Center for
Mathematical Economics
Working Papers

622

September 2019

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http://www.imw.uni-bielefeld.de/wp/
ISSN: 0931-6558

CONVEX SEMIGROUPS ON BANACH LATTICES

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Abstract. In this paper, we investigate convex semigroups on Banach lattices. First, we consider the case, where the Banach lattice is σ-Dedekind complete and satisfies a monotone convergence property, having L^p-spaces in mind as a typical application. Second, we consider monotone convex semigroups on a Banach lattice, which is a Riesz subspace of a σ-Dedekind complete Banach lattice, where we consider the space of bounded uniformly continuous functions as a typical example. In both cases, we prove the invariance of a suitable domain for the generator under the semigroup. As a consequence, we obtain the uniqueness of the semigroup in terms of the generator. The results are discussed in several examples such as semilinear heat equations (g-expectation), nonlinear integro-differential equations (uncertain compound Poisson processes), fully nonlinear partial differential equations (uncertain shift semigroup and G-expectation).

Key words: Convex semigroup, nonlinear Cauchy problem, fully nonlinear PDE, well-posedness and uniqueness, Hamilton-Jacobi-Bellman equations

AMS 2010 Subject Classification: 47H20; 35A02; 35A09

1. Introduction

Given a C_0-semigroup $S = (S(t))_{t\in[0,\infty)}$ of linear operators on a Banach space X with generator $A\colon D(A) \subset X \to X$, it is well known that the domain $D(A)$ is invariant under S, i.e. $S(t)x \in D(A)$ for all $x \in D(A)$ and $t \geq 0$. Moreover, it holds

$$AS(t)x = S(t)Ax \quad \text{for all } x \in D(A) \text{ and } t \geq 0. \tag{1.1}$$

This relation is fundamental in order to prove that the semigroup S is uniquely determined through its generator. The aim of this paper is to establish a relation similar to (1.1) for C_0-semigroups of convex operators on a Banach lattice X in order to prove invariance of the domain and that the semigroup is uniquely specified via its generator.

Convex semigroups arise in a natural way, when considering convex differential equations such as the G-heat equation or more general HJB equations $\partial_t y - Ay = 0$, $u(0) = x$ where $Ay = \sup_\lambda A_\lambda y$. One classical approach to treat such fully nonlinear equations uses the theory of maximal monotone or m-accretive operators (see, e.g., [3], [4], [5], [14], [11] and the references therein). To show that an accretive operator is m-accretive, one has to prove that $1 + h\mathcal{A}$ is surjective for small $h > 0$. However, in many cases it is hard to verify this condition (for instance, it fails for the uncertain shift semigroup on BUC defined in Subsection 4.3). This was one of the reasons for the introduction of viscosity solutions (see the discussion in [11], Section 4). Viscosity solutions are known to exist in many cases (see, e.g., [6], [7], [13]), the proof of uniqueness is rather delicate. In contrast to these classical approaches, we start with the nonlinear semigroup as our

Date: September 5, 2019.
Financial support through the German Research Foundation via CRC 1283 is gratefully acknowledged. We thank Daniel Bartl, Jonas Blessing, Liming Yin and José Miguel Zapata García for helpful discussions and comments.

main object. We study convex C_0-semigroups on Banach lattices, i.e. $S = (S(t))_{t\in[0,\infty)}$ is a family of bounded convex operators $X \to X$, such that, for every $x \in X$, it holds $S(0)x = x$, $S(t+s)x = S(t)S(s)x$ for all $s,t \geq 0$, and $S(t)x \to x$ as $t \downarrow 0$. If $X = L^p(\mu)$ for $p \in [1,\infty)$ and some measure μ, or more generally if X is Dedekind σ-complete and $x_n \to \inf_n x_n$ for all decreasing sequences $(x_n)_n$ in X which are bounded below, we show that the key results from linear semigroup theory extend to the present nonlinear framework. More precisely, defining the generator A by

$$Ax := \lim_{h\downarrow 0} \frac{S(h)x - x}{h} \quad \text{for } x \in D(A),$$

where $D(A) := \big\{x \in X : \lim_{h\downarrow 0} \frac{S(h)x-x}{h} \text{ exists}\big\}$, we show that S leaves the domain $D(A)$ invariant. Moreover, the map $[0,\infty) \to X$, $t \mapsto S(t)x$ is continuously differentiable for all $x \in D(A)$, and the time derivative is given by

$$AS(t)x = S'(t,x)Ax = \inf_{h>0} \frac{S(t)(x + hAx) - S(t)x}{h}.$$

Here, the right-hand side is the directional derivative or Gâteaux derivative of the convex operator $S(t)$ at x in direction Ax. In particular, if $S(t)$ is linear, the Gâteaux derivative simplifies to $S'(t,x)Ax = S(t)Ax$, which is consistent with (1.1). We further show that the generator A is always a closed operator, which uniquely determines the semigroup S on the domain $D(A)$. As a consequence, $y(t) := S(t)x$, for $x \in D(A)$, defines the unique classical solution to the abstract Cauchy problem

$$\text{(CP)} \qquad \begin{cases} y'(t) = Ay(t), & \text{for all } t \geq 0,\\ y(0) = x. \end{cases}$$

In the case of a nonlinear operator of the form $Au = \sup_{\lambda\in\Lambda} A_\lambda u$, where, e.g., A_λ is the generator of a Lévy process for all $\lambda \in \Lambda$, we study the semigroup envelope S, i.e. the smallest semigroup dominating the family of linear semigroups $(S_\lambda)_{\lambda\in\Lambda}$. Following [22], in [10] and [20] the existence of a semigroup envelope, under certain conditions, has been shown for families of semigroups on BUC. Under a suitable boundedness condition, this construction extends to $L^p(\mu)$, which makes our abstract results applicable to the semigroup envelope of certain families of linear C_0-semigroups on $L^p(\mu)$. In general, the obtained domain $D(A)$ will be larger than the natural domain $\bigcap_{\lambda\in\Lambda} D(A_\lambda)$, but we still have – under appropriate assumptions – classical differentiability of the solution for initial values in $D(A)$. We remark that for generators of Lévy processes in BUC under uncertainty, recent results were obtained, e.g., in [10], [12], [18], [20], and [21]. Fully nonlinear equations in the strong L^p-setting were recently considered, e.g., by Krylov in [15], [16], [17].

There are examples of convex C_0-semigroups on the Banach lattice BUC which cannot be extended to $L^p(\mu)$, see e.g. the uncertain shift semigroup in Example 3.14. Since BUC is not Dedekind σ-complete, we consider in the second part of this paper the case, where X is a Riesz subspace of some Dedekind σ-complete Riesz space $\overline{X}$. A typical example for X is BUC. Here, we focus on monotone semigroups that are continuous from above, meaning that $S(t)x_n \downarrow 0$ for all $t \geq 0$, whenever $x_n \downarrow 0$. This additional continuity property allows to extend the semigroup to $X_\delta := \{x \in \overline{X} : x_n \downarrow x \text{ for some bounded sequence } (x_n)_n \text{ in } X\}$. In contrast to the σ-Dedekind complete case, the domain $D(A)$ is, in general, not invariant under convex C_0-semigroups. However, for monotone convex semigroups, the invariance can be achieved by extending the generator. Inspired by the directional derivative, we define the domain $D(A_\delta)$ of the monotone generator A_δ as the set of all $x \in X$ such that for every sequence $(h_n)_n$ in $(0,\infty)$ with $h_n \downarrow 0$ there exists an approximating sequence $(Ax_n)_n$ in X such that

$$\left\| \frac{S(h_n)x - x}{h_n} - A_n x \right\| \to 0 \quad \text{and} \quad A_n x \downarrow y =: A_\delta x.$$

The main results state that a monotone convex C_0-semigroup leaves the domain $D(A_\delta)$ of its monotone generator invariant, and the semigroup is uniquely determined by A_δ on $D(A_\delta)$ if, in addition, the semigroup is continuous from above. As an example, we consider the uncertain shift semigroup, which corresponds to the fully nonlinear PDE $\partial_t y(t) = Ay(t)$, $y(0) = x$, where $Ay := |y'|$ and y' denotes the (weak) space derivative. In that case, it holds $\mathrm{BUC}^1 \subset D(A_\delta) \subset W^{1,\infty}$ and $W^{1,\infty}$ is invariant under the corresponding semigroup. Similarly, for the second-order differential operator $Ax = \frac{1}{2}\max\{\underline{\sigma}x'', \overline{\sigma}x''\}$, where $0 \le \underline{\sigma} \le \overline{\sigma}$, we derive that $W^{2,\infty}$ is invariant under the respective semigroup which corresponds to the G-expectation.

The structure of the paper is as follows. In Section 2 we introduce the setting and state basic results on convex C_0-semigroups which can be derived from the uniform boundedness principle. Section 3 includes the main results on convex C_0-semigroups on σ-Dedekind complete Banach lattices. In particular, we provide invariance of the domain and uniqueness of the semigroup in terms of the generator. The non σ-Dedekind complete case is treated in Section 4. Finally, additional results on bounded convex operators and directional derivatives of convex operators are collected in the appendix.

2. Notation and preliminary results

Let X be a Banach lattice. For an operator $S\colon X \to X$, we define

$$\|S\|_r := \sup_{x \in B(0,r)} \|Sx\|$$

for all $r > 0$, where $B(x_0, r) := \{x \in X \colon \|x - x_0\| \le r\}$ for $x_0 \in X$. We say that an operator $S\colon X \to X$ is *convex* if $S\big(\lambda x + (1-\lambda)y\big) \le \lambda Sx + (1-\lambda)Sy$ for all $\lambda \in [0,1]$, *positive homogeneous* if $S(\lambda x) = \lambda Sx$ for all $\lambda > 0$, *sublinear* if S is convex and positive homogeneous, *monotone* if $x \le y$ implies $Sx \le Sy$ for all $x, y \in X$, and *bounded* if $\|S\|_r < \infty$ for all $r > 0$.

Definition 2.1. A family $S = (S(t))_{t \in [0,\infty)}$ of bounded operators $X \to X$ is called a *semigroup* on X if

(S1) $S(0)x = x$ for all $x \in X$,
(S2) $S(t+s)x = S(t)S(s)x$ for all $x \in X$ and $s, t \in [0,\infty)$.

In this case, we say that S is a C_0-*semigroup* if, additionally,

(S3) $S(t)x \to x$ as $t \downarrow 0$ for all $x \in X$.

We say that S is *monotone*, *convex* or *sublinear* if $S(t)$ is monotone, convex or sublinear for all $t \ge 0$, respectively.

Throughout this article, let S be a convex C_0-semigroup on X. For $t \ge 0$ and $x \in X$, we define the convex operator $S_x(t)\colon X \to X$ by

$$S_x(t)y := S(t)(x+y) - S(t)x.$$

Proposition 2.2. *Let $T > 0$ and $x_0 \in X$. Then, there exist $L \geq 0$ and $r > 0$ such that*

$$\sup_{t\in[0,T]} \|S_x(t)y\| \leq L\|y\|$$

for all $x \in B(x_0, r)$ and $y \in B(0, r)$.

Proof. It suffices to show that

$$\sup_{0\leq t\leq T} \|S(t)x\| < \infty \tag{2.1}$$

for all $x \in X$. Indeed, under (2.1) it follows from Theorem A.8 b) that there exists some $r > 0$ such that $b := \sup_{x\in B(x_0,r)} \sup_{0\leq t\leq T} \|S_x(t)\|_r < \infty$. Since $S_x(t)$ is convex and $S_x(t)0 = 0$, we obtain from Lemma A.1 that

$$\|S_x(t)y\| \leq \tfrac{2b}{r}\|y\|$$

for all $t \in [0, T]$, $x \in B(x_0, r)$ and $y \in B(0, r)$.

In order to prove (2.1), let $x \in X$. Since $S(t)x \to x$ as $t \downarrow 0$, there exists some $n \in \mathbb{N}$ such that

$$R := \sup_{h\in[0,\delta)} \|S(h)x\| < \infty,$$

where $\delta := \frac{T}{n}$. Since $S(t)$ is bounded for all $t \geq 0$, it holds

$$c := \max_{0\leq k\leq n} \|S(k\delta)\|_R < \infty.$$

Now, let $t \in [0, T]$. Then, there exist $k \in \{0, \dots, n\}$ and $h \in [0, \delta)$ such that $t = k\delta + h$. Since $\|S(h)x\| \leq R$, it follows that $\|S(t)x\| = \|S(k\delta)S(h)x\| \leq c$. This proves (2.1) and thus completes the proof. □

Remark 2.3. If S is sublinear, then there exist $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|S(t)x\| \leq Me^{\omega t}\|x\| \tag{2.2}$$

for all $x \in X$ and $t \in [0, \infty)$. Indeed, by Proposition 2.2 and sublinearity of the semigroup S, one has $\sup_{t\in[0,1]} \|S(t)x\| \leq M\|x\|$ for all $x \in X$ and some $M \geq 1$. Set $\omega := \log M$. Then, for all $t \in [0, \infty)$, there exists some $m \in \mathbb{N}$ with $t < m \leq t + 1$. By the semigroup property, it follows that

$$\|S(t)x\| = \left\|S\left(\tfrac{t}{m}\right)^m x\right\| \leq M^m\|x\| \leq M^{t+1}\|x\| = Me^{\omega t}\|x\|$$

for all $x \in X$.

Corollary 2.4. *Let $T > 0$ and $x_0 \in X$. Then, there exist $L \geq 0$ and $r > 0$ such that*

$$\sup_{t\in[0,T]} \|S(t)y - S(t)z\| \leq L\|y - z\|$$

for all $y, z \in B(x_0, r)$.

Proof. By Proposition 2.2, there exist $L \geq 0$ and $r > 0$ such that

$$\sup_{t\in[0,T]} \|S_x(t)y\| \leq L\|y\|$$

for all $x \in B(x_0, 2r)$ and $y \in B(0, 2r)$. Now, let $y, z \in B(x_0, r)$. Then, $y - z \in B(0, 2r)$, and we thus obtain that

$$\sup_{t\in[0,T]} \|S(t)y - S(t)z\| = \sup_{t\in[0,T]} \|S_z(t)(y - z)\| \leq L\|y - z\|,$$

which shows the desired Lipschitz continuity. □

Corollary 2.5. *The map* $[0,\infty)\to X,\ t\mapsto S(t)x$ *is continuous for all* $x\in X$.

Proof. Let $t\geq 0$ and $x\in X$. Then, by Corollary 2.4, there exist $L\geq 0$ and $r>0$ such that

$$\sup_{s\in[0,t+1]}\|S(s)y-S(s)x\|\leq L\|y-x\|$$

for all $y\in B(x,r)$. Moreover, there exists some $\delta\in(0,1]$ such that $\|S(h)x-x\|\leq r$ for all $h\in[0,\delta]$. For $s\geq 0$ with $|s-t|\leq\delta$ it follows that

$$\|S(t)x-S(s)x\|=\|S(s\wedge t)S(|t-s|)x-S(s\wedge t)x\|\leq L\|S(|t-s|)x-x\|\to 0$$

as $s\to t$. □

Corollary 2.6. *Let* $(x_n)_n$ *and* $(y_n)_n$ *be two sequences in* X *with* $x_n\to x\in X$ *and* $y_n\to y\in X$, *and* $(h_n)_n$ *be a sequence in* $(0,\infty)$ *with* $h_n\downarrow 0$. *Then,* $S_{y_n}(h_n)x_n\to x$.

Proof. We first show that $S(h_n)x_n\to x$. By Corollary 2.4, there exist $L\geq 0$ and $r>0$ such that

$$\sup_{t\in[0,1]}\|S(t)z-S(t)x\|\leq L\|z-x\|.$$

for all $z\in B(x,r)$. Hence, for $n\in\mathbb{N}$ sufficiently large, we obtain that

$$\begin{aligned}\|S(h_n)x_n-x\|&\leq\|S(h_n)x_n-S(h_n)x\|+\|S(h_n)x-x\|\\&\leq L\|x_n-x\|+\|S(h_n)x-x\|.\end{aligned}$$

This shows that $S(h_n)x_n\to x$ as $n\to\infty$. As a consequence,

$$S_{y_n}(h_n)x_n=S(h_n)(x_n+y_n)-S(h_n)y_n\to(x+y)-y=x$$

as $n\to\infty$. The proof is complete. □

Proposition 2.7. *Let* $x\in X$ *with*

$$\sup_{h\in(0,h_0]}\left\|\frac{S(h)x-x}{h}\right\|<\infty\quad\textit{for some } h_0>0.$$

Then, the map $[0,\infty)\to X,\ t\mapsto S(t)x$ *is locally Lipschitz continuous, i.e., for every* $T>0$, *there exists some* $L_T\geq 0$ *such that* $\|S(t)x-S(s)x\|\leq L_T|t-s|$ *for all* $s,t\in[0,T]$.

