Faithfully Balanced Modules and Applications in Relative Homological Algebra

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Abstract

This thesis studies faithfully balanced modules, which are faithful modules with the double centralizer property, and their relative version. For finite-dimensional algebras our main tool is the category $\operatorname{cogen}^1(M)$ of modules with a copresentation by summands of finite sums of M on which $\operatorname{Hom}(-, M)$ is exact. For a faithfully balanced module M, the functors $\operatorname{Hom}(-, M)$ and $\operatorname{DHom}(M, -)$ restrict to dualities of some subcategories of the module category and these dualities turn out to be useful in studying (co)tilting and cluster-tilting modules. As examples, we classify faithfully banlanced modules for the path algebra of Dynkin type $\mathbb{A}$ with linear orientation.

Then we turn to study a relative version of faithfully balancedness, which we call 1-**F**-faithful, by using relative homological algebra in the sense of Auslander-Solberg. Following this line, we develop relative versions of the best known classes of faithfully balanced modules (including (co)generators, (co)tilting and cluster tilting modules). Two highlights are the relative cotilting correspondence and the relative (higher) Auslander correspondence, where the first is a generalization of a relative cotilting correspondence of Auslander-Solberg to an involution (as the usual cotilting correspondence is).

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Contents

Introduction

General idea

In mathematics, for example in the area of representation theory of finite-dimensional algebras, a basic problem is to compare objects and to decide when they are equivalent, up to some appropriate equivalence relation. The objects could be, for instance, algebras, modules, or categories of modules. Given two modules over an algebra, one can ask whether they are isomorphic; given two algebras one can ask whether they are isomorphic or whether they have equivalent module categories. When the objects are very complicated, one may consider, instead of "global data" connections between "local data". Given two algebras, one can consider equivalences of subcategories even though the whole module categories are not equivalent. Often, such a subcategory is determined by a module. So it is natural to find a class of modules which could provide us with interesting equivalences between subcategories and such that these equivalences unify some well-studied equivalences that people care about. This thesis attempts to do this and our candidate is the class of faithfully balanced modules.

Symmetries are everywhere in mathematics and faithfully balanced modules can provide us with many symmetric results. In representation theory, a basic and powerful method is to compare the module category of a given algebra Λ with the module category of the endomorphism algebra Γ of some Λ-module M. One may expect to get information of Λ by studying Γ and vice versa. This method works better when M is a faithfully balanced module. In this case, we will see a lot of symmetries between subcategories determined by M as, respectively, Λ-module and Γ-module. Another basic idea in mathematics is to study objects from relative point of view. We will also study a relative version of faithfully balanced module and study the induced equivalences of subcategories. As we will show, in the relative setting, there are many results with stronger symmetry.

Background

Let R be a ring and M a left R-module. We write endomorphisms of ${}_RM$ on the left, thus for two endomorphisms $f, g \in \mathrm{End}_R(M)$ and an element $m \in M$ the image of m under gf is $g(f(m))$. Then M can be considered naturally as a left $\mathrm{End}_R(M)$-module. We say M is *faithful/ balanced/ faithfully balanced* [1] if the natural map of rings $R \to \mathrm{End}_{\mathrm{End}_R(M)}(M)$ is injective/surjective/bijective.

Balanced modules are also known as modules with the double centralizer property and so a faithfully balanced module is the same as a faithful module with the double centralizer property. Let us recall the double centralizer of a subring. Let R be a ring and S a subring of R. The

[1] In some literature (for example [AF92, section 4]), faithfully balancedness is only defined for bimodules. A faithfully balanced module ${}_RM$ in this thesis is called faithful and balanced in loc. cit., which is equivalent to say that ${}_RM_{\mathrm{End}_R(M)^{\mathrm{op}}}$ is a faithfully balanced bimodule.

centralizer of S in R is $\mathbf{C}_R(S) := \{r \in R \mid rs = sr \text{ for all } s \in S\}$. This is a subring of R. The *double centralizer* of S is $\mathbf{C}_R(\mathbf{C}_R(S))$. Clearly, S is a subring of its double centralizer. Now let M be a left R-module and $S := \mathrm{End}_R(M)$ be the endomorphism ring of M acting from the left. Then M is also left S-module. Let $T = \mathrm{End}_S(M)$, then M becomes a left T-module and we have a ring homomorphism

$$\rho : R \to T,\ r \mapsto (m \mapsto rm).$$

On the other hand, since M is a left R-module, there is a ring homomorphism

$$\theta : R \to \mathrm{End}_{\mathbb{Z}}(M),\ r \mapsto (m \mapsto rm).$$

Then $\mathrm{im}(\theta)$ is a subring of $\mathrm{End}_{\mathbb{Z}}(M)$. Its centralizer in $\mathrm{End}_{\mathbb{Z}}(M)$ is

$$\mathbf{C}_{\mathrm{End}_{\mathbb{Z}}(M)}(\mathrm{im}(\theta)) = \{f \mid fi = if \text{ for all } i \in \mathrm{im}(\theta)\} = S,$$

and its double centralizer is

$$\mathbf{C}_{\mathrm{End}_{\mathbb{Z}}(M)}(\mathbf{C}_{\mathrm{End}_{\mathbb{Z}}(M)}(\mathrm{im}(\theta))) = \mathbf{C}_{\mathrm{End}_{\mathbb{Z}}(M)}(S) = T.$$

The module M is said to have the *double centralizer property* if the double centralizer of $\mathrm{im}(\theta)$ in $\mathrm{End}_{\mathbb{Z}}(M)$ is $\mathrm{im}(\theta)$ itself, i.e., if $\mathrm{im}(\theta) = T$. Note that $\mathrm{im}(\theta) = \mathrm{im}(\rho)$, so M has the double centralizer property if and only if ρ is surjective, if and only if ${}_RM$ is balanced. The centralizer and the double centralizer of a subring are also called the commutator and the second commutator, respectively, of this subring.

For general rings, balanced modules and (left) balanced rings (meaning that every left module is balanced) are studied in [Cam70, DR72a, DR72b, DR73]. For finite-dimensional algebras, Thrall [Thr48] gave several generalizations of quasi-Frobenius algebras (=QF algebras=self-injective algebras) by considering faithfulness and balancedness of modules. He defined an algebra Λ to be *QF*-1 if every finitely generated faithful Λ-module if faithfully balanced, and an algebra to be *QF*-3 if it has a minimal faithful module. Then they are further developed in [Mor58b, Tac73, Tac70].

Motivation

In this thesis, we restrict to study finite-dimensional algebras and finite-dimensional modules over them. This thesis is inspired, on the one hand by the correspondences (see Table 1) in representation theory of finite-dimensional algebras (or more generally artin algebras) including the (higher) Auslander correspondence ([Iya07a]) and the (co)tilting correspondence (=Brenner-Butler theorem, see [BB80, Miy86]), on the other hand by relative homological algebra in the sense of Auslander-Solberg ([AS93b]). In each of the correspondences, there always exists a special module that plays an important role. For example, in the (higher) Auslander correspondence there is a cluster-tilting module whereas in the (co)tilting correspondence there is a (co)tilting module. Both of them are faithfully balanced modules with some extra properties. This leads to the study of faithfully balanced modules which play the fundamental roles in all of these correspondences. In their series of papers [AS93c, AS93d, AS93a] Auslander and Solberg studied a special class of faithfully balanced modules - dualizing summands of cotilting modules. As one main result they proved that a dualizing summand of a cotilting module will produce a relative cotilting module for the endomorphism algebra of this dualizing summand and vice versa.

Our motivations are: 1. to understand faithfully balanced modules; 2. to give a unification of some well-known correspondences by using faithfully balanced modules; 3. to understand the interplay between faithfully balanced modules and relative homological algebra in the sense of Auslander-Solberg. For the first motivation, we study some properties of faithfully balanced modules and the dualities induced by them in Chapter 2, and give explicit descriptions of faithfully balanced modules for the path algebra of Dynkin type $\mathbb{A}$ with linear orientations. In order to achieve the latter two purposes, we give a relative version of faithfully balancedness in Chapter 4. Then we will give relative versions of the well-known correspondences in Chapter 5 and 6. In particular, we generalize the main result on relative cotilting modules of Auslander-Solberg and prove a relative version of Iyama's higher Auslander correspondence.

Results

Results in this thesis are only proved for finite-dimensional algebras over a field. However, it is easy to see, most of these results hold for artin algebras.

Consider tuples $(\Lambda, M_1, \ldots, M_t)$ consisting of an algebra Λ and several Λ-modules up to an equivalence relation which identifies two such tuples $(\Lambda, M_1, \ldots, M_t)$ and $(\Lambda', M_1', \ldots, M_t')$ if there is a Morita equivalence from Λ to Λ' which sends each $\mathrm{add}(M_i)$ to $\mathrm{add}(M_i')$. We denote by $[\Lambda, M_1, \ldots, M_t]$ the equivalence class of $(\Lambda, M_1, \ldots, M_t)$. It is easy to see that faithfully balancedness of a module is preserved under this equivalence (cf. [CR72]).

- The assignment $[\Lambda, {}_\Lambda M] \to [\mathrm{End}_\Lambda(M), {}_{\mathrm{End}_\Lambda(M)}M]$ $(*)$ is an involution on the set of pairs $[\Lambda, M]$ with M a faithfully balanced module.

A restriction of $(*)$ to a bijection between two sets of such pairs (or related tuples) will be called a *correspondence*. The correspondence $(*)$ can be viewed as the cornerstone of all the other correspondences in Table 1.

Classical case	**Relative case**
(co)generator correspondence	Corollary 4.3.7 (1) (2)
Morita-Tachikawa correspondence (=generator-cogenerator correspondence)	Corollary 4.3.7 (3)
(co)tilting correspondence (=Brenner-Butler theorem)	Theorem 5.3.2
correspondence of Gorenstein algebras	Corollary 5.4.4
Auslander-Solberg correspondence	Theorem 6.1.2
(higher) Auslander correspondence	Theorem 6.2.3

Table 1

For a module M, we define

$$\mathrm{cogen}^1(M) = \{X \mid \exists\ 0 \to X \to M_0 \to M_1 \text{ exact}, M_i \in \mathrm{add}(M) \text{ and } \mathrm{Hom}_\Lambda(-, M) \text{ exact on it}\}.$$

Dually, one can define $\mathrm{gen}_1(M)$. If ${}_\Lambda M$ is a faithfully balanced module, then we have a duality

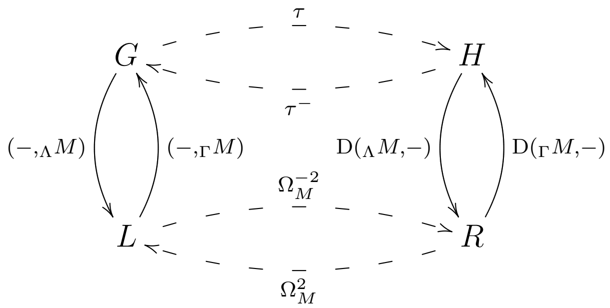
$$\mathrm{Hom}_\Lambda(-, {}_\Lambda M)\colon\ \mathrm{cogen}^1({}_\Lambda M) \longleftrightarrow \mathrm{cogen}^1({}_\Gamma M)\colon\ \mathrm{Hom}_\Gamma(-, {}_\Gamma M)$$

where $\Gamma = \mathrm{End}_\Lambda(M)$. Buan and Solberg [BS98] first observed the symmetry: $\Lambda \in \mathrm{cogen}^1({}_\Lambda M)$ is equivalent to $\mathrm{D}\,\Lambda \in \mathrm{gen}_1({}_\Lambda M)$ and both are equivalent to M being faithfully balanced (see also Lemma 2.2.3). Here $\mathrm{D} = \mathrm{Hom}_K(-, K)$ is the K-dual functor (see section 1.2).

Fix an additive subbifunctor $\mathbf{F} \subseteq \mathrm{Ext}^1_\Lambda(-,-)$ of the form $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ for a generator G and a cogenerator H - this is equivalent to fix the exact structure on finite-dimensional Λ-modules induced by the functor $\mathbf{F}$ (cf. [DRSS99]), meaning an exact sequence is $\mathbf{F}$-exact if and only if it remains exact after applying the functor $\mathrm{Hom}_\Lambda(G, -)$ (or equivalently after applying the functor $\mathrm{Hom}_\Lambda(-, H)$). We define $\mathrm{cogen}^1_{\mathbf{F}}(M) \subseteq \mathrm{cogen}^1(M)$ to be the full subcategory of modules X such that there exists an exact sequence $0 \to X \to M_0 \to M_1$ with $M_0, M_1 \in \mathrm{add}(M)$ and $\mathrm{Hom}_\Lambda(-, H \oplus M)$ is exact on it (analogously we define $\mathrm{gen}^{\mathbf{F}}_1(M)$). We also introduce the notion of 1-$\mathbf{F}$-faithfulness (meaning $G \in \mathrm{cogen}^1_{\mathbf{F}}(M)$) as the relative analogue of the notion of faithfully balancedness. Let ${}_\Lambda M$ be 1-$\mathbf{F}$-faithful, then we have a duality

$$\mathrm{Hom}_\Lambda(-, {}_\Lambda M)\colon\ \mathrm{cogen}^1_{\mathbf{F}^H}(M) \longleftrightarrow \mathrm{cogen}^1_{\mathbf{F}^R}(M)\colon\ \mathrm{Hom}_\Gamma(-, {}_\Gamma M)$$

where $\Gamma = \mathrm{End}_\Lambda(M)$ and $R = \mathrm{D}\,\mathrm{Hom}_\Lambda(M, H)$. There is also a dual version of the above duality which involves the modules G and $L := \mathrm{Hom}_\Lambda(G, M)$. Then we observe the following relationship between G, H and L, R



The upper dashed arrows mean $H = \tau G \oplus \mathrm{D}\,\Lambda$ and $G = \tau^- H \oplus \Lambda$ whereas the lower dashed arrows mean $R = {}_\Gamma M \oplus \Omega_M^{-2} L$ and $L = {}_\Gamma M \oplus \Omega_M^2 R$, where τ is the Auslander-Reiten translation (see the definition before Corollary 1.2.9) and Ω_M^2 and Ω_M^{-2} are defined in Definition 2.3.7. As in the classical case, we have $G \in \mathrm{cogen}^1_{\mathbf{F}}(M)$ is equivalent to $H \in \mathrm{gen}^{\mathbf{F}}_1(M)$ (Theorem 4.2.3).

Let us recall the correspondences in the classical case in the above table based on the correspondence induced by faithfully balanced (abbreviate as f.b.) modules and state the main theorems in this thesis, i.e., relative versions of them.

Generator correspondence and Morita-Tachikawa correspondence

A generator ${}_\Lambda G$ (resp. cogenerator ${}_\Lambda H$) is, by definition, a module such that $\Lambda \in \mathrm{add}\, G$ (resp. $\mathrm{D}\,\Lambda \in \mathrm{add}\, H$) which is automatically a faithfully balanced module. Thus the (co)generator correspondence [Azu66] can be stated as follows.

- The assignment $(*)$ restricts to a one-to-one correspondence
  $\{[\Lambda, G] \mid G \text{ is a generator}\} \overset{1:1}{\longleftrightarrow} \{[\Gamma, P] \mid P \text{ is a f.b. projective module}\}$
  $(\{[\Lambda, H] \mid H \text{ is a cogenerator}\} \overset{1:1}{\longleftrightarrow} \{[\Gamma, I] \mid I \text{ is a f.b. injective module}\})$.

Clearly M is a generator-cogenerator (abbreviate as gen-cogen) if and only if ${}_{\mathrm{End}(M)}M$ is a faithfully balanced projective-injective module. An algebra that has a faithfully balanced projective-injective module is also known as an algebra of dominant dimension ≥ 2. As a special case of generator correspondence the Morita-Tachikawa correspondence [Mor71, Tac70, Rin07] can be expressed as follows.

- The assignment $(*)$ restricts to a one-to-one correspondence
  $\{[\Lambda, M] \mid M \text{ is gen-cogen}\} \overset{1:1}{\longleftrightarrow} \{[\Gamma, M'] \mid M' \text{ is f.b. proj-inj}\}$.

Note that M' is f.b. proj-inj implies M' is the maximal injective summand of Γ. One famous example of the Morita-Tachikawa correspondence is the classical Auslander correspondence [Aus99b]. Another famous example is given by the Schur-Weyl duality ([KSrX01]).

We consider the assignments (AS) (referring to Auslander and Solberg) and its dual version (dual AS):

(AS) The assignment $[\Lambda, {}_\Lambda M, G] \mapsto [\Gamma = \mathrm{End}_\Lambda(M), {}_\Gamma M, L = \mathrm{Hom}_\Lambda(G, M)]$ with M faithfully balanced and G a generator.

(dual AS) The assignment $[\Lambda, {}_\Lambda M, H] \mapsto [\Gamma = \mathrm{End}_\Lambda(M), {}_\Gamma M, R = \mathrm{D}\,\mathrm{Hom}_\Lambda(M, H)]$ with M faithfully balanced and H a cogenerator.

In the relative setting, we have the following result.

Theorem 1. (= *Corollary 4.3.7*)

(1) (*relative generator correspondence*)
The Auslander-Solberg assignment (AS) gives an involution on the set of triples $[\Lambda, M, G]$ *with* $\Lambda \oplus M \in \mathrm{add}(G)$ *and* M *is* 1-$\mathbf{F}_G$*-faithful.*

(2) (*relative cogenerator correspondence*)
The dual Auslander-Solberg assignment (dual AS) givess an involution on the set of triples $[\Lambda, M, H]$ *with* $\mathrm{D}\,\Lambda \oplus M \in \mathrm{add}(H)$ *and* M *is* 1-$\mathbf{F}^H$*-faithful.*

(3) (*relative Morita-Tachikawa correspondence*)
The assignment $[\Lambda, M, G, H] \mapsto [\mathrm{End}_\Lambda(M), M, L = \mathrm{Hom}_\Lambda(G, M), R = \mathrm{D}\,\mathrm{Hom}_\Lambda(M, H)]$ *is a bijection between*

(3a) $[\Lambda, M, G, H]$ *with* $\Lambda \in \mathrm{add}(G), \mathrm{D}\,\Lambda \in \mathrm{add}(H)$, $G = \Lambda \oplus \tau^- H$ *and* $M \in \mathrm{add}(G) \cap \mathrm{add}(H)$ *is* 1-$\mathbf{F}_G$*-faithful, and*

(3b) $[\Gamma, N, L, R]$ *with* L, R *are the ends of a strong* $\mathrm{add}(N)$*-dualizing sequence with* $\Gamma \in \mathrm{add}(L)$ *and* $\mathrm{D}\,\Gamma \in \mathrm{add}(R)$.

See Definition 4.3.1 for the definition of a strong $\mathrm{add}(N)$-dualizing sequence.

(Co)tilting correspondence and correspondence of Gorenstein algebras

A cotilting module ${}_\Lambda T$ is a (faithfully balanced) module such that

(T1) the injective dimension of T is finite, i.e., $\mathrm{id}_\Lambda T < \infty$,

(T2) the module T is self-orthogonal i.e., $\mathrm{Ext}^{>0}_\Lambda(T, T) = 0$, and

(T3) there exists an exact sequence $0 \to T_k \to \cdots \to T_1 \to T_0 \to \mathrm{D}\,\Lambda \to 0$ with $T_i \in \mathrm{add}(T)$.

The cotilting correspondence [BB80, Miy86] is (see Corollary 2.2.7):

- The assignment $(*)$ restricts to an involution on the set $\{[\Lambda, T] \mid T \text{ is a cotilting module}\}$.

The definition of a tilting module is dual to the one of a tilting module and the tilting correspondence can be expressed dually.

The main result on relative cotilting modules (see Definition 5.1.1) of Auslander-Solberg is the following.

Theorem 2. (*[AS93c, Theorem 3.13], [AS93d, Theorem 2.8]*) The assignment (AS) restricts to a bijection between the following two sets of triples

(1) $[\Lambda, M, G]$ with $\Lambda \in \mathrm{add}(G)$, $\mathbf{F} = \mathbf{F}_G$, M is $\mathbf{F}$-cotilting, and

(2) $[\Gamma, N, L]$ with $N \in \mathrm{add}(L)$, $L \in \mathrm{cogen}^1(N)$ and L is a cotilting module.

To generalize this result, we need the 4-tuple assignment

$$[\Lambda, M, L, G] \mapsto [\Gamma, N, \widetilde{L}, \widetilde{G}]$$

with $\Gamma = \mathrm{End}_\Lambda(M)$, $N = {}_\Gamma M$, $\widetilde{L} = \mathrm{Hom}_\Lambda(G, M)$, $\widetilde{G} = \mathrm{Hom}_\Lambda(L, M)$, and its dual

$$[\Lambda, M, R, H] \mapsto [\Gamma, N, \widetilde{R}, \widetilde{H}]$$

with $\Gamma = \mathrm{End}_\Lambda(M)$, $N = {}_\Gamma M$, $\widetilde{R} = \mathrm{D}\,\mathrm{Hom}_\Lambda(M, H)$, $\widetilde{H} = \mathrm{D}\,\mathrm{Hom}_\Lambda(M, R)$. Then we have the following.

Theorem 3. (*=Theorem 5.3.2*)

(1) The 4-tuple assignment restricts to an involution on the set of 4-tuples $[\Lambda, M, L, G]$ satisfying

(1a) $\Lambda \in \mathrm{add}(G)$, $\mathbf{F} = \mathbf{F}_G$,

(1b) L is $\mathbf{F}$-cotilting and M is an $\mathbf{F}$-dualizing summand (that is, $M \in \mathrm{add}(L)$ and $L \in \mathrm{cogen}^1_{\mathbf{F}}(M)$) of L.

(2) The dual 4-tuple assignment restricts to an involution on the set of 4-tuples $[\Lambda, M, R, H]$ satisfying

(2a) $\mathrm{D}\,\Lambda \in \mathrm{add}(H)$, $\mathbf{F} = \mathbf{F}^H$,

(2b) R is $\mathbf{F}$-cotilting and M is an $\mathbf{F}$-codualizing summand of R (that is, $M \in \mathrm{add}(R)$ and $R \in \mathrm{gen}_1^{\mathbf{F}}(M)$).

Furthermore, for an assignment $[\Lambda, M, R, H] \mapsto [\Gamma, {}_\Gamma M, \widetilde{R}, \widetilde{H}]$ we have $\mathrm{id}_{\mathbf{F}^H} R = \mathrm{id}_{\mathbf{F}^{\widetilde{H}}} \widetilde{R}$.

It is well known that a cotilting module will induce a contravariant triangle equivalence, see [Hap88, CPS86]. We prove a relative analogue of this result (Proposition 5.4.1): In the situation of the previous theorem we have a contravariant triangle equivalence between $\mathsf{D}^b_{\mathbf{F}_G}(\Lambda\text{-mod})$ and $\mathsf{D}^b_{\mathbf{F}_{\widetilde{G}}}(\Gamma\text{-mod})$ where $\Gamma = \mathrm{End}_\Lambda(M)$ and $\widetilde{G} = \mathrm{Hom}_\Lambda(L, M)$.

An algebra is Gorenstein if and only if there is a tilting-cotilting module, if and only if every tilting module is cotilting [HU96, AS93a]. The correspondence for Gorenstein algebras, which says that the endomorphism algebra of a tilting-cotilting module of some Gorenstein algebra is again Gorenstein, can now be written as:

- The assignment $(*)$ restricts to an involution on the set $\{[\Lambda, M] \mid M$ is a tilting-cotilting module$\}$.

In the relative setting we have the following.

Theorem 4. (*=Corollary 5.4.4*)
Let $[\Lambda, M, L, G]$ be a 4-tuple satisfying $\Lambda \in \operatorname{add}(G)$, $\mathbf{F} = \mathbf{F}_G$, L is $\mathbf{F}$-cotilting and M is an $\mathbf{F}$-dualizing summand of L and let $[\Gamma, {}_\Gamma M, \widetilde{L}, \widetilde{G}]$ be the corresponding 4-tuple under the 4-tuple assignment. Then Λ is an $\mathbf{F}$-Gorenstein algebra if and only if Γ is an $\widetilde{\mathbf{F}}$-Gorenstein algebra.

Auslander correspondence and Auslander-Solberg correspondence

Recall from [Iya07b, Iya07a] that for $k \geq 1$ a module ${}_\Lambda M$ is called a k-cluster-tilting module if

$$\bigcap_{i=1}^{k-1} \ker \operatorname{Ext}^i_\Lambda(-, M) = \operatorname{add}(M) = \bigcap_{i=1}^{k-1} \ker \operatorname{Ext}^i_\Lambda(M, -).$$

Note that a k-cluster-tilting module is always a generator-cogenerator. It is easy to see that a faithfully balanced module M is k-cluster-tilting if and only if

$$\operatorname{cogen}^1(M) \cap \bigcap_{i=1}^{k-1} \ker \operatorname{Ext}^i_\Lambda(-, M) = \operatorname{add}(M) = \operatorname{gen}_1(M) \cap \bigcap_{i=1}^{k-1} \ker \operatorname{Ext}^i_\Lambda(M, -).$$

Define ${}^{\perp_{1\sim k-1}}M := \bigcap_{i=1}^{k-1} \ker \operatorname{Ext}^i_\Lambda(-, M)$ and $M^{\perp_{1\sim k-1}} := \bigcap_{i=1}^{k-1} \ker \operatorname{Ext}^i_\Lambda(M, -)$, then the higher Auslander correspondence can be expressed as (cf. Theorem 6.2.3):

- The assignment $(*)$ restricts to a one-to-one correspondence between the following two sets
  $\{[\Lambda, M] \mid M$ is f.b. with $\operatorname{cogen}^1(M) \cap {}^{\perp_{1\sim k}}M = \operatorname{add}(M) = \operatorname{gen}_1(M) \cap M^{\perp_{1\sim k-1}}\}$
  and
  $\{[\Gamma, M'] \mid M'$ is f.b. proj-inj with $\Gamma \in \operatorname{cogen}^k(M')$ and $\operatorname{cogen}^k(M') = \operatorname{add}(\Gamma)\}$.

A slightly more general version of the higher Auslander correspondence is the Auslander-Solberg correspondence which is defined by Iyama and Solberg [IS18]. Recall that a module ${}_\Lambda M$ is called a k-precluster-tilting module if M is a generator-cogenerator, $\operatorname{Ext}^i_\Lambda(M, M) = 0$ for $0 < i < k$ and $\tau_k(M), \tau_k^-(M) \in \operatorname{add}(M)$ (see Definition 2.3.10 for the definition of τ_k and τ_k^-). A k-precluster-tilting module is obviously a faithfully balanced module. The Auslander-Solberg correspondence can be described as:

- The assignment $(*)$ restricts to a one-to-one correspondence between the following two sets
  $\{[\Lambda, M] \mid M$ is a k-precluster-tilting module$\}$
  and
  $\{[\Gamma, M'] \mid M'$ is f.b. proj-inj and there exists a strong k-$\operatorname{add}(M')$-dual. seq. with left end Γ and right end $\operatorname{D}\Gamma$ $\}$.

Our relative generalizations are the following.

Theorem 5. (*=Theorem 6.1.2*)
Let ${}_\Lambda M$ be a faithfully balanced module and $\Gamma = \operatorname{End}_\Lambda(M)$. Then the assignment $X, Y \mapsto (X, M), \operatorname{D}(M, Y)$ gives a self-inverse bijection between the following sets of pairs of modules

(1) $\{{}_\Lambda L, {}_\Lambda R \mid {}_\Lambda M \oplus \Lambda \in \operatorname{add}(L), {}_\Lambda M \oplus \mathrm{D}\,\Lambda \in \operatorname{add}(R), L = \tau_k^- R \oplus \Lambda, R = \tau_k L \oplus \mathrm{D}\,\Lambda,$ $\operatorname{Ext}^i_\Lambda(L,R) = 0,\ 1 \leq i \leq k-1$ *such that there exists a strong* $\operatorname{add}({}_\Lambda M)$*-dualizing sequence with left end* L *and right end* $R\}$.

(2) $\{{}_\Gamma G, {}_\Gamma H \mid M \oplus \Gamma \in \operatorname{add}(G), M \oplus \mathrm{D}\,\Gamma \in \operatorname{add}(H), G = \tau^- H \oplus \Gamma, H = \tau G \oplus \mathrm{D}\,\Gamma$ *such that there exists a strong* k-$\operatorname{add}({}_\Gamma M)$*-dualizing sequence with left end* G *and right end* $H\}$.

Theorem 6. (*=Theorem 6.2.3*)
Let $k \geq 1$*. There is a one-to-one correspondence between isomorphism classes of basic* k-(L,R)*-cluster tilting modules* ${}_\Lambda M$ (*for some* L, R*, see Definition 6.2.2*) *and finite-dimensional algebras* Γ *with an exact structure given by* $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ *such that* $\operatorname{domdim}_{\mathbf{F}} \Gamma \geq k+1 \geq \operatorname{gldim}_{\mathbf{F}} \Gamma$*. The correspondence is induced by the assignment*

$$[\Lambda, M, L, R] \mapsto [\Gamma = \operatorname{End}_\Lambda(M), {}_\Gamma M, G = (L, M), H = \mathrm{D}(M, R)].$$

As special cases, if $k = 1$, then the assignment (AS) restricts to involutions on the set of triples

(1) $[\Lambda, M, G]$ with $\Lambda \in \operatorname{add}(G)$, $\mathbf{F} = \mathbf{F}_G$, M *is both* 1-$\mathbf{F}$*-faithful and* $\mathbf{F}$*-projective-injective and* $\operatorname{domdim}_{\mathbf{F}} \Lambda \geq 2 \geq \operatorname{id}_{\mathbf{F}} G$.

(2) $[\Lambda, M, G]$ with $\Lambda \in \operatorname{add}(G)$, $\mathbf{F} = \mathbf{F}_G$, M *is both* 1-$\mathbf{F}$*-faithful and* $\mathbf{F}$*-projective-injective and* $\operatorname{domdim}_{\mathbf{F}} \Lambda \geq 2 \geq \operatorname{gldim}_{\mathbf{F}} \Lambda$.

Chapter 1

Preliminaries

We recall some definitions, constructions and basic results which are necessary for the rest of the thesis. Most of the results here are well-known. For the proofs and unmentioned definitions we refer to readers some standard references. See [ML98] for basic theory of categories and additive categories; [Büh10, Kel96] for exact categories; [ARS95, ASS06] for representation theory of finite-dimensional algebras; [Rot09] for homological algebra and [AS93b, BH62] for relative homological algebra[1]. By a functor (if not clear from the context) we mean a covariant functor.

1.1 Additive categories and exact structures

Let $\mathcal{A}$ be a category. For any two objects A_1 and A_2 of $\mathcal{A}$ we denote by $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$ the set of morphisms from A_1 to A_2. Let R be a commutative ring. A category $\mathcal{A}$ is an R-category if $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$ has the structure of an R-module and the composition maps $\mathrm{Hom}_{\mathcal{A}}(A_2, A_3) \times \mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \to \mathrm{Hom}_{\mathcal{A}}(A_1, A_3)$ are R-bilinear. A $\mathbb{Z}$-category is also known as a *preadditive* category. An object 0 in a preadditive category $\mathcal{A}$ is a *zero* object provided $\mathrm{Hom}_{\mathcal{A}}(0, A) = 0 = \mathrm{Hom}_{\mathcal{A}}(A, 0)$ for all $A \in \mathcal{A}$.

An *additive* category is a preadditive category with finite products (or equivalently[2], finite coproducts). In an additive category finite products and coproducts coincide and they are called *direct sums* (or *biproducts*) in this case. Given a finite set of objects $\{A_1, A_2, \cdots, A_n\}$ in $\mathcal{A}$ we denote by $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ the direct sum of them and each A_i is called a (direct) summand. In particular, if $A_i = A$ for all $i = 1, \cdots, n$ then we write the direct sum as A^n. Binary direct sums can be summarized as the following data:

$$A_1 \underset{p_1}{\overset{i_1}{\rightleftarrows}} A_1 \oplus A_2 \underset{i_2}{\overset{p_2}{\rightleftarrows}} A_2$$

with morphisms i_1, i_2, p_1, p_2 satisfying the identities

$$p_1 i_1 = 1_{A_1},\ p_2 i_2 = 1_{A_2} \text{ and } i_1 p_1 + i_2 p_2 = 1_{A_1 \oplus A_2}.$$

An additive category has a zero object which is isomorphic to the direct sum of the empty set of objects, and zero object in an additive category is unique up to a unique isomorphism. The typical example of an additive category is the category **Ab** (=$\mathbb{Z}$-Mod) of abelian groups.

[1]In this thesis, we only consider relative homological algebra in the sense of Auslander-Solberg.
[2]See for example [ML98, Theorem VIII 2.2].

Fix an additive category $\mathcal{A}$. Then $\mathcal{A}$ is called *Hom-finite* if for any two objects A_1, A_2 the abelian group $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$ is finitely generated. Let K be a field. The category $\mathcal{A}$ is called *K-linear* if the abelain group $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$ has the structure of a K-vector space and the compositions are K-bilinear.

Let $\mathcal{A}, \mathcal{B}$ be additive categories. A functor $F : \mathcal{A} \to \mathcal{B}$ is *additive* if any pair of morphisms $f, f' \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$ satisfies

$$F(f + f') = F(f) + F(f').$$

It is well-known that a functor between additive categories is additive if and only if it preserves finite direct sums (hence zero objects). It is easy to see that $\mathrm{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to \mathbf{Ab}$ and $\mathrm{Hom}_{\mathcal{A}}(-, A) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ are additive functors, where $\mathcal{A}^{\mathrm{op}}$ is the opposite category of $\mathcal{A}$. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be additive categories. A functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is an *additive bifunctor* if $F(A, -) : \mathcal{B} \to \mathcal{C}$ and $F(-, B) : \mathcal{A} \to \mathcal{C}$ are additive functors for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The functor $\mathrm{Hom}_{\mathcal{A}}(-, -) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Ab}$ is an additive bifunctor.

Let $\mathcal{A}$ be an additive category and $\mathcal{I}$ a class of morphisms in $\mathcal{A}$. Then $\mathcal{I}$ is called a (two-sided) ideal of $\mathcal{A}$ if $l(\lambda f + \mu g)r \in \mathcal{I}$ for all $\lambda, \mu \in \mathbb{Z}$, $f, g : A_1 \to A_2$, $r : A' \to A_1$ and $l : A_2 \to A''$. Ideals of $\mathcal{A}$ are in bijection with subfunctors of the functor $\mathrm{Hom}_{\mathcal{A}}(-, -) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Ab}$. For an ideal $\mathcal{I}$, the corresponding subfuntor is defined by assigning to each pair (A_1, A_2) of objects in $\mathcal{A}$ a subgroup $\mathcal{I}(A_1, A_2)$ of $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$ such that $lfr \in \mathcal{I}(A', A'')$ whenever $f \in \mathcal{I}(A_1, A_2)$, $r \in \mathrm{Hom}_{\mathcal{A}}(A', A_1)$ and $l \in \mathrm{Hom}_{\mathcal{A}}(A_2, A'')$. Given an ideal $\mathcal{I}$, we can form the *ideal quotient* $\mathcal{A}/\mathcal{I}$ of $\mathcal{A}$ which is the category with the same objects as $\mathcal{A}$ and the set of morphisms defined via $\mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(A_1, A_2) = \mathrm{Hom}_{\mathcal{A}}(A_1, A_2)/\mathcal{I}(A_1, A_2)$. The ideal quotient of an additive category is again additive.

Krull-Schmidt categories

We fix an additive category $\mathcal{A}$. An object $A \in \mathcal{A}$ is called *indecomposable* if $A \neq 0$ and if $A = A_1 \oplus A_2$ implies $A_1 = 0$ or $A_2 = 0$. An additive category is called a *Krull-Schmidt category* is every object can be written as a finite direct sum of indecomposable objects such that each of the indecomposable summands has local endomorphism ring.

A morphism $f : A \to B$ in $\mathcal{A}$ is called *left minimal* if any endomorphism $g : B \to B$ such that $f = gf$ is an automorphism. Dually, f is called *right minimal* if any endomorphism $h : A \to A$ such that $f = fh$ is an automorphism.

Proposition 1.1.1. *(cf. [ARS95, Theorem 2.2, 2.4]) Let $f : A \to B$ be a morphism in a Krull-Schmidt category.*

(1) There is a decomposition $A \xrightarrow{f = \binom{f'}{0}} B' \oplus B'' = B$ such that f' is left minimal.

(2) There is a decomposition $A = A' \oplus A'' \xrightarrow{f = (f' \ 0)} B$ such that f' is right minimal.

The morphism f' in the above proposition is called the left (resp. right) minimal version of f.

Example 1.1.2. Let Λ be a finite-dimensional algebra over some field K. Then, by Krull-Schmidt theorem (see for instance [AF92, Theorem 12.9]), the category of all finitely generated (left) Λ-modules Λ-mod is a Krull-Schmidt category.

Approximations

Let $\mathcal{A}$ be an additive category and $\mathcal{C}$ a subcategory of $\mathcal{A}$. A morphism $f : A \to C$ with $A \in \mathcal{A}$ and $C \in \mathcal{C}$ is a *left $\mathcal{C}$-approximation* of A if the induced homomorphism of abelian groups $\mathrm{Hom}_{\mathcal{A}}(f, C') : \mathrm{Hom}_{\mathcal{A}}(C, C') \to \mathrm{Hom}_{\mathcal{A}}(A, C')$ is surjective. Dually, a morphism $g : C \to A$ is a *right $\mathcal{C}$-approximation* of A if the induced homomorphism of abelian groups $\mathrm{Hom}_{\mathcal{A}}(C', g) : \mathrm{Hom}_{\mathcal{A}}(C', C) \to \mathrm{Hom}_{\mathcal{A}}(C', A)$ is surjective. A subcategory $\mathcal{C} \subseteq \mathcal{A}$ is called *covariantly* (resp. *contravariantly*) *finite* if any object $A \in \mathcal{A}$ admits a left (resp. right) $\mathcal{C}$-approximation, and it is called *functorially finite* if it is both covariantly and contravariantly finite.

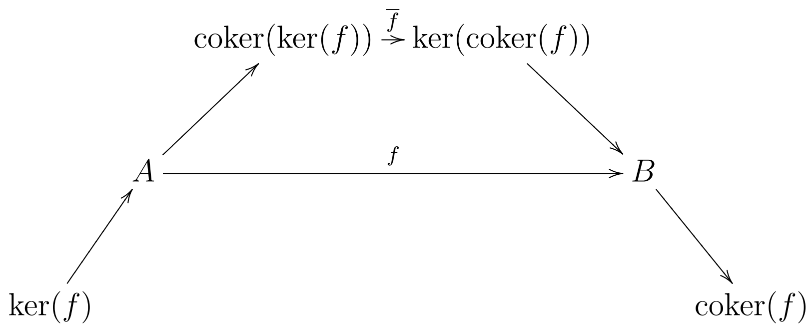
For an object $A \in \mathcal{A}$, we define $\mathrm{add}(A)$ to be the subcategory consisting of all summands of finite direct sums of copies of A. It is well-known that if $\mathcal{A}$ is a Hom-finite additive category then $\mathrm{add}(A)$ is functorially finite.

A morphism $f : A \to C$ with $A \in \mathcal{A}$ and $C \in \mathcal{C}$ is a *minimal left $\mathcal{C}$-approximation* of A if it is a left $\mathcal{C}$-approximation and it is left minimal. Dually, we have the notion of a minimal right $\mathcal{C}$-approximation.

Abelian categories

Let $\mathcal{A}$ be an additive category and $A \xrightarrow{f} B \xrightarrow{g} C$ a pair of composable morphisms in $\mathcal{A}$. Then f is called a *kernel* of g if $gf = 0$ and for any morphism $f' : A' \to B$ such that $gf' = 0$ there exists a unique morphism $\theta : A' \to A$ such that $f = \theta f'$, in this case we define $\ker g := A$. Dually, g is called a *cokernel* of f if $gf = 0$ and for any morphism $g' : A' \to B$ such that $g'f = 0$ there exists a unique morphism $\phi : C \to C'$ such that $g' = \phi g$, in this case we define $\mathrm{coker}\, f := A$. Clearly, a kernel (resp. cokernel) of a morphism is unique up to a unique isomorphism, and so we can say *the* kernel (resp. cokernel). The pair (f, g) is called a *kernel-cokernel pair* if f is the kernel of g and g is the kernel of f.

An *abelian* category is an additive category such that every morphism has a kernel an a cokernel and such that for each morphism $f : A \to B$ the canonical factorization



induces an isomorphism $\overline{f}$. For simplicity, we identify coker(ker(f)) and ker(coker(f)) for a morphism f and define them to be the *image* (denoted by im(f)) of f.

