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On the potential in non-Gaussian chain polymer models

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In this paper, we investigate the potential for a class of non-Gaussian processes so-called generalized grey Brownian motion. We obtain a closed analytic form for the potential as an integral of the *M*-Wright functions and the Green function. In particular, we recover the special cases of Brownian motion and fractional Brownian motion. In addition, we give the connection to a fractional partial differential equation and its the fundamental solution.

KEYWORDS
anomalous diffusions, chain polymer models, chain potential, generalized, grey Brownian motion

MSC CLASSIFICATION
60G22; 82D60

1 | INTRODUCTION

Polymers are consisting of small chemical units that act on each other via different forces. A very simple and well-studied model of an ideal chain is a classical random walk; see, eg, Madras and Slade and Rubinstein and Colby.[1,2] In these models, there are no interactions between the monomers that are far apart along the chain.[2] In that case, it is known that the nearest neighbors are linked via springs; ie, the chain can be considered as a chain of harmonic oscillators.

Of course, to obtain a more realistic polymer model, the suppression of self intersections had to be introduced ("excluded volume"); see Edwards and Westwater[3,4] for a continuum model; for random walks, see Domb and Joyce[5] and references therein. In addition, one has to consider solvent interactions.[2] Individual chain polymer models are hence well studied and widely understood. A continuum limit of those models, that is, where the polymers are modeled by Brownian motion (Bm) paths, led to a deeper understanding in the asymptotic scaling behavior of the chains. The drawback of Bm or random walk models is that they can not reflect long-range forces along the chain without introducing a further potential.

Fractional Brownian motion (fBm) paths have been suggested as a heuristic model,[6] without yet including the "excluded volume effect" although a more proper model would be based on self-avoiding fractional random walks.

The aim of this paper is at first to model the long-range correlations of fBm as a generalized bead-spring model, hence a chain model. For this, we first consider a continuous model, which we then discretize. In the next step, we go beyond the fBm-based models. More precisely, we discuss a generalized interaction that arises from non-Gaussian chain models.

Note: In particular, we consider the logarithm of the corresponding probability density function. Because of its role in polymer physics, we use the term "chain potential" throughout this article, a widely used terminology, such as, eg, in Rubinstein and Colby, Pelissetto and Vicari, and Jannink and des Cloizeaux.[2,7,8]

In that model, the interaction potentials are not only long-range along the chain but can also give rise to multiparticle nonlinear forces between the constituents. The class of underlying random processes is that of generalized grey Brownian motion (ggBm) that will give rise to chain models with nonlinear forces between the constituents and nonergodic dynamics as shown in Molina-Garcí.[9] There also occur higher order interactions; in particular, we give an explicit form for

the three-particle term. The ggBm family of processes generalizes fBm in the sense that it still has stationary increments; thus, it may serve as a generalized continuous polymer model.

In section 2, we shall introduce the required concepts and properties of ggBm so as to then present our results, and in section 3, the chain potentials are derived, and the relation to the fractional partial equations is pointed out.

2 | GENERALIZED GREY BROWNIAN MOTION IN ARBITRARY DIMENSIONS

Before we introduce the definition of ggBm, let us recall two special functions that play a major role in what follows. For $0 < \beta < 1$, the Mittag-Leffler (entire) function E_β is defined by the following:

$$E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad z \in \mathbb{C}, \tag{1}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \quad z \in \mathbb{C}, \, \mathrm{Re}(z) \geq 0$$

is the Euler gamma function. Note that for $n \in \mathbb{N}$, we have $\Gamma(n+1) = n!$.

The M-Wright function is given as a particular choice of the Wright function $W_{\lambda,\mu}$, $\lambda > -1$, $\mu \in \mathbb{C}$ via

$$M_\beta(z) := W_{-\beta, 1-\beta}(-z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(-\beta n + 1 - \beta)}.$$

The special choice $\beta = \frac{1}{2}$ yields the following:

$$M_{\frac{1}{2}}(z) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4}\right); \tag{2}$$

ie, we recover the Gaussian density.

