$. The integral over the matrix U_Y in Eq. (3.15) contributes only as a singular value independent constant. We are now in the position to present the Harish-Chandra/Itzykson–

Zuber integral [79, 80]:

$$\int_{U(N)} d\mu(V)\, \mathrm{e}^{-\mathrm{Tr}\,\Lambda_Q V \Lambda_Y V^\dagger} \propto \frac{\det\left[\mathrm{e}^{-y_j q_k}\right]_{j,k=1}^{N}}{\Delta_N(y_1,\ldots,y_N)\,\Delta_N(q_1,\ldots,q_N)}\,. \tag{3.16}$$

The proportionality constant can be derived by the condition of normalised integral over an entire unitary group $\int_{U(N)} d\mu(V) = 1$ and by making use of the L'Hôpital's rule in the limit of vanishing eigenvalues of Q, cf. [2, App. A]. However, the normalisation of the resulting joint probability distribution function, P_N, as presented in Eq. (3.9) is sufficient for our further discussion. Analogously we would like to present the Berezin-Karpelevich integral.

Let us introduce the external fixed matrix Ω of dimensions $(N+\nu)\times N$ with squared singular values δ_1, …, δ_N. The Berezin-Karpelevich integral [81] can be written as

$$\int_{U(N)\times U(N+\nu)} d\mu(V)\, d\mu(U)\, \mathrm{e}^{2\,\mathrm{Re}\,\mathrm{Tr}\, VYU\Omega} \propto \frac{\det\left[(\delta_j y_k)^{-\frac{\nu}{2}} I_\nu\left(2\sqrt{\delta_j y_k}\right)\right]_{j,k=1}^{N}}{\Delta_N(y_1,\ldots,y_N)\,\Delta_N(\delta_1,\ldots,\delta_N)}\,, \tag{3.17}$$

where I_ν denotes the modified Bessel function of the first kind and which can be introduced via its series representation [82, Eq. 8.445],

$$x^{-\frac{\nu}{2}} I_\nu\left(2\sqrt{x}\right) = \sum_{n=0}^{\infty} \frac{x^n}{n!(n+\nu)!}\,. \tag{3.18}$$

We again left away the proportionality constant in our representation of the Berezin-Karpelevich integral. We would like to underline that the unitary bi-invariance of the integrand on the left hand side in Eq. (3.17) is broken twice, namely from both sides.

Integrals over unitary groups of the type of the Harish-Chandra/Itzykson–Zuber integral as in Eq. (3.16) and the Berezin-Karpelevich integral as in Eq. (3.17) were studied in the 1990s by Guhr and Wettig [83] and Jackson, Şener and Verbaarschot [84]. The key tool of integration over unitary groups in the so-called method of character expansion described by Balantekin in 2000 [85] and later on by Schlittgen and Wettig in 2003 [86] can be seen as a general ansatz in approaching unitary group integrals arising in random matrix models. In addition, we would like to recommend the work of Simon, Moustakas and Marinelli from 2005 [87], where unitary group integrals arising in the complex Wishart ensemble by the inclusion of external fixed matrices are comprised.

Generalisations of these two integrals are thinkable, in particular with respect to the functional dependence under the integral. The so-called zonal polynomials already defined in the 1960s in [88] are of use, in which series expansions of functions with matrix arguments yield integrable structure.

The choice of the random matrix models in this thesis is restricted to this type of integrals over unitary groups. For orthogonal or symplectic groups, analogous integrals are in general not available. Thus, the very first choice of $\beta = 2$ ensembles plays a key role on the integrability of the considered matrix models under inclusion of external fixed matrices.

The unitary bi-invariance is in the Berezin-Karpelevich integral presented in Eq. (3.17) broken twice. The integration in the case of twice broken unitary bi-invariance over the appearing unitary groups containing the eigenvectors from the singular value decomposition are in general not performable. For the doubly-correlated Wishart ensemble, see Eq. (2.7), the joint probability distribution function P_N

for finite N and independent of eigenvectors as defined in Eq. (3.8) is not available. Further methods have to be applied for spectral statistics, in particular the method of supersymmetry was successfully applied recently in [57], yielding the singular value density. However, all k-point correlation functions as defined in Eq. (1.6) are not accessible, due to not performable unitary group integrals.

3.2 Sum of Two Independent Wishart Matrices

In this section we discuss the eigenvalue representation of the sum of two independent Wishart matrices introduced in Sec. 2.1.2 as the two-epoch model due to its application in time-series analysis. It is composed by two independent complex Gaussian matrices X_1 and X_2 of dimensions $N \times (N+\nu_1)$ and $N \times (N+\nu_2)$, respectively. The two matrices X_1 and X_2 are distributed according to the one-side correlated Wishart ensemble as introduced in Eq. (2.5),

$$\mathcal{P}(X_1|\Sigma_A) = c_A \, \mathrm{e}^{-\mathrm{Tr}\,\Sigma_A^{-1} X_1 X_1^\dagger} \quad \text{and} \quad \mathcal{P}(X_2|\Sigma_B) = c_B \, \mathrm{e}^{-\mathrm{Tr}\,\Sigma_B^{-1} X_2 X_2^\dagger}, \tag{3.19}$$

normalised to unity by $c_{A/B} = \pi^{-N\left(N+\nu_{1/2}\right)} \det\left[\Sigma_{A/B}\right]^{-\left(N+\nu_{1/2}\right)}$ with respect to the corresponding flat Lebesgue measure from Eq. (3.2). Here, two external fixed matrices are posed, Σ_A and Σ_B, which are both of dimensions $N \times N$ and positive definite. The additional name of our model, two-epoch model, is chosen due to the geometrical composition of the composed matrix,

$$Y = \begin{pmatrix} X_1 & X_2 \end{pmatrix}, \tag{3.20}$$

which is again a complex Gaussian matrix, but now of dimensions $N \times (N+\nu_1+N+\nu_2)$. The first epoch refers to the entries $\left\{Y_{jk}\right\}_{j=1,\ldots,N}^{k=1,\ldots,N+\nu_1}$ which are distributed according to the probability density $\mathcal{P}(X_1|\Sigma_A)$ and analogously the second epoch refers to the entries $\left\{Y_{jN+\nu_1+k}\right\}_{j=1,\ldots,N}^{k=1,\ldots,N+\nu_2}$, distributed according to the probability density $\mathcal{P}(X_2|\Sigma_B)$. For simpler arrangement we denote the number of zero-modes of the composite matrix Y by

$$\kappa = N+\nu_1+\nu_2 \geq 0, \quad \text{with} \;\; \nu_{1/2} \geq 0. \tag{3.21}$$

We ask for the singular values of Y or equivalently the eigenvalues of the positive matrix

$$H = YY^\dagger = X_1 X_1^\dagger + X_2 X_2^\dagger, \tag{3.22}$$

which is the sum of the positive random matrices $X_1 X_1^\dagger$ and $X_2 X_2^\dagger$. For the probability density for Y we could study the double integral as constructed in Eq. (3.1), where the joint density for both matrices X_1 and X_2 is, due to their independence, given by the multiplication
$\mathcal{P}(X_1, X_2) = \mathcal{P}(X_1|\Sigma_A)\mathcal{P}(X_2|\Sigma_B)$. However, we would like to present the eigenvalue representation of the sum of two independent Wishart matrices by the reference to the results derived by Santosh Kumar in 2014 [61]. He considered an analogous double integral, namely for the probability density for $H = YY^\dagger$ as

$$\mathcal{P}(H) = \int [dX_1]\,[dX_2]\,\mathcal{P}(X_1|\Sigma_A)\,\mathcal{P}(X_2|\Sigma_B)\,\delta\left(H - Y(X_1,X_2)\,Y(X_1,X_2)^\dagger\right). \tag{3.23}$$

From this expression Kumar derived in [61] an expression for the probability density $P(H)$, which has two equivalent representations, corresponding to the commutativity of the sum in the definition of H on the right hand side in Eq. (3.22), as

$$\begin{aligned}\mathcal{P}(H) =& c_H \det[H]^{\kappa} \, \mathrm{e}^{-\mathrm{Tr}\, \Sigma_A^{-1} H} \, {}_1\mathcal{F}_1\left(N+\nu_2; N+\kappa; \left(\Sigma_A^{-1} - \Sigma_B^{-1}\right) H\right) \\ =& c_H \det[H]^{\kappa} \, \mathrm{e}^{-\mathrm{Tr}\, \Sigma_B^{-1} H} \, {}_1\mathcal{F}_1\left(N+\nu_1; N+\kappa; \left(\Sigma_B^{-1} - \Sigma_A^{-1}\right) H\right) . \end{aligned} \tag{3.24}$$

These representations are achievable with the help of zonal polynomials as studied by James 1964 [88] yielding the confluent hypergeometric function of matrix arguments ${}_1\mathcal{F}_1$, cf. [89] for its definition.
The ensemble introduced by Kumar in [61] can be seen as a generalisation of the one-side correlated Wishart ensemble. The one-side correlated Wishart ensemble results from the case of time-independent correlations. Namely, in this case the hypergeometric function of matrix arguments results in a matrix independent factor, ${}_1\mathcal{F}_1 = \text{const.}$ for $\Sigma_A = \Sigma_B$. The one-side correlated Wishart ensemble is well known, see e.g. [52]. Thus, we are interested in non-trivial results going beyond the one-side correlated Wishart ensemble, which is the case with unequal correlations $\Sigma_A \neq \Sigma_B$.
A possible formulation of the two-epoch model, by starting at the one-side correlated Wishart ensemble, can be made via the indicator function, $\Sigma \to \Sigma_A \chi_{[1,N+\nu_1]} + \Sigma_B \chi_{[N+\nu_1+1,N+\kappa]}$. From the view of time-series analysis, the correlations induced by this external fixed matrix depend on time. The time evolution of spatial correlations happens here between two time steps. Between the time point $N+\nu_1$ and the time point $N+\nu_1+1$ a threshold of spatial correlations is assumed. The extension to a multiple-epoch model with the help of the one-side correlated Wishart ensemble can obviously be formulated through an extension of this sum over indicator functions by respecting according thresholds. A possible solution of the multiple-epoch model in terms of a supersymmetric multi-matrix model derived in [1, Eq. (4.20)]. However, an exact solution for all k-point correlation functions as defined in the introduction of this work in Eq. (1.6), is in this very general case not available from this expression and, thus, in the sense of integrability discussed here not solvable. The two-epoch model, for arbitrary external fixed matrices, Σ_A and Σ_B, is, analogously to the multiple-epoch model, not solvable either. Nonetheless, there is a non-trivial limit of the two external fixed matrices preserving $\Sigma_A \neq \Sigma_B$, in which the integrability of the Two-Matrix Model is captured, called *half degeneracy*.
In [61] Kumar derived, in the case of half degeneracy, the joint probability distribution function for the eigenvalues of H distributed by the probability density given in Eq. (3.24). The half degeneracy can be described by total degeneracy of the first covariance matrix and with an arbitrary covariance matrix for the second epoch:

$$\Sigma_A = \sigma_A \mathbb{1}_N \quad \text{and} \quad \Sigma_B = \mathrm{diag}\left(\sigma_{B1}, \ldots, \sigma_{BN}\right) . \tag{3.25}$$

Note that, due to the Haar measure property as discussed in the course of Eq. (3.10), the external fixed matrices Σ_A and Σ_B can be chosen in the random matrix model being diagonal and subsequently the second covariance matrix Σ_B remains, after taking the half degeneracy most general by keeping its rank at its dimension, by N. The eigenvalue decomposition of H or singular value decomposition of Y as defined in Eq. (3.6) involves a unitary group integral comparable to the Harish-Chandra/Itzykson–Zuber integral, cf. Eq. (3.16). Kumar derived the joint probability distribution function for the eigenvalues y_1, …, y_N of H by taking into account the half degeneracy as presented in Eq. (3.25),

which reads

$$P_N(y_1,\ldots,y_N) = \frac{1}{Z_N}\Delta_N(y_1,\ldots,y_N)\det\left[y_i^{\kappa}\,\mathrm{e}^{-\frac{y_i}{\sigma_A}}\,{}_1F_1\left(\nu_2+1;\kappa+1;\left(\sigma_A^{-1}-\sigma_{Bj}^{-1}\right)y_i\right)\right]_{i,j=1}^{N}. \quad (3.26)$$

The joint probability distribution function is up to the normalising constant, Z_N, composed by two determinants: the Vandermonde determinant as defined in Eq. (3.12) and a determinant containing the Kummer's confluent hypergeometric function, ${}_1F_1$, which can be introduced through its series expansion at the origin,

$${}_1F_1(a;b;z) = \sum_{k=0}^{\infty}\frac{a!}{(a+k)!}\frac{(b+k)!}{b!}\frac{z^k}{k!}, \quad (3.27)$$

see [82, Eq. 9.14.1] for the definition of generalised hypergeometric series ${}_pF_q$. The parameters N, ν_2 and κ are dimensional parameters from the random matrix model, whereas the parameters $\{\sigma_A,\sigma_{B1},\ldots,\sigma_{BN}\}$ originate from the external fixed matrices and therefore we call those *external fixed parameters*.

Furthermore, Kumar derived in [61] the corresponding normalising constant Z_N, which is a priori given by an N-fold integral as presented in Eq. (3.9). This N-fold integral results in a determinantal expression,

$$Z_N = N!\det\left[(\kappa+i-1)!\sigma_A^{\kappa+i}\,{}_2F_1\left(\kappa+i+1,\nu_2+1;\kappa+1;\left(\sigma_A^{-1}-\sigma_{Bj}^{-1}\right)\sigma_A\right)\right]_{i,j=1}^{N} \quad (3.28)$$

containing the hypergeometric function, ${}_2F_1$, which analogously to the Kummer's confluent hypergeometric function ${}_1F_1$, can be introduced via its Taylor series expansion at the origin,

$${}_2F_1(a_1,a_2;b;z) = \sum_{k=0}^{\infty}\frac{a_1!}{(a_1+k)!}\frac{a_2!}{(a_2+k)!}\frac{(b+k)!}{b!}\frac{z^k}{k!}. \quad (3.29)$$

For the computation of the normalising constant only one single integral has to be performed,

$$\begin{aligned}&\int_0^{\infty} dy y^{\kappa+i-1}\,\mathrm{e}^{-\frac{y}{\sigma_A}}\,{}_1F_1\left(\nu_2+1;\kappa+1;\left(\sigma_A^{-1}-\sigma_{Bj}^{-1}\right)y\right)\\ &= (\kappa+i-1)!\sigma_A^{\kappa+i}\,{}_2F_1\left(\kappa+i+1,\nu_2+1;\kappa+1;\left(\sigma_A^{-1}-\sigma_{Bj}^{-1}\right)\sigma_A\right), \quad (3.30)\end{aligned}$$

cf. [61, Eq. (16)].

The simplification of an N-fold integral to only one single integral is due to an integration formula derived by Andréief in 1886 [90] and plays a prominent role in RMT, see the review [91]. We restate this formula for an explanation of the normalisation constant form given in Eq (3.28) as well as for later purposes.

For two sequences of integrable functions $\{\varphi_j(y)\}_{j=1,\ldots,N}$ and $\{\psi_j(y)\}_{j=1,\ldots,N}$ on the support $\mathbb{R}_+$ Andréief's integration formula reads

$$\int_0^{\infty} dy_1 \ldots \int_0^{\infty} dy_N \det\left[\varphi_j(y_i)\right]_{i,j=1}^{N}\det\left[\psi_j(y_i)\right]_{i,j=1}^{N} = N!\det\left[\int_0^{\infty} dy\varphi_i(y)\psi_j(y)\right]_{i,j=1}^{N}. \quad (3.31)$$

The computations done by Kumar in [61] include an expression for the one-point correlation function as an $(N+1)\times(N+1)$ determinant. For small matrix dimensions N, ν_2 and κ he confirmed his analytical results with numerical simulations. In addition, in his subsequent work [92] he gave an expression for the k-point correlation function as an $(N+k)\times(N+k)$ determinant. For these results the extended Andréief integration formula can be used, derived recently by Kieburg and Guhr in 2010 [93] and which we will restate at a later stage in this work.
The integrability of the two-epoch model under the half degeneracy is indicated by the results presented in [61, 92]. However, these results are not suitable for the analysis of large matrix dimensions, neither for taking the large-N limit nor to study the issue of universality. In particular, in [1] the authors exploit the determinantal structure of the joint probability distribution function P_N presented in Eq. (3.26) by making use of the Borodin's theory of biorthogonal ensembles [94]. In [1] an expression for all k-point correlation functions is given as a $k\times k$ determinant and by the computation of one single double valued function, called correlation kernel.
We would like to close this section with the discussion of the role of the external parameters connecting to further random matrix models. The one-epoch model plays a crucial role in the understanding of the two-epoch model from the view of time-series analysis. As mentioned in the course of the presentation of the probability density $\mathcal{P}(H)$ given in Eq. (3.24), the one-side correlated Wishart ensemble, as introduced in Eq. (2.5), can be achieved by setting the two external fixed matrices to be equal $\Sigma_A = \Sigma_B$. Let us denote the eigenvalues of $\Sigma_A = \Sigma_B$ by σ_1, ..., σ_N. The joint probability distribution function for the eigenvalues of H in this case can be written as

$$P_N(y_1,\ldots,y_N) = \frac{\Delta_N(y_1,\ldots,y_N)\det\left[y_i^\kappa \mathrm{e}^{-y_i/\sigma_j}\right]_{i,j=1}^N}{N!\det\left[(\kappa+i-1)!\sigma_j^{\kappa+i}\right]_{i,j=1}^N}, \quad \text{for } \Sigma_A = \Sigma_B = \mathrm{diag}(\sigma_1,\ldots,\sigma_N). \tag{3.32}$$

Here, for the normalising constant we again made use of Andréief's integration formula [90] as given in Eq. (3.31) and we took the evaluation of the remaining integral from the list of Laplace transform pairs given in [82, Eq. 17.13.2]. The analogy to the joint probability distribution function in the case of half degeneracy presented in Eq. (3.26) is evident. However, on the level of eigenvalue representations of these two models, there is no direct connection leading from Eq. (3.32) to Eq. (3.26) nor vice versa.
A further limit of the external fixed matrices is thinkable, namely the full degeneracy $\Sigma_A = \Sigma_B = \sigma\mathbb{1}_N$. As discussed in the course of the probability density of H in Eq. (3.24), the confluent hypergeometric function of matrix arguments ${}_1\mathcal{F}_1$ becomes a matrix independent factor for $\Sigma_A = \Sigma_B$. In the case of full degeneracy, the covariance matrix is trivial and the matrix density for H is simply proportional to $\det[H]^\kappa \exp\left[-\sigma^{-1}\mathrm{Tr}\,H\right]$, which defines the standard Wishart-Laguerre ensemble, cf. [52], corresponding to the density presented in Eq. (2.4). We regain the standard Wishart-Laguerre ensemble on the level of the eigenvalue representation of the two-epoch model under the half degeneracy, see Eq. (3.26) by translating the full degeneracy to the external fixed parameters $\sigma_A \to \sigma$ and $\sigma_{Bi} \to \sigma$ for all $i = 1$, ..., N, such that

$$P_N(y_1,\ldots,y_N) = \frac{\Delta_N(y_1,\ldots,y_N)\det\left[y_i^\kappa \mathrm{e}^{-y_i/\sigma} y_i^{j-1}\right]_{i,j=1}^N}{N!\det\left[(\kappa+i+j-1)!\sigma^{\kappa+i+j}\right]_{i,j=1}^N}, \quad \text{for } \Sigma_A = \Sigma_B = \sigma\mathbb{1}_N. \tag{3.33}$$

The relation to the half degenerate case can be seen with the help of the L'Hôpital's rule, which we would like to state here in a generalised form in comparison to [2, Appendix A]. Let us take two Taylor expandable functions $f(y)$ and $g(x)$, then the L'Hôpital's rule, connecting the half degenerate case from Eq. (3.26) with the full degenerate case given in Eq. (3.33), reads

$$\lim_{\delta_1,\ldots,\delta_N\to 0} \frac{\det\left[f\left(\delta_i y_j\right)\right]_{i,j=1}^{N}}{\det\left[g\left(\delta_i x_j\right)\right]_{i,j=1}^{N}} = \frac{\Delta_N\left(y_1,\ldots,y_N\right)}{\Delta_N\left(x_1,\ldots,x_N\right)} \prod_{j=0}^{N-1} \frac{f_j}{g_j}\,, \tag{3.34}$$
$$\text{for } \; f(y) = \sum_{j=0}^{\infty} f_j y^j\,, \quad \text{and } \; g(x) = \sum_{j=0}^{\infty} g_j x^j\,.$$

Thus, the set of external parameters, here $\{\sigma_A, \sigma_{B1}, \ldots, \sigma_{BN}\}$, is to be interpreted as a perturbation of a free solution, which is here the standard Wishart-Laguerre ensemble. It is possible to let the external parameters to degenerate in a different way, as for instance $\sigma_{Bj} \to \sigma_B$ for all $j = 1, \ldots, N$, such that only two parameters are left, $\{\sigma_A, \sigma_B\}$. This case and also further thinkable possibilities of degeneracies in the set of external parameters can be considered at the very last stage of our analysis with no loss of generality.

3.3 Product of Two Coupled Wishart Matrices

The product of two coupled Wishart matrices motivated in Sec. 2.2 comprises two random matrices: X_1 of size $M \times L$ and X_2 of size $M \times N$. Analogously to the previously introduced model, these two matrices are akin to the one-side correlated Wishart ensemble from Eq. (2.5). Two external fixed matrices are of covariance type: W of size $L \times L$ and Q of size $M \times M$. However, the probability density for both random matrices is non-factorisable, $\mathcal{P}(X_1, X_2) \neq \mathcal{P}(X_1)\mathcal{P}(X_2)$, which is the reason for calling them coupled. This coupling is parametrised by a third external fixed matrix: Ω of dimensions $N \times L$. We recall the probability density for X_1 and X_2 from Eq. (2.47):

$$\mathcal{P}(X_1, X_2) = c\, \mathrm{e}^{-\mathrm{Tr}\, W X_1^\dagger X_1 - \mathrm{Tr}\, Q X_2 X_2^\dagger + \mathrm{Tr}\left(\Omega X_1^\dagger X_2 + X_2^\dagger X_1 \Omega^\dagger\right)}\,, \tag{3.35}$$

where the normalisation constant reads:

$$c = \pi^{-M(L+N)} \det[W]^M \det\left[Q \otimes \mathbb{1}_N - \mathbb{1}_M \otimes \Omega W^{-1} \Omega^\dagger\right]. \tag{3.36}$$

The dimensions are confined to the convergence conditions:

$$\kappa = L - N \geq 0 \quad \text{and} \quad \nu = M - N \geq 0\,. \tag{3.37}$$

In this model we ask for the singular values of the matrix product:

$$Y = X_1^\dagger X_2\,, \tag{3.38}$$

which is of dimensions $L \times N$. In principle, the probability density for Y or $YY^\dagger$ could be derived by making use of the matrix integrals as presented in Eq. (3.1) or in Eq. (3.23). However, by following the approach to singular values of the product of multiple indpendent Wishart matrices as derived

in [95], we apply a substitution of the form:

$$[dX_1]\,[dX_2]\mathcal{P}\,(X_1,X_2)|_{X_1\to X_1(Y,X_2)} \to [dY]\,[dX_2]\mathcal{P}\,(Y,X_2)\,, \tag{3.39}$$

and there is no need to perform any integral. This procedure is beneficial if the investigation of joint properties of both matrices, Y and X_2, is of interest. In particular, the derivation of the joint probability distribution function for the singular values of Y and simultaneously of X_2 is amenable. In the principal publication [2] this derivation was presented and will be recalled in this work in Sec. 5.2.1. Here, we would like to mention that the two unitary group integrals presented in the beginning of this chapter are sufficient for this computation, namely the Harish-Chandra/Itzykson–Zuber integral from Eq. (3.16) and the Berezin-Karpelevich integral from Eq. (3.17).
The Berezin-Karpelevich integral appears in the coupling term parametrised through Ω, which is of the form $\exp\left[\mathrm{Tr}\left(\Omega Y + Y^\dagger\Omega^\dagger\right)\right]$. The Harish-Chandra/Itzykson–Zuber integral appears in the Gaussian weight for the matrix X_2 including the covariance type matrix Q and reads $\exp\left[-\mathrm{Tr}\,QX_2X_2^\dagger\right]$. The very first Gaussian weight, $\exp\left[-\mathrm{Tr}\,WY\left(X_2X_2^\dagger\right)^{-1}Y^\dagger\right]$, remains to be considered. This factor breaks the bi-unitary invariance in Y twice and is of an analogous form to the doubly-correlated Wishart ensemble. As it was discussed in Sec. 3.1, these unitary group integrals can't be performed for arbitrary W. Thus, in the very general case, which is: full rank external fixed matrices W, Q and Ω, the k-point correlation functions as defined in the introduction of this work in Eq. (1.6) are not integrable.
This problem of non-integrability can be resolved in a non-trivial sense analogously to the half degeneracy in the two-epoch model, see Eq. (3.25). Thereby, one of our covariance-type matrices is full degenerate, $W \sim \mathbb{1}_{N+\kappa}$, whereas the second covariance-type matrix is kept arbitrary, Q. In addition, the coupling-parametrising matrix, Ω, can be hold arbitrary, too. The external fixed matrices can be introduced in their eigenbasis,

$$W = \alpha\mathbb{1}_{N+\kappa}\,, \quad Q = \mathrm{diag}\,(q_1,\ldots,q_{N+\nu}) \quad \text{and} \quad \Omega\Omega^\dagger = \mathrm{diag}\,(\delta_1,\ldots,\delta_N)\,, \tag{3.40}$$

where we took the degeneracy of the external fixed matrix W into account. The Harish-Chandra/Itzykson–Zuber integral can be applied on the so resulting Gaussian weight, $\exp\left[-\alpha\mathrm{Tr}\left(X_2X_2^\dagger\right)^{-1}Y^\dagger Y\right]$.
In our present model we have the three dimensional parameters N, ν and κ, whereas the set of external fixed parameters involved is $\{\alpha, q_1,\ldots,q_{N+\nu},\delta_1,\ldots,q_N\}$. The normalisation constant from Eq. (3.36) can be expressed in terms of the degeneracy value α, the eigenvalues q_1, …, $q_{N+\nu}$ and in the squared singular values δ_1, …, δ_N and yields by its positiveness, $c > 0$, to the condition:

$$\alpha q_j - \delta_i > 0 \quad \text{and} \quad \delta_i > 0\,, \quad \text{for all} \quad i = 1,\ldots,N \quad \text{and} \quad j = 1,\ldots,N+\nu, \tag{3.41}$$

ensuring convergence of our random matrix model. Here, we also added the obvious positiveness of the singular values of the matrix Ω.
The joint probability distribution function for the squared singular values of Y, which we denote by y_1, …, y_N, and simultaneously for the squared singular values of X_2, which we denote by x_1, …, x_N,

is given after choosing the external fixed matrices as described in Eq. (3.40) by

$$P_N(y_1,\ldots,y_N;x_1,\ldots,x_N) = \frac{1}{N!Z_N}\det\left[1,q_i,\ldots,q_i^{\nu-1},\mathrm{e}^{-q_i x_1},\ldots,\mathrm{e}^{-q_i x_N}\right]_{i=1}^{N+\nu} \times \det\left[\frac{1}{x_k}\left(\frac{\alpha y_j}{x_k}\right)^{\kappa}\mathrm{e}^{-\frac{\alpha y_j}{x_k}}\right]_{k,j=1}^{N} \det\left[(\delta_k y_j)^{-\frac{\kappa}{2}} I_\kappa\left(2\sqrt{\delta_k y_j}\right)\right]_{k,j=1}^{N}, \tag{3.42}$$

which is the statement presented in [2, Thm. 1.1]. In the last determinant, the modified Bessel function of the first kind I_κ appears and was introduced in the course of the Berezin-Karpelevich integral with its Taylor series expansion in Eq. (3.18). The joint probability distribution function for the squared singular values of Y solely can be computed by integrating out all singular values of X_2. Therefore, we have to perform an N-fold integration, $\int dx_1 \ldots \int dx_N P_N(y_1,\ldots,y_N;x_1,\ldots,x_N)$, with an integrand consisting of two determinants. This situation is known to us from Andréief's integration formula from Eq. (3.31) with the difference of extended dimensions of the employed determinants. In the joint probability distribution function for the squared singular values of Y and of X_2 in Eq. (3.42) the first determianant is of dimensions $(N+\nu)\times(N+\nu)$, whereas the first ν columns are not involved in the integration. The extended Andréief integration formula as derived in [93] can be applied to the present case. We recall this formula for our purposes in an adjusted shape.
For two sequences of integrable functions $\{\varphi_j(x)\}_{j=1}^{N+\nu}$ and $\{w_j(x)\}_{j=1}^{N}$ on the support $\mathbb{R}_+$ and a set of distinct parameters $\{q_i\}_{i=1,\ldots,N+\nu}$ the extended Andréief integration formula reads

$$\int_0^\infty dx_1 \ldots \int_0^\infty dx_N \det\left[1,q_i,\ldots,q_i^{\nu-1},\varphi_i(x_1),\ldots,\varphi_i(x_N)\right]_{i=1}^{N+\nu} \det\left[w_j(x_i)\right]_{i,j=1}^{N} = N!\det\left[1,q_i,\ldots,q_i^{\nu-1},\int_0^\infty dx\varphi_i(x)w_1(x),\ldots,\int_0^\infty dx\varphi_i(x)w_N(x)\right]_{i=1}^{N+\nu}. \tag{3.43}$$

The N-fold integration over x_1, …, x_N is reduced to one single integration. The joint probability distribution function for the singular values for the product Y solely was obtained in [2, Cor. 1.2] and reads:

$$P_N(y_1,\ldots,y_N) = \frac{1}{Z_N}\det\left[1,q_i,\ldots,q_i^{\nu-1},\hat{\rho}(y_1,\alpha q_i),\ldots,\hat{\rho}(y_N,\alpha q_i)\right]_{i=1}^{N+\nu}\det\left[\psi(y_k,\delta_j)\right]_{k,j=1}^{N}, \tag{3.44}$$

where we call

$$\hat{\rho}(y,\alpha q_i) = 2(\alpha q_i y)^{\frac{\kappa}{2}} K_\kappa\left(2\sqrt{\alpha q_i y}\right), \quad \text{and} \quad \psi(y,\delta_j) = (\delta_j y)^{-\frac{\kappa}{2}} I_\kappa\left(2\sqrt{\delta_j y}\right). \tag{3.45}$$

Here, we denote the modified Bessel function of the second kind by K_κ that results from this one single integration over x in Eq. (3.44), which can be found in [82, Eq. 3.471.9],

$$2z^{\frac{\kappa}{2}} K_\kappa\left(2\sqrt{z}\right) = \int_0^\infty \frac{dt}{t} t^\kappa \mathrm{e}^{-t-\frac{z}{t}}, \quad \text{for } \operatorname{Re} z > 0, \tag{3.46}$$

where we in addition take into account the reflection symmetry in its index $K_{-\kappa} = K_\kappa$ from [82, Eq. 8.486.16] . The normalising constant remains to be presented.
The computation of the normalising constant Z_N is given by the N-fold integral over all singular values of Y as discussed in Eq. (3.9). We again make use of the extended Andréief integration formula presented above, which reduced the problem to one single integral in an expression of a

$(N+\nu)\times(N+\nu)$ determinant. This integral reads in our case

$$2\left(\frac{\alpha q_i}{\delta_j}\right)^{\frac{\kappa}{2}}\int_0^\infty dy K_\kappa\left(2\sqrt{\alpha q_i y}\right) I_\kappa\left(2\sqrt{\delta_j y}\right) = \frac{1}{\alpha q_i - \delta_j}\,, \tag{3.47}$$

which can be found in [82, Eq. 6.576.7], where one has to make use of the relation between the Bessel functions of the first kind, $I_\kappa(z) = \imath^{-\kappa} J_\kappa(\imath z)$ from [82, Eq. 8.406.3]. For the normalising constant we obtain

$$Z_N = N!\det\left[1, q_i, \ldots, q_i^{\nu-1}, \frac{1}{\alpha q_i - \delta_1}, \ldots, \frac{1}{\alpha q_i - \delta_N}\right]_{i=1}^{N+\nu}. \tag{3.48}$$

