Stability of Traveling Oscillating Fronts in Parabolic Evolution Equations

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Introduction

In mathematics as well as in physics the Ginzburg-Landau equation appears in various applications. A proper justification for this is its role as an amplitude equation describing various phenomena in physics, see [44], [62], [8], [59], [24]. It appears in mathematical models of hydrodynamics, nonlinear optics, superconductivity and phase transition. From the mathematical point of view the interest in the equation is justified by many mathematical phenomena occurring in the equation such as pattern formation. This thesis deals with a special class of such patterns, called traveling oscillating fronts (TOFs). We investigate their long time behavior under small perturbations and prove nonlinear stability with asymptotic phase.
The Ginzburg-Laudau equation in its complex quintic form in one space dimension reads as

$$U_t = \alpha U_{xx} + \mu U + \beta |U|^2 U + \gamma |U|^4 U, \quad x \in \mathbb{R},\, t \geq 0 \tag{QCGL}$$

with complex-valued coefficients $\alpha, \mu, \beta, \gamma \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$ and solution $U : \mathbb{R} \times [0, \infty) \to \mathbb{C}$. It is a special type of a more general class of reaction diffusion equations, which are under consideration in this thesis. These are complex-valued semilinear parabolic equations of the form

$$U_t = \alpha U_{xx} + G(|U|^2)U, \quad x \in \mathbb{R},\, t \geq 0 \tag{0.1}$$

with nonlinearity $G : \mathbb{R} \to \mathbb{C}$ and diffusion coefficient $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$. In case of (QCGL) the nonlinearity G is a quadratic polynomial over $\mathbb{C}$. If G is a polynomial of degree one we obtain the so called cubic complex Ginzburg-Landau equation, see [44]. Other generalized types of Ginzburg-Landau equations containing also first order spatial derivatives of U in the nonlinear reaction term are considered, for instance, in [62]. The existence and uniqueness of solutions of semilinear parabolic equations such as (0.1) is well-known. Details concerning solvability of the equation can be found in the classical book of D. Henry [32] or the book of M. Miklavcic [45], see also [42]. We restrict ourselves to the parabolic case, $\operatorname{Re}\alpha > 0$. The case $\operatorname{Re}\alpha = 0$ belongs to the class of Schrödinger type equations, which has been investigated in the literature, for instance, in [28], [29], [23].

In evolution equations of the type (0.1), especially in (QCGL), many different phenomena occur. There are special solutions of (0.1) which maintain their shape while traveling in space and oscillating in the complex plane. We call them traveling oscillating waves (TOWs). They may also named defects, see [57], or coherent structures, see [62]. Precisely, these are solutions $U_\star$ of (0.1) of the special form

$$U_\star(x,t) = e^{-i\omega t} V_\star(x - ct). \tag{0.2}$$

The parameters $\omega, c \in \mathbb{R}$ are called the frequency and the velocity of the wave respectively and the function $V_\star : \mathbb{R} \to \mathbb{C}$ is called its profile. TOWs occur in many different shapes. There are fronts, pulses and wave trains as well as sources, sinks and spatially periodic fronts, see Figure 0.1. For literature on the classification of TOWs we refer to [57] and [62].

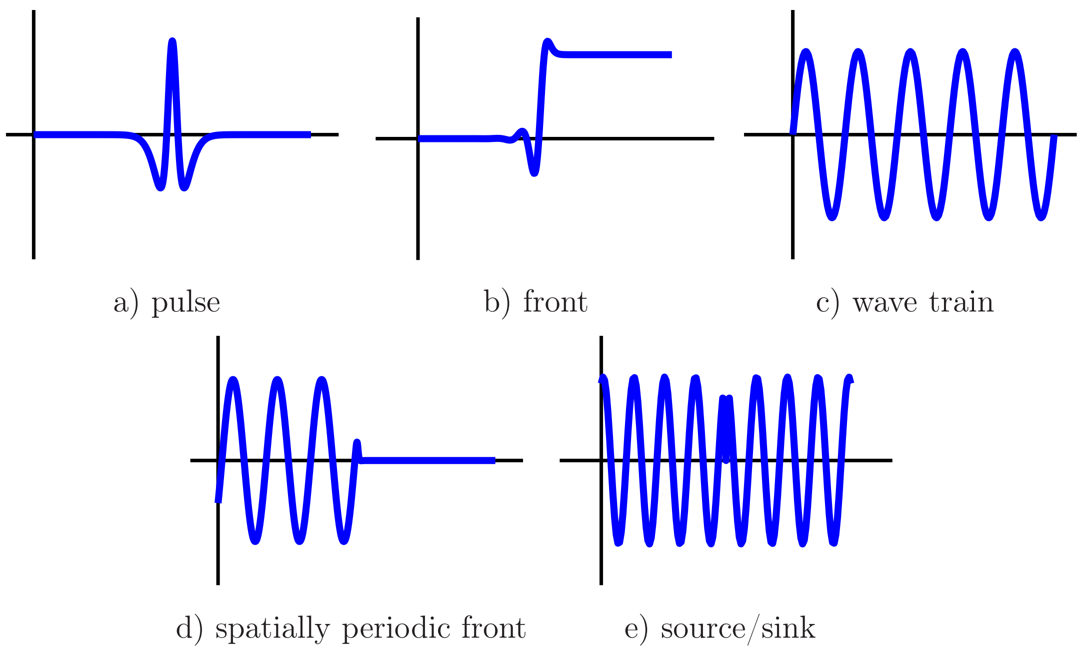


Figure 0.1: Pattern formation in (QCGL).

In the thesis we deal with front solutions, see Figure 0.1 b). A solution (0.1) of the form (0.2) is called a traveling oscillating front (TOF) if the profile satisfies the asymptotic property

$$V_\star(x) \to \begin{cases} r_\infty, & x \to +\infty \\ 0, & x \to -\infty \end{cases} \tag{0.3}$$

for some $r_\infty \in \mathbb{C}$, $r_\infty \neq 0$. These solutions can be interpreted as connecting orbits between the trivial ground state $U \equiv 0$ at $-\infty$ and a spatially constant time periodic solution $U = U(x,t) = r_\infty e^{-i\omega t}$ at $+\infty$. The appearance of TOFs, as well as TOWs, in the equation (0.1) is related by the presence of two symmetries. On the one hand there is a symmetry under translation, i.e. if $U = U(x,t)$ is a solution of (0.1) so is $\tilde{U} = U(x-\tau,t)$ for any $\tau \in \mathbb{R}$. On the other hand we have a symmetry under rotation. This means if U is a solution of (0.1) so is $\tilde{U} = e^{i\theta}U$ for any $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. If an equation has such a symmetry its right-hand side called is equivariant, cf. [15] and [21].

We investigate the long time dynamics of TOFs. In order to do so, it is convenient to transform (0.1) into a equivalent 2-dimensional real-valued system. Let $U = u_1 + iu_2$, $u_i(x,t) \in \mathbb{R}$, $\alpha = \alpha_1 + i\alpha_2$, $\alpha_i \in \mathbb{R}$ and $G = g_1 + ig_2$ with $g_i : \mathbb{R} \to \mathbb{R}$. Then the equivalent real-valued system of (0.1) reads as the semilinear parabolic equation

$$u_t = Au_{xx} + f(u), \quad x \in \mathbb{R},\, t \geq 0 \tag{0.4}$$

where

$$A = \begin{pmatrix} \alpha_1 & -\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix}, \quad f(u) = g(|u|^2)u, \quad g(\cdot) = \begin{pmatrix} g_1(\cdot) & -g_2(\cdot) \\ g_2(\cdot) & g_1(\cdot) \end{pmatrix}. \tag{0.5}$$

Let R_θ denote the rotation matrix in $\mathbb{R}^2$ by the angle $\theta \in S^1$. A traveling oscillating wave of the real-valued system (0.4) is defined as a special solution $u_\star$ of the form

$$u_\star(x,t) = R_{-\omega t} v_\star(x - ct), \tag{0.6}$$

where $v_\star : \mathbb{R} \to \mathbb{R}^2$ is the profile of the wave and ω, c are its frequency and velocity respectively. In addition, the profile $v_\star$ satisfies

$$v_\star(x) \to \begin{cases} v_\infty, & x \to +\infty \\ 0, & x \to -\infty. \end{cases} \tag{0.7}$$

We call the limit at $+\infty$, given by the vector $v_\infty = (\operatorname{Re} r_\infty, \operatorname{Im} r_\infty)^\top \in \mathbb{R}^2$ $v_\infty \neq 0$, the asymptotic rest-state. TOFs can be observed by numerical experiments in the equation (QCGL) in a large set of parameters. An example of such a numerical simulation is shown in Figure 0.2. Since these solutions travel in space and oscillate in the complex plane, it seems natural to transform (0.4) into a co-moving frame. For this purpose, let $u(x,t) = R_{-\omega t} v(\xi, t)$ with the wave coordinate $\xi = x - ct$. Then v solves the so-called co-moving equation

$$v_t = Av_{\xi\xi} + cv_\xi + S_\omega v + f(v), \quad \xi \in \mathbb{R},\, t \geq 0, \tag{0.8}$$

where S_ω is given by the matrix

$$S_\omega := \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}. \tag{0.9}$$

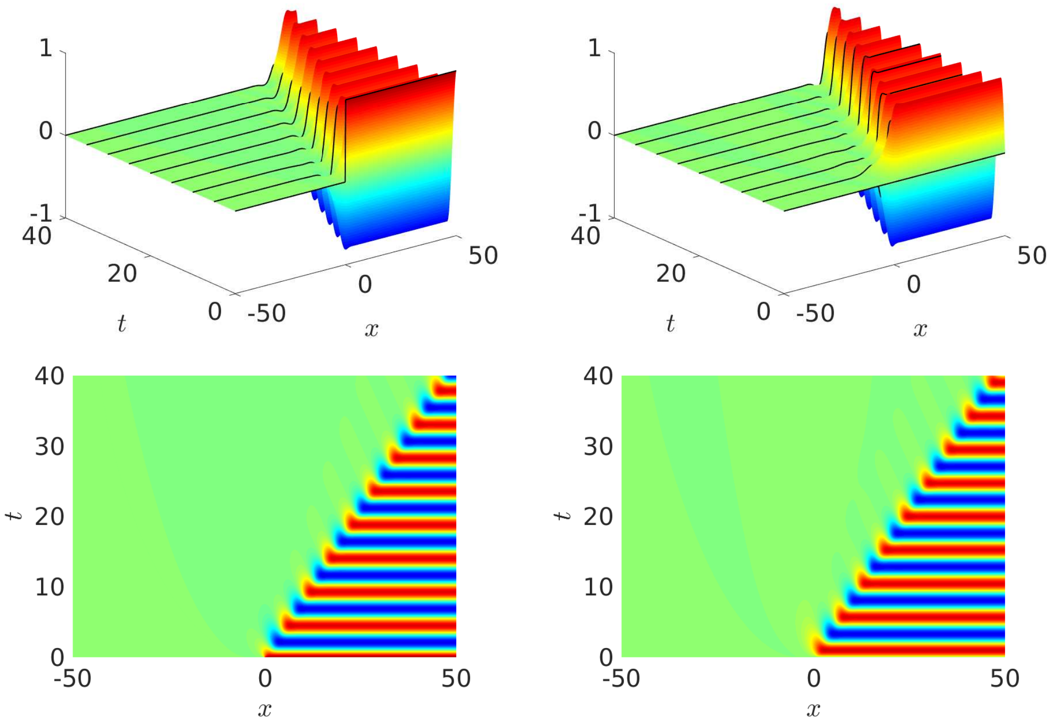


Figure 0.2: Numerical simulation of a TOF in (QCGL) with parameters $\alpha = 1 + \frac{i}{2}$, $\beta = 1 + i$, $\gamma = -1 + i$ and $\mu = -0.1$. Real part (left) and imaginary part (right).

Then the time-independent profile $v_\star$ is a stationary solution of (0.8). Thus it solves the ordinary differential equation (ODE)

$$0 = Av_{xx} + cv_x + S_\omega v + f(v), \quad x \in \mathbb{R}. \tag{0.10}$$

A natural question is whether TOFs as steady-states of (0.8) are stable under small perturbations of the initial data. This is why we are interested in the long time behavior of the solution u of the initial-value problem

$$u_t = Au_{xx} + cu_x + S_\omega u + f(u), \quad u(0) = v_\star + u_0, \tag{0.11}$$

where u_0 is a small initial perturbation. One expects from the numerical experiment in Figure 0.2 that the observed TOF is stable. Otherwise numerical errors should grow in time and the TOF could not be observed. Typically, to show stability one has to consider the linearization of the equation at the steady-state. In the case of TOFs this is the operator

$$Lu = Au_{xx} + cu_x + S_\omega u + Df(v_\star)u \tag{0.12}$$

with the Jacobian Df of the nonlinearity f given by

$$Df(v) = \begin{pmatrix} g_1(|v|^2) + 2g_1'(|v|^2)v_1^2 - 2g_2'(|v|^2)v_1v_2 & 2g_1'(|v|^2)v_1v_2 - g_2(|v|^2) - 2g_2'(|v|^2)v_2^2 \\ g_2(|v|^2) + 2g_2'(|v|^2)v_1^2 + 2g_1'(|v|^2)v_1v_2 & 2g_2'(|v|^2)v_1v_2 + g_1(|v|^2) + 2g_1'(|v|^2)v_2^2 \end{pmatrix}. \tag{0.13}$$

Since the equation (0.8) is equivariant, TOFs always come in families, i.e. there is a whole continuum $\mathcal{O}(v_\star) := \{R_\theta v_\star(\cdot - \tau) : (\theta, \tau) \in S^1 \times \mathbb{R}\}$ of stationary solutions. Therefore the linearization L from (0.12) has a nontrival kernel and one cannot expect stability of $v_\star$ in the classical sense of Lyapunov. One has to weaken the notion of stability in the following sense, cf. [56] and [15]. We say a TOF is nonlinearly stable if for all small initial perturbations u_0 the solution u of (0.11) stays close to the group orbit $\mathcal{O}(v_\star)$ for all positive times. If in addition the solution converges to an element $R_{\theta_\infty} v_\star(\cdot - \tau_\infty)$ of $\mathcal{O}(v_\star)$ as $t \to \infty$, then the TOF is called nonlinearly stable with asymptotic phase. The main results of the thesis state that traveling oscillating fronts are nonlinearly stability with asymptotic phase.

In order to prove nonlinear stability of TOFs, we have to circumvent two major problems. The first one occurs when considering the spectrum of the linearized operator (0.12). A crucial step is to guarantee that the spectrum is included in the strict left half-plane, except for an isolated zero eigenvalue of finite multiplicity caused by the equivariance. In the literature this property is also called linear or spectral stability, cf. [56]. Its importance is explained by the fact that spectral stability implies time decay of the corresponding semigroup $\{e^{tL}\}_{t\geq 0}$ generated by L, cf. [32]. For TOFs it turns out that the essential spectrum of the linearized operator L touches the imaginary axis at the origin. This is due to the so-called dispersion set which is contained in the essential spectrum and which is defined as follows:

$$\sigma_{\mathrm{disp}}(L) = \sigma^-_{\mathrm{disp}}(L) \cup \sigma^+_{\mathrm{disp}}(L), \quad \sigma^\pm_{\mathrm{disp}}(L) := \{s \in \mathbb{C} : \exists \nu \in \mathbb{R} \text{ s.t. } d^\pm(s,\nu) = 0\}, \tag{0.14}$$

where $d^\pm$ is the dispersion relation given by

$$d^\pm(s,\nu) := \det(sI + \nu^2 A - i\nu c I - S_\omega - Df(v_\pm)), \quad v_+ = v_\infty, \quad v_- = 0. \tag{0.15}$$

Here I denotes the identity matrix in $\mathbb{R}^2$. The dispersion set consists of four curves in the complex plane, which typically have the shape of parabolas opened to the left, cf. Figure 0.3, but may also be more complicated. The vertices of the curves are given by the solution of $d^\pm(s_\pm, 0) = 0$. For $d^-(s_\pm, 0) = 0$ these are the values

$$s_\pm = g_1(0) \pm i(g_2(0) + \omega) \in \sigma_{\mathrm{disp}}(L).$$

Thus a necessary condition for spectral stability is $g_1(0) < 0$. Further, $d^+(s_\pm, 0) = 0$ yields

$$s_+ = 2g_1'(|v_\infty|^2)|v_\infty|^2, \quad s_- = 0.$$

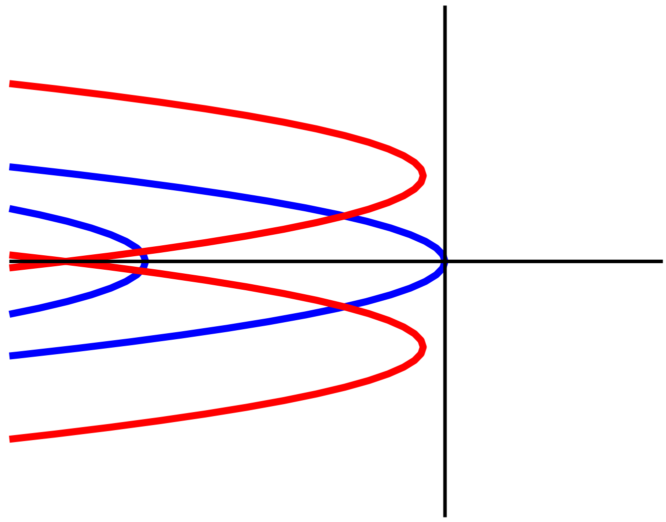


Figure 0.3: The dispersion set $\sigma_{\mathrm{disp}}(L)$ in (QCGL) with $\sigma^+_{\mathrm{disp}}(L)$ (blue) and $\sigma^-_{\mathrm{disp}}(L)$ (red).

Consequently, a second necessary condition is given by $g_1'(|v_\infty|^2) < 0$. But zero is always contained in the dispersion set and the (essential) spectrum touches the imaginary axis at the origin. Therefore, the classical approach to prove nonlinear stability from [32], [36] is not applicable. We overcome this problem by using exponential or polynomial weight functions. In general, let $\eta : \mathbb{R} \to \mathbb{R}$ be a weight function. Then we consider the stability problem on weighted Lebesgue spaces for $1 \le p \le \infty$ defined by

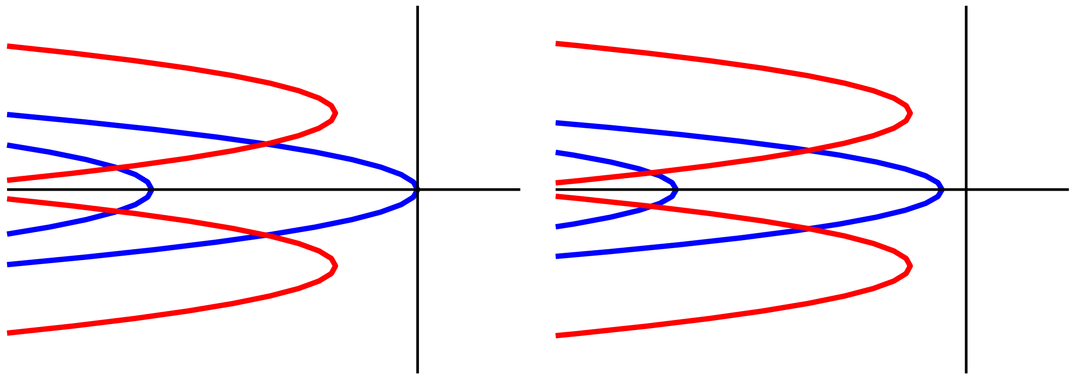
$$L^p_\eta(\mathbb{R},\mathbb{R}^n) := \{u \in L^p(\mathbb{R},\mathbb{R}^n) : \eta u \in L^p(\mathbb{R},\mathbb{R}^n)\}, \quad \|u\|_{L^p_\eta} := \|\eta u\|_{L^p}. \tag{0.16}$$

In the case $p = 2$ we also define the weighted Sobolev spaces for $\ell \in \mathbb{N}$ by

$$\begin{aligned} H^\ell_\eta(\mathbb{R},\mathbb{R}^n) &:= \{u \in L^2_\eta(\mathbb{R},\mathbb{R}^n) \cap H^\ell_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n) : \partial^k u \in L^2_\eta(\mathbb{R},\mathbb{R}^n),\ 1 \le k \le \ell\}, \\ \|u\|^2_{H^\ell_\eta} &:= \sum_{k=0}^{\ell} \|\partial^k u\|^2_{L^2_\eta}. \end{aligned} \tag{0.17}$$

The advantage of using exponential weight functions is that the dispersion set (0.14) is pushed to left of the imaginary axis, cf. Figure 0.4. Therefore, we conclude spectral stability on exponentially weighted spaces and can make use of the approach from [32], [36] to show nonlinear stability. When using polynomial weight functions the dispersion set does not change. However, in polynomially weighted spaces we derive delicate resolvent estimates near the origin using different norms w.r.t. polynomial order. The approach is based on ideas from [35]. Then we are able to show polynomial decay of the semigroup

$\{e^{tL}\}_{t\geq 0}$ w.r.t. norms with different polynomial weights.



a) unweighted/polynomially weighted b) exponentially weighted

Figure 0.4: The dispersion set on unweighted L^2-spaces (left) vs. exponentially weighted L^2_η-spaces (right).

The second problem we have to deal with is caused by the fact that the profile $v_\star$ of a TOF does not decay to zero as $x \to \infty$. Therefore the solution neither lies in the standard L^2-space nor in their weighted versions introduced in (0.16). We have to choose a suitable function space where the stability analysis can be done rigorously. In order to do so, let us assume u to be a smooth solution of (0.8) such that $\rho(t) = \lim_{x\to\infty} u(x,t)$ exists and $u_x(x,t), u_{xx}(x,t) \to 0$ as $x \to \infty$. When formally taking the limit $x \to \infty$ in (0.8) we obtain that ρ solves the ODE

$$\rho'(t) = S_\omega \rho(t) + f(\rho(t)). \tag{0.18}$$

Note that v_∞ must be a stationary solution of (0.18). Now we define a template function

$$\hat{v}(x) := \tfrac{1}{2}\tanh(x) + \tfrac{1}{2}. \tag{0.19}$$

Then we expect the solution u to satisfy $u(t) - \rho(t)\hat{v} \in H^2_\eta(\mathbb{R},\mathbb{R}^2)$. Thus the solution lies in an affine linear space with a time dependent offset given by ρ. This is why we add an additional equation describing the offset ρ via (0.18). We introduce the space

$$X_\eta := \Big\{(u,\rho)^\top : u : \mathbb{R} \to \mathbb{R}^2,\, \rho \in \mathbb{R}^2,\, u - \rho\hat{v} \in L^2_\eta(\mathbb{R},\mathbb{R}^2)\Big\} \tag{0.20}$$

and equip it with the norm $\|(u,\rho)^\top\|^2_{X_\eta} := |\rho|^2 + \|u - \rho\hat{v}\|^2_{L^2_\eta}$. In a canonical manner we also define the smooth analogs, i.e. we set for $\ell \in \mathbb{N}_0$

$$X^\ell_\eta := \big\{(u,\rho)^\top \in X_\eta : u \in H^\ell_{\mathrm{loc}}, \partial^k u \in L^2_\eta,\, 1 \leq k \leq \ell\big\} \tag{0.21}$$

and equip it with the norm $\left\|(u,\rho)^\top\right\|^2_{X^\ell_\eta} := |\rho|^2 + \|u-\rho\hat{v}\|^2_{L^2_\eta} + \sum_{k=1}^{\ell}\|\partial^k u\|^2_{L^2_\eta}$. We set $X^0_\eta := X_\eta$, $Y_\eta := X^2_\eta$ and denote the elements of X^k_η by bold letters, i.e. $\mathbf{u} = (u,\rho)^\top$. Finally, instead of analyzing the initial value problem (0.11), we consider the Cauchy problem on X_η given by

$$\mathbf{u}_t = \mathcal{F}(\mathbf{u}), \quad \mathbf{u}(0) = \mathbf{v}_\star + \mathbf{u}_0, \tag{0.22}$$

where $\mathcal{F}$ is a semilinear operator given by

$$\mathcal{F} : Y_\eta \to X_\eta, \quad \begin{pmatrix} u \\ \rho \end{pmatrix} = \mathbf{u} \mapsto \mathcal{F}(\mathbf{u}) = \begin{pmatrix} Au_{xx} + cu_x + S_\omega u + f(u) \\ S_\omega \rho + f(\rho) \end{pmatrix} \tag{0.23}$$

and $\mathbf{v}_\star = (v_\star, v_\infty)^\top$. It turns out that $v_\star \in v_\infty\hat{v} + H^2_\eta$ and $\mathbf{v}_\star \in Y_\eta$. In addition, since $v_\star$ is a stationary solution of (0.8), we obtain $\mathcal{F}(\mathbf{v}_\star) = 0$. We investigate nonlinear stability with asymptotic phase of $\mathbf{v}_\star$ as a stationary solution of (0.22) in the case of exponential weight functions.

We conclude the introduction by giving an outline of the thesis. Chapter 1 starts with a short overview of the concept of abstract equivariant evolution equations and relative equilibria. The definition of TOWs as well as TOFs is made precise and we collect first observations concerning the determination of the asymptotic rest-state v_∞ and the frequency ω by the nonlinearity g. We conclude the first chapter by stating the assumptions and main results of the thesis in Section 1.3. The first stability result states that under certain assumptions TOFs as stationary solutions of the Cauchy problem (0.22) are nonlinearly stable with asymptotic phase in exponentially weighted spaces. The second result is that TOFs as stationary solutions of (0.8) are nonlinearly stable with asymptotic phase w.r.t. polynomially weighted spaces. Both results are not comparable, since in the polynomial case we have to assume that the initial perturbation decays to zero as $x \to \infty$ whereas we can allow small perturbations at infinity in the exponential case.

In Chapter 2 we study the profile of traveling oscillating fronts as solutions of the stationary co-moving equation (0.10). We use a dynamical systems approach from [62] to derive a first order ODE system in 3 dimensions, which is equivalent to (0.10). Then profiles of TOFs occur as heteroclinic orbits between steady-states of the dynamical system. In this situation we are able to discuss the existence of TOFs by the intersection of stable and unstable manifolds of steady states. In addition, we use the theory of hyperbolic equilibria and exponential dichotomies introduced in [22] to show that the asymptotic convergence in (0.7) is exponentially fast provided certain assumptions are satisfied. This is a crucial step to prove nonlinear stability, since it guarantees exponentially fast convergence of the profile $v_\star$ at $\pm\infty$. In particular, $v_\star \in v_\infty\hat{v} + H^2_\eta$ when η is an exponential weight function.

Chapter 3 covers the nonlinear stability with asymptotic phase in exponentially weighted spaces. The idea of the proof of the main result is similar to the case of traveling waves considered in [32] or the case of rotating patterns from [17]. Nevertheless, since we are working in the spaces X_η we have to take care of the validation of this approach to TOFs. In addition, since the (essential) spectrum of the linearized operator touches the imaginary axis the approach is not directly applicable. We circumvent this problem using exponential weights. Throughout the third chapter we set $\eta = \eta_{\text{exp}}$ where η_{exp} is an exponential weight function given by

$$\eta_{\text{exp}}(x) := e^{\mu\sqrt{x^2+1}}, \quad \mu \geq 0. \tag{0.24}$$

Then η_{exp} is also called a weight function of exponential growth rate $\mu \geq 0$, see [63]. For the sake of notation we will suppress the index and only write η instead of η_{exp}. We describe rotation and translation of elements from X_η by the group action

$$a(\gamma) : X_\eta \to X_\eta, \quad \begin{pmatrix} u \\ \rho \end{pmatrix} \mapsto a(\gamma)\begin{pmatrix} u \\ \rho \end{pmatrix} = \begin{pmatrix} R_{-\theta}v(\cdot - \tau) \\ R_{-\theta}\rho \end{pmatrix}, \tag{0.25}$$

where $\gamma = (\theta, \tau) \in \mathcal{G} = S^1 \times \mathbb{R}$. It follows that $\mathcal{F}$ from (0.23) is equivariant under the group action, i.e. $\mathcal{F}(a(\gamma)\mathbf{u}) = a(\gamma)\mathcal{F}(\mathbf{u})$. The crucial step is to consider the linearized operator given by the linearization of the right hand side in (0.22) at the TOF. It is defined by

$$\mathcal{L} : Y_\eta \to X_\eta, \quad \mathbf{u} \mapsto \mathcal{L}\mathbf{u} = \begin{pmatrix} Au_{xx} + cu_x + S_\omega u + Df(v_\star)u \\ S_\omega\rho + Df(v_\infty)\rho \end{pmatrix}. \tag{0.26}$$

A major part of its spectrum consists of the dispersion set

$$\sigma_{\text{disp},\mu}(\mathcal{L}) = \sigma^-_{\text{disp},\mu}(\mathcal{L}) \cup \sigma^+_{\text{disp},\mu}(\mathcal{L}), \quad \sigma^\pm_{\text{disp},\mu}(\mathcal{L}) := \{s \in \mathbb{C} : \exists \nu \in \mathbb{R} \text{ s.t. } d^\pm_\mu(s,\nu) = 0\}, \tag{0.27}$$

which depends on the exponential growth rate $\mu > 0$. Here $d^\pm_\mu$ is the dispersion relation defined by

$$\begin{aligned} &d^\pm_\mu(s,\nu) := \det(sI + \nu^2 A - i\nu B_\pm(\mu) - C_\pm(\mu)), \quad B_\pm(\mu) = cI \mp 2\mu A, \\ &C_\pm(\mu) = S_\omega + Df(v_\pm) + \mu^2 A \mp c\mu I, \quad v_+ = v_\infty, \quad v_- = 0. \end{aligned} \tag{0.28}$$

For the unweighted case $\mu = 0$ we have $\sigma_{\text{disp},0}(\mathcal{L}) = \sigma_{\text{disp}}(L)$ and the (essential) spectrum touches the imaginary axis. The effect of using exponential weights is that the critical curve $\sigma^+_{\text{disp},\mu}(\mathcal{L})$ of the dispersion set is pushed to the left of the imaginary axis, cf. Figure 0.4. Only an isolated eigenvalue of finite multiplicity remains at the origin. Then the approach from [32] can be used to show nonlinear stability. However, the main work is to ensure that the approach also applies to the larger spaces X_η instead of standard L^2 or L^2_η spaces. In particular, we derive delicate Lipschitz estimates with small Lipschitz constants for the remaining nonlinearities in the spaces X^ℓ_η. In the end, a Gronwall argument from [17] is used to conclude nonlinear stability.

In Chapter 4 we consider the numerical computation of TOFs. We are interested in the computation of the profile and the velocities of the TOFs, which are usually a-priori unknown. By applying a classical finite difference or finite element method to the equation (0.4) the problem occurs, that the TOFs will leave the domain of computation at a certain time. This problem is captured by the so called freezing method from [18], [19], which we apply to our situation in Chapter 4. Further, we prove stability of TOFs in the sense of Lyapunov in the freezing method. We finish the chapter by showing numerical simulations and experiments.

In Chapter 5 we deal with the natural question whether TOFs are nonlinearly stable with asymptotic phase, if the initial perturbation is only polynomially decaying. We consider the nonlinear stability problem on polynomially weighted spaces, which is in contrast to Chapter 3 where we consider exponentially weighted spaces. Throughout Chapter 5 we set $\eta = \eta^k_{\mathrm{poly}}$ for appropriate $k \in \mathbb{N}$ where η_{poly} is a polynomial weight function of linear growth defined by

$$\eta_{\mathrm{poly}}(x) := (x^2+1)^{\frac{1}{2}}. \tag{0.29}$$

In this case we set

$$L^2_k(\mathbb{R},\mathbb{R}^2) = L^2_\eta(\mathbb{R},\mathbb{R}^2), \quad H^\ell_k(\mathbb{R},\mathbb{R}^2) = H^\ell_\eta(\mathbb{R},\mathbb{R}^2), \quad \eta = \eta^k_{\mathrm{poly}}, \quad k,\ell \in \mathbb{N}. \tag{0.30}$$

We consider the perturbed initial value problem (0.11) and assume that u_0 is small in the space H^2_k for sufficiently large $k \in \mathbb{N}$. Then $u_0 \to 0$ as $x \to \infty$ and we obtain $u(x,t) \to v_\infty$ as $x \to \infty$ for all $t \geq 0$. Thus, the offset ρ from (0.18) stays constant in time, i.e. $\rho(t) = v_\infty$ for all $t \geq 0$. Therefore, we seek for a solution u of (0.11) in the affine Banach spaces

$$M_k = \bar{v} + L^2_k, \quad M^\ell_k = \bar{v} + H^\ell_k, \quad \bar{v} := v_\infty \hat{v}. \tag{0.31}$$

To prove nonlinear stability with asymptotic phase, we use the same approach as in Chapter 3, see also [32], [17]. In this case we have to determine the spectrum of the linearized operator L from (0.12) on the space L^2_k. It turns out that for every $k \in \mathbb{N}$ the spectrum of the operator still touches the imaginary axis at the origin, cf. Figure 0.4. Therefore, the classical theory from [32] only gives estimates of the generated semigroup e^{tL} by exponentially increasing terms. In order to circumvent this problem we derive sharp resolvent estimates of the operator L near the origin. We use ideas from [37] and show uniform bounds for the resolvent $(sI - L)^{-1}$ considered as an operator from L^2_{k+3} to L^2_k for s in a crescent Ω_c at the origin, see Figure 0.5.

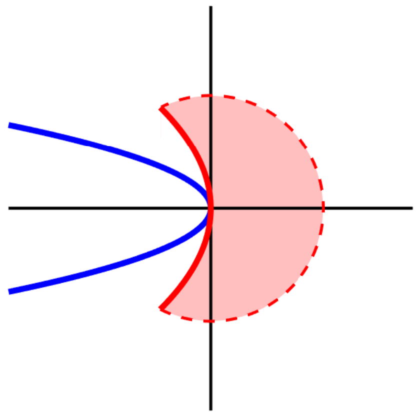


Figure 0.5: The crescent Ω_c.

The loss in the polynomial order will lead to the uniform estimates of the resolvent and then to polynomial estimates of the semigroup mapping from L^2_{k+3} to L^2_k. In the end we show that the loss of the polynomial order caused by the semigroup is compensated by the quadratic nonlinearities. This will lead to nonlinear stability with asymptotic phase of TOFs in polynomially weighted spaces.

We conclude by giving a comment on the main results. Both results are not comparable since the type of admissible perturbations differs. In the exponentially weighted case we can allow perturbations which may not decay to zero as $x \to \infty$ but must converge exponentially fast to some small vector. This is due to the stability with asymptotic phase of the periodic orbit $R_{-\omega t}v_\infty$ of the ODE (0.18) which is guaranteed under our assumptions. In contrast, in the polynomially weighted case we can allow perturbations that converge only with a polynomial rate, but therefore must decay to zero. This is caused by the fact that only in this case we are able to control the remaining nonlinearities w.r.t. polynomial orders. We expect that both results can be combined by taking advantage of the stability behavior of the periodic orbit in (0.18). However, we expect the proof to be much more involved and keep this as an open question.

Chapter 1

Traveling oscillating fronts in evolution equations

1.1 Equivariant evolution equation

We start with a short overview on the concept of equivariant evolution equations and relative equilibria, see for instance [19] and [21]. We consider an abstract evolution equation of the form

$$u_t = F(u), \quad t \geq 0 \tag{1.1}$$

where F is a continuous, densely defined operator on a Banach manifold M modeled over a Banach space X, i.e.

$$F : \mathcal{D}(F) \subset M \to X \tag{1.2}$$

is defined on a dense submanifold $\mathcal{D}(F) = N$ which is modeled over a dense Banach space $Y \subset X$. References for the abstract concepts of manifolds are given by [1], [41]. In many cases, such as traveling waves, the Banach manifold is given by an affine Banach space $M = \hat{v} + X$ and $N = \hat{v} + Y$ for some element $\hat{v}$. Typical examples are $X = L^2(\mathbb{R}, \mathbb{R}^m)$, $Y = H^2(\mathbb{R}, \mathbb{R}^m)$ and $\hat{v} \in C_b^2(\mathbb{R}, \mathbb{R}^m)$ with $\hat{v}_x \in H^1(\mathbb{R}, \mathbb{R}^m)$ when F is a second order semilinear differential operator.
At this point we may let open the precise notion of solution of (1.1) since it strongly depends on the type of the evolution equation and function spaces. However, in our application the following notion of solution is suitable:

Definition 1.1. A function $u \in C([0, t_\infty), N) \cap C^1([0, t_\infty), M)$ is called a solution of (1.1) on $[0, t_\infty)$ with initial value $u_0 \in N$ if for all $t \in [0, t_\infty)$ there hold $u_t(t) = F(u(t))$ in M and $u(0) = u_0$.

Let $(\mathcal{G}, \circ)$ be a Lie group of dimension $\dim \mathcal{G} = n < \infty$ and smooth composition $\circ : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $(\gamma, \tilde{\gamma}) \mapsto \gamma \circ \tilde{\gamma}$. For an introduction into Lie groups we refer to [53]. The unit element of $\mathcal{G}$ is denoted by $\mathbb{1}$ and let $\mathfrak{g} = T_{\mathbb{1}}\mathcal{G}$ be the associated Lie algebra. For the left multiplication we write $L_\gamma : \mathcal{G} \to \mathcal{G}$, $\tilde{\gamma} \mapsto \gamma \circ \tilde{\gamma}$ which is also smooth and its derivative is denoted by $dL_\gamma(\tilde{\gamma}) : T_{\tilde{\gamma}}\mathcal{G} \to T_{\gamma\circ\tilde{\gamma}}\mathcal{G}$. The Lie algebra $\mathfrak{g}$ and the Lie group $\mathcal{G}$ are related via the exponential map $\exp : \mathfrak{g} \to \mathcal{G}$, which can be defined such that $\gamma(t) = \exp(t\mu)$, $\mu \in \mathfrak{g}$ is the unique solution of the initial value problem

$$\gamma_t = dL_\gamma(\mathbb{1})\mu, \quad \gamma(0) = \mathbb{1}. \tag{1.3}$$

The group $\mathcal{G}$ acts on the Banach manifold M via a group action $a(\gamma)$, $\gamma \in \mathcal{G}$. For $v \in M$ it is defined by

$$a(\cdot)v : \mathcal{G} \to M, \quad \gamma \mapsto a(\gamma)v$$

and is assumed to be continuous, satisfying for all $\gamma, \tilde{\gamma} \in \mathcal{G}$ and $v \in M$

$$a(\gamma \circ \tilde{\gamma})v = a(\gamma)a(\tilde{\gamma})v. \quad a(\mathbb{1})v = v, \quad a(\gamma^{-1})v = a(\gamma)^{-1}v.$$

Here $\gamma^{-1} \in \mathcal{G}$ denotes the inverse of an element $\gamma \in \mathcal{G}$, i.e. $\gamma \circ \gamma^{-1} = \mathbb{1}$. Further, we assume that the group action is pathwise continuously differentiable on the Banach manifold N, i.e. for all $v \in N$ the map $a(\cdot)v : \mathcal{G} \to N$ is of class C^1 with derivative

$$d[a(\gamma)v] : T_\gamma\mathcal{G} \to T_{a(\gamma)v}N.$$

Differentiating the relation $a(\gamma \circ \tilde{\gamma})v = a(\gamma)a(\tilde{\gamma})$ w.r.t. $\tilde{\gamma}$ and evaluating at $\tilde{\gamma} = \mathbb{1}$ yields for $\mu \in \mathfrak{g}$

$$d[a(\gamma)v]dL_\gamma(\mathbb{1})\mu = a(\gamma)d[a(\mathbb{1})v]\mu. \tag{1.4}$$

We assume that the operator F is equivariant under the group action $a(\gamma), \gamma \in \mathcal{G}$ according to the following definition:

Definition 1.2. The operator $F : \mathcal{D}(F) = N \subset M \to X$ from (1.2) is called equivariant under the group action a of $\mathcal{G}$ if for all $\gamma \in \mathcal{G}$ and $u \in N$ there hold $a(\gamma)N \subset N$ and

$$a(\gamma)F(u) = F(a(\gamma)u).$$

We transform (1.1) into a co-moving frame via the solution ansatz $u(t) = \exp(t\mu_\star)v(t)$. Plugging this into the equation (1.1) we obtain using (1.3) and (1.4) that v solves the co-moving equation

$$v_t = F(v) - d[a(\mathbb{1})v]\mu_\star. \tag{1.5}$$

We are interested into stationary solutions $v_\star$ of the co-moving equations, i.e.

$$0 = F(v_\star) - d[a(\mathbb{1})v_\star]\mu_\star.$$

Then the corresponding solution $u_\star(t) = a(\exp(t\mu_\star))v_\star$ is a so-called relative equilibrium of the abstract evolution equation (1.1).

Definition 1.3. A solution $u_\star$ on $[0,\infty)$ is called a relative equilibrium of the evolution equation (1.1) if there is $\mu_\star \in \mathfrak{g}$ and $v_\star \in N$ such that for all $t \in [0,\infty)$ there hold

$$u_\star(t) = a(\gamma_\star(t))v_\star, \quad \gamma_\star(t) = \exp(t\mu_\star). \tag{1.6}$$

Sometimes the profile $v_\star$ as well as the whole group orbit $\mathcal{O}(v_\star)$ are called relative equilibria since they define steady-states of the co-moving equation (1.5), see [21]. A natural question arising is, whether the steady-state is stable under small perturbations. In other word, we are interested in the long time behavior of the solution v of (1.5) with initial data $v(0) = v_\star + u_0$ where u_0 is small w.r.t. to some norm $\|\cdot\|$. Since we have a whole continuum of steady-states, asymptotic stability in the classical sense of Lyapunov cannot be expected. The concept of stability is generalized in the following sense, see [15], [19], [56], [36].

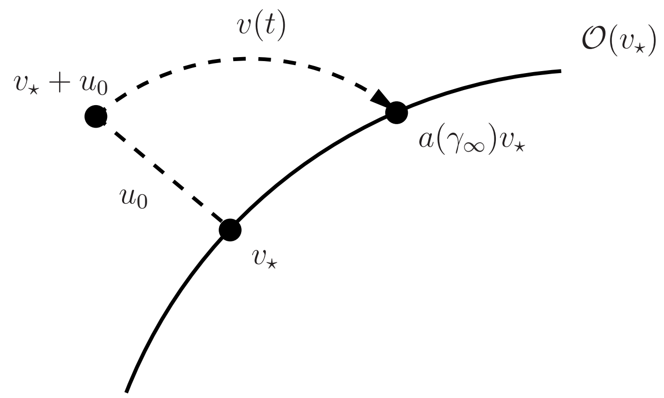


Figure 1.1: Nonlinear stability with asymptotic phase.

Definition 1.4 (Nonlinear stability with asymptotic phase)**.** The relative equilibrium $u_\star$ given by $(v_\star, \mu_\star)$ is called **nonlinearly stable** w.r.t. given norms $\|\cdot\|_1$ and $\|\cdot\|_2$, if for any $\delta > 0$ there exists $\varepsilon > 0$ such that for any initial value $v(0) = v_0$ with $\|v_0 - v_\star\|_1 \leq \varepsilon$ the co-moving equation (1.5) has a unique solution $v(t)$, $t \geq 0$ satisfying for all $t \geq 0$

$$\inf_{\gamma \in \mathcal{G}} \|v(t) - a(\gamma)v_\star\|_2 \leq \delta.$$

If, in addition, there is an asymptotic phase $\gamma_\infty \in \mathcal{G}$ such that

$$\|v(t) - a(\gamma_\infty)v_\star\|_2 \to 0, \quad t \to \infty,$$

then $u_\star$ is called **nonlinearly stable with asymptotic phase**.

1.2 Traveling oscillating waves and fronts

Let us recall the evolution equation (0.4) reading as

$$u_t = Au_{xx} + f(u)$$

with the diffusion matrix A and nonlinearity f given by

$$A = \begin{pmatrix} \alpha_1 & -\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix}, \quad f(u) = g(|u|^2)u, \quad g : \mathbb{R} \to \mathbb{R}^{2,2}, \quad g(\cdot) = \begin{pmatrix} g_1(\cdot) & -g_2(\cdot) \\ g_2(\cdot) & g_1(\cdot) \end{pmatrix}.$$

As mentioned in the introduction there are many different phenomena occurring in equations of the form (0.4). In the thesis we are interested in traveling oscillating fronts (TOFs) for which we give the following precise definition. Recall the rotation matrix R_θ, $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ in $\mathbb{R}^2$ given by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in S^1.$$

Definition 1.5. A solution $u_\star : \mathbb{R} \times [0, \infty) \to \mathbb{R}^2$ of (0.4) of the form

$$u_\star(x,t) = R_{-\omega t} v_\star(x - ct), \quad x \in \mathbb{R},\, t \geq 0 \tag{1.7}$$

with profile $v_\star \in C_b^2(\mathbb{R}, \mathbb{R}^2)$ is called a **traveling oscillating wave (TOW)** of (0.4) with speed $c \in \mathbb{R}$ and frequency $\omega \in \mathbb{R}$. In addition, if the profile $v_\star$ satisfies the asymptotic properties

$$\lim_{\xi \to -\infty} v_\star(\xi) = 0, \quad \lim_{\xi \to \infty} v_\star(\xi) = v_\infty, \tag{1.8}$$

for some $v_\infty \in \mathbb{R}^2 \backslash \{0\}$, then $u_\star$ is called a **traveling oscillating front (TOF)**. In this case the value v_∞ is called the asymptotic rest-state of the TOF.

In other words, TOFs are solutions of (0.4) which connect the zero steady-state as $\xi \to -\infty$ with some non-zero periodic state as $\xi \to \infty$. An illustration of such a solution can be seen in Figure 1.2. Note that by definition a traveling oscillating front of (0.4) is smooth in the sense that

$$u_\star \in C^1([0, \infty), C_b^1(\mathbb{R}, \mathbb{R}^2)) \cap C([0, \infty), C_b^2(\mathbb{R}, \mathbb{R}^2)).$$

To analyze the dynamics of solutions of (0.4), especially TOFs, it is convenient to transform the equation into a co-moving frame. We use the ansatz $u(x,t) = R_{-\omega t} v(\xi, t)$ with the wave variable $\xi = x - ct$. A simple computation shows that the derivatives of u w.r.t. time and space are given by

$$\begin{aligned} u_t(x,t) &= -\omega R_{-\omega t} S_1 v(\xi,t) - c R_{-\omega t} v_\xi(\xi,t) + R_{-\omega t} v_t(\xi,t), \\ u_{xx}(x,t) &= R_{-\omega t} v_{\xi\xi}(\xi,t) \end{aligned}$$

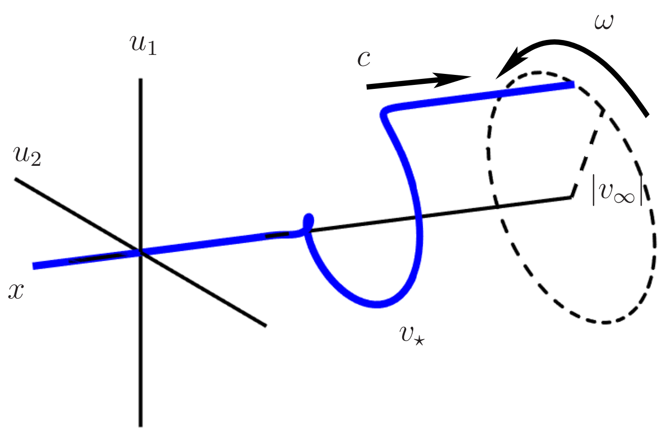


Figure 1.2: Traveling oscillating front.

with the skew-symmetric unit matrix $S_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In particular, since R_θ is a rotation matrix we obtain immediately by invariance of the absolute value and the form of the matrix-valued function g

$$f(u(x,t)) = g(|v(\xi,t)|^2)R_{-\omega t}v(\xi,t) = R_{-\omega t}g(|v(\xi,t)|^2)v(\xi,t).$$

Since $R_{-\omega t}A = AR_{-\omega t}$ we conclude that v is a solution of the co-moving equation (0.8) which reads as

$$v_t = Av_{xx} + cv_x + S_\omega v + f(v), \quad x \in \mathbb{R},\, t \geq 0.$$

Here $S_\omega = \omega S_1$ is given by (0.9). The profile $v_\star$ of a TOF is time independent and hence it is a stationary solution of (0.8), i.e.

$$0 = Av_\star'' + cv_\star' + S_\omega v_\star + f(v_\star).$$

Since the profile $v_\star$ has limits as $x \to \pm\infty$, it seems natural that the derivatives $v_\star', v_\star''$ decay to zero as $x \to \pm\infty$. One observe, if g is at least continuous, that $g(|v_\infty|) = -S_\omega$. Thus, the magnitude of the possible asymptotic rest-states $|v_\infty|$ and the frequency $\omega \in \mathbb{R}$ are determined by the nonlinearity g.

Lemma 1.6. *Let $v_\star \in C_b^2(\mathbb{R},\mathbb{R}^2)$ be the profile of a traveling oscillating front of* (0.4) *with speed $c \in \mathbb{R}$, frequency $\omega \in \mathbb{R}$ and asymptotic rest-state $v_\infty \in \mathbb{R}^2\backslash\{0\}$. Moreover, suppose* $\operatorname{Re}\alpha > 0$ *and $g \in C(\mathbb{R},\mathbb{R}^{2,2})$. Then*

$$g(|v_\infty|^2) = -S_\omega, \quad \lim_{x\to\pm\infty} v_\star'(x) = 0, \quad \lim_{x\to\pm\infty} v_\star''(x) = 0.$$

Proof. Since $v_\star$ is the profile of a traveling oscillating front it solves

$$Av_\star'' + cv_\star' = -S_\omega v_\star - f(v_\star)$$

and the limits $\lim_{\xi\to\pm\infty} v_\star(\xi)$ exist. Setting $h(\xi) = -S_\omega v_\star(\xi) - f(v_\star(\xi))$ to be the right hand side, we obtain $h \in C(\mathbb{R},\mathbb{R}^2)$ and the limits $\lim_{\xi\to\pm\infty} h(\xi)$ exist. Then Lemma D.6 implies

$$\lim_{x\to\pm\infty} v_\star'(x) = 0 = \lim_{x\to\pm\infty} h(x).$$

Moreover, this yields

$$\lim_{x\to\pm\infty} v_\star''(x) = 0.$$

Furthermore,

$$(S_\omega + g(|v_\infty|^2))v_\infty = \lim_{x\to\infty} \big(S_\omega v_\star(x) + f(v_\star(x))\big) = -\lim_{x\to\infty} h(x) = 0.$$

Since $v_\infty \neq 0$ it follows

$$0 \in \sigma(S_\omega + g(|v_\infty|^2)) = \{g_1(|v_\infty|^2) \pm i(\omega + g_2(|v_\infty|^2)\}.$$

Hence, $g(|v_\infty|^2) = -S_\omega$. ◻

Taking the original complex-valued equation (0.1) into account, we observe that the possible asymptotic rest-states v_∞ of a TOF are given by the roots of the real part of the nonlinearity G in the sense that

$$\operatorname{Re} G(|r_\infty|^2) = 0, \quad r_\infty = v_{\infty,1} + iv_{\infty,2}.$$

Moreover, in this case the frequency of the TOF is determined by the imaginary part of G via

$$\operatorname{Im} G(|r_\infty|^2) = -\omega.$$

Remark 1.7. *Let $u_\star$ be a traveling oscillating front of (0.4) with $\omega, c \in \mathbb{R}$ and profile $v_\star$. Then the corresponding solution $U_\star = u_{\star,1} + iu_{\star,2}$ of the complex system (0.1) is of the form*

$$U_\star(x,t) = e^{-i\omega t} V_\star(x - ct), \quad t \geq 0,\, x \in \mathbb{R}.$$

In particular, the profile $V_\star$ has the limiting property

$$\lim_{x\to-\infty} V_\star(x) = 0, \quad \lim_{x\to\infty} V_\star(x) = r_\infty \in \mathbb{C}$$

with $r_\infty = v_{\infty,1} + iv_{\infty,2} \neq 0$. Furthermore, the profile $V_\star$ is a solution of the ODE

$$0 = \alpha V_\star'' + cV_\star' + i\omega V_\star + G(|V_\star|)V_\star.$$

We conclude this section by recalling the template function $\hat{v} \in C_b^\infty(\mathbb{R},\mathbb{R})$ from (0.19) given by $\hat{v}(x) = \frac{1}{2}\tanh(x) + \frac{1}{2}$, $x \in \mathbb{R}$ and note some basic observation concerning $\hat{v}$. Clearly, $\hat{v}(x) \to 0$ as $x \to -\infty$ and $\hat{v}(x) \to 1$ as $x \to \infty$. In particular, the convergence is exponentially fast with rate $0 < \mu < 2$, i.e. we have

$$|\hat{v}(x)| \le e^{2x}, \quad x \le 0, \qquad |\hat{v}(x) - 1| \le e^{-2x}, \quad x \ge 0. \tag{1.9}$$

In addition, for the first and second derivative of $\hat{v}$ we have

$$|\hat{v}_x(x)| \le 2e^{-2|x|}, \quad |\hat{v}_{xx}(x)| \le 4e^{-2|x|}, \quad x \in \mathbb{R}. \tag{1.10}$$

Throughout this thesis we use several notations for the derivative as v_x, v', ∂v. However, the notation will always be clear by the context.

1.3 Assumptions and main results

The thesis deals with the investigation of the stability behavior of traveling oscillating fronts according to Definition 1.5. In this section we state the main results of the thesis. In order to do so, we first state our assumptions on the system and the TOF that guarantees nonlinear stability. The following first assumption relate to the equation (0.4) with (0.5).

Assumption 1. *The equation* (0.4) *with* (0.5) *satisfies*

$$\alpha_1 > 0, \quad g \in C^3(\mathbb{R}, \mathbb{R}^{2,2}), \quad g_1(0) < 0. \tag{A1}$$

The first condition in (A1) is a standard well-posedness assumption for evolution equations of parabolic type, see [32], [45]. The second condition guarantees smoothness of the nonlinearity f in (0.4), i.e. $f \in C^3$. The last condition in (A1) roughly speaking implies the trivial solution of (0.4) to be stable under small perturbations. Since the profile of a traveling oscillating wave tends to zero as $x \to -\infty$ this will be crucial for the stability of the TOF. More precisely, the condition $g_1(0) < 0$ guarantees that $\sigma_{\mathrm{disp}}^-(L)$ from (0.14) is included in the left half-plane, see the red curves in Figure 0.3. As a next step we assume the existence of a TOF in (0.4) which was discussed formally in a larger context by W. van Saarloos et al. in [62] in case of Ginzburg-Landau type equations. A formal discussion of the existence of TOFs in evolution equations of the form (0.1) is done in Chapter 2. However, a rigorous proof on the existence of TOFs is, to our knowledge, unknown in the literature.

Assumption 2. *There is a traveling oscillating front solution $u_\star$ of* (0.4) *with profile $v_\star \in C_b^2(\mathbb{R},\mathbb{R}^2)$, speed $c > 0$, frequency $\omega \in \mathbb{R}$ and asymptotic rest-state $v_\infty = (|v_\infty|, 0)^\top \in \mathbb{R}^2$ which satisfies*

$$g_1'(|v_\infty|^2) < 0. \tag{A2a}$$

To assume $v_\infty = (|v_\infty|, 0)^\top$ is without any loss of generality. The reason is that the profile of the TOF is not unique, since the whole group orbit $\mathcal{O}(v_\star) = \{R_\theta v_\star(\cdot - \tau) : \theta \in S^1,\ \tau \in \mathbb{R}\}$ consists of profiles of the same TOF $u_\star$. This is caused by the equivariance of the equation (0.4). Thus, we can choose the representative of the group orbit which satisfies $v_\star \to v_\infty = (|v_\infty|, 0)^\top$ as $x \to \infty$. Further, the conditions $c > 0$ and $g_1'(|v_\infty|^2) < 0$ are crucial for the stability of TOFs. In particular, $g_1'(|v_\infty|) < 0$ implies that the periodic orbit of the ODE $\zeta' = f(\zeta)$, given by $\zeta_\star(t) = R_{-\omega t} v_\infty$ and describing the evolution of the TOF at $+\infty$, is an asymptotically stable periodic orbit of the ODE.

Remark 1.8. *$\zeta_\star$ is a τ-periodic orbit of the autonomous ODE $\zeta' = f(\zeta)$ with $\tau = \frac{2\pi}{|\omega|}$. Its stability behavior is determined by the linearization given by $\zeta' = Df(\zeta_\star)\zeta$. See the classical Floquet theory, for instance, from [6]. Clearly, the first Floquet multiplier is given by $\mu_1 = 1$ and for the second we have*

$$\mu_2 = \mu_1\mu_2 = e^{\int_0^\tau \operatorname{tr}(Df(\zeta_\star(s)))ds} = e^{2\tau g_1'(|v_\infty|^2)|v_\infty|^2}$$

since (0.13) *and Lemma 1.6 imply*

$$\operatorname{tr}(Df(\zeta_\star(s))) = g_1(|v_\infty|^2) + 2g_1'(|v_\infty|^2)|v_\infty|^2 = 2g_1'(|v_\infty|^2)|v_\infty|^2.$$

Therefore, (A2a) *shows for the second Floquet multiplier $|\mu_2| < 1$ and thus $\zeta_\star$ is an asymptotically stable periodic orbit.*

1.3.1 The exponentially weighted case

The first main result of the thesis deals with the nonlinear stability with asymptotic phase of TOFs in exponentially weighted spaces. The proof of the result is done in Chapter 3. There we choose the weight function η as a weight function of exponential growth rate $\mu > 0$, cf. (0.24), i.e. we set

$$\eta(x) = e^{\mu\sqrt{x^2+1}}, \quad \mu > 0.$$

Recall the weighted Lebesgue and Sobolev spaces L^2_η, H^ℓ_η from (0.16), (0.17) as well as the spaces X_η, X^ℓ_η, Y_η from (0.20), (0.21) and let $\mathbf{v}_\star = (v_\star, v_\infty)^\top$ be given by the profile of the TOF from Assumption 2. We consider the Cauchy problem from (0.22) associated with the nonlinear operator $\mathcal{F}$ from (0.23) with perturbed initial conditions, i.e.

$$\mathbf{u}_t = \mathcal{F}(\mathbf{u}), \quad t > 0, \quad \mathbf{u}(0) = \mathbf{v}_\star + \mathbf{u}_0 \in X_\eta.$$

Definition 1.9. A function $\mathbf{u} : [0, t_\infty) \to X_\eta$ is called a classical solution of the Cauchy problem (0.22) on $[0, t_\infty)$ if

i) $\mathbf{u} \in C((0, t_\infty), Y_\eta) \cap C^1([0, t_\infty), X_\eta)$,

ii) $\mathbf{u}_t(t) = \mathcal{F}(\mathbf{u}(t))$ in X_η for all $t \in [0, t_\infty)$,

iii) $\mathbf{u}(0) = \mathbf{v}_\star + \mathbf{u}_0$.

In the case $t_\infty < \infty$ we also call $\mathbf{u}$ a local classical solution, whereas in the case $t_\infty = \infty$ we also call $\mathbf{u}$ a global classical solution.
We will show in Theorem 2.6 that $\mathbf{v}_\star$ belongs to Y_η as long as μ is sufficiently small. It follows immediately from (0.10) and Lemma 1.6 that $\mathbf{v}_\star$ is a stationary solution of (0.22), i.e.

$$\mathcal{F}(\mathbf{v}_\star) = 0.$$

Now let us consider the group $\mathcal{G} = S^1 \times \mathbb{R}$ with the metric on $\mathcal{G}$ given by

$$d_G(\gamma_1, \gamma_2) = |\gamma_1 - \gamma_2|_G, \quad |\gamma|_G := \min_{k\in\mathbb{Z}} |\theta - 2\pi k| + |\tau|, \quad \gamma = (\theta, \tau). \tag{1.11}$$

We describe rotation and translation on the space X_η by the group action $a(\gamma)$, $\gamma \in \mathcal{G}$ from (0.25). We will prove in Lemma 3.8 that $\mathcal{F}$ is equivariant under the group action $a(\gamma)$, $\gamma \in \mathcal{G}$. Then $\mathbf{v}_\star$ defines a whole continuum of stationary solutions given by the group orbit $\mathcal{O}(\mathbf{v}_\star) = \{a(\gamma)\mathbf{v}_\star : \gamma \in \mathcal{G}\}$, i.e.

$$a(\gamma)\mathcal{F}(\mathbf{v}_\star) = \mathcal{F}(a(\gamma)\mathbf{v}_\star) = 0 \quad \forall \gamma \in \mathcal{G}.$$

To prove nonlinear stability we have to determine the spectrum of the linearized operator from (0.26) reading as

$$\mathcal{L} : Y_\eta \subset X_\eta \to X_\eta, \quad \begin{pmatrix} v \\ \rho \end{pmatrix} \mapsto \mathcal{L}\begin{pmatrix} v \\ \rho \end{pmatrix} = \begin{pmatrix} Av_{xx} + cv_x + S_\omega v + Df(v_\star)v \\ S_\omega \rho + Df(v_\infty)\rho \end{pmatrix}.$$

There are several nonequivalent definitions of the spectrum of a closed operator on a Banach space, see [38], [32], [25]. We use the following definition from [25] using Fredholm index 0 of the operator.

Definition 1.10. Let $\mathcal{T} : X \to Y$ be a closed, densely defined, linear operator with domain $\mathcal{D}(\mathcal{T}) \subset X$. The set

$$\rho(\mathcal{T}) := \{s \in \mathbb{C} : sI - \mathcal{T} : \mathcal{D}(T) \to X \text{ is bijective}\}$$

is called the **resolvent set** of $\mathcal{T}$. Its complement $\sigma(\mathcal{T}) = \mathbb{C}\backslash\rho(\mathcal{T})$ is called the **spectrum** of $\mathcal{T}$ and is decomposed into the **point spectrum**

$$\sigma_{\text{pt}}(\mathcal{T}) := \{s \in \sigma(\mathcal{T}) : sI - \mathcal{T} \text{ is Fredholm of index } 0\}$$

and the **essential spectrum**

$$\sigma_{\text{ess}}(\mathcal{T}) := \sigma(\mathcal{T})\backslash\sigma_{\text{pt}}(\mathcal{T}).$$

For $s \in \rho(\mathcal{T})$ the operator $(sI - \mathcal{T})^{-1} \in L[X, \mathcal{D}(\mathcal{T})]$ is called the **resolvent** of $\mathcal{T}$ at s.

Now recall the dispersion set $\sigma_{\mathrm{disp},\mu}(\mathcal{L}) = \sigma^-_{\mathrm{disp},\mu}(\mathcal{L}) \cup \sigma^+_{\mathrm{disp},\mu}(\mathcal{L})$, which as we will show describes a major part of the spectrum of the linearized operator $\mathcal{L}$ on X_η, cf. Section 3.3. We show that the dispersion set can be represented explicitly depending on the system parameters and the growth rate $\mu \geq 0$ by the curves

$$\begin{aligned}\sigma^+_{\mathrm{disp},\mu}(\mathcal{L}) = \Big\{ s \in \mathbb{C} : s = & -\alpha_1\nu^2 + i(c - 2\alpha_1\mu)\nu + \mu^2\alpha_1 - c\mu + g_1'(|v_\infty|^2)|v_\infty|^2 \\ & \pm \Big[-\alpha_2^2\nu^4 - 4i\alpha_2^2\mu\nu^3 + (6\alpha_2^2\mu^2 + 2\alpha_2 g_2'(|v_\infty|^2)|v_\infty|^2)\nu^2 \\ & + 4i(\alpha_2^2\mu^3 + \mu\alpha_2 g_2'(|v_\infty|^2)|v_\infty|^2)\nu \\ & - \alpha_2^2\mu^4 - 2\alpha_2\mu^2 g_2'(|v_\infty|^2)|v_\infty|^2 + (g_1'(|v_\infty|^2)|v_\infty|^2)^2 \Big]^{\frac{1}{2}} \Big\}\end{aligned} \tag{1.12}$$

and

$$\begin{aligned}\sigma^-_{\mathrm{disp},\mu}(\mathcal{L}) = \Big\{ s \in \mathbb{C} : s = & -\alpha_1\nu^2 + i(c + 2\alpha_1\mu)\nu + \mu^2\alpha_1 + c\mu + g_1(0) \\ & \pm \Big[-\alpha_2^2\nu^4 + 4i\alpha_2^2\mu\nu^3 + (6\alpha_2^2\mu^2 + 2\alpha_2(g_2(0)+\omega))\nu^2 \\ & - 4i\alpha_2(\alpha_2\mu^3 + \mu(g_2(0)+\omega))\nu \\ & - \alpha_2^2\mu^4 - 2(g_2(0)+\omega)\alpha_2\mu^2 - (g_2(0)+\omega)^2 \Big]^{\frac{1}{2}} \Big\}.\end{aligned} \tag{1.13}$$

For the representation we used that (0.13) and Lemma 1.6 imply

$$Df(0) = \begin{pmatrix} g_1(0) & -g_2(0) \\ g_2(0) & g_1(0) \end{pmatrix}, \quad Df(v_\infty) = \begin{pmatrix} 2g_1'(|v_\infty|^2)|v_\infty|^2 & \omega \\ -\omega + 2g_2'(|v_\infty|^2)|v_\infty|^2 & 0 \end{pmatrix}. \tag{1.14}$$

The stability behavior of TOFs strongly depends on the location of the spectrum of the operator $\mathcal{L}$. In particular, we have to show spectral stability of TOFs which means that the whole spectrum of $\mathcal{L}$ on X_η is included in the strict left-half plane except for a zero eigenvalue. Since a major part of the essential spectrum is given by the dispersion set $\sigma_{\mathrm{disp},\mu}(\mathcal{L})$, we assume that there is an exponential growth rate $\mu_{\mathrm{ess}} > 0$ such that for all $0 < \mu < \mu_{\mathrm{ess}}$ the dispersion set is included in the strict left half-plane, cf. Figure 0.4.

Assumption 3 (Spectral condition)**.** *There is $\mu_{\mathrm{ess}} > 0$ such that for all $0 < \mu \leq \mu_{\mathrm{ess}}$ there exists $\beta_0 = \beta_0(\mu) > 0$ with*

$$\operatorname{Re} \sigma_{\mathrm{disp},\mu}(\mathcal{L}) \leq -\beta_0.$$

Using the explicit representations (1.12), (1.13) it is easy to verify Assumption 3 in concrete applications, see Section 4.3. Further, we note that the conditions $g_1(0) < 0$, $c > 0$ and $g_1'(|v_\infty|^2) < 0$ are necessary condition for Assumption 3 to be satisfied. This

can be immediately seen by taking $\nu = 0$ in (1.12), (1.13) describing the vertexes of the dispersion curves.
The last assumption states that there are no further eigenvalues of $\mathcal{L}$ laying to the right of some vertical line with negative real part, except for the zero eigenvalue. Moreover, we assume the algebraic multiplicity of the zero eigenvalue to be at most 2.

Assumption 4 (Eigenvalue Condition)**.** *There is $\gamma > 0$ such that for all $s \in \sigma_{\mathrm{pt}}(\mathcal{L})\backslash\{0\}$ it follows* $\operatorname{Re} s < -\gamma$.
Moreover, $s = 0$ is an eigenvalue of algebraic multiplicity at most 2, i.e.

$$\dim \bigcup_{n=1}^{\infty} \mathcal{N}(\mathcal{L}^n) \leq 2.$$

In contrast to the essential spectrum, the point spectrum does not change when using exponential weights, see [36, Sec. 3.1.1.2] and Section 4.3. The eigenvalue condition, Assumption 4, typically has to be verified numerically. This in done for concrete applications in Section 4.3. Another possibility is given by discussing the roots of the so called Evans function. For details on the Evans function we refer to [3], [36, Chap. 9].

Now we are in the position to formulate the first main result of the thesis. It states that TOFs are nonlinear stable with asymptotic phase.

Theorem 1.11. *Let Assumption 1-4 be satisfied. Then there exists $\varepsilon_0 > 0$ and constants $K, \tilde{\beta}, C_\infty > 0$ such that for all initial perturbations $\mathbf{u}_0 \in Y_\eta$ with $\|\mathbf{u}_0\|_{X^1_\eta} < \varepsilon_0$ equation (0.22) has a unique global solution*

$$\mathbf{u} \in C((0,\infty), Y_\eta) \cap C^1([0,\infty), X_\eta)$$

and there are $\gamma \in C^1([0,\infty), \mathcal{G})$ and $\mathbf{w} \in C((0,\infty), Y_\eta) \cap C^1([0,\infty), X_\eta)$ such that

$$\mathbf{u}(t) = a(\gamma(t))\mathbf{v}_\star + \mathbf{w}(t), \quad t \in [0,\infty). \tag{1.15}$$

Moreover, there is an asymptotic phase $\gamma_\infty = \gamma_\infty(\mathbf{u}_0) \in \mathcal{G}$ with

$$\|\mathbf{w}(t)\|_{X^1_\eta} + |\gamma(t) - \gamma_\infty|_{\mathcal{G}} \leq K e^{-\tilde{\beta} t} \|\mathbf{u}_0\|_{X^1_\eta}, \quad |\gamma_\infty|_{\mathcal{G}} \leq C_\infty \|\mathbf{u}_0\|_{X^1_\eta}. \tag{1.16}$$

Theorem 1.11 is a direct consequence of Theorem 3.29 and their proofs can be found at the end of Section 3.7. We see that Theorem 1.11 implies nonlinear stability with asymptotic phase of traveling oscillating fronts. In particular, the TOF as $\mathbf{u}_\star(t) = a(\omega t, ct)\mathbf{v}_\star$ is a relative equilibrium of the equation

$$\mathbf{u}_t = \begin{pmatrix} A u_{xx} + f(u) \\ f(\rho) \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u \\ \rho \end{pmatrix} \in X_\eta$$

which is nonlinearly stable with asymptotic phase w.r.t. the norms $\|\cdot\|_1 = \|\cdot\|_2 = \|\cdot\|_{X^1_\eta}$, cf. Definition 1.4. It is important to note that we can allow perturbation in Theorem 1.11 which do not decay to zero as $x \to \infty$. This is the benefit we gain by using the space X_η from (0.20) and is due to the stability of the periodic orbit at $+\infty$, cf. (0.18). This is in contrast to the usual results for traveling waves in parabolic PDEs, see [56], [15], where only perturbation in H^1 are allowed.

1.3.2 The polynomially weighted case

The second main result of the thesis states that TOFs are nonlinear stable with asymptotic phase in polynomially weighted spaces. The result is proven in Chapter 5. There we set

$$\eta(x) = (x^2+1)^{\frac{k}{2}}, \quad k \in \mathbb{N}$$

and use the spaces L^2_k and H^ℓ_k from (0.30) as well as the affine linear spaces M_k, M^ℓ_k from (0.31). Then M^ℓ_k can be seen as Banach manifolds modeled over the spaces H^ℓ_k. For this manifold we have a single global chart (M^ℓ_k, χ) with

$$\chi : M^\ell_k \to H^\ell_k, \quad u \mapsto u - \bar{v}. \tag{1.17}$$

Let $v_\star$ be the given TOF from Assumption 2 and consider the perturbed initial value problem on M_k from (0.11) reading as

$$u_t = Au_{xx} + cu_x + S_\omega u + f(u), \quad u(0) = v_\star + u_0.$$

Definition 1.12. A function $u : [0, t_\infty) \to M_k$ for some $k \in \mathbb{N}_0$ is called a classical solution of the initial value problem (0.11) if

i) $u \in C((0,t_\infty), M^2_k) \cap C^1([0,t_\infty), M_k)$,

ii) $u_t(t) = Au_{xx}(t) + cu_x(t) + S_\omega u(t) + f(u(t))$ in L^2_k for all $t \in [0, t_\infty)$,

iii) $u(0) = v_\star + u_0$.

In the case $t_\infty < \infty$ we also call u a local classical solution, whereas in the case $t_\infty = \infty$ we also call u a global classical solution.
As in the case of exponential weights, we have to consider the linearized operator L on L^2_k from (0.12) to prove nonlinear stability with asymptotic phase. The linearized operator is given by

$$L : H^2_k \to L^2_k, \quad u \mapsto Au_{xx} + cu_x + S_\omega u + Df(v_\star)u.$$

Again, the major part of its spectrum is given by the dispersion set $\sigma_{\text{disp}}(L) = \sigma^-_{\text{disp}}(L) \cup \sigma^+_{\text{disp}}(L)$ from (0.14). It can be expressed explicitly as

$$\sigma^+_{\text{disp}}(L) := \Big\{ s \in \mathbb{C} : \exists \nu \in \mathbb{R} \text{ s.t. } s = -\alpha_1\nu^2 + ic\nu + g_1'(|v_\infty|^2)|v_\infty|^2 \\ \pm \left(-\alpha_2^2\nu^4 + 2\alpha_2 g_2'(|v_\infty|^2)|v_\infty|^2\nu^2 + (g_1'(|v_\infty|^2)|v_\infty|^2)^2\right)^{\frac{1}{2}} \Big\}$$

and

$$\sigma^-_{\text{disp}}(L) := \Big\{ s \in \mathbb{C} : \exists \nu \in \mathbb{R} \text{ s.t. } s = -\alpha_1\nu^2 + ic\nu + g_1(0) \\ \pm \left(-\alpha_2^2\nu^4 + 2\alpha_2(g_2(0)+\omega)\nu^2 - (g_2(0)+\omega)^2\right)^{\frac{1}{2}} \Big\}.$$

In this case the dispersion set always touches the imaginary axis at the origin and we cannot expect it to be included in the strict left half-plane. However, to prove nonlinear stability we make the following assumption on the dispersion set which states that the origin is the only point where the imaginary axis is touched by the dispersion set. It can be verified numerically or even analytically by discussing the shape of the dispersion curves.

Assumption 5 (Spectral Condition)**.** *The dispersion set $\sigma_{\text{disp}}(L)$ from* (0.14) *satisfies*

$$\sigma_{\text{disp}}(L) \cap i\mathbb{R} = \{0\}.$$

Further, as in the exponential case we have to assume the following eigenvalue condition concerning the point spectrum of L.

Assumption 6 (Eigenvalue Condition)**.** *Let $L \in \mathcal{C}[L^2]$ from* (0.12)*. Then there is $\gamma > 0$ such that for all $s \in \sigma_{\text{pt}}(L)\backslash\{0\}$ it follows* $\operatorname{Re} s < -\gamma$*. Moreover, there holds*

$$\dim \bigcup_{n=1}^{\infty} \mathcal{N}(L^n) \le 1.$$

In Section 5.3.2 we derive delicate resolvent estimates of the linearized operator w.r.t. different polynomially weighted norms. In order to do so, we consider the piecewise constant coefficient operator L_∞ which is defined by

$$L_\infty : H^2_k \to L^2_k, \quad u \mapsto Au_{xx} + cu_x + C_\pm u, \quad C_\pm(x) = \begin{cases} S_\omega + Df(v_\infty), & x \ge 0, \\ S_\omega + Df(0), & x < 0 \end{cases} \tag{1.18}$$

and it has to satisfy the following non-degeneration assumption:

Assumption 7. *The piecewise constant coefficient operator L_∞ from* (1.18) *satisfies $\mathcal{N}(L_\infty) = \{0\}$.*

We just note that Assumption 7 generically must hold true and can be verified in application using results from Section 5.3.2. For a more detailed discussion we refer to Section 5.3.2. Finally, the last assumption requires the imaginary part of the diffusion coefficient to be sufficiently small. This also effects the geometric shape of the dispersion set at the origin.

Assumption 8. *The imaginary part α_2 of the diffusion coefficient satisfies*

$$\alpha_2 g_2'(|v_\infty|^2)|v_\infty|^2 + \alpha_1 g_1'(|v_\infty|^2)|v_\infty|^2 < 0.$$

Now we are in the position to formulate the second main result of the thesis concerning nonlinear stability of TOFs in polynomially weighted spaces.

Theorem 1.13. *Let Assumption 1, 2 and 5-8 be satisfied. Further, let $m \geq 5$, $k = 3m$. Then there exist $\varepsilon_0 > 0$ and constants $K, C_\infty > 0$ such that for all initial perturbations $u_0 \in H_k^2$ with $\|u_0\|_{H_{2k}^1} < \varepsilon_0$ equation* (0.11) *has a unique global solution*

$$u \in C((0,\infty), M_k^2) \cap C^1([0,\infty), M_k)$$

and there are $\tau \in C^1([0,\infty), \mathbb{R})$ and $w \in C((0,\infty), H_k^2) \cap C^1([0,\infty), L_k^2)$ such that

$$u(t) = v_\star(\cdot - \tau(t)) + w(t), \quad t \in [0,\infty).$$

Moreover, there is an asymptotic phase $\tau_\infty = \tau_\infty(u_0) \in \mathbb{R}$ with

$$\begin{aligned}
\|w(t)\|_{H_k^1} &\leq \frac{K}{(1+t)^{\frac{m-2}{2}}} \|u_0\|_{H_{2k}^1} \\
|\tau(t) - \tau_\infty| &\leq \frac{K}{(1+t)^{\frac{m-4}{2}}} \|u_0\|_{H_{2k}^1}, \quad |\tau_\infty| \leq C_\infty \|u_0\|_{H_{2k}^1}.
\end{aligned}$$

The proof of Theorem (1.13) in done at the end of Section 5.7 and is a consequence of Theorem 5.37. Theorem 1.13 implies nonlinear stability of TOFs with asymptotic phase w.r.t. the norms $\|\cdot\|_1 = \|\cdot\|_2 = \|\cdot\|_{H_1^1 5}$, see Definition 1.4.

Chapter 2

Existence and exponential decay

Before investigating the stability behavior of TOFs, we prove properties of those and discuss their existence in a formal way. In particular, the main goal of this chapter is to show that the convergence of the profile at infinity, see (1.8), must be exponentially fast. In order to do so, we use the approach from [62] and analyze solutions of the stationary co-moving equation, cf. (0.10), reading as

$$0 = Av_{xx} + cv_x + S_\omega v + g(|v|^2)v, \quad x \in \mathbb{R}$$

via a polar-coordinate ansatz. As we have seen in Chapter 1, solutions of the stationary co-moving equation (0.10) define profiles of traveling oscillating waves with speed $c \in \mathbb{R}$ and frequency $\omega \in \mathbb{R}$. If, in addition, the asymptotic properties (1.8) are satisfied, they define profiles of TOFs. We use the following strategy to prove exponentially fast convergence in (1.8). The ansatz shows that the profiles occur as connecting orbits between two hyperbolic fixed points in a first order ODE system. The hyperbolicity of the fixed points then implies, using the theory of exponential dichotomies by W. A. Coppel in [22], that the convergence in (1.8) is exponentially fast.

2.1 A dynamical systems approach

We follow the ideas in [62] and write formally the solution $v \in C_b^2(\mathbb{R}, \mathbb{R}^2)$ of (0.10) in polar coordinates with smooth amplitude and phase

$$v(x) = r(x) \begin{pmatrix} \cos\phi(x) \\ \sin\phi(x) \end{pmatrix}, \quad x \in \mathbb{R} \tag{2.1}$$

where $r \in C_b^2(\mathbb{R}, \mathbb{R}_+)$ and $\phi \in C_b^2(\mathbb{R}, \mathbb{R})$. Hence, r describes the amplitude of the wave solution whereas ϕ describes its phase in $\mathbb{R}^2$ or in the complex plane respectively. If we require v to satisfy the asymptotic behavior (1.8) we conclude that r and ϕ satisfy

$$\lim_{x\to\infty} r(x) = r_\infty, \quad \lim_{x\to\infty} \phi(x) = \phi_\infty$$

with $r_\infty = |v_\infty|$ and $\phi_\infty = \arg(v_\infty)$. For the limit at $-\infty$ we obtain

$$\lim_{x\to-\infty} r(x) = 0.$$

Note that ϕ does not have to decay to zero as $x \to -\infty$. Unfortunately, we have no control of the angle ϕ as x goes to $-\infty$. More precisely, for a general TOF with profile $v_\star$ we do not even know if the angle ϕ converges as x goes to $-\infty$. For that reason, we have to consider the properties of $v_\star$ at $-\infty$ in a different manner than the behavior at ∞ later on. In fact we will only use the polar coordinate ansatz from (2.1) on the positive half-line $\mathbb{R}_+$. On the negative half-line we use the standard first order reduction of (0.10).

However, in what follows we consider the polar coordinate ansatz (2.1). We take first and second derivatives in (2.1) of v w.r.t. x and obtain

$$v_x = R_\phi \begin{pmatrix} r' \\ r\phi' \end{pmatrix}, \quad v_{xx} = R_\phi \begin{pmatrix} r'' - r(\phi')^2 \\ 2r'\phi' + r'\phi'' \end{pmatrix}.$$

Multiply (0.8) by $A^{-1}R_{-\phi}$ and use that the matrices A, $g(|v|^2)$ and $R_{-\phi}$ commute to obtain

$$0 = R_{-\phi}v_{xx} + cA^{-1}R_{-\phi}v_x + A^{-1}S_\omega R_{-\phi}v + g(|v|^2)A^{-1}R_{-\phi}v. \tag{2.2}$$

Here A^{-1} is given by

$$A^{-1} = \begin{pmatrix} \tilde{\alpha}_1 & \tilde{\alpha}_2 \\ -\tilde{\alpha}_2 & \tilde{\alpha}_1 \end{pmatrix} \quad \text{with} \quad \tilde{\alpha}_i = \frac{\alpha_i}{|\alpha|} \quad \text{for} \quad i = 1, 2.$$

A straightforward computation leads to

$$R_{-\phi}v_{xx} = \begin{pmatrix} r'' - r(\phi')^2 \\ 2r'\phi' + r'\phi'' \end{pmatrix}, \quad cA^{-1}R_{-\phi}v_x = c\begin{pmatrix} \tilde{\alpha}_1 r' + \tilde{\alpha}_2 r\phi' \\ -\tilde{\alpha}_2 r' + \tilde{\alpha}_1 r\phi' \end{pmatrix}$$

as well as

$$A^{-1}S_\omega R_{-\phi}v = \begin{pmatrix} \tilde{\alpha}_2 \omega r \\ \tilde{\alpha}_1 \omega r \end{pmatrix} \quad \text{and} \quad g(|v|^2)A^{-1}R_{-\phi}v = \begin{pmatrix} \tilde{\alpha}_1 g_1(|r|^2)r + \tilde{\alpha}_2 g_2(|r|^2)r \\ \tilde{\alpha}_1 g_2(|r|^2)r - \tilde{\alpha}_2 g_1(|r|^2)r \end{pmatrix}.$$

Plugging this into (2.2) yields

$$0 = \begin{pmatrix} r'' - r(\phi')^2 + c\tilde{\alpha}_1 r' + c\tilde{\alpha}_2 r\phi' + \tilde{\alpha}_2\omega r + \tilde{\alpha}_1 g_1(|r|^2)r + \tilde{\alpha}_2 g_2(|r|^2)r \\ 2r'\phi' + r\phi'' - c\tilde{\alpha}_2 r' + c\tilde{\alpha}_1 r\phi' + \tilde{\alpha}_1\omega r + \tilde{\alpha}_1 g_2(|r|^2)r - \tilde{\alpha}_2 g_1(|r|^2)r \end{pmatrix}.$$

Assuming $r(x) \neq 0$ for all $x \in \mathbb{R}$ we introduce, according to [62], the new variables

$$q(x) = \phi'(x), \quad \kappa(x) = \frac{r'(x)}{r(x)}. \tag{2.3}$$

Then, using $\kappa' = \frac{r''}{r} - \kappa^2$, we finally obtain the 3-dimensional ODE system

$$\begin{pmatrix} r \\ \kappa \\ q \end{pmatrix}' = \begin{pmatrix} r\kappa \\ q^2 - \kappa^2 - \tilde{\alpha}_1(c\kappa + g_1(|r|^2)) - \tilde{\alpha}_2(cq + \omega + g_2(|r|^2)) \\ -2\kappa q - \tilde{\alpha}_1(cq + \omega + g_2(|r|^2)) + \tilde{\alpha}_2(c\kappa + g_1(|r|^2)) \end{pmatrix} =: \Gamma(r, \kappa, q). \tag{2.4}$$

Note that Γ can be written as

$$\Gamma(r, \kappa, q) = \begin{pmatrix} r\kappa \\ \begin{pmatrix} q^2 - \kappa^2 \\ -2\kappa q \end{pmatrix} - A^{-1} \begin{pmatrix} c\kappa + g_1(|r|^2) \\ cq + \omega + g_2(|r|^2) \end{pmatrix} \end{pmatrix}.$$

Lemma 2.1. *Let $(r, q, \kappa) \in C^1(\mathbb{R}, \mathbb{R}^3)$ be a solution of (2.4) for some $c, \omega \in \mathbb{R}$. Then there is a family of solutions $v_{\phi_0} \in C^2(\mathbb{R}, \mathbb{R}^2)$, $\phi_0 \in \mathbb{R}$ of (0.8) given by*

$$v_{\phi_0}(x) = r(x) \begin{pmatrix} \cos\phi(x) \\ \sin\phi(x) \end{pmatrix}, \quad \phi(x) = \int_0^x q(s)ds + \phi_0.$$

Proof. Since $q, \kappa \in C^1(\mathbb{R}, \mathbb{R})$ we conclude $r, \phi \in C^2(\mathbb{R}, \mathbb{R})$. Hence $v_{\phi_0} \in C^2(\mathbb{R}, \mathbb{R}^2)$ and the previous calculation shows that v_{ϕ_0} solves (0.10). $\square$

Thus, we have shown that every solution (r, q, κ) of (2.4) defines a solution of (0.8) and therefore the profile of a traveling oscillating wave. Since we are interested in TOFs we now take the asymptotic behavior (1.8) into account. Therefore, we now look for solutions $v \in C_b^2(\mathbb{R}, \mathbb{R}^2)$ of (0.8) with (1.8). Since, $v \equiv 0$ and $v \equiv v_\infty$ are constant solutions to (0.10) it is natural to look for equilibria of (2.4), i.e. let $(\bar{r}, \bar{\kappa}, \bar{q}) \in \mathbb{R}^3$ such that

$$\Gamma(\bar{r}, \bar{\kappa}, \bar{q}) = 0.$$

Then the first equation of (2.4) implies either $\bar{r} = 0$ or $\bar{\kappa} = 0$. Therefore, we distinguish between the two cases. Depending on the fixed point there may be different types of solutions to the equation (0.10).

Corollary 2.2. *Let $(\bar{r}, \bar{\kappa}, \bar{q}) \in \mathbb{R}^3$ be an equilibrium of (2.4).*

i) If $\bar{r} = 0$, then the corresponding family of solutions $v_{\phi_0} \in C_b^2(\mathbb{R}, \mathbb{R}^2)$, $\phi_0 \in \mathbb{R}$ of (0.10) from Lemma 2.1 is given by

$$v_{\phi_0}(x) = 0, \quad x \in \mathbb{R}.$$

ii) If $\bar{\kappa} = 0$, then the corresponding family of solutions $v_{\phi_0} \in C_b^2(\mathbb{R}, \mathbb{R}^2)$, $\phi_0 \in \mathbb{R}$ of (0.10) from Lemma 2.1 is given by

$$v(x) = \bar{r} \begin{pmatrix} \cos(\bar{q}x + \phi_0) \\ \sin(\bar{q}x + \phi_0) \end{pmatrix}, \quad x \in \mathbb{R}.$$

iii) If $\bar\kappa = \bar q = 0$, then the corresponding family of solutions $v_{\phi_0} \in C_b^2(\mathbb{R}, \mathbb{R}^2)$, $\phi_0 \in \mathbb{R}$ of (0.10) *from Lemma 2.1 is given by*

$$v_{\phi_0}(x) = \bar r \begin{pmatrix} \cos(\phi_0) \\ \sin(\phi_0) \end{pmatrix}, \quad x \in \mathbb{R}.$$

iv) Let $(0, \bar\kappa, \bar q)$ and $(\bar r, 0, 0)$ be equilibria of (2.4) *and let $(r, \kappa, q) \in C^1(\mathbb{R}, \mathbb{R}^3)$ be a heteroclinic orbit from $(0, \bar\kappa, \bar q)$ to $(\bar r, 0, 0)$, i.e. (r, κ, q) solves* (2.4) *and*

$$\lim_{x\to-\infty} \begin{pmatrix} r(x) \\ \kappa(x) \\ q(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \bar\kappa \\ \bar q \end{pmatrix}, \quad \lim_{x\to\infty} \begin{pmatrix} r(x) \\ \kappa(x) \\ q(x) \end{pmatrix} = \begin{pmatrix} \bar r \\ 0 \\ 0 \end{pmatrix}.$$

If $q \in L^1([0,\infty), \mathbb{R})$, then $v_{\phi_0} \in C_b^2(\mathbb{R}, \mathbb{R})$, $\phi_0 \in \mathbb{R}$ given by Lemma 2.1 is a profile of a traveling oscillating front of (0.4) *with asymptotic rest-state*

$$v_\infty = \bar r \begin{pmatrix} \cos\phi_\infty \\ \sin\phi_\infty \end{pmatrix}, \quad \phi_\infty = \int_0^\infty q(s)ds + \phi_0.$$

Proof. i), ii) and iii) follow immediately by Lemma 2.1. For iv) we have by Lemma 2.1 that v_{ϕ_0} solves (0.10). Now $q \in L^1([0,\infty), \mathbb{R})$ guarantees that ϕ_∞ exists. Then we obtain

$$\lim_{x\to-\infty} v_{\phi_0}(x) = 0, \quad \lim_{x\to\infty} v_{\phi_0}(x) = v_\infty.$$

Hence $v_{\phi_0}(x)$ is a profile of a traveling oscillating front. □

Corollary 2.2 shows that every connecting orbit between two equilibria $(\bar r, 0, 0)$ and $(0, \bar\kappa, \bar q)$ defines a profile of a TOF, i.e. a solution of (0.8) with (1.8). Conversely, we expect that every profile of a TOF defines such a connecting orbit as well. To see that, assume

$$v_\star(x) = r(x) \begin{pmatrix} \cos\phi(x) \\ \sin\phi(x) \end{pmatrix} \quad \forall x \in \mathbb{R}. \tag{2.5}$$

Then by (1.8) we have

$$r(x) \begin{pmatrix} \cos\phi(x) \\ \sin\phi(x) \end{pmatrix} = v_\star(x) \to 0, \quad x \to -\infty.$$

Thus $r(x) \to 0$ as $x \to -\infty$. Further, we obtain

$$r(x) \begin{pmatrix} \cos\phi(x) \\ \sin\phi(x) \end{pmatrix} = v_\star(x) \to r_\infty \begin{pmatrix} \cos\phi_\infty \\ \sin\phi_\infty \end{pmatrix}, \quad x \to \infty. \tag{2.6}$$

This shows $r(x) \to r_\infty$ as $x \to \infty$. Now by Lemma 1.6 we have $v_\star'(x) \to 0$ as $x \to \infty$. Then we conclude with $v_\star = (v_{\star,1}, v_{\star,2})^\top$

$$r'(x) = \partial_x |v_\star(x)| = \frac{v_{\star,1}'^2(x) + v_{\star,2}'^2(x)}{|v_\star|(x)} \to 0, \quad x \to \infty.$$

This implies

$$\kappa(x) = \frac{r'(x)}{r(x)} \to 0, \quad x \to \infty. \tag{2.7}$$

Finally,

$$r'(x) \begin{pmatrix} \cos\phi(x) \\ \sin\phi(x) \end{pmatrix} + r(x)q(x) \begin{pmatrix} -\sin\phi(x) \\ \cos\phi(x) \end{pmatrix} = v_\star'(x) \to 0, \quad x \to \infty. \tag{2.8}$$

Hence, $q(x) \to 0$ as $x \to \infty$. Summarizing we have shown for the solution of (2.4) given by (r, κ, q) of the profile $v_\star$ that $r(x) \to 0$ as $x \to -\infty$ and

$$(r, \kappa, q) \to (r_\infty, 0, 0), \quad x \to \infty.$$

Assuming $q(x) \to \bar{q}$ and $\kappa(x) \to \bar{\kappa}$ as $x \to \infty$, we see that (r, κ, q) defines a connecting orbit in (2.4). However, the convergence for q, κ at $-\infty$ is only assumed and is an open question.
It turns out that the equilibria of the connecting orbit are hyperbolic. Therefore, the convergence towards the equilibria is in fact exponentially fast. This will be used in Section 2.2 to show that the convergence in (1.8) is exponentially fast as well.

Remark 2.3. *Recall the different phenomena occurring in* (0.4) *and, in particular, in* (QCGL) *from Figure 0.1 such as pulses, wave trains, periodic fronts, sources and sinks. Taking the system* (2.4) *into account, one shows that pulses are given by connecting orbits between equilibria in* (2.4) *with zero amplitude, i.e.* $\bar{r} = 0$. *The stability behavior of pulses was investigated for instance in [58]. Further, a connecting orbit in* (2.4) *of two equilibria* $(0, \bar{\kappa}, \bar{q})$ *to* $(\bar{r}, \bar{q}, 0)$ *with* $\bar{q} \neq 0$ *defines a spatially periodic front, cf. Figure 0.1. At last, a heteroclinc orbit between two equilibria* $(\bar{r}_{1,2}, \bar{q}_{1,2}, 0)$ *with* $\bar{q}_1 < 0 < \bar{q}_2$ *or* $\bar{q}_2 < 0 < \bar{q}_1$ *define sources and sinks. These are connecting orbits between wave trains and are also called Nozaki-Bekki holes, see [46]. The stability behavior of sources was investigated in [10].*

In the beginning of the section we used the formal polar coordinate ansatz (2.1) for the solution of (0.10) with smooth r and ϕ. But the inverse of the polar coordinate transformation may not be globally continuous in the phase ϕ. Nevertheless, since we are interested in the behavior as $x \to \infty$ it will be sufficient to have a transformation for $x \in J = [x_\star, \infty)$ for some $x_\star$ sufficiently large to obtain the system (2.4) on J.