The Mittag-Leffler function E_β is the Laplace transform of the M-Wright function in $\mathbb{R}_+$; hence,

$$E_\beta(-s) = \int_0^\infty e^{-s\tau} M_\beta(\tau) \, d\tau. \tag{3}$$

Definition 1 (see Mura and Mainardi[10] for $d = 1$). Let $0 < \beta < 1$ and $0 < \alpha < 2$ be given. A d-dimensional continuous stochastic process $B^{\beta,\alpha} = \{B^{\beta,\alpha}(t), \, t \geq 0\}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ is a ggBm if:

1. $P(B^{\beta,\alpha}(0) = 0) = 1$, that is ggBm, starts at zero almost surely.
2. Any collection $\{B^{\beta,\alpha}(t_1), \ldots, B^{\beta,\alpha}(t_n)\}$ with $0 \leq t_1 < t_2 < \ldots < t_n < \infty$ has characteristic function given, for any $\theta = (\theta_1, \ldots, \theta_n) \in (\mathbb{R}^d)^n$, where $\theta_j = (\theta_{j,1}, \ldots \theta_{j,d})$

$$\mathbb{E}\left(\exp\left(i \sum_{k=1}^{n} (\theta_k, B^{\beta,\alpha}(t_k))\right)\right) = E_\beta\left(-\frac{1}{2} \sum_{k=1}^{d} (\theta_{\cdot,k}, \Sigma_\alpha \theta_{\cdot,k})\right), \tag{4}$$

where $\mathbb{E}$ denotes the expectation and

$$\Sigma_\alpha = \frac{1}{2} \left(t_k^\alpha + t_j^\alpha - |t_k - t_j|^\alpha\right)_{k,j=1}^{n}.$$

3. The joint probability density function of $(B^{\beta,\alpha}(t_1), \ldots, B^{\beta,\alpha}(t_n))$ is equal to

$$f_\beta(\theta, \Sigma_\alpha) = \frac{(2\pi)^{-\frac{nd}{2}}}{\sqrt{\det \Sigma_\alpha}} \int_0^\infty \tau^{-\frac{nd}{2}} e^{-\frac{1}{2\tau} \sum_{k=1}^d (\theta_{..k}, \Sigma_\alpha^{-1} \theta_{..k})} M_\beta(\tau) d\tau. \tag{5}$$

The ggBm $B^{\beta,\alpha}$ has the following properties:

1. It is an $\frac{\alpha}{2}$ self-similar process with stationary increments.
2. Its characteristic function has the form as follows:

$$\mathbb{E}\left(e^{i(k,B^{\beta,\alpha}(t))}\right) = E_\beta\left(-\frac{|k|^2}{2} t^\alpha\right), \quad k \in \mathbb{R}^d. \tag{6}$$

3. For each $t \geq 0$, the moments of any order are given by the following:

$$\begin{cases} \mathbb{E}(|B^{\beta,\alpha}(t)|^{2n+1}) = 0, \\ \mathbb{E}(|B^{\beta,\alpha}(t)|^{2n}) = \frac{(2n)!}{2^n \Gamma(\beta n+1)} (dt^\alpha)^n. \end{cases}$$

4. The covariance function has the form as follows:

$$\mathbb{E}((B^{\beta,\alpha}(t), B^{\beta,\alpha}(s))) = \frac{d}{2\Gamma(\beta+1)} (t^\alpha + s^\alpha - |t-s|^\alpha), \quad t, s \geq 0. \tag{7}$$

5. For each $t, s \geq 0$, the characteristic function of the increments is as follows:

$$\mathbb{E}\left(e^{i(k,B^{\beta,\alpha}(t)-B^{\beta,\alpha}(s))}\right) = E_\beta\left(-\frac{|k|^2}{2}|t-s|^\alpha\right), \quad k \in \mathbb{R}^d. \tag{8}$$

Remark 1. For $n = 1$, the density $f_\beta(\theta, t)$ is the fundamental solution of the following time fractional differential equation (see Mentrelli and Pagnini [11])

$$\mathbb{D}_t^{2\beta} f_\beta(\theta, t) = \Delta f_\beta(\theta, t), \tag{9}$$

where Δ is the d-dimensional Laplacian in θ and $\mathbb{D}_t^{2\beta}$ is the Caputo-Djrbashian fractional derivative; see Samko et al.[12]