Moreover, this expression was identified in [2] to the degenerate Cauchy determinant, see [96, Lm. 2], which simplifies our normalising constant to

$$Z_N = N!\,\alpha^{\frac{N(N-1)}{2}+N\nu}(-1)^{N(N+\nu-1)}\frac{\Delta_{N+\nu}(q_1,\ldots,q_{N+\nu})\,\Delta_N(\delta_1,\ldots,\delta_N)}{\prod_{i=1}^{N+\nu}\prod_{j=1}^{N}(\alpha q_i - \delta_j)}\,. \tag{3.49}$$

The Vandermonde determinants $\Delta_{N+\nu}$ and Δ_N, as defined in Eq. (3.12), appearing in the numerator, represent all possible differences within each of the sets of external parameters $\{q_i\}_{i=1,\ldots,N+\nu}$ and $\{\delta_j\}_{j=1,\ldots,N}$, respectively. In the denominator, all possible "mixed" differences appear, i.e. all differences between these two sets of external parameters. Through this multiple product in the above mentioned expression, all possible zeros and poles of the normalising constant are exposed.
Two natural limits for the product Y emerge from the motivation described in Sec. 2.2.2. There, the chemical potential μ plays the crucial role in breaking the anti-hermiticity of the Dirac operator: for $\mu\to 0$ the Dirac operator preserves the anti-hermiticity, whereas for $\mu > 0$ this symmetry is broken. The limit of vanishing chemical potential yields the random matrix model created by Shuryak and Verbaarschot presented in Eq. (2.39), where we rescale the weight by the factor of N for our present discussion. The random matrix model created by Osborn, see Eq. (2.45), allows a varying strength of the chemical potential μ. There, the coupling between X_1 and X_2 vanishes for $\mu\to 1$, such that Y represents the product of two independent Wishart matrices. This interpolation property of the chemical potential μ is carried in our model by the external fixed matrices W, Q and Ω and, consequently, in the eigenvalue representation given in Eq. (3.44) by the external parameters $\{\alpha, q_1, \ldots, q_{N+\nu}, \delta_1, \ldots, \delta_N\}$. The interpolation property provided by the chemical potential in studying the singular values of the product of two coupled Wishart matrices was investigated by Akemann and Strahov in 2016 [74, 75]. There, the integrability of the product of coupled Wishart matrices was exposed and a large-N limit was studied at the origin of the spectrum by keeping the interpolating parameter, which is the chemical potential μ, arbitrary. Our product of two coupled Wishart matrices can be brought to the same form as studied by Akemann and Strahov in the above mentioned publications. Namely, with the help of full degeneracy for all fixed matrices, $W = \alpha\mathbb{1}_{N+\kappa}$, $Q = \alpha\mathbb{1}_M$ and $\Omega\Omega^\dagger = \delta^2\mathbb{1}_N$, and by setting the dimensional parameter κ to zero. By comparison, the dependence of these degeneracy values with respect to μ is

$$\alpha = \alpha(\mu) = \frac{1+\mu}{2\mu} \quad \text{and} \quad \delta = \delta(\mu) = \frac{1-\mu}{2\mu} \quad \text{with} \quad \mu\in(0,1]\,. \tag{3.50}$$

An extension out of the full degenerate case was made by Liu in 2017 [76]. He introduced a non-trivial fixed matrix Ω which parametrises the coupling between the two complex Gaussian matrices X_1 and X_2 and he extended the dimension of X_1 to an arbitrary κ. There, it was shown that the product of two coupled Wishart matrices is still integrable by including these extensions and the interpolating large-N limit at the origin was discussed.
In this development of singular value statistics of the product of two coupled Wishart matrices the model studied in the principal publication [2] is generalised in an arbitrary external fixed matrix Q. It was shown, with the help of W, that an extension to further arbitrary external fixed matrices of the product of two coupled Wishart matrices yields out of its integrable structure. In the present work, integrability is shown for finite-N results for singular values under finite-rank perturbations, implying two sets of external parameters $\{q_i\}_{i=1,\ldots,N+\nu}$ and $\{\delta_j\}_{j=1,\ldots,N}$. The interpolating large-N limit at the origin of the spectrum will also be shown in Chap. 5 as a direct gain from the integrability of this model, but it is not the emphasis of the present work.
Integrability of two further random matrix models was exposed in [2]: the product of two independent Wishart matrices with correlations and the generalised Wishart ensemble.
The joint probability distribution function for the singular values of the product of two independent Wishart matrices with correlations is obtained by considering the joint probability distribution function for the singular values of the product of two coupled Wishart matrices by vanishing coupling $\delta_j \to 0$ for all $j = 1, \ldots, N$ in the expression in Eq. (3.44). Therefore, the L'Hôpital's rule derived in [2, App. A] can be applied,

$$\lim_{\delta_1,\ldots,\delta_N \to 0} \frac{\det\left[f\left(\delta_i y_j\right)\right]_{i,j=1}^{N}}{\Delta_N\left(\delta_1,\ldots,\delta_N\right)} = \Delta_N\left(y_1,\ldots,y_N\right) \prod_{j=0}^{N-1} f_j\,, \quad \text{for } f(y) = \sum_{j=0}^{\infty} f_j y^j\,, \tag{3.51}$$

where we benefit from the Vandermonde determinant depending on $\delta_1, \ldots, \delta_N$ in the normalising constant in Eq. (3.49). Due to the Taylor series representation of the modified Bessel function of the first kind given in Eq. (3.18), we can apply the above recalled L'Hôpital's rule and obtain:

$$P_N\left(y_1,\ldots,y_N\right) = \frac{1}{\widetilde{Z}_N} \det\left[1, q_i, \ldots, q_i^{\nu-1}, \hat{\rho}\left(y_1, \alpha q_i\right), \ldots, \hat{\rho}\left(y_N, \alpha q_i\right)\right]_{i=1}^{N+\nu} \Delta_N\left(y_1,\ldots,y_N\right), \tag{3.52}$$
$$\text{for} \quad \delta_1 = \ldots = \delta_N = 0\,,$$

with unchanged elements $\hat{\rho}$ from Eq. (3.45) and with modified normalising constant:

$$\widetilde{Z}_N = N!\,\alpha^{\frac{N(N-1)}{2}+N\nu}(-1)^{N(N+\nu-1)} \frac{\Delta_{N+\nu}\left(q_1,\ldots,q_{N+\nu}\right)\prod_{j=0}^{N-1} j!(\kappa+j)!}{\prod_{i=1}^{N+\nu}\left(\alpha q_i\right)^N}\,. \tag{3.53}$$

The generalised Wishart ensemble was first introduced by Borodin and Péché in 2008 [97]. Soft-edge properties of its singular values under deformation caused by external parameters were of interest. The representation of the generalised Wishart ensemble as a matrix model reads

$$\mathcal{P}(X) = c\,\mathrm{e}^{-\mathrm{Tr}\left(QXX^\dagger + X\Sigma X^\dagger\right)}\,, \tag{3.54}$$

where X is of dimensions $M \times N$ and the normalising constant is $c = \pi^{-MN}\det\left[Q \otimes \mathbb{1}_N + \mathbb{1}_M \otimes \Sigma\right]$. The joint probability distribution function for the singular values of X follows from the expression

given in Eq. (3.42) by integrating out the singular values of Y by keeping the dependence on the singular values of X_2. Thereby, we again can make use of the Andréief integration formula on the N-fold integral, $\int dy_1 \ldots \int dy_N P_N(y_1,\ldots,y_N;x_1,\ldots,x_N)$, where the original formulation as shown in the course of the sum of two independent Wishart matrices in Eq. (3.31) is needed here. Again, only one single integration remains, which reads for our purposes,

$$\int_0^\infty \frac{dy}{x_k} (\delta_j y)^{-\frac{\kappa}{2}} I_\kappa\left(2\sqrt{\delta_j y}\right) \left(\frac{\alpha y}{x_k}\right)^\kappa \mathrm{e}^{-\frac{\alpha y}{x_k}} = \frac{1}{\alpha} \mathrm{e}^{\frac{\delta_j x_k}{\alpha}}, \tag{3.55}$$

and can be found in [82, Eq. 6.631.4]. We obtain the joint probability distribution function for the squared singular values of X drawn from the density presented in Eq. (3.54),

$$P_N(x_1,\ldots,x_N) = \frac{1}{Z_N} \det\left[1, q_i, \ldots, q_i^{\nu-1}, \mathrm{e}^{-q_i x_1}, \ldots, \mathrm{e}^{-q_i x_N}\right]_{i=1}^{N+\nu} \det\left[\frac{1}{\alpha} \mathrm{e}^{\frac{\delta_j x_k}{\alpha}}\right]_{k,j=1}^{N}, \tag{3.56}$$

with unchanged normalising constant Z_N.

The generalised Wishart ensemble for the matrix X is characterised by two fixed matrices Q and Σ. The eigenvalues of the first fixed matrix Q contribute with the eigenvalues q_1, …, $q_{N+\nu}$, whereas the eigenvalues of the second fixed matrix Σ are essentially the squared singular values of the coupling matrix,$-\delta_1/\alpha$, …, $-\delta_N/\alpha$. Note that by studying the generalised Wishart ensemble solely, the eigenvalues of Σ are not conditioned by the positiveness of the squared singular values $\delta_1 > 0$, …, δ_N. For $\nu = 0$ the corresponding probability distribution function was derived by Borodin and Péché [97, Eq. (15)]. Nonetheless, a solution in the sense of a finite-N expression for all k-point correlation functions was not presented.

All these three ensembles were solved in [2] independently. We will show in the present work that the solution of our main random matrix model, which is the product of two coupled Wishart matrices, provides immediately a solution to the two subsequent models and, thus, the integrability of the product of two coupled Wishart matrices involves the integrability of the product of two independent Wishart matrices and of the generalised Wishart ensemble.

4 Determinantal Point Processes

The joint probability distribution functions of the sum of two independent Wishart matrices in Eq. (3.26) and of the product of two coupled Wishart matrices in Eq. (3.44), are stochastic point processes for finite sets of elements. Moreover, these point processes are determinantal, due to the outcome of the solvable integrals over unitary groups emerging from singular value decompositions. The exact solvability of the studied models goes hand in hand with the fact of resulting determinantal correlation functions.
Our considerations are based on the main ideas from the theory of a subclass of determinantal point processes, the so-called *biorthogonal ensembles* introduced by Borodin in 1998 [94]. In particular, we will emphasise the work of Zinn-Justin from 1997 [98, 99] playing a key role in the analysis of the present ensembles. We will rederive the relevant results from the theory of biorthogonal ensembles. Nonetheless, the majority of the content in this chapter is taken from the principal publications [1, 2] and prepares the investiagtion of the said models. For a broader overview about determinantal point processes we recommend the work by Johansson [100] as a connection to random growth processes and RMT, and a detailed overview by Soshnikov [101].

4.1 Determinantal Correlation Functions

A point process is called determinantal if all k-point correlation functions of a sequence of separable and pairwise distinct elements $\{\lambda_j\}_{j=1}^{N}$ can be written as a determinant, comprising a matrix of size $k \times k$ containing one function of two variables K_N which is called *correlation kernel*,

$$R_k\left(\lambda_1, \ldots, \lambda_k\right) = \det\left[K_N\left(\lambda_m, \lambda_n\right)\right]_{m,n=1}^{k} . \tag{4.1}$$

Two particular values for k are special. The determinantal structure on the right hand side results in a simple scalar, if only one single external variable remains, $k = 1$. We call the correlation kernel, evaluated at one variable in both arguments *spectral density*,

$$R_1(\lambda) = K_N(\lambda, \lambda) = \rho_N(\lambda) , \tag{4.2}$$

describing the density of the elements at the position λ. The N-point correlation function, which can be interpreted as the probability to find the elements at positions λ_1, ..., λ_N, is the joint probability distribution function P_N resulting from $k = N$,

$$R_N\left(\lambda_1, \ldots, \lambda_N\right) = \det\left[K_N\left(\lambda_m, \lambda_n\right)\right]_{m,n=1}^{N} = N!\, P_N\left(\lambda_1, \ldots, \lambda_N\right) . \tag{4.3}$$

The solution of a determinantal point process is obviously given by an explicit expression for the correlation kernel K_N. The correlation kernel is not unique. As a direct consequence of the definition

in Eq. (4.1) a gauge invariance of the kernel induces an equivalence class for the solution

$$\left\{\widetilde{K}_N \in \mathcal{K} | \widetilde{K}_N \sim K_N\right\}, \quad \text{with} \quad \sim\colon K_N(x,y) \to \widetilde{K}_N(x,y) = \frac{f(x)}{f(y)} K_N(x,y), \tag{4.4}$$

where f is chosen to be non-zero. Nonetheless, the representation of one member of this equivalence class is sufficient as a statement of the end result.
In view of the definition of the k-point correlation kernel in Eq. (1.6), the integration over $(N-k)$ elements from the joint probability distribution function P_N reduces to an expression that is in its outer structure independent of the matrix dimension N. System specific dependencies are all included in K_N.

4.2 Biorthogonal Ensembles

The joint probability distribution function for the elements $\{\lambda_1, \ldots, \lambda_N\}$ is a biorthogonal ensemble as defined in [94], if it is of the form:

$$P_N(\lambda_1, \ldots, \lambda_N) = \frac{\det\left[\varphi_i(\lambda_j)\right]_{i,j=1}^N \det\left[\psi_j(\lambda_i)\right]_{i,j=1}^N}{N! \det\left[g_{ij}\right]_{i,j=1}^N}, \tag{4.5}$$

where the normalising constant is given by the Gram matrix,

$$g_{ij} = \int_0^\infty d\lambda \, \varphi_i(\lambda) \psi_j(\lambda). \tag{4.6}$$

Here, we took already into account that the support of the elements λ_1, …, λ_N is the positive real line for all models presented in this work.
In [94] it was shown that biorthogonal ensembles belong to determinantal point processes by making use of the reproducing property of the correlation kernel, which reads

$$\int dx \, K_N(x,y) f(x) = f(y) \quad \text{and} \quad \int dy \, K_N(x,y) f(y) = f(x), \tag{4.7}$$

for the appropriate support of the respective integrals. This property was already of great use in solving classical random matrix ensembles, yielding determinantal point processes in their spectral statistics, which belong to biorthogonal ensembles. In particular, the reproducing property together with a bounded one-point function, $\int_0^\infty d\lambda \rho_N(\lambda) < \infty$ can be used for the proof of the relation in Eq. (4) as presented in [40, p. 89], where, in addition, classical random matrix ensembles exhibit hermitian correlation kernels, $K_N(x,y) = (K_N(y,x))^*$. Determiantal point processes arise especially in $\beta = 2$ ensembles of the classical categorisation of random matrix models and enable the access to their spectral statistics with the help of classical orthogonal polynomials.
The essential relation from the biorthogonal ensemble defined in Eq. (4.5) for its solution, as stressed in Borodin's paper, is the *double-sum representation* of the correlation kernel, which reads

$$K_N(x,y) = \sum_{i,j=1}^N \varphi_i(x) c_{ij} \psi_j(y), \quad \text{with} \quad c^\top = g^{-1}. \tag{4.8}$$

The solution for the correlation kernel and, thus, for all k-point correlation functions of a biorthogonal ensemble is brought to a standard problem of linear algebra, which is the inversion of a deterministic matrix, g^{-1}. The inversion of the Gram matrix, in order to compute the correlation kernel of a determinantal point process, was an observation made previously by Zinn-Justin in the course of [98, Eq. (3.7)]. However, the formulation of the general theory by Borodin in [94] can be seen from the historical view as the cornerstone in the analysis of biorthogonal ensembles.
The connection to the method of orthogonal polynomials was made in [94], too, via the invariances of the joint probability distribution function in Eq. (4.5) and is crucial for the analysis of a part of the ensembles studied in this work.
A biorthogonal ensemble is functionally characterised by two sets of functions $\{\varphi_j\}_{j=1,\dots,N}$ and $\{\psi_j\}_{j=1,\dots,N}$. For each of the sets of functions a rescaling can be performed in the form:

$$\varphi_j(x) \to f(x)\varphi_j(x) \quad \text{and} \quad \psi_j(x) \to f^{-1}(x)\psi_j(x)\,, \quad \text{for all} \;\; j = 1,\dots,N\,, \tag{4.9}$$

which keeps the joint probability distribution function invariant. This invariance corresponds to the equivalence class stated in Eq. (4.4). In addition, there is an algebraic invariance.
The linear span of each of the two sets can also be expanded by different sets of functions, where those are given by

$$\phi_k = \sum_{l=1}^{N} \alpha_{kl}\varphi_l \quad \text{and} \quad \chi_k = \sum_{l=1}^{N} \beta_{kl}\psi_l\,, \quad \text{with} \;\; \det[\alpha] \neq 0 \neq \det[\beta]\,, \tag{4.10}$$

such that $\mathrm{Span}\{\varphi_1,\dots,\varphi_N\} = \mathrm{Span}\{\phi_1,\dots,\phi_N\}$ and $\mathrm{Span}\{\psi_1,\dots,\psi_N\} = \mathrm{Span}\{\chi_1,\dots,\chi_N\}$, which keeps the joint probability distribution function invariant. This invariance is equivalent to the invariance of any determinant with respect to elementary row and column operations, and the multi-linearity of a determinant.
By making use of this invariance, the functional dependence in the joint probability distribution function is characterised by two new sets of functions $\{\phi_j\}_{j=1,\dots,N}$ and $\{\chi_j\}_{j=1,\dots,N}$. By comparison, the definition of the Gram matrix in Eq. (4.6), together with the double-sum representation of the correlation kernel in Eq. (4.8) are modified to

$$K_N(x,y) = \sum_{i,j=1}^{N} \phi_i(x)\tilde{c}_{ij}\chi_j(y)\,, \quad \text{with} \;\; \tilde{c}^{\top} = (\alpha g \beta)^{-1}\,. \tag{4.11}$$

With the change from $\{\varphi,\psi\} \to \{\phi,\chi\}$ we define a transformation property for later purposes by keeping the denominator of the joint probability distribution function P_N in Eq. (4.5) fixed,

$$\det[\alpha\beta] = 1\,. \tag{4.12}$$

With this property, any factor which is absorbed by one of the determinants in the expression in Eq. (4.5), needs to be factorised out from the second determinant.
In Eq. (4.11) the problem of the inversion of the Gram matrix g is carried over to the inversion of the new Gram matrix $\alpha g \beta$. The simplest case in which the modified Gram matrix can be inverted is the case of a diagonal Gram matrix. There, we find that the inversion can be done with simple

multiplicative inverses of the diagonal elements,

$$\alpha g \beta = \operatorname{diag}(h_1, \ldots, h_N)\,, \quad \text{and} \;\; \tilde{c} = \operatorname{diag}\left(h_1^{-1}, \ldots, h_N^{-1}\right). \tag{4.13}$$

Having this diagonal matrix, the double-sum representation for the correlation kernel turns to a *single-sum representation*,

$$K_N(x,y) = \sum_{j=1}^{N} \frac{1}{h_j} \phi_j(x) \chi_j(y)\,, \quad \text{with} \;\; h_j \delta_{ij} = \int_0^\infty d\lambda\, \phi_i(\lambda) \chi_j(\lambda)\,, \tag{4.14}$$

where the integral defining the modified Gram matrix turns to a generalised orthogonality condition, called *biorthogonality condition*. The technique of constructing sets of functions orthogonal to each other, in order to obtain a single-sum kernel, is the method of orthogonal polynomials.
Although it was already shown in [94] that a biorthogonal ensemble is a determinanal point process, as well as that the inversion of the Gram matrix provides a solution to the correlation kernel, we would like to present here an alternative approach to the proof of these two statements, that is taken from the principal publication [1].
The definition of the k-point correlation function was introduced in Eq. (1.6) as a multiple integral over $N-k$ elements with the joint probability distribution function of N elements in the integrand. Applying this integration on two determinants, as it is the case for the biorthogonal ensemble defined in Eq. (4.5), we can make use of the extended Andréief integration formula, which was already mentioned in the previous chapter. For our present aim we modify the original formula from [93] to

$$\begin{aligned} &\int_0^\infty d\lambda_{k+1} \ldots \int_0^\infty d\lambda_N \det\left[\varphi_i(\lambda_j)\right]_{i,j=1}^N \det\left[\psi_j(\lambda_i)\right]_{i,j=1}^N \\ &= (N-k)!(-1)^{k^2} \det \begin{bmatrix} 0_{k,k} & \psi_j(\lambda_i)|_{i=1,\ldots,k}^{j=1,\ldots,N} \\ \varphi_i(\lambda_j)|_{i=1,\ldots,N}^{j=1,\ldots,k} & \int_0^\infty d\lambda\, \varphi_i(\lambda)\psi_j(\lambda)|_{i=1,\ldots,N}^{j=1,\ldots,N} \end{bmatrix}, \end{aligned} \tag{4.15}$$

where $0_{k,k}$ stands for the null-valued matrix of the indicated dimensions, being in the left upper block of the determinant. In this relation we left out the combinatorial factor of $N!/(N-k)!$ on the left hand side as well as the normalisation of the biorthogonal ensemble, $1/(N! \det\left[g_{ij}\right]_{i,j=1}^N)$, on the right hand side. In the right lower block of the resulting determinant we find the Gram matrix g from Eq. (4.6). From the well known block determinant identity, $\det \begin{bmatrix} 0 & \psi \\ \varphi & g \end{bmatrix} = \det[g] \det\left[0 - \psi g^{-1} \varphi\right]$, which reduces the size of the resulting determinant to the dimensions of the present null-valued block, we obtain from the $(N-k)$-fold integration from Eq. (1.6) the k-point correlation function

$$R_k(\lambda_1, \ldots, \lambda_k) = (-1)^{k^2} \det \left[-\sum_{i,j=1}^{N} \psi_j(\lambda_n) \left(g^{-1}\right)_{ji} \varphi_i(\lambda_m) \right]_{m,n=1}^{k}. \tag{4.16}$$

The minus sign in front cancels the minus sign in the determinant. We identify the desired expression in Eq. (4.1) with the double-sum representation for the correlation kernel from Eq. (4.8), which finishes the proof.
The approach to the k-point correlation function, by making use of the extended Andréief integral formula, leads us to a further gain, which was presented in [1].

The comparison between the correlation kernel evaluated at one variable in Eq. (4.2), with the definition for the one-point correlation function from the introduction in Eq. (1.6) for $k=1$, allows the conclusion of arriving at the correlation kernel after performing an $(N-1)$-fold integration. By distinguishing the remaining one external variable through its belonging to the first or to the second determinant, we have an $(N-1)$-fold integration representation for the correlation kernel:

$$K_N(x,y) = \int_0^\infty d\lambda_2 \ldots \int_0^\infty d\lambda_N \frac{\det\left[\varphi_i(x)|_{i=1,\ldots,N} \quad \varphi_i(\lambda_j)|_{i=1,\ldots,N}^{j=2,\ldots,N}\right] \det\begin{bmatrix} \psi_j(y)|^{j=1,\ldots,N} \\ \psi_j(\lambda_i)|_{i=2,\ldots,N}^{j=1,\ldots,N} \end{bmatrix}}{(N-1)!\det\left[g_{ij}\right]_{i,j=1}^N}, \quad (4.17)$$

which directly follows from the proof stated above.
Coming back to the double-sum representation for the correlation kernel, see Eq. (4.8), the inversion of the Gram matrix is the central task. In the case of a 2×2 matrix the inversion can be done explicitly, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, such that $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} / (ad-bc)$, which can be proven by a direct multiplication, whereby the matrix A is invertible for $ad-bc = \det A \neq 0$. For larger matrices, of the order $N\times N$ for our purposes, this inversion can be generalised to an algebraic solution [102], which is of the form

$$\left(A^{-1}\right)_{ij} = \frac{(-1)^{i+j}}{\det A} M_{ji}, \quad \text{with} \quad M_{ij} = \det\left[A_{mn}\right]_{1\leq m\neq i\leq N}^{1\leq n\neq j\leq N}, \quad \text{for} \quad i,j=1,\ldots,N, \quad (4.18)$$

where $\det A \neq 0$ and M_{ij} are called *minors*.
The approach to the inverse Gram matrix by computing minors is in general complicating the situation, because we lose the benefit of the reduction of the k-point correlation function being N independent in its outer structure, see Eq. (4.1). We namely would have to compute N^2+1 determinants, whereby N^2 of those are of dimensions $(N-1)\times(N-1)$, and one of those is of dimensions $N\times N$. The dependence on the matrix dimension is not suitable for studying spectral properties of large matrices, unless there exists a simplifying representation of these cumbersome minors. In our main reference here, this case was pointed out for the Gram matrix being a Cauchy matrix [94, Lem. 3.1]. For later purposes we would like to recall this result:

$$g_{ij} = \frac{1}{q_i+\sigma_j}, \quad \text{such that} \quad c_{ij} = \frac{(q_i+\sigma_i)(q_j+\sigma_j)}{q_j+\sigma_i} \prod_{k=1,k\neq j}^N \frac{q_k+\sigma_j}{\sigma_j-\sigma_k} \prod_{k=1,k\neq i}^N \frac{q_i+\sigma_k}{q_i-q_k}, \quad (4.19)$$

with $q_i+\sigma_j \neq 0$ for all $i,j=1,\ldots,N$. In addition, we would like to mention that the earliest work presenting the inversion of the Cauchy matrix is the paper by Schechter in 1959 [103]
However, the inverse Gram matrix can't be generally presented in such a closed form and we aim at an implicit inversion of the Gram matrix. The way of the derivation of the correlation kernel by an implicit inversion of the Gram matrix depends on the main characteristics of the given ensemble, which moves us to distinguish different classes of ensembles.

4.3 Classes of Biorthogonal Ensembles

Biorthogonal ensembles emerging from random matrix models, that include external fixed matrices, have particular characteristics, which will be deepened in this section by reference to different classes.

We can't claim completeness by listing all classes, but several particular ones will help to emphasise the central properties of the present members.
We recall all joint probability distribution functions occurring in this work that are biorthogonal ensembles: the sum of two independent Wishart matrices in Eq. (3.26), the one-side correlated Wishart ensemble in Eq. (3.32), the product of two coupled Wishart matrices in Eq. (3.44), the product of two independent Wishart matrices in Eq. (3.52) and the generalised Wishart ensemble in Eq. (3.56).
When we now focus on the joint probability distribution function of the singular values of the product of two coupled Wishart matrices presented in Eq. (3.44), the extended dimension of the first determinant catches our eye. We introduce the *biorthogonal ensemble of non-equal sizes* as an extension of the definition presented in Eq. (4.5) for $\nu > 0$ by

$$P_N(\lambda_1,\ldots,\lambda_N) = \frac{\det\left[q_{i1}, q_{i2},\ldots,q_{i\nu},\varphi_i(\lambda_1),\ldots,\varphi_i(\lambda_N)\right]_{i=1}^{N+\nu}\det\left[\psi_j(\lambda_i)\right]_{i,j=1}^{N}}{N!\det\left[q_{i1},q_{i2},\ldots,q_{i\nu},g_{i1},\ldots,g_{iN}\right]_{i=1}^{N+\nu}}, \tag{4.20}$$

where the set $\{q_{ij}\}_{j=1,\ldots,N+\nu}^{i=1,\ldots,\nu}$ is kept fixed. The normalisation of the biorthogonal ensemble of non-equal sizes can be proven with the help of the extended Andréief integration formula as presented in Eq. (3.43). A member of this class of biorthogonal ensembles was pointed out by Simon, Moustakas and Marinelli in 2005 [87, Eq. (39)] in the computation of the joint probability distribution function of the singular values of the one-side correlated Wishart ensemble, equivalent to Eq. (2.7) with trivial spatial covariances $\Sigma = \mathbb{1}_N$. Thereby, the rank of the external fixed matrix Π is larger than the rank of the considered random matrix. Analogously, the generalised Wishart ensemble may become a biorthogonal ensemble of non-equal sizes, which was observed by Borodin and Péché in 2008 [97, Eq. (12)], equivalent to Eq. (3.54), where again the rank of one of the fixed matrices is larger than the rank of the considered random matrix. In the product of two coupled Wishart matrices the rank of the external fixed matrix Q with $\nu > 0$ is larger than the rank of the considered random matrix Y yielding the extended dimension of the first determinant in Eq. (3.44).
The extended dimension of the first determinant in Eq. (4.20) dominates the formulation of the solution. In the principal publication [2] the biorthogonal ensemble of non-equal sizes was shown to be indeed a biorthogonal ensemble in the sense of Borodin as defined here in Eq. (4.5). In addition, an analogous double-sum representation for the correlation kernel to Eq. (4.8) has been found:

$$K_N(x,y) = \sum_{i=1}^{N+\nu}\sum_{j=1}^{N}\varphi_i(x)c_{ij}\psi_j(y), \quad \text{with } g^\top c = \mathbb{1}_N. \tag{4.21}$$

The proof that the k-point correlation function obeys the determinantal form, classifying it to be a determinantal point process, see property given in Eq. (4.1), follows from the same chain of arguments as described above from Eq. (4.15) to Eq. (4.17),

$$\begin{aligned} R_k(\lambda_1,\ldots,\lambda_k) =& \frac{\int_0^\infty d\lambda_{k+1}\ldots\int_0^\infty d\lambda_N \det\left[q_{i1},q_{i2},\ldots,q_{i\nu},\varphi_i(\lambda_1),\ldots,\varphi_i(\lambda_N)\right]_{i=1}^{N+\nu}\det\left[\psi_j(\lambda_i)\right]_{i,j=1}^{N}}{(N-k)!\det\left[q_{i1},\ldots,q_{i\nu},g_{i1},\ldots,g_{iN}\right]_{i=1}^{N+\nu}} \\ =& \frac{(-1)^{k^2}\det\begin{bmatrix} 0_{k,k} & 0_{k,\nu} & \psi_j(\lambda_i)|_{i=1,\ldots,k}^{j=1,\ldots,N} \\ \varphi_i(\lambda_j)|_{i=1,\ldots,N+\nu}^{j=1,\ldots,k} & q_{ij}|_{i=1,\ldots,N+\nu}^{j=1,\ldots,\nu} & g_{ij}|_{i=1,\ldots,N+\nu}^{j=1,\ldots,N} \end{bmatrix}}{\det\left[q_{i1},\ldots,q_{i\nu},g_{i1},\ldots,g_{iN}\right]_{i=1}^{N+\nu}}, \end{aligned} \tag{4.22}$$

yielding the double-sum representation of the correlation kernel as stated in Eq. (4.21).
It is not obvious how to compute a rectangular matrix c, here of dimensions $(N+\nu)\times N$, being inverse to the rectangular Gram matrix g of the same dimensions, solely in the multiplication from one side $g^\top c = \mathbb{1}_N$. From the formulation in the latter equation, it is beneficial to define the *Gram type matrix*,

$$G = \begin{pmatrix} q_{11} & \cdots & q_{1\nu} & g_{11} & \cdots & g_{1N} \\ \vdots & & \vdots & \vdots & & \vdots \\ q_{N+\nu 1} & \cdots & q_{N+\nu\nu} & g_{N+\nu 1} & \cdots & g_{N+\nu N} \end{pmatrix}. \tag{4.23}$$

From this construction on, we are able to define its inverse,

$$C^\top = G^{-1}, \quad \text{with} \quad C = \begin{pmatrix} C_{11} & \cdots & C_{1\nu} & c_{11} & \cdots & c_{1N} \\ \vdots & & \vdots & \vdots & & \vdots \\ C_{N+\nu 1} & \cdots & C_{N+\nu\nu} & c_{N+\nu 1} & \cdots & c_{N+\nu N} \end{pmatrix}, \tag{4.24}$$

which will be in use in detailed calculations in the next chapter.
In all random matrix ensembles mentioned above and studied in this work, belonging to biorthogonal ensembles of non-equal sizes, the first ν columns of the Gram type matrix q_{ij} are always given by Vandermonde type terms, see Eq. (3.12),

$$q_{ij} = q_i^{j-1}, \quad \text{for} \quad i = 1,\ldots,N+\nu, \quad \text{and} \quad j = 1,\ldots,\nu. \tag{4.25}$$

However, their specification is not required for the full solution, because these ν columns are absent in the correlation kernel presented in Eq. (4.21). We now turn to a classification of biorthogonal ensembles with respect to the functional dependence on the elements φ_i and ψ_j.
By recalling the joint probability distribution functions of the sum of two independent Wishart matrices from Eq. (3.26), the one-side correlated Wishart ensemble from Eq. (3.32), the standard Wishart-Laguerre ensemble from Eq. (3.33) and the product of two independent Wishart matrices from Eq. (3.52), we observe that one of the singular value dependent determinants is a Vandermonde determinant,

$$P_N(\lambda_1,\ldots,\lambda_N) = \frac{\det\left[\varphi_i(\lambda_j)\right]_{i,j=1}^N \Delta_N(\lambda_1,\ldots,\lambda_N)}{N!\det\left[\int_0^\infty d\lambda \lambda^{j-1}\varphi_i(\lambda)\right]_{i,j=1}^N}. \tag{4.26}$$

as defined in Eq. (3.12). A biorthogonal ensemble in the sense of Borodin, see Eq. (4.5), with the property of polynomial dependence in one of the two involved sets of functions is called *polynomial ensemble*. This class of biorthogonal ensembles was introduced by Kuijlaars and Stivigny in 2014 [104] in particular due to a beneficial feature emerging from the polynomial property, called *Heine formula* [40], contributing to the computation of the correlation kernel for a member of this class. We recall the Heine formula with techniques known to the reader from above discussions.
The Heine formula can be presented by making use of the introduced notion of a single-sum representation of the correlation kernel, see Eq. (4.14). Thereby, the elements χ_j are composed as a linear combination of the elements $\psi_j(\lambda) = \lambda^{j-1}$. Thus, the elements χ_j are polynomials. It is well known, that by choosing these polynomials being monic, the Vandermonde determinant can be arranged with

the help of elementary row and column operations to:

$$\Delta_N(\lambda_1,\ldots,\lambda_N) = \det\left[\chi_j(\lambda_k)\right]_{j,k=1}^N, \quad \text{with} \quad \chi_j(\lambda) = \lambda^{j-1} + \text{lower order terms}, \tag{4.27}$$

which is equivalent to the second equality in Eq. (3.12). In the same equation, the first equality provides a further property that allows us to absorb a characteristic polynomial by the Vandermonde determinant through increased dimension,

$$\Delta_N(\lambda_1,\ldots,\lambda_N)\prod_{j=1}^N(\lambda-\lambda_j) = \Delta_{N+1}(\lambda_1,\ldots,\lambda_N,\lambda)\,. \tag{4.28}$$

We now consider the expectation value of one characteristic polynomial in a polynomial ensemble with biorthogonalised sets of functions χ_j and ϕ_i as in Eq. (4.14),

$$\begin{aligned}\int_0^\infty \prod_{k=1}^N d\lambda_k P_N(\lambda_1,\ldots,\lambda_N)\prod_{j=1}^N(\lambda-\lambda_j) &= \int_0^\infty \prod_{k=1}^N d\lambda_k \frac{\det\left[\phi_i(\lambda_j)\right]_{i,j=1}^N \Delta_{N+1}(\lambda_1,\ldots,\lambda_N,\lambda)}{N!\prod_{j=1}^N h_j} \\ &= \int_0^\infty \prod_{k=1}^N d\lambda_k \frac{\det\left[\phi_i(\lambda_j)\right]_{i,j=1}^N \det\left[\chi_j(\lambda_1),\ldots,\chi_j(\lambda_N),\chi_j(\lambda)\right]_{j=1}^{N+1}}{N!\prod_{j=1}^N h_j}\,.\end{aligned} \tag{4.29}$$

We again can make use of the extended Andréief integration formula in the very last step, such that the integration is involved in a determinantal expression comprising the elements ϕ_i and χ_j. Due to the biorthogonality condition, $\delta_{ij}h_i = \int_0^\infty d\lambda\phi_i(\lambda)\chi_j(\lambda)$, we arrive at one single element $\chi_{N+1}(\lambda)$. Moreover, because of the definition of the expectation value, as defined in the previous chapter in Eq. (3.8), the left hand side in the upper relation can be brought to an expectation value with respect to the corresponding random matrix, instead of its singular values only. Summarising, the Heine formula for polynomial ensembles reads

$$\left\langle \det\left[\lambda\mathbb{1}_N - YY^\dagger\right]\right\rangle_Y = \chi_{N+1}(\lambda)\,, \tag{4.30}$$

with elements $\{\chi_j\}_{j=1,\ldots,N}$ being part of the single-sum representation of the correlation kernel as given in Eq. (4.14), if the joint probability distribution function for the singular values of Y belongs to the class of polynomial ensembles.