Lemma 2.4. *Suppose $v_\star \in C_b^2(\mathbb{R},\mathbb{R}^2)$ to be the profile of a traveling oscillating front. Then there is $x_\star \in \mathbb{R}$ and functions $r \in C_b^2(J,\mathbb{R})$, $\phi \in C^2(J,\mathbb{R})$ with $J = [x_\star,\infty)$ such that for all $x \in J$ there hold*

$$v_\star(x) = r(x)\begin{pmatrix}\cos\phi(x)\\ \sin\phi(x)\end{pmatrix}.$$

Proof. Since $v_\star$ is a traveling oscillating front there is $v_\infty \in \mathbb{R}\backslash\{0\}$ with $v_\star(x) \to v_\infty$ as $x \to \infty$. Suppose w.l.o.g. $v_\infty = (r_\infty, 0)^\top$ for some $r_\infty \in \mathbb{R}$, $r_\infty > 0$. Otherwise consider the rotated profile $R_{-\phi_\infty} v_\star$ with $\phi_\infty \in [0,2\pi)$ such that $v_\infty = r_\infty(\cos\phi_\infty, \sin\phi_\infty)^\top$. Now there is $x_\star \in \mathbb{R}$ such that $v(x) \in \{(z_1,z_2) \in \mathbb{R}^2 : z_1 > 0,\ z_2 \in \mathbb{R}\}$ for all $x \in J = [x_\star,\infty)$. Set $r(x) = |v_\star(x)|$ and $\phi(x) = \arctan\frac{v_2(x)}{v_1(x)}$. Then $r \in C_b^2(J,\mathbb{R})$ and $\phi \in C^2(J,\mathbb{R})$ with (2.1). $\square$

2.2 Exponential decay

In this section we prove in Theorem 2.6 that profile of a traveling oscillating fronts as stationary solutions of (0.8) must converge exponentially fast to 0 and v_∞ as $|x| \to \infty$. For this purpose, we use the theory of hyperbolic equilibria since in the previous section we have seen that profiles of traveling oscillating fronts may occur as connecting orbits between equilibria in the dynamical system (2.4).
So let $(\bar{r},\bar{\kappa},\bar{q}) \in \mathbb{R}^3$ be a equilibrium of (2.4), i.e. $\Gamma(\bar{r},\bar{\kappa},\bar{q}) = 0$. Then the Jacobian at the equilibrium is given by

$$D\Gamma(\bar{r},\bar{\kappa},\bar{q}) = \begin{pmatrix} \bar{\kappa} & \bar{r} & 0\\ -2\tilde{\alpha}_1 g_1'(\bar{r}^2)\bar{r} - 2\tilde{\alpha}_2 g_2'(\bar{r}^2)\bar{r} & -2\bar{\kappa} - \tilde{\alpha}_1 c & 2\bar{q} - \tilde{\alpha}_2 c\\ -2\tilde{\alpha}_1 g_2'(\bar{r}^2)\bar{r} + 2\tilde{\alpha}_2 g_1'(\bar{r}^2)\bar{r} & -2\bar{q} + \tilde{\alpha}_2 c & -2\bar{\kappa} - \tilde{\alpha}_1 c\end{pmatrix}.$$

In fact, we use the theory of exponential dichotomies from [22]. We want to ensure that the system (2.4) has an exponential dichotomy on J. In order to do so, we look for hyperbolic equilibria of (2.4), i.e. the Jacobian at $(\bar{r},\bar{\kappa},\bar{q})$ has no eigenvalues on the imaginary axis. Taking the observations from the previous section into account, we are interested into connecting orbits between equilibria of (2.4). Let us consider the spectrum of the Jacobian at equilibria of (2.4) of the form $y_- = (0,\tilde{\kappa},\tilde{q})$ and $y_+ = (|v_\infty|,0,0)$ and their local stable and unstable manifolds $\mathcal{M}_{\mathfrak{s},\mathfrak{u}}(y_\pm)$. The reason is that a connecting orbit between y_+ and y_- occurs as an intersection of the stable manifold $\mathcal{M}_{\mathfrak{s}}(y_+)$ and the unstable $\mathcal{M}_{\mathfrak{u}}(y_-)$. We want to ensure that $y_\pm$ are hyperbolic.
The Jacobian at y_- reads as

$$D\Gamma(y_-) = \begin{pmatrix}\tilde{\kappa} & 0 & 0\\ 0 & -2\tilde{\kappa} - \tilde{\alpha}_1 c & 2\tilde{q} - \tilde{\alpha}_2 c\\ 0 & -2\tilde{q} + \tilde{\alpha}_2 c & -2\tilde{\kappa} - \tilde{\alpha}_1 c\end{pmatrix}.$$

Therefore the eigenvalues of $D\Gamma(y_-)$ are given by $\{\tilde{\kappa}, -\tilde{\alpha}_1 c - 2\tilde{\kappa} \pm i(2\tilde{q} - \tilde{\alpha}_2 c)\}$. Thus, if $\tilde{\kappa}, c > 0$ then y_- is a hyperbolic equilibrium with

$$\dim \mathcal{M}_{\mathfrak{u}}(y_-) = 1, \quad \dim \mathcal{M}_{\mathfrak{s}}(y_-) = 2. \tag{2.9}$$

For $y_+ = (|v_\infty|, 0, 0)$ we have

$$D\Gamma(y_+) = \begin{pmatrix} 0 & |v_\infty| & 0 \\ -2\tilde{\alpha}_1 g_1'(|v_\infty|^2)|v_\infty| - 2\tilde{\alpha}_2 g_2'(|v_\infty|^2)|v_\infty| & -\tilde{\alpha}_1 c & -\tilde{\alpha}_2 c \\ -2\tilde{\alpha}_1 g_2'(|v_\infty|^2)|v_\infty| + 2\tilde{\alpha}_2 g_1'(|v_\infty|^2)|v_\infty| & \tilde{\alpha}_2 c & -\tilde{\alpha}_1 c \end{pmatrix}.$$

and its characteristic polynomial is given by $\chi(s) = s^3 + a_1 s^2 + a_2 s + a_3$ with

$$\begin{aligned} a_1 &= 2\tilde{\alpha}_1 c, \\ a_2 &= (\tilde{\alpha}_1^2 + \tilde{\alpha}_2^2)c^2 + 2\tilde{\alpha}_1 g_1'(|v_\infty|^2)|v_\infty|^2 + 2\tilde{\alpha}_2 g_2'(|v_\infty|^2)|v_\infty|, \\ a_3 &= 2c(\tilde{\alpha}_1^2 + \tilde{\alpha}_2^2) g_1'(|v_\infty|^2)|v_\infty|^2. \end{aligned} \tag{2.10}$$

For instance, if $\tilde{\alpha}_2 = 0$ then the eigenvalues of the Jacobian are given by

$$\left\{ -\tilde{\alpha}_1 c, -\frac{1}{2}\tilde{\alpha}_1 c \pm \frac{1}{2}\sqrt{\tilde{\alpha}_1^2 c^2 - 8\tilde{\alpha}_1 g_1'(|v_\infty|^2)|v_\infty|^2} \right\}.$$

So if

$$c > 0, \quad g_1'(|v_\infty|^2) < 0$$

then y_+ is a hyperbolic equilibrium with

$$\dim \mathcal{M}_{\mathfrak{s}}(y_+) = 2, \quad \dim \mathcal{M}_{\mathfrak{u}}(y_+) = 1.$$

In fact, the same holds in the case $\alpha_2 \neq 0$:

Lemma 2.5. *Let Assumption 1 and 2 be satisfied. Then $y_+ := (|v_\infty|, 0, 0) \in \mathbb{R}^3$ is a hyperbolic equilibrium of* (2.4), *i.e.*

$$\sigma(D\Gamma(y_+)) \cap i\mathbb{R} = \emptyset.$$

Moreover, the stable and unstable manifolds have the dimensions

$$\dim \mathcal{M}_{\mathfrak{s}}(y_+) = 2, \quad \dim \mathcal{M}_{\mathfrak{u}}(y_+) = 1.$$

Proof. From Lemma 1.6 it follows that y_+ is an equilibrium of (2.4). Assume y_+ not to be hyperbolic. Then there is $\nu \in \mathbb{R}$ such that

$$\chi(i\nu) = -i\nu^3 - a_1\nu^2 + ia_2\nu + a_3 = 0$$

with a_1, a_2, a_3 from (2.10). Then $a_1\nu^2 - a_3 = 0$. This contradicts $a_1 > 0$ and $a_3 < 0$ which follows from Assumption 1 and 2. Thus y_+ is hyperbolic.
We are left with the task of determining the dimensions of the local stable and unstable manifold. For this purpose, we distinguish between two cases. If $a_1a_2 - a_3 \neq 0$ the Routh-Hurwitz criterion from Theorem D.7 states that the number p of zeros of the characteristic polynomial χ in the left half-plane equals to

$$p = 3 - V(1, a_1, a_3(a_1a_2 - a_3)) - V(1, a_1a_2 - a_3) = 2.$$

Here $V(c_1, \dots, c_n)$ is the function counting the sign changes in the sequence $c_1, \dots, c_n$. In the case $a_1a_2 - a_3 = 0$ we have

$$p = 1 + V(1, -a_1) = 2.$$

This shows

$$\dim \mathcal{M}_{\mathfrak{s}}(y_+) = 2, \quad \dim \mathcal{M}_{\mathfrak{u}}(y_+) = 1.$$

□

Before proving exponential decay of TOFs, we discuss briefly the existence of TOFs as connecting orbits between hyperbolic equilibria using counting arguments, see for instance from [11], [13],. We are not able to give a rigorous prove on the existence of TOFs, but the following argumentation shows that one expects TOFs to occur. A connecting orbit between the hyperbolic equilibria y_- and y_+ occurs when there is a nonempty intersection of the unstable manifold $\mathcal{M}_{\mathfrak{u}}(y_-)$ and the stable manifold $\mathcal{M}_{\mathfrak{s}}(y_+)$. Both are submanifolds of $\mathbb{R}^3$ with $\dim \mathcal{M}_{\mathfrak{u}}(y_-) = 1$ and $\dim \mathcal{M}_{\mathfrak{s}}(y_+) = 2$. In addition, both depend continuously on the one dimensional parameter $c \in \mathbb{R}$, cf. (2.4). Following [11], [13] on expects an intersection of the manifolds for an isolated $c \in \mathbb{R}$, see Figure 2.1. To see that, consider the dynamical system (2.4) depending on the parameter $c \in \mathbb{R}$, i.e.

$$y' = \Gamma(y; c), \quad y(x) \in \mathbb{R}^3, \quad c \in \mathbb{R} \tag{2.11}$$

where $y = (r, \kappa, q)$ and $\Gamma(y; c)$ is given by the right hand side of (2.4). Introducing the variable $z = (y, c)$, we can write (2.11) as

$$z' = \gamma(z), \quad \gamma(y, c) = (\Gamma(y; c), 0), \quad z(x) \in \mathbb{R}^{3+1}. \tag{2.12}$$

Further, we assume the existence of hyperbolic equilibria $y_+ = y_+(c) = (|v_\infty|, 0, 0) \in \mathbb{R}^3$ and $y_- = y_-(c) = (0, \tilde{\kappa}, \tilde{q}) \in \mathbb{R}^3$ for $c \in I \subset \mathbb{R}$ such that, according to Lemma 2.5 and (2.9), we have for all $c \in I$

$$\begin{aligned} m_{\mathfrak{s}}^+ &:= \dim \mathcal{M}_{\mathfrak{s}}(y_+) = 2, \quad m_{\mathfrak{u}}^+ := \dim \mathcal{M}_{\mathfrak{u}}(y_+) = 1, \\ m_{\mathfrak{s}}^- &:= \dim \mathcal{M}_{\mathfrak{s}}(y_-) = 1, \quad m_{\mathfrak{u}}^- := \dim \mathcal{M}_{\mathfrak{u}}(y_+) = 2. \end{aligned}$$

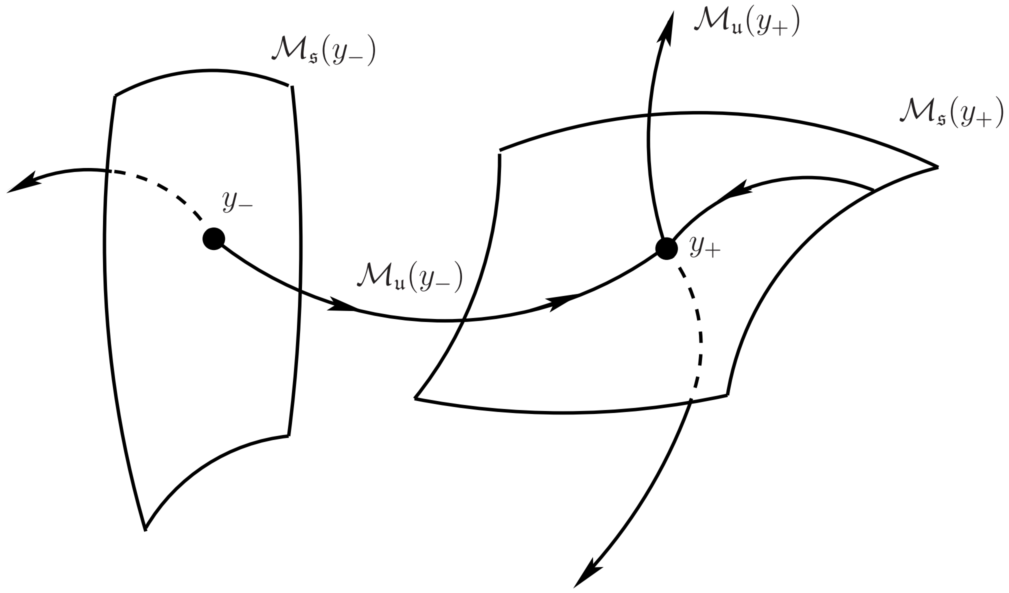


Figure 2.1: A connecting orbit as an intersection of $\mathcal{M}_{\mathfrak{u}}(y_-)$ and $\mathcal{M}_{\mathfrak{s}}(y_+)$ for a certain $c \in \mathbb{R}$.

Now there are invariant manifolds (center-stable/center-unstable) $M_{\mathfrak{u}}^-$ and $M_{\mathfrak{s}}^+$ of the system (2.12) given by

$$M_{\mathfrak{u}}^- = \bigcup_{c \in I} \Big(\mathcal{M}_{\mathfrak{u}}(y_-(c)) \times \{c\} \Big), \quad M_{\mathfrak{s}}^+ = \bigcup_{c \in I} \Big(\mathcal{M}_{\mathfrak{s}}(y_+(c)) \times \{c\} \Big)$$

with $\dim M_{\mathfrak{u}}^- = m_{\mathfrak{u}}^- + 1 = 2$ and $\dim M_{\mathfrak{s}}^+ = m_{\mathfrak{s}}^+ + 1 = 3$. In particular, we have $\dim M_{\mathfrak{s}}^+ + \dim M_{\mathfrak{u}}^- = 5 > 4 = \dim \mathbb{R}^{3+1}$. For that reason we might expect a nonempty intersection of $M_{\mathfrak{s}}^+$ and $M_{\mathfrak{u}}^-$. Further, we assume that $M_{\mathfrak{s}}^+$, $M_{\mathfrak{u}}^-$ are transversal to each other, i.e.

$$T_\xi M_{\mathfrak{s}}^+ + T_\xi M_{\mathfrak{u}}^- = \mathbb{R}^{3+1} \quad \forall \xi \in M_{\mathfrak{s}}^+ \cap M_{\mathfrak{u}}^-. \tag{2.13}$$

Counting dimensions in (2.13) we obtain $\dim (T_\xi M_{\mathfrak{s}}^+ \cap T_\xi M_{\mathfrak{u}}^-) = 1$, $\xi \in M_{\mathfrak{s}}^+ \cap M_{\mathfrak{u}}^-$. Pick $(y_0, c_\star) \in M_{\mathfrak{s}}^+ \cap M_{\mathfrak{u}}^-$ and let $z_\star = (y_\star, c_\star)$ be the solution of

$$z' = \gamma(z), \quad z(0) = (y_0, c_\star).$$

Then $z_\star(x) = (y_\star(x), c_\star) \in M_{\mathfrak{s}}^+ \cap M_{\mathfrak{u}}^-$ for all $x \in \mathbb{R}$ and

$$y_\star(x) \to \begin{cases} y_-(c_\star), & x \to -\infty \\ y_+(c_\star), & x \to \infty \end{cases}.$$

In particular, it follows

$$T_{z_\star(x)}M_{\mathfrak{s}}^+ \cap T_{z_\star(x)}M_{\mathfrak{u}}^- = \operatorname{span}\{z'(x)\} \quad \forall x \in \mathbb{R}$$

and $y_\star$ is an isolated connecting orbit of (2.4) from $y_-(c_\star)$ to $y_+(c_\star)$. From that we are able to construct a profile of TOF with speed $c_\star \in \mathbb{R}$ via Corollary 2.2. We expect that the assumptions in the previous argumentation holds true for at least a large parameter-set of (QCGL).

As a next step, we prove the main result of this chapter, which states that the convergence in (1.8) can only be exponentially fast. This will be an important property in the following chapters, especially for the proof of nonlinear stability. Since the polar coordinate transformation is only valid for the behavior of $v_\star$ at $+\infty$, cf. Lemma 2.4, we are only able to use the system (2.4) and the hyperbolicity of the equilibria from Lemma 2.5 on the positive half-line. On the negative half-line we can make use of a standard transformation to a first order system for $v_\star$ by $w = (v_\star, v_\star')$.

Theorem 2.6. *Let Assumption 1 and 2 be satisfied. Then $v_\star \in C_b^5(\mathbb{R},\mathbb{R}^2)$ and there are constants $K, \mu_\star > 0$ such that*

$$\begin{aligned} |v_\star(x) - v_\infty| + |v_\star'(x)| + |v_\star''(x)| + |v_\star'''(x)| &\le K e^{-\mu_\star x} \quad \forall\, x \ge 0, \\ |v_\star(x)| + |v_\star'(x)| + |v_\star''(x)| + |v_\star'''(x)| &\le K e^{\mu_\star x} \quad \forall\, x \le 0. \end{aligned}$$

Proof. Since $v_\star$ is a profile of a TOF it solves the stationary equation (0.10). Furthermore, by definition and Lemma 1.6, it has the limiting properties (1.8) and $v_\star'(x) \to 0$ as $|x| \to \infty$. In particular, since $f \in C^3(\mathbb{R},\mathbb{R}^2)$ we conclude $v_\star \in C_b^5(\mathbb{R},\mathbb{R}^2)$. Now we show first the estimate on the negative half-line. Set $w = (v_\star, v_\star')^\top$ then w is a solution to the first order system

$$w' = \mathcal{H}(w), \quad \mathcal{H}(w) := \begin{pmatrix} w_2 \\ -A^{-1}\left(cw_2 + S_\omega w_1 + f(w_1)\right) \end{pmatrix}. \tag{2.14}$$

Moreover, $\bar{w} = 0$ is an equilibrium of (2.14), i.e. $\mathcal{H}(0) = 0$, with the Jacobian

$$D\mathcal{H}(0) = \begin{pmatrix} 0 & I_2 \\ -A^{-1}(S_\omega + Df(0)) & -cA^{-1} \end{pmatrix}.$$

Since $Df(0) = g(0)$ we obtain using Assumption 1 for all $v \in \mathbb{C}^2$, $|v| = 1$

$$-\mathrm{Re}\,\left(v^H (S_\omega + Df(0))v\right) = -\mathrm{Re}\,\left(v^H \begin{pmatrix} g_1(0) & -\omega - g_2(0) \\ w + g_2(0) & g_1(0) \end{pmatrix} v\right) = -g_1(0) > 0.$$

Thus the spectral bound of $-(S_\omega + Df(0))$ is positive and, since $c > 0$, Lemma D.1 implies $\bar{w} = 0$ to be a hyperbolic equilibrium of (2.14) with stable and unstable dimensions equal to 2. We denote the eigenvalues of $D\mathcal{H}(0)$ by λ_i for $i = 1, 2, 3, 4$ such that $\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2 < 0 < \operatorname{Re}\lambda_3, \operatorname{Re}\lambda_4$. Now choose a simple connected curve including λ_1, λ_2 and excluding λ_3, λ_4 and let $P_{\mathfrak{s}}^-$ denote the corresponding Riesz projector from Appendix B (B.1). In addition, we define the stable and unstable subspaces

$$X_{\mathfrak{s}}^- := P_{\mathfrak{s}}^-(\mathbb{R}^4), \quad X_{\mathfrak{u}}^- := (I - P_{\mathfrak{s}}^-)(\mathbb{R}^4)$$

as well as $P_{\mathfrak{u}}^- = I - P_{\mathfrak{s}}^-$ the projector onto $X_{\mathfrak{u}}^-$, cf. Proposition B.1.
Since $\lim_{x\to-\infty} w(x) = 0$ there is $x_0 \leq 0$ such that Theorem B.5 implies that there are zero neighborhoods $V_{\mathfrak{s}} \subseteq X_{\mathfrak{s}}^-$, $V_{\mathfrak{u}} \subseteq X_{\mathfrak{u}}^-$, $V \subseteq \mathbb{R}^4$ and a unique $w_{\mathfrak{u}} \in V_{\mathfrak{u}}$ such that $w(x) \in V$, $x \leq x_0$ and $P_{\mathfrak{u}} w(x_0) = w_{\mathfrak{u}}$. Moreover, the boundary value problem

$$\begin{aligned} v' &= \mathcal{H}(v) \quad \text{on} \quad \mathbb{R}_- \\ P_{\mathfrak{u}}^- v(0) &= w_{\mathfrak{u}}, \quad v(x) \in V \quad \forall x \leq 0 \end{aligned} \tag{2.15}$$

has a unique solution $v \in C^4(\mathbb{R}_-, V)$, which satisfies for some $K_1, \mu_1 > 0$ the estimate

$$|v(x)| \leq K_1 e^{\mu_1 x} \quad \forall x \leq 0.$$

Since the solution v of (2.15) is unique we conclude $w(x) = v(x - x_0)$ for all $x \leq x_0$ and

$$|w(x)| = |v(x - x_\star)| \leq K_1 e^{\mu_1 (x - x_0)} \quad \forall x \leq x_0.$$

Now $w = (v_\star, v_\star')^\top \in C_b(\mathbb{R}, \mathbb{R}^4)$. Therefore, we find $K_2, K_3 > 0$ with

$$|v_\star(x)| + |v_\star'(x)| \leq K_2 e^{\mu_1 x} \quad \forall x \leq 0$$

and

$$\begin{aligned} |v_\star''(x)| &= |A^{-1}(c v_\star'(x) + S_\omega v_\star(x) + f(v_\star(x)))| \\ &\leq |A^{-1}| \big(|c||v_\star'(x)| + |\omega||v_\star(x)| + |f(v_\star(x)) - f(0)| \big) \\ &\leq |A^{-1}| \big(|c||v_\star'(x)| + |\omega||v_\star(x)| + L|v_\star(x)| \big) \leq K_3 e^{\mu_1 x} \quad \forall x \leq 0. \end{aligned} \tag{2.16}$$

By differentiating (0.10) we obtain, since $f \in C^3$,

$$|v_\star'''(x)| = |A^{-1}(c v_\star'(x) + S_\omega v_\star'(x) + Df(v_\star(x)) v_\star'(x))| \leq K_4 e^{\mu_1 x} \quad \forall x \leq 0 \tag{2.17}$$

and the estimate on the negative half-line is proven.
Next we show the estimate on the positive half-line. For this purpose take $x_\star \in \mathbb{R}$ from Lemma 2.4 and $r \in C_b^2(J, \mathbb{R})$, $\phi \in C^2(J, \mathbb{R})$, $J := [x_\star, \infty)$ such that $r(x) \neq 0$ and

$$v_\star(x) = r(x) \begin{pmatrix} \cos\phi(x) \\ \sin\phi(x) \end{pmatrix}, \quad x \in J.$$

Defining $q := \phi'$ and $\kappa = \frac{r'}{r}$ the previous calculations in Section 2.1 show that $(r, q, \kappa) \in C^1(\mathbb{R}, \mathbb{R})$ solves (2.4) on J. Now there are $r_\infty > 0$ and $\phi_\infty \in [0, 2\pi)$ such that $v_\infty = r_\infty(\cos\phi_\infty, \sin\phi_\infty)^\top$ holds. Then $r(x) \to r_\infty$ and $\phi(x) \to \phi_\infty$ as $x \to \infty$, see (2.6). Further, $\kappa(x) \to 0$ and $q(x) \to 0$ as $x \to \infty$, cf. (2.7), (2.8). Summarizing there hold

$$(r, \kappa, q) \to y_+ := (r_\infty, 0, 0), \quad x \to \infty.$$

From Lemma 2.5 it follows that y_+ is a hyperbolic equilibrium of (2.4) with stable dimension 2 and unstable dimension 1. Similarly, as in the case on the negative half-line, we denote the eigenvalues of $D\Gamma(y_+)$ by ν_i for $i = 1, 2, 3$ with $\operatorname{Re}\nu_1, \operatorname{Re}\nu_2 < 0 < \operatorname{Re}\nu_3$. Again choose a simple connected curve including ν_1, ν_2 and excluding ν_3 and let $P_{\mathfrak{s}}^+$ denote the corresponding Riesz projector from Appendix B (B.1). Further, define the stable and unstable subspaces

$$X_{\mathfrak{s}}^+ := P_{\mathfrak{s}}^+(\mathbb{R}^3), \quad X_{\mathfrak{u}}^+ := (I - P_{\mathfrak{s}}^+)(\mathbb{R}^3)$$

as well as $P_{\mathfrak{u}}^+ = I - P_{\mathfrak{s}}^+$ the projector onto $X_{\mathfrak{u}}^+$, cf. Proposition B.1. Since $(r, \kappa, q) \to y$ as $x \to \infty$ there is $\xi_0 \geq x_\star$ such that Theorem B.5 implies that there are neighborhoods $U_{\mathfrak{s}} \subseteq X_{\mathfrak{s}}^+$, $U_{\mathfrak{u}} \subseteq X_{\mathfrak{u}}^+$, $U \subseteq \mathbb{R}^3$ of y_+ and a unique $y_{\mathfrak{u}}$ such that $(r, \kappa, q)(x) \in U$, $x \geq \xi_0$ and $P_{\mathfrak{s}}(r, \kappa, q)(\xi_0) = y_{\mathfrak{u}}$. Moreover, the boundary value problem

$$\begin{aligned} u' &= \Gamma(u) \quad \text{on} \quad \mathbb{R}_+ \\ P_{\mathfrak{u}}^+ u(0) &= y_{\mathfrak{u}}, \quad u(x) \in U \quad \forall\, x \geq 0 \end{aligned} \tag{2.18}$$

has a unique solution $u \in C^4([0, \infty), U)$ which satisfies for some $C_1, \mu_2 > 0$

$$|u(x) - y_+| \leq C_1 e^{-\mu_2 x} \quad \forall\, x \geq 0.$$

Since u is the unique solution of (2.18) we conclude $(r, \kappa, q)(x) = u(x - \xi_0)$ for all $x \geq \xi_0$ and thus

$$|(r, \kappa, q)(x) - y_+| = |u(x - \xi_0) - y| \leq C_1 e^{-\mu_2(x - \xi_0)} \quad \forall\, x \geq \xi_0.$$

Therefore,

$$|r(x) - r_\infty|, |q(x)| \leq C_1 e^{-\mu_2(x - \xi_0)} \quad \forall\, x \geq \xi_0.$$

Then it follows

$$|\phi_\infty - \phi(x)| \leq \int_x^\infty |q(s)| ds \leq \frac{C_1}{\mu_2} e^{-\mu_2(x - \xi_0)} \quad \forall\, x \geq \xi_0.$$

Since $\Gamma \in C^3(\mathbb{R}^3, \mathbb{R}^3)$ and therefore locally Lipschitz continuous, we observe

$$\begin{aligned} |(r', \kappa', q')(x)| = |\Gamma(r(x), q(x), \kappa(x)) - \Gamma(y)| &\leq L|(r, \kappa, q)(x) - y| \\ &\leq LC_1 e^{-\mu_2(x - \xi_0)} \quad \forall\, x \geq \xi_0. \end{aligned}$$

Finally, we find $C_2 > 0$ with

$$\begin{aligned}|v_\star(x) - v_\infty| + |v_\star'(x)| &\leq \left| r(x) \begin{pmatrix} \cos\phi(x) \\ \sin\phi(x) \end{pmatrix} - r_\infty \begin{pmatrix} \cos\phi_\infty \\ \sin\phi_\infty \end{pmatrix} \right| + |r'(x)| + |r(x)q(x)| \\ &\leq |r(x) - r_\infty| + |r_\infty||\phi(x) - \phi_\infty| + |r'(x)| + \|r\|_{L^\infty}|q(x)| \\ &\leq C_2 e^{-\mu_2(x-\xi_0)} \quad \forall\, x \geq \xi_0.\end{aligned}$$

Since $v_\star \in C_b^5(\mathbb{R}, \mathbb{R}^2)$ we can choose $C_2 > 0$ such that

$$|v_\star(x) - v_\infty| + |v_\star'(x)| \leq C_2 e^{-\mu_2 x} \quad \forall\, x \geq 0.$$

The estimates for $v_\star''$, $v_\star'''$ follow as in (2.16), (2.17) using the stationary equation (0.10). This proves the claim with $K = \max_i\{K_i, C_i\}$, $\mu = \min_i\{\mu_i\}$. $\square$

We conclude this chapter by recalling $\hat{v}$ from (0.19). This function satisfies for $x \geq 0$

$$\big|\eta(x)(\hat{v}(x) - 1)\big| \leq e^{(\mu-2)x} \tag{2.19}$$

and on the negative half-line for $x \leq 0$

$$\big|\eta(x)\hat{v}(x)\big| \leq e^{(2-\mu)x}. \tag{2.20}$$

Thus, $\hat{v}$ converges exponentially fast to 1 as $x \to \infty$ and decays exponentially fast at $-\infty$ and the convergence is of rate 2. Moreover, for the derivatives we have

$$|\eta(x)\hat{v}_x(x)| \leq 2e^{(\mu-2)|x|} \tag{2.21}$$

as well as

$$|\eta(x)\hat{v}_{xx}(x)| \leq 4e^{(\mu-2)|x|}. \tag{2.22}$$

Throughout the thesis we write $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_- = (-\infty, 0)$.

Proposition 2.7. For $\mu < 2$ and $\hat{v}$ from (0.19) there holds:

$$\begin{gathered}\|\hat{v} - 1\|_{L^2_\eta(\mathbb{R}_+,\mathbb{R})} = \|\hat{v}\|_{L^2_\eta(\mathbb{R}_-,\mathbb{R})} = \frac{1}{\sqrt{2(2-\mu)}}, \\ \|\hat{v}_x\|_{L^2_\eta(\mathbb{R},\mathbb{R})} = \frac{2}{\sqrt{2-\mu}}, \quad \|\hat{v}_{xx}\|_{L^2_\eta(\mathbb{R},\mathbb{R})} = \frac{4}{\sqrt{2-\mu}}.\end{gathered}$$

Proof. The claim follows by integrating and using the estimates (2.19)-(2.22). $\square$

With Proposition 2.7 we conclude together with Theorem 2.6 that the profile $v_\star$ belongs to a shifted L^2_η-space as long as $\mu < 2$ and the shift is given by $v_\infty\hat{v}$. Since the weight function η depends on μ this is only valid for $\mu < \min(\mu_\star, 2)$.

Corollary 2.8. *Let Assumption 1 and 2 be satisfied and* $0 < \mu_\star$ *from Theorem 2.6. Then there is* $0 < \mu < \min(\mu_\star, 2)$ *such that*

$$v_\star \in L^2_\eta(\mathbb{R}, \mathbb{R}^2) + v_\infty \hat{v}, \quad v'_\star, v''_\star \in L^2_\eta(\mathbb{R}, \mathbb{R}^2).$$

Proof. The estimates from Theorem 2.6 imply for $\mu < \mu_\star$

$$\|v_\star\|_{L^2_\eta(\mathbb{R}_-)}, \|v_\star - v_\infty\|_{L^2_\eta(\mathbb{R}_+)}, \|v'_\star\|_{L^2_\eta(\mathbb{R})}, \|v''_\star\|_{L^2_\eta(\mathbb{R})} < \infty.$$

Proposition 2.7 yields with $\mu < 2$

$$\begin{aligned}
&\|v_\star - v_\infty \hat{v}\|_{L^2_\eta(\mathbb{R})} = \|v_\star - v_\infty \hat{v}\|_{L^2_\eta(\mathbb{R}_-)} + \|v_\star - v_\infty \hat{v}\|_{L^2_\eta(\mathbb{R}_+)} \\
&\leq \|v_\star\|_{L^2_\eta(\mathbb{R}_-)} + |v_\infty| \|\hat{v}\|_{L^2_\eta(\mathbb{R}_-)} + \|v_\star - v_\infty\|_{L^2_\eta(\mathbb{R}_+)} + |v_\infty| \|\hat{v} - 1\|_{L^2_\eta(\mathbb{R}_+)} < \infty.
\end{aligned}$$

□

Taking the spaces X^k_η and especially $Y_\eta = X^2_\eta$ from (0.21) into account Corollary 2.8 shows for $\mathbf{v}_\star = (v_\star, v_\infty)^\top$ that $\mathbf{v}_\star \in Y_\eta \subset X_\eta$ as long as $\mu < \min(\mu_\star, 2)$. Moreover, with the group action $a(\gamma)$, $\gamma \in \mathcal{G}$ from (0.25) we also obtain $a(\gamma)\mathbf{v} \in Y_\eta$. This implies that the whole group orbit of $\mathbf{v}_\star$ stays in Y_η, i.e. $\mathcal{O}(v_\star) = \{a(\gamma)v_\star, \gamma \in \mathcal{G}\} \subset Y_\eta \subset X_\eta$.

Chapter 3

Nonlinear stability in exponentially weighted spaces

In this chapter we prove the first main result of the thesis from Theorem 1.11 concerning the nonlinear stability with asymptotic phase of TOFs in exponentially weighted spaces. The strategy of the proof is the same as in the case of traveling waves, see [32], [36], or in the case of rotating solitons, cf. [17]. The proof falls naturally into the following steps:

- Spectral analysis of the linearized operator
- Semigroup estimates
- Decomposition of the nonlinear dynamics
- Estimates of the nonlinearities
- Gronwall estimate of the solution

In the first crucial step, the spectral analysis of the linearized operator, we see that the spectrum of the linearized operator $\mathcal{L}$ from (0.26) touches the imaginary axis at the origin when it is considered on unweighted spaces. In contrast to that, when using exponential weights, we prove that the spectrum is pushed off the imaginary axis. This will imply spectral stability of TOFs in exponentially weighted spaces and time decaying estimates for the analytic semigroup generated by $\mathcal{L}$. After a nonlinear coordinate transformation from [32] the second challenging step is to control the remaining nonlinearities in the spaces X_η from (0.20). In the end, a Gronwall estimate from [17] will lead to Theorem 1.11 and therefore to nonlinear stability of TOFs with asymptotic phase. Before carrying out the spectral analysis of the linearized operator we collect useful properties of exponentially weighted spaces and smoothness of translation and rotation as a group action. In addition, we prove equivariance of the nonlinear operator $\mathcal{F}$ from (0.23).

3.1 Exponentially weighted spaces

Recall $\eta = \eta_{\text{exp}}$ the weight function of exponential growth rate $\mu \geq 0$, cf. [63], [47], from (0.24) which is given by

$$\eta(x) = e^{\mu\sqrt{x^2+1}}. \tag{3.1}$$

Clearly, $\eta \in C^\infty(\mathbb{R}, \mathbb{R})$, $\eta(x) \sim e^{\mu|x|}$ as $|x| \to \infty$ and we have the estimates

$$e^{\mu|x|} \leq \eta(x) \leq C_\mu e^{\mu|x|}, \quad C_\mu = e^\mu.$$

Moreover, for all $x, y \in \mathbb{R}$ we obtain

$$\eta(x+y) = e^{\mu\sqrt{(x+y)^2+1}} \leq e^{\mu|y|}\eta(x). \tag{3.2}$$

The derivatives of η are given by

$$\eta_x(x) = \frac{\mu x}{\sqrt{x^2+1}}\eta(x), \quad |\eta_x(x)| \leq \mu\eta(x), \tag{3.3}$$

$$\eta_{xx}(x) = \left(-\frac{\mu x^2}{(x^2+1)^{\frac{3}{2}}} + \frac{\mu}{\sqrt{x^2+1}} + \frac{\mu^2 x^2}{x^2+1}\right)\eta(x), \quad |\eta_{xx}(x)| \leq \tilde{C}_\mu \eta(x), \tag{3.4}$$

where $\tilde{C}_\mu = 3\max(\mu, \mu^2)$. In addition, using the mean value theorem, we obtain

$$|\eta(x+y) - \eta(x)| \leq |y| \int_0^1 |\eta_x(x+sy)|ds \leq |y|\mu \sup_{\tau \leq |y|} |\eta(x+\tau)| \leq |y|\mu e^{\mu|y|}\eta(x). \tag{3.5}$$

Lets consider the weighted Lebesgue and Sobolev spaces L^2_η and H^k_η from (0.16) and (0.17). The multiplication operator associated with η is an isometry from L^2_η to L^2. On the smooth Sobolev space H^2_η to H^2 the isomorphism is still continuous. We note this in the following lemma.

Lemma 3.1. *Let $m_\eta u = \eta u$ define the multiplication operator associated with η. Then*

i) $m_\eta : L^2_\eta \to L^2$ is an isometric isomorphism,

ii) $m_\eta : H^k_\eta \to H^k$, $k \in \mathbb{N}_0$ is a continuous isomorphism.

Proof. **i)** By definition of $\|\cdot\|_{L^2_\eta}$ the multiplication operator $m_\eta : L^2_\eta \to L^2$ is an isometry and its inverse is given by $m_\eta^{-1} : L^2 \to L^2_\eta$, $u \mapsto \eta^{-1}u$ which is again an isometry. Hence $m_\eta : L^2_\eta \to L^2$ is an isometric isomorphism.
ii) Obviously $m_\eta : H^2_\eta \to H^2$ is linear. The continuity follows by induction over $k \in \mathbb{N}_0$.

The case $k = 0$ is clear by i). Suppose $u \in H_\eta^{k+1}$ and the claim holds true for $k \in \mathbb{N}_0$, i.e. there is $C_k > 0$ such that $\|\eta u\|_{H^k} \le C_k \|u\|_{H_\eta^k}$. Then, using (3.3), (3.4), we have

$$\begin{aligned}\|\partial^{k+1}(\eta u)\|_{L^2} &= \|\partial^k(\eta_x u + \eta u_x)\|_{L^2} \le \mu\|\partial^k(\eta u)\|_{L^2} + \|\partial^k(\eta u_x)\|_{L^2} \\ &\le \mu C_k \|u\|_{H_\eta^k} + C_k\|u_x\|_{H_\eta^k} \le (1+\mu)C_k\|u\|_{H_\eta^{k+1}}.\end{aligned}$$

Thus we find $C_{k+1} > 0$ such that

$$\|\eta u\|^2_{H^{k+1}} = \|\eta u\|^2_{H^k} + \|\partial^{k+1}(\eta u)\|^2_{L^2} \le C^2_{k+1}\|u\|^2_{H_\eta^{k+1}}.$$

Hence $m_\eta : H_\eta^k \to H^k$ is a continuous homomorphism and its inverse is again given by $m_\eta^{-1} : H^k \to H_\eta^k$, $u \mapsto \eta^{-1}u$. Now m_η^{-1} is continuous by the inverse operator theorem. This proves m_η to be a continuous isomorphism. □

For the resolvent estimates of the linearized operator $\mathcal{L}$ we need an integration by parts formula on L_η^2 which is slightly different from the standard integration by parts formula, since derivatives of the weight function η also occur. However, again by (3.3) and (3.4) we can control the derivatives of the weight function.

Lemma 3.2 (Integration by parts in L_η^2)**.** *Let $u, v \in H_\eta^1(\mathbb{R}, \mathbb{R}^n)$. Then there hold the following integration by parts formula:*

$$-(u, v_x)_{L_\eta^2} = (u_x, v)_{L_\eta^2} + 2(\eta_x\eta^{-1}u, v)_{L_\eta^2}.$$

Proof. The claim follows by the standard integration by parts formula and (3.3):

$$\begin{aligned}&-(u, v_x)_{L_\eta^2} = -\int_{\mathbb{R}} \eta^2(x)u(x)v_x(x)dx \\ &= \int_{\mathbb{R}} \eta^2(x)u_x(x)v(x)dx + \int_{\mathbb{R}} 2\eta(x)\eta_x(x)u(x)v(x)dx = (u_x, v)_{L_\eta^2} + 2(\eta_x\eta^{-1}u, v)_{L_\eta^2}.\end{aligned}$$

□

Let $C_0^\infty(\mathbb{R}, \mathbb{R}^2)$ be the set of all C^∞-functions with compact support. For $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^2)$ it is clear that $\eta\varphi$ has compact support. In addition, $\eta\varphi \in C^\infty$ and thus $\varphi \in H_\eta^1$, i.e. $C_0^\infty \subset H_\eta^1$. Since C_0^∞ is dense in H^1 w.r.t. $\|\cdot\|_{H^1}$ one expects that C_0^∞ is also dense in H_η^1 w.r.t. $\|\cdot\|_{H_\eta^1}$. We show this in the next Lemma.

Lemma 3.3. *The set $C_0^\infty(\mathbb{R}, \mathbb{R}^n)$ of infinite differentiable functions with compact support is a dense subset of $H_\eta^1(\mathbb{R}, \mathbb{R}^n)$ w.r.t. $\|\cdot\|_{H_\eta^1}$.*

Proof. Let $u \in H_\eta^1(\mathbb{R}, \mathbb{R}^n)$. Then by Lemma 3.1 $\eta u \in H^1(\mathbb{R}, \mathbb{R}^n)$ and since $C_0^\infty \subset H^1$ is dense there is $(\tilde\varphi_k)_{k\in\mathbb{N}}$ such that $\tilde\varphi_k \to \eta u$ in H^1 for $k \to \infty$. Set $\varphi_k = \eta^{-1}\tilde\varphi_k$. Then $(\varphi_k)_{k\in\mathbb{N}} \subset C_0^\infty$ and for some $C > 0$ there holds

$$\|\varphi_k - u\|_{H_\eta^1} \le C\|\eta\varphi_k - \eta u\|_{H^1} = C\|\tilde\varphi_k - \eta u\|_{H^1} \to 0, \quad k \to \infty.$$

□

We show in this section that the group action $a(\gamma)$, $\gamma \in \mathcal{G}$ from (0.25) is smooth. In particular, we prove that it is at least continuous differentiable. On the one hand the group action describes rotation of an element $\mathbf{v} \in X_\eta$. Since it is just a rotation in the image space it is as smooth as the rotation matrix R_θ which is arbitrary regular. On the other hand the group action describes spatial translation in the argument. This is more delicate since the smoothness of the shift strongly depends on the smoothness of the function. One knows that translation is continuous on L^2 and Lipschitz continuous for functions in H^1. Thus we have to guarantee that the smoothness is conserved under exponential weighting, at least locally.

Lemma 3.4. *i) Let $u \in L^2_\eta$ and $\tau \in \mathbb{R}$. Then*

$$\|u(\cdot + \tau)\|_{L^2_\eta} \leq e^{\mu|\tau|}\|u\|_{L^2_\eta}.$$

ii) Let $u \in H^1_\eta$ and $\tau \in \mathbb{R}$. Then

$$\|u(\cdot + \tau) - u\|_{L^2_\eta} \leq |\tau|e^{\mu|\tau|}\|u_x\|_{L^2_\eta}.$$

iii) Let $u \in L^2_\eta$. Then

$$\|u(\cdot + \tau) - u\|_{L^2_\eta} \to 0, \quad \tau \to 0.$$

Further, the estimate in ii) holds true if u is replaced by $\hat{v}$ from (0.19) *or $v_\star$ from Assumption 2.*

Proof. **i)** Use (3.2) and obtain

$$\begin{aligned}\|u(\cdot + \tau)\|^2_{L^2_\eta} &= \int_{\mathbb{R}} \eta^2(x)|u(x+\tau)|^2 dx = \int_{\mathbb{R}} \eta^2(x-\tau)|u(x)|^2 dx \\ &\leq \int_{\mathbb{R}} e^{2\mu|\tau|}\eta^2(x)|u(x)|^2 dx = e^{2\mu|\tau|}\|u\|^2_{L^2_\eta}.\end{aligned}$$

ii) Suppose $\varphi \in C_0^\infty$. Using Fubini's theorem, the mean value theorem and i) yields

$$\begin{aligned}\|\varphi(\cdot + \tau) - \varphi\|^2_{L^2_\eta} &= \int_{\mathbb{R}} \eta(x)^2|\varphi(x+\tau) - \varphi(x)|^2 dx \\ &\leq |\tau|^2 \int_{\mathbb{R}} \eta(x)^2 \int_0^1 |\varphi_x(x+\tau s)|^2 ds dx \\ &= |\tau|^2 \int_0^1 \|\varphi_x(\cdot + \tau s)\|^2_{L^2_\eta} ds \leq |\tau|^2 e^{2\mu|\tau|}\|\varphi_x\|^2_{L^2_\eta}.\end{aligned} \tag{3.6}$$

By Lemma 3.3, $C_0^\infty \subset H^1_\eta$ is dense and thus there exists a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset C_0^\infty$ such that $\varphi_k \to u$ in H^1_η as $k \to \infty$. Then using the assertion from i), we obtain

$$\begin{aligned}\|u(\cdot+\tau)-u\|_{L^2_\eta} &\le \|u(\cdot+\tau)-\varphi_k(\cdot+\tau)\|_{L^2_\eta} + \|u-\varphi_k\|_{L^2_\eta} + \|\varphi_k(\cdot+\tau)-\varphi_k\|_{L^2_\eta} \\ &\le \|u(\cdot+\tau)-\varphi_k(\cdot+\tau)\|_{L^2_\eta} + \|u-\varphi_k\|_{L^2_\eta} + |\tau|e^{\mu|\tau|}\|\varphi_{k,x}\|_{L^2_\eta} \\ &\le (e^{\mu|\tau|}+1)\|u-\varphi_k\|_{L^2_\eta} + |\tau|e^{\mu|\tau|}\|\varphi_{k,x}\|_{L^2_\eta}.\end{aligned}$$

As $k \to \infty$ we observe

$$\|u(\cdot+\tau)-u\|_{L^2_\eta} \le |\tau|e^{\mu|\tau|}\|u_x\|_{L^2_\eta}.$$

The estimate for $\hat{v}$ follows as in (3.6) since $\hat{v}(\cdot+\tau)-\hat{v} \in L^2_\eta$, $\hat{v} \in C^\infty$ and $\hat{v}_x \in L^2_\eta$. Similarly for $v_\star$.
iii) Suppose $u \in L^2_\eta$. Then, by continuity of the L^2-norm and (3.5),

$$\begin{aligned}\|u(\cdot+\tau)-u\|^2_{L^2_\eta} &= \int_{\mathbb{R}} |\eta(x)u(x+\tau)-\eta(x)u(x)|^2 dx \\ &\le 2\int_{\mathbb{R}} |(\eta(x)-\eta(x+\tau))u(x+\tau)|^2 dx + 2\int_{\mathbb{R}} |\eta u(x+\tau)-\eta u(x)|^2 dx \\ &\le 2\int_{\mathbb{R}} |\eta(x-\tau)-\eta(x)|^2|u(x)|^2 dx + 2\|\eta u(\cdot+\tau)-\eta u\|^2_{L^2} \\ &\le 2\left(|\tau|\mu e^{\mu|\tau|}\right)^2 \int_{\mathbb{R}} \eta^2(x)|u(x)|^2 dx + 2\|\eta u(\cdot+\tau)-\eta u\|^2_{L^2} \\ &= 2\left(|\tau|\mu e^{\mu|\tau|}\right)^2 \|\eta u\|^2_{L^2} + 2\|\eta u(\cdot+\tau)-\eta u\|^2_{L^2} \to 0, \quad \tau \to 0.\end{aligned}$$

□

In the next step we discuss the properties of the product spaces X^k_η, cf. (0.21), which are used for the proof of nonlinear stability. The norm $\|\cdot\|_{X_\eta}$ is induced by the inner product

$$\left(\begin{pmatrix} v \\ \rho \end{pmatrix}, \begin{pmatrix} w \\ \zeta \end{pmatrix}\right)_{X_\eta} := \rho^\top \zeta + (\eta(v-\rho\hat{v}), \eta(w-\zeta\hat{v}))_{L^2}.$$

Thus X_η is in fact a Hilbert space. We note the relation between the weighted spaces X^k_η and their unweighted versions $X^k = X^k_0$.

Lemma 3.5. *The map*

$$\iota : X_\eta \to X, \quad \begin{pmatrix} v \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \eta(v-\rho\hat{v})+\rho\hat{v} \\ \rho \end{pmatrix}$$

is a isometric isomorphism with inverse

$$\iota^{-1} : X \to X_\eta, \quad \begin{pmatrix} v \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \eta^{-1}(v - \rho\hat{v}) + \rho\hat{v} \\ \rho \end{pmatrix}.$$

If X_η is replaced by Y_η and X by Y, respectively, then ι defines a continuous isomorphism.

Proof. Clearly, ι is a linear map between vector spaces and for $\mathbf{v} = (v, \rho)^\top \in X_\eta$ there holds

$$\|\iota\mathbf{v}\|_X^2 = \left\| \begin{pmatrix} \eta(v - \rho\hat{v}) + \rho\hat{v} \\ \rho \end{pmatrix} \right\|_X^2 = |\rho|^2 + \|\eta(v - \rho\hat{v})\|_{L^2}^2 = \|\mathbf{v}\|_{X_\eta}^2.$$

Hence $\iota : X_\eta \to X$ is an isometric homomorphism. The same holds for ι^{-1} and

$$\iota^{-1}\iota \begin{pmatrix} v \\ \rho \end{pmatrix} = \begin{pmatrix} \eta^{-1}[\eta(v - \rho\hat{v}) + \rho\hat{v} - \rho\hat{v}] + \rho\hat{v} \\ \rho \end{pmatrix} = \begin{pmatrix} v \\ \rho \end{pmatrix}.$$

Therefore, $\iota : X_\eta \to X$ is an isometric isomorphism. If X_η is replaced by Y_η and X by Y, respectively, it remains to show that $\iota : Y_\eta \to Y$ is bounded, Then by the inverse mapping theorem ι^{-1} is bounded and ι is a continuous isomorphism. So we estimate using (3.3), (3.4) and Lemma 2.7

$$\begin{aligned}
\|\iota\mathbf{v}\|_Y &= \left\| \begin{pmatrix} \eta(u - \rho\hat{v}) + \rho\hat{v} \\ \rho \end{pmatrix} \right\|_Y \\
&\le |\rho| + \|\eta(u - \rho\hat{v})\|_{L^2} + \|\eta_x(u - \rho\hat{v}) + \eta(u_x - \rho\hat{v}_x) + \rho\hat{v}_x\|_{L^2} \\
&\quad + \|\eta_{xx}(u - \rho\hat{v}) + 2\eta_x(u_x - \rho\hat{v}_x) + \eta(u_{xx} - \rho\hat{v}_{xx}) + \rho\hat{v}_{xx}\|_{L^2} \\
&\le |\rho| + \|\eta(u - \rho\hat{v})\|_{L^2} + \|\eta_x(u - \rho\hat{v})\|_{L^2} + \|\eta u_x\|_{L^2} + \|(1 - \eta)\rho\hat{v}_x\|_{L^2} \\
&\quad + \|\eta_{xx}(u - \rho\hat{v})\|_{L^2} + 2\|\eta_x u_x\|_{L^2} + \|\eta_x\rho\hat{v}_x\|_{L^2} + \|\eta u_{xx}\|_{L^2} + \|(1 - \eta)\rho\hat{v}_{xx}\|_{L^2} \\
&\le C(|\rho| + \|\eta(u - \rho\hat{v})\|_{L^2} + \|\eta u_x\|_{L^2} + \|\eta u_{xx}\|_{L^2}) \le \tilde{C}\|\mathbf{v}\|_{Y_\eta}.
\end{aligned}$$

□

For analyzing the freezing method in Chapter 4 we need the dual space of X_η^1 which we discuss in the following. It is defined in the usual way by

$$X_\eta^{-1} := \Big\{ \psi : X_\eta^1 \to \mathbb{R} \text{ linear and bounded} \Big\}.$$

We equip it with the norm

$$\|\psi\|_{X_\eta^{-1}} := \sup_{\|\mathbf{v}\|_{X_\eta^1} \le 1} \langle \psi, \mathbf{v} \rangle$$

where $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{X_\eta^{-1}\times X_\eta^1}$ denotes the dual pairing. In particular, we sometimes write $\psi(\mathbf{u}) = \langle\psi,\mathbf{u}\rangle_{X_\eta^{-1}\times X_\eta^1}$. Note that every $\mathbf{v}\in X_\eta$ defines a linear functional $\langle\mathbf{v},\cdot\rangle\in X_\eta^{-1}$ via the identification $\langle\mathbf{v},\mathbf{u}\rangle = (\mathbf{v},\mathbf{u})_{X_\eta}$.
Let us briefly relate the space X_η^{-1} using the dual space H_η^{-1} of H_η^1. In order to do so, let $\psi\in X_\eta^{-1}$ be a linear functional on X_η^1. Then $\psi(\cdot,0): H_\eta^1\to\mathbb{R}$ is linear and bounded and thus there is $\psi_1\in H_\eta^{-1}$ such that $\psi(u,0)=\psi_1(u)$ for all $u\in H_\eta^1$. Further, we have $\psi\big((\cdot)\hat{v},\cdot\big):\mathbb{R}^2\to\mathbb{R}$ is linear and bounded. Thus there is $\psi_2\in\mathbb{R}^2$ such that $\psi\big(\rho\hat{v},\rho\big) = \psi_2^\top\rho$ for all $\rho\in\mathbb{R}^2$. Consequently, $\psi\in X_\eta^{-1}$ if there are $\psi_1\in H_\eta^{-1}$ and $\psi_2\in\mathbb{R}^2$ such that

$$\psi\begin{pmatrix}u\\ \rho\end{pmatrix} = \psi\begin{pmatrix}u-\rho\hat{v}\\ 0\end{pmatrix} + \psi\begin{pmatrix}\rho\hat{v}\\ \rho\end{pmatrix} = \psi_1(u-\rho\hat{v}) + \psi_2^\top\rho \quad \forall\begin{pmatrix}u\\ \rho\end{pmatrix}\in X_\eta^1. \tag{3.7}$$

Conversely, for arbitrary $\psi_1\in H_\eta^{-1}$ and $\psi_2\in\mathbb{R}^2$ there is $\psi\in X_\eta^{-1}$ defined by

$$\psi\begin{pmatrix}u\\ \rho\end{pmatrix} := \langle\psi_1, u-\rho\hat{v}\rangle_{H_\eta^{-1}\times H_\eta^1} + \psi_2^\top\rho \quad \forall\begin{pmatrix}u\\ \rho\end{pmatrix}\in X_\eta^1.$$

We conclude the section by considering the second order differential operator

$$\mathcal{L}_0: Y_\eta\to X_\eta,\quad \begin{pmatrix}u\\ \rho\end{pmatrix}\mapsto\begin{pmatrix}Au_{xx}+cu_x\\ 0\end{pmatrix}. \tag{3.8}$$

Then it follows immediately $\mathcal{L}_0\in L[Y_\eta,X_\eta]$. Further, using the integration by parts formula from Lemma 3.2 we obtain for all $(u,\rho)^\top\in Y_\eta$ and $(v,\zeta)^\top\in X_\eta^1$

$$\begin{aligned}\left(\mathcal{L}_0\begin{pmatrix}u\\ \rho\end{pmatrix},\begin{pmatrix}v\\ \zeta\end{pmatrix}\right)_{X_\eta} &= \left(\begin{pmatrix}Au_{xx}+cu_x\\ 0\end{pmatrix},\begin{pmatrix}v\\ \zeta\end{pmatrix}\right)_{X_\eta} = (Au_{xx}+cu_x, v-\zeta\hat{v})_{L_\eta^2}\\ = -(u_x, A^\top(v-\zeta\hat{v})_x)_{L_\eta^2} &+ 2(\eta_x\eta^{-1}u_x, A^\top(v-\zeta\hat{v})_x)_{L_\eta^2} + c(u_x, v-\zeta\hat{v})_{L_\eta^2}.\end{aligned} \tag{3.9}$$

Now the right hand side of (3.9) is even well-defined for $(u,\rho)^\top\in X_\eta^1$. Therefore, we may consider $\mathcal{L}_0$ on X_η^1 as

$$\mathcal{L}_0: X_\eta^1\to X_\eta^{-1},\quad \begin{pmatrix}u\\ \rho\end{pmatrix}\mapsto\mathcal{L}_0\begin{pmatrix}u\\ \rho\end{pmatrix} \tag{3.10}$$

via

$$\begin{aligned}\left\langle\mathcal{L}_0\begin{pmatrix}u\\ \rho\end{pmatrix},\begin{pmatrix}v\\ \zeta\end{pmatrix}\right\rangle_{X_\eta^{-1}\times X_\eta^1} &:= -(u_x, A^\top(v-\zeta\hat{v})_x)_{L_\eta^2}\\ &\quad + 2(\eta_x\eta^{-1}u_x, A^\top(v-\zeta\hat{v})_x)_{L_\eta^2} + c(u_x, v-\zeta\hat{v})_{L_\eta^2}.\end{aligned} \tag{3.11}$$

In particular, the corresponding ψ_1, ψ_2 from (3.7) for $\psi = \mathcal{L}_0 \mathbf{u} \in X_\eta^{-1}$ are given by

$$\langle \psi_1, \cdot \rangle = -(u_x - 2\eta_x \eta^{-1} u_x, A^\top (\cdot)_x)_{L^2_\eta} + c(u_x, \cdot)_{L^2_\eta}, \quad \psi_2 = 0.$$

Finally, we obtain $\mathcal{L}_0 \in L[X^1_\eta, X^{-1}_\eta]$ since

$$\begin{aligned}\|\mathcal{L}_0 \mathbf{u}\|_{X^{-1}_\eta} &= \sup_{\|\mathbf{v}\|_{X^1_\eta} \le 1} \langle \mathcal{L}_0 \mathbf{u}, \mathbf{v} \rangle \\ &\le \sup_{\|\mathbf{v}\|_{X^1_\eta} \le 1} \left\{ (2\mu + 1)|A| \|u_x\|_{L^2_\eta} \|(v - \zeta \hat{v})_x\|_{L^2_\eta} + |c| \|u_x\|_{L^2_\eta} \|v - \zeta \hat{v}\|_{L^2_\eta} \right\} \le C \|\mathbf{u}\|_{X^1_\eta}.\end{aligned}$$

Summarizing we have shown the following lemma:

Lemma 3.6. *For $\mu \ge 0$ the operator $\mathcal{L}_0 : Y_\eta \to X_\eta$ from (3.8) is linear and bounded. Moreover, the operator $\mathcal{L}_0 : X^1_\eta \to X^{-1}_\eta$ from (3.10) with (3.11) is also linear and bounded.*

3.2 Lie group, equivariance and symmetry

In this section we collect smoothness of translation and rotation on the spaces X^k_η as an action of the group $\mathcal{G} := S^1 \times \mathbb{R}$ on the function space X_η with $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. In order to do so, we define the composition on $\mathcal{G}$ via

$$\circ : \mathcal{G} \times \mathcal{G} \to \mathcal{G}, \quad (\gamma_1, \gamma_2) \mapsto ((\theta_1 + \theta_2) \bmod 2\pi, \tau_1 + \tau_2) \tag{3.12}$$

where $\gamma_1 = (\theta_1, \tau_1)$ and $\gamma_2 = (\theta_2, \tau_2)$. Note that usually $\circ$ also denotes the composition of functions. However, the notation will always be clear by the context. We follow the introduction into Lie groups from [53] and the concepts of differentiable manifolds from [1]. Alternative literature on differentiable manifolds can be found for instance in [41]. The group $\mathcal{G}$ is a C^∞-manifold and the composition $\circ : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a C^∞-map as well as the inverse map

$$\text{inv} : \mathcal{G} \to \mathcal{G}, \quad \gamma \mapsto \gamma^{-1} = (-\theta \bmod 2\pi, \tau).$$

Thus $\mathcal{G}$ is Lie group, cf. [53, Chapter 4]. For $\mathcal{G}$ as a manifolds over $\mathbb{R}^2$ we have the two charts $(U, \chi), (\tilde{U}, \tilde{\chi})$ given by

$$U = \{\gamma \in \mathcal{G} : \gamma = (\theta, \tau), \, \theta \in (-\pi, \pi)\}, \quad \chi : U \to \mathbb{R}^2, \quad \gamma \mapsto \chi(\gamma) = (\theta, \tau), \tag{3.13}$$
$$\tilde{U} = \{\gamma \in \mathcal{G} : \gamma = (\theta, \tau), \, \theta \in (0, 2\pi)\}, \quad \tilde{\chi} : \tilde{U} \to \mathbb{R}^2, \quad \gamma \mapsto \tilde{\chi}(\gamma) = (\theta, \tau). \tag{3.14}$$

Then $\chi(U) = (-\pi, \pi) \times \mathbb{R}$ and $\tilde{\chi}(\tilde{U}) = (0, 2\pi) \times \mathbb{R}$. The charts from (3.13), (3.14) will be important in all our considerations of the chapter. In particular, we will work and prove the nonlinear stability in local coordinates. The unit element of $\mathcal{G}$ is given by $\mathbb{1} = (0,0)$ and the tangent space at it $T_{\mathbb{1}}G = \mathfrak{g}$ is the associated Lie algebra of $\mathcal{G}$. Recall the metric $|\cdot|_{\mathcal{G}}$ on $\mathcal{G}$ from (1.11) and we write for $\gamma_1, \gamma_2, \gamma \in G$, in a canonical manner,

$$\gamma_1 + \gamma_2 := \gamma_1 \circ \gamma_2, \quad \gamma_1 - \gamma_2 := \gamma_1 \circ (-\gamma_2), \quad |\gamma| = |\gamma|_G.$$

Furthermore, recall the group action $a(\cdot)\mathbf{v} : \mathcal{G} \to X_\eta$, $\mathbf{v} \in X_\eta$ from (0.25), i.e. a describes rotation and translation of elements from the function space X_η. We now show smoothness of the group action $a(\cdot)\mathbf{v}$ depending on the regularity of the function $\mathbf{v}$. For general $v = (v, \rho)^\top \in X_\eta$, respectively $v = (v, \rho)^\top \in X_\eta^1$, we write throughout the thesis

$$\mathbf{S_1 v} = \begin{pmatrix} S_1 v \\ S_1 \rho \end{pmatrix}, \quad \mathbf{v}_x = \begin{pmatrix} v_x \\ 0 \end{pmatrix}.$$

Lemma 3.7. *The group action*

$$a : \mathcal{G} \to GL[X_\eta], \quad \gamma \mapsto a(\gamma)$$

from (0.25) *is a homomorphism and $a(\gamma)Y_\eta = Y_\eta$, $\gamma \in \mathcal{G}$. Further, for all $\mathbf{v} \in X_\eta$ the map $a(\cdot)\mathbf{v} : \mathcal{G} \to X_\eta$ is continuous and the same holds true if X_η is replaced by Y_η. If $\mathbf{v} = (v, \rho)^\top \in X_\eta^1$ the map $a(\cdot)\mathbf{v} : \mathcal{G} \to X_\eta$ is of class C^1 and for $\gamma \in U$ with $\gamma = \chi^{-1}(z)$ the derivative of $(a(\cdot)\mathbf{v} \circ \chi^{-1}) : \mathbb{R}^2 \to X_\eta$ is given by*

$$(a(\cdot)\mathbf{v} \circ \chi^{-1})'(z) = -(a(\gamma)\mathbf{S_1 v}, a(\gamma)\mathbf{v}_x) \in L[\mathbb{R}^2, X_\eta], \quad \mathbf{S_1 v} = \begin{pmatrix} S_1 v \\ S_1 \rho \end{pmatrix}, \quad \mathbf{v}_x = \begin{pmatrix} v_x \\ 0 \end{pmatrix}.$$

Proof. Let $\mathbf{v} = (v, \rho)^\top \in X_\eta$. Then by using Lemma 3.4 and invariance of the norms under rotation we obtain

$$\begin{aligned}
\|a(\gamma)\mathbf{v}\|_{X_\eta} &\le |\rho| + \|v(\cdot - \tau) - \rho\hat{v}\|_{L^2_\eta} \\
&\le |\rho| + \|v(\cdot - \tau) - \rho\hat{v}(\cdot - \tau)\|_{L^2_\eta} + \|\rho\hat{v}(\cdot - \tau) - \rho\hat{v}\|_{L^2_\eta} \\
&\le |\rho| + e^{\mu|\tau|}\|v - \rho\hat{v}\|_{L^2_\eta} + e^{\mu|\tau|}|\rho|\|\hat{v}_x\|_{L^2_\eta} \le C\|\mathbf{v}\|_{X_\eta}.
\end{aligned}$$

Similarly, for $\mathbf{v} \in Y_\eta$ we have

$$\begin{aligned}
\|a(\gamma)\mathbf{v}\|_{Y_\eta}^2 &\le C\|\mathbf{v}\|_{X_\eta}^2 + \|v_x(\cdot - \tau)\|_{L^2_\eta}^2 + \|v_{xx}(\cdot - \tau)\|_{L^2_\eta}^2 \\
&\le C\|\mathbf{v}\|_{X_\eta}^2 + e^{2\mu|\tau|}\|v_x\|_{L^2_\eta}^2 + e^{2\mu|\tau|}\|v_{xx}\|_{L^2_\eta}^2 \le \tilde{C}\|\mathbf{v}\|_{Y_\eta}^2.
\end{aligned}$$

Thus, $a(\gamma)X_\eta \subset X_\eta$ and $a(\gamma)Y_\eta \subset Y_\eta$. Furthermore, the group action a is a homomorphism since for $\gamma_1, \gamma_2 \in \mathcal{G}$, $(v, \rho)^\top \in X_\eta$ there hold

$$\begin{aligned}
a(\gamma_1)a(\gamma_2)\mathbf{v} &= \begin{pmatrix} R_{\theta_1}R_{\theta_2}v(\cdot - \tau_2 - \tau_1) \\ R_{\theta_1}R_{\theta_2}\rho \end{pmatrix} \\
&= \begin{pmatrix} R_{\theta_1+\theta_2}v(\cdot - (\tau_1 + \tau_2)) \\ R_{\theta_1+\theta_2}\rho \end{pmatrix} = a(\gamma_1 \circ \gamma_2)\mathbf{v}
\end{aligned}$$

where we used $R_{\theta_1}R_{\theta_2} = R_{\theta_1+\theta_2}$. Its inverse is given by $a(\gamma)^{-1} = a(\gamma^{-1})$ and hence $a(\gamma) \in GL[X_\eta]$. In particular, we have shown that $a(\gamma)$ is bounded and $a(\gamma)Y_\eta = Y_\eta$. Let us prove the continuity. Since a is a homomorphism it is sufficient to prove the continuity at $\gamma = \mathbb{1}$. Let $\mathbf{v} = (v,\rho)^\top \in X_\eta$ and $\gamma \in \mathcal{G}$. Then by continuity of $R_{(\cdot)}$ and the shift on L^2_η, cf. Lemma 3.4,

$$
\begin{aligned}
&\|a(\gamma)\mathbf{v} - \mathbf{v}\|_{X_\eta} \le |R_\theta\rho - \rho| + \|R_\theta v(\cdot - \tau) - R_\theta\rho\hat{v} - v + \rho\hat{v}\|_{L^2_\eta} \\
&\le |R_\theta - I||\rho| + \|R_\theta v(\cdot-\tau) - R_\theta\rho\hat{v} - R_\theta v + R_\theta\rho\hat{v}\|_{L^2_\eta} + \|R_\theta v - R_\theta\rho\hat{v} - v + \rho\hat{v}\|_{L^2_\eta} \\
&\le |R_\theta - I|\left(|\rho| + \|v - \rho\hat{v}\|_{L^2_\eta}\right) + \|v(\cdot-\tau) - v\|_{L^2_\eta} \\
&\le |R_\theta - I|\left(|\rho| + \|v - \rho\hat{v}\|_{L^2_\eta}\right) + \|(v-\rho\hat{v})(\cdot-\tau) - (v-\rho\hat{v})\|_{L^2_\eta} + |\rho|\|\hat{v}(\cdot-\tau) - \hat{v}\|_{L^2_\eta} \\
&\to 0, \quad (\theta,\tau) \to 0.
\end{aligned}
$$

Similarly, if $\mathbf{v} \in Y_\eta$, we have $v_x, v_{xx} \in L^2_\eta$ and

$$
\|a(\gamma)\mathbf{v} - \mathbf{v}\|^2_{Y_\eta} = \|a(\gamma)\mathbf{v} - \mathbf{v}\|^2_{X_\eta} + \sum_{\alpha=1}^{2} \|R_\theta\partial^\alpha v(\cdot-\tau) - \partial^\alpha v\|^2_{L^2_\eta} \to 0, \quad (\theta,\tau) \to 0.
$$

Next we show that $a(\cdot)\mathbf{v}$ is of class C^1 if $\mathbf{v} \in Y_\eta$. For this purpose let $\gamma \in U$, $z = (\theta,\tau) = \chi(\gamma) \in \mathbb{R}^2$, $\mathbf{v} = (v,\rho)^\top \in Y_\eta$ and $h = (h_1,h_2) \in \mathbb{R}^2$. From the definition of the chart (U,χ) from (3.13) we see that $\chi^{-1}(z+h) = \chi^{-1}(z) \circ \chi^{-1}(h) = \gamma \circ \chi^{-1}(h)$. Then by the continuity of the group action we have

$$
\begin{aligned}
&\|a(\chi^{-1}(z+h))\mathbf{v} - a(\chi^{-1}(z))\mathbf{v} + h_1 a(\gamma)\mathbf{S_1}\mathbf{v} + h_2 a(\gamma)\mathbf{v}_x\|_{X_\eta} \\
&= \|a(\gamma\circ\chi^{-1}(h))\mathbf{v} - a(\gamma)\mathbf{v} + h_1 a(\gamma)\mathbf{S_1}\mathbf{v} + h_2 a(\gamma)\mathbf{v}_x\|_{X_\eta} \\
&\le C\|a(\chi^{-1}(h))\mathbf{v} - \mathbf{v} - h_1\mathbf{S_1}\mathbf{v} + h_2\mathbf{v}_x\|_{X_\eta} \\
&\le C|R_{-h_1}\rho - \rho + h_1 S_1\rho| \\
&\quad + C\|R_{-h_1}(v(\cdot-h_2) - \rho\hat{v}) - (v-\rho\hat{v}) + h_1 S_1(v-\rho\hat{v}) + h_2 v_x\|_{L^2_\eta}.
\end{aligned}
\tag{3.15}
$$

Using Taylor expansion and $\partial^2_\theta R_\theta = R_\theta$ we observe

$$
R_{-h_1}\rho = \rho - h_1 S_1\rho + \int_0^1 h_1^2(1-\tau)R_{-h_1\tau}d\tau.
$$

Thus,

$$
|R_{-h_1}\rho - \rho + h_1 S_1\rho| \le h_1^2 \int_0^1 (1-\tau)\,|R_{h_1\tau}|\,d\tau|\rho| \le h_1^2|\rho| = o(|h|). \tag{3.16}
$$

Next, we estimate the second term consisting of the L^2_η-norm. For this purpose, we note for $\varphi \in C_0^\infty(\mathbb{R},\mathbb{R}^2)$ and $\tau \in \mathbb{R}$ that Fubini's theorem implies

$$\begin{aligned}
&\|\varphi(\cdot+\tau)-\varphi-\tau\varphi_x\|^2_{L^2_\eta} = |\tau|^2\int_{\mathbb{R}}\eta(x)^2\left|\int_0^1\varphi_x(x+s\tau)ds-\varphi_x(x)\right|^2dx\\
&\le |\tau|^2\int_{\mathbb{R}}\int_0^1\eta^2(x)|\varphi_x(x-s\tau)-\varphi_x(x)|^2dsdx = |\tau|^2\int_0^1\eta(x)^2\|\varphi_x(\cdot-s\tau)-\varphi_x\|^2_{L^2_\eta}ds\\
&\le |\tau|^2\sup_{|s|\le|\tau|}\|\varphi_x(\cdot+s)-\varphi_x\|^2_{L^2_\eta}.
\end{aligned}$$

Since $C_0^\infty \subset H^1_\eta$ is dense, there exists for $v \in H^1_\eta(\mathbb{R},\mathbb{R}^2)$ a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset C_0^\infty(\mathbb{R},\mathbb{R}^2)$ with $\|v-\varphi_k\|_{H^1_\eta} \to 0$ as $k\to\infty$. This implies with $\tau=-h_2$

$$\begin{aligned}
&\|v(\cdot-h_2)-v+h_2v_x\|_{L^2_\eta}\\
&\le \|v(\cdot-h_2)-\varphi_k(\cdot-h_2)\|_{L^2_\eta}+\|v-\varphi_k\|_{L^2_\eta}+|h_2|\|v_x-\varphi_{k,x}\|_{L^2_\eta}\\
&\quad+\|\varphi_k(\cdot-h_2)-\varphi_k+h_2\varphi_{k,x}\|_{L^2_\eta}\\
&\le (1+e^{\mu|h_2|})\|v-\varphi_k\|_{L^2_\eta}+|h_2|\|v_x-\varphi_{k,x}\|_{L^2_\eta}+|h_2|\sup_{s\le|h_2|}\|\varphi_{k,x}(\cdot+s)-\varphi_{k,x}\|_{L^2_\eta}.
\end{aligned}$$

Now let $k\to\infty$ to obtain

$$\|v(\cdot-h_2)-v+h_2v_x\|_{L^2_\eta} \le |h_2|\sup_{s\le|h_2|}\|v_x(\cdot+s)-v_x\|_{L^2_\eta} = o(|h|). \tag{3.17}$$

By frequently adding zero and using triangle inequality, we observe for the second term in (3.15)

$$\begin{aligned}
&\|R_{-h_1}(v(\cdot-h_2)-\rho\hat{v})-(v-\rho\hat{v})+h_1S_1(v-\rho\hat{v})+h_2v_x\|_{L^2_\eta}\\
&\le \underbrace{\|R_{-h_1}(v-\rho\hat{v})(\cdot-h_2)-(v-\rho\hat{v})(\cdot-h_2)+h_1S_1(v-\rho\hat{v})(\cdot-h_2)\|_{L^2_\eta}}_{=:T_1}\\
&\quad+\|R_{-h_1}\rho\hat{v}(\cdot-h_2)-R_{-h_1}\rho\hat{v}+(v-\rho\hat{v})(\cdot-h_2)-h_1S_1(v-\rho\hat{v})(\cdot-h_2)\\
&\qquad-(v-\rho\hat{v})+h_1S_1(v-\rho\hat{v})+h_2v_x\|_{L^2_\eta}\\
&\le T_1+\underbrace{\|(v-\rho\hat{v})(\cdot-h_2)-(v-\rho\hat{v})+h_2(v_x-\rho\hat{v}_x)\|_{L^2_\eta}}_{=:T_2}\\
&\quad+\|h_2\rho\hat{v}_x+R_{-h_1}\rho\hat{v}(\cdot-h_2)-R_{-h_1}\rho\hat{v}-h_1S_1(v-\rho\hat{v})(\cdot-h_2)+h_1S_1(v-\rho\hat{v})\|_{L^2_\eta}\\
&\le T_1+T_2+\underbrace{\|R_{-h_1}[\rho\hat{v}(\cdot-h_2)-\rho\hat{v}+h_2\rho\hat{v}_x]\|_{L^2_\eta}}_{=:T_3}\\
&\quad+\|h_2\rho\hat{v}_x-h_2R_{-h_1}\rho\hat{v}_x-h_1S_1(v-\rho\hat{v})(\cdot-h_2)+h_1S_1(v-\rho\hat{v})\|_{L^2_\eta}
\end{aligned}$$

$$
\begin{aligned}
&\leq T_1 + T_2 + T_3 + \underbrace{\| - h_1 S_1(v - \rho\hat{v})(\cdot - h_2) + h_1 S_1(v - \rho\hat{v}) - h_2(h_1 S_1 v_x - h_1 S_1 \rho\hat{v}_x)\|_{L^2_\eta}}_{=:T_4} \\
&\quad + \|h_2\rho\hat{v}_x - h_2 R_{-h_1}\rho\hat{v}_x + h_2 h_1 S_1 v_x - h_2 h_1 S_1 \rho\hat{v}_x\|_{L^2_\eta} \\
&\leq T_1 + T_2 + T_3 + T_4 + \underbrace{\|R_{-h_1} h_2\rho\hat{v}_x - h_2\rho\hat{v}_x + h_1 S_1 h_2 \rho\hat{v}_x\|_{L^2_\eta}}_{=:T_5} + \underbrace{\|h_2 h_1 S_1 v_x\|_{L^2_\eta}}_{=:T_6} \\
&= T_1 + T_2 + T_3 + T_4 + T_5 + T_6.
\end{aligned}
$$

With (3.16) and Lemma 3.4 we have

$$
\begin{aligned}
T_1 &\leq e^{\mu|h_2|}|R_{-h_1} - I + h_1 S_1|\|v - \rho\hat{v}\|_{L^2_\eta} = o(|h|), \\
T_5 &\leq |R_{-h_1} - I + h_1 S_1|\|h_2\rho\hat{v}_x\|_{L^2_\eta} = o(|h|).
\end{aligned}
$$

Since $v - \rho\hat{v} \in H^1_\eta$, (3.17) implies $T_2, T_3, T_4 = o(|h|)$ and obviously $T_6 = o(|h|)$. Therefore

$$
\frac{1}{|h|}\left\|a(\chi^{-1}(z+h)\mathbf{v} - a(\chi^{-1}(z))\mathbf{v} + h_1 a(\gamma)\mathbf{S_1}\mathbf{v} + h_2 a(\gamma)\mathbf{v}_x\right\|_{X_\eta} = \frac{o(|h|)}{|h|} \to 0, \quad |h| \to 0.
$$

Hence $a(\cdot)\mathbf{v} \circ \chi^{-1} \in C^1(U, X_\eta)$ with derivative

$$
(a(\cdot)\mathbf{v} \circ \chi^{-1})'(z) = -(a(\gamma)\mathbf{S_1}\mathbf{v}, a(\gamma)\mathbf{v}_x).
$$

The same way, one shows that $a(\cdot)\mathbf{v} \circ \tilde{\chi}^{-1} \in C^1(\tilde{U}, X_\eta)$. This proves that $a(\cdot)\mathbf{v} : \mathcal{G} \to X_\eta$ is of class C^1. $\square$

As a consequence of Lemma 3.7 we conclude by the mean value theorem that for any compact set $K \subset U$ and $\mathbf{v} \in X^{k+1}_\eta$, $k = 0, 1$ there is $L > 0$ such that

$$
\|a(\chi^{-1}(z_1))\mathbf{v} - a(\chi^{-1}(z_2))\mathbf{v}\|_{X^k_\eta} \leq L|z_1 - z_2|\|\mathbf{v}\|_{X^{k+1}_\eta} \quad \forall\, z_1, z_2 \in K. \tag{3.18}
$$

Now we take the nonlinear operator $\mathcal{F}$ from (0.23) into account. We prove that it is continuous w.r.t. suitable norms and, in addition, is equivariant under the group action a from (0.25) according to Definition 1.2.

Lemma 3.8. *Let the Assumption 1 be satisfied and* $0 \leq \mu < 2$. *Then* $\mathcal{F} : Y_\eta \to X_\eta$ *from* (0.23) *defines a continuous operator and is equivariant under the group action* $a(\gamma), \gamma \in \mathcal{G}$ *from* (0.25). *Moreover, for every* $\mathbf{v} \in Y_\eta$ *there is* $\delta > 0$ *and* $L_{\mathcal{F}} > 0$ *such that* $\mathbf{w} \in Y_\eta$ *with* $\|\mathbf{v} - \mathbf{w}\|_{Y_\eta} < \delta$ *satisfies*

$$
\|\mathcal{F}(\mathbf{v}) - \mathcal{F}(\mathbf{w})\|_{X^k_\eta} \leq L_{\mathcal{F}}\|\mathbf{v} - \mathbf{w}\|_{X^{k+2}_\eta}, \quad k = -1, 0. \tag{3.19}
$$

Proof. We begin by proving that $\mathcal{F}$ is well defined on Y_η and maps it onto X_η. In what follows $C_{\mathbf{v}} > 0$ denotes a universal constant depending on $\|\mathbf{v}\|_{Y_\eta}$. Let $\mathbf{v} = (v, \rho)^\top \in Y_\eta$. Then $v - \rho\hat{v} \in H^2_\eta$ and by Sobolev embedding, cf. Theorem D.2, we have $v - \rho\hat{v} \in L^\infty$ and therefore $v \in L^\infty$ with $\|v - \rho\hat{v}\|_{L^\infty}, \|v\|_{L^\infty} \le C_{\mathbf{v}}$. To estimate the nonlinear term in $\mathcal{F}$ we split the occurring integral over $\mathbb{R}$ into two integrals over the negative and positive half-line $\mathbb{R}_\pm$. Using Assumption 1, we obtain for every $0 \le \mu < 2$

$$\begin{aligned}
&\|f(v) - f(\rho)\hat{v}\|^2_{L^2_\eta} \\
&\le 2\int_{\mathbb{R}} \eta(x)^2 |g(|v(x)|^2)(v(x) - \rho\hat{v}(x)|^2 dx + 2\int_{\mathbb{R}} \eta(x)^2 |g(|v(x)|^2) - g(|\rho|^2)\rho\hat{v}(x)|^2 dx \\
&\le 2C_{\mathbf{v}}\|v - \rho\hat{v}\|^2_{L^2_\eta} + 2C_{\mathbf{v}}\int_{\mathbb{R}_-} \eta(x)^2|\rho\hat{v}(x)|^2 dx + 2|\rho|^2\int_{\mathbb{R}_+} \eta(x)^2 |g(|v(x)|^2) - g(\rho|^2)|^2 dx \\
&\le 2C_{\mathbf{v}}\|v - \rho\hat{v}\|^2_{L^2_\eta} + 2C_{\mathbf{v}}|\rho|^2\|\hat{v}\|^2_{L^2_\eta(\mathbb{R}_-)} + 2C_{\mathbf{v}}|\rho|^2\int_{\mathbb{R}_+} \Big||v(x)|^2 - |\rho|^2\Big|^2 dx < \infty,
\end{aligned}$$

since Lemma 2.7 yields

$$\begin{aligned}
&\int_{\mathbb{R}_+} \eta(x)^2 \Big||v(x)|^2 - |\rho|^2\Big|^2 dx = \int_{\mathbb{R}_+} \eta(x)^2 \Big|(|v(x)| + |\rho|)(|v(x)| - |\rho|)\Big|^2 dx \\
&\le C_{\mathbf{v}}\int_{\mathbb{R}_+} \eta(x)^2 \Big||v(x)| - |\rho|\Big|^2 dx \\
&\le 2C_{\mathbf{v}}\int_{\mathbb{R}_+} \eta(x)^2 \Big||v(x)| - |\rho|\hat{v}(x)\Big|^2 dx + 2C_{\mathbf{v}}\int_{\mathbb{R}_+} \eta(x)^2 \Big||\rho|\hat{v}(x) - |\rho|\Big|^2 dx \\
&\le 2C_{\mathbf{v}}\int_{\mathbb{R}_+} \eta(x)^2 |v(x) - \rho\hat{v}(x)|^2 dx + 2C_{\mathbf{v}}|\rho|^2\int_{\mathbb{R}_+} \eta(x)^2|\hat{v}(x) - 1|^2 dx \\
&\le 2C_{\mathbf{v}}\|v - \rho\hat{v}\|^2_{L^2_\eta} + 2C_{\mathbf{v}}|\rho|^2\|\hat{v} - 1\|^2_{L^2_\eta(\mathbb{R}_+)} < \infty.
\end{aligned}$$

Taking Lemma 3.6 into account, we have $\mathcal{L}_0 \in L[Y_\eta, X_\eta]$ and we obtain

$$\|\mathcal{F}(\mathbf{v})\|_{X_\eta} \le \|\mathcal{L}_0\mathbf{v}\|_{X_\eta} + \left\|\begin{pmatrix} f(v) \\ f(\rho) \end{pmatrix}\right\|_{X_\eta} \le C\|\mathbf{v}\|_{Y_\eta} + |f(\rho)| + \|f(v) - f(\rho)\hat{v}\|_{L^2_\eta} < \infty.$$

Hence $\mathcal{F} : Y_\eta \to X_\eta$ is well defined and the continuity follows by the Lipschitz estimate (3.19) which is still to be shown. By Lemma 3.7 we have $a(\gamma)Y_\eta = Y_\eta$. Moreover, R_θ commutes with A, S_ω and $g(|\cdot|^2)$. Therefore the equivariance of $\mathcal{F}$ follows by

$$\mathcal{F}(a(\gamma)\mathbf{v}) = \begin{pmatrix} R_\theta[Av_{xx} + cv_x + S_\omega v + g(|v|^2)v](\cdot - \tau) \\ R_\theta[S_\omega\rho + g(|\rho|^2)\rho] \end{pmatrix} = a(\gamma)\mathcal{F}(\mathbf{v}).$$

It is left to show the estimate (3.19). We write $\mathbf{v} = (v, \rho)^\top$ and $\mathbf{w} = (w, \zeta)^\top$. Suppose $\delta > 0$ sufficiently small and let $C > 0$ denote a universal constant depending on $\mathbf{v}$ and δ.

Again, the major task is to estimate the nonlinear term of $\mathcal{F}$, which is done in several steps. As we see, the estimate of the nonlinear term is independent of k. The dependence on k only appears when estimating the main part containing second derivatives. The nonlinear term reads as and is first estimated by

$$\begin{aligned}
&\|f(v) - f(\rho)\hat{v} - f(w) + f(\zeta)\hat{v}\|_{L^2_\eta} = \|g(|v|^2)v - g(|\rho|^2)\rho\hat{v} - g(|w|^2)w + g(|\zeta|^2)\zeta\hat{v}\|_{L^2_\eta} \\
&\le \|g(|\rho|^2)(v - \rho\hat{v}) - g(|\zeta|^2)(w - \zeta\hat{v})\|_{L^2_\eta} + \|(g(|v|^2) - g(|\rho|^2))v - (g(|w|^2) - g(|\zeta|^2))w\|_{L^2_\eta} \\
&=: I_1 + I_2.
\end{aligned}$$

By Assumption 1 we may estimate I_1 by

$$\begin{aligned}
I_1 &\le \|g(|\rho|^2)(v - \rho\hat{v} - w + \zeta\hat{v})\|_{L^2_\eta} + \|(g(|\rho|^2) - g(|\zeta|^2))(w - \zeta\hat{v})\|_{L^2_\eta} \\
&\le |g(|\rho|^2)|\|v - \rho\hat{v} - w + \zeta\hat{v}\|_{L^2_\eta} + |g(|\rho|^2) - g(|\zeta|^2)| \sup_{\mathbf{w}\in B_\delta(\mathbf{v})} \|w - \zeta\hat{v}\|_{L^2_\eta} \\
&\le C\|v - \rho\hat{v} - w + \zeta\hat{v}\|_{L^2_\eta} + C\left||\rho| - |\zeta|^2\right| \\
&\le C\|v - \rho\hat{v} - w + \zeta\hat{v}\|_{L^2_\eta} + C|\rho - \zeta| \le C\|\mathbf{v} - \mathbf{w}\|_{X^1_\eta}.
\end{aligned}$$

Here we used again $||\rho|^2 - |\zeta|^2| \le (|\rho| + |\delta|)||\rho| - |\zeta|| \le C|\rho - \zeta|$. Further, we split I_2 into I_3 and I_4 via

$$\begin{aligned}
I_2 \le\ &\|(g(|v|^2) - g(|\rho|^2))(v - \rho\hat{v}) - (g(|w|^2) - g(|\zeta|^2))(w - \zeta\hat{v}\|_{L^2_\eta} \\
&+ \|(g(|v|^2) - g(|\rho|^2))\rho\hat{v} - (g(|w|^2) - g(|\zeta|^2))\zeta\hat{v}\|_{L^2_\eta} =: I_3 + I_4.
\end{aligned}$$

By Sobolev embedding, cf. Theorem D.2, we have $\|u\|_{L^\infty} \le C\|u\|_{H^1_\eta}$ for all $u \in H^1_\eta$. Therefore, there hold

$$\begin{aligned}
\|v - w\|_{L^\infty} &\le \|v - \rho\hat{v} - w + \zeta\hat{v}\|_{L^\infty} + |\rho - \zeta| \\
&\le C\|v - \rho\hat{v} - w + \zeta\hat{v}\|_{H^1_\eta} + |\rho - \zeta| \\
&\le C\|v - \rho\hat{v} - w + \zeta\hat{v}\|_{L^2_\eta} + \|v_x - w_x\|_{L^2_\eta} + |\rho - \zeta|(1 + \|\hat{v}_x\|_{L^2_\eta}) \\
&\le C\|\mathbf{v} - \mathbf{w}\|_{X^1_\eta}.
\end{aligned}$$

Assumption 1 implies

$$\left|g(|\rho|^2) - g(|\zeta|^2)\right| \le C\left||\rho|^2 - |\zeta|^2\right| \le C|\rho - \zeta|$$

and similarly

$$\|g(|v|^2) - g(|w|^2)\|_{L^\infty} \le C\sup_{x\in\mathbb{R}}\left||v(x)|^2 - |w(x)|^2\right| \le C\|v - w\|_{L^\infty}.$$

Again by Sobolev embedding, cf. Theorem D.2, we have $v \in L^\infty$. Therefore we obtain

$$\begin{aligned} I_3 &\le \|(g(|v|^2) - g(|\rho|^2)(v - \rho\hat{v} - w + \zeta\hat{v})\|_{L^2_\eta} \\ &\quad + \|(g(|v|^2) - g(|\rho|^2) - g(|w|^2) + g(|\zeta|^2))(w - \zeta\hat{v})\|_{L^2_\eta} \\ &\le \|g(|v|^2) - g(|\rho|^2)\|^2_{L^\infty}\|v - \rho\hat{v} - w + \zeta\hat{v}\|_{L^2_\eta} \\ &\quad + C(\|g(|v|^2) - g(|w|^2)\|_{L^\infty} + |\rho - \zeta|)\|w - \zeta\hat{v}\|_{L^2_\eta} \le C\|\mathbf{v} - \mathbf{w}\|_{X^1_\eta}. \end{aligned}$$

We continue in this fashion by splitting I_4 into two terms I_5 and I_6 via

$$\begin{aligned} I_4 &\le \|(g(|v|^2) - g(|\rho|^2))(\rho - \zeta)\hat{v}\|_{L^2_\eta} + \|(g(|v|^2) - g(|\rho|^2) - g(|w|^2) + g(|\zeta|^2))\zeta\hat{v}\|_{L^2_\eta} \\ &=: I_5 + I_6 \end{aligned}$$

and note

$$\|v - \rho\|_{L^2_\eta(\mathbb{R}_+)} \le \|v - \rho\hat{v}\|_{L^2_\eta(\mathbb{R}_+)} + |\rho|\|\hat{v} - 1\|_{L^2_\eta(\mathbb{R}_+)} \le C\|\mathbf{v}\|_{X_\eta}.$$

Then we estimate

$$\begin{aligned} I_5 &= \|(g(|v|^2) - g(|\rho|^2))(\rho - \zeta)\hat{v}\|_{L^2_\eta(\mathbb{R}_-)} + \|(g(|v|^2) - g(|\rho|^2))(\rho - \zeta)\hat{v}\|_{L^2_\eta(\mathbb{R}_+)} \\ &\le C|\rho - \zeta|\|\hat{v}\|_{L^2_\eta(\mathbb{R}_-)} + |\rho - \zeta|\|g(|v|^2) - g(|\rho|^2)\|_{L^2_\eta(\mathbb{R}_+)} \\ &\le C|\rho - \zeta| + C|\rho - \zeta|\|v - \rho\|_{L^2_\eta(\mathbb{R}_+)} \le C|\rho - \zeta| \le C\|\mathbf{v} - \mathbf{w}\|_{X^1_\eta}. \end{aligned}$$

Further, I_6 is decomposed into the integral on the negative and positive half-line denoted by I_7 and I_8:

$$\begin{aligned} I_6 &\le \|(g(|v|^2) - g(|\rho|^2) - g(|w|^2) + g(|\zeta|^2))\zeta\hat{v}\|_{L^2_\eta(\mathbb{R}_-)} \\ &\quad + \|(g(|v|^2) - g(|\rho|^2) - g(|w|^2) + g(|\zeta|^2))\zeta\hat{v}\|_{L^2_\eta(\mathbb{R}_+)} =: I_7 + I_8. \end{aligned}$$

I_7, the integral on the negative half-line, can directly estimated by

$$I_7 \le C\|g(|v|^2) - g(|w|^2)\|_{L^\infty}\|\hat{v}\|_{L^2_\eta(\mathbb{R}_-)} + C\Big|g(|\rho|^2) - g(|\zeta|^2)\Big|\|\hat{v}\|_{L^2_\eta(\mathbb{R}_-)} \le C\|\mathbf{v} - \mathbf{w}\|_{X^1_\eta}.$$

Let for the moment $(\cdot, \cdot)$ denote the standard Euclidean inner product in $\mathbb{R}^2$. For estimating I_8 the following term appears and can be estimated using Cauchy-Schwarz inequality

$$\begin{aligned} &\Big\||v|^2 - |\rho|^2 - |w|^2 + |\zeta|^2\Big\|_{L^2_\eta(\mathbb{R}_+)} = \|(v - \rho, v + \rho) - (w - \zeta, w + \zeta)\|_{L^2_\eta(\mathbb{R}_+)} \\ &\Big\|(v - \rho - w + \zeta, v + \rho)\|_{L^2_\eta(\mathbb{R}_+)} + \|(w - \zeta, v + \rho - w - \zeta)\Big\|_{L^2_\eta(\mathbb{R}_+)} \\ &\le \|v + \rho\|_{L^\infty}\|v - \rho - w + \zeta\|_{L^2_\eta(\mathbb{R}_+)} + \|v + \rho - w - \zeta\|_{L^\infty}\|w - \zeta\|_{L^2_\eta(\mathbb{R}_+)} \\ &\le C\|v - \rho\hat{v} - w + \zeta\hat{v}\|_{L^2_\eta(R_+)} + C\|v - w\|_{L^\infty} + C|\rho - \zeta| \le C\|\mathbf{v} - \mathbf{w}\|_{X^1_\eta}. \end{aligned}$$

Now using the mean value theorem we estimate I_8 by

$$
\begin{aligned}
I_8 &\le C\|g(|v|^2)-g(|\rho|^2)-g(|w|^2)+g(|\zeta|^2)\|_{L^2_\eta(\mathbb{R}_+)}\\
&\le C\Bigg(\int_0^\infty \eta(x)^2\left|\int_0^1 Dg\left(|\rho|^2+\tau\left[|v(x)|^2-|\rho|^2\right]\right)d\tau(|v(x)|^2-|\rho|^2)\right.\\
&\quad\left.+\int_0^1 Dg\left(|\zeta|^2+\tau\left[|w(x)|^2-|\zeta|^2\right]\right)d\tau(|w(x)|^2-|\zeta|^2)\right|^2 dx\Bigg)^{\frac{1}{2}}\\
&\le C\Bigg(\int_0^\infty \eta(x)^2\left|\int_0^1 Dg\left(|\rho|^2+\tau\left[|v(x)|^2-|\rho|^2\right]\right)d\tau(|v(x)|^2-|\rho|^2-|w(x)|^2+|\zeta|^2)\right|^2 dx\Bigg)^{\frac{1}{2}}\\
&\quad+\Bigg(C\int_0^\infty \eta(x)^2\bigg|\int_0^1 \Big(Dg\left(|\rho|^2+\tau\left[|v(x)|^2-|\rho|^2\right]\right)\\
&\quad-Dg\left(|\zeta|^2+\tau\left[|w(x)|^2-|\zeta|^2\right]\right)\Big)d\tau(|w(x)|^2-|\zeta|^2)\bigg|^2 dx\Bigg)^{\frac{1}{2}}\\
&\le C\left\||v|^2-|\rho|^2-|w|^2+|\zeta|^2\right\|_{L^2_\eta(\mathbb{R}_+)}\\
&\quad+C\Bigg(\int_0^\infty \eta(x)^2\int_0^1\left||\rho|^2+\tau\left[|v(x)|^2-|\rho|^2\right]-|\zeta|^2-\tau\left[|w(x)|^2-|\zeta|^2\right]\right|^2 d\tau\left||w(x)|^2-|\zeta|^2\right|^2 dx\Bigg)^{\frac{1}{2}}\\
&\le C\|\mathbf{v}-\mathbf{w}\|_{X^1_\eta}+C\left||\rho|^2-|\zeta|^2\right|\Bigg(\int_0^\infty \eta(x)^2\left||w(x)|^2-|\zeta|^2\right|^2 dx\Bigg)^{\frac{1}{2}}\\
&\quad+\Bigg(C\int_0^\infty \eta(x)^2\left||v(x)|^2-|\rho|^2-|w(x)|^2+|\zeta|^2\right|^2 dx\Bigg)^{\frac{1}{2}}\\
&\le C\|\mathbf{v}-\mathbf{w}\|_{X^1_\eta}+C|\rho-\zeta|\le C\|\mathbf{v}-\mathbf{w}\|_{X^1_\eta}.
\end{aligned}
$$

Summarizing, we have shown

$$\|f(v)-f(\rho)\hat{v}-f(w)+f(\zeta)\hat{v}\|_{L^2_\eta}\le I_1+I_3+I_5+I_7+I_8\le C\|\mathbf{v}-\mathbf{w}\|_{X^1_\eta}.$$

In addition,

$$|f(\rho)-f(\zeta)|\le|g(|\rho|^2)(\rho-\zeta)|+|(g(|\rho|^2)-g(|\zeta|^2))\zeta|\le C|\rho-\zeta|.$$

Now the Lipschitz estimate (3.19) for $k=-1,0$ follows by Lemma 3.6 and

$$
\begin{aligned}
\|\mathcal{F}(\mathbf{v})-\mathcal{F}(\mathbf{w})\|_{X^k_\eta} &\le \|\mathcal{L}_0(\mathbf{v}-\mathbf{w})\|_{X^k_\eta}+\left\|\begin{pmatrix} f(v)-f(w)\\ f(\rho)-f(\zeta)\end{pmatrix}\right\|_{X^k_\eta}\\
&\le C\|\mathbf{v}-\mathbf{w}\|_{X^{k+2}_\eta}+\left\|\begin{pmatrix} f(v)-f(w)\\ f(\rho)-f(\zeta)\end{pmatrix}\right\|_{X_\eta}\\
&\le C\|\mathbf{v}-\mathbf{w}\|_{X^{k+2}_\eta}+C\|\mathbf{v}-\mathbf{w}\|_{X^1_\eta}\le C\|\mathbf{v}-\mathbf{w}\|_{X^{k+2}_\eta}.
\end{aligned}
$$

This completes the proof. □

A simple consequence is that if $\mathbf{v}_\star$ is a stationary solution of (0.22) so is $a(\gamma)\mathbf{v}_\star$ a stationary solution for all $\gamma \in \mathcal{G}$, i.e.

$$\mathcal{F}(a(\gamma)\mathbf{v}_\star) = a(\gamma)\mathcal{F}(\mathbf{v}_\star) = 0.$$

Thus, the whole group orbit $\mathcal{O}(\mathbf{v}_\star)$ consists of stationary solutions of (0.22).