Remark 2. The family $B^{\beta,\alpha}$ forms a class of $\frac{\alpha}{2}$ self-similar processes with stationary increments, which includes the following:

1. For $\beta = \alpha = 1$, the process $\{B^{1,1}(t), t \geq 0\}$ is a standard d-dimensional Bm.
2. For $\beta = 1$ and $0 < \alpha < 2$, $\{B^{1,\alpha}(t), t \geq 0\}$ is a d-dimensional fBm with Hurst parameter $\frac{\alpha}{2}$.
   Note that the one-dimensional fBm with Hurst parameter h is the centered Gaussian process with covariance

$$\mathbb{E}(B_t^h B_s^h) = \frac{1}{2}(t^{2h} + s^{2h} - |t-s|^{2h}).$$

3. For $\alpha = 1$, $\{B^{\beta,1}(t), t \geq 0\}$ is a $\frac{1}{2}$ self-similar non-Gaussian process with the following:

$$\mathbb{E}\left(e^{i(k,B^{\beta,1}(t))}\right) = E_\beta\left(-\frac{|k|^2}{2}t\right), \quad k \in \mathbb{R}^d. \tag{10}$$

4. For $0 < \alpha = \beta < 1$, the process $\{B^\beta(t) := B^{\beta,\beta}(t), t \geq 0\}$ is $\frac{\beta}{2}$-self-similar and is called d-dimensional grey Brownian motion (gBm for short). Its characteristic function is given by the following:

$$\mathbb{E}\left(e^{i(k,B^\beta(t))}\right) = E_\beta\left(-\frac{|k|^2}{2}t^\beta\right), \quad k \in \mathbb{R}^d. \tag{11}$$

For $d = 1$, this process was introduced by W. Schneider.[13,14]

5. For other choices of the parameters β and α, we obtain, in general, non-Gaussian processes.

The $\mathbb{M}$-Wright function with two variables $\mathbb{M}_\beta^1$ of order β (one-dimensional in space) is defined by

$$\mathbb{M}_\beta^1(x,t) := \mathbb{M}_\beta(x,t) := \frac{1}{2}t^{-\beta}M_\beta(|x|t^{-\beta}), \quad 0 < \beta < 1, \ x \in \mathbb{R}, \ t \in \mathbb{R}^+, \tag{12}$$

which is a probability density in x evolving in time t with self-similarity exponent β. The following integral representation for the $\mathbb{M}$-Wright function is valid; see Mainardi et al.[15]

$$\mathbb{M}_{\beta/2}(x,t) = 2 \int_0^\infty \frac{e^{-\frac{x^2}{4\tau}}}{\sqrt{4\pi\tau}} t^{-\beta} M_\beta(\tau t^{-\beta}) d\tau, \quad 0 < \beta \le 1, \ x \in \mathbb{R}. \tag{13}$$

This representation is valid in a more general form; see Mainardi et al,[15, equation (6.3)] but for our purpose, it is sufficient in view of its generalization for $x \in \mathbb{R}^d$. In fact, Equation (13) may be extended to a general spatial dimension d by the extension of the Gaussian function, namely,

$$\mathbb{M}_{\beta/2}^d(x,t) := 2 \int_0^\infty \frac{e^{-\frac{1}{4\tau}|x|^2}}{(4\pi\tau)^{\frac{d}{2}}} t^{-\beta} M_\beta(\tau t^{-\beta}) d\tau, \quad x \in \mathbb{R}^d, \ t \ge 0, \ 0 < \beta \le 1. \tag{14}$$

The function $\mathbb{M}_{\frac{\beta}{2}}^d$ is nothing but the fundamental solution of a time fractional diffusion Equation (9); see Mentrelli and Pagnini[11] for details.

3 | CHAIN POTENTIALS

In this section, we compute the chain potentials associated to the system driven by a ggBm. First, we point out the classical case driven by a Bm that is the sum of harmonic oscillator potentials corresponding to nearest neighbors attraction. We then compute the chain potential for the general non-Gaussian family $B^{\beta,\alpha}$.