Proof. Since the map $[0,\infty)\to X,\ t\mapsto S(t)x$ is continuous by Corollary 2.5, there exists some constant $C_T\geq 0$ such that

$$\sup_{t\in(0,T]}\frac{\|S(t)x-x\|}{t}\leq C_T.$$

By Corollary 2.4, there exist $L\geq 0$ and $r>0$ such that

$$\sup_{t\in[0,T]}\|S(t)y-S(t)z\|\leq L\|y-z\|\quad\text{for all } y,z\in B(x,r).$$

Further, there exists some $n\in\mathbb{N}$ such that $\sup_{h\in[0,\delta]}\|S(h)x-x\|\leq r$, where $\delta:=\frac{T}{n}$. Now, let $L_T:=LC_T$ and $s,t\in[0,T]$ with $s\leq t$. If $t-s\in[0,\delta]$, we have that

$$\|S(t)x-S(s)x\|\leq L\|S(t-s)x-x\|\leq L_T(t-s).$$

In general, there exist $k\in\{0,\dots,n-1\}$ and $h\in[0,\delta]$ such that $t-s=k\delta+h$. Then,

$$\begin{aligned}\|S(t)x-S(s)x\|&\leq\|S(t)x-S(s+k\delta)x\|+\sum_{j=1}^{k}\big\|S(s+j\delta)x-S\big(s+(j-1)\delta\big)x\big\|\\&\leq L_T\big(t-(s+k\delta)\big)+L_Tk\delta=L_T(t-s).\end{aligned}$$

The proof is complete. □

3. Convex semigroups on σ-Dedekind complete Banach lattices

3.1. **The generator and its domain.** In this subsection, we assume that the Banach lattice X is *Dedekind σ-complete*, i.e. any countable non-empty subset of X, which is bounded above, has a supremum. Moreover, we assume that X has the *monotone convergence property*, i.e. for every increasing sequence $(x_n)_n$ which is bounded above one has $\lim_{n\to\infty} \|\sup_{m\in\mathbb{N}} x_m - x_n\| = 0$. A typical examples is given by $X = L^p(\mu)$ for $p \in [1,\infty)$ and some measure μ. Recall that S is a convex C_0-semigroup on X.

Definition 3.1. We define the *generator* $A\colon D(A) \subset X \to X$ of S by

$$D(A) := \left\{ x \in X\colon \frac{S(h)x - x}{h} \text{ is convergent for } h \downarrow 0 \right\} \tag{3.1}$$

and $Ax := \lim_{h\downarrow 0} \frac{S(h)x-x}{h}$ for $x \in D(A)$.

In this subsection, we investigate properties of the generator A and its domain $D(A)$. A fundamental ingredient for the analysis is the directional derivative of a convex operator, see also Appendix B. Fix $t \geq 0$. Since $S(t)\colon X \to X$ is a convex operator, the function

$$\mathbb{R}\setminus\{0\} \to X, \quad h \mapsto \frac{S(t)(x+hy) - S(t)x}{h}$$

is increasing for all $x, y \in X$. In particular,

$$-S_x(t)(-y) \leq \frac{S(t)(x-hy) - S(t)x}{-h} \leq \frac{S(t)(x+hy) - S(t)x}{h} \leq S_x(t)y$$

for $x, y \in X$ and $h \in (0,1]$. Since for all $x, y \in X$ and every sequence $(h_n)_n$ in $(0,\infty)$ with $h_n \to 0$ one has

$$\inf_n \frac{S(t)(x+h_n y) - S(t)x}{h_n} \in X \quad \text{and} \quad \sup_n \frac{S(t)x - S(t)(x-h_n y)}{h_n} \in X,$$

the operators

$$S'_+(t,x)y := \inf_{h>0} \frac{S(t)(x+hy) - S(t)x}{h} \quad \text{and} \quad S'_-(t,x)y := \sup_{h<0} \frac{S(t)(x+hy) - S(t)x}{h} \tag{3.2}$$

are well-defined with values in X. Due to the monotone convergence property one has

$$\left\| S'_\pm(t,x)y \mp \frac{S(t)(x\pm hy) - S(t)x}{h} \right\| \to 0 \quad \text{as } h \downarrow 0. \tag{3.3}$$

If the left and right directional derivatives coincide, then the directional derivative is continuous in time. More precisely, the following holds.

Proposition 3.2. *Suppose that $S'_+(t,x)y = S'_-(t,x)y$ for some $x, y \in X$ and some $t \geq 0$. Then, the maps $[0,\infty) \to X$, $s \mapsto S'_\pm(s,x)y$ are continuous at t. In particular, $\lim_{s\downarrow 0} S'_\pm(s,x)y = y$.*

Proof. Since $S'_-(s,x)y = -S'_+(s,x)(-y)$ for all $s \geq 0$, it suffices to prove the continuity of the map $[0,\infty) \to X$, $s \mapsto S'_+(s,x)y$ at t. For all $s \geq 0$ and $h > 0$, let

$$D_{h,\pm}(s,x)y := \frac{S(s)(x\pm hy) - S(s)x}{\pm h}.$$

By Corollary 2.5, the mapping $[0,\infty) \to X$, $s \mapsto D_{h,\pm}(s,x)y$ is continuous for all $h > 0$. Let $\varepsilon > 0$. By (3.3), there exists some $h_\varepsilon > 0$ with

$$\left\| D_{h_\varepsilon,+}(t,x)y - S'_+(t,x)y \right\| < \frac{\varepsilon}{4} \quad \text{and} \quad \left\| D_{h_\varepsilon,-}(t,x)y - S'_-(t,x)y \right\| < \frac{\varepsilon}{4}.$$

Since the mapping $[0,\infty)\to X,\ s\mapsto D_{h_\varepsilon,\pm}(s,x)y$ is continuous, there exists some $\delta>0$ such that

$$\big\|D_{h_\varepsilon,+}(s,x)y-D_{h_\varepsilon,+}(t,x)y\big\|<\frac{\varepsilon}{4}\quad\text{and}\quad\big\|D_{h_\varepsilon,-}(s,x)y-D_{h_\varepsilon,-}(t,x)y\big\|<\frac{\varepsilon}{4}$$

for all $s\geq 0$ with $|s-t|<\delta$. Hence,

$$\big\|D_{h_\varepsilon,+}(s,x)y-S'_+(t,x)y\big\|<\frac{\varepsilon}{2}\quad\text{and}\quad\big\|D_{h_\varepsilon,-}(s,x)y-S'_-(t,x)y\big\|<\frac{\varepsilon}{2}\tag{3.4}$$

for all $s\geq 0$ with $|s-t|<\delta$. Since $S'_-(s,x)y\leq S'_+(s,x)y$, we obtain that

$$S'_+(s,x)y-S'_-(t,x)y\geq S'_-(s,x)y-S'_-(t,x)y\geq D_{h_\varepsilon,-}(s,x)y-S'_-(t,x)y$$

for all $s\geq 0$. On the other hand,

$$S'_+(s,x)y-S'_+(t,x)y\leq D_{h_\varepsilon,+}(s,x)y-S'_+(t,x)y$$

for all $s\geq 0$. Now, since $S'_+(t,x)y=S'_-(t,x)y$, we obtain that

$$\big|S'_+(s,x)y-S'_+(t,x)y\big|\leq\big|D_{h_\varepsilon,+}(s,x)y-S'_+(t,x)y\big|+\big|D_{h_\varepsilon,-}(s,x)y-S'_-(t,x)y\big|$$

for all $s\geq 0$ and therefore, by (3.4),

$$\big\|S'_+(t,x)y-S'_+(s,x)y\big\|<\varepsilon$$

for all $s\geq 0$ with $|s-t|<\delta$. Since $S(0)=\mathrm{id}_X$ is linear, it follows that

$$S'_+(0,x)=S'_-(0,x)=\mathrm{id}_X$$

and therefore, $\lim_{t\downarrow 0}S'_\pm(t,x)y=S'_\pm(0,x)y=y$. □

It is a straightforward application of Proposition 2.7 that $[0,\infty)\to X,\ t\mapsto S(t)x$ is locally Lipschitz continuous for all $x\in D(A)$. The following first main result states that it is even continuously differentiable on the domain.

Theorem 3.3. *Let $x\in D(A)$ and $t\geq 0$.*

(i) *It holds $S(t)x\in D(A)$ with*

$$AS(t)x=S'_+(t,x)Ax.$$

If $S(t)$ is linear, this results in the well-known relation $AS(t)x=S(t)Ax$.

(ii) *For $t>0$, one has*

$$\lim_{h\downarrow 0}\frac{S(t)x-S(t-h)x}{h}=S'_-(t,x)Ax.$$

(iii) *It holds $S'_+(t,x)Ax=S'_-(t,x)Ax$. The mapping $[0,\infty)\to X,\ s\mapsto S(s)x$ is continuously differentiable and the derivative is given by*

$$\tfrac{\mathrm{d}}{\mathrm{d}s}S(s)x=AS(s)x=S'_\pm(s,x)Ax\quad\textit{for } s\geq 0.$$

(iv) *It holds*

$$S(t)x-x=\int_0^t AS(s)x\,\mathrm{d}s=\int_0^t S'_+(s,x)Ax\,\mathrm{d}s=\int_0^t S'_-(s,x)Ax\,\mathrm{d}s.$$

Proof. (i) Let $t\geq 0$ and $(h_n)_n$ in $(0,\infty)$ with $h_n\downarrow 0$. Then,

$$\frac{S(t+h_n)x-S(t)x}{h_n}-\frac{S(t)(x+h_nAx)-S(t)x}{h_n}=\frac{S(t)S(h_n)x-S(t)(x+h_nAx)}{h_n}.$$

By Corollary 2.4, there exist $L\geq 0$ and $r>0$ such that

$$\|S(t)y-S(t)z\|\leq L\|y-z\|$$

for all $y, z \in B(x, r)$. For $n \in \mathbb{N}$ sufficiently large, we thus obtain that

$$\left\| \frac{S(t)S(h_n)x - S(t)(x + h_n Ax)}{h_n} \right\| \leq L \left\| \frac{S(h_n)x - x}{h_n} - Ax \right\| \to 0.$$

Since, by (3.3),

$$\frac{S(t)(x + h_n Ax) - S(t)x}{h_n} \to S'_+(t, x)Ax,$$

we obtain the assertion.

(ii) Let $t > 0$ and $(h_n)_n$ in $(0, t]$ with $h_n \downarrow 0$. Then,

$$\frac{S(t)x - S(t - h_n)x}{h_n} - \frac{S(t)x - S(t)(x - h_n Ax)}{h_n} = \frac{S(t)(x - h_n Ax) - S(t - h_n)x}{h_n}.$$

Again, by Corollary 2.4, there exist $L \geq 0$ and $r > 0$ such that

$$\sup_{s \in [0,t]} \|S(s)y - S(s)z\| \leq L\|y - z\|$$

for all $y, z \in B(x, r)$. By Corollary 2.6, we have $S(h_n)(x - h_n Ax) \to x$. Hence, for $n \in \mathbb{N}$ sufficiently large, it follows that

$$\left\| \frac{S(t - h_n)S(h_n)(x - h_n Ax) - S(t - h_n)x}{h_n} \right\| \leq L \left\| \frac{S(h_n)(x - h_n Ax) - x}{h_n} \right\|.$$

Using Corollary 2.6 and the convexity of S_x and $S_{x - h_n Ax}$, we find that, for sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} \frac{S(h_n)(x - h_n Ax) - x}{h_n} &= \frac{S_x(h_n)(-h_n Ax)}{h_n} + \frac{S(h_n)x - x}{h_n} \\ &\leq S_x(h_n)(-Ax) + \frac{S(h_n)x - x}{h_n} \to 0 \end{aligned}$$

and

$$\begin{aligned} \frac{x - S(h_n)(x - h_n Ax)}{h_n} &= \frac{S_{x - h_n Ax}(h_n)(h_n Ax)}{h_n} - \frac{S(h_n)x - x}{h_n} \\ &\leq S_{x - h_n Ax}(h_n)(Ax) - \frac{S(h_n)x - x}{h_n} \to 0. \end{aligned}$$

This shows that $\left\|\frac{S(h_n)(x - h_n Ax) - x}{h_n}\right\| \to 0$, which implies that

$$\left\| \frac{S(t)x - S(t - h_n)x}{h_n} - \frac{S(t)x - S(t)(x - h_n Ax)}{h_n} \right\| \to 0.$$

Since, by (3.3),

$$\frac{S(t)x - S(t)(x - h_n Ax)}{h_n} \to S'_-(t, x)Ax,$$

we obtain the assertion.

(iii) By definition, it holds $S'_+(t, x)Ax \geq S'_-(t, x)Ax$, and, for $t = 0$,

$$S'_+(0, x)Ax = S'_-(0, x)Ax = Ax.$$

Therefore, let $t > 0$ and $0 < h \leq t$. Then, by convexity of $S_{S(t-h)x}$, for h sufficiently small, it holds

$$\begin{aligned}\frac{S(t+h)x - S(t)x}{h} &= \frac{S(h)S(t)x - S(h)S(t-h)x}{h} \\ &= \frac{S_{S(t-h)x}(h)\big(S(t)x - S(t-h)x\big)}{h} \\ &\leq S_{S(t-h)x}(h)\bigg(\frac{S(t)x - S(t-h)x}{h}\bigg),\end{aligned}$$

which implies that

$$\begin{aligned}S'_+(t,x)Ax = AS(t)x &= \lim_{h\downarrow 0} \frac{S(t+h)x - S(t)x}{h} \\ &\leq \lim_{h\downarrow 0} S_{S(t-h)x}(h)\bigg(\frac{S(t)x - S(t-h)x}{h}\bigg) \\ &= S'_-(t,x)Ax,\end{aligned}$$

where we used Corollary 2.6 in the last step. Now, Proposition 3.2 yields that the mapping $[0,\infty) \to X$, $s \mapsto S'_+(s,x)Ax$ is continuous.

(iv) This follows directly from (iii) using the fundamental theorem of calculus. □

As in the linear case, the generator of a convex C_0-semigroup is closed.

Proposition 3.4. *The generator A is closed, i.e. for every sequence $(x_n)_n$ in $D(A)$ with $x_n \to x \in X$ and $Ax_n \to y \in X$, one has $x \in D(A)$ and $Ax = y$.*

Proof. First, notice that

$$-S_{x_n}(s)(-Ax_n) \leq S'_+(s,x_n)Ax_n \leq S_{x_n}(s)Ax_n.$$

By Corollary 2.4, there exist $L \geq 0$ and $r > 0$ such that

$$\sup_{s\in[0,1]} \|S(s)w - S(s)z\| \leq L\|w - z\|$$

for all $w, z \in B(x \pm y, r)$. Hence, for $n \in \mathbb{N}$ sufficiently large,

$$\begin{aligned}\|S_{x_n}(s)Ax_n - S_{x_n}(s)y\| &\leq L\|Ax_n - y\| \quad \text{and} \\ \|S_{x_n}(s)(-Ax_n) - S_{x_n}(s)(-y)\| &\leq L\|Ax_n - y\|,\end{aligned}$$

so that

$$\|S'_+(s,x_n)Ax_n - y\| \leq 2L\|Ax_n - y\| + \|S_{x_n}(s)y - y\| + \|S_{x_n}(s)(-y) + y\|$$

for all $s \in [0,1]$. By Theorem 3.3,

$$\frac{S(h)x_n - x_n}{h} - y = \frac{1}{h}\int_0^h \big(S'_+(s,x_n)Ax_n - y\big)\,\mathrm{d}s$$

for all $h > 0$. Hence, for fixed $h \in (0,1]$, we find that

$$\begin{aligned}\bigg\|\frac{S(h)x - x}{h} - y\bigg\| &= \lim_{n\to\infty} \bigg\|\frac{S(h)x_n - x_n}{h} - y\bigg\| \leq \limsup_{n\to\infty} \frac{1}{h}\int_0^h \big\|S'_+(s,x_n)Ax_n - y\big\|\,\mathrm{d}s \\ &\leq \lim_{n\to\infty} 2L\|Ax_n - y\| + \sup_{0\leq s\leq h} \big(\|S_{x_n}(s)y - y\| + \|S_{x_n}(s)(-y) + y\|\big) \\ &= \sup_{0\leq s\leq h} \big(\|S_x(s)y - y\| + \|S_x(s)(-y) + y\|\big).\end{aligned}$$

This shows that

$$\left\|\frac{S(h)x-x}{h}-y\right\| \leq \sup_{0\leq s\leq h}\big(\|S_x(s)y-y\|+\|S_x(s)(-y)+y\|\big)\to 0 \quad \text{as } h\downarrow 0.$$

That is, $x\in D(A)$ with $Ax=y$. □

Theorem 3.3 shows that, for $x\in D(A)$, the function $t\mapsto S(t)x$ is a C^1-solution of the Cauchy problem

$$y'(t)=Ay(t)\ (t>0),\quad y(0)=x.$$

The following theorem is the second main result of this section and shows uniqueness of the solution.

Theorem 3.5. *Let $y\colon [0,\infty)\to X$ be a continuous function with $y(t)\in D(A)$ for all $t\geq 0$ and*

$$\left\|\frac{y(t+h)-y(t)}{h}-Ay(t)\right\|\to 0 \quad \text{as } h\downarrow 0 \quad \text{for all } t\geq 0.$$

Then, $y(t)=S(t)x$ for all $t\geq 0$, where $x:=y(0)$.