Recall that, in an additive category, a morphism $i : A \to B$ is called a *monomorphism* if $ig = 0$ implies $g = 0$. A morphism $i : A \to B$ is a *split monomorphism* if there exists a morphism $j : B \to A$ such that $ji = 1_A$. A morphism $p : B \to C$ is called an *epimorphism* $hp = 0$ imples $h = 0$. A morphism $p : B \to C$ is a *split epimorphism* if there exists a morphism $q : C \to B$ such that $pq = 1_C$. Clearly, any split monomorphism (resp. epimorphism) is a mnomorphism (resp. epimorphism). It is well-known that an aditive category is abelian if and only if every morphism has a kernel and a cokernel, every monomorphism is a kernel and every epimorphism is a cokernel.

Let $\mathcal{A}$ be an abelian category and $A \xrightarrow{f} B \xrightarrow{g} C$ a sequence of morphisms. The sequence is *exact* at B if $\operatorname{im}(f) = \ker(g)$. More generally, a long (possibly infinite) sequence

$$\cdots \to A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \to \cdots$$

is a (*chain*) *complex*[3] if $f_n f_{n+1} = 0$ for all n; it is called exact if it is exact at A_n for all n. In particular, any exact sequence is a complex. A *short exact sequence* is an exact sequence of the following form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

and in this case we say B is an *extension* of C by A. Such a short exact sequence is *split* if f is a split monomorphism, or equivalently g is a split epimorphism. In an abelian category, a sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence if and only if (f, g) is a kernel-cokernel pair.

Let $\mathcal{A}, \mathcal{B}$ be abelian categories. An additive functor $F : \mathcal{A} \to \mathcal{B}$ is *exact* if it sends short exact sequences in $\mathcal{A}$ to short exact sequences in $\mathcal{B}$. In contrast with this, any additive functor sends split short exact sequences to split short exact sequences. An object $P \in \mathcal{A}$ is called projective if the functor $\operatorname{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathbf{Ab}$ is an exact functor, dually, an object $I \in \mathcal{A}$ is called injective if the functor $\operatorname{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ is an exact functor. We denote by $\mathcal{P}(\mathcal{A})$ the subcategory of all projective objects and by $\mathcal{I}(\mathcal{A})$ the subcategory of all injective objects in $\mathcal{A}$.

An abelian category $\mathcal{A}$ is said to have enough projectives (resp. injectives) if for any object $A \in \mathcal{A}$ there exists an epimorphism $P \to A \to 0$ (resp. a monomorphism $0 \to A \to I$) with $P \in \mathcal{P}(\mathcal{A})$ (resp. $I \in \mathcal{I}(\mathcal{A})$). In this case, the map $P \to A$ (resp. $A \to I$) is a right $\mathcal{P}(\mathcal{A})$-approximation (resp. left $\mathcal{I}(\mathcal{A})$-approximation) of A.

Example 1.1.3. (1) Let R be a ring. Then the category of all (left) R-modules R-Mod is an abelian category with enough projectives and injectives.

(2) Let Λ be a finite-dimensional algebra over some filed K. Then the category Λ-mod is a Hom-finite abelian K-category with enough projectives and injectives. The subcategory of projectives is $\mathcal{P}(\Lambda) := \mathcal{P}(\Lambda\text{-mod}) = \operatorname{add}(\Lambda)$ and the subcategory of injectives is $\mathcal{I}(\Lambda) := \mathcal{I}(\Lambda\text{-mod}) = \operatorname{add}(\mathrm{D}\,\Lambda)$, where $\mathrm{D}\,\Lambda$ is the standard K-dual (defined below) of ${}_{\Lambda^{\mathrm{op}}}\Lambda$.

Exact structures

Let $\mathcal{A}$ be an additive category and $\mathcal{E}$ a fixed class of kernel-cokernel pairs in $\mathcal{A}$. If a pair (f, g) belongs to $\mathcal{E}$, then we say f is an *inflation*, g a *deflation* and (f, g) a *conflation*. The class $\mathcal{E}$ is called an *exact structure* on $\mathcal{A}$ if it is closed under isomorphisms and satisfies the following axioms:

(E0) The identity map 1_A is an inflation for all $A \in \mathcal{A}$.

(E0$^{\mathrm{op}}$) The identity map 1_A is a deflation for all $A \in \mathcal{A}$.

(E1) A composition of two inflations is an inflation.

(E1$^{\mathrm{op}}$) A composition of two deflations is a deflation.

[3] A sequence $\cdots \to A^{n-1} \xrightarrow{f^{n-1}} A^n \xrightarrow{f^n} A^{n+1} \to \cdots$ with increasing indices satisfying $f^n f^{n-1} = 0$ for all n is known as a *cochain complex*.

(E2) The push-out of an inflation along an arbitrary morphism exists and yields an inflation.

($E2^{op}$) The pull-back of a deflation along an arbitrary morphism exists and yields a deflation.

An *exact category* is a pair $(\mathcal{A}, \mathcal{E})$ consists of an additive category $\mathcal{A}$ and an exact structure $\mathcal{E}$ on it. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two exact structures on $\mathcal{A}$, then we say $\mathcal{E}_1$ is smaller than $\mathcal{E}_2$ provided that $\mathcal{E}_1 \subseteq \mathcal{E}_2$. This defines a partial order on the class of all exact structures on an additive subcategory.

Example 1.1.4. (1) Let $\mathcal{A}$ be an abelian category, $\mathcal{E}_1 = \{$All short exact sequences in $\mathcal{A}\}$ and $\mathcal{E}_2 = \{$All split short exact sequences in $\mathcal{A}\}$. Then $\mathcal{E}_1$ is the unique maximal exact structure on $\mathcal{A}$ and $\mathcal{E}_2$ is the unique minimal exact structure on $\mathcal{A}$.

(2) Let $\mathcal{A}$ be an additive category and $\mathcal{E} = \{$All split kernel-cokernel pairs in $\mathcal{A}\}$. Then $\mathcal{E}$ is the unique minimal exact structure on $\mathcal{A}$. Rump [Rum11] proved there also exists a unique maximal exact structure on $\mathcal{A}$.

(3) Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ a extension closed subcategory. Let $\mathcal{E}$ the class consisting of all short exact sequences with all the three terms in $\mathcal{B}$. Then $\mathcal{E}$ is an exact structure on the additive category $\mathcal{B}$.

Projectives and injectives. Let $(\mathcal{A}, \mathcal{E})$ and $(\mathcal{A}', \mathcal{E}')$ be exact categories. An additive functor $F : \mathcal{A} \to \mathcal{A}'$ is called *exact* if $F(\mathcal{E}) \subseteq \mathcal{E}'$. An object $P \in \mathcal{A}$ is called *projective* (*with respect to* $\mathcal{E}$) if the functor $\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathbf{Ab}$, here $\mathbf{Ab}$ is the exact category with the exact structure consisting of all short exact sequences. Dually, we can define *injective* objects in an exact category. Projective and injective objects in exact categories are generalizations of Projective and injective objects in abelian categories. An exact category $(\mathcal{A}, \mathcal{E})$ is said to have enough projectives (resp. injectives) if for any object $A \in \mathcal{A}$ there exists a delation $P \to A$ (resp. an inlation $A \to I$) with P projective (resp. I injective).

1.2 Finite-dimensional algebras and modules

Throughout this section, we fix a field K. A (not necessarily commutative, unital) ring Λ is a K-algebra provided Λ is a K-vector space and the multiplication in Λ is compatible with the scalar multiplication, i.e., $k(\lambda\mu) = (k\lambda)\mu = \lambda(k\mu)$ for all $k \in K$ and $\lambda, \mu \in \Lambda$. A finite-dimensional K-algebra is a K-algebra which has finite dimension as a K-vector space.

For a finite-dimensional K-algebra Λ, we denote by Λ-mod the category of finitely generated (= finite-dimensional = finite length) left Λ-modules, and by mod-Λ the category of finitely generated right Λ-modules. Clearly, mod-Λ is the same as Λ^{op}-mod.

Let $M \in \Lambda$-mod. Then there is a corresponding module $\mathrm{Hom}_K(M, K)$ in mod-Λ which is called the K-dual of M. Let $f : M \to N$ be a morphism in Λ-mod, then we have a morphism $\mathrm{Hom}_K(f, K) : \mathrm{Hom}_K(N, K) \to \mathrm{Hom}_K(M, K)$ in mod-Λ and we also have $\mathrm{Hom}_K(fg, K) = \mathrm{Hom}_K(g, K)\,\mathrm{Hom}_K(f, K)$ for any morphism $g : L \to M$. This defines a contravariant functor $\mathrm{D} := \mathrm{Hom}_K(-, K) : \Lambda\text{-mod} \to \text{mod-}\Lambda$. Since the same argument works for right modules, then the fact that the natural morphism $M \to \mathrm{Hom}_K(\mathrm{Hom}_K(M, K), K)$ is an isomorphism for all M induces the following duality:

$$\mathrm{D} : \Lambda\text{-mod} \overset{\sim}{\longleftrightarrow} \text{mod-}\Lambda = \Lambda^{op}\text{-mod} : \mathrm{D}$$

which is known as the standard K-duality.

Let $M \in \Lambda\text{-mod}$ and $\Gamma = \mathrm{End}_\Lambda(M)$ be its endomorphism ring. Then M can be naturally viewed as a left Γ-module and moreover a left Λ-left Γ-bimodule. We write ${}_\Gamma M$ when we consider M as a left Γ-module. We will study the following four contravariant functors:

$$\mathrm{Hom}_\Lambda(-, M)\colon\ \Lambda\text{-mod} \longleftrightarrow \Gamma\text{-mod}\colon\ \mathrm{Hom}_\Gamma(-, M)$$
$$\mathrm{D}\,\mathrm{Hom}_\Lambda(M, -)\colon\ \Lambda\text{-mod} \longleftrightarrow \Gamma\text{-mod}\colon\ \mathrm{D}\,\mathrm{Hom}_\Gamma(M, -).$$

One of the reasons that we would like to study these functors is that they form two adjoint pairs (of contravariant functors). Let us recall the definition of an adjoint pair. Let $\mathcal{A}, \mathcal{B}$ be arbitrary categories and $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ two functors. Then we say (F, G) is an *adjoint pair* if for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ there exists an bijection of sets

$$\mathrm{Hom}_\mathcal{B}(F(A), B) \xrightarrow{\cong} \mathrm{Hom}_\mathcal{A}(A, G(B))$$

which is natural in both A and B. In this case, we also say F is left adjoint to G and G is right adjoint to F. Now let $F : \mathcal{A} \to \mathcal{B}$ be a contravariant functor. Then F is the same as a functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ or a functor $\mathcal{A} \to \mathcal{B}^{\mathrm{op}}$. We say a pair of contravariant functors is an adjoint pair if they form an adjoint pair when viewing as covariant functors.

Example 1.2.1. Let ${}_\Lambda M_\Gamma$ be a bimodule. Then we have an isomorphism

$$\mathrm{Hom}_\Lambda({}_\Lambda M_\Gamma \otimes_\Gamma {}_\Gamma Y, {}_\Lambda X) \xrightarrow{\cong} \mathrm{Hom}_\Gamma({}_\Gamma Y, \mathrm{Hom}_\Lambda({}_\Lambda M_\Gamma, {}_\Lambda X))$$

which is natural in $X \in \Lambda\text{-mod}$ and $Y \in \Gamma\text{-mod}$, i.e., $({}_\Lambda M_\Gamma \otimes_\Gamma -, \mathrm{Hom}_\Lambda({}_\Lambda M_\Gamma, -))$ is an adjoint pair which is usually known as the *Hom-Tensor* adjunction.

Convention In order to keep the formulas and diagrams in reasonable length we will often use the conventions $(-, {}_\Lambda M) := \mathrm{Hom}_\Lambda(-, M)$ and $\mathrm{D}({}_\Lambda M, -) := \mathrm{D}\,\mathrm{Hom}_\Lambda(M, -)$. If there is no ambiguity we may omit the subscript and write $(-, {}_\Lambda M)$ (or $(-, {}_\Gamma M)$) as $(-, M)$.

Lemma 1.2.2. *[AS93c, Lemma 3.3][ARS95, Proposition 2.1] Let* $M \in \Lambda\text{-mod}$ *and* $\Gamma = \mathrm{End}_\Lambda(M)$.

(1) $((-, {}_\Lambda M), (-, {}_\Gamma M))$ *is an adjoint pair of contravariant functors and it restricts to a duality*

$$(-, {}_\Lambda M) : \mathrm{add}({}_\Lambda M) \longleftrightarrow \mathrm{add}(\Gamma) = \mathcal{P}(\Gamma) : (-, {}_\Gamma M).$$

(2) $(\mathrm{D}({}_\Lambda M, -), \mathrm{D}({}_\Gamma M, -))$ *is an adjoint pair of contravariant functors and it restricts to a duality*

$$\mathrm{D}({}_\Lambda M, -) : \mathrm{add}({}_\Lambda M) \longleftrightarrow \mathrm{add}(\mathrm{D}\,\Gamma) = \mathcal{I}(\Gamma) : \mathrm{D}({}_\Gamma M, -).$$

In general, any bimodule ${}_{\Lambda\text{-}\Gamma}M$ gives rise to adjoint pairs of contravariant functors $(-, {}_\Lambda M)$: $\mathrm{add}({}_\Lambda M)$ and $(\mathrm{D}({}_\Lambda M, -), \mathrm{D}({}_\Gamma M, -))$.

Since we have a natural bimodule ${}_{\Lambda\text{-}\Lambda^{\mathrm{op}}}\Lambda = {}_\Lambda\Lambda_\Lambda$, the adjoint pair $((-, {}_\Lambda\Lambda), (-, {}_{\Lambda^{\mathrm{op}}}\Lambda))$ of contravariant functors restricts to a duality

$$(-, {}_\Lambda\Lambda) : \mathrm{add}({}_\Lambda\Lambda) \longleftrightarrow \mathrm{add}({}_{\Lambda^{\mathrm{op}}}\Lambda) : (-, {}_{\Lambda^{\mathrm{op}}}\Lambda)$$

between finitely generated left Λ-modules and finitely generated right Λ-modules. Consider the functor $\nu_\Lambda := \mathrm{D}(-, {}_\Lambda\Lambda) : \Lambda\text{-mod} \to \Lambda\text{-mod}$, then ν_Λ restricts to a duality

$$\mathcal{P}(\Lambda) = \mathrm{add}({}_\Lambda\Lambda) \to \mathrm{add}(\mathrm{D}({}_{\Lambda^{\mathrm{op}}}\Lambda)) = \mathcal{I}(\Lambda)$$

between finitely generated projective left Λ-modules and finitely generated injective left Λ-modules with quasi inverse $\nu_\Lambda^{-1} := (\mathrm{D}(-), {}_{\Lambda^{\mathrm{op}}}\Lambda) \cong (\mathrm{D}({}_{\Lambda^{\mathrm{op}}}\Lambda), -)$. The functor ν_Λ is known as the *Nakayama functor* for Λ-mod. We will write ν_Λ as ν if there is no ambiguity. We have the following important natural isomorphism induced by the Nakayama functor:

Lemma 1.2.3. *[ASS06, Lemma2.11] Let Λ be a finite-dimensional K-algebra. Then we have an isomorphism*

$$\operatorname{Hom}_\Lambda(P, M) \xrightarrow{\cong} \mathrm{D}\operatorname{Hom}_\Lambda(M, \nu P)$$

which is natural in $P \in \mathcal{P}(\Lambda)$ and $M \in \Lambda$-mod.

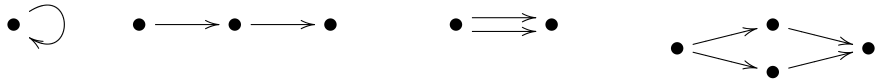
By the Krull-Schmidt theorem, Λ can be decomposed into a direct sum of indecomposable projective modules $\Lambda = P_1 \oplus \cdots \oplus P_n$. An algebra Λ is said to be *basic* if $P_i \ncong P_j$ when $i \neq j$. Similarly, a module M is said to be *basic* if each of its indecomposable summand is of multiplicity 1. Two finite-dimensional algebras Λ and Γ are said to be *Morita equivalent* if Λ-mod and Γ-mod are equivalent. It is well-known that any finite-dimensional algebra Λ is equivalent to a basic algebra and for any module $M \in \Lambda$-mod there is a basic module M' such that $\operatorname{add}(M) = \operatorname{add}(M')$.

Quivers and representations

A (finite) *quiver* $Q = (Q_0, Q_1, s, t)$ consists of

- a finite set Q_0 of vertices;
- a finite set Q_1 of arrows;
- a map $s : Q_1 \to Q_0$ sends each arrow to its source; and
- a map $t : Q_1 \to Q_0$ sends each arrow to its target.

For an arrow α we shall present it as $s(\alpha) \xrightarrow{\alpha} t(\alpha)$ with $s(\alpha), t(\alpha) \in Q_0$. The following are some easy examples of quivers:



Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $i, j \in Q_0$. A *path* from i to j of length l is a sequence $(i|\alpha_1, \alpha_2, \cdots, \alpha_l|j)$ such that each $\alpha_n \in Q_1$ and $s(\alpha_1) = i$, $t(\alpha_n) = s(\alpha_{n+1})$ for all $n \in \{2, \cdots, l\}$ and $t(\alpha_l) = j$. Such a path can be presented as

$$i = s(\alpha_1) \xrightarrow{\alpha_1} t(\alpha_1) \xrightarrow{\alpha_2} \cdots \to s(\alpha_l) \xrightarrow{\alpha_l} t(\alpha_l) = j$$

and will be written as $p = \alpha_l \cdots \alpha_2\alpha_1$. In particular, if $l = 0$ then we must have $i = j$ and the path will be called a *constant* path and denoted by e_i. For a path $p = \alpha_l \cdots \alpha_2\alpha_1$, we define $s(p) = s(\alpha_1)$ and $t(p) = t(\alpha_l)$. A path p such that $s(p) = t(p)$ is called a *oriented cycle* and it is called a *loop* if it is of length 1. A quiver without oriented cycle is called *acyclic.* Given two paths $p = \alpha_l \cdots \alpha_2\alpha_1$ and $p' = \alpha'_m \cdots \alpha'_2\alpha'_1$ such that $t(p') = s(p)$, we denote by $p \cdot p'$ the concatenation of the two paths, i.e.,

$$p \cdot p' = \alpha_l \cdots \alpha_2\alpha_1\alpha'_m \cdots \alpha'_2\alpha'_1.$$

The *path algebra* KQ of Q is the K-algebra whose underlying K-vector space has basis consisting of all paths of length ≥ 0 and with multiplication defined on two basis elements p, p' by

$$pp' = \begin{cases} p \cdot p' & \text{if } t(p') = s(p) \\ 0 & \text{otherwise.} \end{cases}$$

We have the following fundamental result on path algebras:

Lemma 1.2.4. *[ASS06, Lemma 1.4] Let $Q = (Q_0, Q_1, s, t)$ be a (finite) quiver and KQ be its path algebra. Then KQ is an associative unital K-algebra with unit $1 = \sum_{i \in Q_0} e_i$ and it is finite-dimensional if and only if Q has no oriented cycles.*

Let Q be a quiver and KQ be the path algebra of Q. We define R to be ideal of KQ generated by all arrows. An ideal I of KQ is *admissible* if there exists an $m \geq 2$ such that $R^m \subseteq I \subseteq R^2$. If I is an admissible ideal of KQ, then the pair (Q, I) is called a *bound quiver* and the algebra KQ/I is called a *bound quiver algebra* which is always a finite-dimensional algebra ([ASS06, Proposition 2.6]).

Proposition 1.2.5. *[ASS06, Proposition 2.6] Let Q be a quiver and I an admissible ideal of KQ. Then KQ/I is a finite-dimensional K-algebra.*

Remark 1.2.6. If K is algebraically closed, Gabriel proved that every finite dimensional algebra can be obtained as a bounded quiver algebra, see for example [ASS06, Theorem 3.7].

Let Q be a quiver. A (finite-dimensional K-linear) *representation* of Q consists of:

- a finite dimensional K-vector space M_i for each $i \in Q_0$;
- a K-linear map $\varphi_\alpha : M_i \to M_j$ for each arrow $i \xrightarrow{\alpha} j$.

Such a representation is denoted by $M = (M_i, \varphi_\alpha)$. Let $M = (M_i, \varphi_\alpha)$ and $M' = (M'_i, \varphi'_\alpha)$ be two representations of Q. A morphism $f : M \to M'$ is a collection of linear maps $f_i : M_i \to M'_i$, $i \in Q_0$ such that for each arrow $\alpha : i \to j$ we have $f_j \varphi_\alpha = \varphi'_\alpha f_1$, i.e., the following diagram commutes

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi_\alpha} & M_j \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} \\ M'_i & \xrightarrow{\varphi'_\alpha} & M'_j. \end{array}$$

Representations and morphisms between them form a category which we denote by $\operatorname{rep}_K(Q)$. The category $\operatorname{rep}_K(Q)$ is an abelian K-category. For a bounded quiver (Q, I), one can define representations and morpshims in a similar way and then they will form a category $\operatorname{rep}_K(Q, I)$. In this case we have $\operatorname{rep}_K(Q, I) \simeq KQ/I$-mod. For details see [ASS06, Chapter III].

Homological dimensions

Let Λ be a finite-dimensional K-algebra and M be a finitely generated left Λ-module. A *projective resolution* of M is an exact sequence

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

with $P_i \in \mathcal{P}(\Lambda)$. Such a resolution is called *minimal* if $P_0 \to M$ is the minimal right $\mathcal{P}(\Lambda)$-approximation and each map $P_{i+1} \to P_i$ factors through a minimal right $\mathcal{P}(\Lambda)$-approximation $P_{i+1} \to \operatorname{im}(P_{i+1} \to P_i)$. Since Λ-mod is a Krull-Schmidt category with enough projectives, the minimal projective resolution of M always exists. A minimal partial resolution $P_1 \to P_0 \to M \to 0$ is usually called a *minimal projective presentation* of M. If there exists a projective resolution of M such that $P_{n+1} = 0$, then we say M has *projective dimension* at most n and write as $\operatorname{pd}_\Lambda M \leq n$, if moreover $P_n \neq 0$ then we say M has projective dimension n and write as $\operatorname{pd}_\Lambda M = n$. If there is no such a projective resolution of M, then we say the projective dimension of M is infinity write as $\operatorname{pd}_\Lambda M = \infty$. The projective dimension of a module is well-defined, i.e., it does not depend on the choice of projective resolutions. Dually, we have *injective coresolutions* and *injective dimension* $\operatorname{id}_\Lambda M$ for all M. It is well-known that

$$\sup\{\operatorname{pd}_\Lambda M | M \in \Lambda\text{-mod}\} = \sup\{\operatorname{id}_\Lambda M | M \in \Lambda\text{-mod}\}$$

and it is called the (left) *global dimension* of Λ and will be denoted as $\operatorname{gldim} \Lambda$. An algebra Λ is *semisimple* if and only if $\operatorname{gldim} \Lambda = 0$; Λ is *hereditary* i and only if $\operatorname{gldim} \Lambda = 1$.

Proposition 1.2.7. *[ASS06, Proposition A4.7] Let $0 \to L \to M \to N \to 0$ be a short exact sequence in* Λ-mod.

(1) $\operatorname{pd} N \leq \max(\operatorname{pd} M, \operatorname{pd} L + 1)$, *and the equality holds if* $\operatorname{pd} M \neq \operatorname{pd} L$.

(2) $\operatorname{pd} L \leq \max(\operatorname{pd} M, \operatorname{pd} N - 1)$, *and the equality holds if* $\operatorname{pd} M \neq \operatorname{pd} N$.

(3) $\operatorname{pd} M \leq \max(\operatorname{pd} L, \operatorname{pd} N)$, *and the equality holds if* $\operatorname{pd} N \neq \operatorname{pd} L + 1$.

There is an injective version of the above proposition which we shall not state.

Let $\cdots \to M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \to \cdots$ be a complex of Λ-modules, then $\mathsf{H}_i := \ker(f_n)/\operatorname{im}(f_{n+1})$ is called the n-th *homology*[4] group of this complex.

Given a projective resolution $\cdots \to P_{n+1} \xrightarrow{f_{n+1}} P_n \to \cdots \to P_1 \xrightarrow{f_1} P_0 \to M \to 0$ of M, we will get a complex $\cdots \to P_{n+1} \xrightarrow{f_{n+1}} P_n \to \cdots \to P_1 \xrightarrow{f_1} P_0 \to 0$ after deleting the term M. By applying $\operatorname{Hom}_\Lambda(-, N)$ we obtain a complex

$$0 \to (P_0, N) \xrightarrow{(f_1,N)} (P_1, N) \to \cdots \to (P_n, N) \xrightarrow{(f_{n+1},N)} (P_{n+1}) \to \cdots .$$

Consider the cohomology

$$\operatorname{Ext}^i_\Lambda(M, N) := \ker((f_{i+1}, N))/\operatorname{im}((f_i, N))$$

which is known as the i-th *extension group* of M by N. This is well-defined and can be re-obtained by the dual construction involving an injective coresolution of N. Moreover, for each $i \geq 0$ we have an additive bifunctor

$$\operatorname{Ext}^i_\Lambda(-, -) : (\Lambda\text{-mod})^{\text{op}} \times \Lambda\text{-mod} \to \mathbf{Ab}.$$

The functors $\operatorname{Ext}^i_\Lambda(M, -)$ and $\operatorname{Ext}^i_\Lambda(-, N)$ are known as the i-th (right) *derived functors* of the functors $\operatorname{Hom}_\Lambda(M, -)$ and $\operatorname{Hom}_\Lambda(-, N)$.

Elements in the group $\operatorname{Ext}^1_\Lambda(M, N)$ can be identified with equivalent classes of short exact sequences with the equivalence relation as follows. Two exact sequence $\varepsilon : 0 \to N \to E \to M \to 0$ and $\eta : 0 \to N \to E' \to M \to 0$ are equivalent if there exists a map $f : E \to E'$ such that the following diagram commutes

$$\begin{array}{lccccccccc}
\varepsilon : & 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle f} & & \| & & \\
\eta : & 0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

and such a map f must be an isomorphism. We denote by $[\varepsilon]$ the equivalent class of these equivalent short exact sequences.

[4]For cochain complexes we will use the terminology *cohomology* and the symbol H^n.

Given an exact sequence $0 \to A \to B \to C \to 0$ in Λ-mod there is two long exact sequences

$$\begin{array}{l} 0 \longrightarrow (M,A) \longrightarrow (M,B) \longrightarrow (M,C) \longrightarrow \operatorname{Ext}^1(M,A) \longrightarrow \operatorname{Ext}^1(M,B) \longrightarrow \\ \operatorname{Ext}^1(M,C) \longrightarrow \cdots \longrightarrow \operatorname{Ext}^n(M,C) \longrightarrow \operatorname{Ext}^{n+1}(M,A) \longrightarrow \cdots \end{array}$$

$$\begin{array}{l} 0 \longrightarrow (C,N) \longrightarrow (B,N) \longrightarrow (A,N) \longrightarrow \operatorname{Ext}^1(C,N) \longrightarrow \operatorname{Ext}^1(B,N) \longrightarrow \\ \operatorname{Ext}^1(A,N) \longrightarrow \cdots \longrightarrow \operatorname{Ext}^n(A,N) \longrightarrow \operatorname{Ext}^{n+1}(C,N) \longrightarrow \cdots . \end{array}$$

Let $\cdots \to P_{n+1} \xrightarrow{f_{n+1}} P_n \to \cdots \to P_1 \xrightarrow{f_1} P_0 \to M \to 0$ be the minimal projective resolution of M, we define the i-th *syzygy* module of M to be $\Omega^i_\Lambda M := \ker(f_i)$. Dually, we have the i-th *cosyzygy* module $\Omega^{-i}_\Lambda M$ of M.

Now consider the minimal injective coreslution of Λ

$$0 \to \Lambda \to I_0 \to I_1 \to I_2 \to \cdots .$$

If there exists an integer k such that I_i is projective for $0 \leq i \leq k-1$, then we say Λ has *dominant dimension* at least k and write as $\operatorname{domdim}\Lambda \geq k$; if moreover I_k is not projective then we say Λ has dominant dimension k and write as $\operatorname{domdim}\Lambda = k$. If I_i is projective for all $i \geq 0$ then we write $\operatorname{domdim}\Lambda = \infty$. An algebra Λ is called *self-injective* if every injective module is projective, or equivalently the module $\mathrm{D}\,\Lambda$ is projective. Clearly, if Λ is self-injective then $\operatorname{domdim}\Lambda = \infty$.

Auslander-Reiten theory

In the rest of this chapter we fix a finite-dimensional K-algebra Λ.

Let $M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} M$ be the projective presentation of M. Applying the functor $(-,\Lambda)$ yields an exact sequence

$$0 \to (M,\Lambda) \xrightarrow{(f_0,\Lambda)} (P_0,\Lambda) \xrightarrow{(f_1,\Lambda)} (P_1,\Lambda) \to \operatorname{coker}((f_1,\Lambda)) \to 0$$

and we define $\operatorname{coker}((f_1,\Lambda)) \in \Lambda^{\mathrm{op}}$-mod to be the *transpose* of M and denote it by $\operatorname{Tr} M$. The transpose of a module is unique up to isomorphism. It is easy to prove that $\operatorname{Tr} M = 0$ if and only if M is projective, and for M which is not projective $\operatorname{Tr}\operatorname{Tr} M \cong M$.

Let $M, N \in \Lambda$-mod, Let $\mathcal{P}(M,N)$ denote the subset of $\operatorname{Hom}_\Lambda(M,N)$ consisting of all homomorphisms that factor through a projective module. This defines an ideal in the category Λ-mod and so we can form the ideal quotient with objects being the same as objects in Λ-mod and the set of morphisms being $\underline{\operatorname{Hom}}_\Lambda(M,N) := \operatorname{Hom}_\Lambda(M,N)/\mathcal{P}(M,N)$. We denote this ideal quotient by Λ-$\underline{\mathrm{mod}}$, which is known as the *projectively stable category* of Λ-mod. Dually, we have the *injectively stable category* Λ-$\overline{\mathrm{mod}}$ of Λ-mod.

Proposition 1.2.8. *[ARS95, Proposition IV 1.6][ASS06, Proposition 2.2] The assignment* $M \to \operatorname{Tr} M$ *defines a duality*

$$\operatorname{Tr} : \Lambda\text{-}\underline{\mathrm{mod}} \longrightarrow \Lambda^{\mathrm{op}}\text{-}\underline{\mathrm{mod}}.$$

The compositions $\tau := \mathrm{D}\operatorname{Tr}$ and $\tau^- := \operatorname{Tr}\mathrm{D}$ are called the *Auslander-Reiten translations.*

Corollary 1.2.9. *The Auslander-Reiten translations defines a duality*

$$\tau : \Lambda\text{-}\underline{\mathrm{mod}} \longleftrightarrow \Lambda\text{-}\overline{\mathrm{mod}} : \tau^-.$$

Theorem 1.2.10. *[AR75, ASS06] Let Λ be a finite-dimensional K-algebra and $M, N \in \Lambda\text{-mod}$. Then there exist isomorphisms*

$$\mathrm{Ext}^1_\Lambda(M,N) \cong \mathrm{D}\,\underline{\mathrm{Hom}}_\Lambda(\tau^- N, M) \cong \mathrm{D}\,\overline{\mathrm{Hom}}_\Lambda(N, \tau M)$$

which is natural in both variables.

The following lemma first appeared in [Aus99a, Corrolary III 4.2]. We also give a short proof.

Lemma 1.2.11. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of Λ-modules and X be any Λ-module. Then the following are equivalent:*

(1) the sequence $0 \to \mathrm{Hom}_\Lambda(X,A) \to \mathrm{Hom}_\Lambda(X,B) \to \mathrm{Hom}_\Lambda(X,C) \to 0$ is exact;

(2) the sequence $0 \to \mathrm{Hom}_\Lambda(C,\tau X) \to \mathrm{Hom}_\Lambda(B,\tau X) \to \mathrm{Hom}_\Lambda(A,\tau X) \to 0$ is exact.

Proof. If X is projective then there is noting to prove. Assume X is not projective. Let $P_1 \to P_0 \to X \to 0$ be the minimal projective presentation. This gives an exact sequence $0 \to \tau X \to \nu P_1 \to \nu P_0$ whisch is the minimal copresentation of τX. Thus, by Lemma 1.2.3, for any Λ-module Y we have a commutative diagram of exact rows

$$\begin{array}{ccccccccc}
0 \to \mathrm{Hom}_\Lambda(Y,\tau X) & \to & \mathrm{Hom}_\Lambda(Y,\nu P_1) & \to & \mathrm{Hom}_\Lambda(Y,\nu P_0) & & & & \\
 & & \downarrow\cong & & \downarrow\cong & & & & \\
 & & \mathrm{D}\,\mathrm{Hom}_\Lambda(P_1,Y) & \to & \mathrm{D}\,\mathrm{Hom}_\Lambda(P_0,Y) & \to & \mathrm{D}\,\mathrm{Hom}_\Lambda(X,Y) & \to & 0.
\end{array}$$

Take $Y \in \{A,B,C\}$, then the above diagram gives the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}_\Lambda(C,\tau X) & \to & \mathrm{Hom}_\Lambda(B,\tau X) & \to & \mathrm{Hom}_\Lambda(A,\tau X) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{D}\,\mathrm{Hom}_\Lambda(P_1,C) & \to & \mathrm{D}\,\mathrm{Hom}_\Lambda(P_1,B) & \to & \mathrm{D}\,\mathrm{Hom}_\Lambda(P_1,A) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{D}\,\mathrm{Hom}_\Lambda(P_0,C) & \to & \mathrm{D}\,\mathrm{Hom}_\Lambda(P_0,B) & \to & \mathrm{D}\,\mathrm{Hom}_\Lambda(P_0,A) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{D}\,\mathrm{Hom}_\Lambda(X,C) & \to & \mathrm{D}\,\mathrm{Hom}_\Lambda(X,B) & \to & \mathrm{D}\,\mathrm{Hom}_\Lambda(X,A) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with the three columns and the two middle rows exact. By the Snake lemma, the first row is exact if and only if the last row is exact.

□

A morphism $f : A \to B$ is called *left almost split* if it is not split monomorphism and any morphism $A \to M$ which is not split monomorphism factors through f. It is called *minimal left almost split* if it is both left minimal and left almost split. Dually, we have *right almost split* and *minimal right almost split* morphisms. A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is said to be an *almost split exact sequence* (or Auslander-Reiten sequence) if f is left almost split and g is right almost split.

Proposition 1.2.12. *[ARS95, Proposition 1.14] The following are equivalent for a short exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$.

(1) The sequence is an almost split sequence.

(2) f *is minimal left almost split.*

(3) g *is minimal right almost split.*

(4) A *is indecomposable and* g *is right almost split.*

(5) C *is indecomposable and* f *is left almost split.*

(6) $A \cong \tau C$ *and* g *is right almost split.*

(7) $C \cong \tau^{-1} A$ *and* f *is left almost split.*

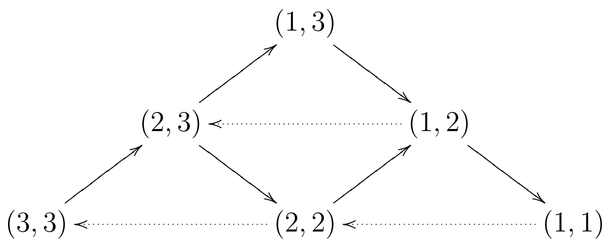
Theorem 1.2.13. *[ARS95, Theorem 1.15]*

(1) If A *is an indecomposable module which is not injective, then there is an almost split exact sequence* $0 \to A \to B \to \tau^{-1}A \to 0$.

(2) If C *is an indecomposable module which is not projective, then there is an almost split exact sequence* $0 \to \tau C \to B \to C \to 0$.

A morphism $f : A \to B$ is said to be *irreducible* if it is neither a split monomorphism nor a split epimorphism and if $f = f_2 f_1$ then either f_1 is a split monomorphism or f_2 is a split epimorphism. Clearly, an irreducible morphism is either a monomorphism or an epimorphism. For a finite-dimensional algebra Λ, there is an associated quiver (maybe infinite), called the *Auslander-Reiten quiver*, defined as follows. Put an vertex for each isomorphism class $[M]$ of indecomposable Λ-module; draw an arrow $[M] \to [N]$ for each irreducible morphism from M to N; draw a dotted arrow from each nonprojective module $[M]$ to $[\tau M]$.

Let $\Lambda = K(1 \to 2 \to 3)$, then indecomposable modules for Λ are indexed by the set $\{(i, j) : 1 \leq i \leq j \leq 3\}$ and the Auslander-Reiten quiver of Λ can be presented the following.



For more details and examples see [ARS95, ASS06].

1.3 Relative homological algebra

We recall from [AS93b] the basics of relative homological algebra. An additive subbifunctor $\mathbf{F}$ of $\mathrm{Ext}^1_\Lambda(-,-)$ is a functor $\mathbf{F} : (\Lambda\text{-mod})^{\mathrm{op}} \times \Lambda\text{-mod} \to \mathbf{Ab}$ such that

- for any $(C, A) \in (\Lambda\text{-mod})^{\mathrm{op}} \times \Lambda\text{-mod}$, $\mathbf{F}(C, A)$ is a subgroup of $\mathrm{Ext}^1_\Lambda(C, A)$,

- for any morphism $(\alpha, \beta) : (C, A) \to (C', A')$ the diagram of group homomorphisms

$$\begin{array}{ccc} \mathbf{F}(C,A) & \hookrightarrow & \operatorname{Ext}^1_\Lambda(C,A) \\ {\scriptstyle \mathbf{F}(\alpha,\beta)} \downarrow & & \downarrow {\scriptstyle \operatorname{Ext}^1_\Lambda(\alpha,\beta)} \\ \mathbf{F}(C',A') & \hookrightarrow & \operatorname{Ext}^1_\Lambda(C',A') \end{array}$$

  is commutative.

A short exact sequence $\varepsilon : 0 \to A \to B \to C \to 0$ is called $\mathbf{F}$-*exact* if $[\varepsilon] \in \mathbf{F}(C, A)$.

Let $\mathcal{X} \subseteq \Lambda\text{-mod}$ be a subcategory, then one can associate two additive subbifunctors $\mathbf{F}_\mathcal{X}, \mathbf{F}^\mathcal{X} \subseteq \operatorname{Ext}^1(-,-)$ to the subcategory $\mathcal{X}$ defined for $(C, A) \in (\Lambda\text{-mod})^{\mathrm{op}} \times \Lambda\text{-mod}$ as follows

$$\begin{aligned} \mathbf{F}^\mathcal{X}(C,A) &= \{0 \to A \to B \to C \to 0 \mid (B,\mathcal{X}) \to (A,\mathcal{X}) \to 0 \ \text{ is exact}\} \\ \mathbf{F}_\mathcal{X}(C,A) &= \{0 \to A \to B \to C \to 0 \mid (\mathcal{X},B) \to (\mathcal{X},C) \to 0 \text{ is exact}\}. \end{aligned}$$

If $\mathcal{X} = \operatorname{add}(M)$ for some module M, then we simply write $\mathbf{F}_{\operatorname{add}(M)}$ as $\mathbf{F}_M$ and $\mathbf{F}^{\operatorname{add}(M)}$ as $\mathbf{F}^M$. Clearly, we have $\mathbf{F}_M = \mathbf{F}_{\Lambda \oplus M}$ and $\mathbf{F}^M = \mathbf{F}^{\mathrm{D}\,\Lambda \oplus M}$. It follows from Lemma 1.2.11 that $\mathbf{F}_M = \mathbf{F}^{\tau M \oplus \mathrm{D}\,\Lambda}$ and $\mathbf{F}^M = \mathbf{F}_{\tau^- M \oplus \Lambda}$.

Let $\mathbf{F} \subseteq \operatorname{Ext}^1(-,-)$ be an additive subbifunctor, we say a monomorphism $f : X \to Y$ is an $\mathbf{F}$-monomorphism if the short exact sequence $0 \to X \xrightarrow{f} Y \to \operatorname{coker} f \to 0$ is $\mathbf{F}$-exact, dually we have $\mathbf{F}$-epimorphism.

Proposition 1.3.1. *[BH62, Theorem 1.1][DRSS99, Proposition 1.4] The following are equivalent for an additive subbifunctor* $\mathbf{F}$ *of* $\operatorname{Ext}^1_\Lambda$.

(1) Compositions of $\mathbf{F}$*-monomorphisms are again* $\mathbf{F}$*-monomorphisms.*

(2) Compositions of $\mathbf{F}$*-epimorphisms are again* $\mathbf{F}$*-epimorphisms.*

(3) The functor $\mathbf{F}(C,-)$ *is half exact*[5] *on* $\mathbf{F}$*-exact sequences for all* $C \in \Lambda\text{-mod}$.