The Heine formula may serve as a key tool in the derivation of the single-sum correlation kernel. However, for the full solution the second set of functions $\{\phi_i\}_{i=1,\ldots,N}$ has to be derived originating from $\{\varphi_i\}_{i=1,\ldots,N}$, too. Therefore, we continue our list with subclasses of polynomial ensembles by specifying the set of functions, $\{\varphi_i\}_{i=1,\ldots,N}$.

The most prominent class of polynomial ensembles are the so-called *orthogonal ensembles* [105], where we find a second Vandermonde determinant in the joint probability distribution function, with $\varphi_i(\lambda) = w(\lambda)\lambda^{i-1}$, such that

$$P_N(\lambda_1,\ldots,\lambda_N) = \frac{\Delta_N(\lambda_1,\ldots,\lambda_N)^2\prod_{j=1}^N w(\lambda_j)}{N!\det\left[\int_0^\infty d\lambda\lambda^{i+j-2}w(\lambda)\right]_{i,j=1}^N}\,. \tag{4.31}$$

The classical $\beta = 2$ random matrix ensembles belong to this class, whereby the *weight function* w and the support of the elements λ_1, …, λ_N differ for different random matrix models, see [40]. It is well

known that the solution of this class of biorthogonal ensembles is given in terms of classical orthogonal polynomials, which are here simply $\phi_j(\lambda) = w(\lambda)\chi_j(\lambda)$.
In the previous chapter we introduced the standard Wishart-Laguerre ensemble in Eq. (3.33) which is a member of the orthogonal ensembles. Its solution is now immediate. Namely, its solution is given in terms of the generalised Laguerre polynomials,

$$\chi_j(\lambda) = (-\sigma)^{j-1} L_{j-1}^{(\kappa)}(\lambda/\sigma)\,, \quad \text{with} \quad L_j^{(\kappa)}(\lambda) = (-\lambda)^j + \text{lower order terms}\,, \tag{4.32}$$

satisfying the orthogonality condition:

$$\delta_{nm} n!(n+\kappa)! = \int_0^\infty d\lambda \lambda^\kappa \mathrm{e}^{-\lambda} L_n^{(\kappa)}(\lambda) L_m^{(\kappa)}(\lambda)\,, \tag{4.33}$$

as it can be taken from [82, Eqs.: 8.970.1, 8.980]. In orthogonal ensembles, the weight function w and the support of the elements play the main role in the derivation of the correlation kernel. Here, without computing any expectation value of the characteristic polynomial, the solution is given, because the corresponding orthogonal polynomials are well known. From the Heine formula we can deduce that the expectation value of one characteristic polynomial with respect to the standard Wishart-Laguerre ensemble, as given in Eq. (2.4) for $\nu = \kappa$, yields the generalised Laguerre polynomials $L_N^{(\kappa)}$.
Orthogonal ensembles appear also beyond classical random matrix models, like the Kravchuk ensemble or the Meixner ensemble, see [100], called *discrete orthogonal ensembles*. However, our focus lies on biorthogonal ensembles dominated by external fixed parameters.
A prominent biorthogonal ensemble, implying a deformation of an orthogonal ensemble with the help of an external fixed parameter, is the so-called Muttalib-Borodin ensemble from 1995 and 1998 [94,106],

$$P_N(\lambda_1, \ldots, \lambda_N) = \frac{\prod_{j=1}^N w(\lambda_j)\, \Delta_N\left(\lambda_1^\theta, \ldots, \lambda_N^\theta\right) \Delta_N(\lambda_1, \ldots, \lambda_N)}{N! \det\left[\int_0^\infty d\lambda \lambda^{i-1+\theta(j-1)} w(\lambda)\right]_{i,j=1}^N}\,, \tag{4.34}$$

where $\theta \in (0,1]$ is the external fixed parameter. This parameter deforms an orthogonal ensemble for $\theta < 1$ smoothly into a polynomial ensemble. This ensemble is still an issue of modern research, see for instance [107]. In this very prominent ensemble we find an example of spectral deformations with the help of an external parameter, but it has an essential difference when compared with the class of biorthogonal ensembles emerging from inclusion of external fixed matrices studied in this work.
The main difference of the Muttalib-Borodin to our ensembles is that one single parameter is deforming each of the elements of the second determinant, $\varphi_1(\lambda) = 1$, $\varphi_2(\lambda) = \lambda^\theta$, $\varphi_3(\lambda) = \lambda^{2\theta}$, …,
$\varphi_N(\lambda) = \lambda^{(N-1)\theta}$, up to the very first one. The order of the elements is naturally given by the quasi polynomial $\lambda^{\theta(i-1)}$ through linearly increased power by increasing the index $i = 1, \ldots, N$. This natural order in the index dependence in the set $\{\varphi_i\}_{i=1,\ldots,N}$ spanning the determinant is lost in the ensembles treated in this work.
The external fixed matrices in our random matrix models are reflected in the corresponding joint probability distribution functions by finite sets of external fixed parameters, i.e. $\{\delta_i\}_{i=1,\ldots,N}$. These sets are spanning the determinants in the according determinantal point processes. The dependence

on these parameters is distinguished by one particular element of this finite set,

$$\varphi_i(\lambda) = \varphi(\lambda, \delta_i)\,, \quad \text{for the finite set of external parameters} \quad \{\delta_i\}_{i=1,\ldots,N}\,. \tag{4.35}$$

The type of the dependence of $\varphi(\lambda, \delta_i)$ in the second argument, on the fixed parameter, is for our purposes a *scaling dependence*, $\varphi(\lambda, \delta_i) \propto f(\delta_i \lambda)$. Further dependencies are thinkable, too: additive dependence $\varphi(\lambda, \delta_i) \propto f(\lambda + \delta_i)$ or a *complementary dependence* $\varphi(\lambda, \delta_i) \propto f\left(\lambda, \{\delta_j\}_{j=1, j\neq i}^{N}\right)$. Note that in the complementary dependence we explicitly make use of the fact that the set of external fixed parameters is consisting of a finite number of elements.
Due to a missing functional order of $\{\varphi(\lambda, \delta_i)\}_{i=1,\ldots,N}$ for $\{\delta_i\}_{i=1,\ldots,N}$ we call this type of determinantal point processes *biorthogonal ensembles with permutational symmetry*, shaped as

$$P_N(\lambda_1, \ldots, \lambda_N) = \frac{\det\left[\varphi(\lambda_j, \delta_i)\right]_{i,j=1}^{N} \det\left[\psi_j(\lambda_i)\right]_{i,j=1}^{N}}{N! \det\left[\int_0^\infty d\lambda\, \varphi(\lambda, \delta_i)\, \psi_j(\lambda)\right]_{i,j=1}^{N}}\,, \quad \text{for the finite set} \quad \{\delta_i\}_{i=1,\ldots,N}\,. \tag{4.36}$$

The Gram matrix has for this ensemble an explicit dependence on the external parameters $g_{ij} = g_j(\delta_i)$, which can be read off from the denominator from the above definition. Obviously, the only way of breaking the permutational symmetry is to introduce an order on the set of external parameters $\{\delta_i\}_{i=1,\ldots,N}$, which immediately induces an order on the set of functions $\{\varphi(\lambda, \delta_i)\}_{i=1,\ldots,N}$.

- We now classify the joint probability distribution functions of the sum of two independent Wishart matrices in Eq. (3.26) and the one-side correlated Wishart ensemble in Eq. (3.32) to be polynomial ensembles with permutational symmetry.

- The product of two independent Wishart matrices in Eq. (3.52) belongs to polynomial ensembles of non-equal sizes with permutational symmetry.

- The product of two coupled Wishart matrices in Eq. (3.44) as well as the generalised Wishart ensemble in Eq. (3.56) belong to biorthogonal ensembles of non-equal sizes with doubly permutational symmetry.

In the literature, biorthogonal ensembles with permutational symmetry have already been studied. In particular, the hermitian random matrix model in an external field belongs to the class of polynomial ensembles with permutational symmetry. It has been solved by Zinn-Justin in 1997 by taking reference to this symmetry [98, 99]. There, a single-sum representation of the correlation kernel was found by making use of the *modified Heine formula*, see [99, Eq. (3.10)], which we will apply to the present ensembles.
A further example in the literature in the study of a member of polynomial ensembles with permutational symmetry can be found in the shifted Wishart matrix, called chiral Gaussian ensemble with external source in [108, 109] from 2006, with the help of so-called multiple orthogonal polynomials. This approach seems cumbersome in view of the modified Heine formula. Nonetheless, the relation given in [109, Eq. (70)] represents an important ansatz for the computation of the corresponding correlation kernel and is comparable to the implicit Gram matrix inversion from the previous example in [98, Eq. (4.4)].
In the next section we will derive the modified Heine formula for polynomial ensembles with permutational symmetry, which can be applied to most of the present biorthogonal ensembles.

In addition, we would like to remark that it is not known to us whether a general approach to biorthogonal ensembles with doubly permutational symmetry exists. The two above mentioned ensembles being members of this class are solved in a system-specific ansatz, up to a simplification resulting from the notion of *biorthogonal two-matrix ensembles*, which we will deepen subsequently.

4.4 Modified Heine Formula

In this section we present a method for the solution of a polynomial ensemble with permutational symmetry as defined in Eq. (4.26), with the property given in Eq. (4.35), which we call modified Heine formula. It can first be found in [99, Eq. (3.10)] and subsequently in the principal publication [1, Eq. (3.14)]. Both derivations are done for ensembles of equal sizes. However, this approach can trivially be extended to a polynomial ensemble with permutational symmetry of non-equal sizes. We recall: a polynomial ensemble with permutational symmetry for the finite set of external parameters $\{\delta_i\}_{i=1}^N$ reads

$$P_N(\lambda_1,\ldots,\lambda_N) = \frac{\det\left[\varphi(\lambda_j,\delta_i)\right]_{i,j=1}^N \Delta_N(\lambda_1,\ldots,\lambda_N)}{N!\det\left[g_j(\delta_i)\right]_{i,j=1}^N}, \quad \text{with} \quad g_j(\delta_i) = \int_0^\infty d\lambda\,\varphi(\lambda,\delta_i)\lambda^{j-1}. \quad (4.37)$$

We are interested in the single-sum representation for the correlation kernel from Eq. (4.14), given by two sets of functions $\{\phi_i\}_{i=1,\ldots,N}$ and $\{\chi_j\}_{j=1,\ldots,N}$, and in the norms $\{h_j\}_{j=1,\ldots,N}$, satisfying the biorthogonality condition $h_j\delta_{ij} = \int_0^\infty d\lambda\,\phi_i(\lambda)\chi_j(\lambda)$.
We make use of the approach to the correlation kernel via the $(N-1)$-fold integration from the formula in Eq. (4.17). Therefore, we distinguish the remaining single external variable through its belonging to the first or to the second determinant: $\det\left[\varphi(x,\delta_i),\varphi(\lambda_2,\delta_i),\ldots,,\varphi(\lambda_N,\delta_i)\right]_{i=1}^N$ and $\Delta_N(y,\lambda_2,\ldots,\lambda_N)$ in the integrand. The first determinant can be expanded with respect to the first column,

$$\det\left[\varphi(x,\delta_i),\varphi(\lambda_2,\delta_i),\ldots,,\varphi(\lambda_N,\delta_i)\right]_{i=1}^N = \sum_{j=1}^N (-1)^{j-1}\varphi(x,\delta_j)\det\left[\varphi(\lambda_l,\delta_k)\right]_{1\le k\ne j\le N}^{2\le l\le N}, \quad (4.38)$$

where in each summand we recognise the appropriate minor. The second determinant is a Vandermonde determinant as defined in Eq. (3.12) and, thus, the y-dependence can be factorised out from the Vandermonde determinant as a product of differences with the remaining eigenvalues,

$$\Delta_N(y,\lambda_2,\ldots,\lambda_N) = \Delta_{N-1}(\lambda_2,\ldots,\lambda_N)\prod_{k=2}^N(\lambda_k - y). \quad (4.39)$$

By comparing to the derivation of the standard Heine formula in the above section in Eq. (4.28), the factorisation of differences of the given elements from the Vandermonde determinant can be interpreted as the key property of the Heine formula motivating the name of the result being derived.
Our approach to the kernel as an $(N-1)$-fold integration was shown with the help of the extended Andréief formula, which reduced the number of integrations to a single one. However, for the present derivation yielding the modified Heine formula, we do not perform any integration, but we give a new interpretation to this multiple integration. Namely, taking the two latter relations into account, the correlation kernel with the ansatz from Eq. (4.17) for the polynomial ensemble with permutational

symmetry as presented in Eq. (4.37) reads

$$K_N(x,y) = \frac{1}{(N-1)!\det\left[g_j(\delta_i)\right]_{i,j=1}^{N}} \int_0^\infty d\lambda_2 \ldots \int_0^\infty d\lambda_N \sum_{j=1}^{N} (-1)^{j-1} \varphi(x,\delta_j) \det\left[\varphi(\lambda_l,\delta_k)\right]_{1\le k\neq j\le N}^{2\le l\le N} \times \Delta_{N-1}(\lambda_2,\ldots,\lambda_N) \prod_{k=2}^{N} (\lambda_k - y) \,. \tag{4.40}$$

The sum over j serves as the only sum in the single-sum representation for the correlation kernel stated in Eq. (4.14), in which

$$K_N(x,y) = \sum_{j=1}^{N} \frac{1}{h_j} \varphi(x,\delta_j)\chi(y,\delta_j) \,, \quad \text{with} \quad h_j\delta_{ij} = \int_0^\infty d\lambda\, \varphi(\lambda,\delta_i)\chi(\lambda,\delta_j) \,, \tag{4.41}$$

where the $(N-1)$-fold integration is implied in the elements:

$$\chi(y,\delta_j) = \int_0^\infty \prod_{k=2}^{N} d\lambda_N \frac{\det\left[\varphi(\lambda_l,\delta_k)\right]_{1\le k\neq j\le N}^{2\le l\le N} \Delta_{N-1}(\lambda_2,\ldots,\lambda_N)}{(N-1)!\det\left[g_l(\delta_k)\right]_{1\le k\neq j\le N}^{2\le l\le N}} \prod_{k=2}^{N} (y-\lambda_k) \,. \tag{4.42}$$

Here, we make use of the above described complementary dependence in the finite set of elements $\{\delta_i\}_{i=1,\ldots,N}$ by excluding the one single element δ_j, such that we obtain the set of $(N-1)$ elements: $\{\delta_1,\ldots,\delta_{j-1},\delta_{j+1},\ldots,\delta_N\}$,

$$\chi(y,\delta_j) = \chi\left(y,\{\delta_i\}_{i=1,i\neq j}^{N}\right) . \tag{4.43}$$

The norms $\{h_j\}_{j=1,\ldots,N}$ result from the condition on $\chi(y,\delta_j)$ being monic and can be expressed in terms of the normalising determinant of the Gram matrix and the $(1,j)$-th minors,

$$h_j = \frac{\det\left[g_j(\delta_i)\right]_{i,j=1}^{N}}{\det\left[g_l(\delta_k)\right]_{1\le k\neq j\le N}^{2\le l\le N}} \,. \tag{4.44}$$

The modified Heine formula can be read of from Eq. (4.42), where we observe that the polynomials χ_j are given as an expectation value with respect to a biorthogonal ensemble of $(N-1)$ elements,

$$\chi(y,\delta_j) = \int_0^\infty \prod_{k=1}^{N-1} d\lambda_k P_{N-1}^{(\neq j)}(\lambda_1,\ldots,\lambda_{N-1}) \prod_{k=1}^{N-1} (y-\lambda_k) \,, \tag{4.45}$$

being nothing else than the ensemble given in the beginning in Eq. (4.37), with the difference of the lower dimension $N \to (N-1)$. This lower dimension emerges through the exclusion of the one single external parameter δ_j in the composition of the elements $\chi(y,\delta_j)$ in Eq. (4.42). In view of a comparison with the standard Heine formula from Eq. (4.30), we observe that a *dimensional reduction* from the expectation value of one characteristic polynomial with respect to the appropriate random matrix $Y^{'}$ has to be performed,

$$\left\langle \det\left[\lambda \mathbb{1}_N - Y^{'} Y^{'\dagger}\right] \right\rangle_{Y^{'},N-1}^{\nu,\kappa,\ldots,\{\delta_i\}_{i=1,i\neq j}^{N}} = \chi\left(\lambda,\{\delta_i\}_{i=1,i\neq j}^{N}\right) . \tag{4.46}$$

On the level of the expectation value with respect to the corresponding random matrix as presented in Eq. (4.46), it is difficult to express the dimensional reduction, which has to be taken for the computation of the polynomials $\chi(\lambda, \delta_j)$. Concurrently, this difficulty does not appear on the level of the corresponding eigenvalue representation, see Eq. (4.45). The dimensional reduction emerges by excluding the jth element of the finite set $\{\delta_i\}_{i=1,\ldots,N}$, whereas all further system specific parameters stay the same, as for example the number of zero modes denoted in our ensembles by ν, κ or ν_1 and ν_2. Thus, the dimensional proportions in the corresponding matrix model have to remain unchanged by taking $N \to (N-1)$.
The modified Heine formula differs from the standard Heine formula in the following sense:

- The polynomials given by the modified Heine formula from Eq. (4.46) are of the degree $(N-1)$ for all $j = 1, \ldots, N$,

$$\chi\left(\lambda, \{\delta_i\}_{i=1, i\neq j}^{N}\right) = \lambda^{N-1} + \text{lower order terms}\,, \tag{4.47}$$

  in contrast to Eq. (4.27).

- The first set of functions $\{\varphi(\lambda, \delta_i)\}_{i=1,\ldots,N}$ remains unchanged for the single-sum representation of the correlation kernel in Eq. (4.41), in contrast to the single-sum representation in Eq. (4.14).

- The norms $\{h_j\}_{j=1,\ldots,N}$ have the explicit representation in terms of determinants of the Gram matrix and its minors in Eq. (4.44), which we obtained as a byproduct.

We conclude that the correlation kernel for the polynomial ensemble with permutational symmetry as defined in Eq. (4.37) has the single-sum representation given in Eq. (4.41). From the point of view of the double-sum representation in Eq. (4.8), we identify the implicit inversion of the Gram matrix in the summation over one of the indices,

$$\sum_{j=1}^{N} c_{ij} y^{j-1} = \frac{1}{h_i} \chi(y, \delta_i)\,, \tag{4.48}$$

which was stated in [1, Eq. (3.14)].

4.5 Biorthogonal Two-Matrix Ensembles

In this section we would like to introduce the notion of biorthogonal two-matrix ensembles, due to the joint probability distribution function of singular values of the product of two coupled Wishart matrices from Eq. (3.42). There, the squared singular values $y_1, \ldots, y_N$ of the product $Y = X_1^\dagger X_2$ and the squared singular values $x_1, \ldots, x_N$ of the factor X_2 are characterised by one joint probability distribution function simultaneously. Two biorthogonal ensembles for each of the two sets $\{y_j\}_{j=1,\ldots,N}$ and $\{x_j\}_{j=1,\ldots,N}$ are obtained by integrating over all $x_1, \ldots, x_N$ as presented in Eq. (3.44), or over $y_1, \ldots, y_N$ as presented in Eq. (3.56), respectively.

We define biorthogonal two-matrix ensembles by

$$P_N(x_1,\ldots,x_N;y_1,\ldots,y_N) \tag{4.49}$$
$$= \frac{\det\left[q_{i1},q_{i2},\ldots,q_{i\nu},\varphi_i(x_1),\ldots,\varphi_i(x_N)\right]_{i=1}^{N+\nu}\det\left[w(x_j,y_i)\right]_{i,j=1}^{N}\det\left[\psi_j(y_i)\right]_{i,j=1}^{N}}{(N!)^2\det\left[q_{i1},q_{i2},\ldots,q_{i\nu},g_{i1},\ldots,g_{iN}\right]_{i=1}^{N+\nu}}.$$

By making use of Andréief's integration formula [90] and the extended Andréief integration formula [93] we obtain two independent biorthogonal ensembles, as defined in Eq. (4.5) after integration over all elements x_1, ..., x_N,

$$P_N^{(Y)}(y_1,\ldots,y_N) = \int_0^\infty dx_1\ldots\int_0^\infty dx_N P_N(x_1,\ldots,x_N;y_1,\ldots,y_N) \tag{4.50}$$
$$= \frac{\det\left[q_{i1},q_{i2},\ldots,q_{i\nu},\hat{\rho}_i(y_1),\ldots,\hat{\rho}_i(y_N)\right]_{i=1}^{N+\nu}\det\left[\psi_j(y_i)\right]_{i,j=1}^{N}}{N!\det\left[q_{i1},q_{i2},\ldots,q_{i\nu},g_{i1},\ldots,g_{iN}\right]_{i=1}^{N+\nu}},$$

or after integration over all elements y_1, ..., y_N,

$$P_N^{(X)}(x_1,\ldots,x_N) = \int_0^\infty dy_1\ldots\int_0^\infty dy_N P_N(x_1,\ldots,x_N;y_1,\ldots,y_N) \tag{4.51}$$
$$= \frac{\det\left[q_{i1},q_{i2},\ldots,q_{i\nu},\varphi_i(x_1),\ldots,\varphi_i(x_N)\right]_{i=1}^{N+\nu}\det\left[\rho_i(x_j)\right]_{i,j=1}^{N}}{N!\det\left[q_{i1},q_{i2},\ldots,q_{i\nu},g_{i1},\ldots,g_{iN}\right]_{i=1}^{N+\nu}}.$$

Thereby, we denote the two sets of functions appearing after these integrations by

$$\rho_j(x) = \int_0^\infty dy w(x,y)\psi_j(y) \quad \text{and} \quad \hat{\rho}_i(y) = \int_0^\infty dx w(x,y)\varphi_i(x). \tag{4.52}$$

In view of the computation of each of the biorthogonal ensembles, P^X and P^Y, the inversion of the Gram matrix is desirable. The notion of biorthogonal two-matrix ensemble as defined in Eq. (4.49) implies one single Gram matrix for both one-matrix ensembles,

$$g_{ij} = \int_0^\infty dx \int_0^\infty dy w(x,y)\varphi_i(x)\psi_j(y), \tag{4.53}$$

which is the main benefit of the introduced notion.
The two resulting correlation kernels,

$$K_N^{(X)}(x_1,x_2) = \sum_{i=1}^{N+\nu}\sum_{j=1}^{N}\varphi_i(x_1)\,c_{ij}\rho_j(x_2) \quad \text{and} \quad K_N^{(Y)}(y_1,y_2) = \sum_{i=1}^{N+\nu}\sum_{j=1}^{N}\hat{\rho}_i(y_1)\,c_{ij}\psi_j(y_2), \tag{4.54}$$

are obtained by one single inversion of the Gram matrix $g^\top c = \mathbb{1}_N$ as presented in the course of Eq. (4.21).
Biorthogonal two-matrix models are not new. In one of our main references a similar definition was formulated [94, Eq. (2.14)]. Moreover, biorthogonal multi-matrix models were already studied by Eynard and Mehta in 1997 [110], where correlations across different sets of eigenvalues were formulated as a generalised determinantal point process. However, the above formulation suffices for our purposes. For further examples of generalised determinantal point processes arising in multi-matrix models see discussions in [100, Sec. 2.3], or in the course of [101, Def. 3'].

5 Spectral Statistics

In this chapter we present the solutions to the two random matrix models: the sum of two independent Wishart matrices and the product of two coupled Wishart matrices. Therefore, we make use of the determinantal point processes arising in the eigenvalue representations, as discussed in Chap. 3, and apply the theory of biorthogonal polynomials to determine the desired correlation kernels characterising all k-point correlation functions, see Chap. 4.
The achieved solutions will be discussed under different points of view, respectively to the two different fields of application presented in Chap. 2. The sum of two independent Wishart matrices is interesting due to global level density, because the distribution for all eigenvalues draws inference to spatial correlations in time series exhibiting two-epoch partitioning. The product of two coupled Wishart matrices is interesting because of its local eigenvalue density at the origin for infinite matrix dimensions, motivated by the weak non-hermiticity regime.
The results presented in this chapter are taken from the principal publications [1, 2]. However, there are additional computations not published yet as acknowledged accordingly.

5.1 Sum of Two Independent Wishart Matrices

Our analysis of the sum of two independent Wishart matrices is presented in three parts, due to its interpretation in the application to time-series analysis as a two-epoch model: one-epoch model, spatial cross-correlations and temporal cross-correlations.
We will derive the exact solution for the correlation kernel of the two-epoch model and, thus, show that this model is integrable. As a direct gain we compute the global level density and compare it to Monte-Carlo simulations.
We have to ask for the difference to the one-epoch model. Therefore, as the first step in our investigation we present the solution to the one-epoch model. The spectral statistics of the two-epoch model will be presented afterwards and we will compare to previous results.
As discussed in the course of the fields of application in Sec. 2.1, the global level density of the sum of two independent Wishart matrices corresponds to spatial cross-correlations in time series. In the third part of our discussion, we turn our focus to temporal cross-correlations in the two-epoch model. The interpretation of these results is obviously interesting in time-series analysis, but it is not the emphasis of the investigation here. The computations to the temporal cross-correlations in the two-epoch model are presented in order to achieve a complete analysis of this model. In addition, this ensemble is again a random two-matrix model under finite-rank perturbations.
All these three random matrix models belong in their eigenvalue representations to the class of polynomial ensembles with permutational symmetry, see Sec. 4.3. By making use of the modified Heine formula we will be able to compute the correlation kernel. Therefore, the expectation value of one single characteristic polynomial is the key quantity. Consequently, our calculations will focus on the

evaluation of the according expectation value.
The technique we will apply for this evaluation is called *superbosonisation formula* [111–115]. This technique can be applied to an arbitrary ratio of characteristic polynomials of the form $\left\langle \prod_{k=1}^{p} (x_k \mathbb{1} - H) / \prod_{l=1}^{q} \left(y_l^{\pm} \mathbb{1} - H\right) \right\rangle_H$, where the variables y_1, ..., y_q have to be regularised, $y_l^{\pm} = y_l \pm \imath\epsilon_l$, due to singularities emerging in the denominator from the eigenvalues of H. Thereby, the expectation value of the ratio of p/q characteristic polynomials is mapped onto a $(q|p)$ supermatrix model. However, for one single characteristic polynomial the superbosoniation formula reduces to a $U(1)$ group integral, which is as a contour integral on the complex plane. For each of the model, this formula will be explained.