Proof. Let $t>0$ and $g(s):=S(t-s)y(s)$ for all $s\in[0,t]$. Fix $s\in[0,t)$. For every $h>0$ with $h\leq t-s$, one has

$$\begin{aligned}\frac{g(s+h)-g(s)}{h}&=\frac{S(t-s-h)y(s+h)-S(t-s)y(s)}{h}\\&=\frac{S_{S(h)y(s)}(t-s-h)\big(y(s+h)-S(h)y(s)\big)}{h}.\end{aligned}$$

By Proposition 2.2, there exist $L\geq 0$ and $r>0$ such that

$$\sup_{\tau\in[0,t]}\|S_x(\tau)z\|\leq L\|z\| \tag{3.5}$$

for all $x\in B(y(s),r)$ and $z\in B(0,r)$. Hence, for h sufficiently small, it follows that

$$\left\|\frac{S_{S(h)y(s)}(t-s-h)\big(y(s+h)-S(h)y(s)\big)}{h}\right\|\leq L\left\|\frac{y(s+h)-S(h)y(s)}{h}\right\|,$$

where we used that $\lim_{h\downarrow 0}y(s+h)=y(s)=\lim_{h\downarrow 0}S(h)y(s)$. Since $y(s)\in D(A)$,

$$\frac{y(s+h)-S(h)y(s)}{h}=\frac{y(s+h)-y(s)}{h}-\frac{S(h)y(s)-y(s)}{h}\to Ay(s)-Ay(s)=0$$

as $h\downarrow 0$. This shows that $\frac{g(s+h)-g(s)}{h}\to 0$ as $h\downarrow 0$.

We next show that the map $g\colon[0,t]\to X$ is continuous. Since its right derivative exists, it follows that $\lim_{h\downarrow 0}g(s+h)=g(s)$ for $s\in[0,t)$. Now, let $s\in(0,t]$ and $h>0$ sufficiently small. Then,

$$\begin{aligned}g(s-h)-g(s)&=S(t-s)S(h)y(s-h)-S(t-s)y(s)\\&=S_{y(s)}(t-s)\big(S(h)y(s-h)-y(s)\big).\end{aligned}$$

Since $y(s-h)\to y(s)$ as $h\downarrow 0$, it follows that $S(h)y(s-h)\to y(s)$ as $h\downarrow 0$ by Corollary 2.6. Together with (3.5), we obtain that $\lim_{h\downarrow 0}g(s-h)=g(s)$.

Finally, fix μ in the dual space X'. Since $\mu g\colon[0,t]\to\mathbb{R}$ is continuous and its right derivative vanishes on $[0,t)$, it follows from [23, Lemma 1.1, Chapter 2] that $[0,t]\to X,\ s\mapsto\mu g(s)$ is constant. In particular, $\mu y(t)=\mu g(t)=\mu g(0)=\mu S(t)x$. This shows that $y(t)=S(t)x$, as X' separates the points of X. □

Remark 3.6. With similar arguments as in the proof the previous theorem, one can show the following statement: Let $y\colon [0,\infty)\to X$ be a continuous function with $y(t)\in D(A)$ for all $t\geq 0$ and

$$\left\|\frac{y(t)-y(t-h)}{h}-Ay(t)\right\|\to 0\quad \text{as } h\downarrow 0\quad \text{for all } t>0.$$

Then, $y(t)=S(t)x$ for all $t\geq 0$ with $x:=y(0)$.

Theorem 3.5 implies that convex semigroups are determined by their generators as soon as the domain is dense.

Corollary 3.7. *Let T be a convex C_0-semigroup with generator $B\subset A$, i.e. $D(B)\subset D(A)$ and $A|_{D(B)}=B$. If $\overline{D(B)}=X$, then $S(t)=T(t)$ for all $t\geq 0$.*

Proof. For every $x\in D(B)$, the mapping $[0,\infty)\to X$, $t\mapsto T(t)x$ satisfies the assumptions of Theorem 3.5. Indeed, $[0,\infty)\to X,\ t\mapsto T(t)x$ is continuous by Corollary 2.5, and, by Theorem 3.3, $T(t)x\in D(B)\subset D(A)$ for all $t\geq 0$ with

$$\lim_{h\downarrow 0}\frac{T(t+h)x-T(t)x}{h}=\lim_{h\downarrow 0}\frac{T(h)T(t)x-T(t)x}{h}=BT(t)x=AT(t)x.$$

By Theorem 3.5, it follows that $T(t)x=S(t)x$ for all $t\geq 0$. Finally, since, by Corollary A.4, the bounded convex functions $T(t)$ and $S(t)$ are continuous and $\overline{D(B)}=X$, it follows that $S(t)=T(t)$ for all $t\geq 0$. ☐

Corollary 3.8. *The abstract Cauchy problem*

$$\text{(CP)}\qquad \begin{cases} y'(t)=Ay(t), & \text{for all } t\geq 0,\\ y(0)=x\end{cases}$$

is (classically) well-posed in the following sense:

(i) *For all $x\in D(A)$, (CP) has a unique classical solution $y\in C^1([0,\infty);X)$ with $y(t)\in D(A)$ for all $t\geq 0$ and $Ay\in C([0,\infty);X)$.*

(ii) *For all $x_0\in D(A)$ and $T>0$, there exist $L\geq 0$ and $r>0$ such that*

$$\sup_{t\in[0,T]}\|y(t,x)-y(t,z)\|<L\|x-z\|\quad \text{for all } x,z\in D(A)\cap B(x_0,r),$$

where $y(\,\cdot\,,x)$ denotes the unique solution to (CP) with initial value $x\in D(A)$.

(iii) *For all $t>0$ and $r>0$, there exists some constant $C\geq 0$ such that*

$$\|y(t,x)\|\leq C\quad \text{for all } x\in D(A) \text{ with } \|x\|\leq r.$$

Proof. By Theorem 3.3 and Theorem 3.5, it follows that, for every $x\in D(A)$, the Cauchy problem (CP) has a unique classical solution $y\in C^1([0,\infty);X)$ with $y(t)\in D(A)$ for all $t\geq 0$ and $Ay\in C([0,\infty);X)$ which is given by $y(t)=S(t)x$. By Corollary 2.4, we obtain (ii), and (iii) is the boundedness of the operator $S(t)$. ☐

Remark 3.9. Assume that for some operator $A_0\colon D(A_0)\subset X\to X$ the abstract Cauchy problem is well-posed in the sense of Corollary 3.8. Let the domain $D(A_0)$ be a dense linear subspace of X, and assume that the map $D(A_0)\to X,\ x\mapsto y(t,x)$ is convex for all $t\geq 0$. Then, there exists a unique convex C_0-semigroup $S=(S(t))_{t\in[0,\infty)}$ with $S(t)x=y(t,x)$ for all $x\in D(A_0)$. Moreover, $A_0\subset A$, where A is the generator of S, and $D(A_0)$ is $S(t)$-invariant for all $t\geq 0$, i.e. $S(t)x\in D(A_0)$ for all $t\geq 0$ and $x\in D(A_0)$.

In fact, we can define the operator $S(t)x:=y(t,x)$ for all $t\geq 0$ and $x\in D(A_0)$. As $S(t)$ is bounded by (iii) and convex, it is Lipschitz on bounded subsets of $D(A_0)$ by

Corollary A.4. Therefore, there exists a unique continuous extension $S(t)\colon X \to X$, which again is bounded and convex. By the uniqueness in (i), the semigroup property for the family $S = (S(t))_{t\in[0,\infty)}$ holds for all $x \in D(A_0)$, and therefore for all $x \in X$. Similarly, the strong continuity follows by $y(\cdot, x) \in C([0,\infty); X)$ for $x \in D(A_0)$ and (ii). Finally, as, for every $x \in D(A_0)$, the function $y(\cdot, x)$ is differentiable at zero with derivative Ax, we obtain $D(A_0) \subset D(A)$ with $A|_{D(A_0)} = A_0$ as well as, by (i), the invariance of $D(A_0)$ under $S(t)$.

In this way, we can construct a convex C_0-semigroup by solving the Cauchy problem only for initial values $x \in D(A_0)$. In applications, one might have $D(A_0)$ being much smaller than $D(A)$.

3.2. **Semigroup envelopes.** In this subsection, let X be a Banach lattice which is *Dedekind super complete*, i.e. every non-empty subset which is bounded above has a countable subset with identical supremum, and satisfies the monotone convergence property (see beginning of this section). The typical example for X is L^p for $1 \leq p < \infty$. For two semigroups S and T on X, we write $S \leq T$ if

$$S(t)x \leq T(t)x \quad \text{for all } t \geq 0 \text{ and } x \in X.$$

Throughout this section, let $(S_\lambda)_{\lambda\in\Lambda}$ be a family of convex monotone semigroups on X. We say that a semigroup S is an *upper bound* of $(S_\lambda)_{\lambda\in\Lambda}$ if $S \geq S_\lambda$ for all $\lambda \in \Lambda$.

Definition 3.10. We call a semigroup S (if existent) the *semigroup envelope* of $(S_\lambda)_{\lambda\in\Lambda}$ if it is the smallest upper bound of $(S_\lambda)_{\lambda\in\Lambda}$, i.e. if S is an upper bound of $(S_\lambda)_{\lambda\in\Lambda}$ and $S \leq T$ for any other upper bound T of $(S_\lambda)_{\lambda\in\Lambda}$.

Notice that the definition of a semigroup envelope already implies its uniqueness. However, the existence of a semigroup envelope is not given in general. In [10] and [20] the existence of a semigroup envelope, under certain conditions, has been shown for families of semigroups on spaces of uniformly continuous functions. This is done following an idea of Nisio [22], who was, to the best of our knowledge, the first to investigate the existence of semigroup envelopes. Moreover, it was shown (cf. [10],[20],[22]) that, for C_0-semigroups, there is a relation between the semigroup envelope, that is the supremum, of a family of semigroups and the pointwise supremum of their generators. In this subsection, we now want to show that the construction of Nisio, which is a pointwise optimization on a finer and finer time-grid, can be realized on Dedekind super complete Banach lattices. Moreover, we show that the ansatz proposed by Nisio is in fact the only way to construct the supremum of a family of semigroups. We further show that, under certain conditions, the semigroup envelope is strongly continuous and a sublinear monotone C_0-semigroup, which makes the results from the previous subsection applicable to the semigroup envelope of certain families of linear C_0-semigroups. In view of the examples in [10] and [20], this could be the starting point for L^p-semigroup theory for a large class of Hamilton-Jacobi-Bellman equations.

In the sequel, we consider finite partitions $P := \{\pi \subset [0,\infty) : 0 \in \pi, \pi \text{ finite}\}$. For a partition $\pi = \{t_0, t_1, \ldots, t_m\} \in P$ with $0 = t_0 < t_1 < \ldots < t_m$ we define $|\pi|_\infty := \max_{j=1,\ldots,m}(t_j - t_{j-1})$. The set of partitions with end-point t is denoted by P_t, i.e. $P_t := \{\pi \in P : \max \pi = t\}$.

Assume that the set $\{S_\lambda(t)x \colon \lambda \in \Lambda\}$ is bounded above for all $x \in X$ and all $t > 0$. Let $x \in X$. Then, we set

$$J_h x := \sup_{\lambda\in\Lambda} S_\lambda(h)x$$

for all $h > 0$ and

$$J_\pi x := J_{t_1-t_0}\cdots J_{t_m-t_{m-1}}x$$

for any partition $\pi = \{t_0, t_1, \ldots, t_m\} \in P$ with $0 = t_0 < t_1 < \ldots < t_m$.

Theorem 3.11. *Assume that, for all $t \geq 0$, there is a bounded operator $C(t)\colon X \to X$ with $J_\pi x \leq C(t)x$ for all $\pi \in P_t$ and $x \in X$. Then, the semigroup envelope $S = (S(t))_{t\in[0,\infty)}$ of $(S_\lambda)_{\lambda\in\Lambda}$ exists, is a convex monotone semigroup, and is given by*

$$S(t)x = \sup_{\pi\in P_t} J_\pi x \tag{3.6}$$

for all $t \geq 0$ and $x \in X$. If $C(t)x \to x$ as $t \downarrow 0$ for all $x \in X$ and S_{λ_0} is a C_0-semigroup for some $\lambda_0 \in \Lambda$, then S is strongly continuous. Moreover, if S_λ is sublinear for all $\lambda \in \Lambda$, then the semigroup envelope S sublinear.

Proof. Clearly, we have that $S_\lambda(h)x \leq J_h x$ for all $\lambda \in \Lambda$, $h > 0$ and all $x \in X$. Moreover, J_h is monotone and convex for all $h \geq 0$ since S_λ is montone and convex for all $\lambda \in \Lambda$. Consequently, J_π is monotone and convex with $S_\lambda(t)x \leq J_\pi x \leq C(t)x$ for all $\lambda \in \Lambda$, $t \geq 0$, $\pi \in P_t$ and $x \in X$, showing that $S = (S(t))_{t\geq 0}$, given by (3.6), is well-defined, monotone, convex and an upper bound of the family $(S_\lambda)_{\lambda\in\Lambda}$. Moreover, one directly sees that S is sublinear as soon as all S_λ are sublinear. From

$$S_{\lambda_0}(t)x \leq S(t)x \leq C(t)x \quad \text{and} \quad S_{\lambda_0}(t)x - x \leq S(t)x - x \leq C(t)x - x,$$

it follows that

$$\|S(t)x\| \leq \|S_{\lambda_0}(t)x\| + \|C(t)x\|$$

and

$$\|S(t)x - x\| \leq \|S_{\lambda_0}(t)x - x\| + \|C(t)x - x\|$$

for all $t \geq 0$, $x \in X$ and some (arbitrary) $\lambda_0 \in \Lambda$. This implies that $S(t)$ is bounded for all $t \geq 0$ and that $\lim_{t\downarrow 0} S(t)x = x$ as soon as $C(t)x \to x$ as $t \downarrow 0$ and S_{λ_0} is a C_0-semigroup for some $\lambda_0 \in \Lambda$. Next, we show that $S = (S(t))_{t\geq 0}$, defined by (3.6), is a semigroup. Clearly, $S(0)x = x$ for all $x \in X$. In order to show that $S(t+s) = S(t)S(s)$ for all $s, t \geq 0$, let $s, t \geq 0$ and $x \in X$. Then, it is easily seen that $S(t+s)x \leq S(t)S(s)x$ since, for all $\pi \in P_{t+s}$,

$$J_\pi x \leq J_{\pi_0} J_{\pi_1} x,$$

where $\pi_0 := \{u \in \pi : u \leq t\} \cup \{t\}$ and $\pi_1 := \{u - t : u \in \pi, u \geq t\} \cup \{0\}$. On the other hand, there exists a sequence $(\pi_n)_n$ in P_s with $S(s)x = \sup_{n\in\mathbb{N}} J_{\pi_n} x$. Defining

$$\pi_n^* := \bigcup_{k=1}^n \pi_k$$

for all $n \in \mathbb{N}$, we obtain that $J_{\pi_n^*} x \to S(s)x$, by the monotone convergence property. Consequently,

$$J_\pi S(s)x = \lim_{n\to\infty} J_\pi J_{\pi_n^*} x \leq S(t+s)x$$

for all $\pi \in P_t$, where, in the first equality, we used the fact that J_π is continuous since it is convex and bounded (see Lemma A.2). Taking the supremum over all $\pi \in P_t$, we obtain that $S(t)S(s)x \leq S(t+s)x$.

Finally, let T be an upper bound of $(S_\lambda)_{\lambda\in\Lambda}$. Then, $J_h x \leq T(h)x$ for all $h > 0$ and all $x \in X$ and consequently $J_\pi x \leq T(t)x$ for all $t \geq 0$, $\pi \in P_t$ and $x \in X$, which shows that $S(t)x \leq T(t)x$ for all $t \geq 0$ and $x \in X$. □

Corollary 3.12. *Let the semigroup T be an upper bound of the family $(S_\lambda)_{\lambda\in\Lambda}$. Then, the semigroup envelope of $(S_\lambda)_{\lambda\in\Lambda}$ exists and is given by* (3.6)*. If T is a C_0-semigroup and S_{λ_0} is a C_0-semigroup for some $\lambda_0 \in \Lambda$, then S is a C_0-semigroup.*

Proof. As we saw in the proof of the previous theorem, $S_\lambda(t)x \le J_\pi x \le T(t)x$ for all $\lambda \in \Lambda$, $t \ge 0$, $\pi \in P_t$ and $x \in X$. Therefore, the upper bound $C(t)$ in the previous theorem can be chosen to be $T(t)$. □

Corollary 3.13. *Let S be the semigroup envelope of the family $(S_\lambda)_{\lambda\in\Lambda}$. Then,*

$$S(t)x = \sup_{\pi\in P_t} J_\pi x$$

for all $t \ge 0$ and $x \in X$.