3.1 | Gaussian case

Let $X = \{X(t), t \ge 0\}$ be a standard Gaussian process in $\mathbb{R}^d$ with covariance

$$\mathbb{E}\left(X_i(t)X_j(s)\right) = R^X(t,s)\delta_{ij}.$$

Denote the discrete increments of X by the following:

$$Y(k) := X(k) - X(k-1), \quad k = 1, \dots, N, \ N \in \mathbb{N}.$$

The density of the $\mathbb{R}^{dN}$-valued random variable $Y = (Y(1), \dots, Y(N))$ may be computed from its characteristic function, namely, for any $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^{dN}$

$$C(\lambda) := \mathbb{E}\left(e^{i(Y,\lambda)}\right) = \mathbb{E}\left(\exp\left(i\sum_{k=1}^N (Y(k), \lambda_k)_{\mathbb{R}^d}\right)\right)$$

$$= \int_{\mathbb{R}^{dN}} \varrho_N^X(y) \exp\left(i\sum_{k=1}^N (y_k, \lambda_k)_{\mathbb{R}^d}\right) dy.$$

If we represent this characteristic function C by

$$C(\lambda) = \int_{\mathbb{R}^{dN}} e^{-H^X(y)} \exp\left(i \sum_{k=1}^N (y_k, \lambda_k)_{\mathbb{R}^d} \right) dy,$$

then by an inverse Fourier transform, we obtain

$$H^X(y) = -\ln\left(\varrho^X(y) \right), \quad y \in \mathbb{R}^{dN}.$$

The function H^X is called chain potential of the system.

1. If X is the Brownian motion B in $\mathbb{R}^d$, putting $t_k = k, k = 1, \ldots, N$, up to an irrelevant constant, the function H^B is given by the following:

$$H^B(y) = \frac{1}{2} \sum_{k=1}^N |y_k|^2 = \frac{1}{2} \sum_{k=1}^N |x_k - x_{k-1}|^2,$$

   where $x_k, k = 1, \ldots, N$ denotes the integrated variables.

2. For the fBm $X = B^h$ with Hurst parameter $h \in (0,1)$ (see Bock et al[16] and a special case below) up to an irrelevant constant

$$H^{B^h}(y) = \frac{1}{2}(y, \Sigma^{-1} y) = \sum_{k,n=1}^N y_k \sigma_{kn} y_n = \sum_{k,n=1}^N g_{kn}(x_k - x_n)^2,$$

   where $\Sigma^{-1} = (\sigma_{kn})_{k,n=1}^N$ denotes the inverse of the covariance matrix of the increments $Y^h(k) = B^h(k) - B^h(k-1)$, $k = 1, \ldots, N$. In addition, it is known that ϱ^X is the fundamental solution of the heat equation; see, eg, Evans[17].

Remark 3.

1. Note that the terms

$$V = \frac{1}{2}|x_k - x_{k-1}|^2$$

   are harmonic oscillator potentials, attracting nearest neighbors. Thus, for Gaussian processes, H^X may be calculated via $H^X = -\ln \varrho^X$ through the characteristic function of the process by an inverse Fourier transform. In addition, H^X will always be a quadratic form, basically the inverse of the covariance matrix $a_{kl} := \mathbb{E}\left((Y(k), Y(l))_{\mathbb{R}^d} \right)$, $k, l = 1, \ldots, N$.