5.1.1 One-Epoch Model

We recall the one-side correlated Wishart ensemble by the probability density of the $N \times (N+\nu)$ complex valued Gaussian random matrix X from Eq. (2.5),

$$\mathcal{P}(X|\Sigma) = \frac{\det\left[\Sigma^{-1}\right]^{N+\nu}}{\pi^{N(N+\nu)}} \mathrm{e}^{-\mathrm{Tr}\,\Sigma^{-1} X X^{\dagger}}, \tag{5.1}$$

where Σ is a fixed and positive definite matrix of dimensions $N \times N$ with eigenvalues σ_1, ..., σ_N, which we assume to be pairwise distinct, without loss of generality. The one-side correlated Wishart ensemble is interpreted here as a one-epoch model with non-trivial spatial correlations,

$$(N+\nu)\,\Sigma = \int [dX]\, \mathcal{P}(X|\Sigma)\, X X^{\dagger}, \tag{5.2}$$

where $[dX]$ denotes the flat Lebesgue measure, see Eq. (3.2). The first moment condition and the covariance matrix are discussed in Sec. 2.1.
For the derivation of the joint probability distribution function of the singular values of X, unitary group elements from according singular value decomposition emerge, which are integrable. Here, the Harish-Chandra/Itzykson–Zuber integral can be applied, see Sec. 3.1. We recall the joint probability distribution function for the squared singular values of X from Eq. (3.32):

$$P_N(x_1, \ldots, x_N) = \frac{\Delta_N(x_1, \ldots, x_N) \det\left[x_i^{\nu} \mathrm{e}^{-x_i/\sigma_j}\right]_{i,j=1}^{N}}{N! \det\left[(\nu+i-1)!\, \sigma_j^{\nu+i}\right]_{i,j=1}^{N}}. \tag{5.3}$$

This joint probability distribution function belongs to the class of polynomial ensembles with permutational symmetry, where we quote Eq. (4.37). From there, we identify the two participating quantaties appearing inside the determinants

$$\varphi(x, \sigma_j) = x^{\nu} \mathrm{e}^{-x/\sigma_j} \quad \text{and} \quad g_i(\sigma_j) = (\nu+i-1)!\, \sigma_j^{\nu+i}. \tag{5.4}$$

Following the considerations in Sec. (4.4), the single-sum representation for the correlation kernel is given by the set of polynomials $\{\chi(x, \sigma_j)\}_{j=1,\ldots,N}$. In these polynomials, by choosing them to be monic, we have the complementary dependence, $\chi(x, \sigma_j) = \chi\left(x, \{\sigma_i\}_{i=1, i\neq j}^{N}\right)$ for the finite set of external parameters $\{\sigma_i\}_{i=1,\ldots,N}$. For their computation we turn to the evaluation of the expectation

value of one single characteristic polynomial,

$$\left\langle \det\left[x\mathbb{1}_N - XX^\dagger\right]\right\rangle_{X,N}^{\nu,\Sigma} = \int [dX]\,\mathcal{P}(X|\Sigma)\det\left[x\mathbb{1}_N - XX^\dagger\right]. \tag{5.5}$$

The dimensional reduction described in the course of Eq. (4.46) will be performed afterwards.
For our purpose, it is beneficial to make use of Sylvester's determinant identity,

$$x^\nu \left\langle \det\left[x\mathbb{1}_N - XX^\dagger\right]\right\rangle_{X,N}^{\nu,\Sigma} = \left\langle \det\left[x\mathbb{1}_{N+\nu} - X^\dagger X\right]\right\rangle_{X,N}^{\nu,\Sigma}, \tag{5.6}$$

allowing to consider an $(N+\nu)\times(N+\nu)$ determinant instead.
The superbosonisation formula, which we want to apply for this determinant, makes use of the so-called Berezin integral [116], which represents reformulation of the determinant. Therefore, we introduce $(N+\nu)$ complex Grassmann variables, which we denote by $v_1, \ldots, v_{N+\nu}$. There are different definitions of complex Grassmann variables that can be set on their algebra. Three conditions on those are sufficient to fix the convention unambiguously,

$$\{v_i, v_j\} = 0\,, \quad \left\{v_i, v_j^*\right\} = 0 \quad \text{and} \quad (v_i^*)^* = -v_i \quad \text{for all } \; i,j = 1,\ldots,N+\nu\,, \tag{5.7}$$

where $\{\ ,\ \}$ denotes the anti-commutator. The third condition is chosen with the purpose to keep the product $v_i^* v_i$ self-adjoint under complex conjugation, which is called *realness condition*. For a simpler arrangement of the illustration in the following calculation we compose a Grassmann variable valued vector:

$$V = \begin{pmatrix} v_1 \\ \vdots \\ v_{N+\nu} \end{pmatrix}, \quad \text{with } \; V^\dagger V = \sum_{i=1}^{N+\nu} v_i^* v_i\,. \tag{5.8}$$

The use of vectors with Grassmann variables in their entries might involve the notion of supervectors. However, we only need the property of the additional minus sign by a transposition of the vector V, which is implicitly defined in the latter equation by introducing the scalar product $V^\dagger V$. Note that the resulting sum of $V^\dagger V$ fulfils the realness condition. One further tool for the Berezin integral has to be introduced, namely the differentiation with respect to Grassmann variables,

$$\int [dV] = \prod_{i=1}^{N+\nu} \frac{\partial}{\partial v_i^*}\frac{\partial}{\partial v_i}, \quad \text{with } \; \frac{\partial}{\partial v_i} v_i = 1 \quad \text{and} \quad \frac{\partial}{\partial v_i} = 0 \quad \text{for all } \; i = 1,\ldots,N+\nu\,. \tag{5.9}$$

On the left hand side, the product of all differentiations is symbolised with an integral, which has conventional reasons as the integration and the differentiation with respect to Grassmann variables are equivalent. We now are able to accomplish the second modification of our expectation value of one characteristic polynomial, in order to prepare the superbosonisation formula. The determinant on the right hand side in Eq. (5.6) can be parametrised through the following Berezin integral,

$$\left\langle \det\left[x\mathbb{1}_{N+\nu} - X^\dagger X\right]\right\rangle_{X,N}^{\nu,\Sigma} = \int [dV]\,\mathrm{e}^{-xV^\dagger V}\left\langle \mathrm{e}^{V^\dagger X^\dagger X V}\right\rangle_{X,N}^{\nu,\Sigma}, \tag{5.10}$$

where we interchanged the order of integrations over $[dV]$ and $[dX]$. It is remarkable that under the expectation value only a Gaussian factor remains. We call the expectation value over the Fourier type factor, $\exp\left[-\mathrm{Tr}\, VV^\dagger X^\dagger X\right]$, *characteristic function*. Note that an additional minus sign appears, if the cyclicity property of the trace operation on Grassmann valued matrices is applied. The evaluation of the expectation value is now equivalent to the performance of Gaussian integral over complex numbers, yielding

$$\left\langle \mathrm{e}^{-\mathrm{Tr}\, VV^\dagger X^\dagger X} \right\rangle_{X,N}^{\nu,\Sigma} = \det\left[\Sigma^{-1}\right]^{N+\nu} \det\left[\Sigma \otimes \mathbb{1}_{N+\nu} + \mathbb{1}_N \otimes VV^\dagger\right]^{-1}, \tag{5.11}$$

where we denote with $\otimes$ the Kronecker product.
From here, the computation of the characteristic function has been mapped from an integration over $N \times (N+\nu)$ complex variables to only $(N+\nu)$ complex Grassmann variables originating from V. Thus, the number of integration variables has been reduced from quadratic dependence in N to a linear dependence only. Moreover, we will now see that the remaining $N+\nu$ integrations over Grassmann variables can be reduced to only one single complex-contour integral.
The application of the superbosonisation formula can be presented in two steps: the *duality formula* and the map of the integration with respect to $[dV]$ to one complex-contour integral.
The duality formula reads for our purposes:

$$\mathrm{Tr}\left(VV^\dagger\right)^k = -\left(V^\dagger V\right)^k, \quad \text{for } k = 1, 2, \ldots. \tag{5.12}$$

For more general situations the duality formula exchanges the trace operation to the so-called supertrace operation, which is useful by the computation of ratios of arbitrary many characteristic polynomials. With this formula invariants of the considered matrix model are mapped to invariants of the corresponding supermatrix model.
In the characteristic function from Eq. (5.11), we observe that the first determinant can be included to the second determinant leading the expression $\det\left[\mathbb{1}_{N(N+\nu)} + \Sigma \otimes VV^\dagger\right]$. By making use of the well known expansion,

$$\det\left[\mathbb{1} - C\right] = \exp\left[-\sum_{k=1}^{\infty} \frac{\mathrm{Tr}\, C^k}{k}\right], \tag{5.13}$$

the duality formula can be applied. Formulating the so obtained expression back into the determinant, we obtain for the characteristic function a determinant comprising an $N \times N$ matrix

$$\left\langle \mathrm{e}^{-\mathrm{Tr}\, VV^\dagger X^\dagger X} \right\rangle_{X,N}^{\nu,\Sigma} = \det\left[\mathbb{1}_N + V^\dagger V \Sigma\right]. \tag{5.14}$$

The dependence on V is a scalar dependence with the realness condition mentioned above, $V^\dagger V$. In addition, the entire Berezin integral, originating from the reformulation of the characteristic polynomial by the parametrisation through Grassmann integration in Eq. (5.10), depends now on V solely through this scalar product, $V^\dagger V$. This situation is suitable for the last step in the superbosonisation formula, which is for our purposes the exchange of the integration over $[dV]$ with one single complex-contour integral. This exchange can be understood via a Taylor expansion.
Let us consider a Taylor expandable function $f(z) = \sum_{k=1}^{N} f^{(k)}(0) z^k / k!$, which we associate with the

integrand under the integration over $[dV]$ with the dependence $f = f\left(V^\dagger V\right)$. The contributing part of f by the integration over $[dV]$ is the one containing the product of all pairs $v_i^* v_i$. Due to the anti-commutation properties of Grassmann variables this is only the case for a certain power in the Taylor expansion. Namely, the Taylor coefficient in the power $\left(V^\dagger V\right)^{N+\nu}$ contributes with the multiplicity of $(N+\nu)!$. This term can be singled out from the Taylor expansion through a complex-contour integral,

$$\int [dV]\, f\left(V^\dagger V\right) = \int [dV]\, f^{(N+\nu)}(0) \prod_{i=1}^{N+\nu} v_i^\dagger v_i = (-1)^{N+\nu} (N+\nu)! \oint_{\gamma_0} \frac{dz}{2\pi\imath z} \frac{f(z)}{z^{N+\nu}} \,. \tag{5.15}$$

Here, the integration contour γ_0 encircles the origin in counter-clockwise direction.
The superbosonisation formula has to be applied to $f(z) = \exp[-xz] \det[\mathbb{1}_N + z\Sigma]$, which brings us to the final expression for the expectation value of the characteristic polynomial from Eq. (5.5),

$$\left\langle \det\left[x\mathbb{1}_N - XX^\dagger\right] \right\rangle_{X,N}^{\nu,\Sigma} = (N+\nu)! \oint_{\gamma_0} \frac{dz}{2\pi\imath z} \frac{\mathrm{e}^z}{z^{N+\nu}} \prod_{i=1}^{N} (x - \sigma_i z) \,, \tag{5.16}$$

where we have absorbed the x dependence by the substitution $z \to -z/x$ and represented the determinant through the product of the eigenvalues of the covariance matrix Σ. The contour γ_0 is enclosing the origin in counter-clockwise direction.
The desired polynomials $\{\chi(x,\sigma_j)\}_{j=1,\ldots,N}$ follow from a dimensional reduction by following the modified Heine formula from Eq. (4.45). Therefore, we exclude the one particular parameter σ_j from the set $\{\sigma_i\}_{i=1,\ldots,N}$ in the expectation value in Eq. (5.16), to obtain the polynomial $\chi(x,\sigma_j)$. The matrix dimension has to be reduced $N \to (N-1)$ accordingly to the reduced set, $\{\sigma_1,\ldots,\sigma_{j-1},\sigma_{j+1},\ldots,\sigma_N\}$, which implies an equivalently modified random matrix $X \to X^{'}$ of dimensions $(N-1)\times(N-1+\nu)$,

$$\chi(x,\sigma_j) = \left\langle \det\left[x\mathbb{1}_{N-1} - X^{'}X^{'\dagger}\right] \right\rangle_{X',N-1}^{\nu,\{\sigma_i\}_{i=1,i\neq j}^{N}} \,. \tag{5.17}$$

In the representation of the expectation value in terms of the corresponding random matrix model, the one-side correlated Wishart ensemble consists of a reduced covariance matrix that a priori does not have to be diagonal. The dimensional reduction excludes one of the eigenvalues of Σ and the corresponding eigenvectors.
However, from the expression obtained from the superbosonisation formula in Eq. (5.16), the dimensional reduction is simple to perform and yields:

$$\chi(x,\sigma_j) = (N-1+\nu)! \oint_{\gamma_0} \frac{dz}{2\pi\imath} \frac{\mathrm{e}^z}{z^{N+\nu}} \prod_{i=1,i\neq j}^{N} (x - \sigma_i z) \,. \tag{5.18}$$

We would like to conclude the above derivation with the statement that a set of polynomials was found, $\{\chi(x,\sigma_j)\}_{j=1,\ldots,N}$ being all of the degree $(N-1)$, which is biorthogonal to the set $\left\{x^\nu \mathrm{e}^{-x/\sigma_j}\right\}_{j=1,\ldots,N}$ as stated in Eq. (4.41). The biorthogonality condition between these two sets can be proven in a direct calculation and leads us to an expression for the norms $\{h_i\}_{i=1,\ldots,N}$.
By employing the explicit expressions for these two sets of functions in the biorthogonality condition,

$\delta_{ij} h_j = \int_0^\infty dx \varphi(x,\sigma_i)\chi(x,\sigma_j)$, two integrals have to be performed,

$$h_j \delta_{ij} = (N-1+\nu)! \oint_{\gamma_0} \frac{dz}{2\pi\imath} \frac{\mathrm{e}^z}{z^{N+\nu}} \int_0^\infty dx x^\nu \mathrm{e}^{-x/\sigma_i} \prod_{k=1,k\neq j}^{N} (x-\sigma_k z)\,, \tag{5.19}$$

where we have interchanged the order of integrations, such that the complex-contour integral from Eq. (5.18) stays in front. With the help of a rescaling of the integration variables, first $x \to zx$, second $z \to \sigma_i z$ we obtain

$$h_j \delta_{ij} = (N-1+\nu)! \sigma_i \prod_{k=1,k\neq j}^{N} (\sigma_j - \sigma_k) \oint_{\gamma_0} \frac{dz}{2\pi\imath} \mathrm{e}^{\sigma_i z} \int_0^\infty dx x^\nu \mathrm{e}^{-zx} \mathtt{f}_j(x)\,, \tag{5.20}$$

where we define the set of polynomials of degree $(N-1)$:

$$\prod_{k=1,k\neq j}^{N} \frac{x-\sigma_k}{\sigma_j - \sigma_k} = \mathtt{f}_j(x) = \sum_{l=1}^{N} \mathtt{f}_{jl} x^{l-1} \quad \text{for } j = 1,\ldots,N\,, \tag{5.21}$$

whose meromorphic properties are of interest. Each of the functions has N fix points on the real line, namely $(N-1)$ zeros at $\mathtt{f}_j\left(\sigma_{i\neq j}\right) = 0$, and it is fixed to one at $\mathtt{f}_j(\sigma_j) = 1$. This property can be summarised with the Kronecker delta,

$$\mathtt{f}_j(\sigma_i) = \delta_{ij} \quad \text{for } i,j = 1,\ldots,N\,. \tag{5.22}$$

Making use of the power series of $\mathtt{f}_j$, the integral over x in Eq. (5.20) can be performed as a simple Laplace transform of the monomial $x^{\nu+l-1}$ [82, Eq. 17.13.2]. Thereby, the Laplace transform holds with a non-negative real part of z, due to singularities emerging in the origin,
$\int_0^\infty dx x^m \mathrm{e}^{-zx} = m!/z^{m+1}$, which compels us to investigate these singularities. However, as we will see, for the resulting complex-contour integration around the origin all so obtained negative powers in z contribute,

$$\begin{aligned} h_j \delta_{ij} =& (N-1+\nu)! \sigma_i \prod_{k=1,k\neq j}^{N} (\sigma_j - \sigma_k) \sum_{l=1}^{N} \mathtt{f}_{jl} (\nu+l-1)! \oint_{\gamma_0} \frac{dz}{2\pi\imath} \frac{\mathrm{e}^{\sigma_i z}}{z^{\nu+l}}\,, \\ =& (N-1+\nu)! \sigma_i^{\nu+1} \prod_{k=1,k\neq j}^{N} (\sigma_j - \sigma_k) \sum_{l=1}^{N} \mathtt{f}_{jl} \sigma_i^{l-1}\,. \end{aligned} \tag{5.23}$$

Due to the Kronecker delta from Eq. (5.22), the norms are immediately given,

$$h_j = (N-1+\nu)! \sigma_j^{\nu+1} \prod_{k=1,k\neq j}^{N} (\sigma_j - \sigma_k)\,. \tag{5.24}$$

With the latter quantity we finish the computation of all needed parts of the single-sum representation of the correlation kernel from Eq. (4.41),

$$K_N(x,y) = \sum_{j=1}^{N} \frac{1}{\sigma_j} (x/\sigma_j)^\nu \mathrm{e}^{-x/\sigma_j} \oint_{\gamma_0} \frac{dz}{2\pi\imath} \frac{\mathrm{e}^z}{z^{N+\nu}} \prod_{k=1,k\neq j}^{N} \frac{y-\sigma_k z}{\sigma_j - \sigma_k}\,. \tag{5.25}$$

Our expression in Eq. (5.25) closes the derivation of the correlation kernel, in which all dependencies are explicit. From here, all k-point correlation functions can be expressed by this correlation kernel and, thus, are known. Nonetheless, we would like to modify the single-sum representation for the correlation kernel to an expression which is suitable for large-N analysis and to encounter degeneracies in the covariance matrix Σ, which completes our aspiration on integrability as introduced in Chap. 1. The single sum in Eq. (5.25) can be expressed as a complex-contour integral encircling σ_1, ..., σ_N. Thereby, we can introduce degeneracies for the eigenvalues of the covariance matrix Σ. For instance, we introduce one single degeneracy value as

$$\sigma_{n+1} = \ldots = \sigma_N = \sigma\,. \tag{5.26}$$

The correlation kernel for the one-side correlated Wishart ensemble takes on the double-contour integral formula:

$$K_N^{(n)}(x,y) = \oint_{\gamma_\Sigma} \frac{ds}{2\pi\imath s} \oint_{\gamma_0} \frac{dz}{2\pi\imath z} \left(\frac{x}{sz}\right)^\nu \frac{\mathrm{e}^{z-x/s}}{y/z-s} \left(\frac{y/z-\sigma}{s-\sigma}\right)^{N-n} \prod_{k=1}^{n} \frac{y/z-\sigma_k}{s-\sigma_k}\,. \tag{5.27}$$

where the contour γ_Σ encircles all eigenvalues of the indicated covariance matrix, $\{\sigma_1,\ldots,\sigma_n,\sigma\}$ in counter-clockwise direction. The two contours γ_0 and γ_Σ are not allowed to cross each other, due to the pole emerging from the difference $y/z-s$ in the denominator. With the help of the residue theorem the exchange of the sum in Eq. (5.25) to the complex-contour integral in Eq. (5.27) can simply be verified.

For the illustration of our final result (5.25) we depict the eigenvalue density of $XX^\dagger$ in Fig. 5.1, which is given by the evaluation of the correlation kernel at equal arguments, $K_N(x,x) = R_1(x)$, see Eq. (4.2). For later purposes, four different lengths of the time series, $N+\nu$, are depicted. In addition, our analytic result is compared to Monte Carlo simulations, where we find a perfect agreement. The global behaviour of the squared singular values of X, or the eigenvalues of $XX^\dagger$ depends strongly on the length of the time series. Reading in the Fig. 5.1 from the left upper diagram over the right upper diagram, then from the left lower to the right lower diagram the length of the time epoch is increased $N+\nu = 40 \to 75 \to 400 \to 750$. Thereby, we keep for the number of observables fixed at $N=9$ as well as the covariance eigenvalues $\{\sigma_1,\ldots,\sigma_9\}$. We observe that the locations of the individual peaks, which correspond to the covariance eigenvalues, move to the right by increasing the length of the time series. In addition, the individual peaks are sharpening, such that the overlap of the distributions of the individual peaks is smaller. For long time-epochs compared to the number of observables the peaks can easily be estimated with

$$R_1(x) \approx \sum_{k=1}^{N} \delta\left(x-(N+\nu)\,\sigma_k\right), \quad \text{for } N+\nu \gg N\,. \tag{5.28}$$

The individual distributions of the peaks are broadened and they still overlap. Nonetheless, the repulsion of the eigenvalues of $XX^\dagger$ separates and shifts those away from the positions estimated in Eq. (5.28) stronger then their increasing widths. In a Gaussian approximation the spacing between neighbouring peaks for large values of $(N+\nu)$ scales linearly, whereas their widths scale as a square root, $\sqrt{N+\nu}$. We observe a sharpening of the individual distributions by a strong shift of those to

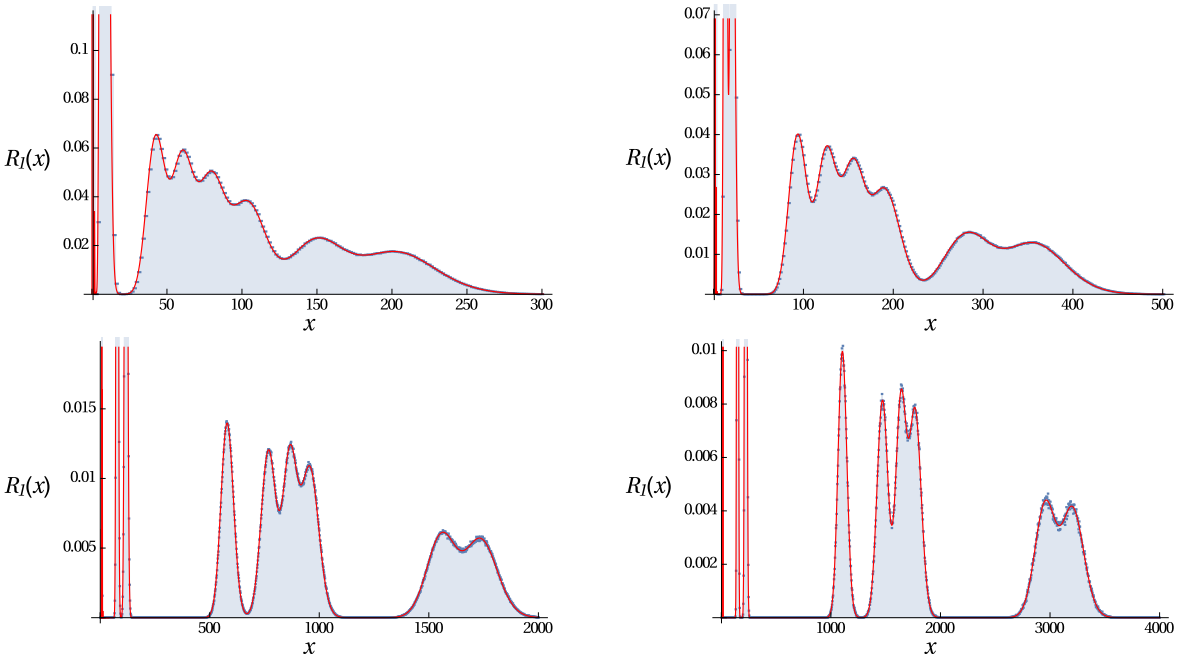

Figure 5.1: Comparison of the analytical result (5.25) for the spectral density (red curves) with Monte Carlo simulations (histograms=shaded area, around 10^6 matrices drawn from the ensemble (5.1) for each of the diagrams). We employed the parameters $N = 9$ with the fixed covariance matrix $\Sigma = \mathrm{diag}(0.02, 0.20, 0.30, 1.50, 2.01, 2.25, 2.27, 4.05, 4.13)$ and four different lengths of the epochs: $N + \nu = 40$ (left upper plot), $N + \nu = 75$ (right upper plot), $N + \nu = 400$ (left lower plot) and $N + \nu = 750$ (right lower plot).

the right on the positive real axis.

At short lengths of the epoch, here 40 and 75, where the number of observables is of comparable order to the number of time steps, $\mathcal{O}(N) \approx \mathcal{O}(N+\nu)$, the distributions of the individual peaks overlap strongly. The shape of the uncorrelated case can be estimated already for $N = 9$ with the limiting distribution, which is the Marchenko-Pastur distribution up to a scaling factor, see Fig. 1.3, although the matrix dimension is of order ten, $\mathcal{O}(N) \sim 10$. The individual peaks represent its deformation caused by the finite-rank perturbation by the non-trivial covariance matrix Σ.

5.1.2 Spatial Cross-Correlations

We now turn to the main result in our analysis of the one-epoch model. We present the derivation of the correlation kernel characterising the spatial cross-correlations between the two epochs X_1 and X_2 from the principal publication [1]. Let us recall the biorthogonal ensemble resulting in the limit of half degeneracy, see Eq. (3.25), being the joint probability distribution function for the eigenvalues of $H = X_1 X_1^\dagger + X_2 X_2^\dagger$ or equivalently the squared singular values of $Y = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$. From Eq. (3.26) and by comparing with the definition for polynomial ensembles with permutational symmetry from Eq. (4.37) it suffices to state two quantities,

$$\begin{aligned} \varphi(x, \delta_j) &= x^\kappa \, \mathrm{e}^{-x/\sigma_A} \, {}_1F_1(\nu_2 + 1; \kappa + 1; \delta_j x) \,, \\ g_j(\delta_i) &= \sigma_A^{\kappa+1} \, (\kappa + j - 1)! \, {}_2F_1(\kappa + j + 1, \nu_2 + 1; \kappa + 1; \delta_i \sigma_A) \,, \end{aligned} \tag{5.29}$$

where we recall that the difference in the number of rows and columns in Y is given by $\kappa = N+\nu_1+\nu_2$, and we use the abbreviation $\delta_j = \sigma_A^{-1} - \sigma_{Bj}^{-1}$.
For the derivation of the corresponding correlation kernel we aim, analogously to the one-epoch model, at the method in Sec. 4.4. Therefore, the set of polynomials $\chi_j(x) = \chi(x,\delta_j)$ as well as the norms $\{h_j\}_{j=1}^N$ are sought. With those quantities the single-sum representation of the correlation kernel is immediately given as stated in Eq. (4.41). Therefore, the expectation value of one single characteristic polynomial after a dimensional reduction is needed,

$$\begin{aligned}&\left\langle \det\left[x\mathbb{1}_N - YY^\dagger\right]\right\rangle_{Y,N}^{\nu_1,\nu_2,\Sigma_B} \\ &= \int [dX_1]\,[dX_2]\,\mathcal{P}(X_1|\Sigma_A)\,\mathcal{P}(X_2|\Sigma_B)\det\left[x\mathbb{1}_N - X_1X_1^\dagger - X_2X_2^\dagger\right],\end{aligned} \tag{5.30}$$

where Σ_A can be taken to be arbitrary in following the application of the superbosonisation formula. In the computation of the expectation value of the characteristic polynomial we follow very closely the approach presented in the previous section regarding the one-epoch model. We first use Sylvester's determinant identity [58], $x^\kappa \left\langle \det\left[x\mathbb{1}_N - YY^\dagger\right]\right\rangle_{Y,N}^{\nu_1,\nu_2,\Sigma_B} = \left\langle \det\left[x\mathbb{1}_{N+\kappa} - Y^\dagger Y\right]\right\rangle_{Y,N}^{\nu_1,\nu_2,\Sigma_B}$, and then we parametrise the resulting $(N+\kappa)\times(N+\kappa)$ determinant by a Berezin integral.
For the parametrisation of the $(N+\kappa)\times(N+\kappa)$ determinant by a Berezin integral $N+\kappa$ complex Grassmann variables are required, v_1, …, $v_{N+\kappa}$, where we use the same conventions as introduced in Sec. 5.1.1. They can be again arranged in a vector V comparable to Eq. (5.8) with one difference. Namely, we divide the Grassmann valued vector V in two subvectors as

$$V = \begin{pmatrix} V_A \\ V_B \end{pmatrix}, \quad \text{with } V_A = \begin{pmatrix} v_1 \\ \vdots \\ v_{N+\nu_1} \end{pmatrix} \quad \text{and } V_B = \begin{pmatrix} v_{N+\nu_1+1} \\ \vdots \\ v_{N+\kappa} \end{pmatrix}. \tag{5.31}$$

The relation between the expectation value of the characteristic polynomial and the characteristic function reads

$$\left\langle \det\left[x\mathbb{1}_{N+\kappa} - Y^\dagger Y\right]\right\rangle_{Y,N}^{\nu_1,\nu_2,\Sigma_B} = \int [dV]\,\mathrm{e}^{-xV^\dagger V}\left\langle \mathrm{e}^{V^\dagger Y^\dagger Y V}\right\rangle_{Y,N}^{\nu_1,\nu_2,\Sigma_B}, \tag{5.32}$$

in analogy to Eq. (5.10). The integration over the two Gaussian weights for X_1 and X_2 yields

$$\begin{aligned}\left\langle \mathrm{e}^{-VV^\dagger Y^\dagger Y}\right\rangle_{Y,N}^{\nu_1,\nu_2,\Sigma_B} &= \det\left[\Sigma_A^{-1}\right]^{N+\nu_1}\det\left[\Sigma_B^{-1}\right]^{N+\nu_2} \\ &\quad\times \det\begin{bmatrix} \Sigma_A^{-1}\otimes\mathbb{1}_{N+\nu_1} + \mathbb{1}_N\otimes V_AV_A^\dagger & \mathbb{1}_N\otimes V_AV_B^\dagger \\ \mathbb{1}_N\otimes V_BV_A^\dagger & \Sigma_B^{-1}\otimes\mathbb{1}_{N+\nu_2} + \mathbb{1}_N\otimes V_BV_B^\dagger \end{bmatrix}^{-1}.\end{aligned} \tag{5.33}$$

For the application of the duality relation from Eq. (5.12) we introduce two steps. First, we make use of the block structure of the $N(N+\kappa)\times N(N+\kappa)$ determinant in the latter expression, $\det\begin{bmatrix} a & d \\ c & b \end{bmatrix} = \det[a]\det[b]\det\left[\mathbb{1} - b^{-1}ca^{-1}d\right]$, where to the first two determinants the approach already

presented can be applied,

$$\begin{aligned}\det[a] =& \det\left[\Sigma_A^{-1}\otimes N_A + \mathbb{1}_N\otimes V_A V_A^\dagger\right] = \det[\Sigma_A]^{-N_A}\det\left[\mathbb{1}_N + V_A^\dagger V_A\Sigma_A\right]^{-1},\\ \det[b] =& \det\left[\Sigma_B^{-1}\otimes N_B + \mathbb{1}_N\otimes V_B V_B^\dagger\right] = \det[\Sigma_B]^{-N_B}\det\left[\mathbb{1}_N + V_B^\dagger V_B\Sigma_B\right]^{-1}. \end{aligned} \tag{5.34}$$

Second, considering the Schur complement in $\det\left[\mathbb{1} - b^{-1}ca^{-1}d\right]$ we are able to expand this determinant again with the help of Eq. (5.13). In these terms we have for each of the segments in the Schur complement:

$$\begin{aligned} d =& \mathbb{1}_N\otimes V_A V_B^\dagger = (\mathbb{1}_N\otimes V_A)\left(\mathbb{1}_N\otimes V_B^\dagger\right),\\ c =& \mathbb{1}_N\otimes V_B V_A^\dagger = (\mathbb{1}_N\otimes V_B)\left(\mathbb{1}_N\otimes V_A^\dagger\right). \end{aligned} \tag{5.35}$$

By the expansion of the determinant, in each of the moments for C, see Eq. (5.13), the factor of $\mathbb{1}_N\otimes V_B^\dagger$ stands at the end of $\operatorname{Tr} C^k$ for all k. This factor can be preponed within the trace operation, as it was done for the one-epoch model. Here we obtain

$$\begin{aligned}\det\left[\mathbb{1} - b^{-1}ca^{-1}d\right] = \det\Big[\mathbb{1}_N &- \left(\mathbb{1}_N\otimes V_B^\dagger\right)\left(\mathbb{1}_N\otimes\mathbb{1}_{N_B} + \Sigma_B\otimes V_B V_B^\dagger\right)^{-1}(\Sigma_B\otimes V_B)\\ &\times\left(\mathbb{1}_N\otimes V_A^\dagger\right)\left(\mathbb{1}_N\otimes\mathbb{1}_{N_A} + \Sigma_A\otimes V_A V_A^\dagger\right)^{-1}(\Sigma_A\otimes V_A)\Big]^{-1}. \end{aligned} \tag{5.36}$$

An additional step has to be implemented in view of the central terms b^{-1} and a^{-1}. These can be expanded in von Neumann series, because then the following commutation is performable,

$$\begin{aligned}\left(\mathbb{1}_N\otimes V_B^\dagger\right)\left(\mathbb{1}_N\otimes\mathbb{1}_{N_B} + \Sigma_B\otimes V_B V_B^\dagger\right)^{-1} &= \left(\mathbb{1}_N\otimes V_B^\dagger\right)\sum_{k=0}^{\infty}(-1)^k\left(\Sigma_B\otimes V_B V_B^\dagger\right)^k\\ &= \sum_{k=0}^{\infty}(-1)^k\left(V_B^\dagger V_B\Sigma_B\right)^{-1}\left(\mathbb{1}_N\otimes V_B^\dagger\right)\\ &= \left(\mathbb{1}_N + V_B^\dagger V_B\Sigma_B\right)^{-1}\left(\mathbb{1}_N\otimes V_B^\dagger\right), \end{aligned} \tag{5.37}$$

and likewise for the second line. By compressing all resulting terms starting from Eq. (5.33) to one single $N\times N$ determinant, we obtain

$$\left\langle \mathrm{e}^{-VV^\dagger Y^\dagger Y}\right\rangle_{Y,N}^{\nu_1,\nu_2,\Sigma_B} = \det\left[\mathbb{1}_N + V_A^\dagger V_A\Sigma_A + V_B^\dagger V_B\Sigma_B\right]. \tag{5.38}$$

Our result for the characteristic function for the two-epoch model is comparable to the one-epoch model result. The $N\times N$ determinant contains an identity matrix with the addition of the covariance matrices weighted by the according scalar products in the Grassmann valued vectors. Comparing to Eq. (5.14) we observe that the extension to the two-epoch model is justified by adding covariance matrices with the according weights. An appropriate generalisation to a multiple-epoch model in the calculation of the expectation value of the characteristic polynomial can be reached with the help of the presented methods. It leads to an expression for the characteristic function of the form $\det\left[\mathbb{1}_N + V_A^\dagger V_A\Sigma_A + V_B^\dagger V_B\Sigma_B + V_C^\dagger V_C\Sigma_C + \ldots\right]$.
The integration over the complex Grassmann variables in the expectation value of the character-

istic polynomial for the two-epoch model can be performed in analogy to the one-epoch model, see Eq. (5.15). Therefore we split the differentiation with respect to the Grassmann variables according to the two vectors, $\int [dV] = \int [dV_A] \int [dV_B]$, and apply the formula (5.15) twice. By taking into account our calculation presented above in this section, we have an integrand depending on two variables, $f(z_1, z_2) = \exp[-xz_1 - xz_2] \det[\mathbb{1}_N + z_1\Sigma_A + z_2\Sigma_B]$, which is Taylor expandable in both arguments. We obtain

$$\begin{aligned} &\left\langle \det\left[x\mathbb{1}_N - YY^\dagger\right]\right\rangle_{Y,N}^{\nu_1,\nu_2,\Sigma_B} \\ &= (N+\nu_1)!\,(N+\nu_2)! \oint_{\gamma_0} \frac{dz_1}{2\pi\imath z_1} \oint_{\gamma_0} \frac{dz_2}{2\pi\imath z_2} \frac{\mathrm{e}^{z_1+z_2}}{z_1^{N+\nu_1} z_2^{N+\nu_2}} \prod_{i=1}^{N} (x - \sigma_A z_1 - \sigma_{Bi} z_2)\,, \end{aligned} \tag{5.39}$$

where we plugged in the half degeneracy, $\Sigma_A = \sigma_A \mathbb{1}_N$ and $\Sigma_B = \operatorname{diag}(\sigma_{B1}, \ldots, \sigma_{BN})$ from Eq. (3.25). Both contours, denoted by γ_0, encircle the origin in counter-clock wise direction and can be taken independently.