3.3 The linearized operator $\mathcal{L}$

to investigate nonlinear stability of traveling oscillating fronts it is essential to analyze the spectrum of the linearized operator from (0.26). It is defined by

$$\mathcal{L} : Y_\eta \to X_\eta, \quad \begin{pmatrix} u \\ \rho \end{pmatrix} \mapsto \mathcal{L}\begin{pmatrix} u \\ \rho \end{pmatrix} = \begin{pmatrix} Au_{xx} + cu_x + S_\omega u + Df(v_\star)u \\ S_\omega \rho + Df(v_\infty)\rho \end{pmatrix}.$$

The linearized operator $\mathcal{L}$ is obtained when taking the Frechét derivative of the nonlinear operator $\mathcal{F}$ from (0.23), i.e. $\mathcal{L} = D\mathcal{F}(\mathbf{v}_\star)$. Then the Cauchy problem (0.22) can be written as a semilinear equation:

$$\mathbf{u}_t = \mathcal{L}\mathbf{u} + \mathcal{N}(\mathbf{u}), \quad \mathbf{u}(0) = \mathbf{u}_0 \in X_\eta,$$

where

$$\mathcal{N}(\mathbf{u}) = \begin{pmatrix} f(u) - Df(v_\star)u \\ f(\rho) - Df(v_\infty)\rho \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u \\ \rho \end{pmatrix}$$

is the remaining nonlinear part. This shows the importance of the linearized operator. The first essential step in proving nonlinear stability is the spectral stability of traveling oscillating fronts. Spectral stability means that the spectrum of the linearization $\mathcal{L}$ is included in the strict left half-plane except for a zero eigenvalue, which is caused by the equivariance. In the case of traveling waves in parabolic evolution equations this can be found, for instance, in [56]. We use the same approach to show spectral stability of traveling oscillating fronts. Here it is important to note that spectral stability can only be obtained in the exponentially weighted spaces, since in the classical unweighted spaces the essential spectrum touches the imaginary axis at the origin and includes the zero eigenvalue. However, by using exponential weights, the spectrum is pushed to left of the imaginary axis and we obtain spectral stability. In particular, the exponential weight causes a spectral gap in the spectrum of the linearized operator. Since the Lie group $\mathcal{G}$ is two dimensional we will see that the isolated zero eigenvalue has in fact at least algebraic multiplicity two. Taking Assumption 4 into account we obtain algebraic multiplicity equal to two. In addition to the spectral stability, we prove that the operator $\mathcal{L}$ is a sectorial operator. Thus, we can apply the classical approach for semilinear parabolic

equations from [32] and [45] to show existence and uniqueness of solutions to (0.22). This is done in Section 3.4 by using estimates for the semigroup generated by $\mathcal{L}$.
In the next lemma we prove the simple observation that $\mathcal{L}$ defines a continuous, linear operator from Y_η to X_η as long as $0 \le \mu < \min(\mu_\star, 2)$ with $\mu_\star$ from Theorem 2.6. In addition, we show in Lemma 3.10 that $\mathcal{L}$ defines a closed linear operator on X_η with $\mathcal{D}(\mathcal{L}) = Y_\eta$. This is a consequence of resolvent estimates for large s. We will then determine the spectrum of $\mathcal{L}$ when considered as a closed operator on X_η.

Lemma 3.9. *Let Assumptions 1 and 2 be satisfied and* $0 \le \mu < \min(\mu_\star, 2)$ *with* $\mu_\star$ *from Theorem 2.6. Then the operator* $\mathcal{L} : Y_\eta \to X_\eta$ *is a continuous, linear operator, i.e.* $\mathcal{L} \in L[Y_\eta, X_\eta]$.

Proof. By Lemma 3.6 it is sufficient to show $\mathcal{L} - \mathcal{L}_0 \in L[X_\eta]$. Let $\mathbf{v} = (v, \rho)^\top \in X_\eta$ and let $C = C(\mathbf{v}_\star) > 0$ denote a universal constant. Then using Assumption 1 and Theorem 2.6 we estimate

$$\begin{aligned}
\|(\mathcal{L} - \mathcal{L}_0)\mathbf{v}\|_{X_\eta} &\le |Df(v_\infty)\rho| + \|Df(v_\star)v - Df(v_\infty)\rho\hat{v}\|_{L^2_\eta} \\
&\le C|\rho| + \|Df(v_\star)(v - \rho\hat{v})\|_{L^2_\eta} + \|(Df(v_\star) - Df(v_\infty))\rho\hat{v}\|_{L^2_\eta} \\
&\le C\|\mathbf{v}\|_{X_\eta} + \|(Df(v_\star) - Df(v_\infty))\rho\hat{v}\|_{L^2_\eta(\mathbb{R}_-)} + \|(Df(v_\star) - Df(v_\infty))\rho\hat{v}\|_{L^2_\eta(\mathbb{R}_+)} \\
&\le C\|\mathbf{v}\|_{X_\eta} + C|\rho| + \left(\int_0^\infty \eta^2(x)|Df(v_\star(x)) - Df(v_\infty)|^2 dx\right)^{\frac{1}{2}} |\rho| \\
&\le C\|\mathbf{v}\|_{X_\eta} + C\|v_\star - v_\infty\|_{L^2_\eta(\mathbb{R}_+)}|\rho| \le C\|\mathbf{v}\|_{X_\eta}.
\end{aligned}$$

Hence the assertion is proven. □

3.3.1 Resolvent estimates

We study the spectrum of the linearized operator $\mathcal{L}$ and are interested in solutions of the resolvent equation

$$(sI - \mathcal{L})\mathbf{u} = \mathbf{r}, \quad s \in \mathbb{C}, \mathbf{r} \in X_\eta. \tag{3.20}$$

In the following we denote the components of $\mathbf{r}$ by $(r, \zeta)^\top$, if necessary. As a next step we show a-priori estimates for solutions $\mathbf{u} \in Y_\eta$ of (3.20) for arbitrary $\mathbf{r} \in X_\eta$ as long as $|s|$ is sufficiently large and $s \in \mathbb{C}$ lies in the exterior of some sector opening to the left, see Figure 3.1. The approach is based on energy estimates from [39], [40].

Lemma 3.10. *Let Assumptions 1 and 2 be satisfied and let* $0 \le \mu < \min(\mu_\star, 2)$ *with* $\mu_\star$ *from Theorem 2.6. Then* $\mathcal{L} : Y_\eta \subset X_\eta \to X_\eta$ *is a closed, densely defined, linear operator on* X_η. *Moreover, there exist* $\varepsilon_0, R_0, C > 0$ *such that for all*

$$s \in \Omega_0 := \left\{ s \in \mathbb{C} : |s| \ge R_0, |\arg(s)| \le \frac{\pi}{2} + \varepsilon_0 \right\}$$

the equation (3.20) *with* $\mathbf{u} \in Y_\eta$ *and* $\mathbf{r} \in X_\eta$ *implies*

$$|s|\|\mathbf{u}\|^2_{X_\eta} + \|u_x\|^2_{L^2_\eta} \le \frac{C}{|s|}\|\mathbf{r}\|^2_{X_\eta} \tag{3.21}$$

$$|s|^2\|\mathbf{u}\|^2_{X_\eta} + |s|\|u_x\|^2_{L^2_\eta} + \|u_{xx}\|^2_{L^2_\eta} \le C\|\mathbf{r}\|^2_{X_\eta}. \tag{3.22}$$

Proof. First we show that (3.22) implies the closedness of $\mathcal{L}$. For this purpose, let $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subset Y_\eta$ with $\mathbf{u}_n \to \mathbf{u}$ in X_η and $\mathcal{L}\mathbf{u}_n \to \mathbf{w}$ in X_η. Pick $s_0 \in \Omega_0$ with $|s_0| \ge 1$. Then (3.22) implies

$$\begin{aligned}\|\mathbf{u}_n - \mathbf{u}_m\|^2_{Y_\eta} &\le |s_0|^2\|\mathbf{u}_n - \mathbf{u}_m\|^2_{X_\eta} + |s_0|\|u_{n,x} - u_{m,x}\|^2_{L^2_\eta} + \|u_{n,xx} - u_{m,xx}\|^2_{L^2_\eta}\\ &\le C_1\|s_0(\mathbf{u}_n - \mathbf{u}_m) - \mathcal{L}\mathbf{u}_n - \mathcal{L}\mathbf{u}_m\|^2_{X_\eta} \to 0, \quad n, m \to \infty.\end{aligned}$$

Thus, $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Y_η and there is $\tilde{\mathbf{u}} \in Y_\eta$ with $\mathbf{u}_n \to \tilde{\mathbf{u}}$ in Y_η. We conclude $\mathbf{u} = \tilde{\mathbf{u}} \in Y_\eta$ and $\mathbf{u}_n \to \mathbf{u}$ in Y_η. Further, using Lemma 3.9 we obtain

$$\begin{aligned}\|\mathcal{L}\mathbf{u} - \mathbf{w}\|_{X_\eta} &\le \|\mathcal{L}(\mathbf{u} - \mathbf{u}_n)\|_{X_\eta} + \|\mathcal{L}\mathbf{u}_n - \mathbf{w}\|_{X_\eta}\\ &\le C_2\|\mathbf{u} - \mathbf{u}_n\|_{Y_\eta} + \|\mathcal{L}\mathbf{u}_n - \mathbf{w}\|_{X_\eta} \to 0, \quad n \to \infty.\end{aligned}$$

Thus $\mathcal{L}\mathbf{u} = \mathbf{w}$ and the closedness is proven.
It is left to show the estimates (3.21) and (3.22). We start with (3.21). For this purpose let $s \in \Omega_0$ with R_0 and ε_0 still to be determined. For the proof we set $C := Df(v_\star)$, $C_\infty := Df(v_\infty)$ as well as $(\cdot,\cdot) = (\cdot,\cdot)_{L^2_\eta(\mathbb{R},\mathbb{R}^2)}$ for the inner product on L^2_η. Take the inner product of (3.20) with $\mathbf{u}$ in X_η to obtain

$$\begin{aligned}(\mathbf{u},\mathbf{r})_{X_\eta} &= (\mathbf{u},(sI - \mathcal{L})\mathbf{u})_{X_\eta} = \left(\begin{pmatrix}u\\ \rho\end{pmatrix}, \begin{pmatrix}su - Au_{xx} - cu_x - S_\omega u - Cu\\ s\rho - S_\omega\rho - C_\infty\rho\end{pmatrix}\right)_{X_\eta}\\ &= \rho^\top(sI - S_\omega - C_\infty)\rho + (u - \rho\hat{v}, su - Au_{xx} - cu_x - S_\omega u - Cu - (s\rho - S_\omega\rho - C_\infty\rho)\hat{v})\\ &= s\|\mathbf{u}\|^2_{X_\eta} - \rho^\top S_\omega\rho - \rho^\top C_\infty\rho\\ &\quad - (u - \rho\hat{v}, Au_{xx})_{L^2_\eta} - c(u - \rho\hat{v}, u_x) - (u - \rho\hat{v}, S_\omega(u - \rho\hat{v})) - (u - \rho\hat{v}, Cu - C_\infty\rho\hat{v}).\end{aligned}$$

The integration by parts formula from Lemma 3.2 leads to

$$\begin{aligned}&s\|\mathbf{u}\|^2_{X_\eta} + (u_x - \rho\hat{v}_x, Au_x)_{L^2_\eta}\\ &= \rho^\top(S_\omega + C_\infty)\rho - 2(\eta'\eta^{-1}(u - \rho\hat{v}), Au_x) + c(u - \rho\hat{v}, u_x)_{L^2_\eta}\\ &+ (u - \rho\hat{v}, S_\omega(u - \rho\hat{v}))_{L^2_\eta} + (u - \rho\hat{v}, Cu - C_\infty\rho\hat{v})_{L^2_\eta} + (\mathbf{u},\mathbf{r})_{X_\eta}.\end{aligned} \tag{3.23}$$

Further we use Cauchy-Schwarz, Young's inequality with $\varepsilon_i > 0$, $i = 1, 2, 3, 4$ and Proposition 2.7 to obtain the estimates

$$
\begin{aligned}
|(u_x - \rho\hat{v}_x, Au_x)| &\leq \|u_x - \rho\hat{v}_x\|_{L^2_\eta}\|Au_x\|_{L^2_\eta} \leq |A|\left(\|u_x\|^2_{L^2_\eta} + \|\rho\hat{v}_x\|_{L^2_\eta}\|u_x\|_{L^2_\eta}\right) \\
&\leq |A|\left(\|u_x\|^2_{L^2_\eta} + \frac{1}{4\varepsilon_1}\|\rho\hat{v}_x\|^2_{L^2_\eta} + \varepsilon_1\|u_x\|^2_{L^2_\eta}\right) \\
&= |A|(1+\varepsilon_1)\|u_x\|^2_{L^2_\eta} + \frac{|A|}{(2-\mu)\varepsilon_1}|\rho|^2,
\end{aligned} \tag{3.24}
$$

$$
\begin{aligned}
|(\eta'\eta^{-1}(u-\rho\hat{v}), Au_x)| &\leq \mu\|u-\rho\hat{v}\|_{L^2_\eta}\|Au_x\|_{L^2_\eta} \\
&\leq \frac{\mu^2|A|}{4\varepsilon_2}\|u-\rho\hat{v}\|^2_{L^2_\eta} + \varepsilon_2|A|\|u_x\|^2_{L^2_\eta},
\end{aligned} \tag{3.25}
$$

$$
|c(u-\rho\hat{v}, u_x)| \leq |c|\|u-\rho\hat{v}\|_{L^2_\eta}\|u_x\|_{L^2_\eta} \leq \frac{|c|}{4\varepsilon_3}\|u-\rho\hat{v}\|^2_{L^2_\eta} + |c|\varepsilon_3\|u_x\|^2_{L^2_\eta}, \tag{3.26}
$$

$$
|(u-\rho\hat{v}, S_\omega(u-\rho\hat{v}))| \leq |\omega|\|u-\rho\hat{v}\|^2_{L^2_\eta}, \tag{3.27}
$$

$$
\begin{aligned}
|(u-\rho\hat{v}, Cu - C_\infty\rho\hat{v})| &\leq \|u-\rho\hat{v}\|_{L^2_\eta}\|Cu - C_\infty\rho\hat{v}\|_{L^2_\eta} \\
&\leq \|C\|_{L^\infty}\|u-\rho\hat{v}\|^2_{L^2_\eta} + \|u-\rho\hat{v}\|_{L^2_\eta}\|(C-C_\infty)\rho\hat{v}\|_{L^2_\eta} \\
&\leq \left(\|C\|_{L^\infty} + \frac{1}{4\varepsilon_4}\right)\|u-\rho\hat{v}\|^2_{L^2_\eta} + \varepsilon_4\|(C-C_\infty)\rho\hat{v}\|^2_{L^2_\eta} \\
&\leq \left(\|C\|_{L^\infty} + \frac{1}{4\varepsilon_4}\right)\|u-\rho\hat{v}\|^2_{L^2_\eta} + K_C\varepsilon_4|\rho|^2.
\end{aligned} \tag{3.28}
$$

To obtain (3.28) we used the fact that Assumption 1 and 2 imply, together with Theorem 2.6 and Proposition 2.7, for some $K_C > 0$ the estimate:

$$
\begin{aligned}
&\|(C-C_\infty)\rho\hat{v}\|^2_{L^2_\eta} \leq \|(C-C_\infty)\rho\hat{v}\|^2_{L^2_\eta(\mathbb{R}_-)} + \|(C-C_\infty)\rho\hat{v}\|^2_{L^2_\eta(\mathbb{R}_+)} \\
&\leq \frac{\|C-C_\infty\|^2_{L^\infty}}{4-2\mu}|\rho|^2 + \int_0^\infty \eta^2(x)|Df(v_\star(x)) - Df(v_\infty)|^2 dx|\rho|^2 \\
&\leq \frac{\|C-C_\infty\|^2_{L^\infty}}{4-2\mu}|\rho|^2 + L^2\|v_\star - v_\infty\|^2_{L^2_\eta}|\rho|^2 \leq K_C|\rho|^2.
\end{aligned}
$$

Take the absolute value in (3.23) and use (3.24)-(3.28) with $\varepsilon_i = 1$ to obtain

$$
|s|\|\mathbf{u}\|^2_{X_\eta} \leq K_0\|u_x\|^2_{L^2_\eta} + K_1\|\mathbf{u}\|^2_{X_\eta} + \|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta}. \tag{3.29}
$$

Here K_0 and K_1 can be chosen as

$$K_0 := 3|A| + |c|, \quad K_1 := \left(\frac{\mu^2}{2} + \frac{1}{2-\mu}\right)|A| + 2|\omega| + 2\|C\|_{L^\infty} + \frac{1+|c|}{4} + K_C.$$

Note that $(u_x - \rho\hat{v}_x, Au_x) = \alpha_1\|u_x\|^2_{L^2_\eta} - (\rho\hat{v}_x, Au_x)$ and the second part can be estimated by

$$|(\rho\hat{v}_x, Au_x)| \leq |A|\|\hat{v}_x\|_{L^2_\eta}|\rho|\|u_x\|_{L^2_\eta} \leq \frac{|A|}{(2-\mu)\varepsilon_5}|\rho|^2 + \varepsilon_5|A|\|v_x\|^2_{L^2_\eta}. \tag{3.30}$$

In contrast to (3.29), when taking real part in (3.23) we obtain by using Cauchy-Schwarz, Young's inequality and (3.25)-(3.28) as well as (3.30) with $\varepsilon_2 = \varepsilon_5 = \frac{\alpha_1}{8|A|}$, $\varepsilon_3 = \frac{\alpha_1}{4|c|}$, $\varepsilon_4 = 1$ the estimate

$$\begin{aligned}\operatorname{Re} s\|\mathbf{u}\|^2_{X_\eta} + \alpha_1\|u_x\|^2_{L^2_\eta} &\leq \big(\varepsilon_5|A| + \varepsilon_2|A| + \varepsilon_3|c|\big)\|u_x\|^2_{L^2_\eta} + K_2\|\mathbf{u}\|^2_{X_\eta} + \|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta} \\ &\leq \frac{\alpha_1}{2}\|u_x\|^2_{L^2_\eta} + K_2\|\mathbf{u}\|^2_{X_\eta} + \|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta},\end{aligned}$$

where K_2 can be chosen as

$$K_2 := \left(4\mu^2 + \frac{8}{2-\mu}\right)\frac{|A|^2}{\alpha_1} + 2|\omega| + 2\|C\|_{L^\infty} + \frac{|c|^2}{\alpha_1} + \frac{1}{4} + K_C.$$

This leads to

$$\operatorname{Re} s\|\mathbf{u}\|^2_{X_\eta} + \frac{\alpha_1}{2}\|u_x\|^2_{L^2_\eta} \leq K_2\|\mathbf{u}\|^2_{X_\eta} + \|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta}. \tag{3.31}$$

The rest of the proof falls naturally into three cases depending on the value of s in the complex plane, cf. Figure 3.1.

Case 1: $\operatorname{Re} s \geq |\operatorname{Im} s|$, $\operatorname{Re} s > 0$, $|s| \geq 2\sqrt{2}K_2$.

We have $0 < \operatorname{Re} s \leq |s| \leq \sqrt{2}\operatorname{Re} s$. Therefore, using (3.31) and Young's inequality with $\varepsilon = \frac{\sqrt{2}}{|s|}$, we obtain

$$\begin{aligned}\frac{|s|}{\sqrt{2}}\|\mathbf{u}\|^2_{X_\eta} + \frac{\alpha_1}{2}\|u_x\|^2_{L^2_\eta} &\leq \frac{|s|}{2\sqrt{2}}\|\mathbf{u}\|^2_{X_\eta} + \|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta} \\ &\leq \frac{|s|}{2\sqrt{2}}\|\mathbf{u}\|^2_{X_\eta} + \frac{|s|}{4\sqrt{2}}\|\mathbf{u}\|^2_{X_\eta} + \frac{\sqrt{2}}{|s|}\|\mathbf{r}\|^2_{X_\eta}.\end{aligned}$$

Thus,

$$\frac{|s|}{4\sqrt{2}}\|\mathbf{u}\|^2_{X_\eta} + \frac{\alpha_1}{2}\|u_x\|^2_{L^2_\eta} \leq \frac{\sqrt{2}}{|s|}\|\mathbf{r}\|^2_{X_\eta}.$$

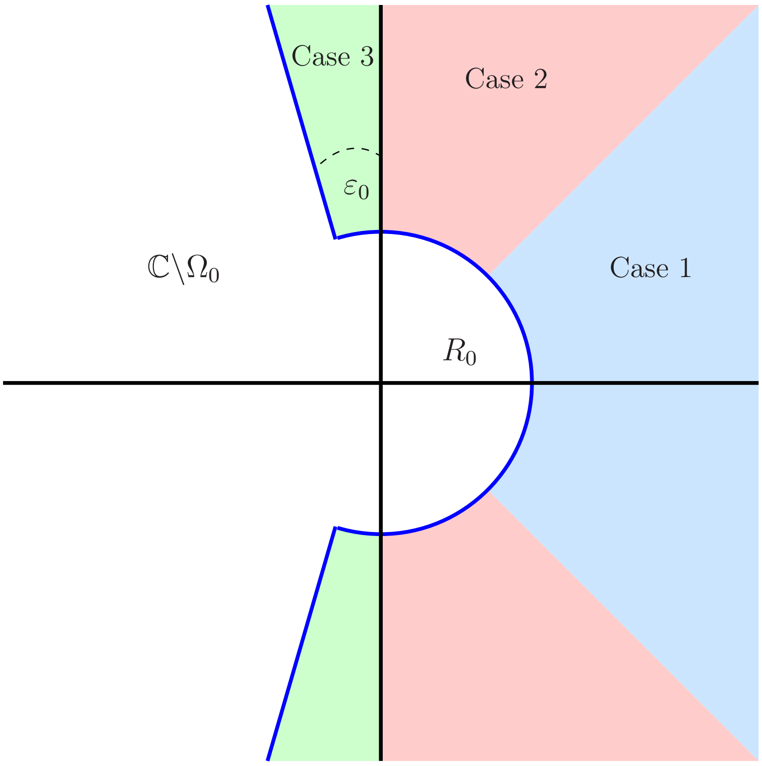


Figure 3.1: The set $\Omega_0 \subset \mathbb{C}$ from Lemma 3.10.

Setting $C_1 = \max(8, 2\sqrt{2}\alpha_1^{-1})$ yields

$$|s|\|\mathbf{u}\|_{X_\eta}^2 + \|u_x\|_{L^2_\eta}^2 \leq \frac{2C_1}{|s|}\|\mathbf{r}\|_{X_\eta}^2.$$

Case 2: $|\operatorname{Im} s| \geq \operatorname{Re} s \geq 0$.
From (3.31) we have

$$\|u_x\|_{L^2_\eta} \leq \frac{2}{\alpha_1}\left(K_2\|\mathbf{u}\|_{X_\eta}^2 + \|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta}\right).$$

Use this in (3.29) to obtain

$$\begin{aligned}
|s|\|\mathbf{u}\|_{X_\eta}^2 &\leq \frac{2K_0}{\alpha_1}\left(K_2\|\mathbf{u}\|_{X_\eta}^2 + \|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta}\right) + K_1\|\mathbf{u}\|_{X_\eta}^2 + \|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta} \\
&= \left(\frac{2K_0K_2}{\alpha_1} + K_1\right)\|\mathbf{u}\|_{X_\eta}^2 + \left(\frac{2K_0}{\alpha_1} + 1\right)\|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta} \\
&\leq K_3\left(\|\mathbf{u}\|_{X_\eta}^2 + \|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta}\right)
\end{aligned}$$

with $K_3 := \max\left(\frac{2K_0K_2}{\alpha_1} + K_1, \frac{2K_0}{\alpha_1} + 1\right)$. Take $|s| > 2K_3$ to observe by Young's inequality with $\varepsilon = |s|^{-1}$

$$|s|\|\mathbf{u}\|^2_{X_\eta} \le \frac{|s|}{2}\|\mathbf{u}\|^2_{X_\eta} + K_3\|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta} \le \frac{|s|}{2}\|\mathbf{u}\|^2_{X_\eta} + \frac{|s|}{4}\|\mathbf{u}\|^2_{X_\eta} + \frac{K_3^2}{|s|}\|\mathbf{r}\|^2_{X_\eta}.$$

Hence

$$\frac{|s|}{4}\|\mathbf{u}\|^2_{X_\eta} \le \frac{K_3^2}{|s|}\|\mathbf{r}\|^2_{X_\eta}. \tag{3.32}$$

Using (3.31), (3.32) and taking $|s| \ge 4K_2$ yields by Young's inequality with $\varepsilon = |s|^{-1}$

$$\begin{aligned}\frac{\alpha_1}{2}\|u_x\|^2_{L^2_\eta} &\le \frac{|s|}{4}\|\mathbf{u}\|^2_{X_\eta} + \|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta} \le \frac{|s|}{4}\|\mathbf{u}\|^2_{X_\eta} + \frac{|s|}{4}\|\mathbf{u}\|^2_{X_\eta} + \frac{1}{|s|}\|\mathbf{r}\|^2_{X_\eta} \\ &= \frac{|s|}{2}\|\mathbf{u}\|^2_{X_\eta} + \frac{1}{|s|}\|\mathbf{r}\|^2_{X_\eta} \le \frac{2K_3^2+1}{|s|}\|\mathbf{r}\|^2_{X_\eta} = \frac{K_4}{|s|}\|\mathbf{r}\|^2_{X_\eta}, \quad K_4 := 2K_3^2 + 1.\end{aligned} \tag{3.33}$$

Combining (3.32) and (3.33) to obtain

$$|s|\|\mathbf{u}\|^2_{X_\eta} + \|u_x\|^2_{L^2_\eta} \le \frac{4K_3^2}{|s|}\|\mathbf{r}\|^2_{X_\eta} + \frac{2K_4}{\alpha_1|s|}\|\mathbf{r}\|^2_{X_\eta} = \frac{C_2}{|s|}\|\mathbf{r}\|^2_{X_\eta}, \quad C_2 := 4K_3^2 + \frac{2K_5}{\alpha_1}.$$

Case 3: $\operatorname{Re} s \le 0$, $|\operatorname{Re} s| \le \varepsilon_0|\operatorname{Im} s|$. Using (3.29) and (3.31) yields

$$\begin{aligned}&|\operatorname{Im} s|\|\mathbf{u}\|^2_{X_\eta} \le |s|\|\mathbf{u}\|^2_{X_\eta} \le K_0\|u_x\|^2_{L^2_\eta} + K_1\|\mathbf{u}\|^2_{X_\eta} + \|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta} \\ &\le \frac{2K_0}{\alpha_1}\left(|\operatorname{Re} s|\|\mathbf{u}\|^2_{X_\eta} + K_2\|\mathbf{u}\|^2_{X_\eta} + \|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta}\right) + K_1\|\mathbf{u}\|^2_{X_\eta} + \|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta}.\end{aligned}$$

Let $0 < \varepsilon_0 < \frac{\alpha_1}{4K_0}$. Then $\frac{2K_0}{\alpha_1}|\operatorname{Re} s| \le \frac{|\operatorname{Im} s|}{2}$ and we obtain

$$\frac{|\operatorname{Im} s|}{2}\|\mathbf{u}\|^2_{X_\eta} \le \left(\frac{2K_0K_2}{\alpha_1} + K_1\right)\|\mathbf{u}\|^2_{X_\eta} + \left(\frac{2K_0}{\alpha_1} + 1\right)\|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta}.$$

Hence

$$|\operatorname{Im} s|\|\mathbf{u}\|^2_{X_\eta} \le K_5\left(\|\mathbf{u}\|^2_{X_\eta} + \|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta}\right), \quad K_5 := \max\left(\frac{4K_0K_2}{\alpha_1} + 2K_1, \frac{4K_0}{\alpha_1} + 2\right).$$

Since $|s| \le \sqrt{1+\varepsilon_0^2}|\operatorname{Im} s|$ we obtain

$$|s|\|\mathbf{u}\|^2_{X_\eta} \le K_5\sqrt{1+\varepsilon_0^2}\left(\|\mathbf{u}\|^2_{X_\eta} + \|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta}\right).$$

Now take $|s| > 2K_5\sqrt{1+\varepsilon_0^2}$ and use Young's inequality with $\varepsilon = |s|^{-1}$ to observe

$$\begin{aligned} |s|\|\mathbf{u}\|_{X_\eta}^2 &\leq \frac{|s|}{2}\|\mathbf{u}\|_{X_\eta}^2 + K_5\sqrt{1+\varepsilon_0^2}\|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta} \\ &\leq \frac{|s|}{2}\|\mathbf{u}\|_{X_\eta}^2 + \frac{|s|}{4}\|\mathbf{u}\|_{X_\eta}^2 + \frac{K_5^2(1+\varepsilon_0^2)}{|s|}\|\mathbf{r}\|_{X_\eta}^2. \end{aligned}$$

This yields

$$\frac{|s|}{4}\|\mathbf{u}\|_{X_\eta}^2 \leq \frac{K_5^2(1+\varepsilon^2)}{|s|}\|\mathbf{r}\|_{X_\eta}^2. \tag{3.34}$$

Take $|s| \geq 2K_2$ in (3.31) and use (3.34) as well as Young's inequality with $\varepsilon = \frac{1}{2|s|}$ to obtain

$$\begin{aligned} \frac{\alpha_1}{2}\|u_x\|_{L_\eta^2}^2 &\leq |\operatorname{Re} s|\|\mathbf{u}\|_{X_\eta}^2 + \frac{|s|}{2}\|\mathbf{u}\|_{X_\eta}^2 + \|\mathbf{u}\|_{X_\eta}\|\mathbf{r}\|_{X_\eta} \\ &\leq |s|\|\mathbf{u}\|_{X_\eta}^2 + \frac{|s|}{2}\|\mathbf{u}\|_{X_\eta}^2 + \frac{|s|}{2}\|\mathbf{u}\|_{X_\eta}^2 + \frac{1}{2|s|}\|\mathbf{r}\|_{X_\eta}^2 \\ &\leq 2|s|\|\mathbf{u}\|_{X_\eta}^2 + \frac{1}{2|s|}\|\mathbf{r}\|_{X_\eta}^2 \leq \frac{K_6}{|s|}\|\mathbf{r}\|_{X_\eta}^2, \quad K_6 := 8K_5^2(1+\varepsilon_0^2) + \tfrac{1}{2}. \end{aligned} \tag{3.35}$$

Combining (3.34) and (3.35) shows

$$|s|\|\mathbf{u}\|_{X_\eta}^2 + \|u_x\|_{L_\eta^2}^2 \leq \frac{C_3}{|s|}\|\mathbf{r}\|_{X_\eta}^2, \quad C_3 := 4K_5^2(1+\varepsilon_0^2) + \frac{2K_6}{\alpha_1}.$$

Hence (3.21) is proven.
It remains to prove (3.22). First note that in (3.28) we have shown

$$\|Cu - C_\infty\rho\hat{v}\|_{L_\eta^2}^2 \leq K_7\|\mathbf{u}\|_{X_k}^2.$$

Now see that (3.20) implies in X_η the equation

$$\begin{pmatrix} u_{xx} \\ 0 \end{pmatrix} = \begin{pmatrix} A^{-1}(-su + cu_x + S_\omega u + Cu + r) \\ A^{-1}(-s\rho + S_\omega\rho + C_\infty\rho + \zeta) \end{pmatrix}.$$

Thus, it is easy to see for $|s| \geq 1$ there is $\tilde{C} > 0$ such that

$$\begin{aligned} \|u_{xx}\|_{L_\eta^2}^2 &\leq \tilde{C}\left(|s|^2\|\mathbf{u}\|_{X_\eta}^2 + \|u_x\|_{L_\eta^2}^2 + \|\mathbf{u}\|_{X_\eta}^2 + \|\mathbf{r}\|_{X_\eta}^2\right) \\ &\leq 2\tilde{C}\left(|s|^2\|\mathbf{u}\|_{X_\eta}^2 + |s|\|u_x\|_{L_\eta^2}^2 + \|\mathbf{r}\|_{X_\eta}^2\right). \end{aligned}$$

Finally the assertion is proven since (3.21) implies for some $C > 0$ the estimate

$$\begin{aligned} &|s|^2\|\mathbf{u}\|_{X_\eta}^2 + |s|\|u_x\|_{L_\eta^2}^2 + \|u_{xx}\|_{L_\eta^2}^2 \\ &\leq (2\tilde{C}+1)\left(|s|^2\|\mathbf{u}\|_{X_\eta}^2 + |s|\|u_x\|_{L_\eta^2}^2\right) + 2\tilde{C}\|\mathbf{r}\|_{X_\eta}^2 \leq C\|\mathbf{r}\|_{X_\eta}^2. \end{aligned}$$

□

The a-priori estimates from Lemma 3.10, in particular (3.22), imply the uniqueness of solutions of the resolvent equation (3.20). Hence the operator $sI - \mathcal{L}$ is one-to-one for $s \in \Omega_0$ and there are no eigenvalues in the region Ω_0 according to Definition 1.10. In order to conclude that Ω_0 is part of the resolvent set we still need to determine the Fredholm properties of $sI - \mathcal{L}$. Below we show that $sI - \mathcal{L}$ is Fredholm of index 0 for $s \in \Omega_0$, if the angle ε_0 in the definition of Ω_0 is sufficiently small. Then we conclude $\Omega_0 \subset \rho(\mathcal{L})$ and the equation (3.20) attains a unique solution for every $\mathbf{r} \in X_\eta$.
In this case Lemma 3.10 implies that the resolvent must decay with rate $|s|^{-1}$ in the operator norm. In particular, there is $C > 0$ such that for all $s \in \Omega_0 \cap \rho(\mathcal{L})$ we have the estimate for the resolvent

$$\left\|(sI - \mathcal{L})^{-1}\mathbf{r}\right\|_{X_\eta^k} \leq C|s|^{\frac{k}{2}-1}\|\mathbf{r}\|_{X_\eta}, \quad k = 0, 1, 2.$$

Hence, if $\Omega_0 \subset \rho(\mathcal{L})$, the operator $\mathcal{L}$ defines a sectorial operator, cf. [32]. Therefore, by the theory from [32], [45], $\mathcal{L}$ generates an analytic semigroup on X_η which is important for proving existence of solutions of (0.22) and the nonlinear stability of traveling oscillating fronts. Additionally to the resolvent estimate (3.22) we now show regularity estimates for the solution of (3.20). Then the semigroup generated by $\mathcal{L}$ is also defined from X_η^1 to X_η^1.

Lemma 3.11. *Let Assumptions 1 and 2 be satisfied and let* $0 \leq \mu < \min(\mu_\star, 2)$ *with* $\mu_\star$ *from Theorem 2.6 and let* Ω_0 *be from Lemma 3.10.*

i) For $s \in \rho(\mathcal{L})$ *there is* $C_1 = C_1(s) > 0$ *such that for all* $\mathbf{r} \in X_\eta^1$ *the equation* (3.20) *has a unique solution* $\mathbf{u} \in X_\eta^3$ *with*

$$\|\mathbf{u}\|_{X_\eta^3} \leq C_1\|\mathbf{r}\|_{X_\eta^1}.$$

ii) There is $C_2 > 0$ *such that for all* $s \in \Omega_0$ *the equation* (3.20) *with* $\mathbf{r} \in X_\eta^1$ *and* $\mathbf{u} \in Y_\eta$ *implies* $\mathbf{u} \in X_\eta^3$ *and*

$$|s|^2\|\mathbf{u}\|_{X_\eta^1}^2 + |s|\|u_{xx}\|_{L_\eta^2}^2 + \|u_{xxx}\|_{L_\eta^2}^2 \leq C_2\|\mathbf{r}\|_{X_\eta^1}^2.$$

Proof. Suppose $\mathbf{r} \in X_\eta^1$ and $s \in \rho(\mathcal{L})$. Then the resolvent equation (3.20) has a unique solution $\mathbf{u} = (u, \rho)^\top \in Y_\eta$ with $\|\mathbf{u}\|_{Y_\eta} \leq C\|\mathbf{r}\|_{X_\eta}$ for some $C > 0$ depending on s. Using Assumption 1 and 2 and Theorem 2.6 we find $\tilde{C} > 0$ such that

$$\begin{aligned}\|D^2f(v_\star)[v_{\star,x}, u]\|_{L_\eta^2} &\leq \|D^2f(v_\star)[v_{\star,x}, u - \rho\hat{v}]\|_{L_\eta^2} + \|D^2f(v_\star)[v_{\star,x}, \hat{v}]\|_{L_\eta^2}|\rho| \\ &\leq \tilde{C}(\|u - \rho\hat{v}\|_{L_\eta^2} + |\rho|) \leq 2\tilde{C}C\|\mathbf{r}\|_{X_\eta}.\end{aligned} \tag{3.36}$$

Hence $D^2f(v_\star)[v_{\star,x}, u] \in L_\eta^2$ and therefore the equation

$$(sI - \mathcal{L})\mathbf{w} = \begin{pmatrix} r_x + D^2f(v_\star)[v_{\star,x}, u] \\ 0 \end{pmatrix} \in X_\eta \tag{3.37}$$

has a solution $\mathbf{w} = (w, \zeta)^\top \in Y_\eta$ Since $(sI - \mathcal{L})\mathbf{u} = \mathbf{r} \in X_\eta^1$ we obtain using integration by parts for all $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{R})$

$$\int_{\mathbb{R}} u_{xx}(x)\varphi_x(x)dx = \int_{\mathbb{R}} A^{-1}[su - cu_x - S_\omega u - Df(v_\star)u - r](x)\varphi_x(x)dx$$
$$= -\int_{\mathbb{R}} A^{-1}[su_x - cu_{xx} - S_\omega u_x - Df(v_\star)u_x - D^2 f(v_\star)[v_{\star,x}, u] - r_x](x)\varphi(x)dx.$$

Thus $u_x \in H_\eta^1 \cap H_{\mathrm{loc}}^2$ with

$$u_{xxx} = A^{-1}[su_x - cu_{xx} - S_\omega u_x - Df(v_\star)u_x - D^2 f(v_\star)[v_{\star,x}, u] - r_x] \in L_\eta^2.$$

Therefore, $u_x \in H_\eta^2$ solves

$$\begin{aligned}(sI - \mathcal{L})\begin{pmatrix} u_x \\ 0 \end{pmatrix} &= \begin{pmatrix} (sI - L)u_x \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \partial_x[su - Au_{xx} - cu_x - S_\omega u - Df(v_\star)u] + D^2 f(v_\star)[v_{\star,x}, u] \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} r_x - D^2 f(v_\star)[v_{\star,x}, u] \\ 0 \end{pmatrix}.\end{aligned}$$

Since $\mathbf{w}$ is the unique solution of (3.37) we conclude $w = u_x$, $\zeta = 0$. This proves i). Now ii) follows by (3.36) and by applying Lemma 3.10 to the equation

$$(sI - \mathcal{L})\begin{pmatrix} u_x \\ 0 \end{pmatrix} = \begin{pmatrix} r_x - D^2 f(v_\star)[v_{\star,x}, u] \\ 0 \end{pmatrix}.$$

□

3.3.2 Fredholm theory and spectral analysis

Our aim is to determine the spectrum of the linearized operator $\mathcal{L}$ on X_η. Assumption 4 together with Lemma 3.10 guarantee that the set $\tilde{\Omega} = \{s \in \mathbb{C} : \mathrm{Re}\, s > -\gamma\} \cup \Omega_0$ does not contain any eigenvalues of $\mathcal{L}$ except the zero eigenvalue. Thus, by Definition 1.10 we conclude that if $s \in \tilde{\Omega}$ such that the operator $sI - \mathcal{L}$ is Fredholm of index 0, then we have $s \in \rho(\mathcal{L})$. Since the Fredholm index of $sI - \mathcal{L}$ stays constant in a small neighborhood of s, cf. [38], we are mainly interested in the set of all s such that $sI - \mathcal{L}$ is not a Fredholm operator. In this section we show that the Fredholm properties of $sI - \mathcal{L}$ on X_η coincide with the variable coefficient operator on L^2 given by

$$L : H_\eta^2 \to L_\eta^2, \quad u \mapsto Au_{xx} + cu_x + S_\omega u + Df(v_\star)u. \tag{3.38}$$

In addition, let us consider the operator

$$L_\eta : H^2 \to L^2, \quad u \mapsto \eta L \eta^{-1} u.$$

A straightforward calculation shows that L_η can be written as a second order differential operator on L^2

$$L_\eta u = A u_{xx} + B_\mu u_x + C_\mu u$$

with coefficients given by

$$B_\mu(x) = cI + \frac{2\mu x}{\sqrt{x^2+1}} A,$$
$$C_\mu(x) = S_\omega + Df(v_\star(x)) + \mu^2 A \frac{x^2}{x^2+1} - c\mu I \frac{x}{\sqrt{x^2+1}} - \mu A \left(\frac{1}{\sqrt{x^2+1}} - \frac{x^2}{(x^2+1)^{\frac{3}{2}}} \right).$$

The next step is to introduce the map

$$\psi : X_\eta \to L^2_\eta \times \mathbb{R}^2, \quad \begin{pmatrix} u \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} u - \rho \hat{v} \\ \rho \end{pmatrix}.$$

Taking the norm $\|(u,\rho)\|^2_{L^2_\eta \times \mathbb{R}} := |\rho|^2 + \|u\|^2_{L^2_\eta}$ on $L^2_\eta \times \mathbb{R}^2$, it follows immediately that ψ defines an isometry from X_η to $L^2_\eta \times \mathbb{R}^2$ and its inverse is given by

$$\psi^{-1} : L^2_\eta \times \mathbb{R}^2 \to X_\eta, \quad \begin{pmatrix} u \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} u + \rho \hat{v} \\ \rho \end{pmatrix}.$$

Lemma 3.12. *The map $\psi : X_\eta \to L^2_\eta \times \mathbb{R}^2$ is an isometric isomorphism. Moreover, if $0 \le \mu < 2$, then $\psi : Y_\eta \to H^2_\eta \times \mathbb{R}^2$ is a homeomorphism.*

Proof. By the previous observation it remains to show the continuity of ψ from Y_η to $H^2_\eta \times \mathbb{R}^2$. From Proposition 2.7 we obtain for $0 \le \mu < 2$ and $\mathbf{u} = (u,\rho)^\top \in Y_\eta$

$$\begin{aligned} \|\psi(\mathbf{u})\|^2_{H^2_\eta \times \mathbb{R}^2} &= |\rho|^2 + \sum_{\alpha=0}^{2} \|\partial^\alpha (u - \rho \hat{v})\|^2_{L^2_\eta} \\ &\le (1 + \|\hat{v}_x\|^2_{L^2_\eta} + \|\hat{v}_{xx}\|^2_{L^2_\eta})|\rho|^2 + \|u - \rho\hat{v}\|^2_{L^2_\eta} + \|u_x\|^2_{L^2_\eta} + \|u_{xx}\|^2_{L^2_\eta} \le C\|\mathbf{u}\|^2_{Y_\eta}. \end{aligned}$$

$\square$

With the homeomorphism ψ we can define the operator

$$\mathcal{L}_\psi : H^2_\eta \times \mathbb{R}^2 \to L^2_\eta \times \mathbb{R}^2, \quad \mathbf{u} \mapsto \psi \mathcal{L} \psi^{-1} \mathbf{u}$$

Again, a straightforward calculation shows that for $\mathbf{u} = (u, \rho)^\top$ the operator $\mathcal{L}_\psi$ can be written as

$$\mathcal{L}_\psi \mathbf{u} = \begin{pmatrix} Au_{xx} + cu_x + S_\omega u + Df(v_\star)u + A\rho\hat{v}_{xx} + c\rho\hat{v}_x + (Df(v_\star) - Df(v_\infty))\,\rho\hat{v} \\ S_\omega\rho + Df(v_\infty)\rho \end{pmatrix}.$$

If Assumption 1 and 2 are satisfied, it follows that $\mathcal{L}_\psi$ defines a closed, linear operator on $L^2_\eta \times \mathbb{R}^2$ with $\mathcal{D}(\mathcal{L}_\psi) = H^2_\eta \times \mathbb{R}^2$. Furthermore, since ψ is a homeomorphism and therefore a Fredholm operator of index 0 we conclude from Lemma A.2 that the Fredholm indices of $sI - \mathcal{L}$ and $sI - \mathcal{L}_\psi$ coincide. The same holds true for the operators $sI - L$ and $sI - L_\eta$ since the multiplication operator associated with η from Lemma 3.1 is a homeomorphism. Furthermore, a compact perturbation argument will show that the variable coefficient operator $sI - L_\eta$ on L^2 has the same Fredholm index as the piecewise constant coefficient operator given by

$$L_{\eta,\infty} : H^2 \to L^2, \quad u \mapsto Au_{xx} + B_{\mu,\infty}u_x + C_{\mu,\infty}u,$$

where

$$B_{\mu,\infty}(x) = \begin{cases} cI + 2\mu A, & x \geq 0 \\ cI - 2\mu A, & x < 0 \end{cases}, \quad C_{\mu,\infty}(x) = \begin{cases} S_\omega + Df(v_\infty) + \mu^2 A - c\mu I, & x \geq 0 \\ S_\omega + Df(0) + \mu^2 A + c\mu I, & x < 0 \end{cases}. \tag{3.39}$$

We note and prove these observations in the following lemma.

Lemma 3.13. *Let Assumption 1 and 2 be satisfied,* $0 \leq \mu \leq \min(\mu_\star, 2)$ *and* $s \in \mathbb{C}$ *with* $\mu_\star$ *from Theorem 2.6. Then the following statements are equivalent:*

i) The operator $(sI - \mathcal{L}) : Y_\eta \to X_\eta$ *is a Fredholm operator of index* k*.*

ii) The operator $(sI - \mathcal{L}_\psi) : H^2_\eta \times \mathbb{R}^2 \to L^2_\eta \times \mathbb{R}^2$ *is a Fredholm operator of index* k*.*

iii) The operator $(sI - L) : H^2_\eta \to L^2_\eta$ *is a Fredholm operator of index* k*.*

iv) The operator $(sI - L_\eta) : H^2 \to L^2$ *is a Fredholm operator of index* k*.*

v) The operator $(sI - L_{\eta,\infty}) : H^2 \to L^2$ *is a Fredholm operator of index* k*.*

Proof. **i)** $\Leftrightarrow$ **ii)**: By Lemma 3.12, the maps $\psi : X_\eta \to L^2_\eta \times \mathbb{R}^2$ and $\psi : Y_\eta \to H^2_\eta \times \mathbb{R}^2$ are homeomorphisms and therefore Fredholm operators of index 0. Thus, the equivalence of i) and ii) follows by Lemma A.2.
iii) $\Leftrightarrow$ **iv)**: By Lemma 3.1, the multiplication operators $m_\eta : L^2_\eta \to L^2$ and $m_\eta : H^2_\eta \to H^2$ are homeomorphisms and therefore Fredholm operators of index 0. Thus, the equivalence

of iii) and iv) follows by Lemma A.2.
ii) $\Leftrightarrow$ iii): The operator $\mathcal{L}_\psi$ can be decomposed into $\mathcal{L}_\psi = \tilde{\mathcal{L}} + K$ where $\tilde{\mathcal{L}}$ is given by

$$\tilde{\mathcal{L}} : H^2_\eta \times \mathbb{R}^2 \to L^2_\eta \times \mathbb{R}^2, \quad \tilde{\mathcal{L}} \begin{pmatrix} u \\ \rho \end{pmatrix} := \begin{pmatrix} Au_{xx} + cu_x + S_\omega u + Df(v_\star)u \\ (S_\omega + Df(v_\infty))\rho \end{pmatrix}$$

and the operator K by

$$K : H^2_\eta \times \mathbb{R}^2 \to L^2_\eta \times \mathbb{R}^2, \quad K \begin{pmatrix} u \\ \rho \end{pmatrix} := \begin{pmatrix} A\rho\hat{v}_{xx} + c\rho\hat{v}_x + (Df(v_\star) - Df(v_\infty))\, \rho\hat{v} \\ 0 \end{pmatrix}.$$

Since $sI - S_\omega - Df(v_\infty) \in \mathbb{R}^{2,2}$ is a Fredholm operator of index 0 on $\mathbb{R}^2$, Lemma A.3 implies that $sI - \tilde{\mathcal{L}}$ is a Fredholm operator of index k if and only if $(sI - L) : H^2_\eta \to L^2_\eta$ is. We show that K is a compact operator. Then the assertion follows by Lemma A.4. To see the compactness of K, let $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subset H^2_\eta \times \mathbb{R}^2$, $\mathbf{u}_n = (u_n, \rho_n)^\top$ be a bounded sequence and let $C > 0$ denote a universal constant. Then there exists a subsequence ρ_{n_k} such that $\rho_{n_k} \to \rho$ as $k \to \infty$. We define

$$w := A\rho\hat{v}_{xx} + c\rho\hat{v}_x + (Df(v_\star) - Df(v_\infty))\, \rho\hat{v}.$$

Then Assumption 1 and Theorem 2.6 imply $w \in L^2_\eta$. Moreover, we have

$$\begin{aligned}
&\|(Df(v_\star) - Df(v_\infty))(\rho_{n_k} - \rho)\hat{v}\|_{L^2_\eta} \\
&\le \|(Df(v_\star) - Df(v_\infty))(\rho_{n_k} - \rho)\hat{v}\|_{L^2_\eta(\mathbb{R}_-)} + \|(Df(v_\star) - Df(v_\infty))(\rho_{n_k} - \rho)\hat{v}\|_{L^2_\eta(\mathbb{R}_+)} \\
&\le C\|\hat{v}\|_{L^2_\eta(\mathbb{R}_-)}|\rho_{n_k} - \rho| + C|\rho_{n_k} - \rho|\|Df(v_\star) - Df(v_\infty)\|_{L^2_\eta(\mathbb{R}_+)} \\
&\le C\|\hat{v}\|_{L^2_\eta(\mathbb{R}_-)}|\rho_{n_k} - \rho| + C|\rho_{n_k} - \rho|\|v_\star - v_\infty\|_{L^2_\eta(\mathbb{R}_+)} \le C|\rho_{n_k} - \rho|.
\end{aligned}$$

This implies for $\mathbf{w} = (w, \rho)^\top \in L^2_\eta \times \mathbb{R}^2$

$$\begin{aligned}
&\|K\mathbf{u}_{n_k} - \mathbf{w}\|_{L^2_\eta \times \mathbb{R}^2} \le |\rho_{n_k} - \rho| + \|A\rho_{n_k}\hat{v}_{xx} + c\rho_{n_k}\hat{v}_x + (Df(v_\star) - Df(v_\infty))\rho_{n_k}\hat{v} - w\|_{L^2_\eta} \\
&= |\rho_{n_k} - \rho| + \|A(\rho_{n_k} - \rho)\hat{v}_{xx} + c(\rho_{n_k} - \rho)\hat{v}_x + (Df(v_\star) - Df(v_\infty))(\rho_{n_k} - \rho)\hat{v}\|_{L^2_\eta} \\
&\le C(1 + \|\hat{v}_{xx}\|_{L^2_\eta} + \|\hat{v}_x\|_{L^2_\eta})|\rho_{n_k} - \rho| + \|(Df(v_\star) - Df(v_\infty))(\rho_{n_k} - \rho)\hat{v}\|_{L^2_\eta} \\
&\le C|\rho_{n_k} - \rho| \to 0, \quad k \to 0.
\end{aligned}$$

Thus, $K\mathbf{u}_{n_k} \to \mathbf{w}$ in $L^2_\eta \times \mathbb{R}^2$ as $k \to \infty$. This shows the compactness of K and the assertion follows by Lemma A.4.
iv) $\Leftrightarrow$ v): Since the operator $\partial_x : H^{k+1} \to H^k$, $k \ge 0$ is bounded, Lemma D.4 and Theorem 2.6 imply that the operator

$$L_\eta - L_{\eta,\infty} : H^2 \to L^2, \quad u \mapsto (B_\mu - B_{\mu,\infty})u_x + (C_\mu - C_{\mu,\infty})u$$

is compact. Hence the assertion follows by Lemma A.4. $\square$

The lemma shows that the Fredholm properties of $sI - \mathcal{L}$ on X_η are determined by the piecewise constant coefficient operator $sI - L_{\eta,\infty}$. Thus, we are interested into the solvabilty of the resolvent equation

$$(sI - L_{\eta,\infty})u = r, \quad u \in H^2,\, r \in L^2,\, s \in \mathbb{C}. \tag{3.40}$$

We use the classical approach, for instance, from [36] or [56], [32], where the solution of (3.40) is constructed using exponential dichotomies. For this purpose we transform (3.40) into a first order system via $w = (u, u')$ and obtain

$$\mathcal{M}(s)w = h, \quad \mathcal{M}(s) = \partial_x - M(s,\cdot), \quad h = (0,r)^\top \tag{3.41}$$

with

$$M(s,x) = \begin{cases} M_+(s), & x \geq 0 \\ M_-(s), & x < 0 \end{cases}, \quad M_\pm(s) = \begin{pmatrix} 0 & I \\ A^{-1}(sI - C_\pm) & -A^{-1}B_\pm \end{pmatrix}$$

and the matrices $B_\pm, C_\pm$ given by

$$B_\pm := cI \mp 2\mu A, \quad C_\pm := S_\omega + Df(v_\pm) + \mu^2 A \mp c\mu I.$$

From [22] we have that the operator $\mathcal{M}(s)$ has an exponential dichotomy on the half-line $\mathbb{R}_\pm$ if and only if the matrix $M_\pm(s)$ is hyperbolic, cf. Proposition B.4. Therefore, we define the set

$$\Omega_F := \{s \in \mathbb{C} : M_+(s) \text{ and } M_-(s) \text{ are hyperbolic}\}.$$

For $s \in \Omega_F$ we denote by $m^\pm_{\mathfrak{s},\mathfrak{u}}(s)$ the dimensions of the stable and unstable subspaces of $M_\pm(s)$, i.e. $m^\pm_{\mathfrak{s}}(s)$ denotes the sum of multiplicities of the eigenvalues of $M_\pm(s)$ with negative real part and $m^\pm_{\mathfrak{u}}(s)$ those with positive real part. Now we have the following classical result which can be found in several texts from the literature. See for instance [36, Lem. 3.1.10] or [48], [49], [56, Sec. 3].

Lemma 3.14. *Let Assumption 1 and 2 be satisfied,* $0 \leq \mu \leq \min(\mu_\star, 2)$ *with* $\mu_\star$ *from Theorem 2.6. Then the operator* $sI - L_{\eta,\infty} : H^2 \to L^2$ *is a Fredholm operator if and only if* $s \in \Omega_F$*. If* $s \in \Omega_F$ *then the Fredholm index is given by*

$$\operatorname{ind}(sI - L_{\eta,\infty}) = m^+_{\mathfrak{s}}(s) - m^-_{\mathfrak{s}}(s).$$

Lemma 3.14 together with Lemma 3.13 imply that $sI - \mathcal{L} : Y_\eta \to X_\eta$ is a Fredholm operator if and only if $s \in \Omega_F$. Moreover, since the matrices $M_\pm$ depend continuously on $s \in \mathbb{C}$ we conclude that the Fredholm index of $sI - \mathcal{L}$ stays constant in any connected component of Ω_F. Recall the dispersion set from (0.27) given by

$$\sigma_{\mathrm{disp},\mu}(\mathcal{L}) = \sigma^-_{\mathrm{disp},\mu}(\mathcal{L}) \cup \sigma^+_{\mathrm{disp},\mu}(\mathcal{L})$$

Then we have the following lemma.

Lemma 3.15. *Let Assumption 1 and 2 be satisfied. Then the set Ω_F is the complement of the dispersion set, i.e. $\Omega_F = \mathbb{C}\backslash\sigma_{\mathrm{disp},\mu}(\mathcal{L})$.*

Proof. First we show that $\det(-\nu^2 A + i\nu B_\pm + C_\pm - sI) = 0$ if and only if $M_\pm(s)$ is not hyperbolic. Assume $M_\pm(s)$ is not hyperbolic, i.e. there exists $\nu \in \mathbb{R}$, $w \in \mathbb{C}^4$, $|w| = 1$ such that $i\nu w = M_\pm(s)w$. This implies, with $w = (w_1, w_2)^\top$, that $i\nu w_1 = w_2$ and hence $w_1 \neq 0$ due to $|w| = 1$. Moreover,

$$-\nu^2 w_1 = A^{-1}(sI - C_\pm)w_1 - A^{-1}B_\pm w_2 = A^{-1}(sI - C_\pm)w_1 - i\nu A^{-1}B_\pm w_1.$$

Hence,

$$(-\nu^2 A + i\nu B_\pm + C_\pm)w_1 = 0.$$

Thus, $\det(-\nu^2 A + i\nu B_\pm + C_\pm - sI) = 0$.
Conversely, suppose $\det(-\nu^2 A + i\nu B_\pm + C_\pm - sI) = 0$. Then there exists $w_1 \in \mathbb{C}^m$ such that $(-\nu^2 A + i\nu B_\pm + C_\pm - sI)w_1 = 0$. Now setting $w_2 = i\nu w_1$ leads to

$$\begin{aligned}
&\begin{pmatrix} 0 & I \\ A^{-1}(sI - C^\pm) & -A^{-1}B^\pm \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_2 \\ A^{-1}(sI - C_\pm)w_1 - A^{-1}B_\pm w_2 \end{pmatrix} \\
&= \begin{pmatrix} i\nu w_1 \\ A^{-1}(sI - C_\pm)w_1 - i\nu A^{-1}B_\pm w_1 \end{pmatrix} = \begin{pmatrix} i\nu w_1 \\ A^{-1}(sI - C_\pm - i\nu B_\pm)w_1 \end{pmatrix} \\
&= \begin{pmatrix} i\nu w_1 \\ -A^{-1}\nu^2 A w_1 \end{pmatrix} = i\nu \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.
\end{aligned}$$

□

It holds true that the Fredholm index of $sI - \mathcal{L}$ stays constant in any connected component of Ω_F, see [36]. Therefore, we are interested in the shape and location of the dispersion set and in particular in the connected components of Ω_F. The Fredholm region Ω_F can be written as

$$\Omega_F = \left\{s \in \mathbb{C} : \det(sI - D_\pm(\nu)) \neq 0 \,\forall\, \nu \in \mathbb{R}\right\}, \quad D_\pm(\nu) := -\nu^2 A + i\nu B_\pm + C_\pm.$$

Hence we look for eigenvalues of the matrix $D_\pm(\nu)$, $\nu \in \mathbb{R}$. Its characteristic polynomial is given by

$$d^\pm(s) = s^2 - \operatorname{tr} D_\pm(\nu)s + \det D_\pm(\nu).$$

The roots of $d^{\pm}(\cdot,\nu)$ can be computed explicitly. We have $s \in \sigma^{+}_{\mathrm{disp},\mu}(\mathcal{L})$ if and only if

$$\begin{aligned} s &= \frac{\operatorname{tr} D_+(\nu)}{2} \pm \sqrt{\frac{(\operatorname{tr} D_+(\nu))^2}{4} - \det D_+(\nu)} \\ &= -\alpha_1\nu^2 + i(c - 2\alpha_1\mu)\nu + \mu^2\alpha_1 - c\mu + g_1'(|v_\infty|^2)|v_\infty|^2 \\ &\quad \pm \Big[-\alpha_2^2\nu^4 - 4i\alpha_2^2\mu\nu^3 + (6\alpha_2^2\mu^2 + 2\alpha_2 g_2'(|v_\infty|^2)|v_\infty|^2)\nu^2 \\ &\quad + 4i(\alpha_2^2\mu^3 + \mu\alpha_2 g_2'(|v_\infty|^2)|v_\infty|^2)\nu \\ &\quad - \alpha_2^2\mu^4 - 2\alpha_2\mu^2 g_2'(|v_\infty|^2)|v_\infty|^2 + (g_1'(|v_\infty|^2)|v_\infty|^2)^2 \Big]^{\frac{1}{2}} \end{aligned} \tag{3.42}$$

and $s \in \sigma^{-}_{\mathrm{disp},\mu}(\mathcal{L})$ if and only if

$$\begin{aligned} s &= \frac{\operatorname{tr} D_-(\nu)}{2} \pm \sqrt{\frac{(\operatorname{tr} D_-(\nu))^2}{4} - \det D_-(\nu)} \\ &= -\alpha_1\nu^2 + i(c + 2\alpha_1\mu)\nu + \mu^2\alpha_1 + c\mu + g_1(0) \\ &\quad \pm \Big[-\alpha_2^2\nu^4 + 4i\alpha_2^2\mu\nu^3 + (6\alpha_2^2\mu^2 + 2\alpha_2(g_2(0)+\omega))\nu^2 \\ &\quad - 4i\alpha_2(\alpha_2\mu^3 + \mu(g_2(0)+\omega))\nu - \alpha_2^2\mu^4 - 2(g_2(0)+\omega)\alpha_2\mu^2 - (g_2(0)+\omega)^2 \Big]^{\frac{1}{2}}. \end{aligned} \tag{3.43}$$

Roughly speaking, these are four curves in the complex plane running from $-\infty - i\infty$ to $-\infty + i\infty$. In the special case $\alpha_2 = 0$ the equations (3.42) and (3.43) simplify to

$$s = -\alpha_1\nu^2 + i(c - 2\alpha_1\mu)\nu + \mu^2\alpha_1 - c\mu + g_1'(|v_\infty|^2)|v_\infty|^2 \pm g_1'(|v_\infty|^2)|v_\infty|^2$$

and

$$s = -\alpha_1\nu^2 + i(c + 2\alpha_1\mu)\nu + \mu^2\alpha_1 + c\mu + g_1(0) \pm i(g_2(0) + \omega).$$

Then the dispersion set consists of four parabolas in the complex plane opened to the left, see 0.4 b).
We are now in the position to formulate and prove the main result of this section describing the essential spectrum of $\mathcal{L}$ on the exponentially weighted space X_η.

Theorem 3.16. *Let the Assumption 1-3 be satisfied and* $0 < \mu < \min(\mu_{\mathrm{ess}}, \mu_\star, 2)$ *with* μ_{ess} *from Assumption 3 and* $\mu_\star$ *from Theorem 2.6. Then there are* $\varepsilon > 0$, $\gamma < 0$ *and a unique connected component* Ω_∞ *of* Ω_F *satisfying:*

i) $\mathcal{S}_{\varepsilon,\gamma} := \left\{ s \in \mathbb{C} : |\arg(s-\gamma)| < \frac{\pi}{2} + \varepsilon,\ s \neq \gamma \right\} \subset \Omega_\infty$.

ii) For all $s \in \Omega_\infty$ *the operator* $sI - \mathcal{L} : Y_\eta \to X_\eta$ *is Fredholm of index* 0.

iii) $\partial\Omega_\infty \subset \sigma_{\mathrm{disp},\mu}(\mathcal{L})$.

iv) $\sigma_{\mathrm{ess}}(\mathcal{L}) \subset \mathbb{C}\backslash\Omega_\infty$.

Proof. **i).** For $s \in \Omega_F$ Lemma 3.13 and Lemma 3.14 imply the operator $sI - \mathcal{L}$ to be a Fredholm operator of index $\mathrm{ind}(sI-\mathcal{L}) = m_{\mathfrak{s}}^+(s) - m_{\mathfrak{s}}^-(s)$. For $s \in \sigma_{\mathrm{disp},\mu}(\mathcal{L})$ we have $\mathrm{Re}\, s \to -\infty$ as $|s| \to \infty$. Thus, $\mathrm{Re}\,\sigma_{\mathrm{disp},\mu}(\mathcal{L}) < \infty$. Now let $s_0 > \mathrm{Re}\,\sigma_{\mathrm{disp},\mu}(\mathcal{L})$. Recall for a matrix $M \in \mathbb{C}^{m,m}$ its lower spectral bound $\alpha(M) := \min\{\mathrm{Re}\,(x^H M x) : |x| = 1\}$ and let s_0 be so large such that

$$\alpha(s_0 I - C_\pm) \geq s_0 - \max\{\mathrm{Re}\,(v^H C_\pm v) : |v| = 1\} > \frac{\mu|\alpha_2|}{\alpha_1}.$$

Then for all $s \geq s_0$ we also have

$$\alpha(sI - C_\pm) \geq s - \max\{\mathrm{Re}\,(x^H C_\pm x) : |x| = 1\} > \frac{\mu|\alpha_2|}{\alpha_1}$$

and since $\alpha(A) = \alpha_1$ we obtain

$$|B_\pm - B_\pm^\top| = |2\mu(A - A^\top)| = 4\mu|\alpha_2| = 4\alpha(A)\frac{\mu|\alpha_2|}{\alpha_1} < 4\alpha(A)\alpha(sI - C_\pm) \quad \forall s \in [s_0, \infty).$$

Now for all $s \in [s_0,\infty)$ Lemma D.1 implies $M_\pm(s)$ to be hyperbolic with $m_{\mathfrak{s}}^\pm(s) = 2$. Thus $sI - \mathcal{L}$ is a Fredholm operator of index 0. Since both $M_\pm$ depend continuously on s we conclude that $m_{\mathfrak{s}}^\pm(s) = 2$ for s in the connected component Ω_∞ of Ω_F containing $[s_0,\infty)$. Thus $sI - \mathcal{L}$ is Fredholm index 0 for $s \in \Omega_\infty$ and $\partial\Omega_\infty \subset \partial\Omega_F = \sigma_{\mathrm{disp},\mu}(\mathcal{L})$. Therefore ii) and iii) hold. Moreover, iv) follows by definition of the essential spectrum, cf. Definition 1.10. It remains to show i). Using ii) it is sufficient to show there is a sector $\mathcal{S}_{\varepsilon,\gamma}$, $\varepsilon > 0$, $\gamma < 0$ with $\sigma_{\mathrm{disp},\mu}(\mathcal{L}) \cap \mathcal{S}_{\varepsilon,\gamma} = \emptyset$. The dispersion set consists of four curves given by the equations (3.42) and (3.43). For each of those we can choose a parametrization $\chi : \mathbb{R} \to \mathbb{C}$ such that (3.42), (3.43) respectively, is equivalent to $\chi(\nu) = s$ and χ is given by

$$\chi(\nu) = -\alpha_1\nu^2 + i\xi_1\nu + \xi_2 \pm \sqrt{-\alpha_2^2\nu^4 + p(\nu)}, \quad \nu \in \mathbb{R},$$

where $\xi_1, \xi_2 \in \mathbb{R}$ and $p(\nu) = i\beta_3\nu^3 + \beta_2\nu^2 + i\beta_1\nu + \beta_0$ is a polynomial of degree 3 over $\mathbb{C}$ with $\beta_i \in \mathbb{R}$. For the derivative of the parametrization there holds

$$\chi'(\nu) = -2\alpha_1\nu + i\xi_1 \mp \frac{2\alpha_2^2\nu^3}{\sqrt{-\alpha_2^2\nu^4 + p(\nu)}} \pm \frac{p'(\nu)}{\sqrt{-\alpha_2^2\nu^4 + p(\nu)}}.$$

Now,

$$\frac{2\alpha_2^2\nu^2}{\sqrt{-\alpha_2^2\nu^4 + p(\nu)}} = \frac{2\alpha_2^2}{\sqrt{-\alpha_2^2 + \nu^{-4}p(\nu)}} \overset{\nu\to\pm\infty}{\longrightarrow} \frac{2\alpha_2^2}{\sqrt{-\alpha_2^2}} = \frac{2\alpha_2^2}{i|\alpha_2|} = -2i|\alpha_2|.$$

Therefore,

$$\nu^{-1}\chi'(\nu) \to -2\alpha_1 \pm 2i|\alpha_2|, \quad \nu \to \pm\infty.$$

Let $\mathcal{T}_{\chi(\nu)} := \frac{\chi'(\nu)}{|\chi'(\nu)|}$ be the tangent vector of the curve at $\chi(\nu)$. Then for $\nu > 0$

$$\mathcal{T}_{\chi(\nu)} = \frac{\nu^{-1}\chi'(\nu)}{|\nu^{-1}\chi'(\nu)|} \to \frac{-\alpha_1 \pm i|\alpha_2|}{|\alpha|}, \quad \nu \to \infty.$$

For $\nu < 0$ we obtain

$$\mathcal{T}_{\chi(\nu)} = -\frac{\nu^{-1}\chi'(\nu)}{|\nu^{-1}\chi'(\nu)|} \to \frac{\alpha_1 \mp i|\alpha_2|}{|\alpha|}, \quad \nu \to -\infty.$$

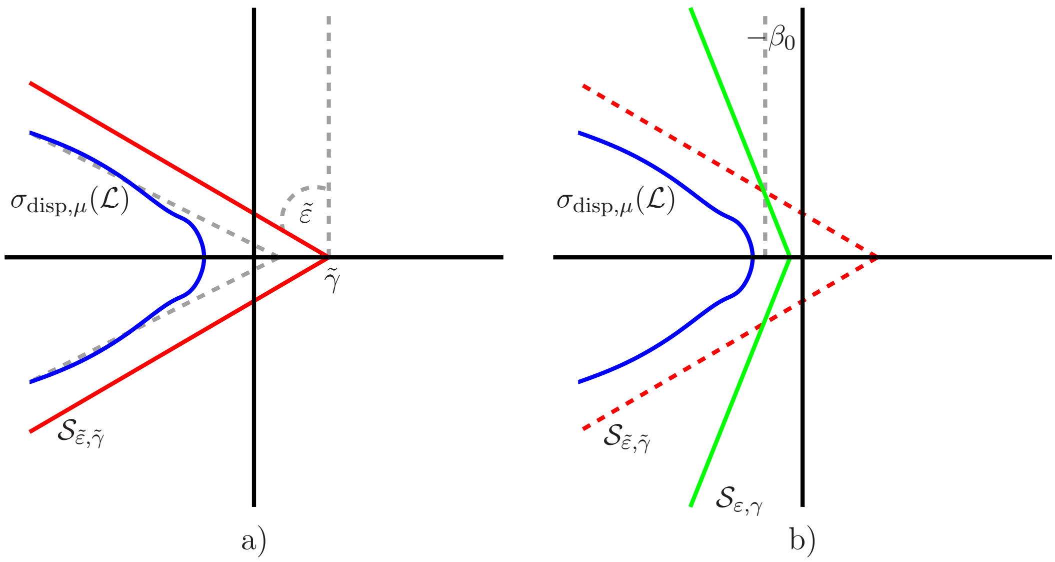


Figure 3.2: Geometric situation in the proof of Theorem 3.16.

Since $\operatorname{Re}\chi(\nu) \to -\infty$ as $\nu \to \pm\infty$, we find a sector $\mathcal{S}_{\tilde{\varepsilon},\tilde{\gamma}}$, $\tilde{\gamma} > 0$, $0 < \tilde{\varepsilon} < \tan^{-1}\left(\frac{\alpha_1}{|\alpha_2|}\right)$ such that $\sigma_{\mathrm{disp},\mu}(\mathcal{L}) \subset (\mathcal{S}_{\tilde{\varepsilon},\tilde{\gamma}})^c$, cf. Figure 3.2 a). Now Assumption 3 implies $\sigma_{\mathrm{disp},\mu}(\mathcal{L}) \subset \{\operatorname{Re} s < -\beta_0\}$. Then for every $-\beta_0 < \gamma < 0$ there is $0 < \varepsilon \leq \tilde{\varepsilon}$ such that $\mathcal{S}_{\varepsilon,\gamma} \cap \sigma_{\mathrm{disp},\mu}(\mathcal{L}) = \emptyset$, cf. Figure 3.2 b). $\square$

Remark 3.17. *To fully describe the essential spectrum, according to Definition 1.10, of the linearized operator $\mathcal{L}$ one can use Lemma 3.14 and compute $m_{\mathfrak{s}}^+(s), m_{\mathfrak{u}}^-(s)$ in the connected components of Ω_F. In the connected components the dimensions $m_{\mathfrak{s}}^+(s), m_{\mathfrak{u}}^-(s)$ stay constant. The dimensions are given by the number of eigenvalues with negative real part of the matrices $M_+(s), M_-(s)$. The Fredholm index in then given by* $\operatorname{ind}(sI - \mathcal{L}) =$

$m_{\mathfrak{s}}^{+}(s) - m_{\mathfrak{u}}^{-}(s)$. *The essential spectrum is shown in Figure 3.3 in case of* (QCGL) *with parameters* $\alpha = 1, \mu = -\frac{1}{8}, \beta = 1 + i, \gamma = -1 + i$ *In this case the matrices are explicitly given by*

$$M_{+}(s) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ s - 2g_1'(|v_\infty|^2)|v_\infty|^2 & 0 & -c & 0 \\ -2g_2'(|v_\infty|^2)|v_\infty|^2 & s & 0 & -c \end{pmatrix}$$

and

$$M_{-}(s) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ s - g_1(0) & -\omega - g_2(0) & -c & 0 \\ \omega + g_2(0) & s - g_1(0) & 0 & -c \end{pmatrix}.$$

The numbers in the connected components indicate the Fredholm index of $sI - \mathcal{L}$. *We*

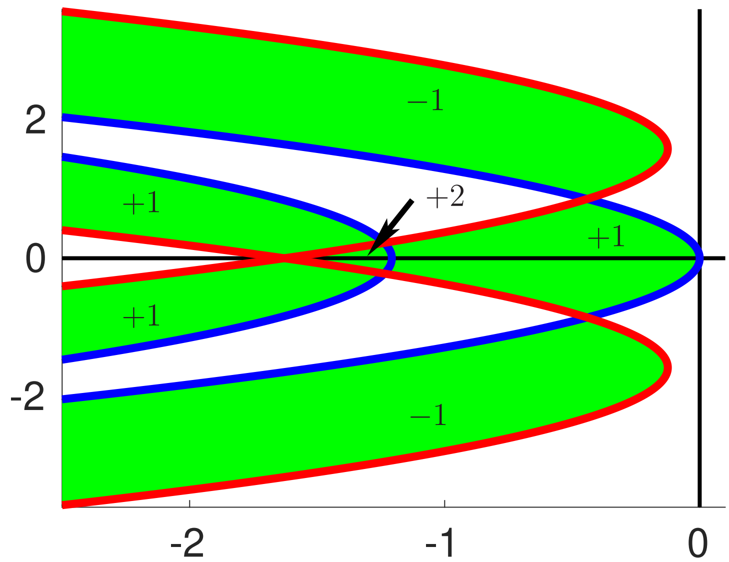


Figure 3.3: The essential spectrum of the linearized operator $\mathcal{L}$ (green) with the dispersion sets (red/blue) in an example of (QCGL). The numbers in the connected component of Ω_F indicate the Fredholm indiex of $sI - \mathcal{L}$.

note that the essential spectrum strongly depends in the choice of its definition which differs in the literature, cf. [25]. However, for the stability behavior of the TOF, the choice of the precise definition has no effects since the essential spectrum is bounded by the dispersion set in any case.

Now we take Assumption 4 into account and conclude the section by studying the zero eigenvalue. Assumption 4 states that zero is an eigenvalue of the linearized operator $\mathcal{L}$ with algebraic multiplicity at most 2. Using the fact that the whole group orbit $a(\gamma)\mathbf{v}_\star$, $\gamma \in \mathcal{G}$ is a stationary solution of the Cauchy problem (0.22) and that the group action $a(\gamma)$ is continuously differentiable, we will see that one finds two linearly independent eigenfunctions of $\mathcal{L}$. Then by Assumption 4 it follows that zero is an eigenvalue of algebraic multiplicity equal to 2.

Lemma 3.18. *Let the Assumption 1-4 be satisfied and* $0 < \mu < \min(\mu_{\mathrm{ess}}, \mu_\star, 2)$ *with* μ_{ess} *from Assumption 3 and* $\mu_\star$ *from Theorem 2.6. Then* $s = 0$ *is an eigenvalue of* $\mathcal{L}$ *with algebraic multiplicity two and linearly independent eigenfunctions given by*

$$\varphi_1 = -\mathbf{S}_1\mathbf{v}_\star = -\begin{pmatrix} S_1 v_\star \\ S_1 v_\infty \end{pmatrix} \in Y_\eta, \quad \varphi_2 = -\mathbf{v}_\star = -\begin{pmatrix} v_{\star,x} \\ 0 \end{pmatrix} \in Y_\eta$$

such that

$$\mathcal{N}(\mathcal{L}) = \operatorname{span}\{\varphi_1, \varphi_2\} =: \Phi.$$

Proof. Clearly, $\varphi_1.\varphi_2$ are linearly independent. Assumption 2 and Theorem 2.6 imply $\varphi_1, \varphi_2 \in Y_\eta$. Thus it remains to show $\mathcal{L}\varphi_i = 0$ for $i = 1, 2$ then the claim follows by Assumption 4. Recall that $(v_\star, v_\infty)^\top$ is a stationary solution of (0.22), i.e.

$$0 = \begin{pmatrix} Av_{\star,xx} + cv_{\star,x} + S_\omega v_\star + f(v_\star) \\ S_\omega v_\infty + f(v_\infty) \end{pmatrix}.$$

By applying the group action $a(\gamma)$ for $\gamma = (\theta, \tau) \in \mathcal{G}$ we obtain

$$0 = \begin{pmatrix} AR_\theta v_{\star,xx}(\cdot - \tau) + cR_\theta v_{\star,x}(\cdot - \tau) + S_\omega R_\theta v_\star(\cdot - \tau) + f(R_\theta v_\star(\cdot - \tau)) \\ S_\omega R_\theta v_\infty + f(R_\theta v_\infty) \end{pmatrix}. \tag{3.44}$$

Differentiating (3.44) w.r.t. θ and evaluating at $(\theta, \tau) = 0$ yields

$$0 = \begin{pmatrix} AS_1 v_{\star,xx} + cS_1 v_{\star,x} + S_\omega S_1 v_\star + Df(v_\star)S_1 v_\star \\ S_\omega S_1 v_\infty + Df(v_\infty)S_1 v_\infty \end{pmatrix} = \mathcal{L}\begin{pmatrix} S_1 v_\star \\ S_1 v_\infty \end{pmatrix} = \mathcal{L}\varphi_1.$$

Further, differentiating (3.44) w.r.t. τ and evaluating at $(\theta, \tau) = 0$ leads to

$$0 = \begin{pmatrix} -Av_{\star,xxx} - cv_{\star,xx} - S_\omega v_{\star,x} - Df(v_\star)v_{\star,x} \\ 0 \end{pmatrix} = -\mathcal{L}\begin{pmatrix} v_{\star,x} \\ 0 \end{pmatrix} = \mathcal{L}\varphi_2.$$

$\square$

By Assumption 4 the half-plane $\{\operatorname{Re} s > \gamma\}$ for some $\gamma < 0$ does not contain any eigenvalues of $\mathcal{L}$ expect for the eigenvalue $s = 0$. Thus we can assume that the sector $\mathcal{S}_{\varepsilon,\gamma}$ from Theorem 3.16 lies in the resolvent set except for the zero eigenvalue.

Corollary 3.19. *Let the Assumptions 1-4 be satisfied and let* $0 < \mu < \min(\mu_{\mathrm{ess}}, \mu_\star, 2)$ *with* μ_{ess} *from Assumption 3 and* $\mu_\star$ *from Theorem 2.6. Then there are* $\varepsilon > 0$, $\gamma < 0$ *such that*

$$\mathcal{S}_{\varepsilon,\gamma} \subset \rho(\mathcal{L}) \cup \{0\}.$$

Proof. The claim is a direct consequence of Theorem 3.16 and Assumption 4 by taking ε and $|\gamma|$ sufficiently small. □

As Corollary 3.19 shows the spectrum of $\mathcal{L}$ is completely included in the strict left half-plane except the zero eigenvalue. However, since it is an isolated eigenvalue of finite multiplicity we are able to block this neutral direction using the projector onto $\mathcal{N}(\mathcal{L})$. This will lead to time decaying estimates of the semigroup on the corresponding orthogonal complement, cf. [32, Thm. 1.5.3.]. For this purpose, we have to take the adjoint of $\mathcal{L}$ into account which will be considered in the following. Since X_η is a Hilbert space we may identify its dual space X_η^* with X_η via the Riesz isomorphism. We define the (abstract) adjoint operator of $\mathcal{L}$, cf. [61, Definition IV.11], by

$$\mathcal{L}^* : \mathcal{D}(\mathcal{L}^*) \subset X_\eta \to X_\eta, \quad \mathbf{u} \mapsto \mathcal{L}^*\mathbf{u}.$$

For a detailed construction and properties of the adjoint operator $\mathcal{L}^*$ we refer to [61, IV.11]. Since $\mathcal{L}$ has a closed range we have, cf. [61, (11-7)],

$$\mathcal{R}(\mathcal{L})^\perp = \mathcal{N}(\mathcal{L}^*), \quad \mathcal{R}(\mathcal{L}) = \mathcal{N}(\mathcal{L}^*)^\perp. \tag{3.45}$$

Lemma 3.20. *Let the Assumptions 1-4 be satisfied and* $0 < \mu < \min(\mu_{\mathrm{ess}}, \mu_\star, 2)$ *with* μ_{ess} *from Assumption 3 and* $\mu_\star$ *from Theorem 2.6. Then there are adjoint eigenfunctions* $\psi_1, \psi_2 \in \mathcal{D}(\mathcal{L}^*)$ *such that*

i) $\mathcal{N}(\mathcal{L}^*) = \mathrm{span}\,\{\psi_1, \psi_2\} =: \Psi$,

ii) $(\psi_i, \varphi_j)_{X_\eta} = \delta_{ij}$, $i, j = 1, 2$,

iii) $X_\eta = \Phi \oplus \Psi^\perp$,

iv) there is a continuous projection $P : X_\eta \to X_\eta$ *onto* Φ, *i.e.*

$$P(\Phi) = \Phi, \quad P(\Psi^\perp) = \{0\}, \quad P^2 = P,$$

which is given by

$$Pv := \sum_{i=1}^{2} (\psi_i, v)_{X_\eta} \varphi_j.$$

v) the subspace $\Psi^\perp \subset X_\eta$ *is invariant under* $\mathcal{L}$*, i.e.* $\mathcal{L}(\Psi^\perp \cap Y_\eta) \subset \Psi^\perp$.

Proof. **i), ii).** Let $(\cdot,\cdot) = (\cdot,\cdot)_{X_\eta}$. $\mathcal{L}$ is a Fredholm operator of index 0. Thus by Lemma A.5 the adjoint $\mathcal{L}^* : \mathcal{D}(\mathcal{L}^*) \to X_\eta$ is Fredholm operator of index 0. This implies by Assumption 4 $\dim \mathcal{N}(\mathcal{L}^*) = \dim \mathcal{N}(\mathcal{L}) = 2$. Then choose linearly independent $\psi_1', \psi_2' \in \mathcal{D}(\mathcal{L}^*)$ such that

$$\mathcal{N}(\mathcal{L}^*) = \operatorname{span}\{\psi_1', \psi_2'\}.$$

Now by Lemma A.2, the operator $\mathcal{L}^n$ is also a Fredholm operator of index 0 on X_η for all $n \in \mathbb{N}$. Thus $(\mathcal{L}^n)^* = (\mathcal{L}^*)^n$ is Fredholm of index 0, which implies by Assumption 4 and Lemma 3.18 that $\dim \mathcal{N}((\mathcal{L}^*)^n) = \dim \mathcal{N}(\mathcal{L}^n) = 2$ for $n \geq 2$. Hence, $\mathcal{L}^*$ has no generalized eigenfunctions and therefore $\psi_1', \psi_2' \notin \mathcal{R}(\mathcal{L}^*)$. Now consider the matrix

$$A := \begin{pmatrix} (\psi_1', \varphi_1) & (\psi_1', \varphi_2) \\ (\psi_2', \varphi_1) & (\psi_2', \varphi_2) \end{pmatrix}.$$

We show that A is invertible. Assume the contrary. Then there is $z = (z_1, z_2)^\top \in \mathbb{R}^2$ with $z^\top A = 0$. This implies

$$(z_1\psi_1' + z_2\psi_2', \varphi_i) = 0, \quad i = 1, 2$$

and therefore $z_1\psi_1' + z_2\psi_2' \in \mathcal{N}(\mathcal{L})^\perp = \mathcal{R}(\mathcal{L}^*)$ due to (3.45). Then we find $w \in X_\eta$ such that $\mathcal{L}^* w = z_1\psi_1' + z_2\psi_2' \in \mathcal{N}(\mathcal{L}^*)$. A contradiction since $\mathcal{L}^*$ has no generalized eigenfunction. Hence A is invertible. Now define

$$\begin{aligned} \psi_1 &= b_1\psi_1' + b_2\psi_2', \quad b = A^{-1}\begin{pmatrix}1\\0\end{pmatrix}, \\ \psi_2 &= c_1\psi_1' + c_2\psi_2', \quad c = A^{-1}\begin{pmatrix}0\\1\end{pmatrix}. \end{aligned}$$

Then ψ_1, ψ_2 are linearly independent with $\mathcal{N}(\mathcal{L}^*) = \operatorname{span}\{\psi_1, \psi_2\}$ and $(\psi_i, \varphi_j) = \delta_{ij}$.
iii). We may write $u \in X_\eta$ as

$$u = u - \sum_{i=1}^{2} (\psi_i, u)\varphi_i + \sum_{i=1}^{2} (\psi_i, u)\varphi_i.$$

Then we have $\sum_{i=1}^2 (\psi_i, u)\varphi_i \in \mathcal{N}(\mathcal{L}) = \Phi$ and due to ii)

$$\left(\psi_j, u - \sum_{i=1}^{2} (\psi_i, u)\varphi_i\right) = (\psi_j, u) - \sum_{i=1}^{2} (\psi_i, u)\,(\psi_j, \varphi_i) = 0$$

Hence $u - \sum_{i=1}^{2}(\psi_i, u)\varphi_i \in \Psi^\perp$. This shows $X_\eta = \Phi \oplus \Psi^\perp$.
iv). Clearly, P is linear. Due to ii) and iii), it follows $P(\Phi) = \{0\}$, $P(\Psi^\perp) = \Psi^\perp$ and $P^2 = P$. Now the continuity of P follows using Cauchy-Schwarz inequality

$$\|P\mathbf{u}\|_{X_\eta} \leq \sum_{i=1}^{2} |(\psi_i, \mathbf{u})| \|\varphi_i\|_{X_\eta} \leq \sum_{i=1}^{2} \|\psi_i\|_{X_\eta} \|\varphi_i\|_{X_\eta} \|\mathbf{u}\|_{X_\eta} \leq C\|\mathbf{u}\|_{X_\eta}.$$

v). By the Fredholm alternative A.11 we have $\mathcal{R}(\mathcal{L}) = \mathcal{N}(\mathcal{L}^*)^\perp = \Psi^\perp$. Thus, $\mathcal{L}(\Psi^\perp \cap Y_\eta) \subset \mathcal{R}(\mathcal{L}) = \Psi^\perp$. □

The subspace $\Psi^\perp \subset X_\eta$ and its intersection with the domain Y_η and the space X_η^1 will be used frequently in the following. For this purpose, we introduce the notation

$$V_\eta := \Psi^\perp \subset X_\eta, \quad V_\eta^1 := \Psi^\perp \cap X_\eta^1, \quad V_\eta^2 := \Psi^\perp \cap Y_\eta. \tag{3.46}$$

3.4 The semigroup $e^{t\mathcal{L}}$

Lemma 3.10 shows that $\mathcal{L}$ defines a sectorial operator on X_η. Using the theory from [32, Chap.1] we conclude that $\mathcal{L}$ generates an analytic semigroup on X_η which will be denoted by $\{e^{t\mathcal{L}}\}_{t\geq 0}$. There are various texts in the literature concerning analytic semigroups, see [50], [9], [42], [52], [26]. We use the semigroup to show existence of a solution of the perturbed problem (0.22) with $\mathbf{u}(0) = \mathbf{v}_\star + \mathbf{u}_0$ and, in addition, the nonlinear stability of $\mathbf{v}_\star$. For this purpose, we need time decaying estimates of the semigroup $e^{t\mathcal{L}}$. But since $0 \in \rho(\mathcal{L})$ the theory from [32] only guarantees estimates by exponentially growing terms, i.e. $\|e^{t\mathcal{L}}\| \leq Ce^{\beta t}$ for $\beta > 0$. This would be sufficient to show local existence of solutions but not for proving nonlinear stability. Taking the projection P from Lemma 3.20 into account, it is possible to show time decaying estimates of the semigroup on the subspace V_η. Roughly speaking, the projection P blocks the zero eigenvalue and therefore the neutral direction spanned by $\mathcal{N}(\mathcal{L})$ in the dynamics of the equation (0.22).