2. For the fBm case, the difference is that the interaction is not anymore confined to nearest neighbors. For small Hurst index, this inverted matrix leads to a long-range attraction along the chain making it more compact than (discretized) Brownian, while for Hurst indices larger than $\frac{1}{2}$, there appears a repulsion of next-to-nearest neighbors, stretching the chain; see Bock et al[16] and Figure 1 that displays the coupling g_{kn} between the central particle and the others along the chain, for $h = 0.3$ and $h = 0.8$.
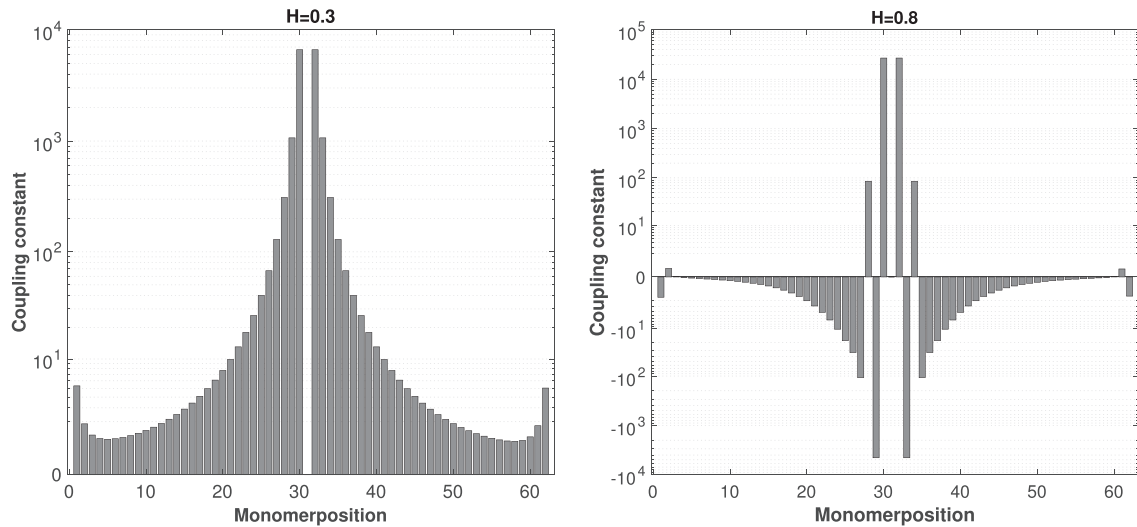
3.2 | A non-Gaussian generalization

In general, for the non-Gaussian case, the chain potential will no more be quadratic. To keep things simple, let us for the moment just look at the case of $N = 2$.

3.2.1 | Chain potential for two-particle interaction

We look at the increment $Y_{kl} = B^{\beta,\alpha}(k) - B^{\beta,\alpha}(l)$ for $0 < l < k < \infty$. The function $H^{\beta,\alpha}(y) = -\ln\left(\varrho^{\beta,\alpha}(y) \right)$ can be computed from the characteristic function of Y_{kl}; ie, for any $\lambda \in \mathbb{R}^d$, we have the following:

$$\mathbb{E}\left(e^{i(\lambda, B^{\beta,\alpha}(k) - B^{\beta,\alpha}(l))} \right) = E_\beta\left(-\frac{1}{2}|\lambda|^2 |k-l|^\alpha \right)$$

$$= \int_{\mathbb{R}^d} \varrho_1^{\beta,\alpha}(y) \exp\left(i(y, \lambda) \right) dy,$$

FIGURE 1 Coupling Constants for small and large Hurst indices in the case of fractional Brownian motion.

through the inverse Fourier transform with $\zeta := |k - l|^\alpha$

$$\varrho_1^{\beta,\alpha}(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\left(-i(y,\lambda)_{\mathbb{R}^d}\right) E_\beta \left(-\frac{1}{2}|\lambda|^2 \zeta\right) d\lambda$$

$$= \frac{1}{(2\pi)^d} \int_0^\infty M_\beta(\tau) \int_{\mathbb{R}^d} \exp\left(-i(y,\lambda)_{\mathbb{R}^d} - \frac{1}{2}|\lambda|^2 \zeta \tau\right) d\lambda \, d\tau.$$

Computing the Gaussian integral and using Equation (14) yields the following:

$$\varrho_1^{\beta,\alpha}(y) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_0^\infty \frac{1}{(\tau\zeta)^{\frac{d}{2}}} \exp\left(-\frac{1}{2\tau\zeta}|y|^2\right) M_\beta(\tau) \, d\tau$$

$$= 2^{\frac{d}{2}} \int_0^\infty \frac{1}{(4\pi\tau)^{\frac{d}{2}}} \exp\left(-\frac{1}{4\tau}|\sqrt{2}y|^2\right) \left(\zeta^{\frac{1}{\beta}}\right)^{-\beta} M_\beta\left(\tau\left(\zeta^{\frac{1}{\beta}}\right)^{-\beta}\right) d\tau$$

$$= 2^{\frac{d}{2}-1} \mathbb{M}_{\frac{\beta}{2}}^d \left(\sqrt{2}y, \zeta^{\frac{1}{\beta}}\right).$$