We now perform the dimensional reduction by following Sec. 4.4, in order to obtain the desired polynomials $\{\chi(x, \delta_j)\}_{j=1,\ldots,N}$. They are all of the degree $(N-1)$ and orthogonal with respect to the set of functions $\{\varphi(x, \delta_j)\}_{j=1,\ldots,N}$. From Eq. (4.45) we have

$$\chi(x, \delta_j) = (\kappa - \nu_2)!\,(N - 1 + \nu_2)! \oint_{\gamma_0} \frac{dz_1}{2\pi\imath z_1} \frac{\mathrm{e}^{z_1}}{z_1^{\kappa-\nu_2}} \oint_{\gamma_0} \frac{dz_2}{2\pi\imath} \frac{\mathrm{e}^{z_2}}{z_2^{N+\nu_2}} \prod_{k=1, k\neq j}^{N} (x - \sigma_A z_1 - \sigma_{Bi} z_2)\,, \tag{5.40}$$

where we recall $\delta_j = \sigma_A^{-1} - \sigma_{Bj}^{-1}$. Here, the dimensions κ and ν_2 were kept fixed whereas the matrix dimension was reduced, $N \to (N-1)$, which required the reduced product $\prod_{k=1,k\neq j}^{N}$.

For the single-sum formulation of the correlation kernel the norms $\{h_j\}_{j=1,\ldots,N}$ have to be computed. In analogy to the one-epoch model, we could use the orthogonality condition between the sets $\{\chi(x, \delta_j)\}_{j=1,\ldots,N}$ and $\{\varphi(x, \delta_j)\}_{j=1,\ldots,N}$. However, for this computation the confluent hypergeometric function ${}_1F_1$ is unfavourable due to the infinite sum in the power expansion. An alternative representation for the confluent hypergeometric function based on elementary functions, exponential function and polynomials of finite degree, was derived in [1, Appendix A]. Still, we prefer to compute the norms $\{h_j\}_{j=1,\ldots,N}$ with the help of the normalising constant [1, Eq. (A.11)], by making use of the expression derived in Eq. (4.44) being a ratio of the determinant of the Gram matrix and its respective minor.

By comparing our normalisation given in terms of the Gram matrix with the normalisation derived in [1, Eq. (A.11)],

$$\det[g_j(\delta_i)]_{i,j=1}^{N} = \Delta_N\left(\sigma_{B1}, \ldots, \sigma_{B_N}\right) \sigma_A^{\kappa-\nu_2} \prod_{k=1}^{N} \sigma_{Bk}^{\nu_2+1} \prod_{l=1}^{N} \frac{(\nu_2 + l - 1)!\,\kappa!}{\nu_2!}\,, \tag{5.41}$$

and by making use of Eq. (4.44) we obtain

$$h_j = \frac{\kappa!}{\nu_2!} (N + \nu_2 - 1)!\, \sigma_{Bj}^{\nu_2+1} \prod_{k=1, k\neq j}^{N} (\sigma_{Bj} - \sigma_{Bk})\,. \tag{5.42}$$

We now can state the single-sum representation for the correlation kernel of the two-epoch model as composed in Eq. (4.41):

$$K_N(x,y) = \frac{\nu_2!\,(\kappa-\nu_2)!}{\kappa!}\sum_{j=1}^{N}\frac{1}{\sigma_{Bj}}\,(x/\sigma_{Bj})^{\kappa}\,\mathrm{e}^{-x/\sigma_A}\,{}_1F_1\left(\nu_2+1;\kappa+1;\delta_j x\right) \tag{5.43}$$
$$\times\oint_{\gamma_0}\frac{dz_1}{2\pi\imath z_1}\frac{\mathrm{e}^{z_1}}{(z_1/\sigma_{Bj})^{\kappa-\nu_2}}\oint_{\gamma_0}\frac{dz_2}{2\pi\imath}\frac{\mathrm{e}^{z_2}}{z_2^{N+\nu_2}}\prod_{k=1,k\neq j}^{N}\frac{y-\sigma_A z_1-\sigma_{Bk}z_2}{\sigma_{Bj}-\sigma_{Bk}}\,.$$

We additionally may represent the one remaining sum in j in the latter expression for the correlation kernel as a contour integral which simplifies the N-dependence and makes the present kernel comparable to the one-epoch model. Let us perform first the following substitutions: $y-z_1\sigma_A=y\zeta$, where the contour for ζ called γ_1 encircles the pole at $\zeta=1$, and after substituting $z_2=z/\zeta$ with the condition of $\mathrm{Re}\,\zeta>0$, we obtain

$$K_N(x,y) = \left(\frac{x}{y}\right)^{\kappa}\frac{\mathrm{e}^{-x/\sigma_A}}{\mathrm{e}^{-y/\sigma_A}}\frac{\nu_2!\,(\kappa-\nu_2)!}{\kappa!}\sum_{j=1}^{N}\frac{1}{\sigma_{Bj}}\,{}_1F_1\left(\nu_2+1;\kappa+1;\left(\sigma_A^{-1}-\sigma_{Bj}^{-1}\right)x\right) \tag{5.44}$$
$$\times\oint_{\gamma_1}\frac{d\zeta}{2\pi\imath(1-\zeta)}\left(\frac{1}{(1-\zeta)\sigma_A}\right)^{\kappa-\nu_2}\oint_{\gamma_0}\frac{dz}{2\pi\imath z}\left(\frac{y\zeta}{z\sigma_{Bj}}\right)^{\nu_2}\mathrm{e}^{z/\zeta-\zeta y/\sigma_A}\prod_{k=1,k\neq j}^{N}\frac{y/z-\sigma_{Bk}}{\sigma_{Bj}-\sigma_{Bk}}\,,$$

where both contours, γ_0 and γ_1, are taken counter-clockwise. The factor, $x^{\kappa}\,\mathrm{e}^{-x/\sigma_A}/y^{\kappa}\,\mathrm{e}^{-y/\sigma_A}$, can be neglected as we still remain in the same equivalence class of correlation kernels as stated in Eq. (4.4). We encircle all covariance eigenvalues σ_{B1}, …, σ_{BN} with the contour integral parametrised with ds counter-clockwise

$$K_N(x,y) = \frac{\nu_2!\,(\kappa-\nu_2)!}{\kappa!}\oint_{\Sigma_B}\frac{ds}{2\pi\imath s}\,{}_1F_1\left(\nu_2+1;\kappa+1;\left(\sigma_A^{-1}-s^{-1}\right)x\right) \tag{5.45}$$
$$\times\oint_{\gamma_1}\frac{d\zeta}{2\pi\imath(1-\zeta)}\left(\frac{1}{(1-\zeta)\sigma_A}\right)^{\kappa-\nu_2}\oint_{\gamma_0}\frac{dz}{2\pi\imath z}\left(\frac{y\zeta}{sz}\right)^{\nu_2}\frac{\mathrm{e}^{z/\zeta-\zeta y/\sigma_A}}{y/z-s}\prod_{k=1}^{N}\frac{y/z-\sigma_{Bk}}{s-\sigma_{Bk}}\,.$$

The contour γ_{Σ_B} encircles all the positive eigenvalues of the covariance matrix Σ_B in positive direction, not crossing the contour γ_0 parametrised by the integration variable z, due to the factor of $(y/z-s)^{-1}$. Because the product standing at the end of the latter expression comprises all parameters σ_{B1}, …, σ_{BN}, this expression for the correlation kernel is suitable for the introduction of degeneracies of the covariance matrix Σ_B.

We illustrate the obtained result by depicting the spectral density, $K_N(x,x)=R_1(x)$, for finite N and with fixed values for the parameter set $\{\sigma_{Bj}\}_{j=1,\ldots,N}$ in Fig. 5.2. The analytical and the numerical solutions to the level density coincide. In these two diagrams two different epoch lengths are imposed. In the left diagram the epoch lengths are comparable to the number of observables, whereas in the right digram long epoch lengths are considered. For long epoch lengths the positions of the individual peaks can be estimated again by

$$R_1(x)\approx\sum_{k=1}^{N}\delta\left(x-(N+\nu_1)\,\sigma_A+(N+\nu_2)\,\sigma_{Bk}\right),\quad\text{for}\;\;N+\nu_1\gg N\;\;\text{and}\;\;N+\nu_2\gg N\,. \tag{5.46}$$

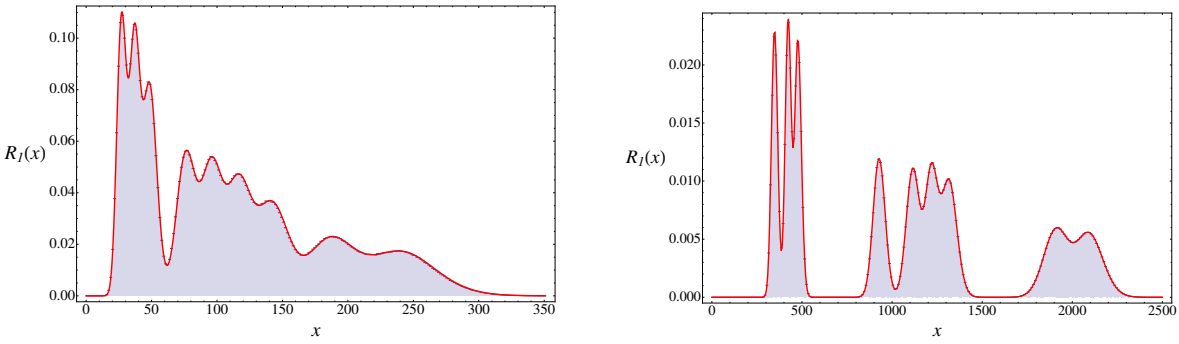

Figure 5.2: Comparison of the analytical result (5.43) for the spectral density (red curves) with Monte Carlo simulations (histograms=shaded area, 10^6 matrices drawn from the ensemble (3.19)) from [1, Fig. 1]. We employed the parameters $N = 9$ with the fixed covariance matrices $\Sigma_A = \mathbb{1}_9$ and $\Sigma_B = \text{diag}(0.02, 0.20, 0.30, 1.50, 2.01, 2.25, 2.27, 4.05, 4.13)$ and the time length of the epochs $(N+\nu_1, N+\nu_2) = (35, 40)$ (left plot) and $(N+\nu_1, N+\nu_2) = (350, 400)$ (right plot).

In this case the separation of the individual eigenvalue distributions is strong, although these individual distributions are broadened by comparison to the left hand side. This effect is already discussed in the previous section after Fig. 5.1.

Let us point out the main differences between the correlation kernel of the one-epoch model from Eq. (5.25) and the correlation kernel of the two-epoch model from Eq. (5.44). These can be seen in the emerging confluent hypergeometric function ${}_1F_1$ in the two-epoch model due to $\nu_2 \neq \kappa$ as well as in the second complex contour integral, such that an interplay between γ_0 and γ_1 has to be involved. The extension from the one- to the two-epoch model is, thus, non-trivial. These and further implications in the correlation kernel of the two-epoch model can't be directly derived from the correlation kernel of the one-epoch model. Hence, the generalisation from the one- to the two-epoch model involves sophisticated methods.

The question for a naive comparison of the global densities of these two models can be asked. The two figures, Fig. 5.1 and Fig. 5.2, allow two interpretations to the comparison of the global spectral density:

- **Lost-memory case**: The second epoch X_2 can be studied as a one single epoch, without the information of the first epoch. Thereby, the second epoch becomes a one-epoch model, $X_2 = X$ and $\Sigma_B = \Sigma$ and X_1 is integrated out. By doing so, the upper left and the lower left diagrams in Fig. 5.1 can be taken for the comparison to the diagrams in Fig. 5.2, due to their number of time steps coinciding with 40 and 400. For both lengths of the time series, the distribution for the three smallest eigenvalues differs strongly between the two models. In the one-epoch model, it stays near the origin, whereas in the two-epoch model these three eigenvalues are pushed away from the origin. In addition, in the case of a long time series, namely with 400 time steps, the repulsion from the origin can be observed for all eigenvalues.

- **Time-independent case**: The entire time series $Y = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$ can be assumed to have time-independent but non-trivial spatial correlations. In this comparison we would set $Y = X$ and $\Sigma_A = \Sigma_B = \Sigma$. Thus, no threshold appears in our system. Fig. 5.2 is comparable with the right upper and the right lower diagrams in Fig. 5.1 with the coinciding number of time steps: 75 and

750. Here, we observe the increased repulsion from the origin for the smallest three eigenvalues for the two-epoch model in analogy to the lost-memory case. However, the entire spectrum of the two-epoch model is narrowly distributed. In the two-epoch model, the increased repulsion from the origin is suppressed for the large eigenvalues.

As discussed above, the time-independent case is analytically not comparable to the two-epoch model, whereas an average over the first epoch in the two-epoch model is thinkable and would yield the lost-memory case. This difficulty is according to our discussion at the end of Sec. 3.2.
From the application point of view, as discussed in the first part of Chap. 2, the experimentalist would like to include empirical data into a random matrix model to model the spectrum of the covariance matrix influenced by statistical fluctuations. Thereby, the statistical fluctuations are of the same order as the eigenvalues of the covariance matrix and, hence, they are overlapping. For a qualitative meaningful figure the experimentalist has to approximate the covariance matrix with the help of the mean, see Eq. (2.8), which implies reproducibility of the given experiment. In this set-up, the time dependence of the experiment is a preliminar knowledge and, thus, the question of the appropriate random matrix model does not arise.

5.1.3 Temporal Cross-Correlations

In this section we present the derivation of the correlation kernel characterising spectral properties of the two-epoch model under the assumption of temporal cross-correlations. Our considerations to this model do not appear in the principal publications nor in further literature, but we use methods, which are well known. The use of unitary group integrals for our present discussion are comprehensively described in [2], whereas the derivation for the correlation kernel follows the method presented in [1].
From Sec. 2.1, the eigenvalues of $YY^\dagger$ with $Y = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$ correspond to spatial cross-correlations in the two-epoch model. The dyadic product, $YY^\dagger$, shares the same spectrum up to null-valued eigenvalues. Still, the dyadic product contains further information as it immediately becomes visible in

$$Y^\dagger Y = \begin{pmatrix} X_1^\dagger X_1 & X_1^\dagger X_2 \\ X_2^\dagger X_1 & X_2^\dagger X_2 \end{pmatrix}. \tag{5.47}$$

Here, in the off-diagonal blocks the product

$$\Upsilon = X_1^\dagger X_2 \tag{5.48}$$

occurs and is non-trivial for unequal covariances $\Sigma_A \neq \Sigma_B$. We interpret the singular value statistics of these off-diagonal blocks as temporal cross-correlations in the two-epoch model.
Note that the eigenvalue statistics in the diagonal blocks, $X_1^\dagger X_1$ and $X_2^\dagger X_2$, respectively, can also be studied. However, they are already known from the one-epoch model, Sec. 5.1.1, due to the independence of X_1 and X_2.
In the literature, products of independent complex Wishart matrices including external fixed matrices are already known, see our discussion in Sec. 3.3. However, due to non-trivial numbers of zero modes ν_1 and ν_2 our particular set-up is new. The product matrix Υ is of dimensions $(N+\nu_1)\times(N+\nu_2)$. The rank of Υ is N. For $\nu_1 \neq 0$, $\nu_2 \neq 0$ and $\nu_1 \neq \nu_2$ explicit expressions for the joint probability

distribution function for the singular values of Υ as well as the corresponding correlation kernel are new results.

In the derivation of the joint probability distribution function of the singular values of Υ, the fixed matrix Σ_A has to be chose degenerate. However, we keep Σ_A arbitrary up to the computational step, at which it becomes immediate why this covariance matrix needs to be chosen degenerate.

We begin our computation with a squaring decomposition of the rectangular matrices X_1 and X_2 by following [117]. Thereby, two random matrices $\tilde{X}_1$ and $\tilde{X}_2$ of dimensions $N \times N$ can be used to represent the spectral properties of X_1 and X_2, respectively. Two unitary matrices are required for the map $X_{1/2} \to \tilde{X}_{1/2}$, where each of those belong to the unitary coset spaces: $U_A \in U(N+\nu_1)/[U(N) \times U(\nu_1)]$ and $U_B \in U(N+\nu_2)/[U(N) \times U(\nu_2)]$, respectively,

$$X_1 = \left(\tilde{X}_1, 0_{N,\nu_1}\right) U_A \quad \text{and} \quad X_2 = \left(\tilde{X}_2, 0_{N,\nu_2}\right) U_B \,, \tag{5.49}$$

where we denote null-valued matrices by $0_{N,\nu_0}$. The singular value statistics of Υ remain unchanged under this map, which can be verified on its characteristic equation by making use of the unitarity of $U_{A/B}$ and cyclicity of the determinant operation,

$$\det\left[x\mathbb{1}_{N+\nu_1} - \Upsilon\Upsilon^\dagger\right] = \det\left[x\mathbb{1}_{N+\nu_1} - \begin{pmatrix} \tilde{\Upsilon}\tilde{\Upsilon}^\dagger & 0_{N,\nu_1} \\ 0_{\nu_1,N} & 0_{\nu_1,\nu_1} \end{pmatrix}\right], \tag{5.50}$$

where we call the emerging product

$$\tilde{\Upsilon} = \tilde{X}_1^\dagger \tilde{X}_2 \,, \tag{5.51}$$

which is of dimensions $N \times N$. From Eq. (5.50) we additionally identify that Υ and $\tilde{\Upsilon}$ share the same singular values.

The probability densities for $\tilde{X}_{1/2}$ are obtained from those of $X_{1/2}$ by implementing the according Jacobian from the Lebesgue measure $\left[dX_{1/2}\right] \to \left[d\tilde{X}_{1/2}\right] d\mu\left(U_{A/B}\right)$, which is essentially given by a determinant of $X_{1/2}X_{1/2}^\dagger$. The probability densities for $\tilde{X}_{1/2}$ are thus weighted Gaussian densities,

$$\mathcal{P}(X_1|\Sigma_A)\,[dX_1] \sim \mathrm{e}^{-\mathrm{Tr}\,\Sigma_A^{-1}\tilde{X}_1\tilde{X}_1^\dagger} \det\left[\tilde{X}_1\tilde{X}_1^\dagger\right]^{\nu_1} \left[d\tilde{X}_1\right] d\mu(U_A) \,, \tag{5.52}$$

and analogously for $\mathcal{P}(X_2|\Sigma_B)\,[dX_2]$. The probability density for Υ is obtained by performing the following substitution,

$$\tilde{X}_1^\dagger \to \tilde{\Upsilon} = \tilde{X}_1^\dagger \tilde{X}_2 \quad \text{and} \quad \tilde{X}_2 \to \tilde{X}_2 \,, \tag{5.53}$$

in the joint densities for $\tilde{X}_1$ and $\tilde{X}_2$. Assuming $\tilde{X}_2$ to be invertible, we find in the probability density for $\tilde{X}_1$ and $\tilde{X}_2$ a random two-matrix model for $\tilde{\Upsilon}$ and $\tilde{X}_2$,

$$\begin{aligned} \mathcal{P}(X_1|\Sigma_A)\,[dX_1]\,\mathcal{P}(X_2|\Sigma_B)\,[dX_2] \sim & \mathrm{e}^{-\mathrm{Tr}\left(\tilde{X}_2^\dagger\Sigma_A\tilde{X}_2\right)^{-1}\tilde{\Upsilon}^\dagger\tilde{\Upsilon}} \det\left[\tilde{\Upsilon}^\dagger\tilde{\Upsilon}\right]^{\nu_1} \left[d\tilde{\Upsilon}\right] \\ & \times \mathrm{e}^{-\mathrm{Tr}\,\Sigma_B^{-1}\tilde{X}_2\tilde{X}_2^\dagger} \det\left[\tilde{X}_2\tilde{X}_2^\dagger\right]^{\nu_2-\nu_1-N} \left[d\tilde{X}_2\right] d\mu(U_A)\, d\mu(U_B) \,. \end{aligned} \tag{5.54}$$

The additional determinantal weight in $\tilde{X}_2$ of the power $-\nu_1 - N$ in the latter expression arises from $\det\left[\tilde{X}_1\tilde{X}_1^\dagger\right]^{\nu_1}\left[d\tilde{X}_1\right] \to \det\left[\tilde{\Upsilon}\tilde{\Upsilon}^\dagger\right]^{\nu_1}\left[d\tilde{\Upsilon}\right]$. The two Haar measures, $d\mu(U_B)$, can simply be integrated

out from the expression in Eq. (5.54) as the integrand does not depend neither on U_A nor on U_B. For the integration over $\tilde{X}_2$ we would like to prepone singular value decompositions, as discussed in Seq. 3.1.
From singular value decompositions, $\tilde{X}_2 = V_X \Lambda_X^{\frac{1}{2}} U_X$ and $\tilde{\Upsilon} = V_\Upsilon \Lambda_\Upsilon^{\frac{1}{2}} U_\Upsilon$, it becomes immediate, why the half degeneracy, $\Sigma_A = \sigma_A \mathbb{1}_N$, has to be assumed if a determinantal point process for the singular values of $\tilde{\Upsilon}$ is desired. From the Lebesgue measures $\left[d\tilde{X}_2\right]$ and $\left[d\tilde{\Upsilon}\right]$ we obtain Haar measures with respect to the involved unitary matrices: V_X, U_X and V_Υ, U_Υ. Integrating out the dependence on these unitary matrices from the expression in Eq. (5.54) we are faced with three breakings of unitary bi-invariance:

- First, in order to keep non-trivial correlations among the entries of $\tilde{X}_2$, its left unitary invariance is broken, $\exp\left[-\mathrm{Tr}\, V_X^\dagger \Sigma_B^{-1} V_X \Lambda_X\right]$.
- Second, the unitary invariance of $\tilde{X}_2$ is again broken from the left by including the first covariance matrix in the exponential, $\exp\left[-\mathrm{Tr}\left(\Lambda_X^{\frac{1}{2}} V_X^\dagger \Sigma_A V_X \Lambda_X^{\frac{1}{2}}\right)^{-1} \tilde{\Upsilon}^\dagger \tilde{\Upsilon}\right]$.
- Third, again in the exponent $\exp\left[-\mathrm{Tr}\left(\Lambda_X^{\frac{1}{2}} V_X^\dagger \Sigma_A V_X \Lambda_X^{\frac{1}{2}}\right)^{-1} \tilde{\Upsilon}^\dagger \tilde{\Upsilon}\right]$ the unitary invariance of $\tilde{\Upsilon}$ is broken from the right .

The only dependence on a unitary matrix factorising from the singular value dependence in the density in Eq. (5.54) is in U_Υ. However, both exponentials are coupled through V_X and the first exponential depends on V_X and on V_Υ. These dependencies cannot be simplified further through the unitary invariance of the involved Haar measures. The integration over these unitary matrices does not lead to a determinantal expression, see for a discussion in a similar case [118]. As in the analysis of spatial cross-correlations, we make use of the half-degeneracy, $\Sigma_A \to \sigma_A \mathbb{1}_N$ and $\Sigma_B = \mathrm{diag}(\sigma_{B1}, \ldots, \sigma_{BN})$, from Eq. (3.25). This degeneracy decouples our two Gaussian weights and ensures determinantal structure from unitary group integrals. We apply the Harish-Chandra/Itzykson–Zuber integral from Eq. (3.16) twice.
The joint probability distribution function for the squared singular values $y_1, \ldots, y_N$ of the matrix $\tilde{\Upsilon}$, or equivalently of the matrix Υ, and simultaneously for the squared singular values of X_2 denoted by $x_1, \ldots, x_N$ is given by

$$\begin{aligned} &P_N(x_1,\ldots,x_N;y_1,\ldots,y_N) \\ &\propto \Delta_N(y_1,\ldots,y_N) \det\left[x_i^{\nu_2-1} \mathrm{e}^{-\frac{x_i}{\sigma_{Bj}}}\right]_{i,j=1}^N \det\left[\left(\frac{y_i}{x_j}\right)^{\nu_1} \mathrm{e}^{-\frac{y_i}{\sigma_A x_j}}\right]_{i,j=1}^N , \end{aligned} \tag{5.55}$$

where we have used that $\Delta_N\left(x_1^{-1},\ldots,x_N^{-1}\right) = \text{const.}\, \Delta_N(x_1,\ldots,x_N) / \prod_{k=1}^N x_k^{N-1}$ and suppressed the normalising constant. Our joint probability distribution function of $\{x_j\}_{j=1,\ldots,N}$ and $\{y_j\}_{j=1,\ldots,N}$ belongs to the class of biorthogonal two-matrix models as discussed in Sec. 4.5. For the joint probability distribution function for $y_1 \ldots, y_N$ we integrate over all $x_1, \ldots, x_N$.
With the help of Andréief's integration formula [90] the N-fold integration,
$\int_0^\infty dx_1 \ldots \int_0^\infty dx_N P_N(x_1,\ldots,x_N;y_1,\ldots,y_N)$ results in one single integral, see Eq. (4.50) with Eq. (4.52).

Thereby, we make use of the formula [82, 3.471.9],

$$\int_0^\infty dx x^{\nu_2-\nu_1-1}\,\mathrm{e}^{-\frac{x}{\sigma_{Bj}}}\,\mathrm{e}^{-\frac{y}{\sigma_A x}} = 2\left(\frac{\sigma_{Bj}y}{\sigma_A}\right)^{\frac{\nu_2-\nu_1}{2}} K_{\nu_2-\nu_1}\left(2\sqrt{\frac{y}{\sigma_A\sigma_{Bj}}}\right), \tag{5.56}$$

where $K_{\nu_2-\nu_1}$ denotes the modified Bessel function of the second kind, which was introduced in Eq. (3.46). We additionally made use of its symmetry in the index, $K_{-\kappa}=K_\kappa$, from [82, Eq. 8.486.16]. The joint probability distribution function for the squared singular values y_1, …, y_N reads

$$P_N(y_1,\ldots,y_N) = \frac{\det\left[\varphi(y_j,\sigma_A\sigma_{Bi})\right]_{i,j=1}^N \Delta_N(y_1,\ldots,y_N)}{N!\det\left[(\nu_1+j-1)!(\nu_2+j-1)!\,(\sigma_A\sigma_{Bi})^j\right]_{i,j=1}^N} \tag{5.57}$$

with

$$\varphi(y,\sigma_A\sigma_{Bj}) = 2\left(\frac{y}{\sigma_A\sigma_{Bj}}\right)^{\frac{\nu_2+\nu_1}{2}} K_{\nu_2-\nu_1}\left(2\sqrt{\frac{y}{\sigma_A\sigma_{Bj}}}\right), \tag{5.58}$$

which is a polynomial ensemble with permutational symmetry as given by Eq. (4.37). The resulting Gram matrix $(\nu_1+j-1)!(\nu_2+j-1)!\,(\sigma_A\sigma_{Bi})^j = \int_0^\infty dy\,\varphi(y,\sigma_A\sigma_{Bi})y^{j-1}$ is given in the literature by the formula [82, Eq. 6.561.16].

From here on we turn to the solution of the given determinantal point process which is its correlation kernel and we follow analogously to the previously solved ensembles the method presented in Sec. 4.4. The given polynomial ensemble with permutational symmetry can be solved by the computation of its single-sum kernel as presented in Eq. (4.41). Therefore, the expectation value of one single characteristic polynomial is needed, which reads in the present case

$$\begin{aligned}&\left\langle\det\left[x\mathbb{1}_N-\tilde{\Upsilon}\tilde{\Upsilon}^\dagger\right]\right\rangle_{\tilde{\Upsilon},N}^{\nu_1,\nu_2,\Sigma_B} \\ &= x^{-\nu_1}\int[dX_1]\,[dX_2]\,\mathcal{P}(X_1|\Sigma_A)\,\mathcal{P}(X_2|\Sigma_B)\det\left[x\mathbb{1}_{N+\nu_1}-X_1^\dagger X_2 X_2^\dagger X_1\right].\end{aligned} \tag{5.59}$$

We extend the dimension of the considered determinant by linearising the operator $\tilde{\Upsilon}\tilde{\Upsilon}^\dagger$,

$$x_A^{\nu_1}x_B^{\nu_2}\left\langle\det\left[x\mathbb{1}_N-\tilde{\Upsilon}\tilde{\Upsilon}^\dagger\right]\right\rangle_{\tilde{\Upsilon},N}^{\nu_1,\nu_2,\Sigma_B} = \left\langle\det\begin{bmatrix}x_A\mathbb{1}_{N+\nu_1} & \Upsilon\\ \Upsilon^\dagger & x_B\mathbb{1}_{N+\nu_2}\end{bmatrix}\right\rangle_{\Upsilon,N}^{\nu_1,\nu_2,\Sigma_B}, \tag{5.60}$$

$$\text{with}\quad x=x_Ax_B,$$

where we came back to the original product matrix Υ from Eq. (5.51). From here, it is immediate that the resulting determinant of dimensions $(2N+\nu_1+\nu_2)\times(2N+\nu_1+\nu_2)$ can be parametrised exactly in the same way as in the analysis of spatial cross-correlations in Sec. 5.1.2. The so obtained Berezin integral reads

$$\begin{aligned}&x_A^{\nu_1}x_B^{\nu_2}\left\langle\det\left[x\mathbb{1}_N-\tilde{\Upsilon}\tilde{\Upsilon}^\dagger\right]\right\rangle_{\tilde{\Upsilon},N}^{\nu_1,\nu_2,\Sigma_B} \\ &= \int[dV]\,\mathrm{e}^{-x_AV_A^\dagger V_A-x_BV_B^\dagger V_B}\left\langle\mathrm{e}^{V_A^\dagger X_1^\dagger X_2V_B+V_B^\dagger X_2^\dagger X_1V_A}\right\rangle_{X_1,X_2,N}^{\nu_1,\nu_2,\Sigma_B}.\end{aligned} \tag{5.61}$$

The expectation value with respect to the two complex Gaussian matrices X_1 and X_2 as well as the duality formula are to be performed analogously to the previous section. We skip these steps, as they are previously presented in detail. The present characteristic function reads

$$\left\langle \mathrm{e}^{V_A^\dagger X_1^\dagger X_2 V_B + V_B^\dagger X_2^\dagger X_1 V_A} \right\rangle_{X_1,X_2,N}^{\nu_1,\nu_2,\Sigma_B} = \det\left[\mathbb{1}_N - V_A^\dagger V_A V_B^\dagger V_B \Sigma_A \Sigma_B\right]. \tag{5.62}$$

As in the previous ensembles, the one-epoch model and spatial cross-correlations in the two-epoch model, we obtain an $N \times N$ determinant. As the main difference, here, for the temporal cross-correlations we obtain a product of the covariance matrices weighted by $V_A^\dagger V_A V_B^\dagger V_B$. For the spatial cross-correlations we could conclude that an extension to more than two epochs can immediately be arranged through an extension of the sum of covariance matrices in the characteristic function. For the temporal cross-correlations a product of more than two covariance matrices in the corresponding characteristic function is thinkable. However, an interpretation of such extension is non-trivial and goes beyond the scope of our considerations.