Theorem 3.21. *Let the Assumption 1-4 be satisfied and $0 < \mu < \min(\mu_{\mathrm{ess}}, \mu_\star, 2)$ with $\mu_\star$ from Theorem 2.6 and μ_{ess} from Assumption 3. Then the linearized operator $\mathcal{L} : Y_\eta \to X_\eta$ generates an analytic semigroup $\{e^{t\mathcal{L}}\}_{t\geq 0}$ on X_η given by*

$$e^{t\mathcal{L}} = \frac{1}{2\pi i}\int_\Gamma e^{ts}(sI - \mathcal{L})^{-1}ds,$$

where Γ is any contour in $\rho(\mathcal{L})$ with $\sigma(\mathcal{L})$ in its interior and $\arg\lambda \to \pm\left(\frac{\pi}{2} + \varepsilon\right)$ as $|\lambda| \to \infty$. $\lambda \in \Gamma$ for some $\varepsilon > 0$.
Moreover, there exist $K \geq 1$, $\beta > 0$ such that for all $t > 0$ and $\mathbf{w} \in V_\eta^\ell$, $\ell = 0, 1$ there hold

$$\|e^{t\mathcal{L}}\mathbf{w}\|_{X_\eta^\ell} \leq Ke^{-\beta t}\|\mathbf{w}\|_{X_\eta^\ell}.$$

Proof. By Lemma 3.10 and Corollary 3.19 the operator $\mathcal{L} : Y_\eta \to X_\eta$ is a sectorial operator, i.e. for all $\gamma > 0$ there is $M \geq 1$ such that

$$\|(sI - \mathcal{L})^{-1}\|_{X_\eta \to X_\eta} \leq \frac{M}{|s-\gamma|} \quad \forall\, s \in S_{\varepsilon_0,\gamma}$$

holds. For fixed $t > 0$ we choose an upwards orientated contour $\Gamma := \Gamma_+ \cup \Gamma_0 \cup \Gamma_-$ with

$$\Gamma_\pm := \{z = \gamma + \tau e^{\pm i\left(\frac{\pi}{2}+\varepsilon\right)},\, \tau \geq t^{-1}\}, \quad \Gamma_0 := \{z = \gamma + t^{-1}e^{i\theta},\, |\theta| \leq \frac{\pi}{2} + \varepsilon\} \tag{3.47}$$

for arbitrary $0 < \varepsilon < \varepsilon_0$, cf. Figure 3.4 a).

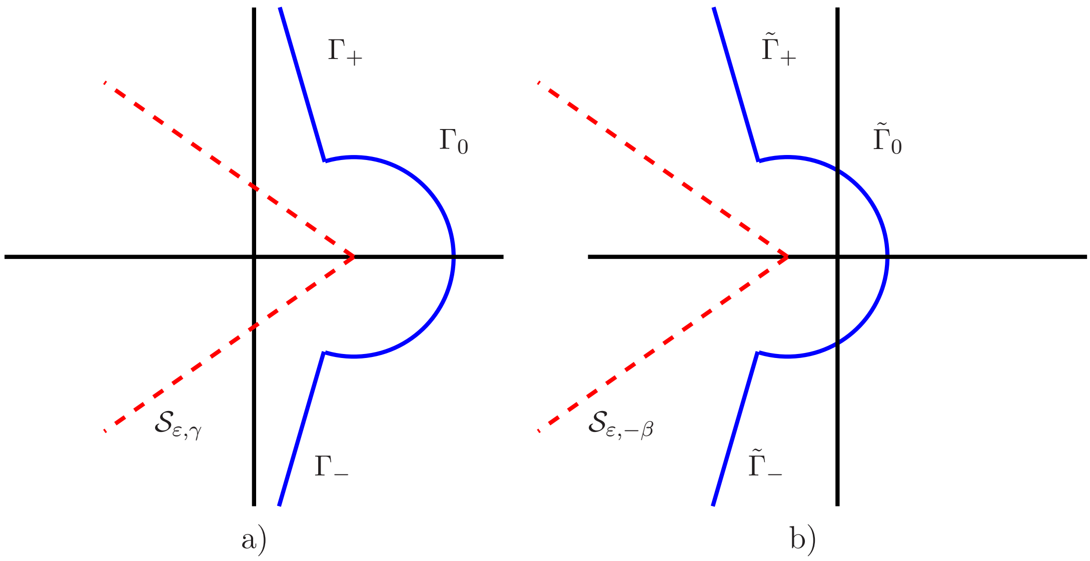


Figure 3.4: The contours $\Gamma, \tilde{\Gamma}$ in the proof of Theorem 3.21.

Then we may estimate

$$\left\|\int_{\Gamma_\pm} e^{ts}(sI - \mathcal{L})^{-1} ds\right\|_{X_\eta \to X_\eta} \leq M e^{t\gamma} \int_{t^{-1}}^{\infty} e^{-t\tau \sin\varepsilon} \tau^{-1} d\tau \leq M e^{t\gamma} \int_{1}^{\infty} e^{-\sigma \sin\varepsilon} \sigma^{-1} d\sigma < \infty$$

and since $\Gamma \subset \rho(\mathcal{L})$

$$\left\|\int_{\Gamma_0} e^{ts}(sI - \mathcal{L})^{-1} ds\right\|_{X_\eta \to X_\eta} \leq C e^{t\gamma} \int_{-\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} e^{\cos\theta} d\theta < \infty.$$

Hence the integral

$$e^{t\mathcal{L}} = \frac{1}{2\pi i} \int_\Gamma e^{ts}(sI - \mathcal{L})^{-1} ds,$$

is absolutely convergent in $L[X_\eta]$. This implies $\{e^{t\mathcal{L}}\}_{t\geq 0}$ to be an analytic semigroup on X_η generated by $\mathcal{L}$, cf. [32, Sec. 1.3]. Moreover, since the integrand is holomorphic in $\rho(\mathcal{L})$, the integral is independent on the choice of the contour Γ satisfying the assumptions.

Consider the restriction of $\mathcal{L}$ on V_η defined by

$$\mathcal{L}_{V_\eta} : V_\eta^2 \to V_\eta, \quad \mathbf{w} \mapsto \mathcal{L}\mathbf{w}.$$

Due to (3.45) we have $V_\eta = \Psi^\perp = \mathcal{N}(\mathcal{L}^*)^\perp = \mathcal{R}(\mathcal{L})$ and thus V_η is closed in X_η. Then $\mathcal{L}_{V_\eta} \in \mathcal{C}[V_\eta]$ with $\mathcal{N}(\mathcal{L}_{V_\eta}) = \{0\}$, $\mathcal{R}(\mathcal{L}_{V_\eta}) = V_\eta$. Therefore, $sI - \mathcal{L}_{V_\eta} : V_\eta^2 \to V_\eta$ is Fredholm of index 0 and $0 \in \rho(\mathcal{L}_{V_\eta})$. Moreover, $\rho(\mathcal{L}) \subset \rho(\mathcal{L}_{V_\eta})$. To see that, take $s \in \rho(\mathcal{L})$. Then the equation $(sI - \mathcal{L})\mathbf{u} = \mathbf{r} \in V_\eta$ has a unique solution $\mathbf{u} \in Y_\eta$. Applying $(I - P)$ to the equation yields $(sI - \mathcal{L})\mathbf{w} = \mathbf{r}$, where $\mathbf{w} = (I - P)\mathbf{u}$. Then $(I - P)\mathbf{u} = \mathbf{u}$ and thus $(sI - \mathcal{L})$ is bounded invertible from V_η^2 to V_η. This shows $s \in \rho(\mathcal{L}_{V_\eta})$. Now we conclude by Corollary 3.19

$$\mathcal{S}_{\varepsilon_0,\gamma_0} \subset \rho(\mathcal{L}_{V_\eta})$$

for some $\varepsilon_0 > 0$, $\gamma_0 < 0$. Using Lemma 3.11 we find $\gamma_0 < -\beta < 0$, $\varepsilon > 0$ and a constants $C_1, C_2 > 0$, $R > 2\beta$ such that for all $\mathbf{w} \in V_\eta^1$ there holds

$$\|(sI - \mathcal{L})^{-1}\mathbf{w}\|_{X_\eta^1} = \|(sI - \mathcal{L}_{V_\eta})^{-1}\mathbf{w}\|_{X_\eta^1} \leq C_1\|\mathbf{w}\|_{X_\eta^1} \quad \forall s \in \mathcal{S}_{\varepsilon,-\beta} \cap B_R(0), \tag{3.48}$$

$$\|(sI - \mathcal{L})^{-1}\mathbf{w}\|_{X_\eta^1} = \|(sI - \mathcal{L}_{V_\eta})^{-1}\mathbf{w}\|_{X_\eta^1} \leq \frac{C_2}{|s|}\|\mathbf{w}\|_{X_\eta^1} \quad \forall s \in \mathcal{S}_{\varepsilon,-\beta}\backslash B_R(0) \tag{3.49}$$

Combining (3.48) and (3.49) we find $M > 0$ such that

$$\|(sI - \mathcal{L})^{-1}\mathbf{w}\|_{X_\eta^1} = \|(sI - \mathcal{L}_{V_\eta})^{-1}\mathbf{w}\|_{X_\eta^1} \leq \frac{M}{|s + \beta|}\|\mathbf{w}\|_{X_\eta^1} \quad \forall s \in \mathcal{S}_{\varepsilon,-\beta}.$$

We choose the contour $\tilde{\Gamma} := \tilde{\Gamma}_+ \cup \tilde{\Gamma}_0 \cup \tilde{\Gamma}_-$ with $\tilde{\Gamma}_\pm, \tilde{\Gamma}_0$ as in (3.47) and $-\beta$ instead of γ, cf. Figure 3.4 b). Then $\Gamma, \tilde{\Gamma} \subset S_{\varepsilon,-\beta} \subset \rho(\mathcal{L}_{V_\eta})$ and we obtain using Cauchy's integral theorem for all $\mathbf{w} \in V_\eta^1$

$$e^{t\mathcal{L}}\mathbf{w} = \int_\Gamma e^{ts}(sI - \mathcal{L})^{-1}\mathbf{w}ds = \int_\Gamma e^{ts}(sI - \mathcal{L}_{V_\eta})^{-1}\mathbf{w}ds = \int_{\tilde{\Gamma}} e^{ts}(sI - \mathcal{L}_{V_\eta})^{-1}\mathbf{w}ds.$$

Then there is $K \geq 1$ such that

$$\begin{aligned}\left\|\int_{\tilde{\Gamma}_\pm} e^{ts}(sI - \mathcal{L}_{V_\eta})^{-1}\mathbf{w}ds\right\|_{X_\eta^1} &\leq Me^{-t\beta}\int_{t^{-1}}^\infty e^{-t\tau\sin\varepsilon}\tau^{-1}d\tau\|\mathbf{w}\|_{X_\eta^1} \\ &\leq Me^{-t\beta}\int_1^\infty e^{-\sigma\sin\varepsilon}\sigma^{-1}d\sigma\|\mathbf{w}\|_{X_\eta^1} \leq \frac{2\pi K}{3}e^{-t\beta}\|\mathbf{w}\|_{X_\eta^1}\end{aligned}$$

and

$$\left\| \int_{\tilde{\Gamma}_0} e^{ts}(sI - \mathcal{L}_{V_\eta})^{-1}\mathbf{w}ds \right\|_{X_\eta^1} \leq Ce^{-t\beta} \int_{-\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} e^{\cos\theta} d\theta \|\mathbf{w}\|_{X_\eta^1} \leq \frac{2\pi K}{3} e^{-t\beta}\|\mathbf{w}\|_{X_\eta^1}.$$

Finally this yields for all $\mathbf{w} \in V_\eta^1$ and $t > 0$ the estimate

$$\|e^{t\mathcal{L}}\mathbf{w}\|_{X_\eta^1} = \left\| \frac{1}{2\pi i} \int_{\tilde{\Gamma}} e^{ts}(sI - \mathcal{L}_{V_\eta})^{-1}\mathbf{w}ds \right\|_{X_\eta^1} \leq Ke^{-t\beta}\|\mathbf{w}\|_{X_\eta^1}.$$

□

3.5 Decomposition of the dynamics

Recall the Cauchy problem (0.22) with perturbed initial data, i.e.

$$\mathbf{u}_t = \mathcal{F}(\mathbf{u}), \quad \mathbf{u}(0) = \mathbf{v}_\star + \mathbf{u}_0.$$

In the previous section we have shown that the semigroup $\{e^{t\mathcal{L}}\}_{t\geq 0}$ is decaying in time on the subspace V_η. In what follows we decompose the dynamics of the solution into a motion along the group orbit of the wave $a(\gamma)\mathbf{v}_\star$, $\gamma \in \mathcal{G}$ and a perturbation $\mathbf{w}$ in the space V_η, cf. [17] and Figure 3.5. Moreover, we write the motion on the group orbit in local coordinates on the manifold $\mathcal{G}$. Precisely, recall the chart (U, χ) on $\mathcal{G}$ from (3.13). For $t \geq 0$ we want to write the solution $\mathbf{u}$ as

$$\mathbf{u}(t) = a(\gamma(t))\mathbf{v}_\star + \mathbf{w}(t), \quad \gamma(t) = \chi^{-1}(z(t)) \in U, \, \mathbf{w}(t) \in V_\eta.$$

Thus z describes the local coordinates of a motion on the group orbit $\mathcal{O}(\mathbf{v}_\star)$ given by γ in the chart (U, χ). Moreover, $\mathbf{w} \in V_\eta$ describes the difference of the solution to the group orbit in the space $V_\eta = \Psi^\perp$. It turns out that the decomposition is unique as long as the solution stays in a small neighborhood of the group orbit and γ stays in U. This will be guaranteed by taking sufficiently small initial perturbations $\mathbf{u}_0$. We follow [17] and start by considering the map

$$\Pi : \chi(U) \subset \mathbb{R}^2 \to \Phi, \quad z \mapsto P(a(\chi^{-1}(z))\mathbf{v}_\star - \mathbf{v}_\star) \tag{3.50}$$

In what follows we often write γ instead of $\chi^{-1}(z)$ for abbreviation.

Lemma 3.22. *Let the Assumptions 1-4 be satisfied and* $0 < \mu < \min(\mu_{\text{ess}}, \mu_\star, 2)$ *with* μ_{ess} *from Assumption 3 and* $\mu_\star$ *from Theorem 2.6. Then there is a zero neighborhood* $W \subset \chi(U)$ *such that the map* $\Pi : W \to \Phi$ *from* (3.50) *is a diffeomorphism. Moreover, there is a zero neighborhood* $V \subset \chi(U) \times V_\eta$ *such that the transformation*

$$T : V \to X_\eta, \quad (z, \mathbf{w}) \mapsto a(\chi^{-1}(z))\mathbf{v}_\star - \mathbf{v}_\star + \mathbf{w}$$

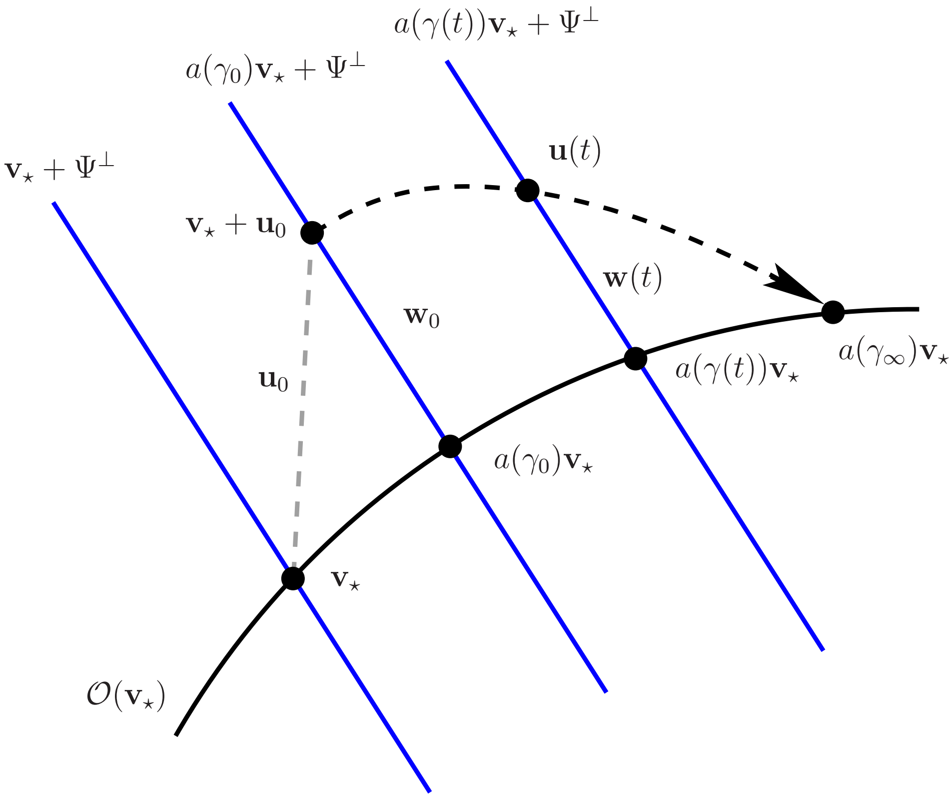


Figure 3.5: Decomposition of the dynamics.

is a diffeomorphism and the solution of $T(z,\mathbf{w}) = \mathbf{v}$ *is given by*

$$z = \Pi^{-1}(P\mathbf{v}), \quad \mathbf{w} = \mathbf{v} + \mathbf{v}_\star - a(\chi^{-1}(z))\mathbf{v}_\star.$$

Proof. Since the group action a is continuous, cf. Lemma 3.7, so is Π and $\Pi(0) = 0$. Using Lemma 3.7 and $P\varphi_i = \varphi_i$, with the eigenfunctions φ_i from Lemma 3.20, we conclude that $\Pi \in C^1(\chi(U), \Phi)$ with derivative

$$\partial_z \Pi(z) y = y_1 a(\gamma)\varphi_1 + y_2 a(\gamma)\varphi_2, \quad y = (y_1, y_2) \in \mathbb{R}^2.$$

Moreover, $\partial_z\Pi(0)$ is invertible on $\Phi = \operatorname{span}\{\varphi_1, \varphi_2\}$. Consider the function $F(z,\mathbf{w}) := \mathbf{w} - \Pi(z)$. Then $F \in C^1(\Phi \times \chi(U), \Phi)$, $F(0,0) = 0$ and $\partial_z F(0,0) = \partial_z \Pi(0)$ is invertible. Now the implicit function theorem D.8 implies Π to be a local diffeomorphism in a zero neighborhood $W \subset \chi(U) \subset \mathbb{R}^2$.
Furthermore, the map T is continuous differentiable w.r.t. $(z,\mathbf{w})$, since a is continuous differentiable, cf. Lemma 3.8, and the derivative at $(z,\mathbf{w}) = (0,0)$ is given by

$$DT(0,0) = \begin{pmatrix} \partial_z \Pi(0) & 0 \\ 0 & I_X \end{pmatrix},$$

where I_X denotes the identity on X_η. Then $DT(0,0)$ is again invertible. Consider $\tilde{F} \in C^1(X_\eta \times \chi(U) \times V_\eta, X_\eta)$ given by $\tilde{F}(\mathbf{v}, z, \mathbf{w}) = \mathbf{v} - T(z, \mathbf{w})$. Then $\tilde{F}(0,0,0) = 0$ and $\partial_{(z,\mathbf{w})}\tilde{F}(0,0,0) = DT(0,0)$ is invertible. Again the implicit function theorem D.8 implies T to be a local diffeomorphism near $(z, \mathbf{w}) = (0,0)$. Finally, we obtain from $T(z, \mathbf{w}) = \mathbf{v}$ the equation

$$\mathbf{w} = \mathbf{v} + \mathbf{v}_\star - a(\chi^{-1}(z))\mathbf{v}_\star. \tag{3.51}$$

Applying P to (3.51) yields $z = \Pi^{-1}(P\mathbf{v})$ and the assertion is proven. □

Assume there is a classical solution $\mathbf{u} \in C((0, t_\infty), Y_\eta) \cap C^1([0, t_\infty), X_\eta)$ of (0.22), cf. Definition 1.9, such that

$$\|\mathbf{u}(t) - \mathbf{v}_\star\|_{X_\eta} < \delta, \quad \forall t \in [0, t_\infty).$$

Let $\delta > 0$ be sufficiently small such that Lemma 3.22 guarantees that the map T stays invertible on $B_\delta(0) \subset X_\eta$. Then there exist $\mathbf{w} : [0, t_\infty) \to V_\eta$ and $z : [0, t_\infty) \to \mathbb{R}^2$ such that

$$\mathbf{u}(t) - \mathbf{v}_\star = T(z(t), \mathbf{w}(t)), \quad \forall t \in [0, t_\infty).$$

Since T is a local diffeomorphism we conclude $\mathbf{w} \in C([0, t_\infty), V_\eta^2) \cap C^1([0, t_\infty), V_\eta)$ and $z \in C^1([0, t_\infty), \chi(U))$. By Lemma 3.22 we obtain the decomposition of the solution $\mathbf{u}$ via

$$\mathbf{u}(t) = a(\chi^{-1}(z(t)))\mathbf{v}_\star + \mathbf{w}(t), \quad \forall t \in [0, t_\infty).$$

Taking the initial condition from (0.22) into account yields for $t = 0$

$$\mathbf{v}_\star + \mathbf{u}_0 = \mathbf{u}(0) = a(\chi^{-1}(z(0)))\mathbf{v}_\star + \mathbf{w}(0),$$

which leads to

$$\mathbf{u}_0 = T(z(0), \mathbf{w}(0)).$$

Thus the initial values for $z, \mathbf{w}$ are given by

$$z(0) = \Pi^{-1}(P\mathbf{u}_0) =: z_0, \quad \mathbf{w}(0) = \mathbf{u}_0 + \mathbf{v}_\star - a(\chi^{-1}(z(0)))\mathbf{v}_\star =: \mathbf{w}_0. \tag{3.52}$$

Now let $z(t) = (\theta(t), \tau(t))$ then Lemma 3.7 and the chain rule imply

$$\frac{d}{dt}(a(\cdot)\mathbf{v}_\star \circ \chi^{-1})(z) = a(\chi^{-1}(z))\varphi_1\theta_t + a(\chi^{-1}(z))\varphi_2\tau_t.$$

Recall $\mathcal{L}_0$ from (3.8). Since $\mathbf{u}$ solves (0.22) we have with $\mathbf{w} = (w, \zeta)^\top$, $\gamma = \chi^{-1}(z)$

$$
\begin{aligned}
0 &= \mathbf{u}_t - \mathcal{L}_0 \mathbf{u} - \begin{pmatrix} f(u) \\ f(\rho) \end{pmatrix} \\
&= \frac{d}{dt}(a(\cdot)\mathbf{v}_\star \circ \chi^{-1})(z) + \mathbf{w}_t - \mathcal{L}_0[a(\gamma)\mathbf{v}_\star] - \mathcal{L}_0 \mathbf{w} - \begin{pmatrix} f(R_\theta v_\star(\cdot - \tau) + w) \\ f(R_\theta v_\infty + \zeta) \end{pmatrix} \\
&= \frac{d}{dt}(a(\cdot)\mathbf{v}_\star \circ \chi^{-1})(z) + \mathbf{w}_t - a(\gamma)\mathcal{L}_0 \mathbf{v}_\star - \mathcal{L}_0 \mathbf{w} - \begin{pmatrix} f(R_\theta v_\star(\cdot - \tau) + w) \\ f(R_\theta v_\infty + \zeta) \end{pmatrix}.
\end{aligned}
$$

The equivariance of $\mathcal{F}$ from Lemma 3.8 implies

$$
-a(\gamma)\mathcal{L}_0 \mathbf{v}_\star = \begin{pmatrix} f(R_\theta v_\star(\cdot - \tau)) \\ f(R_\theta v_\infty) \end{pmatrix} \quad \forall \gamma =\in \mathcal{G}.
$$

Therefore by taking the linearized operator $\mathcal{L}$ from (0.26) into account we finally observe

$$
\mathbf{w}_t = \mathcal{L}\mathbf{w} - a(\chi^{-1}(z))\varphi_1 \theta_t - a(\chi^{-1}(z))\varphi_2 \tau_t + r^{[f]}(z, \mathbf{w}) \tag{3.53}
$$

where the remainder $r^{[f]}$ is given by

$$
r^{[f]}(z, \mathbf{w}) := \begin{pmatrix} f(R_\theta v_\star(\cdot - \tau) + w) \\ f(R_\theta v_\infty + \zeta) \end{pmatrix} - \begin{pmatrix} f(R_\theta v_\star(\cdot - \tau)) \\ f(R_\theta v_\infty) \end{pmatrix} - \begin{pmatrix} Df(v_\star)w \\ Df(v_\infty)\zeta \end{pmatrix}, \quad z = (\theta, \tau). \tag{3.54}
$$

Since $\mathbf{w}(t) \in V_\eta$, $t \in [0, t_\infty)$ we have by applying the projector P to (3.53)

$$
0 = Pr^{[f]}(z, \mathbf{w}) - Pa(\chi^{-1}(z))\varphi_1 \theta_t - Pa(\chi^{-1}(z))\varphi_2 \tau_t. \tag{3.55}
$$

Let us write (3.55) as an explicit ODE for $z = (\theta, \tau)$.

Lemma 3.23. *Let the Assumptions 1-4 be satisfied and $0 < \mu < \min(\mu_{\mathrm{ess}}, \mu_\star, 2)$ with μ_{ess} from Assumption 3 and $\mu_\star$ from Theorem 2.6. Then for every $z \in \mathbb{R}^2$ the map*

$$
S(z) : \mathbb{R}^2 \to \Phi, \quad y \mapsto Pa(\chi^{-1}(z))\varphi_1 y_1 + Pa(\chi^{-1}(z))\varphi_2 y_2
$$

is a continuous, linear map and continuously differentiable w.r.t. z, i.e. $S \in C^1(\mathbb{R}^2, L[\mathbb{R}^2, \Phi])$. Moreover, there is a zero neighborhood $V \subset \mathbb{R}^2$ such that $S(z)^{-1}$ exists for all $z \in V$ and depends continuously on z.

Proof. By Lemma 3.7, it follows that $S(z)$ is continuous and linear and since $\varphi_1, \varphi_2 \in Y_\eta$ is continuously differentiable in z. It remains to show the same for $S(z)^{-1}$. Let $\mathbf{w} \in \Phi$. Applying $(\psi_1, \cdot), (\psi_2, \cdot)$ with $(\cdot, \mathrm{cot}) = (\cdot, \cdot)_{X_\eta}$ to the equation $S(z)y = \mathbf{w}$ yields

$$
M(z)y = \begin{pmatrix} (\psi_1, \mathbf{w}) \\ (\psi_2, \mathbf{w}) \end{pmatrix}, \quad M(z) = \begin{pmatrix} (\psi_1, Pa(\chi^{-1}(z))\varphi_1) & (\psi_1, Pa(\chi^{-1}(z))\varphi_2) \\ (\psi_2, Pa(\chi^{-1}(z))\varphi_1) & (\psi_2, Pa(\chi^{-1}(z))\varphi_2) \end{pmatrix}. \tag{3.56}
$$

Now $M(0) = I$ and, by Lemma 3.7, $M(z)$ is C^1. Then there exists a zero neighborhood $V \subset \mathbb{R}^2$ such that $M(z)^{-1}$ exists and is also C^1. Now $S(z)^{-1} \in L[\Phi, \mathbb{R}^2]$ is given by

$$S(z)^{-1}\mathbf{w} = M(z)^{-1} \begin{pmatrix} (\psi_1, \mathbf{w}) \\ (\psi_2, \mathbf{w}) \end{pmatrix}$$

and $S(\cdot)^{-1} \in C^1(V, L[\Phi, \mathbb{R}^2])$. □

By Lemma 3.23 we obtain from (3.55) and (3.52) the ODE for z

$$z_t = r^{[z]}(z, \mathbf{w}), \quad z(0) = \Pi^{-1}(P\mathbf{u}_0), \tag{3.57}$$

where $r^{[z]}(\cdot, \mathbf{w}) : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$r^{[z]}(z, \mathbf{w}) := S(z)^{-1} P r^{[f]}(z, \mathbf{w}). \tag{3.58}$$

Applying $(I - P)$ to (3.53) and using (3.57) yields

$$\begin{aligned} \mathbf{w}_t &= \mathcal{L}\mathbf{w} + (I-P) r^{[f]}(z, \mathbf{w}) - (I-P)(a(\cdot)\mathbf{v}_\star \circ \chi^{-1})(z) S(z)^{-1} P r^{[f]}(z, \mathbf{w}) \\ &=: \mathcal{L}\mathbf{w} + r^{[w]}(z, \mathbf{w}). \end{aligned}$$

with the remainder $r^{[w]}$ given by

$$r^{[w]}(z, \mathbf{w}) := \Big((I-P) - (I-P)(a(\cdot)\mathbf{v}_\star \circ \chi^{-1})(z) S(z)^{-1} P \Big) r^{[f]}(z, \mathbf{w}). \tag{3.59}$$

Finally, we obtain the transformed system for w, γ

$$\mathbf{w}_t = \mathcal{L}\mathbf{w} + r^{[w]}(z, \mathbf{w}), \qquad \mathbf{w}(0) = \mathbf{u}_0 + \mathbf{v}_\star - a(\Pi^{-1}(P\mathbf{u}_0))\mathbf{v}_\star =: \mathbf{w}_0 \tag{3.60}$$

$$z_t = r^{[z]}(z, \mathbf{w}), \qquad z(0) = \Pi^{-1}(P\mathbf{u}_0) =: z_0. \tag{3.61}$$

According to Definition 1.9 we define classical solutions to the system (3.60), (3.61).

Definition 3.24. A pair $(z, \mathbf{w})$ is called a classical solution of (3.60), (3.61) on $[0, t_\infty)$ for some $t_\infty > 0$ if

i) $\mathbf{w} \in C((0, t_\infty), V_\eta^2) \cap C^1([0, t_\infty), V_\eta)$ and $z \in C^1([0, t_\infty), \mathbb{R}^2)$.

ii) $\mathbf{w}_t(t) = \mathcal{L}\mathbf{w}(t) + r^{[w]}(z(t), \mathbf{w}(t))$ and $z_t(t) = r^{[z]}(z(t), \mathbf{w}(t))$ for every $t \in [0, t_\infty)$.

iii) $\mathbf{w}(0) = \mathbf{w}_0$ and $z(0) = z_0$.

If $t_\infty = \infty$ we will call $(z, \mathbf{w})$ a global classical solution of (3.60), (3.61), whereas for $t_\infty < \infty$ we will call $(z, \mathbf{w})$ a local classical solution of (3.60), (3.61).

3.6 Estimates of nonlinearities in weighted spaces

To study solutions of the system (3.60), (3.61) we will use the semigroup $e^{t\mathcal{L}}$ and need to control the remaining nonlinearities $r^{[f]}, r^{[w]}, r^{[z]}$ from (3.54), (3.59) and (3.58). In this section we derive Lipschitz estimates with small Lipschitz constants for the nonlinearities in the space X^1_η.

Lemma 3.25. *Let the Assumptions 1-4 be satisfied and $0 < \mu < \min(\mu_{\mathrm{ess}}, \mu_\star, 2)$ with μ_{ess} from Assumption 3 and $\mu_\star$ from Theorem 2.6. Then there exist $\delta > 0$ and constants $C_0, C_1, C_2, C_3, C_4 > 0$ such that for all $z, z_1, z_2 \in B_\delta(0) \subset \mathbb{R}^2$ and $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2 \in B_\delta(0) \subset X^1_\eta$ there hold*

$$
\begin{aligned}
&i)\ \ \ \|r^{[f]}(z,\mathbf{w}_1) - r^{[f]}(z,\mathbf{w}_2)\|_{X^1_\eta} \le C_0 \left(|z| + \max\left\{\|\mathbf{w}_1\|_{X^1_\eta}, \|\mathbf{w}_2\|_{X^1_\eta}\right\}\right) \|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta},\\
&ii)\ \ \|r^{[f]}(z_1,\mathbf{w}) - r^{[f]}(z_2,\mathbf{w})\|_{X^1_\eta} \le C_1 |z_1 - z_2|,\\
&iii)\ \|r^{[w]}(z,\mathbf{w}_1) - r^{[w]}(z,\mathbf{w}_2)\|_{X^1_\eta} \le C_2 \left(|z| + \max\left\{\|\mathbf{w}_1\|_{X^1_\eta}, \|\mathbf{w}_2\|_{X^1_\eta}\right\}\right) \|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta},\\
&iv)\ \ \|r^{[w]}(z_1,\mathbf{w}_2) - r^{[w]}(z_2,\mathbf{w}_2)\|_{X^1_\eta} \le C_3 \left(|z_1 - z_2| + \|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta}\right),\\
&v)\ \ \ |r^{[z]}(z_1,\mathbf{w}_1) - r^{[z]}(z_2,\mathbf{w}_2)| \le C_4 \left(|z_1 - z_2| + \|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta}\right).
\end{aligned}
$$

Note that since $r^{[f]}(z, 0) = 0$ the estimates i) and iii) also imply boundedness of the nonlinearities $r^{[f]}, r^{[w]}$.

Proof. For the proof let $C > 0$ denote a universal constant and let $\delta > 0$ be so small such that $B_\delta(0) \subset \chi(U)$ with (U, χ) from (3.13). Moreover, let $\gamma = \chi(z) = (\theta, \tau)$, $\mathbf{w} = (w, \zeta)^\top$ as well as $\gamma_i = \chi(z_i) = (\theta_i, \tau_i)$, $\mathbf{w}_i = (w_i, \zeta_i)^\top$, $i = 1, 2$. For the sake of notation we write $a(\gamma)v = R_\theta v(\cdot - \tau)$ for a function $v : \mathbb{R} \to \mathbb{R}^2$. For a matrix-valued function $M : \mathbb{R} \to \mathbb{R}^{2,2}$ we write $\|M\|_{L^\infty} = \||M|\|_{L^\infty(\mathbb{R},\mathbb{R})}$, $\|M\|_{L^2_\eta} = \||M|\|_{L^2_\eta(\mathbb{R},\mathbb{R})}$ for some matrix norm $|\cdot|$ on $\mathbb{R}^{2,2}$.

i). We have by definition and the triangle inequality

$$
\begin{aligned}
&\|r^{[f]}(z,\mathbf{w}_1) - r^{[f]}(z,\mathbf{w}_2)\|_{X^1_\eta}\\
&\le |f(R_\theta v_\infty + \zeta_1) - f(R_\theta v_\infty + \zeta_2) - Df(v_\infty)(\zeta_1 - \zeta_2)|\\
&\quad + \|f(a(\gamma)v_\star + w_1) - f(a(\gamma)v_\star + w_2) - Df(v_\star)(w_1 - w_2)\\
&\qquad\quad - \hat{v}[f(R_\theta v_\infty + \zeta_1) - f(R_\theta v_\infty + \zeta_2) - Df(v_\infty)(\zeta_1 - \zeta_2)]\|_{L^2_\eta}\\
&\quad + \|\partial_x[f(a(\gamma)v_\star + w_1) - f(a(\gamma)v_\star + w_2) - Df(v_\star)(w_1 - w_2)]\|_{L^2_\eta} =: T_1 + T_2 + T_3.
\end{aligned}
$$

In the following we frequently use the mean value theorem and the fact that Assumption 1 states $f \in C^3$. Note the following estimates, which follow by Sobolev embedding

Theorem D.2, Proposition 2.7 and (3.18),

$$\begin{aligned}&\|Df(a(\gamma)v_\star)-Df(v_\star)\|_{L^\infty}\le C\|a(\gamma)v_\star-v_\star\|_{L^\infty}\\&\le C\|a(\gamma)v_\star-R_\theta v_\infty\hat v-(v_\star-v_\infty\hat v)\|_{L^\infty}+C|R_\theta v_\infty-v_\infty|\\&\le C\|a(\gamma)v_\star-R_\theta v_\infty\hat v-(v_\star-v_\infty\hat v)\|_{H^1}+C|R_\theta v_\infty-v_\infty|\\&\le C\|a(\chi^{-1}(z))\mathbf{v}_\star-\mathbf{v}_\star\|_{X^1_\eta}\le C|z|\|\mathbf{v}_\star\|_{Y_\eta}\le C|z|,\end{aligned}\tag{3.62}$$

$$|Df(R_\theta v_\infty)-Df(v_\infty)|\le C|z|,\tag{3.63}$$

$$\begin{aligned}&\|a(\gamma)v_\star-R_\theta v_\infty-(v_\star-v_\infty)\|_{L^2_\eta(\mathbb{R}_+)}\\&\le\|a(\gamma)v_\star-R_\theta v_\infty\hat v-(v_\star-v_\infty\hat v)\|_{L^2_\eta(\mathbb{R}_+)}+|R_\theta v_\infty-v_\infty|\|\hat v-1\|_{L^2_\eta(\mathbb{R}_+)}\\&\le C\|a(\chi^{-1}(z))\mathbf{v}_\star-\mathbf{v}_\star\|_{X_\eta}\le C|z|\|\mathbf{v}_\star\|_{X^1_\eta}+C|z|\le C|z|,\end{aligned}\tag{3.64}$$

$$\|v_\star-v_\infty\|_{L^2_\eta(\mathbb{R}_+)}\le\|v_\star-v_\infty\hat v\|_{L^2_\eta(\mathbb{R}_+)}+|v_\infty|\|\hat v-1\|_{L^2_\eta(\mathbb{R}_+)}\le C.\tag{3.65}$$

T_1 can be estimated by

$$\begin{aligned}&T_1=|f(R_\theta v_\infty+\zeta_1)-f(R_\theta v_\infty+\zeta_2)-Df(v_\infty)(\zeta_1-\zeta_2)|\\&\le\int_0^1|Df(R_\theta v_\infty+\zeta_2+(\zeta_1-\zeta_2)s)-Df(v_\infty)|ds|\zeta_1-\zeta_2|\\&\le\left(\int_0^1|Df(R_\theta v_\infty+\zeta_2+(\zeta_1-\zeta_2)s)-Df(R_\theta v_\infty)|ds+|Df(R_\theta v_\infty)-Df(v_\infty)|\right)|\zeta_1-\zeta_2|\\&\le C\left(\int_0^1|\zeta_2-(\zeta_1-\zeta_2)s|ds+|R_\theta v_\infty-v_\infty|\right)|\zeta_1-\zeta_2|\\&\le C\left(|z|+\max\{|\zeta_1|,|\zeta_2|\}\right)|\zeta_1-\zeta_2|\le C\left(|z|+\max\left\{\|\mathbf{w}_1\|_{X^1_\eta},\|\mathbf{w}_2\|_{X^1_\eta}\right\}\right)\|\mathbf{w}_1-\mathbf{w}_2\|_{X^1_\eta}.\end{aligned}$$

For T_2 we have

$$\begin{aligned}T_2=\;&\|f(a(\gamma)v_\star+w_1)-f(a(\gamma)v_\star+w_2)-Df(v_\star)(w_1-w_2)\\&-\hat v[f(R_\theta v_\infty+\zeta_1)-f(R_\theta v_\infty+\zeta_2)-Df(v_\infty)(\zeta_1-\zeta_2)]\|_{L^2_\eta}\\=\;&\Big\|\int_0^1 Df(a(\gamma)v_\star+w_2+(w_1-w_2)s)-Df(v_\star)ds(w_1-w_2)\\&-\hat v\int_0^1 Df(R_\theta v_\infty+\zeta_2+(\zeta_1-\zeta_2)s)-Df(v_\infty)ds(\zeta_1-\zeta_2)\Big\|_{L^2_\eta}\\\le\;&\Big\|\int_0^1 Df(a(\gamma)v_\star+w_2+(w_1-w_2)s)-Df(a(\gamma)v_\star)ds(w_1-w_2)\\&-\hat v\int_0^1 Df(R_\theta v_\infty+\zeta_2+(\zeta_1-\zeta_2)s)-Df(R_\theta v_\infty)ds(\zeta_1-\zeta_2)\Big\|_{L^2_\eta}\\&+\|[Df(a(\gamma)v_\star)-Df(v_\star)](w_1-w_2)-\hat v[Df(R_\theta v_\infty)-Df(v_\infty)](\zeta_1-\zeta_2)\|_{L^2_\eta}\\=:\;&T_4+T_5.\end{aligned}$$

We estimate T_5 by two terms

$$\begin{aligned}T_5 \leq\; &\|[Df(a(\gamma)v_\star) - Df(v_\star)](w_1 - \hat{v}\zeta_1 - w_2 + \hat{v}\zeta_2)\|_{L^2_\eta} \\ &+ \|[Df(a(\gamma)v_\star) - Df(v_\star) - Df(R_\theta v_\infty) + Df(v_\infty)](\zeta_1 - \zeta_2)\hat{v}\|_{L^2_\eta} =: T_6 + T_7.\end{aligned}$$

Using (3.62) we have

$$T_6 \leq C|z|\|w_1 - \hat{v}\zeta_1 - w_2 + \hat{v}\zeta_2\|_{L^2_\eta} \leq C|z|\|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta}.$$

Next, we bound T_7 by

$$\begin{aligned}T_7 \leq\; &\|[Df(a(\gamma)v_\star) - Df(v_\star) - Df(R_\theta v_\infty) + Df(v_\infty)](\zeta_1\hat{v} - \zeta_2\hat{v})\|_{L^2_\eta(\mathbb{R}_-)} \\ &+ \|[Df(a(\gamma)v_\star) - Df(v_\star) - Df(R_\theta v_\infty) + Df(v_\infty)](\zeta_1\hat{v} - \zeta_2\hat{v})\|_{L^2_\eta(\mathbb{R}_+)} =: T_8 + T_9\end{aligned}$$

and (3.62), (3.63) together with Proposition 2.7 imply

$$\begin{aligned}T_8 &\leq \|Df(a(\gamma)v_\star) - Df(v_\star) - Df(R_\theta v_\infty) + Df(v_\infty)\|_{L^\infty}|\zeta_1 - \zeta_2|\|\hat{v}\|_{L^2_\eta(\mathbb{R}_-)} \\ &\leq C|z||\zeta_1 - \zeta_2| \leq C|z|\|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta}.\end{aligned}$$

Use the abbreviations $\chi_1(s) := v_\star + s(a(\gamma)v_\star - v_\star)$, $\chi_2(s) := v_\infty + s(R_\theta v_\infty - v_\infty)$, $s \in [0,1]$. Then use (3.64), (3.65) to obtain

$$\begin{aligned}T_9 &= \|[Df(a(\gamma)v_\star) - Df(v_\star) - Df(R_\theta v_\infty) + Df(v_\infty)](\zeta_1 - \zeta_2)\hat{v}\|_{L^2_\eta(\mathbb{R}_+)} \\ &\leq \Big\| \int_0^1 D^2 f(\chi_1(s))[a(\gamma)v_\star - v_\star, (\zeta_1 - \zeta_2)\hat{v}]ds \\ &\qquad - \int_0^1 D^2 f(\chi_2(s))[R_\theta v_\infty - v_\infty, (\zeta_1 - \zeta_2)\hat{v}]ds \Big\|_{L^2_\eta(\mathbb{R}_+)} \\ &\leq \Big\| \int_0^1 D^2 f(\chi_1(s))[a(\gamma)v_\star - v_\star - R_\theta v_\infty + v_\infty, (\zeta_1 - \zeta_2)\hat{v}]ds \Big\|_{L^2_\eta(\mathbb{R}_+)} \\ &\qquad + \Big\| \int_0^1 [D^2 f(\chi_1(s)) - D^2 f(\chi_2(s))][R_\theta v_\infty + v_\infty, (\zeta_1 - \zeta_2)\hat{v}]ds \Big\|_{L^2_\eta(\mathbb{R}_+)} \\ &\leq C\|a(\gamma)v_\star - R_\theta v_\infty - (v_\star - v_\infty)\|_{L^2_\eta(\mathbb{R}_+)}|\zeta_1 - \zeta_2| \\ &\qquad + C\Big\| \int_0^1 \chi_1(s) - \chi_2(s)ds \Big\|_{L^2_\eta(\mathbb{R}_+)} |R_\theta v_\infty - v_\infty||\zeta_1 - \zeta_2| \\ &\leq C|z||\zeta_1 - \zeta_2| \leq C|z|\|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta}.\end{aligned}$$

To estimate T_4 use the abbreviations $w(s) := w_2 + (w_1 - w_2)s$, $\zeta(s) := \zeta_2 + (\zeta_1 - \zeta_2)s$,

$s \in [0,1]$ and obtain

$$\begin{aligned}
T_4 &= \Big\| \int_0^1 Df(a(\gamma)v_\star + w(s)) - Df(a(\gamma)v_\star) ds (w_1 - w_2) \\
&\qquad - \hat{v} \int_0^1 Df(R_\theta v_\infty + \zeta(s)) - Df(R_\theta v_\infty) ds (\zeta_1 - \zeta_2) \Big\|_{L^2_\eta} \\
&\le \Big\| \int_0^1 Df(a(\gamma)v_\star + w(s)) - Df(a(\gamma)v_\star) ds (w_1 - \zeta_1 \hat{v} - w_2 + \zeta_2 \hat{v}) \Big\|_{L^2_\eta} \\
&\quad + \Big\| \int_0^1 Df(a(\gamma)v_\star + w(s)) - Df(a(\gamma)v_\star) \\
&\qquad - Df(R_\theta v_\infty + \zeta(s)) + Df(R_\theta v_\infty) ds (\zeta_1 - \zeta_2)\hat{v} \Big\|_{L^2_\eta} =: T_{10} + T_{11}.
\end{aligned}$$

Now for every $s \in [0,1]$ we have

$$\begin{aligned}
&\|Df(a(\gamma)v_\star + w(s)) - Df(a(\gamma)v_\star)\|_{L^\infty} \le C \|w_2 + s(w_1 - w_2)\|_{L^\infty} \\
&\le C \max\{\|w_1\|_{L^\infty}, \|w_2\|_{L^\infty}\} \le C \max\left\{\|\mathbf{w}_1\|_{X^1_\eta}, \|\mathbf{w}_2\|_{X^1_\eta}\right\},
\end{aligned} \tag{3.66}$$

where we used that the Sobolev embedding Theorem D.2 implies for $i \in \{1,2\}$

$$\|w_i\|_{L^\infty} \le \|w_i - \zeta_i \hat{v}\|_{L^\infty} + |\zeta_i| \le C\|w_i - \zeta_i \hat{v}\|_{H^1} + |\zeta_i| \le C \|\mathbf{w}_i\|_{X^1_\eta}.$$

Now (3.66) yields

$$\begin{aligned}
T_{10} &\le \int_0^1 \|Df(a(\gamma)v_\star + w(s)) - Df(a(\gamma)v_\star)\|_{L^\infty} ds \|w_1 - \zeta_1 \hat{v} - w_2 + \zeta_2 \hat{v}\|_{L^2_\eta} \\
&\le C \max\left\{\|\mathbf{w}_1\|_{X^1_\eta}, \|\mathbf{w}_2\|_{X^1_\eta}\right\} \|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&T_{11} \\
&\le \Big\| \int_0^1 Df(a(\gamma)v_\star + w(s)) - Df(a(\gamma)v_\star) - Df(R_\theta v_\infty + \zeta(s)) + Df(R_\theta v_\infty) ds (\zeta_1 - \zeta_2)\hat{v} \Big\|_{L^2_\eta(\mathbb{R}_-)} \\
&+ \Big\| \int_0^1 Df(a(\gamma)v_\star + w(s)) - Df(a(\gamma)v_\star) - Df(R_\theta v_\infty + \zeta(s)) + Df(R_\theta v_\infty) ds (\zeta_1 - \zeta_2)\hat{v} \Big\|_{L^2_\eta(\mathbb{R}_+)} \\
&=: T_{12} + T_{13}.
\end{aligned}$$

We write $\kappa(s) := a(\gamma)v_\star + w(s) - R_\theta v_\infty - \zeta(s)$. Then for $s \in [0,1]$ there holds

$$\begin{aligned}
&\Big\| Df(a(\gamma)v_\star + w(s)) - Df(a(\gamma)v_\star) - Df(R_\theta v_\infty + \zeta(s)) + Df(R_\theta v_\infty)\Big\|_{L^\infty} \\
&= \Big\| \int_0^1 D^2 f(R_\theta v_\infty + \zeta(s) + \kappa(s)\tau)[\kappa(s), \cdot] \\
&\qquad - D^2 f(R_\theta v_\infty + (a(\gamma)v_\star - R_\theta v_\infty)\tau)[a(\gamma)v_\star - R_\theta v_\infty, \cdot] d\tau \Big\|_{L^\infty} \\
&\le \Big\| \int_0^1 D^2 f(R_\theta v_\infty + \zeta(s) + \kappa(s)\tau)[w(s) - \zeta(s), \cdot] d\tau \Big\|_{L^\infty} \\
&\quad + \Big\| \int_0^1 \big(D^2 f(R_\theta v_\infty + \zeta(s) + \kappa(s)\tau) \\
&\qquad - D^2 f(R_\theta v_\infty + (a(\gamma)v_\star - R_\theta v_\infty)\tau)\big)[a(\gamma)v_\star - R_\theta v_\infty, \cdot] d\tau \Big\|_{L^\infty} \\
&\le C\|w(s) - \zeta(s)\|_{L^\infty} + C\int_0^1 \|\zeta(s) - (w(s) - \zeta(s))\tau\|_{L^\infty} d\tau \\
&\le C \max\left\{\|w_1\|_{L^\infty}, \|w_2\|_{L^\infty}\right\} \le C\max\left\{\|\mathbf{w}_1\|_{X^1_\eta}, \|\mathbf{w}_2\|_{X^1_\eta}\right\},
\end{aligned}$$

where we used $|\zeta_i| \le \|w_i\|_{L^\infty}$, $i = 1, 2$. Thus

$$T_{12} \le C\max\left\{\|\mathbf{w}_1\|_{X^1_\eta}, \|\mathbf{w}_2\|_{X^1_\eta}\right\} |\zeta_1 - \zeta_2|.$$

Similarly, for every $s \in [0,1]$,

$$\begin{aligned}
&\Big\| Df(a(\gamma)v_\star + w(s)) - Df(a(\gamma)v_\star) - Df(R_\theta v_\infty + \zeta(s)) + Df(R_\theta v_\infty)\Big\|_{L^2_\eta(\mathbb{R}_+)} \\
&\le C\|w(s) - \zeta(s)\|_{L^2_\eta(\mathbb{R}_+)} + C\int_0^1 \|\zeta(s) - (w(s) - \zeta(s))\tau\|_{L^\infty} d\tau \|a(\gamma)v_\star - R_\theta v_\infty\|_{L^2_\eta(\mathbb{R}_+)} \\
&\le C\max\left\{\|w_1 - \zeta_1\|_{L^2_\eta(\mathbb{R}_+)}, \|w_2 - \zeta_2\|_{L^2_\eta(\mathbb{R}_+)}\right\} + C\max\left\{\|w_1\|_{L^\infty}, \|w_2\|_{L^\infty}\right\} \\
&\le C\max\left\{\|\mathbf{w}_1\|_{X^1_\eta}, \|\mathbf{w}_2\|_{X^1_\eta}\right\}.
\end{aligned}$$

This yields an estimate for T_{13}

$$T_{13} \le C\max\left\{\|\mathbf{w}_1\|_{X^1_\eta}, \|\mathbf{w}_2\|_{X^1_\eta}\right\} |\zeta_1 - \zeta_2|.$$

Summarizing, we have shown

$$\begin{aligned}
&\|f(a(\gamma)v_\star + w_1) - f(a(\gamma)v_\star + w_2) - Df(v_\star)(w_1 - w_2) \\
&\qquad - \hat{v}(R_\theta v_\infty + \zeta_1) - f(R_\theta v_\infty + \zeta_2) - Df(v_\infty)(\zeta_1 - \zeta_2)]\hat{v}\|_{L^2_\eta} \\
&\le C\left(|z| + \max\left\{\|\mathbf{w}_1\|_{X^1_\eta}, \|\mathbf{w}_2\|_{X^1_\eta}\right\}\right) \|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta}.
\end{aligned}$$

It remains to estimate the derivative T_3. We have

$$\begin{aligned}
&\left\|\partial_x\Big[f(a(\gamma)v_\star + w_1) - f(a(\gamma)v_\star + w_2) - Df(v_\star)(w_1 - w_2)\Big]\right\|_{L^2_\eta} \\
&= \|Df(a(\gamma)v_\star + w_1)w_{1,x} + Df(a(\gamma)v_\star + w_1)a(\gamma)v_{\star,x} \\
&\quad - Df(a(\gamma)v_\star + w_2)w_{2,x} - Df(a(\gamma)v_\star + w_2)a(\gamma)v_{\star,x} \\
&\quad - D^2 f(v_\star)[w_1 - w_2, v_{\star,x}] - Df(v_\star)(w_1 - w_2)_x\|_{L^2_\eta} \\
&= \underbrace{\|[Df(a(\gamma)v_\star + w_1) - Df(a(\gamma)v_\star + w_2)]w_{1,x}\|_{L^2_\eta}}_{=:I_1} \\
&\quad + \underbrace{\|[Df(a(\gamma)v_\star + w_2) - Df(v_\star)](w_1 - w_2)_x\|_{L^2_\eta}}_{=:I_2} \\
&\quad + \underbrace{\|[Df(a(\gamma)v_\star + w_1) - Df(a(\gamma)v_\star + w_2)](a(\gamma)v_\star - v_\star)_x\|_{L^2_\eta}}_{=:I_3} \\
&\quad + \left\|[Df(a(\gamma)v_\star + w_1) - Df(a(\gamma)v_\star + w_2)]v_{\star,x} - D^2 f(v_\star)[w_1 - w_2, v_{\star,x}]\right\|_{L^2_\eta} \\
&= I_1 + I_2 + I_3 + \underbrace{\left\|\int_0^1 D^2 f(a(\gamma)v_\star + w_2 + (w_1 - w_2)s) - D^2 f(v_\star)ds[w_1 - w_2, v_{\star,x}]\right\|_{L^2_\eta}}_{=:I_4} \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Now

$$\begin{aligned}
I_1 &\le \|Df(a(\gamma)v_\star + w_1) - Df(a(\gamma)v_\star + w_2)\|_{L^\infty}\|w_{1,x}\|_{L^2_\eta} \\
&\le C\|w_1 - w_2\|_{L^\infty}\|w_{1,x}\|_{L^2_\eta} \le C \max\left\{\|\mathbf{w}_1\|_{X^1_\eta}, \|\mathbf{w}_2\|_{X^1_\eta}\right\}\|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta}.
\end{aligned}$$

In the same fashion using Lemma 3.7 and (3.18) we obtain

$$\begin{aligned}
I_2 &\le \|Df(a(\gamma)v_\star + w_2) - Df(v_\star)\|_{L^\infty}\|(w_1 - w_2)_x\|_{L^2_\eta} \\
&\le C\left(\|a(\gamma)v_\star - v_\star\|_{L^\infty} + \|w_2\|_{L^\infty}\right)\|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta} \\
&\le C\left(|z| + \max\left\{\|\mathbf{w}_1\|_{X^1_\eta}, \|\mathbf{w}_2\|_{X^1_\eta}\right\}\right)\|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta}
\end{aligned}$$

and for I_3

$$\begin{aligned}
I_3 &\le \|Df(a(\gamma)v_\star + w_1) - Df(a(\gamma)v_\star + w_2)\|_{L^\infty}\|a(\gamma)v_{\star,x} - v_{\star,x}\|_{L^2_\eta} \\
&\le C\|w_1 - w_2\|_{L^\infty}\|a(\gamma)v_{\star,x} - v_{\star,x}\|_{L^2_\eta} \le C|z|\|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta}.
\end{aligned}$$

Thus it remains to estimate I_4. We have

$$\begin{aligned}
I_4 &\le C\left(\|a(\gamma)v_\star - v_\star\|_{L^\infty} + \max\{\|w_1\|_{L^\infty}, \|w_2\|_{L^\infty}\}\right)\|w_1 - w_2\|_{L^\infty}\|v_{\star,x}\|_{L^2_\eta} \\
&\le C\left(|z| + \max\left\{\|\mathbf{w}_1\|_{X^1_\eta}, \|\mathbf{w}_2\|_{X^1_\eta}\right\}\right)\|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta}.
\end{aligned}$$

Hence

$$\begin{aligned} T_3 &= \left\| \Big[f(a(\gamma)v_\star + w_1) - f(a(\gamma)v_\star + w_2) - Df(v_\star)(w_1 - w_2) \Big]_x \right\|_{L^2_\eta}^2 \\ &\le C \left(|z| + \max\left\{ \|\mathbf{w}_1\|_{X^1_\eta}, \|\mathbf{w}_2\|_{X^1_\eta} \right\} \right) \|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta}. \end{aligned}$$

Finally we have shown

$$\left\| r^{[f]}(z, w_1, \zeta_1) - r^{[f]}(z, w_2, \zeta_2) \right\|_{X^1_\eta} \le C_0 \left(|z| + \max\left\{ \|\mathbf{w}_1\|_{X^1_\eta}, \|\mathbf{w}_2\|_{X^1_\eta} \right\} \right) \|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta}.$$

ii). As in i) we frequently use the mean value theorem and the smoothness of f from Assumption 1. First, we estimate

$$\begin{aligned} &\|r^{[f]}(z_1, \mathbf{w}) - r^{[f]}(z_2, \mathbf{w})\|_{X^1_\eta} \\ &= \left\| \begin{pmatrix} f(a(\gamma_1)v_\star + w) - f(a(\gamma_1)v_\star) - f(a(\gamma_2)v_\star + w) - f(a(\gamma_2)v_\star) \\ f(R_{\theta_1}v_\infty + \zeta) - f(R_{\theta_1}v_\infty) - f(R_{\theta_2}v_\infty + \zeta) - f(R_{\theta_2}v_\infty) \end{pmatrix} \right\|_{X^1_\eta} \\ &\le \underbrace{|f(R_{\theta_1}v_\infty + \zeta) - f(R_{\theta_2}v_\infty + \zeta)|}_{=:J_1} + \underbrace{|f(R_{\theta_1}v_\infty) - f(R_{\theta_2}v_\infty)|}_{=:J_2} \\ &\quad + \underbrace{\|f(a(\gamma_1)v_\star + w) - f(a(\gamma_2)v_\star + w) - \hat{v}[f(R_{\theta_1}v_\infty + \zeta) - f(R_{\theta_2}v_\infty + \zeta)]\|_{L^2_\eta}}_{=:J_3} \\ &\quad + \underbrace{\|f(a(\gamma_1)v_\star) - f(a(\gamma_2)v_\star) - \hat{v}(f(R_{\theta_1}v_\infty) - f(R_{\theta_2}v_\infty))\|_{L^2_\eta}}_{=:J_4} \\ &\quad + \underbrace{\|\partial_x[f(a(\gamma_1)v_\star + w) - f(a(\gamma_2)v_\star + w)]\|_{L^2_\eta}}_{=:J_5} + \underbrace{\|\partial_x[f(a(\gamma_1)v_\star) - f(a(\gamma_2)v_\star)]\|_{L^2_\eta}}_{=:J_6}. \end{aligned}$$

Using the Lipschitz estimate (3.18) we have

$$J_1 = |f(R_{\theta_1}v_\infty + \zeta) - f(R_{\theta_2}v_\infty + \zeta)| \le C|R_{\theta_1}v_\infty - R_{\theta_2}v_\infty| \le C|z_1 - z_2|$$

and the same holds true for $\zeta = 0$. Thus

$$J_2 \le C|z_1 - z_2|.$$

Write $\kappa_1(s) := a(\gamma_2)v_\star + w + (a(\gamma_1)v_\star - a(\gamma_2)v_\star)s$ and $\kappa_2(s) := R_{\theta_2}v_\infty + \zeta + (R_{\theta_1}v_\infty -$

$R_{\theta_2}v_\infty)s$, $s \in [0,1]$ and obtain for J_3

$$\begin{aligned}
J_3 &= \|f(a(\gamma_1)v_\star + w) - f(a(\gamma_2)v_\star + w) - \hat{v}[f(R_{\theta_1}v_\infty + \zeta) + f(R_{\theta_2}v_\infty + \zeta)]\|_{L^2_\eta} \\
&= \Big\| \int_0^1 Df(a(\gamma_2)v_\star + w + (a(\gamma_1)v_\star - a(\gamma_2)v_\star)s)(a(\gamma_1)v_\star - a(\gamma_2)v_\star)ds \\
&\quad - \hat{v}\int_0^1 Df(R_{\theta_2}v_\infty + \zeta + (R_{\theta_1}v_\infty - R_{\theta_2}v_\infty)s)(R_{\theta_1}v_\infty - R_{\theta_2}v_\infty)ds\Big\|_{L^2_\eta} \\
&\le \Big\| \int_0^1 Df(\kappa_1(s))(a(\gamma_1)v_\star - R_{\theta_1}v_\infty\hat{v} - a(\gamma_2)v_\star + R_{\theta_2}v_\infty\hat{v})ds\Big\|_{L^2_\eta} \\
&\quad + \Big\| \int_0^1 [Df(\kappa_1(s)) - Df(\kappa_2(s))](R_{\theta_1}v_\infty\hat{v} - R_{\theta_2}v_\infty\hat{v})ds\Big\|_{L^2_\eta} =: J_7 + J_8.
\end{aligned}$$

We estimate J_7, using (3.18), by

$$\begin{aligned}
J_7 &\le C\|a(\gamma_1)v_\star - R_{\theta_1}v_\infty\hat{v} - a(\gamma_2)v_\star + R_{\theta_2}v_\infty\hat{v}\|_{L^2_\eta} \\
&\le C\|a(\chi^{-1}(z_1))\mathbf{v}_\star - a(\chi^{-1}(z_2))\mathbf{v}_\star\|_{X_\eta} \le C|z_1 - z_2|.
\end{aligned}$$

We bound J_8 by two terms

$$\begin{aligned}
J_8 &\le \Big\| \int_0^1 [Df(\kappa_1(s)) - Df(\kappa_2(s))](R_{\theta_1}v_\infty\hat{v} - R_{\theta_2}v_\infty\hat{v})ds\Big\|_{L^2_\eta(\mathbb{R}_-)} \\
&\quad + \Big\| \int_0^1 [Df(\kappa_1(s)) - Df(\kappa_2(s))](R_{\theta_1}v_\infty\hat{v} - R_{\theta_2}v_\infty\hat{v})ds\Big\|_{L^2_\eta(\mathbb{R}_+)} = J_9 + J_{10}.
\end{aligned}$$

Then

$$J_9 \le C\|\hat{v}\|_{L^2_\eta(\mathbb{R}_-)}|R_{\theta_1}v_\infty - R_{\theta_2}v_\infty| \le C|z_1 - z_2|$$

and for J_{10}

$$J_{10} \le C|R_{\theta_1}v_\infty - R_{\theta_2}v_\infty| \int_0^1 \|\kappa_1(s) - \kappa_2(s)\|_{L^2_\eta} ds \le C|z_1 - z_2|.$$

Thus we have shown

$$J_3 \le C|z_1 - z_2|.$$

In particular the estimates hold for $w = 0$, $\zeta = 0$. Therefore we also have shown

$$J_4 \le C|z_1 - z_2|$$

and it remains to estimate the spatial derivatives J_5 and J_6. We note that for arbitrary $u \in L^2_\eta$ we have by Sobolev embedding, cf. Theorem D.2, and Lemma 3.4

$$\begin{aligned}
&\|[Df(a(\gamma_1)v_\star + w) - Df(a(\gamma_2)v_\star + w)]u\|_{L^2_\eta} \leq C\|a(\gamma_1)v_\star - a(\gamma_2)v_\star\|_{L^\infty}\|u\|_{L^2_\eta} \\
&\leq C\|u\|_{L^2_\eta}\left(\|a(\gamma_1)v_\star - R_{\theta_1}v_\infty\hat{v} - a(\gamma_2)v_\star + R_{\theta_2}v_\infty\hat{v}\|_{L^\infty} + \|R_{\theta_1}v_\infty\hat{v} - R_{\theta_2}v_\infty\hat{v}\|_{L^\infty}\right) \\
&\leq C\|u\|_{L^2_\eta}|z_1 - z_2|.
\end{aligned}$$

This implies with (3.18)

$$\begin{aligned}
J_5 &\leq \|[Df(a(\gamma_1)v_\star + w) - Df(a(\gamma_2)v_\star + w)]w_x\|_{L^2_\eta} \\
&\quad + \|Df(a(\gamma_1)v_\star + w)a(\gamma_1)v_{\star,x} - Df(a(\gamma_2)v_\star + w)a(\gamma_2)v_{\star,x}\|_{L^2_\eta} \\
&\leq C\|w_x\|_{L^2_\eta}|z_1 - z_2| + \|[Df(a(\gamma_1)v_\star + w) - Df(a(\gamma_2)v_\star + w)]a(\gamma_1)v_{\star,x}\|_{L^2_\eta} \\
&\quad + C\|a(\gamma_1)v_{\star,x} - a(\gamma_2)v_{\star,x}\|_{L^2_\eta} \\
&\leq C\left(\|w_x\|_{L^2_\eta} + \|a(\gamma_1)v_{\star,x}\|_{L^2_\eta}\right)|z_1 - z_2| + C\|a(\gamma_1)v_{\star,x} - a(\gamma_2)v_{\star,x}\|_{L^2_\eta} \leq C|z_1 - z_2|.
\end{aligned}$$

In particular the same holds true for $w = 0$ and we observe

$$J_6 \leq C|z_1 - z_2|.$$

Summarizing, we have

$$\|r^{[f]}(z_1, \mathbf{w}) - r^{[f]}(z_2, \mathbf{w})\|_{X^1_\eta} \leq C_1|z_1 - z_2|.$$

iii). Using the continuity of the derivative of the group action, cf. Lemma 3.7, the continuity of the projector P from Lemma 3.20 and Lemma 3.23 to see that there is $C > 0$ such that

$$\left\|\Big((I - P) - (I - P)\big(a(\cdot)\mathbf{v}_\star \circ \chi^{-1}\big)(z)S(z)^{-1}P\Big)\mathbf{u}\right\|_{X^1_\eta} \leq C\|\mathbf{u}\|_{X^1_\eta} \quad \forall \mathbf{u} \in X^1_\eta.$$

Now the claim follows from i).

iv). By Lemma 3.7, Lemma 3.23 we have that $\big(a(\cdot)\mathbf{v}_\star \circ \chi^{-1}\big)(z)S(z)^{-1}$ is continuously differentiable in z. Therefore we have

$$\|\big(a(\cdot)\mathbf{v}_\star \circ \chi^{-1}\big)(z_1)S(z_1)^{-1}\mathbf{w} - \big(a(\cdot)\mathbf{v}_\star \circ \chi^{-1}\big)(z_2)S(\gamma_2)^{-1}\mathbf{w}\|_{X^1_\eta} \leq C|z_1 - z_2|\|\mathbf{w}\|_{X^1_\eta}. \tag{3.67}$$

Then use (3.67) and i) to obtain

$$\begin{aligned}
&\|r^{[w]}(z_1, \mathbf{w}) - r^{[w]}(z_2, \mathbf{w})\|_{X^1_\eta} \leq C\|r^{[f]}(z_1, \mathbf{w}) - r^{[f]}(z_2, \mathbf{w})\|_{X^1_\eta} \\
&+ \|\big(a(\cdot)\mathbf{v}_\star \circ \chi^{-1}\big)(z_1)S(z_1)^{-1}Pr^{[f]}(z_1, \mathbf{w}) - \big(a(\cdot)\mathbf{v}_\star \circ \chi^{-1}\big)(z_2)S(z_2)^{-1}Pr^{[f]}(z_1, \mathbf{w})\|_{X^1_\eta} \\
&\leq C|z_1 - z_2|.
\end{aligned}$$

Now we conclude by using ii) and iii)

$$
\begin{aligned}
&\|r^{[w]}(z_1,\mathbf{w}_1) - r^{[w]}(z_2,\mathbf{w}_2)\|_{X^1_\eta} \\
&\leq \|r^{[w]}(z_1,\mathbf{w}_1) - r^{[w]}(z_2,\mathbf{w}_1)\|_{X^1_\eta} + \|r^{[w]}(z_2,\mathbf{w}_1) - r^{[w]}(z_2,\mathbf{w}_2)\|_{X^1_\eta} \\
&\leq C_3\left(|z_1 - z_2| + \|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta}\right).
\end{aligned}
$$

v). We conclude from Lemma 3.23 that $S(z)^{-1}$ is locally Lipschitz w.r.t. z. Then, similarly as in iv), we obtain

$$
\begin{aligned}
&\left|r^{[z]}(z_1,\mathbf{w}_1) - r^{[z]}(z_2,\mathbf{w}_2)\right| \\
&= \left|S(z_1)^{-1}Pr^{[f]}(z_1,\mathbf{w}_1) - S(z_2)^{-1}Pr^{[f]}(z_2,\mathbf{w}_2)\right| \\
&\leq C\left\|r^{[f]}(z_1,\mathbf{w}_1) - r^{[f]}(z_2,\mathbf{w}_2)\right\|_{X^1_\eta} + \left|(S(z_1)^{-1} - S(z_2)^{-1})Pr^{[f]}(z_2,\mathbf{w}_2)\right| \\
&\leq C_4\left(|z_1 - z_2| + \|\mathbf{w}_1 - \mathbf{w}_2\|_{X^1_\eta}\right).
\end{aligned}
$$

□

3.7 Nonlinear stability theorem

In the section we prove under the Assumptions 1-4 the first main results of the thesis - the nonlinear stability of traveling oscillating fronts in exponentially weighted spaces. In particular, we show Theorem 1.11 and the idea of its proof is as follows. We need to assume that the initial perturbation $\mathbf{u}_0$ in (0.22) is sufficiently small. In Section 3.5 we have seen that the equation (0.22) can be decomposed by a nonlinear coordinate transformation into the system (3.60), (3.61) if the solution of (0.22) stays close to the profile $\mathbf{v}_\star$. Then the first step is to show existence of a local mild solution $(z,\mathbf{w})$ of the system (3.60), (3.61), cf. Definition 3.26. This means we show that the corresponding integral equations

$$\mathbf{w}(t) = e^{t\mathcal{L}}\mathbf{w}_0 + \int_0^t e^{(t-s)\mathcal{L}}r^{[w]}(z(s),\mathbf{w}(s))ds, \tag{3.68}$$

$$z(t) = z(0) + \int_0^t r^{[z]}(z(s),\mathbf{w}(s))ds \tag{3.69}$$

have a unique solution for small time. This is done by using a contraction argument in Lemma 3.27 and the estimates of the semigroup and nonlinearities from Theorem 3.21 and Lemma 3.25. The strategy is similar as in [17] but we remark that the approach from [17] is not completely rigorous since the presence of the chart is omitted. However, following our procedure one can carry out the proof in [17] by taking the manifold property of the group into account. The results are then obtained by working in local charts.

Definition 3.26. A solution $(z, \mathbf{w}) \in C([0, t_\infty), \mathbb{R}^2 \times V^1_\eta)$ of the integral equations (3.68), (3.69) on $0 \leq t < t_\infty$ for some $t_\infty > 0$ is called a **mild solution** of (3.60), (3.61) on $[0, t_\infty)$.

In the case $t_\infty = \infty$ the we will call the solution $(z, \mathbf{w})$ global mild solution, whereas for $t < \infty$ we will call $(z, \mathbf{w})$ a local mild solution of (3.60), (3.61). We equip the product space $\mathbb{R}^2 \times X^1_\eta$ with the norm

$$\|(z, \mathbf{w})\|_{\mathbb{R}^2 \times X^1_\eta} := |z| + \|w\|_{X^1_\eta}.$$

Lemma 3.27 (Local existence and uniqueness)**.** *Let the Assumptions 1-4 be satisfied and* $0 < \mu < \min\{\mu_{\mathrm{ess}}, \mu_\star, 2\}$ *with* μ_{ess} *from Assumption 3 and* $\mu_\star$ *from Theorem 2.6. Further, let* K *be from Theorem 3.21 and* δ *be from Lemma 3.25. Then for every* $0 < \varepsilon_1 < \delta$ *and* $0 < 2K\varepsilon_0 \leq \delta$ *there is* $t_\star = t_\star(\varepsilon_0, \varepsilon_1) > 0$ *such that for all initial values* $(z_0, \mathbf{w}_0) \in \mathbb{R}^2 \times V^1_\eta$ *with*

$$\|\mathbf{w}_0\|_{X^1_\eta} < \varepsilon_0, \quad |z_0| < \varepsilon_1$$

there exists a unique local mild solution $(z, \mathbf{w})$ *of* (3.60), (3.61) *on* $[0, t_\star)$ *with*

$$\|\mathbf{w}(t)\|_{X^1_\eta} \leq 2K\varepsilon_0, \quad |z(t)| \leq 2\varepsilon_1, \quad t \in [0, t_\star).$$

In particular, $t_\star$ *can be taken uniformly for* $(z_0, \mathbf{w}_0) \in B_{\varepsilon_1}(0) \times B_{\varepsilon_0}(0)$.

Proof. Take $\beta > 0$ from Theorem 3.21 and C_i from Lemma 3.25. Choose $t_\star$ so small such that the following conditions are satisfied:

$$t_\star < \frac{\varepsilon_1}{2C_4\varepsilon_1 + 2KC_4\varepsilon_0}, \quad C_4 t_\star + \frac{2KC_3}{\beta}(1 - e^{-\beta t_\star}) < 1. \tag{3.70}$$

Note that $t_\star$ can be taken uniformly for $(z_0, \mathbf{w}_0) \in B_{\varepsilon_1}(0) \times B_{\varepsilon_0}(0)$. The proof follows a contraction argument in the space $Z := C([0, t_\star), \mathbb{R}^2 \times V^1_\eta)$ equipped with the norm $\|(z, \mathbf{w})\|_Z := \sup_{t \in [0, t_\star)}\{|z(t)| + \|\mathbf{w}(t)\|_{X^1_\eta}\}$. Define the map

$$\Upsilon : Z \to Z, \quad (z, w) \mapsto \begin{pmatrix} z_0 + \int_0^{(\cdot)} r^{[z]}(z(s), \mathbf{w}(s))ds \\ e^{(\cdot)\mathcal{L}}\mathbf{w}_0 + \int_0^{(\cdot)} e^{(\cdot - s)\mathcal{L}} r^{[w]}(z(s), \mathbf{w}(s))ds \end{pmatrix}$$

given by the right hand side of (3.68), (3.69). We show that Υ is a contraction on the closed set

$$B := \{(z, \mathbf{w}) \in Z : \|\mathbf{w}(t)\|_{X^1_\eta} \leq 2K\varepsilon_0, \ |z(t)| \leq 2\varepsilon_1, \ t \in [0, t_\star)\} \subset Z.$$

Let $(z, \mathbf{w}) \in B$. By using the estimates from Theorem 3.21, Lemma 3.25 and (3.70) we obtain for all $0 \le t < t_\star$

$$\begin{aligned}
&\left\| e^{t\mathcal{L}}\mathbf{w}_0 + \int_0^t e^{(t-s)\mathcal{L}} r^{[w]}(z(s), \mathbf{w}(s))ds \right\|_{X^1_\eta} \\
&\le Ke^{-\beta t}\varepsilon_0 + K \int_0^t e^{-\beta(t-s)} \| r^{[w]}(z(s), \mathbf{w}(s))\|_{X^1_\eta} ds \\
&\le Ke^{-\beta t}\varepsilon_0 + KC_3 \int_0^t e^{-\beta(t-s)} \|\mathbf{w}(s)\|_{X^1_\eta} ds \\
&\le K\varepsilon_0 + \frac{2K^2C_3\varepsilon_0}{\beta}(1 - e^{-\beta t_\star}) \le 2K\varepsilon_0.
\end{aligned}$$

and

$$\begin{aligned}
\left| z_0 + \int_0^t r^{[z]}(z(s), \mathbf{w}(s))ds \right| &\le \varepsilon_1 + \int_0^t |r^{[z]}(z(s), \mathbf{w}(s))| ds \\
&\le \varepsilon_1 + C_4 \int_0^t |z(s)| + \|\mathbf{w}(s)\|_{X^1_\eta} ds \\
&\le \varepsilon_1 + C_4(2\varepsilon_1 + 2K\varepsilon_0)t_\star \le 2\varepsilon_1
\end{aligned}$$

Hence Υ maps B into itself. Further, for $(z_1, \mathbf{w}_1), (z_2, \mathbf{w}_2) \in B$ and $0 \le t < t_\star$ we can estimate

$$\begin{aligned}
\|\Upsilon(z_1, \mathbf{w}_1) - \Upsilon(z_2, \mathbf{w}_2)\|_Z \le \sup_{t \in [0, t_\star)} &\Big\{ \int_0^t |r^{[z]}(z_1(s), \mathbf{w}_1(s)) - r^{[z]}(z_2(s), \mathbf{w}_2(s))| ds \\
&+ \int_0^t Ke^{-\beta(t-s)} \|r^{[w]}(z_1(s), \mathbf{w}_1(s)) - r^{[w]}(z_2(s), \mathbf{w}_2(s))\|_{X^1_\eta} ds \Big\} \\
&\le \left(C_4 t_\star + \frac{KC_3}{\beta}(1 - e^{-\beta t_\star}) \right) \|(z_1 - z_2, \mathbf{w}_1 - \mathbf{w}_2)\|_Z \\
&< \|(z_1 - z_2, \mathbf{w}_1 - \mathbf{w}_2)\|_Z.
\end{aligned}$$

Thus Υ is a contraction in B. Therefore, there exists a unique $(z, \mathbf{w}) \in B \subset C([0, t_\star), \mathbb{R}^2 \times V^1_\eta)$ such that (3.68), (3.69) hold. □

As a next step, we use a Gronwall argument to show that the local mild solution from Lemma 3.27 can be extended to a global mild solution and that the perturbation $\mathbf{w}$ decays to zero as $t \to \infty$. This will imply that z converge to some z_∞. In the end, we conclude that the mild solution has more regularity and is a classical solution, cf. Definition 3.24. In addition, if the initial perturbation is small then the solution stays in a small neighborhood of $\mathbf{v}_\star$. Thus $(z, \mathbf{w})$ transform into a classical solution $\mathbf{u}$ of (0.22), which converge to the profile $\mathbf{v}_\star$ with asymptotic phase given by $\gamma_\infty = \chi(z_\infty)$. As mentioned we use the following Gronwall estimate, which can be found in [17, Lemma 6.3].

Lemma 3.28. *Suppose* $\varepsilon, C, \tilde{C}, \beta > 0$ *such that*

$$C \geq 1, \quad \varepsilon \leq \frac{\beta}{16\tilde{C}C}$$

and let $\varphi \in C([0, t_\infty), [0, \infty))$ *for some* $0 < t_\infty \leq \infty$ *satisfying*

$$\varphi(t) \leq C\varepsilon e^{-\beta t} + \tilde{C} \int_0^t e^{-\beta(t-s)} \left(\varphi(s)^2 + \varepsilon\varphi(s)\right) ds, \quad \forall t \in [0, t_\infty).$$

Then for all $0 \leq t < t_\infty$ *there hold*

$$\varphi(t) \leq 2C\varepsilon e^{-\frac{3}{4}\beta t}.$$

Proof. The estimate is satisfied for $t = 0$. Let

$$T := \sup \left\{ t \in [0, t_\infty) : \varphi(s) \leq 2C\varepsilon e^{-\frac{3}{4}\beta s} \, \forall s \in [0, t) \right\}.$$

Then $T > 0$. Assume $T < t_\infty$. Since $\varphi \in C([0, t_\infty), \mathbb{R}_+)$ we obtain

$$\begin{aligned}
2C\varepsilon e^{-\frac{3}{4}\beta T} = \varphi(T) &\leq C\varepsilon e^{-\frac{3}{4}\beta T} + 2C\tilde{C}\varepsilon^2 e^{-\beta T} \int_0^T e^{\frac{1}{4}\beta s} + 2Ce^{-\frac{1}{2}\beta s} ds \\
&= C\varepsilon e^{-\frac{3}{4}\beta T} + 2C\tilde{C}\varepsilon^2 e^{-\beta T} \left(\frac{4}{\beta}(e^{\frac{1}{4}\beta T} - 1) + \frac{4C}{\beta}(1 - e^{-\frac{1}{2}\beta T}) \right) \\
&< 2C\varepsilon e^{-\frac{3}{4}\beta T} \left(\frac{1}{2} + \frac{4\tilde{C}\varepsilon}{\beta} + \frac{4\tilde{C}C\varepsilon}{\beta} \right) \leq 2C\varepsilon e^{-\frac{3}{4}\beta T}.
\end{aligned}$$

A contradiction. Thus $T = t_\infty$ and the assertion is proven. $\square$

Now we are in the situation to prove the stability result for the $(z, \mathbf{w})$-system (3.60), (3.61). The regularity of the solution will follow by classical results from [5] and [32], cf. Theorem C.3. As in [5], for a Hölder exponent $\alpha \in (0, 1)$ we denote by C^α the space of Hölder continuous functions and by $C^{1+\alpha}$ the space of differentiable functions with Hölder continuous derivative. Recall the notion of a classical solution $(z, \mathbf{w})$ from Definition 3.24.

Theorem 3.29. *Let the Assumptions 1-4 be satisfied and* $0 < \mu < \min\{\mu_{\text{ess}}, \mu_\star, 2\}$ *with* μ_{ess} *from Assumption 3 and* $\mu_\star$ *from Theorem 2.6. Then there exist* $\varepsilon > 0$ *and constants* $K_0 \geq 1$, $\tilde{\beta} > 0$ *such that for all initial values* $(z_0, \mathbf{w}_0) \in \mathcal{G} \times V_\eta^2$ *with* $\|(z_0, \mathbf{w}_0)\|_{\mathbb{R}^2 \times X_\eta^1} < \varepsilon$ *there hold:*

i) The system (3.60), (3.61) *has a unique classical solution*

$$\mathbf{w} \in C^{\alpha}((0,\infty), V_\eta^2) \cap C^{1+\alpha}((0,\infty), V_\eta) \cap C^1([0,\infty), V_\eta), \quad z \in C^1([0,\infty), \mathbb{R}^2)$$

for arbitrary $\alpha \in (0,1)$.

ii) There exists $z_\infty = z_\infty(z_0, \mathbf{w}_0) \in \mathbb{R}^2$ *such that for all* $t \geq 0$

$$\|\mathbf{w}(t)\|_{X_\eta^1} + |z(t) - z_\infty| \leq K_0 e^{-\tilde{\beta} t} \|(z_0, \mathbf{w}_0)\|_{\mathbb{R}^2 \times X_\eta^1}, \quad |z_\infty| \leq (K_0 + 1)\|(z_0, \mathbf{w}_0)\|_{\mathbb{R}^2 \times X_\eta^1}.$$

Proof. Recall K, β from Theorem 3.21 and δ, C_i from Lemma 3.25. Now choose $\varepsilon, \tilde{\varepsilon} > 0$ such that $0 < 2K\tilde{\varepsilon} < \delta$ and

$$\varepsilon < \min\left(\frac{\delta}{C_z}, \frac{\tilde{\varepsilon}}{4K}, \frac{\beta}{16K^2 C_2 C_z}\right), \quad C_z > 2 + \frac{16 C_4 K}{3\beta}. \tag{3.71}$$

We abbreviate $\xi_0 := \|(z_0, w_0)\|_{\mathbb{R}^2 \times X_\eta^1} < \varepsilon$. Let

$$t_\infty := \sup\Big\{T > 0 : \exists (z, w) \text{ local mild solution of (3.60), (3.61) on } [0,T)$$
$$\|\mathbf{w}(t)\|_{X_\eta^1} \leq K\tilde{\varepsilon},\ |z(t)| \leq C_z \xi_0,\ t \in [0,T)\Big\}.$$

Then Lemma 3.27 with $\varepsilon_0 = \tilde{\varepsilon}$ and $\varepsilon_1 = \frac{C_z \xi_0}{2} < \delta$ implies $t_\infty \geq t_\star = t_\star(\varepsilon_0, \varepsilon_1)$. Using Theorem 3.21 and Lemma 3.25 we estimate for all $0 \leq t < t_\infty$

$$\begin{aligned}
\|\mathbf{w}(t)\|_{X_\eta^1} &\leq \|e^{t\mathcal{L}} \mathbf{w}_0\|_{X_\eta^1} + \int_0^t \|e^{(t-s)\mathcal{L}} r^{[w]}(z(s), \mathbf{w}(s))\|_{X_\eta^1} ds \\
&\leq K e^{-\beta t} \|\mathbf{w}_0\|_{X_\eta^1} + \int_0^t e^{-\beta(t-s)} \|r^{[w]}(z(s), \mathbf{w}(s)\|_{X_\eta^1} ds \\
&\leq K e^{-\beta t} \|\mathbf{w}_0\|_{X_\eta^1} + K C_2 \int_0^t e^{-\beta(t-s)} \left(|z(s)| + \|\mathbf{w}(s)\|_{X_\eta^1}\right) \|\mathbf{w}(s)\|_{X_\eta^1} ds \\
&\leq K e^{-\beta t} \xi_0 + K C_2 C_z \int_0^t e^{-\beta(t-s)} \left(\xi_0 + \|\mathbf{w}(s)\|_{X_\eta^1}\right) \|\mathbf{w}(s)\|_{X_\eta^1} ds.
\end{aligned}$$

Then the Gronwall estimate in Lemma 3.28 implies due to (3.71)

$$\|\mathbf{w}(t)\|_{X_\eta^1} \leq 2K e^{-\frac{3}{4}\beta t} \xi_0 < 2K e^{-\frac{3}{4}\beta t} \varepsilon < \frac{\tilde{\varepsilon}}{2}, \quad t \in [0, t_\infty). \tag{3.72}$$

This yields

$$\begin{aligned}
|z(t)| &\leq |z_0| + \int_0^t |r^{[z]}(z(s), \mathbf{w}(s))| ds \leq \xi_0 + C_4 \int_0^t \|\mathbf{w}(s)\|_{X_\eta^1} ds \\
&\leq \xi_0 + 2 C_4 K \xi_0 \int_0^t e^{-\frac{3}{4}\beta s} ds \leq \xi_0 + \frac{8 C_4 K}{3\beta} \xi_0 < \frac{C_z \xi_0}{2}, \quad t \in [0, t_\infty).
\end{aligned} \tag{3.73}$$

Next, we show that $t_\infty = \infty$. For this purpose, assume the contrary, i.e. $t_\infty < \infty$. Then the estimates (3.72), (3.73) imply

$$\|\mathbf{w}(t_\infty - \tfrac{1}{2}t_\star)\|_{X^1_\eta} < \frac{\tilde{\varepsilon}}{2} = \varepsilon_0, \quad |z(t_\infty - \tfrac{1}{2}t_\star)| < \frac{C_z\xi_0}{2} = \varepsilon_1.$$

Now we can apply Lemma 3.27 once again to the integral equations (3.68), (3.69) with $\mathbf{w}_0 = \mathbf{w}(t_\infty - \frac{1}{2}t_\star)$ and $z_0 = z(t_\infty - \frac{1}{2}t_\star)$ and obtain a solution $(\tilde{z}, \tilde{\mathbf{w}})$ of (3.68), (3.69) on $[0, t_\star)$ with

$$\begin{aligned}
&\tilde{\mathbf{w}}(0) = \mathbf{w}(t_\infty - \tfrac{1}{2}t_\star), && \|\mathbf{w}(t)\|_{X^1_\eta} \le K\tilde{\varepsilon}, && t \in [0, t_\star) \\
&\tilde{z}(0) = z(t_\infty - \tfrac{1}{2}t_\star), && |z(t)| \le C_z\xi_0, && t \in [0, t_\star).
\end{aligned}$$

Define

$$(\bar{z}, \bar{\mathbf{w}})(t) := \begin{cases} (z, \mathbf{w})(t), & t \in [0, t_\infty - \frac{1}{2}t_\star] \\ (\tilde{z}, \tilde{\mathbf{w}})(t - t_\infty + \frac{1}{2}t_\star), & t \in (t_\infty - \frac{1}{2}t_\star, t_\infty + \frac{1}{2}t_\star). \end{cases}$$

Then $(\bar{z}, \bar{\mathbf{w}})$ is a local mild solution on $[0, t_\infty + \frac{1}{2}t_\star)$ with $\|\bar{\mathbf{w}}(t)\|_{X^1_\eta} \le K\tilde{\varepsilon}$ and $|\bar{z}(t)| \le C_z\xi_0$. A contradiction to the definition of t_∞. Hence $t_\infty = \infty$ and (3.72) holds on $[0, \infty)$. We see that the integral

$$z_\infty := z_0 + \int_0^\infty r^{[z]}(z(s), \mathbf{w}(s))ds$$

exists since

$$\begin{aligned}
|z(t) - z_\infty| &\le \int_t^\infty |r^{[z]}(z(s), \mathbf{w}(s))|ds \\
&\le C_4 \int_t^\infty \|\mathbf{w}(s)\|_{X^1_\eta} \le 2KC_4\xi_0 \int_t^\infty e^{-\frac{3}{4}\beta s}ds = \frac{8KC_4}{3\beta}e^{-\frac{3}{4}\beta t}\xi_0.
\end{aligned}$$

Thus the first estimate in ii) is proven with $K_0 = 2K + \frac{8KC_4}{3\beta}$ and $\tilde{\beta} = \frac{3}{4}\beta$. The second estimate is obtained by

$$|z_\infty| \le |z(0) - z_\infty| + |z_0| \le (K_0 + 1)\xi_0.$$

Hence ii) is proven and it remains to show the regularity of $(z, \mathbf{w})$. By Lemma 3.27 we have $r^{[z]} \in C(V, \mathbb{R}^2)$, $V = B_\delta(0) \times B_\delta(0) \subset \mathbb{R}^2 \times X^1_\eta$ and, since $(z, \mathbf{w}) \in C([0, \infty), \mathbb{R}^2 \times V^1_\eta)$ with $|z(t)|, \|\mathbf{w}(t)\|_{X^1_\eta} < \delta$, there hold $r^{[z]}(z(\cdot), w(\cdot)) \in C([0, \infty), \mathbb{R}^2)$. Thus $z \in C^1([0, \infty), \mathbb{R}^2)$. Furthermore, consider the equation

$$\mathbf{u}(t) = \mathcal{L}\mathbf{u}(t) + r(t), \quad t > 0, \quad \mathbf{u}(0) = \mathbf{w}_0, \tag{3.74}$$

where $r(t) := r^{[w]}(z(t), \mathbf{w}(t))$. Suppose $0 \le s \le t < \infty$. Then by Lemma 3.27 we find some $C > 0$ such that

$$\begin{aligned}
\|r(t) - r(s)\|_{X_\eta} &= \|r^{[w]}(z(t), \mathbf{w}(t)) - r^{[w]}(z(s), \mathbf{w}(s))\|_{X_\eta} \\
&\le C_3 \left(|z(t) - z(s)| + \|\mathbf{w}(t) - \mathbf{w}(s)\|_{X^1_\eta} \right) \\
&\le C_3 \left(\int_s^t |r^{[z]}(z(\sigma), \mathbf{w}(\sigma))| d\sigma + \int_s^t \|r^{[w]}(z(\sigma), \mathbf{w}(\sigma))\|_{X^1_\eta} d\sigma \right) \\
&\le C_3 \bigg(C_4 \int_s^t \|\mathbf{w}(\sigma)\|_{X^1_\eta} d\sigma + C_2 \int_s^t |z(\sigma)| + \|\mathbf{w}(\sigma)\|_{X^1_\eta} d\sigma \bigg) \le C(t-s).
\end{aligned}$$

This implies $r \in C^\alpha([0, \infty), X_\eta)$ for every $\alpha \in (0, 1)$. Moreover, for arbitrary $s > 0$ there hold

$$\int_0^s \|r(t)\|_{X_\eta} dt = \int_0^s \|r^{[w]}(z(t), \mathbf{w}(t))\|_{X_\eta} dt \le C_3 \int_0^s \|\mathbf{w}(t)\|_{X^1_\eta} dt < \infty.$$

Now Theorem C.3 implies

$$\mathbf{u}(t) = e^{t\mathcal{L}} \mathbf{w}_0 + \int_0^t e^{(t-s)\mathcal{L}} r(s) ds$$

solves (3.74) and $\mathbf{u} \in C^\alpha((0, \infty), V^2_\eta) \cap C^{1+\alpha}((0, \infty), V_\eta) \cap C^1([0, \infty), V_\eta)$. But

$$\mathbf{u}(t) = e^{t\mathcal{L}} \mathbf{w}_0 + \int_0^t e^{(t-s)\mathcal{L}} r(s) ds = e^{t\mathcal{L}} \mathbf{w}_0 + \int_0^t e^{(t-s)\mathcal{L}} r^{[w]}(z(s), \mathbf{w}(s)) ds = \mathbf{w}(t).$$

Hence, for all $\alpha \in (0, 1)$

$$\mathbf{w}(t) \in C^\alpha((0, \infty), V^2_\eta) \cap C^{1+\alpha}((0, \infty), V_\eta) \cap C^1([0, \infty), V_\eta).$$

□

The final step is to ensure that the solution $(z, \mathbf{w})$ from Theorem 3.29 stays in a small zero neighborhood where the nonlinear coordinate transformation T from Lemma 3.22 is diffeomorphic. Thanks to the stability estimates in Theorem 3.29 ii) this is guaranteed if the initial values are sufficiently small. Hence if $\mathbf{u}_0$ in (0.22) is sufficiently small the solution $(z, \mathbf{w})$ is equivalent to a solution $\mathbf{u}$ of (0.22), which converges to the group orbit of $\mathbf{v}_\star$ with an asymptotic phase. Moreover, the solution $\mathbf{u}$ stays in the neighborhood of the group orbit for all positive times. This proves nonlinear stability with asymptotic phase of the traveling oscillating front.

