Finiteness properties of split extensions of linear groups

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Abstract

We investigate presentation problems for certain split extensions of discrete matrix groups. In the soluble front, we classify finitely presented Abels groups over arbitrary commutative rings R in terms of their ranks and the Borel subgroup $\mathbf{B}_2^\circ(R) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \leq \mathrm{SL}_2(R)$. In the classical set-up, we prove that, under mild conditions, a parabolic subgroup of a classical group is relatively finitely presented with respect to its extended Levi factor. This yields, in particular, a partial classification of finitely presented S-arithmetic parabolics. Furthermore, we consider higher dimensional finiteness properties and establish an upper bound on the finiteness length of groups that admit certain representations with soluble image.

Deutsche Zusammenfassung

In der vorliegenden Dissertation werden Präsentationen semi-direkter Produkte diskreter Matrizengruppen untersucht. Im auflösbaren Fall zeigen wir in Abhängigkeit vom Rang und von der Borel-Untergruppe $\mathbf{B}_2^\circ(R) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \leq \mathrm{SL}_2(R)$, welche Abels-Gruppen über dem Ring R endlich präsentiert sind. Außerdem beweisen wir unter leichten Voraussetzungen, dass eine parabolische Untergruppe einer klassischen Gruppe relativ endlich präsentiert bezüglich ihres erweiterten Levi-Faktors ist. Dies liefert insbesondere eine partielle Klassifizierung endlich präsentierter S-arithmetischer parabolischer Gruppen. Desweiteren studieren wir hochdimensionale Endlichkeitseigenschaften und zeigen, dass die Endlichkeitslänge von Gruppen mit gewissen auflösbaren Darstellungen nach oben durch die Endlichkeitslänge von $\mathbf{B}_2^\circ(R)$ beschränkt ist.

Keywords: Group theory, split extensions, finiteness properties, Abels groups, Chevalley–Demazure groups, parabolic subgroups.

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Contents

Introduction

A group G is called *linear* if it admits an embedding $G \hookrightarrow \mathrm{GL}_n(\mathbb{K})$ into a general linear group over some field $\mathbb{K}$. Linear groups with finite generating sets have a strong presence. Examples from geometry and topology include fundamental groups of closed surfaces, lattices in many semi-simple Lie groups, and conjecturally all compact 3-manifold groups. Also, free groups, Coxeter groups and braid groups of finite rank are among the linear examples occurring in the intersection between algebra, geometry and topology. From number theory, further linear examples include class groups of algebraic integers, units in Hasse domains, and arithmetic lattices.

The study of finitely generated linear groups is thus of interest to many areas. There has been a great deal of research on the structure of such groups; cf. [51, Chapter 26] for some important examples. A major topic in the theory concerns generators and relations. By their very nature, all groups mentioned in the first paragraph admit finite presentations[1], which raises the following question: which discrete linear groups are *finitely presented*? Loosely speaking this means that, besides the finite generating set, there exist finitely many equations between the generators that completely determine the given group up to isomorphism.

To answer the broad question above, it is natural to first investigate what happens with important families of discrete linear groups. The most prominent examples of such groups are perhaps those lying in the class of *S-arithmetic groups*, such as $\mathrm{SL}_n(\mathbb{Z})$, $\mathrm{SO}_{2n}(\mathbb{F}_p[t])$, $\mathbb{P}\mathrm{Sp}_{2n}(\mathbb{Z}[i])$, $\mathrm{GL}_n(\mathbb{Z}[1/m])$, and $\mathrm{SO}_n(\mathbb{F}_q[t,t^{-1}])$; see Section **1.1.4** for a formal definition. Now, if a finitely generated group G is known to be linear over a global field $\mathbb{K}$, then there exists an S-arithmetic subgroup $\mathbf{G}(\mathcal{O}_S) \leq \mathrm{GL}_n(\mathbb{K})$ such that $G \hookrightarrow \mathbf{G}(\mathcal{O}_S)$.

Besides containing many known finitely generated linear groups, S-arithmetic groups are important in their own right. Indeed, the interest in such groups dates back to the work of C. F. Gauß. In the *Disquisitiones arithmeticae*, Gauß investigated—in modern terms—the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper-half Euclidean plane in connection with his work on quadratic

[1] In the case of 3-manifolds, recall that any 3-manifold group known to be finitely generated is automatically finitely presented by a result of G. P. Scott [84].

forms; see [54, Abschnitt V]. The theory of S-arithmetic groups and associated geometric objects has since been a fruitful area of research; see [73] for an introduction and overview.

The natural problem of classifying which S-arithmetic groups are finitely presented is an ongoing major topic. About six decades ago, A. Borel and Harish-Chandra [23], followed by M. S. Raghunathan [78], analyzed the action of S-arithmetic groups on certain symmetric spaces in the case where the matrix entries lie in the ring of integers of an algebraic number field—for instance, $\mathrm{SL}_n(\mathbb{Z})$ or $\mathbb{P}\mathrm{Sp}_{2n}(\mathbb{Z}[i])$. They concluded, among other things, that such groups are finitely presented. However, H. Nagao shook the theory by showing that the S-arithmetic group $\mathrm{SL}_2(\mathbb{F}_q[t])$ is not even finitely generated. J.-P. Serre reproved Nagao's result from the point of view of groups acting on one-dimensional Euclidean buildings [86]. This general strategy of analyzing actions of S-arithmetic groups on buildings and symmetric spaces to extract information on the groups themselves was carried out intensively since then. After important partial results—refer to the introductions of [33, 37, 36] for an overview—the efforts culminated in the following.

Theorem 0.1 (Borel–Serre [24], Kneser, Borel–Tits, and Abels [3], Behr [14], Bux [33])**.** *Let Γ be an S-arithmetic subgroup of a split linear algebraic group $\mathbf{G}$ over a global field $\mathbb{K}$. Assume $\mathbf{G}$ is either reductive or a Borel subgroup of a reductive group. If $\mathrm{char}(\mathbb{K}) = 0$, then Γ is finitely presented. If $\mathrm{char}(\mathbb{K}) > 0$, then Γ is finitely presented if and only if $|S|$ and the local ranks of $\mathbf{G}$ are large enough.*

The results above raise some questions. Regarding the underlying group scheme $\mathbf{G}$, what can one say if $\mathbf{G}$ is entirely contained in a Borel subgroup of a reductive group, or if $\mathbf{G}$ sits between such a Borel subgroup and its reductive overgroup? Another question concerns the underlying base ring. Firstly, Theorem 0.1 is not unified in the sense that different proofs are required depending on the characteristic of the base ring. Secondly, there are natural representations of important finitely generated groups into non-S-arithmetic matrix groups. For instance, the Burau representation of the braid group B_3 on three strands gives an embedding $B_3 \hookrightarrow \mathrm{GL}_3(\mathbb{Z}[t, t^{-1}])$.

The above discussion leads to the following. In the sequel, R denotes a commutative ring with unity, unless stated otherwise. By a *matrix group* we mean an affine group subscheme $\mathbf{G}$ of some GL_n, defined over $\mathbb{Z}$.

Question 0.2. Suppose an affine reductive $\mathbb{Z}$-group scheme $\mathbf{H} \leq \mathrm{GL}_n$ and a Borel $\mathbb{Z}$-subgroup $\mathbf{B} \leq \mathbf{H}$ are given. For which rings R (not necessarily integral domains)...

- **i.** ...and *proper* matrix subgroups $\mathbf{G} < \mathbf{B}$ is the group $\mathbf{G}(R)$ finitely presented?
- **ii.** ...and matrix groups $\mathbf{G}$ of the form $\mathbf{B} \leq \mathbf{G} < \mathbf{H}$ is $\mathbf{G}(R)$ finitely presented?

The most important examples of (non-nilpotent) matrix groups covered by Question 0.2 are arguably the soluble group schemes $\{\mathbf{A}_n\}_{n\geq 2}$ of Herbert Abels (case i; see Section 0.1.1) and parabolic subgroups $\mathcal{P}$ of reductive groups (case ii; see Section 0.1.2). The main results of this thesis include the following contributions towards Question 0.2.

Theorem A. *An Abels group $\mathbf{A}_n(R)$ is finitely presented if and only n is large enough and the group $\mathbf{B}_2^\circ(R) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \leq \mathrm{SL}_2(R)$ is finitely presented.*

Theorem B. *Let $\mathcal{G}$ be a classical group associated to a reduced, irreducible root system Φ of rank at least two. Let I be a proper subset of simple roots and suppose the triple (R, Φ, I) satisfies the* **QG** *condition. Then the standard parabolic subgroup $\mathcal{P}_I(R) \leq \mathcal{G}(R)$ associated to I is finitely presented if and only if its* extended Levi factor *$\mathcal{LE}_I(R)$ is so.*

The common feature shared by Abels groups and standard parabolics is the fact that they decompose as semi-direct products, i.e. they are split extensions of matrix groups. Before explaining the terminology and origins of Theorems A and B in Sections 0.1.1 and 0.1.2, respectively, we discuss in Section 0.1 some generalities and known difficulties regarding presentations of split extensions.

The necessary conditions and assumptions occurring in Theorems A and B involve generators and relators for the Borel subgroup of rank one

$$\mathbf{B}_2^\circ(R) = \left\{ \left(\begin{smallmatrix} u & r \\ 0 & u^{-1} \end{smallmatrix}\right) \mid u \in R^\times, r \in R \right\} \leq \mathrm{SL}_2(R).$$

This is part of a more general phenomenon involving representations with soluble images as well as finiteness properties that generalize finite presentability. Out third main result explains the above mentioned phenomenon and is an ingredient in the proofs of Theorems A and B.

Theorem C. *Let R be a commutative ring with unity. Suppose a group G retracts onto the group of R-points $\mathfrak{X}(R) \rtimes \mathcal{H}(R)$ of a connected soluble matrix group $\mathfrak{X} \rtimes \mathcal{H}$, where $\mathfrak{X}$ and $\mathcal{H}$ denote an elementary root subgroup and a maximal torus, respectively, of a classical matrix group. Then the* finiteness length *of G is bounded above by that of $\mathbf{B}_2^\circ(R)$, that is, $\phi(G) \leq \phi(\mathbf{B}_2^\circ(R))$.*

The finiteness length and the origins of Theorem C are discussed in Section 0.2, whereas Theorem C itself is proved in Chapter 2. Using the language of finiteness length, Theorems A and B are more precisely stated (and proved) in Chapters 3 and 4, respectively.

0.1 Presentations of split extensions

A group retract is a homomorphism $r : G \to Q$ which admits a homomorphism $\iota : Q \to G$ as a section. Thus, a retract is just another name for the *split extension* (or *semi-direct product*) $G \cong N \rtimes Q$, where $N = \ker(r)$ and Q acts on N via conjugation. To achieve finite presentability of G one must, of course, assume it to be finitely generated. This means that N must be finitely generated *as a normal subgroup* of G and that the quotient Q has to be finitely generated as well. But one needs also assume that Q is finitely presented due to the following observation.

Lemma 0.3 (Stallings [101, Lemma 1.3])**.** *If Q is a retract of a finitely presented group, then Q is also finitely presented.*

One might ask: does this collection of necessary assumptions suffice to guarantee that G is finitely presented? Before looking at examples, we remark that G will be finitely presented whenever both N and Q are so. However, N often does not even admit a finite generating set *as a group*.

***Examples* 0.4.** The following split extensions $G = N \rtimes Q$ all fulfill the above listed necessary conditions for finite presentability, with N infinitely generated (as a group). The question is whether Q acts 'strongly enough' on N as to allow for G to be finitely presented. The theory of metabelian groups shows that this can be a delicate matter:

i. Let $M_1 = (\mathbb{F}_3[t, t^{-1}], +) = M_2$, that is, M_1 and M_2 are isomorphic copies of the underlying additive group of the ring of Laurent polynomials with coefficients in the finite field with three elements. Given $n \in \mathbb{N} \cup \{\infty\}$, denote by C_n the cyclic group of order n. Consider

$$G = (M_1 \times M_2) \rtimes C_\infty^2 = (\mathbb{F}_3[t, t^{-1}] \times \mathbb{F}_3[t, t^{-1}]) \rtimes \langle a, b \mid [a, b] = 1\rangle$$

with the action

$$M_1 \times M_2 \times C_\infty^2 \ni ((r, s), a^\ell b^m) \mapsto (rt^\ell, st^m).$$

Although G fulfils the necessary conditions for finite presentability, the action of the quotient C_∞^2 on the abelian normal subgroup $M_1 \times M_2$ is 'weak': a result due to Bux [32] implies that not even the factors $M_1 \rtimes C_\infty^2 \cong M_2 \rtimes C_\infty^2$ can be finitely presented. Hence, neither is the full group $G = (M_1 \times M_2) \rtimes C_\infty^2$ itself.

ii. The next example is due to H. Abels, R. Bieri, and R. Strebel. Consider the split extension $G = (M_1 \times M_2) \rtimes (C_\infty \times C_2)$, where M_1 is isomorphic to the $(C_\infty \times C_2)$-module $\mathbb{Z}[1/2]$ with the action of $(C_\infty \times C_2) = \langle a, b \mid b^2 = [a, b] = 1\rangle$ given by

$$M_1 \times (C_\infty \times C_2) \ni (r, (a^n, b^\epsilon)) \mapsto (-1)^\epsilon 2^{-n} r,$$

and M_2 is isomorphic to the $(C_\infty \times C_2)$-module $\mathbb{Z}[1/2]$ with action

$$M_2 \times (C_\infty \times C_2) \ni (r, (a^n, b^\epsilon)) \mapsto (-1)^\epsilon 2^n r.$$

In this case, the action of the quotient $C_\infty \times C_2$ on the normal subgroup $M_1 \times M_2$ is 'better' than in the previous example because both factors $M_1 \rtimes (C_\infty \times C_2)$ and $M_2 \rtimes (C_\infty \times C_2)$ are finitely presented by a theorem of Abels [3]. However, [20, Theorem 5.1] implies that $G = (M_1 \times M_2) \rtimes (C_\infty \times C_2)$ cannot be finitely presented.

iii. The following group was considered by Baumslag–Bridson–Holt–Miller [12]. Let $m = pq \in \mathbb{N}$ with p, q coprime. This time, M_1 is the C_∞^2-module $\mathbb{Z}[1/m]$ with action

$$M_1 \times C_\infty^2 \ni (r, (a^k, b^\ell)) \mapsto p^k q^\ell r,$$

and M_2 is the C_∞^2-module $\mathbb{Z}[1/m]$ with action

$$M_2 \times C_\infty^2 \ni (r, (a^k, b^\ell)) \mapsto p^k q^{-\ell} r.$$

Once again, the factors $M_1 \rtimes (C_\infty \times C_\infty)$ and $M_2 \rtimes (C_\infty \times C_\infty)$ are finitely presented. Moreover, the given C_∞^2-action on $M_1 \times M_2$ is 'even better' in the sense that the full group $G = (M_1 \times M_2) \rtimes C_\infty^2$ even has finitely generated second homology [12, Example 3]. Nevertheless, G still admits no finite presentations [12, p. 21].

The examples discussed in **0.4** suggest that one might need an *ad hoc* analysis of group actions to be able to obtain qualitative results on presentations of split extensions of groups belonging to a given class.

0.1.1 The group schemes of Herbert Abels

For every natural number $n \geq 2$, consider the following $\mathbb{Z}$-subscheme of the general linear group.

$$\mathbf{A}_n = \begin{pmatrix} 1 & * & \cdots & \cdots & * \\ 0 & * & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * & * \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \leq \mathrm{GL}_n .$$

Interest in the infinite family $\{\mathbf{A}_n\}_{n\geq 2}$, nowadays known as Abels' groups, was sparked in the late seventies when Herbert Abels [2] published a proof of finite presentability of the group $\mathbf{A}_4(\mathbb{Z}[1/p])$, where p is any prime number; see also [3, 0.2.7 and 0.2.14]. Abels' groups emerged as counterexamples to answers to long-standing problems in group theory and later became a source of construction of groups with curious properties; see [46, 38, 19, 13] for recent examples.

Not long after Abels announced that $\mathbf{A}_4(\mathbb{Z}[1/p])$ is finitely presented, Ralph Strebel went on to generalize Abels' result in the manuscript [94], which never got to be published and only came to our attention after Theorem **A** was established. Strebel gives necessary and sufficient conditions for subgroups of $\mathbf{A}_n(R)$, defined by

$$A_n(R,Q) = \left\{ g \in \mathbf{A}_n(R) \mid \text{the diagonal entries of } g \text{ belong to } Q \leq R^\times \right\},$$

to be finitely presented. (In particular, $\mathbf{A}_n(R) = A_n(R, R^\times)$.) Remeslennikov [unpublished] also verified that $\mathbf{A}_4(\mathbb{Z}[x, x^{-1}, (x+1)^{-1}])$ has a finite presentation—a similar example is treated in detail in Strebel's manuscript.

In the mid eighties, S. Holz and A. N. Lyul'ko proved independently that $\mathbf{A}_n(\mathbb{Z}[1/p])$ and $\mathbf{A}_n(\mathbb{Z}[1/m])$, respectively, are finitely presented as well, for all $n, m, p \in \mathbb{N}$ with $n \geq 4$ and p prime [57, Anhang], [67]. Their techniques differ from Strebel's in that they consider large matrix subgroups of $\mathbf{A}_n$ and relations among them to check for finite presentability of the overgroup. In [103], S. Witzel generalizes the family $\{\mathbf{A}_n\}_{n\geq 2}$ and proves, in particular, that most such groups over $\mathbb{Z}[1/p]$, for p an odd prime, are also finitely presented and have varying Bredon finiteness properties.

Besides those examples in characteristic zero and Strebel's manuscript, the only published case of a finitely presented Abels group over a torsion ring is also S-arithmetic. Y. de Cornulier and R. Tessera proved, in particular, that $\mathbf{A}_4(\mathbb{F}_p[t, t^{-1}, (t-1)^{-1}])$ admits a finite presentation [47, Corollary 1.7].

Theorem **A** thus generalizes the above mentioned results of Abels, Remeslennikov, Holz, Lyul'ko, and Cornulier–Tessera. As for the differences between our methods, Strebel's proof [94] is more algebraic and direct: after establishing necessary conditions, he proves them to be sufficient by explicitly constructing a convenient finite presentation of $A_n(R,Q) = \mathbf{U}_n(R) \rtimes Q^{n-2}$. The proof of Theorem **A** given here follows an alternative route: it has a topological disguise and uses horospherical subgroups and nerve complexes of Abels–Holz [2, 3, 57, 5], the early Σ-invariant for metabelian groups of Bieri–Strebel [20], and K. S. Brown's criterion for finite presentability [28]; see Chapter **3** for details.

We remark that, in the S-arithmetic case, Theorem **A** and results of Holz [57], Bieri [17], and Abels–Brown [4] show that Abels' schemes yield families of affine $\mathbb{Z}$-group schemes whose S-arithmetic groups have varying finiteness lengths; see Section **3.2** for details and a conjecture.

0.1.2 Parabolic subgroups

Parabolic subgroups play an important role, for example, in the structure theory of algebraic groups and in the theory of buildings; see e.g. [22, 48, 6]. By a *classical group* we mean a matrix group $\mathbf{G} \leq \mathrm{GL}_n$ which is either GL_n itself (for some $n \geq 2$) or a universal Chevalley–Demazure group scheme, such as SL_n, Sp_{2n}, or SO_n; see Section **1.1.3** for more on such functors. The protagonists of the present section are the so-called standard parabolic subgroups of classical groups; refer to Chapter **4** for the formal definition.

Instead of introducing a large amount of notation to define such groups here, we find as useful as instructive to have the following working examples in mind.

***Example* 0.5** (Parabolics in GL_n)**.** For the purpose of this introduction, we can think of parabolic subgroups of a general linear group GL_n as its subgroups of *block upper triangular* matrices, such as

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \text{ in } \mathrm{GL}_4 \text{ and } \begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}, \begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix} \text{ in } \mathrm{GL}_5\,.$$

Pictorially, a standard parabolic $\mathbf{P} \leq \mathrm{GL}_n$ is thus of the form

$$\mathbf{P} = \begin{pmatrix} \mathbf{n_1} \times \mathbf{n_1} & * & \cdots & * \\ 0 & \mathbf{n_2} \times \mathbf{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \mathbf{n_k} \times \mathbf{n_k} \end{pmatrix},$$

where $k \leq n$ is the number of diagonal (square) blocks and the i-th block consists of $n_i \times n_i$ matrices.

Going back to presentation problems, the starting point for parabolics is the well-known *Levi decomposition* [48, Exposé XXVI]. This describes a standard parabolic as a split extension of its *unipotent radical* by its *Levi factor*. In our working example, the unipotent radical of $\mathbf{P}$ is

$$\mathcal{U} = \begin{pmatrix} \mathbf{1}_{n_1} & * & \cdots & * \\ 0 & \mathbf{1}_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \mathbf{1}_{n_k} \end{pmatrix} \trianglelefteq \mathbf{P},$$

where $\mathbf{1}_n$ denotes the $n \times n$ identity matrix, and the Levi factor is the block diagonal

$$\mathcal{L} = \begin{pmatrix} \mathbf{n_1} \times \mathbf{n_1} & 0 & \cdots & 0 \\ 0 & \mathbf{n_2} \times \mathbf{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{n_k} \times \mathbf{n_k} \end{pmatrix} \leq \mathbf{P},$$

Keeping in mind Examples 0.4, we ask whether the Levi factor acts strongly enough on the unipotent radical as to detect finite presentability of the full parabolic. Now, expectations on the Levi factor are high. In fact, important structural and representation theoretical results regarding this action are known to hold when the base ring is 'not bad'; see e.g. [9, 90]. It might well be the case that, under mild assumptions on the base ring R, the Levi factor $\mathcal{L}(R)$ detects whether its parabolic overgroup $\mathbf{P}(R)$ is finitely presented. The following example briefly outlines the phenomena behind Theorem **B** and its proof.