Therefore, up to a constant, the function $H^{\beta,\alpha}$ is given by the following:

$$H^{\beta,\alpha}(y) = -\ln\left(\mathbb{M}_{\frac{\beta}{2}}^d \left(\sqrt{2}y, \zeta^{\frac{1}{\beta}}\right)\right), \quad y \in \mathbb{R}^d.$$

Remark 4.

1. In Figure 2, we plot $H^{\beta,\alpha}$ for different values of β and $\alpha = \frac{1}{3}$ (other values of α do not produce any essential difference in the shape of the plots) assuming a time length $|k - l| = 1$.
2. It follows from Remark 1 that the density $\varrho_1^{\beta,\alpha}$ is the fundamental solution of a time fractional differential equation in $\mathbb{R}^d$.
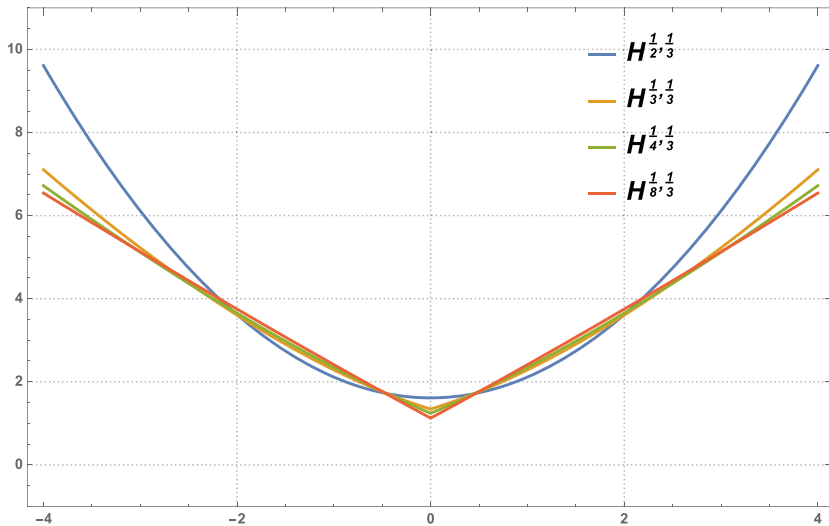
FIGURE 2 Chain potential $H^{\beta,\alpha}$ for $d = 1$; $|k - l| = 1$ and $\beta = 1, \frac{1}{3}, \frac{1}{4}$, and $\frac{1}{8}$ [Colour figure can be viewed at wileyonlinelibrary.com]

3.2.2 | Chain potential for $(N + 1)$-particle interactions

In general, for an arbitrary $N \in \mathbb{N}$, $H^{\beta,\alpha}$ may be computed using the same technique, namely considering the vectors

$$Y(k) := Y^{\beta,\alpha}(k) := B^{\beta,\alpha}(k) - B^{\beta,\alpha}(k - 1), \quad k = 1, \dots, N.$$

The characteristic function of $Y = \left(Y^{\beta,\alpha}(1), \dots, Y^{\beta,\alpha}(N)\right)$, for any $\lambda = (\lambda_1, \dots, \lambda_N) \in (\mathbb{R}^d)^N$, is given by the following:

$$\mathbb{E}\left(e^{i \sum_{k=1}^{N} (\lambda_k, Y^{\beta,\alpha}(k))_{\mathbb{R}^d}}\right) = E_\beta\left(-\frac{1}{2} \sum_{k=1}^{N} (\lambda_k, Q\lambda_k)_{\mathbb{R}^d}\right),$$

where $Q := Q^{\beta,\alpha} = (a_{kn})_{k,n=1}^{d}$ is the covariance matrix of Y given by the following:

$$a_{kn} = \mathbb{E}\left((Y(k), Y(n))\right) = \frac{d}{2\Gamma(\beta + 1)}\left[|k - 1 - n|^\alpha + |n - 1 - k|^\alpha - 2|k - n|^\alpha\right].$$