We now turn to the superbosonisation formula, which has to be applied twice on $f(z_1,z_2) = \exp[-x_A z_1 - x_B z_2]\det[\mathbb{1}_N - z_1 z_2 \Sigma_A \Sigma_B]$. By taking into account the half degeneracy, $\Sigma_A \to \sigma_A \mathbb{1}_N$ and $\Sigma_B = \mathrm{diag}(\sigma_{B1},\ldots,\sigma_{BN})$, from Eq. (3.25), we obtain

$$\begin{aligned} &\left\langle \det\left[x\mathbb{1}_N - \tilde{\Upsilon}\tilde{\Upsilon}^\dagger\right]\right\rangle_{\tilde{\Upsilon},N}^{\nu_1,\nu_2,\Sigma_B} \\ &= (N+\nu_1)!(N+\nu_2)! \oint_{\gamma_0} \frac{dz_1}{2\pi\imath z_1} \oint_{\gamma_0} \frac{dz_2}{2\pi\imath z_2} \frac{\mathrm{e}^{z_1+z_2}}{z_1^{N+\nu_1} z_2^{N+\nu_2}} \prod_{i=1}^{N} (x - z_1 z_2 \sigma_A \sigma_{Bi}), \end{aligned} \tag{5.63}$$

where we plugged in $x = x_A x_B$ from Eq. (5.60). The two contour integrals specified by γ_0 are taken counter-clockwise around the origin and independently of each other. Nonetheless, the two countour integrals in z_1 and z_2 are not factorisable due to the product $\prod_{i=1}^{N}(x - z_1 z_2 \sigma_A \sigma_{Bi})$.

For the wanted polynomials $\{\chi(x,\sigma_A\sigma_{Bj})\}_{j=1,\ldots,N}$, we perform the dimensional reduction according to Eq. (4.45), yielding

$$\begin{aligned} &\chi(x,\sigma_A\sigma_{Bj}) \\ &= (N-1+\nu_1)!(N-1+\nu_2)! \oint_{\gamma_0} \frac{dz_1}{2\pi\imath z_1} \oint_{\gamma_0} \frac{dz_2}{2\pi\imath z_2} \frac{\mathrm{e}^{z_1+z_2}}{z_1^{\nu_1} z_2^{\nu_2}} \prod_{k=1,k\neq j}^{N} \left(\frac{x}{z_1 z_2} - \sigma_A \sigma_{Bk}\right), \end{aligned} \tag{5.64}$$

where both contours are taken around the origin in counter clock-wise direction.

As a simplifying step we remove this coupling from the complementary product. It can be done by considering substitutions of the type $z_1 \to z(z_1) = s z_1$ and $z_2 \to s(z_2) = z_2$. We are able to make use of the modified Bessel function of the first kind,

$$z^{-\frac{n}{2}} I_n\left(2\sqrt{z}\right) = \oint_{\gamma_0} \frac{ds}{2\pi\imath s} s^{-n}\, \mathrm{e}^{s+\frac{z}{s}}, \quad \text{for } \mathrm{Re}\, n > -1, \tag{5.65}$$

where this integral representation follows from the Taylor series from Eq. (3.18) and by applying the residue theorem, such that only one single contour integral remains,

$$\begin{aligned}&\chi(x,\sigma_A\sigma_{Bj}) \\ &= (N-1+\nu_1)!\,(N-1+\nu_2)!\oint_{\gamma_0}\frac{dz}{2\pi\imath z}z^{-\frac{\nu_2+\nu_1}{2}}I_{\nu_2-\nu_1}\left(2\sqrt{z}\right)\prod_{k=1,k\neq j}^{N}\left(\frac{x}{z}-\sigma_A\sigma_{Bk}\right).\end{aligned} \tag{5.66}$$

Note that the modified Bessel function of the second kind possesses the symmetry $I_{-n}=(-1)^n I_n$, from [82, Eq. 8.404.2].
It remains to compute the norms h_j in order to obtain the entire correlation kernel in the derivation from Eq. (4.41). Therefore we make use of the orthogonality condition,
$\delta_{ij}h_j=\int_0^\infty dx\varphi(x,\sigma_A\sigma_{Bi})\,\chi(x,\sigma_A\sigma_{Bj})$, in analogy to the one-epoch model in Sec. 5.1.1. We have to perform two integrals,

$$\begin{aligned}h_j\delta_{ij} =&2\,(N-1+\nu_1)!\,(N-1+\nu_2)!\int_0^\infty dx\left(\frac{x}{\sigma_A\sigma_{Bi}}\right)^{\frac{\nu_2+\nu_1}{2}}K_{\nu_2-\nu_1}\left(2\sqrt{\frac{x}{\sigma_A\sigma_{Bi}}}\right)\\ &\times\oint_{\gamma_0}\frac{dz}{2\pi\imath z}z^{-\frac{\nu_2+\nu_1}{2}}I_{\nu_2-\nu_1}\left(2\sqrt{z}\right)\prod_{k=1,k\neq j}^{N}\left(\frac{x}{z}-\sigma_A\sigma_{Bk}\right).\end{aligned} \tag{5.67}$$

Comparing to the calculation of the norms in the one-epoch model case, the construction of the polynomials $\mathtt{f}_j$ by considering their meromorphic properties was the key tool, see Eq. (5.21). Here we have

$$\mathtt{f}_j(u)=\prod_{k=1,k\neq j}^{N}\frac{u-\sigma_A\sigma_{Bk}}{\sigma_A\sigma_{Bj}-\sigma_A\sigma_{Bk}}, \tag{5.68}$$

where they obviously satisfy the Kronecker-Delta property for the finite set $\{\sigma_A\sigma_{Bi}\}_{i=1,\ldots,N}$,

$$\mathtt{f}_j(\sigma_A\sigma_{Bi})=\delta_{ij}\quad\text{for }\;i,j=1,\ldots,N. \tag{5.69}$$

By performing the substitution $z\to xz$, the polynomials $\{\mathtt{f}_j\}_{j=1,\ldots,N}$ emerge under the integral with $1/z$ in their arguments,

$$\begin{aligned}h_j\delta_{ij} =&2\,(N-1+\nu_1)!\,(N-1+\nu_2)!\,(\sigma_A\sigma_{Bi})^{-\frac{\nu_2+\nu_1}{2}}\prod_{k=1,k\neq j}^{N}(\sigma_A\sigma_{Bj}-\sigma_A\sigma_{Bk})\\ &\times\int_0^\infty dx K_{\nu_2-\nu_1}\left(2\sqrt{\frac{x}{\sigma_A\sigma_{Bi}}}\right)\oint_{\gamma_0}\frac{dz}{2\pi\imath z}z^{-\frac{\nu_2+\nu_1}{2}}I_{\nu_2-\nu_1}\left(2\sqrt{xz}\right)\mathtt{f}_j\left(\frac{1}{z}\right).\end{aligned} \tag{5.70}$$

The coupling between the integrals in x and z has been moved by this substitution to the argument of the I-Bessel function, which facilitates the x integral. Namely, we now can change the order of integration and are allowed to make use of the formula [82, Eq. 6.576.7],

$$\int_0^\infty dx K_{\nu_2-\nu_1}\left(2\sqrt{\frac{x}{\sigma_A\sigma_{Bi}}}\right)I_{\nu_2-\nu_1}\left(2\sqrt{xz}\right)=\frac{1}{2}\left(\sigma_A\sigma_{Bi}z\right)^{\frac{\nu_2-\nu_1}{2}}\frac{1}{(\sigma_A\sigma_{Bi})^{-1}-z}. \tag{5.71}$$

It remains to evaluate the contour integral around the origin,

$$h_j\delta_{ij} = (N-1+\nu_1)!\,(N-1+\nu_2)!\,(\sigma_A\sigma_{Bi})^{-\nu_1+1}\prod_{k=1,k\neq j}^{N}(\sigma_A\sigma_{Bj}-\sigma_A\sigma_{Bk}) \times \oint_{\gamma_0}\frac{dz}{2\pi\imath z}\frac{1}{z^{\nu_1+1}}\frac{\mathtt{f}_j\left(\frac{1}{z}\right)}{\frac{1}{z}-\sigma_A\sigma_{Bi}}\,. \tag{5.72}$$

The integrand is holomorphic up to $z=0$ and a possible singularity in $z=1/\sigma_A\sigma_{Bi}$, depending if $i=j$ or $i\neq j$, which is not encircled. Using the Riemann surface as a sphere through the substitution $z\to 1/u$ we have

$$\oint_{\gamma_0}\frac{dz}{2\pi\imath z}\frac{1}{z^{\nu_1+1}}\frac{\mathtt{f}_j\left(\frac{1}{z}\right)}{\frac{1}{z}-\sigma_A\sigma_{Bi}} = \oint_{\gamma_{\sigma_A\sigma_{Bi}}}\frac{du}{2\pi\imath}\frac{u^{\nu_1}\mathtt{f}_j(u)}{u-\sigma_A\sigma_{Bi}}\,, \tag{5.73}$$

where the contour integral $\gamma_{\sigma_A\sigma_{Bi}}$ encircles the pole at $\sigma_A\sigma_{Bi}$ in counter-clockwise direction. The integrand contains the singularity at $\sigma_A\sigma_{Bi}$ only once and thus by making use of the residue theorem we evaluate the numerator at this value, $(\sigma_A\sigma_{Bi})^{\nu_1}\,\mathtt{f}_j(\sigma_A\sigma_{Bi})$. This evaluation yields us the Kronecker-Delta property presented in Eq. (5.69). The norms h_j are given by

$$h_j = (N-1+\nu_1)!\,(N-1+\nu_2)!\,\sigma_A\sigma_{Bj}\prod_{k=1,k\neq j}^{N}(\sigma_A\sigma_{Bj}-\sigma_A\sigma_{Bk})\,. \tag{5.74}$$

The single-sum representation for the correlation kernel for the squared singular values y_1, …, y_N distributed by the joint probability distribution function from Eq. (5.57) reads

$$K_N(x,y) = \sum_{j=1}^{N}\frac{1}{\sigma_A\sigma_{Bj}}2\left(\frac{x}{\sigma_A\sigma_{Bj}}\right)^{\frac{\nu_2+\nu_1}{2}}K_{\nu_2-\nu_1}\left(2\sqrt{\frac{x}{\sigma_A\sigma_{Bj}}}\right) \times \oint_{\gamma_0}\frac{dz}{2\pi\imath}\frac{I_{\nu_2-\nu_1}(2\sqrt{z})}{z^{N+\frac{\nu_2+\nu_1}{2}}}\prod_{k=1,k\neq j}^{N}\frac{y-z\,\sigma_A\sigma_{Bk}}{\sigma_A\sigma_{Bj}-\sigma_A\sigma_{Bk}}\,, \tag{5.75}$$

which was the main goal in this section.
Analogously to the correlation kernels derived previously, the remaining single-sum in j in the latter equation can be rewritten to a contour integral enclosing all fixed parameters $\sigma_A\sigma_{B1}$, …, $\sigma_A\sigma_{BN}$. In addition, we can introduce degeneracies in the set $\{\sigma_A\sigma_{Bj}\}_{j=1,\ldots,N}$. For illustration, a degeneracy can be introduced by

$$\sigma_A\sigma_{Bn+1} = \ldots, \sigma_A\sigma_{BN} = \sigma_A\sigma_B\,, \quad \text{with } \sigma_A\sigma_B > 0 \tag{5.76}$$

leading to the correlation kernel:

$$K_N(x,y) = \oint_{\gamma_{\Sigma_B}}\frac{ds}{2\pi\imath s}\oint_{\gamma_0}\frac{dz}{2\pi\imath z}\left(\frac{x}{sz}\right)^{\frac{\nu_2+\nu_1}{2}}\frac{I_{\nu_2-\nu_1}(2\sqrt{z})\,2K_{\nu_2-\nu_1}\left(2\sqrt{x/s}\right)}{y/z-s} \times\left(\frac{y/z-\sigma_A\sigma_{Bk}}{s-\sigma_A\sigma_{Bk}}\right)^{N-n}\prod_{k=1}^{n}\frac{y/z-\sigma_A\sigma_{Bk}}{s-\sigma_A\sigma_{Bk}}\,, \tag{5.77}$$

where γ_{Σ_B} encircles all $\{\sigma_A\sigma_{B1},\ldots,\sigma_A\sigma_{Bn},\sigma_A\sigma_B\}$ in counter-clockwise direction but not enclosing the origin. Due to the coupling $(y/z-s)^{-1}$, tha paths γ_{Σ_B} and γ_0 are not crossing. In Fig. 5.3 one possible choice of the countour integrals γ_0 and γ_{Σ_B} is depicted. This expression for the correlation kernel is a

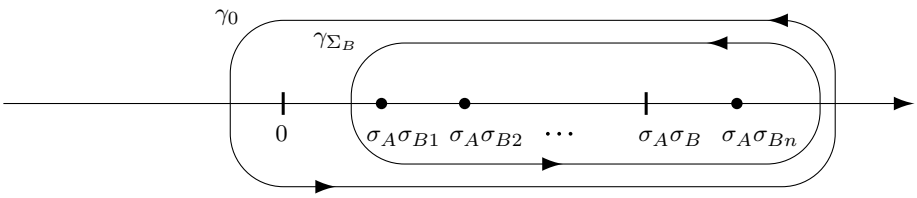

Figure 5.3: Nested choice of the contours γ_{Σ_B} and γ_0. For the second choice of non-intersecting contours that is non-nested γ_0 is including only the origin, but none of the $\{\sigma_A\sigma_{B1},\ldots,\sigma_A\sigma_{Bn},\sigma_A\sigma_B\}$. Because γ_{Σ_B} is excluding the origin we don't have a third choice here, where γ_0 is lying inside γ_{Σ_B}.

new result. It is a generalisation of the correlation kernel of the product of two independent Wishart matrices from Claeys, Keuijlaars and Wang in 2015 [119, Thm. 2.8] generalised by an additional dimension in the indices of the modified Bessel functions, $I_{\nu_2-\nu_1}$, $K_{\nu_2-\nu_1}$ and in the power of the additional weight $(x/sz)^{\frac{\nu_2+\nu_1}{2}}$. The finite-rank perturbation caused by the pair wise distinct external fixed parameters $\sigma_A\sigma_{B1}$, ..., $\sigma_A\sigma_{Bn}$ has been implemented in the principal publication [2, Thm. 3.8] with $M=N$ by taking into account the degeneracy value $\sigma_A\sigma_B$ in the degenerated $N-n$ parameters and $\sigma_A\sigma_{Bj}=\alpha q_j$ in the non-degenerated n parameters.
We close the discussion about the two-epoch model. In the first part in Sec. 5.1.1 we studied the singular value statistics of the one-epoch model as a preliminary consideration. In the second part in Sec. 5.1.2 we studied the spatial cross-correlations in the two-epoch model and in Sec. 5.1.3 the temporal cross-correlations in the same model. Thereby, the half degeneracy for the two-epoch model $\Sigma_A=\sigma_A\mathbb{1}_N$ and $\Sigma_B=\mathrm{diag}(\sigma_{B1},\ldots,\sigma_{BN})$ had to be taken into account to obtain determinantal point processes. Nonetheless, it represents unequal covariances for the two epochs.
All three determinantal point processes belong to the class of polynomial ensembles with permutational symmetry. Thus, we could make use of the modified Heine formula from Sec. 4.4 in order to derive the corresponding correlation kernels. For the desired polynomials, $\chi(x,\sigma_j)$, $\chi(x,\delta_j)$ and $\chi(x,\sigma_A\sigma_{Bj})$, we made use of the superbosonisation formula yielding the Eqs.: (5.18), (5.40) and (5.66), respectively. After the derivations of the respective weights, h_j, the single-sum representation of the particular correlation kernels could be presented. In addition, the single sum in each of the correlation kernels were mapped onto a contour integral, respective to the ensemble in Eqs.: (5.27), (5.45) and (5.77). These analytically obtained expressions are the main results in our analysis of the two-epoch model and reflect the integrability of the considered models.

5.2 Product of Two Coupled Wishart Matrices

In this section we study singular value statistics of the product Y of two complex Wishart matrices X_1 and X_2, which are coupled and carry correlations induced by fixed matrices. Our discussion is presented in three parts.

First, we derive the eigenvalue representation of this model, for which we make use of unitary group integrals from Chap. 3. The so obtained joint probability distribution function belongs to biorthogonal two-matrix ensembles and, thus, it determines two sets of singular values, namely of Y and of X_2. In addition, by setting the coupling to zero we obtain the product of two independent Wishart matrices comparable to our results in temporal cross-correlations from previous analysis. Second, for all these cases we will derive the respective correlation kernels with the help of one single computation and a case analysis afterwards. Thereby, we will refer to our preparatory work from Chap. 4. Third, our focus in the spectral statistics of this model lies on the limiting kernel in the weak non-hermiticity regime motivated by the application in QCD. We will discuss the interpolating property of this kernel under the inclusion of external parameters.
Our results are based on the princpial publication [2]. However, the use of the notion of biorthogonal two-matrix ensembles is new and simplifies the approach to the emerging point processes.
Let us recall the main quantities of our present model from Sec. 3.3. The probability density for X_1 and X_2 being of sizes $(N+\nu)\times(N+\kappa)$ and $(N+\nu)\times N$, respectively, is drawn by

$$\mathcal{P}(X_1,X_2) = c\,\mathrm{e}^{-\mathrm{Tr}\,WX_1^\dagger X_1 - \mathrm{Tr}\,QX_2X_2^\dagger + \mathrm{Tr}\left(\Omega X_1^\dagger X_2 + X_2^\dagger X_1\Omega\right)}\,. \tag{5.78}$$

Here, the normalising constant is $c = \pi^{-(N+\nu)(2N+\kappa)}\det[W]^{N+\nu}\det\left[Q\otimes\mathbb{1}_N - \mathbb{1}_{N+\nu}\otimes\Omega W^{-1}\Omega^\dagger\right]$ and we keep the additional parameters in the sizes of the matrices positive, fixed and independent of N,

$$\kappa \geq 0 \quad \text{and} \quad \nu \geq 0\,. \tag{5.79}$$

Three fixed matrices are introduced in the probability density for X_1 and X_2. Namely, two matrices of covariance type, W of size $(N+\kappa)\times(N+\kappa)$ and Q of size $(N+\nu)\times(N+\nu)$, are given in the Gaussian weights. The coupling weight is parametrised by the fixed matrix Ω, which is of size $N\times(N+\kappa)$.
We ask for the singular value statistics of the product

$$Y = X_1^\dagger X_2\,, \tag{5.80}$$

which is of dimensions $(N+\kappa)\times N$.
We will briefly sketch the derivation for the joint probability distribution function for the squared singular values y_1, …, y_N of Y in emphasising the key steps. Thereby, it will become clear that the first covariance matrix has to be assumed degenerated if a determinantal point process is to be obtained,

$$W = \alpha\mathbb{1}_{N+\kappa}\,, \quad \text{with} \quad \alpha > 0\,. \tag{5.81}$$

Nonetheless, in the derivation of the joint probability distribution function of $\{y_j\}_{j=1,\ldots,N}$ we will keep W arbitrary as long as as possible.
The normalisation constant, $c>0$, provides a condition on W, Q and Ω. Let us denote the eigenvalues of Q by q_1, …, $q_{N+\nu}$ and the squared singular values of Ω by δ_1, …, δ_N. Together with the degeneracy of W from Eq. (5.81) the convergence condition reads

$$\alpha q_i - \delta_j > 0 \quad \text{and} \quad \delta_j > 0\,, \quad \text{for all} \quad i=1,\ldots,N+\nu\,, \quad \text{and} \quad j=1,\ldots,N\,. \tag{5.82}$$

The two fixed matrices Q and Ω can, up to the convergence condition, be taken arbitrarily. Thus, on the level of determinantal point processes these two sets of external parameters are implemented.

5.2.1 Eigenvalue Representation

For the derivation of the joint probability distribution function for the squared singular values y_1, …, y_N of the product $Y = X_1^\dagger X_2$ we consider the probability density for X_1 and X_2. We multiply the probability density from Eq. (5.78) with the corresponding Lebesgue measure as introduced in Eq. (3.2). In addition, we apply the squaring decomposition on X_2 analogously to Eq. (5.49),

$$X_2 = U \begin{pmatrix} \tilde{X}_2 \\ 0_{\nu,N} \end{pmatrix} \tag{5.83}$$

including a Jacobian of the form $[dX_2] = \det\left[\tilde{X}_2\tilde{X}_2^\dagger\right]^\nu \left[d\tilde{X}_2\right] d\mu(U)$, where $d\mu$ denotes the Haar measure for the unitary matrix U belonging to the coset space $U(N+\nu)/[U(N)\times U(\nu)]$. The resulting matrix $\tilde{X}_2$ is of quadratic size, $N\times N$, and shares the singular values with X_2.
From the above deomposition of X_2 we obtain in the coupling term parametrised by Ω in Eq. (5.78) the multiplication of X_1 with U, which yields a substructure

$$U^\dagger X_1 \equiv \begin{pmatrix} \hat{X}_1 \\ \hat{X}_0 \end{pmatrix}, \tag{5.84}$$

which factorises in the $\hat{X}_0$ dependence in the probability density as

$$\begin{aligned} \mathcal{P}(X_1,X_2)\,[dX_1]\,[dX_2] = c\,\mathrm{e}^{-\mathrm{Tr}\,W\hat{X}_0^\dagger\hat{X}_0}\left[d\hat{X}_0\right]\mathrm{e}^{-\mathrm{Tr}\,W\hat{X}_1^\dagger\hat{X}_1}\,\mathrm{e}^{-\mathrm{Tr}\,QU\begin{pmatrix}\tilde{X}_2\tilde{X}_2^\dagger & 0_{N,\nu}\\ 0_{\nu,N} & 0_{\nu,\nu}\end{pmatrix}U^\dagger} \\ \times\,\mathrm{e}^{\mathrm{Tr}\left(\Omega\hat{X}_1^\dagger\tilde{X}_2+\tilde{X}_2^\dagger\hat{X}_1\Omega^\dagger\right)}\left[d\hat{X}_1\right]\det\left[\tilde{X}_2\tilde{X}_2^\dagger\right]^\nu\left[d\tilde{X}_2\right]d\mu(U)\,. \end{aligned} \tag{5.85}$$

We observe that the $\hat{X}_0$-dependent part contributes to the normalising constant solely. By plugging Eq. (5.83) and Eq. (5.84) into the composition of the product matrix, $Y = \hat{X}_1^\dagger\tilde{X}_2$, no $\hat{X}_0$-dependent term remains.
We perform now the substitution

$$\hat{X}_1 \to Y = \hat{X}_1^\dagger\tilde{X}_2 \quad \text{and} \quad \tilde{X}_2 \to \tilde{X}_2\,, \tag{5.86}$$

yielding the probability density for Y and $\hat{X}_2$ from Eq. (5.85). By taking into account the so emerging Jacobian, $\left[d\hat{X}_1\right] = \det\left[\tilde{X}_2\tilde{X}_2^\dagger\right]^{-\kappa-N}[dY]$ we obtain

$$\begin{aligned} \mathcal{P}\left(Y,\tilde{X}_2\right)[dY]\left[d\tilde{X}_2\right] = c'\,\mathrm{e}^{-\mathrm{Tr}\,WY\left(\tilde{X}_2^\dagger\tilde{X}_2\right)^{-1}Y^\dagger}\,\mathrm{e}^{-\mathrm{Tr}\,QU\begin{pmatrix}\tilde{X}_2\tilde{X}_2^\dagger & 0_{N,\nu}\\ 0_{\nu,N} & 0_{\nu,\nu}\end{pmatrix}U^\dagger} \\ \times\,\mathrm{e}^{\mathrm{Tr}\,(\Omega Y+Y^\dagger\Omega^\dagger)}\det\left[\tilde{X}_2\tilde{X}_2^\dagger\right]^{\nu-\kappa-N}[dY]\left[d\tilde{X}_2\right]d\mu(U)\,, \end{aligned} \tag{5.87}$$

where the integration over $\hat{X}_0$ is implied in the modified proportionality constant c'.
The latter expression is closely comparable to the probability measure of $\tilde{\Upsilon}$ and $\tilde{X}_2$ from the pre-

vious section presented in Eq. (5.54) regarding temporal cross-correlations in the two-epoch model. Thereby, the degeneracy of the covariance matrix $\Sigma_A \to \sigma_A \mathbb{1}_N$ was crucial in order to obtain a determinantal point process. Here, the Gaussian weight $\exp\left[-\mathrm{Tr}\, WY\left(\tilde{X}_2^\dagger \tilde{X}_2\right)^{-1} Y^\dagger\right]$ is breaking the unitary bi-invariance in Y. The application of unitary group integrals arising from the corresponding singular value decomposition on this term is intractable for a non-degenerate covariance matrix W. Hence, we continue from here on with the degeneracy, $W \to \alpha \mathbb{1}_{N+\kappa}$ with $\alpha > 0$.
The joint probability distribution function for y_1, ..., y_N and concurrently for the squared singular values of X_2, which we denote by x_1, ..., x_N, can be derived from their singular value decompositions, $Y = V_Y \begin{pmatrix} \Lambda_Y^{\frac{1}{2}} \\ 0_{\kappa,N} \end{pmatrix} U_Y$ and $\tilde{X}_2 = V_X \Lambda_X^{\frac{1}{2}} U_X$. Thereby, the two Lebesgue measures factorise as presented in Eq. (3.11) yielding the appropriate Haar measures in V_Y, U_Y, V_X and U_X as well as squared Vandermonde determinants in the sets $\{y_j\}_{j=1,\ldots,N}$ and $\{x_j\}_{j=1,\ldots,N}$ with remaining Lebesgue measures, $[d\Lambda_Y]$ and $[d\Lambda_X]$.
Five unitary matrices are involved in our computation: U, V_Y, U_Y, V_X and U_X. By making use of the invariance of the Haar measure $d\mu$ under the conjugation by an elemenet of the appropriate unitary coset space, all dependencies on unitary matrices can be reduced to three known integrals.

- First, by shifting $U_X \to U_X U_Y$ the integral over $U_X \in U(N)$ can be performed with the help of the Harish-Chandra/Itzykson–Zuber integral, see Eq. (3.16),

$$\int_{U(N)} d\mu\,(U_X)\, \mathrm{e}^{-\alpha \mathrm{Tr}\, \Lambda_Y U_X^\dagger \Lambda_X^{-1} U_X} \propto \frac{\det\left[\mathrm{e}^{-\alpha \frac{y_j}{x_i}}\right]_{i,j=1}^N}{\Delta_N\,(y_1,\ldots,y_N)\,\Delta_N\,(x_1,\ldots,x_N)} \prod_{i=1}^N x_i^{N-1}, \tag{5.88}$$

  where we made use of $\Delta_N\left(x_1^{-1},\ldots,x_N^{-1}\right) = \mathrm{const.}\, \Delta_N\,(x_1,\ldots,x_N) \,/ \prod_{i=1}^N x_i^{N-1}$.

- Second, the unitary group integrals over V_Y and U_Y can be performed with the help of the Berezin-Karpelevich integral, see Eq. (3.17). Thereby, the invariance of the corresponding Haar measures allows us to carry out the singular value decomposition of the coupling parametrsing matrix $\Omega = V_\Omega \begin{pmatrix} \Lambda_\Omega^{\frac{1}{2}} & 0_{N,\kappa} \end{pmatrix} U_\Omega$. The so appearing unitary matrices can be absorbed through the shifts: $V_Y \to U_\Omega^\dagger V_Y$ and $U_Y \to U_Y V_\Omega^\dagger$. The remaining integral reads

$$\begin{aligned} &\int_{U(N+\kappa)\times U(N)/U(1)^N} d\mu\,(V_Y)\, d\mu\,(U_Y)\, \mathrm{e}^{2\mathrm{Re}\mathrm{Tr}\left[\begin{pmatrix} \Lambda_\Omega^{\frac{1}{2}} & 0_{N,\kappa} \end{pmatrix} V_Y \begin{pmatrix} \Lambda_Y^{\frac{1}{2}} \\ 0_{\kappa,N} \end{pmatrix} U_Y\right]} \\ &\propto \frac{\det\left[y_j^{-\frac{\kappa}{2}} I_\kappa\left(2\sqrt{\delta_i y_j}\right)\right]_{i,j=1}^N}{\Delta_N\,(\delta_1,\ldots,\delta_N)\,\Delta_N\,(y_1,\ldots,y_N)}. \end{aligned} \tag{5.89}$$

- Third, the integration over the unitary matrices U and V_X remains. These unitary matrices are involved in the Gaussian weight $\exp\left[-\mathrm{Tr}\, QU \begin{pmatrix} V_X \Lambda_X V_X^\dagger & 0_{N,\nu} \\ 0_{\nu,N} & 0_{\nu,\nu} \end{pmatrix} U^\dagger\right]$. Again, the fixed matrix can be diagonalised $Q = V_Q \Lambda_Q V_Q^\dagger$ and the V_Q-dependence can be absorbed by the invariance of the Haar measure by shifting $U \to V_Q^\dagger U$. However, the application of the Harish-

Chandra/Itzykson–Zuber integral from Eq. (3.16) requires an appropriate domain, which is $U(N+\nu)$*, because the fixed matrix Q is of rank $(N+\nu)$.
The Haar measure $d\mu(U)$ is defined on the coset space $U(N+\nu)/[U(N)\times U(\nu)]$, whereas $d\mu(V_X)$ is defined on $U(N)/U(1)^N$. For the appropriate domain we introduce a complementary unitary matrix V_ν defined on $U(\nu)$ such that the product

$$\tilde{U} = U \begin{pmatrix} V_X & 0_{N,\nu} \\ 0_{\nu,N} & V_\nu \end{pmatrix}, \tag{5.90}$$

factorises from $\begin{pmatrix} \Lambda_X & 0_{N,\nu} \\ 0_{\nu,N} & 0_{\nu,\nu} \end{pmatrix}$. We denote the integration over $d\mu(U)$, $d\mu(V_X)$ and $d\mu(V_\nu)$ by $d\mu\left(\tilde{U}\right)$ taken over $U(N+\nu)/U(1)^N$. This unitary coset space is sufficient for the application of the Harish-Chandra/Itzykson–Zuber integral,

$$\begin{aligned} &\int_{U(N+\nu)/U(1)^N} d\mu\left(\tilde{U}\right) \mathrm{e}^{-\mathrm{Tr}\,\Lambda_Q \tilde{U} \begin{pmatrix} \Lambda_X & 0_{N,\nu} \\ 0_{\nu,N} & 0_{\nu,\nu} \end{pmatrix} \tilde{U}^\dagger} \\ &\qquad \propto \frac{\det\left[1, q_j, \ldots, q_j^{\nu-1}, \mathrm{e}^{-q_j x_1}, \ldots, \mathrm{e}^{-q_j x_N}\right]_{j=1}^{N+\nu}}{\Delta_{N+\nu}(q_1,\ldots,q_{N+\nu})\,\Delta_N(x_1,\ldots,x_N)\prod_{i=1}^N x_i^\nu}. \end{aligned} \tag{5.91}$$

Note that this is not the standard Harish-Chandra/Itzykson–Zuber integral, but a degenerate limit of it. However, it can be derived in a simple way by making use of L'Hôpital's rule, see Eq. (3.51). Therefore, the standard Harish-Chandra/Itzykson–Zuber integral from Eq. (3.16) can be considered and a part of the participating elements are taken to zero.