3.3. **Convolution semigroups on L^p.** Let $d \in \mathbb{N}$. In [10], the semigroup envelope, discussed in the previous subsection, has been constructed for a wide class of Lévy processes. In [10, Example 3.2], the authors consider families $(S_\lambda)_{\lambda\in\Lambda}$ of semigroups on the space $\mathrm{BUC} = \mathrm{BUC}(\mathbb{R}^d)$ of bounded uniformly continuous functions, which are indexed by a Lévy triplet $\lambda = (b, \Sigma, \mu)$. Recall that a Lévy triplet (b, Σ, μ) consists of a vector $b \in \mathbb{R}^d$, a symmetric positive semidefinite matrix $\Sigma \in \mathbb{R}^{d\times d}$ and a Lévy measure μ on $\mathbb{R}^d$. For each Lévy triplet λ, the semigroup S_λ is the one generated by the transition kernels of a Lévy process with Lévy triplet λ. More precisely,

$$\big(S_\lambda(t)x\big)(u) := \mathbb{E}\big[x(u + L_t^\lambda)\big] \tag{3.7}$$

for $t \ge 0$, $x \in \mathrm{BUC}$ and $u \in \mathbb{R}^d$, where L_t^λ is a Lévy process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Lévy triplet λ. In [10, Example 3.2], it was shown that, under the condition

$$\sup_{(b,\Sigma,\mu)\in\Lambda} |b| + |\Sigma| + \int_{\mathbb{R}^d\setminus\{0\}} 1 \wedge |y|^2 \,\mathrm{d}\mu(y) < \infty, \tag{3.8}$$

the semigroup envelope S_{BUC} for the family $(S_\lambda)_{\lambda\in\Lambda}$ exists and that in this case (cf. [10, Lemma 5.10])

$$\lim_{h\downarrow 0} \left\| \frac{S_{\mathrm{BUC}}(h)x - x}{h} - \sup_{\lambda\in\Lambda} A_\lambda x \right\|_\infty = 0 \quad \text{for } x \in \mathrm{BUC}^2. \tag{3.9}$$

Here, $\mathrm{BUC}^2 = \mathrm{BUC}^2(\mathbb{R}^d)$ is the space of all twice differentiable functions with bounded uniformly continuous derivatives up to order 2 and A_λ is the generator of the semigroup S_λ for each $\lambda \in \Lambda$. Notice that the setup in [10] is not contained in the setup of the previous subsection since BUC is not Dedekind super complete and does not satisfy the monotone convergence property. Recall that, for each Lévy triplet λ, (3.7) also gives rise to a linear monotone C_0-semigroup on $L^p = L^p(\mathbb{R}^d)$, which will again be denoted by S_λ (cf. [2, Theorem 3.4.2]). Therefore, the question arises if under a similar condition as (3.8), the semigoup envelope of the family $(S_\lambda)_{\lambda\in\Lambda}$ can be constructed on L^p. In general, the answer to this question is negative as the following example shows.

Example 3.14 (Uncertain shift semigroup)**.** Let $d = 1$ and $(S_\lambda(t)x)(u) := x(u + t\lambda)$ for $\lambda \in \Lambda := [-1, 1]$, $t \ge 0$, $x \in L^p(\mathbb{R})$ and $u \in \mathbb{R}$. Then, for $x \in L^p(\mathbb{R})$ given by $x(u) = |u|^{-1/2p} 1_{[-1,1]}(u)$,

$$\sup_{\lambda\in\Lambda} (S_\lambda(t)x)(u) = \infty \quad \text{for all } t \ge 0 \text{ and } u \in [-t, t].$$

Therefore, the set $\{S_\lambda(t)x \colon \lambda \in \Lambda\}$ does not have a least upper bound in L^p for all $t > 0$. In particular, the semigroup envelope of the family $(S_\lambda)_{\lambda\in\Lambda}$ does not exist although the set Λ satisfies condition (3.8).

In view of the previous example, additional conditions are required in order to guarantee the existence of the semigroup envelope on L^p. In the sequel, let C_c^∞ denote the space of all C^∞-functions $x\colon \mathbb{R}^d \to \mathbb{R}$ with compact support $\operatorname{supp} x$.

Theorem 3.15. *Let Λ be a non-empty set of Lévy triplets that satisfies* (3.8)*.*

(i) *Assume that, for each $t > 0$, there exists a bounded operator $C(t)\colon L^p \to L^p$ with*
$$|J_\pi x| \leq C(t)x \quad \text{for all } t > 0,\ \pi \in P_t \text{ and } x \in L^p. \tag{3.10}$$
Then, the semigroup envelope S of $(S_\lambda)_{\lambda\in\Lambda}$ exists, and is a monotone sublinear semigroup.

(ii) *In addition to* (3.10)*, assume that*
$$\sup_{\lambda\in\Lambda} A_\lambda x \in L^p \quad \text{for all } x \in C_c^\infty \tag{3.11}$$
and that, for every $x \in C_c^\infty$ and every $\varepsilon > 0$, there exists a compact set $K \subset \mathbb{R}^d$ with $\operatorname{supp} x \subset K$ and
$$\limsup_{h\downarrow 0} \left(\int_{\mathbb{R}^d\setminus K} \frac{\big|\big(C(h)x\big)(u)\big|^p}{h}\,\mathrm{d}u \right)^{1/p} \leq \varepsilon. \tag{3.12}$$
Then, the semigroup S is a C_0-semigroup, $C_c^\infty \subset D(A)$ and
$$Ax = \sup_{\lambda\in\Lambda} A_\lambda x$$
for all $x \in C_c^\infty$, where A denotes the generator of S.

Proof. (i) By Theorem 3.11, it is clear that (3.10) implies the existence of the semigroup envelope S and that the latter is monotone and sublinear.

(ii) Let $x \in C_c^\infty$. We show that $x \in D(A)$ with $Ax = \sup_{\lambda\in\Lambda} A_\lambda x =: Bx$. Let $\varepsilon > 0$. By (3.11) and (3.12), there exists some compact set $K \subset \mathbb{R}^d$ with $\operatorname{supp} x \subset K$ and
$$\left(\int_{\mathbb{R}^d\setminus K} \big|\big(Bx\big)(u)\big|^p \mathrm{d}u \right)^{1/p} < \frac{\varepsilon}{3} \quad \text{and} \quad \left(\int_{\mathbb{R}^d\setminus K} \frac{\big|\big(C(h)x\big)(u)\big|^p}{h}\,\mathrm{d}u \right)^{1/p} < \frac{\varepsilon}{3}$$
for $h > 0$ sufficiently small. Since $x \in C_c \subset \mathrm{BUC}^2 \cap L^p$, it follows that $S(t)x = S_{\mathrm{BUC}}(t)x$ for all $t \geq 0$. Hence, by (3.9),
$$\begin{aligned} \left\| \frac{S(h)x - x}{h} - Bx \right\|_p &\leq \operatorname{vol}(K)^{1/p} \left\| \frac{S(h)x - x}{h} - Bx \right\|_\infty + \left(\int_{\mathbb{R}^d\setminus K} \big|\big(Bx\big)(u)\big|^p \mathrm{d}u \right)^{1/p} \\ &\quad + \left(\int_{\mathbb{R}^d\setminus K} \frac{\big|\big(S(h)x\big)(u)\big|^p}{h}\,\mathrm{d}u \right)^{1/p} \\ &< \varepsilon \end{aligned}$$
for $h > 0$ sufficiently small, where $\operatorname{vol}(K)$ denotes the Lebesgue measure of K.

In particular, $\|S(h)x - x\|_p \to 0$ for all $x \in C_c^\infty$. Since C_c^∞ is dense in L^p and $S(t)\colon L^p \to L^p$ is continuous, this implies the strong continuity of S. □

Notice that the semigroup envelope from the previous theorem is exactly the extension of the semigroup envelope on BUC, constructed in [10], to the space L^p. More precisely, for each $t \geq 0$, the operator $S(t)$ is the unique bounded monotone sublinear operator $L^p \to L^p$ with $S(t)x = S_{\mathrm{BUC}}(t)x$ for all $x \in \mathrm{BUC} \cap L^p$. We will now give two examples of Lévy semigroups $(S_\lambda)_{\lambda\in\Lambda}$, where the semigroup envelope exists on L^p. The first one is a semilinear version of Example 3.14. The problem in Example 3.14

arises due to shifting sufficiently integrable poles. In order to treat this problem, one first has to smoothen a given function $x \in L^p$ via a suitable normal distribution and then shift the smooth version of x. This results in the following example.

Example 3.16 (g-expectation)**.** Let $d \in \mathbb{N}$, $p \in [1,\infty)$, and

$$\varphi_\lambda(t,z) := (2\pi t)^{-d/2} e^{-\frac{|z+\lambda t|^2}{2t}} \quad \text{for } \lambda, z \in \mathbb{R}^d \text{ and } t > 0.$$

For $\lambda \in \mathbb{R}^d$, we consider the linear C_0-semigroup $S_\lambda = (S_\lambda(t))_{t\in[0,\infty)}$ in $L^p = L^p(\mathbb{R}^d)$ given by $S_\lambda(0)x = x$ and

$$\big(S_\lambda(t)x\big)(u) := \int_{\mathbb{R}^d} x(v)\varphi_\lambda(t,u-v)\,\mathrm{d}v = \big(x * \varphi_\lambda(t,\,\cdot\,)\big)(u) = \mathbb{E}\big[x(u + W_t + \lambda t)\big]$$

for all $t > 0$, $x \in L^p$ and $u \in \mathbb{R}^d$, where $(W_t)_{t\in[0,\infty)}$ is a d-dimensional Brownian Motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For each $\lambda \in \Lambda$, the generator A_λ of S_λ is given by $D(A_\lambda) = W^{2,p}$ and

$$A_\lambda x = \tfrac{1}{2}\Delta x + \lambda \cdot \nabla x \quad \text{for } x \in W^{2,p},$$

where Δ denotes the Laplacian, '$\cdot$' is the scalar product in $\mathbb{R}^d$, and $W^{2,p} = W^{2,p}(\mathbb{R}^d)$ stands for the L^p-Sobolev space of order 2 (see also [19, Theorem 3.1.3] for the generation of a C_0-semigroup in L^p and [25, Theorem 31.5] for the connection between generator and Lévy triplet). Now, let $\Lambda \subset \mathbb{R}^d$ be a bounded and non-empty, and define

$$\big(J_h x\big)(u) := \sup_{\lambda\in\Lambda} \big(S_\lambda(h)x\big)(u) \quad \text{for } h \geq 0,\ x \in L^p \text{ and } u \in \mathbb{R}^d. \tag{3.13}$$

Notice that, for $h > 0$, $S_\lambda(h)x \in \mathrm{BUC}$ for all $x \in L^p$, which is why the supremum in (3.13) can be understood pointwise for $h > 0$.

We show that the conditions of Theorem 3.15 are satisfied. For the construction of an upper bound, we use the relation

$$\varphi_\lambda(h, u-v) = e^{-\lambda\cdot(u-v) - h|\lambda|^2/2}\varphi_0(h, u-v)$$

for all $\lambda \in \mathbb{R}^d$, $h > 0$ and $u, v \in \mathbb{R}^d$. With this and Hölder's inequality, it follows that

$$\begin{aligned}
\big|J_h x\big|(u) &= \left|\sup_{\lambda\in\Lambda}\int_{\mathbb{R}^d} x(v) e^{-\lambda\cdot(u-v)-h|\lambda|^2/2}\varphi_0(h,u-v)\,\mathrm{d}v\right| \\
&= \left|\sup_{\lambda\in\Lambda}\mathbb{E}\Big[x(u+W_h)e^{-\lambda\cdot W_h - h|\lambda|^2/2}\Big]\right| \\
&\leq \Big(\mathbb{E}\big[|x(u+W_h)|^p\big]\Big)^{1/p} \sup_{\lambda\in\Lambda}\Big(e^{-qh|\lambda|^2/2}\mathbb{E}\big[e^{-q\lambda\cdot W_h}\big]\Big)^{1/q} \\
&= \Big(\mathbb{E}\big[|x(u+W_h)|^p\big]\Big)^{1/p} \sup_{\lambda\in\Lambda} e^{(q-1)h|\lambda|^2/2} \\
&= \Big(\mathbb{E}\big[|x(u+W_h)|^p\big]\Big)^{1/p} e^{(q-1)h\overline{\lambda}^2/2} =: \big(C(h)x\big)(u),
\end{aligned}$$

where $\overline{\lambda} := \sup_{\lambda\in\Lambda}|\lambda|$ and $\frac{1}{p} + \frac{1}{q} = 1$. As

$$\big[(C(h)x)(u)\big]^p = e^{qh\overline{\lambda}^2/2}\big[|x|^p * \varphi_0(h,\,\cdot\,)\big](u),$$

we obtain that $C(h_1)C(h_2) = C(h_1+h_2)$ for $h_1, h_2 > 0$. Therefore,

$$|J_\pi x| \leq C(t_1 - t_0)\cdots C(t_m - t_{m-1})x = C(t_m)x$$

for any partition $\pi = \{t_0, t_1, \ldots, t_m\} \in P$ with $0 = t_0 < t_1 < \ldots < t_m$. By Fubini's theorem,

$$\|C(h)x\|_p^p = e^{qh\overline{\lambda}^2/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x(u-v)|^p \varphi_0(h,v)\,\mathrm{d}v\,\mathrm{d}u = e^{qh\overline{\lambda}^2/2}\|x\|_p^p$$

for all $h > 0$ and $x \in L^p$, showing that $C(h)\colon L^p \to L^p$ is bounded.

Now, let $x \in C_c^\infty$. We consider

$$(Bx)(u) := \sup_{\lambda\in\Lambda}(A_\lambda x)(u) = \tfrac{1}{2}\Delta x(u) + \sup_{\lambda\in\Lambda} \lambda \cdot \nabla x(u) \tag{3.14}$$

for $u \in \mathbb{R}^d$. As, for every $\lambda \in \Lambda$ and $u \in \mathbb{R}^d$,

$$|\lambda \cdot \nabla x(u)| \leq \sum_{j=1}^d |\lambda_j|\,|\partial_j x(u)| \leq \overline{\lambda} \sum_{j=1}^d |\partial_j x(u)|,$$

we obtain

$$\|Bx\|_{L^p} \leq C\big(\|\Delta x\|_{L^p} + \overline{\lambda}\|\nabla x\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}\big) \leq C\max\{1,\overline{\lambda}\}\|x\|_{W^{2,p}}, \tag{3.15}$$

with a constant C independent of x and Λ, which shows, in particular, that $Bx \in L^p$ for all $x \in C_c^\infty$.

It remains to verify (3.12). Let $x \in C_c^\infty$, and choose a compact set $K \subset \mathbb{R}^d$ with $\{u+v : u \in \operatorname{supp} x,\ |v| \leq 1\} \subset K$. For $u \in \mathbb{R}^d \setminus K$, we obtain $x(u+W_h) = 0$ if $|W_h| \leq 1$, and therefore,

$$\big(|x|^p * \varphi_0(h,\,\cdot\,)\big)(u) = \mathbb{E}\big(|x(u+W_h)|^p\big) = \mathbb{E}\big(\mathbf{1}_{\{|W_h|>1\}}|x(u+W_h)|^p\big).$$

By Fubini's theorem and Markov's inequality, for any $s > 2$,

$$\begin{aligned}\frac{1}{h}\int_{\mathbb{R}^d\setminus K} \mathbb{E}\big(\mathbf{1}_{\{|W_h|>1\}}|x(u+W_h)|^p\big)\,\mathrm{d}u &= \frac{1}{h}\mathbb{E}\bigg[\mathbf{1}_{\{|W_h|>1\}}\int_{\mathbb{R}^d\setminus K}|x(u+W_h)|^p\,\mathrm{d}u\bigg]\\ \leq \frac{1}{h}\|x\|_p^p\,\mathbb{P}(|W_h|>1) = \frac{1}{h}\|x\|_p^p\,\mathbb{P}\big(|W_1| > h^{-1/2}\big) &\leq h^{s/2-1}\mathbb{E}\big[|W_1|^s\big] \to 0\end{aligned}$$

as $h \downarrow 0$. By definition of $C(h)$, it follows that $\frac{1}{h}\int_{\mathbb{R}^d\setminus K}\big|\big(C(h)x\big)(u)\big|^p\,\mathrm{d}u \to 0$ as $h \downarrow 0$.

We have seen that all conditions of Theorem 3.15 are satisfied, and therefore the semigroup envelope $S = (S(t))_{t\in[0,\infty)}$ of $(S_\lambda)_{\lambda\in\Lambda}$ exists, and is a sublinear monotone C_0-semigroup.

As the map $\mathbb{R}^d \to \mathbb{R}$, $z \mapsto \sup_{\lambda\in\Lambda}\lambda\cdot z$ is Lipschitz (which follows, e.g., by Lemma A.7), the same holds for the nonlinearity

$$F\colon W^{1,p} \to L^p,\ x \mapsto \sup_{\lambda\in\Lambda}\lambda\cdot\nabla x,$$

where $W^{1,p} = W^{1,p}(\mathbb{R}^d)$ denotes the L^p-Sobolev space of order 1. In particular, the operator $B\colon W^{2,p} \to L^p$, $x \mapsto \sup_{\lambda\in\Lambda} A_\lambda x$, is well-defined and Lipschitz. Now let $x \in W^{2,p}$, and let $(x_n)_n$ be a sequence in C_c^∞ with $\|x - x_n\|_{W^{2,p}} \to 0$. By the Lipschitz continuity of B, we see that $(Bx_n)_n$ is a Cauchy sequence in L^p and therefore convergent. By Theorem 3.15, we have $Ax = Bx$ for all $x \in C_c^\infty$, and as the generator A of S is closed due to Proposition 3.4, we obtain $x \in D(A)$. Therefore, we see that $W^{2,p} \subset D(A)$. In particular, we obtain a unique classical solution to the Cauchy problem in the sense of Corollary 3.8 for all initial values in $D(A)$.