(4) The functor $\mathbf{F}(-,A)$ *is half exact on* $\mathbf{F}$*-exact sequences for all* $A \in \Lambda\text{-mod}$.

An additive subbifunctor $\mathbf{F}$ of Ext^1 is called *closed* if it satisfies the above equivalent conditions. The functors $\mathbf{F}_\mathcal{X}, \mathbf{F}^\mathcal{X}$ are closed for any subcategory $\mathcal{X}$.

Proposition 1.3.2. *[DRSS99, Corollary 1.6] There is a one-to-one correspondence between*

(1) the class of closed additive subbifunctors of $\operatorname{Ext}^1_\Lambda$*;*

(2) exact structures on $\Lambda\text{-mod}$.

Explicitly, to each closed additive subbifunctor $\mathbf{F}$ *we attach an exact structure*

$$\mathcal{E}_\mathbf{F} := \{\text{all } \mathbf{F}\text{-exact sequences}\}$$

and to each exact structure $\mathcal{E}$ *we attach a closed additive subbifunctor* $\mathbf{F}$ *defined via*

$$\mathbf{F}(C,A) := \{\varepsilon : 0 \to A \to B \to C \to 0 | \varepsilon \in \mathcal{E}\}.$$

[5] A functor $\mathbf{F}$ is called half exact if it sends an short exact sequence $0 \to A \to B \to C \to 0$ to an exact sequence $\mathbf{F}(A) \to \mathbf{F}(B) \to \mathbf{F}(C)$.

We say $\mathbf{F}$ has enough projectives (resp. injectives) if the corresponding exact category has enough projectives (resp. injectives). We denote by $\mathcal{P}(\mathbf{F})$ (resp. $\mathcal{I}(\mathbf{F})$) the subcategory of projectives (resp. injectives) of $\mathbf{F}$. By [AS93b, Proposition 1.7], we have that $\mathbf{F}^M, \mathbf{F}_M$ are both additive subbifunctors of $\operatorname{Ext}^1_\Lambda(-,-)$ with enough projectives and enough injectives, and $\mathcal{P}(\mathbf{F}^M) = \operatorname{add}(\tau^- M \oplus \Lambda)$, $\mathcal{I}(\mathbf{F}^M) = \operatorname{add}(M \oplus \operatorname{D}\Lambda)$, $\mathcal{P}(\mathbf{F}_M) = \operatorname{add}(M \oplus \Lambda)$ and $\mathcal{I}(\mathbf{F}_M) = \operatorname{add}(\tau M \oplus \operatorname{D}\Lambda)$. Therefore, one can define, as in the abelian case, for $\mathbf{F} \in \{\mathbf{F}^M, \mathbf{F}_M\}$ the derived functors $\operatorname{Ext}^i_{\mathbf{F}}(-,-)$, $i \geq 1$, which are defined by using $\mathbf{F}$-injective coresolutions or $\mathbf{F}$-projective resolutions. In particular, we have a natural isomorphism of functors $\operatorname{Ext}^1_{\mathbf{F}} \cong \mathbf{F}$. Also, we have the notions of $\mathbf{F}$-projective dimension, $\mathbf{F}$-injctive dimension and $\mathbf{F}$-global dimension which we denote by $\operatorname{pd}_{\mathbf{F}}$, $\operatorname{id}_{\mathbf{F}}$ and $\operatorname{gldim}_{\mathbf{F}}$, respectively. Many arguments work for the standard homological algebra also work for relative homological algebra, for example, there is a relative version of Lemma 1.2.7. We may freely use the results that can be proved with the same arguments as the standard case.

According to [AS93c], the existence of $\mathbf{F}$-cotilting modules is equivalent to $\mathbf{F}$ is of the form $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ for a generator G and a cogenerator H, and in this case $H = \tau G \oplus \operatorname{D}\Lambda$ and $G = \tau^- H \oplus \Lambda$. Such a functor is called an additive subbifunctor (of $\operatorname{Ext}^1_\Lambda$) of *finite type*. As one of our main results, we will prove the relative (co)tilting correspondence. So, in this thesis, we will only consider the additive subbifunctors of finite type. If Λ is representation finite, i.e., there are only finitely many indecomposable Λ-modules up to isomorphism, then there is a largest generator G which is the direct sum (taken over a representative set of nonisomorphic indecomposables) of all the indecomposables. We refer to this G the *Auslander generator* of Λ. Note that G is also the largest cogenerator and we have $\mathbf{F}_G = \mathbf{F}^G$.

Chapter 2

Faithfully balanced modules

2.1 Categories generated or cogenerated by a module

Let M be a Λ-module. The subcategories $\mathrm{cogen}(M) := \{N \mid \exists \text{ monomorphism } 0 \to N \to M_0 \text{ with } M_0 \in \mathrm{add}(M)\}$ and $\mathrm{gen}(M) := \{N \mid \exists \text{ epimorphism } M_0 \to N \to 0 \text{ with } M_0 \in \mathrm{add}(M)\}$ are well-studied and play an important role in classical tilting theory (cf. [ASS06, Chapter VI]). We are going to study the "higher" versions of them. For every non-negative integer k we associate to a module M two full subcategories of Λ-mod

$$\mathrm{cogen}^k(M) := \left\{ N \;\middle|\; \begin{array}{c} \exists \text{ exact seq. } 0 \to N \to M_0 \to \cdots \to M_k \text{ with } M_i \in \mathrm{add}(M), \text{ and s.t.} \\ (M_k, M) \to \cdots \to (M_0, M) \to (N, M) \to 0 \text{ is exact} \end{array} \right\}$$

$$\mathrm{gen}_k(M) := \left\{ N \;\middle|\; \begin{array}{c} \exists \text{ exact seq. } M_k \to \cdots \to M_0 \to N \to 0 \text{ with } M_i \in \mathrm{add}(M), \text{ and s.t.} \\ (M, M_k) \to \cdots \to (M, M_0) \to (M, N) \to 0 \text{ is exact} \end{array} \right\}.$$

We define $\mathrm{cogen}^\infty(M)$ to be the full subcategory of Λ-mod consisting of modules N such that there exists an exact sequence

$$0 \to N \xrightarrow{f_0} M_0 \xrightarrow{f_1} M_1 \cdots \xrightarrow{f_n} M_n \to \cdots$$

such that f_i factors through $\mathrm{coker}\, f_{i-1} \to M_i$ which is a minimal left $\mathrm{add}(M)$-approximation for every $i \geq 0$. The definition of $\mathrm{gen}_\infty(M)$ is dual. It is easy to see that $\mathrm{cogen}^0(M) = \mathrm{cogen}(M)$ and $\mathrm{gen}_0(M) = \mathrm{gen}(M)$.

The following lemma will be used frequently, the case $k = 0$ is well known and can be found in [ASS06, Lemma VI 1.8].

Lemma 2.1.1. *Let* $1 \leq k \leq \infty$.

(1) The following are equivalent for $N \in \Lambda$-mod.

  (1a) $N \in \mathrm{cogen}^k(M)$.

  (1b) The natural map $N \to \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Lambda(N, M), M) = ((N, M), M)$, $n \mapsto (f \mapsto f(n))$ *is an isomorphism and* $\mathrm{Ext}^i_\Gamma(\mathrm{Hom}_\Lambda(N, M), M) = 0$ *for* $1 \leq i \leq k-1$.

(2) The following are equivalent for $N \in \Lambda$-mod.

  (2a) $N \in \mathrm{gen}_k(M)$.

  (2b) The natural map $\mathrm{D}(M, \mathrm{D}(M, N)) \cong \mathrm{Hom}_\Lambda(M, N) \otimes_\Gamma M \to N$, $f \otimes m \mapsto f(m)$ *is an isomorphism and* $\mathrm{Ext}^i_\Gamma(M, \mathrm{D}(M, N)) = 0$ *for* $1 \leq i \leq k-1$.

Proof. (1) Let $N \in \operatorname{cogen}^k(M)$, that means we have an exact sequence

$$0 \to N \to M_0 \to \cdots \to M_k$$

with $M_i \in \operatorname{add}(M)$ and such that the functor $\operatorname{Hom}_\Lambda(-, M)$ is exact on it, i.e., we get an exact sequence

$$(M_k, M) \to \cdots \to (M_0, M) \to (N, M) \to 0.$$

This sequence is a projective resolution of $\operatorname{Hom}_\Lambda(N, M)$ as a left Γ-module. Applying the functor $\operatorname{Hom}_\Gamma(-, M)$ to it yields a complex

$$0 \to ((N, M), M) \to ((M_0, M), M) \to \cdots \to ((M_k, M), M).$$

Now, consider the natural map $N \to \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(N, M), M)$, this gives a commutative diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & ((N,M),M) & \longrightarrow & ((M_0,M),M) & \longrightarrow & \cdots & \longrightarrow & ((M_k,M),M) \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
0 & \longrightarrow & N & \longrightarrow & M_0 & \longrightarrow & \cdots & \longrightarrow & M_k
\end{array}$$

The map $M' \to \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(M', M), M)$ is an isomorphism for $M' \in \operatorname{add}(M)$ because it is in the case of $M' = M$. This implies that all vertical maps are isomorphisms, in particular $N \to \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(M, N), M)$ is an isomorphism and since the second row is exact, the complex in the first row is also exact. This implies $\operatorname{Ext}^i_\Gamma(\operatorname{Hom}_\Lambda(N, M), M) = 0$ for $1 \leq i \leq k-1$.
For the other direction, by Lemma 1.2.2 (1) we can take a projective resolution of $\operatorname{Hom}_\Lambda(N, M)$ as a left Γ-module as follows

$$(M_k, M) \to \cdots \to (M_0, M) \to (N, M) \to 0$$

and apply $\operatorname{Hom}_\Gamma(-, M)$ to compute $\operatorname{Ext}^i_\Gamma(\operatorname{Hom}_\Lambda(N, M), M)$, $1 \leq i \leq k-1$. Since by assumption $\operatorname{Ext}^i_\Gamma(\operatorname{Hom}_\Lambda(N, M), M) = 0$, $1 \leq i \leq k-1$ and $N \to \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(N, M), M)$ is an isomorphism. The complex gives an exact sequence

$$0 \to N \to M_0 \to \cdots \to M_k.$$

If we apply $\operatorname{Hom}_\Lambda(-, M)$ to this sequence we get the projective resolution from before, so it is exact which shows that N is in $\operatorname{cogen}^k(M)$.

(2) By using the facts that $N \in \operatorname{gen}_k(M)$ if and only if $\mathrm{D}\,N \in \operatorname{cogen}^k(\mathrm{D}\,M)$ and $\operatorname{End}_{\Lambda^{op}}(\mathrm{D}\,M) \cong \operatorname{End}_\Lambda(M)^{op}$, we see that the statement (2) can be deduced from the right module version of (1).

□

Corollary 2.1.2. *For $1 \leq k \leq \infty$, the categories $\operatorname{cogen}^k(M)$ and $\operatorname{gen}_k(M)$ are closed under direct sums and summands. Furthermore, we have*

$$\operatorname{cogen}^\infty(M) = \bigcap_{1 \leq k < \infty} \operatorname{cogen}^k(M), \quad \operatorname{gen}_\infty(M) = \bigcap_{1 \leq k < \infty} \operatorname{gen}_k(M).$$

We will need the following useful lemma which already appeared for the specific situation of a relative cotilting module in [AS93c, Lemma 3.3 (b)] and [AS93c, Proposition 3.7].

Lemma 2.1.3. *Let $M \in \Lambda$-mod and $\Gamma = \mathrm{End}_\Lambda(M)$.*

(1) A module $X \in \mathrm{cogen}^1(M)$ if and only if the natural map

$$\mathrm{Hom}_\Lambda(Y, X) \to \mathrm{Hom}_\Gamma((X, M), (Y, M))$$

is an isomorphism for all $Y \in \Lambda$-mod. Furthermore, in this case we have

$$\nu_\Gamma(X, M) = \mathrm{D}((X, M), (M, M)) \cong \mathrm{D}(M, X).$$

Dually, a module $Y \in \mathrm{gen}_1(M)$ if and only if the natural map

$$\mathrm{Hom}_\Lambda(Y, X) \to \mathrm{Hom}_\Gamma(\mathrm{D}(M, X), \mathrm{D}(M, Y))$$

is an isomorphism for all $X \in \Lambda$-mod. Furthermore, in this case

$$\nu_\Gamma^- \mathrm{D}(M, Y) = (\mathrm{D}(M, M), \mathrm{D}(M, Y)) \cong (Y, M).$$

(2) For $k \geq 1$, $X \in \mathrm{cogen}^{k+1}(M)$ if and only if the natural maps

$$\mathrm{Ext}^i_\Lambda(Y, X) \to \mathrm{Ext}^i_\Gamma((X, M), (Y, M)),\ \ 0 \leq i \leq k$$

are isomorphisms for all $Y \in \bigcap_{i=1}^k \ker \mathrm{Ext}^i_\Lambda(-, M)$.
Dually, $Y \in \mathrm{gen}_{k+1}(M)$ if and only if the natural maps

$$\mathrm{Ext}^i_\Lambda(Y, X) \to \mathrm{Ext}^i_\Gamma(\mathrm{D}(M, X), \mathrm{D}(M, Y)),\ \ 0 \leq i \leq k$$

are isomorphisms for all $X \in \bigcap_{i=1}^k \ker \mathrm{Ext}^i_\Lambda(M, -)$.

Proof. (1) Assume $X \in \mathrm{cogen}^1(M)$, then there exists an exact sequence $0 \to X \to M_0 \to M_1$ such that $M_i \in \mathrm{add}(M)$ and $\mathrm{Hom}_\Lambda(-, M)$ is exact on it. We apply $\mathrm{Hom}_\Lambda(Y, -)$ to get an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(Y, X) \to \mathrm{Hom}_\Lambda(Y, M_0) \to \mathrm{Hom}_\Lambda(Y, M_1).$$

Now, we consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (Y, X) & \longrightarrow & (Y, M_0) & \longrightarrow & (Y, M_1) \\ & & \downarrow{\scriptstyle (-,M)} & & \downarrow{\scriptstyle (-,M)}\,\cong & & \downarrow{\scriptstyle (-,M)}\,\cong \\ 0 & \longrightarrow & ((X, M), (Y, M)) & \longrightarrow & ((M_0, M), (Y, M)) & \longrightarrow & ((M_1, M), (Y, M)). \end{array}$$

The second row also can be obtained by applying first $\mathrm{Hom}_\Lambda(-, M)$ then $\mathrm{Hom}_\Gamma(-, (Y, M))$ to the exact sequence $0 \to X \to M_0 \to M_1$, so it remains exact. The induced isomorphism of the kernels is the map in the claim. Conversely, by taking $Y = \Lambda$ we obtain a natural isomorphism $X \xrightarrow{\cong} ((X, M), M)$ which implies $X \in \mathrm{cogen}^1(M)$.

(2) Assume $X \in \mathrm{cogen}^{k+1}(M)$, then we have an exact sequence $0 \to X \to M_0 \to \cdots \to M_{k+1}$ such that $M_i \in \mathrm{add}(M)$ and $\mathrm{Hom}_\Lambda(-, M)$ is exact on it. Applying $\mathrm{Hom}_\Lambda(-, M)$ yields an exact sequence $(M_{k+1}, M) \to \cdots \to (M_0, M) \to (X, M) \to 0$ which is a projective resolution of (X, M) as a left Γ-module. Now assume $Y \in \bigcap_{i=1}^k \ker \mathrm{Ext}^i_\Lambda(-, M)$. To compute $\mathrm{Ext}^i_\Gamma((X, M), (Y, M))$ for $1 \leq i \leq k$ we apply $\mathrm{Hom}_\Gamma(-, (Y, M))$ to this projective resolution and delete the term $((X, M), (Y, M))$ to get a complex

$$\cdots \to 0 \to ((M_0, M), (Y, M)) \to ((M_1, M), (Y, M)) \to \cdots \to ((M_{k+1}, M), (Y, M)) \to 0 \to \cdots$$

which fits into the following commutative diagram

$$\begin{array}{ccccccccccc}
\cdots \to 0 & \to & (Y, M_0) & \to & (Y, M_1) & \to & \cdots & \to & (Y, M_{k+1}) & \to & 0 \to \cdots \\
 & & \downarrow{\scriptstyle (-,M)} \cong & & \downarrow{\scriptstyle (-,M)} \cong & & & & \downarrow{\scriptstyle (-,M)} \cong & & \\
\cdots \to 0 & \to & ((M_0, M), (Y, M)) & \to & ((M_1, M), (Y, M)) & \to & \cdots & \to & ((M_{k+1}, M), (Y, M)) & \to & 0 \to \cdots
\end{array}$$

where the complex in the first row is obtained by applying $\mathrm{Hom}_\Lambda(Y, -)$ to $0 \to X \to M_0 \to \cdots \to M_{k+1}$ and deleting the term (Y, X). Our assumption $Y \in \bigcap_{i=1}^{k} \ker \mathrm{Ext}_\Lambda^i(-, M)$ implies that the i-th cohomology of the first row is $\mathrm{Ext}_\Lambda^i(Y, X)$. Now the isomorphism of the two complexes induces the claimed natural isomorphisms. To prove the other implication, just take $Y = \Lambda$.

□

We also have the following subcategories of Λ-mod that are closely related to $\mathrm{cogen}^k(M)$ and $\mathrm{gen}_k(M)$:

$$\mathrm{copres}^k(M) := \{N \mid \exists \text{ exact seq. } 0 \to N \to M_0 \to \cdots \to M_k \text{ with } M_i \in \mathrm{add}(M)\}$$
$$\mathrm{pres}_k(M) := \{N \mid \exists \text{ exact seq. } M_k \to \cdots \to M_0 \to N \to 0 \text{ with } M_i \in \mathrm{add}(M)\}.$$

These subcategories are useful in characterizing tilting and cotilting modules (see [Wei10]). It follows from the definitions that $\mathrm{cogen}^0(M) = \mathrm{cogen}(M) = \mathrm{copres}^0(M)$ and $\mathrm{cogen}^k(M) \subseteq \mathrm{copres}^k(M)$ for any M and $k \geq 1$. In particular, if M is injective then $\mathrm{cogen}^k(M) = \mathrm{copres}^k(M)$ for any $k \geq 0$.

Remark 2.1.4. By using results in [GT96, Section 3], one can easily deduce that the subcategories $\mathrm{copres}^k(M)$ and $\mathrm{pres}_k(M)$ are functorially finite for any $k \geq 0$ and $M \in \Lambda$-mod. However, to the author's knowledge, it is unknown for an arbitrary M whether $\mathrm{cogen}^k(M)$ or $\mathrm{gen}_k(M)$ is functorially finite (even contravariantly finiteness or covariantly finiteness is unknown) when $k > 1$.

2.2 Faithfully balanced modules and dualities

Faithfully balanced modules can be defined for any ring. For finite-dimensional algebras, Lemma 2.1.1 allows us to give the following internal definition.

Definition 2.2.1. A Λ-module M is called *faithfully balanced* if $\Lambda \in \mathrm{cogen}^1(M)$.

Note that the above definition also makes sense for right modules. The following surprising and also well-known result says every module becomes faithfully balanced when considered as a module over its endomorphism ring.

Lemma 2.2.2. *[AF92, Proposition 4.12] [AS93a, Lemma 2.2] Let* $M \in \Lambda$-mod *and* $\Gamma = \mathrm{End}_\Lambda(M)$ *and consider* M *as a left* Γ*-module. Then* ${}_\Gamma M$ *is faithfully balanced.*

In [BS98], a faithfully balanced module is also known as a module of faithful dimension at least 2. The following lemma (the same as [BS98, Proposition 2.2]), which characterizes modules of faithful dimension at least $k + 1$, can be obtained as an immediate consequence of Lemma 2.1.1.

Lemma 2.2.3. *The following are equivalent for every* $1 \leq k \leq \infty$.

(1) $\Lambda \in \operatorname{cogen}^k(M)$.

(2) The natural map $\Lambda \to \operatorname{End}_\Gamma(M)$ *is an isomorphism and* $\operatorname{Ext}^i_\Gamma(M,M) = 0$, $1 \leq i \leq k-1$.

(3) $\mathrm{D}\,\Lambda \in \operatorname{gen}_k(M)$.

Proof. The equivalence between (1) and (2) is a special case of Lemma 2.1.1. The equivalence to (3) follows again by seeing that the equivalence between (1) and (2) also works for right modules. Then pass with the duality from the right module statement for (1) to (3). □

The following dualities will play a fundamental role.

Lemma 2.2.4. *Let M be a faithfully balanced Λ-module and $\Gamma = \operatorname{End}_\Lambda(M)$. Then the functors $(-, {}_\Lambda M)\colon \Lambda\text{-mod} \longleftrightarrow \Gamma\text{-mod}\colon (-, {}_\Gamma M)$ restrict to a duality of categories*

$$\operatorname{cogen}^1({}_\Lambda M) \longleftrightarrow \operatorname{cogen}^1({}_\Gamma M).$$

They restrict further to a duality

$$\operatorname{cogen}^k({}_\Lambda M) \longleftrightarrow \operatorname{cogen}^1({}_\Gamma M) \cap \bigcap_{i=1}^{k-1} \ker \operatorname{Ext}^i_\Gamma(-, {}_\Gamma M).$$

Dually, the functors $\mathrm{D}({}_\Lambda M, -)\colon \Lambda\text{-mod} \longleftrightarrow \Gamma\text{-mod}\colon \mathrm{D}({}_\Gamma M, -)$ restrict to a duality of categories

$$\operatorname{gen}_1({}_\Lambda M) \longleftrightarrow \operatorname{gen}_1({}_\Gamma M).$$

They restrict further to a duality

$$\operatorname{gen}_k({}_\Lambda M) \longleftrightarrow \operatorname{gen}_1({}_\Gamma M) \cap \bigcap_{i=1}^{k-1} \ker \operatorname{Ext}^i_\Gamma({}_\Gamma M, -).$$

Proof. By Lemma 2.1.3 the functor $(-, {}_\Lambda M)$ is fully faithful on $\operatorname{cogen}^1({}_\Lambda M)$. Let ${}_{\Lambda\text{-}\Gamma}M$ be a Λ-Γ-bimodule, ${}_\Lambda N$ a left Λ-module and ${}_\Gamma N'$ a left Γ-module. We denote by $\alpha_N\colon N \to ((N,M),M)$ and $\alpha_{N'}\colon N' \to ((N',M),M)$ the two natural maps. Then the compositions

$$(N,M) \xrightarrow{\alpha_{(N,M)}} (((N,M),M),M) \xrightarrow{(\alpha_N,M)} (N,M)$$
$$(N,M) \xrightarrow{\alpha_{(N',M)}} (((N',M),M),M) \xrightarrow{(\alpha_{N'},M)} (N',M)$$

are both identities, since by Lemma 1.2.2 the functors $(-, {}_\Lambda M)$ and $(-, {}_\Gamma M)$ form an adjoint pair. Therefore, if α_N (resp. $\alpha_{N'}$) is an isomorphism, then so is $\alpha_{(N,M)}$ (resp. $\alpha_{(N',M)}$). Since M is faithfully balanced the dualities follow from Lemma 2.1.1. □

Remark 2.2.5. (1) We also have the above dualities when M is a balanced Λ-module. To see this, we observe that if ${}_\Lambda M$ is balanced then it is faithfully balanced as a $\Lambda/\operatorname{ann}(M)$-module, where $\operatorname{ann}(M)$ is the annihilator ideal of M, and $\Gamma = \operatorname{End}_\Lambda(M) \cong \operatorname{End}_{\Lambda/\operatorname{ann}(M)}(M)$ as algebras. We may identify $\operatorname{cogen}^1({}_\Lambda M)$ and $\operatorname{cogen}^1({}_{\Lambda/\operatorname{ann}(M)}M)$ since any $\Lambda/\operatorname{ann}(M)$-module is naturally a Λ-module and if $X \in \operatorname{cogen}^1({}_\Lambda M)$ then we must have $\operatorname{ann}(M)X = 0$. Since ${}_\Gamma M$ is always faithfully balanced by Lemma 2.2.2, we have the desired duality

$$\operatorname{cogen}^1({}_\Lambda M) = \operatorname{cogen}^1({}_{\Lambda/\operatorname{ann}(M)}M) \longleftrightarrow \operatorname{cogen}^1({}_\Gamma M)).$$

In general, for an arbitrary Λ-module M, we have a duality

$$\operatorname{cogen}^1({}_\Lambda M) \longleftrightarrow \{Y \in \Gamma\text{-mod} \,|\, Y \xrightarrow{\cong} ((Y, {}_\Gamma M), {}_\Lambda M)\}$$

where $\Gamma = \operatorname{End}_\Lambda(M)$. We have similar dualities for $k \geq 2$.

(2) We have already seen in Lemma 2.1.1 that $\mathrm{cogen}^1(M)$ consists of the modules N such that α_N is an isomorphism. It is also straightforward to see that $\mathrm{cogen}(M)$ consists of the modules N with α_N a monomorphism.

If we now consider a faithfully balanced Λ-module M, $\Gamma = \mathrm{End}_\Lambda(M)$ and $\mathrm{Im}(-, M)$ the essential image of the functor $(-, M)$, then we have

$$\mathrm{cogen}^1({}_\Gamma M) \subseteq \mathrm{Im}(-, M) \subseteq \mathrm{cogen}({}_\Gamma M).$$

Let $\mathrm{Im}(-, M)_\oplus$ be the full subcategory of Γ-mod whose objects are summands of modules in $\mathrm{Im}(-, M)$. Then it is easy to see from the previous proof that $\mathrm{Im}(-, M)_\oplus$ consists of those modules N such that α_N is a split monomorphism.

If ${}_\Lambda M$ is a cogenerator, then $\mathrm{Im}(-, M) = \mathrm{cogen}^1({}_\Gamma M)$ and in particular $\mathrm{Im}(-, M)$ is closed under summands in this case.

Corollary 2.2.6. *Let $k \geq 1$. Let $M \in \Lambda\text{-mod}$ be faithfully balanced and assume $\mathrm{id}\,{}_\Gamma M \leq k-1$, then we have*

$$\mathrm{cogen}^k(M) = \mathrm{cogen}^{k+1}(M) = \cdots = \mathrm{cogen}^\infty(M).$$

Corollary 2.2.7. *Let $k \geq 1$ and M be a faithfully balanced Λ-module and $\mathrm{Ext}^i_\Lambda(M, M) = 0$ for $1 \leq i \leq k-1$. Then we have*

(1) The functors $(-, {}_\Lambda M), (-, {}_\Gamma M)$ restrict to a duality

$$\{M' \in \mathrm{add}({}_\Lambda M) \mid \mathrm{pd}\, M' \leq k\} \longleftrightarrow \{P \in \mathrm{add}(\Gamma) \mid \Omega_M^{-(k+1)} P = 0\}.$$

(2) The functors $\mathrm{D}({}_\Lambda M, -), \mathrm{D}({}_\Gamma M, -)$ restrict to a duality

$$\{M' \in \mathrm{add}({}_\Lambda M) \mid \mathrm{id}\, M' \leq k\} \longleftrightarrow \{J \in \mathrm{add}(\mathrm{D}\,\Gamma) \mid \Omega_M^{(k+1)} J = 0\}.$$

Proof. It is straightforward to check that the duality from Lemma 2.2.4 restricts to these equivalences. □

A result of Morita

We recall a result of Morita which is very useful in constructing new faithfully balanced modules from old ones.

Lemma 2.2.8. *Let $M, X \in \Lambda\text{-mod}$. Assume M is faithful and X is indecomposable. If $M \oplus X$ is (faithfully) balanced, then we have either $X \in \mathrm{gen}(M)$ or $X \in \mathrm{cogen}(M)$.*

Proof. Let $E = \mathrm{End}_\Lambda(X)$, then E is a local ring and hence there is a unique simple E-module, say S. Define

$$X_1 = \sum_{f: M \to X} \mathrm{im}(f) \text{ and } X_0 = \bigcap_{g: X \to M} \ker(g).$$

Then X_1 and X_0 are left-Λ-left-E-subbimodules of X. By definition, we have $X \in \mathrm{gen}(M)$ if and only if $X_1 = X$ and $X \in \mathrm{cogen}(M)$ if and only if $X_0 = 0$. Now assume $X_1 \neq X$ and $X_0 \neq 0$. Then $X/X_1 \neq 0$ and hence has S as a quotient. This implies $\mathrm{Hom}_E(X/X_1, X_0) \neq 0$. Thus there exists a non-zero E-endomorphism $\theta : X \to X$ such that $X_1 \subseteq \ker(\theta)$ and $\mathrm{im}(\theta) \subseteq X_0$. Let $\Gamma = \mathrm{End}_\Lambda(M)$, then we have

$$\mathrm{End}_\Lambda(M \oplus X) = \begin{pmatrix} \Gamma & \mathrm{Hom}_\Lambda(X, M) \\ \mathrm{Hom}_\Lambda(M, X) & E \end{pmatrix},$$

and $M \oplus X$ is a left Λ-left $\mathrm{End}_\Lambda(M\oplus X)$-bimodule. We claim that $\begin{pmatrix} 0 & 0 \\ 0 & \theta \end{pmatrix}$ is an $\mathrm{End}_\Lambda(M\oplus X)$-endomorphism of $M \oplus X$, that is, for any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{End}_\Lambda(M \oplus X)$ we have

$$\begin{pmatrix} 0 & 0 \\ 0 & \theta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \theta \end{pmatrix}.$$

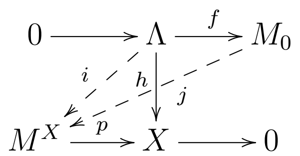
To prove the claim we need to show $\theta c = 0$, $b\theta = 0$ and $\theta d = d\theta$. Now $\mathrm{im}(c) \subseteq X_1 \subseteq \ker(\theta)$ gives $\theta c = 0$, $\mathrm{im}(\theta) \subseteq X_0$ gives $b\theta = 0$ and the fact that θ is an E-endomorphism gives $\theta d = d\theta$. By assumption, $M \oplus X$ is balanced and this implies that the action of $\begin{pmatrix} 0 & 0 \\ 0 & \theta \end{pmatrix}$ is given by the multiplication of some element $\lambda \in \Lambda$. Now we must have $\lambda M = 0$ which forces $\lambda = 0$ since M is faithful as a Λ-module. Thus we have $\theta = 0$, a contradiction. □

Lemma 2.2.9. *Let M be faithfully balanced. If either $X \in \mathrm{gen}(M)$ or $X \in \mathrm{cogen}(M)$, then $M \oplus X$ is also faithfully balanced.*

Proof. We will prove the case $X \in \mathrm{gen}(M)$ and the proof of the case $X \in \mathrm{cogen}(M)$ is dual. Since M is faithfully balanced, there is an exact sequence

$$0 \to \Lambda \xrightarrow{f} M_0 \xrightarrow{g} M_1$$

such that f and $\mathrm{coker}(f) \to M_1$ are minimal left $\mathrm{add}(M)$-approximations. We claim that the map f is also a left $\mathrm{add}(M \oplus X)$-approximation. To this end, it is enough to show that any map $h : \Lambda \to X$ factors through f. Consider the following diagram



where p is the minimal right $\mathrm{add}(M)$-approximation of X. Since $X \in \mathrm{gen}(M)$, p is an epimorphism and so there is an $i : \Lambda \to M^X$ such that $h = pi$. Then i factors as $i = jf$ and we have $h = pi = (pj)f$. This proves the claim. Now since $\mathrm{coker}(f) \in \mathrm{cogen}(M) \subseteq \mathrm{cogen}(M \oplus X)$ we conclude that $\Lambda \in \mathrm{cogen}^1(M \oplus X)$. This proves $M \oplus X$ is faithfully balanced. □

As an immediate consequence of Lemma 2.2.8 and Lemma 2.2.9 we have the following

Corollary 2.2.10. *[Mor58a, Theorem 1.1]*
Let $M \in \Lambda$-mod be faithfully balanced and $X \in \Lambda$-mod indecomposable. Then the following are equivalent:

(1) $M \oplus X$ is faithfully balanced;

(2) $X \in \mathrm{gen}(M)$ or $X \in \mathrm{cogen}(M)$.

In particular, $M \oplus P \oplus I$ is faithfully balanced for every projective module P and injective module I.

2.3 Dualizing summands and the Auslander-Solberg assignment

Dualizing summands

Auslander and Solberg introduced (in [AS93d, section 2]) the following notion.

Definition 2.3.1. Let $M, L \in \Lambda$-mod and assume M is a summand of L. We say M is a *dualizing summand* of L if $L \in \operatorname{cogen}^1(M)$. For $k \geq 0$, we say M is a k-dualizing summand if $L \in \operatorname{cogen}^k(M)$. Thus a dualizing summand of L is the same as a 1-dualizing summand of L.

By using the duality from Lemma 2.2.4 it is easy to find modules having a given faithfully balanced module as a dualizing summand.

Corollary 2.3.2. *Let M be a faithfully balanced Λ-module and $\Gamma = \operatorname{End}(M)$. Then the assignments $G \mapsto (G, M), L \mapsto (L, M)$ give inverse bijections between*

(1) isomorphism classes of ${}_\Lambda G \in \operatorname{cogen}^1(M)$ with $\Lambda \in \operatorname{add}(G)$, and

(2) isomorphism classes of modules $L \in \Gamma$-mod having ${}_\Gamma M$ as a dualizing summand.

Lemma 2.3.3. *Let $M, L \in \Lambda$-mod, $\Gamma = \operatorname{End}_\Lambda(M)$ and assume M is a summand of L. Then M is a dualizing summand of L if and only if $\operatorname{cogen}^1(L) = \operatorname{cogen}^1(M)$.*

Proof. The "if" part is obvious. For the "only if" part, assume M is a dualizing summand of L and $X \in \operatorname{cogen}^1(L)$. Then there exists an exact sequence

$$0 \to X \to L_0^X \to L_1^X$$

with $L_i^X \in \operatorname{add}(L)$ and $(-, L)$ exact on it. We apply $(-, {}_\Lambda M)$ to it and the resulting complex remains exact, since $M \in \operatorname{add}(L)$. Now apply $(-, {}_\Gamma M)$ to see $X \cong ((X, M), M)$. This proves $\operatorname{cogen}^1(L) \subseteq \operatorname{cogen}^1(M)$. To prove $\operatorname{cogen}^1(M) \subseteq \operatorname{cogen}^1(L)$, take any $Y \in \operatorname{cogen}^1(M)$ and take the minimal left $\operatorname{add}(L)$-approximations $f : Y \to L_0^Y$ and $\operatorname{coker} f \to L_1^Y$. Then we get a complex

$$0 \to Y \xrightarrow{f} L_0^Y \to L_1^Y.$$

We need to show it is exact. By construction, we will obtain an exact sequence

$$(L_1^Y, M) \to (L_0^Y, M) \to (Y, M) \to 0$$

after applying $(-, {}_\Lambda M)$. Now apply $(-, {}_\Gamma M)$ to yield an exact sequence

$$0 \to ((Y, M), M) \to ((L_0^Y, M), M) \to ((L_1^Y, M), M)$$

which is naturally isomorphic to the complex $0 \to Y \xrightarrow{f} L_0^Y \to L_1^Y$, as desired. □

We observe the following

Lemma 2.3.4. *Let $M, N \in \Lambda$-mod and $L = M \oplus N$. For $k \geq 1$, if $N \in \operatorname{cogen}^k(M)$ (i.e., M is a k-dualizing summand of L), then M is faithfully balanced if and only if L is faithfully balanced. In this case we have $\operatorname{cogen}^k(M) = \operatorname{cogen}^k(L)$. Furthermore, if additionally $\operatorname{copres}^k(L) = \operatorname{cogen}^k(L)$, then we also have $\operatorname{copres}^k(M) = \operatorname{cogen}^k(M)$.*

Proof. According to Lemma 2.3.3, we may assume $k > 1$. Since $L \in \text{cogen}^k(M) \subseteq \text{cogen}^1(M)$, it follows from Lemma 2.3.3 that $\text{cogen}^1(L) = \text{cogen}^1(M)$ and hence M is faithfully balanced if and only if L is faithfully balanced.

Let us from now on assume that M, L are faithfully balanced. We want to see that $\text{cogen}^k(L) = \text{cogen}^k(M)$. Let $\Gamma = \text{End}_\Lambda(M)$. Since $L \in \text{cogen}^1(M)$ we can find a generator $G \in \Gamma\text{-mod}$ such that $L = (G, M)$ by Corollary 2.3.2. We observe that $L \in \text{cogen}^k(M)$ implies $\text{Ext}^i_\Gamma((L, M), M) = \text{Ext}^i_\Gamma(G, M) = 0$ for $1 \leq i \leq k-1$. In other words, ${}_\Gamma M \in \bigcap_{i=1}^{k-1} \ker \text{Ext}^i_\Gamma(G, -)$. But since G is a generator we have that $\text{gen}_{k+1}(G) = \Gamma\text{-mod}$. We set $B = \text{End}_\Lambda(L) \cong \text{End}_\Gamma(G)^{op}$ and take $X \in \text{cogen}^k({}_\Lambda M)$. Now, observe

$$(X, L) \cong (((X, M), M), (G, M)) \cong (G, (X, M))$$

is an isomorphism of left B-modules. The dual statement in Lemma 2.1.3 (2) gives that we have natural isomorphisms

$$\text{Ext}^i_\Gamma((X, M), M) \to \text{Ext}^i_B((G, (X, M)), (G, M)) \cong \text{Ext}^i_B((X, L), L)$$

for $1 \leq i \leq k-1$. This implies by Lemma 2.1.1 that $\text{cogen}^k(M) = \text{cogen}^k(L)$. Furthermore, since $\text{cogen}^k(M) \subseteq \text{copres}^k(M) \subseteq \text{copres}^k(L)$ are always fulfilled, an equality $\text{cogen}^k(M) = \text{copres}^k(L)$ implies they are all equal. □

Example 2.3.5. Let H be a cogenerator, then every summand of H of the form $\text{D}\,\Lambda \oplus X$ is a k-dualizing summand for every $k \geq 0$.

The Auslander-Solberg assignment

Now we look at triples (Λ, M, G) where Λ is a finite-dimensional algebra and M and G are finite-dimensional left Λ-modules. We define the following equivalence relation between these triples: (Λ, M, G) is equivalent to (Λ', M', G') if there is a Morita equivalence $\Lambda\text{-mod} \to \Lambda'\text{-mod}$ restricting to equivalences $\text{add}(M) \to \text{add}(M')$ and $\text{add}(G) \to \text{add}(G')$. We denote by $[\Lambda, M, G]$ the equivalence class of a triple.

Definition 2.3.6. We consider the following assignment

$$[\Lambda, M, G] \mapsto [\Gamma, N, L]$$

with $\Gamma = \text{End}(M)$, $N = {}_\Gamma M$, $L = (G, M)$ and call this the Auslander-Solberg assignment. There is a dual assignment

$$[\Lambda, M, H] \mapsto [\Gamma, N, R]$$

with Γ, N as before and $R = \text{D}(M, H)$ which we call the dual Auslander-Solberg assignment.

From Corollary 2.3.2 we see that the Auslander-Solberg assignment gives a one-to-one correspondence between the following

(1) $[\Lambda, M, G]$ with $\Lambda \in \text{add}(G)$, $G \in \text{cogen}^1(M)$,

(2) $[\Gamma, N, L]$ with $N \in \text{add}(L)$, $\Gamma \oplus L \in \text{cogen}^1(N)$.

The previous bijection has an obvious dual version using the dual Auslander-Solberg assignment and gen, H and R instead of cogen, G and L, respectively.

We are going to refine this assignment, our first refinement needs the following notation.

Definition 2.3.7. We denote for Λ-modules M and X by $\Omega_M X$ the kernel of the minimal right $\operatorname{add}(M)$-approximation $M_X \to X$. For $k \geq 1$ we define inductively:

$$\Omega_M^k X := \begin{cases} \Omega_M X & \text{if } k = 1, \\ \Omega_M(\Omega_M^{k-1} X) & \text{if } k \geq 2. \end{cases}$$

Dually, we define $\Omega_M^- X$ as the cokernel of a minimal left $\operatorname{add}(M)$-approximation $X \to M^X$ and $\Omega_M^{-k} X$ inductively.