***Example* 0.6.** Let $\mathbf{P}_1, \mathbf{P}_2$ be the following parabolic subgroups of GL_{12}.

$$\mathbf{P}_1 = \begin{pmatrix} \mathbf{1{\times}1} & * & \cdots & * \\ 0 & \mathbf{5{\times}5} & \ddots & \vdots \\ \vdots & \ddots & \mathbf{1{\times}1} & * \\ 0 & \cdots & 0 & \mathbf{5{\times}5} \end{pmatrix} \text{ and } \mathbf{P}_2 = \begin{pmatrix} \mathbf{5{\times}5} & * & \cdots & * \\ 0 & \mathbf{1{\times}1} & * & \vdots \\ \vdots & 0 & \mathbf{1{\times}1} & * \\ 0 & \cdots & 0 & \mathbf{5{\times}5} \end{pmatrix}.$$

Now consider the groups above over the ring $\mathbb{Z}[t, t^{-1}]$ of integer Laurent polynomials. In Example **4.3** we shall prove that both groups $\mathbf{P}_1(\mathbb{Z}[t, t^{-1}])$ and $\mathbf{P}_2(\mathbb{Z}[t, t^{-1}])$ fulfil the necessary conditions for finite presentability. Now, the techniques from Chapter **4** will show that, to a finite presentation of the Levi factor $\mathcal{L}_1(\mathbb{Z}[t, t^{-1}])$ of $\mathbf{P}_1(\mathbb{Z}[t, t^{-1}])$, we need only add finitely many normal generators of the unipotent radical $\mathcal{U}_1(\mathbb{Z}[t, t^{-1}]) \trianglelefteq \mathbf{P}_1(\mathbb{Z}[t, t^{-1}])$ and finitely many (mostly commutator and conjugation) relations to construct a presentation for the full group $\mathbf{P}_1(\mathbb{Z}[t, t^{-1}]) = \mathcal{U}_1(\mathbb{Z}[t, t^{-1}]) \rtimes \mathcal{L}_1(\mathbb{Z}[t, t^{-1}])$. In other words, $\mathbf{P}_1(\mathbb{Z}[t, t^{-1}])$ is relatively finitely presented with respect to its Levi factor, giving meaning to the 'strength' of the action of $\mathcal{L}_1(\mathbb{Z}[t, t^{-1}])$ on $\mathcal{U}_1(\mathbb{Z}[t, t^{-1}])$. In particular, $\mathbf{P}_1(\mathbb{Z}[t, t^{-1}])$ is finitely presented, which gives support to the expectations on the Levi factor.

So far, so good. The situation for $\mathbf{P}_2(\mathbb{Z}[t, t^{-1}])$, however, is more delicate. We observe that such group admits the following retract.

$$\mathbf{P}_2(\mathbb{Z}[t, t^{-1}]) \twoheadrightarrow \begin{pmatrix} \mathbf{1}_5 & 0 & \cdots & 0 \\ 0 & * & * & \vdots \\ \vdots & 0 & * & 0 \\ 0 & \cdots & 0 & \mathbf{1}_5 \end{pmatrix} \cong \mathbf{B}_2(\mathbb{Z}[t, t^{-1}]) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \leq \mathrm{GL}_2(\mathbb{Z}[t, t^{-1}]).$$

The retract $\mathbf{B}_2(\mathbb{Z}[t, t^{-1}])$ clearly contains the matrices $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Thus, $\mathbf{B}_2(\mathbb{Z}[t, t^{-1}])$ cannot be finitely presented by a result due to Krstić and McCool [64, Section 4], whence $\mathbf{P}_2(\mathbb{Z}[t, t^{-1}])$ itself also cannot be finitely presented by Stallings' Lemma **0.3**. This shows that the Levi factor $\mathcal{L}_2(\mathbb{Z}[t, t^{-1}])$ of $\mathbf{P}_2(\mathbb{Z}[t, t^{-1}])$ does not act strongly enough on the unipotent radical $\mathcal{U}_2(\mathbb{Z}[t, t^{-1}])$ in order to encode finite presentability of the whole parabolic $\mathbf{P}_2(\mathbb{Z}[t, t^{-1}]) = \mathcal{U}_2(\mathbb{Z}[t, t^{-1}]) \rtimes \mathcal{L}_2(\mathbb{Z}[t, t^{-1}])$.

Example **0.6** shows that, even for very similar parabolics, the Levi factor can fail to detect finite presentability of its overgroup. The issues that might arise, however, come from possible retracts of the given parabolic. The observation leading to Theorem **B** is the following. If the obstructive quotient $\mathbf{B}_2(\mathbb{Z}[t, t^{-1}])$ of $\mathbf{P}_2(\mathbb{Z}[t, t^{-1}])$ were itself finitely presented, the techniques from Chapter **4** (used for $\mathbf{P}_1(\mathbb{Z}[t, t^{-1}])$ in particular) would have equally well worked for $\mathbf{P}_2(\mathbb{Z}[t, t^{-1}])$ by also taking into account the action of $\mathbf{B}_2(\mathbb{Z}[t, t^{-1}])$—besides $\mathcal{L}_2(\mathbb{Z}[t, t^{-1}])$—on the normal subgroup $\mathcal{U}_2(\mathbb{Z}[t, t^{-1}])$.

The idea to remedy the situation is to look at the shape of a parabolic subgroup $\mathbf{P}$ (equivalently, to look at adjacency relations between the roots defining $\mathbf{P}$) in order to construct a subgroup encoding the desired information.

***Example* 0.7** (A new retract)**.** Given a standard parabolic $\mathbf{P} \leq \mathrm{GL}_n$, we define its *extended Levi factor* $\mathcal{LE} \leq \mathbf{P}$ to be its subgroup generated by the diagonal *square blocks*—which form the Levi factor itself—and the *triangular blocks* right above the diagonal—which will produce the obstructive quotients. For instance, in $\mathbf{P}_1$ we only find square blocks, namely the 1×1, 5×5, 1×1, and 5×5 blocks which compose the Levi factor. Thus, its extended Levi factor $\mathcal{LE}_1(\mathbb{Z}[t, t^{-1}])$ is nothing but the Levi factor $\mathcal{L}_1(\mathbb{Z}[t, t^{-1}])$ itself.

$$\mathcal{LE}_1(\mathbb{Z}[t,t^{-1}]) = \mathcal{L}_1(\mathbb{Z}[t,t^{-1}]) = \begin{pmatrix} \mathbf{1\times 1} & 0 & \cdots & 0 \\ 0 & \mathbf{5\times 5} & \ddots & \vdots \\ \vdots & \ddots & \mathbf{1\times 1} & 0 \\ 0 & \cdots & 0 & \mathbf{5\times 5} \end{pmatrix}.$$

In turn, in $\mathbf{P}_2(\mathbb{Z}[t, t^{-1}])$ we have the usual square blocks but we also find a single triangular block over the diagonal, as shown below.

$$\begin{pmatrix} \mathbf{5\times 5} & 0 & \cdots & 0 \\ 0 & \mathbf{1\times 1} & \ddots & \vdots \\ \vdots & \ddots & \mathbf{1\times 1} & 0 \\ 0 & \cdots & 0 & \mathbf{5\times 5} \end{pmatrix} \text{ and } \begin{pmatrix} \mathbf{1}_5 & 0 & \cdots & 0 \\ 0 & * & * & \vdots \\ \vdots & 0 & * & 0 \\ 0 & \cdots & 0 & \mathbf{1}_5 \end{pmatrix}.$$

Thus, the extended Levi factor $\mathcal{LE}_2(\mathbb{Z}[t, t^{-1}]) \leq \mathbf{P}_2(\mathbb{Z}[t, t^{-1}])$ is the subgroup

$$\mathcal{LE}_2(\mathbb{Z}[t,t^{-1}]) = \begin{pmatrix} \mathbf{5\times 5} & 0 & \cdots & 0 \\ 0 & \mathbf{1\times 1} & * & \vdots \\ \vdots & 0 & \mathbf{1\times 1} & 0 \\ 0 & \cdots & 0 & \mathbf{5\times 5} \end{pmatrix} \leq \mathbf{P}_2(\mathbb{Z}[t,t^{-1}]).$$

The extended Levi factor of a parabolic $\mathcal{P}(R)$ is thus a subgroup containing both the Levi factor and the (possible) obstructive quotients of $\mathcal{P}(R)$ as retracts. Theorem **B** is proved by showing that, under mild technical assumptions on the base ring and root systems, a standard parabolic is relatively finitely presented with respect to its extended Levi factor. In other words, Theorem **B** reduces the question of whether $\mathcal{P}(R)$ is finitely presented to the same question for the more tractable, proper subgroup $\mathcal{LE}(R) \leq \mathcal{P}(R)$. See Chapter 4 for details.

The proof of Theorem **B** uses generators and relators à la Steinberg [93, 92] and is elementary in the sense that it is done purely by means of elementary calculations—that is, commutator calculus paired with the well-known commutator formulae of Chevalley. The technical **QG** condition from Theorem **B**, defined in Chapter **4**, derives from presentation properties for $\mathbf{B}_2^{\circ}(R)$ and from the so-called **NVB** condition. The latter is a common assumption often made in the literature when one deals with commutators in Chevalley–Demazure groups; compare, for example, [90, 9, 91].

In Section **4.3**, we combine Theorem **B** with Theorem **0.1** to obtain a partial classification of finitely presented S-arithmetic parabolics. This not only establishes finite presentability of S-arithmetic groups in new cases, but also recovers some known results. Furthermore, this points to a higher dimensional conjecture; see Theorem **4.20** and Conjecture **4.24**.

0.2 A view towards higher dimensions: finiteness length & retracts

The properties considered in the previous sections, i.e. being finitely generated or finitely presented, fit into a larger topological framework considered by Charles T. C. Wall in the sixties [101].

Definition 0.8. The *finiteness length* $\phi(G)$ of a group G is the supremum of the $n \in \mathbb{Z}_{\geq 0}$ for which G admits a classifying space with finite n skeleton.

The quantity $\phi(G)$ has three immediate applications: it is a quasi-isometry invariant of the given group [8]; if $\phi(G) \geq n$ then G has, *at least* up to dimension n, finitely generated (co)homology [30]; and $\phi(G)$ recovers familiar algebraic properties including finite presentability [79]. To be precise, the group G is finitely generated if and only if $\phi(G) \geq 1$, and G admits a finite presentation if and only if $\phi(G) \geq 2$; furthermore, G being finitely identified is equivalent to $\phi(G) \geq 3$; see Section **1.2**.

Thus, already establishing lower bounds on the finiteness length can be a tricky issue—as evidenced by Example **0.4** and Theorems **0.1**, **A**, and **B**—and this has useful implications on the group structure. As a simple example, Theorems **B** and **C** show that, despite deep similarities, the parabolics $\mathbf{P}_1(\mathbb{Z}[t, t^{-1}])$ and $\mathbf{P}_2(\mathbb{Z}[t, t^{-1}])$ from Example **0.6** are *not* quasi-isometric. Distinguishing groups via their finiteness lengths can be interpreted as part of the ongoing program of distinguishing discrete groups up to quasi-isometry. (This program was initiated by M. Gromov and gave birth to (modern) geometric group theory.) The results in the remaining chapters of this thesis are more precisely stated and proved using the language of finiteness length. The examples and open problems discussed in the sequel also concern the computation of the finiteness length of the groups we are interested in.

We mentioned in passing that the necessary conditions involving Theorems **A** and **B** concern the Borel subgroup $\mathbf{B}_2^{\circ}(R) \leq \mathrm{SL}_2(R)$ of rank one.

What happens is that Theorem **C** holds for Abels groups $\mathbf{A}_n(R)$ (for $n \geq 4$) and certain parabolics $\mathcal{P}_I(R)$ (depending on the shape of I in the Dynkin diagram), showing that their finiteness lengths are bounded above by $\phi(\mathbf{B}_2^\circ(R))$. The class of groups for which Theorem **C** applies is quite large. The most notable examples are perhaps certain groups of type (R) studied by M. Demazure and A. Grothendieck in the sixties [48, Exposé XXII, Cap. 5]; see Section **2.2** for further examples and details.

To put Theorem **C** into a broader perspective, we recall the following generalization of Theorem **0.1**.

Theorem 0.9 (Borel–Serre [24], Tiemeyer [98], Bux [33], Bux–Köhl–Witzel [36])**.** *Let Γ be an S-arithmetic subgroup of a split linear algebraic group $\mathbf{G}$ defined over a global field $\mathbb{K}$. If $\mathbf{G}$ is either reductive or a Borel subgroup of a reductive group, then $\phi(\Gamma)$ is known and can be computed depending on* $\mathrm{char}(\mathbb{K})$*, on $|S|$, and on the local ranks of $\mathbf{G}$.*

Among the collection above is the Rank Theorem [36], which implies that the finiteness length of S-arithmetic subgroups of classical groups in positive characteristic grows as the rank of the underlying root system increases. This means that taking a classical group $\mathbf{G}$ whose matrices are very large yields $\phi(\mathbf{G}(\mathcal{O}_S)) \gg 0$. For generators and relations this is also observed in algebraic K-theory for arbitrary rings [56, Section 4.3]. On the other hand, Strebel observed that the finiteness length of a soluble linear group is not necessarily large if the size of its matrices is big [94, 95]. Further examples were later given by Bux [33] in the S-arithmetic set-up. Theorem **C** provides a sufficient condition for a matrix group to belong to the extreme case of not having better finiteness properties even if its matrices are very large.

Inspiration for Theorem **C** came from Borel subgroups $\mathcal{B}$ of Chevalley–Demazure groups investigated by Kai-Uwe Bux in his Ph.D. thesis [33] in the S-arithmetic set-up. The main result of [33] establishes $\phi(\mathcal{B}(\mathcal{O}_S)) = |S| - 1$ if $\mathcal{O}_S$ is a Dedekind ring of arithmetic type and positive characteristic. This makes precise for this class of groups the non-dependency of $\phi(\mathcal{B}(\mathcal{O}_S))$ on the size of the matrices in $\mathcal{B}$. The proof is geometric and first establishes the upper bound $\phi(\mathcal{B}(\mathcal{O}_S)) \leq |S| - 1$ [33, Theorem 5.1]. This inequality was obtained by applying Brown's criterion [29] to the simultaneous (diagonal) action of $\mathcal{B}(\mathcal{O}_S)$ on a product of $|S|$ trees found in the product of the Bruhat–Tits buildings associated to the completions $\mathcal{G}(\mathrm{Frac}(\mathcal{O}_S)_v)$ for each place $v \in S$. On the other hand, the number $|S| - 1$ had been shown to equal $\phi(\mathbf{B}_2^\circ(\mathcal{O}_S))$ in a simpler example [33, Corollary 3.5]. Our goal was to give an easier, purely algebraic explanation for the inequality $\phi(\mathcal{B}(\mathcal{O}_S)) \leq \phi(\mathbf{B}_2^\circ(\mathcal{O}_S))$, which would likely extend to larger classes of rings. And this was in fact the case; see Chapter **2** for the rather elementary proof of Theorem **C**. As a by-product, Theorem **C** pairs up with results due to M. Bestvina, A. Eskin and K. Wortman [16], and G. Gandini [53] yielding a new proof of (a generalization of) Bux's equality; see Theorem **2.11**.

Chapter 1

Background and preliminaries

In this chapter we recall and summarize basic facts on classical linear groups and homotopical finiteness properties to be used throughout. All of the material presented here is standard. Our notation for linear groups closely follows those of Steinberg [93] and Silvester [87], and the basics on the general linear group can be found e.g. in [56]. Throughout this text we assume familiarity with root systems and Dynkin diagrams; see, for instance, [26, Chapter 6] or [58, Chapter 3]. We refer the reader to the classics [42, 93, 48] for a detailed account on Chevalley–Demazure group schemes and their classification. The results on finiteness properties listed here can be found in standard books and articles on the subject, such as [18, 79, 8, 55]. Specifically regarding generators and relators, we assume familiarity with standard tools from combinatorial group theory. The results on group presentations needed here are invoked without further comments; refer e.g. to [43, 82]. The reader familiar with those topics might prefer to skip this chapter altogether.

1.1 Matrices and classical groups

In this work we are interested in well-known concrete matrix groups with entries in arbitrary commutative rings with unity.

Definition 1.1. A *classical group* is an affine group scheme $\mathbf{G} \leq \mathrm{GL}_n$, defined over $\mathbb{Z}$, that belongs to the following set.

$$\{\mathrm{GL}_n, \mathcal{G}_\Phi^{\mathrm{sc}} \mid n \in \mathbb{N}_{\geq 2}, \Phi \text{ reduced irreducible root system}\},$$

where $\mathcal{G}_\Phi^{\mathrm{sc}}$ denotes the universal Chevalley–Demazure group scheme associated to the root system Φ; see Definition **1.5**.

The simplest example of Chevalley–Demazure group scheme is perhaps the special linear group $\mathrm{SL}_n = \mathcal{G}^{\mathrm{sc}}_{\mathrm{A}_{n-1}}$.

In what follows we fix most of the notation to be used throughout while listing well-known facts on matrices and classical groups to be used later on.

1.1.1 The general linear group and elementary matrices

Given $i, j \in \{1, \ldots, n\}$ with $i \neq j$, we denote by $\mathrm{E}_{ij}(r)$ the matrix of $\mathrm{GL}_n(R)$ whose only non-zero entry is $r \in R$ in the position (i, j). The corresponding *elementary matrix* (also called elementary transvection) is defined as $e_{ij}(r) = \mathbf{1}_n + \mathrm{E}_{ij}(r)$, where $\mathbf{1}_n$ denotes the $n \times n$ identity matrix. For example, in $\mathrm{GL}_{12}(\mathbb{Z}[t, t^{-1}])$,

$$e_{67}(-t^{-1}) = \begin{pmatrix} \mathbf{1}_5 & 0 & \cdots & 0 \\ 0 & 1 & -t^{-1} & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & \mathbf{1}_5 \end{pmatrix} = \mathbf{1}_{12} + \mathrm{E}_{67}(-t^{-1}).$$

The subgroup of $\mathrm{GL}_n(R)$ generated by all elementary matrices in a fixed position (i, j) is denoted by $\mathbf{E}_{ij}(R)$. For instance,

$$\mathbf{E}_{67}(\mathbb{Z}[t, t^{-1}]) = \langle \{ e_{67}(r) \mid r \in \mathbb{Z}[t, t^{-1}] \} \rangle \leq \mathrm{GL}_{12}(\mathbb{Z}[t, t^{-1}]).$$

Elementary matrices and commutators between them have the following properties, which are easily checked.

$$e_{ij}(r)e_{ij}(s) = e_{ij}(r + s), \; [e_{ij}(r), e_{kl}(s)^{-1}] = [e_{ij}(r), e_{kl}(s)]^{-1}, \text{ and}$$

$$[e_{ij}(r), e_{kl}(s)] = \begin{cases} e_{il}(rs) & \text{if } j = k, \\ 1 & \text{if } i \neq l, k \neq j. \end{cases} \tag{1.1}$$

In particular, we see that each subgroup $\mathbf{E}_{ij}(R)$ is isomorphic to the underlying additive group $\mathbb{G}_a(R) = (R, +) \cong \{ \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix} \right) \mid r \in R \}$. The group generated by *all* elementary matrices is denoted simply by $E_n(R)$, i.e.

$$E_n(R) = \langle \{ e_{ij}(r) \in \mathrm{GL}_n(R) \mid r \in R, \, i \neq j \} \rangle.$$

Since every elementary matrix has determinant one, we have that $E_n(R)$ is in fact a subgroup of the special linear group $\mathrm{SL}_n(R)$, whence we call $E_n(R)$ the *elementary subgroup of* $\mathrm{SL}_n(R)$.

There are further useful relations and subgroups in $\mathrm{GL}_n(R)$. Given units $u_1, \ldots, u_n \in R^\times$, let $\mathrm{Diag}(u_1, \ldots, u_n)$ denote the following diagonal matrix.

$$\mathrm{Diag}(u_1, \ldots, u_n) = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & u_n \end{pmatrix} \in \mathrm{GL}_n(R).$$

Direct matrix computations show that

$$\mathrm{Diag}(a_1,\ldots,a_n)\,\mathrm{Diag}(b_1,\ldots,b_n) = \mathrm{Diag}(a_1b_1,\ldots,a_nb_n).$$

Thus, for each $j \in \{1,\ldots,n\}$, the group $D_j(R) = \{\mathrm{Diag}(u_1,\ldots,u_n) \mid u_i = 1 \text{ if } i \neq j\} \leq \mathrm{GL}_n(R)$ is isomorphic to the group of units $\mathbb{G}_m(R) = (R^\times,\cdot) = \mathrm{GL}_1(R)$. The subgroup $\mathbf{D}_n(R) \leq \mathrm{GL}_n(R)$ of diagonal matrices is defined as

$$\mathbf{D}_n(R) = \langle\{\mathrm{Diag}(u_1,\ldots,u_n) \mid u_1,\ldots,u_n \in R^\times\}\rangle.$$

In particular, $\mathbf{D}_n(R) = \prod_{j=1}^n D_j(R) \cong \mathbb{G}_m(R)^n$. The matrix group scheme $\mathbf{D}_n \cong \mathbb{G}_m^n$, which is defined over $\mathbb{Z}$, is also known as the standard (maximal) torus of GL_n. The following relations between diagonal and elementary matrices are easily verified.

$$\mathrm{Diag}(u_1,\ldots,u_n)e_{ij}(r)\,\mathrm{Diag}(u_1,\ldots,u_n)^{-1} = e_{ij}(u_iu_j^{-1}r). \tag{1.2}$$

The subgroup of $\mathrm{GL}_n(R)$ generated by all diagonal and elementary matrices is known as the *general elementary linear group*, denoted $\mathrm{GE}_n(R)$. (This group is sometimes also called the *elementary subgroup of* $\mathrm{GL}_n(R)$.) This important object carries a lot of information on the base ring R and lies at the heart of algebraic K-theory. Depending on R, the groups $\mathrm{GE}_n(R)$ and $\mathrm{GL}_n(R)$ need not coincide, though they do in many important cases—for instance, when R is a field. We refer the reader to [44, 56] for more on GE_n and K-theory. Presentations for $\mathrm{GE}_n(R)$—which, of course, depend on the base ring R—were given by Silvester in [87].

Further subgroups of $\mathrm{GL}_n(R)$ play an important role in its structure theory, namely the subgroups $\mathbf{B}_n(R)$ (resp. $\mathbf{B}_n^-(R)$) of upper (resp. lower) triangular matrices. That is,

$$\mathbf{B}_n(R) = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix} \text{ and } \mathbf{B}_n^-(R) = \begin{pmatrix} * & 0 & \cdots & 0 \\ * & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & * \end{pmatrix}.$$

The groups $\mathbf{B}_n(R)$ and $\mathbf{B}_n^-(R)$ typically show up in the investigation of matrix decompositions from linear algebra, but also in stronger structural results such as the Bruhat decomposition; see e.g. [22, 56, 6]. The $\mathbb{Z}$-group subscheme $\mathbf{B}_n$ shall also be called the *standard Borel subgroup* of the classical group GL_n. In the theory of buildings, the conjugates of $\mathbf{B}_n(R)$ (for R a field) are the Borel subgroups of $\mathrm{GL}_n(R)$ and correspond precisely to the stabilizers of chambers in the Tits building associated to $\mathrm{GL}_n(R)$ [6].