Notice that we did not use results from PDE theory in order to obtain the well-posedness of the Cauchy problem. As the nonlinearity F is Lipschitz continuous as a map from $W^{1,p}$ to L^p, it can be shown that all assumptions of [19, Prop. 7.1.10 (iii)] are satisfied. Therefore, for every $x \in W^{2,p}$ there exists a solution $y \in C^1([0,\infty); L^p)$ with $y(t) \in W^{2,p}$ for all $t \geq 0$ that solves the Cauchy problem

$$y'(t) = By(t) \quad \text{for all } t > 0, \quad y(0) = x.$$

By Theorem 3.5, it follows that $y(t) = S(t)x$ for all $t \geq 0$ and $x \in W^{2,p}$. In particular, $W^{2,p}$ is $S(t)$-invariant for all $t \geq 0$. Therefore, S is the unique continuous extension of the solution operator $x \mapsto y(\cdot, x)$, which is defined on $W^{2,p}$.

Remark 3.17. In the above examples, we consider the uncertain shift semigroup and the uncertain shift with known volatility (g-expectation). For the case of an uncertain volatility matrix λ (G-expectation) and the corresponding fully nonlinear operator

$$\big(Ax\big)(u) = \frac{1}{2} \sup_{\lambda \in \Lambda} \operatorname{tr}\big(\lambda \nabla^2 x(u)\big) = \sup_{\lambda \in \Lambda} \frac{1}{2} \sum_{i,j=1}^{d} \lambda_{ij} \partial_{ij} x(u),$$

the existence of the semigroup envelope in L^p seems to be an open problem.

Example 3.18 (Compound Poisson processes)**.** Let $\mu\colon \mathcal{B}(\mathbb{R}^d) \to [0,1]$ be a fixed probability measure. For $\lambda \geq 0$, $t \geq 0$, $x \in L^p$ and $u \in \mathbb{R}^d$, let

$$\big(S_\lambda(t)x\big)(u) := e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} x(u + v_1 + \ldots + v_n) \, \mathrm{d}\mu(v_1) \cdots \mathrm{d}\mu(v_n).$$

Then, S_λ is the semigroup corresponding to a compound Poisson process with intensity $\lambda \geq 0$ and jump distribution μ. Now, let $\Lambda \subset [0,\infty)$ be bounded, $\underline{\lambda} := \inf \Lambda$ and $\overline{\lambda} := \sup \Lambda$. Let

$$J_h x := \sup_{\lambda \in \Lambda} S_\lambda(h)x \quad \text{for } h \geq 0 \text{ and } x \in L^p.$$

Then, by Jensen's inequality,

$$\begin{aligned} \big|J_h x\big|(u) &\leq \left(\sup_{\lambda \in \Lambda} e^{-\lambda h} \sum_{n=0}^{\infty} \frac{(\lambda h)^n}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} |x(u + v_1 + \ldots + v_n)|^p \, \mathrm{d}\mu(v_1) \cdots \mathrm{d}\mu(v_n) \right)^{1/p} \\ &\leq e^{\left(\overline{\lambda} - \underline{\lambda}\right)h} \Big(\big(S_{\overline{\lambda}}(h)|x|^p\big)(u)\Big)^{1/p} =: \big(C(h)x\big)(u) \end{aligned}$$

for all $h \geq 0$, $x \in L^p$ and $u \in \mathbb{R}^d$. As before, we see that $C(h_1)C(h_2) = C(h_1 + h_2)$ for all $h_1, h_2 > 0$ and

$$|J_\pi x| \leq C(t_1 - t_0) \cdots C(t_m - t_{m-1})x = C(t_m)x$$

for any partition $\pi = \{t_0, t_1, \ldots, t_m\} \in P$ with $0 = t_0 < t_1 < \ldots < t_m$. Again, by Fubini's theorem,

$$\|C(h)x\|_p = e^{\left(\overline{\lambda} - \underline{\lambda}\right)h} \|x\|_p$$

for all $h \geq 0$ and $x \in L^p$, showing that $C(h)\colon L^p \to L^p$ is bounded. Let $x \in C_c^\infty$. It remains to show that $\frac{1}{h} \int_{\mathbb{R}^d \setminus K} \big|\big(C(h)x\big)(u)\big|^p \, \mathrm{d}u < \varepsilon$ for $h > 0$ sufficiently small. However, this follows from the fact that

$$\int_{\mathbb{R}^d} \left| \frac{\big(S_{\overline{\lambda}}(h)|x|^p\big)(u) - |x(u)|^p}{h} - \overline{\lambda} \int_{\mathbb{R}^d} |x(u+v)|^p - |x(u)|^p \, \mathrm{d}\mu(v) \right| \mathrm{d}u \to 0 \quad \text{as } h \downarrow 0.$$

By Theorem 3.15, the semigroup envelope $S = (S(t))_{t \in [0,\infty)}$ of $(S_\lambda)_{\lambda \in \Lambda}$ exists, and is a monotone, bounded and sublinear C_0-semigroup. Let $B\colon L^p \to L^p$ be given by

$$(Bx)(u) := \sup_{\lambda \in \Lambda} \lambda \int_{\mathbb{R}^d} x(u+v) - x(v) \, \mathrm{d}\mu(v) \quad \text{for } x \in L^p \text{ and } u \in \mathbb{R}^d.$$

Then, we have $A = B$ on C_c^∞ by Theorem 3.15. Since B is bounded and sublinear, and thus globally Lipschitz (see Lemma A.7), A is closed by Proposition 3.4 and C_c^∞ is dense in L^p, it follows that $D(A) = L^p$ and therefore $A = B$. In particular, we obtain a classical solution in the sense of Corollary 3.8 for all initial values $x \in L^p$.

Finally, we remark that due to the global Lipschitz continuity of B, we can also apply the theorem of Picard-Lindelöf to obtain a unique solution $y(\cdot, x)$ to the abstract initial value problem

$$y'(t) = By(t) \quad \text{for } t > 0, \quad y(0) = x,$$

for all $x \in L^p$. By Theorem 3.5, it follows that $y(t,x) = S(t)x$ for all $t \geq 0$ and $x \in L^p$.

4. The non σ-Dedekind complete case

In this section, we consider convex semigroups on Banach lattices which are not σ-Dedekind complete. As we have seen in Example 3.14, the uncertain shift semigroup cannot be defined on L^p, but we will see below that it is a convex C_0-semigroup on the space BUC of all bounded uniformly continuous functions. Another example, we are going to discuss in this section, is the G-expectation, which is the solution to a fully nonlinear version of the heat equation.

We assume that X is a Banach lattice which is a Riesz subspace of a Dedekind σ-complete Riesz space $\bar{X}$. For a sequence $(x_n)_n$ in X, we write $x_n \downarrow x$ if $(x_n)_n$ is decreasing, bounded from below, and $x = \inf_n x_n \in \bar{X}$. A typical example is the space BUC as a subspace of the space $\mathcal{L}^\infty$ of all bounded measurable functions. We define

$$X_\delta := \left\{ x \in \bar{X} \colon x_n \downarrow x \text{ for some sequence } (x_n)_n \text{ in } X \right\}.$$

Let M be the space of all positive linear functionals $\mu \colon X \to \mathbb{R}$ which are continuous from above, i.e. $\mu x_n \downarrow 0$ for every sequence $(x_n)_n$ in X such that $x_n \downarrow 0$. Every $\mu \in M$ has a unique extension $\mu \colon X_\delta \to \mathbb{R}$ which is continuous from above, i.e. $\mu x_n \downarrow \mu x$ for every sequence $(x_n)_n$ in X_δ such that $x_n \downarrow x \in X_\delta$, see e.g. [9, Lemma 3.9]. We assume that M separates the points of X_δ, i.e. for every $x, y \in X_\delta$ with $x \neq y$ there exists some $\mu \in M$ with $\mu x \neq \mu y$.

Definition 4.1. A monotone semigroup S is called *continuous from above* if $S(t)x_n \downarrow S(t)0$ for all $t \in [0,\infty)$ and every sequence $(x_n)_n$ in X with $x_n \downarrow 0$.

4.1. Invariant domains.

As before, let S be a convex semigroup on X. In contrast to Section 3, where the Banach lattice X is Dedekind σ-complete and has the monotone convergence property, the domain

$$D(A) := \left\{ x \in X \colon \frac{S(h)x - x}{h} \text{ is convergent in } X \text{ for } h \downarrow 0 \right\}$$

is in general not invariant under the semigroup. We therefore introduce the following modified versions of the domain.

Definition 4.2. The domain $D(A_\delta)$ of the *monotone generator* A_δ is defined as the set of all $x \in X$ such that, for every $(h_n)_n$ in $(0,\infty)$ with $h_n \downarrow 0$, there exists a sequence $(A_n x)_n$ in X and some $y \in X_\delta$ such that

$$\left\| \frac{S(h_n)x - x}{h_n} - A_n x \right\| \to 0 \quad \text{and} \quad A_n x \downarrow y. \tag{4.1}$$

We define the monotone generator $A_\delta \colon D(A_\delta) \subset X \to X_\delta$ of S by $A_\delta x := y$ for $x \in D(A_\delta)$, where y is the limit in (4.1), which is uniquely determined by Lemma B.1.

Definition 4.3. The *Lipschitz set* of the semigroup S is defined as

$$D_L := \left\{x \in X\colon \sup_{h\in(0,h_0]} \left\|\frac{S(h)x - x}{h}\right\| < \infty \quad \text{for some } h_0 > 0\right\}. \tag{4.2}$$

We further define the *symmetric Lipschitz set* of the semigroup S by

$$D_L^s := \big\{x \in X\colon x, -x \in D_L\big\}.$$

Then the following holds.

Lemma 4.4. *One has $D(A) \subset D(A_\delta) \subset D_L$, and $A_\delta|_{D(A)} = A$. If X is Dedekind σ-complete and has the monotone convergence property, then $D(A) = D(A_\delta)$.*

Proof. We first assume that $x \in D(A)$. Then, for every $h_n \downarrow 0$ and $A_n x := Ax$ for all $n \in \mathbb{N}$, one has

$$\left\|\frac{S(h_n)x - x}{h_n} - A_n x\right\| \to 0,$$

which shows that $x \in D(A_\delta)$ with $A_\delta x = Ax$.

We next assume that $x \in D(A_\delta)$. Then, there exists some $h_0 > 0$ such that

$$\sup_{h\in(0,h_0]} \left\|\frac{S(h)x - x}{h}\right\| < \infty.$$

Otherwise, there exists a sequence $h_n \downarrow 0$ such that $\left\|\frac{S(h_n)x-x}{h_n}\right\| \geq n$ for all n. Since $x \in D(A_\delta)$ there exists a bounded decreasing sequence $(A_n x)_n$ in X such that $A_n x \downarrow A_\delta x$ and

$$\left\|\frac{S(h_n)x - x}{h_n} - A_n x\right\| \to 0.$$

But then,

$$\sup_n \left\|\frac{S(h_n)x - x}{h_n}\right\| \leq \sup_n \left\|\frac{S(h_n)x - x}{h_n} - A_n x\right\| + \sup_n \|A_n x\| < \infty,$$

which is a contradiction. This shows that $x \in D_L$. If, in addition, X is σ-Dedekind complete and has the monotone convergence property, then $A_\delta x \in X$ and $\|A_n x - A_\delta x\| \to 0$, so that $\frac{S(h_n)x-x}{h_n} \to A_\delta x$ which shows that $D(A_\delta) = D(A)$. □

For every $x \in X$ and $y \in X_\delta$, the directional derivative is defined as

$$S'_+(t,x)y = \inf_{h>0} \frac{S(t)(x+hy) - S(t)x}{h} \in X_\delta.$$

For further details on the directional derivative we refer to Appendix B. The main result of this subsection is that both, $D(A_\delta)$ and D_L, are invariant under the semigroup, and states regularity properties in the time variable t.

Theorem 4.5. *For every $x \in D_L$ one has*

(i) *$S(t)x \in D_L$ for all $t \in [0,\infty)$,*
(ii) *for every $\mu \in M$ there is a locally bounded measurable function $f_\mu\colon [0,\infty) \to \mathbb{R}$ with $\mu S(t)x = \mu x + \int_0^t f_\mu(s)\,ds$ for all $x \in D(A_\delta)$ and $t \geq 0$.*

For every $x \in D(A)$ it holds

(iii) *$S(t)x \in D(A_\delta)$ for all $t \geq 0$ with $A_\delta S(t)x = S'_+(t,x)A_\delta x$,*
(iv) *$\mu S(t)x = \mu x + \int_0^t \mu S'_+(s,x)A_\delta x\,ds$ for every $\mu \in M$ and all $t \geq 0$. In particular, $f_\mu(s) = \mu S'_+(s,x)A_\delta x$ for almost every $s \in [0,\infty)$.*

Moreover, (iii) *and* (iv) *hold for all* $x \in D(A_\delta)$ *if, in addition, the semigroup is monotone and continuous from above.*

Proof. (i) Fix $t \geq 0$. By Corollary 2.4 there exist $L \geq 0$ and $r > 0$ such that

$$\|S(t)(y+x) - S(t)x\| \leq L\|y\|$$

for all $y \in B(x,r)$. Since $S(h)x \to x$ as $h \downarrow 0$, it follows that

$$\left\|\frac{S(h)S(t)x - S(t)x}{h}\right\| = \left\|\frac{S(t)S(h)x - S(t)x}{h}\right\| \leq L\left\|\frac{S(h)x - x}{h}\right\| < \infty$$

for all $h \in (0, h_0']$ and some $h_0' > 0$.

(ii) Since $x \in D_L$, it follows from Proposition 2.7 that the map $[0,\infty) \to X, t \mapsto S(t)x$ is locally Lipschitz continuous. Fix $\mu \in M$. Since μ is continuous on X, see e.g. [1, Theorem 9.6], the map $[0,\infty) \to \mathbb{R}$, $t \mapsto \mu S(t)x$ is also locally Lipschitz continuous and is therefore in $W^{1,\infty}_{\text{loc}}\big([0,\infty)\big)$ by Lebesgue's theorem. That is, there exists a locally bounded measurable function $f_\mu \colon [0,\infty) \to \mathbb{R}$ with $\mu S(t)x = \mu x + \int_0^t f_\mu(s)\,\mathrm{d}s$.

(iii) Fix $t > 0$, let $(h_n)_n$ be a sequence in $(0,\infty)$ with $h_n \downarrow 0$, and $x \in D(A)$. By Corollary 2.4, there exists some $L > 0$ such that

$$\begin{aligned}
&\left\|\frac{S(t+h_n)x - S(t)x}{h_n} - \frac{S(t)(x + h_nAx) - S(t)x}{h_n}\right\| = \left\|\frac{S(t)S(h_n)x - S(t)(x+h_nAx)}{h_n}\right\| \\
&\leq L\left\|\frac{S(h_n)x - x - h_nAx}{h_n}\right\| = L\left\|\frac{S(h_n)x - x}{h_n} - Ax\right\| \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Moreover, the sequence

$$A_n\big(S(t)x\big) := \frac{S(t)(x+h_nAx) - S(t)x}{h_n}$$

is decreasing and satisfies $A_n(S(t)x) \downarrow S_+'(t,x)Ax$. This shows that $S(t)x \in D(A_\delta)$ with $A_\delta S(t)x = S_+'(t,x)Ax$. Recall that $Ax = A_\delta x$ for all $x \in D(A)$ by Lemma 4.4.

If in addition, S is monotone, continuous from above, and $x \in D(A_\delta)$, then there exists a bounded decreasing sequence $(A_nx)_n$ in X such that

$$\left\|\frac{S(h_n)x - x}{h_n} - A_nx\right\| \to 0 \quad \text{and} \quad A_nx \downarrow A_\delta x.$$

By Corollary 2.4, there exists some $L > 0$ such that

$$\left\|\frac{S(t+h_n)x - S(t)x}{h_n} - \frac{S(t)(x+h_nA_nx) - S(t)x}{h_n}\right\| \leq L\left\|\frac{S(h_n)x - x}{h_n} - A_nx\right\| \to 0$$

as $n \to \infty$. By Lemma B.4, the sequence $(A_nS(t)x)$ given by

$$A_nS(t)x := \frac{S(t)(x+h_nA_nx) - S(t)x}{h_n}$$

is decreasing and satisfies $A_nS(t)x \downarrow S_+'(t,x)A_\delta x$. This shows that $S(t)x \in D(A_\delta)$ with $A_\delta S(t)x = S_+'(t,x)A_\delta x$.