Proof of Theorem 1.11. Take W, V from Lemma 3.22 and let $\delta > 0$ such that

$$B_\delta := \{\mathbf{u} \in X_\eta : \|\mathbf{u}\|_{X_\eta} \le \delta\}$$

satisfies $B_\delta \subset T(V)$ and $P(B_\delta) \subset \Pi(W)$. In particular, $T : T^{-1}(B_\delta) \to B_\delta$ and $\Pi : \Pi^{-1}(P(B_\delta)) \to P(B_\delta)$ are diffeomorphic. Then there is $C_\Pi > 0$ such that

$$|\Pi^{-1}(P\mathbf{v})| \le C_\Pi \|\mathbf{v}\|_{X_\eta} \quad \forall \mathbf{v} \in B_\delta.$$

Now we take $\varepsilon > 0$ from Theorem 3.29 so small such that the solution $(z, \mathbf{w})$ of (3.60), (3.61) satisfies $(z(t), \mathbf{w}(t)) \in T^{-1}(B_\delta)$ and $z(t) \in \Pi^{-1}(P(B_\delta))$ for all $t \in [0, \infty)$. Further, let $C \ge 1$ be such that Lemma 3.7 and (3.18) imply

$$\|a(\chi^{-1}(z_1))\mathbf{v}_\star - a(\chi^{-1}(z_2))\mathbf{v}_\star\|_{X^1_\eta} \le C|z_1 - z_2| \quad \forall z_1, z_2 \in \Pi^{-1}(P(B_\delta)).$$

Choose

$$\varepsilon_0 < \min\left(\frac{\delta}{4C\tilde{C}K_0 + \tilde{C}K_0 + CC_\Pi}, \varepsilon\tilde{C}^{-1}, \frac{\pi}{K_0 + C_\infty}\right), \quad \tilde{C} := C_\Pi(1 + C) + 1.$$

with K_0, C_∞ from Theorem 3.29 and define

$$(z_0, \mathbf{w}_0) := T^{-1}(\mathbf{u}_0) = (\Pi^{-1}(P\mathbf{u}_0), \mathbf{u}_0 + \mathbf{v}_\star - a(\chi^{-1}(z_0))\mathbf{v}_\star).$$

Then $|z_0| \le C_\Pi \|\mathbf{u}_0\|_{X_\eta}$ and

$$\begin{aligned} \|(z_0, \mathbf{w}_0)\|_{\mathbb{R}^2 \times X^1_\eta} &= |z_0| + \|\mathbf{w}_0\|_{X^1_\eta} \\ &\le |z_0| + \|a(\chi^{-1}(z_0))\mathbf{v}_\star - \mathbf{v}_\star\|_{X^1_\eta} + \|\mathbf{u}_0\|_{X^1_\eta} \le \tilde{C}\|\mathbf{u}_0\|_{X^1_\eta} \le \tilde{C}\varepsilon_0 < \varepsilon. \end{aligned} \tag{3.75}$$

Moreover, Theorem 3.29 implies there exist $z \in C^1([0, \infty), \mathbb{R}^2)$ and $\mathbf{w} \in C((0, \infty), V^2_\eta) \cap C^1((0, \infty), V_\eta)$ such that $(z, \mathbf{w})$ solves (3.60), (3.61) with $z(0) = z_0$, $\mathbf{w}(0) = \mathbf{w}_0$ and

$$\|\mathbf{w}(t)\|_{X^1_\eta} \le K_0\varepsilon_0, \quad |z(t)| \le |z(t) - z_\infty| + |z_\infty| \le (K_0 + C_\infty)\varepsilon_0 < \pi, \quad t \in [0, \infty).$$

Hence $z(t) \in U$ for all $t \in [0, \infty)$ and we define $\gamma(t) = \chi^{-1}(z(t)) \in C^1([0, \infty), \mathcal{G})$. Set

$$\mathbf{u}(t) = a(\gamma(t))\mathbf{v}_\star + \mathbf{w}(t), \quad t \in [0, \infty).$$

Then $\mathbf{u} \in C((0, \infty), Y_\eta) \cap C^1([0, \infty), X_\eta)$ and since $\varepsilon_0 < \delta$ Lemma 3.22 implies

$$\begin{aligned} \mathbf{u}(0) &= a(\gamma(0))\mathbf{v}_\star + \mathbf{w}(0) = a(\chi^{-1}(z_0))\mathbf{v}_\star - \mathbf{v}_\star + \mathbf{w}_0 + \mathbf{v}_\star \\ &= T(z_0, \mathbf{w}_0) + \mathbf{v}_\star = \mathbf{u}_0 + \mathbf{v}_\star. \end{aligned}$$

For $t \in (0,\infty)$ we obtain with $\mathbf{u} = (u,\rho)^\top$ and $\mathbf{w} = (w,\zeta)^\top$

$$\begin{aligned}
&\mathbf{u}_t(t) - \mathcal{L}_0\mathbf{u}(t) - \begin{pmatrix} f(u(t)) \\ f(\rho(t)) \end{pmatrix} \\
&= [(a(\cdot)\mathbf{v}_\star \circ \chi^{-1})'(z(t))]z_t(t) + \mathbf{w}_t(t) - \mathcal{L}_0 a(\gamma(t))\mathbf{v}_\star - \mathcal{L}_0\mathbf{w}(t) - \begin{pmatrix} f(a(\gamma(t))v_\star + w) \\ f(a(\gamma(t))v_\infty + \zeta) \end{pmatrix} \\
&= [(a(\cdot)\mathbf{v}_\star \circ \chi^{-1})'(z(t))]z_t(t) + \mathbf{w}_t(t) - \mathcal{L}w(t) - r^{[f]}(z(t),\mathbf{w}(t)) \\
&= \mathbf{w}_t(t) - \mathcal{L}\mathbf{w}(t) + (I-P)[(a(\cdot)\mathbf{v}_\star \circ \chi^{-1})'(z(t))]z_t(t) - (I-P)r^{[f]}(z(t),\mathbf{w}(t)) \\
&\quad + P[(a(\cdot)\mathbf{v}_\star \circ \chi^{-1})'(z(t))]z_t(t) - Pr^{[f]}(z(t),\mathbf{w}(t)) \\
&= \mathbf{w}_t(t) - \mathcal{L}\mathbf{w}(t) - r^{[w]}(z(t),\mathbf{w}(t)) = 0.
\end{aligned}$$

Hence, $\mathbf{u}$ solves (0.22). Further, recall the metric $d_{\mathcal{G}}(\gamma_1,\gamma_2) = |\gamma_1 - \gamma_2|_{\mathcal{G}}$ on $\mathcal{G}$ from (1.11). With $\gamma_\infty = \chi^{-1}(z_\infty)$ we have by Theorem 3.29

$$\begin{aligned}
\|\mathbf{w}(t)\|_{X^1_\eta} + |\gamma(t) - \gamma_\infty|_G &\le \|\mathbf{w}(t)\|_{X^1_\eta} + |z(t) - z_\infty| \\
&\le K_0 e^{-\tilde{\beta}t}\|(z_0,\mathbf{w}_0)\|_{\mathbb{R}^2\times X^1_\eta} \le K e^{-\tilde{\beta}t}\|\mathbf{u}_0\|_{X^1_\eta}
\end{aligned}$$

with $K = \tilde{C}K_0$. In addition,

$$|\gamma_\infty|_{\mathcal{G}} \le |\gamma_0|_{\mathcal{G}} + |\gamma_0 - \gamma_\infty|_{\mathcal{G}} \le |z_0| + |z_0 - z_\infty| \le C_\infty\|\mathbf{u}_0\|_{X^1_\eta}, \quad C_\infty = C_\Pi + \tilde{C}K_0.$$

Finally, we show uniqueness of $\mathbf{u}$. For this purpose, we have

$$\|\mathbf{u}(t) - \mathbf{v}_\star\|_{X_\eta} \le C|z(t) - z_\infty| + \|\mathbf{w}(t)\|_{X_\eta} + C|z_\infty| \le ((C+1)K + CC_\infty)\varepsilon_0 \le \frac{\delta}{2}.$$

Let $\tilde{\mathbf{u}}$ be another solution of (0.22) on $[0,T)$ for some $T > 0$. Let

$$\tau := \sup\{t \in [0,T) : \|\tilde{\mathbf{u}} - \mathbf{v}_\star\|_{X_\eta} \le \delta \text{ on } [0,t)\}.$$

Then there is a solution $(\tilde{z},\tilde{\mathbf{w}})$ of (3.60), (3.61) on $[0,\tau)$ such that $T(\tilde{z}(t),\tilde{\mathbf{w}}(t)) = \tilde{\mathbf{u}}(t) - \mathbf{v}_\star$ and thus $\tilde{\mathbf{u}}(t) = a(\tilde{\gamma}(t))\mathbf{v}_\star + \tilde{\mathbf{w}}(t)$, $\tilde{\gamma}(t) = \chi^{-1}(\tilde{z}(t))$. But since $(z,\mathbf{w})$ is unique we conclude $(\tilde{z},\tilde{\mathbf{w}}) = (z,\mathbf{w})$ and $\mathbf{u}(t) = \tilde{\mathbf{u}}(t)$ on $[0,\tau)$. Now assume $\tau < T$. Then for all $t \in [0,\tau)$

$$\frac{\delta}{2} \ge \|\mathbf{u}(t) - \mathbf{v}_\star\|_{X_\eta} = \|\tilde{\mathbf{u}}(t) - \mathbf{v}_\star\|_{X_\eta}.$$

Since the right-hand side converges to δ as $t \to \tau$, we arrive at a contradiction.

$\square$

In particular, in the proof of Theorem 1.11 we have shown the following corollary concerning the local coordinates in the chart (U,χ) of the motion on the group orbit γ and the asymptotic phase γ_∞. This will be useful in Chapter 4.

Corollary 3.30. *Let the Assumptions of Theorem 1.11 be satisfied and let $\varepsilon_0 > 0$ be sufficiently small. Then γ_∞, γ from Theorem 1.11 satisfy $\gamma_\infty, \gamma(t) \in U$ for all $t \geq 0$ and have local coordinates $z_\infty \in \mathbb{R}^2$ and $z \in C^1([0,\infty), \mathbb{R}^2)$, i.e.*

$$\gamma(t) = \chi^{-1}(z(t)), \quad \gamma_\infty = \chi^{-1}(z_\infty), \quad \gamma_\infty \circ \gamma^{-1} = \chi^{-1}(z_\infty - z(t)), \quad t \geq 0.$$

Moreover, there hold

$$|z(t) - z_\infty| \leq K e^{-\tilde{\beta} t} \|\mathbf{u}_0\|_{Z_\eta^1}, \quad |z_\infty| \leq C_\infty \|\mathbf{u}_0\|_{X_\eta^1}$$

with $K, \tilde{\beta}, C_\infty$ from Theorem 1.11.

Proof. The assertions follows by the proof of Theorem 1.11 and the definition of the chart (U, χ) from (3.13). □

Chapter 4

Freezing traveling oscillating fronts

In this chapter we apply the concept of the freezing method from [18], [19] to traveling oscillating fronts. We develop a method to compute TOFs numerically. When starting a finite difference or finite element computation to solve the equation (0.4) numerically and to observe the formation of TOFs, two basic problems occur. First, one has to truncate the spatial domain of computation to a finite interval. But since TOFs are traveling in space, the wave will leave the computational domain at a certain time. Second, the frequency ω and translation velocity c are unknown a-priori. So on the one hand, we are naturally interested in the velocities and on the other hand we cannot make use of the co-moving equation (0.8) for which the profile becomes stationary. The freezing method solves both problems. The idea is to transform (0.4) into a co-moving frame via $\mathbf{u}(t) = a(\gamma(t))\mathbf{v}(t)$ with the new variable γ for which one has to solve additional equations. The number of additional degrees of freedom equals the dimension of the Lie group $\mathcal{G}$. They are compensated by a corresponding number of algebraic constraints resulting in a well-posed problem.

We start by applying the abstract concept of the freezing method to TOFs and obtain a partial differential algebraic equation (PDAE). We discuss how to choose the phase condition and how to obtain a well-posed problem called the freezing system. According to Chapter 3 we show that TOFs are stable steady states of the freezing system. We prove stability of TOFs for the freezing systems using the results from Chapter 3. For this purpose, we will use the approach from [55] and [54] where the stability of traveling waves in the freezing method was shown for first order hyperbolic systems. In the end, we conclude the chapter with numerical experiments.

4.1 The freezing method

We derive the freezing system on the unweighted spaces X^ℓ, $\ell \in \mathbb{N}_0$, i.e. $\eta = 0$. Hence $X \simeq L^2 \times \mathbb{R}^2$. To formulate the freezing system no weights are necessary, since for the moment we do not ask for stability of TOFs in the system. We consider the Cauchy-Problem on X associated with (0.4) for $\mathbf{u} = (u, \rho)^\top$ reading as

$$\mathbf{u}_t = \begin{pmatrix} Au_{xx} + f(u) \\ f(\rho) \end{pmatrix} =: \mathcal{F}_0(\mathbf{u}), \quad t > 0, \quad \mathbf{u}(0) = \mathbf{u}_0 \in X. \tag{4.1}$$

It easy to see that $\mathcal{F}_0$ defines a closed, densely defined, linear operator on X with $D(\mathcal{F}_0) = Y$. Moreover, recall the Lie group $\mathcal{G} = S^1 \times \mathbb{R}$ acting on X via the group action $a(\gamma)$, $\gamma = (\theta, \tau) \in \mathcal{G}$ from (0.25). Further, $T_\gamma \mathcal{G}$ denotes the tangent space of $\mathcal{G}$ at γ and the associated Lie algebra $\mathfrak{g}$ is given by the tangent space at the unit element $\mathbb{1}$, i.e. $\mathfrak{g} = T_{\mathbb{1}}\mathcal{G}$. By Lemma 3.8, for every $\mathbf{v} = (v, \rho)^\top \in X^1$ the group action $a(\cdot)\mathbf{v} : \mathcal{G} \to X$ is of class C^1 and we denote its derivative (tangent) at $\gamma \in \mathcal{G}$ by

$$d[a(\gamma)\mathbf{v}] : T_\gamma G \to X, \quad \nu \mapsto d[a(\gamma)\mathbf{v}]\nu.$$

The left-multiplication by an element $\gamma \in \mathcal{G}$ on $\mathcal{G}$ is defined as the map

$$L_\gamma : \mathcal{G} \to \mathcal{G}, \quad \tilde{\gamma} \mapsto L_\gamma(\tilde{\gamma}) = \gamma \circ \tilde{\gamma}$$

and is of class C^∞. Its derivative (tangent) is denoted by

$$dL_\gamma(\tilde{\gamma}) : T_{\tilde{\gamma}}\mathcal{G} \to T_\gamma \mathcal{G}, \quad \nu \mapsto dL_\gamma(\tilde{\gamma})\nu.$$

In the case $\tilde{\gamma} = \mathbb{1}$ we have $dL_\gamma(\mathbb{1}) : \mathfrak{g} \to T_\gamma\mathcal{G}$ and $dL_\gamma(\mathbb{1})$ defines a homeomorphism from $\mathfrak{g}$ to $T_\gamma\mathcal{G}$, see [1], [53].
The operator $\mathcal{F}_0$ is equivariant under the group action $a(\gamma)$, i.e. $a(\gamma)Y \subset Y$ and $a(\gamma)\mathcal{F}_0(\mathbf{v}) = \mathcal{F}_0(a(\gamma)\mathbf{v})$. We assume $\mathbf{u} = (u, \zeta)^\top$ to be a solution of the Cauchy problem (4.1) and transform it into a co-moving frame via the ansatz

$$\mathbf{u}(t) = a(\gamma(t))\mathbf{v}(t), \quad t \geq 0.$$

Then we obtain by using the equivariance of $\mathcal{F}_0$

$$a(\gamma)\mathcal{F}_0(\mathbf{v}) = \mathcal{F}_0(\mathbf{u}) = \mathbf{u}_t = d[a(\gamma)\mathbf{v}]\gamma_t + a(\gamma)\mathbf{v}_t. \tag{4.2}$$

Since $a(\cdot)$ is a homomorphism we have for $\gamma, \tilde{\gamma} \in \mathcal{G}$

$$a(\gamma)a(\tilde{\gamma})\mathbf{v} = a(\gamma \circ \tilde{\gamma})\mathbf{v}$$

which upon taking the $\tilde{\gamma}$-derivative leads to

$$a(\gamma)d[a(\tilde{\gamma})\mathbf{v}]\nu = d[a(\gamma \circ \tilde{\gamma})\mathbf{v}]dL_\gamma(\tilde{\gamma})\nu \quad \forall \nu \in T_{\tilde{\gamma}}\mathcal{G}.$$

In particular, for $\tilde{\gamma} = \mathbb{1}$ we obtain

$$d[a(\gamma)\mathbf{v}]dL_\gamma(\mathbb{1})\nu = a(\gamma)d[a(\mathbb{1})\mathbf{v}]\nu \quad \forall \nu \in \mathfrak{g}.$$

Introducing the new variable $\mu(t) \in \mathfrak{g}$ via $\gamma_t(t) = dL_{\gamma(t)}(\mathbb{1})\mu(t)$, we conclude from (4.2)

$$\mathbf{v}_t = \mathcal{F}_0(\mathbf{v}) - d[a(\mathbb{1})\mathbf{v}]\mu_t. \tag{4.3}$$

To compensate the additional degrees of freedom in the μ-variable, we require an additional algebraic constraint, which is called the **phase condition**. In general, it is given by a map

$$\psi : X \times \mathfrak{g} \to \mathbb{R}^2, \quad (\mathbf{v}, \mu) \mapsto \psi(\mathbf{v}, \mu).$$

This leads to the so called **freezing system** reading as

$$\begin{aligned} \mathbf{v}_t &= \mathcal{F}_0(\mathbf{v}) - d[a(\mathbb{1})\mathbf{v}]\mu_t, \quad \mathbf{v}(0) = \mathbf{v}_0, && \text{(4.4a)}\\ 0 &= \psi(\mathbf{v}; \mu), && \text{(4.4b)}\\ \gamma_t &= dL_\gamma(\mathbb{1})\mu, \quad \gamma(0) = \mathbb{1}. && \text{(4.4c)} \end{aligned}$$

Note that (4.4c) describes the position of the wave and is decoupled from (4.4a), (4.4b). In order to analyze solutions of the freezing system and using the stability results from the previous chapter we have to formulate the system (4.4) in the local charts from (3.13), (3.14). In particular, we use the representation of the derivative of the group action from Lemma 3.7. In addition, this is necessary to give a concrete expression for the freezing system which we can solve numerically later on. For this purpose, we note that the Lie algebra $\mathfrak{g}$ turns into a linear space via the derivative of the chart $d\chi(\mathbb{1}) : \mathfrak{g} \to \mathbb{R}^2$, which is one-to-one and onto, see [1, Sec. 3.3] or [53, Sec. 4.1]. Now taking the derivative of $a(\gamma)\mathbf{v} = (a(\cdot)\mathbf{v} \circ \chi^{-1})(\chi(\gamma))$ w.r.t. γ and evaluating at $\gamma = \mathbb{1}$ to obtain the local representation

$$d[a(\mathbb{1})\mathbf{v}] = (a(\cdot) \circ \chi^{-1})'(0)d\chi(\mathbb{1}) = -(\mathbf{S}_1\mathbf{v}, \mathbf{v}_x)d\chi(\mathbb{1}).$$

Next we set $\nu(t) = d\chi(\mathbb{1})\mu(t) \in \mathbb{R}^2$ and define $\tilde{\psi} : X \times \mathbb{R}^2 \to \mathbb{R}^2$ via $\tilde{\psi}(\mathbf{v}, \nu) = \psi(\mathbf{v}, \mu)$. Then we obtain the freezing system in local coordinates reading as the initial value problem with $\mathbf{v} = (v, \rho)^\top$

$$\begin{aligned} \mathbf{v}_t &= \begin{pmatrix} Av_{xx} + \nu_2 v_x + \nu_1 S_1 v + f(v) \\ \nu_1 S_1 \rho + f(\rho) \end{pmatrix}, \quad \mathbf{v}(0) = \mathbf{v}_0, && \text{(4.5a)}\\ 0 &= \tilde{\psi}(\mathbf{v}; \nu), && \text{(4.5b)}\\ \gamma_t &= dL_\gamma(\mathbb{1})d\chi(\mathbb{1})^{-1}\nu, \quad \gamma(0) = \mathbb{1}. && \text{(4.5c)} \end{aligned}$$

Again (4.5c) is decoupled from (4.5a), (4.5b) and can be computed in a post process. It remains to specify the phase condition $\tilde{\psi}$. There are several ways to choose the phase condition, cf. [18, Sec. 2.3]. A possibility is to choose a fixed template function $\hat{\mathbf{w}} = (\hat{w}, \hat{\zeta})^\top \in X^1$ and require that $\hat{\mathbf{w}}$ is the closest point on the group orbit $\mathcal{O}\hat{\mathbf{w}}$ to the solution $\mathbf{v}$ of the PDE (4.5a) w.r.t. the X-norm, i.e. for all $t > 0$ we require

$$\min_{\gamma \in G} \|a(\gamma)\hat{\mathbf{w}} - \mathbf{v}(t)\|_X^2 = \|\hat{\mathbf{w}} - \mathbf{v}(t)\|_X^2.$$

The first order necessary condition is

$$\frac{d}{d\gamma}\Big[\|a(\gamma)\hat{\mathbf{w}} - \mathbf{v}(t)\|_X^2\Big]_{\gamma=\mathbb{1}} = 0$$

and therefore

$$\langle d[a(\mathbb{1})\hat{\mathbf{w}}]d\chi(\mathbb{1})^{-1}\nu, \hat{\mathbf{w}} - \mathbf{v}(t)\rangle_X = 0 \quad \forall\, \nu \in \mathbb{R}^2.$$

Using Lemma 3.8, this yields

$$0 = \Psi_{\text{fix}}(\hat{\mathbf{w}} - \mathbf{v}), \quad \Psi_{\text{fix}}(\mathbf{u}) = \begin{pmatrix} (\mathbf{S}_1\hat{\mathbf{w}}, \mathbf{u})_X \\ (\hat{\mathbf{w}}_x, \mathbf{u})_X \end{pmatrix}. \tag{4.6}$$

The condition (4.6) with $\Psi_{\text{fix}} \in L[X, \mathbb{R}^2]$ is called the **fixed phase condition**, cf. [18]. The inner products defining Ψ_{fix} can be written explicitly as

$$\Psi_{\text{fix}}(\hat{\mathbf{w}} - \mathbf{v}) = \begin{pmatrix} (S_1\hat{\zeta})^\top(\hat{\zeta} - \rho) + (S_1(\hat{w} - \hat{\zeta}\hat{v}), (\hat{w} - \hat{\zeta}\hat{v}) - (v - \rho\hat{v}))_{L^2} \\ (\hat{w}_x, (\hat{w} - \hat{\zeta}\hat{v}) - (v - \rho\hat{v}))_{L^2} \end{pmatrix} = 0.$$

Now we replace $\tilde{\psi}$ in (4.5b) by Ψ_{fix} and obtain the freezing system with the fixed phase condition

$$\mathbf{v}_t = \begin{pmatrix} Av_{xx} + \nu_2 v_x + \nu_1 S_1 v + f(v) \\ \nu_1 S_1 \rho + f(\rho) \end{pmatrix}, \quad \mathbf{v}(0) = \mathbf{v}_0, \tag{4.7a}$$

$$0 = \Psi_{\text{fix}}(\hat{\mathbf{w}} - \mathbf{v}), \tag{4.7b}$$

$$\gamma_t = dL_\gamma(\mathbb{1})d\chi(\mathbb{1})^{-1}\nu, \quad \gamma(0) = \mathbb{1}. \tag{4.7c}$$

The two equation (4.7a), (4.7b) define a partial differential algebraic equation (PDAE) of index 2, cf. [30]. In order to see that see algebraic constraint (4.7b) is of index 2 we take the first derivative of the first component of (4.7b) w.r.t. t and obtain using (4.7a)

$$\begin{aligned} 0 &= (\mathbf{S}_1\hat{\mathbf{w}}, \mathbf{v}_t)_X = (S_1\hat{\zeta})^\top\rho_t + (S_1(\hat{w} - \hat{\zeta}\hat{v}), v_t - \rho_t\hat{v})_{L^2} \\ &= \nu_1\Big((S_1\hat{\zeta})^\top S_1\rho + (S_1(\hat{w} - \hat{\zeta}\hat{v}), S_1(v - \rho\hat{v}))_{L^2}\Big) + \nu_2(S_1(\hat{w} - \hat{\zeta}\hat{v}), v_x)_{L^2} \\ &\quad + (S_1\hat{\zeta})^\top f(\rho) + (S_1(\hat{w} - \hat{\zeta}\hat{v}), Av_{xx} + f(v) - f(\rho)\hat{v})_{L^2} \\ &= \nu_1(\mathbf{S}_1\hat{\mathbf{w}}, \mathbf{S}_1\mathbf{v})_X + \nu_2(\mathbf{S}_1\hat{\mathbf{w}}, \mathbf{v}_x)_X \\ &\quad + (S_1\hat{\zeta})^\top f(\rho) + (S_1(\hat{w} - \hat{\zeta}\hat{v}), Av_{xx} + f(v) - f(\rho)\hat{v})_{L^2}. \end{aligned} \tag{4.8}$$

Differentiating the second component of (4.7b) w.r.t. t yields

$$\begin{aligned} 0 &= (\hat{\mathbf{w}}_x, \mathbf{v}_t)_X = (\hat{w}_x, v_t - \rho_t \hat{v})_{L^2} \\ &= \nu_1(\hat{w}_x, S_1(v - \rho\hat{v}))_{L^2} + \nu_2(\hat{w}_x, v_x)_{L^2} + (\hat{w}_x, Av_{xx} + f(v) - f(\rho)\hat{v})_{L^2} \\ &= \nu_1(\hat{\mathbf{w}}_x, \mathbf{S}_1\mathbf{v})_X + \nu_2(\hat{\mathbf{w}}_x, \mathbf{v}_x)_X + (\hat{w}_x, Av_{xx} + f(v) - f(\rho)\hat{v})_{L^2}. \end{aligned} \tag{4.9}$$

Combining (4.8), (4.9) yields

$$Q_{\text{fix}}(\mathbf{v})\nu = \begin{pmatrix} (S_1\hat{\zeta})^\top f(\rho) + (S_1(\hat{w} - \hat{\zeta}\hat{v}), Av_{xx} + f(v) - f(\rho)\hat{v})_{L^2} \\ (\hat{w}_x, Av_{xx} + f(v) - f(\rho)\hat{v})_{L^2} \end{pmatrix} \tag{4.10}$$

with

$$Q_{\text{fix}}(\mathbf{v}) = -\begin{pmatrix} (\mathbf{S}_1\hat{\mathbf{w}}, \mathbf{S}_1\mathbf{v})_X & (\mathbf{S}_1\hat{\mathbf{w}}, \mathbf{v}_x)_X \\ (\hat{\mathbf{w}}_x, \mathbf{S}_1\mathbf{v})_X & (\hat{\mathbf{w}}_x, \mathbf{v}_x)_X \end{pmatrix}. \tag{4.11}$$

Assuming that $Q_{\text{fix}}(\mathbf{v})$ is invertible for all $t \geq 0$ we can write (4.10) explicitly for ν which shows that (4.7a), (4.7b) is a PDAE of index 2, cf. [30]. In application, one has to choose the template $\hat{\mathbf{w}}$ such that $Q_{\text{fix}}(\mathbf{v}_0)$ is invertible. Then $Q_{\text{fix}}(\mathbf{v})$ is invertible as long as the time evolution of $\mathbf{v}$ is small. As we will see in the next section, this will be the case when we start with $\mathbf{v}_0$ sufficiently close to the profile of the TOF $\mathbf{v}_\star$. In Section 4.3 we will use (4.10) to solve the freezing system (4.7) numerically.

4.2 Stability of the freezing system

We assume Assumption 1-4 and consider the freezing system as the PDAE with perturbed initial conditions of the TOF $\mathbf{v}_\star$

$$\mathbf{v}_t = \begin{pmatrix} Av_{xx} + \nu_2 v_x + \nu_1 S_1 v + f(v) \\ \nu_1 S_1 \rho + f(\rho) \end{pmatrix}, \quad \mathbf{v}(0) = \mathbf{v}_\star + \mathbf{u}_0, \tag{4.12a}$$

$$0 = \Psi(\hat{\mathbf{w}} - \mathbf{v}), \tag{4.12b}$$

where $\hat{\mathbf{w}} \in X^1_\eta$ is a template function and $\Psi : X_\eta \to \mathbb{R}^2$ a two dimensional linear functional on X_η. The phase condition Ψ can be chosen as the fixed phase condition Ψ_{fix} from (4.6) minimizing the distance of the solution of the PDE to the group orbit of the template function. However, in the system (4.12) we allow a general phase condition satisfying appropriate assumptions, cf. Assumption 9. The reconstruction of the position can be written as the differential equation on the manifold $\mathcal{G}$

$$\gamma_t = dL_\gamma(\mathbb{1})d\chi(\mathbb{1})^{-1}\nu, \quad \gamma(0) = \mathbb{1}, \tag{4.13}$$

which is decoupled from the PDAE (4.12).
By Assumption 2 there is a TOF of (0.4) with profile $\mathbf{v}_\star$, frequency ω and speed c.

In particular, we have $\mathcal{F}(\mathbf{v}_\star) = 0$, where $\mathcal{F}$ is the nonlinear operator from (0.23). Let $\nu_\star := (\omega, c)$ and let us assume that $\mathbf{v}_\star$ satisfies the phase condition, i.e. $\Psi(\hat{\mathbf{w}} - \mathbf{v}_\star) = 0$. Then we conclude that $(\mathbf{v}_\star, \nu_\star)$ is a stationary solution of (4.12) with $\mathbf{u}_0 = 0$. Thus we can ask for stability of the solution $(\mathbf{v}_\star, \nu_\star)$. In particular, we are interested in the long time behavior of the solution $(\mathbf{v}, \nu)$ of (4.12) if the initial perturbation $\mathbf{u}_0$ is small. We will prove a stability result of the solution $(\mathbf{v}_\star, \nu_\star)$ of the freezing system by using the nonlinear stability with asymptotic phase of $\mathbf{v}_\star$ from Theorem 1.11. In order to do so, we use the following notion of a solution.

Definition 4.1. A pair $(\mathbf{v}, \nu)$ is called a classical solution of the PDAE (4.12) on $[0, t_\infty)$ if

i) $\mathbf{v} \in C((0, t_\infty), Y_\eta) \cap C^1([0, t_\infty), X_\eta)$ and $\nu = (\nu_1, \nu_2) \in C([0, t_\infty), \mathbb{R}^2)$,

ii) $(\mathbf{v}, \nu)$ solves the PDE (4.12a) pointwise for all $t \in [0, t_\infty)$ in X_η,

iii) the algebraic constraint (4.12b) is satisfied for all $t \in [0, t_\infty)$,

iv) $\mathbf{v}(0) = \mathbf{v}_\star + \mathbf{u}_0 \in X_\eta$.

If a classical solution exists it must satisfy the algebraic constraint (4.12b) at $t = 0$. Thus we will require the consistency condition of the initial value $\Psi(\hat{\mathbf{w}} - \mathbf{v}_\star - \mathbf{u}_0) = 0$.

Definition 4.2. The initial value $\mathbf{v}_\star + \mathbf{u}_0$ in (4.12a) is called consistent if

$$\Psi(\hat{\mathbf{w}} - \mathbf{v}_\star - \mathbf{u}_0) = 0. \tag{4.14}$$

The condition (4.14) seems to be very restrictive regarding the initial data $\mathbf{u}_0$. But it is not, since we did not specify the representative $\mathbf{v}_\star$ of the group orbit $\mathcal{O}(\mathbf{v}_\star)$. In other words, for arbitrary $\mathbf{u}_0$ one finds generically some representative $\mathbf{v}_\star$ of the group orbit $\mathcal{O}(\mathbf{v}_\star)$ such that (4.14) is satisfied.

We make the following assumption on the phase condition Ψ:

Assumption 9. *$\Psi : X_\eta \to \mathbb{R}^2$ is a linear bounded functional on X_η and satisfies for some $C_\Psi > 0$ the estimate*

$$|\Psi(\mathbf{v})| \le C_\Psi \|\mathbf{v}\|_{X^{-1}} \quad \forall\, \mathbf{v} \in X_\eta. \tag{4.15}$$

Moreover, with $\Psi = (\Psi_1, \Psi_2)^\top$ the matrix

$$\begin{pmatrix} \Psi_1(\mathbf{S}_1\mathbf{v}_\star) & \Psi_1(\mathbf{v}_{\star,x}) \\ \Psi_2(\mathbf{S}_1\mathbf{v}_\star) & \Psi_2(\mathbf{v}_{\star,x}) \end{pmatrix}$$

is invertible and there is $\hat{\mathbf{w}} = (\hat{w}, \hat{\zeta})^\top \in X_\eta^1$ with

$$\Psi(\hat{\mathbf{w}} - \mathbf{v}_\star) = 0.$$

Note that the fixed phase condition Ψ_{fix} from (4.6) satisfies Assumption 9 if we choose an appropriate $\hat{\mathbf{w}} \in Y \cap X^1_\eta$ (e.g. $\hat{\mathbf{w}} = \mathbf{v}_\star$). To see that (4.15) is satisfied, we can use the Gelfand triplet property of $X^1 \subset X \subset X^{-1}$ and estimate

$$|\Psi_{\text{fix},1}(\mathbf{v})| = |(\mathbf{S}_1\hat{\mathbf{w}}, \mathbf{v})_X| = |\langle \mathbf{S}_1\hat{\mathbf{w}}, \mathbf{v}\rangle_{X^1\times X^{-1}}| \le \|\mathbf{S}_1\hat{\mathbf{w}}\|_{X^1}\|\mathbf{v}\|_{X^{-1}}$$

and

$$|\Psi_{\text{fix},2}(\mathbf{v})| = |(\hat{\mathbf{w}}_x, \mathbf{v})_X| = |\langle \hat{\mathbf{w}}_x, \mathbf{v}\rangle_{X^1\times X^{-1}}| \le \|\hat{\mathbf{w}}_x\|_{X^1}\|\mathbf{v}\|_{X^{-1}}.$$

Remark 4.3. *For the proof of the stability in this section it would be sufficient to require $\Psi \in L[X^{-1}_\eta, \mathbb{R}^2]$. But in this case one is forced to extend the fixed phase condition to a linear bounded functional $\tilde{\Psi}_{\text{fix}}$ on X^{-1}_η such that $\tilde{\Psi}_{\text{fix}} \in L[X^{-1}_\eta, \mathbb{R}^2]$. For that reason, we decided to work with Assumption 9 such that we can keep in mind Ψ to be the fixed phase condition from* (4.6)*.*

In what follows, we prove under Assumption 1-4 and 9 that the freezing system (4.12) attains a unique classical solution and that the stationary solution $(\mathbf{v}_\star, \nu_\star)$ with $\nu_\star = (\omega, c)$ is asymptotically stable in the classical sense of Lyapunov. For this purpose, we make the following solution ansatz for the solution of the PDAE (4.12a)

$$\mathbf{v}(t) = a(\gamma^{-1}(t))\mathbf{u}(t), \quad \gamma(t) = \chi^{-1}(-z(t)) \quad t \ge 0, \tag{4.16}$$

where $\mathbf{u}$ is a solution to the Cauchy problem (0.22) with initial value $\mathbf{u}(0) = \mathbf{v}_\star + \mathbf{u}_0$ and some $z \in C^1([0,\infty), \mathbb{R}^2)$. Hence, z are the local coordinates of some group element $\gamma(t) \in G$ in the chart (U, χ), i.e. $\gamma(t) \in U \subset G$. Note that by definition of the chart (U, χ) we have $\gamma(t) \in U$ if and only if $\gamma(t)^{-1} \in U$ and $\gamma^{-1}(t) = \chi^{-1}(-z(t))$. We often write γ^{-1} instead of $\chi^{-1}(-z)$ and γ instead of $\chi^{-1}(z)$. The initial value in (4.12a) implies $\gamma(0) = \mathbb{1}$ and therefore $z(0) = 0$. Plugging the ansatz (4.16) into (4.12a), we obtain with $z_t = (\theta_t, \tau_t)$

$$\begin{aligned}
&a(\gamma^{-1})\mathbf{u}_x\tau_t + a(\gamma^{-1})\mathbf{S}_1\mathbf{u}\theta_t + a(\gamma^{-1})\mathcal{F}(\mathbf{u}) \\
&= \frac{d}{dt}[(a(\cdot)\mathbf{u} \circ \chi^{-1})(-z)] + a(\gamma^{-1})\mathbf{u}_t \\
&= \mathbf{v}_t = \begin{pmatrix} Av_{xx} + \nu_2 v_x + \nu_1 S_1 v + f(v) \\ \nu_1 S_1\rho + f(\rho) \end{pmatrix} \\
&= a(\gamma^{-1})\mathcal{F}(\mathbf{u}) + a(\gamma^{-1})(\nu_2 - c)\mathbf{u}_x + a(\gamma^{-1})(\nu_1 - \omega)\mathbf{S}_1\mathbf{u}.
\end{aligned} \tag{4.17}$$

This determines ν via $\nu = z_t + \nu_\star$. We define the map

$$Q : X^1_\eta \to \mathbb{R}^{2,2}, \quad \mathbf{v} \mapsto Q(\mathbf{v}) := -\begin{pmatrix} \Psi_1(\mathbf{S}_1\mathbf{v}) & \Psi_1(\mathbf{v}_x) \\ \Psi_2(\mathbf{S}_1\mathbf{v}) & \Psi_2(\mathbf{v}_x) \end{pmatrix}. \tag{4.18}$$

By Assumption 9 we have $Q(\mathbf{v}_\star)$ to be non-singular. Now taking the time derivative of the algebraic constraint yields

$$\begin{aligned}\frac{d}{dt}\Psi(\hat{\mathbf{w}}-\mathbf{v}) &= -\Psi(\mathbf{v}_t) = -\Psi(a(\gamma^{-1})\mathbf{S}_1\mathbf{u})\theta_t - \Psi(a(\gamma^{-1})\mathbf{u}_x)\tau_t - \Psi(a(\gamma^{-1})\mathcal{F}(\mathbf{u})) \\ &= -\begin{pmatrix}\Psi_1(a(\gamma^{-1})\mathbf{S}_1\mathbf{u}) & \Psi_1(a(\gamma^{-1})\mathbf{u}_x) \\ \Psi_2(a(\gamma^{-1})\mathbf{S}_1\mathbf{u}) & \Psi_2(a(\gamma^{-1})\mathbf{u}_x)\end{pmatrix} z_t - \Psi(a(\gamma^{-1})\mathcal{F}(\mathbf{u})) \\ &= Q(a(\gamma^{-1})\mathbf{u})z_t - \Psi(a(\gamma^{-1})\mathcal{F}(\mathbf{u})).\end{aligned} \tag{4.19}$$

Thus, if $z \in C^1([0,\infty),\mathbb{R}^2)$ is a solution of the ODE

$$Q(a(\chi^{-1}(-z))\mathbf{u})z_t = \Psi\big(a(\chi^{-1}(-z))\mathcal{F}(\mathbf{u})\big), \quad z(0)=0 \tag{4.20}$$

the algebraic constraint is constant in time and the consistency of the initial value, cf. Definition 4.2, implies

$$\Psi(\hat{\mathbf{w}}-\mathbf{v}(t)) = 0 \quad \forall t \geq 0.$$

Thus, $\mathbf{v}$ is classical solution of the freezing system (4.12). Now the idea is to study solutions of the ODE (4.20) and construct a solution to the PDAE (4.12) via the ansatz (4.16).

Lemma 4.4. *Let Assumption 1-4 and Assumption 9 be satisfied. Further, let $0 < \mu < \min\{\mu_{\text{ess}}, \mu_\star, 2\}$ with μ_{ess} from Assumption 3 and $\mu_\star$ from Theorem 2.6. Then there are $\varepsilon, \delta > 0$ such that for all $\mathbf{u}_0 \in Y_\eta$ with $\|\mathbf{u}_0\|_{X^1_\eta} < \varepsilon$ and $t \in [0,\infty)$ there hold*

$$Q(a(\chi^{-1}(\cdot))\mathbf{u}) \in C^1(B_\delta(0), GL(\mathbb{R}^2)),$$

where $\mathbf{u}$ is the solution from Theorem 1.11.

Proof. First let $\varepsilon > 0$ be so small such that Theorem 1.11 applies. By Assumption 9 the matrix $Q(\mathbf{v}_\star)$ is invertible and continuously differentiable in $\mathbf{v}_\star$, since it is linear in $\mathbf{v}_\star$. As a consequence of the implicit function theorem there is $\delta_Q > 0$ such that $Q(\mathbf{v})$ is invertible whenever $\|\mathbf{v}_\star - \mathbf{v}\|_{X_\eta} < \delta_Q$. If this is the case, $Q(\mathbf{v})$ itself and the inverse are continuously differentiable in $\mathbf{v}$. So we show $\|a(\chi^{-1}(z))\mathbf{u}(t) - \mathbf{v}_\star\|_{X_\eta} \leq \delta_Q$ for all $t \geq 0$ and $z \in B_\delta(0)$. By Lemma 3.7 and (3.18), there are $\delta_0 > 0$ and a constant $C > 0$ such that

$$\|a(\chi^{-1}(z))\mathbf{v}\|_{X_\eta} \leq C\|\mathbf{v}\|_{X_\eta} \quad \forall\, z \in B_{\delta_0}(0),\ \mathbf{v} \in X_\eta, \tag{4.21}$$

$$\|a(\chi^{-1}(z_1))\mathbf{v}_\star - a(\chi^{-1}(z_2))\mathbf{v}_\star|_{X_\eta} \leq C|z_1 - z_2|\|\mathbf{v}_\star\|_{X^1_\eta} \quad \forall\, z_1, z_2 \in B_{\delta_0}(0). \tag{4.22}$$

Take C_∞, K from Theorem 1.11. Now choose δ, ε sufficiently small such that $0 < \delta < \delta_0$, $0 < \varepsilon < C_\infty^{-1}\delta_0$ and

$$CK\varepsilon + C^2C_\infty\varepsilon\|\mathbf{v}_\star\|_{X^1_\eta} + C^2\delta\|\mathbf{v}_\star\|_{X^1_\eta} < \delta_Q.$$

Then, using Theorem 1.11, Corollary 3.30, (3.18), (4.21), (4.22), we obtain for all $z \in B_\delta(0)$, $\mathbf{u}_0 \in Y_\eta$ with $\|\mathbf{u}_0\|_{X_\eta} < \varepsilon$ and $t \in [0,\infty)$ the estimate

$$\begin{aligned}\|a(\chi^{-1}(z))\mathbf{u}(t) - \mathbf{v}_\star\|_{X_\eta} &\leq C\|\mathbf{u}(t) - a(\chi^{-1}(-z))\mathbf{v}_\star\|_{X_\eta} \\ &\leq C\|\mathbf{u}(t) - a(\gamma_\infty)\mathbf{v}_\star\|_{X_\eta} + C\|a(\chi^{-1}(z_\infty))\mathbf{v}_\star - a(\chi^{-1}(-z))\mathbf{v}_\star\|_{X_\eta} \\ &\leq CKe^{-\tilde{\beta}t}\varepsilon + C^2\left(|z_\infty| + |z|\right)\|\mathbf{v}_\star\|_{X_\eta^1} \\ &\leq CK\varepsilon + C^2C_\infty\varepsilon\|\mathbf{v}_\star\|_{X_\eta^1} + C^2\delta\|\mathbf{v}_\star\|_{X_\eta^1} < \delta_Q.\end{aligned}$$

Now the assertion is proven, since $\mathbf{u}(t) \in Y_\eta$ and the group action $(a(\cdot)\mathbf{u}(t) \circ \chi^{-1})$ is continuously differentiable. ☐

By the previous lemma we can write the ODE (4.20) as an explicit ODE for z with a continuous right-hand side. Then we obtain local existence by using Peano's existence theorem. However, since $\mathcal{F}(\mathbf{u})$ only belongs to X_η the group action $(a(\cdot)\mathcal{F}(\mathbf{u}) \circ \chi^{-1})$ is only continuous and not Lipschitz continuous in z. Therefore, we do not have uniqueness of the solution. This will be concluded in a further step. In order to do so, we use the approach from [55], [54].

Lemma 4.5 (Local existence)**.** *Let Assumption 1-4 and Assumption 9 be satisfied. Further, let $0 < \mu < \min\{\mu_{\mathrm{ess}}, \mu_\star, 2\}$ with μ_{ess} from Assumption 3 and $\mu_\star$ from Theorem 2.6 and let $\varepsilon > 0$ be given as in Lemma 4.4. Then for all $\mathbf{u}_0 \in Y_\eta$ with $\|\mathbf{u}_0\|_{X_\eta^1} < \varepsilon$ there is $t_0 = t_0(\mathbf{u}_0) > 0$ such that the ODE (4.20) has a solution $z \in C^1([0,t_0),\mathbb{R}^2)$.*

Proof. By Lemma 4.4, $Q(a(\chi^{-1}(\cdot))\mathbf{u}(t))$ is invertible for all $z \in B_\delta(0)$, $t \in [0,\infty)$. Then the ODE 4.20 can be rewritten as

$$z_t = r(t,z), \quad z(0) = \mathbb{1} \tag{4.23}$$

with the right-hand side is given by

$$r : [0,\infty) \times B_\delta(0) \to \mathbb{R}^{,} \quad (t,z) \mapsto Q\big(a(\chi^{-1}(-z))\mathbf{u}(t)\big)^{-1}\Psi\big(a(\chi^{-1}(-z))\mathcal{F}(\mathbf{u}(t))\big).$$

By Assumption 9, Theorem 1.11, Lemma 3.8 and Lemma 4.4 it follows $r \in C([0,\infty) \times B_\delta(0), \mathbb{R}^2)$. Now the claim is a consequence of Peano's existence theorem. ☐

As a next step we show that the solution $z \in C([0,t_0),\mathbb{R}^2)$ from Lemma 4.5 exists for all times, i.e. we have $t_0 = \infty$. For this purpose we need the following lemma.

Lemma 4.6. *Let Assumption 1-4 and Assumption 9 be satisfied. Further, let $0 < \mu < \min\{\mu_{\mathrm{ess}}, \mu_\star, 2\}$ with μ_{ess} from Assumption 3 and $\mu_\star$ from Theorem 2.6. Then the map*

$$\Lambda : \mathbb{R}^2 \to \mathbb{R}^2, \quad z \mapsto \Lambda(z) := \Psi(a(\chi^{-1}(z))\mathbf{v}_\star - \mathbf{v}_\star)$$

is a local C^1-diffeomorphism near 0.

Proof. We have $\Lambda(0) = 0$ and $\partial_z \Lambda(0) = Q(\mathbf{v}_\star)$ which is invertible by Assumption 9. Hence the assertion is a consequence of the implicit function theorem D.8. □

Now we conclude global existence of the solution from Lemma 4.5.

Lemma 4.7 (Global existence). *Let Assumption 1-4 and Assumption 9 be satisfied. Further, let $0 < \mu < \min\{\mu_{\mathrm{ess}}, \mu_\star, 2\}$ with μ_{ess} from Assumption 3 and $\mu_\star$ from Theorem 2.6 and let $\delta > 0$ be from Lemma 4.4. Then there is $\varepsilon > 0$ such that for all $\mathbf{u}_0 \in Y_\eta$ with $\|\mathbf{u}_0\|_{X^1_\eta} < \varepsilon$ the ODE* (4.20) *has a solution $z \in C^1([0,\infty), B_{\delta/2}(0))$.*

Proof. First take $\varepsilon > 0$ so small such that Lemma 4.4 and Lemma 4.5 as well as Theorem 1.11 apply. Let $z \in C^1([0,t_\infty), \mathbb{R}^2)$ be the maximal extension of the local solution from Lemma 4.5 in $B_\delta(0)$ and assume $t_\infty < \infty$. Then $|z(t)| \to \delta$ as $t \to t_\infty$. The ansatz (4.16) and the previous calculation (4.19) show

$$\Psi(\hat{\mathbf{w}} - a(\chi^{-1}(-z(t)))\mathbf{u}(t)) = 0 \quad \forall\, t \in [0, t_\infty)$$

which implies together with Assumption 9

$$\Psi(\mathbf{v}_\star - a(\chi^{-1}(-z(t)))\mathbf{u}(t)) = 0 \quad \forall\, t \in [0, t_\infty).$$

Choose $0 < \delta_\Lambda < \delta$ such that Lemma 4.6 implies and thus $\Lambda^{-1} : B_{\delta_\Lambda}(\mathbb{1}) \to \Lambda^{-1}(B_{\delta_\Lambda}(\mathbb{1}))$ is diffeomorphic. Then, since $\Lambda(0) = 0$, there is $C_\Lambda > 0$ such that

$$|\Lambda^{-1}(y)| \le C_\Lambda |y| \quad \forall\, y \in B_{\delta_\Lambda}(0). \tag{4.24}$$

Now choose $\varepsilon > 0$ so small such that

$$(C_\Lambda C_\Psi C K + C_\infty)\varepsilon \le \frac{\delta_\Lambda}{2}$$

with C from (4.21) and K, C_∞ from Theorem 1.11. Let z_∞ be from Corollary 3.30. Then we have $|z_\infty| \le C_\infty \varepsilon < \frac{\delta_\Lambda}{2}$ and

$$\chi^{-1}(z_\infty - z(t)) = \gamma_\infty \circ \gamma^{-1}(t).$$

We obtain for all $0 \le t < t_\infty$

$$\begin{aligned}
|z(t)| &\le |z_\infty| + \left|z_\infty - z(t)\right| \le C_\infty \varepsilon + \left|\Lambda^{-1}\big(\Psi(a(\chi^{-1}(z_\infty - z(t)))\mathbf{v}_\star - \mathbf{v}_\star)\big)\right| \\
&\le C_\infty \varepsilon + C_\Lambda \left|\Psi(a(\gamma_\infty \circ \gamma^{-1}(t))\mathbf{v}_\star - \mathbf{v}_\star) - \Psi(\mathbf{v}_\star - a(\gamma^{-1}(t))\mathbf{u}(t))\right| \\
&\le C_\infty \varepsilon + C_\Lambda C_\Psi \left\|a(\gamma_\infty \circ \gamma^{-1}(t))\mathbf{v}_\star - a(\gamma^{-1}(t))\mathbf{u}(t)\right\|_{X_\eta} \\
&\le C_\infty \varepsilon + C_\Lambda C_\Psi C \|a(\gamma_\infty)\mathbf{v}_\star - \mathbf{u}(t)\|_{X_\eta} \le C_\infty \varepsilon + C_\Lambda C_\Psi C K \varepsilon \le \frac{\delta_\Lambda}{2} \le \frac{\delta}{2},
\end{aligned}$$

where we used the estimate from Theorem 1.11 and Lemma 3.7. Since z is continuous on $[0, t_\infty)$ this contradicts $|z(t)| \to \delta$ as $t \to t_\infty$. Hence $t_\infty = \infty$ and the assertion is proven. □

It remains to show uniqueness of the global solution z from Lemma 4.7 without using classical Lipschitz continuity. As the next Lemma will show, we are able to prove special Lipschitz-like estimates in a small neighborhood of the solution z from Lemma 4.7 which will yield the uniqueness of the solution, cf. [55], [54].

Lemma 4.8. *Let Assumption 1-4 and Assumption 9 be satisfied. Further, let $0 < \mu < \min\{\mu_{\mathrm{ess}}, \mu_\star, 2\}$ with μ_{ess} from Assumption 3 and $\mu_\star$ from Theorem 2.6 and let $z \in C^1([0,\infty), B_{\delta/2}(0))$ be the solution of the ODE (4.20) from Lemma 4.7 and let*

$$H : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}^2, \quad (t, \tilde{z}) \mapsto \tilde{z} - Q\big(a(\chi^{-1}(-z(t)))\mathbf{u}(t)\big)^{-1}\Psi(\hat{\mathbf{w}} - a(\chi^{-1}(-\tilde{z}))\mathbf{u}(t)).$$

Then there is $\varepsilon_z > 0$ such that for all $t \in [0,\infty)$ there hold

$$|H(t,\tilde{z}) - H(t, z(t))| \leq \frac{1}{2}|\tilde{z} - z(t)| \quad \forall\, \tilde{z} \in B_{\varepsilon_z}(z(t)).$$

Proof. By Theorem 1.11, $\mathbf{u}(t)$ is uniformly bounded for $t \in [0,\infty)$. Let $\tilde{\delta} = \frac{3}{4}\delta$. By Lemma 4.4 there is $C_Q > 0$ such that

$$\begin{aligned}
&\left|Q\big(a(\chi^{-1}(z))\mathbf{u}(t)\big)^{-1}\right| \leq C_Q \quad \forall\, t \in [0,\infty),\ z \in B_{\tilde{\delta}}(0),\\
&\big|Q(a(\chi^{-1}(z_1))\mathbf{u}(t)) - Q(a(\chi^{-1}(z_2))\mathbf{u}(t))\big| \leq C_Q|z_1 - z_2| \quad \forall\, t \in [0,\infty),\ z_1, z_2 \in B_{\tilde{\delta}(0)}.
\end{aligned}$$

Now let

$$\varepsilon_z < \min\left(\frac{\delta}{4}, \frac{1}{2C_Q^2}\right)$$

and $\tilde{z} \in B_{\varepsilon_z}(z(t))$, $t \in [0,\infty)$. Then

$$|z(t)| \leq \frac{\delta}{2} < \tilde{\delta}, \quad |\tilde{z}| \leq |\tilde{z} - z(t)| + |z(t)| \leq \varepsilon_z + \frac{\delta}{2} < \tilde{\delta}$$

and we estimate, using the mean value theorem and $\partial_z\Psi((a(\cdot)\mathbf{u} \circ \chi^{-1})(z)) = -Q(a(\cdot)\mathbf{u} \circ \chi^{-1})(z))$,

$$\begin{aligned}
&|H(t,\tilde{z}) - H(t,z(t))|\\
&\leq C_Q\left|Q(a(\chi^{-1}(-z(t)))\mathbf{u}(t))(\tilde{z} - z(t)) - \Psi(a(\chi^{-1}(-z(t))\mathbf{u}(t) - a(\chi^{-1}(-\tilde{z}))\mathbf{u}(t))\right|\\
&= C_Q\Big|Q(a(\chi^{-1}(-z(t)))\mathbf{u}(t))(\tilde{z} - z(t))\\
&\qquad - \int_0^1 Q(a(\chi^{-1}(-z(t) + (z(t) - \tilde{z})\tau))\mathbf{u}(t))(\tilde{z} - z(t))d\tau\Big|\\
&\leq C_Q \int_0^1 \left|Q(a(\chi^{-1}(-z(t)))\mathbf{u}(t)) - Q(a(\chi^{-1}(-z(t) + (z(t) - \tilde{z})\tau))\mathbf{u}(t))\right| d\tau |\tilde{z} - z(t)|\\
&\leq C_Q^2\varepsilon_z|\tilde{z} - z(t)| \leq \frac{1}{2}|\tilde{z} - z(t)|.
\end{aligned}$$

□

Now we are in the situation to conclude the uniqueness of z.

Lemma 4.9 (Uniqueness)**.** *Let Assumption 1-4 and Assumption 9 be satisfied. Further, let $0 < \mu < \min\{\mu_{\text{ess}}, \mu_\star, 2\}$ with μ_{ess} from Assumption 3 and $\mu_\star$ from Theorem 2.6. Then there is $\varepsilon > 0$ such that for all $\mathbf{u}_0 \in Y_\eta$ with $\|\mathbf{u}_0\|_{X^1_\eta} < \varepsilon$ the ODE (4.20) has a unique global solution $z \in C^1([0,\infty), \mathbb{R}^2)$.*

Proof. Take $z \in C^1([0,\infty), B_{\delta/2}(\mathbb{1}))$ from Lemma 4.7 and let $\tilde{z} \in C^1([0,t_0), \mathbb{R}^2)$ be another maximal extended solution of (4.20). Define

$$t_\infty := \sup\{T \in [0,t_0) : \tilde{z}(t) = z(t)\,\forall\, 0 \le t < T\}. \tag{4.25}$$

Assume $t_\infty < t_0$. By continuity of the solutions there is $\delta > 0$ with $t_\infty + \delta \le t_0$ such that

$$|\tilde{z}(t) - z(t)| \le \varepsilon_z \quad \forall\, t \in [t_\infty - \delta, t_\infty + \delta]$$

where ε_z is from Lemma 4.8. Now since $\tilde{z}, z$ solve (4.20) there holds for all $0 \le t < t_0$

$$\Psi(\hat{\mathbf{w}} - a(\chi^{-1}(-z(t)))\mathbf{u}(t)) = \Psi(\hat{\mathbf{w}} - a(\chi^{-1}(-\tilde{z}(t)))\mathbf{u}(t)) = 0$$

and thus Lemma 4.8 implies for all $t \le t_\infty + \delta$

$$|\tilde{z}(t) - z(t)| = |H(t, \tilde{z}(t)) - H(t, z(t))| \le \frac{1}{2}|\tilde{z}(t) - z(t)|.$$

Therefore, $\tilde{z}(t) = z(t)$ for all $t \le t_\infty + \delta$. This contradicts (4.25). Hence $t_\infty = t_0 = \infty$. □

Finally by using the ansatz (4.16) we obtain the following stability result for TOFs in the freezing system.

Theorem 4.10 (Stability of the freezing system)**.** *Let Assumption 1-4 and Assumption 9 be satisfied. Further, let $0 < \mu < \min\{\mu_{\text{ess}}, \mu_\star, 2\}$ with μ_{ess} from Assumption 3 and $\mu_\star$ from Theorem 2.6.. Then there is $\varepsilon > 0$ such that for all consistent initial values $\mathbf{u}_0 \in Y_\eta$ with $\|\mathbf{u}_0\|_{X^1_\eta} < \varepsilon$ and $\nu_0 \in \mathbb{R}^2$ the PDAE (4.12) has a unique classical solution $(\mathbf{v}, \nu)$ on $[0,\infty)$. Moreover, there are $K, \tilde{\beta} > 0$ such that for all $t \ge 0$ there hold*

$$\|\mathbf{v}(t) - \mathbf{v}_\star\|_{X^1_\eta} + |\nu(t) - \nu_\star| \le K e^{-\tilde{\beta} t}\|\mathbf{u}_0\|_{X^1_\eta}. \tag{4.26}$$

Proof. Take $\varepsilon > 0$ so small such that Theorem 1.11 and Lemma 4.9 apply. Define $\mathbf{v}$ via the ansatz (4.16) with $\mathbf{u} \in C([0,\infty), Y_\eta) \cap C^1([0,\infty), X_\eta)$ to be the solution from Theorem 1.11 and $z \in C^1([0,\infty), \mathbb{R}^2)$ to be from Lemma 4.9. Then $\mathbf{v} \in C([0,\infty), Y_\eta) \cap C^1([0,\infty), X_\eta)$. Moreover, set $\nu = z_t + \nu_\star$. The anstaz (4.16) and the calculation (4.17), (4.19) show that the pair $(\mathbf{v}, \nu)$ solves (4.12) pointwise. Moreover, since $z(0) = 0$ the initial value from (4.12a) is satisfied. Hence $(\mathbf{v},\nu)$ is a classical solution of the PDAE (4.12). Now let $(\tilde{\mathbf{v}},\tilde{\nu})$ be another classical solution of the PDAE (4.12). We define

$$\tilde{z}(t)=\int_0^t \nu_\star-\tilde{\nu}(s)ds,\quad \mathbf{w}(t)=a(\chi^{-1}(-\tilde{z}(t))(t))\tilde{\mathbf{v}}(t),\quad t\geq 0.$$

Then $\tilde{z}(0)=0$ and $\mathbf{w}(0)=\mathbf{v}_\star+\mathbf{u}_0$. Furthermore, writing $\tilde{\gamma}=\chi^{-1}(-\tilde{z})$,

$$\begin{aligned}\mathbf{w}_t&=(\omega-\tilde{\nu}_1)a(\tilde{\gamma})\mathbf{S}_1\tilde{\mathbf{v}}+(c-\tilde{\nu}_2)a(\tilde{\gamma})\tilde{\mathbf{v}}_x+a(\tilde{\gamma})\tilde{\mathbf{v}}_t\\&=\omega\mathbf{S}_1\mathbf{w}+c\mathbf{w}_x+a(\tilde{\gamma})\begin{pmatrix}A\tilde{v}_{xx}+f(\tilde{v})\\ f(\tilde{\rho})\end{pmatrix}=\mathcal{F}(\mathbf{w}).\end{aligned}$$

Thus $\mathbf{w}$ solves (0.22) and Theorem 1.11 states $\mathbf{w}=\mathbf{u}$. Moreover, the calculation (4.19) shows $\tilde{z}$ solves (4.20) and we conclude $z=\tilde{z}$, see Lemma 4.9. Then $\nu=\tilde{\nu}$ and $\mathbf{v}=\tilde{\mathbf{v}}$. Thus $(\mathbf{v},\nu)$ is the unique solution of the PDAE (4.12) and it remains to show the exponential estimate (4.26). For this purpose, let $\|\mathbf{u}_0\|_{X^1_\eta}=:\varepsilon_0<\varepsilon$ and use Assumption 9, Theorem 1.11 and take z_∞ from Corollary 3.30 and C,C_Λ from (4.21), (4.22) and (4.24) to estimate

$$\begin{aligned}|z_\infty-z(t)|&=\left|\Lambda^{-1}(\Psi(a(\chi^{-1}(z_\infty-z(t)))\mathbf{v}_\star-\mathbf{v}_\star))\right|\\&\leq C_\Lambda\left|\Psi(a(\chi^{-1}(z_\infty-z(t)))\mathbf{v}_\star-\mathbf{v}_\star)-\Psi(a(\gamma^{-1}(t))\mathbf{u}(t)-\mathbf{v}_\star)\right|\\&\leq C_\Lambda C_\Psi\|a(\gamma_\infty\circ\gamma^{-1}(t))\mathbf{v}_\star-a(\gamma^{-1}(t))\mathbf{u}(t)\|_{X^{-1}}\\&\leq C_\Lambda C_\Psi\|a(\gamma_\infty\circ\gamma^{-1}(t))\mathbf{v}_\star-a(\gamma^{-1}(t))\mathbf{u}(t)\|_{X_\eta}\\&\leq C_\Lambda C_\Psi C\|a(\gamma_\infty)\mathbf{v}_\star-\mathbf{u}(t)\|_{X_\eta}\leq C_\Lambda C_\Psi CKe^{-\tilde{\beta}t}\varepsilon_0.\end{aligned}$$

Further, we obtain

$$\begin{aligned}\|\mathbf{v}(t)-\mathbf{v}_\star\|_{X^1_\eta}&=\|a(\gamma^{-1}(t))\mathbf{u}(t)-\mathbf{v}_\star\|_{X^1_\eta}\\&\leq\|a(\gamma^{-1}(t))\mathbf{u}(t)-a(\gamma_\infty\circ\gamma^{-1}(t))\mathbf{v}_\star\|_{X^1_\eta}+\|a(\gamma_\infty\circ\gamma^{-1}(t))\mathbf{v}_\star-\mathbf{v}_\star\|_{X^1_\eta}\\&\leq\|a(\gamma^{-1}(t))\mathbf{u}(t)-a(\gamma_\infty\circ\gamma^{-1}(t))\mathbf{v}_\star\|_{X^1_\eta}+\|a(\chi^{-1}(z_\infty-z(t)))\mathbf{v}_\star-\mathbf{v}_\star\|_{X^1_\eta}\\&\leq C\|\mathbf{u}(t)-a(\gamma_\infty)\mathbf{v}_\star\|_{X^1_\eta}+C|z_\infty-z(t)|\|\mathbf{v}_\star\|_{Y_\eta}\\&\leq CKe^{-\tilde{\beta}t}\varepsilon_0+C_\Lambda C_\Psi C^2Ke^{-\tilde{\beta}t}\varepsilon_0\|\mathbf{v}_\star\|_{Y_\eta}\leq\frac{\tilde{K}}{2}e^{-\tilde{\beta}t}\varepsilon_0.\end{aligned}$$

Use the Lipschitz continuity of $\mathcal{F}:X^1\to X^{-1}$ from Lemma 3.8 and Assumption 9 to obtain

$$\begin{aligned}|\nu(t)-\nu_\star|&=|z_t(t)|\leq C_Q|\Psi(a(\chi^{-1}(-z(t)))\mathcal{F}(\mathbf{u}(t)))|=C_Q|\Psi(\mathcal{F}(\mathbf{v}(t)))|\\&\leq C_QC_\Psi\|\mathcal{F}(\mathbf{v}(t))\|_{X^{-1}}=C_QC_\Psi\|\mathcal{F}(\mathbf{v}(t))-\mathcal{F}(\mathbf{v}_\star)\|_{X^{-1}}\\&\leq C_QC_\Psi L_{\mathcal{F}}\|\mathbf{v}(t)-\mathbf{v}_\star\|_{X^1}\leq C_QC_\Psi L_{\mathcal{F}}Ke^{-\tilde{\beta}t}\varepsilon_0=\frac{\tilde{K}}{2}e^{-\tilde{\beta}t}\varepsilon_0.\end{aligned}$$

□

4.3 Numerical simulations and experiments

In this section we perform numerical simulations and experiments concerning TOFs. We show how TOFs can be observed by solving (0.4) using numerical methods. As a prototype for the equation (0.4) we choose the quintic Ginzburg-Landau equation (QCGL). We solve the equation (QCGL) itself as well as the corresponding freezing system from Section 4.1. We conclude the section by calculating the spectrum of the corresponding linearized operator and verify the assumptions from Section 1.3 that guarantees nonlinear stability of TOFs. In particular, we determine the point spectrum of the linearized operator and discuss the shape of the essential spectrum in applications.

4.3.1 Computing traveling oscillating fronts

Let us consider (QCGL) reading as

$$U_t = \alpha U_{xx} + \mu U + \beta |U|^2 U + \gamma |U|^4 U, \quad x \in \mathbb{R},\, t > 0$$

for $U(x,t) \in \mathbb{C}$ with initial data $U(\cdot,0) = U_0$ and parameter $\alpha, \mu, \beta, \gamma \in \mathbb{C}$. We set $\alpha = \alpha_1 + i\alpha_2$, $\mu = \mu_1 + i\mu_2$, $\beta = \beta_1 + i\beta_2$, $\gamma = \gamma_1 + i\gamma_2$ with real coefficients $\alpha_i, \mu_i, \beta_i, \gamma_i \in \mathbb{R}$. Then the corresponding real-valued system for $u = (\operatorname{Re} U, \operatorname{Im} U)^\top$ is given by (0.4), i.e.

$$u_t = A u_{xx} + g(|u|^2)u, \quad x \in \mathbb{R},\, t > 0,$$

with initial value $u(\cdot,0) = u_0$, $u_0 = (\operatorname{Re} U_0, \operatorname{Im} U_0)^\top$. Here A and g are given by (0.5) with

$$g(|u|^2) = \begin{pmatrix} g_1(|u|^2) & -g_2(|u|^2) \\ g_2(|u|^2) & g_1(|u|^2) \end{pmatrix}, \quad g_i(|u|^2) = \mu_i + \beta_i |u|^2 + \gamma_i |u|^4, \quad i = 1,2. \tag{4.27}$$

As an example we choose the parameters

$$\alpha = 1, \quad \mu = -\frac{1}{8}, \quad \beta = 1 + i, \quad \gamma = -1 + i. \tag{4.28}$$

We look for a TOF in the system (0.4) with the special nonlinearity g given by (4.27). Before solving (0.4) with (4.27) we discuss a-priori properties of a TOF. Using Lemma 1.6, we can calculate the possible asymptotic rest-state v_∞ and frequency $\omega \in \mathbb{R}$ of the TOF a-priori since by Lemma 1.6 there must hold $g_1(|v_\infty|^2) = 0$ and $\omega = -g_2(|v_\infty|^2)$. For this purpose, let $r_\infty := |v_\infty|^2$. Then there holds

$$0 = g_1(r_\infty) = \gamma_1 r_\infty^2 + \beta_1 r_\infty + \mu_1.$$

Hence there are at most two possible solutions

$$r_\infty^\pm = \frac{\beta_1}{2\gamma_1} \pm \frac{1}{2\gamma_1}\sqrt{\beta_1^2 - 4\gamma_1\mu_1}.$$

Taking Assumption 2 into account, we have

$$g_1'(r_\infty^\pm) = \beta_1 + 2\gamma_1 r_\infty = \pm\sqrt{\beta_1^2 - 4\gamma_1\mu_1}.$$

Therefore, $g_1'(r_\infty^+) < 0$ and $g_1'(r_\infty^-) > 0$. By Assumption 2 we have for a stable TOF $r_\infty = r_\infty^+$ and with the parameters from (4.28) we obtain

$$|v_\infty| = \sqrt{\frac{\beta_1}{2\gamma_1} + \frac{1}{2\gamma_1}\sqrt{\beta_1^2 - 4\gamma_1\mu_1}} = \frac{\sqrt{2+\sqrt{2}}}{2} \approx 0.9239. \tag{4.29}$$

In addition, the corresponding frequency of the TOF is given by

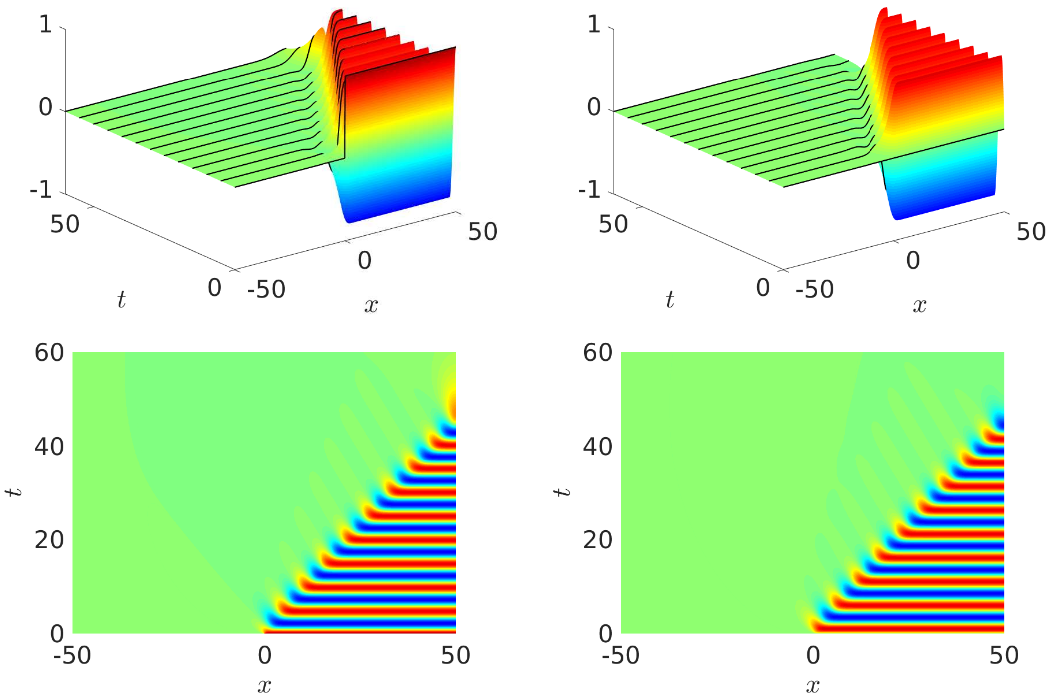


Figure 4.1: Numerical simulation of a TOF in (QCGL) with parameters from (4.28). Real part (left), imaginary part (right).

$$\omega = -g_2(r_\infty) = -\mu_2 - \beta_2 r_\infty - \gamma_2 r_\infty^2 = -\frac{7}{8} - \frac{\sqrt{2}}{2} \approx -1.5821. \tag{4.30}$$

To solve (0.4) numerically as an initial value problem, we have to truncate the equation to a bounded domain of computation $\Omega = [-L, L]$ and then solve the equation using, for instance, finite difference methods. In this case we will choose homogeneous Neumann boundary conditions for the equation, i.e. $u_x(x,t) = 0$, $x \in \partial\Omega$, $t > 0$. As an initial value we set $u(\cdot,0) = (u_0,0)^\top$ with $u_0(x) = \frac{1}{2}\tanh(1000x) + \frac{1}{2}$. In Figure 4.1 we see the results of a finite difference approximation of the solution for $L = 50$ with spatial step size $\Delta x = 0.1$. For the time integration we used the implicit Euler method to avoid restrictions on the step size in time, which is chosen to be $\Delta t = 0.1$. The implicit equations are solved using Newton's method with a tolerance of 10^{-5}.

We see that after a short time period the solution has the shape of a TOF. Taking Theorem 1.11 into account, we expect that the solution is an approximation of a TOF in the equation since it converges to the TOF. We see that the front travels in space with positive velocity $c > 0$ and the asymptotic rest-state is approximately $|v_\infty| \approx 0.9239$ as calculated in (4.29). However, the velocity c and the frequency ω cannot be determined by the numerical results precisely without additional effort. Moreover, we see that the TOF leaves the domain of computation after a certain time period and vanishes. In order to avoid this and to compute the velocities precisely, we apply the freezing method to this example in the next section.

TOFs can be observed in a large set of parameters for (QCGL). As a second example of a TOF for different parameter we refer to Figure 0.2. In addition, we consider the Ginzburg-Landau equation with an extra septic term, see [20], reading as

$$U_t = \alpha U_{xx} + \mu U + \beta|U|^2U + \gamma|U|^4U + \delta|U|^6U \quad x \in \mathbb{R},\, t > 0. \tag{4.31}$$

In this case the nonlinearity g is a cubic polynomial in $|u|^2$. The corresponding real-valued system reads as (0.4) with the nonlinearity

$$g(|u|^2) = \begin{pmatrix} g_1(|u|^2) & -g_2(|u|^2) \\ g_2(|u|^2) & g_1(|u|^2) \end{pmatrix}, \quad g_i(|u|^2) = \mu_i + \beta_i|u|^2 + \gamma_i|u|^4 + \delta_i|u|^6, \quad i = 1,2. \tag{4.32}$$

We choose the parameter set

$$\alpha = 1 + \frac{i}{2}, \quad \mu = -\frac{1}{10}(1-i), \quad \beta = 1+i, \quad \gamma = 1+i, \quad \delta = -1+i. \tag{4.33}$$

In this case we have $|v_\infty| \approx 1.2608$ and $\omega \approx 8.2323$. The numerical results are shown in Figure 4.2. We see that in this equation TOFs occur as well and expect them to be stable. In particular, the experiments in this section show the existence and the stability properties of TOFs from in Chapter 3.

4.3.2 Freezing traveling oscillating fronts

Now we apply the freezing method to the first example from Section 4.3.1, cf. (0.4), (4.27), with parameter (4.28) and compute the profile and velocities of a TOF numerically. For this purpose, recall the freezing system from Section 4.1 with the fixed phase

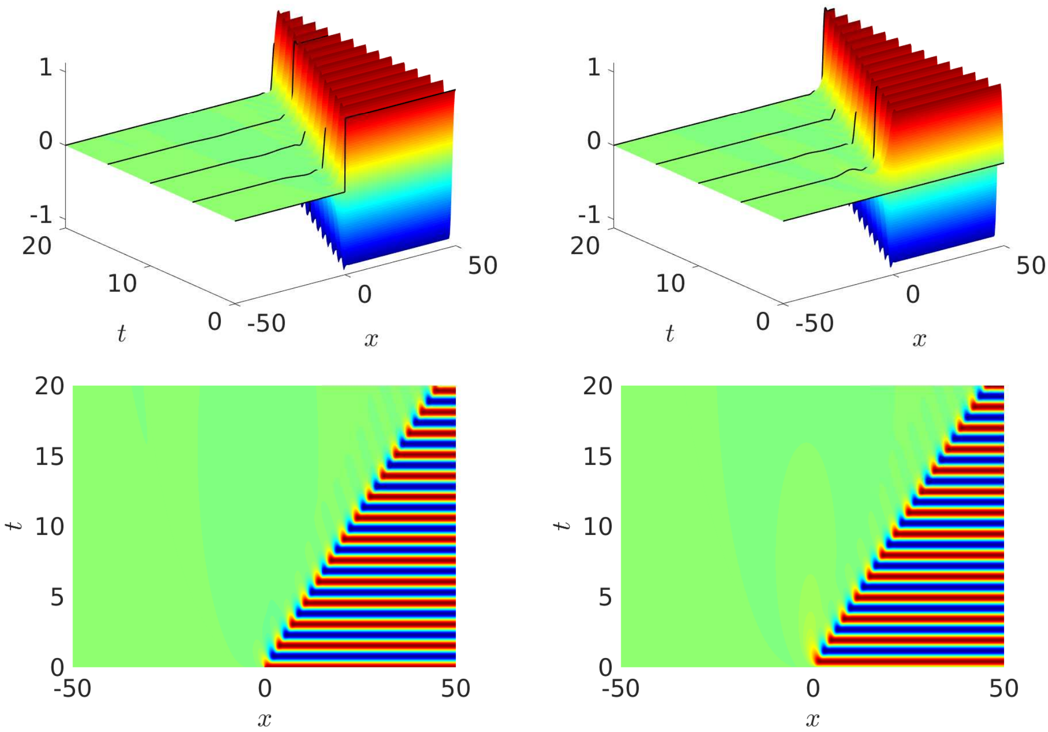


Figure 4.2: Numerical simulation of a TOF in (4.31) with parameters from (4.33). Real part (left), imaginary part (right).

condition (4.7) reading as

$$\begin{aligned} \mathbf{v}_t &= \begin{pmatrix} Av_{xx} + \nu_2 v_x + \nu_1 S_1 v + f(v) \\ \nu_1 S_1 \rho + f(\rho) \end{pmatrix}, \quad \begin{pmatrix} v(0) \\ \rho(0) \end{pmatrix} = \mathbf{u}_0, \\ 0 &= \Psi_{\text{fix}}(\hat{\mathbf{w}} - \mathbf{v}), \\ \gamma_t &= dL_\gamma(\mathbb{1})d\chi(\mathbb{1})^{-1}\nu, \quad \gamma(0) = \mathbb{1} \end{aligned}$$

with initial value $\mathbf{u}_0 = (u_0, \rho_0)^\top$. To compute the solution of (4.7a), (4.7b) we use a finite difference discretization in space and for time integration we use the following algorithm to compute the solution $(\mathbf{v}^{n+1}, \nu^{n+1})$ at the next time step $t_{n+1} = t_n + \Delta t$ with $\mathbf{v}^{n+1} = (v^{n+1}, \rho^{n+1})^\top$ from a current state $(\mathbf{v}^n, \nu^n)$ at time t_n:

1. Given a solution $(\mathbf{v}^n, \nu^n)$ with $\mathbf{v}^n = (v^n, \rho^n)^\top$ of (4.7) at time t_n.

2. Compute v^{n+1} as an implicit Euler step of the finite difference discretization of the equation $v_t = Av_{xx} + \nu_1 v_x + \nu_2 S_1 v + f(v)$ with $\nu = \nu^n$ and initial value v^n on a truncated domain $\Omega = [-L, L]$ and step size Δt.

3. Set $\rho^{n+1} = v^{n+1}(L)$.

4. Compute ν^{n+1} from the linear system (4.10) with $(v, \rho) = (v^{n+1}, \rho^{n+1})$.

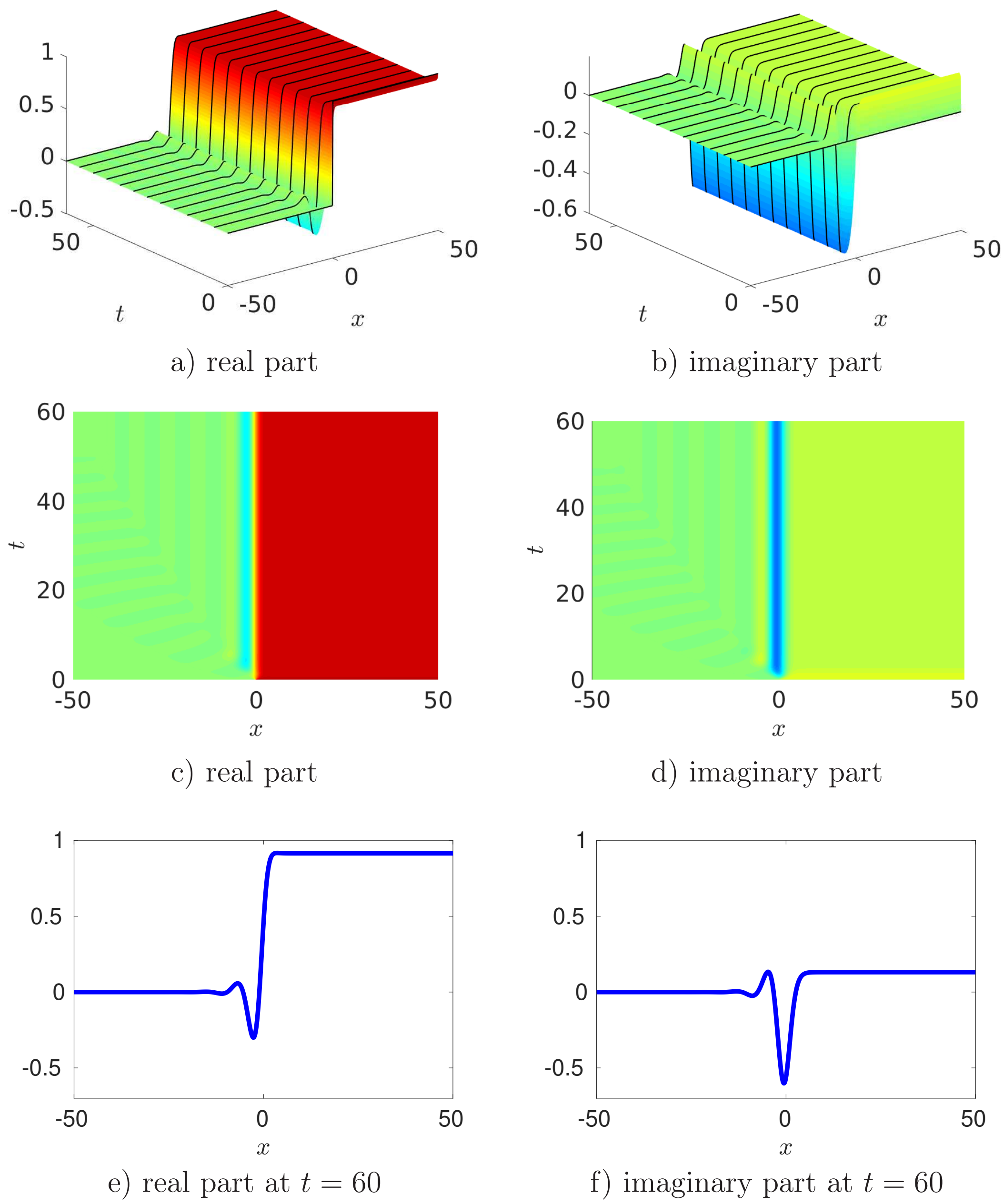
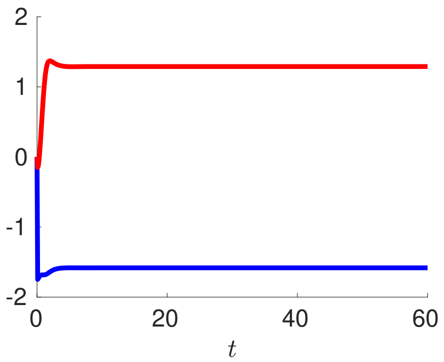


Figure 4.3: Numerical simulation of the freezing method in (QCGL) with parameters from (4.28).



g) frequency ω (blue) and speed c (red)

Figure 4.4: Frequency ω (blue) and velocity c (red) in the numerical simulation of the freezing method in (QCGL) with parameters from (4.28).

We apply this algorithm to (QCGL) with the parameter set (4.28) from the previous Section 4.3.1 and initial value $u_0(x) = (\frac{1}{2}\tanh(1000x) + \frac{1}{2}, 0)^\top$. The results are shown in Figure 4.3 and Figure 4.4. The spatial step size is chosen to be $\Delta x = 0.1$ and the domain of computation is $\Omega = [-50, 50]$. For the time discetization we use $\Delta t = 0.1$ and the implicit equations are solved with Newton's methods using a tolerance of 10^{-5}. We see that the numerical solution of the freezing method converge to the profile of the TOF as Theorem 4.10 guarantees. In addition, the variable ν converges to the frequency and velocity of the TOF which are numerically given by $\omega \approx -1.5821$ and $c \approx 1.29$ respectively. Note that the numerical value of the frequency coincides with the a-priori calculated value in (4.30). In particular, the freezing method is a powerful tool to compute the profile of a TOF as well as its frequency and velocity. Moreover, it enables us to pursue the TOF for arbitrary times without increasing the computational domain and therefore the numerical effort.

4.3.3 Numerical spectrum

We conclude Chapter 4 by computing the spectrum of the linearized operator $\mathcal{L}$ from (0.26) numerically and discuss the geometric shape of its essential spectrum in applications. In particular, we verify numerically Assumption 3 and 4 which guarantee nonlinear stability with asymptotic phase by Theorem 1.11.

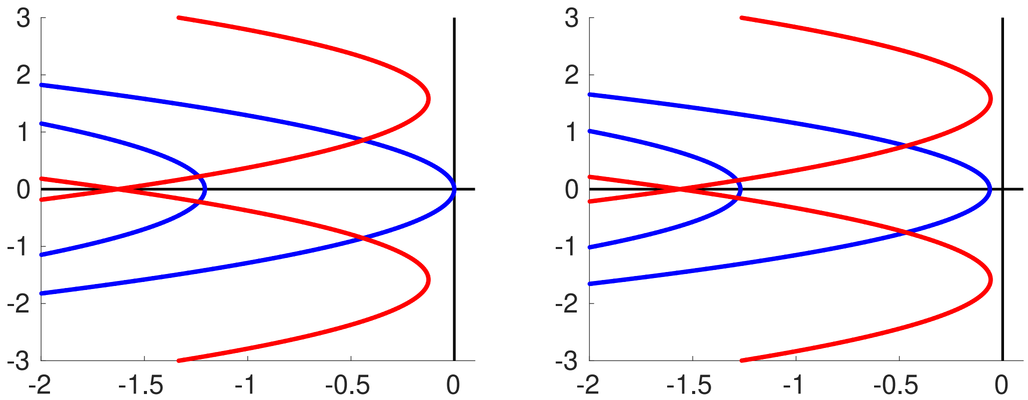
Let us first consider the essential spectrum and the dispersion set $\sigma_{\mathrm{disp},\mu}(\mathcal{L})$ from (1.12), (1.13). We want to verify Assumption 4 in applications for (QCGL) with the parameter set from (4.28). One can discuss the geometry of the dispersion curves analytically, cf. Section 3.3. However, we are interested in a numerical visualization of the curves. In the case of (QCGL) with (4.28) the dispersion set consists of

$$\sigma^+_{\mathrm{disp},\tilde{\mu}}(\mathcal{L}) = \big\{s \in \mathbb{C} : \\ s = -\nu^2 + i(c+2\tilde{\mu})\nu + \tilde{\mu}^2 - c\tilde{\mu} + g_1'(|v_\infty|^2)|v_\infty|^2 \pm |g_1'(|v_\infty|^2)|v_\infty|^2|\big\}$$

and

$$\sigma^-_{\mathrm{disp},\tilde{\mu}}(\mathcal{L}) = \big\{s \in \mathbb{C} : s = -\nu^2 + i(c+2\tilde{\mu})\nu + \tilde{\mu}^2 + c\tilde{\mu} + \mu_1 \pm i\omega\big\}.$$

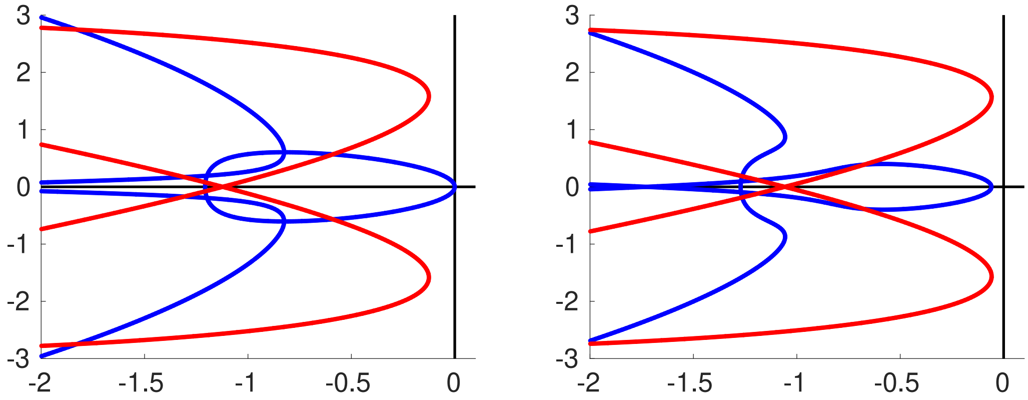
It is easy to see that the dispersion set describes four parabolas in the complex plane opened to the left. Since we have $c > 0$, $\mu_1 < 0$ and $g_1'(|v_\infty|^2) = \beta_1 + 2\gamma_1|v_\infty|^2 = 1 - 2|v_\infty|^2 < 0$, Assumption 4 is satisfied. In particular, in Figure 4.5 the parabolas are shown in the cases $\tilde{\mu} = 0$ and $\tilde{\mu} = 0.05$. In the latter case, we see that the dispersion curves are included in the strict left half-plane.



a) $\tilde{\mu} = 0$ b) $\tilde{\mu} = 0.05$

Figure 4.5: The dispersion sets $\sigma^+_{\mathrm{disp},\tilde{\mu}}(\mathcal{L})$ (blue) and $\sigma^-_{\mathrm{disp},\tilde{\mu}}(\mathcal{L})$ (red) in (QCGL) with (4.28).

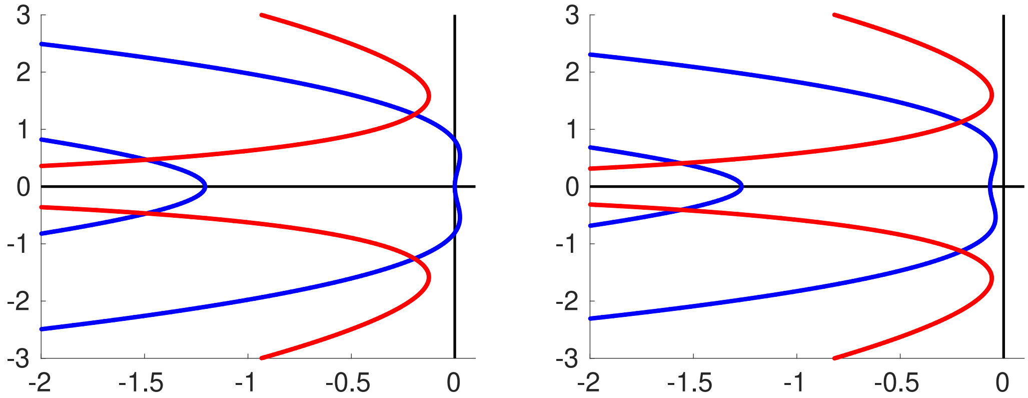
In the example (4.28) the dispersion curves are given by four parabolas, since the imaginary part of the diffusion coefficient vanishes. From (1.12) and (1.13) we see that the dispersion curves can be much more complicated if there is a non-vanishing imaginary part of the diffusion coefficient, i.e. $\alpha_2 \neq 0$. As an example for this case we use the parameters (4.28) but set $\alpha_2 = -\frac{3}{10}$. The dispersion curves in this case are shown in Figure 4.6, again for the exponential growth rates $\tilde{\mu} = 0$ and $\tilde{\mu} = 0.05$. Also in this case the essential spectrum is included in the strict left half-plane if the exponential growth rate is chosen to be $\tilde{\mu} = 0.05$. This strongly depends on the magnitude and sign of the imaginary part of the diffusion coefficient. It might happen that the curve $\sigma_{\mathrm{disp},\mu}(\mathcal{L})$ forms a dovetail due to the fourth order terms, see Figure 4.7. Then Assumption 4 is violated. But as Figure 4.7 shows there also might be an exponential growth rate such that the



a) $\tilde{\mu} = 0$ b) $\tilde{\mu} = 0.05$

Figure 4.6: The dispersion sets $\sigma^{+}_{\mathrm{disp},\tilde{\mu}}(\mathcal{L})$ (blue) and $\sigma^{-}_{\mathrm{disp},\tilde{\mu}}(\mathcal{L})$ (red) in (QCGL) with (4.28) but $\alpha = 1 - \frac{3}{10}i$.

dispersion set is still included in the left half-plane. We expect that our stability results also apply in this case. However, one has to be careful using the exponential growth rates μ.



a) $\tilde{\mu} = 0$ b) $\tilde{\mu} = 0.05$

Figure 4.7: The dispersion sets $\sigma^{+}_{\mathrm{disp},\tilde{\mu}}(\mathcal{L})$ (blue) and $\sigma^{-}_{\mathrm{disp},\tilde{\mu}}(\mathcal{L})$ (red) in (QCGL) with (4.28) but $\alpha = 1 + \frac{3}{10}i$.

Now let us consider the point spectrum of the linearized operator $\mathcal{L}$ and verify Assumption 3. In Section 3.3 we have shown that for the essential spectrum it holds $\sigma_{\mathrm{ess}}(\mathcal{L}) = \sigma_{\mathrm{ess}}(L)$ for the operators $\mathcal{L} \in \mathcal{C}[X_\eta]$ and $L \in \mathcal{C}[L^2_\eta]$. As we will see, a similar relation holds true for the point spectrum. For this purpose, recall $\mathcal{L}$ defined by (0.26) reading as

$$\mathcal{L}: Y_\eta \to X_\eta, \quad \begin{pmatrix} u \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} Au_{xx} + cu_x + S_\omega u + Df(v_\star)u \\ S_\omega \rho Df(v_\infty)\rho \end{pmatrix}$$

and L from (0.12) reading as

$$L: H^2_\eta \to L^2_\eta, \quad u \mapsto Au_{xx} + cu_x + S_\omega u + Df(v_\star)u.$$

It follows that if $s \in \mathbb{C}$ is an eigenvalue of L with eigenfunction $u_0 \in H^2_\eta$ then s is also an eigenvalue of $\mathcal{L}$ with eigenfunction $\mathbf{u_0} = (u_0, 0)^\top \in Y_\eta$, i.e. $\sigma_{\mathrm{pt}}(L) \subset \sigma_{\mathrm{pt}}(\mathcal{L})$. Conversely, let $s \in \mathbb{C}$ be an eigenvalue of $\mathcal{L}$ with eigenfunction $\mathbf{u_0} = (u_0, \rho_0)^\top$. If $\rho_0 = 0$ then s is also an eigenvalue of L with eigenfunction u_0. Now assume $\rho_0 \neq 0$. Then we have $(sI - S_\omega - Df(v_\infty))\rho_0 = 0$ and thus either $s = 2g_1'(|v_\infty|^2)|v_\infty|^2$ or $s = 0$. But $0 \in \sigma(L)$ and we obtain

$$\sigma_{\mathrm{pt}}(\mathcal{L}) \subset \sigma_{\mathrm{pt}}(L) \cup \{2g_1'(|v_\infty|^2)|v_\infty|^2\}.$$

The possible eigenvalue $2g_1'(|v_\infty|^2)|v_\infty|^2$ is of no interest since it is included in the strict left half-plane by Assumption 2. So neglecting this additional eigenvalue it is sufficient to compute the spectrum of the operator L instead of the spectrum of the operator $\mathcal{L}$ to verify Assumption 4. Moreover, we have

$$\sigma(\mathcal{L}) \subset \sigma(L) \cup \{2g_1'(|v_\infty|^2)|v_\infty|^2\}.$$

Now if $s \in \sigma_{\mathrm{pt}}(L)$ for $L \in \mathcal{C}[L^2_\eta]$ with eigenfunction $u_0 \in Y_\eta$, then $u_0 \in Y$ and $s \in \sigma_{\mathrm{pt}}(L)$ for $L \in \mathcal{C}[L^2]$. In particular, the point spectrum does not move by taking exponential weights into account. More precisely, if s belongs to the point spectrum of L on L^2 then s belongs to the point spectrum of L on L^2_η, unless the essential spectrum has moved to encompass s, cf. [36, Sec. 3.1.1.2]. Thus to verify that the point spectrum of L, respectively $\mathcal{L}$, is included in the strict left half-plane it is sufficient to compute the spectrum on L^2 instead of L^2_η.
Finally, let us compute the point spectrum of L, respectively $\mathcal{L}$, numerically in the case of (QCGL) with (4.28). In order to do so, we use a finite difference approximation of L on the truncated domain $\Omega = [-1000, 1000]$ with spatial step size $\Delta x = 0.1$ and periodic boundary conditions. The numerical results are shown in Figure 4.8 and Figure 4.9.