In the case of matrices with determinant one, the standard Borel subgroup of $\mathrm{SL}_n(R)$ is the subgroup of upper triangular matrices

$$\mathbf{B}_n^\circ(R) = \mathbf{B}_n(R) \cap \mathrm{SL}_n(R).$$

We remark that both $\mathbf{B}_n(R)$ and $\mathbf{B}_n^\circ(R)$ decompose as semi-direct products by **(1.2)** and **(1.1)**. More precisely,

$$\mathbf{B}_n(R) = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \rtimes \begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & * \end{pmatrix}.$$

(Analogously for $\mathbf{B}_n^\circ(R)$). Consequently, $\mathbf{B}_n(R)$ and $\mathbf{B}_n^\circ(R)$ retract onto a copy of the group of units $\mathbb{G}_m(R)$ as follows.

$$\mathbf{B}_n(R) \twoheadrightarrow \begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \cong \mathbb{G}_m(R)$$

and

$$\mathbf{B}_n^\circ(R) \twoheadrightarrow \left\{ \begin{pmatrix} u & 0 & \cdots & \cdots & 0 \\ 0 & u^{-1} & \ddots & & \vdots \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathrm{SL}_n(R) \;\middle|\; u \in R^\times \right\} \cong \mathbb{G}_m(R) \text{ for } n \geq 2.$$

1.1.2 Unitriangular groups and some commutator calculus

Any elementary matrix $e_{ij}(r)$ is either upper or lower unitriangular—the former being the case whenever $i < j$. The subgroup $\mathbf{U}_n(R) \leq \mathrm{GL}_n(R)$ of upper unitriangular matrices, is easily seen to be nilpotent, of nilpotency class $n-1$, by **(1.1)**. In fact, its lower central series is given below.

$$\mathbf{U}_n(R) = \mathbf{E}_1(R) \trianglerighteq \cdots \trianglerighteq \mathbf{E}_{n-1}(R) \trianglerighteq 1,$$

where, for each $k \in \{1, \ldots, n-1\}$, the normal subgroup $\mathbf{E}_k(R)$ is given by

$$\mathbf{E}_k(R) = \langle \{e_{ij}(r) \in \mathrm{GL}_n(R) \mid r \in R, 1 \leq i < j \leq n, \text{ and } |j-i| \geq k\} \rangle.$$

In fact, by carefully inspecting the indices and using the relations between elementary matrices, one has that

$$\mathbf{E}_k(R)/\mathbf{E}_{k+1}(R) \cong \prod_{i=1}^{n-k} \mathbf{E}_{i,i+k}(R) \cong \prod_{i=1}^{n-k} \mathbb{G}_a(R).$$

Presentation problems for $\mathbf{U}_n(R)$ have been considered many times in the literature, most notably in the case where R is a field and in connection to buildings and amalgams; see e.g. [99, Appendix 2], [49], and [6, Chapters 7 and 8]. As a warm-up for some of the arguments to be used throughout this thesis, we spell out below a 'canonical' presentation for the subgroup $\mathbf{U}_n(R) \leq \mathrm{GL}_n(R)$, obtained via commutator calculus with elementary matrices. Before stating the result, we recall some well-known commutator identities, which can be verified directly, and fix the notation to be used for the remainder of this section.

Lemma 1.2. *Let G be a group and let $a, b, c \in G$. Then*

$$[ab, c] = a[b, c]a^{-1}[a, c] \tag{1.3}$$

and

$$[cac^{-1}, [b, c]] \cdot [bcb^{-1}, [a, b]] \cdot [aba^{-1}, [c, a]] = 1. \qquad \text{(Hall's identity)}$$

Fix $T \subseteq R$ a generating set, containing the unit 1, for the underlying additive group $\mathbb{G}_a(R)$ of the base ring R. That is, we view R as a quotient of the free abelian group $\bigoplus\limits_{t \in T} \mathbb{Z}t$.

We fix furthermore $\mathcal{R} \subseteq \bigoplus_{t \in T} \mathbb{Z}t$ a set of additive defining relators of R. In other words, $\mathcal{R}$ is a set of expressions $\left\{ \sum\limits_{\ell} a_\ell t_\ell \mid a_\ell \in \mathbb{Z}, t_\ell \in T \right\} \subseteq \bigoplus\limits_{t \in T} \mathbb{Z}t$, where all but finitely many a_ℓ's are zero, and such that

$$\mathbb{G}_a(R) \cong \frac{\bigoplus\limits_{t \in T} \mathbb{Z}t}{\langle \mathcal{R} \rangle}.$$

For every pair $t, s \in T$ of additive generators, we choose an expression $m(t, s) = m(s, t) \in \bigoplus_{t \in T} \mathbb{Z}t$ such that the image of $m(s, t)$ in R under the given projection $\bigoplus_{t \in T} \mathbb{Z}t \twoheadrightarrow R$ equals the products ts and st. In case $t = 1$, we take $m(1, s)$ to be s itself, i.e. $m(1, s) = s = m(s, 1)$.

With such expressions $m(t, s)$ chosen for all additive generators $t, s \in T$, we extend $m : T \times T \to \bigoplus\limits_{t \in T} \mathbb{Z}t$ to the whole $\mathbb{Z}$-module $\bigoplus\limits_{t \in T} \mathbb{Z}t$ by linearity. In other words, given arbitrary expressions $r, s \in \bigoplus\limits_{t \in T} \mathbb{Z}t$, say

$$r = \sum_\lambda a_\lambda t_\lambda \quad \text{and} \quad s = \sum_\mu b_\mu t_\mu \quad \text{in} \quad \bigoplus_{t \in T} \mathbb{Z}t,$$

we define $m(r, s) \in \bigoplus\limits_{t \in T} \mathbb{Z}t$ as

$$m(r, s) = \sum_\lambda \sum_\mu a_\lambda b_\mu m(t_\lambda, t_\mu).$$

In particular, the additive expressions $m(r, s)$ for the pairs r, s satisfy the equalities

$$m(r, s) = m(s, r)$$

and

$$m(r, 1) = m(1, r) = r.$$

Notice furthermore that the image of $m(r, s)$ in the ring R equals the product of the images of r and s in R.

Lemma 1.3. *With the notation above, the group $\mathbf{U}_n(R) \leq \mathrm{GL}_n(R)$ admits a presentation $\mathbf{U}_n(R) = \langle \mathcal{Y} \mid \mathcal{S} \rangle$ with generating set*

$$\mathcal{Y} = \{e_{ij}(t) \mid t \in T, 1 \leq i < j \leq n\},$$

and a set of defining relators $\mathcal{S}$ given as follows. For all (i,j) with $1 \leq i < j \leq n$ and all pairs $t, s \in T$,

$$[e_{ij}(t), e_{kl}(s)] = \begin{cases} \prod_u e_{il}(u)^{a_u}, & \text{if } j = k; \\ 1, & \text{if } i \neq l, k \neq j, \end{cases} \tag{1.4}$$

where $m(t,s) = \sum_u a_u u \in \bigoplus_{t \in T} \mathbb{Z}t$ is the fixed expression attached to the pair t, s as in the previous page.
For all (i,j) with $1 \leq i < j \leq n$,

$$\prod_\ell e_{ij}(t_\ell)^{a_\ell} = 1 \text{ for each } \sum_\ell a_\ell t_\ell \in \mathcal{R} \subseteq \bigoplus_{t \in T} \mathbb{Z}t. \tag{1.5}$$

The set $\mathcal{S}$ is defined as the set of all relations **(1.4)** *and* **(1.5)** *given above.*

Proof. Let $\widetilde{U}$ be the group defined by the presentation above—to make everything explicit, we draw tildes $\sim$ over the given generators of $\widetilde{U}$, i.e. $\widetilde{e}_{ij}(t)$, and keep the notation $e_{ij}(r)$ for the actual elementary matrices of $\mathbf{U}_n(R) \leq \mathrm{GL}_n(R)$. Consider the obvious homomorphism $f : \widetilde{U} \to \mathbf{U}_n(R)$ sending $\widetilde{e}_{ij}(t)$ to $e_{ij}(t)$. We prove $\widetilde{U}$ to be isomorphic to $\mathbf{U}_n(R)$ via f by inspecting the lower central series of $\widetilde{U}$.

We observe that each subgroup $\widetilde{U}_{ij} = \langle \{\widetilde{e}_{ij}(t) \mid t \in T\} \rangle \leq \widetilde{U}$ is abelian since $[\widetilde{e}_{ij}(t), \widetilde{e}_{ij}(s)] = 1$ for all $t, s \in T$ by **(1.4)**. Moreover, the restriction of f to $\widetilde{U}_{ij}$ takes values in the subgroup $\mathbf{E}_{ij}(R) \leq \mathbf{U}_n(R)$, by definition. But then it follows from **(1.5)** and von Dyck's theorem that $f|_{\widetilde{U}_{ij}} : \widetilde{U}_{ij} \to \mathbf{E}_{ij}(R)$ is a surjection because $\mathbf{E}_{ij}(R) \cong \mathbb{G}_a(R) = (\bigoplus_{t \in T} \mathbb{Z}t)/\langle \mathcal{R} \rangle$. In particular, f itself is surjective.

We claim that the restrictions $f|_{\widetilde{U}_{ij}}$ are in fact isomorphisms onto their images. Recall that $\mathbf{E}_{ij}(R)$ is canonically isomorphic to $\mathbb{G}_a(R)$ via the obvious assignment $e_{ij}(r) \mapsto r$. Define a map $\varphi_{ij} : \mathbb{G}_a(R) \to \widetilde{U}_{ij}$ as follows. Given $r \in \mathbb{G}_a(R)$, pick *any* pre-image $\sum_\lambda a_\lambda t_\lambda \in \bigoplus_{t \in T} \mathbb{Z}t$ of r under the given natural projection $\bigoplus_{t \in T} \mathbb{Z}t \twoheadrightarrow \mathbb{G}_a(R)$ and set $\varphi_{ij}(r) = \prod_\lambda \widetilde{e}_{ij}(t_\lambda)^{a_\lambda}$. It is easy to see that $\varphi_{ij}(r+s) = \varphi_{ij}(r)\varphi_{ij}(s)$ for all $r, s \in \mathbb{G}_a(R)$, by the very definition of φ_{ij}. Secondly, φ_{ij} is in fact well-defined. Indeed, if $\sum_\lambda a_\lambda t_\lambda$ and $\sum_\mu b_\mu t_\mu$ are two pre-images of r in $\bigoplus_{t \in T} \mathbb{Z}t$, then there exist finitely many expressions

$$x_1 = \sum_{\eta_1} x^1_{\eta_1} t_{\eta_1}, \ \ldots, \ x_k = \sum_{\eta_k} x^k_{\eta_k} t_{\eta_k} \in \mathcal{R}$$

such that

$$\sum_\lambda a_\lambda t_\lambda = x_1 + \cdots + x_k + \sum_\mu b_\mu t_\mu \text{ in } \bigoplus_{t\in T} \mathbb{Z}t.$$

Thus,

$$\begin{aligned}
\varphi_{ij}(r) &= \prod_\lambda \widetilde{e}_{ij}(t_\lambda)^{a_\lambda} \\
&= \left(\prod_{\eta_1} \widetilde{e}_{ij}(t_{\eta_1})^{x^1_{\eta_1}}\right) \cdots \left(\prod_{\eta_k} \widetilde{e}_{ij}(t_{\eta_k})^{x^k_{\eta_k}}\right) \left(\prod_\mu \widetilde{e}_{ij}(t_\mu)^{b_\mu}\right) \\
&\overset{(1.5)}{=} \prod_\mu \widetilde{e}_{ij}(t_\mu)^{b_\mu},
\end{aligned}$$

as desired. Thus, identifying $\mathbf{E}_{ij}(R) \cong \mathbb{G}_a(R)$ as before, the maps $\varphi_{ij} : \mathbf{E}_{ij}(R) \cong \mathbb{G}_a(R) \to \widetilde{U}_{ij}$ are homomorphisms satisfying $\varphi_{ij} \circ f|_{\widetilde{U}_{ij}} = id_{\widetilde{U}_{ij}}$. Therefore, each $f|_{\widetilde{U}_{ij}} : \widetilde{U}_{ij} \to \mathbf{E}_{ij}(R)$ is an isomorphism.

We remark that the maps $\varphi_{ij} : \mathbf{E}_{ij}(R) \cong \mathbb{G}_a(R) \to \widetilde{U}_{ij}$ satisfy

$$\varphi_{ij}(x(y+z)) = \varphi_{ij}(xy)\varphi_{ij}(xz) = \varphi_{ij}(yx)\varphi_{ij}(zx) = \varphi_{ij}((y+z)x) \quad (1.6)$$

for all $x, y, z \in \mathbb{G}_a(R)$ because φ_{ij} is an isomorphism onto its image and the same holds in the domain $\mathbb{G}_a(R) \cong (\bigoplus_{t\in T} \mathbb{Z}t)/\langle \mathcal{R}\rangle$. Furthermore, we claim that, for all i, j with $1 \le i < j \le n$ and all $r, s \in \mathbb{G}_a(R)$,

$$[\varphi_{ij}(r), \varphi_{kl}(s)] = \begin{cases} \varphi_{il}(rs) & \text{if } j = k; \\ 1 & \text{if } i \neq l, j \neq k. \end{cases} \quad (1.7)$$

In effect, Equation (**1.7**) holds for $r, s \in T$ by (**1.4**) and the definitions of φ_{il} and $m(r,s)$. For arbitrary $r, s \in \mathbb{G}_a(R)$, pick pre-images $\sum_{\ell=1}^L a_\ell t_\ell$ and $\sum_{m=1}^M b_m t_m$ of r and s, respectively, in $\bigoplus_{t\in T} \mathbb{Z}t$. Assume, without loss of generality, that $a_1 \neq 0$ and moreover $a_1 > 0$ (the proof for $a_1 < 0$ is analogous). Write $r' = (a_1 - 1)t_1 + \sum_{\ell=2}^L a_\ell t_\ell$. By induction on the sum $\sum_{\ell=1}^L |a_\ell| + \sum_{m=1}^M |b_m|$ and repeated use of (**1.3**), one has that

$$\begin{aligned}
[\varphi_{ij}(r), \varphi_{kl}(s)] &\overset{\text{Def.}}{=} \left[\prod_{\ell=1}^L \widetilde{e}_{ij}(t_\ell)^{a_\ell}, \prod_{m=1}^M \widetilde{e}_{kl}(t_m)^{b_m}\right] \\
&\overset{(1.3)}{=} \widetilde{e}_{ij}(t_1) \cdot \left[\widetilde{e}_{ij}(t_1)^{a_1-1} \prod_{\ell=2}^L \widetilde{e}_{ij}(t_\ell)^{a_\ell}, \prod_{m=1}^M \widetilde{e}_{kl}(t_m)^{b_m}\right] \cdot \\
&\quad \cdot \widetilde{e}_{ij}(t_1)^{-1} \cdot \left[\widetilde{e}_{ij}(t_1), \prod_{m=1}^M \widetilde{e}_{kl}(t_m)^{b_m}\right] \\
&\overset{\text{Def.}}{=} \varphi_{ij}(t_1)[\varphi_{ij}(r'), \varphi_{kl}(s)]\varphi_{ij}(t_1)^{-1}[\varphi_{ij}(t_1), \varphi_{kl}(s)] \\
&\overset{\text{induction}}{=} \begin{cases} \varphi_{il}(r's)\varphi_{il}(t_1 s) & \text{if } j = k; \\ 1 & \text{if } i \neq l, j \neq k. \end{cases}
\end{aligned}$$

Since

$$\varphi_{il}(r's)\varphi_{il}(t_1 s) = \varphi_{il}(r's + t_1 s) = \varphi_{il}((t_1 + r')s) = \varphi_{il}(rs)$$

by **(1.6)**, the claim follows.

We have thus shown that the usual commutator relations **(1.1)** hold in $\widetilde{U}$ by identifying $e_{ij}(r) \mapsto r \mapsto \varphi_{ij}(r)$. Now, the naive claim would be that the function $\varphi : \mathbf{U}_n(R) \to \widetilde{U}$ induced by the assignments $e_{ij}(r) \mapsto \varphi_{ij}(r)$ is a homomorphism, which would yield a left inverse to the epimorphism $f : \widetilde{U} \twoheadrightarrow \mathbf{U}_n(R)$. However, it is a priori not clear why φ should be a homomorphism at all since it is defined locally. This is why we turn to the lower central series of $\mathbf{U}_n(R)$ and $\widetilde{U}$.

For every $k \in \{1, \ldots, n-1\}$ we let $\widetilde{U}_k$ denote the subgroup $\widetilde{U}_k = \langle \widetilde{U}_{ij} : |j-i| \geq k \rangle$. By definition we have $\widetilde{U}_k \supset \widetilde{U}_{k+1}$ for all k. The commutator relations **(1.4)** imply that each $\widetilde{U}_{ij}$ is normal in $\widetilde{U}$ and that each factor $\widetilde{U}_k/\widetilde{U}_{k+1}$ is of the form

$$\frac{\widetilde{U}_k}{\widetilde{U}_{k+1}} \cong \prod_{i=1}^{n-k} \widetilde{U}_{i,i+k}.$$

Moreover,

$$\widetilde{U}_{n-1} = \widetilde{U}_{1,n} \cong \mathbb{G}_a(R) \cong \mathbf{E}_{1,n}(R) = \mathbf{E}_{n-1}(R)$$

and $\widetilde{U}_n = 1$, again by **(1.4)**. Now, any element $g \in \widetilde{U}_k$ can be written uniquely as a product

$$g = u_1 \cdots u_{n-k} h,$$

where each u_i belongs to $\widetilde{U}_{i,i+k}$ and h belongs to $\widetilde{U}_{k+1}$. This is proved by reverse induction on $k = n-1, n-2, \ldots, 1$ and again repeated use of **(1.4)**; see e.g. [93, p. 21] for similar computations. Thus, it follows (again from **(1.4)**) that f induces, for every $k \in \{1, \ldots, n-1\}$, an epimorphism $\overline{f}|_k$ defined by

$$\overline{f}|_k : \frac{\widetilde{U}_k}{\widetilde{U}_{k+1}} \twoheadrightarrow \frac{\mathbf{E}_k(R)}{\mathbf{E}_{k+1}(R)}$$
$$\widetilde{e}_{ij}(t)\widetilde{U}_{k+1} \mapsto e_{ij}(t)\mathbf{E}_{k+1}(R).$$

Conversely, the maps φ_{ij} induce homomorphisms from $\mathbf{E}_k(R)/\mathbf{E}_{k+1}(R)$ to $\widetilde{U}_k/\widetilde{U}_{k+1}$ as follows. We first observe that the image of φ_{ij} lies in $\widetilde{U}_{|j-i|}$ by **(1.7)**. Now, identifying

$$\frac{\mathbf{E}_k(R)}{\mathbf{E}_{k+1}(R)} \cong \prod_{i=1}^{n-k} \mathbf{E}_{i,i+k}(R) \cong \prod_{i+1}^{n-k} \mathbb{G}_a(R),$$

we define a map $\overline{\varphi}_k : \mathbf{E}_k(R)/\mathbf{E}_{k+1}(R) \cong \prod_{i=1}^{n-k} \mathbb{G}_a(R) \to \widetilde{U}_k/\widetilde{U}_{k+1}$ by sending

$$\prod_{i=1}^{n-k} \mathbb{G}_a(R) \ni (r_1, \ldots, r_{n-k}) \mapsto \varphi_{1,1+k}(r_1) \cdots \varphi_{n-k,n}(r_{n-k})\widetilde{U}_{k+1}.$$

By (1.7) and reverse induction on $k = n-1, n-2, \ldots, 1$, it follows that the maps $\overline{\varphi}_k$ are homomorphisms. Since, by definition, $\overline{\varphi}_k \circ \overline{f}_k = id_{\widetilde{U}_k/\widetilde{U}_{k+1}}$ for all k, we have that the maps $\overline{f}_k$ are isomorphisms.

Finally, reverse induction on k and commutativity of the following diagrams

$$\begin{array}{ccccc}
\widetilde{U}_{k+1} & \hookrightarrow & \widetilde{U}_k & \twoheadrightarrow & \widetilde{U}_k/\widetilde{U}_{k+1} \\
\downarrow f|_{\widetilde{U}_{k+1}} & & \downarrow f|_{\widetilde{U}_k} & & \downarrow \overline{f}_k \\
\mathbf{E}_{k+1}(R) & \hookrightarrow & \mathbf{E}_k(R) & \twoheadrightarrow & \mathbf{E}_k(R)/\mathbf{E}_{k+1}(R)
\end{array}$$

yields that f is an isomorphism, as required. □

1.1.3 Chevalley–Demazure group schemes

Chevalley groups play a paramount role in the theories of algebraic groups and finite simple groups, and have been intensively studied in the last six decades. A Chevalley–Demazure group scheme is a representable functor (cf. [102, 62]) from the category of commutative rings to the category of groups which is uniquely associated to a complex, connected, semi-simple Lie group and to a lattice of weights of the corresponding Lie algebra. We recall below the general construction of Chevalley–Demazure group schemes over $\mathbb{Z}$ along the lines of Abe [1] and Kostant [62] and state in the sequel a precise definition of such functors.

Let $G_{\mathbb{C}}$ be a complex, connected, semi-simple Lie group, and let $\mathfrak{g}$ be its Lie algebra with a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and associated reduced root system $\Phi \subseteq \mathfrak{h}^*$. In his seminal Tohoku paper, Claude Chevalley established the following.

Theorem 1.4 (Chevalley [41]). *There exist non-zero elements $X_\alpha \in \mathfrak{g}$, where α runs over Φ, with the following properties.*

i. *Given $\alpha, \beta \in \Phi$ with $\alpha \neq -\beta$, if $\alpha + \beta \in \Phi$, then*

$$[X_\alpha, X_\beta] = \pm(m+1)X_{\alpha+\beta},$$

where m is the largest integer for which $\beta - m\alpha \in \Phi$; otherwise $[X_\alpha, X_\beta] = 0$;

ii. *$X_\alpha \in \mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{ad}(H)X = \alpha(H)X \ \forall H \in \mathfrak{h}\}$, i.e. each vector X_α belongs to the weight space of $\mathfrak{h}$ under the adjoint representation;*

iii. *Setting* $H_\alpha = [X_\alpha, X_\alpha]$ *and* $(\alpha, \beta) := 2\frac{\langle \alpha,\beta\rangle}{\langle \beta,\beta\rangle} \in \mathbb{Z}$ *for* $\alpha, \beta \in \Phi$, *one has that* $H_\alpha \in \mathfrak{h} \setminus \{0\}$ *and* $[H_\alpha, X_\beta] = (\beta, \alpha)X_\beta$;

iv. $\{H_\alpha\}_{\alpha\in\Phi}$ *spans* $\mathfrak{h}$, *the set* $\{H_\alpha, X_\alpha\}_{\alpha\in\Phi}$ *is a basis for* $\mathfrak{g}$, *and there is a decomposition* $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha)$.