After these three integrals, all dependencies on unitary matrices are integrated out. The normalising constants play only a secondary role, because the resulting determinantal point process can be normalised by a direct integration. The joint probability distribution function for the squared singular values of Y and of X_2 was already presented in Sec. 3.3 in detail. We want to recall the main cornerstones.
The squared singular values y_1, …, y_N of Y and concurrently the squared singular values x_1, …, x_N of X_2 are distributed by

$$\begin{aligned} &P_N(x_1,\ldots,x_N;y_1,\ldots,y_N) \\ &= \frac{\det\left[1, q_i, \ldots, q_i^{\nu-1}, \varphi(x_1,\alpha q_i), \ldots, \varphi_i(x_N,\alpha q_i)\right]_{i=1}^{N+\nu} \det\left[w(x_j,y_i)\right]_{i,j=1}^N \det\left[\psi(y_i,\delta_j)\right]_{i,j=1}^N}{(N!)^2 \det\left[1, q_i, \ldots, q_i^{\nu-1}, g_{i1}, \ldots, g_{iN}\right]_{i=1}^{N+\nu}}, \end{aligned} \tag{5.92}$$

where the singular value dependent terms are given by

$$\varphi(x,\alpha q_i) = \mathrm{e}^{-q_i x}, \quad \psi(y,\delta_j) = (\delta_j y)^{-\frac{\kappa}{2}} I_\kappa\left(2\sqrt{\delta_j y}\right), \quad \text{and} \quad w(x,y) = \frac{1}{x}\left(\frac{\alpha y}{x}\right)^\kappa \mathrm{e}^{-\frac{\alpha y}{x}}, \tag{5.93}$$

*The application of the Harish-Chandra/Itzykson–Zuber integral is also possible on the domain of $U(N+\nu)/U(1)^{N+\nu}$, where the extension to the full space $U(N+\nu)$ is trivially given by integrating over $(N+\nu)$ independent unitary spaces $U(1)$.

and we already implied that $q_{ij} = q_i^{j-1}$ for $i = 1, \ldots, (N+\nu)$ and $j = 1, \ldots, \nu$. In addition, we emphasise the dependence on the external parameter in the second argument of the elements, $\{\varphi_i(x) = \varphi(x, \alpha q_i)\}_{i=1,\ldots,N+\nu}$ and $\{\psi_j(y) = \psi(y, \delta_j)\}_{j=1,\ldots,N}$.
Making use of the notion of biorthogonal two-matrix ensembles from Sec. 4.5, the joint probability distribution function for the squared singular values, of Y solely, reads

$$P_N^{(Y)}(y_1, \ldots, y_N) = \frac{\det\left[1, q_i, \ldots, q_i^{\nu-1}, \hat{\rho}(y_1, \alpha q_i), \ldots, \hat{\rho}(y_N, \alpha q_i)\right]_{i=1}^{N+\nu} \det\left[\psi(y_i, \delta_j)\right]_{i,j=1}^{N}}{N! \det\left[1, q_i, \ldots, q_i^{\nu-1}, g_{i1}, \ldots, g_{iN}\right]_{i=1}^{N+\nu}}, \tag{5.94}$$

whereas the joint probability distribution function only for the squared singular values of X_2 reads

$$P_N^{(X)}(x_1, \ldots, x_N) = \frac{\det\left[1, q_i, \ldots, q_i^{\nu-1}, \varphi(x_1, \alpha q_i), \ldots, \varphi(x_N, \alpha q_i)\right]_{i=1}^{N+\nu} \det\left[\rho(x_i, \delta_j)\right]_{i,j=1}^{N}}{N! \det\left[1, q_i, \ldots, q_i^{\nu-1}, g_{i1}, \ldots, g_{iN}\right]_{i=1}^{N+\nu}}. \tag{5.95}$$

with

$$\hat{\rho}(y, \alpha q_i) = 2(\alpha q_i y)^{\frac{\kappa}{2}} K_\kappa\left(2\sqrt{\alpha q_i y}\right) \quad \text{and} \quad \rho(x, \delta_j) = \frac{1}{\alpha} \mathrm{e}^{\frac{\delta_j x}{\alpha}}, \tag{5.96}$$

for all $i = 1, \ldots, (N+\nu)$ and $j = 1, \ldots, N$. The normalisation is given by the Gram matrix

$$g_{ij} = \frac{1}{\alpha q_i - \delta_j}, \quad \text{for } i = 1, \ldots, N+\nu \quad \text{and } j = 1, \ldots, N. \tag{5.97}$$

We would like to emphasise that the joint probability distribution function $P_N^{(X)}$ can be identified with the generalised Wishart ensemble introduced in Eq. (3.54) by setting $\Sigma = -\Omega\Omega^\dagger/\alpha$. By this identification, the eigenvalues of Σ are all conditioned by the positiveness of the singular values, $\delta_j > 0$ for all j.
Once again, both determinantal point processes, $P_N^{(Y)}$ and $P_N^{(X)}$, share the same Gram matrix. The computation of the correlation kernel of one of these ensembles, $K_N^{(Y)}$ or $K_N^{(X)}$, yields immediately the correlation kernel of the second ensemble as stated with the help of Eq. (4.54).
In addition, the product of two independent Wishart matrices can be obtained from $P_N^{(Y)}$ by setting all coupling parameters to zero, $\delta_j \to 0$ for all j, which reads

$$\begin{aligned} &\lim_{\delta_1, \ldots, \delta_N \to 0} P_N^{(Y)}(y_1, \ldots, y_N) \\ &= \frac{1}{\widetilde{Z}_N} \det\left[1, q_i, \ldots, q_i^{\nu-1}, \hat{\rho}(y_1, \alpha q_i), \ldots, \hat{\rho}(y_N, \alpha q_i)\right]_{i=1}^{N+\nu} \Delta_N(y_1, \ldots, y_N), \end{aligned} \tag{5.98}$$

as shown in the course of Eq. (3.52). This biorthogonal ensemble is a polynomial ensemble with permutational symmetry and its correlation kernel can be derived with methods presented in the previous section. However, this ensemble results from $\lim_{\delta_1, \ldots, \delta_N \to 0} P_N^{(Y)}$, such that we achieve its solution by investigating $\lim_{\delta_1, \ldots, \delta_N \to 0} K_N^{(Y)}$.
Note that our previously derived correlation kernel from Eq. (5.75) can only be applied to the determinantal point process from Eq. (5.98) by setting $\nu_1 = \nu_2 = \kappa = \nu = 0$ and by identifying $\alpha q_i = (\sigma_A \sigma_{Bi})^{-1}$ for $i = 1, \ldots, N$.

5.2.2 Correlation Kernels for Finite Matrix Dimensions

Here, we derive the correlation kernels $K_N^{(Y)}$, $K_N^{(X)}$ and $\lim_{\delta_1,\ldots,\delta_N\to 0} K_N^{(Y)}$ for finite matrix dimension $N<\infty$, corresponding to the point processes of: the product of two coupled Wishart matrices from Eq. (5.94), the generalised Wishart ensemble from Eq. (5.95) and the product of two independent Wishart matrices from Eq. (5.98), respectively. The derivation of the correlation kernels implies the inversion of the Gram matrix g.
For $\nu=0$, the Gram matrix can be identified to the Cauchy matrix, see Eq. (4.19), where already its inverse has been presented. In our case, $g_{ij}=(\alpha q_i-\delta_j)^{-1}$ the inverse, $c^\top=g^{-1}$, reads

$$c_{ij}=\frac{(\alpha q_i-\delta_i)\,(\alpha q_j-\delta_j)}{\alpha q_j-\delta_i}\prod_{k=1,k\neq j}^{N}\frac{\delta_j-\alpha q_k}{\delta_j-\delta_k}\prod_{k=1,k\neq i}^{N}\frac{\alpha q_i-\delta_k}{\alpha q_i-\alpha q_k}\,,\quad\text{for }\ \nu=0\,. \tag{5.99}$$

For $\nu>0$, the three present point processes belong to the class of biorthgonal ensembles of non-equal sizes. The inversion of the Gram matrix g requires the investiagtion of its extension, which is the Gram type matrix G as described in the beginning of Sec. 4.3,

$$G=\begin{pmatrix} 1 & q_1 & \ldots & q_1^{\nu-1} & \frac{1}{\alpha q_1-\delta_1} & \ldots & \frac{1}{\alpha q_1-\delta_N}\\ \vdots & \vdots & & \vdots & \vdots & & \vdots\\ 1 & q_{N+\nu} & \ldots & q_{N+\nu}^{\nu-1} & \frac{1}{\alpha q_{N+\nu}-\delta_1} & \ldots & \frac{1}{\alpha q_{N+\nu}-\delta_N}\end{pmatrix}. \tag{5.100}$$

The inversion of the Gram type matrix is non-trivial and its inverse, $C^\top=G^{-1}$, can only be presented as an implicit expression in the correlation kernels.
The inverse of the Gram type matrix satisfies obviously the condition $CG^\top=\mathbb{1}_{N+\nu}$. We rewrite this condition to a Kronecker-Delta relation and insert the explicit Gram matrix from Eq. (5.100),

$$\sum_{k=1}^{\nu}q_i^{k-1}C_{jk}+\sum_{k=1}^{N}\frac{1}{\alpha q_i-\delta_k}c_{jk}=\delta_{ij}\,,\quad\text{for }\ \nu\geq 0\,,\quad\text{and }\ i,j=1,\ldots,N+\nu \tag{5.101}$$

We observe in the Kronecker-Delta relation that the dependence on q_i is composed by a potency series up to the degree $(\nu-1)$ and a sum of rational functions having single poles at $\alpha q_i=\delta_j$ for all possible i and j. Kronecka relations of this type emerged in the derivation of the correlation kernels corresponding to the one-epoch model, Eq. (5.22), and the temporal cross-correlations in the two-epoch model, Eq. (5.69), where only potency series appeared. Still, an analogous definition of a set of meromorphic functions can be established,

$$\mathtt{f}_j(\eta)=\sum_{k=1}^{\nu}\left(\frac{\eta}{\alpha}\right)^{k-1}C_{jk}+\sum_{k=1}^{N}\frac{1}{\eta-\delta_k}c_{jk}\,,\quad\text{for }\ j=1,\ldots,M\,, \tag{5.102}$$

obeying

$$\mathtt{f}_j\,(\alpha q_i)=\delta_{ij}\,,\quad\text{for }\ i,j=1,\ldots,N+\nu\,. \tag{5.103}$$

The analysis of meromorphic properties of this set of functions $\{\mathtt{f}_j\}_{j=1,\ldots,N+\nu}$ leads us to their representation as products of their zeros and poles. The finite set of external parameters $\{\delta_j\}_{j=1,\ldots,N}$, where they are assumed to be pair wise distinct, represents single poles of all functions $\mathtt{f}_1$, …, $\mathtt{f}_{N+\nu}$.

The finite set of external parameters $\{\alpha q_i\}_{i=1,\ldots,N+\nu}$, where they are assumed to be pair wise distinct, represents zeros of the functions up to coinciding indices $\mathtt{f}_j(\alpha q_i) = 0$ for $i \neq j$. The coinciding index case, $\mathtt{f}_j(\alpha q_j) = 1$, fixes a possible proportionality constant, such that we conclude

$$\mathtt{f}_j(\eta) = \prod_{k=1,k\neq j}^{N+\nu} \frac{\eta - \alpha q_k}{\alpha q_j - \alpha q_k} \prod_{k=1}^{N} \frac{\alpha q_j - \delta_k}{\eta - \delta_k}, \quad \text{for } j = 1,\ldots,N+\nu. \tag{5.104}$$

The representation of the sum involving the inverse of the Gram type matrix C in Eq. (5.101) with the help of the meromorphic functions given by Eq. (5.104) can be interpreted as an implicit inversion of the Gram type matrix G.

We make use of the implicit inversion of the Gram type matrix by extending one of the two sums in the double-sum representation of the correlation kernel from Eq. (4.21). From there, the correlation kernel for the product of two coupled matrices reads for our purposes

$$K_N^{(Y)}(y_1, y_2) = \sum_{k=1}^{N+\nu} \hat{\rho}(y_1, \alpha q_k) \sum_{l=1}^{N} c_{kl} \psi(y_2, \delta_l). \tag{5.105}$$

In order to make use of Eq. (5.101), we extend the sum over l by trivial-zero equations, with the help of

$$0 = \oint_{\gamma_0} \frac{d\eta}{2\pi\imath} \left(\frac{\eta}{\alpha}\right)^{l-1} \psi(y_2, \eta) C_{kl}, \quad \text{for } l = 1,\ldots,\nu, \tag{5.106}$$

where the path γ_0 encircles the origin in counter-clock wise direction. The sum over l in the double-sum representation of the correlation kernel can now be rewritten to

$$\sum_{l=1}^{N} c_{kl} \psi(y_2, \delta_l) = \sum_{l=1}^{\nu} \oint_{\gamma_0} \frac{d\eta}{2\pi\imath} \left(\frac{\eta}{\alpha}\right)^{l-1} \psi(y_2, \eta) C_{kl} + \sum_{l=1}^{N} \oint_{\gamma_{\delta_l}} \frac{d\eta}{2\pi\imath} \frac{\psi(y_2, \eta)}{\eta - \delta_l} c_{kl}, \tag{5.107}$$

where we additionally made use of the residue theorem in the second argument in the elements $\{\psi(y, \delta_j)\}_{j=1,\ldots,N}$,

$$\psi(y_2, \delta_l) = \oint_{\delta_l} \frac{d\eta}{2\pi\imath} \frac{\psi(y_2, \eta)}{\eta - \delta_l}, \quad \text{for } l = 1,\ldots,N. \tag{5.108}$$

Here, the path γ_{δ_l} encircles only the pole at δ_l in counter-clockwise direction. The order of the sum in k and of the contour integrals can be interchanged by introducing closed path γ_δ encircling the origin and all elements $\{\delta_l\}_{l=1,\ldots,N}$,

$$\sum_{l=1}^{N} \psi(y_2, \delta_l) c_{kl} = \oint_{\gamma_\delta} \frac{d\eta}{2\pi\imath} \psi(y_2, \eta) \left(\sum_{l=1}^{\nu} \left(\frac{\eta}{\alpha}\right)^{l-1} C_{kl} + \sum_{l=1}^{N} \frac{1}{\eta - \delta_l} c_{kl} \right) \tag{5.109}$$

We identify the sum over l under the brackets to the meromorphic function $\mathtt{f}_k(\eta)$ from Eq. (5.102), such that the correlation kernel reads

$$K_N^{(Y)}(y_1, y_2) = \oint_{\gamma_\delta} \frac{d\eta}{2\pi\imath} \psi(y_2, \eta) \sum_{k=1}^{N+\nu} \hat{\rho}(y_1, \alpha q_k) \mathtt{f}_k(\eta). \tag{5.110}$$

The representation of the meromorphic functions $\mathtt{f}_1$, …, $\mathtt{f}_{N+\nu}$ in terms of their zeros and poles from Eq. (5.104) can be plugged in. Thereby, the elements $\{\alpha q_k\}_{k=1,\ldots,N+\nu}$ can all be enclosed by a contour integral individually,

$$\sum_{k=1}^{N+\nu} \hat{\rho}\,(y_1, \alpha q_k)\, \mathtt{f}_k\,(\eta) = \sum_{k=1}^{N+\nu} \oint_{\gamma_{\alpha q_k}} \frac{d\zeta}{2\pi\imath} \frac{\hat{\rho}\,(y_1,\zeta)}{\zeta - \alpha q_k} \prod_{l=1}^{N} \frac{\zeta - \delta_l}{\eta - \delta_l} \prod_{l=1, l\neq k}^{N+\nu} \frac{\eta - \alpha q_l}{\zeta - \alpha q_l}\,, \tag{5.111}$$

which follows by the residue theorem. Similarly to the contour integral γ_δ above, all elements αq_1, …, $\alpha q_{N+\nu}$ can be enclosed by one single contour integral $\gamma_{\alpha q}$

$$\sum_{k=1}^{N+\nu} \hat{\rho}\,(y_1, \alpha q_k)\, \mathtt{f}_k\,(\eta) = \oint_{\gamma_{\alpha q}} \frac{d\zeta}{2\pi\imath} \frac{\hat{\rho}\,(y_1,\zeta)}{\eta - \zeta} \prod_{l=1}^{N} \frac{\zeta - \delta_l}{\eta - \delta_l} \prod_{l=1}^{N+\nu} \frac{\eta - \alpha q_l}{\zeta - \alpha q_l}\,, \tag{5.112}$$

completing the complementary product $\prod_{l=1,l\neq k}^{N+\nu}$. Note that this representation for the sum over the elements $\{\alpha q_i\}_{i=1,\ldots,N+\nu}$ is only allowed for $\eta \neq \zeta$. The correlation kernel obeys the form

$$K_N^{(Y)}(y_1, y_2) = \oint_{\gamma_\delta} \frac{d\eta}{2\pi\imath} \oint_{\gamma_{\alpha q}} \frac{d\zeta}{2\pi\imath} \frac{\hat{\rho}\,(y_1,\zeta)\,\psi\,(y_2,\eta)}{\eta - \zeta} \prod_{l=1}^{N} \frac{\zeta - \delta_l}{\eta - \delta_l} \prod_{l=1}^{N+\nu} \frac{\eta - \alpha q_l}{\zeta - \alpha q_l}\,, \tag{5.113}$$

where the paths γ_δ and $\gamma_{\alpha q}$ are not crossing. From the convergence condition from Eq. (5.82) the two areas of $\{\delta_j\}_{j=1,\ldots,N}$ and $\{\alpha q_j\}_{j=1,\ldots,N+\nu}$ are separated and, thus, three possible choices for the paths γ_δ and $\gamma_{\alpha q}$ arise.
First, $\gamma_{\alpha q}$ may enclose the set $\{\alpha q_j\}_{j=1,\ldots,N+\nu}$ and the entire path γ_δ. We call this case *nested*. Second, and analogously, a nested case is if γ_δ encloses, apart from the set $\{\delta_j\}_{j=1,\ldots,N}$ and the origin, also the entire path $\gamma_{\alpha q}$. Third, a non-nested case can be constructed, if $\gamma_{\alpha q}$ and γ_δ are separated paths not encircling each other. Two possible choices for the contours are depicted in Fig. 5.4. For all these choices, the path γ_δ has to enclose the origin, due to the trivial-zero equations composed in Eq. (5.106). This representation of the correlation kernel is according to [2, Thm. 1.3] and is the main result in this section. It enables us to write down the three desired correlation kernels with possible degeneracies in the external fixed parameters, $\{\alpha q_i\}_{i=1,\ldots,N+\nu}$ and $\{\delta_j\}_{j=1,\ldots,N}$.
Let us introduce the degeneracies

$$\delta_{n+1} = \ldots = \delta_N = \delta\,, \quad \text{and} \;\; \alpha q_{m+1} = \ldots, \alpha q_{N+\nu} = \alpha q\,, \tag{5.114}$$

where the convergence condition from Eq. (5.82) translates to

$$\alpha q_i - \delta > 0\,, \quad \alpha q - \delta_j > 0 \quad \text{and} \;\; \alpha q - \delta > 0\,, \quad \text{for} \;\; i = 1,\ldots,m \quad \text{and} \;\; j = 1,\ldots,n\,. \tag{5.115}$$

We are now in the position to present the main result.
The correlation kernel for the product of two coupled Wishart matrices obeys the double-contour

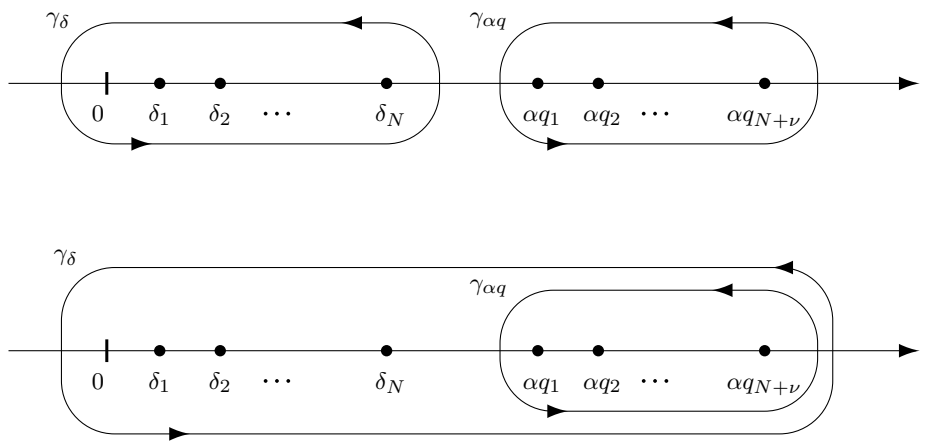

Figure 5.4: Possible choices for the integration contours γ_δ and $\gamma_{\alpha q}$ in (5.113): non-nested contours (top) and nested contours (bottom). For simplicity we have ordered the parameters δ_1, ... δ_N and αq_1, ..., $\alpha q_{N+\nu}$. We do not display a third possible choice which is nested, where $\gamma_{\alpha q}$ encircles γ_δ analogously to nested case depicted (bottom).

integral representation

$$K_N^{(Y,n,m)}(y_1,y_2) = 2\left(\frac{y_1}{y_2}\right)^{\frac{\kappa}{2}} \oint_{\gamma_\delta} \frac{d\eta}{2\pi\imath} \oint_{\gamma_{\alpha q}} \frac{d\zeta}{2\pi\imath} \frac{K_\kappa\left(2\sqrt{\zeta y_1}\right) I_\kappa\left(2\sqrt{\eta y_2}\right)}{\eta-\zeta} \times \left(\frac{\zeta}{\eta}\right)^{\frac{\kappa}{2}} \left(\frac{\zeta-\delta}{\eta-\delta}\right)^{N-n} \left(\frac{\eta-\alpha q}{\zeta-\alpha q}\right)^{N+\nu-m} \prod_{l=1}^{n} \frac{\zeta-\delta_l}{\eta-\delta_l} \prod_{l=1}^{m} \frac{\eta-\alpha q_l}{\zeta-\alpha q_l}, \tag{5.116}$$

where we denote the number of non-degenerate parameters on the top-index in $K_N^{(Y,m,n)}(y_1,y_2)$. The two contour integrals, γ_δ and $\gamma_{\alpha q}$, encircle $\{0,\delta_1,\ldots,\delta_n,\delta\}$ and $\{\alpha q_1,\ldots,\alpha q_m,\alpha q\}$, respectively, counter clock-wise in one of the three above described choices.

The correlation kernel of the product of two coupled Wishart matrices was first derived in [74, Thm. 3.2]. For a comparison to our result, all external parameters have to be chosen degenerated $m=0$ and $n=0$ and the dimensions of the matrix X_1 has to be reduced with $\kappa=0$. In [74] the correlation kernel is represented with the help of the method of orthogonal polynomials. The comparison between these two representations of the correlation kernel of two coupled Wishart matrices was carried out by the same authors recently in [120, p. 39]. The representation of this correlation kernel in terms of two contour integrals was derived in [76, Thm. 1.2], where coupling parameters $\{\delta_j\}_{j=1,\ldots,N}$ were introduced as well as the dimension of X_1 was extended for $\kappa>0$. This representation coincides immediately with Eq. (5.116) for $m=0$. Thus, our result is new due to its extension for $m\neq 0$ derived in the principal publication [2].

In addition, from the correlation kernel $K_N^{(Y,n,m)}(y_1,y_2)$ two further correlation kernels immediately follow.

By exchanging the elements $\hat{\rho}(y_1,\zeta)\to\varphi(x_1,\zeta)$ and $\psi(y_2,\eta)\to\rho(x_2,\eta)$ in the double-contour integral representation for the correlation kernel in Eq. (5.113) we obtain the correlation kernel for the gener-

alised Wishart ensemble $K_N^{(X)}$. By taking into account the degeneracies introduced in Eq. (5.114) it reads

$$K_N^{(X,n,m)}(x_1,x_2) = \frac{1}{\alpha}\oint_{\gamma_\delta}\frac{d\eta}{2\pi\imath}\oint_{\gamma_{\alpha q}}\frac{d\zeta}{2\pi\imath}\frac{\mathrm{e}^{-\frac{1}{\alpha}(\zeta x_1-\eta x_2)}}{\eta-\zeta} \times\left(\frac{\zeta-\delta}{\eta-\delta}\right)^{N-n}\left(\frac{\eta-\alpha q}{\zeta-\alpha q}\right)^{N+\nu-m}\prod_{l=1}^{n}\frac{\zeta-\delta_l}{\eta-\delta_l}\prod_{l=1}^{m}\frac{\eta-\alpha q_l}{\zeta-\alpha q_l}\,. \tag{5.117}$$

The correlation kernel of the product of two independent Wishart matrices results from $K_N^{(Y)}$ from Eq. (5.113) by setting all coupling parameters to zero,

$$\lim_{\delta_1,\ldots,\delta_N\to 0}K_N^{(Y)}(y_1,y_2) = \left(\frac{y_1}{y_2}\right)^{\frac{\kappa}{2}}\oint_{\gamma_0}\frac{d\eta}{2\pi\imath}\oint_{\gamma_{\alpha q}}\frac{d\zeta}{2\pi\imath}\frac{I_\kappa\left(2\sqrt{\eta y_2}\right)2K_\kappa\left(2\sqrt{\zeta y_1}\right)}{\eta-\zeta} \times\left(\frac{\zeta}{\eta}\right)^{\frac{\kappa}{2}+N}\left(\frac{\eta-\alpha q}{\zeta-\alpha q}\right)^{N+\nu-m}\prod_{l=1}^{m}\frac{\eta-\alpha q_l}{\zeta-\alpha q_l}\,, \tag{5.118}$$

where we took into account the degeneracy $\alpha q_{n+1} = \ldots = \alpha q_{N+\nu} = \alpha q$ and the contour integral γ_δ reduced to γ_0.

5.2.3 Interpolating Hard-Edge Limit

In this section we turn to a microscopic limit in spectral statistics motivated by the weak non-hermiticity regime discussed in the course of applications in Sec. 2.2.3. This microscopic limit aims to the origin of the spectrum by implying the limit of infinite matrix dimensions, $N \to \infty$. Due to positiveness of singular values, their support is limited by the origin as a hard cut-off. The terminology of *hard edge* is to be interpreted with respect to the zero probability of the presence of a singular value on the negative real line.

From the application in QCD from Sec. 2.2.2, the coupling between X_1 and X_2, parameterised by $\{\delta_j\}_{j=1,\ldots,n}$ and the degeneracy value δ in the correlation kernel from Eq. (5.116), represents the inclusion of the chemical potential μ breaking the hermiticity of the Dirac operator. For the desired weak breaking, the coupling has to depend on the matrix dimension, N, before the large-N limit is taken. Simultaneously, the second set of parameters $\{\alpha q_1,\ldots,\alpha q_m,\alpha q\}$ has to depend on the matrix dimension N, too, due to the positiveness relation presented in Eq. (5.115).

In the hard-edge limit, the correlation kernel of the product of two coupled Wishart matrices yields a limiting kernel $\mathbb{K}_{\mathrm{II}}^{(n,m)}$, from which we can reach two further limiting kernels. We express this interpolating property with the help of one single parameter $\hat{\tau} \in [0,\infty)$. For vanishing interpolating parameter $\hat{\tau} \to 0^+$ we arrive at the Meijer G-kernel $\mathbb{K}_{\mathrm{I}}^{(0,m)}$, whereas for infinite interpolating parameter $\hat{\tau} \to \infty$ we arrive at the Bessel kernel $\mathbb{K}_{\mathrm{III}}^{(n,m)}$.

In the principal publication [2] all three limiting kernels $\mathbb{K}_{\mathrm{I}}^{(0,m)}$, $\mathbb{K}_{\mathrm{II}}^{(n,m)}$ and $\mathbb{K}_{\mathrm{III}}^{(n,m)}$ were derived independently from the finite-N results presented above. We restate their double-contour representation:

- The Bessel kernel derived in [2, Sec. 4.1] from the finite-N correlation kernel for the generalised Wishart ensemble, shown in the previous section in Eq. (5.117), reads

$$\mathbb{K}_{\mathrm{III}}^{(n,m)}(y_1,y_2) = \left(\frac{y_1}{y_2}\right)^{\kappa} \oint_{\Gamma_{out}} \frac{d\hat{v}}{2\pi\imath} \oint_{\Gamma_{in}} \frac{d\hat{u}}{2\pi\imath} \frac{\mathrm{e}^{y_1\hat{u}-y_2\hat{v}}}{\hat{u}-\hat{v}} \times \mathrm{e}^{-\frac{1}{\hat{u}}+\frac{1}{\hat{v}}} \left(\frac{\hat{v}}{\hat{u}}\right)^{\nu+n-m} \prod_{l=1}^{n} \frac{\hat{u}-\hat{\pi}_l}{\hat{v}-\hat{\pi}_l} \prod_{k=1}^{m} \frac{\hat{v}-\hat{\theta}_k}{\hat{u}-\hat{\theta}_k}, \tag{5.119}$$

  where the external parameters are restricted by

$$\hat{\theta}_k \in \bigcap_{l=1}^{n} (-\infty, \hat{\pi}_l], \quad \hat{\pi}_j \in [0,\infty), \quad \text{for all } k=1,\dots,m \quad \text{and } j=1,\dots,n. \tag{5.120}$$

  The two contour-integrals Γ_{in} and Γ_{out} are not crossing each other and are nested in a way depicted in Fig. 5.5. Thereby, Γ_{in} encircles the parameters $\hat{\theta}_1$, …, $\hat{\theta}_m$ and the origin, whereas Γ_{out} encircles the parameters $\hat{\pi}_1$, …, $\hat{\pi}_n$ and Γ_{in}. Without finite-rank perturbations, the limiting

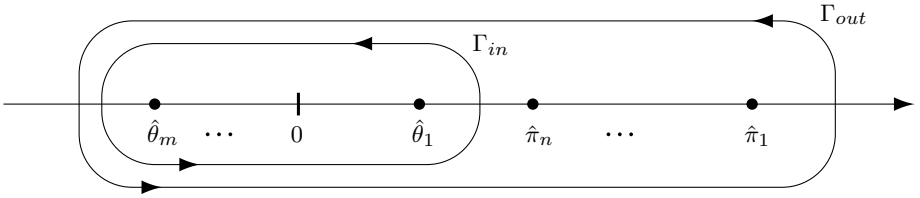

Figure 5.5: The contours Γ_{in} and Γ_{out} parametrised by $\hat{u}$ and $\hat{v}$, respectively, for the Bessel kernel, $\mathbb{K}_{\mathrm{III}}^{(n,m)}$, presented in (5.119). The contour Γ_{in} encircles the origin as well as all parameters $\hat{\theta}_1$, …, $\hat{\theta}_m$ bounded from above by the parameter set $\{\hat{\pi}_l\}_{l=1,\dots,n}$. The contour Γ_{out} encircles Γ_{in} by not crossing it and all parameters $\hat{\pi}_1$, …, $\hat{\pi}_n$, which lie on the positive real line.

  kernel $\mathbb{K}_{\mathrm{III}}^{(0.0)}$ coincides with the Bessel kernel found in [108]. The Bessel kernel perturbed by one set of parameters $\mathbb{K}_{\mathrm{III}}^{(n,0)}$ was found in [76] and for $n=0$ it agrees with $\mathbb{K}_{\mathrm{III}}^{(0.0)}$. The findings in [2] generalise known results for $m>0$.

- The interpolating kernel derived in [2, Sec. 4.3] from the finite-N correlation kernel for the product of two coupled Wishart matrices, shown in the previous section in Eq. (5.116), reads

$$\mathbb{K}_{\mathrm{II}}^{(n,m)}(y_1,y_2;\hat{\tau}) = \left(\frac{y_1}{y_2}\right)^{\frac{\kappa}{2}} \oint_{\Gamma_{out}} \frac{d\hat{v}}{2\pi\imath} \oint_{\Gamma_{in}} \frac{d\hat{u}}{2\pi\imath} \frac{2K_\kappa\left(2\sqrt{(\hat{\tau}^{-1}-\hat{u})\,y_1}\right) I_\kappa\left(2\sqrt{(\hat{\tau}^{-1}-\hat{v})\,y_2}\right)}{\hat{u}-\hat{v}} \times \left(\frac{\hat{u}-\hat{\tau}^{-1}}{\hat{v}-\hat{\tau}^{-1}}\right)^{\frac{\kappa}{2}} \mathrm{e}^{-\frac{1}{\hat{u}}+\frac{1}{\hat{v}}} \left(\frac{\hat{v}}{\hat{u}}\right)^{\nu+n-m} \prod_{l=1}^{n} \frac{\hat{u}-\hat{\pi}_l}{\hat{v}-\hat{\pi}_l} \prod_{k=1}^{m} \frac{\hat{v}-\hat{\theta}_k}{\hat{u}-\hat{\theta}_k}, \tag{5.121}$$

  where the external parameters are restricted by

$$\hat{\theta}_k \in \bigcap_{l=1}^{n} (-\infty, \hat{\pi}_l], \quad \hat{\pi}_j \in \left[0, \hat{\tau}^{-1}\right), \quad \text{for all } k=1,\dots,m \quad \text{and } j=1,\dots,n. \tag{5.122}$$

The contour Γ_{in} is taken counterclockwise encircling $\left\{\hat{\theta}_1,\ldots,\hat{\theta}_m,0\right\}$, whereas the contour Γ_{out} encircles $\left\{\hat{\pi}_1,\ldots,\hat{\pi}_n,\hat{\tau}^{-1}\right\}$ and Γ_{in} as well counterclockwise as depicted in Fig. 5.6. This limiting kernel generalises the interpolating kernel from [75], where the two sets of parameters $\left\{\hat{\theta}_k\right\}_{k=1,\ldots,m}$ and $\{\hat{\pi}_j\}_{j=1,\ldots,n}$ are absent, $m=0$ and $n=0$. In [76] this interpolating kernel was derived in the presence of the set $\{\hat{\pi}_j\}_{j=1,\ldots,n}$ in the same representation, $m=0$ and $n>0$. Thus, the interpolating kernel presented in Eq. (5.121) is a generalisation of known results by the finite-rank perturbation for $m>0$ and $n>0$.