(iv) Since $x \in D(A_\delta)$, it follows from Lemma 4.4 that $x \in D_L$. Fix $\mu \in M$. By (ii) one has

$$\mu S(t)x = \mu x + \int_0^t f_\mu(s)\,ds$$

for all $t \geq 0$. In particular, $t \mapsto \mu S(t)x$ is differentiable almost everywhere. Since μ is continuous from above it follows from the previous step (iii) that the derivative is a.e. given by

$$f_\mu(t) = \lim_{h \downarrow 0} \frac{\mu S(t+h)x - \mu S(t)x}{h} = \mu A_\delta S(t)x = \mu S'_+(t,x)A_\delta x.$$

The proof is complete. □

For the symmetric Lipschitz set of a sublinear monotone semigroup, we have the following proposition.

Proposition 4.6. *Let S be sublinear and monotone. Then, the symmetric Lipschitz set D_L^s is a linear subspace of X. If*

$$-S(s)\big(-S(t)x\big) \geq S(t)\big(-S(s)(-x)\big) \quad \text{for all } s,t \geq 0 \text{ and } x \in X, \tag{4.3}$$

then $S(t)x \in D_L^s$ for all $t \geq 0$ and $x \in D_L^s$.

Proof. The sublinearity of S implies that

$$S(t)(x+\lambda y) - (x+\lambda y) \leq S(t)x - x + \lambda\big(S(t)y - y\big)$$

and

$$-S(t)(x+\lambda y) + x + \lambda y \leq S(t)(-x) + x + \lambda\big(S(t)(-y) + y\big)$$

for all $x, y \in X$ and $\lambda > 0$. Consequently,

$$\|S(t)(x+\lambda y)-(x+\lambda y)\| \leq \|S(t)x-x\|+\|S(t)(-x)+x\|+\lambda\big(\|S(t)y-y\|+\|S(t)(-y)+y\|\big)$$

for all $x, y \in X$ and $\lambda > 0$, which shows that $x + \lambda y \in D_L^s$ for all $x, y \in D_L^s$ and $\lambda > 0$. Since $-x \in D_L^s$ for all $x \in D_L^s$, it follows that D_L^s is a linear subspace of X.

Now, let $x \in D_L^s$ and $t \geq 0$. Since $S(t)$ is sublinear and bounded, it is globally Lipschitz with some Lipschitz constant $L > 0$ (see Lemma A.7). Therefore,

$$\|S(h)S(t)x - S(t)x\| \leq L\|S(h)x - x\|,$$

i.e. $S(t)x \in D_L$. It remains to show that $-S(t)x \in D_L$. First, observe that

$$-S(t)x - S(h)\big(-S(t)x\big) \leq -S(t)x + S(h)S(t)x \leq S(t)\big(S(h)x - x\big)$$

and, by (4.3),

$$S(h)\big(-S(t)x\big) + S(t)x \leq -S(t)\big(-S(t)(-x)\big) + S(t)x \leq S(t)\big(S(h)(-x) + x\big).$$

Therefore,

$$\big\|S(h)\big(-S(t)x\big) + S(t)x\big\| \leq L\Big(\|S(h)x - x\| + \big\|\big(S(h)(-x) + x\big)\big\|\Big),$$

which shows that $-S(t)x \in D_L$. □

Example 4.7. Let S be a translation-invariant sublinear monotone semigroup on the space $\mathrm{BUC} = \mathrm{BUC}(G)$, where G is an abelian group with a translation invariant metric d such that (G,d) is separable and complete. Here, *translation invariant* means that

$$\big(S(t)x(u+\cdot)\big)(0) = \big(S(t)x\big)(u) \quad \text{for all } x \in \mathrm{BUC},\ u \in G \text{ and } t \geq 0.$$

The space BUC of all bounded uniformly continuous functions $x\colon G \to \mathbb{R}$ is endowed with the supremum norm $\|x\|_\infty := \sup_{u \in G}|x(u)|$. Under mild continuity assumptions, the semigroup has a dual representation

$$\big(S(t)x\big)(u) = \sup_{\mu \in \mathcal{P}_t} \int_G x(u+v)\,\mathrm{d}\mu_t(v) \quad \text{for all } x \in \mathrm{BUC},\ u \in G \text{ and } t \geq 0. \tag{4.4}$$

where $\mathcal{P}_t$ is a convex set of Borel measures on G for all $t \geq 0$. For further details on dual representations we refer to [9]. For instance, such a dual representation holds if the semigroup is continuous from above. An example, where the semigroup S satisfies (4.4), is the semigroup envelope S_{BUC} of a family $(S_\lambda)_{\lambda\in\Lambda}$ of linear convolution semigroups of the form (3.7) satisfying (3.8) (see Section 3.3). For further examples, we refer to [10]. Notice that, under (4.4),

$$-\big(S(t)(-x)\big)(u) = \inf_{\mu\in\mathcal{P}_t} \int_G x(u+v)\,\mathrm{d}\mu_t(v) \quad \text{for all } x \in \mathrm{BUC},\ u \in G \text{ and } t \geq 0.$$

Then, for $x \in \mathrm{BUC}$, $u \in G$, $\mu_t \in \mathcal{P}_t$ and $\mu_s \in \mathcal{P}_s$, it follows from (4.4) and Fubini's theorem that

$$\begin{aligned}\int_G \big(S(t)x\big)(u+v)\,\mathrm{d}\mu_s(v) &\geq \int_G\int_G x(u+v+w)\,\mathrm{d}\mu_t(w)\,\mathrm{d}\mu_s(v)\\ &= \int_G\int_G x(u+v+w)\,\mathrm{d}\mu_s(v)\,\mathrm{d}\mu_t(w)\\ &\geq \int_G -\big(S(s)(-x)\big)(u+w)\,\mathrm{d}\mu_t(w).\end{aligned}$$

Taking the infimum over all $\mu_s \in \mathcal{P}_t$ and supremum over all $\mu_t \in \mathcal{P}_s$ yields

$$-S(s)\big(-S(t)x\big) \geq S(t)\big(-S(s)(x)\big).$$

By Proposition 4.6, we thus find that D_L^s is $S(t)$-invariant for all $t \geq 0$.

Remark 4.8. Consider the setup of the previous example. Given $C \geq 0$ and $h_0 > 0$, let $D_L^s(C,h_0)$ denote the set of all $x \in D_L^s$ such that $\|S(h)x - x\|_\infty \leq Ch$ and $\|S(h)(-x) + x\|_\infty \leq Ch$ for all $h \in [0,h_0]$. Let $x \in D_L^s(C,h_0)$ and ν be a Borel probability measure on G. Then, one has $x_\nu \in D_L^s(C,h_0)$, where $x_\nu(u) := \int_G x(u+v)\,\nu(dv)$. In fact, by a Banach space valued version of Jensen's inequality (see e.g. [10] or [20]) and the translation invariane of S,

$$\begin{aligned}S(h)x_\nu - x_\nu &= S(h)\bigg(\int_G x(\,\cdot\,+v)\,\mathrm{d}\nu(v)\bigg) - x_\nu \leq \int_G \big(S(h)x\big)(\,\cdot\,+v)\,\mathrm{d}\nu(v) - x_\nu\\ &= \int_G \big(S(h)x\big)(\,\cdot\,+v) - x(\,\cdot\,+v)\,\mathrm{d}\nu(v) \leq Ch\end{aligned}$$

for all $h \geq 0$. In a similar way, it follows that

$$S(h)(-x_\nu) + x_\nu \leq \int_G \big(S(h)(-x)\big)(\,\cdot\,+v) + x(\,\cdot\,+v)\,\mathrm{d}\nu(v) \leq Ch$$

for all $h \in [0,h_0]$. Combining these two estimates yields that

$$\big\|S(h)x_\nu - x_\nu\big\|_\infty \leq Ch \quad \text{and} \quad \big\|S(h)(-x_\nu) + x_\nu\big\|_\infty \leq Ch$$

for all $h \in [0,h_0]$, i.e. $x_\nu \in D_L^s(C,h_0)$.

4.2. **Uniqueness.** Now, we are ready to state the main result of this paper. We show that a convex semigroup is uniquely determined on $D(A_\delta)$ through its generator A_δ if the semigroup is, in addition, monotone and continuous from above.

Theorem 4.9. *Let S be a convex monotone C_0-semigroup on X which is continuous from above with monotone generator A_δ. Let $y\colon [0,\infty) \to X$ be a continuous function* *with $y(t) \in D(A_\delta)$ for all $t \geq 0$, and assume that, for all $t \geq 0$ and $(h_n)_n$ in $(0,\infty)$ with $h_n \downarrow 0$, there exists a bounded decreasing sequence $(B_n y(t))_n$ in X such that*

$$\left\| \frac{y(t+h_n) - y(t)}{h_n} - B_n y(t) \right\| \to 0 \quad \textit{and} \quad B_n y(t) \downarrow A_\delta y(t).$$

Then, $y(t) = S(t)x$ for all $t \geq 0$, where $x := y(0)$.

Proof. Let $t > 0$ and $g(s) := S(t-s)y(s)$ for all $s \in [0,t]$. Fix $s \in (0,t)$. For every $h > 0$ with $h < t-s$ one has

$$\begin{aligned} \frac{g(s+h)-g(s)}{h} &= \frac{S(t-s-h)y(s+h) - S(t-s)y(s)}{h} \\ &= \frac{S(t-s-h)y(s+h) - S(t-s-h)y(s)}{h} \\ &\qquad - \frac{S(t-s-h)S(h)y(s) - S(t-s-h)y(s)}{h}. \end{aligned}$$

Let $(h_n)_n$ in $(0,\infty)$ with $h_n \downarrow 0$ and $\mu \in M$. By assumption, for $y := y(s) \in D(A_\delta)$, there exists a bounded decreasing sequence $(B_n y)_n$ with

$$\left\| \frac{y(s+h_n) - y(s)}{h_n} - B_n y \right\| \to 0 \quad \text{and} \quad B_n y \downarrow A_\delta y. \tag{4.5}$$

Define

$$\nu_n z := \frac{\mu S(t-s-h_n)(y+h_n z) - \mu S(t-s-h_n)y}{h_n} \quad \text{and} \quad \nu z := \limsup_{n\to\infty} \nu_n z$$

for every $z \in X_\delta$ and all n for which $t-s-h_n > 0$, where we take the unique extension of S to X_δ given by Lemma B.2. By Corollary 2.4, there exists some $L > 0$ such that

$$\left\| \frac{S(t-s-h_n)y(s+h_n) - S(t-s-h_n)\big(y + h_n B_n y\big)}{h_n} \right\| \leq L \left\| \frac{y(s+h_n) - y}{h_n} - B_n y \right\| \to 0$$

as $n \to \infty$. Therefore, we conclude that

$$\limsup_{n\to\infty} \mu\left(\frac{S(t-s-h_n)y(s+h_n) - S(t-s-h_n)y}{h_n} \right) = \limsup_{n\to\infty} \nu_n B_n y. \tag{4.6}$$

We next show that

$$\limsup_{n\to\infty} \nu_n B_n y = \nu A_\delta y. \tag{4.7}$$

To that end, we first show

$$\nu z \leq \inf_{h>0} \frac{\mu S(t-s)(y+hz) - \mu S(t-s)y}{h} \tag{4.8}$$

for all $z \in X$. Indeed, for every $\varepsilon > 0$, there exists some $h_0 > 0$ and, by Corollary 2.5 there exists some $m_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\inf_{h>0} \frac{\mu S(t-s)(y+hz) - \mu S(t-s)y}{h} + 2\varepsilon \geq \frac{\mu S(t-s)(y+h_0 z) - \mu S(t-s)y}{h_0} + \varepsilon \\ &\geq \frac{\mu S(t-s-h_m)(y+h_0 z) - \mu S(t-s-h_m)y}{h_0} \end{aligned}$$

for all $m \geq m_0$. Hence, for all $n \geq m_0$ which satisfy $h_n \leq h_0$ one has

$$\begin{aligned}\inf_{h>0} &\frac{\mu S(t-s)(y+hz)-\mu S(t-s)y}{h}+2\varepsilon \\ &\geq \frac{\mu S(t-s-h_n)(y+h_nz)-\mu S(t-s-h_n)y}{h_n}=\nu_n z,\end{aligned}$$

which shows (4.8) by taking the limit superior as $n\to\infty$ and letting $\varepsilon\downarrow 0$. As a consequence of (4.8), it follows that ν is continuous from above. Indeed, for every $z_n\downarrow 0$ one has

$$0\leq \inf_n \nu z_n \leq \inf_{h>0}\inf_n \frac{\mu S(t-s)(y+hz_n)-\mu S(t-s)y}{h}=0$$

so that $\nu z_n\downarrow 0$. Hence, for every $\varepsilon>0$ there exist $n_0,m_0\in\mathbb{N}$ such that

$$\nu A_\delta y+2\varepsilon \geq \nu B_{n_0}y+\varepsilon \geq \nu_m B_{n_0}y\geq \nu_m B_m y$$

for all $m\geq m_0\vee n_0$, where the last inequality follows by monotonicity of ν_m. This shows that

$$\nu A_\delta y\geq \limsup_{n\to\infty}\nu_n B_n y.$$

Further, $\nu A_\delta y=\limsup_{n\to\infty}\nu_n A_\delta y\leq \limsup_{n\to\infty}\nu_n B_n y$ by monotonicity of ν_n, which proves (4.7).

Since $y=y(s)\in D(A_\delta)$, it follows from (4.1) that there exists a bounded decreasing sequence $(A_n y)_n$ with

$$\left\|\frac{S(h_n)y-y}{h_n}-A_n y\right\|\to 0\quad\text{and}\quad A_n y\downarrow A_\delta y.$$

By the same arguments as before we get,

$$\limsup_{n\to\infty}\mu\bigg(\frac{S(t-s-h_n)S(h_n)y-S(t-s-h_n)y}{h_n}\bigg)=\limsup_{n\to\infty}\nu_n A_n y=\nu A_\delta y. \tag{4.9}$$

Hence, in combination with (4.6) and (4.7) we get

$$\begin{aligned}&\limsup_{n\to\infty}\mu\bigg(\frac{S(t-s-h_n)y(s+h_n)-S(t-s-h_n)y(s)}{h_n}\bigg)\\ &=\limsup_{n\to\infty}\mu\bigg(\frac{S(t-s-h_n)S(h_n)y(s)-S(t-s-h_n)y(s)}{h_n}\bigg)\end{aligned}\tag{4.10}$$

for every sequence $(h_n)_n$ in $(0,\infty)$ with $h_n\downarrow 0$ and all $\mu\in M$. As a consequence, we conclude that

$$\frac{\mu g(s+h_n)-\mu g(s)}{h_n}\to 0 \tag{4.11}$$

for every sequence $(h_n)_n$ in $(0,\infty)$ with $h_n\downarrow 0$ and all $\mu\in M$. Indeed, by passing to a subsequence $(n_k)_k$, we may assume that

$$\limsup_{n\to\infty}\frac{\mu g(s+h_n)-\mu g(s)}{h_n}=\lim_{k\to\infty}\frac{\mu g(s+h_{n_k})-\mu g(s)}{h_{n_k}}.$$

By passing to another subsequence, which we still denote by $(n_k)_k$, we can further assume that

$$\begin{aligned}&\liminf_{k\to\infty}\mu\bigg(\frac{S(t-s-h_{n_k})S(h_{n_k})y(s)-S(t-s-h_{n_k})y(s)}{h_{n_k}}\bigg)\\ &=\limsup_{k\to\infty}\mu\bigg(\frac{S(t-s-h_{n_k})S(h_{n_k})y(s)-S(t-s-h_{n_k})y(s)}{h_{n_k}}\bigg).\end{aligned}\tag{4.12}$$

Then, by applying the equality (4.10) to the subsequence $(h_{n_k})_k$ we obtain

$$\begin{aligned}\limsup_{n\to\infty} \frac{\mu g(s+h_n)-\mu g(s)}{h_n} &= \lim_{k\to\infty} \frac{\mu g(s+h_{n_k})-\mu g(s)}{h_{n_k}} \\ \le \limsup_{k\to\infty} \mu\bigg(&\frac{S(t-s-h_{n_k})y(s+h_{n_k})-S(t-s-h_{n_k})y(s)}{h_{n_k}}\bigg) \\ &- \liminf_{k\to\infty} \mu\bigg(\frac{S(t-s-h_{n_k})S(h_{n_k})y(s)-S(t-s-h_{n_k})y(s)}{h_{n_k}}\bigg) = 0,\end{aligned}$$

where the last equality follows from (4.10) and (4.12). With similar arguments, we also obtain $\liminf_{n\to\infty} \frac{\mu g(s+h_n)-\mu g(s)}{h_n} \ge 0$, which shows (4.11).