A basis $\{H_\alpha, X_\alpha\}_{\alpha\in\Phi}$ for $\mathfrak{g}$ as above is known as a Chevalley basis and the $\mathbb{Z}$-Lie ring $\mathfrak{g}_\mathbb{Z}$ generated by it is sometimes called a Chevalley lattice. Using $\mathfrak{g}_\mathbb{Z}$ alone, one may already proceed to construct the first examples of Chevalley–Demazure groups over fields, namely those of adjoint type. Such groups yield, for instance, infinite families of simple groups (both finite and infinite). These were the groups introduced in [41], later popularized as Chevalley groups; see e.g. [39]. The next step to construct more general group schemes is to allow for different representations of $\mathfrak{g}$.

Let $P_{sc} = \{\chi \in \mathfrak{h} \mid \chi(H) \in \mathbb{Z}\ \forall H \in \mathfrak{h}\}$ be the lattice of weights of $\mathfrak{h}$ and let $P_{ad} = \mathrm{span}_\mathbb{Z}(\Phi) \subseteq P_{sc}$ be the root lattice. If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a faithful representation of $\mathfrak{g}$, then $P_{ad} \subseteq P_\rho \subseteq P_{sc}$, where $P_\rho = \{\chi \in \mathfrak{h}^* \mid V_\chi \neq \{0\}\}$ denotes the lattice of weights of the representation ρ. (Recall that $V_\chi = \{X \in V \mid \chi(H)X = \rho(H)X\ \forall H \in \mathfrak{h}\}$.) Conversely, given $P \subseteq \mathfrak{h}^*$ with $P_{ad} \subseteq P \subseteq P_{sc}$, there exists a faithful representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ such that $P_\rho = P$; see, for instance, [80, 21, 59].

Fix a lattice $P := P_\rho$ as above. From Kostant's construction [62, Thm. 1 and Cor. 1], one can define a $\mathbb{Z}$-lattice B_ρ in the universal enveloping algebra $U(\mathfrak{g})$ and a certain family F of ideals of B_ρ [62, Section 1.3 and p. 98] such that

$$\mathbb{Z}[G_\mathbb{C}, P] := \{f \in \mathrm{Hom}(B_\rho, \mathbb{Z}) \mid f \text{ vanishes on some } I \in F\}$$

is a Hopf algebra over $\mathbb{Z}$ with the following properties:

i. $\mathbb{Z}[G_\mathbb{C}, P]$ is a finitely generated integral domain;

ii. The coordinate ring $\mathbb{C}[G_\mathbb{C}]$ is isomorphic to the Hopf algebra $\mathbb{Z}[G_\mathbb{C}, P] \otimes_\mathbb{Z} \mathbb{C}$.

In particular, we get a representable functor $\mathcal{G}_\Phi^P := \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}[G_\mathbb{C}, P], -)$ from the category of commutative rings with unity to the category of groups. Since the Lie group $G_\mathbb{C}$ and the representation ρ are determined, up to isomorphism, by the root system Φ and the lattice P, respectively, we see that $\mathcal{G}_\Phi^P$ depends only on Φ and P up to isomorphism. Moreover, by **(ii)** we recover $G_\mathbb{C} \cong \mathcal{G}_\Phi^P(\mathbb{C})$ as the group of $\mathbb{C}$-points of $\mathcal{G}_\Phi^P$. The functor $\mathcal{G}_\Phi^P$ also inherits some properties of $G_\mathbb{C}$. Namely, it is semi-simple (in the sense of Demazure–Grothendieck [48]) and contains a maximal torus of rank $\mathrm{rk}(\Phi)$ defined over $\mathbb{Z}$. Demazure's theorem [48, Exposé XXIII, Cor. 5.4] ensures that $\mathcal{G}_\Phi^P$ is unique up to isomorphism. A detailed proof of existence is also given in [48, Exposé XXV]. (See also Lusztig's recent approach [66] to the construction of Kostant.) We summarize the discussion with the following.

Definition/Theorem 1.5 (Chevalley, Ree, Demazure [42, 80, 48, 62])**.** Given a reduced root system Φ and a lattice P with $P_{ad} \subseteq P \subseteq P_{sc}$, the *Chevalley–Demazure group scheme* of type (Φ, P) is the split, semi-simple, affine group scheme $\mathcal{G}_\Phi^P$ defined over $\mathbb{Z}$ such that, for any field k, the split, semi-simple, linear algebraic group of type Φ and defined over k is isomorphic to $\mathcal{G}_\Phi^P \otimes_{\mathbb{Z}} k$.

By a *Chevalley–Demazure group* we mean the group of R-points $\mathcal{G}_\Phi^P(R)$ of some Chevalley–Demazure group scheme $\mathcal{G}_\Phi^P$ for some commutative ring R with unity. Of course, the two extreme cases of P deserve special names. If $P = P_{ad}$, the root lattice, then $\mathcal{G}_\Phi^{P_{ad}}$ is said to be of *adjoint type* and we write $\mathcal{G}_\Phi^{P_{ad}} =: \mathcal{G}_\Phi^{ad}$ in case Φ is, in addition, irreducible. If $P = P_{sc}$, the full lattice of weights of $\mathfrak{g}$, then $\mathcal{G}_\Phi^P$ is of *simply-connected type.* If, moreover, Φ is irreducible, then $\mathcal{G}_\Phi^{P_{sc}}$ is called *universal*, and we write $\mathcal{G}_\Phi^{\mathrm{sc}} := \mathcal{G}_\Phi^{P_{sc}}$.

The group scheme $\mathcal{G}_\Phi^P$ has the following properties. Let y be an independent variable. For each $\alpha \in \Phi$ we get a monomorphism of the additive group scheme $\mathbb{G}_a = \mathrm{Hom}(\mathbb{Z}[y], -)$ into $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G_{\mathbb{C}}, P], -) = \mathcal{G}_\Phi^P$. Fix a ring R. Given an element $r \in (R, +) = \mathbb{G}_a(R)$, we denote its image under the map above by $x_\alpha(r) \in \mathcal{G}_\Phi^P(R)$. The *unipotent root subgroup* associated to α is defined as $\mathfrak{X}_\alpha(R) := \langle x_\alpha(r) \mid r \in R \rangle \leq \mathcal{G}_\Phi^P(R)$, which is isomorphic to $\mathbb{G}_a(R)$. Furthermore, the map $\mathbb{G}_a \hookrightarrow \mathcal{G}_\Phi^P$ can be chosen so that

$$\mathrm{SL}_2(R) \ni \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right) \mapsto x_\alpha(r) \text{ and } \mathrm{SL}_2(R) \ni \left(\begin{smallmatrix} 1 & 0 \\ r & 1 \end{smallmatrix}\right) \mapsto x_{-\alpha}(r).$$

In particular, if $P = P_{sc}$, we obtain an isomorphism from the subgroup of elementary matrices $\langle \mathbf{E}_{12}(R), \mathbf{E}_{21}(R) \rangle \leq \mathrm{SL}_2(R)$ to the subgroup $\langle \mathfrak{X}_\alpha(R), \mathfrak{X}_{-\alpha}(R) \rangle \leq \mathcal{G}_\Phi^{\mathrm{sc}}(R)$. Accordingly, we define the *elementary subgroup* E_Φ^P of $\mathcal{G}_\Phi^P$ to be its subgroup generated by all unipotent root elements, that is

$$E_\Phi^P(R) = \langle \mathfrak{X}_\alpha(R) \,:\, \alpha \in \Phi \rangle \leq \mathcal{G}_\Phi^P(R).$$

In particular, $E_{\mathrm{A}_{n-1}}^{sc}(R) = E_n(R) \leq \mathrm{SL}_n(R)$. In the Chevalley–Demazure setting, $E_\Phi^P(R)$ is the analogue of the elementary subgroup GE_n of GL_n. The groups $E_\Phi^P(R)$ and $\mathcal{G}_\Phi^P(R)$ need not coincide in general, but they are known to be equal in some important cases—perhaps most prominently in the case where $\mathcal{G}_\Phi^P$ is universal and R is a field.

The maps from $\mathbb{G}_a$ into the $\mathfrak{X}_\alpha \leq \mathcal{G}_\Phi^P$ as above also induce, for each $\alpha \in \Phi$, an embedding of the multiplicative group $\mathbb{G}_m \cong \left(\begin{smallmatrix} * & 0 \\ 0 & *^{-1} \end{smallmatrix}\right) \hookrightarrow \mathcal{G}_\Phi^P$. Given a unit $u \in (R^\times, \cdot) = \mathbb{G}_m(R)$ we denote by $h_\alpha(u)$ the image of the matrix $\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right) \in \mathrm{SL}_2(R)$ under the map above. We call $\mathcal{H}_\alpha(R) := \langle \{h_\alpha(u) \mid u \in R^\times\} \rangle \leq \mathcal{G}_\Phi^P(R)$ a *semi-simple root subgroup*, which is a subtorus of $\mathcal{G}_\Phi^P(R)$. One of the main features of a universal group is that $\mathcal{H}(R) := \langle \mathcal{H}_\alpha(R) \mid \alpha \in \Phi \rangle$ is a maximal split torus of $\mathcal{G}_\Phi^{\mathrm{sc}}(R)$, defined over $\mathbb{Z}$.

Two root subgroups $\mathfrak{X}_\alpha, \mathfrak{X}_\beta$ with $\alpha \neq -\beta$ are related by the *Chevalley commutator formula* (or *Chevalley relations*). If $x_\alpha(r) \in \mathfrak{X}_\alpha(R)$ and

$x_\beta(s) \in \mathfrak{X}_\beta(R)$, then

$$[x_\alpha(r), x_\beta(s)] = \begin{cases} \prod_{\substack{m,n>0 \\ m\alpha+n\beta\in\Phi}} x_{m\alpha+n\beta}(r^m s^n)^{C^{\alpha,\beta}_{m,n}} & \text{if } \alpha+\beta\in\Phi, \\ 1 & \text{otherwise,} \end{cases} \tag{1.8}$$

where the powers $C^{\alpha,\beta}_{m,n}$, called *structure constants*, always belong to $\{0, \pm1, \pm2, \pm3\}$ and do not depend on r nor on s, but rather on α, on β, and on the chosen total order on the set of simple roots $\Delta \subset \Phi$. The formulae above generalize the commutator relations **(1.1)** that we saw earlier for the general linear group.

***Example* 1.6.** Suppose $\mathrm{rk}(\Phi) = n-1 \geq 2$. Then $E^{sc}_{\mathrm{A}_{n-1}}(R) = E_n(R) \leq \mathrm{SL}_n(R)$, the subgroup of $\mathrm{GL}_n(R)$ generated by all elementary transvections. A set of simple roots of A_{n-1} is given by $\Delta = \{\alpha_1, \dots, \alpha_{n-1}\}$ for $\alpha_i = v_i - v_{i+1}$ and $1 \leq i \leq n-1$, where $\{v_j\}_{j=1}^n \subseteq \mathbb{R}^n$ is the canonical basis.
Via the usual identification $e_{i,i+1}(r) \longleftrightarrow x_{\alpha_i}(r)$ of elementary matrices with unipotent root elements, we iteratively recover all unipotent root subgroups as well as the commutator formulae **(1.8)** in type A_{n-1}. For instance, we can see that $x_{\alpha_i+\alpha_{i+1}}(r) = [e_{i,i+1}(r), e_{i+1,i+2}(1)]$, and the commutator formulae assume the simpler form

$$[x_\alpha(r), x_\beta(s)] = \begin{cases} x_{\alpha+\beta}(rs) & \text{if } \alpha+\beta\in\Phi; \\ 1 & \text{otherwise.} \end{cases}$$

These are precisely the same relations shown in **(1.1)**.

Steinberg derives in [93, Chapter 3] a series of consequences of the commutator formulae, nowadays known as Steinberg relations. Among those, we highlight the ones that relate the subtori $\mathcal{H}_\beta$ to the root subgroups $\mathfrak{X}_\alpha$. Given $h_\beta(u) \in \mathcal{H}_\beta(R)$ and $x_\alpha(r) \in \mathfrak{X}_\alpha(R)$, the following conjugation relation holds.

$$h_\beta(u) x_\alpha(r) h_\beta(u)^{-1} = x_\alpha(u^{(\alpha,\beta)} r), \tag{1.9}$$

where $(\alpha, \beta) \in \{0, \pm1, \pm2, \pm3\}$ is the corresponding Cartan integer from Chevalley's Theorem **1.4**. The relations above are the analogues of the diagonal relations **(1.2)** seen before for the general linear group.

Let W be the Weyl group associated to Φ. The Steinberg relations **(1.9)** behave well with respect to the W-action on the roots Φ. More precisely, let $\alpha \in \Phi \subseteq \mathbb{R}^{\mathrm{rk}(\Phi)}$ and let $r_\alpha \in W$ be the associated reflection. The group W has a canonical copy in E^P_Φ obtained via the assignment

$$r_\alpha \mapsto w_\alpha := x_\alpha(1) x_{-\alpha}(1)^{-1} x_\alpha(1) = \text{ image of } \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in E_2,$$

under the map $E_2 \to \langle \mathfrak{X}_\alpha, \mathfrak{X}_{-\alpha} \rangle$ above. With the above notation, given arbitrary roots $\beta, \gamma \in \Phi$, one has

$$\begin{aligned} h_{r_\alpha(\gamma)}(v) x_{r_\alpha(\beta)}(s) h_{r_\alpha(\gamma)}(v)^{-1} &= w_\alpha (h_\gamma(v) x_\beta(s)^{\pm 1} h_\gamma(v)^{-1}) w_\alpha^{-1} \\ &= x_{r_\alpha(\beta)}(v^{(\beta,\gamma)} s)^{\pm 1}, \end{aligned} \tag{1.10}$$

where the sign ± 1 above does not depend on $v \in R^\times$ nor on $s \in R$. We shall sometimes refer to the relations above as Weyl relations.

Similarly to subgroups of triangular matrices in GL_n, the Borel subgroups of Chevalley–Demazure groups $\mathcal{G}_\Phi^P$ play an important role in their structure theory. Results such as the Bruhat decomposition hold equally well for $\mathcal{G}_\Phi^P$; see e.g. [22, 48]. For our purposes, we define the *standard Borel subgroup* $\mathcal{B}_\Phi$ of the universal Chevalley–Demazure group $\mathcal{G}_\Phi^{\mathrm{sc}}$ as

$$\mathcal{B}_\Phi(R) = \langle \mathcal{H}_\alpha(R), \mathfrak{X}_\alpha(R) \ : \ \alpha \in \Phi^+ \rangle \leq \mathcal{G}_\Phi^{\mathrm{sc}}(R).$$

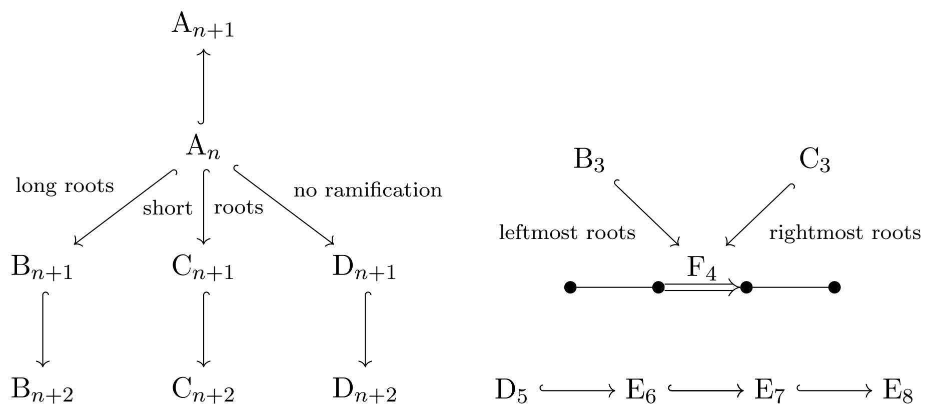
In particular, $\mathcal{B}_{\mathrm{A}_{n-1}}(R) = \mathbf{B}_n^\circ(R)$.

The explicit construction of reduced, irreducible root systems leads to the following.

***Remark* 1.7.** For every $n \geq 1$ there exist the following $\mathbb{Z}$-embeddings of Chevalley–Demazure group schemes:

$$\mathcal{G}^{\mathrm{sc}}_{\mathrm{A}_n} \hookrightarrow \mathcal{G}^{\mathrm{sc}}_{\mathrm{A}_{n+1}}, \quad \mathcal{G}^{\mathrm{sc}}_{\mathrm{A}_n} \hookrightarrow \mathcal{G}^{\mathrm{sc}}_{\mathrm{B}_{n+1}}, \quad \mathcal{G}^{\mathrm{sc}}_{\mathrm{A}_n} \hookrightarrow \mathcal{G}^{\mathrm{sc}}_{\mathrm{C}_{n+1}}, \quad \mathcal{G}^{\mathrm{sc}}_{\mathrm{A}_n} \hookrightarrow \mathcal{G}^{\mathrm{sc}}_{\mathrm{D}_{n+1}} \ (n \geq 3),$$
$$\mathcal{G}^{\mathrm{sc}}_{\mathrm{B}_n} \hookrightarrow \mathcal{G}^{\mathrm{sc}}_{\mathrm{B}_{n+1}} \ (n \geq 2), \quad \mathcal{G}^{\mathrm{sc}}_{\mathrm{B}_3} \hookrightarrow \mathcal{G}^{\mathrm{sc}}_{\mathrm{F}_4}, \quad \mathcal{G}^{\mathrm{sc}}_{\mathrm{C}_n} \hookrightarrow \mathcal{G}^{\mathrm{sc}}_{\mathrm{C}_{n+1}} \ (n \geq 2), \quad \mathcal{G}^{\mathrm{sc}}_{\mathrm{C}_3} \hookrightarrow \mathcal{G}^{\mathrm{sc}}_{\mathrm{F}_4},$$
$$\mathcal{G}^{\mathrm{sc}}_{\mathrm{D}_n} \hookrightarrow \mathcal{G}^{\mathrm{sc}}_{\mathrm{D}_{n+1}} \ (n \geq 4), \quad \mathcal{G}^{\mathrm{sc}}_{\mathrm{D}_5} \hookrightarrow \mathcal{G}^{\mathrm{sc}}_{\mathrm{E}_6}, \quad \mathcal{G}^{\mathrm{sc}}_{\mathrm{E}_6} \hookrightarrow \mathcal{G}^{\mathrm{sc}}_{\mathrm{E}_7}, \quad \mathcal{G}^{\mathrm{sc}}_{\mathrm{E}_7} \hookrightarrow \mathcal{G}^{\mathrm{sc}}_{\mathrm{E}_8}.$$

Proof. This follows immediately from Theorem **1.5** and the following natural embeddings of Dynkin diagrams.



□

There are, of course, many other embeddings of Chevalley–Demazure group schemes into one another besides those listed on Remark **1.7**, though we will not need them in this work.

1.1.4 S-arithmetic groups

The most important examples of matrix groups to which the main results of this thesis apply are the so-called S-arithmetic groups. Below we briefly recall their definition—we refer the reader e.g. to [76, 69, 73] for a proper introduction to arithmetic lattices and their S-arithmetic counterparts. As usual, we draw from [22, 48] standard results on linear algebraic groups.

Let $\mathbf{G}$ be a linear algebraic group defined over a global field $\mathbb{K}$, i.e. a finite extension either of the rational numbers $\mathbb{Q}$ or of a function field $\mathbb{F}_q(t)$ with coefficients in a finite field $\mathbb{F}_q$. In what follows, S denotes a *finite* set of places of $\mathbb{K}$—a further standing assumption is that S contains all the archimedean places and that $S \neq \varnothing$ if $\mathrm{char}(\mathbb{K}) > 0$. Recall that the *ring of S-integers* $\mathcal{O}_S \subseteq \mathbb{K}$ is the subring

$$\mathcal{O}_S = \{x \in \mathbb{K} \ : \ |x|_v \leq 1 \text{ for all } [v] \notin S\} \, ;$$

see e.g. [56, p. 86]. In this set-up, S is sometimes called a Hasse set of valuations on $\mathbb{K}$ and $\mathcal{O}_S$ is also known in the literature as a *Dedekind ring of arithmetic type*. Loosely speaking, $\mathcal{O}_S$ is the subring of $\mathbb{K}$ of all elements which are 'integers' except possibly at S. Typical examples of such rings include: $\mathbb{Z}[\frac{1}{p_1 \cdots p_n}]$, the ring of rational integers whose denominators have divisors only in $\{p_1, \ldots, p_n\} \subset \mathbb{N}$; the ring $\mathbb{F}_q[t, t^{-1}]$ of Laurent polynomials with coefficients in a finite field $\mathbb{F}_q$; and $\mathcal{O}_{\mathbb{L}}$, the ring of integers of an algebraic number field $\mathbb{L}$.

Definition 1.8. A subgroup $\Gamma \leq \mathbf{G}$ is called *S-arithmetic* if it is commensurable with $\rho^{-1}(\mathrm{GL}_n(\mathcal{O}_S)) \leq \mathbf{G}$ for some faithful $\mathbb{K}$-representation $\rho : \mathbf{G} \hookrightarrow \mathrm{GL}_n$.

Besides the ones seen in the introduction, examples of S-arithmetic groups are scattered everywhere around this work. Indeed, given a *matrix group* $\mathbf{G}$, that is, an affine $\mathbb{Z}$-group subscheme $\mathbf{G} \leq \mathrm{GL}_n$, we can always take the group of R-points $\mathbf{G}(R)$ for any commutative ring R with unity. In particular, $\mathbf{G}(\mathcal{O}_S)$ is an S-arithmetic subgroup of $\mathbf{G}$ considered as an algebraic group over $\mathbb{K} = \mathrm{Frac}(\mathcal{O}_S)$.