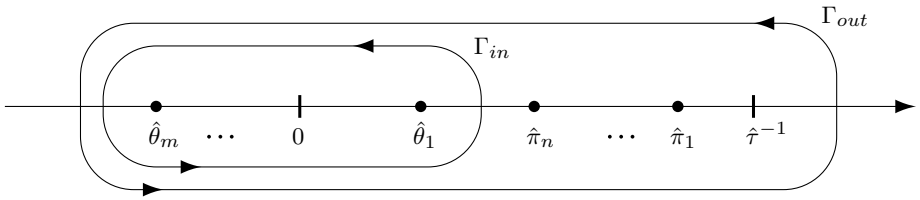

Figure 5.6: The contours Γ_{in} and Γ_{out} parametrised by $\hat{u}$ and $\hat{v}$, respectively, for the interpolating kernel, $\mathbb{K}_{\mathrm{II}}^{(n,m)}$, presented in (5.121). The contour Γ_{in} encircles the origin as well as all parameters $\hat{\theta}_1$, …, $\hat{\theta}_m$ bounded from above by the parameter set $\{\hat{\pi}_l\}_{l=1,\ldots,n}$. The contour Γ_{out} encircles Γ_{in} by not crossing it and all parameters $\hat{\pi}_1$, …, $\hat{\pi}_n$, which lie on the positive real line, and the interpolation parameter $\hat{\tau}^{-1}$ limiting $\left\{\hat{\pi}_j\right\}_{j=1,\ldots,n}$ from above.

- The Meijer G-kernel derived in [2, Sec. 4.2] from the finite-N correlation kernel for the product of two independent Wishart matrices, shown in the previous section in Eq. (5.118), reads

$$\begin{aligned}\mathbb{K}_{\mathrm{I}}^{(0,m)}(y_1,y_2) = \int_0^\infty \frac{dt}{t} \oint_0 \frac{ds}{2\pi\imath s} \left(\frac{t}{s}\right)^\kappa \mathrm{e}^{s-t} \oint_{\Gamma_{out}} \frac{d\hat{v}}{2\pi\imath} \oint_{\Gamma_{in}} \frac{d\hat{u}}{2\pi\imath} \frac{\mathrm{e}^{\frac{\hat{u}y_1}{t}-\frac{\hat{v}y_2}{s}}}{\hat{u}-\hat{v}} \\ \times \mathrm{e}^{-\frac{1}{\hat{u}}+\frac{1}{\hat{v}}} \left(\frac{\hat{v}}{\hat{u}}\right)^{\nu-m} \prod_{k=1}^m \frac{\hat{v}-\hat{\theta}_k}{\hat{u}-\hat{\theta}_k},\end{aligned} \tag{5.123}$$

where the external parameters $\left\{\hat{\theta}_k\right\}_{k=1,\ldots,m}$ are restricted by

$$\hat{\theta}_k \in (-\infty,0]\,, \quad \text{for all } \; k=1,\ldots,m\,. \tag{5.124}$$

The contour Γ_{in} encircles counterclockwise the parameters $\hat{\theta}_1$, …, $\hat{\theta}_m$ and the origin, whereas the contour Γ_{out} encircles the contour Γ_{in} as depicted in Fig. 5.7. The two exterior integrals over dt and ds can be done by using Fubini's theorem and the relations [82, Eqs. 8.406-7] with the already introduced integrals from Eq. (5.65) and Eq. (3.46). However, we would like to keep them unperformed below for a clear computation.
This kernel generalises the Meijer G-kernel resulting in the hard-edge limit of two independent matrices [121] by the finite set of external parameters $\left\{\hat{\theta}_k\right\}_{k=1,\ldots,m}$. In addition, in [119] it was shown that it can be written as a double integral of the Bessel kernel. This property extends for finite-rank perturbations and is shown here with the help of the two exterior integrals over dt and ds.

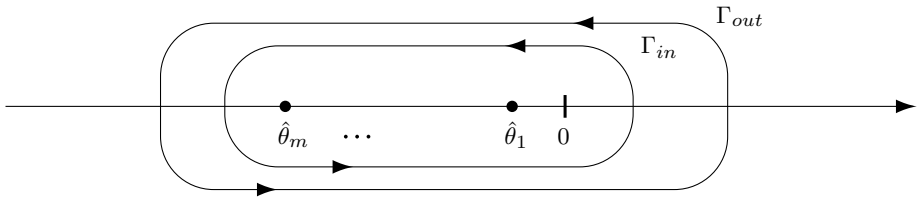

Figure 5.7: The contours Γ_{in} and Γ_{out} parametrised by $\hat{u}$ and $\hat{v}$, respectively, for the Meijer-G kernel, $\mathbb{K}_{\mathrm{I}}^{(0,m)}$, presented in (5.123). The contour Γ_{in} encircles the origin as well as all parameters $\hat{\theta}_1$, ..., $\hat{\theta}_m$, which lie on the negative real line. The contour Γ_{out} encircles Γ_{in} by not crossing it.

First, we derive the interpolating kernel $\mathbb{K}_{\mathrm{II}}^{(n,m)}(y_1,y_2;\hat{\tau})$ from our finite-N result to the product of two coupled Wishart matrices from Eq. (5.116), which is the most general limit in view of the three limiting kernels. Second, we present the interpolating property of the kernel $\mathbb{K}_{\mathrm{II}}^{(n,m)}(y_1,y_2;\hat{\tau})$, which is very descriptive by the analysis of vanishing interpolation parameter on the one hand and by the analysis of infinite interpolation parameter on the other hand.

We start our considerations at the double-contour integral representation for the correlation kernel of the product of two coupled Wishart matrices from Eq. (5.116). There, the two sets of parameters $\{0,\delta_1,\ldots,\delta_n,\delta\}$ and $\{\alpha q_1,\ldots,\alpha q_m,\alpha q\}_{k=1,\ldots,m}$ are enclosed by two contours over $d\eta$ and $d\zeta$, respectively. The conditions on these sets of parameters are given by the inequalities presented in Eq. (5.115) as well as $\delta_j>0$ for all j and $\delta>0$, due to positiveness of singular values of the fixed matrix Ω. The positiveness of the difference of the degeneracy values, $\alpha q-\delta>0$, yields to us a natural scale, on which our parameters are living as well as fixpoints in their distributions. Thus, as a preliminary step, we express the system of the two contour integrals and the external parameters with the help of a unit-less construction.

We map the two sets of external parameters $\{\delta_j,\alpha q_k\}_{j=1,\ldots,n;k=1,\ldots,m}\to\{\pi_j,\theta\}_{j=1,\ldots,n;k=1,\ldots,m}$ with the help of

$$\theta_k=\frac{\alpha q-\alpha q_k}{\alpha q-\delta}\quad\text{and}\quad\pi_l=\frac{\alpha q-\delta_l}{\alpha q-\delta},\quad\text{with}\quad k=1,\ldots,m,\quad\text{and}\quad l=1,\ldots,n. \tag{5.125}$$

Here, the corresponding degeneracy values are mapped as $\{\delta,\alpha q\}\to\{1,0\}$. The lower boundary of the parameters $\delta_j>0$ for all j as well as $\delta>0$ is mapped to an upper boundary, $\alpha q/(\alpha q-\delta)$. All these conditions are summarised in the supports of the new parameters as

$$\theta_k\in\bigcap_{l=1}^{n}(-\infty,\pi_l)\quad\text{and}\quad\pi_j\in\left(0,\frac{\alpha q}{\alpha q-\delta}\right),\quad\text{for all}\quad k=1,\ldots,m\quad\text{and}\quad j=1,\ldots,n \tag{5.126}$$

The upper limit of the support of $\{\pi_l\}_{l=1,\ldots,n}$ is the parameter that will later on provide the interpolation parameter,

$$\tau^{-1}=\frac{\alpha q}{\alpha q-\delta}. \tag{5.127}$$

The contours, γ_δ and $\gamma_{\alpha q}$, enclosing these sets of parameters and the degeneracy values, can accordingly be transformed with the help of the following substitution,

$$\zeta = \alpha q - (\alpha q - \delta)\, u\,, \quad \text{and} \quad \eta = \alpha q - (\alpha q - \delta)\, v\,, \tag{5.128}$$

where we call the new two contours Γ_π and Γ_θ respectively to the enclosed sets of parameters. In addition, for a smooth interpolation property of $\mathbb{K}_{\mathrm{II}}^{(n,m)}$, it is beneficial to choose the proper contour constellation, which is a nested one. All these changes substituted in the expression from Eq. (5.116) result in

$$\begin{aligned} K_N^{(Y)}(y_1,y_2) =& \alpha q\tau \oint_{\Gamma_\pi} \frac{dv}{2\pi\imath} \oint_{\Gamma_\theta} \frac{du}{2\pi\imath} \frac{\hat{\rho}\left(y_1, \alpha q\tau\left(\tau^{-1}-u\right)\right)\psi\left(y_2, \alpha q\tau\left(\tau^{-1}-v\right)\right)}{u-v} \\ &\times \left(\frac{1-\frac{1}{u}}{1-\frac{1}{v}}\right)^{N-n} \left(\frac{v}{u}\right)^{\nu+n-m} \prod_{l=1}^{n} \frac{u-\pi_l}{v-\pi_l} \prod_{k=1}^{m} \frac{v-\theta_k}{u-\theta_k}\,, \end{aligned} \tag{5.129}$$

where the inner contour Γ_θ encloses counterclockwise the set $\{\theta_1,\ldots,\theta_m,0\}$, whereas the outer contour Γ_π encloses counterclockwise the contour Γ_θ and the set $\left\{\pi_1,\ldots,\pi_n,1,\tau^{-1}\right\}$, not crossing each other. We depict this situation in Fig. 5.4. Here, we have left the explicit expressions for the involved

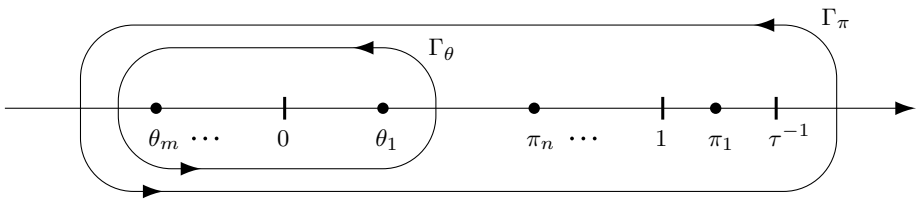

Figure 5.8: The contours Γ_θ and Γ_π resulting from the substitution (5.128) are shown. The poles θ_k are centred around the degenerate value αq that has been mapped to the origin, and likewise the poles π_j are centred around the degenerate value δ mapped to unity. The upper boundary of the support of $\{\pi_l\}$ is mapped to τ^{-1} also encircled by Γ_π.

functions ψ and $\hat{\rho}$ given in terms of the modified Bessel functions of the first kind I_κ and of the second kind K_κ, respectively, from Eqs. (5.93) and (5.96).

In view of the large-N limit of the above correlation kernel at the hard edge, we need to center the arguments of the kernel to the origin and then zoom with an appropriate N scaling to the typical scale on which the singular values live. The centering procedure is not noticeable as $K_N^{(Y)}(y_1-0,y_2-0) = K_N^{(Y)}(y_1,y_2)$. However, the appropriate scaling in N is crucial.

Therefore, we would like to refer to [76], where the same representation of the interpolation kernel was found for $n>0$ and $m=0$. The decisive property is the only N dependence in the correlation kernel presented in Eq. (5.129) in the term: $\left(1-u^{-1}\right)^{N-n}/\left(1-v^{-1}\right)^{N-n}$. In order to generate convergence in this term, we perform an N-dependent substitution,

$$u = N\hat{u} \quad \text{and} \quad v = N\hat{v}\,, \tag{5.130}$$

which allows to make use of the well-known formula, $\lim_{N\to\infty}(1+x/N)^N = \exp(x)$, by taking the large-$N$ limit. This substitution plugged into the expression in Eq. (5.129), and by scaling the arguments of the correlation kernel suitably, yields

$$\frac{1}{\alpha q\tau N}K_N^{(Y)}\left(\frac{y_1}{\alpha q\tau N},\frac{y_2}{\alpha qN}\right) = \oint_{\Gamma_\pi}\frac{d\hat{v}}{2\pi\imath}\oint_{\Gamma_\theta}\frac{d\hat{u}}{2\pi\imath}\frac{\hat{\rho}\left(\left(N^{-1}\tau^{-1}-\hat{u}\right)y_1\right)\psi\left(\left(N^{-1}\tau^{-1}-\hat{v}\right)y_2\right)}{\hat{u}-\hat{v}} \times\left(\frac{1-\frac{1}{N\hat{u}}}{1-\frac{1}{N\hat{v}}}\right)^{N-n}\left(\frac{\hat{v}}{\hat{u}}\right)^{\nu+n-m}\prod_{l=1}^{n}\frac{\hat{u}-\frac{1}{N}\pi_l}{\hat{v}-\frac{1}{N}\pi_l}\prod_{k=1}^{m}\frac{\hat{v}-\frac{1}{N}\theta_k}{\hat{u}-\frac{1}{N}\theta_k}, \tag{5.131}$$

where we observe the so-called *homogeneous scaling* of the correlation kernel. Here, we additionally made use of the particular dependence of the elements $\hat{\rho}$ and ψ in their two arguments as

$$\hat{\rho}(y,\zeta) = \hat{\rho}(\zeta y) \quad \text{and} \quad \psi(y,\eta) = \psi(\eta y)\,. \tag{5.132}$$

By taking the large-N limit, Lebesgue's dominated convergence theorem allows the use of the above mentioned formula,

$$\lim_{N\to\infty}\left(\frac{1-\frac{1}{N\hat{u}}}{1-\frac{1}{N\hat{v}}}\right)^{N-n} = \mathrm{e}^{-\frac{1}{\hat{u}}+\frac{1}{\hat{v}}}\,. \tag{5.133}$$

Concurrently, all parameters θ_1, …, θ_m, π_1, …, π_n, as well as the upper boundary τ^{-1}, are scaled by the factor of N^{-1} and vanish in the large-N limit, unless these parameters depend on the matrix dimension N in a countering way.

In [2, Sec. 4], the Meijer-G kernel and the Bessel kernel under finite-rank perturbations, as presented in Eq. (5.123) and Eq. (5.119), were derived independently from the interpolating kernel. We observe that for all these limiting kernels the parameters persist in the large-N limit for the following limiting values,

$$\hat{\theta}_k = \lim_{N\to\infty}\frac{1}{N}\theta_k\,, \quad \text{and} \quad \hat{\pi}_l = \lim_{N\to\infty}\frac{1}{N}\pi_l\,, \quad \text{for all} \quad k=1,\ldots,m \quad \text{and} \quad j=1,\ldots,n \tag{5.134}$$

as well as for the bounding parameter for the interpolating kernel,

$$\hat{\tau}^{-1} = \lim_{N\to\infty}\frac{1}{N}\tau^{-1}\,. \tag{5.135}$$

The interpolating limit presented in Eq. (5.121) results from the large-N limit of the kernel from Eq. (5.131) by taking into account the limiting values for the parameters $\hat{\theta}_1$, …, $\hat{\theta}_m$, $\hat{\pi}_1$, …, $\hat{\pi}_n$, and $\hat{\tau}^{-1}$ as

$$\lim_{N\to\infty}\frac{1}{\alpha q\tau N}K_N^{(Y)}\left(\frac{y_1}{\alpha q\tau N},\frac{y_2}{\alpha q\tau N}\right) = \mathbb{K}_{\mathrm{II}}^{(n,m)}(y_1,y_2;\hat{\tau})\,, \tag{5.136}$$

where we recall $\hat{\rho}(y) = 2y^{\kappa/2}K_\kappa\left(2\sqrt{y}\right)$ and $\psi(y) = y^{-\kappa/2}I_\kappa\left(2\sqrt{y}\right)$. Note, that the unity from the contour-constellation in Fig. 5.8 is not noticeable after the large-N limit depicted in Fig. 5.6, where the contours are mapped as $\{\Gamma_\theta,\Gamma_\pi\}\to\{\Gamma_{in},\Gamma_{out}\}$. All parameters are spread out with the scale of N, whereas the unity is obviously fixed.

The derivation of the two further limiting kernels, the Meijer G-kernel and the Bessel kernel presented

in Eq. (5.123) and Eq. (5.119), can be done in an analogous way by implying the respective circumstances for the parameters.
We turn to the interpolating property of $\mathbb{K}_{\mathrm{II}}^{(n,m)}$ with the help of the interpolation parameter $\hat{\tau}^{-1}$. The interesting property of the limiting kernel is that the dependence on the interpolation parameter is explicitly given only in the arguments of the elements $\hat{\rho}$ and ψ. Indeed, the two modified Bessel functions I_κ and K_κ play a crucial role by reaching $\mathbb{K}_{\mathrm{I}}$ or $\mathbb{K}_{\mathrm{III}}$.
The Meijer G-kernel results from the product of two independent Wishart matrices, thus, the coupling parameterised by Ω vanishes in this case. The vanishing coupling translates to vanishing parameters $\{\delta_1,\ldots,\delta_n,\delta\}$, which after the large-$N$ limit corresponds to the limit of $\hat{\tau}\to\infty$. By comparing the contours from the interpolating kernel to the contours of the Meijer G-kernel, see Figs.: 5.6 and 5.7, we observe that the support for the parameters $\hat{\pi}_1$, …, $\hat{\pi}_n$ shrinks to zero and the remaining parameters $\hat{\theta}_1$, …, $\hat{\theta}_m$ are bounded from above by the origin. Simultaneously, the support of the set $\left\{\hat{\theta}_k\right\}_{k=1,\ldots,m}$ is limited from above by those zero-tending parameters. In our expression for $\mathbb{K}_{\mathrm{II}}$ in Eq. (5.121) we have to evaluate the elements $\hat{\rho}$ and ψ at negative arguments, which can be done with the help of their integral representations for modified Bessel functions from Eq. (3.46) and from Eq. (5.65),

$$\hat{\rho}(-y_1\hat{u}) = \int_0^\infty \frac{dt}{t} t^\kappa \,\mathrm{e}^{-t+\frac{\hat{u}y_1}{t}}\,, \quad \text{and} \quad \psi(-y_2\hat{v}) = \oint_0 \frac{ds}{2\pi\imath s} s^{-\kappa}\,\mathrm{e}^{s-\frac{\hat{v}y_2}{s}}\,, \tag{5.137}$$

and the integrals over ds and dt can be reordered due to Fubini's theorem. With this we achieve the expression from Eq. (5.123),

$$\lim_{\hat{\tau}\to\infty} \mathbb{K}_{\mathrm{II}}^{(n,m)}(y_1,y_2;\hat{\tau}) = \mathbb{K}_{\mathrm{I}}^{(0,m)}(y_1,y_2)\,. \tag{5.138}$$

We thus obtain the interpolating property of $\mathbb{K}_{\mathrm{II}}\to\mathbb{K}_{\mathrm{I}}$ by limiting the support of the parameter set $\{\hat{\pi}_l\}_{l=1,\ldots,n}$. In contrast, the perturbed Bessel kernel results from opening this support to infinity.
The interpolating parameter can be sent to zero, $\hat{\tau}\to 0^+$. The support of $\hat{\pi}_l$ becomes $(0,\infty)$. Therefore, our elements $\hat{\rho}$ and ψ need to be expanded for large arguments. The modified Bessel functions have the following asymptotic behaviour for large z, see [82, Eq.8.451.5-6],

$$I_\kappa(z)\sim\frac{e^z}{\sqrt{2\pi z}} \text{ for } |\mathrm{arg}(z)|\leq\frac{1}{2}\pi-\beta \quad \text{and} \quad K_\kappa(z)\sim\sqrt{\frac{\pi}{2z}}e^{-z} \text{ for } |\mathrm{arg}(z)|\leq\frac{3}{2}\pi-\beta\,. \tag{5.139}$$

In order to use this, we need to scale the arguments of the kernel with $\hat{\tau}^{-1}$. Namely then, using the Taylor expansion for the square root $\sqrt{1-\hat{\tau}\hat{u}}=1-\hat{\tau}\hat{u}/2+\ldots$ and analogously with $\hat{v}$, we have:

$$\hat{\rho}\left(\frac{y_1}{\hat{\tau}^2}(1-\hat{\tau}\hat{u})\right)\psi\left(\frac{y_2}{\hat{\tau}^2}(1-\hat{\tau}\hat{v})\right) = \hat{\tau}\,\mathrm{e}^{\frac{2}{\hat{\tau}}(\sqrt{y_2}-\sqrt{y_1})}\frac{1}{2}(y_1y_2)^{-\frac{1}{4}}\left(\frac{y_1}{y_2}\right)^{\frac{\kappa}{2}}\mathrm{e}^{\sqrt{y_1}\hat{u}-\sqrt{y_2}\hat{v}}+\ldots\,, \tag{5.140}$$

$$\text{for } \hat{\tau}\ll 1\,,$$

such that the Bessel kernel from Eq. (5.119) is reached with

$$\lim_{\hat{\tau}\to 0^+}\frac{1}{\hat{\tau}}\mathbb{K}_{\mathrm{II}}^{(n,m)}\left(\frac{y_1}{\hat{\tau}},\frac{y_2}{\hat{\tau}};\hat{\tau}\right)\mathrm{e}^{-\frac{2}{\hat{\tau}}(\sqrt{y_2}-\sqrt{y_1})} = \frac{1}{2}(y_1y_2)^{-\frac{1}{4}}\,\mathbb{K}_{\mathrm{III}}^{(n,m)}(\sqrt{y_1},\sqrt{y_2})\,, \tag{5.141}$$

where we again could make use of homogeneous scaling appearing now in $\mathbb{K}_{\mathrm{II}}^{(n,m)}$. Note that the interpolating parameter $\hat{\tau}^{-1}$ is included in the encircled area of the integral Γ_{out} in the interpolating

kernel, see Fig. 5.6. However, it does not contribute as a singularity and can leave the integrated area without any additional terms.
We would like to close this section with a brief discussion about the chemical potential μ motivated by the application field of QCD.
The chemical potential μ enters the random matrix models described in Chap. 2 in the probability densities of the matrices X_1 and X_2 as a prefactor in the exponential weights. By comparing with the full degeneracy of the external fixed matrices $W = \alpha \mathbb{1}_{N+\kappa}$, $Q = q\mathbb{1}_{N+\nu}$ and $\Omega\Omega^\dagger = \delta \mathbb{1}_N$, we identify the degeneracy values to

$$q = \alpha = \alpha(\mu) = \frac{1+\mu}{2\mu} \quad \text{and} \quad \delta = \delta(\mu) = \left(\frac{1-\mu}{2\mu}\right)^2 . \tag{5.142}$$

In our computations, the chemical potential is subtly implied in the interpolation parameter $\hat{\tau}^{-1}$. Let us express from the given $\mu \in (0,1]$ dependence in the degeneracy values the limiting interpolation parameter from Eq. (5.135) in leading order,

$$\frac{1}{\hat{\tau}} = \lim_{N\to\infty} \frac{1}{4\mu N}\left(1 + \mathcal{O}(\mu)\right) . \tag{5.143}$$

A finite interpolation parameter can only be obtained if the chemical potential depends on the matrix dimension $\mu = \mu(N)$. The interpolating kernel results from $\mu(N) \sim N^{-1}$ such that the limit $\lim_{N\to\infty} \mu(N)N \to \hat{\tau}/4$ is finite.
The strong non-hermiticity case is obtained when the chemical potential reaches its maximal value $\mu = 1$. This case corresponds to the limit $\mu(N)N \to \infty$, such that $\hat{\tau} \to \infty$ yields the Meijer G-kernel, Eq. (5.138).
In the case of full-hermiticity the chemical potential tends to zero, $\mu \to 0^+$. Here, the dependence of the chemical potential with respect to the matrix dimension has to be chosen to $\mu(N)N \to 0^+$. We thus obtain the limit of $\hat{\tau} \to 0^+$, which yields the Bessel kernel as presented in Eq. (5.141).
These findings are according to [75, Thm 1.6]. The weak non-hermiticity regime is, thus, an intermediate regime of the parameter $\mu \in (0,1)$ and translates in our system to a finite $\hat{\tau} \in (0,\infty)$ after taking the limit of infinite matrix dimensions.

6 Summary and Outlook

In this thesis we presented an approach to the solution of random two-matrix models comprising complex Wishart matrices. Our discussions were made on two models on the basis of the principal publications. These two presented models extend the chiral Gaussian Unitary Ensemble in a different way. The first model is a sum of two independent matrices and the second model is a product of two coupled matrices. We showed the integrability by explicitly constructing the solution of all k-point correlation functions of their spectra for finite matrix dimensions. The main challenge in the presented investigation regarding the integrability of this class of random matrix models was the addition of external fixed matrices. These external fixed matrices occur in the ensembles as finite sets of parameters, which can be interpreted as finite-rank perturbations. Without external fixed matrices both models have known solutions, whereas the inclusion of these finite sets in our computations represent further reaching and, thus, new results. Our considerations were motivated by two fields of application: time-series analysis and Quantum Chromodynamics.

From the time-series analysis point of view our analysis in regard to the sum of two independent Wishart matrices with unequal covariances was used to study spatial cross-correlations in a two-epoch setting. With the help of this ensemble, statistical fluctuations are modeled for long time series in the scope of time-dependent spatial correlations. This situation becomes important when system specific correlations, represented by external fixed matrices, are of the same order as the statistical fluctuations, represented by random matrices. The external fixed matrices are to be identified with covariance matrices, which vary depending on the epoch.

The time dependence in the two-epoch model was presented in comparison to the one-epoch model where, in contrast, constant covariances for the entire time series are assumed. We showed that on the scale of their global level densities there is a qualitative difference by assuming time dependent covariances, which was one of the main conclusions. As a bridge passage to the next model, a section was addressed to temporal cross-correlations in the two-epoch setting. There, the product of two independent Wishart matrices was studied. In this particular setting our exact results for this model for finite matrix dimensions are new. By comparison with random matrix models from the literature it is an extension by the tuned index of the participating Bessel functions as well as by the finite set of external parameters. The interpretation of these results in time series was not fully carried out due to lack of time. The emphasis of this part lies again on the integrability of this two-matrix model, which we showed through the computation of the corresponding correlation kernel.

The product of two coupled Wishart matrices with correlations was motivated by the deepest infrared limit of Quantum Chromodynamics in Euclidean space-time. We focused on the singular-value rather than on the eigenvalue properties of the arising Dirac operator in the weak non-hermiticity regime. There, the local spectral density at the origin was studied, which implied the limit of infinite matrix dimensions of the exact analytical solution for this model. In this hard-edge scaling limit we derived

the crucial quantity, which is the interpolating kernel. We showed that the interpolating kernel interpolates between the Bessel kernel and the Meijer G-kernel. The Bessel kernel is known to occur in the one-matrix case, whereas the product of random matrices yields the Meijer G-kernel. Our results extend known results by finite sets of external parameters and, thus, are finite-rank perturbations in all the three limiting kernels.

The hermiticity of the Dirac operator modeled by our random matrix model is broken by a finite value of the chemical potential. In comparison to the literature, it was known that in the case of full hermiticity, where the chemical potential tends to zero, the Bessel kernel results from the interpolating kernel. On the other side, the interpolating kernel converts to the Meijer G-kernel in the strong non-hermiticity limit, in which the chemical potential reaches the unity. In our model, this situation is characterised by the interplay of two sets of external parameters. Thereby, we showed that the external parameters persist the limit of large matrix dimensions if a suitable dependence is chosen. Their convergence to non-zero values provides the desired non-trivial perturbation of the limiting kernels implying their deformations. These deformations extend the three limiting kernels without changing the principles of the interpolating mechanism. Hence, we broadened the universality of these kernels in the context of this mechanism by deformations originating from external fixed matrices.

The derivation of the solutions for both models has been divided in three steps. First, we discussed the eigenvalue representation of the present models, being the joint probability distribution function for all eigenvalues, which are mainly characterised by the presence of external parameters. Second, in the treated class of random matrix models the corresponding eigenvalues follow the laws of determinantal point processes, allowing to reason that all needed information is contained by one single function of two arguments: the correlation kernel. Third, a case-specific approach was derived for the computation of the correlation kernel, depending on the particular class of biorthogonal ensembles, yielding the complete spectral statistics.

For all akin random matrix models, by posing an equivalent question, the search of the eigenvalue representation implies the challenge of performing the corresponding unitary group integrals. The fundamental integrals opening integrability in this class of random matrix models are: the Harish-Chandra/Itzykson–Zuber and the Berezin-Karpelevich integrals. In the investigation of the sum of two independent Wishart matrices, the desired joint probability distribution function was known from the literature. Nonetheless, in our computations to temporal cross-correlations and, later, in the product of two coupled Wishart matrices, we applied repeatedly these two unitary group integrals. Moreover, we modified these integrals with respect to the rank of the emerging sets of fixed parameters. The unitary group integrals are solved in the sense that only determinantal expressions remain, which allow the analysis of the emerging point processes.

The eigenvalues associated to random two-matrix models comprising complex Wishart random matrices and external, fixed matrices belong to very specific biorthogonal ensembles. We discussed polynomial ensembles with permutational symmetry by including one external fixed matrix to a model of orthogonal type. This class of biorthogonal ensembles appeared multiple times in this thesis, where we made use of the modified Heine formula for the computation of the particular correlation kernel. Building on the experience of the inclusion of one external fixed matrix, we continued with the inclusion of two external fixed matrices. Here, a general approach to a biorthogonal ensemble with doubly permutational symmetry is missing. Nonetheless, two ensembles of this class have been solved:

the product of two coupled Wishart matrices and the generalised Wishart ensemble. For these two ensembles we made use of the notion of biorthogonal two-matrix ensembles reducing the computations to the derivation of one single correlation kernel.

In view of further research on integrability of random matrix models under finite-rank perturbations, the knowledge in two issues presented here has to be emphasised. First, the external fixed matrices included in random matrix models require solvable group integrals. In each of the presented ensembles any additional external fixed matrix would lead to an unsolvable problem. All unitary group integrals arising from singular value decomposition of the present random matrices have been used to involve external fixed matrices. Second, the investigation of the point processes, comprising finite sets of external parameters, was made comprehensively with respect to the solutions of the unitary group integrals, which are determinantal expressions.

What happens if the unitary group integrals are not solvable? The common technique to solve integrals of this type is the so-called character expansion. Thereby, the unitary group integrals are solved by the orthogonality of representations of unitary group elements. What remains are sums over irreducible representations, which can be resummed with the help of the Cauchy-Binet formula if the given conditions are suitable. It is remarkable that the determinantal structure of the solution does not depend on the sum over irreducible representations, but on Weyl's character formula. Even so, for an undoable resummation, the expressions consist of determinants. It is an open problem to extend the presented concept of integrability to determinantal point processes being in non-resummable convolution.

Staying on the presented concept of integrability, further external fixed matrices can only be included if we extend the considered ensembles from random two- to random multi-matrix models. The two-epoch model can a priori be extended to a multiple-epoch model comprising an arbitrary number of external fixed matrices. However, all covariance matrices have to be chosen degenerate up to one single covariance matrix to ensure the integrability of this model. The product of two coupled Wishart matrices can be extended to the product of an arbitrary number of coupled Wishart matrices. Recently, progress for this model has been done by Akemann and Strahov [120]. However, the unitary group integrals are mainly in use for the coupling of the involved random matrices, such that by ensuring its integrability only two external fixed matrices can be involved. Hence, the challenge is to find a multi-matrix model which allows to incorporate more than two external fixed matrices, while keeping integrability.

For the next step directly following from our investigations, we recommend to introduce characteristic polynomials into the product of two coupled Wishart matrices. With the help of the presented methods it might be possible to derive the exact analytical solution for the finite-N correlation kernel characterising all possible correlation functions in the point process of the $2N$ singular values. In the literature we find already recent solutions to such biorthogonal ensembles for one-matrix models, by Fyodorov, Grela and Strahov in [122]. Products of characteristic polynomials may deform the behaviour of the local level density and, hence, be very interesting in studying the universality of the interpolating mechanism in the product of two coupled Wishart matrices at the hard edge limit as a biorthogonal two-matrix ensemble.

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