We see that there are no eigenvalues in the right half-plane or on the imaginary axis expect for the zero eigenvalue. Moreover, there are even no eigenvalues in left half-plane. Therefore we expect the point spectrum of $\mathcal{L}$ on X to be empty. The isolated eigenvalues between the dispersion curves belong to the essential spectrum since in these regions the operator $sI - \mathcal{L}$ is not Fredholm of index 0. This can be seen by computing the corresponding Morse indices, cf. [36] and Figure 3.3. In particular, the dispersion curves from the numerical spectrum do not fit exactly to the dispersion set calculated in

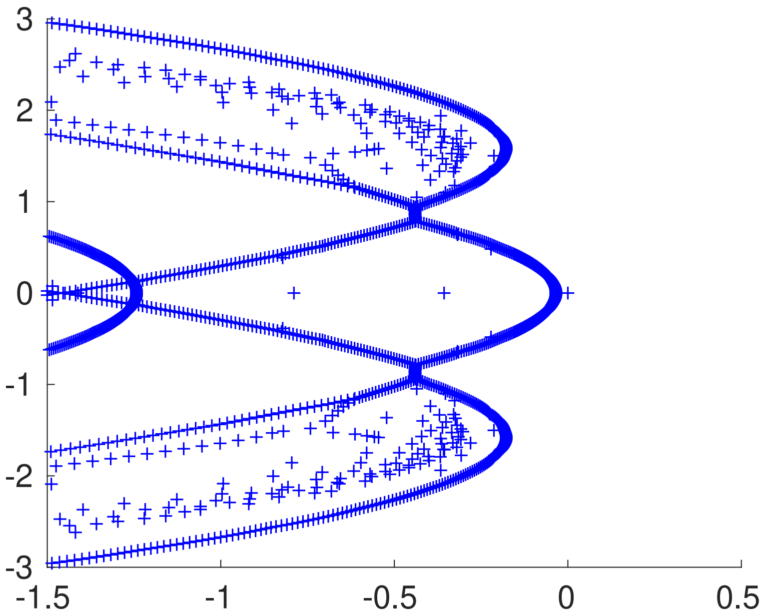


Figure 4.8: Numerical spectrum of the linearized operator.

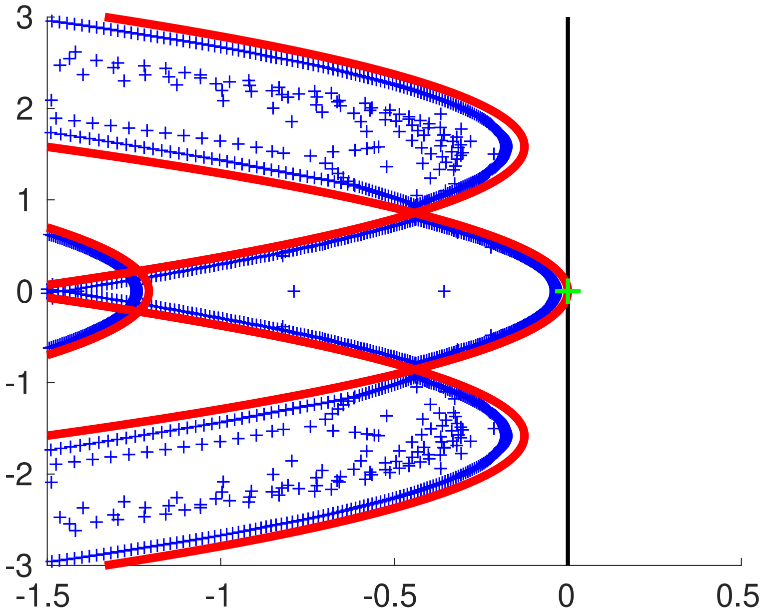


Figure 4.9: Numerical spectrum of the linearized operator with dispersion set (red) and zero eigenvalue (green).

(1.12) and (1.13). The reason for this is that we approximate the operator $\mathcal{L}$ by an finite difference approximation on a large, but bounded, domain. As a result the parabolas from the dispersion set are approximated by ellipses depending on the size of the domain of computation. This can be seen by considering the whole spectrum of the linearized operator on a truncated domain, see Figure 4.10.

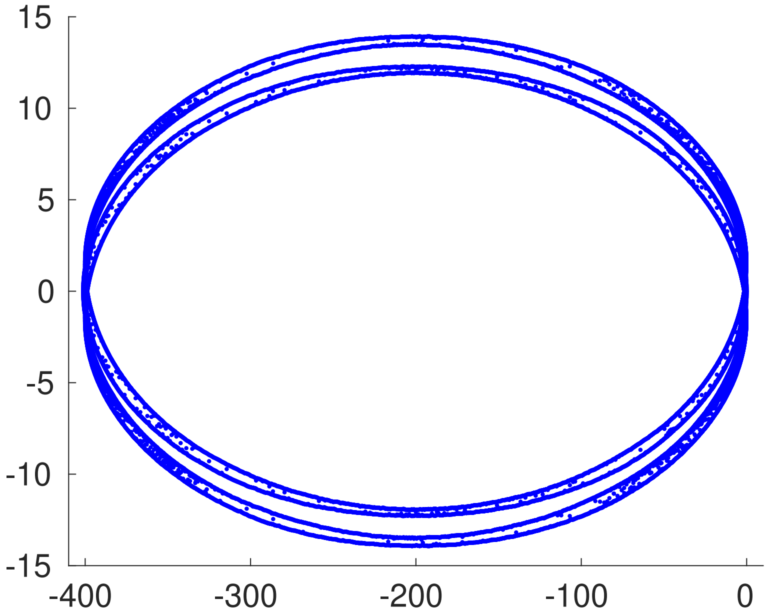


Figure 4.10: Whole numerical spectrum of the linearized operator on the truncated domain with periodic boundary conditions.

Chapter 5

Stability in polynomially weighted spaces

In Chapter 3 we proved a nonlinear stability result for TOFs when the perturbation u_0 of the initial data $u(0) = v_\star + u_0$ converges exponentially fast to some limit r_∞ at $+\infty$ and to zero at $-\infty$. A natural question arises whether the assumption on the initial perturbation can be weakened from exponential to polynomial decay. In this chapter we prove a nonlinear stability result for polynomially decaying initial perturbations, see Theorem 1.13. In this case we have to assume $r_\infty = 0$, i.e. $u_0 \in H^1_\eta$ with a polynomial weight function η.
Throughout the chapter we set $\eta = \eta_{\mathrm{poly}}$ with the polynomial weight function from (0.29) reading as

$$\eta_{\mathrm{poly}}(x) = (x^2+1)^{\frac{1}{2}}.$$

and consider the weighted spaces L^2_k, H^ℓ_k from (0.30). We assume the existence of a TOF with profile $v_\star$ and speeds (ω, c) and consider the perturbed co-moving equation from (0.11)

$$u_t = Au_{xx} + cu_x + S_\omega u + f(u), \quad u(0) = v_\star + u_0$$

with an initial perturbation $u_0 \in H^2_k$ for some $k \in \mathbb{N}$. Since $u_0 \to 0$ at $\pm\infty$ we expect that the limit at $+\infty$ of the solution u of (0.11) stays constant in the time evolution, cf. (0.18). In particular, a TOF with frequency ω can be seen as a traveling wave solution of the co-rotated equation

$$u_t = Au_{xx} + S_\omega u + f(u).$$

For that reason we seek for solutions of (0.11) in the affine linear spaces, cf. (0.31)

$$M_k = \bar{v} + L^2_k, \quad M^\ell_k = \bar{v} + H^\ell_k, \quad \bar{v} = v_\infty \hat{v}.$$

5.1 Polynomially weighted Sobolev spaces

Before investigating the nonlinear stability we collect some properties concerning the weighted spaces L^2_k, H^ℓ_k from (0.30). The function $\eta = \eta_{\mathrm{poly}} \in C^\infty(\mathbb{R}, \mathbb{R})$ from (0.29) is a function of linear growth, i.e. $\eta(x) \sim |x|$. More precisely, for $|x| > 1$, $k \in \mathbb{N}_0$ the following estimates hold true

$$|x|^k \leq \eta(x)^k \leq 2^{\frac{k}{2}}|x|^k.$$

Furthermore, we note the first and second derivative

$$\eta_x(x) = x(x^2+1)^{-\frac{1}{2}}, \quad \eta_{xx}(x) = (x^2+1)^{-\frac{3}{2}}.$$

Then $|\eta_x(x)|, |\eta_{xx}(x)| \leq 1$, $x \in \mathbb{R}$ and the function spaces L^2_k, H^ℓ_k are Hilbert spaces with the inner products

$$(u,v)_{L^2_k} = (\eta^k u, \eta^k v)_{L^2}, \quad (u,v)_{H^\ell_k} = (\eta^k u, \eta^k v)_{H^\ell}.$$

Moreover, H^ℓ_k is dense in L^2_k and since $\eta^k(x) \leq \eta^\ell(x)$ for all $x \in \mathbb{R}$ as long as $k \leq \ell$, it follows immediately that $L^2_\ell \subset L^2_k$ and the inclusion is dense as well. As in Chapter 3 we consider the multiplication operator $m_k u = \eta^k u$ for $u \in H^\ell_k$, $k \in \mathbb{N}_0$, $\ell = 1, 2$. Similarly to Lemma 3.1 we have that m_k defines a continuous isomorphism from H^ℓ_k to H^ℓ.

Lemma 5.1. *Let $k \in \mathbb{N}_0$ and $m_k u = \eta^k u$ define the multiplication operator associates with η^k. Then*

i) $m_k : L^2_k \to L^2$ is an isometric isomorphism.

ii) $m_k : H^\ell_k \to H^\ell$, $\ell = 1, 2$ is a continuous isomorphism.

Remark 5.2. *It also holds true that $m_k : H^\ell_k \to H^\ell$ is a continuous isomorphism for arbitrary $\ell \in \mathbb{N}$. However, the proof is more involved and we are only interested in the cases $\ell = 0, 1, 2$ as in Lemma 5.1.*

Proof. We show that $m_k : H^\ell_k \to H^\ell$, $\ell = 1, 2$ is continuous. Then the claim follows as in the proof of Lemma 3.1. First let $u \in H^1_k$. Then

$$\|(\eta^k u)_x\|_{L^2} = \|k\eta^{k-1}\eta_x u + \eta^k u_x\|_{L^2} \leq k\|\eta^{k-1}u\|_{L^2} + \|\eta^k u_x\|_{L^2} \leq (k+1)\|u\|_{H^1_k}.$$

Thus,

$$\|\eta^k u\|^2_{H^1} = \|u\|^2_{L^2_k} + \|\partial(\eta^k u)\|^2_{L^2} \leq (k^2+2k+2)\|u\|^2_{H^1_k}.$$

For $u \in H^2_k$ we have

$$\begin{aligned}\|(\eta u)_{xx}\|_{L^2} &= \|k(k-1)\eta^{k-2}\eta_x^2 u + k\eta^{k-1}\eta_{xx}u + 2k\eta^{k-1}\eta_x u_x + \eta^k u_{xx}\|_{L^2} \\ &\leq (k(k-1)+k)\|u\|_{L^2_k} + 2k\|u_x\|_{L^2_k} + \|u_{xx}\|_{L^2_k} \leq (k+1)^2\|u\|_{H^2_k}.\end{aligned}$$

$\square$

To show resolvent estimates of the linearized operator later on we need a integration by parts formula in L^2_k which is slightly different from the standard integration by parts formula in L^2.

Lemma 5.3. *Suppose $u, v \in H^1_k(\mathbb{R}, \mathbb{R}^n)$. Then there holds the following integration by parts formula*

$$-(u, v_x)_{L^2_k} = (u_x, v)_{L^2_k} + 2k(\eta^{-1}\eta_x u, v)_{L^2_k}.$$

Moreover, if $u, v \in H^2_k(\mathbb{R}, \mathbb{R}^n)$, then there holds

$$(u, v_{xx})_{L^2_k} = (u_{xx}, v)_{L^2_k} + 4k(\eta^{-1}\eta_x u_x, v)_{L^2_k} + (4k^2 - 2k)(\eta^{-2}\eta_x^2 u, v)_{L^2_k} + 2k(\eta^{-1}\eta_{xx} u, v)_{L^2_k}.$$

Proof. We write $(\cdot, \cdot)_{L^2_k} = (\cdot, \cdot)$. The assertion is a direct consequence of the standard integration by parts formula in L^2, since

$$\begin{aligned}
-(u, v_x) &= -\int_{\mathbb{R}} \eta^{2k}(x) u(x)^\top v_x(x) dx = \int_{\mathbb{R}} \partial_x(\eta^{2k}(x) u(x))^\top v(x) dx \\
&= \int_{\mathbb{R}} \eta^{2k}(x) u_x(x)^\top v(x) dx + 2k \int_{\mathbb{R}} \eta^{2k-1}(x) \eta_x(x) u(x)^\top v(x) dx \\
&= (u_x, v) + 2k(\eta^{-1}\eta_x u, v).
\end{aligned}$$

The second formula follows by applying integration by parts in L^2_η twice. We obtain

$$\begin{aligned}
(u, v_{xx}) &= -(u_x, v_x) - 2k(\eta^{-1}\eta_x u, v_x) \\
&= (u_{xx}, v) + 2k(\eta^{-1}\eta_x u_x, v) + 2k[(2k-1)(\eta^{-2}\eta_x^2 u, v) + (\eta^{-1}\eta_{xx} u, v) + (\eta^{-1}\eta_x u_x, v)] \\
&= (u_{xx}, v) + 4k(\eta^{-1}\eta_x u_x, v) + (4k^2 - 2k)(\eta^{-2}\eta_x^2 u, v) + 2k(\eta^{-1}\eta_{xx} u, v).
\end{aligned}$$

□

In Lemma 3.3 we have proven that the space $C_0^\infty(\mathbb{R}, \mathbb{R}^n)$ of smooth function with compact support are dense in $H^1_\eta(\mathbb{R}, \mathbb{R}^n)$ if η is an exponential weight function as in (0.24). However, the proof of Lemma 3.3 is independent of the choice of the weight function η and we conclude that $C_0^\infty(\mathbb{R}, \mathbb{R}^n)$ is dense in $H^1_k(\mathbb{R}, \mathbb{R}^n)$. Since we look for traveling waves in the space L^2_k we have to collect some smoothness properties of the shift $u \mapsto u(\cdot - \tau)$, $\tau \in \mathbb{R}$ on the polynomially weighted spaces. As in the exponential case, cf. Lemma 3.4, it turns out that the shift is continuous on L^2_k and locally Lipschitz continuous on H^1_k.

Lemma 5.4. *Suppose $k \in \mathbb{N}_0$.*

i) If $u \in L^2_k$ and $\tau \in \mathbb{R}$, then

$$\|u(\cdot - \tau)\|_{L^2_k} \leq C_\tau^k \|u\|_{L^2_k}, \quad C_\tau := 1 + |\tau|.$$

ii) If $u \in H^1_k$ and $\tau \in \mathbb{R}$, then

$$\|u(\cdot - \tau) - u\|_{L^2_k} \leq C^k_\tau |\tau| \|u_x\|_{L^2_k}.$$

iii) If $u \in L^2_k$, then

$$\|u(\cdot - \tau) - u\|_{L^2_k} \to 0 \quad \text{as} \quad \tau \to 0.$$

Further, the estimate in ii) holds true if u is replaced by $\hat{v}$ from (0.19) *or $v_\star$ from Assumption 2.*

Proof. The case $k = 0$ is the usual case in L^2. Thus, let $k \geq 1$. We only show i) then ii) and iii) follow exactly as in the proof of Lemma 3.4.
First note for all $x, \tau \in \mathbb{R}$

$$\frac{\eta(x+\tau)^2}{\eta(x)^2} = \frac{x^2 + 2\tau x + \tau^2 + 1}{x^2 + 1} \leq 1 + 2|\tau| \frac{|x|}{x^2+1} + \frac{\tau^2}{x^2+1} \leq 1 + 2|\tau| + \tau^2 = (1 + |\tau|)^2.$$

Now we obtain

$$\|u(\cdot - \tau)\|^2_{L^2_k} = \int_{\mathbb{R}} \eta^{2k}(x+\tau)|u(x)|^2 dx \leq (1+|\tau|)^{2k} \int_{\mathbb{R}} \eta^{2k}(x)|u(x)|^2 dx = C^{2k}_\tau \|u\|^2_{L^2_k}.$$

□

5.2 Group action and equivariance

We consider the group $\mathbb{R}$ with the canonical composition

$$\circ : \mathbb{R} \times \mathbb{R} \to G, \quad (\tau_1, \tau_2) \mapsto \tau_1 + \tau_2.$$

Here our Lie group is simply the additive group $\mathbb{R}$. In contrast to Chapter 3 its structure as a manifold is trivial since it is a linear space. Therefore we do not have to work in charts since their definition is trivial. We let $\mathbb{R}$ act on the affine Hilbert space M_k via the shift

$$\mathfrak{a}(\tau) : M_k \to M_k, \quad v \mapsto \mathfrak{a}(\tau)v := v(\cdot - \tau). \tag{5.1}$$

We continue in the same fashion as in Chapter 3 and study the smoothness properties of this action.

Lemma 5.5. *For $k \in \mathbb{N}_0$, $\ell = 0,1,2$, and $v \in M_k^\ell$ the group action*

$$\mathfrak{a}(\cdot)v : \mathbb{R} \to M_k^\ell, \quad \tau \mapsto v(\cdot - \tau)$$

is continuous. If $v \in M_k^1$ then the group action $\mathfrak{a}(\cdot)v : \mathbb{R} \to M_k$ is of class C^1 and its derivative has the local representation

$$d[\mathfrak{a}(\tau)v)] : \mathbb{R} \to L_k^2, \quad h \mapsto -v_x(\cdot - \tau)h. \tag{5.2}$$

Proof. Recall $\bar{v} = v_\infty \hat{v}$ from (0.31). Then using Lemma 5.4 and the chart (1.17) we obtain for $v \in M_k^\ell$

$$\begin{aligned}
\|\mathfrak{a}(\tau)v - \bar{v}\|_{M_k^\ell} &\leq \|v(\cdot - \tau) - \bar{v}\|_{L_k^2} + \sum_{i=1}^{\ell} \|\partial^i v(\cdot - \tau)\|_{L_k^2} \\
&\leq \|(v - \bar{v})(\cdot - \tau)\|_{L_k^2} + |v_\infty| \|\hat{v}(\cdot - \tau) - \hat{v}\|_{L_k^2} + \sum_{i=1}^{\ell} \|\partial^i v(\cdot - \tau)\|_{L_k^2} \\
&\leq C_\tau^k \Big(\|v - \bar{v}\|_{L_k^2} + |\tau| \|\hat{v}_x\|_{L_k^2} |v_\infty| + \sum_{i=1}^{\ell} \|\partial^i v\|_{L_k^2} \Big) < \infty.
\end{aligned}$$

Hence $\mathfrak{a}(\cdot)v$ maps $\mathbb{R}$ into M_k^ℓ. Similarly, by Lemma 5.4 $\mathfrak{a}(\cdot)v$ is continuous since

$$\begin{aligned}
\|\mathfrak{a}(\tau)v - v\|_{H_k^\ell} = \|v(\cdot - \tau) - v\|_{H_k^\ell} &\leq \|(v - \bar{v})(\cdot - \tau) - (v - \bar{v})\|_{L_k^2} \\
&+ |v_\infty| \|\hat{v}(\cdot - \tau) - \hat{v}\|_{L_k^2} + \sum_{i=1}^{\ell} \|\partial^i v(\cdot - \tau)\|_{L_k^2} \to 0, \quad \tau \to 0.
\end{aligned}$$

It remains to show that $\mathfrak{a}(\cdot)v$ is of class C^1 if $v \in M_k^1$. As in the proof of Lemma 3.7, cf. (3.17), one shows for $u \in H_k^1$

$$\|u(\cdot - h) - u - hu_x\|_{L_k^2} = o(|h|)$$

as $h \to 0$ as well as

$$\|\hat{v}(\cdot - h) - \hat{v} - h\hat{v}_x\|_{L_k^2} = o(|h|).$$

Then we conclude for $v \in M_k^1$ using the chart (M_k^1, χ) from (1.17)

$$\begin{aligned}
&\|\chi(\mathfrak{a}(\tau + h)v) - \chi(\mathfrak{a}(\tau)v) - hv_x(\cdot - \tau)\|_{L_k^2} \\
&= \|v(\cdot - \tau - h) - v(\cdot - \tau) - hv_x(\cdot - \tau)\|_{L_k^2} \leq C_\tau^k \|v(\cdot - h) - v - hv_x\|_{L_k^2} \\
&\leq C_\tau^k \|(v - \bar{v})(\cdot - h) - (v - \bar{v}) - h(v - \bar{v})_x\|_{L_k^2} + C_\tau^k |v_\infty| \|\hat{v}(\cdot - h) - \hat{v} - h\hat{v}_x\|_{L_k^2} = o(|h|).
\end{aligned}$$

This proves $\mathfrak{a}(\cdot)v$ to be of class C^1 and its derivative has the local representation in the chart (M_k^1, χ) given by (5.2). □

The right hand side of (0.11) is given by the nonlinear operator

$$F : M_k^2 \to L_k^2, \quad u \mapsto Au_{xx} + cu_x + S_\omega u + f(u).$$

As a next step we show that F is well-defined, i.e. maps M_k^2 into L_k^2, and is continuous.

Lemma 5.6. *Let Assumption 1, 2 be satisfied and $k \in \mathbb{N}_0$. Then $F : M_k^2 \to L_k^2$ from (0.23) defines a continuous operator.*

Proof. Let $u, v \in M_k^2$. Then by Sobolev embedding, cf. Theorem D.2, we have $u, v \in L^\infty$. Since $f \in C^3$ this yields, using the mean value theorem, for some $K > 0$

$$\begin{aligned}\|f(u) - f(v)\|_{L_k^2}^2 &\leq \int_{\mathbb{R}} \eta^{2k}(x) \Big| \int_0^1 f'(u(x) + \tau(u(x) - v(x))(u(x) - v(x)) d\tau \Big|^2 dx \\ &\leq K \max(\|u\|_{L^\infty}^2, \|v\|_{L^\infty}^2) \|u - v\|_{L_k^2}^2.\end{aligned}$$

Next, we note from Lemma 1.6 the equality $f(v_\infty) = g(|v_\infty|^2)v_\infty = -S_\omega v_\infty$ and from Proposition 2.7 that $\|\hat{v}\|_{L_k^2(\mathbb{R}_-)}, \|\hat{v} - 1\|_{L_k^2(\mathbb{R}_+)} < \infty$. Then use $f \in C^3$ from Assumption 1 and $\bar{v} = v_\infty \hat{v}$ to obtain for $u \in M_k^2$ and some $C > 0$

$$\begin{aligned}&\|S_\omega u + f(u)\|_{L_k^2} \leq \|S_\omega(u - \bar{v})\|_{L_k^2} + \|f(u) - f(\bar{v})\|_{L_k^2} + \|S_\omega \bar{v} - f(\bar{v})\|_{L_k^2} \\ &\leq |\omega| \|u - \bar{v}\|_{L_k^2} + C\|u - \bar{v}\|_{L_k^2} + \|S_\omega \bar{v}\|_{L_k^2(\mathbb{R}_-)} + \|f(\bar{v}) - f(0)\|_{L_k^2(\mathbb{R}_-)} \\ &\qquad + \|S_\omega \bar{v} - f(\bar{v})\|_{L_k^2(\mathbb{R}_+)} \\ &\leq (|\omega| + C)(\|u - \bar{v}\|_{L_k^2} + |v_\infty| \|\hat{v}\|_{L_k^2(\mathbb{R}_-)}) + \|S_\omega(\bar{v} - v_\infty) + f(\bar{v}) - f(v_\infty)\|_{L_k^2(\mathbb{R}_+)} \\ &\leq (|\omega| + C)(\|u - \bar{v}\|_{L_k^2} + |v_\infty| \|\hat{v}\|_{L_k^2(\mathbb{R}_-)}) + |\omega| |v_\infty| \|\hat{v} - 1\|_{L_k^2(\mathbb{R}_+)} \\ &\qquad + C|v_\infty| \|\hat{v} - 1\|_{L_k^2(\mathbb{R}_+)} < \infty.\end{aligned}$$

Thus, F maps M_k^2 into L_k^2. For the continuity pick $u, v \in M_k^2$. Then

$$\begin{aligned}&\|F(u) - F(v)\|_{L_k^2} = |A| \|u_{xx} - v_{xx}\|_{L_k^2} + |c| \|u_x - v_x\|_{L_k^2} + |\omega| \|u - v\|_{L_k^2} + \|f(u) - f(v)\|_{L_k^2} \\ &\leq |A| \|u_{xx} - v_{xx}\|_{L_k^2} + |c| \|u_x - v_x\|_{L_k^2} + |\omega| \|u - v\|_{L_k^2} + K \max(\|u\|_{L^\infty}^2, \|v\|_{L^\infty}^2) \|u - v\|_{L_k^2}^2 \\ &\to 0, \quad \|u - v\|_{H_k^2} \to 0.\end{aligned}$$

□

5.3 The linearized operator

In this section we discuss the spectral properties of the linearized operator L from (0.12) considered on the polynomially weighted spaces L_k^2, i.e.

$$L : H_k^2 \subset L_k^2 \to L_k^2, \quad u \mapsto Lu = Au_{xx} + cu_x + S_\omega u + Df(v_\star)u. \tag{5.3}$$

The operator occurs when linearizing the right hand side of (0.11) at the TOF given by the profile $v_\star$. In particular, we can write (0.11) as the semilinear parabolic equation

$$u_t = Lu + N(u), \quad u(0) = v_\star + u_0$$

with nonlinear part $N(u) = f(u) - Df(v_\star)u$. As in Chapter 3 one of the main steps is to show existence and time decaying estimates of the semigroup $\{e^{tL}\}_{t>0}$ generated by L. It turns out that the essential spectrum of L touches the imaginary axis, cf. Figure 0.3.

5.3.1 Resolvent estimates for large $|s|$

We are interested in the solution of the resolvent equation

$$(sI - L)u = r, \quad s \in \mathbb{C},\, r \in L^2_k. \tag{5.4}$$

As in the exponential case we start with a-priori estimates for solutions $u \in H^2_k$ of (5.4) for arbitrary $r \in L^2_k$ as long as $|s|$ is sufficiently large and $s \in \mathbb{C}$ lies in the exterior of some sector with an appropriate angle opened to the left, cf. Figure 3.1.

Lemma 5.7. *Let Assumption 1, 2 be satisfied and $k \geq 0$. Then $L : H^2_k \subset L^2_k \to L^2_k$ is a closed, densely defined, linear operator on L^2_k. Moreover, there exist $\varepsilon_0, R_0, C > 0$ such that for all*

$$s \in \Omega_0 := \left\{ s \in \mathbb{C} : |s| \geq R_0, |\arg(s)| \leq \frac{\pi}{2} + \varepsilon_0 \right\}$$

the equation (5.4) with $u \in H^2_k$ and $r \in L^2_k$ implies

$$|s| \|u\|^2_{L^2_k} + \|u_x\|^2_{L^2_k} \leq \frac{C}{|s|} \|r\|^2_{L^2_k} \tag{5.5}$$

$$|s|^2 \|u\|^2_{L^2_k} + |s| \|u_x\|^2_{L^2_k} + \|u_{xx}\|^2_{L^2_k} \leq C \|r\|^2_{L^2_k}. \tag{5.6}$$

Proof. The proof is almost the same as the one of Lemma 3.10. Therefore, we only note the main steps that differ. First of all, note that the closedness of L follows from (5.6) as in the proof of Lemma 3.10. We write $(\cdot,\cdot) = (\cdot,\cdot)_{L^2_k}$ and take the inner product of (5.4) with u in L^2_k to obtain

$$(u, r) = s\|u\|^2_{L^2_k} - (u, Au_{xx}) - c(u, u_x) - (u, S_\omega u) - (u, Df(v_\star)u).$$

Now the integration by parts formula from Lemma 5.3 leads to

$$s\|u\|^2_{L^2_k} + (u_x, Au_x) = -2k(\eta^{-1}\eta_x u, Au_x) + c(u, u_x) + (u, (S_\omega + Df(v_\star)u) + (u, r). \tag{5.7}$$

Now since $|\eta^{-1}(x)\eta_x(x)| \le 1$, $|Df(v_\star(x))| \le K_f$ for all $x \in \mathbb{R}$ and some $K_f > 0$ we obtain using Cauchy-Schwarz and Young's inequality with $\varepsilon_i > 0$, $i = 1, 2$

$$\begin{aligned} &|(u_x, Au_x)| \le |A| \|u_x\|_{L^2_k}^2, \quad |2k(\eta^{-1}\eta_x u, Au_x)| \le \frac{k^2|A|}{\varepsilon_1}\|u\|_{L^2_k}^2 + \varepsilon_1 |A| \|u_x\|_{L^2_k}^2, \\ &|c(u, u_x)| \le \frac{|c|}{4\varepsilon_2}\|u\|_{L^2_k}^2 + \varepsilon_2 |c| \|u_x\|_{L^2_k}^2, \quad (u, (S_\omega + Df(v_\star)u) \le (|\omega| + K_f)\|u\|_{L^2_k}^2. \end{aligned} \tag{5.8}$$

Taking absolute value in (5.7) and using (5.8) with $\varepsilon_i = 1$ yields

$$|s| \|u\|_{L^2_k}^2 \le K_0 \|u_x\|_{L^2_k}^2 + K_1 \|u\|_{L^2_k}^2 + \|u\|_{L^2_k} \|r\|_{L^2_k} \tag{5.9}$$

with $K_0 = 2|A| + |c|$ and $K_1 = k^2|A| + \frac{|c|}{4} + |\omega| + K_f$. Note that $(u_x, Au_x) = \alpha_1 \|u_x\|_{L^2_k}$ holds. Taking real part in (5.7) and using (5.8) with $\varepsilon_1 = \frac{\alpha_1}{4|A|}$ and $\varepsilon_2 = \frac{\alpha_1}{4|c|}$ leads to

$$\operatorname{Re} s \|u\|_{L^2_k}^2 + \alpha_1 \|u_x\|_{L^2_k}^2 \le \frac{\alpha_1}{2} \|u_x\|_{L^2_k}^2 + K_2 \|u\|_{L^2_k}^2 + \|u\|_{L^2_k} \|r\|_{L^2_k}$$

with $K_2 = \frac{4k^2|A|^2}{\alpha_1} + \frac{|c|^2}{\alpha_1} + |\omega| + K_f$. Then we have

$$\operatorname{Re} s \|u\|_{L^2_k}^2 + \frac{\alpha_1}{2} \|u_x\|_{L^2_k}^2 \le K_2 \|u\|_{L^2_k}^2 + \|u\|_{L^2_k} \|r\|_{L^2_k} \tag{5.10}$$

Now the claim follows exactly as in the proof of Lemma 3.10 using the estimates (5.9), (5.10). □

As in Section 3.3 we continue by determining the essential spectrum of L on L^2_k. In particular, we prove $sI - L$ to be Fredholm of index 0 for s to the right of the dispersion set $\sigma_{\text{disp}}(L)$ from (0.14). This is done using the classical results from the spectral theory of second order differential operators, cf. [32], [56], [36]. We proceed in the same fashion as in Section 3.3. We show that $sI - L$ is Fredholm in L^2_k if and only if the operator $sI - L_k$ is Fredholm in L^2 with

$$L_k : H^2 \to L^2, \quad u \mapsto \eta^k L \eta^{-k} u \tag{5.11}$$

and the Fredholm indices coincide. Then L_k has the form of a second order differential operator given by

$$L_k u = A u_{xx} + B_k u_x + C_k u_x$$

with coefficients

$$B_k = -2k\eta^{-1}\eta_x A + cI, \quad C_k = \big((k^2 + k)(\eta^{-1}\eta_x)^2 - k\eta^{-1}\eta_{xx}\big)A - k\eta^{-1}\eta_x cI + S_\omega + Df(v_\star).$$

Since $|\eta^{-1}\eta_x|, |\eta^{-1}\eta_{xx}| \to 0$ as $x \to \pm\infty$ we note their limits

$$B_k(x) \to cI, \quad C_k(x) \to S_\omega + Df(v_\pm), \quad x \to \pm\infty, \quad v_+ = v_\infty, \quad v_- = 0. \tag{5.12}$$

In view of these limits we also consider the piecewise constant coefficient operator

$$L_\infty : H^2 \to L^2, \quad u \mapsto Au_{xx} + cu_x + C_\pm u, \quad C_\pm(x) = \begin{cases} S_\omega + Df(v_+), & x \geq 0 \\ S_\omega + Df(v_-), & x < 0. \end{cases} \tag{5.13}$$

Lemma 5.8. *Let Assumption 1, 2 be satisfied and $k \in \mathbb{N}_0$. Then the following statements are equivalent:*

i) The operator $sI - L : H^2_k \to L^2_k$ is a Fredholm operator of index ℓ.

ii) The operator $sI - L_k : H^2 \to L^2$ is a Fredholm operator of index ℓ.

iii) The operator $sI - L_\infty : H^2 \to L^2$ is a Fredholm operator of index ℓ.

Proof. **i) $\Leftrightarrow$ ii)** The claim follows since the multiplication operator associated with η^k is a homeomorphism, cf. Lemma 5.1.
ii) $\Leftrightarrow$ iii) The assertion follows exactly as in the proof of (3.13) using a compact perturbation argument and (5.12). $\square$

Recall the dispersion set $\sigma_{\text{disp}}(L)$ from (0.14) given by

$$\sigma_{\text{disp}}(L) = \sigma^-_{\text{disp}}(L) \cup \sigma^+_{\text{disp}}(L), \quad \sigma^\pm_{\text{disp}}(L) := \{s \in \mathbb{C} : \exists \nu \in \mathbb{R} \text{ s.t. } d^\pm(s,\nu) = 0\},$$

where $d^\pm$ is given by

$$d^\pm(s,\nu) := \det(sI + \nu^2 A - i\nu cI - S_\omega - Df(v_\pm)), \quad v_+ = v_\infty, \quad v_- = 0. \tag{5.14}$$

A straightforward computation shows $d^+(s,\nu) = 0$ if and only if

$$s = -\alpha_1\nu^2 + ic\nu + g_1'(|v_\infty|^2)|v_\infty|^2 \pm \left(-\alpha_2^2\nu^4 + 2\alpha_2 g_2'(|v_\infty|^2)|v_\infty|^2\nu^2 + (g_1'(|v_\infty|^2)|v_\infty|^2)^2\right)^{\frac{1}{2}}$$

as well as $d^-(s,\nu) = 0$ if and only if

$$s = -\alpha_1\nu^2 + ic\nu + g_1(0) \pm \left(-\alpha_2^2\nu^4 + 2\alpha_2(g_2(0)+\omega)\nu^2 - (g_2(0)+\omega)^2\right)^{\frac{1}{2}}.$$

As in Section 3.3 we have by the classical results, for instance from [36], [32], that the piecewise constant coefficient operator $sI - L_\infty$ is Fredholm if and only if $s \notin \sigma_{\text{disp}}(L)$ and the same holds true for $sI - L$, cf. Lemma 5.8. Summarizing we have the following theorem and its proof follows as in Theorem 3.16.

Theorem 5.9. *Let Assumption 1, 2 and 5 be satisfied. Then there are $\varepsilon, \gamma > 0$ and a unique connected component Ω_∞ of $\mathbb{C}\backslash\sigma_{\mathrm{disp}}(L)$ satisfying for all $k \in \mathbb{N}_0$:*

i) $\mathcal{S}_{\varepsilon,\gamma} := \left\{ s \in \mathbb{C} : |\arg(s-\gamma)| < \frac{\pi}{2} + \varepsilon,\ s \neq \gamma \right\} \subset \Omega_\infty$.

ii) For all $s \in \Omega_\infty$ the operator $sI - L : H^2_k \to L^2_k$ is Fredholm of index 0.

iii) $\partial\Omega_\infty \subset \sigma_{\mathrm{disp}}(L)$.

iv) $\sigma_{\mathrm{ess}}(L) \subset \mathbb{C}\backslash\Omega_\infty$.

Proof. The assertion follows in the same way as in the proof of Theorem 3.16 using Lemma 5.8. □

From Theorem 5.9 and Lemma 5.7 we conclude that $L : H^2_k \to L^2_k$ is a sectorial operator. But since its essential spectrum touches the imaginary axis at the origin we can only derive estimates of the corresponding semigroup by exponentially increasing terms. To show time decaying estimates for the semigroup we need to show delicate resolvent estimates near the origin. In order to do so, the strategy is as follows. First, we discuss the piecewise constant operator L_∞ given by

$$L_\infty u = Au_{xx} + cu_x + C_\pm u$$

with $C_\pm$ from (5.13). In particular, we are interested into the solution of the resolvent equation

$$(sI - L_\infty)u = r \tag{5.15}$$

for small s. We use the concepts of exponential dichotomies, cf. [22], and the concepts of exponential trichotomies, cf. [31] and [13], to construct solutions of the equation (5.15). See also Appendix B. Now we have $0 \in \sigma_{\mathrm{ess}}(L_\infty)$ when L_∞ is considered as closed operator on L^2_k. Thus, L_∞ is not a Fredholm operator and not invertible on L^2_k. However, using ideas from [37] we derive resolvent estimates for u in L^2_k as long as $r \in L^2_{k+2}$ and show that L_∞ is invertible from L^2_k to L^2_{k+2}. A compact perturbation argument in Section 5.3.3 will show the that the linearized operator L considered from L^2_k to L^2_{k+2} is a Fredholm operator of index 0. Using Fredholm index 0 and roughness of exponential trichotomies under small perturbations, will lead to sharp resolvent estimates for the linearized operator L in Section 5.3.4.

5.3.2 Resolvent estimates for the piecewise constant operator L_∞

Let us start by considering the piecewise constant coefficient operator L_∞ and its resolvent equation (5.15). L_∞ belongs to a large class of general second order differential operators of the form

$$\mathcal{T}u = Au'' + Bu' + Cu, \quad A, B \in \mathbb{R}^{n,n},\ x^\top Ax > 0,\ x \neq 0, \quad C \in L^\infty(\mathbb{R}, \mathbb{R}^n) \tag{5.16}$$

which we consider on the polynomial weighted spaces L^2_k, $k \in \mathbb{N}_0$.

Lemma 5.10. *For all $k \in \mathbb{N}_0$ the linear second order differential operator*

$$\mathcal{T} : H^2_k \subset L^2_k \to L^2_k$$

given by (5.16) *is a closed, densely defined, linear operator on L^2_k, i.e. $\mathcal{T} \in \mathcal{C}[L^2_k]$.*

Proof. Clearly, $\mathcal{T}$ is densely defined and linear. Thus it is left to show the closedness. Let $u_n \in H^2_k$ with $u_n \to u$ in L^2_k and $\mathcal{T}u_n \to h$ in L^2_k. We define $w_n := \eta^k u_n \in H^2$ and $w := \eta^k u \in L^2$. Then we have $w_n \to w$ in L^2, cf. Lemma 5.1. Moreover,

$$\eta^k \mathcal{T} \eta^{-k} w_n \to \eta^k h =: r \in L^2$$

and for $v \in H^2$ there hold

$$\eta^k \mathcal{T} \eta^{-k} v = Av'' + \tilde{B}v' + \tilde{C}v =: \tilde{\mathcal{T}}v$$

with

$$\tilde{B} = B - 2k\eta^{-1}\eta_x A, \quad \tilde{C} = C + (k^2+k)(\eta^{-1}\eta_x)^2 A - k\eta^{-1}\eta_{xx}A - k\eta^{-1}\eta_x B.$$

Since $\tilde{B}, \tilde{C} \in L^\infty$, the operator $\tilde{\mathcal{T}} : H^2 \subset L^2 \to L^2$ is closed. Now we have $w_n \in H^2$, $w_n \to w$ in L^2 and $\tilde{\mathcal{T}}w_n \to r$ in L^2. Thus $w \in H^2$ and $\tilde{\mathcal{T}}w = r$. This implies $u \in H^2_k$, cf. Lemma 5.1, and $\mathcal{T}u = h$ in L^2_k since

$$\eta^k h = r = \tilde{\mathcal{T}}w = \eta^k \mathcal{T} \eta^{-k} w = \eta^k \mathcal{T} u.$$

□

As a next step we consider $\mathcal{T}$ as on operator from L^2_k to L^2_{k+2} and determines its domain $\mathcal{D}(\mathcal{T})$ such that $\mathcal{T}$ gets to a closed, densely defined, linear operator.

Lemma 5.11. *For all $k \in \mathbb{N}_0$ the operator*

$$\mathcal{T} : \mathcal{D}(\mathcal{T}) \subset L^2_k \to L^2_{k+2}$$

given by (5.16) *with*

$$\mathcal{D}(\mathcal{T}) := \{u \in H^1_k \cap H^2_{\mathrm{loc}} : \mathcal{T}u \in L^2_{k+2}\}$$

is a closed, densely defined, linear operator from L^2_k to L^2_{k+2}, i.e. $\mathcal{T} \in \mathcal{C}[L^2_k, L^2_{k+2}]$.

Proof. It is clear that $\mathcal{T}$ is linear. Moreover, $C_0^\infty \subset \mathcal{D}(\mathcal{T})$ and C_0^∞ is dense in L^2_k. Thus $\mathcal{T}$ is densely defined on L^2_k. Thus it remains to show the closedness. Let $u_n \in \mathcal{D}(\mathcal{T})$ such that $u_n \to u$ in L^2_k and $\mathcal{T}u_n =: h_n \to h$ in L^2_{k+2}. Since $u_n \in \mathcal{D}(\mathcal{T})$, $h_n \in L^2_{k+2}$ and $B, C \in L^\infty$ it follows

$$\|h_n - Bu_n' - Cu_n\|_{L^2_k} \le \|h_n\|_{L^2_k} + \|B\|_\infty \|u_n'\|_{L^2_k} + \|C\|_\infty \|u\|_{L^2_k} < \infty.$$

Hence, we have

$$u_n'' = A^{-1}[h_n - Bu_n' - Cu_n] \in L_k^2$$

and therefore $u_n \in H_k^2$. By Lemma 5.10, $\mathcal{T} : H_k^2 \subset L_k^2 \to L_k^2$ is a closed operator on L_k^2. Thus we conclude by the closedness $u \in H_k^2 \subset H_k^1 \cap H_{\mathrm{loc}}^2$ and $\mathcal{T}u = h$ in L_k^2. Since $h \in L_{k+2}^2$ we obtain $u \in \mathcal{D}(\mathcal{T})$ with $\mathcal{T}u = h$. This shows the claim. □

Summarizing we have shown that L_∞ with $\mathcal{D}(L_\infty) = H_k^2$ defines is closed. Moreover, if we define

$$L_\infty : \mathcal{D}(L_\infty) \subset L_k^2 \to L_{k+2}^2, \quad \mathcal{D}(L_\infty) = \{u \in H_k^1 \cap H_{\mathrm{loc}}^2 : L_\infty u \in L_{k+2}^2\}.$$

then $L_\infty \in \mathcal{C}[L_k^2, L_{k+2}^2]$.

Corollary 5.12. *Let Assumption 1, 2 be satisfied and $k \in \mathbb{N}_0$. Then the operator $L_\infty : \mathcal{D}(L_\infty) \subset L_k^2 \to L_{k+2}^2$ with $\mathcal{D}(L_\infty)$ is a closed, densely defined, linear operator, i.e. $L_\infty \in \mathcal{C}[L_k^2, L_{k+2}^2]$.*

Now we discuss solution of (5.15). For this purpose, we transform (5.15) into a first order system via $Y = (u, u')^\top$ and obtain

$$Y' - M_\infty(s, \cdot)Y = R, \quad R = (0, r)^\top. \tag{5.17}$$

with

$$M_\infty(s, x) = \begin{cases} M_+(s), & x \geq 0 \\ M_-(s), & x < 0 \end{cases}, \quad M_\pm(s) = \begin{pmatrix} 0 & I_2 \\ A^{-1}(sI - C_\pm) & -cA^{-1} \end{pmatrix}.$$

To show that L_∞ is invertible it would be sufficient to consider (5.15) for $s = 0$. But since we want to show uniform estimates in a neighborhood of zero we consider the general case $s \in \Omega_\infty$ and $|s|$ sufficiently small. We choose $\varepsilon > 0$ sufficiently small and let $s \in B_\varepsilon(0)$. Since $B_\varepsilon(0) \cap \sigma_{\mathrm{disp}}^-(L) = \emptyset$ we conclude $M_-(s)$ to be hyperbolic with stable and unstable dimensions $m_{\mathfrak{s}}^-(s) = m_{\mathfrak{u}}^-(s) = 2$, cf. Figure 5.1 and Figure 5.2. Note that the complex conjugated pairs in Figure 5.1 and Figure 5.2 may also build a double eigenvalue and do not have to be separated. But $\lambda_3^+(s), \lambda_4^+(s)$ are simple eigenvalues.
In particular, there are $\sigma_{\mathfrak{s}}(M_-(s)), \sigma_{\mathfrak{u}}(M_-(s))$ uniformly bounded away from the imaginary axis and such that

$$\operatorname{Re} \sigma_{\mathfrak{s}}(M_-(s)) < 0 < \sigma_{\mathfrak{u}}(M_-(s)), \quad \sigma_{\mathfrak{s}}(M_-(s)) \cup \sigma_{\mathfrak{u}}(M_-(s)) = \sigma(M_-(s)).$$

Let $P_{\mathfrak{s}}^-(s), P_{\mathfrak{u}}^-(s)$ be the corresponding Riesz projectors then following [22] the operator $\partial_x - M_\infty(s, \cdot)$ has an exponential dichotomy on $\mathbb{R}_-$ with data (K, α^-, β^-), $\alpha^- < 0 < \beta^-$ such that

$$\begin{aligned} |e^{(x-y)M_-(s)} P_{\mathfrak{s}}^-(s)| &\leq K e^{\alpha^-(x-y)}, \quad y \leq x < 0, \\ |e^{(x-y)M_-(s)} P_{\mathfrak{u}}^-(s)| &\leq K e^{\beta^-(x-y)}, \quad x \leq y < 0. \end{aligned} \tag{5.18}$$

In particular, since M_- is analytic in $s \in B_\varepsilon(0)$ the projectors $P^-_{\mathfrak{s},\mathfrak{u}}$ are analytic and the data (K, α^-, β^-) can be chosen independent on s. In addition, the invariant subspaces $\mathcal{R}(P^-_{\mathfrak{s}}(s)), \mathcal{R}(P^-_{\mathfrak{u}}(s))$ are spanned by matrices

$$\begin{aligned} V^-_{\mathfrak{s}}(s) &= (v^-_1(s), v^-_2(s)) \in \mathbb{C}^{4,2}, \quad \mathcal{R}(V^-_{\mathfrak{s}}(s)) = \mathcal{R}(P^-_{\mathfrak{s}}(s)), \\ V^-_{\mathfrak{u}}(s) &= (v^-_3(s), v^-_4(s)) \in \mathbb{C}^{4,2}, \quad \mathcal{R}(V^-_{\mathfrak{u}}(s)) = \mathcal{R}(P^-_{\mathfrak{u}}(s)) \end{aligned} \tag{5.19}$$

and we find $w^-_i(s)$, $i = 1, \ldots, 4$ spanning the corresponding left invariant subspaces

$$\begin{aligned} &W^-_{\mathfrak{s}}(s) = (w^-_1(s), w^-_2(s)) \in \mathbb{C}^{4,2}, \quad W^-_{\mathfrak{u}}(s) = (w^-_3(s), w^-_3(s)) \in \mathbb{C}^{4,2}, \\ &(W^-_{\mathfrak{s}}(s), W^-_{\mathfrak{u}}(s))^H (V^-_{\mathfrak{s}}(s), V^-_{\mathfrak{u}}(s)) = I_4, \\ &P^-_{\mathfrak{s}}(s) = V^-_{\mathfrak{s}}(s) W^-_{\mathfrak{s}}(s)^H, \quad P^-_{\mathfrak{u}}(s) = V^-_{\mathfrak{u}}(s) W^-_{\mathfrak{u}}(s)^H. \end{aligned}$$

Now we consider $M_+(s)$, $s \in B_\varepsilon(0)$ which is also analytic in s. But since $0 \in \sigma^+_{\mathrm{disp}}(L)$,

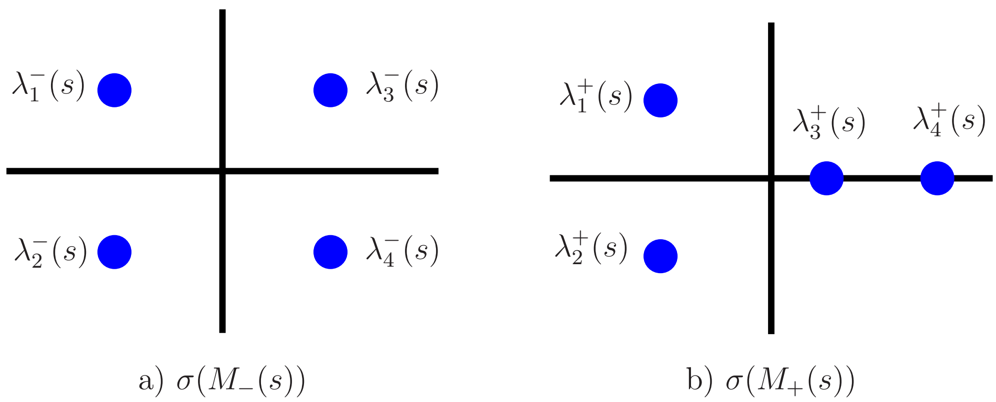


Figure 5.1: Eigenvalues $\lambda^\pm_1(s), \lambda^\pm_2(s), \lambda^\pm_3(s), \lambda^\pm_4(s)$ from left to right of $M_\pm(s)$ with $s \in \Omega_\infty$.

the matrix $M_+(0)$ has a simple zero eigenvalue, since

$$M_+(0) = \begin{pmatrix} 0 & I_2 \\ -A^{-1}C_+ & -cA^{-1} \end{pmatrix}, \quad C_+ = \begin{pmatrix} 2g_1'(|v_\infty|^2)|v_\infty|^2 & 0 \\ 2g_2'(|v_\infty|^2)|v_\infty|^2 & 0 \end{pmatrix}.$$

Its characteristic polynomial is given by

$$\chi(\lambda) = \lambda p(\lambda), \quad p(\lambda) = \lambda^3 + 2c\tilde{\alpha}_1\lambda^2 + \xi_1\lambda + \xi_2$$

where $\tilde{\alpha}_i = |\alpha|^{-1}\alpha_i$, $i = 1, 2$ and

$$\begin{aligned} \xi_1 &= (\tilde{\alpha}_1 + \tilde{\alpha}_2)c^2 + 2\tilde{\alpha}_1 g_1'(|v_\infty|^2)|v_\infty|^2 + 2\tilde{\alpha}_2 g_2'(|v_\infty|^2)|v_\infty|^2, \\ \xi_2 &= 2c(\tilde{\alpha}_1^2 + \tilde{\alpha}_2^2) g_1'(|v_\infty|^2)|v_\infty|^2. \end{aligned}$$

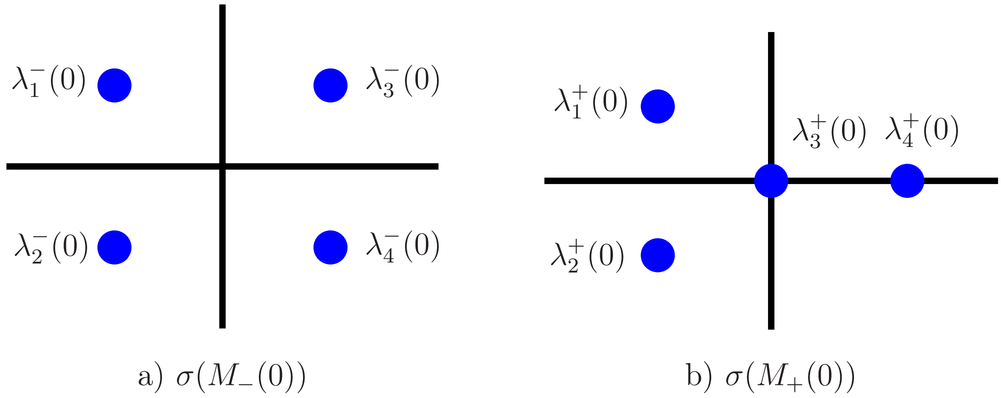


Figure 5.2: Eigenvalues $\lambda_1^\pm(0), \lambda_2^\pm(0), \lambda_3^\pm(0), \lambda_4^\pm(0)$ from left to right of $M_\pm(0)$.

Now Assumption 1, 2 and Assumption 8 imply $\xi_1 > 0$ and $\xi_2 < 0$. The Hurwitz determinants δ_i of the polynomial p, cf. Lemma D.7, satisfy

$$\delta_0 = 1, \quad \delta_1 = 2c\tilde{\alpha}_1 > 0, \quad \delta_2 = 2c\tilde{\alpha}_1\xi_1 - \xi_2 > 0, \quad \delta_3 = \xi_2\delta_2 < 0.$$

Then Lemma D.7 implies $m_{\mathfrak{s}}^+(0) = 2$ and $m_{\mathfrak{u}}^+(0) = 1$, cf. Figure 5.2. So we conclude there are $\sigma_{\mathfrak{s}}(M_+(s))$ and two simple eigenvalues $\lambda_3^+(s)$, $\lambda_4^+(s)$ such that

$$\begin{aligned}
&\sigma_{\mathfrak{s}}(M_+(s)) \cup \{\lambda_3^+(s), \lambda_4^+(s)\} = \sigma(M_+(s)), \\
&\operatorname{Re}\sigma_{\mathfrak{s}}(M_+(0)) < 0 = \lambda_3^+(0) < \lambda_4^+(0) \\
&\operatorname{Re}\sigma_{\mathfrak{s}}(M_+(s)) < 0 < \lambda_3^+(s) < \lambda_4^+(s), \quad s \in B_\varepsilon(0)\backslash\{0\}.
\end{aligned}$$

In addition, $\sigma_{\mathfrak{s}}(M_+(s))$ and $\lambda_4^+(s)$ are uniformly bounded away from the imaginary axis and λ_3^+ depends analytically in s. Further, let $v_3^+(s), v_4^+(s) \in \mathbb{C}^4$ be the corresponding eigenvectors of $\lambda_3^+(s), \lambda_4^+(s)$, i.e.

$$(\lambda_i^+(s)I - M_+(s))v_i^+(s) = 0, \quad |v_i^+| = 1, \quad i = 3, 4,$$

and $w_3^+(s), w_4^+(s)$ the corresponding left eigenvectors, i.e.

$$w_i^+(s)^H(\lambda_i^+(s)I - M_+(s)) = 0, \quad i = 3, 4,$$

such that the normalization $w_i^+(s)^H v_j^+(s) = \delta_{ij}$ holds. Let $P_{\mathfrak{s}}^+(s)$ be the Riesz projector associated with $\sigma_{\mathfrak{s}}(M_+(s))$ and let

$$P_{\mathfrak{c}}^+(s) = v_3^+(s)w_3^+(s)^H, \quad P_{\mathfrak{u}}^+(s) = v_4^+(s)w_4^+(s)^H.$$

Then the operator $\partial_x - M_\infty(s, \cdot)$ has an exponential trichotomy on $\mathbb{R}_+$ with data $(K, \alpha^+, \nu(s), \beta^+)$, $\alpha^+ < 0 < \beta^+$, $\nu(s) = \operatorname{Re}\lambda_3^+(s)$ such that for all $x, y \in \mathbb{R}_+$ there hold

$$\begin{aligned}
&|e^{(x-y)M_+(s)}P_{\mathfrak{s}}^+(s)| \le Ke^{\alpha^+(x-y)}, \quad |e^{(x-y)M_+(s)}P_{\mathfrak{c}}^+(s)| \le Ke^{\nu(s)(s)(x-y)}, \quad y \le x, \\
&|e^{(x-y)M_+(s)}P_{\mathfrak{u}}^+(s)| \le Ke^{\beta^+(x-y)}, \quad |e^{(x-y)M_+(s)}P_{\mathfrak{c}}^+(s)| \le Ke^{\nu(s)(x-y)}, \quad x \le y.
\end{aligned} \tag{5.20}$$

In particular, the projectors P_κ^+ depend analytically in $s \in B_\varepsilon(0)$. The invariant subspace $\mathcal{R}(P_\mathfrak{s}^+(s))$ is spanned by a matrix

$$V_\mathfrak{s}^+(s) = (v_1^+(s), v_2^+(s)) \in \mathbb{C}^{4,2}, \quad \mathcal{R}(V_\mathfrak{s}^+(s)) = \mathcal{R}(P_\mathfrak{s}^+(s)) \tag{5.21}$$

and we find $w_i^+(s)$, $i = 1, 2$ spanning the corresponding left invariant subspace

$$\begin{aligned} &W_\mathfrak{s}^+(s) = (w_1^+(s), w_2^+(s)) \in \mathbb{C}^{4,2}, \quad P_\mathfrak{s}^+(s) = V_\mathfrak{s}^+(s) W_\mathfrak{s}^+(s)^H, \\ &(W_\mathfrak{s}^+(s), w_3^+(s), w_4^+(s))^H (V_\mathfrak{s}^+(s), v_3^+(s), v_4^+(s)) = I_4. \end{aligned}$$

Using Assumption 7 we have the decomposition of $\mathbb{C}^4$

$$\mathbb{C}^4 = \mathcal{R}(P_\mathfrak{s}^+(s)) \oplus \mathcal{R}(P_\mathfrak{u}^-(s)) = \operatorname{span}\{v_1^+(s), v_2^+(s), v_3^-(s), v_4^-(s)\} \quad \forall\, s \in B_\varepsilon(0). \tag{5.22}$$

Remark 5.13. *In order to verify Assumption 7 in applications, it is much simpler to verify* (5.22). *Both statements are equivalent and closely related to the so called Evans function, cf. [3], [36]. For $s \in B_\varepsilon(0) \cap \Omega_\infty$ it is defined as*

$$\mathcal{E}(s) = \det(v_1^+(s), v_2^+(s), v_3^-(s), v_4^-(s)).$$

Then Assumption 7 and (5.22) *are equivalent to the fact that the Evans function does not vanish as $s \to 0$.*

As a next step we discuss the behavior of the critical eigenvalue $\lambda_3^+(s)$ as $s \to 0$ and the geometry of the dispersion set $\sigma_{\mathrm{disp}}(L)$ at the origin. We prove that it is possible to place a parabola between the dispersion set and the imaginary axis locally at the origin, cf. Figure 5.3.

Lemma 5.14. *Let Assumption 1, 2, 5 and 8 be satisfied and $k \in \mathbb{N}_0$. Then there are $a_\star < 0 < \delta$ such that the curve $\Gamma_c = \{\varphi(\tau) : |\tau| < \delta\}$ with*

$$\varphi : (-\delta, \delta) \to \mathbb{C}, \quad t \mapsto a_\star \tau^2 + i\tau$$

satisfies $\Gamma_c \subset \Omega_\infty \cup \{0\}$ with Ω_∞ from Lemma 5.9. Moreover, there exist $0 < \varepsilon < |\varphi(\delta)|$, $C > 0$ and a crescent Ω_c, defined as the closure of the unique connected component of $B_\varepsilon(0) \backslash \Gamma_c$ containing $(0, \varepsilon)$, such that for all $s \in \Omega_c$ there holds

$$|\lambda_3^+(s)|^2 \leq C \operatorname{Re} \lambda_3^+(s). \tag{5.23}$$

In addition, the derivatives of λ_3^+ w.r.t. s at $s = 0$ are given by

$$\partial_s \lambda_3^+(0) = \frac{1}{c}, \quad \partial_s^2 \lambda_3^+(0) = \frac{4(\alpha_1 g_1'(|v_\infty|^2)|v_\infty|^2 + \alpha_2 g_2'(|v_\infty|^2)|v_\infty|^2)}{|c^3 g_1'(|v_\infty|^2)|v_\infty|^2|}. \tag{5.24}$$

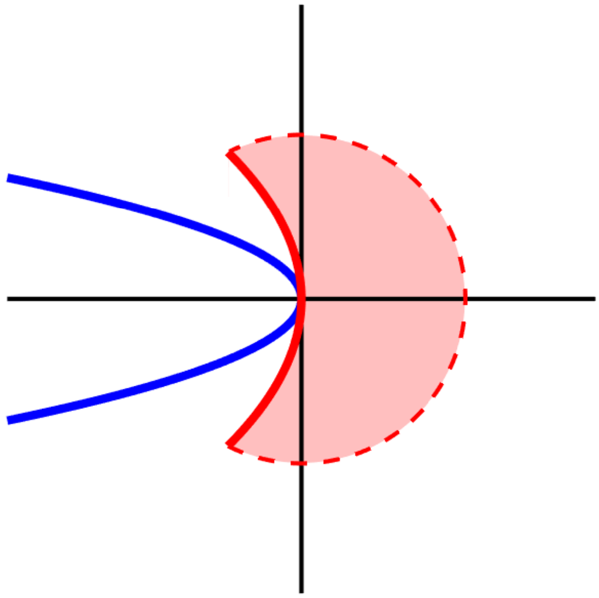


Figure 5.3: The crescent Ω_c.

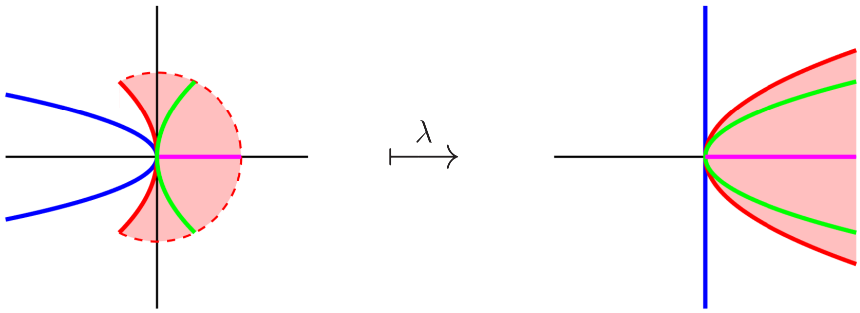


Figure 5.4: Geometric situation in the proof of Lemma 5.14.

In particular, for arbitrary $\varepsilon > 0$ the crescent Ω_c is uniquely defined, since the set $\overline{B_\varepsilon(0)} \backslash \Gamma_c$ consists of exactly two connected components where only the right component includes $(0, \varepsilon)$. Throughout the rest of the chapter the crescent Ω_c will be frequently chosen sufficiently small, i.e. we frequently assume w.l.o.g. that ε from 5.14 is sufficiently small.

Proof. Let $\lambda(s) = \lambda_3^+(s)$. Then for $s > 0$ we have $\lambda(s) > 0$ and λ is analytic for $s \in B_\varepsilon(0)$, i.e.

$$\lambda(s) = \lambda'(0)s + \tfrac{1}{2}\lambda''(0)s^2 + \mathcal{O}(|s|^3). \tag{5.25}$$

In particular, $\lambda'(0), \lambda''(0) \in \mathbb{R}$ since $\lambda(s) \in \mathbb{R}, s \in \mathbb{R} \cap \Omega_c$. Further we set $2\kappa_1 := g_1'(|v_\infty|^2)|v_\infty|^2$ and $2\kappa_2 := g_2'(|v_\infty|^2)|v_\infty|^2$ then

$$C_+ = \begin{pmatrix} \kappa_1 & 0 \\ \kappa_2 & 0 \end{pmatrix}, \quad C_+ v_0 = 0, \quad w_0^\top C_+ = 0, \quad v_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad w_0 = \begin{pmatrix} -\kappa_2 \\ \kappa_1 \end{pmatrix}.$$

Next we compute $\lambda'(0)$ and $\lambda''(0)$. For this purpose, let $D(\lambda) := \lambda^2 A + c\lambda I + C_+$ and

$$T(v, \lambda, s) = \begin{pmatrix} D(\lambda)v - sv \\ w_0^\top v - \kappa_1 \end{pmatrix}.$$

Then $T(v_0, 0, 0) = 0$ and

$$D_{(v,\lambda)} T(v_0, 0, 0) = \begin{pmatrix} C_+ & cv_0 \\ w_0^\top & 0 \end{pmatrix} = \begin{pmatrix} \kappa_1 & 0 & 0 \\ \kappa_2 & 0 & c \\ -\kappa_2 & \kappa_1 & 0 \end{pmatrix}$$

which is invertible by Assumption 1 and 2. As a consequence of the implicit function theorem D.8 there is $\varepsilon > 0$ and holomorphic $v : B_\varepsilon(0) \to \mathbb{C}^2$ such that

$$0 = T(v(s), \lambda(s), s) \quad \forall s \in B_\varepsilon(0). \tag{5.26}$$

Differentiating (5.26) w.r.t. s once and evaluating at $s = 0$ yields

$$0 = c\lambda'(0)v_0 + C_+ v'(0) - v_0, \quad 0 = w_0^\top v'(0). \tag{5.27}$$

Solving (5.27) for $\lambda'(0)$ leads with Assumption 2 to

$$\lambda'(0) = \frac{1}{c} > 0.$$

Further, by differentiating (5.26) w.r.t. s twice and evaluating at $s = 0$ we obtain

$$0 = 2c^{-2} A v_0 + c\lambda''(0) v_0 + C_+ v''(0), \quad 0 = w_0^\top v''(0). \tag{5.28}$$

Solving (5.28) for $\lambda''(0)$ yields with Assumption 2 and 8

$$\lambda''(0) = \frac{2(\alpha_1\kappa_1 + \alpha_2\kappa_2)}{c^3|\kappa_1|} < 0.$$

Next, pick $\nu > 0$ and define

$$\tilde{\Omega} := \{s \in \Omega_c : |\mathrm{Im}\, s|^2 \le \nu |\mathrm{Re}\, s|\},$$

cf. Figure 5.4.Then $|s|^2 \leq (\varepsilon+\nu)|\mathrm{Re}\, s|$ for all $s \in \tilde{\Omega}$. Using (5.25) we have

$$\mathrm{Re}\,\lambda(s) = \mathrm{Re}\, s\lambda'(0) + \tfrac{1}{2}(\mathrm{Re}\, s)^2\lambda''(0) - \tfrac{1}{2}(\mathrm{Im}\, s)^2\lambda''(0) + \mathcal{O}(|s|^3).$$

Since $\lambda'(0) > 0$, $\lambda''(0) < 0$ we find $\tilde{C} > 0$ such that for all $s \in \tilde{\Omega}$

$$\begin{aligned} |\mathrm{Re}\, s| &\leq \frac{1}{\lambda'(0)}\left|\mathrm{Re}\,\lambda(s) - \lambda''(0)(\mathrm{Re}\, s)^2 + \lambda''(0)(\mathrm{Im}\, s)^2\right| + \mathcal{O}(|s|^3) \\ &\leq \frac{1}{\lambda'(0)}|\mathrm{Re}\,\lambda(s)| + \frac{|\lambda''(0)|}{\lambda'(0)}\varepsilon|\mathrm{Re}\, s| + \varepsilon\tilde{C}|\mathrm{Re}\, s|. \end{aligned}$$

Taking $\varepsilon > 0$ sufficiently small we find $C_1 > 0$ such that for all $s \in \tilde{\Omega}$

$$|\mathrm{Re}\, s| \leq C_1|\mathrm{Re}\,\lambda(s)|.$$

Then we find $C_2 > 0$ such that the imaginary part satisfies

$$\begin{aligned} |\mathrm{Im}\,\lambda(s)|^2 &\leq 2|\mathrm{Im}\, s\lambda'(0) + 2(\mathrm{Re}\, s)(\mathrm{Im}\, s)\lambda''(0)|^2 + \mathcal{O}(|s|^6) \\ &\leq 4|\mathrm{Im}\, s|^2\lambda'(0)^2 + 8|\mathrm{Re}\, s||\mathrm{Im}\, s|^2|\lambda'(0)\lambda''(0)| + 8|\mathrm{Re}\, s|^2|\mathrm{Im}\, s|^2 + \mathcal{O}(|s|^6) \\ &\leq C_2|\mathrm{Re}\, s| \leq C_1C_2|\mathrm{Re}\,\lambda(s)|. \end{aligned}$$

Hence the estimate (5.23) holds for all $s \in \tilde{\Omega}$. Now we choose

$$a_\star = \frac{\alpha_1\kappa_1 + \alpha_2\kappa_2}{2c^2|\kappa_1|}.$$

and let $\varepsilon > 0$ be sufficiently small. Then for all $s \in \Omega_c\backslash\tilde{\Omega}$ there is $\tau \in [-\varepsilon, \varepsilon]$ and $a_\star < a < \nu$ such that $s = a\tau^2 + i\tau$, cf. Figure 5.4. Then

$$\lambda(s) = (a\tau^2 + i\tau)\lambda'(0) - \tfrac{1}{2}\tau^2\lambda''(0) + \mathcal{O}(|\tau|^3)$$

and we find $C > 0$ independent in τ, a such that

$$\begin{aligned} \frac{|\mathrm{Im}\,\lambda(s)|^2}{|\mathrm{Re}\,\lambda(s)|} &= \frac{\lambda'(0)^2 + \mathcal{O}(|\tau|^2)}{|-\frac{1}{2}\lambda''(0) + a\lambda'(0) + \mathcal{O}(|\tau|)|} \leq \frac{\lambda'(0)^2 + \mathcal{O}(|\tau|^2)}{|-\frac{1}{2}\lambda''(0) + a_\star\lambda'(0) + \mathcal{O}(|\tau|)|} \\ &\leq \frac{2|\kappa_1| + \mathcal{O}(|\tau|^2)}{|\alpha_1\kappa_1 + \alpha_2\kappa_2| + \mathcal{O}(|\tau|)} \leq C. \end{aligned}$$

Now the assertion is proven. □

We follow an approach similar to [35] and construct for given $R \in L^2_{k+2}$, $k \in \mathbb{N}_0$ a solution $Y_\infty(s,\cdot) \in L^2_k$ of (5.17) via Green's functions. Suppose $\zeta_+(s) \in \mathcal{R}(P^+_{\mathfrak{s}}(s))$, $x \in \mathbb{R}_+$ and define

$$Y^+_\infty(s,x) := e^{xM_+(s)}\zeta_+(s) + \int_0^\infty G^+_s(x,y)R(y)dy \tag{5.29}$$

where $G_s^+ \in C_b(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{C}^{4,4})$ is a Green's function defined by

$$G_s^+(x,y) = \begin{cases} e^{(x-y)M_+(s)} P_{\mathfrak{s}}^+(s), & 0 \le y \le x \\ -e^{(x-y)M_+(s)}(P_{\mathfrak{c}}^+(s) + P_{\mathfrak{u}}^+(s)), & 0 \le x < y. \end{cases}$$

For $\zeta_-(s) \in \mathcal{R}(P_{\mathfrak{u}}^-(s))$ and $x \in \mathbb{R}_-$ set

$$Y_\infty^-(s,x) := e^{xM_-(s)}\zeta_-(s) + \int_{-\infty}^0 G_s^-(x,y)R(y)dy \tag{5.30}$$

where $G_s^- \in C_b(\mathbb{R}_- \times \mathbb{R}_-, \mathbb{C}^{4,4})$ is a Green's function given by

$$G_s^-(x,y) = \begin{cases} -e^{(x-y)M_-(s)} P_{\mathfrak{u}}^-(s), & x \le y \le 0 \\ e^{(x-y)M_-(s)} P_{\mathfrak{s}}^-(s), & y < x \le 0 \end{cases}.$$

Note that $Y_\infty^\pm(s,\cdot)$ can be represented as

$$\begin{aligned} Y_\infty^+(s,x) &= e^{xM_+(s)}\zeta_+(s) + \int_0^x e^{(x-y)M_+(s)} P_{\mathfrak{s}}^+(s)R(y)dy \\ &\quad - e^{\lambda_3^+(s)x} v_3^+(s) \int_x^\infty e^{-\lambda_3^+(s)y} w_3^+(s)^H R(y)dy - \int_x^\infty e^{(x-y)M_+(s)} P_{\mathfrak{u}}^+(s)R(y)dy, \\ Y_\infty^-(s,x) &= e^{xM_-(s)}\zeta_-(s) + \int_{-\infty}^x e^{(x-y)M_-(s)} P_{\mathfrak{s}}^-(s)R(y)dy - \int_x^0 e^{(x-y)M_-(s)} P_{\mathfrak{u}}^-(s)R(y)dy. \end{aligned} \tag{5.31}$$

So since $R \in L_{k+2}^2$ it follows $Y_\infty^\pm(s,\cdot) \in H_{\mathrm{loc}}^1(\mathbb{R}_\pm, \mathbb{C}^4)$. Moreover, $Y_\infty^\pm(s,\cdot)$ solve (5.17) on $\mathbb{R}_\pm$ in the weak sense. This follows by taking the derivative

$$\begin{aligned} \partial_x Y_\infty^+(s,x) &= M_+(s)e^{xM_+(s)}\zeta_+(s) + P_{\mathfrak{s}}^+(s)R(x) + M_+(s)\int_0^x e^{(x-y)M_+(s)} P_{\mathfrak{s}}^+(s)R(y)dy \\ &\quad + (P_{\mathfrak{c}}^+(s) + P_{\mathfrak{u}}^+(s))R(x) - M_+(s)\int_x^\infty e^{(x-y)M_+(s)}(P_{\mathfrak{c}}^+(s) + P_{\mathfrak{u}}^+(s))R(y)dy \\ &= M_+(s)Y_\infty^+(s,x) + R(x) \end{aligned}$$

and similarly

$$\begin{aligned} \partial_x Y_\infty^-(s,x) &= M_-(s)e^{xM_-(s)}\zeta_-(s) + P_{\mathfrak{s}}^-(s)R(x) + M_-(s)\int_{-\infty}^x e^{(x-y)M_-(s)} P_{\mathfrak{s}}^-(s)R(y)dy \\ &\quad + P_{\mathfrak{u}}^-(s)R(x) - M_-(s)\int_x^0 e^{(x-y)M_-(s)} P_{\mathfrak{u}}^-(s)R(y)dy \\ &= M_-(s)Y_\infty^-(s,x) + R(x). \end{aligned}$$

We want to choose $\zeta_+(s) \in \mathcal{R}(P_{\mathfrak{s}}^+(s))$ and $\zeta_-(s) \in \mathcal{R}(P_{\mathfrak{u}}^-(s))$ such that

$$Y_\infty(s,x) = \begin{cases} Y_\infty^+(s,x), & x \geq 0 \\ Y_\infty^-(s,x), & x < 0 \end{cases} \tag{5.32}$$

is continuous in $x = 0$ and therefore globally continuous. For this purpose set $\Phi(s) = (v_1^+(s), v_2^+(s), v_3^-(s), v_4^-(s))$. Assumption 7 implies $\det \Phi(s) \neq 0$ for all $s \in B_\varepsilon(0)$ and we define

$$Q_{\mathfrak{s}}^+(s) = V_{\mathfrak{s}}^+(s)\Psi_+(s)^H, \quad Q_{\mathfrak{u}}^-(s) = V_{\mathfrak{u}}^-(s)\Psi_-(s)^H, \quad (\Psi_+(s), \Psi_-(s)) = \Phi(s)^{-H}.$$

Then for all $s \in B_\varepsilon(0)$ we have

$$\begin{aligned} &\mathcal{R}(Q_{\mathfrak{s}}^+(s)) = \mathcal{R}(P_{\mathfrak{s}}^+(s)), && \mathcal{R}(Q_{\mathfrak{u}}^-(s)) = \mathcal{R}(P_{\mathfrak{u}}^-(s)), \\ &\mathbb{C}^4 = \mathcal{R}(Q_{\mathfrak{s}}^+(s)) \oplus \mathcal{R}(Q_{\mathfrak{u}}^-(s)), && I = Q_{\mathfrak{s}}^+(s) + Q_{\mathfrak{u}}^-(s), \\ &Q_{\mathfrak{s}}^+(s)Q_{\mathfrak{u}}^-(s) = Q_{\mathfrak{s}}^+(s)Q_{\mathfrak{u}}^-(s) = 0. \end{aligned} \tag{5.33}$$

Moreover there is $C > 0$ such that for all $s \in B_\varepsilon(0)$ we have

$$|Q_{\mathfrak{s}}^+(s)|, |Q_{\mathfrak{u}}^-(s)| \leq C. \tag{5.34}$$

Now let

$$G_s(y) = \begin{cases} G_s^+(0,y), & x \geq 0 \\ G_s^-(0,y), & x < 0 \end{cases}.$$

Then $G_s \in C_b(\mathbb{R}, \mathbb{C}^{4,4})$ and we define

$$\begin{aligned} \zeta_-(s) &:= -Q_{\mathfrak{u}}^-(s) \int_{\mathbb{R}} G_s(y)R(y)dy \in \mathcal{R}(P_{\mathfrak{u}}^-(s)), \\ \zeta_+(s) &:= Q_{\mathfrak{s}}^+(s) \int_{\mathbb{R}} G_s(y)R(y)dy \in \mathcal{R}(P_{\mathfrak{s}}^+(s)). \end{aligned} \tag{5.35}$$

This implies using (5.33)

$$\begin{aligned} &Y_\infty^-(s,0) - Y_\infty^+(s,0) \\ &= \zeta_-(s) - \zeta_+(s) + \int_{-\infty}^0 e^{-yM_-(s)} P_{\mathfrak{s}}^-(s)R(y)dy + \int_0^\infty e^{-yM_+(s)} P_{\mathfrak{u}}^-(s)R(y)dy \\ &= \zeta_-(s) - \zeta_+(s) + \int_{\mathbb{R}} G_s(y)R(y)dy \\ &= -\int_{\mathbb{R}} G_s(y)R(y)dy + \int_{-\infty}^\infty G_s(y)R(y)dy = 0. \end{aligned}$$

Thus $Y_\infty(s,\cdot) \in C_b(\mathbb{R}, \mathbb{C}^4)$. Moreover, this implies $Y_\infty(s,\cdot) \in H^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^4)$ solves (5.17) on whole $\mathbb{R}$ in the weak sense. To estimate $Y_\infty(s,\cdot)$ in $\|\cdot\|_{H^1_k}$ we use the estimates (5.20), (5.18) from the exponential dichotomy and trichotomy on $\mathbb{R}_\pm$ and the following technical and delicate estimates from [37, Lem. 3.2].

Lemma 5.15. *For every $\beta_0 > 0$ and $k \in \mathbb{N}$ there is $C = C(k, \beta_0) > 0$ such that for all $0 < \beta \leq \beta_0$ the following estimates hold:*

$$
\begin{aligned}
&|x|^k e^{\beta x} \int_x^\infty \frac{e^{-\beta y}}{|y|^k} dy \leq \frac{C}{\beta}, && x \geq 1, && |x|^k e^{-\beta x} \int_{-\infty}^x \frac{e^{\beta y}}{|y|^k} dy \leq \frac{C}{\beta}, && x \leq -1,\\
&|x|^k e^{\beta x} \int_x^\infty \frac{e^{-\beta y}}{|y|^{k+1}} dy \leq C, && x \geq 1, && |x|^k e^{-\beta x} \int_{-\infty}^x \frac{e^{\beta y}}{|y|^{k+1}} dy \leq C, && x \leq -1,\\
&|x|^k e^{-\beta x} \int_1^x \frac{e^{\beta y}}{|y|^k} dy \leq \frac{C}{\beta^2}, && x \geq 1, && |x|^k e^{\beta x} \int_x^{-1} \frac{e^{-\beta y}}{|y|^k} dy \leq \frac{C}{\beta^2}, && x \leq -1,\\
&|x|^k e^{-\beta x} \int_1^x \frac{e^{\beta y}}{|y|^{k+1}} dy \leq \frac{C}{\beta}, && x \geq 1, && |x|^k e^{\beta x} \int_x^{-1} \frac{e^{-\beta y}}{|y|^{k+1}} dy \leq \frac{C}{\beta}, && x \leq -1.
\end{aligned}
$$

Proof. Note that the second column follows by the first and replacing x by $-x$. The first two lines for $x \geq 1$ are obtained by

$$|x|^k e^{\beta x} \int_x^\infty \frac{e^{-\beta y}}{|y|^k} dy \leq e^{\beta x} \int_x^\infty e^{-\beta y} dy \leq \frac{1}{\beta}$$

and

$$|x|^k e^{\beta x} \int_x^\infty \frac{e^{-\beta y}}{|y|^{k+1}} dy \leq x^k \int_x^\infty y^{-k-1} dy \leq \frac{1}{k}.$$

For the third line we use series expansion of the exponential function

$$\frac{e^{\beta y}}{y^k} = \frac{\beta^{k-1} x^{-1}}{(k-1)!} + \sum_{\substack{n=0\\ n \neq k-1}}^\infty \frac{\beta^n y^{n-k}}{n!}.$$

Integrating over $(1, x)$, using $\log(x) \leq x$ and $\frac{n}{n-k-1} \to 1$, $n \to \infty$ yields for some $C > 0$

$$
\begin{aligned}
|x|^k \int_1^x \frac{e^{\beta y}}{|y|^k} dy &\leq \frac{\beta^{k-1} \log(x) x^k}{(k-1)!} + x^k \sum_{\substack{n=0\\ n \neq k-1}}^\infty \frac{\beta^n x^{n-k+1}}{n!(n-k+1)}\\
&\leq \frac{k(k+1)}{\beta^2} \frac{(\beta x)^{k+1}}{(k+1)!} + \frac{1}{\beta} \sum_{\substack{n=0\\ n \neq k-1}}^\infty \frac{(\beta x)^{n+1}}{(n+1)!} \frac{(n+1)}{(n-k+1)} \leq \frac{C}{\beta^2} e^{\beta x}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|x|^k \int_1^x \frac{e^{\beta y}}{|y|^{k+1}} dy &\leq \frac{\beta^k \log(x) x^k}{k!} + x^k \sum_{\substack{n=0\\ n \neq k}}^\infty \frac{\beta^n x^{n-k}}{n!(n-k)}\\
&\leq \frac{(k+1)}{\beta} \frac{(\beta x)^{k+1}}{(k+1)!} + \sum_{\substack{n=0\\ n \neq k}}^\infty \frac{(\beta x)^n}{n!(n-k)} \leq \frac{C}{\beta} e^{\beta x}.
\end{aligned}
$$

□

Now we use the estimates from Lemma 5.15 to derive delicate resolvent estimates for the operator $L_\infty : \mathcal{D}(L_\infty) \subset L^2_k \to L^2_{k+2}$ in the crescent Ω_c. In particular, we show that the equation (5.15) has a unique solution in L^2_k if the right hand side is from L^2_{k+2}.

Lemma 5.16. *Let Assumption 1, 2, 5, 7 and 8 be satisfied and $k \in \mathbb{N}_0$. Then there is $C > 0$ such that for all $R \in L^2_{k+2}$ and $s \in \Omega_c$ the function $Y_\infty(s,\cdot) \in H^1_{\mathrm{loc}}(\mathbb{R},\mathbb{C}^4)$ from (5.32) with (5.29), (5.30) and (5.35) is a solution in L^2_k and satisfies the estimate*

$$\|Y_\infty(s,\cdot)\|_{L^2_k} + \|Y'_\infty(s,\cdot)\|_{L^2_{k+1}} \leq C\|R\|_{L^2_{k+2}}. \tag{5.36}$$

Proof. We have already shown that $Y_\infty(s,\cdot) \in H^1_{\mathrm{loc}}$ solves (5.17). Thus it remains to show the estimate (5.36). For this purpose we frequently use the estimates in (5.20), (5.18), Lemma 5.15, Cauchy-Schwarz inequality and the explicit representation of $Y^\pm_\infty(s,\cdot)$ from (5.31). Let $C > 0$ denote a universal constant independent on s. Then by (5.34) we have

$$\begin{aligned}
|\zeta_\pm(s)|^2 &\leq C\left|\int_{\mathbb{R}} G_s(y)R(y)dy\right|^2 \leq C\left|\int_{-\infty}^0 e^{-\alpha^- y}|R(y)|dy\right|^2 + C\left|\int_0^\infty e^{-\nu(s)y}|R(y)|dy\right|^2 \\
&\leq C\int_{-\infty}^0 \frac{e^{-2\alpha^- y}}{\eta^{2(k+2)}(y)}dy\|R\|^2_{L^2_{k+2}} + C\int_0^\infty \frac{e^{-2\nu(s)y}}{\eta^{2(k+2)}(y)}dy\|R\|^2_{L^2_{k+2}} \\
&\leq C\int_{\mathbb{R}} \frac{1}{\eta^{2(k+2)}(y)}dy\|R\|^2_{L^2_{k+2}} \leq C\|R\|^2_{L^2_{k+2}}.
\end{aligned}$$

Recall the representation of $Y^\pm_\infty(s,\cdot)$ from (5.31) and $\nu(s) = \operatorname{Re}\lambda^+_3(s)$. We estimate the λ^+_3 term for $x \geq 1$ by

$$\begin{aligned}
&\left|\int_x^\infty e^{\lambda^+_3(s)(x-y)}v^+_3(s)w^+_3(s)^H R(y)dy\right|^2 \leq C\left|\int_x^\infty \frac{e^{\nu(s)(x-y)}}{\eta^{k+2}(y)}\eta^{k+2}(y)|R(y)|dy\right|^2 \\
&\leq C\int_x^\infty \frac{e^{2\nu(s)(x-y)}}{|y|^{2(k+2)}}dy\|R\|^2_{L^2_{k+2}} \leq \frac{C}{\nu(s)^\ell}|x|^{-2k-\ell-3}\|R\|^2_{L^2_{k+2}} \\
&\leq \frac{C\eta^{-2(k+\ell)}(x)}{\nu(s)^\ell}|x|^{-3+\ell}\|R\|^2_{L^2_{k+2}}, \quad \ell = 0,1.
\end{aligned} \tag{5.37}$$

Further, for $x \geq 1$

$$\begin{aligned}
&\left|\int_x^\infty e^{(x-y)M_+(s)}P^+_{\mathfrak{u}}(s)R(y)dy\right|^2 \leq C\left|\int_x^\infty \frac{e^{\beta^+(x-y)}}{\eta^{k+2}(y)}\eta^{k+2}(y)|R(y)|dy\right|^2 \\
&\leq C\int_x^\infty \frac{e^{2\beta^+(x-y)}}{|y|^{2(k+2)}}dy\|R\|^2_{L^2_{k+2}} \leq \frac{C}{\beta^+}|x|^{-2k-4}\|R\|^2_{L^2_{k+2}} \\
&\leq C\eta^{-2(k+1)}(x)|x|^{-2}\|R\|^2_{L^2_{k+2}}
\end{aligned} \tag{5.38}$$

and

$$
\begin{aligned}
&\left|\int_0^x e^{(x-y)M_+(s)}P_{\mathfrak{s}}^+(s)R(y)dy\right|^2 \le C\left|\int_0^x \frac{e^{\alpha^+(x-y)}}{\eta^{k+2}(y)}\eta^{k+2}(y)|R(y)|dy\right|^2\\
&\le Ce^{2\alpha^+x}\|R\|^2_{L^2_{k+2}} + C\int_1^x \frac{e^{2\alpha^+(x-y)}}{|y|^{2(k+2)}}dy\|R\|^2_{L^2_{k+2}}\\
&\le \frac{C}{(\alpha^+)^2}|x|^{-2k-4}\|R\|^2_{L^2_{k+2}} \le C\eta^{-2(k+1)}(x)|x|^{-2}\|R\|^2_{L^2_{k+2}}.
\end{aligned}
\tag{5.39}
$$

On the negative half-line $x \le -1$ we estimate

$$
\begin{aligned}
&\left|\int_{-\infty}^x e^{(x-y)M_-(s)}P_{\mathfrak{s}}^-(s)R(y)dy\right|^2 \le C\left|\int_{-\infty}^x \frac{e^{\alpha^-(x-y)}}{\eta^{k+2}(y)}\eta^{k+2}(y)|R(y)|dy\right|^2\\
&\le C\int_{-\infty}^x \frac{e^{2\alpha^-(x-y)}}{|y|^{2(k+2)}}dy\|R\|^2_{L^2_{k+2}} \le \frac{C}{|\alpha^-|}|x|^{-2k-4}\|R\|^2_{L^2_{k+2}}\\
&\le C\eta^{-2(k+1)}(x)|x|^{-2}\|R\|^2_{L^2_{k+2}}
\end{aligned}
\tag{5.40}
$$

and

$$
\begin{aligned}
&\left|\int_x^0 e^{(x-y)M_-(s)}P_{\mathfrak{u}}^-(s)R(y)dy\right|^2 \le C\left|\int_x^0 \frac{e^{\beta^-(x-y)}}{\eta^{k+2}(y)}\eta^{k+2}(y)|R(y)|dy\right|^2\\
&\le Ce^{2\beta^-x}\|R\|^2_{L^2_{k+2}} + C\int_x^{-1} \frac{e^{2\beta^-(x-y)}}{|y|^{2(k+2)}}dy\|R\|^2_{L^2_{k+2}}\\
&\le \frac{C}{(\beta^-)^2}|x|^{-2k-4}\|R\|^2_{L^2_{k+2}} \le C\eta^{-2(k+1)}(x)|x|^{-2}\|R\|^2_{L^2_{k+2}}.
\end{aligned}
\tag{5.41}
$$

Now the L^2_k-estimate follows by (5.37) with $\ell = 0$ and (5.38), (5.39), (5.40), (5.41), since

$$
\begin{aligned}
\int_1^\infty \eta^{2k}(x)|Y_\infty(s,x)|^2dx \le{}& C\int_1^\infty \eta^{2k}(x)e^{\alpha^+x}dx\|R\|^2_{L^2_{k+2}}\\
&+ C\int_1^\infty \eta^{2k}(x)\left|\int_0^x e^{(x-y)M_+(s)}P_{\mathfrak{s}}^+(s)R(y)dy\right|^2 dx\\
&+ C\int_1^\infty \eta^{2k}(x)\left|\int_x^\infty e^{\lambda_3^+(s)(x-y)}v_3^+(s)w_3^+(s)^HR(y)dy\right|^2 dx\\
&+ C\int_1^\infty \eta^{2k}(x)\left|\int_x^\infty e^{(x-y)M_+(s)}P_{\mathfrak{u}}^+(s)R(y)dy\right|^2 dx\\
\le{}& C\|R\|^2_{L^2_{k+2}} + C\int_1^\infty |x|^{-3}dx\|R\|^2_{L^2_{k+2}} \le C\|R\|^2_{L^2_{k+2}}
\end{aligned}
$$

and

$$\begin{aligned}\int_{-\infty}^{-1}\eta^{2k}(x)|Y_\infty(s,x)|^2dx &\le C\int_{-\infty}^{-1}\eta^{2k}(x)e^{\beta^- x}dx\|R\|^2_{L^2_{k+2}}\\ &\quad + C\int_{-\infty}^{-1}\eta^{2k}(x)\left|\int_0^x e^{(x-y)M_-(s)}P_{\mathfrak{u}}^-(s)R(y)dy\right|^2dx\\ &\quad + C\int_{-\infty}^{-1}\eta^{2k}(x)\left|\int_x^\infty e^{(x-y)M_-(s)}P_{\mathfrak{s}}^-(s)R(y)dy\right|^2dx\\ &\le C\|R\|^2_{L^2_{k+2}} + C\int_{-\infty}^{-1}|x|^{-4}dx\|R\|^2_{L^2_{k+2}} \le C\|R\|^2_{L^2_{k+2}}.\end{aligned}$$

Since $Y_\infty(s,\cdot)\in H^1_{\mathrm{loc}}$ this shows

$$\|Y_\infty(s,\cdot)\|_{L^2_k}\le C\|R\|_{L^2_{k+2}}.$$

The derivatives $\partial_x Y^\pm_\infty(s,\cdot)$ are given by

$$\begin{aligned}\partial_x Y^+_\infty(s,x) &= M_+(s)e^{xM_+(s)}\zeta_+(s) + R(x) + M_+(s)\int_0^x e^{(x-y)M_+(s)}P_{\mathfrak{s}}^+(s)R(y)dy\\ &\quad - \lambda_3^+(s)e^{\lambda_3^+(s)x}v_3^+(s)\int_x^\infty e^{-\lambda_3^+(s)y}w_3^+(s)^H R(y)dy\\ &\quad - M_+(s)\int_x^\infty e^{(x-y)M_+(s)}P_{\mathfrak{u}}^+(s)R(y)dy,\\ \partial_x Y^-_\infty(s,x) &= M_-(s)e^{xM_-(s)}\zeta_-(s) + R(x) + M_-(s)\int_{-\infty}^x e^{(x-y)M_-(s)}P_{\mathfrak{s}}^-(s)R(y)dy\\ &\quad - M_-(s)\int_x^0 e^{(x-y)M_-(s)}P_{\mathfrak{u}}^-(s)R(y)dy.\end{aligned}$$

Thus, use (5.37) with $\ell=1$ and (5.38), (5.39), (5.40), (5.41) to obtain

$$\begin{aligned}&\int_1^\infty \eta^{2(k+1)}(x)|Y'_\infty(s,x)|^2dx \le C\int_1^\infty \eta^{2(k+1)}(x)e^{\alpha^+ x}dx\|R\|^2_{L^2_{k+2}} + C\|R\|_{L^2_{k+2}}\\ &\quad + C\int_1^\infty \eta^{2(k+1)}(x)\left|\int_0^x e^{(x-y)M_+(s)}P_{\mathfrak{s}}^+(s)R(y)dy\right|^2dx\\ &\quad + C\int_1^\infty |\lambda_3^+(s)|^2\eta^{2(k+1)}(x)\left|\int_x^\infty e^{\lambda_3^+(s)(x-y)}v_3^+(s)w_3^+(s)^H R(y)dy\right|^2dx\\ &\quad + C\int_1^\infty \eta^{2(k+1)}(x)\left|\int_x^\infty e^{(x-y)M_+(s)}P_{\mathfrak{u}}^+(s)R(y)dy\right|^2dx\\ &\le C\|R\|^2_{L^2_k} + C\int_1^\infty |x|^{-2}dx\|R\|^2_{L^2_{k+2}} + C\frac{|\lambda_3^+(s)|^2}{\operatorname{Re}\lambda_3^+(s)}\int_1^\infty |x|^{-2}dx\|R\|_{L^2_{k+2}}\end{aligned}$$

$$\leq C\left(1+\frac{|\lambda_3^+(s)|^2}{\operatorname{Re}\lambda_3^+(s)}\right)\|R\|^2_{L^2_{k+2}}$$

and

$$\begin{aligned}\int_{-\infty}^{-1}\eta^{2(k+1)}|Y_\infty'(s,x)|^2dx &\leq C\int_{-\infty}^{-1}\eta^{2(k+1)}(x)e^{\beta^- x}dx\|R\|^2_{L^2_{k+2}}+C\|R\|^2_{L^2_{k+2}}\\ &\quad + C\int_{-\infty}^{-1}\eta^{2(k+1)}(x)\left|\int_0^x e^{(x-y)M_-(s)}P_{\mathfrak{s}}^-(s)R(y)dy\right|^2dx\\ &\quad + C\int_{-\infty}^{-1}\eta^{2(k+1)}(x)\left|\int_x^\infty e^{(x-y)M_-(s)}P_{\mathfrak{u}}^-(s)R(y)dy\right|^2dx\\ &\leq C\|R\|^2_{L^2_k}+C\int_{-\infty}^{-1}|x|^{-2}dx\|R\|^2_{L^2_{k+2}}\leq C\|R\|^2_{L^2_{k+2}}.\end{aligned}$$

By Lemma 5.14 we have $\frac{|\lambda_3^+(s)|^2}{\operatorname{Re}\lambda_3^+(s)}\leq C$ uniformly in Ω_c. This implies

$$\|Y_\infty'(s,\cdot)\|_{L^2_{k+1}}\leq C\|R\|_{L^2_{k+2}}.$$

□

As a consequence of Lemma 5.16 we have existence of a solution of (5.15) in $\mathcal{D}(L_\infty)\subset L^2_k$ if $r\in L^2_{k+2}$. Furthermore, the solution can be bounded by the right hand side r uniformly in Ω_c. Moreover, by Assumption 7 we conclude that the solution is unique as long as $s\in\Omega_c$ is sufficiently small.