Since μ is continuous on X, see e.g. [1, Theorem 9.6], it follows by the same arguments as in the proof of Theorem 3.5 that $s\mapsto \mu g(s)$ is continuous on $[0,t]$. By [23, Lemma 1.1, Chapter 2] we conclude that the map $s\mapsto \mu g(s)$ is constant on $[0,t]$, since it is continuous and its right derivative vanishes on $[0,t)$. In particular, $\mu y(t)=\mu g(t)=\mu g(0)=\mu S(t)y(0)$ for all $\mu\in M$. This shows that $y(t)=S(t)y(0)$ as M separates the points of X. □

Corollary 4.10. *Let S be a convex monotone C_0-semigroup on X which is continuous from above with monotone generator A_δ, and let T be a convex C_0-semigroup on X with generator B and monotone generator B_δ such that $B_\delta\subset A_\delta$. If $\overline{D(B)}=X$, then $S(t)=T(t)$ for all $t\ge 0$.*

Proof. For every $x\in D(B)$, the mapping $y\colon[0,\infty)\to X$, $y(t):=T(t)x$ satisfies the assumptions of Theorem 4.9. Indeed, $y(0)=x$ by definition, $t\mapsto y(t)$ is continuous by Corollary 2.5, and $y(t)\in D(B_\delta)\subset D(A_\delta)$ by Theorem 4.5 with

$$\left\|\tfrac{y(t+h_n)-y(t)}{h_n}-B_ny(t)\right\|\to 0 \quad\text{and}\quad B_ny(t)\downarrow B_\delta y(t)=A_\delta y(t)$$

where $B_ny(t):=\frac{T(t)(x+h_nBx)-T(t)x}{h_n}$ for all $n\in\mathbb{N}$. Hence, by Theorem 4.9, it follows that $T(t)x=y(t)=S(t)x$ for all $t\ge0$. Since, by Corollary A.4, the bounded convex functions $T(t)$ and $S(t)$ are continuous, and $\overline{D(B)}=X$, it holds $S(t)=T(t)$ for all $t\ge 0$. □

4.3. **The uncertain shift semigroup on** BUC**.** Let G be a convex set endowed with a metric $d\colon G\times G\to[0,\infty)$. We assume that, for every $u,v\in G$ and $\lambda\in(0,1)$, there exists some $\lambda(u,v)\in G$ such that $d(u,\lambda(u,v))=\lambda d(u,v)$ and $d(\lambda(u,v),v)=(1-\lambda)d(u,v)$. The space of all bounded uniformly continuous functions $x\colon G\to\mathbb{R}$ is denoted by $\mathrm{BUC}=\mathrm{BUC}(G)$ and endowed with the supremum norm $\|x\|_\infty:=\sup_{u\in G}|x(u)|$. Notice that BUC is a Riesz subspace of the Dedekind σ-complete Riesz space $\mathcal{L}^\infty$ of all bounded Borel measurable functions $x\colon G\to\mathbb{R}$. On $\mathcal{L}^\infty$, we consider the partial order $x\le y$ whenever $x(u)\le y(u)$ for all $u\in G$.

The *uncertain shift semigroup* S on BUC is defined by

$$\big(S(t)x\big)(u):=\sup_{d(u,v)\le t} x(v)\quad\text{for all } x\in\mathrm{BUC},\ u\in G \text{ and } t\ge 0.$$

Lemma 4.11. *S is a sublinear monotone C_0-semigroup on* BUC*. Moreover,*

$$D_L=D_L^s=\mathrm{Lip}_b,$$

where $\mathrm{Lip}_b=\mathrm{Lip}_b(G)$ is the space of all bounded Lipschitz continuous functions $G\to\mathbb{R}$.

Proof. We first show that $S(t)\colon \mathrm{BUC} \to \mathrm{BUC}$ is well-defined and bounded. To this end, fix $x \in \mathrm{BUC}$. Since

$$|S(t)x(u)| \le \sup_{d(u,v)\le t} |x(v)| = \|x\|_\infty \quad \text{for all } u \in G,$$

it follows that $\|S(t)x\|_\infty \le \|x\|_\infty$. Fix $\varepsilon > 0$ and $\delta > 0$ such that $|x(u) - x(v)| \le \varepsilon$ for all $u, v \in G$ with $d(u,v) \le \delta$. Let $u, v \in G$ with $d(u,v) \le \delta$ and $w \in G$ with $d(u,w) \le t$. Then, for $\lambda := \frac{t}{t+\delta}$, one has

$$d\big(v, \lambda(v,w)\big) = \lambda d(v,w) \le \lambda(t+\delta) = t$$

and

$$d\big(w, \lambda(v,w)\big) = (1-\lambda) d(v,w) \le (1-\lambda)(t+\delta) = \delta$$

Hence,

$$x(w) - \big(S(t)x\big)(v) \le x(w) - x\big(\lambda(v,w)\big) \le \varepsilon.$$

Taking the supremum over all $w \in G$ with $d(u,w) \le t$, it follows that

$$\big(S(t)x\big)(u) - \big(S(t)x\big)(v) \le \varepsilon.$$

By a symmetry argument, we obtain that $|S(t)x(u) - S(t)x(v)| \le \varepsilon$, showing that $S(t)x$ is uniformly continuous with the same modulus of continuity as x. We thus have shown that $S(t)\colon \mathrm{BUC} \to \mathrm{BUC}$ is well-defined and bounded. By definition, each $S(t)$ is sublinear and monotone, and $S(0)x = x$ for all $x \in \mathrm{BUC}$. Moreover, for $t \le \delta$, one has

$$\big|\big(S(t)x\big)(u) - x(u)\big| \le \sup_{d(u,v)\le t} |x(v) - x(u)| \le \varepsilon$$

for all $u \in G$, i.e. $\|S(t)x - x\|_\infty \le \varepsilon$ for all $t \le \delta$, which shows that S is strongly continuous. It remains to show that S satisfies the semigroup property. Let $s, t \ge 0$. Further, let $u \in G$ and $w \in G$ with $d(u,w) \le s+t$. Then, for $\lambda := \frac{t}{s+t}$, it holds

$$d\big(w, \lambda(u,w)\big) = (1-\lambda) d(u,w) \le s$$

and

$$d\big(u, \lambda(u,w)\big) = \lambda d(u,w) \le t.$$

Hence,

$$x(w) \le \sup_{d(\lambda(u,w),v)\le s} x(v) = \big(S(s)x\big)\big(\lambda(u,w)\big) \le \sup_{d(u,v)\le t} \big(S(s)x\big)(v) = \big(S(t)S(s)x\big)(u).$$

Taking the supremum over all $w \in G$ with $d(u,w) \le s+t$, it follows that

$$\big(S(s+t)x\big)(u) \le \big(S(t)S(s)x\big)(u).$$

Now, let $w \in G$ with $d(u,w) \le t$. Then, there exists a sequence $(w_n)_n$ in G with $d(w,w_n) \le s$ and $x(w_n) \to \big(S(s)x\big)(w)$. Then,

$$\big(S(s)x\big)(w) = \lim_{n\to\infty} x(w_n) \le \sup_{d(u,v)\le s+t} x(v) = \big(S(s+t)x\big)(u).$$

Taking the supremum over all $w \in G$ with $d(u,w) \le t$, yields that

$$\big(S(t)S(s)x\big)(u) \le \big(S(s+t)x\big)(u).$$

Altogether, we have shown that S is a sublinear monotone C_0-semigroup on BUC.

Now, let $x \in D_L$. Then, there exist $h_0 > 0$ and $C \ge 0$ such that $\|S(h)x - x\|_\infty \le Ch$ for all $h \in [0, h_0]$. Hence, for all $u, v \in G$ with $d(u,v) =: h \le h_0$,

$$x(u) - x(v) \le \big(S(h)x\big)(v) - x(v) \quad \text{and} \quad x(v) - x(u) \le \big(S(h)x\big)(u) - x(u).$$

This implies that $|x(u) - x(v)| \leq \|S(h)x - x\|_\infty \leq Ch = Cd(u, v)$. Since $x \in \mathrm{BUC}$ is bounded, it follows that $x \in \mathrm{Lip}_b$. On the other hand, if $x \in \mathrm{Lip}_b \subset \mathrm{BUC}$ with Lipschitz constant $C > 0$, it follows that

$$\|\big(S(h)x\big)(u) - x(u)\| \leq \sup_{d(u,v)\leq h} |x(v) - x(u)| \leq Cd(u, v) \leq Ch$$

for all $u \in G$ and $h \geq 0$. Therefore $x \in D_L$. Since $-x \in \mathrm{Lip}_b$ for all $x \in \mathrm{Lip}_b$, it follows that $\mathrm{Lip}_b \subset D_L^s$. Since, by definition, $D_L^s \subset D_L$, the assertion follows. □

We now specialize on the case, where $G = \mathbb{R}$ with the Euclidian distance $d(u, v) = |u - v|$. In this case, the uncertain shift semigroup is given by

$$\big(S(t)x\big)(u) = \sup_{|v|\leq t} x(u + v)$$

for all $u \in \mathbb{R}$ and $t \in [0, \infty)$. By Lemma 4.11, it follows that S is a sublinear monotone C_0-semigroup on BUC. In addition, by Dini's lemma, it is continuous from above. Denote by $A_\delta \colon D(A_\delta) \subset \mathrm{BUC} \to \mathrm{BUC}_\delta$ the monotone generator of S. Notice that BUC_δ is the space of all bounded upper semicontinuous functions $\mathbb{R} \to \mathbb{R}$. Moreover, by Lemma 4.11, we have that $D_L = D_L^s = W^{1,\infty}$. Recall that the space of all Lipschitz continuous functions coincides with the space $W^{1,\infty} = W^{1,\infty}(\mathbb{R})$ of all functions with weak derivative $x' \in L^\infty = L^\infty(\mathbb{R})$ (w.r.t. the Lebesgue measure). As usual, we denote by $\mathrm{BUC}^1 = \mathrm{BUC}^1(\mathbb{R})$ the set of all $x \in \mathrm{BUC}$ which are differentiable with $x' \in \mathrm{BUC}$.

Proposition 4.12. *Let $G = \mathbb{R}$. Then, $\mathrm{BUC}^1 \subset D(A) \subset D(A_\delta) \subset D_L = D_L^s = W^{1,\infty}$. In particular, $S(t)x \in W^{1,\infty}$ for every $x \in W^{1,\infty}$ and all $t \geq 0$. Further, for $x \in D(A_\delta)$, one has $A_\delta x = |x'|$ almost everywhere.*

Proof. If $x \in \mathrm{BUC}^1$, it follows from Taylor's theorem that

$$\left\| \frac{S(h)x - x}{h} - |x'| \right\|_\infty \to 0 \quad \text{as } h \downarrow 0.$$

Hence, by Lemma 4.4 and Lemma 4.11,

$$\mathrm{BUC}^1 \subset D(A) \subset D(A_\delta) \subset D_L = D_L^s = W^{1,\infty}.$$

In particular, $W^{1,\infty}$ is invariant under the uncertain shift semigroup by Theorem 4.5.

Let $x \in W^{1,\infty}$. By Rademacher's theorem the function x is differentiable almost everywhere. If x is differentiable at u, then

$$\begin{aligned}\lim_{h\downarrow 0} \frac{\big(S(h)x\big)(u) - x(u)}{h} &= \lim_{h\downarrow 0} \sup_{|v|\leq h} \frac{x(u + v) - x(u)}{h} = \lim_{h\downarrow 0} \sup_{|v|=h} \frac{x(u + v) - x(u)}{h} \\ &= |x'(u)|.\end{aligned}$$

Since, for $x \in D(A_\delta)$, one has

$$\big(A_\delta x\big)(u) = \lim_{h\downarrow 0} \frac{\big(S(h)x\big)(u) - x(u)}{h}$$

for all $u \in \mathbb{R}^d$, we conclude that $A_\delta x = |x'|$ almost everywhere. Here, x' is understood as the weak derivative in L^∞. □

4.4. **The symmetric Lipschitz set of the G-expectation.** We consider the G-expectation on $\mathrm{BUC} = \mathrm{BUC}(\mathbb{R})$, which corresponds to the sublinear semigroup

$$\big(S(t)x\big)(u) := \sup_{\underline{\sigma}\leq\sigma\leq\overline{\sigma}} \mathbb{E}\Big[x\Big(u + \int_0^t \sigma_s\, dW_s\Big)\Big] \quad \text{for } x \in \mathrm{BUC},\ u \in G \text{ and } t \geq 0,$$

where W is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and the supremum is taken over all progressively measurable processes with values in $[\underline{\sigma}, \overline{\sigma}]$, see e.g. [8] and [24] for an overview on G-expectations. We assume that $0 \leq \underline{\sigma} \leq \overline{\sigma}$. One can verify that S is a translation invariant sublinear C_0-semigroup on BUC which is continuous from above. Moreover, an application of Itô's formula shows that

$$\lim_{h\downarrow 0} \frac{S(h)x - x}{h} = \frac{1}{2}\max\big\{\underline{\sigma}x'', \overline{\sigma}x''\big\}$$

for all $x \in \mathrm{BUC}^2 = \mathrm{BUC}^2(\mathbb{R})$.

Fix $x \in D_L^s$. By definition of the symmetric Lipschitz set, there exist $C > 0$ and $h_0 > 0$ such that $x \in D_L^s(C, h_0)$. For every $\delta > 0$, define $x_\delta(u) := \int_{\mathbb{R}} x(u+v)\, \nu_\delta(dv)$, where ν_δ is the normal distribution $\mathcal{N}(0,\delta)$ with mean zero and variance δ. Then, $x_\delta \in \mathrm{BUC}^2$ for all $\delta > 0$, and $\|x_\delta - x\|_\infty \to 0$ as $\delta \downarrow 0$. In view of Remark 4.8, one has

$$S(h)x_\delta - x_\delta \leq Ch \quad \text{and} \quad -S(h)(-x_\delta) - x_\delta \geq -Ch$$

for all $h \in [0, h_0]$ and $\delta > 0$. Hence, letting $h \downarrow 0$, it follows that

$$\frac{1}{2}\overline{\sigma}x_\delta'' \leq C \quad \text{and} \quad \frac{1}{2}\underline{\sigma}x_\delta'' \geq -C.$$

This shows that $\|x_\delta''\|_\infty$ is uniformly bounded in $\delta > 0$. Hence, there exists a sequence $\delta_n \downarrow 0$ such that $\int_u^v x_{\delta_n}''(z) - y(z)\,\mathrm{d}z \to 0$ for all $u, v \in \mathbb{R}$ with $u < v$ and some $y \in L^\infty$ w.r.t. the Lebesgue measure. By the dominated convergence theorem, we get

$$\begin{aligned} x(u+h) - x(u) &= \lim_{n\to\infty}\Big(x_{\delta_n}(u+h) - x_{\delta_n}(u)\Big) \\ &= \lim_{n\to\infty}\Big(hx_{\delta_n}'(u) + \int_u^{u+h}\int_u^v x_{\delta_n}''(z)\,\mathrm{d}z\,\mathrm{d}v\Big) \\ &= \Big(\lim_{n\to\infty} hx_{\delta_n}'(u)\Big) + \int_u^{u+h}\int_u^v y(z)\,\mathrm{d}z\,\mathrm{d}v \end{aligned}$$

for all $u \in \mathbb{R}$ and $h > 0$. In particular, x is differentiable with $x'(t) = \lim_{n\to\infty} x_{\delta_n}'(t)$ and second weak derivative $x'' = y$, i.e. $x \in W^{2,\infty}$. This shows that $D_L^s = W^{2,\infty}$. As an application of Proposition 4.6, it follows that $S(t)x \in W^{2,\infty}$ for all $t \geq 0$ and $x \in W^{2,\infty}$. Notice that we do not assume that $\underline{\sigma} > 0$, which is a standard assumption in PDE theory for obtaining regularity results in Hölder spaces (cf. [24, Appendix C, §4] for a short survey).

Appendix A. Bounded convex operators

Let X and Y be Banach lattices. For an operator $S\colon X \to Y$, we define $S_x\colon X \to Y$ by $S_x y := S(x+y) - Sx$ for all $x, y \in X$. Recall that $S\colon X \to Y$ is bounded, if $\|S\|_r < \infty$ for all $r > 0$, where

$$\|S\|_r := \sup_{x\in B(0,r)} \|Sx\|.$$

Here, $B(x_0, r) := \{x \in X \colon \|x - x_0\| \leq r\}$ for $x_0 \in X$ and $r > 0$.

Lemma A.1. *Let $S\colon X\to Y$ be convex with $S0=0$ and $r>0$ with $b:=\|S\|_r<\infty$. Then,*

$$\|Sx\|\leq \tfrac{2b}{r}\|x\|$$

for all $x\in B(0,r)$.

Proof. Let $x\in B(0,r)$. For $x=0$, the statement holds by assumption. For $x\neq 0$, the convexity of S implies that

$$Sx\leq \tfrac{\|x\|}{r}S\Big(\tfrac{r}{\|x\|}x\Big)\quad\text{and}\quad Sx\geq -S(-x)\geq -\tfrac{\|x\|}{r}S\Big(-\tfrac{r}{\|x\|}x\Big),$$

and therefore,

$$\|Sx\|\leq \tfrac{\|x\|}{r}\Big(\big\|S\big(\tfrac{r}{\|x\|}x\big)\big\|+\big\|S\big(-\tfrac{r}{\|x\|}x\big)\big\|\Big)\leq \tfrac{2b}{r}\|x\|.$$

□

The following two lemmas aim to clarify the difference between convex continuous and convex bounded operators.