Definition 2.3.8. Let k be a non-negative integer and $L, M, R \in \Lambda\text{-mod}$. An exact sequence

$$0 \to L \to M_0 \to M_1 \to \cdots \to M_k \to R \to 0$$

is called a k-$\operatorname{add}(M)$-dualizing sequence from L to R if

(i) $M_i \in \operatorname{add}(M)$ for $i \in \{0, \ldots, k\}$,

(ii) the functors $(-, M)$ and $\mathrm{D}(M, -)$ are exact on it,

(iii) $\operatorname{add}(R) = \operatorname{add}(M \oplus \Omega_M^{-(k+1)} L)$ and $\operatorname{add}(L) = \operatorname{add}(M \oplus \Omega_M^{k+1} R)$.

In this case we say L is the left end and R is the right end of this k-$\operatorname{add}(M)$-dualizing sequence.

This has the following consequences for the ideal quotient categories $\operatorname{add}(L)/\operatorname{add}(M)$ and $\operatorname{add}(R)/\operatorname{add}(M)$:

Lemma 2.3.9. *Let $0 \to L \to M_0 \to M_1 \to \cdots \to M_k \to R \to 0$ be a k-$\operatorname{add}(M)$-dualizing sequence from L to R for some $k \geq 0$ in Λ-mod. Then we have an equivalence*

$$\Omega_M^{-(k+1)} : \operatorname{add}(L)/\operatorname{add}(M) \longleftrightarrow \operatorname{add}(R)/\operatorname{add}(M) : \Omega_M^{k+1}.$$

Proof. We claim that given a short exact sequence $\eta : 0 \to U \xrightarrow{f} M_0 \xrightarrow{g} V \to 0$ with $M_0 \in \operatorname{add}(M)$ and such that the functors $(-, M)$ and $(M, -)$ are exact on it, then we have an equivalence

$$\Omega_M^{-1} : \operatorname{add}(U)/\operatorname{add}(M) \leftrightarrow \operatorname{add}(V)/\operatorname{add}(M) : \Omega_M^1.$$

Take a map $\alpha : X \to Y$ in $\operatorname{add}(U)$ and consider the following commutative diagram

$$\begin{array}{rccccccccc} \eta^X : & 0 & \longrightarrow & X & \xrightarrow{f^X} & M_0^X & \xrightarrow{g^X} & \Omega_M^{-1} X & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ \eta^Y : & 0 & \longrightarrow & Y & \xrightarrow{f^Y} & M_0^Y & \xrightarrow{g^Y} & \Omega_M^{-1} Y & \longrightarrow & 0 \end{array}$$

where η^X and η^Y are both direct summands of η by our assumption. In particular, we have $\Omega_M \Omega_M^{-1} X \cong X$ and $\Omega_M \Omega_M^{-1} Y \cong Y$ in $\operatorname{add}(U)/\operatorname{add}(M)$. Assume there is another map $\beta' : M_0^X \to M_0^Y$ such that $\beta' f^X = f^Y \alpha$ and denote by γ' the induced map on cokernels. Then we have $(\beta - \beta') f^X = 0$ and thus there exists a unique $\theta : \Omega_M^{-1} X \to M_0^Y$ such that $\beta - \beta' = \theta g^X$. It follows that $\gamma - \gamma' = g^Y \theta$. Now assume α factors as $\alpha = \alpha_2 \alpha_1$ though an object $M' \in \operatorname{add}(M)$, then since f^X is a left $\operatorname{add}(M)$-approximation there is a map $\phi : M_0^X \to M'$ such that $\alpha_1 = \phi f^X$. Thus we have $\alpha = \alpha_2 \alpha_1 = (\alpha_2 \phi) f^X$, and a diagram chasing gives a map $\psi : \Omega_M^{-1} X \to M_0^Y$ such that $\gamma = g^Y \psi$. These proves that the map

$$\operatorname{Hom}_{\operatorname{add}(U)/\operatorname{add}(M)}(X, Y) \to \operatorname{Hom}_{\operatorname{add}(V)/\operatorname{add}(M)}(\Omega_M^{-1} X, \Omega_M^{-1} Y), \alpha \mapsto \gamma$$

is well defined. Similarly, we have a map

$$\mathrm{Hom}_{\mathrm{add}(V)/\,\mathrm{add}(M)}(\Omega_M^{-1}X, \Omega_M^{-1}Y) \to \mathrm{Hom}_{\mathrm{add}(U)/\,\mathrm{add}(M)}(X, Y), \gamma \mapsto \alpha.$$

Clearly, these two maps are mutually inverse and this proves the claim. Now the lemma follows by induction on k. □

Definition 2.3.10. Let $X \in \Lambda$-mod and $k \geq 1$ be an integer. We define

$$\tau_k X = \tau(\Omega_\Lambda^{k-1}X) \quad \text{and} \quad \tau_k^- X = \tau^-(\Omega_\Lambda^{-(k-1)}X).$$

We occasionally use the conventions

$$X^{\perp_{1\sim k}} := \bigcap_{i=1}^{k} \ker \mathrm{Ext}^i(X, -) \quad \text{and} \quad {}^{1\sim k\perp}X := \bigcap_{i=1}^{k} \ker \mathrm{Ext}^i(-, X).$$

Lemma 2.3.11. *Let M be a faithfully balanced Λ-module and $\Gamma = \mathrm{End}_\Lambda(M)$. Then, for $k \geq 1$, the assignment $X, Y \mapsto (X, M), \mathrm{D}(M, Y)$ gives a self-inverse bijection (up to seeing X, Y as Λ or as Γ-modules) between the following sets of pairs of Λ-modules and Γ-modules*

$$\{{}_\Lambda G, {}_\Lambda H \mid \begin{array}{c} G = \tau_k^- H \oplus \Lambda \in \mathrm{cogen}^1(M) \cap {}^{1\sim(k-1)\perp}M \\ H = \tau_k G \oplus \mathrm{D}\,\Lambda \in \mathrm{gen}_1(M) \cap M^{\perp_{1\sim(k-1)}} \end{array}\}$$

and

$$\{{}_\Gamma L, {}_\Gamma R \mid \exists \;\; a\ k\text{-}\ {}_\Gamma M\text{-dualizing sequence from } L \text{ to } R\}.$$

If all modules are basic, we have $\mathrm{D}(M, \tau_k G) = \Omega_M^{-(k+1)}(G, M)$.

Proof. The bijection follows from Lemma 2.2.4 and the observation $\nu_\Gamma(M', M) = \mathrm{D}(M, M')$ for every $M' \in \mathrm{add}(M) \subseteq \mathrm{cogen}^1(M)$ from Lemma 2.1.3. The rest statements are obvious. □

Corollary 2.3.12. *Let G, H be as in the bijection of Lemma 2.3.11, then we have an equivalence*

$$\tau_k : \underline{\mathrm{add}(G)} \longleftrightarrow \overline{\mathrm{add}(H)} : \tau_k^-,$$

where $\underline{\mathrm{add}(G)}$ (resp. $\overline{\mathrm{add}(H)}$) denotes the projective (resp. injective) stable category.

Proof. This follows from the equivalence of Lemma 2.3.9 by pre- and postcomposing with $(-, M)$ and $\mathrm{D}(M, -)$ and then use Lemma 2.3.11. □

Example 2.3.13. Obviously triples $[\Lambda, M, M]$ correspond to triples $[\Gamma, N, \Gamma]$ and since M is a generator we conclude that N is a projective Γ-module. The module M is a generator-cogenerator (i.e., $\Lambda \oplus \mathrm{D}\,\Lambda \in \mathrm{add}(M)$) if and only if N is projective-injective. Furthermore, $\mathrm{add}(M) = \mathrm{add}(\tau_k M \oplus \Lambda)$ and M being a generator-cogenerator with $\mathrm{Ext}^i(M, M) = 0$, $1 \leq i \leq k-1$ corresponds to a dualizing sequence $0 \to \Gamma \to N_0 \to \cdots \to N_k \to \mathrm{D}\,\Gamma \to 0$ with N_i projective-injective and $\mathcal{P}(G)$ a projective generator, $\mathcal{I}(\Gamma)$ an injective cogenerator. But this is equivalent to Γ being k-minimal Auslander-Gorenstein which means by definition $\mathrm{id}\,{}_\Gamma\Gamma \leq k+1 \leq \mathrm{domdim}_\Gamma\Gamma$. These algebras have been studied by Iyama and Solberg in [IS18].

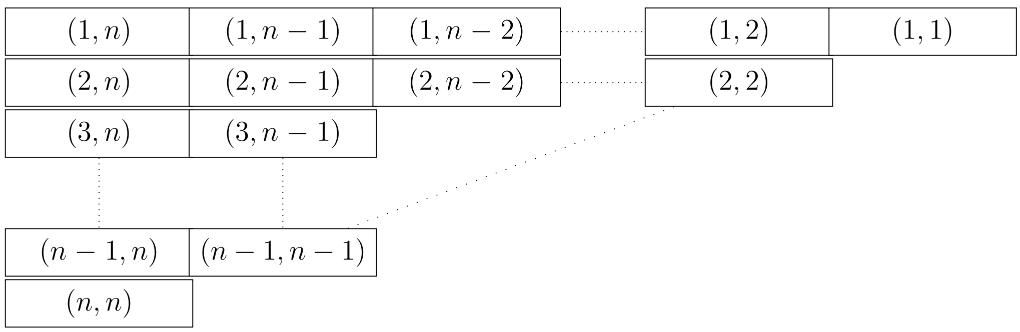
Chapter 3

Faithfully balanced modules for $K(\vec{\mathbb{A}}_n)$

In this chapter, we will study and classify faithfully balanced modules for the algebra $\Lambda_n = K(1 \to 2 \to \cdots \to n)$ which is isomorphic to the $n \times n$ lower triangular matrix algebra over K. This chapter is based on an earlier version of [CBMRS19].

3.1 Combinatorics of faithfully balanced modules

The indecomposable modules for Λ_n are indexed by the set $I_n = \{(i, j) : 1 \le i \le j \le n\}$, which we display as the blocks of a Young diagram



The element (i, j) corresponds to the module M_{ij} with top and socle the simple modules $S[i]$ and $S[j]$. The left hand column is the indecomposable projective modules, the top row is the indecomposable injective modules and the modules M_{ii} are the simple modules $S[i]$. The Auslander-Reiten quiver is the same picture, with irreducible maps going vertically and to the right, and the Auslander-Reiten translation $\tau = \mathrm{D}\,\mathrm{Tr}$ takes each module M_{ij} with $j < n$ to $M_{i+1,j+1}$. By a *leaf* we mean an element of the set $L_n = \{(1, 0), (2, 1), \ldots, (n + 1, n)\}$. We define *cohooks* for $(i, j) \in I_n$ and *virtual cohooks* for $(i, j) \in L_n$ by the formula

$$\mathrm{cohook}(i, j) = \{M_{kj} : 1 \le k < i\} \cup \{M_{i\ell} : n \ge \ell > j\}.$$

Definition 3.1.1. Let M be a finite-dimensional Λ_n-module. We say M fulfills the cohook-condition if the following conditions are fulfilled:

(C1) $M_{1,n} \in \mathrm{add}(M)$;

(C2) $\mathrm{cohook}(i, i - 1) \cap \mathrm{add}(M) \neq \emptyset$, $2 \le i \le n$;

(C3) for every indecomposable $M_{i,j} \in \mathrm{add}(M)$ with $M_{i,j} \neq M_{1,n}$ we have $\mathrm{cohook}(i, j) \cap \mathrm{add}(M) \neq \emptyset$.

It is clear that if a module M satisfies (C1) then $\mathrm{cohook}(i,i-1)\cap\mathrm{add}(M)\neq\emptyset$ for $i=1,n+1$. To characterize faithfully balanced modules for Λ_n we need the following construction of minimal approximations of projectives.

Let M be a basic module with $P_1 = M_{1,n}\in\mathrm{add}(M)$. Fix $i\in\{2,\dots,n\}$, we define a summand M_{J_i} of M as follows. Let J_i be the maximal subset of the set $S=\{(a,b)\mid a\leq i\leq b\}$ fulfilling:

- If $M_{a,b}\in\mathrm{add}(M)$ with $(1,n)\neq(a,b)\in S$, then $J_i\cap\{(x,y)\mid x\leq a, y\leq b\}=\{(a,b)\}$.

By assumption we have $J_i=\{(i_1,j_1),(i_2,j_2),\dots,(i_t,j_t)\}$ with $i\geq i_1>i_2>\cdots>i_{t-1}>i_t\geq 1$. We define

$$M_{J_i}:=\begin{cases}\bigoplus_{(a,b)\in J_i}M_{a,b} & \text{if } j_t=n\\ (\bigoplus_{(a,b)\in J_i}M_{a,b})\oplus M_{1,n} & \text{if } j_t<n.\end{cases}$$

Without loss of generality we may assume $j_t=n$. For every pair (i_s,j_s) we fix a nonzero homomorphism $f_s\colon P_i=M_{i,n}\to M_{i_s,j_s}$ which is the composition of irreducible morphisms in the Auslander-Reiten quiver. Note that f_t is a monomorphism. For f_1 we have an exact sequence

$$M_{i,n}\xrightarrow{f_1}M_{i_1,j_1}\xrightarrow{g_{11}}N_1\to 0$$

and for every successive pair (f_s,f_{s+1}) we have an exact sequence

$$M_{i,n}\xrightarrow{\binom{f_s}{f_{s+1}}}M_{i_s,j_s}\oplus M_{i_{s+1},j_{s+1}}\xrightarrow{(g_{s+1,s}\ g_{s+1,s+1})}N_{s+1}\to 0.$$

These exact sequences give rise to a short exact sequence

$$0\to P_i\xrightarrow{f}M_{J_i}\xrightarrow{g}N(i)\to 0$$

where

$$f=\begin{pmatrix}f_1\\ f_2\\ f_3\\ \vdots\\ f_t\end{pmatrix}\qquad g=\begin{pmatrix}g_{11} & & & & \\ g_{21} & g_{22} & & & \\ & g_{32} & g_{33} & & \\ & & \ddots & \ddots & \\ & & & g_{t,t-1} & g_{t,t}\end{pmatrix}$$

and $N(i)=\bigoplus_{s=1}^t N_s$. Moreover, we have $N_1=M^{\epsilon}_{i_1,i-1}$ with $\epsilon=1-\delta_{i,i_1}$ (i.e., $\epsilon\in\{0,1\}$ with $\epsilon=0$ if and only if $i_1=i$) and $N_s=M_{i_{s+1},j_s}$ for $1<s\leq t$.

Lemma 3.1.2. *The map f is the minimal left $\mathrm{add}(M)$-approximation of P_i.*

Proof. Note that $\mathrm{Hom}(P_i,M_{c,d})=0$ if $c>i$ or $d<i$. For an indecomposable summand $M_{a,b}$ of M, if $M_{a,b}$ is not a summand of M_{J_i} then there exists some $(i_s,j_s)\in J_i$ such that $M_{a,b}$ is a quotient of M_{i_s,j_s} or M_{i_s,j_s} is a submodule of $M_{a,b}$ and hence any homomorphism from P_i to $M_{a,b}$ factors through f_s. This shows that f is a left $\mathrm{add}(M)$-approximation of P_i. Note that the minimal $\mathrm{add}(M)$-approximation must be a summand of M_{J_i}. By our construction, none of the maps f_s factor through another one. Hence f is the minimal $\mathrm{add}(M)$-approximation. □

Theorem 3.1.3. *For a finite dimensional Λ_n-module M the following are equivalent:*

(1) M is faithfully balanced,

(2) M fulfills the cohook condition.

Proof. Assume that M does not fulfill the cohook condition. We show that M is then not faithfully balanced. If $P_1 \notin \operatorname{add}(M)$, then M is not faithfully balanced. If there is an $i \in \{2, 3, \dots, n\}$ with $\operatorname{cohook}(i, i-1) \cap \operatorname{add}(M) = \emptyset$, then in the previous construction $J_i = \{(i_1, j_1), \dots, (i_t, j_t) = (i_t, n)\}$ with $1 \le i_1 < i$ and therefore $M_{i_1,i-1} \in \operatorname{add}(N(i))$. But the assumption implies $M_{i_1,i-1} \notin \operatorname{cogen}(M)$ and therefore $N(i) \notin \operatorname{cogen}(M)$. If there is $P_1 \neq M_{i,j} \in \operatorname{add}(M)$ with $\operatorname{cohook}(i, j) \cap \operatorname{add}(M) = \emptyset$. We can apply the previous construction, we observe $J_i = \{(i, j) = (i_1, j_1), \dots, (i_t, j_t) = (i_t, n)\}$ and therefore $N(i)$ has a summand $M_{i_2,j}$ with $i_2 < i$ but by assumption $M_{i_2,j} \notin \operatorname{cogen}(M)$. This completes the proof of one implication.

Assume now that M fulfills the cohook condition. We need to see that $N(2), \dots, N(n) \in \operatorname{cogen}(M)$. Assume there is $i \in \{2, \dots, n\}$ minimal with $N(i) \notin \operatorname{cogen}(M)$. Let J_ℓ, $2 \le \ell \le n$ be the sets of indices from the previous construction. Let $J_i = \{(i_1, j_1), \dots, (i_t, j_t) = (i_t, n)\}$ with $i \ge i_1 > i_2 > \dots > i_t \ge 1$.

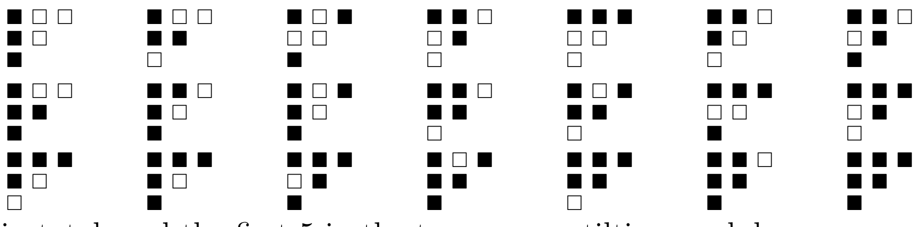
Case 1: Assume $1 < i_1 < i$, then we have $J_i \subset J_{i-1}$ and $M_{i_{s+1},j_s} \in \operatorname{add} N(i-1) \subset \operatorname{cogen}(M)$, $1 \le s \le t-1$. So $N(i) \notin \operatorname{cogen}(M)$ implies $M_{i_1,i-1} \notin \operatorname{cogen}(M)$. Since we have $\operatorname{cohook}(i, i-1) \cap \operatorname{add}(M) \neq \emptyset$, we conclude that $i_1 \neq i-1$ and that there is an $M_{x,i-1} \in \operatorname{add}(M)$ with $i_1 < x \le i-1$. Now, by definition we have $(x, i-1), (i_1, j_1)$ are the first two entries in J_x, this implies $M_{i_1,i-1} \in \operatorname{add} N(x) \subset \operatorname{cogen}(M)$ by assumption since $x < i$.

Case 2: $i_1 = 1$. In this case we have $J_i = \{(1, n)\}$ and $N(i) = M_{1,i-1}$. By assumption $\operatorname{cohook}(i, i-1) \cap \operatorname{add}(M) \neq \emptyset$ and $J_i = \{(1, n)\}$ we conclude that there is an $1 \le x \le i-1$ with $M_{x,i-1} \in \operatorname{add}(M)$. Since we assume $N(i) = M_{1,i-1} \notin \operatorname{add}(M)$ we deduce $x > 1$. Now we have $J_x = \{(x, i-1), (1, n)\}$ and therefore $N(x) = M_{1,i-1}$ which is in $\operatorname{cogen}(M)$ by assumption.

Case 3: $i_1 = i$. In this case $(i_2, j_2), \dots, (i_t, j_t) = (i_t, n)$ are in J_{i-1} and $M_{i_{s+1},j_s} \in \operatorname{add} N(i-1)$ for $2 \le s \le t-1$. So $N(i) \notin \operatorname{cogen}(M)$ implies $M_{i_2,j_1} \notin \operatorname{cogen}(M)$. But $\operatorname{cohook}(i_1, j_1) \cap \operatorname{add}(M) \neq \emptyset$ implies that there is $i_2 < x < i_1$ with $M_{x,j_1} \in \operatorname{add}(M)$. Now again, $(x, j_1), (i_2, j_2)$ are the first two entries in J_x and therefore $M_{i_2,j_1} \in \operatorname{add} N(x) \subset \operatorname{cogen}(M)$ by assumption since $x < i$.

□

Example 3.1.4. The following is a complete list of basic faithfully balanced modules for Λ_3, where each small Young diagram represents a faithfully balanced module with indecomposable summands the black boxes ■.



There are 21 in total, and the first 5 in the top row are tilting modules.

3.2 Counting faithfully balanced modules

Let M be a Λ_n-module, we denote by $|M|$ the number of nonisomorphic indecomposable summands of M. Consider the subsets

$$P_{n,i} := \Big\{ M \in \Lambda_n\text{-}\operatorname{mod} \mid M \text{ is basic with } |M| = i \text{ and satisfies (C1) and (C3)} \Big\}$$

$$K_{n,i} := \Big\{ M \in \Lambda_n\text{-}\operatorname{mod} \mid M \text{ is basic with } |M| = i \text{ and satisfies the cohook condition} \Big\}$$

and define $p_{n,i} = \#P_{n,i}$, $k_{n,i} = \#K_{n,i}$, $p_n(X) = \sum_{i\geq 1} p_{n,i}X^i$, and $k_n(X) = \sum_{i\geq 1} k_{n,i}X^i$. We also consider the subsets

$$P_{n,i}^{[-r]} := \left\{ M \in \Lambda_n\text{- mod} \;\middle|\; \begin{array}{l} M \text{ is basic with } |M| = i,\ M \text{ satisfies (C1) and (C3)} \\ \text{but fails (C2) for some fixed r leaves} \end{array} \right\}$$

$$P_{n,i}^{[r]} := \left\{ M \in \Lambda_n\text{- mod} \;\middle|\; \begin{array}{l} M \text{ is basic with } |M| = i,\ M \text{ satisfies (C1) and (C3)} \\ \text{and also satisfies (C2) for some fixed } r \text{ leaves} \end{array} \right\}$$

and define $p_{n,i}^{[-r]} = \#P_{n,i}^{[-r]}$ and $p_{n,i}^{[r]} = \#P_{n,i}^{[r]}$. If $M \in P_{n,i}^{[-r]}$, by removing the given r virtual cohooks M can be viewed as a Λ_{n-r}-module which satisfies (C1) and (C3). This shows $p_{n,i}^{[-r]} = p_{n-r,i}$. By using the inclusion-exclusion priciple we conclude

$$p_{n,i}^{[r]} = \sum_{j=0}^{r}(-1)^j\binom{r}{j}p_{n,i}^{[-j]} = \sum_{j=0}^{r}(-1)^j\binom{r}{j}p_{n-j,i}.$$

Lemma 3.2.1. *For $n \geq 1$ we have*

(1) $k_n(X) = \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}p_{n-j}(X)$;

(2) $p_n(X) = \sum_{j=0}^{n-1}\binom{n-1}{j}k_{n-j}(X)$.

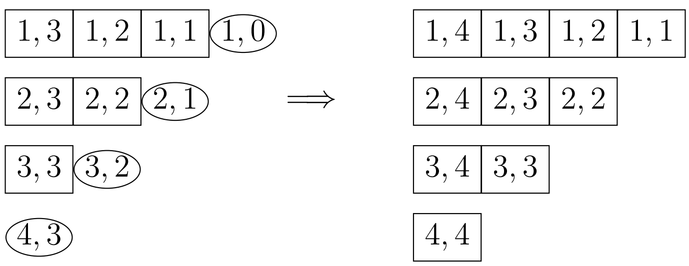
In particular, we have $k_1(X) = X = p_1(X)$.

Proof. By the definition of $k_{n,i}$ we have $k_{n,i} = p_{n,i}^{[n-1]} = \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}p_{n-j,i}$. So

$$\begin{aligned} k_n(X) &= \sum_{i\geq 1}\Big(\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}p_{n-j,i}\Big)X^i \\ &= \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}p_{n-j}(X). \end{aligned}$$

This proves (1), and (2) follows from (1) and the fact $k_1(X) = p_1(X)$. □

The AR-quiver of Λ_{n+1} can be obtained by extending the AR-quiver of Λ_n with leaves by sending each module $M_{i,j}$ (including the leaves) to a module $M_{i,j+1} \in \Lambda_{n+1}$-mod. As before, we don't display the arrows. The following example shows how to get the AR-quiver of Λ_4 from the AR-quiver of Λ_3 with leaves.



Lemma 3.2.2. *For $n \geq 1$ we have*

$$p_{n+1}(X) = (1+X)^2\sum_{r=0}^{n-1}\binom{n-1}{r}\Big(\sum_{j=0}^{r}(-1)^j\binom{r}{j}p_{n-j}(X)X^r\Big).$$

Proof. By definition $p_{n+1}(X) = \sum_{i\geq 1} p_{n+1,i}X^i$. Note that for each i, $P_{n+1,i}$ can be written as a disjoint union

$$P_{n+1,i} = P^{(a)}_{n+1,i} \sqcup P^{(b)}_{n+1,i} \sqcup P^{(c)}_{n+1,i} \sqcup P^{(d)}_{n+1,i}$$

where

$$\begin{aligned}
P^{(a)}_{n+1,i} &= \{M \in P_{n+1,i} \mid M_{n+1,n+1} \notin \operatorname{add}(M) \text{ and } M_{1,1} \notin \operatorname{add}(M)\}\\
P^{(b)}_{n+1,i} &= \{M \in P_{n+1,i} \mid M_{n+1,n+1} \notin \operatorname{add}(M) \text{ but } M_{1,1} \in \operatorname{add}(M)\}\\
P^{(c)}_{n+1,i} &= \{M \in P_{n+1,i} \mid M_{n+1,n+1} \in \operatorname{add}(M) \text{ but } M_{1,1} \notin \operatorname{add}(M)\}\\
P^{(d)}_{n+1,i} &= \{M \in P_{n+1,i} \mid M_{n+1,n+1} \in \operatorname{add}(M) \text{ and } M_{1,1} \in \operatorname{add}(M)\}.
\end{aligned}$$

Define $p^{(a)}_{n+1,i} = \#P^{(a)}_{n+1,i}$, $p^{(b)}_{n+1,i} = \#P^{(b)}_{n+1,i}$, $p^{(c)}_{n+1,i} = \#P^{(c)}_{n+1,i}$ and $p^{(d)}_{n+1,i} = \#P^{(d)}_{n+1,i}$. Then it is clear that $p^{(b)}_{n+1,i} = p^{(c)}_{n+1,i}$ and $p^{(a)}_{n+1,i} = p^{(b)}_{n+1,i+1} = p^{(c)}_{n+1,i+1} = p^{(d)}_{n+1,i+2}$ for all $i \geq 1$. Moreover, we have

$$p^{(a)}_{n+1,i} = \sum_{r=0}^{n-1} \binom{n-1}{r} \left(\sum_{j=0}^{r} (-1)^j \binom{r}{j} p^{[-j]}_{n,i-r}\right) = \sum_{r=0}^{n-1} \binom{n-1}{r} \left(\sum_{j=0}^{r} (-1)^j \binom{r}{j} p_{n-j,i-r}\right).$$

Hence

$$\begin{aligned}
p_{n+1}(X) &= \sum_{i\geq 1} p^{(a)}_{n+1,i}X^i + \sum_{i\geq 2} p^{(b)}_{n+1,i}X^i + \sum_{i\geq 2} p^{(c)}_{n+1,i}X^i + \sum_{i\geq 3} p^{(d)}_{n+1,i}X^i\\
&= \sum_{i\geq 1} p^{(a)}_{n+1,i}X^i + \sum_{i\geq 1} p^{(b)}_{n+1,i+1}X^iX + \sum_{i\geq 1} p^{(c)}_{n+1,i+1}X^iX + \sum_{i\geq 1} p^{(d)}_{n+1,i+2}X^iX^2\\
&= (1+X)^2 \left(\sum_{i\geq 1} p^{(a)}_{n+1,i}X^i\right)\\
&= (1+X)^2 \sum_{r=0}^{n-1} \binom{n-1}{r} \left(\sum_{j=0}^{r} (-1)^j \binom{r}{j} p_{n-j}(X)X^r\right).
\end{aligned}$$

□

Theorem 3.2.3. *We have*

$$k_n(X) = \prod_{j=1}^{n} ((1+X)^j - 1)$$

for every $n \geq 1$.

Proof. Let $h_n(X) = \prod_{j=1}^{n}((1+X)^j - 1)$, then $k_1(X) = h_1(X)$. Assume $k_m(X) = h_m(X)$ holds for all $m \le n$. Then we need to show $k_{n+1}(X) = h_{n+1}(X)$. Indeed, we have

$$
\begin{aligned}
k_{n+1}(X) &= p_{n+1}(X) + \sum_{j=1}^{n} (-1)^j \binom{n}{j} p_{n+1-j}(X) && \text{(by Lemma 3.2.1 (1))}\\
&= p_{n+1}(X) - \sum_{l=0}^{n-1} (-1)^l \binom{n}{l+1} p_{n-l}(X) && \text{(by setting } l = j-1\text{)}\\
&= (1+X)^2 \sum_{r=0}^{n-1} \binom{n-1}{r} X^r (\sum_{l=0}^{r} (-1)^l \binom{r}{l} p_{n-l}(X)) && \text{(by Lemma 3.2.2)}\\
&\quad - \sum_{l=0}^{n-1} (-1)^l \binom{n}{l+1} p_{n-l}(X)\\
&= \sum_{l=0}^{n-1} (-1)^l ((1+X)^2 \sum_{r=0}^{n-1} \binom{n-1}{r} X^r - \binom{n}{l+1}) && \text{(by Lemma 3.2.1 (2))}\\
&\quad \cdot (\sum_{s=0}^{n-1-l} \binom{n-1-l}{s} k_{n-l-s}(X))\\
&= \sum_{t=0}^{n-1} \sum_{l=0}^{t} (-1)^l (1+X)^2 (\sum_{r=l}^{n-1} \binom{r}{l} \binom{n-1}{r} X^r) \binom{n-1-l}{t-l} k_{n-t}(X) && \text{(by setting } t = s+l\text{)}\\
&\quad - \sum_{t=0}^{n-1} \sum_{l=0}^{t} (-1)^l \binom{n}{l+1} \binom{n-1-l}{t-l} k_{n-t}(X).
\end{aligned}
$$

Note that we have the following identities

$$
\begin{aligned}
\sum_{r=l}^{n-1} \binom{r}{l} \binom{n-1}{r} X^r &= \binom{n-1}{l} X^l (1+X)^{n-1-l}\\
\sum_{l=0}^{t} (-1)^l (1+X)^2 (\binom{n-1}{l} X^l (1+X)^{n-1-l}) \binom{n-1-l}{s} &= \binom{n-1}{t} (1+X)^{n+1-t}\\
\sum_{l=0}^{t} (-1)^l \binom{n}{l+1} \binom{n-1-l}{s} &= \binom{n}{t+1}.
\end{aligned}
$$

So we can continue to write $k_{n+1}(X)$ as

$$
\begin{aligned}
&\sum_{t=0}^{n-1} (\binom{n-1}{t} (1+X)^{n+1-t} - \binom{n}{t+1}) h_{n-t}(X)\\
&= \sum_{t=0}^{n-1} (\binom{n-1}{t} (1+X)^{n+1-t} - \binom{n-1}{t}) h_{n-t}(X) - \sum_{t=0}^{n-2} \binom{n-1}{t+1} h_{n-t}(X)\\
&= \sum_{t=0}^{n-1} \binom{n-1}{t} h_{n+1-t}(X) - \sum_{t=0}^{n-2} \binom{n-1}{t+1} h_{n-t}(X)\\
&= h_{n+1}(X)
\end{aligned}
$$

as desired. □

We denote by $fb(\Lambda_n)$ the set of isomorphism classes of basic faithfully balanced modules, and by $fb(\Lambda_n)_{\leq q}$ the set of isomorphism classes of faithfully balanced modules with each indecomposable summand has a multiplicity of at most q.

Corollary 3.2.4. *For Λ_n we have the following.*

(1) The number of basic faithfully balanced modules is $\prod_{i=1}^{n}(2^i - 1)$.

(2) $|fb(\Lambda_n)_{\leq q}| = \sum_{i\geq 1} k_{n,i}q^i = \prod_{i=1}^{n}((q+1)^i - 1)$.

(3) $k_{n,i} = \sum_{(j_1,j_2,\dots,j_n):\, 1\leq j_r\leq r, \sum_{r=1}^{n} j_r = i} \binom{1}{j_1}\binom{2}{j_2}\cdots\binom{n}{j_n}$.

(4) Any basic faithfully balanced module for Λ_n has at least n indecomposable summands, and the number of basic faithfully balanced modules with n indecomposable summands is $n!$.

(5) The direct sum of all indecomposable modules is a faithfully balanced module with $N = n(n+1)/2$ indecomposable summands; there are $N-1$ basic faithfully balanced modules with $N-1$ summands.

3.3 Poset structures

Let Λ be any finite-dimensional algebra. We say that a Λ-module M is gen_1-*critical* if any proper summand N of M has $\text{gen}_1(N) \neq \text{gen}_1(M)$; similarly for cogen^1-critical. We say that a module is *minimal faithfully balanced* if it is faithfully balanced and any proper direct summand is not faithfully balanced. Clearly any minimal faithfully balanced module is gen_1- and cogen^1-critical.

Proposition 3.3.1. *If T is a 1-tilting module, i.e. T has projective dimension ≤ 1, then T is gen_1-critical. If in addition Λ is hereditary, then T is minimal faithfully balanced.*

Proof. For the first part of the theorem, we prove a stronger result that every basic rigid module T with $\text{pd}\, T \leq 1$ is is gen_1-critical. Assume $T = M \oplus N$ and $\text{gen}_1(M) = \text{gen}_1(T)$. Then we have $N \in \text{gen}_1(M)$ and so there is an exact sequence $M_1 \to M_0 \to N \to 0$ with $M_0, M_1 \in \text{add}(M)$ and $\text{Hom}(M,-)$ exact on it. Thus we obtain two short exact sequences

$$0 \to X_1 \to M_1 \to X_0 \to 0,$$
$$0 \to X_0 \to M_0 \to N \to 0.$$

Applying $\text{Hom}(N,-)$ to the first exact sequence yields an exact sequence

$$0 = \text{Ext}^1(N, M_1) \to \text{Ext}^1(N, X_0) \to \text{Ext}^2(N, X_1) = 0$$

since T is rigid and $\text{pd}\, N \leq \text{pd}\, T \leq 1$. This means the second short exact sequence is split and so $N \in \text{add}(M)$. It follows that $\text{add}(M) = \text{add}(T)$ and therefore $M = T$ since T is basic.

Now suppose that T is a basic tilting module and Λ is hereditary. Let M be a faithfully balanced summand of T. Then we have two exact sequences

$$0 \to \Lambda \to M_0 \to X \to 0,$$
$$0 \to X \to M_1 \to Y \to 0$$

with $M_i \in \text{add}(M)$ such that $\text{Hom}_\Lambda(-, M)$ is exact on both short exact sequences. It is straightforward to check that $T' = M \oplus X$ is a tilting module. By definition $T' \in \text{gen}(T) \cap$

$\mathrm{cogen}(T) = T^{\perp} \cap {}^{\perp}T$, so $T \oplus T'$ is rigid and since tilting modules are maximal rigid we conclude $\mathrm{add}(T) = \mathrm{add}(T')$. By applying $\mathrm{Hom}(-, M)$ to the second short exact sequence we conclude $\mathrm{Ext}^1(Y, M) = 0$. By applying $\mathrm{Hom}(Y, -)$ to the first exact sequence we conclude $\mathrm{Ext}^1(Y, X) = 0$ and therefore the second short exact sequence splits, so $X \in \mathrm{add}(M)$. This implies $\mathrm{add}(T) = \mathrm{add}(M \oplus X) = \mathrm{add}(M)$. Since T is basic, we deduce that $M = T$. □

Recall [AS80] that a module $X \in \mathrm{add}(M)$ is a *splitting projective* if every epimorphism $M' \to X$ with $M' \in \mathrm{add}(M)$ is a split epimorphism, and it is a *splitting injective* if every monomorphism $X \to M'$ is a split monomorphism. We write M^g for the direct sum of one copy of each of the splitting projective summands of M and M^c for the direct sum of one copy of each of the splitting injective summands of M. By [AS80, Theorem 2.3], $\mathrm{add}(M^g)$ is a minimal cover for $\mathrm{add}(M)$, so M^g is a minimal summand of M with $\mathrm{gen}(M^g) = \mathrm{gen}(M)$, and it is unique up to isomorphism with this property. Similarly for M^c with $\mathrm{cogen}(M^c) = \mathrm{cogen}(M)$.

Lemma 3.3.2. *If M is a minimal faithfully balanced module for Λ_n, then any indecomposable summand X of M is a summand of M^g or M^c, and X is a summand of both if and only if X is projective-injective. Thus*

$$M \oplus P_1 \cong M^g \oplus M^c.$$

Proof. Since M is faithfully balanced, by condition (C2) in Definition 3.1.1, every indecomposable summand X of M which is not projective-injective is a proper submodule or quotient of another summand of M. Thus X cannot be a summand of both M^g and M^c. On the other hand, if X is a summand of neither, then it is both a proper submodule and quotient of other summands of M. But then the complement of X still satisfies the conditions of Definition 3.1.1, so is faithfully balanced, contradicting minimality. □

Lemma 3.3.3. *Let M be a minimal faithfully balanced module for Λ_n. If N is a module with $\mathrm{gen}(N) \cap \mathrm{cogen}(N) = \mathrm{gen}(M) \cap \mathrm{cogen}(M)$, then N is faithfully balanced and M is a summand of N.*

Proof. Clearly $\mathrm{gen}(\mathrm{gen}(M) \cap \mathrm{cogen}(M)) = \mathrm{gen}(M)$ and $\mathrm{cogen}(\mathrm{gen}(M) \cap \mathrm{cogen}(M)) = \mathrm{cogen}(M)$, so we have $\mathrm{gen}(N) = \mathrm{gen}(M)$ and $\mathrm{cogen}(N) = \mathrm{cogen}(M)$. By the uniqueness of minimal covers and cocovers, $M^g \cong N^g$ and $N^c \cong M^c$. By Lemma 3.3.2, we conclude that M is a summand of N. Now N is faithfully balanced by Corollary 2.2.10. □

Define $fb(n) := \{M \in fb(\Lambda_n) \mid |M| = n\}$. We consider some possible partial orders on $fb(n)$. We define the following relations on minimal faithfully balanced modules

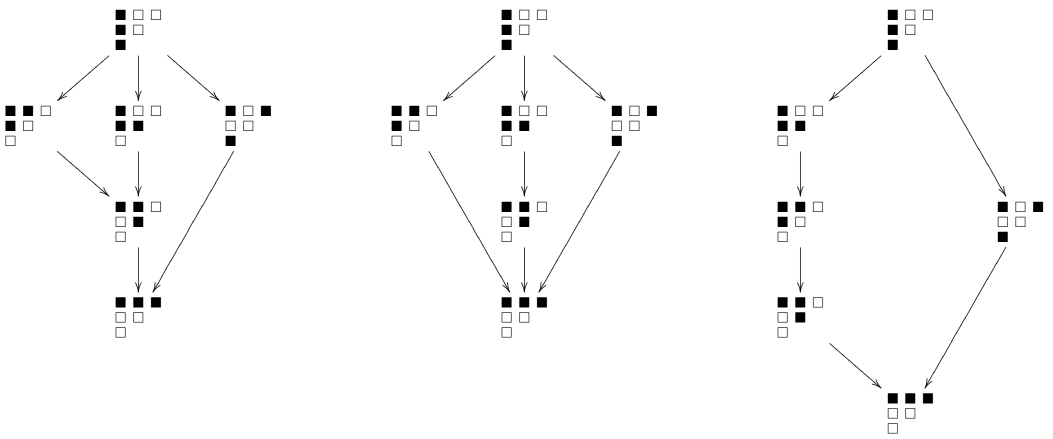
$$\begin{aligned} N \trianglelefteq M &: \Leftrightarrow \quad \mathrm{cogen}(N) \subseteq \mathrm{cogen}(M), \ \ \mathrm{gen}(N) \supseteq \mathrm{gen}(M) \\ N \leq M &: \Leftrightarrow \quad \mathrm{cogen}^1(N) \subseteq \mathrm{cogen}^1(M), \ \ \mathrm{gen}_1(N) \supseteq \mathrm{gen}_1(M). \end{aligned}$$

They are clearly reflexive and transitive, by Lemma 3.3.3 and Corollary 3.2.4(5) they are also antisymmetric and therefore a partial order.

Example 3.3.4. We have the following Hasse diagrams[1] of $fb(3)$. First for the inclusion of

[1]In this thesis, we draw an arrow $x \to y$ in the Hasse diagram of some poset P if y covers x (i.e., $X < y$ and there is no z such that $x < z < y$) in P.

the cogen[1]-categories, then $\leq$ and $\trianglelefteq$ respectively.