Corollary 5.17. *Let the Assumption 1, 2, 5, 7 and 8 be satisfied and $k\in\mathbb{N}_0$. Then there is $C>0$ such that for all $s\in\Omega_c$ and $r\in L^2_{k+2}$ the equation* (5.15) *has a unique solution $u\in\mathcal{D}(L_\infty)\subset L^2_k$ and satisfies the estimate*

$$\|u\|_{L^2_k}+\|u_x\|_{L^2_{k+1}}\leq C\|f\|_{L^2_{k+2}}.$$

In particular, the operator $L_\infty:\mathcal{D}(L_\infty)\subset L^2_k\to L^2_{k+2}$ is invertible.

Proof. The case $s=0$ follows by Lemma 5.16 and Assumption 7. If $s\neq 0$ and $|s|<\varepsilon$ for some ε sufficiently small, then s is not an eigenvalue of $L_\infty\in\mathcal{C}[L^2]$, cf. Assumption 7. Hence the operator $sI-L_\infty\in\mathcal{C}[L^2]$ is one-to-one. The function $Y(s,\cdot)=(w_1,w_2)^\top\in L^2_k$ from Lemma 5.16 solves (5.17). Thus $(sI-L_\infty)w_1=r$ and since $sI-L_\infty\in\mathcal{C}[L^2]$ is one-to-one w_1 is unique in L^2. The estimate follows by Lemma 5.16. □

5.3.3 Fredholm theory: the variable coefficient operator L

We are interested in the variable coefficient operator L from (0.12). By Lemma 5.7 we have $L \in \mathcal{C}[L_k^2]$ with $\mathcal{D}(L) = H_k^2$. But $0 \in \sigma_{\text{ess}}(L)$ and therefore $L \in \mathcal{C}[L_k^2]$ is not a Fredholm operator. Typically, L has no closed range.

Remark 5.18. *Assume $L \in \mathcal{C}[L_k^2]$ has closed range. Since $\dim \mathcal{N}(L) < \infty$, due to Assumption 6, we have that L is at least a semi-Fredholm operator. But since $sI - L$, $s > 0$ is Fredholm of index 0 we conclude by Lemma A.6 that L is a Fredholm operator of index 0. This contradicts $0 \in \sigma_{\text{ess}}(L)$ and therefore the range of L cannot be closed.*

The aim of the section is to prove that L becomes a Fredholm operator of index 0 when considered as a closed operator from $\mathcal{D}(L) \subset L_k^2 \to L_{k+2}^2$. This is done by using a perturbation argument from [38] and the fact that Corollary 5.17 implies the operator $L_\infty \in \mathcal{C}[L_k^2, L_{k+2}^2]$ to be Fredholm of index 0.

Lemma 5.19. *Let Assumption 1, 2, 5, 7 and 8 be satisfied and $k \in \mathbb{N}_0$. Then the operator $L : \mathcal{D}(L) \subset L_k^2 \to L_{k+2}^2$ with $\mathcal{D}(L) = \{u \in H_k^1 \cap H_{\text{loc}}^2 : Lu \in L_{k+2}^2\}$ is a closed, densely defined, linear operator from L_k^2 to L_{k+2}^2. Moreover, $L \in \mathcal{C}[L_k^2, L_{k+2}^2]$ is a Fredholm operator of index 0.*

Proof. The closedness follows from Lemma 5.11. The Fredholm property follows by showing $(L_\infty - L)$ to be L_∞-compact and Lemma A.10. It is immediately clear that, $\mathcal{D}(L_\infty) = \mathcal{D}(L) \subset \mathcal{D}(L_\infty - L)$. It remains to show that $(L_\infty - L)L_\infty^{-1} \in L[L_{k+2}^2]$ is compact. For this purpose, recall

$$(L_\infty - L)u = (C - C_\pm)u, \quad C(x) = S_\omega + Df(v_\star(x)), \quad C_\pm(x) = \begin{cases} S_\omega + Df(v_\infty) & x \geq 0 \\ S_\omega + Df(0), & x < 0 \end{cases}.$$

Thus $L_\infty - L$ is a multiplication operator associated with $C - C_\pm \in L^\infty(\mathbb{R}, \mathbb{R}^{2,2})$. By Theorem 2.6 there are $K, \mu_\star > 0$ such that

$$|Df(v_\star(x)) - Df(v_\infty)| \leq C|v_\star(x) - v_\infty| \leq Ke^{-\mu_\star x}, \quad x \geq 0$$

and

$$|Df(v_\star(x))| \leq Ke^{\mu_\star x}, \quad x \leq 0.$$

Therefore the multiplication operator given by

$$m : H^1(\mathbb{R}, \mathbb{R}^2) \to L^2(\mathbb{R}, \mathbb{R}^2), \quad u \mapsto \eta^2(C - C_\pm)u$$

satisfies the assumption of Lemma D.4. Hence it is compact. This implies $\eta^k \eta^2 (L_\infty - L)\eta^{-k} = \eta^2(C - C_\pm)$ to be compact from H^1 to L^2 and thus, by Lemma 5.1, the operator $L_\infty - L : H_k^1 \to L_{k+2}^2$ is compact. By Lemma 5.16, $L_\infty^{-1} \in L[L_{k+2}^2, \mathcal{D}(L_\infty)] \subset L[L_{k+2}^2, H_k^1]$. Hence, $(L_\infty - L)L_\infty^{-1} \in \mathcal{C}[L_{k+2}^2]$ is compact and the assertion is proven. □

By Assumption 6 the kernel $\mathcal{N}(L) \subset L^2_k$ has dimension one. As in Lemma 3.18 we have $Lv_{\star,x} = 0$ with $v_{\star,x} \in L^2_k$ and thus $\varphi = v_{\star.x}$ is an eigenfunction, i.e.

$$\mathcal{N}(L) = \operatorname{span}\{\varphi\}.$$

Now we take the (abstract) adjoint operator $L^* : \mathcal{D}(L^*) \subset L^2_{k+2} \to L^2_k$ into account. By the Fredholm alternative we conclude that its kernel $\mathcal{N}(L^*)$ is spanned by an eigenfunction and has also dimension one. Moreover, since L, L^* are closed, densely defined, linear operators between Hilbert spaces we obtain, cf. [61, (11-7)],

$$\mathcal{N}(L)^\perp = \mathcal{R}(L^*) \tag{5.42}$$

where the orthogonal complement is taken w.r.t. $(\cdot,\cdot)_{L^2_k}$.

Lemma 5.20. *Let Assumption 1, 2 and 5-8 be satisfied and $k \in \mathbb{N}_0$. Then there is an adjoint eigenfunction $\psi \in \mathcal{D}(L^*)$ such that*

i) $\mathcal{N}(L^*) = \operatorname{span}\{\psi\} =: \Psi$,

ii) $(\psi, \varphi)_{L^2_k} = 1$,

iii) $L^2_k = \Phi \oplus \Psi^\perp$ *where the orthogonal complement is taken w.r.t.* $(\cdot,\cdot)_{L^2_k}$,

iv) there is a continuous projection $P : L^2_k \to L^2_k$ onto Φ, i.e.

$$P(\Phi) = \Phi, \quad P(\Psi^\perp) = \{0\}, \quad P^2 = P$$

which is given by

$$Pv := (\psi, v)_{L^2_k}\varphi.$$

v) the subspace $\Psi^\perp \subset L^2_k$ is invariant under L, i.e. $L(\Psi^\perp \cap H^2_k) \subset \Psi^\perp$.

Proof. We only prove ii). The other assertions follow exactly as in the proof of Lemma 3.20 and using ii). So by i) we have an eigenfunction $\psi \in \mathcal{D}(L^*) \subset L^2_{k+2}$ and L^* has no generalized eigenfunction. Assume $(\psi, \varphi)_{L^2_k} = 0$. From (5.42) we conclude $\psi \in \mathcal{N}(L)^\perp = \mathcal{R}(L^*)$. Thus there is $u \in L^2_{k+2}$ with $Lu = \varphi$. This is a contradiction and we can normalize ψ such that $(\psi, \varphi)_{L^2_k} = 1$. □

By construction of the projector P it is clear that

$$PLu = 0, \quad \forall u \in H^2_k. \tag{5.43}$$

Since $\Psi^\perp = \mathcal{R}(I-P)$ and its intersection with the smooth spaces H^ℓ_k are frequently used in the following we introduce the notation

$$V_k = \mathcal{R}(I-P), \quad V^1_k = \mathcal{R}(I-P) \cap H^1_k, \quad V^2_k = \mathcal{R}(I-P) \cap H^2_k. \tag{5.44}$$

Remark 5.21. *According to Chapter 3 there hold $LS_1v_\star = 0$ where L is applied to C^2 functions with classical derivatives. However, we have $S_1v_\star \notin L^2_k$. Hence, $S_1v_\star$ cannot be an eigenfunction and therefore is not part of the kernel of L.*

By construction we have that $\mathcal{D}(L) \subset H^1_k$ and the graph norm $\|\cdot\|_{\mathcal{D}(L)}$ of L is defined by

$$\|v\|_{\mathcal{D}(L)} = \|v\|_{L^2_k} + \|Lv\|_{L^2_{k+2}}.$$

Clearly, since $L \in \mathcal{C}[L^2_k, L^2_{k+2}]$ we have $(\mathcal{D}(L), \|\cdot\|_{\mathcal{D}(L)})$ is a Banach space. In particular, the inclusion $\mathcal{D}(L) \subset H^1_k$ is continuous. In order to see this, let $v \in \mathcal{D}(L)$ and pick $s \in \Omega_0$, $|s| > 1$ with Ω_0 from Lemma 5.7. Moreover, (5.5) implies

$$\|v\|^2_{H^1_k} \le \frac{C}{|s|}\|(sI - L)v\|^2_{L^2_k} \le 2C|s|\|v\|^2_{L^2_k} + 2C\|Lv\|^2_{L^2_k} \le \tilde{C}^2\|v\|^2_{\mathcal{D}(L)}. \tag{5.45}$$

Now we take the projector P from Lemma 5.20 into account. Then $L^{-1} : (I-P)L^2_{k+2} \to (I-P)\mathcal{D}(L)$ exists and is bounded. Using (5.45), we find $K > 0$ such that for all $u \in \mathcal{D}(L)$ there holds

$$\|(I-P)u\|_{H^1_k} \le \tilde{C}\|(I-P)u\|_{\mathcal{D}(L)} \le K\|L(I-P)u\|_{L^2_{k+2}} = K\|(I-P)Lu\|_{L^2_{k+2}}. \tag{5.46}$$

5.3.4 Resolvent estimates for small |s|

Recall the crescent Ω_c from Lemma 5.14, cf. Figure 5.3. In this section we derive estimates for the solution of the resolvent equation

$$(sI - L)u = r \in L^2_{k+2}, \quad s \in \Omega_c. \tag{5.47}$$

We transform (5.47) into a first order system via $Y = (u, u')$ and obtain

$$Y' - M(s,\cdot)Y = R \tag{5.48}$$

with $R = (0, r)^\top \in L^2_{k+2}$ and

$$M(s,x) = \begin{pmatrix} 0 & I_2 \\ A^{-1}(sI - C(x)) & -cA^{-1} \end{pmatrix}, \quad C = S_\omega + Df(v_\star).$$

We denote by $\mathcal{S}_s : \mathbb{R}^2 \to \mathbb{C}^{2,2}$ the solution operator of (5.48), i.e. the function

$$Y(s,x) := \mathcal{S}_s(x,y)\xi_0, \quad x, y \in \mathbb{R}$$

is the solution of the initial value problem

$$Y' - M(s,\cdot)Y = 0, \quad Y(y) = \xi_0. \tag{5.49}$$

The approach is similar to the one from Section 5.3.2. There we used the exponential dichotomy on $\mathbb{R}_-$ and the exponential trichotomy on $\mathbb{R}_+$ of the first order piecewise constant operator $\partial_x - M_\infty(s,\cdot)$ from (5.17) to construct solutions of (5.15) via Green's functions. We want to do the same for the first order variable coefficient operator $\partial_x - M(s,\cdot)$ from (5.48). Using the roughness of exponential dichotomies under small perturbations from [22], cf. Lemma B.3, we can immediately conclude that $\partial_x - M(s,\cdot)$ also has an exponential dichotomy on $\mathbb{R}_-$ with data arbitrary close to the data of $\partial_x - M_\infty(s,\cdot)$. However, in the case of an exponential trichotomy the exact exponential behavior of the center part is usually not preserved under small perturbation, see [31]. In [13, Lem. 2.5] it was shown that under some additional assumption the exponential behavior of the center part is preserved. In our case the additional assumption cannot be verified. Therefore we prove a generalization of [13, Lem. 2.5] and show in particular that the additional assumption can be neglected.

Theorem 5.22 (Roughness theorem). *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathcal{A}(s) = \partial_x - A(s,\cdot)$, $s \in \Omega$, $A \in C_b(\Omega \times J, \mathbb{K}^{n,n})$, $J = [0,\infty)$ have an ordinary exponential trichotomy on J for every $s \in \Omega$ with data $(K(s), \alpha(s), \nu(s), \beta(s))$, $\sup_{s\in\Omega} K(s) < \infty$ depending continuously/analytically on $s \in \Omega$ and projectors $P_\kappa(s,x)$, $\kappa = \mathfrak{s}, \mathfrak{c}, \mathfrak{u}$, $x \in J$ depending continuously/analytically on $s \in \Omega$. Further assume that $P_\mathfrak{c}(s,x)$ is of rank $m_\mathfrak{c} = 1$ and has the form $P_\mathfrak{c}(s,x) = z(s,x)\psi(s,x)^\top$ where $\mathcal{A}z(s,\cdot) = 0$ and there are $C_1, C_2 > 0$ such that*

$$C_1 \le e^{-\nu(s)x}|z(s,x)| \le C_2, \quad C_1 \le e^{\nu(s)x}|\psi(s,x)| \le C_2 \quad \forall\, x \in J,\ s \in \Omega.$$

Let $B \in C(J, \mathbb{K}^{n,n})$ with

$$|B(x)| \le C_B e^{-\varepsilon x}, \quad x \in J$$

for some $0 < 2\varepsilon < \inf_{s\in\Omega} \min(\nu(s) - \alpha(s), \beta(s) - \nu(s))$. Then the perturbed operator $\tilde{\mathcal{A}}(s) = \mathcal{A}(s) - B$ has an ordinary exponential trichotomy on J with data $(\tilde{K}, \tilde{\alpha}(s), \nu(s), \tilde{\beta}(s))$ depending continuous/analytically on $s \in \Omega$ and with $\tilde{K}$ is independent of $s \in \Omega$. Specifically, $\tilde{\alpha}(s)$ and $\tilde{\beta}(s)$ are given by

$$\tilde{\alpha}(s) = \alpha(s) + 2\delta K(s), \quad \tilde{\beta}(s) = \beta(s) - 2\delta K(s)$$

where

$$\delta \le \frac{\varepsilon}{4\max(K_\infty, K_\infty^2)}, \quad K_\infty = \sup_{s\in\Omega} K(s).$$

Proof. In the following $C = C(\varepsilon) > 0$ denote constants that are independent of $s \in \Omega$. Choose $x_0 \in [0,\infty)$ such that the following condition hold

$$4\max(K_\infty, K_\infty^2) C_B e^{-\varepsilon x_0} \le \varepsilon. \tag{5.50}$$

Then for all $s \in \Omega$ we have with $\delta := C_B e^{-\varepsilon x_0} \geq \sup_{x \geq x_0} |B(x)|$:

$$\ell := \frac{K_\infty \delta}{\varepsilon} < 1, \quad \frac{8\delta K(s)^2}{\nu(s) - \alpha(s)} < 1, \quad \frac{8\delta K(s)^2}{\beta(s) - \nu(s)} < 1 \tag{5.51}$$

and

$$2\delta K(s) \leq \frac{\varepsilon}{2} < \frac{1}{4} \min\left(\nu(s) - \alpha(s), \beta(s) - \nu(s)\right). \tag{5.52}$$

Now consider the shifted operator $\mathcal{A}_\nu(s) = \mathcal{A}(s) - \nu(s)I$ with corresponding solution operator $S_{\nu,s}(x,y)$, $x, y \in J$, i.e. $w(s,x) = S_{\nu,s}(x,y)w_0$ solves the initial value problem $L_\nu(s)w = 0$, $w(y) = w_0$. Let $P_\kappa(s,y)$, $y \in J$ denote the projectors of the ordinary exponential trichotomy of $\mathcal{A}(s)$ on J. Then we have the estimates

$$\begin{aligned} &|S_{\nu,s}(x,y)P_{\mathfrak{s}}(s,y)| \leq K(s)e^{(\alpha(s)-\nu(s))(x-y)}, \quad x \geq y \geq 0, \\ &|S_{\nu,s}(x,y)(P_{\mathfrak{c}}(s,y) + P_{\mathfrak{u}}(s,y))| \leq K(s), \quad 0 \leq x \leq y. \end{aligned}$$

Let $J_{x_0} = [x_0, \infty)$ and $\tilde{w} \in C_b(J_{x_0}, \mathbb{K}^n)$. We define

$$\begin{aligned} T(s,\tilde{w})(x) = &\int_{x_0}^{x} S_{\nu,s}(x,y)P_{\mathfrak{s}}(s,y)B(y)\tilde{w}(y)dy \\ &- \int_{x_0}^{x} S_{\nu,s}(x,y)(P_{\mathfrak{c}}(s,y) + P_{\mathfrak{u}}(s,y))B(y)\tilde{w}(y)dy. \end{aligned}$$

Note that by assumption we have $\nu(s) - \alpha(s) - \varepsilon > 0$. Then

$$\begin{aligned} |T(s,\tilde{w})(x)| &\leq K(s)C_B e^{(\alpha(s)-\nu(s))x} \int_{x_0}^{x} e^{-(\alpha(s)-\nu(s)+\varepsilon)y} dy \|\tilde{w}\|_\infty + K(s)C_B \int_x^\infty e^{-\varepsilon y} dy \|\tilde{w}\|_\infty \\ &\leq \frac{K(s)C_B}{\nu(s)-\alpha(s)-\varepsilon} e^{-\varepsilon x}\|\tilde{w}\|_\infty + \frac{K(s)C_B}{\varepsilon} e^{-\varepsilon x}\|\tilde{w}\|_\infty \leq C_T e^{-\varepsilon x}\|\tilde{w}\|_\infty \end{aligned} \tag{5.53}$$

for some $C_T > 0$ sufficiently large and independent on s. Thus T maps $\Omega \times C_b(J_{x_0}, \mathbb{K}^n)$ into $C_b(J_{x_0}, \mathbb{K}^n)$. Moreover, for $w_1, w_2 \in C_b(J_{x_0}, \mathbb{K}^n)$ we have

$$\begin{aligned} &|T(s,w_1)(x) - T(s,w_2)(x)| \\ &\leq K(s) \int_{x_0}^{x} |B(y)| dy \|w_1 - w_2\|_\infty + K(s) \int_x^\infty |B(y)| dy \|w_1 - w_2\|_\infty \\ &\leq K_\infty C_B \int_{x_0}^{\infty} e^{-\varepsilon y} dy = \ell \|w_1 - w_2\|_\infty. \end{aligned}$$

Hence by (5.51) $T(s,\cdot)$ is a contraction on $C_b(J_{x_0},\mathbb{K}^n)$. Let $w(s,x) = e^{-\nu(s)x}z(s,x)$ then, by the contraction theorem, there is a unique $\tilde{w}(s,\cdot) \in C_b(J_{x_0},\mathbb{K}^n)$ such that

$$\tilde{w}(s,x) = w(s,x) + T(s,\tilde{w}(s,\cdot))(x).$$

In addition, since $T(s,0) = 0$ we have the a-priori bound

$$\|\tilde{w}(s,\cdot)\|_\infty \le \frac{1}{1-\ell}\|w(s,\cdot)\|_\infty \le \frac{C_2}{1-\ell}.$$

Since $T(s,\cdot)$ depends continuously/analytically on s we conclude, using the implicit function theorem, cf. Theorem D.8, that $\tilde{w}(s,\cdot)$ depends continuous/analytically on s. Moreover, for $x \in J_{x_0}$ there hold

$$\begin{aligned}\tilde{w}'(s,x) &= w'(s,x) + P_{\mathfrak{s}}(s,x)B(x)\tilde{w}(s,x) + (P_{\mathfrak{c}}(s,x) + P_{\mathfrak{u}}(s,x))B(x)\tilde{w}(s,x)\\ &\quad + (A(s,x) - \nu(s)I)T(s,\tilde{w}(s,\cdot))(x)\\ &= (A(s,x) - \nu(s)I)\tilde{w}(s,x) + B(x)\tilde{w}(s,x).\end{aligned}$$

Thus $\mathcal{A}_\nu(s)\tilde{w}(s,\cdot) - B\tilde{w}(s,\cdot) = 0$ on J_{x_0} and using (5.53) we obtain

$$|w(s,x) - \tilde{w}(s,x)| = |T(s,\tilde{w}(s,\cdot))(x)| \le C_T e^{-\varepsilon x}\|\tilde{w}(s,\cdot)\|_\infty \le Ce^{-\varepsilon x} \quad \forall\, x \in J_{x_0}.$$

Now the operator $\mathcal{A}_\nu(s)$ has a shifted exponential dichotomy on J_{x_0} with data $(K(s),\alpha(s)-\nu(s),0)$ and projectors $Q_{\mathfrak{s}}(s,x) = P_{\mathfrak{s}}(s,x)$, $Q_{\mathfrak{u}}(s,x) = P_{\mathfrak{c}}(s,x) + P_{\mathfrak{u}}(s,x)$ as well as a shifted exponential dichotomy on J_{x_0} with data $(K(s),0,\beta(s)-\nu(s))$ and projectors $R_{\mathfrak{s}}(s,x) = P_{\mathfrak{s}}(s,x) + P_{\mathfrak{c}}(s,x)$, $R_{\mathfrak{u}}(s,x) = P_{\mathfrak{u}}(s,x)$. By (5.51) we can apply Lemma B.3 and obtain that $\tilde{\mathcal{A}}_\nu(s) = \mathcal{A}_\nu(s) - B$ has a shifted exponential dichotomy on J_{x_0} with data $(\frac{5}{2}K(s)^2,\tilde{\alpha}(s),\tilde{\nu}_1(s))$ and projectors $\tilde{Q}_\kappa(s,x)$, $\kappa = \mathfrak{s},\mathfrak{u}$ where

$$\tilde{\alpha}(s) = \alpha(s) - \nu(s) + 2\delta K(s), \quad \tilde{\nu}_1(s) = -2\delta K(s), \quad \tilde{\alpha}(s) < \tilde{\nu}_1(s) < 0.$$

In addition, $\tilde{\mathcal{A}}_\nu(s) = \mathcal{A}_\nu(s) - B$ has a shifted exponential dichotomy on J_{x_0} with data $(\frac{5}{2}K(s),\tilde{\nu}_2(s),\tilde{\beta}(s))$ and projectors $\tilde{R}_\kappa(s,x)$, $\kappa = \mathfrak{s},\mathfrak{u}$ where

$$\tilde{\nu}_2(s) = 2\delta K(s), \quad \tilde{\beta}(s) = \beta(s) - \nu(s) - 2\delta K(s), \quad 0 < \tilde{\nu}_2(s) < \tilde{\beta}(s).$$

Then $\mathcal{R}(\tilde{Q}_{\mathfrak{s}}(s,x_0)) \subset \mathcal{R}(\tilde{R}_{\mathfrak{s}}(s,x_0))$ and we claim that the codimension is equal to 1. On the one hand $\tilde{w}(s,x_0) \in \mathcal{R}(\tilde{R}_{\mathfrak{s}}(s,x_0))$ since

$$\begin{aligned}|\tilde{R}_{\mathfrak{u}}(s,x_0)\tilde{w}(s,x_0)| &= |\tilde{S}_{\nu,s}(x_0,x)\tilde{R}_{\mathfrak{u}}(s,x)\tilde{S}_{\nu,s}(x,x_0)\tilde{w}(s,x_0)|\\ &\le \tfrac{5}{2}K(s)e^{\tilde{\beta}(s)(x_0-x)}\tilde{S}_{\nu,s}(x,x_0)\tilde{w}(s,x_0)|\\ &\le \tfrac{5}{2}K(s)e^{\tilde{\beta}(s)(x_0-x)}|\tilde{w}(s,x)| \to 0, \quad x \to \infty.\end{aligned}$$

On the other hand, assume $\tilde{w}(s,x_0) \in \mathcal{R}(\tilde{Q}_{\mathfrak{s}}(s,x_0))$. Then

$$
\begin{aligned}
0 \neq |w(s,x_0)| &= |S_{\nu,s}(x_0,x)w(s,x)| \leq K(s)|w(s,x)| \\
&\leq K(s)|w(s,x)-\tilde{w}(s,x)| + K(s)|\tilde{S}_{\nu,s}(x,x_0)\tilde{Q}_{\mathfrak{s}}(s,x_0)\tilde{w}(s,x_0)| \\
&\leq K(s)C_T(s)e^{-\varepsilon x}\|\tilde{w}(s,\cdot)\|_\infty + \tfrac{5}{2}K(s)^2 e^{\tilde{\alpha}(s)(x-x_0)}\|\tilde{w}(s,\cdot)\|_\infty \to 0, \quad x \to \infty.
\end{aligned}
$$

This is a contradiction and thus $\mathcal{R}(\tilde{Q}_{\mathfrak{s}}(s,x_0)) \subset \mathcal{R}(\tilde{R}_{\mathfrak{s}}(s,x_0))$ with codimension 1. In addition, we have $\mathcal{N}(\tilde{R}_{\mathfrak{s}}(s,x_0)) \subset \mathcal{N}(\tilde{Q}_{\mathfrak{s}}(s,x_0))$ and $\tilde{Q}_{\mathfrak{s}}(s,x_0)\tilde{R}_{\mathfrak{u}}(s,x_0) = \tilde{R}_{\mathfrak{u}}(s,x_0)\tilde{Q}_{\mathfrak{s}}(s,x_0) = 0$. Now take $\tilde{\psi}(s,x_0)$ such that

$$
\tilde{\psi}(s,x_0)^H\tilde{w}(s,x_0) = 1, \quad \tilde{\psi}(s,x_0)^H\tilde{Q}_{\mathfrak{s}}(s,x_0) = \tilde{\psi}(s,x_0)^H\tilde{R}_{\mathfrak{u}}(s,x_0) = 0.
$$

Further, set $\tilde{\psi}(s,x) = \tilde{S}^H_{\nu,s}(x,x_0)\tilde{\psi}(s,x_0)$, where $\tilde{S}^H_{\nu,s}$ denotes the solution operator of the adjoint $\tilde{\mathcal{A}}_{\nu,s}(s)^*$. Then $\tilde{P}_{\mathfrak{c}}(s,x) = \tilde{w}(s,x)\tilde{\psi}(s,x)^H$ is the projector onto $\operatorname{span}\{\tilde{w}(s,x)\}$ satisfying

$$
\begin{aligned}
&\mathbb{K}^n = \mathcal{R}(\tilde{Q}_{\mathfrak{s}}(s,x)) \oplus \mathcal{R}(\tilde{P}_{\mathfrak{c}}(s,x)) \oplus \mathcal{R}(\tilde{R}_{\mathfrak{u}}(s,x)), \\
&\tilde{P}_{\mathfrak{c}}(s,x)\tilde{Q}_{\mathfrak{s}}(s,x) = \tilde{Q}_{\mathfrak{s}}(s,x)\tilde{P}_{\mathfrak{c}}(s,x) = 0, \\
&\tilde{P}_{\mathfrak{c}}(s,x)\tilde{R}_{\mathfrak{u}}(s,x) = \tilde{R}_{\mathfrak{u}}(s,x)\tilde{P}_{\mathfrak{c}}(s,x) = 0.
\end{aligned}
$$

Summarizing with $\tilde{P}_{\mathfrak{s}}(s,x) = \tilde{Q}_{\mathfrak{s}}(s,x)$ and $\tilde{P}_{\mathfrak{u}}(s,x) = \tilde{R}_{\mathfrak{u}}(s,x)$ we obtain

$$
I = \tilde{P}_{\mathfrak{s}}(s,x) + \tilde{P}_{\mathfrak{c}}(s,x) + \tilde{P}_{\mathfrak{u}}(s,x).
$$

Let $\mu_1(s) := \tilde{\nu}_1(s) - \alpha(s) + \nu(s)$. Then by (5.52)

$$
\mu_1(s) - \varepsilon = \tilde{\nu}_1(s) - \alpha(s) + \nu(s) - \varepsilon > -2\delta K_\infty + \varepsilon > \frac{\varepsilon}{2} > 0
$$

and the estimate in Lemma B.3 yields

$$
\begin{aligned}
|P_{\mathfrak{s}}(s,x) - \tilde{P}_{\mathfrak{s}}(s,x)| &= |Q_{\mathfrak{s}}(s,x) - \tilde{Q}_{\mathfrak{s}}(s,x)| \leq 5CK(s)^3\int_{x_0}^\infty e^{-\mu_1(s)|x-y|}e^{-\varepsilon y}dy \\
&= 5C_BK(s)^3\int_{x_0}^x e^{-\mu_1(s)(x-y)}e^{-\varepsilon y}dy + 5C_BK(s)^3\int_x^\infty e^{\mu_1(s)(x-y)}e^{-\varepsilon y}dy \\
&= \frac{5C_BK(s)^3e^{-\mu_1(s)x}}{\mu_1(s)-\varepsilon}\left(e^{(\mu_1(s)-\varepsilon)x} - e^{(\mu_1(s)-\varepsilon)x_0}\right) + \frac{5C_BK(s)^3}{\mu_1(s)+\varepsilon}e^{-\varepsilon x} \\
&\leq \frac{10C_BK(s)^3}{\mu_1(s)-\varepsilon}e^{-\varepsilon x} = \frac{10C_BK(s)^3}{\nu(s)-\alpha(s)-2\delta K(s)-\varepsilon}e^{-\varepsilon x} \leq Ce^{-\varepsilon x}.
\end{aligned}
$$

Further, we have by (5.52)

$$
\tilde{\beta}(s) - \varepsilon = \beta(s) - \nu(s) - 2\delta K(s) - \varepsilon > \varepsilon - 2\delta K(s) > \frac{\varepsilon}{2}
$$

and thus, using the estimate in Lemma B.3,

$$
\begin{aligned}
|P_{\mathfrak{u}}(s,x) - \tilde{P}_{\mathfrak{u}}(s,x)| = |R_{\mathfrak{u}}(s,x) - \tilde{R}_{\mathfrak{u}}(s,x)| &\le 5C_B K(s)^3 \int_{x_0}^{\infty} e^{-\tilde{\beta}(s)|x-y|} e^{-\varepsilon y} dy \\
&\le \frac{10 C_B K(s)^3}{\tilde{\beta}(s) - \varepsilon} e^{-\varepsilon x} \le C e^{-\varepsilon x}.
\end{aligned}
$$

This implies

$$
|P_{\mathfrak{c}}(s,x) - \tilde{P}_{\mathfrak{c}}(s,x)| \le \left(\frac{10 C_B K(s)^3}{\mu_1(s) - \varepsilon} + \frac{10 C_B K(s)^3}{\tilde{\beta}(s) - \varepsilon} \right) e^{-\varepsilon x} \le C e^{-\varepsilon x}.
$$

Now

$$
|\tilde{w}(s,x)| \ge |w(s,x)| - |\tilde{w}(s,x) - w(s,x)| \ge C_1 - C e^{-\varepsilon x_0} \ge \tilde{C}^{-1}
$$

Note that for instance in Frobenius norm there hold $|w\psi^H|_F = |w||\psi|$ for all w, ψ. Thus, we obtain

$$
\begin{aligned}
|\psi(s,x) - \tilde{\psi}(s,x)| &= |\tilde{w}(s,x)|^{-1} |\tilde{w}(s,x)(\psi(s,x) - \tilde{\psi}(s,x))^H| \\
&\le \tilde{C} |\tilde{w}(s,x)(\psi(s,x) - \tilde{\psi}(s,x))^H| \\
&\le \tilde{C} |P_{\mathfrak{c}}(s,x) - \tilde{P}_{\mathfrak{c}}(s,x)| + \tilde{C} |\tilde{w}(s,x) - w(s,x)||\psi(s,x)| \le C e^{-\varepsilon x}.
\end{aligned}
$$

Hence,

$$
|\tilde{\psi}(s,x)| \le |\psi(s,x)| + |\tilde{\psi}(s,x) - \psi(s,x)| \le C_2 + C e^{-\varepsilon x_0}.
$$

This implies for all $x, y \in [x_0, \infty)$

$$
|\tilde{S}_{\nu,s}(x,y)\tilde{P}_{\mathfrak{c}}(y)| = |\tilde{w}(s,x)||\tilde{\psi}(s,y)| \le \frac{C_2(C_2 + C e^{-\varepsilon x})}{1 - \ell} = \bar{K}.
$$

Finally we have shown that the operator $\tilde{\mathcal{A}}(s) = \mathcal{A}(s) - B$ has an ordinary exponential trichotomy on $J_{x_0} = [x_0, \infty)$ with data $(K_0, \tilde{\alpha}(s), \nu(s), \tilde{\beta}(s))$ where

$$
K_0 = \max\left(\frac{5}{2} K_\infty, \bar{K} \right), \quad \tilde{\alpha}(s) = \alpha(s) + 2\delta K(s), \quad \tilde{\beta}(s) = \beta(s) - 2\delta K(s).
$$

Since $A \in C_b(\Omega \times J, \mathbb{K}^{n,n})$ we find $\tilde{K}$ sufficiently large such that $\tilde{\mathcal{A}}(s) = \mathcal{A}(s) - B$ has an ordinary exponential trichotomy on $J = [0, \infty)$ with data $(\tilde{K}, \tilde{\alpha}(s), \nu(s), \tilde{\beta}(s))$. $\square$

Armed with this tool we are now in the position to conclude that the operator $\partial_x - M(s, \cdot)$ has also an exponential trichotomy on $\mathbb{R}_+$ with projectors denoted by $\mathcal{P}^\pm(s)$ and the exponential rate of the center part is given by $\nu(s) = \operatorname{Re} \lambda_3^+(s)$. As in Section 5.3.2 we denote by $m_\kappa^\pm(s)$ the ranks of the projectors $\mathcal{P}_\kappa^\pm(s)$.

Lemma 5.23. *Let Assumption 1, 2 and 5-8 be satisfied and let* $\nu(s) = \operatorname{Re}\lambda_3^+(s)$. *Then there is* $\varepsilon > 0$ *and constants* $K > 0$, $\tilde{\alpha}^\pm < 0 < \tilde{\beta}^\pm$ *such that for all* $s \in B_\varepsilon(0)$ *there hold:*

i) The operator $\partial_x - M(s,\cdot)$ *has an ordinary exponential trichotomy on* $\mathbb{R}_+$ *with data* $(K, \tilde{\alpha}^+, \nu(s), \tilde{\beta}^+)$ *and projector-valued functions*

$$\mathcal{P}_\kappa^+ : B_\varepsilon(0) \times \mathbb{R}_+ \to \mathbb{C}^{4,4}, \quad \kappa = \mathfrak{s}, \mathfrak{c}, \mathfrak{u}, \quad m_\mathfrak{s}^+(s) = 2, \quad m_\mathfrak{u}^+(s) = m_\mathfrak{c}^+(s) = 1$$

depending analytically on $s \in B_\varepsilon(0)$ *and such that for all* $x, y \in \mathbb{R}_+$ *there hold*

$$\begin{aligned} &\mathcal{S}_s(x,y)\mathcal{P}_\kappa^+(s,y) = \mathcal{P}_\kappa^+(s,x)\mathcal{S}_s(x,y), && \kappa = \mathfrak{s}, \mathfrak{c}, \mathfrak{u} \\ &|\mathcal{S}_s(x,y)\mathcal{P}_\mathfrak{s}^+(s,y)| \le K e^{\tilde{\alpha}^+(x-y)}, && |\mathcal{S}_s(x,y)\mathcal{P}_\mathfrak{c}^+(s,y)| \le K e^{\nu(s)(x-y)}, \quad x \ge y, \\ &|\mathcal{S}_s(x,y)\mathcal{P}_\mathfrak{u}^+(s,y)| \le K e^{\tilde{\beta}^+(x-y)}, && |\mathcal{S}_s(x,y)\mathcal{P}_\mathfrak{c}^+(s,y)| \le K e^{\nu(s)(x-y)}, \quad x \le y. \end{aligned}$$

ii) The operator $\partial_x - M(s,\cdot)$ *has an exponential dichotomy on* $\mathbb{R}_-$ *with data* $(K, \tilde{\alpha}^-, \tilde{\beta}^-)$ *and projector-valued functions*

$$\mathcal{P}_\kappa^- : B_\varepsilon(0) \times \mathbb{R}_- \to \mathbb{C}^{4,4}, \quad \kappa = \mathfrak{s}, \mathfrak{u}, \quad m_\mathfrak{s}^-(s) = m_\mathfrak{u}^-(s) = 2, \quad (s,x) \in B_\varepsilon(0) \times \mathbb{R}_-$$

depending analytically on $s \in B_\varepsilon(0)$ *and such that for all* $x, y \in \mathbb{R}_-$ *there hold*

$$\begin{aligned} &\mathcal{S}_s(x,y)\mathcal{P}_\kappa^-(s,y) = \mathcal{P}_\kappa^-(s,x)\mathcal{S}_s(x,y), && \kappa = \mathfrak{s}, \mathfrak{u}, \\ &|\mathcal{S}_s(x,y)\mathcal{P}_\mathfrak{s}^-(s,y)| \le K e^{\tilde{\alpha}^-(x-y)}, && x \ge y, \\ &|\mathcal{S}_s(x,y)\mathcal{P}_\mathfrak{u}^-(s,y)| \le K e^{\tilde{\beta}^-(x-y)}, && x \le y. \end{aligned}$$

Proof. **i).** For all $s \in B_\varepsilon(0)$ the operator $\partial_x - M_\infty(s,\cdot)$ has an ordinary exponential trichotomy on $\mathbb{R}_+$ with data $(K, \alpha^+, \nu(s), \beta^+)$ where $K > 0$ can be taken independent on $s \in B_\varepsilon(0)$ and projectors $P_\kappa(s,\cdot)$, $\kappa = \mathfrak{s}, \mathfrak{c}, \mathfrak{u}$ depending analytically on $s \in B_\varepsilon(0)$ and given by

$$P_\mathfrak{s}(s,x) = V_\mathfrak{s}^+(s)W_\mathfrak{s}^+(s)^H, \quad P_\mathfrak{c}(s,x) = v_3^+(s)w_3^+(s)^H, \quad P_\mathfrak{u}(s,x) = v_4^+(s)w_4^+(s)^H.$$

Now $P_\mathfrak{c}(s,x)$ is of rank $m_\mathfrak{c}(s) = 1$ and can be written as $P_\mathfrak{c}(s,x) = z(s,x)\psi(s,x)^H$ where $z(s,x) = e^{\lambda_3^+(s)x}v_3(s)$, $\psi(s,x) = e^{-\overline{\lambda_3^+(s)x}}w_3(s)$. Then there are $C_1, C_2 > 0$ such that for all $s \in \Omega_c$

$$C_1 = |v_3(s)| = e^{-\nu(s)x}|z(s,x)|, \quad C_2 = |w_3(s)| = e^{\nu(s)x}|\psi(s,x)|.$$

Further set $B(x) = M_\infty(s,x) - M(s,x)$. Then B is independent of s and by Theorem 2.6 we have for some $C > 0$

$$|B(x)| = |A^{-1}||Df(v_\infty) - Df(v_\star(x))| \le Ce^{-\mu_\star x}, \quad x \in \mathbb{R}_+.$$

Take $\mu_\star$ so small such that

$$0 < 2\mu_\star < \inf_{s\in B_\varepsilon(0)}\{\min(\nu(s)-\alpha^+, \beta^+-\nu(s))\}, \quad \frac{\mu_\star}{\max(K,K^2)} < \min(|\alpha^+|, \beta^+).$$

Then we can apply Lemma 5.22 and obtain that the perturbed operator $\partial_x - M(s,\cdot)$ has an ordinary exponential trichotomy with data $(\tilde{K}, \tilde{\alpha}^+, \nu(s), \tilde{\beta}^+)$ given by

$$\tilde{\alpha}^+ = \alpha^+ + \frac{\mu_\star}{2\max(K,K^2)}, \quad \tilde{\beta}^+ = \beta^+ - \frac{\mu_\star}{2\max(K,K^2)},$$

and constant $\tilde{K} > 0$ which is independent on $s \in B_\varepsilon(0)$, and projectors $\mathcal{P}_\kappa^+$, $\kappa = \mathfrak{s}, \mathfrak{c}, \mathfrak{u}$ depending analytically on $s \in B_\varepsilon(0)$.
ii). By (5.18) the operator $\partial_x - M_\infty(s,\cdot)$ has an exponential dichotomy on $\mathbb{R}_-$ for all $s \in B_\varepsilon(0)$. Now we have for $x \le 0$ and $s \in B_\varepsilon(0)$ using Theorem 2.6 and Assumption 2

$$|M_\infty(s,x) - M(s,x)| \le |A^{-1}||Df(0) - Df(v_\star(x))| \le Ce^{\mu_\star x}.$$

Then the claim is a consequence of Lemma B.3. □

Now we are able to construct a solution of (5.48) via Green's functions in the same fashion as in Section 5.3.2 for the piecewise constant coefficient system (5.17). For $\zeta_+(s) \in \mathcal{R}(\mathcal{P}_\mathfrak{s}^+(s,0))$, $x \in \mathbb{R}_+$ and $s \in \Omega_c$ let

$$Y_+(s,x) = \mathcal{S}_s(x,0)\zeta_+(s) + \int_0^\infty G_s^+(x,y)R(y)dy \tag{5.54}$$

with the Green's function

$$G_s^+(x,y) = \begin{cases} \mathcal{S}_s(x,y)\mathcal{P}_\mathfrak{s}^+(s,y), & 0 \le y \le x \\ -\mathcal{S}_s(x,y)(\mathcal{P}_\mathfrak{c}^+(s,y) + \mathcal{P}_\mathfrak{u}^+(s,y)), & 0 \le x < y \end{cases}.$$

For $\zeta_-(s) \in \mathcal{R}(\mathcal{P}_\mathfrak{u}^-(s,0))$, $x \in \mathbb{R}_-$ and $s \in \Omega_c$ let

$$Y_-(s,x) = \mathcal{S}_s(x,0)\zeta_-(s) + \int_{-\infty}^0 G_s^-(x,y)R(y)dy \tag{5.55}$$

with the Green's function

$$G_s^-(x,y) = \begin{cases} -\mathcal{S}_s(x,y)\mathcal{P}_\mathfrak{u}^-(s,y), & x \le y \le 0 \\ \mathcal{S}_s(x,y)\mathcal{P}_\mathfrak{s}^-(s,y), & y < x \le 0 \end{cases}.$$

Then it is easy to verify that $Y_\pm(s,\cdot)$ solves (5.48) on $\mathbb{R}_\pm$. Moreover, the solutions can be represented in the following form:

$$\begin{aligned} Y_+(s,x) = {} & \mathcal{S}_s(x,0)\zeta_+(s) + \int_0^x \mathcal{S}_s(x,y)\mathcal{P}_\mathfrak{s}^+(s,y)R(y)dy \\ & + \int_x^\infty \mathcal{S}_s(x,y)\mathcal{P}_\mathfrak{c}^+(s,y)R(y)dy + \int_x^\infty \mathcal{S}_s(x,y)\mathcal{P}_\mathfrak{u}^+(s,y)R(y)dy \end{aligned}$$

and

$$Y_-(s,x) = \mathcal{S}_s(x,0)\zeta_-(s) + \int_{-\infty}^{x} \mathcal{S}_s(x,y)\mathcal{P}_{\mathfrak{s}}^-(s,y)R(y)dy + \int_{x}^{0} \mathcal{S}_s(x,y)\mathcal{P}_{\mathfrak{u}}^-(s,y)R(y)dy.$$

It is clear, since $R \in L^2_{k+2}$, that $Y_\pm(s,\cdot) \in H^1_{\mathrm{loc}}(\mathbb{R}_\pm, \mathbb{C}^4)$. By Assumption 6 we can assume for all $s \in \Omega_c\backslash\{0\}$ the equation

$$(sI - L)u = 0, \quad u \in L^2 \tag{5.56}$$

has no solution $u \in L^2$ except the trivial one $u = 0$. Otherwise decrease ε. Moreover, (5.56) with $s = 0$ has only one nontrivial solution in L^2 given by $v_{\star,x}$, i.e. $Lv_{\star,x} = 0$. Note that also $LS_1 v_\star = 0$, but $S_1 v_\star \notin L^2$. By Lemma 5.23 we have the projectors $\mathcal{P}_{\mathfrak{s}}^+(s) = \mathcal{P}_{\mathfrak{s}}^+(s,0)$ and $\mathcal{P}_{\mathfrak{u}}^-(s) = \mathcal{P}_{\mathfrak{u}}^-(s,0)$ depending analytically on $s \in B_\varepsilon(0)$ and satisfying the decompositions

$$\mathbb{C}^4 = \mathcal{R}(\mathcal{P}_{\mathfrak{s}}^+(s,0)) \oplus \mathcal{R}(I - \mathcal{P}_{\mathfrak{s}}^+(s,0)), \quad \mathbb{C}^4 = \mathcal{R}(\mathcal{P}_{\mathfrak{u}}^-(s,0)) \oplus \mathcal{R}(I - \mathcal{P}_{\mathfrak{u}}^-(s,0)).$$

However, since $\mathcal{N}(L)$ is non-trivial the decomposition

$$\mathbb{C}^4 = \mathcal{R}(\mathcal{P}_{\mathfrak{s}}^+(s,0)) \oplus \mathcal{R}(\mathcal{P}_{\mathfrak{u}}^-(s,0))$$

holds true for $s \in \Omega_c\backslash\{0\}$ but fails for $s = 0$. In particular, the projectors $\mathcal{P}_{\mathfrak{s}}^+(s,0), \mathcal{P}_{\mathfrak{u}}^-(s,0)$ are unbounded as $s \to 0$. In the next step we show that the projectors can be chosen such that the singularity at $s = 0$ is of order one. It turns out that this is sufficient to derive suitable resolvent estimates since the singularity only act on a finite dimensional subspace given by the range of the projector P from Lemma 5.20.

Lemma 5.24. *Let Assumption 1, 2 and 5-8 be satisfied. Then there is $\varepsilon > 0$ such that for $s \in B_\varepsilon(0)\backslash\{0\}$ there are projectors $\mathcal{Q}_{\mathfrak{s}}^+(s), \mathcal{Q}_{\mathfrak{u}}^-(s)$ depending analytically on $s \in B_\varepsilon(0)\backslash\{0\}$ and satisfying*

$$\begin{aligned} &\mathcal{R}(\mathcal{Q}_{\mathfrak{s}}^+(s)) = \mathcal{R}(\mathcal{P}_{\mathfrak{s}}^+(s,0)), \quad \mathcal{R}(\mathcal{Q}_{\mathfrak{u}}^-(s)) = \mathcal{R}(\mathcal{P}_{\mathfrak{u}}^-(s,0)), \\ &\mathbb{C}^4 = \mathcal{R}(\mathcal{Q}_{\mathfrak{s}}^+(s)) \oplus \mathcal{R}(\mathcal{Q}_{\mathfrak{u}}^-(s)). \end{aligned}$$

Moreover, there is $C > 0$ such that for all $s \in B_\varepsilon(0)\backslash\{0\}$ there hold the estimate

$$|\mathcal{Q}_{\mathfrak{s}}^+(s)|, |\mathcal{Q}_{\mathfrak{u}}^-(s)| \le \frac{C}{|s|}. \tag{5.57}$$

Proof. In the proof we fix $x = 0$ and neglect the dependence of the projectors on x, i.e. we write $\mathcal{P}_{\mathfrak{s},\mathfrak{u}}^\pm(s) = \mathcal{P}_{\mathfrak{s},\mathfrak{u}}^\pm(s,0)$ We choose a basis $\{\varphi_1, \varphi_2\}$ of $\mathcal{R}(\mathcal{P}_{\mathfrak{s}}^+(0))$ and $\{\varphi_3, \varphi_4\}$ of $\mathcal{R}(\mathcal{P}_{\mathfrak{u}}^-(0))$. W.l.o.g. we can assume $\varphi_1 = \varphi_3$ and for

$$J_0(x) = \begin{pmatrix} v_{\star,x}(x) \\ v_{\star,xx}(x) \end{pmatrix}.$$

we have $J_0(0) = \varphi_1 = \varphi_3$. Let ε be sufficiently small. Then for $s \in B_\varepsilon(0)$ we have, since $\mathcal{P}^{\pm}_{\mathfrak{s},\mathfrak{u}}(s)$ are analytic in s,

$$(\|\mathcal{P}^+_{\mathfrak{s}}(0)\| + \|\mathcal{P}^+_{\mathfrak{s}}(s)\|)\|\mathcal{P}^+_{\mathfrak{s}}(0) - \mathcal{P}^+_{\mathfrak{s}}(s)\| < 1,$$
$$(\|\mathcal{P}^-_{\mathfrak{u}}(0)\| + \|\mathcal{P}^-_{\mathfrak{u}}(s)\|)\|\mathcal{P}^-_{\mathfrak{u}}(0) - \mathcal{P}^-_{\mathfrak{u}}(s)\| < 1.$$

Then Lemma A.12 implies with $I + H = P^+_{\mathfrak{s}}(s)P^+_{\mathfrak{s}}(0) + (I - P^+_{\mathfrak{s}}(s))(I - P^+_{\mathfrak{s}}(0))$, and $I + H = P^-_{\mathfrak{u}}(s)P^-_{\mathfrak{u}}(0) + (I - P^-_{\mathfrak{u}}(s))(I - P^-_{\mathfrak{u}}(0))$ respectively, that

$$\varphi_i(s) = \mathcal{P}^+_{\mathfrak{s}}(s)\varphi_i, \quad i = 1, 2, \qquad \varphi_j(s) = \mathcal{P}^-_{\mathfrak{u}}(s)\varphi_j, \quad j = 3, 4$$

form a basis of $\mathcal{R}(\mathcal{P}^+_{\mathfrak{s}}(s))$, and $\mathcal{R}(\mathcal{P}^-_{\mathfrak{u}}(s))$ respectively, and are analytic in $s \in B_\varepsilon(0)$. Now let $\Phi(s) = (\Phi_+(s), \Phi_-(s))$ with $\Phi_+(s) = (\varphi_1(s), \varphi_2(s))$ and $\Phi_-(s) = (\varphi_3(s), \varphi_4(s))$. By Assumption 6 we have $\mathcal{N}(L) = \operatorname{span}\{v_{\star,x}\}$ and $\mathcal{N}(L^2) = \{0\}$. Further, $\det(\Phi(0)) = 0$ and since Φ is analytic in $s \in B_\varepsilon(0)$ and Assumption 6 we conclude $\Phi(s) \neq 0$ for $s \neq 0$. Moreover, with $v = (1, 0, -1, 0)^\top$ it follows $\Phi(0)v = 0$ and since $v_{\star,x}$ is the only eigenfunction of L it follows $\mathcal{N}(\Phi(0)) = \operatorname{span}\{v\}$. Thus $\mathcal{R}(\Phi(0)) \subset \mathbb{C}^4$ with codimension equal to 1 and there is $w \in \mathbb{C}^4$ such that $w^H\Phi(0) = 0$. Next we show that $w^H\Phi'(0)v \neq 0$. For this purpose assume the contrary. Then $\Phi'(0)v \in \mathcal{R}(\Phi(0)) = \mathcal{R}(\mathcal{P}^+_{\mathfrak{s}}(0)) + \mathcal{R}(\mathcal{P}^-_{\mathfrak{u}}(0))$. Now define for $i = 1, 2, 3, 4$ the Jost solutions, cf. [36, Ch. 9],

$$J_i(s, x) = \mathcal{S}_s(x, 0)\varphi_i(s), \quad x \in \mathbb{R}.$$

Then $J_1(0, x) = J_3(0, x) = J_0(x)$, $J_i(s, \cdot)$ are analytic in s, solve (5.48) with $R = 0$ and satisfy the estimates

$$|J_{1,2}(s, x)| \leq \tilde{K}e^{\tilde{\alpha}^+ x}, \quad x \geq 0, \qquad |J_{3,4}(s, x)| \leq \tilde{K}e^{\tilde{\beta}^- x}, \quad x \leq 0 \tag{5.58}$$

where $\tilde{K}, \tilde{\alpha}^+, \tilde{\beta}^-$ are given by Lemma 5.23. Moreover,

$$\Phi'(s) = (\partial_s J_1, \partial_s J_2, \partial_s J_3, \partial_s J_4)(s, 0).$$

By differentiating (5.48) w.r.t. s, we obtain that $\partial_s J_1(0, \cdot)$ and $\partial_s J_3(0, \cdot)$ solve the inhomogeneous equation

$$Y' - M(0, \cdot)Y = \begin{pmatrix} 0 & 0 \\ A^{-1} & 0 \end{pmatrix} J_0. \tag{5.59}$$

Using Cauchy's integral formula and (5.58) we obtain for $x \geq 0$

$$|\partial_s J_1(0, x)| = \left| \frac{1}{2\pi i} \int_{\partial B_{\frac{\varepsilon}{2}}(0)} \frac{J_1(\lambda, x)}{\lambda^2} d\lambda \right| \leq \frac{2}{\varepsilon}\tilde{K}e^{\tilde{\alpha}^+ x} \tag{5.60}$$

as well as for $x \le 0$

$$|\partial_s J_3(0,x)| = \left| \frac{1}{2\pi i} \int_{\partial B_{\frac{\varepsilon}{2}}(0)} \frac{J_3(\lambda,x)}{\lambda^2} d\lambda \right| \le \frac{2}{\varepsilon} \tilde{K} e^{\tilde{\beta}^- x}. \tag{5.61}$$

Now since $\Phi'(0)v \in \mathcal{R}(\mathcal{P}_{\mathfrak{s}}^+(0)) + \mathcal{R}(\mathcal{P}_{\mathfrak{u}}^-(0)) = \operatorname{span}\{J_0(0,0), J_2(0,0), J_4(0,0)\}$ and $v = (1,0,-1,0)^\top$ we find $\gamma_i \in \mathbb{R}$, $i = 1,2,3$ such that

$$\partial_s J_1(0,0) - \partial_s J_3(0,0) = \Phi'(0)v = \gamma_1 J_2(0,0) + \gamma_2 J_4(0,0) + \gamma_3 J_0(0,0).$$

Now setting

$$Y(x) = \begin{cases} \gamma_1 J_2(0,x) + \gamma_3 J_0(0,x) - \partial_s J_1(0,x), & x \ge 0 \\ -\gamma_2 J_4(0,x) - \partial_s J_3(0,x), & x < 0 \end{cases}.$$

Then Y is continuous and solves (5.59). Using (5.58), (5.60) and (5.61) there is $C > 0$ such that

$$|Y(x)| \le C e^{\tilde{\alpha}^+ x}, \quad x \ge 0, \qquad |Y(x)| \le C e^{\tilde{\beta}^- x}, \quad x < 0.$$

Hence $Y \in H^1$. Let $Y = (y_1, y_2)^\top$ with $y_i(x) \in \mathbb{C}^2$, $i = 1,2$. Then we obtain from (5.59) and a short calculation that $Ly_1 = v_{\star,x}$. Thus y_1 defines a generalized eigenfunction of L and we arrive at a contradiction. Hence $w^H \Phi'(0)v \ne 0$ and we can normalize w, v such that $|v| = 1$ and $w^H \Phi'(0)v = 1$. Now we can apply Keldysh's Theorem D.3 and find a holomorphic function $\Gamma : B_\varepsilon(0) \to \mathbb{C}^{4,4}$ with ε again sufficiently small such that for all $s \in B_\varepsilon(0) \backslash \{0\}$ there hold

$$\Phi(s)^{-1} = \frac{1}{s} v w^H + \Gamma(s). \tag{5.62}$$

Now let

$$\Psi_+(s) = \Phi(s)^{-H} \begin{pmatrix} I_2 \\ 0 \end{pmatrix}, \quad \Psi_-(s) = \Phi(s)^{-H} \begin{pmatrix} 0 \\ I_2 \end{pmatrix}$$

and define the projectors

$$\mathcal{Q}_{\mathfrak{s}}^+(s) = \Phi_+(s)\Psi_+(s)^H, \quad \mathcal{Q}_{\mathfrak{u}}^-(s) = \Phi_-(s)\Psi_-(s)^H.$$

Then an elementary calculation shows

$$\mathcal{Q}_{\mathfrak{s}}^+(s)\mathcal{Q}_{\mathfrak{u}}^-(s) = \mathcal{Q}_{\mathfrak{u}}^-(s)\mathcal{Q}_{\mathfrak{s}}^+(s) = 0.$$

This yields for all $s \in B_\varepsilon(0)\backslash\{0\}$

$$\begin{aligned}&\mathcal{R}(\mathcal{Q}_{\mathfrak{s}}^{+}(s)) = \mathcal{R}(\mathcal{P}_{\mathfrak{s}}^{+}(s)), \quad \mathcal{R}(\mathcal{Q}_{\mathfrak{u}}^{-}(s)) = \mathcal{R}(\mathcal{P}_{\mathfrak{u}}^{-}(s)),\\ &\mathbb{C}^4 = \mathcal{R}(\mathcal{Q}_{\mathfrak{s}}^{+}(s)) \oplus \mathcal{R}(\mathcal{Q}_{\mathfrak{u}}^{-}(s)), \quad I = \mathcal{Q}_{\mathfrak{s}}^{+}(s) + \mathcal{Q}_{\mathfrak{u}}^{-}(s).\end{aligned}$$

Moreover there is $C > 0$ such that for all $s \in B_\varepsilon(0)\backslash\{0\}$ we have by (5.62)

$$|\mathcal{Q}_{\mathfrak{s}}^{+}(s)|, |\mathcal{Q}_{\mathfrak{u}}^{-}(s)| \le \frac{C}{|s|}.$$

□

As in Section 5.3.2 we now choose $\zeta_+(s), \zeta_-(s)$ in (5.54), (5.55) such that the function

$$Y(s,x) = \begin{cases} Y_+(s,x), & x \ge 0 \\ Y_-(s,x), & x < 0 \end{cases} \tag{5.63}$$

is continuous in $x = 0$. Then $Y \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^4)$ and solves (5.48) on $\mathbb{R}$. For that purpose take

$$\begin{aligned}\zeta_+(s) &= \mathcal{Q}_{\mathfrak{s}}^{+}(s) \int_{\mathbb{R}} G_s(y) R(y) dy \in \mathcal{R}(\mathcal{P}_{\mathfrak{s}}^{+}(s,0)),\\ \zeta_-(s) &= -\mathcal{Q}_{\mathfrak{u}}^{-}(s) \int_{\mathbb{R}} G_s(y) R(y) dy \in \mathcal{R}(\mathcal{P}_{\mathfrak{u}}^{-}(s,0))\end{aligned} \tag{5.64}$$

with

$$G_s(y) = \begin{cases} -G_s^{+}(0,y), & y \ge 0 \\ G_s^{-}(0,y), & y < 0 \end{cases}.$$

Then the previous construction of the projectors $Q_{\mathfrak{s}}^{+}, Q_{\mathfrak{u}}^{-}$ in Lemma 5.24 implies

$$\begin{aligned}Y_+(s,0) - Y_-(s,0) &= \zeta_+(s) - \zeta_-(s) + \int_0^\infty G_s^{+}(0,y) R(y) dy - \int_{-\infty}^0 G_s^{-}(0,y) R(y) dy\\ &= \zeta_+(s) - \zeta_-(s) - \int_{\mathbb{R}} G_s(y) R(y) dy = 0.\end{aligned}$$

Hence $Y(s,\cdot)$ from (5.63) is continuous at $x = 0$. Roughly speaking, we see by Lemma 5.24 that the singularity at $s = 0$ of the resolvent $(sI - L)^{-1}$ caused by the single 'eigenvalue' $s = 0$ is of order one. However, the essential spectrum still touches the imaginary axis. As in the case of L_∞ in Lemma 5.16 we are able to preserve the weak singularity $|s|^{-1}$ by choosing different polynomial weights.

Lemma 5.25. *Let Assumption 1, 2 and 5-8 be satisfied and $k \in \mathbb{N}_0$. Then there is $C > 0$ such that for all $R \in L^2_{k+1}$ and $s \in \Omega_c \backslash \{0\}$ the function $Y(s, \cdot) \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^4)$ from (5.63) is a solution in L^2_k of (5.48) with*

$$\|Y(s,\cdot)\|_{L^2_k} \leq \frac{C}{|s|}\|R\|_{L^2_{k+1}}. \tag{5.65}$$

Proof. We have already seen that $Y \in H^1_{\mathrm{loc}}$ is a solution of (5.48) and is continuous. Therefore, it remains to show the estimate (5.65). We frequently use the estimates from Lemma 5.15 and Lemma 5.23 and the Cauchy-Schwarz inequality. Recall $Y_\pm$ from (5.54), (5.55) and let $C > 0$ denote a universal constant independent on s. Moreover, let Ω_c be sufficiently small in the sense that ε is sufficiently small in the definition of Ω_c from Lemma 5.14. By Lemma 5.14, cf. (5.24), we have $\partial_s \lambda_3^+(0) \neq 0$. Since λ_3^+ is analytic in s we obtain for some $C_1 > 0$ for all $s \in \Omega_c \backslash \{0\}$

$$\frac{|s|}{|\lambda_3^+(s)|} = \frac{|s|}{|s\partial_s\lambda_3^+(0) + \mathcal{O}(|s|^2)|} \leq \frac{1}{|\partial_s\lambda_3^+(0) + \mathcal{O}(|s|)|} \leq C_1.$$

Then using Lemma 5.14 we obtain

$$|s| \leq C_1|\lambda_3^+(s)| \leq C\sqrt{\operatorname{Re}\lambda_3^+(s)} = C\sqrt{\nu(s)}. \tag{5.66}$$

Let us estimate $Y_\pm$. Using $|G_s(y)| \leq K$ for all $(s,y) \in \Omega_c \times \mathbb{R}$, Lemma 5.24 and (5.64) we observe

$$|\zeta_\pm(s)|^2 \leq \frac{C}{|s|^2}\left|\int_{\mathbb{R}} G_s(y)R(y)dy\right|^2 \leq \frac{C}{|s|^2}\int_{\mathbb{R}} K^2\eta^{-2(k+1)}(y)dy\|R\|^2_{L^2_{k+1}} \leq \frac{C}{|s|^2}\|R\|^2_{L^2_{k+1}}.$$

This implies

$$\int_0^\infty \eta^{2k}(x)|\mathcal{S}_s(0,x)\zeta_+(s)|^2dx \leq \frac{C}{|s|^2}\|R\|^2_{L^2_{k+1}}\int_0^\infty \eta^{2k}(x)e^{\tilde{\alpha}^+x}dx \leq \frac{C}{|s|^2}\|R\|^2_{L^2_{k+1}} \tag{5.67}$$

as well as

$$\int_{-\infty}^0 \eta^{2k}(x)|\mathcal{S}_s(0,x)\zeta_-(s)|^2dx \leq \frac{C}{|s|^2}\|R\|^2_{L^2_{k+1}}\int_{-\infty}^0 \eta^{2k}(x)e^{\tilde{\beta}^+x}dx \leq \frac{C}{|s|^2}\|R\|^2_{L^2_{k+1}}. \tag{5.68}$$

Next we estimate $Y(s,\cdot)$ at $+\infty$. Let $x \geq 1$ and use Cauchy-Schwarz inequality and Lemma 5.15 to obtain

$$\begin{aligned}\eta^{2k}(x)\left|\int_x^\infty \mathcal{S}_s(x,y)\mathcal{P}^+_{\mathfrak{c}}(s,y)R(y)dy\right|^2 &\leq C\int_x^\infty \frac{|x|^{2k}}{|y|^{2(k+1)}}e^{2\nu(s)(x-y)}dy\|R\|^2_{L^2_{k+1}} \\ &\leq \frac{C}{\nu(s)}|x|^{-2}\|R\|^2_{L^2_{k+1}}\end{aligned} \tag{5.69}$$

as well as since $\tilde{\beta}^+ > 0$

$$\begin{aligned}\eta^{2k}(x)\left|\int_x^\infty \mathcal{S}_s(x,y)\mathcal{P}_{\mathfrak{u}}^+(s,y)R(y)dy\right|^2 &\le C\int_x^\infty \frac{|x|^{2k}}{|y|^{2(k+1)}}e^{2\tilde{\beta}^+(x-y)}dy\|R\|^2_{L^2_{k+1}} \\ &\le C|x|^{-2}\|R\|^2_{L^2_{k+1}}.\end{aligned} \tag{5.70}$$

Moreover, since $\tilde{\alpha}^+ < 0$, we have

$$\begin{aligned}&\eta^{2k}(x)\left|\int_0^x \mathcal{S}_s(x,y)\mathcal{P}_{\mathfrak{s}}^+(s,y)R(y)dy\right|^2 \\ &\le \eta^{2k}(x)\left|\int_0^1 \mathcal{S}_s(x,y)\mathcal{P}_{\mathfrak{s}}^+(s,y)R(y)dy\right|^2 + \eta^{2k}(x)\left|\int_1^x \mathcal{S}_s(x,y)\mathcal{P}_{\mathfrak{s}}^+(s,y)R(y)dy\right|^2 \\ &\le C\eta^{2k}(x)e^{2\tilde{\alpha}^+x}\|R\|^2_{L^2_{k+1}} + C\int_1^x \frac{|x|^{2k}}{|y|^{2(k+1)}}e^{2\tilde{\alpha}^+(x-y)}dy\|R\|^2_{L^2_{k+1}} \le C|x|^{-2}\|R\|^2_{L^2_{k+1}}.\end{aligned} \tag{5.71}$$

Now (5.67), (5.69), (5.70), (5.71) imply

$$\begin{aligned}\int_1^\infty \eta^{2k}(x)|Y(s,x)|^2dx &\le \frac{C}{|s|^2}\|R\|^2_{L^2_{k+1}} + C\int_1^\infty \eta^{2k}(x)\left|\int_0^x \mathcal{S}_s(x,y)\mathcal{P}_{\mathfrak{s}}^+(s,y)R(y)dy\right|^2 dx \\ &\quad + C\int_1^\infty \eta^{2k}(x)\left|\int_x^\infty \mathcal{S}_s(x,y)\mathcal{P}_{\mathfrak{c}}^+(s,y)R(y)dy\right|^2 dx \\ &\quad + C\int_1^\infty \eta^{2k}(x)\left|\int_x^\infty \mathcal{S}_s(x,y)\mathcal{P}_{\mathfrak{c}}^+(s,y)R(y)dy\right|^2 dx \\ &\le \frac{C}{|s|^2}\|R\|^2_{L^2_{k+1}} + \left(2C + \frac{C}{\nu(s)}\right)\int_1^\infty |x|^{-2}dx\|R\|^2_{L^2_{k+1}} \\ &\le \left(\frac{C}{|s|^2} + 2C + \frac{C}{\nu(s)}\right)\|R\|^2_{L^2_{k+1}} \le \frac{C}{|s|^2}\|R\|_{L^2_{k+1}}.\end{aligned} \tag{5.72}$$

So it remains to estimate $Y(s,\cdot)$ at $-\infty$. For that purpose let $x \le -1$. Then we obtain using again Cauchy-Schwarz inequality, Lemma 5.15 and $\tilde{\alpha}^- < 0$

$$\begin{aligned}\eta^{2k}(x)\left|\int_{-\infty}^x \mathcal{S}_s(x,y)\mathcal{P}_{\mathfrak{s}}^-(s,y)R(y)dy\right|^2 &\le C\int_{-\infty}^x \frac{|x|^{2k}}{|y|^{2(k+1)}}e^{2\tilde{\alpha}^-(x-y)}dy\|R\|^2_{L^2_{k+1}} \\ &\le C|x|^{-2}\|R\|^2_{L^2_{k+1}}\end{aligned} \tag{5.73}$$

as well as since $0 < \tilde{\beta}^-$

$$\begin{aligned} &\eta^{2k}(x)\left|\int_x^0 \mathcal{S}_s(x,y)\mathcal{P}_{\mathfrak{u}}^-(s,y)R(y)dy\right|^2 \\ &\leq C\eta^{2k}(x)\left|\int_{-1}^0 e^{\tilde{\beta}^-(x-y)}|R(y)|dy\right|^2 + C\eta^{2k}(x)\left|\int_{-\infty}^{-1} e^{\tilde{\beta}^-(x-y)}|R(y)|dy\right|^2 \\ &\leq C\eta^{2k}(x)e^{\tilde{\beta}^- x}\|R\|^2_{L^2_{k+1}} + C\int_{-\infty}^{-1}\frac{|x|^{2k}}{|y|^{2(k+1)}}e^{2\tilde{\beta}^-(x-y)}dy\|R\|^2_{L^2_{k+1}} \leq C|x|^{-2}\|R\|_{L^2_{k+1}}. \end{aligned} \tag{5.74}$$

Now (5.66), (5.68), (5.73) and (5.74) imply

$$\begin{aligned} \int_{-\infty}^{-1}\eta^{2k}(x)|Y(s,\cdot)|^2dx \leq \frac{C}{|s|^2}\|R\|^2_{L^2_{k+1}} &+ \int_{-\infty}^{-1}\eta^{2k}(x)\left|\int_{-\infty}^{x}\mathcal{S}_s(x,y)\mathcal{P}_{\mathfrak{s}}^-(s,y)R(y)dy\right|^2 dx \\ &+ \int_{-\infty}^{-1}\eta^{2k}(x)\left|\int_x^0 \mathcal{S}_s(x,y)\mathcal{P}_{\mathfrak{u}}^-(s,y)R(y)dy\right|^2 dx \\ &\leq \frac{C}{|s|^2}\|R\|^2_{L^2_{k+1}} + 2C\int_{-\infty}^{-1}|x|^{-2}dx\|R\|^2_{L^2_{k+1}} \leq \frac{C}{|s|^2}\|R\|^2_{L^2_{k+1}}. \end{aligned} \tag{5.75}$$

Since $Y(s,\cdot)$ is continuous it is easily seen that

$$\int_{-1}^{1}\eta^{2k}(x)|Y(s,x)|^2dx \leq \frac{C}{|s|^2}\|R\|^2_{L^2_{k+1}}.$$

Thus with (5.72) and (5.75) we obtain (5.65) after taking square root

$$\|Y(s,\cdot)\|_{L^2_k} \leq \frac{C}{|s|}\|R\|_{L^2_{k+1}}.$$

$\square$

The Lemma now implies the following resolvent estimate for L on L^2_k for $s \in \Omega_c\backslash\{0\}$.

Corollary 5.26. *There is $C > 0$ such that for all $s \in \Omega_c\backslash\{0\}$ and $r \in L^2_{k+1}$ the equation $(sI - L)u = r$ has a unique solution $u \in H^2_{k+1}$ satisfying*

$$\|u\|_{L^2_k} \leq \frac{C}{|s|}\|r\|_{L^2_{k+1}}.$$

Proof. Since $\Omega_c\backslash\{0\} \subset \rho(L)$ there is a unique solution $u \in H^2_{k+1}$ of $(sI - L)u = r$ for all $r \in L^2_{k+1}$. In particular, $r \in L^2$ and u is unique in L^2. Take $Y(s,\cdot) = (w_1, w_2)^\top$ from Lemma 5.25. Then $(sI - L)w_1 = r$. Thus $u = w_1$ and the estimate is direct consequence of Lemma 5.25. $\square$

Next we take the projector P onto $\mathcal{N}(L)$ and the space $V_k = \Psi^\perp \cap L^2_k$ from (5.44) into account and prove the major result of this section, which gives sharp resolvent estimates for $sI - L$ in the crescent Ω_c. The estimates are the essential ingredients to derive time decaying estimates for the semigroup generated by L. Using a perturbation argument similar as in [12, Lem. B2], we prove that the weak singularity of $(sI - L)^{-1}$ of order $|s|^{-1}$ is caused by the nontrivial kernel $\mathcal{N}(L)$ and only acts on a subspace given by the range of the projector P onto $\mathcal{N}(L)$ from Lemma 5.20.

Lemma 5.27. *Let Assumption 1, 2 and 5-8 be satisfied and $k \in \mathbb{N}_0$. Then there is $C > 0$ such that for all $r \in L^2_{k+3}$ and $s \in \Omega_c \backslash \{0\}$ the solution $u \in H^2_{k+3}$ of (5.47) satisfies the estimate*

$$\|u\|_{H^1_k} \leq \frac{1}{|s|}\|Pr\|_{L^2_k} + C\|(I-P)r\|_{L^2_{k+3}}. \tag{5.76}$$

In particular, if $r \in V_{k+3}$ then

$$\|u\|_{H^1_k} \leq C\|r\|_{L^2_{k+3}} \tag{5.77}$$

uniformly for $s \in \Omega_c$.

Proof. We approximate $(sI - L)$ by the operator

$$\tilde{L}(s) := sP - L(I-P).$$

Then

$$\tilde{L}(s) - (sI - L) = s(P - I) + LP = -s(I-P). \tag{5.78}$$

Since $\Omega_c \backslash \{0\} \subset \rho(L)$ the equation $(sI - L)u = r \in L^2_{k+3}$ with $s \neq 0$ has a unique solution $u \in H^2_{k+3}$. Recall that $PLu = LPu = 0$ by construction of P, cf. (5.43). We obtain

$$|s|\|Pu\|_{H^1_k} = \|(sI-L)Pu\|_{H^1_k} = \|Pr\|_{H^1_k}. \tag{5.79}$$

By Lemma 5.19 L is a Fredholm operator of index 0 from L^2_k to L^2_{k+2}. Now we use the estimate (5.46) to obtain for some $K > 0$

$$\|\tilde{L}(s)(I-P)u\|_{L^2_{k+2}} = \|L(I-P)u\|_{L^2_{k+2}} \geq K\|(I-P)u\|_{H^1_k}.$$

This yields with (5.78)

$$\begin{aligned} \|(I-P)r\|_{L^2_{k+2}} &= \|(sI-L)(I-P)u\|_{L^2_{k+2}} \\ &\geq \|\tilde{L}(s)(I-P)u\|_{L^2_{k+2}} - \|[\tilde{L}(s) - (sI-L)](I-P)u\|_{L^2_{k+2}} \\ &\geq K\|(I-P)u\|_{H^1_k} - |s|\|(I-P)u\|_{L^2_{k+2}}. \end{aligned} \tag{5.80}$$

Using (5.79), (5.80) we obtain

$$\begin{aligned}\|u\|_{H^1_k} &\leq \|Pu\|_{H^1_k} + \|(I-P)u\|_{H^1_k} \\ &\leq \frac{1}{|s|}\|Pr\|_{H^1_k} + C|s|\|(I-P)u\|_{L^2_{k+2}} + C\|(I-P)r\|_{L^2_{k+2}}.\end{aligned}$$

Now by Corollary 5.26 we have $|s|\|(I-P)u\|_{L^2_{k+2}} \leq C\|(I-P)r\|_{L^2_{k+3}}$. Thus,

$$\|u\|_{H^1_k} \leq \frac{1}{|s|}\|Pr\|_{H^1_k} + C\|(I-P)r\|_{L^2_{k+3}} \leq \frac{C}{|s|}\|Pr\|_{L^2_k} + C\|(I-P)r\|_{L^2_{k+3}}.$$

□

5.4 The semigroup e^{tL}

From Lemma 5.7 and Theorem 5.9 we conclude that the linearized operator L generates an analytic semigroup on L^2_k. It is denoted by $\{e^{tL}\}_{t\geq 0}$ and will be used to show existence of a solution to (0.11) with $u(0) = v_\star + u_0$. To conclude also nonlinear stability of the solution we need time decaying estimates of the semigroup which will be proven using the delicate resolvent estimates from Lemma 5.27. For this purpose, recall the subspaces V_k from (5.44).

Theorem 5.28. *Let Assumption 1, 2 and 5-8 be satisfied and $k \in \mathbb{N}_0$. Then the linearized operator $L : H^2_k \to L^2_k$ generates an analytic semigroup $\{e^{tL}\}_{t\geq 0}$ on L^2_k given by*

$$e^{tL} = \frac{1}{2\pi i}\int_\Gamma e^{ts}(sI-L)^{-1}ds,$$

where Γ is any contour in $\rho(L)$ with $\arg\lambda \to \pm\left(\frac{\pi}{2}+\varepsilon\right)$ as $|\lambda| \to \infty$ for some $\varepsilon > 0$. In addition, there is $K \geq 1$ and $\beta > 0$ such that for all $t > 0$ and $\ell = 0, 1$ there hold

$$\|e^{tL}u\|_{H^\ell_k} \leq Ke^{\beta t}\|u\|_{H^\ell_k}, \quad \|e^{tL}u\|_{H^1_k} \leq \frac{K}{\sqrt{t}}e^{\beta t}\|u\|_{L^2_k}. \tag{5.81}$$

Moreover, for $m \in \mathbb{N}$ there is $C_m \geq 1$ such that for all $u \in V_{k+3m}$ there hold

$$\|e^{tL}u\|_{L^2_k} \leq \frac{C_m}{(1+t)^{\frac{m}{2}}}\|u\|_{L^2_{k+3m}}, \tag{5.82}$$

$$\|e^{tL}u\|_{H^1_k} \leq \frac{C_m}{t^{\frac{m}{2}}}\|u\|_{L^2_{k+3m}}, \tag{5.83}$$

$$\|e^{tL}u\|_{H^1_k} \leq \frac{C_m}{(1+t)^{\frac{m}{2}}}\|u\|_{H^1_{k+3m}}. \tag{5.84}$$

Proof. As in the proof of Theorem 3.21 (see also [32]) it follows from Lemma 5.7 and Theorem 5.9 that L generates an analytic semigroup on L_k^2 and the estimates (5.81) hold. So it remains to show the estimates (5.82), (5.83) and (5.84). For this purpose we first consider the case $m = 1$. Take Ω_c, Γ_c from Lemma 5.14 and such that Lemma 5.27 applies to Ω_c and let $C > 0$ denote a universal constant. Let $\gamma = a_\star\delta^2 + i\delta$ and $\varepsilon = \cos^{-1}(\frac{\delta}{|\gamma|})$ with $a_\star, \delta$ defining Γ_c. Set for sufficiently small $\varepsilon_0 > 0$

$$\Gamma_+ := \{\gamma + \tau e^{i\left(\frac{\pi}{2}+\varepsilon_0\right)},\ \tau \geq 0\}, \quad \Gamma_- := \{\overline{\gamma} + \tau e^{-i\left(\frac{\pi}{2}+\varepsilon_0\right)},\ \tau \geq 0\},$$
$$\Gamma_0 := \{|\gamma|e^{i\theta},\ |\theta| \leq \frac{\pi}{2} + \varepsilon\}$$

and set $\Gamma = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$ being a contour running in upward direction. W.l.o.g. $\Gamma \subset \rho(L)$

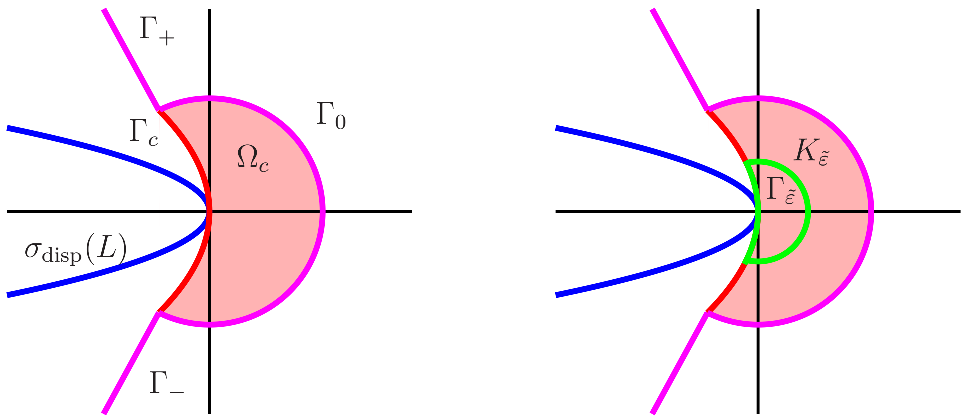


Figure 5.5: The contour Γ (left) and $\Gamma_{\tilde{\varepsilon}}$, $K_{\tilde{\varepsilon}}$ (right) in the proof of Theorem 5.28.

and there are no eigenvalues of L to the right of Γ in the complex plane. Otherwise increase $a_\star < 0$ and decrease $\varepsilon_0 > 0$. Since e^{tL} is independent of the choice of the contour we have

$$e^{tL} = \frac{1}{2\pi i}\int_\Gamma e^{ts}(sI - L)^{-1} ds.$$

In particular, $\partial\Omega_c = \Gamma_c \cup \Gamma_0$. Take arbitrary $\tilde{\varepsilon} > 0$ and set $K_{\tilde{\varepsilon}} = \partial B_{\tilde{\varepsilon}}(0) \cap \Omega_c$ and $\Gamma_{\tilde{\varepsilon}} := \{z \in \Gamma, |z| \leq \tilde{\varepsilon}\} \cup K_{\tilde{\varepsilon}}$. Then the Cauchy integral theorem implies for $u \in V_{k+3} = \mathcal{R}(I - P) \cap L_{k+3}^2$

$$\int_{\partial\Omega_c} e^{ts}(sI - L)^{-1} u ds = \int_{\Gamma_{\tilde{\varepsilon}}} e^{ts}(sI - L)^{-1} u ds,$$

where the integrals running clockwise. By Lemma 5.27 we have for all $s \in \Gamma_{\tilde{\varepsilon}}$, $0 < \tilde{\varepsilon} < |\gamma|$ the uniform estimate

$$\|(sI - L)^{-1}u\|_{H^1_k} \leq C\|u\|_{L^2_{k+3}} \quad \forall\, u \in V_{k+3}. \tag{5.85}$$

Using (5.85) and parameterizing $\Gamma_{\tilde{\varepsilon}}$ yields by a straight forward calculation for $u \in V_{k+3}$

$$\left\| \int_{\Gamma_{\tilde{\varepsilon}}} e^{ts}(sI - L)^{-1}u ds \right\|_{H^1_k} \to 0, \quad \tilde{\varepsilon} \to 0.$$

This shows

$$\int_{\partial\Omega_c} e^{ts}(sI - L)^{-1}u ds = 0 \quad \forall\, u \in V_{k+3}.$$

Then we conclude

$$e^{tL}u = \frac{1}{2\pi i} \int_{\tilde{\Gamma}} e^{ts}(sI - L)^{-1}u ds \quad \forall\, u \in V_{k+3}$$

where $\tilde{\Gamma} = \Gamma_- \cup \Gamma_c \cup \Gamma_+$ is the contour running in upwards direction. Now since $\Gamma_\pm \subset \rho(L)$ and using Lemma 5.27 and Lemma 5.7 we find $C \geq 0$ such that

$$\|(sI - L)^{-1}u\|_{H^1_k} \leq C\|u\|_{L^2_{k+3}} \quad \forall\, s \in \tilde{\Gamma}, u \in V_{k+3}.$$

Now let $t \geq 1$ and $u \in V_{k+3}$. Then we observe since $a_\star < 0$

$$\begin{aligned}
\left\| \int_{\Gamma_\pm} e^{ts}(sI - L)^{-1}u ds \right\|_{H^1_k} &\leq C\|u\|_{L^2_{k+3}} e^{ta_\star\delta^2} \int_0^\infty e^{t\tau\cos(\frac{\pi}{2}+\varepsilon_0))} d\tau \\
&\leq \frac{C}{t}\|u\|_{L^2_{k+3}} e^{ta_\star\delta^2} \int_0^\infty e^{-s|\cos(\frac{\pi}{2}+\varepsilon_0)|} ds \leq \frac{C}{t}\|u\|_{L^2_{k+3}}.
\end{aligned}$$

Moreover, there holds

$$\begin{aligned}
&\left\| \int_{\Gamma_c} e^{ts}(sI - L)^{-1}u ds \right\|_{H^1_k} \leq C\|u\|_{L^2_{k+3}} \int_{-\delta}^{\delta} e^{a_\star t\tau^2} |2a_\star\tau + i| d\tau \leq C\|u\|_{L^2_{k+3}} \int_{-\delta}^{\delta} e^{a_\star t\tau^2} d\tau \\
&= \frac{C}{\sqrt{t}}\|u\|_{L^2_{k+3}} \int_0^{\sqrt{t}\delta} e^{a_\star s^2} ds \leq \frac{C}{\sqrt{t}}\|u\|_{L^2_{k+3}} \int_0^\infty e^{a_\star s^2} ds \leq \frac{C}{\sqrt{t}}\|u\|_{L^2_{k+3}}.
\end{aligned}$$

Now using (5.81) for $t \in (0, 1)$, we obtain for all $t > 0$ and $u \in V_{k+3}$

$$\|e^{tL}u\|_{L^2_k} \leq \frac{C_1}{\sqrt{1+t}}\|u\|_{L^2_{k+3}}, \quad \|e^{tL}u\|_{H^1_k} \leq \frac{C_1}{\sqrt{t}}\|u\|_{L^2_{k+3}}, \quad \|e^{tL}u\|_{H^1_k} \leq \frac{C_1}{\sqrt{1+t}}\|u\|_{H^1_{k+3}}. \tag{5.86}$$

Now the general case, $m \in \mathbb{N}$, in (5.82) follows by applying (5.86) m-times:

$$\|e^{tL}u\|_{L^2_k} = \left\| \left(e^{\frac{t}{m}L}\right)^m u \right\|_{L^2_k}$$
$$\leq \frac{C_1}{(1+\frac{1}{m}t)^{\frac{1}{2}}} \left\| \left(e^{\frac{t}{m}L}\right)^{m-1} u \right\|_{L^2_{k+3}} \leq \cdots \leq \frac{C_1^m}{(1+\frac{1}{m}t)^{\frac{m}{2}}} \|u\|_{L^2_{k+3m}} \leq \frac{C_m}{(1+t)^{\frac{m}{2}}} \|u\|_{L^2_{k+3m}}$$

and similarly for (5.83) and (5.84). $\square$

5.5 Decomposition of the dynamics

Recall the co-moving equation with perturbed initial data from (0.11) reading as

$$u_t = Au_{xx} + cu_x + S_\omega u + f(u), \quad u(0) = v_\star + u_0.$$

As in Chapter 3 the next step is to decompose the dynamics of the solution of (0.11) by a nonlinear coordinate transformation, cf. Section 3.5. In particular, the solution is written as a motion along the group orbit $\mathcal{O}(v_\star)$ described by a group element $\tau(t) \in \mathbb{R}$ and a perturbation w in the space V_k for appropriate $k \in \mathbb{N}$, cf. Figure 5.6. For $t \geq 0$ we want to write the solution $u(t) \in M_k$ as

$$u(t) = v_\star(\cdot - \tau(t)) + w(t), \quad \tau(t) \in \mathbb{R}, \quad w(t) \in V_k.$$

This transformation will be unique as long as the solution u stays close to the group orbit $\mathcal{O}(v_\star)$. As in Chapter 3 this will be guaranteed by taking sufficiently small initial perturbations u_0. Since the procedure is very close to the one from Section 3.5 we only give the main steps of the proofs during this section.

We start by considering the map

$$\Pi : \mathbb{R} \to \Phi, \quad \tau \mapsto P(v_\star(\cdot - \tau) - v_\star). \tag{5.87}$$

Lemma 5.29. *Let Assumption 1, 2 and 5-8 be satisfied and $k \in \mathbb{N}_0$. Then there is a zero neighborhood $W \subset \mathbb{R}$ such that the map $\Pi : W \to \Pi(W) \subset \Phi$ from (5.87) is a local diffeomorphism. Moreover, there is a zero neighborhood $V \subset \mathbb{R} \times V_k$ such that the transformation*

$$T : V \to T(V) \subset L^2_k, \quad (\tau, w) \mapsto v_\star(\cdot - \tau) - v_\star + w$$

is a local diffeomorphism and the solution of $T(\tau, w) = v$ is given by

$$\tau = \Pi^{-1}(Pv), \quad w = v + v_\star - v_\star(\cdot - \tau).$$

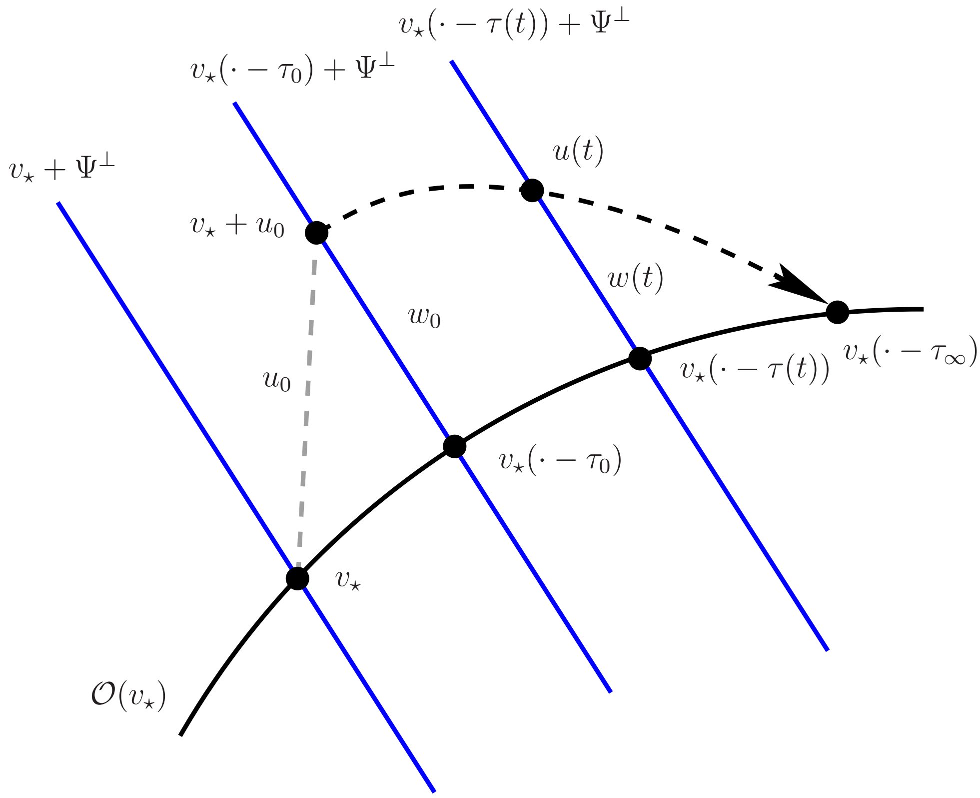


Figure 5.6: Decomposition of the dynamics.