Lemma A.2. *Let $S\colon X\to Y$ be convex. Then, the following statements are equivalent:*

(i) *S is continuous.*
(ii) *For all $x\in X$, there exists some $r>0$ such that $\|S_x\|_r<\infty$.*

Proof. Let $x\in X$ and $r>0$ with $b:=\|S_x\|_r<\infty$. Then, since S_x is convex with $S_x(0)=0$, we obtain from Lemma A.1 that

$$\|S_xy\|\leq \tfrac{2b}{r}\|y\|\quad\text{for all } y\in B(0,r).$$

This shows that S_x is continuous at 0, i.e. S is continuous at x.

Now, assume that there exists some $x\in X$ such that $\|S_x\|_r=\infty$ for all $r>0$. Then, there exists a sequence $(y_n)_n$ in X with $y_n\to 0$ and $\|S_xy_n\|\geq n$. Therefore, the sequence $(S_xy_n)_n$ in Y is unbounded, and thus not convergent. This shows that S_x is not continuous at 0, i.e. S is not continuous at x. □

Lemma A.3. *Let $S\colon X\to Y$. Then, the following statements are equivalent:*

(i) *S is bounded.*
(ii) *For all $x\in X$ and all $r>0$, it holds $\|S_x\|_r<\infty$.*

Proof. Clearly, (ii) implies (i) by considering $x=0$ in (ii). Therefore, assume that S is bounded. Then, for every $x\in X$ and $r>0$, one has $\|S_x\|_r\leq 2\|S\|_{\|x\|+r}<\infty$. □

Corollary A.4. *Let $S\colon X\to Y$ be bounded and convex. Then, S is Lipschitz on bounded subsets, i.e. for every $r>0$, there exists some $L>0$ such that $\|Sx-Sy\|\leq L\|x-y\|$ for all $x,y\in B(0,r)$.*

Proof. Let $x,y\in B(0,r)$, so that $x-y\in B(0,2r)$. As in the proof of Lemma A.3, it follows that

$$\|S_x\|_{2r}\leq 2\|S\|_{\|x\|+2r}\leq 2\|S\|_{3r}=:b.$$

Hence, it follows from Lemma A.1 that $\|Sy-Sx\|=\|S_x(y-x)\|\leq \frac{b}{r}\|y-x\|$. □

In the previous two lemmas, we have seen that, for a convex operator $S\colon X\to Y$, boundedness implies continuity. The following example shows that a convex and continuous operator $S\colon X\to Y$ is not necessarily bounded.

Example A.5. Let $X = c_0 := \big\{(x_n) \text{ in } \mathbb{R}\colon x_n \to 0 \text{ as } n \to \infty\big\}$ be endowed with the supremum norm $\|\cdot\|_\infty$ and $Y = \mathbb{R}$. Then, X and Y are two Banach lattices. We define $S\colon X \to Y$ by

$$Sx := \sup_{n\in\mathbb{N}} |x_n|^n.$$

Notice that S is well-defined, since for every $x \in X$, there exists some $n_0 \in \mathbb{N}$ such that $|x_n| \le 1$ for all $n \in \mathbb{N}$ with $n \ge n_0$. We first show that $S\colon X \to Y$ is convex. For $\lambda \in [0,1]$ and $x, y \in X$, one has

$$\big|\lambda x_n + (1-\lambda)y_n\big|^n \le \lambda|x_n|^n + (1-\lambda)|y_n|^n$$

for all $n \in \mathbb{N}$, which implies that

$$S\big(\lambda x + (1-\lambda)y\big) = \sup_{n\in\mathbb{N}} \big|\lambda x_n + (1-\lambda)y_n\big|^n \le \lambda Sx + (1-\lambda)Sy.$$

Next, we show that S is continuous. Let $x \in X$ and $\varepsilon \in (0,1]$. Then, there exists $n_0 \in \mathbb{N}$ such that $|x_n| \le \frac{\varepsilon}{3}$ for all $n \in \mathbb{N}$ with $n \ge n_0$. Now, let $y \in X$ with $\|x-y\|_\infty \le \frac{\varepsilon}{3}$ and $\|x-y\|_\infty$ is sufficiently small such that

$$\big||x_n|^n - |y_n|^n\big| \le \varepsilon \quad \text{for all } n \in \mathbb{N} \text{ with } n < n_0.$$

For $n \in \mathbb{N}$ with $n \ge n_0$, one has

$$|x_n| + |y_n| \le 2|x_n| + \|x-y\|_\infty \le \varepsilon.$$

Hence, for all $n \in \mathbb{N}$ with $n \ge n_0$,

$$\big||x_n|^n - |y_n|^n\big| \le |x_n|^n + |y_n|^n \le |x_n| + |y_n| \le \varepsilon.$$

Altogether,

$$|Sx - Sy| \le \sup_{n\in\mathbb{N}} \big||x_n|^n - |y_n|^n\big| \le \varepsilon.$$

So far, we have shown that $S\colon X \to Y$ is convex and continuous. However, S is not bounded. To that end, let e_k denote the k-th unit vector. Then, $2e_k \in B(0,2)$ for all $k \in \mathbb{N}$, but $S(2e_k) = 2^k \to \infty$.

In the sublinear case, the notions of continuity and boundedness are equivalent.

Lemma A.6. *Let $S\colon X \to Y$ be sublinear. Then, S is bounded if and only if it is continuous if and only if it is continuous at 0.*

Proof. We have already seen that boundedness implies continuity. Therefore, assume that S is continuous at 0. Then, there exists some $r > 0$ such that $\|S\|_r < \infty$. Since S is positive homogeneous, it follows that $\|S\|_r < \infty$ for all $r > 0$. □

Lemma A.7. *Let $S\colon X \to Y$ be sublinear and continuous. Then S is Lipschitz, i.e. there exists some $L > 0$ such that $\|Sx - Sy\| \le L\|x-y\|$ for all $x, y \in X$.*

Proof. Let $L := 2\|S\|_1$ which is finite by Lemma A.6. Fix $x, y \in X$. By sublinearity, it holds

$$Sx - Sy \le S(x-y) \le |S(x-y)| + |S(y-x)|.$$

By a symmetry argument, it follows that

$$|Sx - Sy| \le |S(x-y)| + |S(y-x)|.$$

Hence,

$$\|Sx - Sy\| \le \|S(x-y)\| + \|S(y-x)\| \le L\|x-y\|.$$

□

The results in Section 2 strongly rely on the following uniform boundedness principle for convex continuous operators.

Theorem A.8. *Let $\mathcal{S}$ be a family of convex continuous operators $X \to Y$. Assume that $\sup_{S\in\mathcal{S}} \|Sx\| < \infty$ for all $x \in X$.*

(i) *There exists some $r > 0$ such that*

$$\sup_{S\in\mathcal{S}} \|S\|_r < \infty.$$

(ii) *For every $x_0 \in X$, there exists some $r > 0$ such that*

$$\sup_{x\in B(x_0,r)} \sup_{S\in\mathcal{S}} \|S_x\|_r < \infty.$$

Proof. (i) By the uniform boundedness principle, there exist $c > 0$, $x_1 \in X$ and $r > 0$ such that

$$\|Sx\| \leq \tfrac{2c}{3}$$

for all $S \in \mathcal{S}$ and $x \in B(x_1, 4r)$. If $x_1 = 0$, the proof is finished. Hence, assume that $x_1 \neq 0$ and define

$$x_0 := \Big(1 - \tfrac{2r}{\|x_1\|}\Big)x_1.$$

Since $\|x_0 - x_1\| \leq 2r$, it follows that $B(x_0, 2r) \subset B(x_1, 4r)$. By assumption,

$$d := \sup_{S\in\mathcal{S}} \tfrac{1}{2}\|S(-x_0)\| + 2\big\|S\big(\tfrac{x_0}{2}\big)\big\| < \infty.$$

Now, let $x \in B(0, r)$ and $S \in \mathcal{S}$. Then,

$$Sx = S\big(\tfrac{x_0+2x}{2} - \tfrac{x_0}{2}\big) \leq \tfrac{1}{2}\big(S(x_0 + 2x) + S(-x_0)\big)$$

and

$$2S\big(\tfrac{x_0}{2}\big) - S(x_0 - x) = 2S\big(\tfrac{x+(x_0-x)}{2}\big) - S(x_0 - x) \leq Sx.$$

We thus obtain that

$$\begin{aligned}\|Sx\| &\leq \tfrac{1}{2}\big\|S(x_0 + 2x) + S(-x_0)\big\| + \big\|2S\big(\tfrac{x_0}{2}\big) - S(x_0 - x)\big\| \\ &\leq \tfrac{1}{2}\|S(x_0 + 2x)\| + \|S(x_0 - x)\| + \tfrac{1}{2}\|S(-x_0)\| + 2\big\|S\big(\tfrac{x_0}{2}\big)\big\| \\ &\leq c + d.\end{aligned}$$

(ii) Let $x_0 \in X$. Then, $\sup_{S\in\mathcal{S}} \|S_{x_0}x\| < \infty$ for all $x \in X$. By part a), there exist $b \geq 0$ and $r > 0$ such that

$$\sup_{S\in\mathcal{S}} \|S_{x_0}\|_{2r} \leq \tfrac{b}{2}.$$

Now, let $S \in \mathcal{S}$, $x \in B(x_0, r)$ and $y \in B(0, r)$. Then, $x + y \in B(x_0, 2r)$ and

$$S_x y = S_{x_0}(x + y - x_0) - S_{x_0}(x - x_0).$$

Therefore, $\|S_x y\| \leq \|S_{x_0}(x + y - x_0)\| + \|S_{x_0}(x - x_0)\| \leq b$. □

Appendix B. Directional derivatives of convex operators

We are in the setting of Section 4, i.e. X is a Banach lattice which is a Riesz subspace of a Dedekind σ-complete Riesz space $\bar{X}$. Let M be the space of all positive linear functionals $\mu\colon X_\delta \to \mathbb{R}$ which are continuous from above. We assume that M separates the points of X_δ.

Lemma B.1. *Let $(x_n)_n$ be a sequence in X. If $(y_n)_n$ and $(z_n)_n$ are decreasing sequences in X which are bounded from below such that $\|x_n - y_n\| \to 0$ and $\|x_n - z_n\| \to 0$, then $\inf_n y_n = \inf_n z_n$.*

Proof. Fix $\mu \in M$. Since μ is continuous on X, see e.g. [1, Theorem 9.6], one has

$$\mu(y_n - z_n) = \mu(y_n - x_n) + \mu(x_n - z_n) \to 0,$$

which shows that

$$\mu\Big(\inf_n y_n\Big) = \lim_{n\to\infty} \mu y_n + \lim_{n\to\infty} \mu(z_n - y_n) = \lim_{n\to\infty} \mu z_n = \mu\Big(\inf_n z_n\Big).$$

Since $\inf_n y_n, \inf_n z_n \in X_\delta$ and M separates the points of X_δ, it follows that $\inf_n y_n = \inf_n z_n$. □

Lemma B.2. *Let $S\colon X \to X$ be a convex monotone operator which is continuous from above. Then, it has a unique monotone convex extension $S\colon X_\delta \to X_\delta$ which is continuous from above.*

Proof. For each $\mu \in M$, the convex monotone functional $\mu S\colon X \to \mathbb{R}$ is continuous from above. Thus, by [9, Lemma 3.9], it has a unique extension to a convex monotone functional $\mu S\colon X_\delta \to \mathbb{R}$ which is continuous from above.

Fix $x \in X_\delta$. For $(x_n)_n$ and $(y_n)_n$ in X with $x_n \downarrow x$ and $y_n \downarrow x$, one has

$$\mu\Big(\inf_n Sx_n\Big) = \inf_n \mu Sx_n = \mu S\Big(\inf_n x_n\Big) = \mu S\Big(\inf_n y_n\Big) = \inf_n \mu S y_n = \mu\Big(\inf_n Sy_n\Big),$$

so that $Sx := \inf_n Sx_n$ is well defined as M separates the points of X_δ. Then, S is convex and continuous from above as

$$\mu\Big(\inf_n Sx_n\Big) = \inf_n \mu Sx_n = \mu Sx$$

for every $(x_n)_n$ in X_δ with $x_n \downarrow x \in X_\delta$. Moreover, if $\tilde{S}$ is another extension which is continuous from above, then $\tilde{S}x = \lim_{n\to\infty} \tilde{S}x_n = \lim_{n\to\infty} Sx_n = Sx$ for every $(x_n)_n$ in X with $x_n \downarrow x \in X_\delta$, which shows that such an extension is unique. □

Let $S\colon X \to X$ be a convex operator. Then, the function

$$\mathbb{R}\setminus\{0\} \to X, \quad h \mapsto \frac{S(x+hy) - Sx}{h}$$

is increasing for all $x, y \in X$. Hence, for all $x \in X$, the operators

$$S'_+(x)y := \inf_{h>0} \frac{S(x+hy) - Sx}{h} \quad \text{and} \quad S'_-(x)y := \sup_{h<0} \frac{S(x+hy) - Sx}{h} \tag{B.1}$$

for $y \in X$ are well-defined with values in $\bar{X}$ since

$$S'_+(x)y = \inf_{n\in\mathbb{N}} \frac{S(x+h_n y) - Sx}{h_n} \in X_\delta \quad \text{and} \quad S'_-(x)y = \sup_{n\in\mathbb{N}} \frac{Sx - S(x - h_n y)}{h_n} \in -X_\delta$$

for every sequence $(h_n)_n$ in $(0,\infty)$ with $h_n \to 0$. The following properties follow directly from the definition.

Remark B.3. For all $x, y \in X$ one has

(i) $S'_-(x)y = -S'_+(x)(-y)$,
(ii) $S'_-(x)y \le S'_+(x)y$,
(iii) $S'_+(x)y = S'_-(x)y = Sy$, if S is linear.

If $S\colon X \to X$ is a convex monotone operator which is continuous from above, then by Lemma B.2 it has a unique convex monotone extension $S\colon X_\delta \to X_\delta$ which is continuous from above. Therefore, $S(x+hy) \in X_\delta$ for all $y \in X_\delta$ and $h > 0$. Hence, $S'_+(x)$ extends to

$$S'_+(x)\colon X_\delta \to X_\delta, \quad y \mapsto \inf_{h>0} \frac{S(x+hy) - Sx}{h}$$

for all $x \in X$.

Lemma B.4. *Let $S\colon X \to X$ be a convex monotone operator which is continuous from above. For every $x \in X$, the mapping $S'_+(x)$ has the following properties:*

(i) *$S'_+(x)y \le S_x y$ for all $y \in X_\delta$,*
(ii) *$S'_+(x)\colon X_\delta \to X_\delta$ is convex and positive homogeneous,*
(iii) *$S'_+(x)$ is continuous from above,*
(iv) *$\frac{S(x+h_n y_n)-Sx}{h_n} \downarrow S'_+(x)y$, for all sequences (h_n) in $(0,\infty)$ and (y_n) in X_δ which satisfy $h_n \downarrow 0$ and $y_n \downarrow y \in X_\delta$.*

Proof. (i) For every $y \in X_\delta$, one has $S'_+(x)y \le S(x+y) - S(x) = S_x(y)$.

(ii) For $\varepsilon > 0$, $\mu \in M$, and $\lambda \in [0,1]$ there exists some $h > 0$ such that

$$\begin{aligned}
&\mu\big(\lambda S'_+(x)y_1 + (1-\lambda)S'_+(x)y_2\big) + \varepsilon \\
&\ge \lambda \frac{\mu S(x+hy_1) - \mu S(x)}{h} + (1-\lambda)\frac{\mu S(x+hy_2) - \mu S(x)}{h} \\
&\ge \frac{\mu S\big(x + h(\lambda y_1 + (1-\lambda)y_2)\big) - \mu S(x)}{h} \ge \mu S'_+(x)\big(\lambda y_1 + (1-\lambda)y_2\big).
\end{aligned}$$

This shows that $S'_+(x)$ is convex on X_δ. Moreoever, for $\lambda > 0$ and $y \in X_\delta$ it holds

$$S'_+(x)(\lambda y) = \inf_{h>0} \frac{S(x+\lambda h y) - Sx}{h} = \lambda \inf_{h>0}\left(\frac{S(x+\lambda h y) - Sx}{\lambda h}\right) = \lambda S'_+(x)y.$$

(iii) For every $y_n \downarrow y$ one has

$$\inf_n S'_+(x)y_n = \inf_{h>0}\inf_n \frac{S(x+hy_n) - S(x)}{h} = \inf_{h>0} \frac{S(x+hy) - S(x)}{h} = S'_+(x)y.$$

(iv) Fix $\varepsilon > 0$, and $\mu \in M$. By definition of S'_+ and continuity from above of S, there exist $n_0, m_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\mu S'_+(x)y + 2\varepsilon &\ge \frac{\mu S(x + h_{n_0} y) - \mu Sx}{h_{n_0}} + \varepsilon \\
&\ge \frac{\mu S(x + h_{n_0} y_{m_0}) - \mu Sx}{h_{n_0}} \ge \frac{\mu S(x + h_{n_1} y_{n_1}) - \mu Sx}{h_{n_1}}
\end{aligned}$$

for $n_1 := n_0 \vee m_0$. This shows that $\frac{S(x+h_n y_n)-Sx}{h_n} \downarrow S'_+(x)y$. □

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