Of course, $\mathbf{G}(\mathcal{O}_S)$ is not the only S-arithmetic subgroup of $\mathbf{G}$—different $\mathbb{K}$-embeddings $\theta : \mathbf{G} \hookrightarrow \mathrm{GL}_m$ usually yield non-isomorphic S-arithmetic subgroups of $\mathbf{G}$. However, all S-arithmetic subgroups of a given linear algebraic group lie in the same commensurability class; see, for instance, [69, Section 3.1].

1.2 Basics on the finiteness length

As highlighted in the introduction, the finiteness length is useful for many reasons. Perhaps one of the most important ones is the fact that it is a quasi-isometry invariant. In a recent remarkable paper, Skipper, Witzel and Zaremsky used such invariant to construct infinitely many quasi-isometry classes of finitely presented simple groups [88].

Lemma 1.9. *Let G and H be quasi-isometric groups. Then $\phi(G) = \phi(H)$. In particular, if H and G are commensurable, then $\phi(G) = \phi(H)$.*

Refer to [8] for a proof of Lemma **1.9**. As pointed out in Section **1.1.4**, all S-arithmetic subgroups of a given linear algebraic group $\mathbf{G}$ are commensurable. In particular, if $\rho : \mathbf{G} \hookrightarrow \mathrm{GL}_n$ is *any* $\mathrm{Frac}(\mathcal{O}_S)$-embedding, Lemma **1.9** implies that $\phi(\Gamma) = \phi(\rho(\mathbf{G}) \cap \mathrm{GL}_n(\mathcal{O}_S))$ for all S-arithmetic subgroups $\Gamma \leq \mathbf{G}$.

In fortunate cases, the finiteness length of a group is handed to us by nature.

***Example* 1.10.** Since a classifying space for a finite group can be constructed from a compact presentation 2-complex by inductively adding finitely many cells in each dimension to kill higher homotopy groups, it follows that all finite groups have unbounded finiteness length. In symbols, $\phi(G) = \infty$ whenever $|G| < \infty$.

***Example* 1.11.** Using Lemma **1.9**, it is easy to check that finitely generated abelian groups also have unbounded finiteness length. Indeed, if A is such a group, then Lemma **1.9** implies $\phi(A) = \phi(A/\mathrm{tor}(A))$, where $\mathrm{tor}(A)$ is the torsion part of A. But $A/\mathrm{tor}(A)$ is just a free abelian group of finite rank, i.e. it is isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{Z}_{\geq 0}$. Since $\mathbb{Z}^n$ has the n-torus as a classifying space, it follows that $\phi(A/\mathrm{tor}(A)) = \infty$, as claimed.

***Example* 1.12.** More generally, all the groups G from the first paragraph of the introduction satisfy $\phi(G) = \infty$; see e.g. [79, 6].

The finiteness length can also be defined in terms of Wall's homotopical finiteness properties [101]. A group G is said to be *of type* F_n if it admits a classifying space with finite n-skeleton. Thus, $\phi(G)$ is the largest n for which G is of type F_n—in case G is of type F_n for all n (equivalently, if its finiteness length is unbounded), we say that G is of type F_∞ (resp. we write $\phi(G) = \infty$).

All groups are of type F_0. Considering algebraic finiteness properties [79], one has the following. By looking, for instance, at Cayley graphs, one shows that G is of type F_1 if and only if it admits a finite generating set—in particular, G must be countable. Passing to presentation 2-complexes, which are quotients of Cayley complexes by the group action, one sees that G is of type F_2 if and only if it is finitely presented. Furthermore, attaching 3-cells to a compact presentation 2-complex to kill its second homotopy, one proves that G is of type F_3 if and only if it is finitely identified [75]. This shows how lower bounds on the finiteness length recover familiar properties.

The following result shows how finiteness properties behave under group extensions.

Lemma 1.13. *Consider a short exact sequence*

$$N \hookrightarrow G \twoheadrightarrow Q.$$

If N and Q are of type F_n, then so is G. In case N is of type F_{n-1} and G is of type F_n, then Q is of type F_n. If the sequence splits and G is of type F_n, then the retract Q is also of type F_n.

We refer the reader e.g. to [79, Theorems 4 and 6] for a proof of the above. Lemma **1.13** can be used to give useful bounds on the finiteness length of groups which fit into short exact sequences.

Corollary 1.14. *Given a short exact sequence*

$$N \hookrightarrow G \twoheadrightarrow Q,$$

the following hold.

i. *If $\phi(Q) = \infty$, then $\phi(N) \leq \phi(G)$.*

ii. *If $\phi(N) = \infty$, then $\phi(Q) = \phi(G)$.*

iii. *If the sequence splits, then $\phi(G) \leq \phi(Q)$.*

iv. *If the sequence splits trivially, i.e. $G = N \times Q$, then $\phi(G) = \min\{\phi(N), \phi(Q)\}$.*

Proof. If Q (resp. N) enjoys all finiteness properties F_n, then G inherits all finiteness properties from N (resp. from Q) by Lemma **1.13**, whence **(i)** and **(ii)** follow. Part **(iii)** is just the third claim of Lemma **1.13** restated in the language of finiteness length. By **(iii)**, one has $\phi(N \times Q) \leq \min\{\phi(N), \phi(Q)\}$. If both N and Q are of type F_n, then so is $N \times Q$ by the first claim of Lemma **1.13**, whence $\phi(N \times Q) \geq \min\{\phi(N), \phi(Q)\}$. □

Chapter 2

The retraction tool

In this chapter we give an upper bound on the finiteness length of groups which admit certain split soluble representations, and present some consequences of this result. Typical examples to which the theorem below applies include many soluble (non-nilpotent) linear groups and some parabolic subgroups of classical groups.

The first main result of this thesis is the following.

Theorem 2.1 (Theorem C, restated)**.** *Suppose a group* Γ *retracts onto a soluble matrix group* $\mathfrak{X}(R) \rtimes \mathcal{H}(R) \leq \mathcal{G}(R)$*, where* $\mathfrak{X}$ *and* $\mathcal{H}$ *denote, respectively, a unipotent root subgroup and a maximal torus of a classical matrix group* $\mathcal{G}$*. Then* $\phi(\Gamma) \leq \phi(\mathbf{B}_2^\circ(R))$*.*

Recall that [33, Corollary 3.5 and Theorem 5.1] show that $\phi(\mathcal{B}(\mathcal{O}_S)) \leq \phi(\mathbf{B}_2^\circ(\mathcal{O}_S))$ in the case where $\mathcal{B}$ is a Borel subgroup of a Chevalley–Demazure group scheme and $\mathcal{O}_S$ is an S-arithmetic ring in positive characteristic. Theorem 2.1 thus generalizes Bux's inequality to a much wider class of groups, but now with a fairly elementary proof, to be given below in Section 2.1. In Section 2.2 we give examples of groups for which Theorem 2.1 holds. We also combine Theorem 2.1 with some known results to give a new proof of (a generalization of) the main result of [33].

2.1 Proof of Theorem 2.1

The hypotheses of the theorem already yield an obvious bound on the finiteness length of the given groups by Corollary 1.14. Indeed, in the notation of Theorem 2.1, Corollary 1.14**(iii)** shows that $\phi(\Gamma) \leq \phi(\mathfrak{X}(R) \rtimes \mathcal{H}(R))$. The actual work thus consists of proving that the finiteness length of $\mathfrak{X}(R) \rtimes \mathcal{H}(R)$ is no greater than the desired value, namely the finiteness length of the standard Borel subgroup $\mathbf{B}_2^\circ(R) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \leq \mathrm{SL}_2(R)$ of rank one.

We begin with the following observation, which is well-known in the S-arithmetic case.

Lemma 2.2. *For any commutative ring R with unity, the standard Borel subgroups $\mathbf{B}_n(R) \leq \mathrm{GL}_n(R)$ and $\mathbf{B}_n^\circ(R) \leq \mathrm{SL}_n(R)$ have the same finiteness length, which in turn is no greater than $\phi(\mathbf{B}_2^\circ(R))$.*

Proof. Though Lemma 2.2 is stated for arbitrary rings, the proof presented here is essentially Bux's proof in the S-arithmetic case in positive characteristic; see [33, Remark 3.6].

If $|R^\times| = 1$, then there are no diagonal entries other than 1, whence $\mathbf{B}_n(R) = \mathbf{B}_n^\circ(R) = \mathbf{U}_n(R)$, which trivially implies equality of the finiteness lengths. We may thus assume that R has at least two units. Recall that both $\mathbf{B}_n(R)$ and $\mathbf{B}_n^\circ(R)$ retract onto $\mathbb{G}_m(R)$. Thus, if $\mathbb{G}_m(R)$ is *not* finitely generated, then $\phi(\mathbf{B}_n(R)) = \phi(\mathbf{B}_n^\circ(R)) = \phi(\mathbf{B}_2^\circ(R)) = 0$. Suppose from now on that $\mathbb{G}_m(R)$ is finitely generated.

Consider the central subgroups $\mathbf{Z}_n(R) \leq \mathbf{B}_n(R)$ and $\mathbf{Z}_n^\circ(R) \leq \mathbf{B}_n^\circ(R)$ given by

$$\mathbf{Z}_n(R) = \left\{\mathrm{Diag}(u, \ldots, u) \in \mathrm{GL}_n(R) \mid u \in R^\times\right\} = \left\{u\mathbf{1}_n \mid u \in R^\times\right\} \cong \mathbb{G}_m(R)$$

and

$$\mathbf{Z}_n^\circ(R) = \mathbf{Z}_n(R) \cap \mathbf{B}_n^\circ(R) = \left\{u\mathbf{1}_n \mid u \in R^\times \text{ and } u^n = 1\right\} \cong \mu_n(R),$$

respectively, where $\mu_n(R)$ denotes the group of n-th roots of unity of R. (Remark: Since R is an arbitrary commutative ring with unity, the groups above need not coincide with the centers of their overgroups.) Using the determinant map and passing to projective groups (that is, factoring out the central subgroups above) we obtain the following commutative diagram of short exact sequences.

$$\begin{array}{ccccc}
\mathbf{Z}_n^\circ(R) & \hookrightarrow & \mathbf{Z}_n(R) & \twoheadrightarrow & \mathrm{pow}_n(\mathbb{G}_m(R)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{B}_n^\circ(R) & \hookrightarrow & \mathbf{B}_n(R) & \xrightarrow{\det} & \mathbb{G}_m(R) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{P}\mathbf{B}_n^\circ(R) & \hookrightarrow & \mathbb{P}\mathbf{B}_n(R) & \twoheadrightarrow & \frac{\mathbb{G}_m(R)}{\mathrm{pow}_n(\mathbb{G}_m(R))}
\end{array},$$

where the map $\mathrm{pow}_n : \mathbf{Z}_n(R) \to \mathbb{G}_m(R)$ means taking n-th powers, i.e. $\mathrm{pow}_n(u \cdot \mathbf{1}_n) = u^n$. Since $\mathbb{G}_m(R)$ is a finitely generated (abelian) group, we have that the groups of the top row and right-most column have finiteness lengths equal to ∞, whence we obtain, by Corollary 1.14,

$$\phi(\mathbb{P}\mathbf{B}_n^\circ(R)) = \phi(\mathbf{B}_n^\circ(R)) \leq \phi(\mathbf{B}_n(R)) = \phi(\mathbb{P}\mathbf{B}_n(R)) \geq \phi(\mathbb{P}\mathbf{B}_n^\circ(R)).$$

Since the group $\mathbb{G}_m(R)/\mathrm{pow}_n(\mathbb{G}_m(R))$ of the bottom right corner is a (finitely generated) torsion abelian group, it is finite, from which the equality

$$\phi(\mathbb{P}\mathbf{B}_n(R)) = \phi(\mathbb{P}\mathbf{B}_n^\circ(R))$$

follows, by Lemma 1.9. Together with the inequalities above, this concludes the proof of the first claim.

Finally, any $\mathbf{B}_n(R)$ retracts onto $\mathbf{B}_2(R)$ via the map

$$\mathbf{B}_n(R) = \begin{pmatrix} * & * & * & \cdots & * \\ 0 & * & * & \ddots & \vdots \\ 0 & 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ 0 & \cdots & \cdots & 0 & * \end{pmatrix} \longrightarrow\!\!\!\!\rightarrow \begin{pmatrix} * & * & 0 & \cdots & 0 \\ 0 & * & 0 & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \cong \mathbf{B}_2(R),$$

which yields the second claim by Corollary **1.14(iii)** and the equality $\phi(\mathbf{B}_2^\circ(R)) = \phi(\mathbf{B}_2(R))$ established above. □

Recall that, in the notation of Theorem 2.1, it suffices to show that $\phi(\mathfrak{X}(R) \rtimes \mathcal{H}(R)) \leq \phi(\mathbf{B}_2^\circ(R))$. To this end we shall use the standard matrix representations of classical groups and apply Corollary 1.14 repeatedly. To do so, however, we have to assume that the group of units $\mathbb{G}_m(R)$ of the underlying base ring R is finitely generated, as we did at some stage during the proof of Lemma 2.2. This assumption is in fact harmless.

***Remark* 2.3.** If the group of units $\mathbb{G}_m(R)$ is *not* finitely generated, then Theorem 2.1 holds.

Proof. In this case we have that

$$0 \leq \phi(\Gamma) \leq \phi(\mathfrak{X}(R) \rtimes \mathcal{H}(R)) \leq \phi(\mathcal{H}(R)) \leq \phi(\mathbb{G}_m(R)) = 0$$

and

$$0 \leq \phi(\mathbf{B}_2^\circ(R)) \leq \phi(\{\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right) \mid u \in R^\times\}) = \phi(\mathbb{G}_m(R)) = 0$$

by Corollary **1.14(iii)**, because both the torus $\mathcal{H}(R)$ and $\mathbf{B}_2^\circ(R)$ retract onto $\mathbb{G}_m(R)$. □

In face of Remark 2.3 we may (and do) **assume, for the remainder of this section, that $\mathbb{G}_m(R)$ is finitely generated**.

Proceeding with the proof of Theorem 2.1, we warm-up by considering the easier case where the underlying classical group $\mathcal{G}$ containing $\mathfrak{X} \rtimes \mathcal{H}$ is the general linear group itself, which will set the tune for the remaining cases. (Recall that $\mathcal{H}$ is a maximal torus of $\mathcal{G}$.)

Proposition 2.4. *Theorem 2.1 holds if $\mathcal{G} = \mathrm{GL}_n$.*

Proof. Here we take a matrix representation of $\mathcal{G} = \mathrm{GL}_n$ such that the given soluble subgroup $\mathfrak{X} \rtimes \mathcal{H}$ is upper triangular. In this case, the maximal torus $\mathcal{H}$ is the subgroup of diagonal matrices of GL_n, i.e.

$$\mathcal{H}(R) = \mathbf{D}_n(R) = \prod_{i=1}^{n} D_i(R) \leq \mathrm{GL}_n(R),$$

and $\mathfrak{X}$ is identified with a subgroup of all elementary matrices in a single fixed position, say (i, j) with $i < j$. That is,

$$\mathfrak{X}(R) = \mathbf{E}_{ij}(R) = \langle \{e_{ij}(r) \mid r \in R\} \rangle \leq \mathrm{GL}_n(R).$$

Recall that the action of the torus $\mathcal{H}(R) = \mathbf{D}_n(R)$ on the unipotent root subgroup $\mathfrak{X}(R) = \mathbf{E}_{ij}(R)$ is given by the diagonal relations **(1.2)**. But such relations also imply the decomposition

$$\begin{aligned} \mathfrak{X}(R) \rtimes \mathcal{H}(R) = \mathbf{E}_{ij}(R) \rtimes \mathbf{D}_n(R) &= \langle \mathbf{E}_{ij}(R), D_i(R), D_j(R) \rangle \times \prod_{i \neq k \neq j} D_k(R) \\ &\cong \mathbf{B}_2(R) \times \mathbb{G}_m(R)^{n-2} \end{aligned}$$

because all diagonal subgroups $D_k(R)$ with $k \neq i, j$ act trivially on the elementary matrices $e_{ij}(r)$. Since we are assuming $\mathbb{G}_m(R)$ to be finitely generated, it follows from Corollary **1.14(iv)** and Lemma **2.2** that

$$\begin{aligned} \phi(\mathfrak{X}(R) \rtimes \mathcal{H}(R)) = \min \left\{ \phi(\mathbf{B}_2(R)), \phi(\mathbb{G}_m(R)^{n-2}) \right\} &= \min \left\{ \phi(\mathbf{B}_2^{\circ}(R)), \infty \right\} \\ &= \phi(\mathbf{B}_2^{\circ}(R)). \end{aligned}$$

□

Having solved the initial case of GL_n, we now investigate the situation where the classical group scheme $\mathcal{G}$ in the statement of Theorem **2.1** is a universal Chevalley–Demazure group scheme, say $\mathcal{G} = \mathcal{G}_{\Phi}^{\mathrm{sc}}$ with underlying root system Φ associated to the given maximal torus $\mathcal{H} \leq \mathcal{G}_{\Phi}^{\mathrm{sc}}$ and with a fixed set of simple roots $\Delta \subset \Phi$. In this case we have that

$$\mathcal{H}(R) = \prod_{\alpha \in \Delta} \mathcal{H}_{\alpha}(R),$$

and $\mathfrak{X}$ is the unipotent root subgroup associated to some (positive) root $\eta \in \Phi^+$, that is,

$$\mathfrak{X}(R) = \mathfrak{X}_{\eta}(R) = \langle x_{\eta}(r) \mid r \in R \rangle.$$

The proof proceeds on a case-by-case analysis on the root system Φ and the given root $\eta \in \Phi^+$. Instead of diving into all possible combinations, however, some obvious reductions can be done.

Lemma 2.5. *If Theorem* **2.1** *holds whenever $\mathcal{G}$ is a universal Chevalley–Demazure group scheme $\mathcal{G}_\Phi^{\mathrm{sc}}$ of rank at most four and $\mathfrak{X} = \mathfrak{X}_\eta$ with $\eta \in \Phi^+$ simple, then it holds when $\mathcal{G}$ is any universal Chevalley–Demazure group scheme.*

Proof. Write $\mathfrak{X}(R) = \mathfrak{X}_\eta(R)$ and $\mathcal{H}(R) = \prod_{\alpha\in\Delta} \mathcal{H}_\alpha(R)$ as above. By the Weyl relations **(1.10)**, we can find an element w in the Weyl group W associated to Φ and a corresponding element ω in the image of W in $\mathcal{G}_\Phi^{\mathrm{sc}}(R)$ such that $w(\eta) \in \Phi^+$ is a simple root and

$$\omega(\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R))\omega^{-1} \cong \mathfrak{X}_{w(\eta)}(R) \rtimes \mathcal{H}(R).$$

(The conjugation above takes place in the overgroup $\mathcal{G}_\Phi^{\mathrm{sc}}(R)$.) We may thus assume $\eta \in \Phi^+$ to be simple. From the Steinberg relations **(1.9)** we have that

$$\mathfrak{X}(R) \rtimes \mathcal{H}(R) = \left(\mathfrak{X}_\eta(R) \rtimes \left(\prod_{\substack{\alpha\in\Delta \\ \langle\eta,\alpha\rangle\neq 0}} \mathcal{H}_\alpha(R) \right) \right) \times \prod_{\substack{\beta\in\Delta \\ \langle\eta,\beta\rangle=0}} \mathcal{H}_\beta(R),$$

which implies that $\phi(\mathfrak{X}(R) \rtimes \mathcal{H}(R)) = \phi(\mathfrak{X}_\eta(R) \rtimes \mathcal{H}^\circ(R))$ by Corollary **1.14(iv)**, where

$$\mathcal{H}^\circ(R) = \prod_{\substack{\alpha\in\Delta \\ \langle\eta,\alpha\rangle\neq 0}} \mathcal{H}_\alpha(R).$$

Inspecting the Dynkin diagrams for (reduced, irreducible) root systems, it follows that the number of simple roots $\alpha \in \Delta$ for which $\langle \eta, \alpha\rangle \neq 0$ is at most four. Since the semi-simple root subgroups generating the torus $\mathcal{H}^\circ(R)$ are precisely those which might not commute with $\mathfrak{X}_\eta(R)$, the embeddings of Remark **1.7** yield the result. □

Thus, in view of Remark **2.3**, Proposition **2.4** and Lemma **2.5**, the proof of Theorem **2.1** will be complete once we establish the following.

Proposition 2.6. *Theorem* **2.1** *holds whenever $\mathcal{G}$ is a universal Chevalley–Demazure group scheme $\mathcal{G}_\Phi^{\mathrm{sc}}$ with*

$$\Phi \in \{A_1, A_2, C_2, G_2, A_3, B_3, C_3, D_4\}$$

and $\mathfrak{X} \rtimes \mathcal{H}$ is of the form

$$\mathfrak{X} \rtimes \mathcal{H} = \mathfrak{X}_\eta \rtimes \left(\prod_{\substack{\alpha\in\Delta \\ \langle\eta,\alpha\rangle\neq 0}} \mathcal{H}_\alpha \right)$$

with $\eta \in \Phi^+$ simple.

Proof. This time we do not avoid the case-by-case analysis, though the idea of the proof is quite simple. In each case, we find a matrix group $G(\Phi, \eta, R)$ satisfying $\phi(G(\Phi, \eta, R)) = \phi(\mathbf{B}_2^\circ(R))$ and which fits in a short exact sequence

$$\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R) \hookrightarrow G(\Phi, \eta, R) \twoheadrightarrow Q(\Phi, \eta, R)$$

where $Q(\Phi, \eta, R)$ is finitely generated abelian. In fact, $G(\Phi, \eta, R)$ is often taken to be $\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)$ itself so that $Q(\Phi, \eta, R)$ is trivial in many cases. The proposition thus follows from Corollary **1.14(i)**.

To construct the matrix groups $G(\Phi, \eta, R)$ above, we use mostly Ree's matrix representations of classical groups [80] as worked out by Carter in [39]. (Recall that the case of Type B_2 was cleared by Dieudonné [50] after left open in Ree's paper.) In the exceptional case G_2 we follow Seligman's identification from [85]. We remark, however, that Seligman's numbering of indices agrees with that of Carter's for G_2 as a subalgebra of B_3.