Consider the poset $(fb(n), \trianglelefteq)$. A module L is a common lower bound of M and N in $(fb(n), \trianglelefteq)$ if and only if $\mathrm{cogen}(L) \subseteq \mathrm{cogen}(M) \cap \mathrm{cogen}(N)$ and $\mathrm{gen}(L) \supseteq \mathrm{gen}(M) \cup \mathrm{gen}(N)$. For any two elements M and N in $(fb(n), \trianglelefteq)$ the module Λ_n is always a common lower bound of them.

Proposition 3.3.5. *The poset $(fb(n), \trianglelefteq)$ is a lattice for all $n \geq 1$.*

Proof. For any two elements M and N in $fb(n)$, we will construct the meet of them and then apply [Sta12, Proposition 3.3.1]. Consider the basic module C such that $\mathrm{add}(C)$ is the minimal cocover of $\mathrm{cogen}(M) \cap \mathrm{cogen}(N)$ and the basic module G such that $\mathrm{add}(G)$ is the minimal cover of $\mathrm{gen}(M) \cup \mathrm{gen}(N)$. Then we have $\mathrm{cogen}(C) = \mathrm{cogen}(M) \cap \mathrm{cogen}(N)$ and $\mathrm{gen}(G) = \mathrm{gen}(M) \cup \mathrm{gen}(N)$, also we have $C \in \mathrm{gen}(G)$. Note that $|C| \leq n$ and the equality holds if and only if $C = \mathrm{D}\,\Lambda_n = M = N$. Now we complete C to a faithfully balanced module with exactly n indecomposable summands. If $|C| = n$ then we are done. Assume $|C| = t < n$. Note that G can be written as

$$G = G_{i_1} \oplus G_{i_2} \oplus \cdots \oplus G_{i_s}$$

such that $1 = i_1 < i_2 < \cdots < i_s \leq n$ and $\mathrm{top}(G_{i_\alpha}) = S[i_\alpha]$ for all $1 \leq \alpha \leq s$. Clearly, we have $G_{i_1} = M_{1n}$. For each $2 \leq \alpha \leq s$, we take the indecomposable module M_{i_α} with $\mathrm{top}(M_{i_\alpha}) = S[i_\alpha]$ and having the following properties:

(P1) M_{i_α} is a submodule of C,

(P2) G_{i_α} is a quotient of M_{i_α},

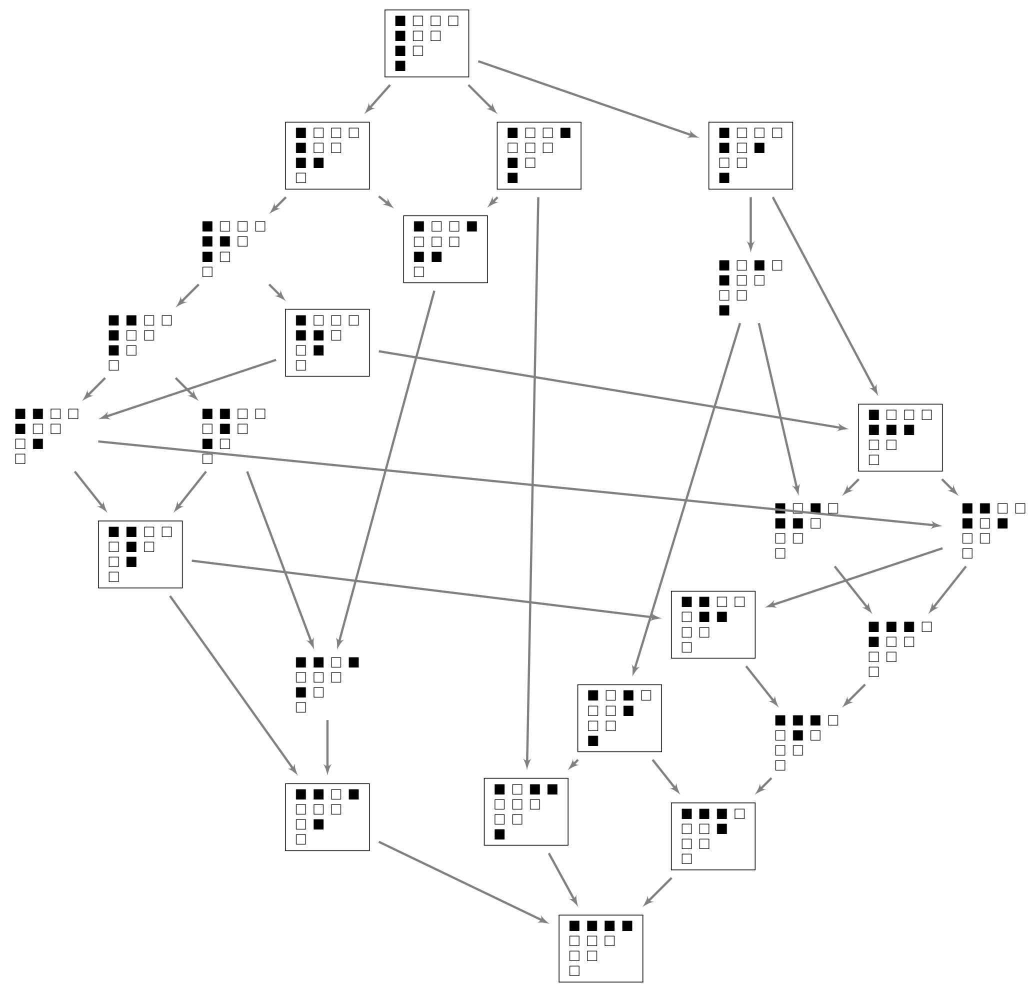
(P3) M_{i_α} has minimal length with respect to (P1) and (P2).

Define $G' = \bigoplus_{2 \leq \alpha \leq s} M_{i_\alpha}$ and $L = C \oplus G'$, then we have $C \in \mathrm{gen}(G) \subseteq \mathrm{gen}(G' \oplus M_{1n})$. We claim that $L \in fb(n)$, and in this case we have $\mathrm{cogen}(L) = \mathrm{cogen}(C)$ and $\mathrm{gen}(L) = \mathrm{gen}(G' \oplus M_{1n})$ which imply that L is a common lower bound of M and N. We first examine the conditions in Theorem 3.1.3 for L to show it is faithfully balanced. It follows from the construction that (C1) and (C2) are satisfied. Consider $\mathrm{cohook}(i, i-1)$ for $2 \leq i \leq n$. If $S[i-1] \in \mathrm{cogen}(L)$ then we have $\mathrm{cohook}(i, i-1) \cap \mathrm{add}(L) \neq \emptyset$. If $S[i-1] \notin \mathrm{cogen}(L) = \mathrm{cogen}(C)$, then it is not in $\mathrm{cogen}(M)$ or $\mathrm{cogen}(N)$. Without loss of generality we may assume $S[i-1] \notin \mathrm{cogen}(M)$, then we must have $S[i] \in \mathrm{gen}(M)$ since M is faithfully balanced. Thus we have $S[i] \in \mathrm{gen}(G') \subseteq \mathrm{gen}(L)$ and $\mathrm{cohook}(i, i-1) \cap \mathrm{add}(L) \neq \emptyset$. This proves that L also satisfies (C3). Now the inclusion-exclusion principle gives $0 = (n-1)-(t-1)-(s-1)+u$, where u is the number of virtual cohooks that each one of which contains an indecomposable summand of C and an indecomposable summand of G'. If $u \neq 0$, then by the construction of G' we know that there exists some i such that $\mathrm{cohook}(i, i-1)$ contains an indecomposable summand of C and an indecomposable summand of G. But this contradicts the fact that $M, N \in fb(n)$. Hence we have $u = 0$ and $|L| = t + (s-1) = n$, as desired.

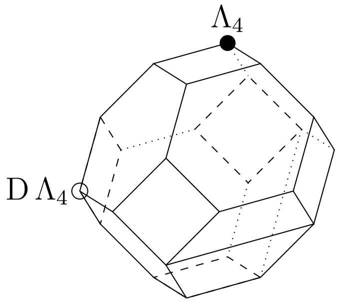
Assume $L' \in fb(n)$ is also a common lower bound of M, N. Then we have $\mathrm{cogen}(L') \subset \mathrm{cogen}(M) \cap \mathrm{cogen}(N) = \mathrm{cogen}(L) = \mathrm{cogen}(C)$ and hence every indecomposable summand of L' is a submodule of C. We claim that $L' \trianglelefteq L$. To prove this, it is enough to show that $\mathrm{gen}(L') \supseteq \mathrm{gen}(L)$ and it reduces further to show that $M_{i_\alpha} \in \mathrm{gen}(L')$ for all $2 \leq \alpha \leq s$. Since L' is a common lower bound of M, N, we have $G_{i_\alpha} \in \mathrm{gen}(L')$ for all $2 \leq \alpha \leq s$. Now assume there is an α such that $M_{i_\alpha} \notin \mathrm{gen}(L')$, then there must exists an indecomposable module $M'_{i_\alpha} \in \mathrm{add}(L')$ such that $\mathrm{top}(M'_{i_\alpha}) = S[i_\alpha]$, G_{i_α} is a quotuent of M'_{i_α} and $l(M'_{i_\alpha}) < l(M_{i_\alpha})$. But this contradicts the minimality of M_{i_α}. Thus we have $L' \trianglelefteq L$ and this proves L is the meet of M and N. □

Example 3.3.6. The following table gives an examples of the construction in the above result for $n = 4$. We also give the Hasse diagram of $(fb(4), \trianglelefteq)$. The vertices in the boxes are the cotilting modules.

M	N	C	G	G'	L
■■□■ □□□ □■ □	■■■□ □□■ □□ □	■■□□ □□□ □□ □	■□□□ □□■ □■ □	□□□□ □■□ □■ □	■■□□ □■□ □■ □



The underlying graph of the Hasse diagram can be visualized as a truncated octahedron with two disected hexagons as indicated in the picture below:



Proposition 3.3.7. *The poset $(fb(n), \leq)$ is not a lattice when $n \geq 4$.*

Proof. Consider modules $M = P[1] \oplus P[2] \oplus \cdots \oplus P[n-2] \oplus M_{2,n-1} \oplus M_{1,n-2}$ and $N = P[1] \oplus P[2] \oplus \cdots \oplus P[n-2] \oplus M_{2,n-1} \oplus M_{2,n-2}$. It's easy to check $M, N \in fb(n)$. Moreover, we have

$$\mathrm{cogen}^1(M) = \mathrm{add}(\Lambda_n \oplus M_{2,n-1} \oplus M_{1,n-2} \oplus M_{n-1,n-1})$$

and

$$\mathrm{cogen}^1(N) = \mathrm{add}(\Lambda_n \oplus M_{2,n-1} \oplus M_{2,n-2} \oplus M_{n-1,n-1}).$$

Thus

$$\mathrm{cogen}^1(M) \cap \mathrm{cogen}^1(N) = \mathrm{add}(\Lambda_n \oplus M_{2,n-1} \oplus M_{n-1,n-1}).$$

If L is a common lower bound of M and N, then we must have $\mathrm{add}(\Lambda_n) \subseteq \mathrm{cogen}^1(L) \subseteq \mathrm{cogen}^1(M) \cap \mathrm{cogen}^1(N)$ and $\mathrm{gen}_1(L) \supseteq \mathrm{gen}_1(M) \cup \mathrm{gen}_1(N)$. Note that there are exactly four subcategories sitting between $\mathrm{add}(\Lambda_n)$ and $\mathrm{cogen}^1(M) \cap \mathrm{cogen}^1(N)$:

$$\mathrm{add}(\Lambda_n),\ \mathrm{add}(\Lambda_n \oplus M_{n-1,n-1}),\ \mathrm{add}(\Lambda_n \oplus M_{2,n-1}),\ \mathrm{add}(\Lambda_n \oplus M_{2,n-1} \oplus M_{n-1,n-1}),$$

and they correspond to basic modules

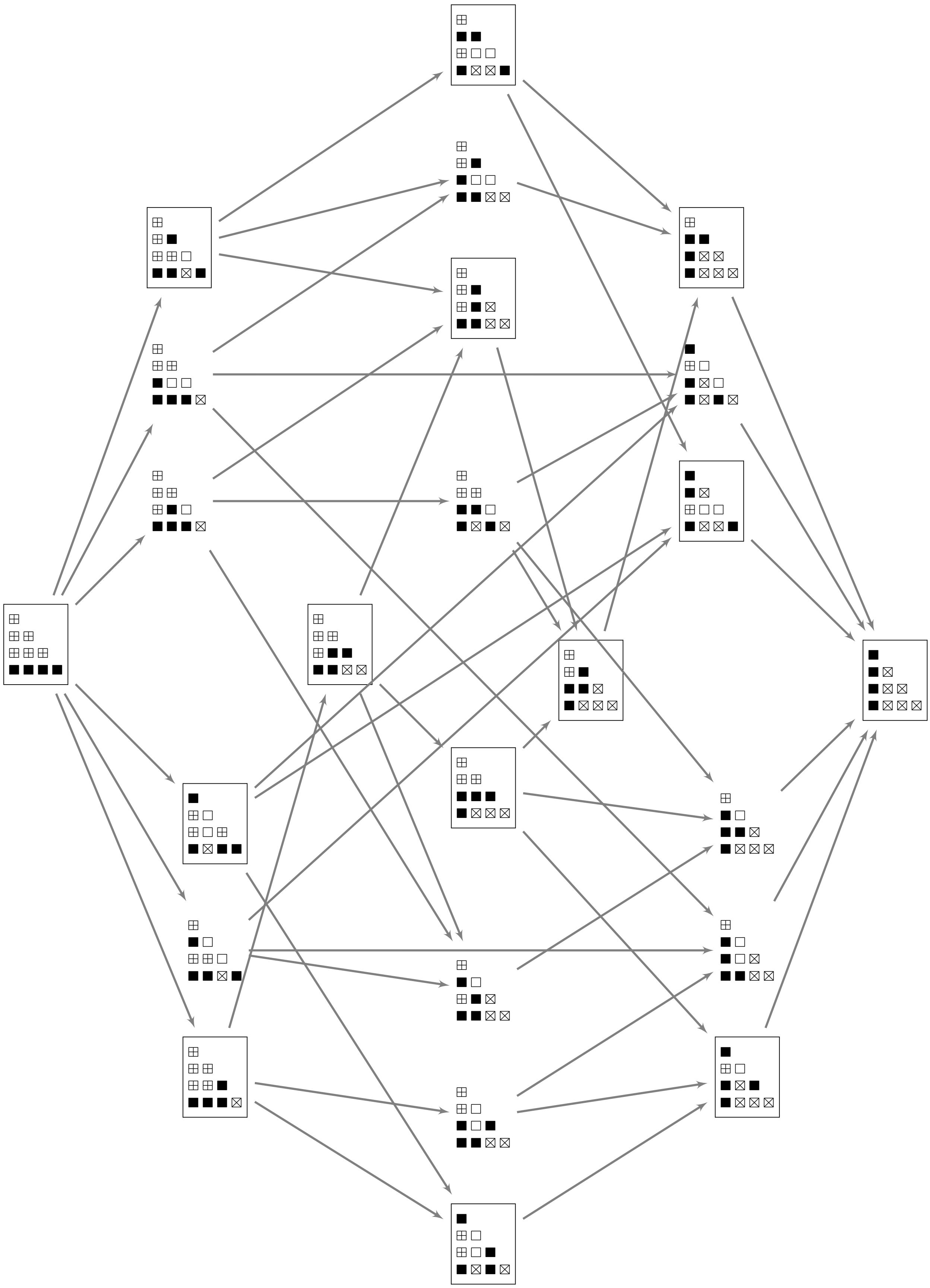
$$\Lambda_n,\ E := P[1] \oplus P[2] \oplus \cdots \oplus P[n-1] \oplus M_{n-1,n-1},$$

$$L_1 := P[1] \oplus P[2] \oplus \cdots \oplus P[n-1] \oplus M_{2,n-1},\ L_2 := P[1] \oplus P[2] \oplus \cdots \oplus P[n-2] \oplus M_{2,n-1} \oplus M_{n-1,n-1}$$

in $fb(n)$, respectively, in the sense that each of these subcategories is the cogen^1 subcategory of the corresponding module. It is straightforward to check the four modules are all common lower bounds of M and N and $\Lambda_n < E < L_2$ and $\Lambda_n < L_1$. But L_1 and L_2 are not comparable since we have $P[n-1] \in \mathrm{gen}_1(L_1) \setminus \mathrm{gen}_1(L_2)$ and $M_{2,n-1} \in \mathrm{gen}_1(L_2) \setminus \mathrm{gen}_1(L_1)$. This means M and N have no smallest common lower bound. $\square$

Example 3.3.8. We give the Hasse diagramm of $(fb(4), \leq)$ on the next page. The vertices in the boxes are the cotilting modules. In the Hasse diagram we denote the modules in $\mathrm{cogen}^1(M) \setminus \mathrm{add}(M)$ by ⊠ and the ones in $\mathrm{gen}_1(M) \setminus \mathrm{add}(M)$ by ⊞. The following table gives concrete examples of modules considered in the above result, and their relation can be easily seen from the Hasse diagram.

M	N	L_1	L_2	E
■ ⊞ ■ ⊞ ■ ■ □ ⊠ ⊠ ⊠	■ ⊞ ⊞ ⊞ ■ ■ ■ ⊠ ⊠ ⊠	■ ⊞ ⊞ ⊞ ■ ■ ⊞ ■ □ ⊠	■ ⊞ ⊞ ⊞ ■ ■ ⊞ ⊠ ■ ⊠	■ ⊞ ⊞ ⊞ ■ ⊞ ⊞ ■ ■ ⊠



Chapter 4

The relative version of faithfully balancedness

4.1 On categories relatively cogenerated by a module

Let $M \in \Lambda$-mod. We recall that one can associate two additive subbifunctors $\mathbf{F}_M, \mathbf{F}^M \subseteq \operatorname{Ext}^1_\Lambda(-,-)$ to the subcategory $\operatorname{add}(M)$ defined for $(C,A) \in (\Lambda\text{-mod})^{op} \times \Lambda$-mod as follows

$$\mathbf{F}^M(C,A) = \{0 \to A \to B \to C \to 0 \mid \operatorname{Hom}_\Lambda(-,M) \text{ is exact on it}\}$$
$$\mathbf{F}_M(C,A) = \{0 \to A \to B \to C \to 0 \mid \operatorname{Hom}_\Lambda(M,-) \text{ is exact on it}\}.$$

An exact sequence in Λ-modules is $\mathbf{F}^M$ exact if and only if $\operatorname{Hom}_\Lambda(-,M)$ is exact on it and the category $\mathcal{I}(\mathbf{F}_M) = \operatorname{add}(M \oplus \operatorname{D}\Lambda)$ is the category of $\mathbf{F}_M$-injectives.
An exact sequence in Λ-modules is $\mathbf{F}_M$ exact if and only if $\operatorname{Hom}(M,-)$ is exact on it and the category $\mathcal{P}(\mathbf{F}_M) = \operatorname{add}(M \oplus \Lambda)$ is the category of $\mathbf{F}_M$-projectives.

In the two new exact structures, we have

(1) $\operatorname{cogen}^k(M)$ is the category of modules N such that there exists an $\mathbf{F}^M$-exact sequence

$$0 \to N \to M_0 \to \cdots \to M_k$$

with $M_i \in \operatorname{add}(M)$. Since M is $\mathbf{F}^M$-injective, this sequence can be seen as the beginning of an $\mathbf{F}^M$-injective coresolution.

(2) $\operatorname{gen}_k(M)$ is the category of modules N such that there exists an $\mathbf{F}_M$-exact sequence

$$M_k \to \cdots \to M_0 \to N \to 0$$

with $M_i \in \operatorname{add}(M)$. Since M is $\mathbf{F}_M$-projective, this sequence can be seen as the beginning of an $\mathbf{F}_M$-projective resolution.

We define two new full subcategories of Λ-mod

$$\operatorname{cogen}^k_{\mathbf{F}}(M) := \left\{ N \;\middle|\; \begin{array}{c} \exists\, \mathbf{F}\text{-exact seq. } 0 \to N \to M_0 \to \cdots \to M_k \text{ with } M_i \in \operatorname{add}(M), \text{ and s.t.} \\ \operatorname{Hom}(M_k, M) \to \cdots \to \operatorname{Hom}(M_0, M) \to \operatorname{Hom}(N,M) \to 0 \quad \text{is exact} \end{array} \right\}$$
$$\operatorname{gen}^{\mathbf{F}}_k(M) := \left\{ N \;\middle|\; \begin{array}{c} \exists\, \mathbf{F}\text{-exact seq } M_k \to \cdots \to M_0 \to N \to 0 \text{with } M_i \in \operatorname{add}(M), \text{ and s.t.} \\ \operatorname{Hom}(M, M_k) \to \cdots \to \operatorname{Hom}(M, M_0) \to \operatorname{Hom}(M,N) \to 0 \quad \text{is exact} \end{array} \right\}.$$

Similarly, we can define $\operatorname{copres}^k_{\mathbf{F}}(M)$ and $\operatorname{pres}^{\mathbf{F}}_k(M)$. Then, we have $\operatorname{cogen}^k(M) = \operatorname{copres}^k_{\mathbf{F}^M}(M) = \operatorname{cogen}^k_{\mathbf{F}^M}(M)$ and $\operatorname{gen}_k(M) = \operatorname{pres}^{\mathbf{F}_M}_k(M) = \operatorname{gen}^{\mathbf{F}_M}_k(M)$.

Example 4.1.1. Let $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ for a generator G and a cogenerator H and M be a module with $\operatorname{Ext}^i_\Lambda(G,M) = 0$ (resp. $\operatorname{Ext}^i_\Lambda(M,H) = 0$), $1 \le i \le k+1$ for some $k \ge 0$. Then one has

$$\operatorname{cogen}^k_{\mathbf{F}}(M) = \operatorname{cogen}^k(M) \cap \bigcap_{i=1}^{k+1} \ker \operatorname{Ext}^i_\Lambda(G,-) \quad (\text{resp. } \operatorname{gen}^{\mathbf{F}}_k(M) = \operatorname{gen}_k(M) \cap \bigcap_{i=1}^{k+1} \ker \operatorname{Ext}^i_\Lambda(-,H)\ ).$$

Lemma 4.1.2. *Let $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ for a generator G and a cogenerator H. A module $Z \in \operatorname{cogen}^k(M)$ is in $\operatorname{cogen}^k_{\mathbf{F}}(M)$ if and only if the short exact sequences*

$$0 \to \Omega_M^{-i} Z \xrightarrow{f_i} M_i \to \Omega_M^{-(i+1)} Z \to 0$$

with f_i minimal left $\operatorname{add}(M)$-approximation are $\mathbf{F}$-exact for $0 \le i \le k$.

Proof. It is enough to observe the following: If $f\colon X \to Y$ is an $\mathbf{F}$-monomorphism and $\mathbf{F} = \mathbf{F}^H$, then this is equivalent to (f,H) being surjective. So if an $\mathbf{F}$-monomorphism f factors as $f = \alpha\beta$, then β also has to be an $\mathbf{F}$-monomorphism. □

Remark 4.1.3. Let M be any module and $k \ge 0$, then $\operatorname{cogen}^k(M) = \operatorname{cogen}^k_{\mathbf{F}^M}(M) = \operatorname{copres}^k_{\mathbf{F}^M}(M)$ is closed under summands and is $\mathbf{F}^M$-extension closed since M is $\mathbf{F}^M$-injective.

For $k \ge 1$, it is closed under kernels of $\mathbf{F}^M$-epimorphisms $X \to Y$ with $X, Y \in \operatorname{cogen}^k(M)$. For $k = \infty$ it is also closed under cokernels of $\mathbf{F}^M$-monomorphisms $X \to Y$ with $X, Y \in \operatorname{cogen}^\infty(M)$. So, one can define the derived category $\mathsf{D}^b_{\mathbf{F}^M}(\operatorname{cogen}^k(M))$, see [Nee90, Kel96]. It is completely unknown which informations these encode.

Embedding into an abelian category

We fix $\Delta = \operatorname{End}_\Lambda(G)^{op}$ and $e \in \Delta$ the projection onto the summand Λ (resp. $\Sigma = \operatorname{End}_\Lambda(H)^{op}$ and $\varepsilon \in \Sigma$ the projection onto $\mathrm{D}\,\Lambda$), $r = \operatorname{Hom}_\Lambda(G,-)$ then we have a pair (e,r) of adjoint functors (resp. $\ell = \Sigma\varepsilon \otimes_\Lambda - = \mathrm{D}\operatorname{Hom}_\Lambda(-,H)$, then we have an adjoint pair (ε, ℓ))

$$e\colon \Delta\text{-mod} \rightleftarrows \Lambda\text{-mod}\colon r \quad (\text{resp. } \ell\colon \Lambda\text{-mod} \rightleftarrows \Sigma\text{-mod}\colon \varepsilon)$$

with e is exact and r is fully faithful, maps $\mathbf{F}$-exact sequences to exact sequences and $\operatorname{add}(G)$ to $\operatorname{add}(\Delta)$. In particular, it maps $\mathbf{F}$-projective resolutions to projective resolutions and we get induced isomorphisms

$$\operatorname{Ext}^i_{\mathbf{F}}(M,N) \to \operatorname{Ext}^i_\Delta(r(M), r(N)), \quad i \ge 0.$$

Dually, ε is exact, ℓ is fully faithful, maps $\mathbf{F}$-exact sequences to exact sequences and $\operatorname{add}(H)$ to $\operatorname{add}(\mathrm{D}\,\Sigma)$, it maps $\mathbf{F}$-injective resolutions to injective resolutions and induces isomorphisms on the Ext-groups $\operatorname{Ext}^i_{\mathbf{F}}(M,N) \to \operatorname{Ext}^i_\Sigma(\ell(M), \ell(N)), i \ge 0$. We have

$$\operatorname{Im} r = \operatorname{cogen}^1(\mathrm{D}(e\Delta)) \text{ and } \operatorname{Im} \ell = \operatorname{gen}_1(\Sigma\varepsilon).$$

It is also easy to see: If T is a relative tilting Λ-module, then $r(T)$ is a tilting Δ-module: Conversely, every tilting Δ-module in $\operatorname{cogen}^1(J)$ restricts under e to a relative tilting module. This gives a bijection, respecting the partial order (given by inclusion of perpendicular categories).
If C is a relative cotilting module then $\ell(C)$ is a cotilting Σ-module and every cotilting module in $\operatorname{Im} \ell = \operatorname{gen}^1(\Sigma\varepsilon)$ restricts under ε to a relative cotilting module.

The following lemma is originally proved by Auslander and Solberg in [AS93c].

Lemma 4.1.4. *Keep notations as above. We have*

$$\operatorname{gldim}_{\mathbf{F}} \Lambda \leq \operatorname{gldim} \Delta \leq \operatorname{gldim}_{\mathbf{F}} \Lambda + 2.$$

Proof. For the first inequality, if $\operatorname{gldim} \Delta = \infty$ then there is nothing to prove. Assume $\operatorname{gldim} \Delta = k < \infty$, then we have $\operatorname{Ext}^i_{\mathbf{F}}(M, N) \cong \operatorname{Ext}^i_{\Delta}(r(M), r(N)) = 0$ for all $i > k$. Thus $\operatorname{gldim}_{\mathbf{F}} \Lambda \leq \operatorname{gldim} \Delta$. For the second inequality, if $\operatorname{gldim} \Lambda = \infty$ then we are done. Assume $\operatorname{gldim}_{\mathbf{F}} \Lambda = k < \infty$. For an arbitrary $X \in \Delta$-mod, one can choose a projective presentation $(G, G_1) \xrightarrow{(G,f)} (G, G_0) \to X \to 0$ of X. This exact sequence comes from a map $f : G_1 \to G_0$ in Λ-mod with $G_0, G_1 \in \operatorname{add}(G)$. Since $\operatorname{gldim}_{\mathbf{F}} \Lambda = k$, there exists an **F**-projective resolution

$$0 \to G_{k+2} \to \cdots \to G_3 \to G_2 \to \ker(f) \to 0$$

of $\ker(f)$. Thus we have a projective resolution

$$0 \to (G, G_{k+2}) \to \cdots \to (G, G_3) \to (G, G_2) \to (G, \ker(f)) \to 0$$

of $(G, \ker(f))$ in Δ-mod. This gives rise to a projective resolution

$$0 \to (G, G_{k+2}) \to \cdots \to (G, G_3) \to (G, G_2) \to (G, G_1) \to (G, G_0) \to X \to 0$$

of X. This proves $\operatorname{gldim} \Delta \leq k + 2$. □

Lemma 4.1.5. *Let Λ and Δ be as above and $k \geq 0$. Then the following are equivalent:*

(1) $\operatorname{pd}_{\mathbf{F}} \mathrm{D}\Lambda \leq k$ *and* $\operatorname{gldim} \Delta \leq k + 2$*,*

(2) $\operatorname{gldim}_{\mathbf{F}} \Lambda \leq k$*,*

(3) $\operatorname{id}_{\mathbf{F}} \Lambda \leq k$ *and* $\operatorname{gldim} \Sigma \leq k + 2$*.*

Proof. (1) ⇒ (2): Let $J = \mathrm{D}(e\Delta)$. Clearly, $\operatorname{gldim}_{\mathbf{F}} \Lambda \leq k$ if and only if

$$\operatorname{Ext}^{k+1}_{\Delta}(\operatorname{cogen}^1(J), \operatorname{cogen}^1(J)) = 0.$$

We have $J = \mathrm{D}(e\Delta) = r(\mathrm{D}\Lambda)$ and it is easily seen that $\operatorname{pd}_{\mathbf{F}} \mathrm{D}\Lambda \leq k$ is equivalent to $\operatorname{pd}_{\Delta} J \leq k$. We claim the stronger implication: $\operatorname{gldim} \Delta \leq k + 2$ and $\operatorname{pd} J \leq k$ implies

$$\operatorname{Ext}^{k+1}(\operatorname{cogen}^1(J), \Delta\text{-mod}) = 0$$

(i.e., $\operatorname{pd} X \leq k$ for all $X \in \operatorname{cogen}^1(J)$).
If we have an exact sequence $0 \to A \to J_0 \to B \to 0$ with $J_0 \in \operatorname{add}(J)$ and we apply a functor $(-, Y)$ then we get a dimension shift $\operatorname{Ext}^i(A, Y) \cong \operatorname{Ext}^{i+1}(B, Y)$ for all $i \geq k+1$. In particular, we have for $X \in \operatorname{cogen}^1(J)$,

$$\operatorname{Ext}^{k+1}(X, Y) \cong \operatorname{Ext}^{k+2}(\Omega^{-} X, Y) \cong \operatorname{Ext}^{k+3}(\Omega^{-2} X, Y) = 0$$

since we assume that $\operatorname{gldim} \Delta \leq k + 2$.

(2) ⇒ (1) Clearly, if $\operatorname{gldim}_{\mathbf{F}} \Lambda \leq k$, then $\operatorname{pd}_{\mathbf{F}} \mathrm{D}\Lambda \leq k$. By Lemma 4.1.4, we also have

$$\operatorname{gldim} \Delta \leq \operatorname{gldim}_{\mathbf{F}} \Lambda + 2 \leq k + 2.$$

The equivalence of (2) and (3) is proven analogously. □

Example 4.1.6. Let $\Lambda = K(1 \to 2 \to \cdots \to n)$. Then there are 2^N with $N = \sum_{k=1}^{n-1} k$ basic generators G. The minimal **F**-global dimension is 0 which is obtained if and only of G is the Auslander generator. The maximal **F**-global dimension is $n - 1$ (cf. Example 6.3.2, (4)).

4.2 The relative version of faithfully balancedness

Recall that for a finite-dimensional algebra Λ, a module ${}_\Lambda M$ is faithful if and only if $\Lambda \in \operatorname{cogen}(M) = \operatorname{cogen}^0(M)$, and it is faithfully balanced if and only if $\Lambda \in \operatorname{cogen}^1(M)$. So it makes sense to call a faithful module 0-faithful and call a faithfully balanced module 1-faithful. Of course one can define the notion of k-faithful module for any non-negative integer k. Since in the relative setting balancedness doesn't make sense, we introduce the following definition.

Definition 4.2.1. Let $\mathbf{F} \subseteq \operatorname{Ext}^1_\Lambda(-,-)$ be an additive subbifunctor of finite type and k a non-negative integer. We say a module M is k-$\mathbf{F}$-faithful if $\mathcal{P}(\mathbf{F}) \subseteq \operatorname{cogen}^k_{\mathbf{F}}(M)$. In particular, a 1-$\mathbf{F}_\Lambda$-faithful module is just a faithfully balanced module.

Easy examples of 1-$\mathbf{F}$-faithful modules are $\mathbf{F}$-(co)tilting modules (see section 8) and modules which have G or H as a summand. Here is an other easy example.

Example 4.2.2. (1) Let Λ be a finite-dimensional algebra, $P_1, \dots, P_n$ its indecomposable projectives and assume that there is a subset $I \subseteq \{1, \dots, n\}$ such that $M := \bigoplus_{i\in I} \bigoplus_{j\geq 0} \tau^{-j} P_i$ is finite-dimensional and faithfully balanced. Then $G = M \oplus \Lambda$ and $H = M \oplus \mathrm{D}\,\Lambda$ fulfill $\mathbf{F}_G = \mathbf{F}^H =: \mathbf{F}$. Clearly, we have $G \in \operatorname{cogen}^1_{\mathbf{F}}(M)$, so M is 1-$\mathbf{F}$-faithful.

(2) Let Λ be a basic Nakayama algebra and assume $M = \bigoplus_{X:\ \text{indec, not simple}} X$ is faithfully balanced [1]. Let $G = M \oplus \bigoplus_{P_i:\ \text{simple proj}} P_i$ and $H = M \oplus \bigoplus_{I_i:\ \text{simple inj}} I_i$. Then we claim:

$$\{1\text{-}\mathbf{F}\text{-faithful modules}\} = \{M' \oplus S \mid S \text{ semi-simple}, \operatorname{add}(M') = \operatorname{add}(M)\}$$

Since M is $\mathbf{F}$-projective-injective, M has to be summand of every 1-$\mathbf{F}$-faithful module. On the other hand, let S be a semi-simple module, we want to see that $M \oplus S$ is 1-$\mathbf{F}$-faithful. Assume that there is a simple projective $P \notin \operatorname{add}(S)$, since $(P, S) = 0 = (S, P)$ we have that the minimal left $\operatorname{add}(M \oplus S)$ equals the minimal left $\operatorname{add}(M)$ and the minimal left $\operatorname{add}(H)$-approximation, in particular $G \in \operatorname{cogen}_{\mathbf{F}}(M \oplus S)$. Now, we look at the cokernel of the approximation $X = \Omega^-_{M\oplus S} P$, since M is faithfully balanced we have $X \in \operatorname{cogen}(M)$, in particular X has no simple injective summand. So, every simple summand $S' \notin \operatorname{add}(S)$ of X has a minimal left $\operatorname{add}(H)$-approximation which coincides with a minimal left $\operatorname{add}(M)$- and $\operatorname{add}(M \oplus S)$-approximation which is an $\mathbf{F}$-monomorphism and therefore, we conclude that $G \in \operatorname{cogen}^1_{\mathbf{F}}(M \oplus S)$.

The main result of this section is the following

Theorem 4.2.3. *Let $\mathbf{F} \subseteq \operatorname{Ext}^1(-,-)$ be an addtive subbifunctor of the form $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ for a generator G and a cogenerator H. The following are equivalent for every module M and every $k \geq 0$.*

(1) $G \in \operatorname{cogen}^k_{\mathbf{F}}(M)$.

(2) $H \in \operatorname{gen}^{\mathbf{F}}_k(M)$.

Let $M \in \Lambda$-mod and $\Gamma = \operatorname{End}_\Lambda(M)$. We define

$$\Sigma = \operatorname{End}_\Lambda(H) \quad \text{and} \quad \Delta = \operatorname{End}_\Lambda(G).$$

[1] this is the case if Λ has no simple projective-injective and $\tau^- S$ is not simple injective for every S simple projective - for example $\mathbb{A}_n$ fulfills this for $n \geq 3$.

We first remark that generators and cogenerators are faithfully balanced, in particular this applies to H and G and we have

$$\begin{aligned}
\Lambda\text{-mod} &= \operatorname{cogen}(H) &&= \operatorname{cogen}^1(H) &&= \operatorname{cogen}^2(H) &&= \cdots &&= \operatorname{cogen}^\infty(H)\\
\Lambda\text{-mod} &= \operatorname{gen}(G) &&= \operatorname{gen}_1(G) &&= \operatorname{gen}_2(G) &&= \cdots &&= \operatorname{gen}_\infty(G).
\end{aligned}$$

By Lemma 2.2.4 we have dualities of categories

$$\begin{aligned}
(-, {}_\Lambda H)\colon \quad & \Lambda\text{-mod} \longleftrightarrow \operatorname{cogen}^1({}_\Sigma H) && : (-, {}_\Sigma H)\\
\mathrm{D}({}_\Lambda G, -)\colon \quad & \Lambda\text{-mod} \longleftrightarrow \operatorname{gen}_1({}_\Delta G) && : \mathrm{D}({}_\Delta G, -).
\end{aligned}$$

The key step in the proof is given by the following lemma.

Lemma 4.2.4. *Keep the above notations. For $0 \le k \le \infty$ we have*

(1) The following are equivalent

- *(1a)* $N \in \operatorname{cogen}^k_{\mathbf{F}}(M)$.
- *(1b)* ${}_\Sigma(N, H) \in \operatorname{gen}_k({}_\Sigma(M, H))$.
- *(1c) Consider the natural map* $(M, H) \otimes_\Gamma (N, M) \to (N, H)$, $f \otimes g \mapsto f \circ g$.
  - *(i) For $k = 0$: It is an epimorphism .*
  - *(ii) For $k \ge 1$: It is an isomorphism and* $\operatorname{Ext}^i_\Gamma((N, M), \mathrm{D}(M, H)) = 0$ *for* $1 \le i \le k - 1$.

(2) The following are equivalent

- *(2a)* $N \in \operatorname{gen}^{\mathbf{F}}_k(M)$.
- *(2b)* $(G, N)_\Delta \in \operatorname{gen}_k((G, M)_\Delta)$.
- *(2c) Consider the natural map* $(M, N) \otimes_\Gamma (G, M) \to (G, N)$, $f \otimes g \mapsto f \circ g$.
  - *(i) For $k = 0$: It is an epimorphism.*
  - *(ii) For $k \ge 1$: It is an isomorphism and* $\operatorname{Ext}^i_\Gamma((G, M), \mathrm{D}(M, N)) = 0$ *for* $1 \le i \le k - 1$.

Proof. It is easy to see the equivalence of (1a) and (1b) using that the duality $(-, H)$ restricts to a duality of categories

$$(-, {}_\Lambda H)\colon \operatorname{cogen}^k_{\mathbf{F}}(M) \longleftrightarrow \operatorname{cogen}^1({}_\Sigma H) \cap \operatorname{gen}_k({}_\Sigma(M, H))\colon (-, {}_\Sigma H).$$

To see that the map from the right to the left is well-defined it is important to observe that ${}_\Sigma H$ is an injective module (since H is a cogenerator), therefore the functor $(-, {}_\Sigma H)$ is exact. Similarly, it is easy to see the equivalence of (2a) and (2b) using the second equivalence mentioned above. For the equivalence of (1b) and (1c) we translate the statement of (1c) into the characterization from Lemma 2.1.1. The most important observation is the following $E := \operatorname{End}_\Sigma((M, H)) = \Gamma^{\mathrm{op}}$. The natural map from Lemma 2.1.1 (for the category $\operatorname{gen}_{k\ \Sigma}(M, H)$) is:

$$\begin{aligned}
\operatorname{Hom}_\Sigma((M, H), (N, H)) \otimes_E (M, H) &\to (N, H)\\
f \otimes g &\mapsto f(g).
\end{aligned}$$

First observe $E = \Gamma^{\mathrm{op}}$ means left (resp. right) E-modules are naturally right (resp. left) Γ-modules and $X_E \otimes_E {}_E L \cong L_\Gamma \otimes_\Gamma {}_\Gamma X$. Secondly, since H is a cogenerator we have

$$\operatorname{Hom}_\Sigma((M, H), (N, H)) = (N, M).$$

With this identifications the map from before becomes the natural map mentioned in (1c). The equivalence of (2b) and (2c) is analogue. We set $C = \mathrm{End}_\Delta((G,M)) = \Gamma$. By lemma 2.1.1 we have to look at the natural map

$$\begin{aligned}\mathrm{Hom}_\Delta((G,M),(G,N)) \otimes_C (G,M) &\to (G,N)\\ f\otimes g &\mapsto f(g).\end{aligned}$$

We have an isomorphism of right Γ-modules since G is a generator

$$\mathrm{Hom}_\Delta((G,M),(G,N)) = (M,N).$$

With this identifications the map from before becomes the natural map in (2c). □

We observe that the proof of Theorem 4.2.3 is a direct consequence of the previous lemma: By setting $N = G$ in part (1) and $N = H$ in part (2), we obtain the same maps in (1c), (2c) and therefore the claim follows.