Proof. The proof follows as in Lemma 3.22 and we only note the important steps. We have $\Pi(0) = 0$ and Π is continuously differentiable with derivative $D\Pi(0) = -v_{\star,x} \neq 0$, cf. Lemma 5.5. Thus, we conclude using the implicit function theorem D.8 that Π is a local diffeomorphism near $\tau = 0$. The same holds true for T since it is continuously differentiable with derivative at $(\tau, w) = 0$

$$DT(0,0) = \begin{pmatrix} D\Pi(0) & 0 \\ 0 & I \end{pmatrix}$$

which is invertible. The rest of the proof follows as in in the proof of Lemma 3.22. $\square$

Next we assume there is a classical solution $u \in C([0, t_\infty), M_k^2) \cap C^1([0, t_\infty), M_k)$ of (0.11) on $[0, t_\infty)$ for some $k \in \mathbb{N}$ satisfying

$$\|u(t) - v_\star\|_{L_k^2} < \delta \quad \forall t \in [0, t_\infty).$$

Let δ be sufficiently small such that Lemma 5.5 guarantees that the map T stays invertible on $B_\delta(0) \subset L_k^2$. Then we have for all $t \in [0, t_\infty)$

$$u(t) - v_\star = T(\tau(t), w(t))$$

with $w : [0, t_\infty) \to V_k$ and $\tau : [0, t_\infty) \to \mathbb{R}$. Then, since T is diffeomorphic, we conclude $w \in C([0, t_\infty), V_k^2) \cap C^1([0, t_\infty), V_k)$ and $\tau \in C^1([0, t_\infty), \mathbb{R})$ and the decomposition

$$u(t) = v_\star(\cdot - \tau(t)) + w(t) \tag{5.88}$$

holds for all $t \in [0, t_\infty)$. Since u solves (0.11) we obtain for $t = 0$

$$u_0 + v_\star = u(0) = v_\star(\cdot - \tau(0)) + w(0)$$

which yields

$$\tau(0) = \Pi^{-1}(Pu_0) =: \tau_0, \quad w(0) = u_0 + v_\star - v_\star(\cdot - \tau_0) =: w_0. \tag{5.89}$$

Let $L_0 u := Au_{xx} + cu_x + S_\omega u$. Then using the chain rule and the local representation of the derivative of the group action from Lemma 5.5, see (5.2), we obtain for $t \in (0, t_\infty)$

$$\begin{aligned}
0 &= u_t(t) - F(u(t)) = u_t(t) - L_0 u(t) - f(u(t)) \\
&= \frac{d}{dt} v_\star(\cdot - \tau) + w_t - L_0 v_\star(\cdot - \tau) - L_0 w - f(v_\star(\cdot - \tau) + w) \\
&= -v_{\star,x}(\cdot - \tau)\tau_t + w_t - L_0 v_\star(\cdot - \tau) - L_0 w - f(v_\star(\cdot - \tau) + w) \\
&= -v_{\star,x}(\cdot - \tau)\tau_t + w_t - L_0 v_\star(\cdot - \tau) - Lw + Df(v_\star)w - f(v_\star(\cdot - \tau) + w).
\end{aligned}$$

Since $L_0 v_\star(\cdot - \tau) + f(v_\star(\cdot - \tau)) = F(v_\star(\cdot - \tau)) = 0$ we observe

$$w_t = Lw + v_{\star,x}(\cdot - \tau)\tau_t + r^{[f]}(\tau, w) \tag{5.90}$$

where

$$r^{[f]}(\tau, w) := f(v_\star(\cdot - \tau) + w) - f(v_\star(\cdot - \tau)) - Df(v_\star)w. \tag{5.91}$$

Applying the projector P to (5.90) yields

$$0 = Pv_{\star,x}(\cdot - \tau)\tau_t + Pr^{[f]}(\tau, w). \tag{5.92}$$

Hence τ is determined by the ODE (5.92). As a next step, we want to write the ODE in an explicit form.

Lemma 5.30. *Let Assumption 1, 2 and 5-8 be satisfied and $k \in \mathbb{N}_0$. Then for $\tau \in \mathbb{R}$ the map*

$$S(\tau) : \mathbb{R} \to \Phi, \quad \mu \mapsto -Pv_{\star,x}(\cdot - \tau)\mu$$

satisfies $S(\tau) \in L[\mathbb{R}, \Phi]$. Moreover, $S(\cdot) \in C^1(\mathbb{R}, L[\mathbb{R}, \Phi])$ and there is a zero neighborhood $V \subset \mathbb{R}$ such that $S(\tau)^{-1}$ exists for $\tau \in V$ and $S(\cdot)^{-1} \in C^1(V, L[\Phi, \mathbb{R}])$.

Proof. Follows as in the proof of Lemma 3.23. □

With the use of Lemma 5.30 we can write (5.92) together with the initial condition as an initial-value problem for τ

$$\tau_t = r^{[\tau]}(\tau, w), \quad \tau_0 = \Pi^{-1}(Pu_0)$$

where

$$r^{[\tau]}(\tau, w) := S(\tau)^{-1} P r^{[f]}(\tau, w). \tag{5.93}$$

Now we apply the projector $I - P$ to (5.90) and obtain

$$w_t = Lw + (I-P)v_{\star,x}(\cdot - \tau) r^{[\tau]}(\tau, w) + (I-P) r^{[f]}(\tau, w) = Lw + r^{[w]}(\tau, w)$$

where

$$r^{[w]}(\tau, w) := \left[(I-P) + (I-P)v_{\star,x}(\cdot - \tau) S(\tau)^{-1} P\right] r^{[f]}(\tau, w). \tag{5.94}$$

Summarizing we have shown that the new coordinates (τ, w) solve the initial-value problem

$$w_t = Lw + r^{[w]}(\tau, w), \quad w(0) = u_0 + v_\star - v_\star(\cdot - \tau_0) =: w_0 \tag{5.95}$$

$$\tau_t = r^{[\tau]}(\tau, w), \quad \tau(0) = \Pi^{-1}(Pu_0) =: \tau_0 \tag{5.96}$$

as long as the decomposition (5.88) is valid.

Definition 5.31. A pair (τ, w) is called a classical solution of (5.95), (5.96) on $[0, t_\infty)$ for some $t_\infty > 0$ if there is $k \in \mathbb{N}$ with

i) $w \in C((0, t_\infty), V_k^2) \cap C^1([0, t_\infty), V_k)$ and $\tau \in C^1([0, t_\infty), \mathbb{R})$.

ii) $w_t(t) = Lw(t) + r^{[w]}(\tau(t), w(t))$ and $\tau_t(t) = r^{[\tau]}(\tau(t), w(t))$ for every $t \in [0, t_\infty)$.

iii) $w(0) = w_0$ and $\tau(0) = \tau_0$.

If $t_\infty = \infty$ we will call (τ, w) a global classical solution of (5.95), (5.96), whereas for $t_\infty < \infty$ we will call (τ, w) a local classical solution of (5.95), (5.96).

5.6 Estimates of nonlinearities

As in Section 3.6 the next step is to show Lipschitz estimates of the remaining nonlinearities $r^{[f]}, r^{[w]}, r^{[\tau]}$ from (5.91), (5.94) and (5.93). But in contrast to the exponential case in Section 3.6 we have to show sharper estimates since by Theorem 5.28 we have a loss in the polynomial order. However, this is captured by the fact that the remainders $r^{[f]}, r^{[w]}, r^{[\tau]}$ are actually Taylor-remainders of second order. This gives us additional polynomial orders to compensate the loss of the semigroup.

Lemma 5.32. *Let Assumption 1, 2 and 5-8 be satisfied and $k \in \mathbb{N}_0$. Then there is $\delta > 0$ and constants $C_0, C_1, C_2, C_3, C_4 > 0$ such that for all $\tau, \tau_1, \tau_2 \in B_\delta(0) \subset \mathbb{R}$ and $w, w_1, w_2 \in B_\delta(0) \subset H^1_k$ there hold:*

i) $\left\|r^{[f]}(\tau, w_1) - r^{[f]}(\tau, w_2)\right\|_{H^1_{2k}} \leq C_0 \left(|\tau| + \max(\|w_1\|_{H^1_k}, \|w_2\|_{H^1_k})\right) \|w_1 - w_2\|_{H^1_k}$,

ii) $\left\|r^{[f]}(\tau_1, w) - r^{[f]}(\tau_2, w)\right\|_{H^1_{2k}} \leq C_1 |\tau_1 - \tau_2|$,

iii) $\left\|r^{[w]}(\tau, w_1) - r^{[w]}(\tau, w_2)\right\|_{H^1_{2k}} \leq C_2 \left(|\tau| + \max(\|w_1\|_{H^1_k}, \|w_2\|_{H^1_k})\right) \|w_1 - w_2\|_{H^1_k}$,

iv) $\left\|r^{[w]}(\tau_1, w_1) - r^{[w]}(\tau_2, w_2)\right\|_{H^1_{2k}} \leq C_3 \left(|\tau_1 - \tau_2| + \|w_1 - w_2\|_{H^1_k}\right)$,

v) $\left|r^{[\tau]}(\tau_1, w_1) - r^{[\tau]}(\tau_2, w_2)\right| \leq C_4 \left(|\tau_1 - \tau_2| + \|w_1 - w_2\|_{H^1_k}\right)$.

Proof. The proof is similar to the one of Lemma 3.25 and we denote by $C > 0$ a universal constant. For a matrix-valued function $M : \mathbb{R} \to \mathbb{R}^{2,2}$ we write $\|M\|_{L^\infty} = \||M|\|_{L^\infty}$ and $\|M\|_{L^2_k} = \||M|\|_{L^2_k}$ for some matrix norm $|\cdot|$ on $\mathbb{R}^{2,2}$. We frequently use Sobolev embedding, cf. Theorem D.2, Lemma 5.4 and the exponential estimates of the profile $v_\star$ from Theorem 2.6. The key idea is to use the following estimates to gain the better behavior w.r.t. to the polynomial order. Let $v \in H^1_k$, $u \in L^2_k$ then

$$\begin{aligned} \||v||u|\|^2_{L^2_{2k}} &= \int_{\mathbb{R}} \eta^{4k}(x)|v(x)|^2|u(x)|^2 dx = \int_{\mathbb{R}} |\eta^k(x)v(x)|^2|\eta^k(x)u(x)|^2 dx \\ &\leq \|v\|^2_{L^\infty_k}\|u\|^2_{L^2_k} \leq \|v\|^2_{H^1_k}\|u\|^2_{L^2_k}. \end{aligned} \tag{5.97}$$

and

$$\||v_\star(\cdot - \tau) - v_\star||v|\|^2_{L^2_{2k}} \leq \|v\|^2_{L^\infty} \int_{\mathbb{R}} \eta^{4k}(x)|v_\star(x-\tau) - v_\star(x)|^2 dx \leq C|\tau|^2\|v\|^2_{H^1_k}. \tag{5.98}$$

We frequently use the estimates (5.97) and (5.98) with different functions u, v depending on w_1, w_2 and their derivatives $w_{1,x}, w_{2,x}$. Now we can proceed in the same fashion as in the proof of Lemma 3.25.

i). Let $\kappa(s) := v_\star - v_\star(\cdot - \tau) - w_2 + s(w_1 - w_2)$. For the remainder $r^{[f]}$ we use the intermediate value theorem and the estimates (5.97), (5.98) to estimate

$$\begin{aligned} &\|r^{[f]}(\tau, w_1) - r^{[f]}(\tau, w_2)\|_{L^2_{2k}} \\ &= \|f(v_\star(\cdot - \tau) + w_1) - f(v_\star(\cdot - \tau) + w_2) - Df(v_\star)(w_1 - w_2)\|_{L^2_{2k}} \\ &\leq \int_0^1 \|[Df(v_\star(\cdot - \tau) + w_2 + s(w_1 - w_2)) - Df(v_\star)](w_1 - w_2)\|_{L^2_{2k}} ds \\ &\leq \int_0^1 \int_0^1 \|D^2 f(v_\star + \sigma\kappa(s))[\kappa(s), w_1 - w_2]\|_{L^2_{2k}} d\sigma ds \end{aligned}$$

$$\begin{aligned}
&\leq C\Big(\||v_\star(\cdot-\tau)-v_\star|\,|w_1-w_2|\|_{L^2_{2k}}+\||w_2|\,|w_1-w_2|\|_{L^2_{2k}}+\||w_1-w_2|^2\|_{L^2_{2k}}\Big)\\
&\leq C\Big(|\tau|+\max(\|w_1\|_{H^1_k},\|w_2\|_{H^1_k})\Big)\|w_1-w_2\|_{H^1_k}.
\end{aligned}$$

Next we estimate the derivative

$$\begin{aligned}
&\|\partial_x[r^{[f]}(\tau,w_1)-r^{[f]}(\tau,w_2))]\|_{L^2_{2k}}\\
&=\|[Df(v_\star(\cdot-\tau)+w_1)-Df(v_\star(\cdot-\tau)+w_2)]v_{\star,x}(\cdot-\tau)-D^2f(v_\star)[v_{\star,x},w_1-w_2]\\
&\quad+[Df(v_\star(\cdot-\tau)+w_1))-Df(v_\star)]w_{1,x}-[Df(v_\star(\cdot-\tau)+w_2))-Df(v_\star)]w_{2,x}\|_{L^2_{2k}}.
\end{aligned} \tag{5.99}$$

With $\tilde{\kappa}(s):=v_\star(\cdot-\tau)+w_2+s(w_1-w_2)$ the first term can be estimated by

$$\begin{aligned}
&\|[Df(v_\star(\cdot-\tau)+w_1)-Df(v_\star(\cdot-\tau)+w_2)]v_{\star,x}(\cdot-\tau)-D^2f(v_\star)[v_{\star,x},w_1-w_2]\|_{L^2_{2k}}\\
&\leq\int_0^1\|D^2f(\tilde{\kappa}(s))[v_{\star,x}(\cdot-\tau),w_1-w_2]-D^2f(v_\star)[v_{\star,x},w_1-w_2]\|_{L^2_{2k}}ds\\
&\leq\int_0^1\|D^2f(\tilde{\kappa}(s))[v_{\star,x}(\cdot-\tau)-v_{\star,x},w_1-w_2]\|_{L^2_{2k}}ds\\
&\quad+\int_0^1\|\big(D^2f(\tilde{\kappa}(s))-D^2f(v_\star)\big)[v_{\star,x},w_1-w_2]\|_{L^2_{2k}}ds\\
&\leq\int_0^1\|D^2f(\tilde{\kappa}(s))[v_{\star,x}(\cdot-\tau)-v_{\star,x},w_1-w_2]\|_{L^2_{2k}}ds\\
&\quad+\int_0^1\int_0^1\|D^3f(v_\star+\sigma(\tilde{\kappa}(s)-v_\star))[\tilde{\kappa}(s)-v_\star,v_{\star,x},w_1-w_2]\|_{L^2_{2k}}d\sigma ds\\
&\leq C\Big(\|v_{\star,x}(\cdot-\tau)-v_{\star,x}\|_{H^1_{2k}}\|w_1-w_2\|_{L^\infty}\\
&\qquad+\||w_2||w_1-w_2|\|_{L^2_{2k}}+\||w_1-w_2|^2\|_{L^2_{2k}}\Big)\\
&\leq C\Big(|\tau|+\max(\|w_1\|_{H^1_k},\|w_2\|_{H^1_k})\Big)\|w_1-w_2\|_{H^1_k}.
\end{aligned}$$

For the second term in (5.99) we use the abbreviation $\kappa_i(s)=v_\star+s(v_\star(\cdot-\tau)-v_\star+w_i)$, $i=1,2$ and estimate by frequently adding zero

$$\begin{aligned}
&\|[Df(v_\star(\cdot-\tau)+w_1))-Df(v_\star)]w_{1,x}-[Df(v_\star(\cdot-\tau)+w_2))-Df(v_\star)]w_{2,x}\|_{L^2_{2k}}\\
&\leq\int_0^1\|D^2f(\kappa_1(s))[v_\star(\cdot-\tau)-v_\star+w_1,w_{1,x}]-D^2f(\kappa_2(s))[v_\star(\cdot-\tau)-v_\star+w_2,w_{2,x}]\|_{L^2_{2k}}ds\\
&\leq\int_0^1\|\big(D^2f(\kappa_1(s))-D^2f(\kappa_2(s))\big)[v_\star(\cdot-\tau)-v_\star+w_1,w_{1,x}]\|_{L^2_{2k}}ds\\
&\quad+\int_0^1\|D^2f(\kappa_2(s))[w_1-w_2,w_{1,x}]\|_{L^2_{2k}}ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 \|D^2 f(\kappa_2(s))[v_\star(\cdot-\tau)-v_\star+w_2,(w_1-w_2)_x]\|_{L^2_{2k}}ds\\
&\le \int_0^1\int_0^1 \|D^3 f(\kappa_2(s)+\sigma(\kappa_1(s)-\kappa_2(s)))[\kappa_1(s)-\kappa_2(s),v_\star(\cdot-\tau)-v_\star+w_1,w_{1,x}]\|_{L^2_{2k}}d\sigma ds\\
&\quad+\int_0^1 \|D^2 f(\kappa_2(s))[w_1-w_2,w_{1,x}]\|_{L^2_{2k}}ds\\
&\quad+\int_0^1 \|D^2 f(\kappa_2(s))[v_\star(\cdot-\tau)-v_\star+w_2,(w_1-w_2)_x]\|_{L^2_{2k}}ds\\
&\le C\left(\||w_1-w_2||w_{1,x}|\|_{L^2_{2k}}+\||v_\star(\cdot-\tau)-v_\star||(w_1-w_2)_x|\|_{L^2_{2k}}+\||w_2||(w_1-w_2)_x|\|_{L^2_{2k}}\right)\\
&\le C\left(|\tau|+\max(\|w_1\|_{H^1_k},\|w_2\|_{H^1_k})\right)\|w_1-w_2\|_{H^1_k}.
\end{aligned}
$$

Putting things together we have shown

$$\left\|r^{[f]}(\tau,w_1)-r^{[f]}(\tau,w_2)\right\|_{H^1_{2k}}\le C_0\left(|\tau|+\max(\|w_1\|_{H^1_k},\|w_2\|_{H^1_k})\right)\|w_1-w_2\|_{H^1_k}.$$

ii). Using Theorem 2.6 and the intermediate value theorem yields

$$
\begin{aligned}
\|r^{[f]}(\tau_1,w)-r^{[f]}(\tau_2,w)\|_{L^2_{2k}} &= \|f(v_\star(\cdot-\tau_1)+w)-f(v_\star(\cdot-\tau_2)+w)\|_{L^2_{2k}}\\
&\le \|v_\star(\cdot-\tau_1)-v_\star(\cdot-\tau_2)\|_{L^2_{2k}}\le C|\tau_1-\tau_2|.
\end{aligned}
$$

iii)-v). The estimates iii), iv) and v) follow exactly as in the proof of Lemma 3.25 by using i) and Lemma 5.30. ☐

Remark 5.33. *Up to this point excluding the estimates of the nonlinearities, we could have done all the analysis using the space X_η as in Chapter 3 but with a polynomial weight function. In particular, for $X_k = X_\eta$ with $\eta=\eta^k_{\text{poly}}$ it is possible to derive estimates of the semigroup $e^{t\mathcal{L}}$ as in Theorem 5.28 for the linearized operator from* (0.26) *considered as $\mathcal{L}:\mathcal{D}(\mathcal{L})\subset X_k\to X_{k+2}$. However, when proceeding as in Chapter 3 we were not able to compensate the loss of the polynomial orders by estimates of the nonlinearities. Especially, we could not prove a result as in Lemma 3.25 in the case of polynomial weight functions.*

5.7 Nonlinear stability theorem in polynomial spaces

In this section we prove the second main result of the thesis - nonlinear stability with asymptotic phase of traveling oscillating fronts in polynomially weighted spaces, cf. Theorem 1.13. The proof follows the same strategy as in the case of exponentially weighted spaces, see Chapter 3. We consider the decomposed system (5.95), (5.96) which was derived by a nonlinear coordinate transformation in Section 5.5. We begin by showing existence of a local mild solution of (5.95), (5.96). For this purpose consider the corresponding integral equations

$$\begin{aligned} w(t) &= e^{tL}w_0 + \int_0^t e^{(t-s)L} r^{[w]}(\tau(s), w(s))ds, \\ \tau(t) &= \tau_0 + \int_0^t r^{[\tau]}(\tau(s), w(s))ds. \end{aligned} \tag{5.100}$$

Definition 5.34. A solution $(\tau, w) \in C([0, t_\infty), \mathbb{R} \times V_k^1)$ for some $k \in \mathbb{N}_0$ of the integral equations (5.100) on $0 \le t < t_\infty$ for some $t_\infty > 0$ is called a **mild solution** of (5.95), (5.96) on $[0, t_\infty)$.

In the case $t_\infty = \infty$ the we will call the solution $(\tau, \mathbf{w})$ global mild solution, whereas for $t < \infty$ we will call $(\tau, \mathbf{w})$ a local mild solution of of (5.95), (5.96). To prove existence of a local mild solution we use the classical semigroup estimates (5.81) and the Lipschitz estimates from Lemma 5.32. Furthermore, we obtain a-priori estimates for the solution. As in Chapter 3 we equip the product space $\mathbb{R} \times H_k^1$ with the norm

$$\|(\tau, w)\|_{\mathbb{R}\times H_k^1} := |\tau| + \|w\|_{H_k^1}.$$

Lemma 5.35 (Local existence and uniqueness). *Let the Assumption 1, 2 and 5-8 be satisfied and $k \in \mathbb{N}_0$. Further, let $K > 0$ be from Theorem 5.28 and $\delta > 0$ from Lemma 5.32. Then for every $0 < \varepsilon_1 < \delta$ and $0 < 3K\varepsilon_0 \le \delta$ there is $t_\star = t_\star(\varepsilon_0, \varepsilon_1) > 0$ such that for all initial values $(\tau_0, w_0) \in \mathbb{R} \times V_k^1$ with*

$$\|w_0\|_{H_k^1} \le \varepsilon_0, \quad |\tau(t)| \le \varepsilon_1$$

there exists a unique local mild solution $(\tau, w) \in C([0, t_\star), \mathbb{R} \times V_k^1)$ of (5.100) with

$$\|w(t)\|_{H_k^1} \le 2K\varepsilon_0, \quad |\tau(t)| \le 2\varepsilon_1, \quad t \in [0, t_\star).$$

In particular, $t_\star$ can be taken uniformly for $(\tau_0, w_0) \in B_{\varepsilon_1}(0) \times B_{\varepsilon_0}(0)$.

Proof. Take $\beta > 0$ from (5.28) and $C_i > 0$ from Lemma 5.32. Now choose $t_\star$ so small such that the following conditions are satisfied:

$$t_\star < \frac{\varepsilon_1}{2C_4\varepsilon_1 + 2KC_4\varepsilon_0}, \quad \frac{1}{2}e^{\beta t_\star} + C_4 t_\star + \frac{KC_3}{\beta}(e^{\beta t_\star} - 1) < 1. \tag{5.101}$$

Note that $t_\star$ can be taken uniformly for $(\tau_0, w_0) \in B_{\varepsilon_1}(0) \times B_{\varepsilon_0}(0)$. The proof follows a contraction argument in the space $Z := C([0, t_\star), \mathbb{R} \times V_k^1)$ equipped with the norm $\|(\tau, w)\|_Z := \sup_{t\in[0,t_\star)}\{|\tau(t)| + \|w(t)\|_{H_k^1}\}$. Define the map

$$\Upsilon : Z \to Z, \quad (\tau, w) \mapsto \begin{pmatrix} \tau_0 + \int_0^{(\cdot)} r^{[\tau]}(\tau(s), w(s))ds \\ e^{(\cdot)L}w_0 + \int_0^{(\cdot)} e^{(\cdot-s)L} r^{[w]}(\tau(s), w(s))ds \end{pmatrix}$$

given by the right-hand side of (5.100). We show that Υ is a contraction on the closed set

$$B := \{(\tau, w) \in Z : \|w(t)\|_{H^1_k} \le 2K\varepsilon_0,\ |\tau(t)| \le 2\varepsilon_1,\ t \in [0, t_\star)\} \subset Z.$$

Let $(\tau, w) \in B$. By using the estimates from (5.81), Lemma 5.32 and the conditions (5.101) we obtain for all $0 \le t < t_\star$

$$\begin{aligned}
\left\| e^{tL} w_0 + \int_0^t e^{(t-s)L} r^{[w]}(\tau(s), w(s)) ds \right\|_{H^1_k} &\le K e^{\beta t} \varepsilon_0 + K \int_0^t e^{\beta(t-s)} \|r^{[w]}(\tau(s), w(s))\|_{H^1_k} ds \\
&\le K e^{\beta t} \varepsilon_0 + K C_3 \int_0^t e^{\beta(t-s)} \|w(s)\|_{H^1_k} ds \\
&\le K e^{\beta t_\star} \varepsilon_0 + \frac{2K^2 C_3 \varepsilon_0}{\beta} (e^{\beta t_\star} - 1) \le 2K\varepsilon_0.
\end{aligned}$$

and

$$\begin{aligned}
\left| \tau_0 + \int_0^t r^{[\tau]}(\tau(s), w(s)) ds \right| &\le \varepsilon_1 + \int_0^t |r^{[\tau]}(\tau(s), w(s))| ds \\
&\le \varepsilon_1 + C_4 \int_0^t |\tau(s)| + \|w(s)\|_{H^1_k} ds \\
&\le \varepsilon_1 + (2C_4 \varepsilon_1 + 2K C_4 \varepsilon_0) t_\star \le 2\varepsilon_1.
\end{aligned}$$

Hence Υ maps B into itself. Further, for $(\tau_1, w_1), (\tau_2, w_2) \in B$ and $0 \le t \le t_\star$ we can estimate

$$\begin{aligned}
\|\Upsilon(\tau_1, w_1) - \Upsilon(\tau_2, w_2)\|_Z \le \sup_{t \in [0, t_\star)} \Big\{ &\int_0^t |r^{[\tau]}(\tau_1(s), w_1(s)) - r^{[\tau]}(\tau_2(s), w_2(s))| ds \\
&+ \int_0^t K e^{\beta(t-s)} \|r^{[w]}(\tau_1(s), w_1(s)) - r^{[w]}(\tau_2(s), w_2(s))\|_{H^1_k} ds \Big\} \\
&\le \left(C_4 t_\star + \frac{K C_3}{\beta} (e^{\beta t_\star} - 1) \right) \|(\tau_1 - \tau_2, w_1 - w_2)\|_Z \\
&< \|(\tau_1 - \tau_2, w_1 - w_2)\|_Z.
\end{aligned}$$

Thus Υ is a contraction on B and the assertion is proved. $\square$

As in the proof of nonlinear stability in Chapter 3 the next step is to use a Gronwall estimate to show that the unique local mild solution from Lemma 5.35 exists for all times and, in addition, converges to some element of the group orbit $\mathcal{O}(v_\star)$. Since the estimates of the semigroup from Theorem 5.28, see (5.84), consist of polynomial terms we need a Gronwall estimate including polynomial integral kernels.

Lemma 5.36. *Let $0 \leq 2q \leq m-2$, $C, \tilde{C}, \varepsilon > 0$ such that*

$$C \geq 1, \quad \varepsilon \leq \frac{1}{9CC_1\tilde{C}}, \quad C_1 = \frac{2^q m}{m-2}$$

and let $\varphi \in C([0,T), \mathbb{R}_+)$ for some $T > 0$ satisfying

$$\varphi(t) \leq \frac{C\varepsilon}{(1+t)^{\frac{m}{2}}} + \tilde{C} \int_0^t \frac{1}{(1+t-s)^{\frac{m}{2}}} (\varepsilon + \varphi(s))\varphi(s)ds \quad \forall t \in [0,T).$$

Then for all $0 \leq t < T$ there holds

$$\varphi(t) \leq \frac{3C\varepsilon}{(1+t)^q}.$$

Proof. Since $1 < q+1 \leq \frac{m}{2}$ and $m \geq 2$ for all $t \geq 0$ there holds

$$\begin{aligned}
&\int_0^t \frac{(1+t)^q}{(1+s)^q(1+t-s)^{\frac{m}{2}}} ds = \int_0^1 \frac{t(1+t)^q}{(1+\tau t)^q (1+(1-\tau)t)^{\frac{m}{2}}} d\tau \\
&\leq \int_0^{\frac{1}{2}} \frac{(1+t)^{q+1}}{(1+\tau t)^q(1+(1-\tau)t)^{\frac{m}{2}}} d\tau + \int_{\frac{1}{2}}^1 \frac{t(1+t)^q}{(1+\tau t)^q(1+(1-\tau)t)^{\frac{m}{2}}} d\tau \\
&\leq \int_0^{\frac{1}{2}} \frac{(1+t)^{q+1}}{(1+\frac{1}{2}t)^{\frac{m}{2}}} d\tau + \frac{(1+t)^q}{(1+\frac{1}{2}t)^q} \int_{\frac{1}{2}}^1 \frac{t}{(1+(1-\tau)t)^{\frac{m}{2}}} d\tau \\
&= \frac{(1+t)^{q+1}}{2(1+\frac{1}{2}t)^{\frac{m}{2}}} + \frac{2(1+t)^q}{(m-2)(1+\frac{1}{2}t)^q} \left(1 - (1+\tfrac{1}{2}t)^{\frac{2-m}{2}}\right) \\
&\leq \frac{(1+t)^{q+1}}{2(1+\frac{1}{2}t)^{\frac{m}{2}}} + \frac{2(1+t)^q}{(m-2)(1+\frac{1}{2}t)^q} \leq 2^q(1+\tfrac{2}{m-2}) = \frac{2^q m}{m-2} =: C_1
\end{aligned}$$

where the last step uses the bound

$$\frac{(1+t)^q}{(1+\frac{1}{2}t)^q} \leq 2^q, \quad \forall t \geq 0,\, q \in \mathbb{N}.$$

Now let

$$\tau := \sup \left\{ t_\infty \in [0,T) : \varphi(t) \leq \frac{3C\varepsilon}{(1+t)^q},\, \forall t \in [0, t_\infty) \right\} > 0.$$

Assume $\tau < T$. Since $\varphi \in C([0,T), \mathbb{R}_+)$ we obtain

$$\begin{aligned}
3C\varepsilon &= (1+\tau)^q \varphi(\tau) \leq C\varepsilon + K \int_0^\tau \frac{(1+\tau)^q}{(1+\tau-s)^{\frac{m}{2}}} (\varepsilon + \varphi(s))\varphi(s) ds \\
&< C\varepsilon + 3CK\varepsilon^2 \int_0^\tau \frac{(1+\tau)^q}{(1+s)^q(1+\tau-s)^{\frac{m}{2}}} ds + 9C^2K\varepsilon^2 \int_0^\tau \frac{(1+\tau)^q}{(1+s)^{2q}(1+\tau-s)^{\frac{m}{2}}} ds \\
&\leq C\varepsilon + 3CC_1K\varepsilon^2 + 9C_1C^2K\varepsilon^2 < 3C\varepsilon.
\end{aligned}$$

This is a contradiction. Thus $\tau = T$ and the assertion is proven. ∎

Now we are in the situation to prove the stability result for the (τ, w)-system (5.95), (5.96). As in the exponential case, the regularity of the solution will again follow by classical results from [5] and [32], cf. Theorem C.3.

Theorem 5.37. *Let Assumption 1, 2 and 5-8 be satisfied and let $m \geq 5$, $k = 3m$. Then there is $\varepsilon > 0$ and constants $K_1, K_2 \geq 1$ such that for all initial values $(\tau_0, w_0) \in \mathbb{R} \times V_k^2$ with $\|(\tau_0, w_0)\|_{\mathbb{R}\times H^1_{2k}} < \varepsilon$ there hold:*

i) The system (5.95), (5.96) *has a unique global classical solution*

$$w \in C^\alpha((0,\infty), V_k^2) \cap C^{1+\alpha}((0,\infty), V_k) \cap C^1([0,\infty), V_k), \quad \tau \in C^1([0,\infty), \mathbb{R}).$$

for arbitrary $\alpha \in (0,1)$.

ii) There exists $\tau_\infty = \tau_\infty(\tau_0, w_0) \in \mathbb{R}$ such that for all $t \geq 0$

$$\|w(t)\|_{H^1_k} \leq \frac{K_1}{(1+t)^{\frac{m-2}{2}}} \|(\tau_0, w_0)\|_{\mathbb{R}\times H^2_{2k}},$$
$$|\tau(t) - \tau_\infty| \leq \frac{K_2}{(1+t)^{\frac{m-4}{2}}} \|(\tau_0, w_0)\|_{\mathbb{R}\times H^2_{2k}}, \quad |\tau_\infty| \leq (K_2+1)\|(\tau_0, w_0)\|_{\mathbb{R}\times H^2_{2k}}.$$

Proof. Recall $K, C_m \geq 1$ from Theorem 5.28 and δ, C_i from Lemma 5.32. Now choose $\varepsilon, \tilde{\varepsilon} > 0$ such that $0 < 2K\tilde{\varepsilon} < \delta$ and

$$\varepsilon < \min\left(\frac{\delta}{C_\tau}, \frac{\tilde{\varepsilon}}{6C_m}, \frac{2^{-\frac{m-2}{2}}(m-2)}{9mC_m^2 C_2 C_\tau}\right), \quad C_\tau := 2 + \frac{12 C_m C_4}{m-4}. \tag{5.102}$$

We abbreviate $\xi_0 := \|(\tau_0, w_0)\|_{\mathbb{R}\times H^1_{2k}} < \varepsilon$. Let

$$t_\infty := \sup\Big\{T > 0 : \exists(\tau, w) \text{ local mild solution of (5.100) on } [0,T),$$
$$\|w(t)\|_{H^1_k} \leq K\tilde{\varepsilon},\ |\tau(t)| \leq C_\tau \xi_0,\ t \in [0,T)\Big\}.$$

Then Lemma 5.35 with $\varepsilon_0 = \tilde{\varepsilon}$ and $\varepsilon_1 = \frac{C_\tau \xi_0}{2} < \delta$ implies $t_\infty \geq t_\star = t_\star(\varepsilon_0, \varepsilon_1)$. Using Theorem 5.28 and Lemma 5.32 we estimate for all $0 \leq t < t_\infty$, since $k + 3m = 2k$,

$$\begin{aligned}
\|w(t)\|_{H^1_k} &\leq \|e^{tL} w_0\|_{H^1_k} + \int_0^t \|e^{(t-s)L} r^{[w]}(\tau(s), w(s))\|_{H^1_k} ds \\
&\leq \frac{C_m}{(1+t)^{\frac{m}{2}}}\|w_0\|_{H^1_{2k}} + \int_0^t \frac{C_m}{(1+t-s)^{\frac{m}{2}}}\|r^{[w]}(\tau(s), w(s)\|_{H^1_{2k}} ds \\
&\leq \frac{C_m}{(1+t)^{\frac{m}{2}}}\|w_0\|_{H^1_{2k}} + C_m C_2 \int_0^t \frac{1}{(1+t-s)^{\frac{m}{2}}}\left(|\tau(s)| + \|w(s)\|_{H^1_k}\right)\|w(s)\|_{H^1_k} ds \\
&\leq \frac{C_m}{(1+t)^{\frac{m}{2}}}\xi_0 + C_m C_2 C_\tau \int_0^t \frac{1}{(1+t-s)^{\frac{m}{2}}}\left(\xi_0 + \|w(s)\|_{H^1_k}\right)\|w(s)\|_{H^1_k} ds.
\end{aligned}$$

Then the Gronwall estimate in Lemma 5.36 with $q = \frac{m-2}{2}$ implies due to (5.102)

$$\|w(t)\|_{H^1_k} \le \frac{3C_m\xi_0}{(1+t)^{\frac{m-2}{2}}} \le \frac{\tilde{\varepsilon}}{2}, \quad t \in [0, t_\infty). \tag{5.103}$$

This yields, due to $m \ge 5$ and (5.102),

$$\begin{aligned}
|\tau(t)| &\le |\tau_0| + \int_0^t |r^{[\tau]}(\tau(s), w(s))| ds \le \xi_0 + C_4 \int_0^t \|w(s)\|_{H^1_k} ds \\
&\le \xi_0 + 3C_4C_m\xi_0 \int_0^t (1+s)^{-\frac{m-2}{2}} ds \\
&\le \xi_0 + \frac{6C_4C_m\xi_0}{m-4} = \frac{C_\tau\xi_0}{2}, \quad t \in [0, t_\infty).
\end{aligned} \tag{5.104}$$

Next we show that $t_\infty = \infty$. For this purpose, assume the contrary, i.e. $t_\infty < \infty$. Then the estimates (5.103), (5.104) imply

$$\|w(t_\infty - \tfrac{1}{2}t_\star)\|_{H^1_k} \le \frac{\tilde{\varepsilon}}{2} = \varepsilon_0, \quad |\tau(t_\infty - \tfrac{1}{2}t_\star)| \le \frac{C_\tau\xi_0}{2} = \varepsilon_1.$$

Now we can apply Lemma 5.35 once again to the integral equation (5.100) with $w_0 = w(t_\infty - \frac{1}{2}t_\star)$ and $\tau_0 = \tau(t_\infty - \frac{1}{2}t_\star)$ and obtain a solution $(\tilde{\tau}, \tilde{w})$ of (5.100) on $[0, t_\star)$ with

$$\begin{aligned}
&\tilde{w}(0) = w(t_\infty - \tfrac{1}{2}t_\star), \quad \|w(t)\|_{H^1_k} \le K\tilde{\varepsilon}, \quad t \in [0, t_\star) \\
&\tilde{\tau}(0) = \tau(t_\infty - \tfrac{1}{2}t_\star), \quad |\tau(t)| \le C_\tau\xi_0, \qquad t \in [0, t_\star).
\end{aligned}$$

Define

$$(\bar{\tau}, \bar{w})(t) := \begin{cases} (\tau, w)(t), & t \in [0, t_\infty - \frac{1}{2}t_\star] \\ (\tilde{\tau}, \tilde{w})(t - t_\infty + \frac{1}{2}t_\star), & t \in (t_\infty - \frac{1}{2}t_\star, t_\infty + \frac{1}{2}t_\star) \end{cases}.$$

Then $(\bar{\tau}, \bar{w})$ is a local mild solution on $[0, t_\infty + \frac{1}{2}t_\star)$ with $\|\bar{w}(t)\|_{H^1_k} \le K\tilde{\varepsilon}$ and $|\bar{\tau}(t)| \le C_\tau\varepsilon$. A contradiction to the definition of t_∞. Hence $t_\infty = \infty$ and (5.103) holds on $[0, \infty)$. The estimate (5.103) yields that the following integral

$$\tau_\infty := \tau_0 + \int_0^\infty r^{[\tau]}(\tau(s), w(s)) ds$$

exists and satisfies the estimate

$$\begin{aligned}
|\tau(t) - \tau_\infty| &\le \int_t^\infty |r^{[\tau]}(\tau(s), w(s))| ds \\
&\le C_4 \int_t^\infty \|w(s)\|_{H^1_k} \le 3C_mC_4\xi_0 \int_t^\infty (1+s)^{-\frac{m-2}{2}} ds = \frac{6C_mC_4\xi_0}{m-4}(1+t)^{-\frac{m-4}{2}}.
\end{aligned}$$

Hence the first two estimates in ii) are proven with $K_1 = 3C_m$ and $K_2 = \frac{6C_mC_4}{m-4}$. The third estimate is obtained by

$$|\tau_\infty| \leq |\tau(0) - \tau_\infty| + |\tau_0| \leq (K_2 + 1)\xi_0.$$

Hence ii) is proven and it remains to show the regularity of (τ, w). By Lemma 5.32 we have $r^{[\tau]} \in C(V, \mathbb{R})$, $V = B_\delta(0) \times B_\delta(0) \subset \mathbb{R} \times H_k^1$ and, since $(\tau, w) \in C([0,\infty), \mathbb{R} \times V_k^1)$, there hold $r^{[\tau]}(\tau(\cdot), w(\cdot)) \in C([0,\infty), \mathbb{R})$. Thus $\tau \in C^1([0,\infty), \mathbb{R})$. Furthermore, consider the equation

$$u(t) = Lu(t) + r(t), \quad t > 0, \quad u(0) = w_0 \tag{5.105}$$

where $r(t) := r^{[w]}(\tau(t), w(t))$. Suppose $0 \leq s \leq t < \infty$. Then by Lemma 5.32 we find some $C > 0$ such that

$$\begin{aligned}
\|r(t) - r(s)\|_{L_k^2} &= \|r^{[w]}(\tau(t), w(t)) - r^{[w]}(\tau(s), w(s))\|_{L_k^2} \\
&\leq C_3 \left(|\tau(t) - \tau(s)| + \|w(t) - w(s)\|_{H_k^1} \right) \\
&\leq C_3 \left(\int_s^t |r^{[\tau]}(\tau(\sigma), w(\sigma))| d\sigma + \int_s^t \|r^{[w]}(\tau(\sigma), w(\sigma))\|_{H_k^1} d\sigma \right) \\
&\leq C_3 \left(C_4 \int_s^t \|w(\sigma)\|_{H_k^1} d\sigma + C_2 \int_s^t |\tau(\sigma)| + \|w(\sigma)\|_{H_k^1} d\sigma \right) \leq C(t-s).
\end{aligned}$$

This implies $r \in C^\alpha([0,\infty), L_k^2)$ for every $\alpha \in (0,1)$. Moreover, for arbitrary $s > 0$ there hold

$$\int_0^s \|r(t)\|_{L_k^2} dt = \int_0^s \|r^{[w]}(\tau(t), w(t))\|_{L_k^2} dt \leq C_3 \int_0^s \|w(t)\|_{H_k^1} dt < \infty.$$

Now Theorem C.3 implies

$$u(t) = e^{tL} w_0 + \int_0^t e^{(t-s)L} r(s) ds$$

solves (5.105) and $u \in C^\alpha((0,\infty), V_k^2) \cap C^{1+\alpha}((0,\infty), V_k) \cap C^1([0,\infty), V_k)$. Therefore, we have for all $t \geq 0$

$$u(t) = e^{tL} w_0 + \int_0^t e^{(t-s)L} r(s) ds = e^{tL} w_0 + \int_0^t e^{(t-s)L} r^{[w]}(\tau(s), w(s)) ds = w(t).$$

Hence, for all $\alpha \in (0,1)$

$$w(t) \in C^\alpha((0,\infty), V_k^2) \cap C^{1+\alpha}((0,\infty), V_k) \cap C^1([0,\infty), V_k).$$

□

Remark 5.38. *Let us briefly explain why we have to choose $m \geq 5$ and $k = 3m$ in Theorem 5.37. We want to derive stability with asymptotic phase, i.e. $\tau(t) \to \tau_\infty$ as $t \to \infty$. Lets say roughly $|\tau(t) - \tau_\infty| \sim t^{-\frac{1}{2}}$. Following the proof of Theorem 5.37 we then need to show $\|w(t)\| \sim t^{-\frac{3}{2}}$ since it is integrated once. Following the Gronwall estimate from Lemma 5.36 we need to choose $m \geq 5$ in Theorem 5.28. Finally, to compensate the loss of the polynomial order, caused by the semigroup, by the nonlinearities, cf. Lemma 5.32, we have to choose $k = 3m$. This yields at least $u_0 \in H^2_{15}$ and $\|u_0\|_{H^1_{30}} < \varepsilon$ as a smallness condition on the perturbation in* (0.11). *We just note that the polynomial orders are not optimal and the results may be increased w.r.t. the polynomial orders.*

Finally, we prove our second main result by reconstructing a solution of (0.11) via the nonlinear coordinate transformation from Section 5.5 and Theorem 5.37. We ensure that the nonlinear coordinate transformation is valid by taking a sufficiently small initial perturbation u_0.

Proof of Theorem 1.13. Take W, V from Lemma 5.5 and let $\delta > 0$ such that

$$B_\delta := \{u \in L^2_k : \|u\|_{L^2_k} \leq \delta\},$$

satisfies $B_\delta \subset T(V)$ and $P(B_\delta) \subset \Pi(W)$. In particular, $T : T^{-1}(B_\delta) \to B_\delta$ and $\Pi : \Pi^{-1}(P(B_\delta)) \to P(B_\delta)$ are diffeomorphic. Then there is $C_\Pi > 0$ such that

$$\left|\Pi^{-1}(Pv)\right| \leq C_\Pi \|v\|_{L^2_k} \quad \forall v \in B_\delta.$$

Now we take $\varepsilon > 0$ from Theorem 5.37 so small such that the solution (τ, w) of (5.95), (5.96) satisfies $(\tau(t), w(t)) \in T^{-1}(B_\delta)$ and $\tau(t) \in \Pi^{-1}(P(B_\delta))$ for all $t \in [0, \infty)$. Further let $C \geq 1$ be such that Lemma 5.4 i) and ii) imply

$$\|v_\star(\cdot - \tau_1) - v_\star(\cdot - \tau_2)\|_{H^1_{2k}} \leq C|\tau_1 - \tau_2| \quad \forall \tau_1, \tau_2 \in \Pi^{-1}(P(B_\delta)).$$

Choose

$$\varepsilon_0 < \min\left(\frac{\delta}{\tilde{C}\max\{K_1, K_2\}(4C+2) + 2CC_\Pi}, \frac{\varepsilon}{\tilde{C}}\right), \quad \tilde{C} := C_\Pi(1 + C) + 1.$$

with K_1, K_2 from Theorem 5.37 and define

$$(\tau_0, w_0) := T^{-1}(u_0) = (\Pi^{-1}(Pu_0), u_0 + v_\star - v_\star(\cdot - \tau_0)).$$

Then $|\tau_0| \leq C_\Pi \|u_0\|_{L^2_k}$ and

$$\begin{aligned} \|(\tau_0, w_0)\|_{\mathbb{R}\times H^1_{2k}} &= |\tau_0| + \|w_0\|_{H^1_{2k}} \\ &\leq |\tau_0| + \|v_\star(\cdot - \tau_0) - v_\star\|_{H^1_{2k}} + \|u_0\|_{H^1_{2k}} \leq \tilde{C}\|u_0\|_{H^1_{2k}} \leq \tilde{C}\varepsilon_0 < \varepsilon. \end{aligned} \tag{5.106}$$

Moreover, Theorem 5.37 implies there exist $\tau \in C^1([0,\infty),\mathbb{R})$ and $w \in C((0,\infty),V_k^2) \cap C^1((0,\infty),V_k)$ such that (τ,w) solves (5.95), (5.96) with $\tau(0)=\tau_0$, $w(0)=w_0$ and

$$\|w(t)\|_{H_k^1} \le K_1\varepsilon_0, \quad |\tau(t)| \le |\tau(t)-\tau_\infty| + |\tau_\infty| \le (2K_2+1)\varepsilon_0, \quad t\in[0,\infty).$$

Set

$$u(t) = v_\star(\cdot - \tau(t)) + w(t), \quad t\in[0,\infty).$$

Then using the chart (M_k^ℓ,χ), $\ell=0,2$ form (1.17) we conclude $u \in C((0,\infty),M_k^2) \cap C^1([0,\infty),M_k)$. Since $\varepsilon_0<\delta$ Lemma 5.5 implies

$$u(0) = v_\star(\cdot-\tau(0)) + w(0) = T(\tau_0,w_0) + v_\star = u_0 + v_\star.$$

For $t\in(0,\infty)$ we obtain since (τ,w) solve (5.95), (5.96)

$$\begin{aligned}
&u_t(t) - L_0u(t) - f(u(t))\\
&= -v_{\star,x}(\cdot-\tau(t))\tau_t(t) + w_t(t) - L_0w(t) - f(v_\star(\cdot-\tau(t))+w(t)) + f(v_\star(\cdot-\tau(t))\\
&= -v_{\star,x}(\cdot-\tau(t))\tau_t(t) + w_t(t) - Lw(t) - r^{[f]}(\tau(t),w(t))\\
&= w_t(t) - Lw(t) - (I-P)v_{\star,x}(\cdot-\tau(t))\tau_t(t) - (I-P)r^{[f]}(\tau(t),w(t))\\
&\quad - Pv_{\star,x}(\cdot-\tau(t))\tau_t(t) - Pr^{[f]}(z(t),w(t))\\
&= w_t(t) - Lw(t) - r^{[w]}(\tau(t),w(t)) = 0.
\end{aligned}$$

Hence, u solves (0.11). Further, (5.106) and Theorem 5.37 show there is $\tau_\infty\in\mathbb{R}$ such that

$$\begin{aligned}
\|w(t)\|_{H_k^1} &\le K_1(1+t)^{-\frac{m-2}{2}}\|(\tau_0,\mathbf{w}_0)\|_{\mathbb{R}^2\times H_{2k}^1} \le K(1+t)^{-\frac{m-2}{2}}\|u_0\|_{H_{2k}^1},\\
|\tau(t)-\tau_\infty| &\le K_2(1+t)^{-\frac{m-4}{2}}\|(\tau_0,\mathbf{w}_0)\|_{\mathbb{R}^2\times H_{2k}^1} \le K(1+t)^{-\frac{m-4}{2}}\|u_0\|_{H_{2k}^1},
\end{aligned}$$

with $K=\tilde{C}\max\{K_1,K_2\}$. Furthermore,

$$|\tau_\infty| \le |\tau_0| + |\tau_0-\tau_\infty| \le C_\infty\|u_0\|_{H_{2k}^1}, \quad C_\infty = C_\Pi + K.$$

Finally, we show uniqueness of u. For that purpose, since $\tau(t),\tau_\infty\in\Pi^{-1}(P(B_\delta))$ and Theorem 5.37 we have

$$\begin{aligned}
\|u(t)-v_\star\|_{L_k^2} &\le C|\tau(t)-\tau_\infty| + \|w(t)\|_{L_k^2} + C|\tau_\infty| \le ((C+1)K + CC_\infty)\varepsilon_0\\
&= (\tilde{C}\max\{K_1,K_2\}(2C+1) + CC_\Pi)\varepsilon_0 \le \frac{\delta}{2}.
\end{aligned}$$

Let $\tilde{u}$ be another solution of (0.11) on $[0,T)$ for some $T>0$. Let

$$t_0 := \sup\{t \in [0,T) : \|\tilde{u} - v_\star\|_{L^2_k} \le \delta \text{ on } [0,t)\}.$$

Then there is a solution $(\tilde{\tau},\tilde{w})$ of (5.95), (5.96) on $[0,t_0)$ such that $T(\tilde{\tau}(t),\tilde{w}(t)) = \tilde{u}(t) - v_\star$ and thus $\tilde{u}(t) = v_\star(\cdot - \tilde{\tau}(t)) + \tilde{w}(t)$. But since (τ,w) is unique we conclude $(\tilde{\tau},\tilde{w}) = (\tau,w)$ and $u(t) = \tilde{u}(t)$ on $[0,t_0)$. Now assume $t_0 < T$. Then for all $t \in [0,t_0)$

$$\frac{\delta}{2} \ge \|u(t) - v_\star\|_{L^2_k} = \|\tilde{u}(t) - v_\star\|_{L^2_k}.$$

Since the right-hand side converges to δ as $t \to t_0$, we arrive at a contradiction. $\square$

Appendix A

Functional analysis and Fredholm theory

First we collect some basic definitions related to linear operator on Banach spaces. Let X, Y be Banach spaces. The set of all linear, bounded operators $T : X \to Y$ is denoted by $L[X, Y]$. In the case $X = Y$ we write $L[X, X] = L[X]$. The set of all closed, linear operators is denoted by $\mathcal{C}[X, Y]$ and $\mathcal{C}[X]$ respectively. The kernel of an operator T is denoted by $\mathcal{N}(T)$ and its range by $\mathcal{R}(T)$.

Definition A.1. An operator $T \in L[X, Y]$ is called a Fredholm operator if

i) $\dim \mathcal{N}(T) < \infty$,

ii) $\operatorname{codim}(\mathcal{R}(T), Y) < \infty$,

iii) $\mathcal{R}(T)$ is closed in Y.

The number

$$\operatorname{ind}(T) := \dim \mathcal{N}(T) - \operatorname{codim}(\mathcal{R}(T), Y)$$

is called the Fredholm index of T. If only $\dim \mathcal{N}(T)$ or $\operatorname{codim}(\mathcal{R}(T), Y)$ is infinite T is called a semi-Fredholm operator. In this case $\operatorname{ind}(T) = \pm\infty$.

Clearly, every semi-Fredholm operator with $\operatorname{ind}(T) < \infty$ is a Fredholm operator. Now we collect some properties concerning Fredholm operators. The results can be found in several texts from the literature, see [38], [61], [4], [25] and [33]

Lemma A.2 ([33, Thm. 25.9])**.** *Let X, Y, Z be Banach spaces, $T : X \to Y$ and $S : Y \to Z$ be Fredholm operators. Then $S \circ T : X \to Z$ is a Fredholm operator of index $\operatorname{ind}(S \circ T) = \operatorname{ind}(S) + \operatorname{ind}(T)$.*

Lemma A.3. *Let X_1, X_2, Y_1, Y_2 be Banach spaces, $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$ be Fredholm operators of index* $\operatorname{ind}(T_1)$, $\operatorname{ind}(T_2)$. *Then the operator*

$$\begin{aligned}(T_1 \times T_2) : X_1 \times X_2 &\to Y_1 \times Y_2\\ (u_1, u_2) &\mapsto (T_1 u_1, T_2 u_2)\end{aligned}$$

is a Fredholm operator of index $\operatorname{ind}(T_1 \times T_2) = \operatorname{ind}(T_1) + \operatorname{ind}(T_2)$.

Proof. Since T_1, T_2 are Fredholm operators, it is clear that $(T_1 \times T_2)$ is a linear bounded operator from $X_1 \times X_2$ to $Y_1 \times Y_2$. We have that $R(T_1)$ is closed in Y_1 and $R(T_2)$ is closed in Y_2, hence $R(T_1 \times T_2) = R(T_1) \times R(T_2)$ is closed in $Y_1 \times Y_2$. Moreover, $N(T_1 \times T_2) = N(T_1) \times N(T_2)$ and

$$(Y_1 \times Y_2)/R(T_1 \times T_2) = (Y_1 \times Y_2)/(R(T_1) \times R(T_2)) = Y_1/R(T_1) \times Y_2/R(T_2).$$

Hence

$$\begin{aligned}&\dim N(T_1 \times T_2) = \dim\left(N(T_1) \times N(T_2)\right) = \dim N(T_1) + \dim N(T_2) < \infty,\\ &\operatorname{codim}(R(T_1 \times T_2), Y_1 \times Y_2) = \dim\left(Y_1/R(T_1)\right) + \dim\left(Y_2/R(T_2)\right) < \infty\end{aligned}$$

which proves that $T_1 \times T_2$ is Fredholm operator of index

$$\begin{aligned}\operatorname{ind}(T_1 \times T_2) = {}& \dim N(T_1 \times T_2) - \operatorname{codim}(R(T_1 \times T_2), Y_1 \times Y_2)\\ & \dim N(T_1) - \dim\left(Y_1/R(T_1)\right) + \dim N(T_2) - \dim\left(Y_2/R(T_2)\right)\\ & \operatorname{ind}(T_1) + \operatorname{ind}(T_2).\end{aligned}$$

□

Lemma A.4 ([33, Cor. 25.11])**.** *Let $T : X \to Y$ be a Fredholm operator and $K : X \to Y$ be compact. Then $T + K$ is a Fredholm operator.*

Lemma A.5 ([38, Chap. IV, Cor. 5.29])**.** *Let $T : X \to Y$ be a semi-Fredholm operator. Then the adjoint operator $T^* : \mathcal{D}(T^*) \subset Y^* \to X^*$ is a semi-Fredholm operator with* $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$.

Lemma A.6 ([38, Chap. IV, Thm. 5.31])**.** *Let $T : X \to Y$ be a semi-Fredholm operator. Then there exists $\varepsilon_0 > 0$ such that for all $0 < |\kappa| < \varepsilon_0$ the operator $T + \kappa I$ is a semi-Fredholm operator of index* $\operatorname{ind}(T + \kappa I) = \operatorname{ind}(T)$

Lemma A.7. *Let X, Y be Banach spaces, $T : \mathcal{D} \subset X \to Y$ be a closed densely defined linear operator and $S \in L[X, Y]$. Then $T + S : \mathcal{D} \subset X \to Y$ is a closed densely defined linear operator.*

Proof. Clearly, $L = T + S : \mathcal{D} \to Y$ is densely defined and linear. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}$, $u_n \xrightarrow{X} u$, $Lu_n \xrightarrow{Y} y$. Since $S \in L[X,Y]$ we have $Su_n \xrightarrow{Y} Su$ and therefore

$$Tu_n = Lu_n - Su_n \xrightarrow{Y} y - Su =: w.$$

In addition, since T is closed it follows $u \in \mathcal{D}$ and $Tu = w$. Thus, $Lu = (T+S)u = w + Su = y$. $\square$

Definition A.8. Let $\mathcal{L}_0 : \mathcal{D}(\mathcal{L}_0) \subset X \to Y$ and $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset X \to Z$. Then $\mathcal{L}$ is called relatively compact w.r.t. $\mathcal{L}_0$ or $\mathcal{L}_0$-compact if $\mathcal{D}(\mathcal{L}_0) \subset \mathcal{D}(\mathcal{L})$ and for any bounded sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(\mathcal{L}_0)$ such that $\{\mathcal{L}_0 u_n\}_{n\in\mathbb{N}} \subset Y$ is also bounded. The sequence $\{\mathcal{L}u_n\}_{n\in\mathbb{N}} \subset Z$ has a convergent subsequence.

Lemma A.9. *Let $\mathcal{L}_0 : \mathcal{D}(\mathcal{L}_0) \subset X \to Y$ and $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset X \to Y$ with $\mathcal{D}(\mathcal{L}_0) \subset \mathcal{D}(\mathcal{L})$. If $(\mathcal{L}_0 - \mathcal{L})\mathcal{L}_0^{-1} : Y \to Y$ is compact then $(\mathcal{L}_0 - \mathcal{L})$ is $\mathcal{L}_0$-compact.*

Proof. $\{u_n\} \subset \mathcal{D}(\mathcal{L}_0)$, $\{\mathcal{L}_0 u_n\} \subset Y$ are bounded. Set $w_n = \mathcal{L}_0 u_n$. Since $(\mathcal{L}_0 - \mathcal{L})\mathcal{L}_0^{-1}$ is compact, the sequence $(\mathcal{L}_0 - \mathcal{L})\mathcal{L}_0^{-1} w_n$ has a convergent subsequence in Y, thus $(\mathcal{L}_0 - \mathcal{L})u_n$ has convergent subsequence. $\square$

Lemma A.10 ([38, Chap. IV, Thm. 5.22])**.** *Let $\mathcal{L}_0 \in \mathcal{C}[X,Y]$ be a semi-Fredholm operator. And let $\mathcal{L}$ be a $\mathcal{L}_0$-compact operator from X to Y. Then $\mathcal{T} = \mathcal{L}_0 + \mathcal{L}$ is a semi-Fredholm operator and*

$$\operatorname{ind}(\mathcal{T}) = \operatorname{ind}(\mathcal{L}_0).$$

Theorem A.11 (Fredholm alternative, [33, Chap. VII.25], [36, Thm. 2.2.1])**.** *Suppose $T : X \to Y$ is a Fredholm operator of index 0. Then either the homogeneous equation*

$$Tu = 0$$

has only the trivial solution $u = 0$, or the homogeneous equation has $\dim \mathcal{N}(T) = n$ linearly independent solutions $u_1, \ldots, u_n \in X$. In the latter case the inhomogeneous equation

$$Tu = r$$

has at least on solution if and only if $\langle w, r\rangle = 0$ for all $w \in \mathcal{N}(T^*)$, i.e. $r \in \mathcal{N}(T^*)^\perp$.

We conclude by considering projectors.

Lemma A.12 ([16, Prop. 8.5])**.** *Let X be a Banach space and $P, Q \in L[X]$ be projectors satisfying*

$$(\|P\| + \|Q\|)\, \|P - Q\| < 1.$$

Then

$$I + H = PQ + (I - P)(I - Q) \in L[X]$$

is a homeomorphism in X *which maps* $\mathcal{R}(Q)$ *resp.* $\mathcal{N}(Q)$ *homeomorphically into* $\mathcal{R}(P)$ *resp.* $\mathcal{N}(P)$.

Proof. It is easy to see that

$$H = P(Q - P) + (P - Q)Q.$$

Therefore $\|H\| \leq (\|P\| + \|Q\|)\|P - Q\| < 1$. Then $I + H$ is a homeomorphism. Moreover, $(I + H)Q = PQ$ and $(I + H)(I - Q) = (I - P)(I - Q)$ which shows $(I + H)\mathcal{R}(Q) \subset \mathcal{R}(P)$ and $(I + H)\mathcal{N}(Q) \subset \mathcal{N}(P)$. Now let $v \in \mathcal{R}(P)$ and let $u = (I + H)^{-1}v$. Then

$$Pv = v = (I + H)u = PQu + (I - P)(I - Q)u.$$

Apply $I - P$ to obtain $(I - P)(I - Q)u = 0$. We conclude

$$v = PQu = (I + H)Qu \in (I + H)\mathcal{R}(Q).$$

Thus, $(I + H)\mathcal{R}(Q) = \mathcal{R}(P)$. Similarly, one shows $\mathcal{N}(P) = (I + H)\mathcal{N}(Q)$. □

Appendix B

Exponential dichotomies and hyperbolic equilibria

In this section we collect results from the theory of exponential dichotomies, see [22], and hyperbolic equilibria. Further, we use exponential trichotomies as in [13], [31]. Some of the results are originally taken from lectures on given by W.-J. Beyn at Bielefeld University in 2014/2015 and 2017, see [16]. The results are well-known and can also be found in the literature [22], [34], [60].
Consider a matrix $A \in \mathbb{C}^{m,m}$ and its spectrum $\sigma(A) \subset \mathbb{C}$ and decompose it into

$$\sigma(A) = \sigma_{\mathfrak{s}}(A) \cup \sigma_{\mathfrak{u}}(A)$$

where $\sigma_{\mathfrak{s}}(A) \cap \sigma_{\mathfrak{u}}(A) = \emptyset$. Now let $\Gamma \subset \mathbb{C}$ be a contour with $\sigma_{\mathfrak{s}}(A)$ in its interior and $\sigma_{\mathfrak{u}}(A)$ in its exterior. Then Cauchy's Integral Formula states

$$\frac{1}{2\pi i}\int_{\Gamma}(z-\lambda)^{-1}dz = \begin{cases} 1, & \lambda \in \sigma_{\mathfrak{s}}(A), \\ 0, & \lambda \in \sigma_{\mathfrak{u}}(A). \end{cases}$$

Now the matrix

$$P := \frac{1}{2\pi i}\int_{\Gamma}(zI-A)^{-1}dz \in \mathbb{C}^{m,m} \tag{B.1}$$

is called the **Riesz projector** associated with $\sigma_{\mathfrak{s}}(A)$.

Proposition B.1 ([22], [16, Prop. 4.4])**.** The Riesz projector P is independent of the choice of the contour Γ. Further, P is the unique projector satisfying for

$$X_{\mathfrak{s}} = P(\mathbb{C}^m), \quad X_{\mathfrak{u}} = (I-P)(\mathbb{C}^m)$$

the properties

$$\begin{aligned} &\mathbb{C}^m = X_{\mathfrak{s}} \oplus X_{\mathfrak{u}}, \quad A(X_{\mathfrak{s}}) \subseteq X_{\mathfrak{s}}, \quad A(X_{\mathfrak{u}}) \subseteq X_{\mathfrak{u}}, \\ &\sigma(A_{|X_{\mathfrak{s}}}) = \sigma_{\mathfrak{s}}(A), \quad \sigma(A_{|X_{\mathfrak{u}}}) = \sigma_{\mathfrak{u}}(A). \end{aligned}$$

In the case $A \in \mathbb{R}^{m,m}$ the properties hold with $\mathbb{R}^m$ instead of $\mathbb{C}^m$.

We consider a linear differential operator

$$Lz = z' - A(x)z \tag{B.2}$$

for some $A \in C(J, \mathbb{R}^{n,n})$, $z \in C^1(J, \mathbb{R}^n)$ and $J \subset \mathbb{R}$. For $x, y \in J$ we denote by $S(x,y)$ the solution operator (B.2), i.e. the function $z(x) = S(x,y)z_0$ solves the initial value problem $z' = A(x)z$, $z(y) = z_0$.

Definition B.2. The linear differential operator $L(x) = \partial_x - A(x)$, $A \in C(J, \mathbb{R}^{m,m})$, $J \subset \mathbb{R}$ has a **shifted exponential dichotomy** on J with exponents $\alpha < \beta$ if there is a constant $K > 0$ and projectors $P_\kappa(x)$, $x \in J$, $\kappa = \mathfrak{s}, \mathfrak{u}$ of rank m_κ such that $P_{\mathfrak{s}} + P_{\mathfrak{u}} = I$ in J and such that for all x, y in J there hold

$$\begin{aligned} &S(x,y)P_\kappa(y) = P_\kappa(x)S(x,y), \quad \kappa = \mathfrak{s}, \mathfrak{u}, \\ &|S(x,y)P_{\mathfrak{s}}(y)| \le Ke^{\alpha(x-y)}, \quad x \ge y, \\ &|S(x,y)P_{\mathfrak{u}}(y)| \le Ke^{\beta(x-y)}, \quad x \le y. \end{aligned} \tag{B.3}$$

In the case $\alpha < 0 < \beta$, L is said to have an **exponential dichotomy** on J. If (B.3) holds with $\alpha = \beta = 0$ then L is said to have an **ordinary dichotomy** on J. We call (K, α, β) the data of the dichotomy.

Lemma B.3 (Roughness of shifted exponential dichotomies, [13, Prop. 2.3] ,[22, Prop. 4.1])**.** *Let L have a shifted exponential dichotomy on $J = [\tau, \infty)$ with data (K, α, β) and projectors P_κ, $\kappa = \mathfrak{s}, \mathfrak{u}$. Let $B \in C(J, \mathbb{R}^{n,n})$ satisfy*

$$\frac{8K^2\delta}{\beta - \alpha} < 1, \quad \delta = \sup_{x \ge \tau} |B(x)|.$$

Then the perturbed operator $\tilde{L} = L - B$ has a shifted exponential dichotomy on $J = [\tau, \infty)$ with data

$$\tilde{\alpha} = \alpha + 2\delta K < \tilde{\beta} = \beta - 2\delta K,$$

constant $\tilde{K} = \frac{5}{2}K^2$ and projectors $\tilde{P}_\kappa$, $\kappa = \mathfrak{s}, \mathfrak{u}$ satisfying

$$|P_\kappa(x) - \tilde{P}_\kappa(x)| \le 5K^3 \int_\tau^\infty e^{-(\tilde{\beta}-\alpha)|x-y|}|B(y)|dy, \quad \kappa = \mathfrak{s}, \mathfrak{u}.$$

In addition, if the data (K, α, β) and projectors P_κ, $\kappa = \mathfrak{s}, \mathfrak{u}$ depend continuously/analytically on some parameter $s \in \Omega$ for an open domain $\Omega \subset \mathbb{K}^n$ and B is independent of s then the data $(\tilde{K}, \tilde{\alpha}, \tilde{\beta})$ and the projectors $\tilde{P}_\kappa$, $\kappa = \mathfrak{s}, \mathfrak{u}$ depend continuously/analytically on s.

Proposition B.4 ([22, Chap. 2])**.** Suppose $L = \partial_t - A$ with $A \in \mathbb{K}^{m,m}$ and $J = [x_0, \infty)$. Then L has an exponential dichotomy on J if and only if A is hyperbolic, i.e. $\sigma(A) \cap i\mathbb{R} = \emptyset$.

Now consider the initial value problem

$$z' = f(z), \quad z(0) = z_0, \quad f \in C^k(\Omega, \mathbb{R}^m), \quad \Omega \subset \mathbb{R}^d \tag{B.4}$$

and suppose $\bar{z}$ is a hyperbolic equilibrium, i.e. $f(\bar{z}) = 0$ and $\sigma(Df(\bar{z})) \cap i\mathbb{R} = \emptyset$. Moreover, let $P_\mathfrak{s}$ we the Riesz projector associated with $\sigma_\mathfrak{s}(Df(\bar{z})) = \{s \in \sigma(Df(\bar{z})) : \operatorname{Re} s < 0\}$ and $P_\mathfrak{u} = I - P_\mathfrak{s}$. Moreover, we denote the solution of (B.4) by $z(t, z_0)$, $t \in J(z_0) \subset \mathbb{R}$ on the maximal interval of existence $J(z_0)$. Let $V \subset \mathbb{R}^d$ be a neighborhood of $\bar{z}$. Then we define the local stable and unstable manifolds

$$\begin{aligned}
\mathcal{M}_\mathfrak{s}^V(\bar{z}) &:= \{z_0 \in V : [0, \infty) \subset J(z_0),\, z(t) \in V \,\forall\, t \geq 0,\, \lim_{t \to \infty} z(t) = \bar{z}\}, \\
\mathcal{M}_\mathfrak{u}^V(\bar{z}) &:= \{z_0 \in V : (-\infty, 0] \subset J(z_0),\, z(t) \in V \,\forall\, t \leq 0,\, \lim_{t \to -\infty} z(t) = \bar{z}\}.
\end{aligned}$$

The following theorem about the local stable and unstable manifolds holds.

Theorem B.5 (Local stable/unstable manifold theorem, [60, Thm. 7.6], [16, Thm. 4.9])**.** *There are neighborhoods of* $\bar{z}$

$$V_\mathfrak{s} \subset X_\mathfrak{s}, \quad V_\mathfrak{u} \subset X_\mathfrak{u}, \quad V \subset \mathbb{R}^m$$

with $V_\mathfrak{s} \oplus V_\mathfrak{u} \subset V$ *such that the following holds:*

i) For every $z_\mathfrak{s} \subset V_\mathfrak{s}$ *the boundary value problem*

$$\begin{aligned}
z' &= f(z) \quad \text{on} \quad [0, \infty), \\
P_\mathfrak{s} z(0) &= z_\mathfrak{s}, \quad z(t) \in V \quad \forall\, t \geq 0
\end{aligned}$$

has a unique solution $z(\cdot, z_\mathfrak{s}) \in C^{k+1}([0, \infty), V)$ *and there are* $K, \mu > 0$ *such that*

$$|z(t, z_\mathfrak{s}) - \bar{z}| \leq K e^{-\mu t} \quad \forall\, t \geq 0.$$

ii) For every $z_\mathfrak{u} \subset V_\mathfrak{u}$ *the boundary value problem*

$$\begin{aligned}
z' &= f(z) \quad \text{on} \quad (-\infty, 0], \\
P_\mathfrak{u} z(0) &= z_\mathfrak{u}, \quad z(t) \in V \quad \forall\, t \leq 0
\end{aligned}$$

has a unique solution $z(\cdot, z_\mathfrak{u}) \in C^{k+1}((-\infty, 0], V)$ *and there are* $K, \mu > 0$ *such that*

$$|z(t, z_\mathfrak{u}) - \bar{z}| \leq K e^{\mu t} \quad \forall\, t \leq 0.$$

Definition B.6. The linear differential operator $L(x) = \partial_x - A(x)$, $A \in C(J, \mathbb{R}^{m,m})$, $J \subset \mathbb{R}$ has an ordinary exponential trichotomy with exponents $\alpha < \nu < \beta$ if there is a constant $K > 0$ and projectors $P_\kappa(x)$, $x \in J$, $\kappa = \mathfrak{s}, \mathfrak{c}, \mathfrak{u}$ of rank m_κ such that $P_\mathfrak{s} + P_\mathfrak{c} + P_\mathfrak{u} = I$ in J and such that for all x, y in J there hold

$$\begin{aligned}
&S(x,y)P_\kappa(y) = P_\kappa(x)S(x,y), \quad \kappa = \mathfrak{s}, \mathfrak{c}, \mathfrak{u},\\
&|S(x,y)P_\mathfrak{s}(y)| \le Ke^{\alpha(x-y)}, \quad |S(x,y)P_\mathfrak{c}(y)| \le Ke^{\nu(x-y)}, \quad x \ge y,\\
&|S(x,y)P_\mathfrak{u}(y)| \le Ke^{\beta(x-y)}, \quad |S(x,y)P_\mathfrak{c}(y)| \le Ke^{\nu(x-y)}, \quad x \le y.
\end{aligned}$$

We call (K, α, ν, β) the data of the ordinary exponential trichotomy.

Appendix C

Semilinear parabolic equations

In this section we collect results concerning solutions of semilinear parabolic equations and their regularity. References are [32] and [5]. Suppose X, Y are Banach spaces where $Y \subset X$ is dense and $T \in \mathbb{R}_+ \cup \{\infty\}$.

Theorem C.1 ([32, Thm. 3.2.2])**.** *Suppose $A : Y \to X$ is a sectorial operator, $u_0 \in X$ and $f \in C^\alpha((0,T), X)$ for some $\alpha \in (0,1)$. Further let $\int_0^\rho \|f(t)\|_X dt < \infty$ for some $\rho > 0$. Then there exists a unique $u \in C([0,T), X) \cap C^1((0,T), X)$ with $u(t) \in Y$ for $0 < t < T$ satisfying*

$$\begin{aligned} u'(t) &= Au(t) + f(t), \quad 0 < t < T, \\ u(0) &= u_0, \end{aligned}$$

namely

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s) ds.$$

Theorem C.2 ([5, p.43, Thm. 1.2.1])**.** *Suppose $A : Y \to X$ is a sectorial operator, $u_0 \in X$ and $f \in C^\alpha([0,T), X)$ for some $\alpha \in (0,1)$. Then there exists a unique*

$$u \in C([0,T), X) \cap C^\alpha((0,T), Y) \cap C^{1+\alpha}((0,T), X)$$

satisfying

$$\begin{aligned} u'(t) &= Au(t) + f(t), \quad 0 < t < T, \\ u(0) &= u_0. \end{aligned}$$

In addition, if $u_0 \in Y$ then $u \in C^1([0,T), X)$.

Theorem C.3. *Suppose $A : Y \to X$ is a sectorial operator, $u_0 \in Y$ and $f \in C^\alpha([0,T), X)$ for some $\alpha \in (0,1)$. Further let $\int_0^\rho \|f(t)\|_X dt < \infty$ for some $\rho > 0$. Then there exists a unique*

$$u \in C^\alpha((0,T), Y) \cap C^{1+\alpha}((0,T), X) \cap C^1([0,T), X)$$

satisfying

$$\begin{aligned} u'(t) &= Au(t) + f(t), \quad 0 < t < T, \\ u(0) &= u_0, \end{aligned}$$

namely

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s) ds.$$

Proof. Theorem C.2 implies there exists a unique $\tilde{u} \in C^\alpha((0,T), Y) \cap C^{1+\alpha}((0,T), X) \cap C^1([0,T), X)$, s.t. $\tilde{u}' = A\tilde{u} + f$ in $(0,T)$ and $u(0) = u_0$. Moreover, Theorem C.1 implies that the function

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s) ds$$

is the unique solution of

$$\begin{aligned} u'(t) &= Au(t) + f(t), \quad 0 < t < T, \\ u(0) &= u_0. \end{aligned}$$

in $u \in C^1((0,T), X)$ with $u(t) \in Y$. Since $\tilde{u} \in C^1((0,T), X)$ and $\tilde{u}(t) \in Y$, it follows $\tilde{u} = u$ for all $0 \leq t < T$. $\square$

Appendix D

Miscellaneous

In this section we collect classical tools from different areas which are used in the thesis.

For a matrix $A \in \mathbb{C}^{m,m}$ we define the lower spectral bound

$$\alpha(A) := \min\left\{\operatorname{Re}(x^H A x) : |x| = 1\right\}.$$

Lemma D.1 ([16, Lem. 6.3])**.** *Let $A, B \in \mathbb{R}^{m,m}$ and $C \in \mathbb{C}^{m,m}$ with $\alpha(A), \alpha(C) > 0$ and*

$$|B - B^\top|^2 < 4\alpha(A)\alpha(C).$$

Then the matrix

$$M = \begin{pmatrix} 0 & I \\ A^{-1}C & -A^{-1}B \end{pmatrix}$$

is hyperbolic with $m_{\mathfrak{s}} = m_{\mathfrak{u}} = m$.

Theorem D.2 (Sobolev embedding, [2, Thm. 4.12])**.** *Let $n, k \in \mathbb{N}$ and $1 \le p \le q \le \infty$ with $kp > n$. Then the inclusion $W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ is continuous, i.e. there is $C > 0$ such that*

$$\|u\|_{L^q(\mathbb{R}^n)} \le C\|u\|_{W^{k,p}(\mathbb{R}^n)} \quad \forall u \in W^{k,p}(\mathbb{R}^n).$$

We now consider nonlinear eigenvalue problems. The proof of the following result in a more general version can be found in [43, Thm. 1.6.5]. We use and state here the simpler version from [14, Thm. 2.4] concerning simple eigenvalues.

Theorem D.3 (Keldysh)**.** *Let $\Omega \subset \mathbb{C}$ open, $T : \Omega \to \mathbb{C}^{m,m}$ be holomorphic in Ω, $\lambda \in \Omega$ be a simple eigenvalue of T and $v, w \in C^m$ satisfy*

$$T(\lambda)v = 0 = w^H T(\lambda), \quad w^H T'(\lambda)v = 1, \quad |v| = 1.$$

Then there is a neighborhood $U \subset \Omega$ of λ and a holomorphic function $\Gamma : U \to \mathbb{C}^{m,m}$ such that

$$T(z)^{-1} = \frac{1}{z-\lambda} v w^H + \Gamma(z), \quad z \in U \backslash \{\lambda\}.$$

Lemma D.4. *Let $B \in L^\infty(\mathbb{R}, \mathbb{R}^{n,n})$ satisfy*

$$\sup_{|x|>R} |B(x)| \to 0, \quad R \to \infty.$$

Then the multiplication operator m_B associated with B given by

$$m_B : H^1(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n), \quad u \mapsto Bu$$

is compact.

The proof of Lemma D.4 requires the following classical result concerning compactness in L^p. It goes back to M. Riesz and can be found in [4, Thm. 2.16].

Lemma D.5. *Let $1 \le p \le \infty$. Then $K \subset L^p(\mathbb{R}^n, \mathbb{R}^m)$ is relatively compact if and only if*

i) $\sup_{u \in K} \|u\|_{L^p(\mathbb{R}^n)} < \infty$,

ii) $\sup_{u \in K} \|u(\cdot + h) - u\|_{L^p(\mathbb{R}^n)} \to 0$ *as* $|h| \to 0$,

iii) $\sup_{u \in K} \|u\|_{L^p(\mathbb{R}^n \backslash B_R(0))} \to 0$ *as* $R \to \infty$.

Proof of Lemma D.4. Let $K = \{Bu, \|u\|_{H^1} \le 1\}$ be the image of the unit Ball under m_B. Clearly, $\sup_{u \in K} \|u\|_{L^2(\mathbb{R})} < \infty$ and

$$\sup_{\|u\|_{H^1} \le 1} \int_{|x|>R} |B(x)u(x)|^2 dx \le \sup_{|x|>R} |B(x)|^2 \to 0, \quad R \to \infty.$$

Now let $h \le h_0 < 1$. We estimate

$$\begin{aligned}
&\sup_{\|u\|_{H^1} \le 1} \int_{|x|\le R} |B(x+h)u(x+h) - B(x)u(x)|^2 dx \\
&\le \sup_{\|u\|_{H^1} \le 1} \left\{ \int_{|x|\le R} |B(x+h)|^2 |u(x+h) - u(x)|^2 dx + \int_{|x|\le R} |B(x+h) - B(x)|^2 |u(x)|^2 dx \right\} \\
&\le \sup_{\|u\|_{H^1} \le 1} \left\{ \|B\|_{L^\infty}^2 h^2 \|u_x\|_{L^2}^2 + \|u\|_{L^\infty}^2 \|B(\cdot + h) - B\|_{L^2([-R,R])} \right\} \\
&\le \|B\|_{L^\infty}^2 h^2 + \|B(\cdot + h) - B\|_{L^2([-R,R])}.
\end{aligned}$$

Moreover,

$$
\begin{aligned}
&\sup_{\|u\|_{H^1}\leq 1}\int_{|x|\geq R}|B(x+h)u(x+h)-B(x)u(x)|^2dx\\
&\leq \sup_{\|u\|_{H^1}\leq 1}\left\{\int_{|x|\geq R}|B(x+h)|^2|u(x+h)-u(x)|^2dx+\int_{|x|\geq R}|B(x+h)-B(x)|^2|u(x)|^2dx\right\}\\
&\leq \sup_{\|u\|_{H^1}\leq 1}\left\{\|B\|_{L^\infty}^2h^2\|u_x\|_{L^2}^2+2\sup_{|x|>R-h}|B(x)|^2\|u\|_{L^2}^2\right\}\leq \|B\|_{L^\infty}^2h^2+\sup_{|x|>R-1}|B(x)|^2.
\end{aligned}
$$

Now for arbitrary $\varepsilon>0$ there are $R>0$ and $h_0<1$ such that for all $h<h_0<1$

$$
\sup_{|x|>R-1}|B(x)|^2<\frac{\varepsilon}{2},\quad 2\|B\|_{L^\infty}^2h^2+\|B(\cdot+h)-B\|_{L^2([-R,R])}<\frac{\varepsilon}{2}.
$$

Then for all $h\leq h_0$ there holds

$$
\sup_{\|u\|_{H^1}\leq 1}\int_{\mathbb{R}}|B(x+h)u(x+h)-B(x)u(x)|^2dx<\varepsilon.
$$

As a consequence of Lemma D.5 the set K is relatively compact in L^2 and the assertion is proven. $\square$

Lemma D.6 ([16, Lem. 2.23])**.** *Let $A\in\mathbb{R}^{m,m}$ have only eigenvalues with positive real part and suppose $v\in C^2(\mathbb{R},\mathbb{R}^m)$ and $c\in\mathbb{R}$ solve the second order ODE*

$$
Av''+cv'=h\in C(\mathbb{R},\mathbb{R}^m),
$$

such that both limits $\lim_{x\to\pm\infty}h(x)$ and $\lim_{x\to\pm\infty}v(x)$ exist. Then

$$
\lim_{x\to\pm\infty}h(x)=0=\lim_{x\to\pm\infty}v'(x).
$$

Consider a real polynomial

$$
f(z)=z^3+a_1z^2+a_2z+a_3,\quad a_i\in\mathbb{R},i=1,2,3 \tag{D.1}
$$

and define the corresponding Hurwitz determinants

$$
\delta_0=1,\quad \delta_1=a_1,\quad \delta_2=a_1a_2-a_3,\quad \delta_3=a_3\delta_2.
$$

Then the following theorem holds, see [51, Thm. 11.4.5] or [27, Chap. V. Thm.4, Thm. 5].

Theorem D.7 (Routh-Hurwitz Theorem, [51, Thm. 11.4.5])**.** *Let f from* (D.1) *have no root on the imaginary axis and $\delta_i \neq 0$ for all $i = 1, 2, 3$. Then the number of roots of f in the left half-plane is given by*

$$p = 3 - V(1, \delta_1, \delta_3) - V(1, \delta_2)$$

where $V(a_1, ..., a_n)$ is the function counting the variations of signs in the sequence $a_1, ..., a_n$. If $\delta_2 = 0$ and $\delta_1 \neq 0$, we have

$$p = 1 + V(1, -\delta_1).$$

Theorem D.8 (Implicit Function Theorem, [7, VII Thm. 8.2])**.** *Let E_1, E_2, F be Banach spaces, $\Omega \subset E_1 \times E_2$ an open subset and $f \in C^q(\Omega, F)$. Further let $(x_0, y_0) \in \Omega$ with*

$$f(x_0, y_0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0, y_0) \in L[E_2, F] \quad \text{invertible.}$$

Then there are neighborhoods $U \subset \Omega$ of (x_0, y_0), $V \subset E_1$ of x_0 and a unique $g \in C^q(V, E_2)$ such that $f(x, y) = 0$ holds for $(x, y) \in U$ if and only if $y = g(x)$ for $x \in V$. Moreover,

$$\partial_x g(x) = -\left[\frac{\partial f}{\partial y}(x, g(x))\right]^{-1} \frac{\partial f}{\partial x}(x, g(x)).$$

Bibliography

[1] R. Abraham, J. E. Marsden, and T. S. Ratiu. *Manifolds, tensor analysis, and applications*, volume 75 of *Applied mathematical sciences*. Springer, New York, 2nd edition, 1988.

[2] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and applied mathematics*. Acad. Press, Amsterdam, 2nd edition, 2008.

[3] J. Alexander, R. Gardner, and C. Jones. A topological invariant arising in the stability analysis of travelling waves. *Journal für die Reine und Angewandte Mathematik, 410, 167*, 1990.

[4] H. W. Alt. *Lineare Funktionalanalysis*. Springer-Lehrbuch Masterclass. Springer, Berlin, Heidelberg, 6th edition, 2012.

[5] H. Amann. *Linear and quasilinear parabolic problems*. Monographs in mathematics. Birkhäuser, Basel.

[6] H. Amann. *Ordinary differential equations*, volume 13 of *de Gruyter studies in mathematics*. de Gruyter, Berlin, 1990.

[7] H. Amann and J. Escher. *Analysis II*. Grundstudium Mathematik. Birkhäuser Verlag, Basel, 2nd edition, 2006.

[8] I. S. Aranson and L. Kramer. The world of the complex Ginzburg-Landau equation. *Reviews of Modern Physics, 74, 1, 99*, 2002.

[9] W. Arendt. *Vector valued Laplace transforms and Cauchy problems*, volume 96 of *Monographs in mathematics*. Birkhäuser, Basel, 2001.

[10] M. Beck, T. T. Nguyen, B. Sandstede, and K. Zumbrun. Nonlinear stability of source defects in the complex Ginzburg-Landau equation. *Nonlinearity, 27, 4, 739*, 2014.

[11] W.-J. Beyn. The numerical computation of cennecting orbits in dynamic-systems. *IMA Journal of Numerical Analysis, 10, 3, 379*, 1990.

[12] W.-J. Beyn. Numerical analysis of homoclinic orbits emanating from a Takens-Bogdanov point. *IMA Journal of Numerical Analysis, 14, 3, 381*, 1994.

[13] W.-J. Beyn. On well-posed problems for connecting orbits in dynamical systems chaotic numerics. *Contemporary Mathematics, 172, 131*, 1994.

[14] W.-J. Beyn. An integral method for solving nonlinear eigenvalue problems. *Linear Algebra and its Applications, 436, 10, 3839*, 2012.

[15] W.-J. Beyn. *Current Challenges in Stability Issues for Numerical Differential Equations*, volume 2082 of *Lecture Notes in Mathematics*. Springer International Publishing, 2014.

[16] W.-J. Beyn. Manuscript on Waves in Evolution Equations, Bielefeld University. *https://www.math.uni-bielefeld.de/~beyn/AG_Numerik/html/en/teaching*, 2017.

[17] W.-J. Beyn and J. Lorenz. Nonlinear stability of rotating patterns. *Dynamics of Partial Differential Equations, 5, 4, 349*, 2008.

[18] W.-J. Beyn and V. Thümmler. Freezing solutions of equivariant evolution equations. *SIAM Journal on Applied Dynamical Systems, 3, 2, 85*, 2004.

[19] W.-J. Beyn and V. Thümmler. Dynamics of patterns in nonlinear equivariant pdes. *GAMM-Mitteilungen, 32, 1, 7*, 2009.

[20] U. Bortolozzo, M. G. Clerc, C. Falcon, S. Residori, and R. Rojas. Localized states in bistable pattern-forming systems. *Physical Review Letters, 96, 21*, 2006.

[21] P. Chossat and R. Lauterbach. *Methods in equivariant bifurcations and dynamical systems*, volume 15 of *Advanced series nonlinear dynamics*. World Scientific, Singapore, 2000.

[22] W. A. Coppel. *Dichotomies in stability theory*, volume 629 of *Lecture notes in mathematics Series: Australian National University, Canberra*. Springer, Berlin, 1978.

[23] S. Dieckmann. *Dynamics of patterns in equivariant Hamiltonian partial differential equations*. PhD Thesis. Bielefeld, 2017.

[24] W. Eckhaus and G. Iooss. Strong selection or rejection of spatially periodic patterns in degenerate bifurcations. *Physica D, 39, 1, 124*, 1989.

[25] D. E. Edmunds and W. D. Evans. *Spectral theory and differential operators*. Oxford mathematical monographs. Oxford University Press, Oxford, 2nd edition, 2018.

[26] K.-J. Engel and R. Nagel, editors. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate texts in mathematics*. Springer, New York, 2000.

[27] F. R. Gantmacher. *Applications of the theory of matrices*. Interscience Publ., New York, 1959.

[28] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry. i. *Journal of Functional Analysis, 74, 1, 160*, 1987.

[29] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry. ii. *Journal of Functional Analysis, 94, 2, 308*, 1990.

[30] E. Hairer. *Solving Ordinary Differential Equations II*, volume 14 of *Springer Series in Computational Mathematics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

[31] J. K. Hale and X.-B. Lin. Heteroclinic orbits for retarded functional-differential equations. *Journal of Differential Equations, 65, 2, 175*, 1986.

[32] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture notes in mathematics*. Springer, Berlin, 1981.

[33] F. Hirzebruch and W. Scharlau. *Einführung in die Funktionalanalysis*, volume 296 : Mathematik of *BI-Hochschultaschenbücher*. Bibliograph. Institut, Mannheim, 1971.

[34] M. C. Irwin. *Smooth dynamical systems*, volume 94 of *Pure and applied mathematics*. Acad. Press, London, 1980.

[35] T. Kapitula. On the stability of travelling waves in weighted L^∞ spaces. *Journal of Differential Equations, 112, 1, 179*, 1994.

[36] T. Kapitula and K. Promislow. *Spectral and dynamical stability of nonlinear waves*, volume 185 of *Applied mathematical sciences*. Springer, New York, 2013.

[37] T. M. Kapitula. Stability of weak shocks in λ–ω systems. *Indiana University Mathematics Journal, 40, 4, 1193*, 1991.

[38] T. Kato. *Perturbation theory for linear operators*, volume 132. Springer, Berlin, 1966.

[39] G. Kreiss, H.-O. Kreiss, and N. Anders Petersson. On the convergence to steady state of solutions of nonlinear hyperbolic- parabolic systems. *SIAM Journal on Numerical Analysis, 31, 6, 1577*, 1994.

[40] H.-O. Kreiss and J. Lorenz. Stability for time-dependent differential equations. *Acta Numerica, 203*, 1998.

[41] S. Lang. *Introduction to differentiable manifolds.* Interscience Publ., New York, 3rd edition, 1967.

[42] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*, volume 16 of *Progress in nonlinear differential equations and their applications.* Birkhäuser, Basel, 1995.

[43] R. Mennicken and M. Möller. *Non-self adjoint boundary eigenvalue problems*, volume 192 of *North-Holland mathematics studies.* Elsevier, Amsterdam, 1st edition, 2003.

[44] A. Mielke. The Ginzburg-Landau equation in its role as a modulation equation. *Handbook of dynamical systems, Vol. 2*, 2002.

[45] M. Miklavcic. *Applied functional analysis and partial differential equations.* World Scientific, Singapore, 1998.

[46] K. Nozaki and N. Bekki. Exact solutions of the generalized Ginzburg-Landau equation. *Journal of the Physical Society of Japan, 53, 5, 1581*, 1984.

[47] D. Otten. Exponentially weighted resolvent estimates for complex Ornstein-Uhlenbeck systems. *Journal of Evolution Equations, 15, 4*, 2015.

[48] K. J. Palmer. Exponential dichotomies and transversal homoclinic points. *Journal of Differential Equations, 55, 2, 225*, 1984.

[49] K. J. Palmer. Exponential dichotomies and fredholm operators. *Proceedings of the American Mathematical Society, 104, 1, 149*, 1988.

[50] A. Pazy. *Semi-groups of linear operators and applications to partial differential equations*, volume 10 of *Lecture notes, University of Maryland, Department of Mathematics.* College Park, Md., 1974.

[51] Q. I. Rahman and G. Schmeisser. *Analytic theory of polynomials*, volume 26 of *London Mathematical Society monographs.* Clarendon Press, Oxford, 2002.

[52] M. Renardy and R. C. Rogers. *An introduction to partial differential equations*, volume 13 of *Texts in applied mathematics.* Springer, New York, 2nd edition, 2009.

[53] W. Rossmann. *Lie groups*, volume 5 of *Oxford graduate texts in mathematics.* Oxford Univ. Press, Oxford, 2006.

[54] J. Rottmann-Matthes. *Computation and stability of patterns in hyperbolic-parabolic Systems.* Berichte aus der Mathematik. Shaker, Aachen, 2010.

[55] J. Rottmann-Matthes. Stability and freezing of nonlinear waves in first order hyperbolic pdes. *Journal of Dynamics and Differential Equations, 24, 2, 341*, 2012.

[56] B. Sandstede. Stability of travelling waves. *Handbook of Dynamical Systems, 2, 983*, 2002.

[57] B. Sandstede and A. Scheel. Defects in oscillatory media: Toward a classification. *SIAM Journal on Applied Dynamical Systems, 3, 1, 1*, 2004.

[58] G. Schneider. Existence and stability of modulating pulse-solutions in a phenomenological model of nonlinear optics. *Physica D, 140, 3-4, 283*, 2000.

[59] A. Shepeleva. On the validity of the degenerate Ginzburg-Landau equation. *Mathematical Methods in the Applied Science, 20, 14, 1239*, 1997.

[60] T. C. Sideris. *Ordinary Differential Equations and Dynamical Systems*, volume 2 of *Atlantis Studies in Differential Equations.* Atlantis Press, Paris, 2013.

[61] A. E. Taylor and D. C. Lay. *Introduction to functional analysis.* Wiley, New York, 2nd edition, 1980.

[62] W. van Saarloos and P. Hohenberg. Fronts, pulses, sources and sinks in generalized complex Ginzburg-Landau equations. *Physica D, 56, 4, 303*, 1992.

[63] S. V. Zelik and A. Mielke. *Multi-pulse evolution and space time chaos in dissipative systems*, volume 925 of *Memoirs of the American Mathematical Society.* American Mathematical Soc., Providence, RI, 2009.