Type A: We identify the scheme $\mathcal{G}_{A_n}^{\mathrm{sc}}$ with SL_{n+1} so that the soluble subgroup $\mathfrak{X}_\eta \rtimes \mathcal{H} \leq \mathrm{SL}_{n+1}$ is upper triangular and the given maximal torus $\mathcal{H}$ of SL_{n+1} is the subgroup of diagonal matrices. Now, if $\mathrm{rk}(\Phi) = 1$, then there is nothing to check, for in this case $\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)$ itself is isomorphic to $\mathbf{B}_2^\circ(R)$. If $\Phi = \mathrm{A}_2$, identify $\mathfrak{X}_\eta(R)$ with the root subgroup $\mathbf{E}_{12}(R) \leq \mathrm{SL}_3(R)$ so that

$$\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R) = \left\{ \begin{pmatrix} a & r & 0 \\ 0 & b & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix} \in \mathrm{SL}_3(R) \,\middle|\, a, b \in R^\times, r \in R \right\}.$$

It follows that $\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)$ is isomorphic to $\mathbf{B}_2(R)$ via

$$\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R) \ni \begin{pmatrix} a & r & 0 \\ 0 & b & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix} \longmapsto \begin{pmatrix} a & r \\ 0 & b \end{pmatrix} \in \mathbf{B}_2(R) \leq \mathrm{GL}_2(R).$$

(Recall that $\mathbf{B}_2(R)$ has the same finiteness length as $\mathbf{B}_2^\circ(R)$ by Lemma **2.2**.) Concluding the case of Type A, if $\Phi = \mathrm{A}_3$ we identify $\mathfrak{X}_\eta(R)$ with the root subgroup $\mathbf{E}_{23}(R) \leq \mathrm{SL}_4(R)$, which gives

$$\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R) = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & r & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & (abc)^{-1} \end{pmatrix} \in \mathrm{SL}_3(R) \,\middle|\, a, b, c \in R^\times, r \in R \right\}.$$

Here, $\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)$ is isomorphic to the group $\mathbf{B}_2(R) \times \mathbb{G}_m(R)$ via the map

$$\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R) \ni \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & r & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & (abc)^{-1} \end{pmatrix} \longmapsto \left(\begin{pmatrix} b & r \\ 0 & c \end{pmatrix}, a \right) \in \mathbf{B}_2(R) \times \mathbb{G}_m(R).$$

Thus, $\phi(\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)) = \phi(\mathbf{B}_2^\circ(R))$ by Corollary **1.14(iv)** and Lemma **2.2**. In the notation given in the beginning of the proof, we have defined the groups $G(\mathrm{A}_1, \eta, R)$, $G(\mathrm{A}_2, \eta, R)$ and $G(\mathrm{A}_3, \eta, R)$ to be $\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)$.

Type C: Suppose $\Phi = \mathrm{C}_n$. Following Ree and Carter we identify $\mathcal{G}^{\mathrm{sc}}_{\mathrm{C}_n}$ with the symplectic group $\mathrm{Sp}_{2n} \leq \mathrm{SL}_{2n}$. If $\Phi = \mathrm{C}_2$, denote by $\Delta = \{\alpha, \beta\}$ the set of simple roots, where α is short and β is long. The unipotent root subgroups are given by

$$\mathfrak{X}_\alpha(R) = \langle \{ e_{12}(r) e_{43}(r)^{-1} \in \mathrm{SL}_4(R) \mid r \in R \} \rangle$$

and

$$\mathfrak{X}_\beta(R) = \mathbf{E}_{24}(R) = \langle \{ e_{24}(r) \in \mathrm{SL}_4(R) \mid r \in R \} \rangle,$$

whereas the maximal torus $\mathcal{H}(R)$ is the diagonal subgroup

$$\mathcal{H}(R) = \langle \mathcal{H}_\alpha(R), \mathcal{H}_\beta(R) \rangle = \langle \{ \mathrm{Diag}(a, a^{-1}, a^{-1}, a), \\ \mathrm{Diag}(1, b, 1, b^{-1}) \in \mathrm{SL}_4(R) \mid a, b \in R^\times \} \rangle.$$

Now, if $\eta = \alpha$ (that is, if η is a short root), then $\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)$ is the group

$$\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R) = \left\{ \begin{pmatrix} u & r & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & u^{-1} & 0 \\ 0 & 0 & -r & v^{-1} \end{pmatrix} \in \mathrm{Sp}_4(R) \,\middle|\, u, v \in R^\times, r \in R \right\}.$$

Hence, $\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)$ is isomorphic to $\mathbf{B}_2(R)$ via

$$\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R) \ni \begin{pmatrix} u & r & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & u^{-1} & 0 \\ 0 & 0 & -r & v^{-1} \end{pmatrix} \longmapsto \begin{pmatrix} u & r \\ 0 & v \end{pmatrix} \in \mathbf{B}_2(R),$$

which yields $\phi(\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)) = \phi(\mathbf{B}_2^\circ(R))$ by Lemma **2.2**. On the other hand, if $\eta = \beta$ (i.e. η is long), then $\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)$ is given by

$$\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R) = \left\{ \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & v & 0 & r \\ 0 & 0 & u^{-1} & 0 \\ 0 & 0 & 0 & v^{-1} \end{pmatrix} \in \mathrm{Sp}_4(R) \,\middle|\, u, v \in R^\times, r \in R \right\},$$

which is isomorphic to $\mathbf{B}_2^\circ(R) \times \mathbb{G}_m(R)$ via

$$\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R) \ni \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & v & 0 & r \\ 0 & 0 & u^{-1} & 0 \\ 0 & 0 & 0 & v^{-1} \end{pmatrix} \longmapsto \left(\begin{pmatrix} v & r \\ 0 & v^{-1} \end{pmatrix}, u \right) \in \mathbf{B}_2^\circ(R) \times \mathbb{G}_m(R).$$

Thus, $\phi(\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)) = \phi(\mathbf{B}_2^\circ(R))$ by Corollary **1.14(iv)**. Again we have simply defined $G(\mathrm{C}_2, \eta, R) = \mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)$ for any choice of $\eta \in \mathrm{C}_2$ simple.

Lastly, assume $\Phi = \mathrm{C}_3$ and denote its set of simple roots by $\Delta = \{\alpha_1, \alpha_2, \beta\}$, where β is the long root. We have $\mathcal{G}^{\mathrm{sc}}_{\mathrm{C}_3} = \mathrm{Sp}_6$ with the root subgroups given by the following matrix subgroups.

$$\mathfrak{X}_{\alpha_1}(R) = \langle \{ e_{12}(r) e_{54}(r)^{-1} \in \mathrm{SL}_6(R) \mid r \in R \} \rangle,$$

$$\mathfrak{X}_{\alpha_2}(R) = \langle \{ e_{23}(r) e_{65}(r)^{-1} \in \mathrm{SL}_6(R) \mid r \in R \} \rangle,$$

$$\mathfrak{X}_\beta(R) = \mathbf{E}_{36}(R) = \langle \{e_{36}(r) \in \mathrm{SL}_6(R) \mid r \in R\}\rangle,$$

and

$$\mathcal{H}(R) = \langle \mathcal{H}_{\alpha_1}(R), \mathcal{H}_{\alpha_2}(R), \mathcal{H}_\beta(R)\rangle = \langle \{\mathrm{Diag}(a_1, a_1^{-1}, 1, a_1^{-1}, a_1, 1),$$
$$\mathrm{Diag}(1, a_2, a_2^{-1}, 1, a_2^{-1}, a_2), \mathrm{Diag}(1,1,b,1,1,b^{-1}) \in \mathrm{SL}_6(R) \mid a_1, a_2, b \in R^\times\}\rangle.$$

Here we are only interested in the case where η is the central root α_2, for otherwise η would be orthogonal to one of the other simple roots. Thus,

$$\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R) = \left\{ \begin{pmatrix} u & 0 & 0 & 0 & 0 & 0 \\ 0 & v & r & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & 0 \\ 0 & 0 & 0 & u^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & v^{-1} & 0 \\ 0 & 0 & 0 & 0 & -r & w^{-1} \end{pmatrix} \in \mathrm{Sp}_6(R) \;\middle|\; u, v, w \in R^\times, r \in R \right\}.$$

The isomorphism

$$\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R) \ni \begin{pmatrix} u & 0 & 0 & 0 & 0 & 0 \\ 0 & v & r & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & 0 \\ 0 & 0 & 0 & u^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & v^{-1} & 0 \\ 0 & 0 & 0 & 0 & -r & w^{-1} \end{pmatrix} \longmapsto \left(\left(\begin{smallmatrix} v & r \\ 0 & w \end{smallmatrix}\right), u\right) \in \mathbf{B}_2(R) \times \mathbb{G}_m(R)$$

then yields $\phi(\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)) = \phi(\mathbf{B}_2(R)) = \phi(\mathbf{B}_2^\circ(R))$ by Corollary **1.14(iv)** and Lemma **2.2**—once again, $G(\mathrm{C}_3, \eta, R) = \mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)$ does the job.

Type D: The case of maximal rank concerns the root system $\Phi = \mathrm{D}_4$, with set of simple roots $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and the given simple root η being equal to the central root α_2 which is not orthogonal to any other simple root. Here, $\mathcal{G}^{\mathrm{sc}}_{\mathrm{D}_4} = \mathrm{SO}_8$. Following Ree and Carter, the root subgroups and the maximal torus are given as follows.

$$\mathfrak{X}_{\alpha_1}(R) = \langle \{e_{12}(r)e_{65}^{-1} \in \mathrm{SL}_8(R) \mid r \in R\}\rangle,$$
$$\mathfrak{X}_{\alpha_2}(R) = \langle \{e_{23}(r)e_{76}^{-1} \in \mathrm{SL}_8(R) \mid r \in R\}\rangle,$$
$$\mathfrak{X}_{\alpha_3}(R) = \langle \{e_{34}(r)e_{87}^{-1} \in \mathrm{SL}_8(R) \mid r \in R\}\rangle,$$
$$\mathfrak{X}_{\alpha_4}(R) = \langle \{e_{38}(r)e_{47}^{-1} \in \mathrm{SL}_8(R) \mid r \in R\}\rangle,$$

and

$$\begin{aligned}\mathcal{H}(R) =& \langle \mathcal{H}_{\alpha_1}(R),\, \mathcal{H}_{\alpha_2}(R),\, \mathcal{H}_{\alpha_3}(R),\, \mathcal{H}_{\alpha_4}(R)\rangle \\ =& \langle \{\mathrm{Diag}(a_1, a_1^{-1}, 1, 1, a_1^{-1}, a_1, 1, 1), \quad \mathrm{Diag}(1, a_2, a_2^{-1}, 1, 1, a_2^{-1}, a_2, 1), \\ & \mathrm{Diag}(1, 1, a_3, a_3^{-1}, 1, 1, a_3^{-1}, a_3), \quad \mathrm{Diag}(1, 1, a_4, a_4, 1, 1, a_4^{-1}, a_4^{-1}) \\ & \in \mathrm{SL}_8(R) \mid a_1, a_2, a_3, a_4 \in R^\times\}\rangle.\end{aligned}$$

The torus $\mathcal{H}(R)$ is a subgroup of the diagonal group $T(R)$ given by

$$T(R) = \langle \{\mathrm{Diag}(u, v, w, x, u^{-1}, v^{-1}, w^{-1}, x^{-1}) \in \mathrm{SL}_8(R) \mid u, v, w, x \in R^\times\}\rangle.$$

Recall that $\eta = \alpha_2$, the central root. Set $G(\mathrm{D}_4, \alpha_2, R) = \mathfrak{X}_\eta(R) \rtimes T(R)$. We have a short exact sequence

$$\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R) \hookrightarrow G(\mathrm{D}_4, \alpha_2, R) \twoheadrightarrow \frac{T(R)}{\mathcal{H}(R)},$$

from which we deduce $\phi(\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)) \leq \phi(G(\mathrm{D}_4, \alpha_2, R)$ by Corollary **1.14(i)** because the quotient $T(R)/\mathcal{H}(R)$ is finitely generated abelian. But $G(\mathrm{D}_4, \alpha_2, R)$ is isomorphic to $\mathbf{B}_2(R) \times \mathbb{G}_m(R)^2$ via

$$G(\mathrm{D}_4, \alpha_2, R) \ni \begin{pmatrix} u & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & v^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -r & w^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^{-1} \end{pmatrix} \longmapsto \left(\left(\begin{smallmatrix} v & r \\ 0 & w \end{smallmatrix} \right), u, x \right),$$

whence $\phi(G(\mathrm{D}_4, \alpha_2, R)) = \phi(\mathbf{B}_2(R)) = \phi(\mathbf{B}_2^\circ(R))$ by Corollary **1.14(iv)** and Lemma **2.2**.

Types B and G: For simplicity, we approach the remaining cases at once since we shall use the embedding of the group of type G_2 into the special orthogonal group of type B_3. For convenience, we assume from now on that the base ring R has $\mathrm{char}(R) \neq 2$ in order to simplify the choice of a symmetric matrix preserved by the elements of $\mathcal{G}^{\mathrm{sc}}_{\mathrm{B}_3}(R) = \mathrm{SO}_7(R)$. This assumption is harmless, for the proof in the case $\mathrm{char}(R) = 2$ follows analogously (after a change of basis) using Dieudounné's matrix representation [50] since the underlying quadratic form preserved by the ambient group SO_7 is the same.

Denote by $\Delta = \{\alpha_1, \alpha_2, \beta\}$ the set of simple roots of B_3, where β is the short root. The root subgroups are given below.

$$\mathfrak{X}_{\alpha_1}(R) = \langle \big\{ e_{23}(r) e_{65}(r)^{-1} \in \mathrm{SL}_7(R) \mid r \in R \big\} \rangle,$$

$$\mathfrak{X}_{\alpha_2}(R) = \langle \big\{ e_{34}(r) e_{76}(r)^{-1} \in \mathrm{SL}_7(R) \mid r \in R \big\} \rangle,$$

$$\mathfrak{X}_{\beta}(R) = \langle \{ \exp(r \cdot (2 \cdot \mathrm{E}_{41} - \mathrm{E}_{17})) \in \mathrm{SL}_7(R) \mid r \in R \} \rangle,$$

$$\mathcal{H}_{\alpha_1}(R) = \langle \big\{ \mathrm{Diag}(1, a_1, a_1^{-1}, 1, a_1^{-1}, a_1, 1) \in \mathrm{SL}_7(R) \mid a_1 \in R^\times \big\} \rangle,$$

$$\mathcal{H}_{\alpha_2}(R) = \langle \big\{ \mathrm{Diag}(1, 1, a_2, a_2^{-1}, 1, a_2^{-1}, a_2) \in \mathrm{SL}_7(R) \mid a_2 \in R^\times \big\} \rangle,$$

and

$$\mathcal{H}_{\beta}(R) = \langle \big\{ \mathrm{Diag}(1, 1, 1, b^2, 1, 1, b^{-2}) \in \mathrm{SL}_7(R) \mid b \in R^\times \big\} \rangle.$$

Now let $\Lambda = \{\alpha, \gamma\}$ denote the set of simple roots of G_2, where γ is the short root. In the identification above, the embedding of G_2 into B_3 maps the long root $\alpha \in \mathrm{G}_2$ to the (long) root $\alpha_1 \in \mathrm{B}_3$, and the root subgroups of $\mathcal{G}^{\mathrm{sc}}_{\mathrm{G}_2} \leq \mathcal{G}^{\mathrm{sc}}_{\mathrm{B}_3} = \mathrm{SO}_7$ are listed below.

$$\mathfrak{X}_\alpha(R) = \mathfrak{X}_{\alpha_1}(R),$$

$$\mathfrak{X}_\gamma(R) = \langle\{\exp(r\cdot(2\cdot \mathrm{E}_{12} + \mathrm{E}_{37} - \mathrm{E}_{46} - \mathrm{E}_{51})) \in \mathrm{SL}_7(R) \mid r \in R\}\rangle,$$

$$\mathcal{H}_\alpha(R) = \mathcal{H}_{\alpha_1}(R),$$

and

$$\mathcal{H}_\gamma(R) = \langle\big\{\mathrm{Diag}(1, c^{-2}, c, c, c^2, c^{-1}, c^{-1}) \in \mathrm{SL}_7(R) \mid c \in R^\times\big\}\rangle.$$

We now return to the soluble subgroup $\mathfrak{X}_\eta \rtimes \mathcal{H} \le \mathcal{G}_\Phi^{\mathrm{sc}}$. In the case $\Phi = \mathrm{B}_3$, the maximal torus $\mathcal{H}(R)$ is the diagonal subgroup $\mathcal{H}(R) = \langle\mathcal{H}_{\alpha_1}(R), \mathcal{H}_{\alpha_2}(R), \mathcal{H}_\beta(R)\rangle$ and η is the middle simple root α_2 which is not orthogonal to the other simple roots, so that $\mathfrak{X}_\eta(R) = \mathfrak{X}_{\alpha_2}(R)$. Let $T(R)$ be the diagonal group

$$T(R) = \langle\big\{\mathrm{Diag}(1, u, v, w, u^{-1}, v^{-1}, w^{-1}) \in \mathrm{SL}_7(R) \mid u, v, w \in R^\times\big\}\rangle.$$

Setting $G(\mathrm{B}_3, \alpha_2, R) = \mathfrak{X}_\eta(R) \rtimes T(R)$ we obtain a short exact sequence

$$\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R) \hookrightarrow G(\mathrm{B}_3, \alpha_2, R) \twoheadrightarrow \frac{T(R)}{\mathcal{H}(R)},$$

which gives $\phi(\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)) \le \phi(G(\mathrm{B}_3, \alpha_2, R))$ by Corollary **1.14(i)**. But the isomorphism $G(\mathrm{B}_3, \alpha_2, R) \cong \mathbf{B}_2(R) \times \mathbb{G}_m(R)$ given by

$$G(\mathrm{B}_3, \alpha_2, R) \ni \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&u&0&0&0&0&0\\ 0&0&v&r&0&0&0\\ 0&0&0&w&0&0&0\\ 0&0&0&0&u^{-1}&0&0\\ 0&0&0&0&0&v^{-1}&0\\ 0&0&0&0&0&-r&w^{-1} \end{pmatrix} \longmapsto \left(\left(\begin{smallmatrix} v & r\\ 0 & w\end{smallmatrix}\right), u\right),$$

yields $\phi(G(\mathrm{B}_3, \alpha_2, R)) = \phi(\mathbf{B}_2(R) = \phi(\mathbf{B}_2^\circ(R))$ by Corollary **1.14(iv)** and Lemma **2.2**.

Suppose now that $\Phi = \mathrm{G}_2$. The maximal torus $\mathcal{H}(R)$ is the diagonal subgroup $\mathcal{H}(R) = \langle\mathcal{H}_\alpha(R), \mathcal{H}_\gamma(R)\rangle$. This time we consider the diagonal subgroup $T(R)$ given by

$$T(R) = \langle\big\{\mathrm{Diag}(1, u, v, u^{-1}v^{-1}, u^{-1}, v^{-1}, uv) \in \mathrm{SL}_7(R) \mid u, v \in R^\times\big\}\rangle$$

and let $G(\mathrm{G}_2, \eta, R) = \mathfrak{X}_\eta(R) \rtimes T(R)$, again obtaining a short exact sequence $\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R) \hookrightarrow G(\mathrm{G}_2, \eta, R) \twoheadrightarrow T(R)/\mathcal{H}(R)$ giving $\phi(\mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)) \le \phi(G(\mathrm{G}_2, \eta, R))$. If η is the long root $\alpha = \alpha_1$, then the map

$$G(\mathrm{G}_2, \eta, R) = \mathfrak{X}_{\alpha_1}(R) \rtimes T(R) \ni \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&u&r&0&0&0&0\\ 0&0&v&0&0&0&0\\ 0&0&0&u^{-1}v^{-1}&0&0&0\\ 0&0&0&0&u^{-1}&0&0\\ 0&0&0&0&-r&v^{-1}&0\\ 0&0&0&0&0&0&uv \end{pmatrix} \longmapsto \left(\begin{smallmatrix} u & r\\ 0 & v\end{smallmatrix}\right)$$

yields an isomorphism $G(\mathrm{G}_2, \eta, R) \cong \mathbf{B}_2(R)$, whence $\phi(G(\mathrm{G}_2, \eta, R)) = \phi(\mathbf{B}_2^\circ(R))$ by Lemma 2.2. If η is the short root γ, we observe that

$$G(\mathrm{G}_2, \eta, R) = \mathfrak{X}_\gamma(R) \rtimes \mathcal{H}(R) \cong (\mathfrak{X}_\gamma(R) \rtimes \mathcal{H}_\alpha(R)) \times \mathbb{G}_m(R)$$

because for any $x_\gamma(r) = \exp(r \cdot (2 \cdot \mathrm{E}_{12} + \mathrm{E}_{37} - \mathrm{E}_{46} - \mathrm{E}_{51})) \in \mathfrak{X}_\gamma(R)$ and $d = \mathrm{Diag}(1, u, v, u^{-1}v^{-1}, u^{-1}, v^{-1}, uv) \in T(R)$ the following holds.

$$dx_\gamma(r)d^{-1} = d \begin{pmatrix} 1 & 2r & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 & 0 & -r & 0 \\ -r & -r^2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} d^{-1} = \begin{pmatrix} 1 & u^{-1}2r & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & u^{-1}r \\ 0 & 0 & 0 & 1 & 0 & -u^{-1}r & 0 \\ -u^{-1}r & -u^{-2}r^2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence, $\phi(G(\mathrm{G}_2, \eta, R)) = \phi(\mathfrak{X}_\gamma(R) \rtimes \mathcal{H}_\alpha(R))$. The latter group, however, is isomorphic to the matrix group $\{\left(\begin{smallmatrix} 1 & r \\ 0 & u \end{smallmatrix}\right) \in \mathrm{GL}_2(R) \mid r \in R, u \in R^\times\}$, which in turn is isomorphic to $\mathbb{G}_a(R) \rtimes \mathbb{G}_m(R) = \{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)\} \leq \mathrm{GL}_2(R)$ by inverting the action of the diagonal matrices on the unipotent part. However, due to our standing assumption that $\mathbb{G}_m(R)$ is finitely generated, we have that the group $\mathbb{G}_a(R) \rtimes \mathbb{G}_m(R)$ described above is commensurable with $\mathbf{B}_2^\circ(R)$. Indeed, $\mathbf{B}_2^\circ(R)$ contains a subgroup of finite index which is isomorphic to a group of the form

$$\left\{\left(\begin{smallmatrix} u^2 & r \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(R) \mid u \in S^\times, r \in R\right\}$$

for some (torsion-free) subgroup of units $S \subseteq R^\times$. Since the group above is a subgroup of finite index of $\mathbb{G}_a(R) \rtimes \mathbb{G}_m(R)$, the claim follows. Thus,

$$\phi(\mathfrak{X}_\eta(R) \rtimes \mathcal{H}_\alpha(R)) = \phi(\mathbb{G}_a(R) \rtimes \mathbb{G}_m(R)) = \phi(\mathbf{B}_2^\circ(R))$$

by Lemma 1.9. This finishes the proof of the proposition and thus of Theorem 2.1. □

2.2 Applications

In this section, we give some concrete examples of groups for which Theorem 2.1 holds. We also combine the theorem to important results due to Bestvina–Eskin–Wortman and Gandini in order to generalize (and obtain a new proof of) Bux's main result in [33]. We begin with the examples, some of which will show up in the following chapters.