Inverting the Fourier transform, denoting $\|y\|_Q^2 := \sum_{k=1}^{N} (y_k, Qy_k)_{\mathbb{R}^d}$, the density $\varrho_N^{\beta,\alpha}$ of Y has the form

$$\varrho_N^{\beta,\alpha}(y) = \frac{1}{(2\pi)^{\frac{dN}{2}}(\det Q)^{\frac{1}{2}}}\int_0^\infty \frac{1}{\tau^{\frac{dN}{2}}} \exp\left(-\frac{1}{2\tau}\|y\|_Q^2\right) M_\beta(\tau)\,d\tau.$$

Hence, the chain potential is as follows:

$$H^{\beta,\alpha}(y) = -\ln\left(\varrho_N^{\beta,\alpha}(y)\right).$$

Let us consider the special case of the three-particle $(N = 2)$ interaction in dimension $d = 1$. The previous result gives the chain potential:

$$\varrho_3^{\beta,\alpha}(y) = \frac{1}{(2\pi)^d(\det Q)^{\frac{1}{2}}}\int_0^\infty \frac{1}{\tau} \exp\left(-\frac{1}{2\tau}\|y\|_Q^2\right) M_\beta(\tau)\,d\tau.$$

For special values of the parameter β, we may compute in a closed form the density $\varrho_3^{\beta,\alpha}$ shown in Table 1.

Here, K_ν, $G_{p,q}^{m,n}$, and Ai are the Bessel K, Meijer G, and Airy functions, respectively; see Olver et al.[18]

4 | OUTLOOK

The specific form of the higher order interactions that arise for $\beta < 1$, based on a Taylor expansion of $\ln(\varrho^{\beta,\alpha}(y))$, would be an interesting subject of investigation, at least for some special cases. Because of the type of the special functions in the

TABLE 1 Density $\varrho_3^{\beta,\alpha}$ for special values of the parameter β

	$M_\beta(\tau)$	$\varrho_3^{\beta,\alpha}(y)$
$\beta = 0$	$e^{-\tau}$	$\frac{1}{\pi \det(Q)^{\frac{1}{2}}} K_0\left(\sqrt{2}\|y\|_Q\right)$
$\beta = \frac{1}{3}$	$3^{\frac{2}{3}} \mathrm{Ai}(3^{-\frac{1}{3}}\tau)$	$\frac{1}{8\pi^3 \det(Q)^{\frac{1}{2}}} G_{5,0}^{0,0}\left(\frac{\|y\|_Q^6}{5832} \middle| 0,0,\frac{1}{3},\frac{1}{3},\frac{2}{3}\right)$
$\beta = \frac{1}{2}$	$\frac{1}{\sqrt{\pi}} e^{-\frac{\tau^2}{4}}$	$\frac{1}{(2\pi)^2 \det(Q)^{\frac{1}{2}}} G_{3,0}^{0,0}\left(\frac{\|y\|_Q^4}{64} \middle| 0,0,\frac{1}{2}\right)$
$\beta = 1$	$\delta_1(\tau)$	$\frac{1}{2\pi \det(Q)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\|y\|_Q^2\right)$

table above, such Taylor expansion is not directly at hand. The dynamics of a polymer chain has irreversible aspects leading to decorrelation of configurational properties for a longtime limit. For simple mechanical models with time reversible equations of motion, the irreversibility is a consequence of the chaotic nature of the dynamics which, for a many body system, is expected to result in ergodic mixing. There is a well-known connection between ggBm and anomalous diffusions; see Mura and Mainardi.[10] It shown in Schwarzl et al[19] that anomalous diffusions are having nonergodicity. Indeed, it is easy to prove that ggBm lacks ergodicity whenever we have $\beta \neq 1$, hence in the non-Gaussian case. A physical and mathematical study the consequences of the non-Gaussianity are of special interest.

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