Lemma 4.2.5. *Let $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ for a generator G and a cogenerator H. Let $M \in \Lambda$-mod, $\Gamma = \mathrm{End}_\Lambda(M)$, $L = (G,M)$ and $R = \mathrm{D}(M,H)$.*
If we assume that $\Lambda \in \mathrm{cogen}^1_{\mathbf{F}}(M)$ and $H \in \mathrm{gen}_1(M)$ then the duality $(-,{}_\Lambda M)\colon \mathrm{cogen}^1(M) \leftrightarrow \mathrm{cogen}^1(M)\colon (-,{}_\Gamma M)$ restricts to a duality $\mathrm{cogen}^1_{\mathbf{F}^H}({}_\Lambda M) \leftrightarrow \mathrm{cogen}^1_{\mathbf{F}^R}({}_\Gamma M)$. Furthermore, it restricts to a duality

$$(-,{}_\Lambda M)\colon\ \mathrm{cogen}^k_{\mathbf{F}^H}(M) \longleftrightarrow \mathrm{cogen}^1_{\mathbf{F}^R}(M) \cap \bigcap_{i=1}^{k-1} \ker \mathrm{Ext}^i_\Gamma(-,R)\colon (-,{}_\Gamma M)$$

In particular, $G \in \mathrm{cogen}^k_{\mathbf{F}^H}(M)$ is equivalent to $L \in \mathrm{cogen}^1_{\mathbf{F}^R}(M)$ and $\mathrm{Ext}^i_\Gamma(L,R) = 0$ for $1 \le i \le k-1$.

Proof. Since $H \in \mathrm{gen}_1(M)$ we have that $\mathrm{D}(M,R) = \mathrm{D}(M,\mathrm{D}(M,H)) \to H$ is an isomorphism. So, it is enough to proof that $(-,{}_\Lambda M)$ maps $\mathrm{cogen}^1_{\mathbf{F}^H}(M)$ to $\mathrm{cogen}^1_{\mathbf{F}^R}(M)$ and use $R \in \mathrm{gen}^1({}_\Gamma M)$ to get the quasi-inverse by symmetry.

Let $X \in \mathrm{cogen}^1_{\mathbf{F}^H}(M)$. We choose a projective presentation $P_1 \to P_0 \to X \to 0$. By applying $(-,{}_\Lambda M)$ we get an exact sequence of Γ-modules $0 \to (X,M) \to (P_0,M) \to (P_1,M)$ is with $(P_i,M) \in \mathrm{add}(M)$. We apply $(-,R)$ to get a complex

$$((P_1,M),R) \to ((P_0,M),R) \to ((X,M),R) \to 0.$$

We would like to see that it is exact. By Hom-Tensor adjunction it identifies with the first row in the following commutative diagram

$$\begin{array}{ccccccc}
\mathrm{D}[(M,H)\otimes_\Gamma (P_1,M)] & \longrightarrow & \mathrm{D}[(M,H)\otimes_\Gamma (P_0,M)] & \longrightarrow & \mathrm{D}[(M,H)\otimes_\Gamma (X,M)] & \longrightarrow & 0\\
\uparrow & & \uparrow & & \uparrow & & \\
\mathrm{D}(P_1,H) & \longrightarrow & \mathrm{D}(P_0,H) & \longrightarrow & \mathrm{D}(X,H) & \longrightarrow & 0
\end{array}$$

Note that the arrows up are the dual of the natural maps $(M,H)\otimes_\Gamma (Y,M) \to (Y,H)$ given by $f\otimes g \mapsto f\circ g$. By Lemma 4.2.4 we know that this natural map is an isomorphism if and only if $Y \in \mathrm{cogen}^1_{\mathbf{F}^H}(M)$. By assumption we have $P_1, P_0, X \in \mathrm{cogen}^1_{\mathbf{F}^H}(M)$ and the first row identifies with the complex in the second row. But the exactness of the second row follows since $\mathrm{D}(-,H)$ is right exact. This proves $(X,M) \in \mathrm{cogen}^1_{\mathbf{F}^R}(M)$. For the symmetry, we need to see $\Gamma \in \mathrm{cogen}^1_{\mathbf{F}^R}(M)$. But $\Gamma = (M,M)$ and $M \in \mathrm{cogen}^1_{\mathbf{F}^H}(M)$ implies the claim by the argument just given.

The further restriction follows directly from Lemma 4.2.4.

□

Of course there is a dual version of the previous lemma which we will leave out.

If ${}_\Lambda M$ is 1-$\mathbf{F}^H$-faithful, then ${}_\Gamma M$ does not have to be 1-$\mathbf{F}^R$-faithful (with $\Gamma = \mathrm{End}_\Lambda(M)$ and $R = \mathrm{D}(M,H)$). We give an example for this.

Example 4.2.6. Let Λ be the path algebra of $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ modulo the relation $\beta\alpha = 0$. Let $G = \Lambda \oplus S_1$, $H = \mathrm{D}\,\Lambda \oplus S_2$ and $\mathbf{F} = \mathbf{F}^H = \mathbf{F}_G$. Then $M := G$ is clearly 1-$\mathbf{F}^H$-faithful and $\mathrm{pd}_{\mathbf{F}} M = 0$. Let us look at $\Gamma = \mathrm{End}_\Lambda(M)$, since we have irreducible morphisms

$$S_3 \to {\scriptstyle\begin{smallmatrix}2\\3\end{smallmatrix}} \to {\scriptstyle\begin{smallmatrix}1\\2\end{smallmatrix}} \to S_1$$

we can identify it with the following bound path algebra

$$d \to c \to b \to a$$

modulo all path of length 2. We have

$${}_\Gamma M = (P_1, M) \oplus (P_2, M) \oplus (P_3, M) = P_b \oplus P_c \oplus P_d$$

and

$${}_\Gamma R := \mathrm{D}(M,H) = {}_\Gamma M \oplus \mathrm{D}(M, S_2).$$

We apply $\mathrm{D}(M,-)$ to an injective coresolution $0 \to S_2 \to I_2 \to I_1$ to obtain a projective presentation

$$P_b = (P_1, M) \to P_c = (P_2, M) \to \mathrm{D}(M, S_2) \to 0.$$

This implies $\mathrm{D}(M,S_2) \cong S_c$ and therefore $\tau^- R = \tau^- S_c = S_d$. It is easy to see that $S_d \notin \mathrm{cogen}({}_\Gamma M)$ implying $\tau^- R \notin \mathrm{cogen}^1_{\mathbf{F}^R}(M)$. This shows ${}_\Gamma M$ is not 1-$\mathbf{F}^R$-faithful.

Thus the property of being 1-$\mathbf{F}$-faithful is not as nicely symmetric as being faithfully balanced. Nevertheless, we can get the symmetry again if we restrict to the following special case.

Proposition 4.2.7. *Let $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ for a generator G and a cogenerator H. Let M be a faithfully balanced Λ-module, $\Gamma = \mathrm{End}_\Lambda(M)$, $L = (G, M)$ and $R = \mathrm{D}(M,H)$.*
If $M \in \mathrm{add}(H)$ (or equivalently, $\mathrm{D}\,\Gamma \in \mathrm{add}(R)$), then the following are equivalent:

(1) ${}_\Lambda M$ is 1-$\mathbf{F}^H$-faithful.

(2) ${}_\Gamma M$ is 1-$\mathbf{F}^R$-faithful.

Dually, if $M \in \mathrm{add}(G)$, then ${}_\Lambda M$ is 1-$\mathbf{F}_G$-faithful if and only if ${}_\Gamma M$ is 1-$\mathbf{F}_L$-faithful.

Proof. We assume $M \in \mathrm{add}(H)$. Assume $G \in \mathrm{cogen}^1_{\mathbf{F}^H}(M)$, we have to see $\tau^- R \in \mathrm{cogen}^1_{\mathbf{F}^R}(M)$. Since $H \in \mathrm{gen}_1^{\mathbf{F}^H}(M)$ implies that we have an $\mathbf{F}$-exact sequence

$$0 \to \Omega^2_M H \to M_1 \to M_0 \to H \to 0$$

with $M_i \in \mathrm{add}(M)$. Since $M \in \mathrm{add}(H)$, this implies $\Omega^2_M H \in \mathrm{cogen}^1_{\mathbf{F}}(M)$. We apply $\mathrm{D}(M,-)$ to the last three terms of the four term sequence and obtain an injective copresentation of R. We apply $(-,M)$ to the first three terms and observe and get an exact sequence

$$(M_0, M) \to (M_1, M) \to (\Omega^2_M H, M) = \tau^- R \to 0$$

in particular this proves the claim. □

4.3 Strong dualizing sequences

Definition 4.3.1. Let $0 \to L \to M_0 \to M_1 \to \cdots \to M_k \to R \to 0$ be a k-add(M)-dualizing sequence in Γ-mod for some non-negative integer k. We say it is strong if $\mathrm{D}(L, -)$ is exact on it.

We can characterize it as follows.

Lemma 4.3.2. *A k-add(M)-dualizing sequence as in the above definition is strong if and only if one (equivalently all) of the following equivalent statement is fulfilled:*

(1) $\mathrm{D}(L, -)$ is exact on it, i.e., it is an $\mathbf{F}_L$-exact sequence (or equivalently, $R \in \mathrm{gen}_k^{\mathbf{F}_L}(M)$).

(2) $(-, R)$ is exact on it, i.e., it is an $\mathbf{F}^R$-exact sequence (or equivalently, $L \in \mathrm{cogen}^k_{\mathbf{F}^R}(M)$).

(3) Consider the natural map $(M, R) \otimes_\Lambda (L, M) \to (L, R)$, where $\Lambda = \mathrm{End}_\Gamma(M)$.

(i) For $k = 0$: It is an epimorphism .

(ii) For $k \geq 1$: It is an isomorphism and $\mathrm{Ext}^i_\Lambda((L, M), \mathrm{D}(M, R)) = 0$ for $1 \leq i \leq k - 1$.

Proof. We will prove (1) and (3) are equivalent and the equivalence of (2) and (3) can be proved dually.

We consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \mathrm{D}(L,R) & \xrightarrow{i'} & \mathrm{D}(L,M_k) & \xrightarrow{f} & \mathrm{D}(L,M_{k-1}) & \to \cdots \to & \mathrm{D}(L,M_0) \\
& \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
0 \to & \mathrm{D}((M,R)\otimes(L,M)) & \xrightarrow{j} & \mathrm{D}((M,M_k)\otimes(L,M)) & \xrightarrow{g} & \mathrm{D}((M,M_{k-1})\otimes(L,M)) & \to \cdots \to & \mathrm{D}((M,M_0)\otimes(L,M)) \\
& \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
0 \to & ((L,M),\mathrm{D}(M,R)) & \to & ((L,M),\mathrm{D}(M,M_k)) & \to & ((L,M),\mathrm{D}(M,M_{k-1})) & \to \cdots \to & ((L,M),\mathrm{D}(M,M_0)).
\end{array}$$

Assume (1), then the first row is exact. Since the functor $((L, M), -)$ is left exact, the sequence

$$0 \to \mathrm{D}((M, R) \otimes (L, M)) \to \mathrm{D}((M, M_k) \otimes (L, M)) \to \mathrm{D}((M, M_{k-1}) \otimes (L, M))$$

is exact. For $k = 0$, we have ji is a monomorphism and so is i. This shows the natural map $(M, R) \otimes_\Lambda (L, M) \to (L, R)$ is an epimorphism. For $k \geq 1$, we have an induced isomorphism on kernels

$$\mathrm{D}(L, R) = \ker f \xrightarrow{\cong} \ker g = \mathrm{D}((M, R) \otimes (L, M)).$$

This proves the natural map $(M, R) \otimes_\Lambda (L, M) \to (L, R)$ is an isomorphism. Now the exactness of the first row implies the exactness of the last row which is equivalent to

$$\mathrm{Ext}^i_\Lambda((L, M), \mathrm{D}(M, R)) = 0$$

for $1 \leq i \leq k - 1$. Conversely, assume (3). If $k = 0$, then the map i is a monomorphism and so is i'. If $k \geq 1$, then the last row is exact and the natural map $(M, R) \otimes_\Lambda (L, M) \to (L, R)$ is an isomorphism will imply the first row is isomorphisc to the last row. So we have, in both cases, that the first row is exact. Since the functor $\mathrm{D}(L, -)$ is right exact, (1) follows from the exactness of the first row. $\square$

Remark 4.3.3. From the proof of the above lemma we see that for any X if $N \in \operatorname{cogen}^1_{\mathbf{F}^X}(M)$ then the natural map $(M, X) \otimes (N, M) \to (N, X)$ is an isomorphism. The converse holds true if X is a cogenerator (cf. Lemma 4.2.4). Similarly, we have if $N \in \operatorname{gen}_1^{\mathbf{F}_X}(M)$ then the natural map $(M, N) \otimes (X, M) \to (X, N)$ is an isomorphism.

Lemma 4.3.4. *Let Γ be a finite-dimensional algebra and $0 \to L \to M_0 \to \cdots \to M_k \to R \to 0$ be a k-add(M)-dualizing sequence of Γ-modules with M faithfully balanced. Define $\Lambda = \operatorname{End}_\Gamma(M)$, $G = (L, M)$ and $H = \mathrm{D}(M, R)$. If $\Gamma \in \operatorname{cogen}^1_{\mathbf{F}^R}(M)$ and $R \in \operatorname{gen}_1(M)$ then for every $k \geq 1$ the functor $(-, M)$ restricts to a duality*

$$\operatorname{cogen}^k_{\mathbf{F}^R}(M) \longleftrightarrow \operatorname{cogen}^1_{\mathbf{F}^H}(M) \cap \bigcap_{i=1}^{k-1} \ker \operatorname{Ext}^i_\Lambda(-, H \oplus M).$$

In particular, $L \in \operatorname{cogen}^k_{\mathbf{F}^R}(M)$ is equivalent to $\operatorname{Ext}^i_\Lambda(G, H \oplus M) = 0$, $1 \leq i \leq k-1$.

Proof. The case $k = 1$ follows directly from Lemma 4.2.5. For $k > 1$ we note that $\mathbf{F}^R = \mathbf{F}^{R \oplus \mathrm{D}\Gamma}$ and then apply Lemma 4.2.5 using the cogenerator $R \oplus \mathrm{D}\Gamma$ (in place of H). □

Lemma 4.3.5. *Let M be a faithfully balanced Λ-module and $\Gamma = \operatorname{End}_\Lambda(M)$. Let $k \geq 1$. Then, the assignment $X, Y \mapsto (X, M), \mathrm{D}(M, Y)$ gives a self-inverse bijection (up to seeing X, Y as Λ or as Γ-modules) between the following sets of pairs of Λ-modules and Γ-modules*

$$\left\{ {}_\Lambda G, {}_\Lambda H \;\middle|\; \begin{array}{c} G = \tau_k^- H \oplus \Lambda \in \operatorname{cogen}^1_{\mathbf{F}^H}(M) \cap {}^{1\sim(k-1)\perp}(M \oplus H) \\ H = \tau_k G \oplus \mathrm{D}\Lambda \in \operatorname{gen}_1^{\mathbf{F}_G}(M) \cap (M \oplus G)^{\perp 1\sim(k-1)} \end{array} \right\}$$

and

$$\{ {}_\Gamma L, {}_\Gamma R \mid \exists \text{ a strong } k\text{-}{}_\Gamma M\text{-dualizing sequence from } L \text{ to } R \}.$$

Proof. This follows from Lemma 2.3.11, Lemma 4.3.2 and Lemma 4.3.4. □

Example 4.3.6. Let M be a faithfully balanced Λ-module and assume that it has a summand $X \oplus \tau^- X$ with X not injective. We define $G = \Lambda \oplus \tau^- X$, $H = \mathrm{D}\Lambda \oplus X$ and $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$. Then, by definition we have $G \in \operatorname{cogen}^1(M) = \operatorname{cogen}^1_{\mathbf{F}}(M)$ and $H \in \operatorname{gen}_1(M) = \operatorname{gen}_1^{\mathbf{F}}(M)$. Therefore, we obtain for $\Gamma = \operatorname{End}_\Lambda(M)$ a strong add$({}_\Gamma M)$-dualizing sequence with a projective-plus-M left end and an injective-plus-M right end.

Now, we can formulate a relative version of the generator/ cogenerator and Morita-Tachikawa correspondence.

Corollary 4.3.7. *(1) (relative generator correspondence)*
The Auslander-Solberg assignment $[\Lambda, M, G] \mapsto [\operatorname{End}(M), M, (G, M)]$ is an involution on the set of triples $[\Lambda, M, G]$ with $\Lambda \oplus M \in \operatorname{add}(G)$ and M is 1-$\mathbf{F}_G$-faithful.

(2) (relative cogenerator correspondence)
The dual Auslander-Solberg assignment $[\Lambda, M, H] \mapsto [\operatorname{End}(M), M, \mathrm{D}(M, H)]$ is an involution on the set of triples $[\Lambda, M, H]$ with $\mathrm{D}\Lambda \oplus M \in \operatorname{add}(H)$ and M is 1-$\mathbf{F}^H$-faithful.

(3) (relative Morita-Tachikawa correspondence)
The assignment $[\Lambda, M, G, H] \mapsto [\operatorname{End}(M), M, L = (G, M), R = \mathrm{D}(M, H)]$ is a bijection between

* *$[\Lambda, M, G, H]$ with $\Lambda \in \operatorname{add}(G), \mathrm{D}\Lambda \in \operatorname{add}(H)$, $G = \Lambda \oplus \tau^- H$ and $M \in \operatorname{add}(G) \cap \operatorname{add}(H)$ is 1-$\mathbf{F}_G$-faithful, and*
* *$[\Gamma, N, L, R]$ with L, R are the ends of a strong add(N)-dualizing sequence with $\Gamma \in \operatorname{add}(L)$ and $\mathrm{D}\Gamma \in \operatorname{add}(R)$.*

4.4 F-dualizing summands

Of course, we can also consider relative dualizing summands.

Definition 4.4.1. Let $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$, $M, L \in \Lambda$-mod and assume M is a summand of L. We say M is an $\mathbf{F}$-dualizing summand of L if $L \in \operatorname{cogen}^1_{\mathbf{F}}(M)$. For $k \geq 0$, we say it is a k-$\mathbf{F}$-dualizing summand if $L \in \operatorname{cogen}^k_{\mathbf{F}}(M)$.

Relative dualizing summands have the properties which we expect from them:

Lemma 4.4.2. *Let $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ and M, N be Λ-modules and $L = M \oplus N$, $k \geq 1$. If $N \in \operatorname{cogen}^k_{\mathbf{F}}(M)$ (i.e., M is k-$\mathbf{F}$-dualizing summand of L), then M is 1-$\mathbf{F}$-faithful if and only if L is 1-$\mathbf{F}$-faithful.*
If $H \in \operatorname{gen}_1(M)$, then $\operatorname{cogen}^k_{\mathbf{F}}(M) = \operatorname{cogen}^k_{\mathbf{F}}(L)$. Furthermore, in this case if also $\operatorname{copres}^k_{\mathbf{F}}(L) = \operatorname{cogen}^k_{\mathbf{F}}(L)$ then we have $\operatorname{copres}^k_{\mathbf{F}}(M) = \operatorname{cogen}^k_{\mathbf{F}}(M)$.
In particular, if M is 1-$\mathbf{F}$-faithful, then $M \oplus P \oplus I$ is 1-$\mathbf{F}$-faithful for every $\mathbf{F}$-projective module P and $\mathbf{F}$-injective module I.

Proof. Let $\Sigma = \operatorname{End}_\Lambda(H)$. We consider the duality for M from Lemma 4.2.4:

$$(-, H)\colon \operatorname{cogen}^k_{\mathbf{F}}(M) \longleftrightarrow \operatorname{cogen}^1({}_\Sigma H) \cap \operatorname{gen}_k({}_\Sigma(M, H))\colon (-, H)$$

and also for L we have

$$(-, H)\colon \operatorname{cogen}^k_{\mathbf{F}}(L) \longleftrightarrow \operatorname{cogen}^1({}_\Sigma H) \cap \operatorname{gen}_k({}_\Sigma(L, H))\colon (-, H).$$

Since (M, H) is a summand of (L, H) and $(L, H) \in \operatorname{gen}_k(M, H)$ follows that $\operatorname{gen}_1(L, H) \subseteq \operatorname{gen}_1(M, H)$ (dual argument to 1-dualizing summand situation).
Furthermore, we claim: if $H \in \operatorname{gen}_1(M)$, then ${}_\Sigma(M, H)$ is faithfully balanced (and therefore, the claim follows from the dual of Lemma 2.3.4 and using the duality from above again). So, assume there is an exact sequence $M_1 \to M_0 \to H \to 0$ with $M_i \in \operatorname{add}(M)$ and $(M, -)$ exact on it. Apply $(-, H)$ to it, to obtain an exact sequence $0 \to \Sigma \to (M_0, H) \to (M_1, H)$. Apply $(-, (M, H))$ to it and using $((X, H), (Y, H)) = (Y, X)$ for all Λ-modules X, Y you can identify the result with the complex $(M, M_1) \to (M, M_0) \to (M, H) \to 0$ which we know is exact since $H \in \operatorname{gen}_1(M)$. This proves $\Sigma \in \operatorname{cogen}^1((M, H))$ and therefore the claim. The remaining claims are proven as in Lemma 2.3.4. □

Example 4.4.3. Let G be a generator and $\mathbf{F} = \mathbf{F}_G$. Then a 1-$\mathbf{F}$-faithful summand of G is the same as an $\mathbf{F}$-dualizing summand of G. These are easily determined as follows, let $H = \mathrm{D}\,\Lambda \oplus \tau G$ and $P_1 \to P_0 \to H \to 0$ a minimal $\mathbf{F}$-presentation with $P_i \in \operatorname{add}(G)$. Then, the 1-$\mathbf{F}$-faithful summands of G are the summands P of G with $P_1 \oplus P_0 \in \operatorname{add}(P)$. Of course, with a dual statement one can find the 1-$\mathbf{F}$-faithful (i.e., the $\mathbf{F}$-codualizing) summands of H.

Chapter 5

Relative cotilting theory

5.1 Relative cotilting modules

Relative cotilting modules are introduced in [AS93c].

Definition 5.1.1. Let $\mathbf{F} = \mathbf{F}^H \subseteq \mathrm{Ext}^1_\Lambda$ be an additive subbifunctor with H a cogenerator. We call a Λ-module C a k-$\mathbf{F}$-cotilting module if

(i) it is $\mathbf{F}$-self-orthogonal (i.e., $\mathrm{Ext}^{>0}_{\mathbf{F}}(C, C) = 0$),

(ii) $\mathrm{id}_{\mathbf{F}}\, C \leq k$, and

(iii) there is an $\mathbf{F}$-exact sequence $0 \to C_k \to \cdots \to C_1 \to C_0 \to H \to 0$ with $C_i \in \mathrm{add}(C)$.

We recall a result of Wei. Partially, it is already proven in [AR91a].

Theorem 5.1.2. *([Wei10, Theorem 3.10]) Let $\mathbf{F} \subseteq \mathrm{Ext}^1(-,-)$ be an additive subbifunctor with enough projectives and injectives, C be a Λ-module and let $k \geq 1$. Then the following are equivalent*

(1) C is a k-$\mathbf{F}$-cotilting module.

(2) $\mathrm{copres}^{k-1}_{\mathbf{F}}(C) = \bigcap_{i \geq 1} \ker \mathrm{Ext}^i_{\mathbf{F}}(-, C)$.

In this case, we also have $\mathrm{copres}^{k-1}_{\mathbf{F}}(C) = \mathrm{cogen}^{k-1}_{\mathbf{F}}(C)$ *and*

$$\mathrm{cogen}^{k-1}_{\mathbf{F}}(C) = \mathrm{cogen}^{k}_{\mathbf{F}}(C) = \mathrm{cogen}^{k+1}_{\mathbf{F}}(C) = \cdots = \mathrm{cogen}^{\infty}_{\mathbf{F}}(C).$$

Lemma 5.1.3. *Let $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H \subseteq \mathrm{Ext}^1(-,-)$ be an additive subbifunctor with enough projectives and injectives. Let $k \geq 1$ and M an $\mathbf{F}$-self-orthogonal module. If $\mathrm{id}_{\mathbf{F}}\, M \leq 1$ and $H \in \mathrm{gen}^{\mathbf{F}}_{k-1}(M)$, then $C = M \oplus \Omega^k_M H$ is a k-$\mathbf{F}$-cotilting module. Furthermore, we have*

$$\mathrm{cogen}^{k-1}_{\mathbf{F}}(M) = \bigcap_{i \geq 1} \ker \mathrm{Ext}^i_{\mathbf{F}}(-, C).$$

Then M is an $(k-1)$-$\mathbf{F}$-dualizing summand of C.

Proof. We first prove that C is a k-$\mathbf{F}$-cotilting module. It is straightforward to check $\mathrm{id}_{\mathbf{F}}\, C \leq k$ by induction on k. Now we check C is $\mathbf{F}$-self-orthogonal. By using the definition of $\Omega^k_M H$ by approximations we easily check $\mathrm{Ext}^i_{\mathbf{F}}(M, \Omega^k_M H) = 0$ for all $i \geq 1$. Then use the fact that M is $\mathbf{F}$-self-orthogonal we have

$$\mathrm{Ext}^i_{\mathbf{F}}(\Omega^k_M H, M) \cong \mathrm{Ext}^{i+1}_{\mathbf{F}}(\Omega^{k-1}_M H, M) = 0$$

for all $i \geq 1$, where the last equality holds since $\operatorname{id}_{\mathbf{F}} M \leq 1$. To see $\operatorname{Ext}^i_{\mathbf{F}}(\Omega^k_M H, \Omega^k_M H) = 0$ for all $i \geq 1$ we use $\operatorname{Ext}^i_{\mathbf{F}}(M, \Omega^k_M H) = 0$ for all $i \geq 1$ and $\operatorname{id}_{\mathbf{F}} C \leq k$. More precisely, We apply the functor $(-, \Omega^k_M H)$ to the $\mathbf{F}$-exact sequences $0 \to \Omega^t_M H \to M_{t-1} \to \Omega^{t-1}_M H \to 0$, with $M_{t-1} \in \operatorname{add}(M)$, to conclude

$$\operatorname{Ext}^i_{\mathbf{F}}(\Omega^k_M H, \Omega^k_M H) \cong \operatorname{Ext}^{i+1}_{\mathbf{F}}(\Omega^{k-1}_M H, \Omega^k_M H) \cong \cdots \cong \operatorname{Ext}^{i+k}_{\mathbf{F}}(H, \Omega^k_M H) = 0$$

since $\operatorname{id}_{\mathbf{F}} C \leq k$. Together with $H \in \operatorname{gen}^{\mathbf{F}}_{k-1}(M)$, we conclude that C is an k-$\mathbf{F}$-cotilting module.

Furthermore, it is easy to check $\operatorname{cogen}^{k-1}_{\mathbf{F}}(M) \subseteq \bigcap_{i \geq 1} \ker \operatorname{Ext}^i_{\mathbf{F}}(-, C)$ by using the fact that $\operatorname{id}_{\mathbf{F}} C \leq k$. We prove the other inclusion by induction over k.

For $k = 1$. By definition we have $C \in \operatorname{cogen}_{\mathbf{F}}(M)$ and this implies

$$\ker \operatorname{Ext}^1_{\mathbf{F}}(-, C) = \operatorname{cogen}_{\mathbf{F}}(C) \subseteq \operatorname{cogen}_{\mathbf{F}}(M)$$

by Theorem 5.1.2.

For $k \geq 2$. Since C does depend on k we denote it in this part of the proof by C_k. We divide the proof into the follwing parts.

(i) $\bigcap_{i \geq 1} \ker \operatorname{Ext}^i_{\mathbf{F}}(-, C_k) \subseteq \bigcap_{i \geq 1} \ker \operatorname{Ext}^i_{\mathbf{F}}(-, C_{k-1})$. This is easy to see that there is an $\mathbf{F}$-exact sequence $0 \to C_k \to M' \to C_{k-1} \to 0$ with $M' \in \operatorname{add}(M)$.

(ii) By induction hypothesis we may assume $\bigcap_{i \geq 1} \ker \operatorname{Ext}^i_{\mathbf{F}}(-, C_k) \subseteq \operatorname{cogen}^{k-2}_{\mathbf{F}}(M)$. Let $X \in \bigcap_{i \geq 1} \ker \operatorname{Ext}^i_{\mathbf{F}}(-, C_k)$, so there exists an $\mathbf{F}$-exact sequence

$$0 \to X \to M^0 \to \cdots \to M^{k-2} \to Z \to 0$$

with $M^i \in \operatorname{add}(M)$ and $(-, M)$ exact on it. We claim $Z \in \operatorname{cogen}_{\mathbf{F}}(M) = \ker \operatorname{Ext}^1_{\mathbf{F}}(-, C_1)$. We split the sequence up into short $\mathbf{F}$-exact sequences

$$0 \to X^t \to M^t \to X^{t+1} \to 0, \quad 0 \leq t \leq k-2$$

with $X := X^0, Z := X^{k-1}$. Since $(-, M)$ is exact on the sequence for $t = k-2$, we conclude $\operatorname{Ext}^1_{\mathbf{F}}(Z, M) = 0$. So, it is enough to see $\operatorname{Ext}^1_{\mathbf{F}}(Z, \Omega^1_M H) = 0$.

(iii) $\operatorname{Ext}^1_{\mathbf{F}}(Z, \Omega^1_M H) \cong \operatorname{Ext}^k_{\mathbf{F}}(Z, \Omega^k_M H)$. Applying $(Z, -)$ to the sequences

$$0 \to \Omega^t_M H \to M_{t-1} \to \Omega^{t-1}_M H \to 0$$

to conclude $\operatorname{Ext}^i_{\mathbf{F}}(Z, \Omega^{t-1}_M H) \cong \operatorname{Ext}^{i+1}_{\mathbf{F}}(Z, \Omega^t_M H)$ for all $i \geq 1$. Applying this iteratively gives (iii).

(iv) $\operatorname{Ext}^k_{\mathbf{F}}(Z, \Omega^k_M H) \cong \operatorname{Ext}^1(X, \Omega^k_M H)$. Applying $(-, \Omega^k_M H)$ to the short exact sequences $0 \to X^t \to M^t \to X^{t+1} \to 0$ yields $\operatorname{Ext}^i_{\mathbf{F}}(X^t, \Omega^k_M H) \cong \operatorname{Ext}^{i+1}_{\mathbf{F}}(X^{t+1}, \Omega^k_M H)$ for all $i \geq 1$. Applying this iteratively gives (iv).

But since $X \in \bigcap_{i \geq 1} \ker \operatorname{Ext}^i_{\mathbf{F}}(-, C_k)$ we have $\operatorname{Ext}^1(X, \Omega^k_M H) = 0$ and therefore, using (iii) and (iv) this implies $\operatorname{Ext}^1_{\mathbf{F}}(Z, \Omega^1_M H) = 0$. □

Remark 5.1.4. If C is a 1-$\mathbf{F}$-cotilting module and M an $\mathbf{F}$-dualizing summand, then we have $M = C$. Therefore, non-trivial $\mathbf{F}$-dualizing summands only appear in the theory of $\mathbf{F}$-cotilting modules with $\operatorname{id}_{\mathbf{F}} > 1$.

Example 5.1.5. Let $M \in \Lambda$-mod be rigid (i.e., $\operatorname{Ext}^1_\Lambda(M, M) = 0$) and also $X := \Omega^- M$ be rigid, then for $H = X \oplus \mathrm{D}\Lambda$, $\mathbf{F} = \mathbf{F}^H$ we have $\operatorname{id}_{\mathbf{F}} M \leq 1$, M is $\mathbf{F}$-self-orthogonal and $X \in \operatorname{gen}_1(M)$. If we now assume additionally that M is faithfully balanced and $\operatorname{Ext}^{1,2}(M \oplus X, X) = 0$, then we have

$$H \in \operatorname{gen}^{\mathbf{F}}_1(M) = \operatorname{gen}_1(M) \cap \bigcap_{i=1}^{2} \ker \operatorname{Ext}^i(-, X)$$

(cf. Example 4.1.1) implying that M is 1-$\mathbf{F}$-faithful. In particular, we have then $C := M \oplus \Omega_M H$ is a 1-$\mathbf{F}$-cotilting module with

$$\operatorname{cogen}_{\mathbf{F}}(M) = \bigcap_{i \geq 1} \ker \operatorname{Ext}^i_{\mathbf{F}}(-, C).$$

Example 5.1.6. Let X be an arbitrary faithfully balanced module and $k \geq 1$. If $\tau X \in \operatorname{cogen}^{k-1}(X)$, then $\operatorname{cogen}^{k-1}(X)$ is the $\mathbf{F}^X$-perpendicular category $\bigcap_{i \geq 1} \ker \operatorname{Ext}^i_{\mathbf{F}^X}(-, C)$ for the $\mathbf{F}^X$-k-coltilting module $C = X \oplus \Omega^k_X \operatorname{D} \Lambda$. If $\operatorname{add}(X)$ is, for example, τ-stable then $\tau X \in \operatorname{cogen}^{k-1}(X)$.

More generally we will study the $\mathbf{F}$-cotilting modules obtained from a 1-$\mathbf{F}$-faithful $\mathbf{F}$-injective module as special cotilting modules.

Let us fix an $\mathbf{F}$-exact resolution by $\mathbf{F}$-projectives of H (with $\operatorname{add}(H) = \mathcal{I}(\mathbf{F})$)

$$\cdots \to P_2 \to P_1 \to P_0 \to H \to 0.$$

Then we obtain the relative version of [IZ18, Theorem 1.1] as follows, let $\operatorname{cotilt}^{\mathbf{F}}_n(\Lambda)$ be the set of basic isomorphism classes of n-$\mathbf{F}$-cotilting Λ-modules. It is naturally a poset with respect $C \leq C'$ if and only if $C \in \bigcap_{i \geq 1} \ker \operatorname{Ext}^i_{\mathbf{F}}(-, C')$.

Lemma 5.1.7. *Let $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ and $n \geq 1$, we define $P := \bigoplus_{j=0}^{n-1} P_j$. If $\operatorname{id}_{\mathbf{F}} P \leq n$ and $\operatorname{id}_{\mathbf{F}} \Omega^n_P H \leq n$, then $C = P \oplus \Omega^n_P H$ is an n-$\mathbf{F}$-cotilting module and it is the minimum element in $\operatorname{cotilt}^{\mathbf{F}}_n(\Lambda)$. Furthermore, if $\operatorname{id}_{\mathbf{F}} P_j \leq j+1$, $1 \leq j \leq n-1$, then $\operatorname{id}_{\mathbf{F}} P \oplus \Omega^n_P H \leq n$.*

Proof. We check that $\operatorname{id}_{\mathbf{F}} C \leq n$ implies that C is $\mathbf{F}$-self-orthogonal. Observe that $\Omega^n_P H = \Omega^n_{\mathbf{F}} H$ and let $i \geq 1$, then we have

$$\operatorname{Ext}^i_{\mathbf{F}}(C, C) = \operatorname{Ext}^i_{\mathbf{F}}(\Omega^n_{\mathbf{F}} H, C) = \operatorname{Ext}^{i+n}_{\mathbf{F}}(H, C) = 0$$

since $\operatorname{id}_{\mathbf{F}} C \leq n$. Since the last condition is fulfilled by definition of C, we can conclude that C is an n-$\mathbf{F}$-cotilting module.

If $L \in \operatorname{cotilt}^{\mathbf{F}}_n(\Lambda)$, then we have by definition of C that

$$\operatorname{Ext}^i_{\mathbf{F}}(C, L) = \operatorname{Ext}^{i+n}_{\mathbf{F}}(H, L) = 0$$

since $\operatorname{id}_{\mathbf{F}} L \leq n$. Therefore C is the minimum.

The last claim is a straightforward induction on n. For $n = 1$ the claim follows from the previous lemma. For the induction step apply $(-, M)$ to the $\mathbf{F}$-exact sequence

$$0 \to \Omega^n_P H \to P_{n-1} \to \Omega^{n-1}_P H \to 0.$$

By hypothesis $\operatorname{id}_{\mathbf{F}} P_{n-1} \leq n$ and $\operatorname{id}_{\mathbf{F}} \Omega^{n-1}_P H \leq n-1$, we conclude $\operatorname{id}_{\mathbf{F}} \Omega^n_P H \leq n$. □

5.2 The 4-tuple assignment

Now we consider 4-tuples (Λ, M, L, G) with Λ a finite-dimensional algebra and M, L, G finite-dimensional Λ-modules. We define the following equivalence relation between these 4-tuples: (Λ, M, L, G) is equivalent to (Λ', M', L', G') if there is an equivalence of categories $\Lambda\text{-mod} \xrightarrow{\sim} \Lambda'\text{-mod}$ restricting to equivalences $\operatorname{add}(M) \xrightarrow{\sim} \operatorname{add}(M')$, $\operatorname{add}(L) \xrightarrow{\sim} \operatorname{add}(L')$ and $\operatorname{add}(G) \xrightarrow{\sim} \operatorname{add}(G')$. We denote by $[\Lambda, M, L, G]$ the equivalence class of a 4-tuple and we may assume the algebra and all the modules appearing in the equivalence class to be basic.

To establish a relative version of cotilting correspondence which is an involution, we will need the following definition.

Definition 5.2.1. We define the following assignment

$$[\Lambda, M, L, G] \mapsto [\Gamma, N, \widetilde{L}, \widetilde{G}]$$

with $\Gamma = \mathrm{End}_\Lambda(M)$, $N = {}_\Gamma M$, $\widetilde{L} = (G, M)$, $\widetilde{G} = (L, M)$ and call this the balanced Auslander-Solberg assignment or just the 4-tuple assignment.
The dual 4-tuple assignment is the following

$$[\Lambda, M, R, H] \mapsto [\Gamma, N, \widetilde{R}, \widetilde{H}]$$

with $\Gamma = \mathrm{End}(M)$, $N = {}_\Gamma M$, $\widetilde{R} = \mathrm{D}(M, H)$, $\widetilde{H} = \mathrm{D}(M, R)$. Since, we will always consider pairs (G, H) and (L, R) which determine each other, we will in later proofs combine the two assignments into a 6-tuple assignment

$$[\Lambda, M, L, R, G, H] \mapsto [\Gamma, N, \widetilde{L}, \widetilde{R}, \widetilde{G}, \widetilde{H}]$$

with $\Gamma = \mathrm{End}(M)$, $N = {}_\Gamma M$, $\widetilde{L} = (G, M)$, $\widetilde{R} = \mathrm{D}(M, H)$, $\widetilde{G} = (L, M)$, $\widetilde{H} = \mathrm{D}(M, R)$.

Lemma 5.2.2. *Keep the above notations. Then we have*

(1) The 4-tuple assignment restricts to an involution on the set of 4-tuples $[\Lambda, M, L, G]$ with $\Lambda \in \mathrm{add}(G)$, $\mathbf{F} = \mathbf{F}_G$, M is 1-$\mathbf{F}$-faithful, M is an $\mathbf{F}$-dualizing summand of L and L is the left end of an $\mathbf{F}$-exact strong $\mathrm{add}(M)$-dualizing sequence.

(2) The dual 4-tuple assignment restricts to an involution on the set of 4-tuples $[\Lambda, M, R, H]$ with $\mathrm{D}\,\Lambda \in \mathrm{add}(H)$, $\mathbf{F} = \mathbf{F}^H$, M is 1-$\mathbf{F}$-faithful, M is an $\mathbf{F}$-codualizing summand of R and R is the right end of an $\mathbf{F}$-exact strong M-dualizing sequence.