***Example* 2.7.** The soluble affine $\mathbb{Z}$-group subscheme $\mathbf{A}_4 \leq \mathrm{GL}_4$ below retracts onto $\mathbf{B}_2$ as follows.

$$\mathbf{A}_4(R) = \begin{pmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \twoheadrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cong \mathbf{B}_2(R) \leq \mathrm{GL}_2(R).$$

Hence $\phi(\mathbf{A}_4(R)) \leq \phi(\mathbf{B}_2^\circ(R))$ by Theorem 2.1. In fact, soluble matrix groups are the 'obvious' candidates to apply Theorem 2.1.

***Example* 2.8.** On the other hand, it is not hard to find non-soluble examples for which Theorem 2.1 holds. Consider the following subgroup of $\mathrm{SL}_{12}(\mathbb{Z}[t,t^{-1}])$.

$$\mathcal{P}_2(\mathbb{Z}[t,t^{-1}]) = \begin{pmatrix} * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \end{pmatrix} \leq \mathrm{SL}_{12}(\mathbb{Z}[t,t^{-1}]).$$

Although $\mathcal{P}_2(\mathbb{Z}[t,t^{-1}])$ itself does not retract onto $\mathbf{B}_2^\circ(\mathbb{Z}[t,t^{-1}])$, it does admit a retract onto the following soluble subgroup of $\mathrm{SL}_4(\mathbb{Z}[t,t^{-1}])$.

$$\begin{pmatrix} \mathbf{1}_4 & 0 & \cdots & & \cdots & 0 \\ 0 & * & 0 & 0 & \ddots & \vdots \\ \vdots & 0 & * & * & 0 & \\ & 0 & 0 & * & 0 & \vdots \\ \vdots & \ddots & 0 & 0 & * & 0 \\ 0 & \cdots & & \cdots & 0 & \mathbf{1}_4 \end{pmatrix} \cong \mathbf{E}_{23}(\mathbb{Z}[t,t^{-1}]) \rtimes (\mathbf{D}_4(\mathbb{Z}[t,t^{-1}]) \cap \mathrm{SL}_4(\mathbb{Z}[t,t^{-1}])).$$

This yields $\phi(\mathcal{P}_2(\mathbb{Z}[t,t^{-1}])) \leq 1$ by Theorem 2.1 and [64, Section 4]. In particular, $\mathcal{P}_2(\mathbb{Z}[t,t^{-1}])$ can never be finitely presented.

***Example* 2.9.** Classical results also yield many non-linear examples for which Theorem 2.1 applies, though for rather trivial reasons.
For instance, suppose R is such that $|R^\times| \geq 2$ with a generator $s \in R^\times$ of even or unbounded order. Consider a Borel subgroup $\mathcal{B}_\Phi(R)$ of a Chevalley–Demazure group $\mathcal{G}_\Phi^{\mathrm{sc}}(R)$; cf. Section 1.1.3 for definitions and notation. Then we can always find an epimorphism $\Psi : \mathcal{B}_\Phi(R) \twoheadrightarrow C_2 = \langle x \mid x^2 \rangle$. For example, take the map induced by

$$h_\alpha(s) \mapsto x \text{ for some fixed simple root } \alpha \in \Phi^+,$$
$$h_\delta(s') \mapsto 1 \text{ whenever } R^\times \ni s' \neq s \text{ or } \delta \neq \alpha,$$
$$x_\delta(r) \mapsto 1 \text{ for all } r \in R \text{ and } \delta \in \Phi.$$

Consequently, there is an action

$$\psi : \mathcal{B}_\Phi(R) \to \mathrm{Aut}(\mathrm{BS}(\ell,m))$$

on a Baumslag–Solitar group $\mathrm{BS}(\ell,m) = \langle a,t \mid t^{-1}a^\ell t = a^m \rangle$ by sending $\mathcal{B}_\Phi(R)$ onto the subgroup $C_2 \cong \langle \beta \rangle \leq \mathrm{Aut}(\mathrm{BS}(\ell,m))$ via Ψ, where β is the involution given by $a \mapsto a^{-1}$ and $t \mapsto t$. Thus, the group

$$G = \mathrm{BS}(\ell,m) \rtimes_\psi \mathcal{B}_\Phi(R)$$

is an example to which Theorem 2.1 applies and which is non-linear whenever $\ell, m \in \mathbb{Z}\setminus\{0,1\}$ are coprime. This is because such a $\mathrm{BS}(\ell, m)$ is not residually finite; refer e.g. to [45] for some well-known results on Bausmalg–Solitar groups and their automorphisms. (In fact, we even have $\phi(G) = \phi(\mathcal{B}_{\Phi}(R))$ by Corollary 1.14 because finitely generated one-relator groups have aspherical presentation 2-complexes [52].)

Most notably, we can apply Theorem 2.1 to the following series of examples, which relate to groups of type (R). These were studied by M. Demazure and A. Grothendieck in the sixties and generalize parabolic subgroups of reductive algebraic groups; see [48, Exposé XXII, Cap. 5].

Corollary 2.10. *Let* $\mathbf{G}$ *be an affine group scheme defined over* $\mathbb{Z}$ *and let* $\mathbf{H} \leq \mathcal{G}$ *be a* $\mathbb{Z}$*-subgroup, of type* (R) *with soluble geometric fibers, of a classical group* $\mathcal{G}$*. If there exists a* $\mathbb{Z}$*-retract* $r : \mathbf{G} \to \mathbf{H}$*, then* $\phi(\mathbf{G}(R)) \leq \phi(\mathbf{B}_2^{\circ}(R))$ *for any commutative ring* R *with unity.*

Proof. For every R as in the statement, the given $\mathbb{Z}$-retract yields a group retract $r : \mathbf{G}(R) \twoheadrightarrow \mathbf{H}(R)$, whence $\phi(\mathbf{G}(R)) \leq \phi(\mathbf{H}(R))$. We show that $\phi(\mathbf{H}(R))$ must be bounded above by $\phi(\mathbf{B}_2^{\circ}(R))$.

Since $\mathbf{H} \leq \mathcal{G}$ is of type (R) with soluble geometric fibers, it follows from [48, Exposé XXII, Cor. 5.6.5] that $\mathbf{H}$ is the semi-direct product $\mathbf{H} = \mathcal{U}_{\mathbf{H}} \rtimes \mathcal{T} \leq \mathcal{G}$, where $\mathcal{T}$ is the maximal torus of $\mathcal{G}$ and $\mathcal{U}_{\mathbf{H}}$ is the unipotent radical of (the soluble group) $\mathbf{H}$. Moreover, there exists some positive root α in the underlying root system of $\mathcal{G}$ (with respect to the given torus $\mathcal{T}$) for which the unipotent root subgroup $\mathfrak{X}_{\alpha} \leq \mathcal{G}$ is contained in $\mathcal{U}_{\mathbf{H}}$. Now, $\mathcal{U}_{\mathbf{H}}(R)$ itself is generated by unipotent root subgroups $\mathfrak{X}_{\beta}(R)$ for $\beta \in \Lambda$, where Λ is some subset of simple roots of $\mathcal{G}$. Thus, the map sending each $\mathfrak{X}_{\beta}(R)$ pointwise to 1 for all $\beta \neq \alpha$ and which is constant on $\mathcal{T}(R)$ and on $\mathfrak{X}_{\alpha}(R)$ induces a further retract

$$\mathbf{H}(R) = \mathcal{U}_{\mathbf{H}}(R) \rtimes \mathcal{T}(R) \longrightarrow\!\!\!\!\!\rightarrow \mathfrak{X}_{\alpha}(R) \rtimes \mathcal{T}(R) \leq \mathcal{G}(R),$$

yielding $\phi(\mathbf{H}(R)) \leq \phi(\mathfrak{X}_{\alpha}(R) \rtimes \mathcal{T}(R))$. It follows from Theorem 2.1 that $\phi(\mathfrak{X}_{\alpha}(R) \rtimes \mathcal{T}(R)) \leq \phi(\mathbf{B}_2^{\circ}(R))$. □

We now prove the following generalization of Bux's main theorem in [33].

Corollary 2.11. *Suppose* $\mathbf{P}$ *is a proper parabolic subgroup of a non-commutative, connected, reductive, split linear algebraic group* $\mathbf{G} \leq \mathrm{GL}_n$ *defined over a global field* $\mathbb{K}$*. Denote by* $\mathbf{U_P}$ *the unipotent radical of* $\mathbf{P}$ *and by* $\mathbf{T_P}$ *the maximal torus of* $\mathbf{G}$ *contained in* $\mathbf{P}$*. For any* S*-arithmetic subgroup* $\Gamma \leq \mathbf{U_P} \rtimes \mathbf{T_P}$*, the following inequalities hold.*

$$|S| - 1 \leq \phi(\Gamma) \leq \phi(\mathbf{B}_2^{\circ}(\mathcal{O}_S)).$$

In particular, if $\mathbb{K}$ *has positive characteristic and* $\mathbf{P} = \mathbf{B} = \mathbf{U_B} \rtimes \mathbf{T_B}$ *is a Borel subgroup of* $\mathbf{G}$*, then* $\phi(\Gamma) = |S| - 1$*.*

Proof. We first reduce the problem to the case where $\mathbf{G}$ is a universal Chevalley–Demazure group scheme. The group $\mathbf{G}$ fits into the following diagram of linear algebraic groups over $\mathbb{K}$.

$$\begin{array}{ccccc} & & & & \mathcal{G} \\ & & & & \Big\downarrow\!\!\!\downarrow f \\ \mathbf{G} & \twoheadleftarrow & \mathcal{R}\mathbf{G} \times \mathbf{G}' & \twoheadrightarrow & \mathbf{G}' \end{array}$$

In the diagram, $\mathbf{G}'$ is semi-simple, $\mathcal{R}\mathbf{G}$ is the radical of $\mathbf{G}$ and the upper group $\mathcal{G}$ is simply-connected. The left-most projection is an almost direct product. Since $\mathbf{G}$ and $\mathcal{G}$ are split and semi-simple, the projection given by the central isogeny f is a covering of Chevalley–Demazure group schemes, with $\mathcal{G}$ being universal. By taking the corresponding diagram intersected with parabolic subgroups and recalling that the unipotent subgroups of $\mathbf{G}$, $\mathbf{G}'$ and $\mathcal{G}$ are isomorphic, it follows that S-arithmetic subgroups of $\mathbf{P} \leq \mathbf{G}$ have the same finiteness length as the S-arithmetic subgroups of the corresponding parabolic $\mathcal{P} \leq \mathcal{G}$ (see e.g. the steps in [14, 2.6(c)]; notice that Satz 1 cited by Behr holds regardless of characteristic). Since S-arithmetic subgroups of a given linear algebraic group are commensurable, we may restrict ourselves to the S-arithmetic group $\mathcal{P}(\mathcal{O}_S) \leq \mathcal{G}(\mathcal{O}_S)$.

In the set-up above, the first inequality follows from [16, Proposition 10] and [29], and the second inequality follows from Corollary **2.10**. Now suppose $\mathrm{char}(\mathcal{O}_S) > 0$. Since $\mathbf{B}_2^\circ(\mathcal{O}_S) \supseteq \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \cong \mathbb{G}_a(\mathcal{O}_S)$, one has that $\mathbf{B}_2^\circ(\mathcal{O}_S)$ has no bounds on the orders of its finite subgroups due to the additive structure of $\mathcal{O}_S$; see e.g. [74, Section 23]). But $\mathbf{B}_2^\circ(\mathcal{O}_S)$ acts by cell-permuting homeomorphisms on the product of $|S|$ Bruhat–Tits trees, each such tree being associated to the locally compact group $\mathrm{SL}_2(\mathrm{Frac}(\mathcal{O}_S)_v)$ for $v \in S$; cf. [86]. Since the stabilizers of this action are finite [31, Section 3.3], it follows that $\mathbf{B}_2^\circ(\mathcal{O}_S)$ belongs to Kropholler's $\mathbf{H}\mathfrak{F}$ class [63]. Thus, Gandini's theorem applies [53], yielding $\phi(\mathbf{B}_2^\circ(\mathcal{O}_S)) < |S|$, which finishes the proof. □

Corollary **2.11** gives, in particular, a shorter (and the author dares say simpler) proof of Bux's equality [33, Theorem A]. Of course, Corollary **2.11** is known to be "uninteresting" in characteristic zero—the reader familiar with the theory of S-arithmetic groups recalls that, in this case, $\phi(\mathcal{B}(\mathcal{O}_S))$ does not depend on the cardinality of S by the Kneser–Tiemeyer local-global principle [98, Theorem 3.1], and moreover that $\phi(\mathbf{B}_2^\circ(\mathcal{O}_S)) = \infty$ [98, Corollary 4.5]. Nevertheless, the charm of Corollary **2.11** lies in the independency of characteristic and in the content of the three theorems used to prove it, namely: isoperimetric inequalities in higher dimensions for S-arithmetic lattices [16], a homotopical obstruction intrinsic to the group schemes considered (Cor. **2.10**), and a geometric obstruction occurring for many groups which act nicely on finite-dimensional contractible complexes [53].

Chapter 3

Finite presentability of Herbert Abels' groups

From the fifties to the eighties, the general theory of finitely presented soluble groups underwent major progress thanks to the works of Baumslag, Bieri, Groves, P. Hall, Remeslennikov, Strebel, and many others; see, for instance, [65, Chap. 11], the survey [95], and [15, Appendix A]. Even for small solubility class, the area still has quite challenging open problems, such as the Σ^m-conjecture [61].

In the fifties, Philip Hall asked whether the image of any finitely presented soluble group is itself finitely presented. Herbert Abels proved this to be false by giving the following, remarkably simple counterexample.

Theorem 3.1 (Abels [2]). *The soluble linear group*

$$\mathbf{A}_4(\mathbb{Z}[1/p]) = \begin{pmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \leq \mathrm{GL}_4(\mathbb{Z}[1/p])$$

is a counterexample to Philip Hall's problem.

That $\mathbf{A}_4(\mathbb{Z}[1/p])$ has a non-finitely presented homomorphic image is easy to see: by commutator and diagonal relations **(1.1)** and **(1.2)**, its center $Z(\mathbf{A}_4(\mathbb{Z}[1/p]))$ is the subgroup of upper right-most elementary matrices

$$Z(\mathbf{A}_4(\mathbb{Z}[1/p])) = \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, $Z(\mathbf{A}_4(\mathbb{Z}[1/p]))$ is isomorphic to the additive group $\mathbb{G}_a(\mathbb{Z}[1/p])$, which is infinitely generated. In particular, $\mathbf{A}_4(\mathbb{Z}[1/p])$ modulo its center can never be finitely presented. The tricky part of Abels' theorem thus consisted in proving that $\mathbf{A}_4(\mathbb{Z}[1/p])$ is finitely presented.

The example above promptly led to generalizations which also found applications in many areas, as highlighted in the introduction. The group schemes of Herbert Abels are the infinite family $\{\mathbf{A}_n\}_{n\geq 2}$ of matrix groups given by

$$\mathbf{A}_n := \begin{pmatrix} 1 & * & \cdots & \cdots & * \\ 0 & * & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * & * \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \leq \mathrm{GL}_n .$$

We observe that $\mathbf{A}_n(R)$ decomposes as a semi-direct product $\mathbf{A}_n(R) = \mathbf{U}_n(R) \rtimes \mathbf{T}_n(R)$, where

$$\mathbf{U}_n(R) = \begin{pmatrix} 1 & * & \cdots & \cdots & * \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & * \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \ \mathbf{T}_n(R) = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & * & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} = \mathbf{A}_n(R) \cap \mathbf{D}_n(R).$$

Just as with $\mathbf{A}_4(\mathbb{Z}[1/p])$, it is easy to see that the center of $\mathbf{A}_n(R)$ is the additive group $Z(\mathbf{A}_n(R)) = \mathbf{E}_{1n}(R) \cong \mathbb{G}_a(R)$ generated by all elementary matrices in the upper right corner.

The generalizations and applications mentioned above typically relate to, or make use of, presentations of Abels groups. In this chapter, we classify which such groups are finitely presented by proving the following, precise version of Theorem **A**.

Theorem 3.2 (Theorem **A**, restated)**.** *Let R be a commutative ring with unity. If R is* not *finitely generated as a ring, then $\phi(\mathbf{A}_n(R)) = 0$ for all $n \geq 2$. Otherwise, the following hold.*

i. $\phi(\mathbf{A}_2(R)) > 0$ *if and only if R is finitely generated* as a $\mathbb{Z}$-module, *in which case $\phi(\mathbf{A}_n(R)) = \phi(\mathbf{B}_2^\circ(R)) = \infty$ for all $n \geq 2$.*

ii. *If R is* infinitely generated *as a $\mathbb{Z}$-module, then $\phi(\mathbf{A}_3(R)) = \min\{1, \phi(\mathbf{B}_2^\circ(R))\}$.*

iii. *Suppose $n \geq 4$ and that R is infinitely generated as a $\mathbb{Z}$-module. Then $\phi(\mathbf{A}_n(R)) \leq \phi(\mathbf{B}_2^\circ(R))$ and, given $\ell \in \{1, 2\}$, one has that $\phi(\mathbf{B}_2^\circ(R)) \geq \ell$ implies $\phi(\mathbf{A}_n(R)) \geq \ell$.*

Parts **(ii)** and **(iii)** were established by Ralph Strebel independently in an unpublished manuscript [94]. Our proof is less algebraic than Strebel's and employs tools that were not available to him at the time, namely the early Σ-theory for metabelian groups due to Strebel himself and Bieri [20], and horospherical subgroups considered by Holz in his thesis [57].

The first claims of Theorem **3.2** are well-known and follow from standard methods:

Proof of Theorem **3.2** *(except for* **(iii)***).* If $\phi(\mathbf{A}_n(R)) > 0$ for some $n \geq 3$, then $\mathbf{A}_n(R)$ and its retract $\mathbb{G}_a(R) \rtimes \mathbb{G}_m(R)$ are finitely generated, where the action of $\mathbb{G}_m(R)$ on $\mathbb{G}_a(R)$ is given by multiplication. Thus, $\mathbb{G}_m(R)$ is a finitely generated abelian group and $\mathbb{G}_a(R)$ is finitely generated as a $\mathbb{Z}[\mathbb{G}_m(R)]$-module, which shows that R is finitely generated as a ring. This deals with the very first claim of the theorem, except possibly when $n = 2$. Now, if $\phi(\mathbf{A}_2(R)) > 0$, then $\mathbf{A}_2(R) \cong \mathbb{G}_a(R)$ is finitely generated as a $\mathbb{Z}$-module. This implies, for every $n \geq 2$, the following: the unipotent radical $\mathbf{U}_n(R)$ of $\mathbf{A}_n(R)$ is a finitely generated nilpotent group and thus has $\phi(\mathbf{U}_n(R)) = \infty$; the group of units $\mathbb{G}_m(R)$ of R is itself finitely generated by Samuel's generalization of Dirichlet's Unit Theorem [83, Section 4.7], whence the diagonal subgroups $\mathbf{D}_n(R)$ and $\mathbf{T}_n(R) \leq \mathbf{A}_n(R)$ also have $\phi(\mathbf{T}_n(R)) = \phi(\mathbf{D}_n(R)) = \infty$. Since $\mathbf{A}_n(R) = \mathbf{U}_n(R) \rtimes \mathbf{T}_n(R)$—and $\phi(\mathbf{B}_2^\circ(R)) = \phi(\mathbf{B}_2(R))$ by Lemma **2.2**—it follows from Corollary **1.14** that $\phi(\mathbf{A}_n(R)) = \phi(\mathbf{B}_2^\circ(R)) = \infty$.

Assume from now on that R is *not* finitely generated as a $\mathbb{Z}$-module. In this case, $\mathbf{A}_3(R)$ can never be finitely presented. Indeed, if $\mathbb{G}_m(R)$ is finite, then $\phi(\mathbf{A}_3(R)) = \phi(\mathbf{U}_3(R)) = 0$. In case $\mathbb{G}_m(R)$ has torsion-free rank at least one and if $\mathbf{A}_3(R)$ were finitely presented, then its metabelian quotient $\mathbf{A}_3(R)/Z(\mathbf{A}_3(R)) = \mathbf{A}_3(R)/\mathbf{E}_{13}(R)$ would also be finitely presented by [20, Corollary 5.6]. But the complement of the Σ-invariant [20] of the $\mathbb{Z}[\mathbf{T}_1(R)]$-module $\mathbf{U}_3(R)/\mathbf{E}_{13}(R) \cong \mathbf{E}_{12}(R) \oplus \mathbf{E}_{23}(R)$ is easily seen to contain antipodal points. This implies that $\mathbf{U}_3(R)/\mathbf{E}_{13}(R)$ is not tame as a $\mathbb{Z}[\mathbf{T}_1(R)]$-module, which contradicts [20, Theorem 5.1].

To prove the equality in **(ii)**, we first note that $\phi(\mathbf{A}_3(R)) = 0 = \phi(\mathbf{B}_2^\circ(R))$ if $\mathbb{G}_m(R)$ is finite or not finitely generated (e.g. as in the beginning of the proof of Lemma **2.2**). Assuming $\mathbb{G}_m(R)$ to be (infinite and) finitely generated, we see that $\mathbf{A}_3(R)$ retracts onto a group commensurable with $\mathbf{B}_2^\circ(R)$ just as in the end of the proof of Proposition **2.6** (case G_2 with η short). Thus, $\phi(\mathbf{A}_3(R)) \leq \phi(\mathbf{B}_2^\circ(R))$. □

The tricky part of Theorem **3.2** is thus **(iii)**. Its proof goes as follows. The first inequality follows from Theorem **2.1** because $\mathbf{A}_n(R)$, with $n \geq 4$, retracts onto the Borel subgroup $\mathbf{B}_2(R) \leq \mathrm{GL}_2(R)$, as shown below.

$$\mathbf{A}_n(R) = \begin{pmatrix} 1 & * & \cdots & \cdots & * \\ 0 & * & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & * & * \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \longrightarrow\!\!\!\!\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & * & * & 0 & \ddots & & \vdots \\ 0 & 0 & * & 0 & \ddots & \ddots & \vdots \\ \vdots & 0 & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & & \cdots & 0 & 0 & 1 \end{pmatrix} \cong \mathbf{B}_2(R).$$

It is also not hard to see that $\mathbf{A}_n(R)$ is finitely generated whenever $\mathbf{B}_2^\circ(R)$ is so. Indeed, $\phi(\mathbf{B}_2^\circ(R)) \geq 1$ implies that the subgroups

$\mathbf{E}_{ij}(R) \rtimes \mathbf{T}_n(R) \leq \mathbf{A}_n(R)$, with $1 \leq i < j \leq n$, admit finite generating sets; confer e.g. the last paragraph of the previous proof. But the unipotent part $\mathbf{U}_n(R) \leq \mathbf{A}_n(R)$ is itself generated by all elementary matrices $e_{ij}(r)$ for $r \in R$ and $1 \leq i < j \leq n$. Thus, given a finite set of generators for $\mathbf{B}_2^\circ(R)$, one can construct a finite set of generators for $\mathbf{U}_n(R) \rtimes \mathbf{T}_n(R) = \mathbf{A}_n(R)$.