Proof. We take a 6-tuple $[\Lambda, M, L, R, G, H]$ with $\Lambda \in \mathrm{add}(G), \mathrm{D}\,\Lambda \in \mathrm{add}(H)$, $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$, M 1-$\mathbf{F}$-faithful and there is an $\mathbf{F}$-exact strong M-dualizing sequence

$$0 \to L \to M_0 \to M_1 \to R \to 0.$$

We want to see that applying the 6-tuple assignment gives an involution. So consider a 6-tuple $[\Gamma, N, \widetilde{L}, \widetilde{R}, \widetilde{G}, \widetilde{H}]$ with $\Gamma = \mathrm{End}(M)$, $N = {}_\Gamma M$, $\widetilde{L} = (G, M)$, $\widetilde{R} = \mathrm{D}(M, H)$, $\widetilde{G} = (L, M)$, $\widetilde{H} = \mathrm{D}(M, R)$. Clearly, $\Gamma \in \mathrm{add}(\widetilde{G})$, $\mathrm{D}\,\Gamma \in \mathrm{add}(\widetilde{H})$ since $M \in \mathrm{add}(L) \cap \mathrm{add}(R)$ and since L and R are ends of an $\mathrm{add}({}_\Lambda M)$-dualizing sequence we have $\mathbf{F}_{\widetilde{G}} = \mathbf{F}^{\widetilde{H}} =: \widetilde{\mathbf{F}}$. Since L is left end of a strong $\mathrm{add}(M)$-dualizing sequence, we have by Lemma 4.3.5 that $\widetilde{G} = (L, M) \in \mathrm{cogen}^1_{\widetilde{\mathbf{F}}}(N)$, this means N is 1-$\widetilde{\mathbf{F}}$-faithful. Since M is 1-$\mathbf{F}$-faithful we get a strong $\mathrm{add}(N)$-dualizing sequence

$$0 \to \widetilde{L} \to \widetilde{N_0} \to \widetilde{N_1} \to \widetilde{R} \to 0$$

by Lemma 4.3.5. It split off the summand

$$0 \to N \xrightarrow{1} N \xrightarrow{0} N \xrightarrow{1} N \to 0$$

and obtain an exact sequence

$$0 \to \widetilde{L}' \to N_0 \to N_1 \to \widetilde{R}' \to 0 \qquad (*)$$

with $N_i \in \mathrm{add}(N)$. The only missing property is that $(*)$ is $\widetilde{\mathbf{F}}$-exact. We first observe that $N_i = \mathrm{D}(M, I_i)$ with $0 \to H' \to I_1 \to I_0$ is an injective copresentation, $H = \mathrm{D}\,\Lambda \oplus H'$. Since $(\widetilde{G}, -) = ((L, M), -)$ is left exact, it is enough to check that it is also right exact on $(*)$.

Now, since $L \in \operatorname{cogen}^1_{\mathbf{F}^H}(M)$ we have a natural isomorphism $\mathrm{D}(L,H) \to ((L,M),\mathrm{D}(M,H))$ by Lemma 4.2.4 (1). In particular, we have a natural isomorphism

$$(\widetilde{G}, N_i) = ((L,M),\mathrm{D}(M,I_i)) \to \mathrm{D}(L,I_i)$$

since $I_i \in \operatorname{add}(H)$. This means when we apply $(\widetilde{G},-)$ to the last three nonzero terms of $(*)$ we get an exact sequence which identifies under the just mentioned natural isomorphism with

$$\mathrm{D}(L,I_0) \to \mathrm{D}(L,I_1) \to \mathrm{D}(L,H') \to 0$$

and this is exact. □

5.3 Relative cotilting correspondence

We give a generalization of the cotilting correspondence to a relative set-up together with a relative dualizing summand - this is a generalization of Auslander-Solberg's main results in [AS93c, AS93d] which we reobtain as a corollary. We will use the 4-tuple assignments for our theorem (see Definition 5.2.1, Lemma 5.2.2).

As before, we fix an additive subbifunctor $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ of $\operatorname{Ext}^1_\Lambda(-,-)$ for some generator G and cogenerator H.

Define

$$\mathsf{K}^{+,b}_{\mathbf{F}}(\operatorname{add}(H)) = \{Y \in \mathsf{K}^+(\operatorname{add}(H)) \mid \exists\, n \in \mathbb{Z} \text{ such that } \mathsf{H}^i(\operatorname{Hom}_\Lambda(Y,H)) = 0 \text{ for } i \geq n\}$$

then we have $\mathsf{D}^b_{\mathbf{F}}(\Lambda\text{-mod}) \simeq \mathsf{K}^{+,b}_{\mathbf{F}}(\operatorname{add}(H))$ as triangulated categories, where $\mathsf{D}^b_{\mathbf{F}}(\Lambda\text{-mod})$ is the bounded derived category of the exact category Λ-mod with the exact structure induced by $\mathbf{F}$. For more on the derived category of an exact category we refer to [Nee90, Kel96, Pan16]. As in the standard case, one can prove that an $\mathbf{F}$-self-orthogonal Λ-module L is an $\mathbf{F}$-cotilting module if and only if $\mathsf{Thick}(L) = \mathsf{K}^b(\operatorname{add}(H))$ where by $\mathsf{Thick}(L)$ we mean the smallest triangulated subcategory of $\mathsf{K}^b(\operatorname{add}(H))$ which contains L and closed under direct summands. We also have the following lemma which can be proved by the same argument in the standard case (cf. [CHU94, AI12]).

Lemma 5.3.1. *Let $L = M \oplus U$ be a basic $\mathbf{F}$-cotilting module.*

(1) If there exists an $\mathbf{F}$-exact sequence $0 \to U \xrightarrow{f} M_0 \to V \to 0$ with f the left minimal $\operatorname{add}(M)$-approximation of U, then $M \oplus V$ is a basic $\mathbf{F}$-cotilting module with $\operatorname{id}_{\mathbf{F}}(M\oplus V) \leq \operatorname{id}_{\mathbf{F}} L$. Furthermore, this $\mathbf{F}$-exact sequence (after adding 1_M to f and its cokernel) gives rise to a strong 0-$\operatorname{add}(M)$-dualizing sequence with $\operatorname{Ext}^i(U\oplus M, V\oplus M) = 0$ for $i \geq 1$.

(2) If there exists an $\mathbf{F}$-exact sequence $0 \to V \to M_1 \xrightarrow{g} U \to 0$ with g the right minimal $\operatorname{add}(M)$-approximation of U, then $M \oplus V$ is a basic $\mathbf{F}$-cotilting module with $\operatorname{id}_{\mathbf{F}}(M \oplus V) \leq \operatorname{id}_{\mathbf{F}} L + 1$. Again this gives rise to a strong 0-$\operatorname{add}(M)$-dualizing sequence with $\operatorname{Ext}^i(V\oplus M, U\oplus M) = 0$ for $i \geq 1$.

Now we are ready to present our improvement of Auslander and Solberg's results. Recall, the 4-tuple assignment

$$[\Lambda, M, L, G] \mapsto [\Gamma, {}_\Gamma M, \widetilde{L}, \widetilde{G}]$$

where $\Gamma = \operatorname{End}_\Lambda(M)$, $\widetilde{L} = (G,M)$ and $\widetilde{G} = (L,M)$. We also consider the dual 4-tuple assignment

$$[\Lambda, M, R, H] \mapsto [\Gamma, {}_\Gamma M, \widetilde{R}, \widetilde{H}]$$

where $\Gamma = \operatorname{End}_\Lambda(M)$, $\widetilde{R} = \mathrm{D}(M,H)$ and $\widetilde{H} = \mathrm{D}(M,R)$.

Theorem 5.3.2. *Keep the above notations. Then we have*

(1) The 4-tuple assignment restricts to an involution on the set of 4-tuples $[\Lambda, M, L, G]$ satisfying

(1a) $\Lambda \in \operatorname{add}(G)$, $\mathbf{F} = \mathbf{F}_G$,

(1b) L is $\mathbf{F}$-cotilting and M is an $\mathbf{F}$-dualizing summand of L.

(2) The dual 4-tuple assignment restricts to an involution on the set of 4-tuples $[\Lambda, M, R, H]$ satisfying

(2a) $\mathrm{D}\,\Lambda \in \operatorname{add}(H)$, $\mathbf{F} = \mathbf{F}^H$,

(2b) R is $\mathbf{F}$-cotilting and M is an $\mathbf{F}$-codualizing summand of R (that is, $M \in \operatorname{add}(R)$ and $R \in \operatorname{gen}_1^{\mathbf{F}}(M)$).

Furthermore, for an assignment $[\Lambda, M, R, H] \mapsto [\Gamma, {}_\Gamma M, \widetilde{R}, \widetilde{H}]$ we have $\operatorname{id}_{\mathbf{F}^H} R = \operatorname{id}_{\mathbf{F}^{\widetilde{H}}} \widetilde{R}$.

Proof. We prove (1) and (2) together.

We want to use Lemma 5.2.2, so we first prove that (1b) (or (2b)) implies that M is 1-$\mathbf{F}$-faithful. To prove M is 1-$\mathbf{F}$-faithful we need to show the natural map $(M, H) \otimes_\Gamma (G, M) \to (G, H)$ is an isomorphism, where $\Gamma = \operatorname{End}_\Lambda(M)$. Since L is $\mathbf{F}$-cotilting it is 1-$\mathbf{F}$-faithful and thus the natural map $(L, H) \otimes_B (G, L) \to (G, H)$ is an isomorphism, where $B = \operatorname{End}_\Lambda(L)$. By Lemma 4.2.4 (1), M being an $\mathbf{F}$-dualizing summand of L is equivalent to that the natural map $(M, H) \otimes_\Gamma (L, M) \to (L, H)$ is an isomorphism. Hence we have

$$(M, H) \otimes_\Gamma (G, M) \xrightarrow{\cong} ((M, H) \otimes_\Gamma (L, M)) \otimes_B (G, L) \xrightarrow{\cong} (L, H) \otimes_B (G, L) \xrightarrow{\cong} (G, H)$$

as desired. Since L is $\mathbf{F}$-cotilting and M is an $\mathbf{F}$-dualizing summand of L, we have an $\mathbf{F}$-exact strong $\operatorname{add}(M)$-dualizing sequence $0 \to L \to M_0 \to M_1 \to R \to 0$ with $M_i \in \operatorname{add}(M)$. By Lemma 5.3.1 we see that R is also an $\mathbf{F}$-cotilting module. Now, by Lemma 5.2.2 the 6-tuple assignment restricts to an involution on the set of 6-tuples $[\Lambda, M, L, R, G, H]$ satisfying the conditions (1a), (1b), (2a) and (2b) if we prove that $\widetilde{R} := \mathrm{D}(M, H)$ and $\widetilde{L} := (G, M)$ are $\widetilde{\mathbf{F}}$-cotilting modules, where $\widetilde{\mathbf{F}} := \mathbf{F}_{\widetilde{G}} = \mathbf{F}^{\widetilde{H}}$, $\widetilde{G} = (L, M)$ and $\widetilde{H} = \mathrm{D}(M, R)$.

Assume $\operatorname{id}_{\mathbf{F}} R = n$, then we have $\mathbf{F}$-exact sequences

$$0 \to R \to H^0 \to H^1 \to \cdots \to H^{n-1} \to H^n \to 0 \qquad (*)$$

and

$$0 \to R_n \to R_{n-1} \to \cdots \to R_1 \to R_0 \to H \to 0. \qquad (**)$$

The functor $(M, -)$ is exact on both $(*)$ and $(**)$. Applying $\mathrm{D}(M, -)$ to $(**)$ we get an exact sequence

$$0 \to \mathrm{D}(M, H) = \widetilde{R} \to \mathrm{D}(M, R_0) \to \mathrm{D}(M, R_1) \to \cdots \to \mathrm{D}(M, R_{n-1}) \to \mathrm{D}(M, R_n) \to 0 \qquad (\star\star)$$

of Γ-modules, where each $\mathrm{D}(M, R_i) \in \operatorname{add}(\widetilde{H})$ is an $\widetilde{\mathbf{F}}$-injective module. We claim that this sequence is $\widetilde{\mathbf{F}}$-exact which will imply that $(\star\star)$ is an $\widetilde{\mathbf{F}}$-injective resolution of $\widetilde{R}$ and so $\operatorname{id}_{\widetilde{\mathbf{F}}} \widetilde{R} \leq n$. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & ((L, M), \mathrm{D}(M, H)) & \to & ((L, M), \mathrm{D}(M, R_0)) & \to \cdots \to & ((L, M), \mathrm{D}(M, R_n)) & \to & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \to & \mathrm{D}((M, H) \otimes (L, M)) & \to & \mathrm{D}((M, R_0) \otimes (L, M)) & \to \cdots \to & \mathrm{D}((M, R_n) \otimes (L, M)) & \to & 0 \\
 & & \uparrow\cong & & \uparrow\cong & & \uparrow\cong & & \\
0 & \to & \mathrm{D}(L, H) & \to & \mathrm{D}(L, R_0) & \to \cdots \to & \mathrm{D}(L, R_n) & \to & 0
\end{array}$$

The first row and the second row are naturally isomorphic by the Hom-Tensor adjunction, the second row and the last row are naturally isomorphic because $H, R \in \mathrm{gen}_1^{\mathbf{F}_L}(M)$. The last row is obtained by applying the functor $\mathrm{D}(L, -)$ to $(**)$ and it is exact . Hence the first row is exact and the claim follows.

Similarly, apply the functor $\mathrm{D}(M, -)$ to $(*)$ we will get an $\widetilde{\mathbf{F}}$-exact sequence

$$0 \to \mathrm{D}(M, H_n) \to \mathrm{D}(M, H_{n-1}) \to \cdots \to \mathrm{D}(M, H_1) \to \mathrm{D}(M, H_0) \to \mathrm{D}(M, R) = \widetilde{H} \to 0 \qquad (\star)$$

with $\mathrm{D}(M, H_i) \in \mathrm{add}(\widetilde{R})$. Now applying the functor $\mathrm{D}(\widetilde{R}, -) = \mathrm{D}((M, H), -)$ to $(\star\star)$ we will get the first row of the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathrm{D}(M,H), \mathrm{D}(M,H)) & \longrightarrow & (\mathrm{D}(M,H), \mathrm{D}(M,R_0)) & \longrightarrow \cdots \longrightarrow & (\mathrm{D}(M,H), \mathrm{D}(M,R_n)) & \longrightarrow & 0 \\
 & & \cong \uparrow \mathrm{D}(M,-) & & \cong \uparrow \mathrm{D}(M,-) & & \cong \uparrow \mathrm{D}(M,-) & & \\
0 & \longrightarrow & (H, H) & \longrightarrow & (R_0, H) & \longrightarrow \cdots \longrightarrow & (R_n, H) & \longrightarrow & 0
\end{array}$$

The lower row is exact because $(**)$ is $\mathbf{F}$-exact and the vertical arrows are isomorphisms because $H, R \in \mathrm{gen}_1(M)$. Therefore the upper row is exact and this means $\mathrm{Ext}^i_{\widetilde{\mathbf{F}}}(\widetilde{R}, \widetilde{R}) = 0$ for $i > 0$. Combining $(\star)$ and $(\star\star)$, we see that $\widetilde{R}$ is an $\widetilde{\mathbf{F}}$-cotilting module. According to the proof of Lemma 5.2.2, there is a strong $\mathrm{add}({}_\Gamma M)$-dualizing sequnce $0 \to \widetilde{L} \to \widetilde{M_0} \to \widetilde{M_1} \to \widetilde{R} \to 0$ with $\widetilde{M_i} \in \mathrm{add}({}_\Gamma M)$. Again by Lemma 5.3.1, we conclude that $\widetilde{L}$ is an $\widetilde{\mathbf{F}}$-cotilting module.

Finally, since the dual 4-tuple assignment restricts to an involution we have $\mathrm{id}_{\mathbf{F}} R = \mathrm{id}_{\widetilde{\mathbf{F}}} \widetilde{R}$. □

Corollary 5.3.3. *(1) The functors $(-, {}_\Lambda M) : \Lambda\text{-mod} \longleftrightarrow \Gamma\text{-mod} : (-, {}_\Gamma M)$ restrict to dualities ${}^{0<\perp_{\mathbf{F}}}L \longleftrightarrow {}^{0<\perp_{\widetilde{\mathbf{F}}}}\widetilde{L}$ and ${}^{0<\perp_{\mathbf{F}}}R \longleftrightarrow {}^{0<\perp_{\widetilde{\mathbf{F}}}}\widetilde{R}$.*

(2) We have $\mathrm{id}_{\mathbf{F}} R \le \mathrm{id}_{\mathbf{F}} L \le \mathrm{id}_{\mathbf{F}} R + 2$ and $\mathrm{id}_{\widetilde{\mathbf{F}}} \widetilde{R} \le \mathrm{id}_{\widetilde{\mathbf{F}}} \widetilde{L} \le \mathrm{id}_{\widetilde{\mathbf{F}}} \widetilde{R} + 2$.

Proof. (1) Given $X \in {}^{0<\perp_{\mathbf{F}}}L$ we need to show that $(X, {}_\Lambda M) \in {}^{0<\perp_{\widetilde{\mathbf{F}}}}\widetilde{L}$ and it is enought to show $(X, {}_\Lambda M) \in \mathrm{copres}^\infty_{\widetilde{\mathbf{F}}}(\widetilde{L})$ by Theorem 5.1.2. Taking an $\mathbf{F}$-projective resolution $\cdots \to P_1 \to P_0 \to X \to 0$ of X and applying $(-, {}_\Lambda M)$ to get a complex

$$0 \to (X, M) \to (P_0, M) \to (P_1, M) \to \cdots .$$

A standard argument shows that it is $\widetilde{\mathbf{F}}$-exact and therefore $(X, {}_\Lambda M) \in \mathrm{copres}^\infty_{\widetilde{\mathbf{F}}}(\widetilde{L})$. Now given $Y \in {}^{0<\perp_{\mathbf{F}}}R$ we will prove that $(Y, {}_\Lambda M) \in {}^{0<\perp_{\widetilde{\mathbf{F}}}}\widetilde{R}$. Applying $((Y, M), -)$ to the $\widetilde{\mathbf{F}}$-injective resolution $(\star\star)$ of $\widetilde{R}$ gives a complex

$$0 \to ((Y, M), \widetilde{R}) \to ((Y, M), \mathrm{D}(M, R_0)) \to \cdots \to ((Y, M), \mathrm{D}(M, R_n)) \to 0.$$

One can easily check that it is in fact exact and thus $(Y, {}_\Lambda M) \in {}^{0<\perp_{\widetilde{\mathbf{F}}}}\widetilde{R}$.
(2) follows from Lemma 5.3.1. □

Remark 5.3.4. In particular, if we take $M = L$ to be the trivial $\mathbf{F}$-dualizing summand then we have $[\Lambda, L, L, G] \mapsto [\Gamma, {}_\Gamma L, \widetilde{L} = (G, L), \Gamma]$ and thus $\widetilde{L}$ is a cotilting Γ-module, ${}_\Gamma L$ is a dualizing summand of $\widetilde{L}$ and $\mathrm{id}_{\mathbf{F}} L \le \mathrm{id}_\Gamma \widetilde{L} \le \mathrm{id}_{\mathbf{F}} L + 2$. This gives [AS93c, Theorem 3.13]. The fact that the 4-tuple assignment restricts to an involution gives [AS93d, Theorem 2.8].

5.4 Consequences of the relative cotilting correspondence

Derived equivalence induced by an F-dualizing summand

Let $[\Lambda, M, L, G]$ be a 4-tuple satisfying $\Lambda \in \operatorname{add}(G)$, $\mathbf{F} = \mathbf{F}_G$, L is $\mathbf{F}$-cotilting and M is an $\mathbf{F}$-dualizing summand of L. Then by Theorem 5.3.2 the 4-tuple assignment gives a 4-tuple $[\Gamma = \operatorname{End}_\Lambda(M), {}_\Gamma M, \widetilde{L} = (G, M), \widetilde{G} = (L, M)]$ satisfying $\Gamma \in \operatorname{add}(\widetilde{G})$, $\widetilde{\mathbf{F}} = \mathbf{F}_{\widetilde{G}}$, $\widetilde{L}$ is $\widetilde{\mathbf{F}}$-cotilting and ${}_\Gamma M$ is an $\widetilde{\mathbf{F}}$-dualizing summand of $\widetilde{L}$. We consider the derived categories of exact categories $\mathsf{D}^b_{\mathbf{F}}(\Lambda\text{-mod})$ and $\mathsf{D}^b_{\widetilde{\mathbf{F}}}(\Gamma\text{-mod})$ and we will show the functors $(-, {}_\Lambda M)$ and $(-, {}_\Gamma M)$ induce a duality between triangulated categories $\mathsf{D}^b_{\mathbf{F}}(\Lambda\text{-mod})$ and $\mathsf{D}^b_{\widetilde{\mathbf{F}}}(\Gamma\text{-mod})$.

Proposition 5.4.1. *Let $[\Lambda, M, L, G]$ be a 4-tuple such that $\Lambda \in \operatorname{add}(G)$, $\mathbf{F} = \mathbf{F}_G$, L is $\mathbf{F}$-cotilting and M is an $\mathbf{F}$-dualizing summand of L and let $[\Gamma, {}_\Gamma M, \widetilde{L}, \widetilde{G}]$ be the corresponding 4-tuple under the 4-tuple assignment. Then the functors $(-, {}_\Lambda M)$ and $(-, {}_\Gamma M)$ induce a contravariant triangle equivalence between $\mathsf{D}^b_{\mathbf{F}}(\Lambda\text{-mod})$ and $\mathsf{D}^b_{\widetilde{\mathbf{F}}}(\Gamma\text{-mod})$.*

Proof. Let $B = \operatorname{End}_\Lambda(L)$ and $\widetilde{B} = \operatorname{End}_\Gamma(\widetilde{L})$, then $C := (G, L)$ is a cotilting B-module and $\widetilde{C} := (\widetilde{G}, \widetilde{L})$ is a cotilting $\widetilde{B}$-module. By [Bua01, Proposition 4.4.3] , the functor $(-, {}_\Lambda L)$ induces a contravariant triangle equivalence between $\mathsf{D}^b_{\mathbf{F}}(\Lambda\text{-mod})$ and $\mathsf{D}^b(B\text{-mod})$ and the functor $(-, {}_\Gamma \widetilde{L})$ induces a contravariant triangle equivalence between $\mathsf{D}^b_{\widetilde{\mathbf{F}}}(\Gamma\text{-mod})$ and $\mathsf{D}^b(\widetilde{B}\text{-mod})$.

We note that by Lemma 2.1.3 (1) the composition

$$\operatorname{End}_B(C) = ((G, L), (G, L)) \xrightarrow{\cong} \operatorname{End}_\Lambda(G)^{op} \xrightarrow{\cong} ((G, M), (G, M)) = \operatorname{End}_\Gamma(\widetilde{L}) = \widetilde{B}$$

is an isomorphism of algebras. Similarly, we have $\operatorname{End}_{\widetilde{B}}(\widetilde{C}) \cong \operatorname{End}_\Gamma(\widetilde{G})^{op} \cong B$. Since ${}_B C$ is cotilting, ${}_{\widetilde{B}} C$ is also cotilting and we have

$${}_{\widetilde{B}} C = (B, {}_B C) = ((L, L), (G, L)) \xrightarrow{\cong} (G, L) \xrightarrow{\cong} ((L, M), (G, M)) = (\widetilde{G}, \widetilde{L}) = {}_{\widetilde{B}} \widetilde{C}$$

by Lemma 2.1.3 (1). It follows that the functors $(-, {}_B C)$ and $(-, {}_{\widetilde{B}} \widetilde{C})$ induce a contravariant triangle equivalence between $\mathsf{D}^b(B\text{-mod})$ and $\mathsf{D}^b(\widetilde{B}\text{-mod})$. The desired contravariant triangle equivalence follows by combining this duality and the above triangle dualities. □

Remark 5.4.2. As the above proof suggests, there exist triangle equivalences $\mathsf{D}^b_{\mathbf{F}}(\Lambda\text{-mod}) \simeq \mathsf{D}^b(\widetilde{B}\text{-mod})$ and $\mathsf{D}^b_{\mathbf{F}}(\Gamma\text{-mod}) \simeq \mathsf{D}^b(B\text{-mod})$. The dual version of Proposition 5.4.1 shows that an $\mathbf{F}$-codualizing summand of an $\mathbf{F}$-tilting module will induce a relative derived equivalence.

F-Gorenstein algebra

Recall that an algebra Λ is called *Gorenstein* if $\operatorname{id}({}_\Lambda\Lambda) < \infty$ and $\operatorname{id}(\Lambda_\Lambda) < \infty$. Define

$$\mathcal{P}^\infty(\Lambda) = \{X \in \Lambda\text{-mod} \mid \operatorname{pd}_\Lambda X < \infty\} \text{ and } \mathcal{I}^\infty(\Lambda) = \{Y \in \Lambda\text{-mod} \mid \operatorname{id}_\Lambda Y < \infty\}.$$

Then Λ being Gorenstein is equivalent to $\mathcal{P}^\infty(\Lambda) = \mathcal{I}^\infty(\Lambda)$. Let $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ be a subbifunctor of $\operatorname{Ext}^1_\Lambda$ and define

$$\mathcal{P}^\infty(\mathbf{F}) = \{X \in \Lambda\text{-mod} \mid \operatorname{pd}_{\mathbf{F}} X < \infty\} \text{ and } \mathcal{I}^\infty(\mathbf{F}) = \{Y \in \Lambda\text{-mod} \mid \operatorname{id}_{\mathbf{F}} Y < \infty\}.$$

Following [AS93a] we call an algebra $\mathbf{F}$-*Gorenstein* if $\mathcal{P}^\infty(\mathbf{F}) = \mathcal{I}^\infty(\mathbf{F})$, and $\mathbf{F}$-Gorenstein algebras can be chcaracterized as follows.

Lemma 5.4.3. (*[AS93a, Proposition 3.3]*)

(1) An algebra Λ is $\mathbf{F}$-Gorenstein if and only if there exists an $\mathbf{F}$-cotilting $\mathbf{F}$-tilting module.

(2) An algebra Λ is $\mathbf{F}$-Gorenstein if and only if every $\mathbf{F}$-cotilting module is $\mathbf{F}$-tilting and every $\mathbf{F}$-tilting module is $\mathbf{F}$-cotilting.

Corollary 5.4.4. *Let $[\Lambda, M, L, G]$ be a 4-tuple satisfying $\Lambda \in \mathrm{add}(G)$, $\mathbf{F} = \mathbf{F}_G$, L is $\mathbf{F}$-cotilting and M is an $\mathbf{F}$-dualizing summand of L and let $[\Gamma, {}_\Gamma M, \widetilde{L}, \widetilde{G}]$ be the corresponding 4-tuple under the 4-tuple assignment. Then Λ is an $\mathbf{F}$-Gorenstein algebra if and only if Γ is an $\widetilde{\mathbf{F}}$-Gorenstein algebra.*

Proof. Consider the 6-tuple assignment $[\Lambda, M, L, R, G, H] \mapsto [\Gamma, {}_\Gamma M, \widetilde{L}, \widetilde{R}, \widetilde{G}, \widetilde{H}]$ as in the proof of Theorem 5.3.2. Then L, R are $\mathbf{F}$-cotilting modules and $\widetilde{L}, \widetilde{R}$ are $\widetilde{\mathbf{F}}$-cotilting modules. By Lemma 5.4.3, Λ is $\mathbf{F}$-Gorenstein if and only if L and R are $\mathbf{F}$-tilting modules, if and only if $\widetilde{L}, \widetilde{R}$ are $\widetilde{\mathbf{F}}$-tilting modules by the tilting version of Theorem 5.3.2, if and only if Γ is $\widetilde{\mathbf{F}}$-Gorenstein by Lemma 5.4.3 again. □

Remark 5.4.5. (1) The tilting version of Theorem 5.3.2 implies that $\mathrm{pd}_{\widetilde{\mathbf{F}}} \widetilde{L} = \mathrm{pd}_{\mathbf{F}} L$, $\mathrm{pd}_{\mathbf{F}} L \leq \mathrm{pd}_{\mathbf{F}} R \leq \mathrm{pd}_{\mathbf{F}} L + 2$ and $\mathrm{pd}_{\widetilde{\mathbf{F}}} \widetilde{L} \leq \mathrm{pd}_{\widetilde{\mathbf{F}}} \widetilde{R} \leq \mathrm{pd}_{\widetilde{\mathbf{F}}} \widetilde{L} + 2$. Now by using [AS93a, Proposition 3.4], we see that $\mathrm{pd}_{\widetilde{\mathbf{F}}} \widetilde{G} = \mathrm{id}_{\widetilde{\mathbf{F}}} \widetilde{H} \leq \mathrm{pd}_{\widetilde{\mathbf{F}}} \widetilde{L} + \mathrm{id}_{\widetilde{\mathbf{F}}} \widetilde{L} \leq \mathrm{pd}_{\mathbf{F}} L + \mathrm{id}_{\mathbf{F}} R + 2$.

(2) In particular, if we take $M = L$ then the above result gives [AS93a, Proposition 3.1 and Proposition 3.6].

5.5 Special cotilting

We assume throughout this section that $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ for a generator G and a cogenerator H. The easiest situation where relative dualizing summands appear in relative cotilting modules are when these summands are 1-$\mathbf{F}$-faithful $\mathbf{F}$-injective modules.

Definition 5.5.1. Let C be an $\mathbf{F}$-cotilting module of $\mathrm{id}_{\mathbf{F}} C \leq r$. We say that C is special if it has an $\mathbf{F}$-injective $(r-1)$-$\mathbf{F}$-dualizing summand I. This is equivalent to an $\mathbf{F}$-injective summand I of C such that $\mathrm{cogen}^{r-1}_{\mathbf{F}}(C) = \mathrm{cogen}^{r-1}_{\mathbf{F}}(I)$ by Lemma 4.4.2. We sometimes call C I-special if it is special with respect to the $\mathbf{F}$-injective I.
Dually, we say an $\mathbf{F}$-tilting module T of $\mathrm{pd}_{\mathbf{F}} T \leq r$ is special if it has a $\mathbf{F}$-projective summand P such that $\mathrm{gen}^{\mathbf{F}}_{r-1}(T) = \mathrm{gen}^{\mathbf{F}}_{r-1}(P)$.

We look at a minimal $\mathbf{F}$-injective $\mathbf{F}$-coresolution of G

$$0 \to G \to I_0 \to I_1 \to I_2 \to \cdots$$

and define $J_n = \bigoplus_{t \leq n} I_t$ (so in particular we have $G \in \mathrm{cogen}^n_{\mathbf{F}}(J_n)$)

Theorem 5.5.2. *Let $r \geq 1$. We consider the following three finite sets.*

(1) Isomorphism classes of basic special cotilting modules of $\mathrm{id}_{\mathbf{F}} \leq r$.

(2) Isomorphism classes of basic $\mathbf{F}$-injective modules I with $G \in \mathrm{cogen}^{r-1}_{\mathbf{F}}(I)$.

(3) Isomorphism classes of basic $I \in \mathrm{add}(H)$ with $J_{r-1} \in \mathrm{add}(I)$.

Then the sets (2) *and* (3) *are equal. Mapping* C *to its maximal* F*-injective summand gives a bijection between* (1) *and* (2). *The inverse is given by mapping* I *to* $C_{I,r} := I \oplus \Omega_I^r H$.

Proof. Assume $J_{r-1} \in \operatorname{add}(I) \subset \operatorname{add}(H)$, then clearly $G \in \operatorname{cogen}_{\mathbf{F}}^{r-1}(J_{r-1}) \subset \operatorname{cogen}_{\mathbf{F}}^{r-1}(I)$ and we conclude that (3) is a subset of (2). So assume $I \in \operatorname{add}(H)$ with $G \in \operatorname{cogen}_{\mathbf{F}}^{r-1}(I)$. Since the minimal **F**-injective **F**-exact r-copresentation (of G) must be a summand of any other **F**-injective **F**-exact r-copresentation, it follows that $J_{r-1} \in \operatorname{add}(I)$ and therefore the sets (2) and (3) are equal.
So let C be an I-special r-**F**-cotilting module and let J be its maximal injective summand - of course $I \in \operatorname{add}(J)$ and clearly $\operatorname{copres}_{\mathbf{F}}^{r-1}(I) \subseteq \operatorname{copres}_{\mathbf{F}}^{r-1}(J) \subseteq \operatorname{copres}_{\mathbf{F}}^{r-1}(C)$. Since I, J are **F**-injective and C is r-**F**-cotilting we conclude that these inclusions of subcategories coincide with $\operatorname{cogen}_{\mathbf{F}}^{r-1}(I) \subseteq \operatorname{cogen}_{\mathbf{F}}^{r-1}(J) \subseteq \operatorname{cogen}_{\mathbf{F}}^{r-1}(C)$. Since C is I-special it follows that they are all equal, in particular $J \in \operatorname{cogen}_{\mathbf{F}}(I)$ implies $J \in \operatorname{add}(I)$ and therefore $\operatorname{add}(I) = \operatorname{add}(J)$. This means the map is well-defined. It follows from lemma 5.1.3 that the assignment $I \mapsto C_I = I \oplus \Omega_I^r H$ is the inverse map. □

Let $\Sigma_{\mathbf{F}}^r(\Lambda)$ be the finite subposet of the poset of isomorphism classes of basic **F**-cotilting modules of $\operatorname{id}_{\mathbf{F}} \leq r$, where the partial order is given by inclusion of perpendicular categories... Let $\operatorname{add}_{J_{r-1}}(H)$ be the lattice given by isomorphism classes of basic summands I of H such that $J_{r-1} \in \operatorname{add}(I)$. The partial order is just given by inclusion of summands, the meet and join are defined in the obvious way. In particular, if $H = J_{r-1} \oplus X$ with $|X| = t$, then the lattice $\operatorname{add}_{J_{r-1}}(H)$ is isomorphic to the power set $\mathcal{P}(\{1, 2, \dots, t\})$ which is a poset with respect to inclusion and a lattice with respect to intersection and union (sometimes also referred to as a t-dimensional cube).

Corollary 5.5.3. *The finite poset* $\Sigma_{\mathbf{F}}^r(\Lambda)$ *is a lattice and the bijection from the previous theorem gives a lattice isomorphism*

$$\Sigma_{\mathbf{F}}^r(\Lambda) \to \operatorname{add}_{J_{r-1}}(H).$$

We also observe that if an I-special r-**F**-cotilting module C has an $(r-1)$-**F**-dualizing summand M, then $I \in \operatorname{add}(M)$.

We give now several little applications, in particular connecting it with the other parts of the article.

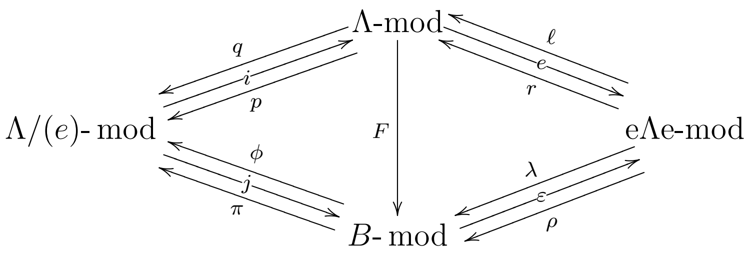
Examples and applications

(1) Non-relative special tilting has been defined in [PS17] and many special cases had been considered before, as APR-tilting and BB-tilting [BGfP73], [BB80], [APR79], n-APR-tilts [IO11] or flip-flops for posets [Lad07]. Any endomorphism ring of a generator has a canonical special cotilt, this has been used to define desingularizations of orbit closures and quiver Grassmannians in [CIFR13], [CBS17], [PS18].

(2) We explain that (non-relative) special cotilting naturally gives two recollements relating the cotilted algebras: Let I be a $(k-1)$-faithful injective Λ-module for $k \geq 1$ and $C = C_{I,k} = I \oplus \Omega_I^k \operatorname{D}\Lambda$ the I-special k-cotilting module. Then Ω_I^k is an equivalence of categories $\operatorname{add}\operatorname{D}\Lambda/\operatorname{add} I \to \operatorname{add} C/\operatorname{add} I$ with quasi-inverse Ω_I^{-k} (this follows from [AR91b, Theorem 5.2] with $\mathcal{X} = \operatorname{add}(I)$). Let $B = \operatorname{End}_\Lambda(C)^{op}$, then ${}_B\operatorname{D}C$ is special k-tilting module with respect to the $(k-1)$-faithful projective module $P = (C, I)$. Let $P = B\varepsilon$ and $I = \operatorname{D}(e\Lambda)$ for idempotents $e \in \Lambda, \varepsilon \in B$. Then the equivalence Ω_I^k induced an isomorphism of algebras

$$(\Lambda/(e))^{op} = \operatorname{End}_{\operatorname{add}\operatorname{D}\Lambda/\operatorname{add} I}(\operatorname{D}\Lambda) \cong \operatorname{End}_{\operatorname{add} C/\operatorname{add} I}(C) = (B/(\varepsilon))^{op}$$

Observe also $e\Lambda e \cong \operatorname{End}_\Lambda(I)^{op} \cong \varepsilon B \varepsilon$, therefore we have two recollements with isomorphic ends induced by the idempotents e, ε.



Furthermore, the cotilting functor $F := \mathrm{D}(-, C)$ commutes with the following functors from the recollements $\varepsilon \circ F = e, F \circ \ell = \lambda$.

(3) The standard cogenerator correspondence says that the assignment $[\Lambda, {}_\Lambda E] \mapsto [\Gamma, {}_\Gamma I]$ defined by $\Gamma = \operatorname{End}(E), I = {}_\Gamma E$ gives a bijection between

(a) $[\Lambda, {}_\Lambda E]$ with $\mathrm{D}\,\Lambda \in \operatorname{add} E$.

(b) $[\Gamma, {}_\Gamma I]$ with I injective and $\Gamma \in \operatorname{cogen}^1(I)$.

Let us denote C_I to be the special 2-cotilting Γ-module which exists in situation (b). Then the AS-assignment $[\Gamma, {}_\Gamma I, C_I] \mapsto [\Lambda, {}_\Lambda E, G]$ with $G = (C_I, I)$ gives a natural extension of the cogenerator correspondence to a bijection between the following.

(a') $[\Lambda, {}_\Lambda E, G]$ with $\mathrm{D}\,\Lambda \in \operatorname{add} E$ and E is an $\mathbf{F}_G$-cotilting module.

(b') $[\Gamma, {}_\Gamma I, C_I]$ with I injective and $\Gamma \in \operatorname{cogen}^1(I)$, C_I 2-cotilting with $\operatorname{cogen}^1(C_I) = \operatorname{cogen}^1(I)$.

This can be generalized to the 4-tuple assignment as follows:

Example of the relative cotilting correspondence using special cotilting

This is our main example for Theorem 5.3.2. Let us look at the 5-tuple assignment $[\Lambda, I, L, G, H] \mapsto [\Gamma = \operatorname{End}_\Lambda(I), {}_\Gamma I, \widetilde{L} = (G, I), \widetilde{G} = (L, I), \widetilde{H} = \mathrm{D}(I, H)]$. Then this gives a involution on the following 5-tuples $[\Lambda, I, L, G, H]$ with $\Lambda \in \operatorname{add}(G), \mathrm{D}\,\Lambda \oplus I \in \operatorname{add}(H)$, $\mathbf{F} = \mathbf{F}^H = \mathbf{F}_G$ and L is an I-special 2-$\mathbf{F}$-cotilting module.

The proof goes as follows: By Theorem 5.3.2 we know that $\widetilde{L}$ is again an $\widetilde{\mathbf{F}}$-cotilting module with $\widetilde{\mathbf{F}} = \mathbf{F}^{\widetilde{H}}$ and has an $\widetilde{F}$-dualizing summand ${}_\Gamma I$. So we need to see that $\operatorname{id}_{\widetilde{\mathbf{F}}} \widetilde{L} \leq 2$, then $\widetilde{L}$ is the (uniquely determined) ${}_\Gamma I$-special 2-$\mathbf{F}$-cotilting module. Recall that the assumption ensures that we have an $\mathbf{F}$-exact strong I-dualizing sequence $0 \to L \to I_0 \to I_1 \to H \to 0$ with $I_j \in \operatorname{add}(I)$, so we can see $R := H$ as the right end of it. This has been used to show that for $\widetilde{G} := (L, I), \widetilde{H} = \mathrm{D}(I, H)$ we have $\widetilde{\mathbf{F}} = \mathbf{F}^{\widetilde{H}} = \mathbf{F}_{\widetilde{G}}$. Now, apply $(-, I)$ to a minimal projective presentation of G and $\mathrm{D}(I, -)$ to a minimal injective copresentation of H to obtain an $\widetilde{\mathbf{F}}$-exact, strong ${}_\Gamma I$-dualizing sequence with left end $(G, I) = \widetilde{L}$ and right end $\mathrm{D}(I, H) = \widetilde{H}$. This ensures that $\operatorname{id}_{\widetilde{\mathbf{F}}} \widetilde{L} \leq 2$ and therefore $\widetilde{L}$ is an ${}_\Gamma I$-special 2-$\widetilde{\mathbf{F}}$-cotilting.

We remark that special r-(co)tilting requires an $\mathbf{F}$-injective $(r-1)$-$\mathbf{F}$-dualizing summand. In our previously considered assignments we looked only at 1-$\mathbf{F}$-dualizing summands, that is why our example only works for $r = 2$.

Mutation and dualizing sequences induce special tilts on endomorphism rings

Lemma 5.5.4. *Let* $0 \to L \to M_0 \to \cdots \to M_k \to R \to 0$ *be an* $\mathbf{F}$*-exact strong* k*-*M*-dualizing sequence with* $\mathrm{Ext}^j_{\mathbf{F}}(L,R) = 0$ *for* $j \geq 1$ *and* L,R *be* $\mathbf{F}$*-self-orthogonal. Let* $B = \mathrm{End}(L)$ *and* $A = \mathrm{End}(R)$*. Then* $T = (L,R)$ *is a special* k*-tilting* A*-module with respect to* $P = (M,R)$ *and* $C = \mathrm{D}(L,R)$ *is a special* k*-cotilting* B*-module with respect to* $I = \mathrm{D}(L,M)$*. Furthermore, we have* $\mathrm{End}_A(T) \cong B^{op}$ *and* $\mathrm{End}_B(C) \cong A^{op}$*.*

Proof. Apply $(-,R)$ to the strong dualizing sequence, setting $P_i = {}_A(M_i,R) \in \mathrm{add}(P)$, we get an exact sequence of A-modules

$$0 \to A \to P_k \to \cdots \to P_0 \to T \to 0.$$

This shows $\mathrm{pd}\, T \leq k$ and A has an $\mathrm{add}(T)$-resolution with all middle terms in $\mathrm{add}(P)$ ($\subseteq \mathrm{add}(T)$). Since the dualizing sequence is strong and by assumption $L \in {}^{\mathbf{F},1\leq\perp}R \cap \mathrm{cogen}^{\infty}_{\mathbf{F}}(R)$, we can use Lemma 2.1.3,(2) to get an isomorphism $\mathrm{Ext}^j_{\mathbf{F}}(L,L) \to \mathrm{Ext}^j_A(T,T)$. Since L is $\mathbf{F}$-self-orthogonal, the module T is self-orthogonal. This implies that T is a special k-tilting module with respect to P. Similarly, one can show that C is a special k-cotilting module with respect to I. The last claim follows from Lemma 2.1.3,(1). □

Passing to endomorphism rings of special cotilting modules

Recall, that in the non-relative case the *Brenner-Butler* assignment (BB): $[\Sigma, J, L'] \mapsto [B = \mathrm{End}_\Sigma(L'), \mathrm{D}(L',J), {}_BL']$ maps J-special t-cotilting Σ-modules L' to a $\mathrm{D}(L',J)$-special t-cotilting B-module and this assignment is an involution on these triples.
We explain how this relates to relative special cotilting: Let H be a basic cogenerator, $\Sigma = \mathrm{End}_\Lambda(H)^{op}$ and $\varepsilon \in \Sigma$ the projection onto the summand $\mathrm{D}\,\Lambda$, then we have a pair of adjoint functors

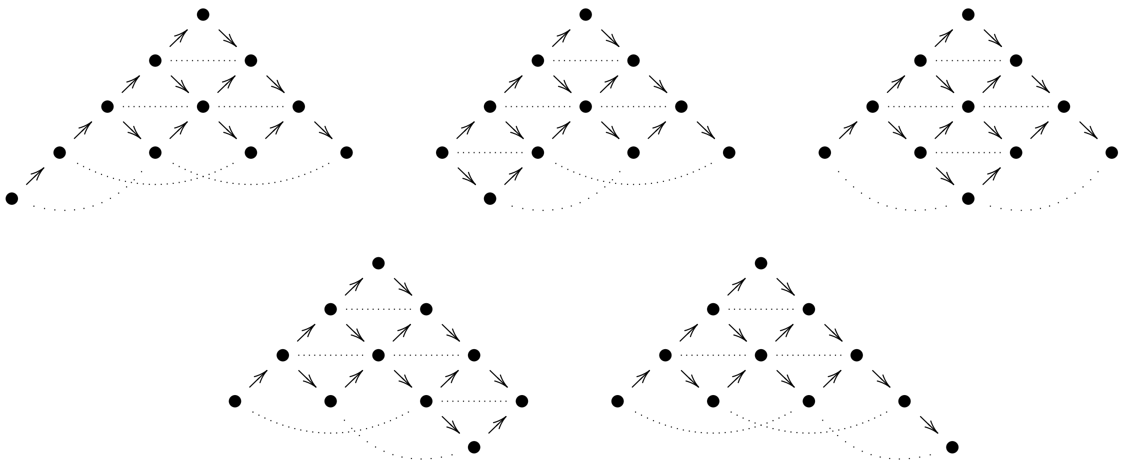
$$\ell = \mathrm{D}(-,H)\colon\ \Lambda\text{-mod} \rightleftarrows \Sigma\text{-mod}\colon \varepsilon = (\Sigma\varepsilon, -)$$

with $\mathrm{Im}\,\ell = \mathrm{gen}_1(\Sigma\varepsilon)$. As always we set $\mathbf{F} = \mathbf{F}^H$. Then for $I \in \mathrm{add}(H)$ we have $\ell(I) \in \mathrm{add}(\mathrm{D}\,\Sigma)$ and: $H \in \mathrm{gen}^{\mathbf{F}}_{t-1}(I) \Leftrightarrow \mathrm{D}\,\Sigma \in \mathrm{gen}_{t-1}(\ell(I)), \bigoplus_{j\geq 1}^t \Omega^j_{\ell(I)}\,\mathrm{D}\,\Sigma \in \mathrm{gen}_1(\Sigma\varepsilon)$.

The assignment $[\Lambda, I, L, H] \mapsto [\Sigma = \mathrm{End}_\Lambda(H)^{op}, \ell(I), \ell(L)]$ injects an I-special t-$\mathbf{F}^H$-cotilting modules L to an $\ell(I)$-special t-cotilting Σ-module $\ell(L)$. Any J-special t-cotilting Σ-module L' for some $J \in \mathrm{add}(\mathrm{D}\,\Sigma)$ is in the image of this assignment if and only if $\bigoplus_{j\geq 1}^t \Omega^j_J\,\mathrm{D}\,\Sigma \in \mathrm{gen}_1(\Sigma\varepsilon)$.
The assignment $[\Lambda, I, L, H] \mapsto [B = \mathrm{End}_\Lambda(L), \mathrm{D}(L,I), \mathrm{D}(L,H)]$ injects an I-special t-$\mathbf{F}^H$-cotilting modules L to an $\mathrm{D}(L,I)$-special t-cotilting B-module $\mathrm{D}(L,H)$. In fact, combining the assignments we get a commuting triangle as follows

$$\begin{array}{ccc} & [\Lambda, I, L, H] & \\ {\scriptstyle \ell}\swarrow & & \searrow{\scriptstyle \mathrm{D}(L,-)} \\ [\Sigma = \mathrm{End}_\Lambda(H)^{op}, \ell(I), \ell(L)] & \overset{(BB)}{\longleftrightarrow} & [B = \mathrm{End}_\Lambda(L), \mathrm{D}(L,I), \mathrm{D}(L,H)] \end{array}$$

Example 5.5.5. Here are the endomorphism rings of the special cotilts of the relative Auslander algebras for $\Lambda = K(1 \to 2 \to 3 \to 4 \to 5)$, $G = P_5 \oplus M, M = \bigoplus_{i=1}^4 \bigoplus_{j\geq 0} \tau^{-j} P_i$, $\mathbf{F} = \mathbf{F}_G$ from Example 6.3.2, (4). We choose $I = M \in \mathrm{add}(H)$, then the M-special tilting and cotilting modules conincide with: $G, M \oplus S_4, M \oplus S_3, M \oplus S_2, H$. Their respective endomorphism ring is shown by the quiver with relations below.



Chapter 6

Relative Auslander correspondence and F-Auslander algebras

We generalize the notion of dominant dimension to the relative setting.

Definition 6.0.1. Let Γ be a finite-dimensional algebra and $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ for a generator ${}_\Gamma G$ and a cogenerator ${}_\Gamma H$. Consider the minimal $\mathbf{F}$-coresolution of G by $\mathbf{F}$-injectives

$$0 \to G \to H_0 \to H_1 \to H_2 \to \cdots .$$

We define $\operatorname{domdim}_{\mathbf{F}} \Gamma = k$ if there exists an integer k such that $H_i \in \operatorname{add}(G)$ for $0 \leq i \leq k-1$ and $H_k \notin \operatorname{add}(G)$. If $H_i \in \operatorname{add}(G)$ for all $i \geq 0$ then we define $\operatorname{domdim}_{\mathbf{F}} \Gamma = \infty$.

Remark 6.0.2. As is in the classical case, our definition of $\mathbf{F}$-dominant dimension is left-right symmetric in the following sense: A functor $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ determines a functor $\mathbf{F}_{\mathrm{D}\,H} = \mathbf{F}^{\mathrm{D}\,G} =:$ $\mathbf{F}^*$ in the category Γ^{op}-mod and vice versa, and $\operatorname{domdim}_{\mathbf{F}} \Gamma = k$ if and only if $\operatorname{domdim}_{\mathbf{F}^*} \Gamma^{op} = k$.

6.1 Relative Auslander-Solberg correspondence

Lemma 6.1.1. *Let $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ with G and H basic and assume ${}_\Gamma M$ is a module such that $\operatorname{add}(M) = \operatorname{add}(G) \cap \operatorname{add}(H)$. Then the following are equivalent for every $k \geq 1$.*

(1) There is an $\mathbf{F}$-exact sequence $0 \to G \to M_0 \to M_1 \to \cdots \to M_k \to H \to 0$ with $M_i \in \operatorname{add}(M)$.

(2) $\operatorname{domdim}_{\mathbf{F}} \Gamma \geq k+1 \geq \operatorname{id}_{\mathbf{F}} G$.

(3) $\operatorname{domdim}_{\mathbf{F}} \Gamma \geq k+1 \geq \operatorname{pd}_{\mathbf{F}} H$.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious. We prove (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) is dual.

Assume (2) then we have an $\mathbf{F}$-exact sequence

$$0 \to G \to M_0 \to M_1 \to \cdots \to M_{k-1} \to M'_k \to H' \to 0$$

with $M_i \in \operatorname{add}(M)$ for $0 \leq i \leq k-1$, $M'_k \in \operatorname{add}(M)$ and $H' \in \operatorname{add}(H)$. We may assume this $\mathbf{F}$-exact sequence is a successive composition of minimal left $\operatorname{add}(M)$-approximations of the cokernels. By the dual version of Lemma 5.1.3 we have $M \oplus H'$ is an $\mathbf{F}$-tilting module with $\operatorname{id}_{\mathbf{F}}(M \oplus H') = 0$. By Lemma 5.3.1 (1) we know that $M \oplus H'$ is basic and hence $M \oplus H' = H$. Now the desired $\mathbf{F}$-exact sequence in (1) can be obtained by adding $M \xrightarrow{1} M$ to $M'_k \to H'$. $\square$

Theorem 6.1.2. (*relative Auslander-Solberg correspondence*)
Let ${}_\Lambda M$ *be a faithfully balanced module and* $\Gamma = \operatorname{End}_\Lambda(M)$. *Then the assignment* $X, Y \mapsto (X, M), \mathrm{D}(M, Y)$ *gives a self-inverse bijection between the following sets of pairs of modules*

(1) $\{{}_\Lambda L, {}_\Lambda R \mid {}_\Lambda M \oplus \Lambda \in \operatorname{add}(L), {}_\Lambda M \oplus \mathrm{D}\Lambda \in \operatorname{add}(R), L = \tau_k^- R \oplus \Lambda, R = \tau_k L \oplus \mathrm{D}\Lambda,$ $\operatorname{Ext}^i_\Lambda(L, R) = 0$, $1 \leq i \leq k-1$ *such that there exists a strong* $\operatorname{add}({}_\Lambda M)$*-dualizing sequence with left end* L *and right end* $R\}$.

(2) $\{{}_\Gamma G, {}_\Gamma H \mid M \oplus \Gamma \in \operatorname{add}(G), M \oplus \mathrm{D}\Gamma \in \operatorname{add}(H), G = \tau^- H \oplus \Gamma, H = \tau G \oplus \mathrm{D}\Gamma$ *such that there exists a strong* k-$\operatorname{add}({}_\Gamma M)$*-dualizing sequence with left end* G *and right end* $H\}$.

Proof. Combine Lemma 4.3.5 and Lemma 6.1.1. □

6.2 Relative Auslander correspondence

Lemma 6.2.1. *Let* $k \geq 1$ *and assume* $\operatorname{domdim}_{\mathbf{F}} \Gamma \geq k+1$. *Let* ${}_\Gamma M$ *be a module with* $\operatorname{add}(M) = \operatorname{add}(G) \cap \operatorname{add}(H)$. *Then we have* $\operatorname{cogen}^k_{\mathbf{F}}(M) = \Omega^{k+1}_{\mathbf{F}}(\Gamma\text{-mod})$ *and* $\operatorname{gen}^{\mathbf{F}}_k(M) = \Omega^{-(k+1)}_{\mathbf{F}}(\Gamma\text{-mod})$. *Furthermore, the following are equivalent:*

(1) $\operatorname{cogen}^k_{\mathbf{F}}(M) = \operatorname{add}(G)$.

(2) $\operatorname{gen}^{\mathbf{F}}_k(M) = \operatorname{add}(H)$.

(3) $\operatorname{gldim}_{\mathbf{F}} \Gamma \leq k+1$.

Proof. Since $\operatorname{domdim}_{\mathbf{F}} \Gamma \geq k+1$ and $\operatorname{add}(M) = \operatorname{add}(H) \cap \operatorname{add}(G)$, we have clearly $\operatorname{add}(G) \subseteq \operatorname{cogen}^k_{\mathbf{F}}(M) \subseteq \Omega^{k+1}_{\mathbf{F}}(\Gamma\text{-mod})$. On the other hand, we proved in Lemma 5.1.3 that in this case

$$\operatorname{cogen}^k_{\mathbf{F}}(M) = \bigcap_{i \geq 1} \ker \operatorname{Ext}^i_{\mathbf{F}}(-, C)$$

for $C = M \oplus \Omega^{k+1}_M H$ and $\operatorname{id}_{\mathbf{F}} C \leq k+1$. So given $X \in \Omega^{k+1}_{\mathbf{F}}(\Gamma\text{-mod})$, there is an $Y \in \Gamma\text{-mod}$ such that $X = \Omega^{k+1}_{\mathbf{F}} Y$ and then by dimension shift for $i \geq 1$

$$\operatorname{Ext}^i_{\mathbf{F}}(X, C) = \operatorname{Ext}^{i+k+1}_{\mathbf{F}}(Y, C) = 0$$

since $\operatorname{id}_{\mathbf{F}} C \leq k+1$. In particular, $X \in \operatorname{cogen}^k_{\mathbf{F}}(M)$. One can prove $\operatorname{gen}^{\mathbf{F}}_k(M) = \Omega^{-(k+1)}_{\mathbf{F}}(\Gamma\text{-mod})$ with the dual argument.

Now clearly, $\operatorname{gldim}_{\mathbf{F}} \Gamma \leq k+1$ is equivalent to $\Omega^{k+1}_{\mathbf{F}}(\Gamma\text{-mod}) \subseteq \operatorname{add}(G)$ and by the just proved result, we conclude it is equivalent to (1). The equivalence of (3) and (2) can be proven with the analogous argument. □

Definition 6.2.2. Let $M \in \Lambda\text{-mod}$ and assume that there is a strong $\operatorname{add}(M)$-dualizing sequence with left end L and right end R.

We say that M is a k-(L, R)-cluster tilting module if

(i) $\Lambda \in \operatorname{cogen}^1_{\mathbf{F}^R}(M)$ and $\mathrm{D}\Lambda \in \operatorname{gen}^{\mathbf{F}^L}_1(M)$,

(ii) $\operatorname{cogen}^1_{\mathbf{F}^R}(M) \cap \bigcap_{i=1}^{k-1} \ker \operatorname{Ext}^i_\Lambda(-, R) = \operatorname{add}(L)$ and $\operatorname{gen}^{\mathbf{F}^L}_1(M) \cap \bigcap_{i=1}^{k-1} \ker \operatorname{Ext}^i_\Lambda(L, -) = \operatorname{add}(R)$.

Let Γ be a finite-dimensional algebra and $\mathbf{F} = \mathbf{F}_G$ for a generator G. Then we say Γ is an k-$\mathbf{F}$-Auslander algebra if $\operatorname{domdim}_{\mathbf{F}} \Gamma \geq k+1 \geq \operatorname{gldim}_{\mathbf{F}} \Gamma$.

Theorem 6.2.3. (*relative Auslander correspondence*)
Let $k \geq 1$. There is a one-to-one correspondence between isomorphism classes of basic k-(L,R)-cluster tilting modules ${}_\Lambda M$ (for some L, R) and finite-dimensional algebras Γ with an exact structure given by $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ such that $\operatorname{domdim}_{\mathbf{F}} \Gamma \geq k+1 \geq \operatorname{gldim}_{\mathbf{F}} \Gamma$. The correspondence is induced by the assignment

$$[\Lambda, M, L, R] \mapsto [\Gamma = \operatorname{End}_\Lambda(M), {}_\Gamma M, G = (L, M), H = \mathrm{D}(M, R)].$$

Proof. Let M be an k-(L,R)-cluster tilting module and $\Gamma = \operatorname{End}_\Lambda(M)$, $G = (L,M), H = \mathrm{D}(M,R)$ and $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$. Since $L \in \operatorname{cogen}^1_{\mathbf{F}^R}(M) \cap \bigcap_{i=1}^{k-1} \ker \operatorname{Ext}^i_\Lambda(-,R)$, we have $G \in \operatorname{cogen}^k_{\mathbf{F}}(M)$ by Lemma 4.2.5. Similarly, from $\Lambda \in \operatorname{add}(L), \mathrm{D}\,\Lambda \in \operatorname{add}(R)$ we conclude that ${}_\Gamma M \in \operatorname{add}(G) \cap \operatorname{add}(H)$ and therefore $\operatorname{domdim}_{\mathbf{F}} \Gamma \geq k+1$. By the same lemma, we also have $\operatorname{cogen}^k_{\mathbf{F}}(M) = \operatorname{add}(G)$ and therefore by Lemma 6.2.1 $\operatorname{gldim}_{\mathbf{F}} \Gamma \leq k+1$.
Conversely, by Lemma 6.2.1 and Lemma 4.2.5 we can also conclude the other implication. $\square$

6.3 Examples of F-Auslander algebras

The easiest example can be found for $k = 1$. Here, for a 1-cluster tilting pair (L,R) with respect to M we have $G = L$ is a generator, $H = R$ is a cogenerator with $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ and the definition shortens to a module M such that $\operatorname{cogen}^1_{\mathbf{F}}(M) = \operatorname{add}(G)$ and $\operatorname{gen}^{\mathbf{F}}_1(M) = \operatorname{add}(H)$ is fulfilled.

Here are some easy examples of 1-**F**-Auslander algebras.

Example 6.3.1. (1) Let $\mathbf{F} = \mathbf{F}_\Lambda$ and M be a projective-injective module such that $\operatorname{cogen}^1(M) = \operatorname{add}(\Lambda)$ and $\operatorname{gen}_1(M) = \operatorname{add}(\mathrm{D}\,\Lambda)$. Then, by Lemma 6.2.1 it is easy to see that this is equivalent to $\operatorname{domdim} \Lambda \geq 2 \geq \operatorname{gldim} \Lambda$ and it is well-known that this characterizes Λ to be an Auslander algebra.

(2) Assume $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$ and $G = H$ is a generator-cogenerator, in this case we say Λ is **F**-selfinjective. A classification of **F**-selfinjective algebras can be found in [AS93a, section 5]. For example, if G is an Auslander generator (= 1-cluster tilting module), this is fulfilled. Then, if we choose $M = G = H$, then we have $\operatorname{cogen}^1_{\mathbf{F}}(M) = \operatorname{cogen}^1(M) = \operatorname{add}(M) = \operatorname{gen}^1(M) = \operatorname{gen}^{\mathbf{F}}_1(M)$ and this gives us another example.

(3) Let Γ be the path algebra of $1 \to 2 \to 3$ and let $M = P_2 \oplus P_1 \oplus I_2$. We define $G := \Gamma \oplus M$ and $H := \mathrm{D}\,\Gamma \oplus M$, then it is easy to see $\mathbf{F}_G = \mathbf{F}^H =: \mathbf{F}$ and $\operatorname{cogen}^1_{\mathbf{F}}(M) = \operatorname{add}(G), \operatorname{gen}^{\mathbf{F}}_1(M) = \operatorname{add}(H)$.

(4) Let Γ be the path algebra of the following quiver:

$$\begin{array}{ccccc} & & 1 & & \\ & & \downarrow & & \\ 3 & \longrightarrow & 2 & \longrightarrow & 4. \end{array}$$

Let $M := P_2 \oplus \tau^- P_2 \oplus \tau^{-2} P_2$, $G = M \oplus P_1 \oplus P_3 \oplus P_4, H = M \oplus I_1 \oplus I_3 \oplus I_4$ and $\mathbf{F} := \mathbf{F}_G = \mathbf{F}^H$. Then we have **F**-exact sequences

$$0 \to P_4 \to P_2 \to \tau^- P_2 \to I_4 \to 0$$
$$0 \to P_3 \to \tau^- P_2 \to \tau^{-2} P_2 \to I_3 \to 0$$
$$0 \to P_1 \to \tau^- P_2 \to \tau^{-2} P_2 \to I_1 \to 0$$

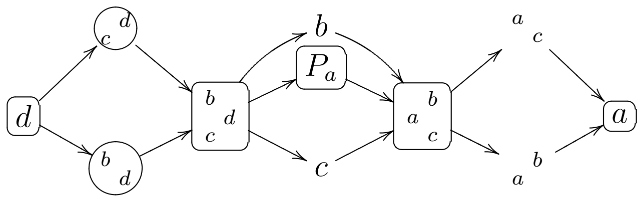
which show $\operatorname{domdim}_{\mathbf{F}} \Gamma = 2$. It also easy to see that $2 = \max_X\{\operatorname{pd}_{\mathbf{F}} X\}(= \operatorname{gldim}_{\mathbf{F}} \Gamma)$, since the three missing indecomposables which are not in $\operatorname{add} G$ or $\operatorname{add} H$ are 2, $\frac{1}{2}$, $\frac{3}{2}$ which appear as cosyzygies of the three injectives in the $\mathbf{F}$-exact sequences and so all have $\operatorname{pd}_{\mathbf{F}} = 1$. We have $\Lambda = \operatorname{End}_\Gamma(M)$ is given by the following quiver with relations

$$\bullet \overset{\beta}{\underset{\alpha}{\rightrightarrows}} \bullet \overset{\delta}{\underset{\gamma}{\rightrightarrows}} \bullet \qquad \gamma\alpha = \delta\beta = \delta\alpha + \gamma\beta = 0.$$

(5) Let Λ be the path algebra modulo the relations:

$$\begin{array}{ccccc} & {}^{\alpha}\nearrow & b & \searrow^{\beta} & \\ a & & & & d, \\ & {}_{\gamma}\searrow & c & \nearrow_{\delta} & \end{array} \qquad \beta\alpha - \delta\gamma = 0.$$

Its Auslander-Reiten quiver is drawn in the following graphic, in the square boxes you find $M = P_d \oplus {\begin{smallmatrix} b & \\ & d \\ c & \end{smallmatrix}} \oplus P_a \oplus {\begin{smallmatrix} & b \\ a & \\ & c \end{smallmatrix}} \oplus I_a$ and together with the remaining circled modules $G = M \oplus P_b \oplus P_c$



It is very easy to show that M is faithfully balanced and $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^{M \oplus \mathrm{D}\Lambda}$ fulfills $\operatorname{domdim}_{\mathbf{F}} \Lambda = 2 = \operatorname{gldim}_{\mathbf{F}} \Lambda$. Now we look at $\Gamma = \operatorname{End}_\Lambda(M)$, this is given by the path algebra of the following quiver with the overlapping zero-relations

$$\begin{array}{ccccc} & & 3 & & \\ & {}^{\gamma}\nearrow & & \searrow^{\delta} & \\ 1 \xrightarrow{\alpha} 2 & & \xrightarrow{\beta} & & 4 \xrightarrow{\varepsilon} 5, \end{array} \qquad \beta\alpha = 0 = \delta\gamma\alpha, \quad \varepsilon\beta = 0 = \varepsilon\delta\gamma$$

the vertices $1, 2, 3, 4, 5$ correspond to the summands d, ${\begin{smallmatrix} b & \\ & d \\ c & \end{smallmatrix}}$, P_a, ${\begin{smallmatrix} & b \\ a & \\ & c \end{smallmatrix}}$, a in the given order. To calculate ${}_\Gamma M = (\Lambda, M) = \mathrm{D}(M, \mathrm{D}\Lambda)$ we look at its four indecomposable summands

$$P_3 = (P_a, M) = \mathrm{D}(M, I_a) = I_5 \qquad I_3 = \mathrm{D}(M, I_d) = (P_d, M) = P_1$$
$$(P_b, M) = \mathrm{D}(M, I_b) \qquad (P_c, M) = \mathrm{D}(M, I_c)$$

then we apply $(-, M)$ to the $\mathbf{F}$-exact sequence $0 \to P_b \to {\begin{smallmatrix} b & \\ & d \\ c & \end{smallmatrix}} \to {\begin{smallmatrix} & b \\ a & \\ & c \end{smallmatrix}} \to I_b \to 0$ and obtain a projective presentation $0 \to P_5 \to P_4 \to P_2 \to (P_b, M) \to 0$ (and similar for (P_c, M)). From this we conclude that (P_b, M), (P_c, M) are two regular modules in different homogeneous tubes for the full subquiver $\widetilde{A}_2$, more precisely:

$$(P_b, M) =: R_0: \quad \begin{array}{ccccc} & & K & & \\ & {}^{1}\nearrow & & \searrow^{1} & \\ 0 \to K & & \xrightarrow{0} & & K \to 0 \end{array} \qquad (P_c, M) =: R_1: \quad \begin{array}{ccccc} & & K & & \\ & {}^{1}\nearrow & & \searrow^{1} & \\ 0 \to K & & \xrightarrow{1} & & K \to 0 \end{array}$$

then we have $\tau^{(\pm)} R_j = R_j$, $j = 0, 1$. We set now $\widetilde{G} = P_2 \oplus P_4 \oplus P_5 \oplus M$, $\widetilde{H} = I_1 \oplus I_2 \oplus I_4 \oplus M$ and define $\widetilde{\mathbf{F}} := \mathbf{F}_{{}_\Gamma\widetilde{G}} = \mathbf{F}^{{}_\Gamma\widetilde{H}}$, observe that $\operatorname{add}({}_\Gamma M) = \operatorname{add}(\widetilde{G}) \cap \operatorname{add}(\widetilde{H})$. The following sequences are $\widetilde{\mathbf{F}}$-exact (setting $R_{01} = R_0 \oplus R_1$)

$$0 \longrightarrow P_2 \longrightarrow R_{01} \longrightarrow I_3 \longrightarrow I_1 \longrightarrow 0$$
$$0 \longrightarrow P_4 \longrightarrow P_3 \longrightarrow I_3 \longrightarrow I_2 \longrightarrow 0$$
$$0 \longrightarrow P_5 \longrightarrow P_3 \longrightarrow R_{01} \longrightarrow I_4 \longrightarrow 0$$

We deduce $\operatorname{domdim}_{\widetilde{\mathbf{F}}} \Gamma = 2 = \operatorname{id}_{\widetilde{\mathbf{F}}} \mathrm{D}\,\Gamma$. We have $\operatorname{End}_\Gamma(\widetilde{G})^{op} \cong \operatorname{End}_\Lambda(G)$ has gldim ≤ 4 since $\operatorname{gldim}_{\mathbf{F}} \Lambda \leq 2$ by Lemma 4.1.5, therefore by the same argument and the observation $2 = \operatorname{id}_{\widetilde{\mathbf{F}}} \mathrm{D}\,\Gamma$ we conclude $\operatorname{gldim}_{\widetilde{\mathbf{F}}} \Gamma \leq 2$.

Here are examples of higher **F**-Auslander algebras.

Example 6.3.2. (1) A k-(L,R)-cluster tilting module M with $L = M = R$ is just the same as a k-cluster tilting module in the sense of [Iya08]. In this case, $\Gamma = \operatorname{End}_\Lambda(M)$, $G = (M,M) = \Gamma$, $H = \mathrm{D}(M,M) = \mathrm{D}\,\Gamma$, so $\mathbf{F} = \mathbf{F}_\Gamma$ and so $\operatorname{domdim}_{\mathbf{F}} \Gamma = \operatorname{domdim} \Gamma$, $\operatorname{gldim}_{\mathbf{F}} \Gamma = \operatorname{gldim} \Gamma$ and we reobtain a higher Auslander algebra (this is the Krull-dimension zero case of Iyama's Auslander correspondence, see [Iya07a]).

(2) Let Γ be the path algebra of $1 \to 2 \to \cdots \to n$. Let $M_t := \bigoplus_{i \neq t} \bigoplus_{j \geq 0} \tau^{-j} P_i$, $G_t = M_t \oplus P_t$, $\mathbf{F}_t = \mathbf{F}_{G_t}$, $1 < t \leq n$, then Γ has the structure of a $(t-2)$-$\mathbf{F}_t$-Auslander algebra for $t \geq 3$ and for $t = 2$ we have $\operatorname{domdim}_{\mathbf{F}_2} \Lambda = 1 = \operatorname{gldim}_{\mathbf{F}_2} \Lambda$. For large n we have that $\Lambda_3 = \operatorname{End}_\Lambda(M_3)$ is a representation-infinite algebra with an **F**-Auslander structure.

(3) We consider the following quiver (of Dynkin type E_6) Q

$$\begin{array}{ccccccccc} & & & & f & & & & \\ & & & & \uparrow & & & & \\ a & \to & b & \to & c & \to & d & \to & e \end{array}$$

For $x \in \{a,b,d,e,f\}$ we define $M_x = \bigoplus_{y \neq x} \bigoplus_{j \geq 0} \tau^{-j} P_y$, $G_x = M_x \oplus P_x$, $\mathbf{F}_x = \mathbf{F}_{G_x}$. Then an inspection if the AR-quiver gives the following for the path algebra $\Gamma = KQ$: Γ is a 2-$\mathbf{F}_a$- and 2-$\mathbf{F}_b$-Auslander algebra, a 4-$\mathbf{F}_d$- and 4-$\mathbf{F}_f$-Auslander algebra and 6-$\mathbf{F}_e$-Auslander algebra.

(4) Let $\Gamma = K(1 \to 2 \to \cdots \to n)$ for some integer $n > 3$ and we define $M := \bigoplus_{i=1}^{n-1} \bigoplus_{j \geq 0} \tau^{-j} P_i$, $G = M \oplus P_n$, $H = M \oplus I_1$ and $\mathbf{F} = \mathbf{F}_G = \mathbf{F}^H$. We find the minimal **F**-projective resolution of I_1 (which is also the minimal **F**-injective resolution of P_n) as follows

$$0 \to P_n \to \begin{smallmatrix} n-1 \\ n \end{smallmatrix} \to \begin{smallmatrix} n-2 \\ n-1 \end{smallmatrix} \to \cdots \to \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \to I_1 \to 0 \quad (*)$$

from this we conclude $\operatorname{pd}_{\mathbf{F}} \mathrm{D}\,\Gamma = n-1$ and $\operatorname{domdim}_{\mathbf{F}} G = n-1$. One can easily see that the highest $\operatorname{pd}_{\mathbf{F}}$ is obtained at an injective module and therefore $\operatorname{gldim}_{\mathbf{F}} \Gamma = n-1$, so we have an $(n-2)$-**F**-Auslander algebra.

Let $\Lambda = \operatorname{End}_\Gamma(M)$, we denote by $P_{[M_i]}, I_{[M_i]}, S_{[M_i]}$ the projective, injective and semi-simple Λ-module associated to $M_i \in \operatorname{add}(M)$. Let $L = {}_\Lambda(G,M) = \Lambda \oplus (P_n, M)$, $R = \mathrm{D}(M,H) = \mathrm{D}\,\Lambda \oplus \mathrm{D}(M, I_1)$ and ${}_\Lambda M \in \operatorname{add}(L) \cap \operatorname{add}(R)$. Then we have $\Pi := (\bigoplus_{1<j<n} P_j, M) = \mathrm{D}(M, \bigoplus_{1<j<n} I_j)$ is a projective-injective Λ-module, $M = \Pi \oplus P_{[P_1]} \oplus I_{[P_1]}$, $L = \Lambda \oplus I_{[P_1]}$, $R = \mathrm{D}\,\Lambda \oplus P_{[P_1]}$. We verify $(\Pi, P_{[P_1]}) = ((S_i, M), P_{[P_1]}) = (I_{[P_1]}, P_{[P_1]}) = 0$ for $3 \leq i \leq n$ and $(I_1, M) = 0$, $(S_2, M) = S_{[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}]}$. We apply $(-,M)$ to $(*)$ and obtain an exact sequence of Λ-modules

$$0 \to 0 = (I_1, M) \to P_{[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}]} \to P_{[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}]} \to \cdots \to P_{[\begin{smallmatrix} n-1 \\ n \end{smallmatrix}]} \to I_{[P_1]} \to 0 \quad (**)$$

This implies $\operatorname{pd} I_{[P_1]} = n-2$. Now, apply $(-, P_{[P_1]})$ to $(**)$ and obtain $K = (S_{[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}]}, P_{[P_1]}) \cong \operatorname{Ext}^1_\Lambda((S_3, M), P_{[P_1]}) = \operatorname{Ext}^{n-2}_\Lambda(I_{[P_1]}, P_{[P_1]})$.

We would like to see that ${}_\Lambda M$ is a $(n-2)$-(L,R)-cluster tilting module with respect to L and R as before. Since we easily verify $\operatorname{cogen}^1_{F^H}({}_\Gamma M) = \operatorname{add}(G \oplus \bigoplus_{3 \leq i < n} S_i)$ and $(S_i, M) =$

$\Omega^{n-i} I_{[P_1]}$ by the exact sequence $(**)$, we have $\operatorname{cogen}^1_{F^R}(M) = \operatorname{add}(L \oplus \bigoplus_{3\leq i<n} \Omega^{n-i} I_{[P_1]})$. Now, we conclude

$$\begin{aligned}
&\operatorname{cogen}^1_{F^R}(M) \cap \bigcap_{i=1}^{n-3} \ker \operatorname{Ext}^i_\Lambda(-, R) \\
&= \operatorname{add}(L \oplus \bigoplus_{3\leq i<n} \Omega^{n-i} I_{[P_1]}) \cap \bigcap_{i=1}^{n-3} \ker \operatorname{Ext}^i_\Lambda(-, P_{[P_1]}) \\
&= \operatorname{add}(L)
\end{aligned}$$

where we use the calculation of $\operatorname{Ext}^j_\Lambda(I_{[P_1]}, P_{[P_1]})$, $j \geq 1$ from before.

There are further examples of converting $K\mathbb{A}_n$ into a relative Auslander algebra. Here is another family of these:

Example 6.3.3. We fix $\Gamma = K(1 \to 2 \to \cdots \to n)$ for some integer $n \geq 3$ and we will also allow quotients by certain admissible 2-sided ideals I. Our aim is to describe a family of $\mathbf{F}$-Auslander algebras which *interpolate* between Iyama's example [Iya08, Example 2.4] and the usual exact structure on Γ-mod. We study the following class of generators $G_\ell := \Gamma \oplus \bigoplus_{1\leq i\leq \ell} \bigoplus_{j>0} \tau^{-j} P_i$, $1 < \ell < n-1$ [1] .

1. If $(n-\ell+1)|n$ (or equivalently, $(n-\ell+1)|(\ell-1)$), then Γ is a $(2\frac{\ell-1}{n-\ell+1})$-minimal $\mathbf{F}_\ell$-Auslander-Gorenstein algebra (i.e., $\operatorname{domdim}_{\mathbf{F}_\ell} \Gamma \geq 2\frac{\ell-1}{n-\ell+1}+1 \geq \operatorname{id}_{\mathbf{F}_\ell} G_\ell$), where $\mathbf{F}_\ell = \mathbf{F}_{G_\ell}$. If $\ell < n-1$ and $n-\ell+1$ does not divide n, then Γ is not a minimal $\mathbf{F}_\ell$-Auslander-Gorenstein algebra.
   proof: For $\ell \leq k \leq n$ we look at the $\mathbf{F}_\ell$-injective resolution of P_k and here we keep track the sequence of tops (they are all simple) of the $\mathbf{F}_\ell$-injectives appearing, it fulfills $a_1 = \ell, a_2 = k-(n-\ell-1), a_t = a_{t-2} - (n-\ell+1)$ for all $t \geq 3$. Now, the condition to be a minimal $\mathbf{F}_\ell$-Auslander-Gorenstein algebra is equivalent to that there is one t (for all k) such that $a_t = 1$. Since t has to work for all k (and $\ell < n-1$), we conclude that t has to be uneven, say $t = 2s+1$ (then it is an $2s$-minimal $\mathbf{F}_\ell$-Auslander-Gorenstein algebra). Now, the recursion tells us $1 = a_t = a_{t-2} - (n-\ell+1) = a_{t-2s} - s(n-\ell+1) = \ell - s(n-\ell+1)$, so it follows $s = \frac{\ell-1}{n-\ell+1}$.

2. But from the shape of the Auslander-Reiten quiver of Γ we can conclude that the maximal $\operatorname{pd}_{\mathbf{F}_\ell}$ is obtained at an injective module, therefore $\operatorname{gldim}_{\mathbf{F}_\ell} \Gamma = \operatorname{pd}_{\mathbf{F}_\ell} \mathrm{D}\,\Gamma$ and we have:
   Γ is a k-$\mathbf{F}_\ell$-Auslander algebra (for some k) if and only if $(n-\ell+1)|n$ and in this case $k = 2(\frac{\ell-1}{n-\ell+1})$.

3. Assume I is a 2-sided admissible ideal with $\{X \mid IX = 0\} \subseteq \{X \mid \dim_K X \geq n-\ell+2\}$. We define $\overline{G}_\ell := \Gamma/I \otimes_\Gamma G_\ell$ is a generator for Γ/I and we set $\overline{\mathbf{F}}_\ell := \mathbf{F}_{\overline{G}_\ell}$. Since we can use the *same* $\mathbf{F}$-projective and $\mathbf{F}$-injective resolutions (because of the choice of the ideal) we have: Γ is a k-$\mathbf{F}_\ell$-Auslander algebra) if and only if Γ/I is an k-$\overline{\mathbf{F}}_\ell$-Auslander algebra) In particular, if we set $I = \operatorname{rad}^{n-\ell+1}(\Gamma)$, then we have $\overline{G}_\ell = \Gamma/I$ and if $(n-\ell+1)|n$ then we get a (non-relative) $2(\frac{\ell-1}{n-\ell+1})$-Auslander algebra.
   If we allow $\ell = n-1$ (cf. previous example), this describes the $(n-1)$-Auslander algebra of Iyama [Iya08, Example 2.4].

[1] For $\ell = 1$ we have $\mathbf{F}_1 = \operatorname{Ext}^1_\Gamma$ and observe $\operatorname{domdim}\Gamma = 1 = \operatorname{gldim}\Gamma$; $\ell = n, n-1$ are already studied in the previous examples

Conceptually the same family can be defined more generally for Nakayama algebras, we explain this in the selfinjective Nakayama algebra case:

Example 6.3.4. Let C_n be the oriented cycle quiver with arrows $i \to i+1 (\text{mod n})$ and $J \subseteq KC_n$ be the ideal generated by the arrows, $N \in \mathbb{N}$, we define $\Gamma := KC_n/J^N$ (this is a selfinjective Nakayama algebra). Let $n-\ell+1 < N$ and $M_{\ell\ell \geq n-\ell+1}$ be the direct sum of all modules of vector space dimension $\geq n-\ell+1$ and let X_n be the direct sum of all modules having S_n as a composition factor and vector space dimension $< n-\ell+1$, we define $G_\ell = M_{\ell\ell \geq n-\ell+1} \oplus X_n$ and $\mathbf{F}_\ell = \mathbf{F}_{G_\ell}$. Then G_n is the Auslander generator and for $\ell = n-1$ we have Γ is an $(n-2)$-$\mathbf{F}_{n-1}$- Auslander algebra. Moreover, for $1 < \ell < n-1$ we have Γ is a k-$\mathbf{F}_\ell$-Auslander algebra (for some k) if and only if $(n-\ell+1)|n$, and in this case $k = 2\frac{\ell-1}{n-\ell+1}$. The proof is exactly the same as in the previous example.

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