Now, for each pair (n, R), where n is a natural number greater than three and R is a commutative ring with unity, we construct a finite-dimensional connected simplicial complex $CC(\mathscr{H}(n,R))$ on which $\mathbf{A}_n(R)$ acts cocompactly by cell-permuting homeomorphisms. Generalizing a result due to Stephan Holz, we show that the space $CC(\mathscr{H}(n,R))$ is always simply-connected. Using Σ-theory for metabelian groups [20], we prove that all cell stabilizers of the given action $\mathbf{A}_n(R) \curvearrowright CC(\mathscr{H}(n,R))$ are finitely presented whenever $\mathbf{B}_2^\circ(R)$ is so. We finish off the proof by invoking the following well-known criterion whose final form below is due to K. S. Brown.

Theorem 3.3 ([28])**.** *Let G be a group acting by cell-permuting homeomorphisms on a CW-complex X such that (a) all vertex-stabilizers are finitely presented; (b) all edge-stabilizers are finitely generated; and (c) the G-action on the 2-skeleton $X^{(2)}$ is cocompact. Then G is finitely presented.*

Section **3.1** is devoted to the construction of the space $CC(\mathscr{H}(n,R))$. The above mentioned properties of the action and of the space are shown in Sections **3.1.1** and **3.1.2**.

3.1 A space for $\mathbf{A}_n(R)$

Recall that a covering of a given set X is a collection of subsets $\{X_\lambda\}_{\lambda\in\Lambda}$ of X whose union is the whole of X, i.e. $X = \cup_{\lambda\in\Lambda} X_\lambda$. The *nerve of the covering* $\{X_\lambda\}_{\lambda\in\Lambda}$ is the simplicial complex $N(\{X_\lambda\}_{\lambda\in\Lambda})$ defined as follows. Its vertices are the sets X_λ for $\lambda \in \Lambda$, and $k+1$ vertices $X_{\lambda_0}, X_{\lambda_1}, \ldots, X_{\lambda_k}$ span a k-simplex whenever the intersection of all such X_{λ_i} is non-empty, i.e. $\cap_{i=0}^k X_{\lambda_i} \neq \varnothing$.

In [57, 5], Stephan Holz and Herbert Abels investigate nerve complexes attached to groups as follows. Fixing a family of subgroups, they take the nerve of the covering of the group by all cosets of subgroups of the given family. (Such spaces are also called *coset posets* or *coset complexes* in the literature.) More precisely, given a group G and a family $\mathscr{H} = \{H_\lambda\}_{\lambda\in\Lambda}$ of subgroups of G, let $\mathfrak{H}$ denote the covering $\mathfrak{H} = \{gH \mid g \in G,\ H \in \mathscr{H}\}$ of G by all (left) cosets of all members of $\mathscr{H}$. The coset complex $CC(\mathscr{H})$ is defined as the nerve of the covering $N(\mathfrak{H})$. In particular, if the family $\mathscr{H}$ is finite, one has that $CC(\mathscr{H})$ is $(|\mathscr{H}|-1)$-dimensional.

The inspiration for considering such spaces came primarily from the theory of buildings. For example, if G is a group with a BN-pair (G, B, N, S) [6, Chapter 6], then the coset poset $CC(\mathscr{H})$ associated to the family $\mathscr{H}$ of all maximal standard parabolic subgroups of G is by definition the building $\Delta(G,B)$ associated to the system (G,B,N,S); see e.g. [6, Section 6.2]. However, such complexes show up in many other contexts, e.g. Deligne complexes [40], Bass–Serre theory [86], Σ-invariants of right-angled Artin groups [71], and higher generating families of pure [35] and general braid groups [34].

Since the vertices of $CC(\mathscr{H})$ are (left) cosets of subgroups of G, it follows that G has a natural action on $CC(\mathscr{H})$ by cell-permuting homeomorphisms, namely the action induced by left multiplication on the cosets gH for $g \in G$ and $H \in \mathscr{H}$.

Going back to the groups of Abels, consider the following $\mathbb{Z}$-subschemes of $\mathbf{A}_n$ for $n \geq 4$.

$$H_1 = \begin{pmatrix} 1 & * & \cdots & * & 0 \\ 0 & * & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & * & \vdots \\ 0 & & 0 & * & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \; H_2 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & * & * & \cdots & * \\ \vdots & \ddots & \ddots & * & \vdots \\ 0 & & 0 & * & * \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \; \text{and } H_3 = \begin{pmatrix} 1 & * & 0 & \cdots & 0 \\ 0 & * & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & & 0 & * & * \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

For $n = 4$ we consider in addition the following the following $\mathbb{Z}$-subscheme.

$$H_4 = \begin{pmatrix} 1 & 0 & * & 0 \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The unipotent radicals of the matrix groups above—i.e. the intersections of each H_i with the group of upper unitriangular matrices $\mathbf{U}_n \leq \mathrm{GL}_n$—are examples of group schemes arising from (maximal) contracting subgroups; see e.g. [57, 3, 11]. To see this, consider the locally compact group $\mathbf{A}_n(\mathbb{K})$ for $\mathbb{K}$ a non-archimedean local field. In this case, each unipotent radical $\mathcal{U}_i(\mathbb{K}) = H_i(\mathbb{K}) \cap \mathbf{U}_n(\mathbb{K})$ is the contracting subgroup associated to the automorphism given by conjugation by some element t contained in the torus $\mathbf{T}_n(\mathbb{K})$. Holz shows [57, 2.7.3 and 2.7.4] that this defines a unipotent group scheme over $\mathbb{Z}$ depending on $t \in \mathbf{T}_n(\mathbb{K})$. Following Abels we call the schemes H_i above *horospherical* and their unipotent radicals $\mathcal{U}_i = H_i \cap \mathbf{U}_n$ *contracting* subgroups.

Let R be an arbitrary commutative ring with unity. For $n \geq 4$, let $\mathscr{H}(n,R)$ denote the family of (R-points of) horospherical subgroups of $\mathbf{A}_n(R)$ given by

$$\mathscr{H}(n,R) = \begin{cases} \{H_1(R),\, H_2(R),\, H_3(R),\, H_4(R)\}, & \text{if } n = 4; \\ \{H_1(R),\, H_2(R),\, H_3(R)\} & \text{otherwise.} \end{cases}$$

We also let

$$\mathfrak{H}(n,R) = \{gH \mid g \in \mathbf{A}_n(R), H \in \mathscr{H}(n,R)\}.$$

In the notation above, the space we shall consider is the nerve complex

$$CC(\mathscr{H}(n,R)) = N(\mathfrak{H}(n,R))$$

associated to the covering of $\mathbf{A}_n(R)$ by the left cosets $\mathfrak{H}(n,R)$ of the horospherical subgroups listed above. As mentioned previously, the group $\mathbf{A}_n(R)$ acts on the simplicial complex $CC(\mathscr{H}(n,R))$ by cell-permuting homeomorphisms via left multiplication on the vertices.

3.1.1 Fundamental domain and cell-stabilizers

The complex $CC(\mathscr{H}(n,R))$ has many useful features. Most of the facts we are about to list here and in Section **3.1.2** hold for arbitrary groups and coset complexes with similar properties. To be precise, we shall only need the facts that the chosen family $\mathscr{H}(n,R)$ is finite, the group $\mathbf{A}_n(R)$ is a split extension $\mathbf{A}_n(R) = \mathbf{U}_n(R) \rtimes \mathbf{T}_n(R)$, and the contracting subgroups $\mathcal{U}_i(R) = H_i(R) \cap \mathbf{U}_n(R)$ (as well as the intersections of any number of contracting subgroups) are all $\mathbf{T}_n(R)$-invariant.

Lemma 3.4. *The complex $CC(\mathscr{H}(n,R))$ is colorable and homogeneous. The given action of $\mathbf{A}_n(R)$ on $CC(\mathscr{H}(n,R))$ is type-preserving and cocompact. Any cell-stabilizer is isomorphic to a finite intersection of members of $\mathscr{H}(n,R)$.*

Proof. Since the intersection of cosets in a group is a coset of the intersection of the underlying subgroups, it follows that the complex $CC(\mathscr{H}(n,R))$ is *homogeneous*. That is to say, every simplex is contained in a simplex of dimension $k = |\mathscr{H}(n,R)| - 1$ and every maximal simplex has dimension exactly k. (Note that $CC(\mathscr{H}(n,R))$ is a chamber complex if $n \geq 5$.) We observe that $CC(\mathscr{H}(n,R))$ is colored with *types* (or *colors*) given precisely by the family of subgroups $\mathscr{H}(n,R)$. Also, the given action of $\mathbf{A}_n(R)$ on $CC(\mathscr{H}(n,R))$ is type-preserving and transitive on the set of maximal simplices of $CC(\mathscr{H}(n,R))$. Thus, the maximal simplex given by the intersection

$$\bigcap_{H \in \mathscr{H}(n,r)} H$$

is a fundamental domain for the $\mathbf{A}_n(R)$-action. In particular, since $|\mathscr{H}(n,R)|$ is finite, it follows that the action of $\mathbf{A}_n(R)$ is cocompact.

The stabilizers of the $\mathbf{A}_n(R)$-action are also easy to determine. For instance, given a maximal simplex $\sigma = \{g_1 H_1(R),\, g_2 H_2(R),\, g_3 H_3(R)\}$ in $CC(\mathscr{H}(n,R))$ with $n \geq 5$, there exists $g \in \mathbf{A}_n(R)$ such that $\sigma = g \cdot \{H_1(R),\, H_2(R),\, H_3(R)\}$. A group element $h \in \mathbf{A}_n(R)$ fixes σ if and only if $h \in g(H_1(R) \cap H_2(R) \cap H_3(R))g^{-1}$. A similar argument shows that a cell-stabilizer of $CC(\mathscr{H}(n,R))$ for any $n \geq 4$ is a conjugate of some intersection of subgroups that belong to the family $\mathscr{H}(n,R)$. □

The existence of a single simplex as fundamental domain and the property that cell-stabilizers are conjugates of finite intersections of members of a fixed family of subgroups actually characterize coset complexes; see e.g. [35, Observation A.4 and Proposition A.5].

Having determined the cell-stabilizers, we now prove that they are finitely presented whenever we need them to be.

Proposition 3.5. *Suppose $\mathbf{B}_2^{\circ}(R)$ is finitely presented and $n \geq 4$. Then any finite intersection of members of $\mathscr{H}(n,R)$ is finitely presented.*

Proof. We shall prove that, under the given assumption, the vertex-stabilizers are finitely presented. It will be clear from the arguments below that the same holds for stabilizers of higher dimensional cells.

By Lemma 3.4, we need only show that the members of $\mathscr{H}(n,R)$ are finitely presented. By the commutator **(1.1)** and diagonal relations **(1.2)**, we see that the 'last-column subgroup' $\mathcal{C}_{n-1}$ of H_1, given by

$$\mathcal{C}_{n-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & * & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots \\ \vdots & & \ddots & 1 & * & \vdots \\ & & & 0 & * & 0 \\ 0 & \cdots & & \cdots & 0 & 1 \end{pmatrix},$$

is normal in H_1. The quotient $H_1/\mathcal{C}_{n-1}$ is isomorphic to the subgroup

$$Q_{n-1} = \begin{pmatrix} 1 & * & \cdots & * & 0 & 0 \\ 0 & * & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & * & \vdots & \vdots \\ \vdots & & \ddots & * & 0 & \vdots \\ & & & 0 & 1 & 0 \\ 0 & \cdots & & \cdots & 0 & 1 \end{pmatrix} \leq H_1.$$

Now, the column subgroup $\mathcal{C}_{n-1}$ is itself finitely presented. Indeed, since $\mathbf{B}_2^{\circ}(R)$ is finitely presented, then so is the commensurable group

$$\mathbb{G}_a(R) \rtimes \mathbb{G}_m(R) \cong \{ \left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right) \} \leq \mathrm{GL}_2(R),$$

as in the end of the proof of Proposition 2.6 (case G_2 with η short). In particular, the $\mathbb{G}_m(R)$-module $\mathbb{G}_a(R)$ with the given action

$$\mathbb{G}_m(R) \times \mathbb{G}_a(R) \ni (u,r) \mapsto u^{-1}r$$

is tame [20, Theorem 5.1]. But $\mathcal{C}_{n-1}$ is isomorphic to $(\mathbb{G}_a(R))^{n-2} \rtimes \mathbb{G}_m(R)$, where the action of $\mathbb{G}_m(R)$ on each copy of the $\mathbb{G}_m(R)$-module $\mathbb{G}_a(R)$ is the multiplication shown above, and the action $\mathbb{G}_m(R) \curvearrowright (\mathbb{G}_a(R))^{n-2}$ is just the diagonal action. Thus, by [20, Proposition 2.5(i)] it follows that $(\mathbb{G}_a(R))^{n-2}$ is a tame $\mathbb{G}_m(R)$-module, which implies—again by [20, Theorem 5.1]—that

$\mathcal{C}_{n-1} \cong (\mathbb{G}_a(R))^{n-2} \rtimes \mathbb{G}_m(R)$ is finitely presented. We have shown that H_1 fits into a (split) short exact sequence

$$\mathcal{C}_{n-1} \hookrightarrow H_1 \twoheadrightarrow Q_{n-1}$$

where $\mathcal{C}_{n-1}$ is finitely presented. Decomposing Q_{n-1} similarly via the last column, as we did with H_1, a simple induction on n shows that Q_{n-1} itself is also finitely presented. It follows from Lemma **1.13** that H_1 is finitely presented.

By considering the 'first-row subgroup'

$$\mathcal{R}_{n-1} = \begin{pmatrix} 1 & 0 & & \cdots & & 0 \\ 0 & * & * & \cdots & * & * \\ \vdots & 0 & 1 & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ & & & 0 & 1 & 0 \\ 0 & \cdots & & \cdots & 0 & 1 \end{pmatrix}$$

of H_2, which is also normal by **(1.1)** and **(1.2)**, an entirely analogous argument using $\mathbb{G}_a(R) \rtimes \mathbb{G}_m(R) \cong \{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)\} \leq \mathrm{GL}_2(R)$ shows that H_2 is also finitely presented.

The case of H_3 is even easier since it is the direct product

$$H_3 = \begin{pmatrix} 1 & * & 0 & \cdots & 0 & 0 \\ 0 & * & 0 & \iddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots \\ \vdots & & \ddots & * & 0 & \vdots \\ & & & 0 & 1 & 0 \\ 0 & \cdots & & \cdots & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & 0 & \vdots \\ & & & 0 & * & * \\ 0 & \cdots & & \cdots & 0 & 1 \end{pmatrix},$$

and both factors on the right-hand side are finitely presented because both groups $\{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)\}, \{\left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right)\} \leq \mathrm{GL}_2(R)$ are so.

Establishing finite presentability of $H_4 \leq \mathbf{A}_4(R)$ is slightly different. Consider the subgroups

$$\Gamma_1 = \left(\begin{smallmatrix} 1 & 0 & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \text{ and } \Gamma_2 = \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)$$

and let $p_1 : \Gamma_1 \twoheadrightarrow Q$ and $p_2 : \Gamma_2 \twoheadrightarrow Q$ denote the natural projections onto the diagonal subgroup

$$Q = \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right).$$

With this notation, we have that H_4 is isomorphic to the fiber product

$$P = \{(g, h) \in \Gamma_1 \times \Gamma_2 \mid p_1(g) = p_2(h)\}.$$

We observe now that Γ_1 and Γ_2 are finitely presented—i.e. of homotopical type F_2—since

$$\ker(p_1) = \left(\begin{smallmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \cong \mathcal{C}_3 \leq H_1 \text{ and } \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \cong \{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)\} \leq \mathrm{GL}_2(R)$$

are so. (In particular, $\ker(p_1)$ is of type F_1.) Furthermore, the finite presentability of $\mathbf{B}_2^\circ(R)$ implies that Q is a finitely generated abelian group—in particular, it is of type F_3; cf. Section 1.2. Therefore, the fiber product $P \cong H_4$ is finitely presented by the (asymmetric) 1-2-3-Theorem [27, Theorem B].

Entirely analogous arguments for the non-trivial finite intersections of members of $\mathscr{H}(n,R)$ show that all such groups are also finitely presented, which concludes the proof of the proposition. □

3.1.2 Connectivity properties

The following observation is due to Stephan Holz. To prove it one considers the homotopy equivalences given in [5, Theorem 1.4].

Lemma 3.6 ([57, Korollar 5.18])**.** *Let $G = N \rtimes Q$ and suppose $\mathscr{H}$ is a family of Q-invariant subgroups of N. Then there exists a homotopy equivalence between the coset complex $CC(\mathscr{H})$ of $\mathscr{H}$* with respect to *N and the coset complex $CC(\{H \rtimes Q \mid H \in \mathscr{H}\})$* with respect to whole group *G.*

Corollary 3.7. *Let $\mathscr{H}_u(n,R)$ denote the family of unipotent radicals $\mathcal{U}_i(R) = H_i(R) \cap \mathbf{U}_n(R)$ of all members $H_i(R) \in \mathscr{H}(n,R)$ and write*

$$\mathfrak{H}_u(n,R) = \{v\mathcal{U}(R) \mid v \in \mathbf{U}_n(R), \mathcal{U}(R) \in \mathscr{H}_u(n,R)\}.$$

Then the spaces $CC(\mathscr{H}_u(n,R)) = N(\mathfrak{H}_u(n,R))$ (with respect to $\mathbf{U}_n(R)$) and $CC(\mathscr{H}(n,R))$ (with respect to $\mathbf{A}_n(R)$) are homotopy equivalent.

Proof. This follows at once from Lemma 3.6 since the $\mathbf{T}_n(R)$-action by conjugation preserves each $\mathcal{U}_i(R)$ by the diagonal relations (1.2). □

Thus, to show that $CC(\mathscr{H}(n,R))$ is connected and simply-connected, it suffices to prove that the coset complex $CC(\mathscr{H}_u(n,R))$ of contracting subgroups, with cosets taken *in the unipotent radical* $\mathbf{U}_n(R)$, is connected and simply-connected. To do so we take advantage of the algebraic meaning of connectivity properties of coset complexes.

Recall that the colimit $\mathrm{colim} F$ of a diagram $F : I \to \mathrm{Grp}$ from a small category I to the category of groups is a group K together with a family of maps $\Psi = \{\psi_O : F(O) \to K\}_{O \in \mathrm{Obj}(I)}$ satisfying the following properties.

- $\psi_P \circ F(f) = \psi_O$ for all $f \in \mathrm{Hom}(O,P)$;
- If (K', Ψ') is another pair also satisfying the conditions above, then there exists a unique group homomorphism $\varphi : K \to K'$ such that $\varphi \circ \psi_O = \psi'_O$ for all $O \in \mathrm{Obj}(I)$.

In this case we write $K = \operatorname{colim} F$, omitting the maps Ψ. Typical examples of colimits in the category of groups are amalgamated free products. In general, suppose $\mathscr{H}$ is a family of subgroups of a given group. This induces a diagram $F_{\mathscr{H}} : I_{\mathscr{H}} \to \mathrm{Grp}$ by defining the category $I_{\mathscr{H}}$ to be the poset given by members of $\mathscr{H}$ and their pairwise intersections, ordered by inclusion. For example, if $\mathscr{H} = \{A, B\}$ with $A, B \leq G$ and $C = A \cap B$, then $F_{\mathscr{H}}$ is just the usual diagram

$$\begin{array}{ccc} C & \hookrightarrow & B \\ \downarrow & & \\ A & & \end{array}$$

and the colimit $\operatorname{colim} F_{\mathscr{H}}$ is simply the push-out (or amalgamated product) $\operatorname{colim} F_{\mathscr{H}} = A *_C B$.

Theorem 3.8 (Abels–Holz [5, Thm. 2.4])**.** *Let $\mathscr{H}$ be a family of subgroups of a group G and let $\pi : \operatorname{colim} F_{\mathscr{H}} \to G$ denote the natural map from the colimit of $F_{\mathscr{H}}$ to G. Then the coset complex $CC(\mathscr{H})$ is connected if and only if π is surjective, and $CC(\mathscr{H})$ is additionally simply-connected if and only if π is an isomorphism.*

Using the above we will obtain the last ingredient to finish the proof of Theorem **3.2(iii)** once we establish the following generalization of a result due to Stephan Holz [57, Proposition A.3].

Proposition 3.9. *For every $n \geq 4$ one has that $\mathbf{U}_n(R) \cong \operatorname{colim} F_{\mathscr{H}_u(n,R)}$.*

Proof. The idea is to write down a convenient presentation for $\mathbf{U}_n(R)$ which shows that it is the desired colimit. To do so, we first write down presentations for the members of $\mathscr{H}_u(n, R)$. For the course of this proof we fix (and follow strictly) the notation of Lemma **1.3**. In particular, $T \subseteq R$ will denote an arbitrary, but fixed, additive generating set for $(R, +) = \mathbb{G}_a(R)$ containing 1. As in Lemma **1.3**, we fix $\mathcal{R}$ a set of additive defining relators of $\mathbb{G}_a(R)$.

We observe that $\mathcal{U}_3(R)$ and $\mathcal{U}_4(R)$ are abelian, by the commutator relations (**1.1**). It is also easy to see that $\mathcal{U}_1(R) \cong \mathbf{U}_{n-1}(R) \cong \mathcal{U}_2(R)$ by translating the indices of elementary matrices accordingly. Thus, we have the following presentations.

$$\mathcal{U}_1(R) = \langle \{e_{ij}(t) \mid t \in T,\ 1 \leq i < j \leq n-1\} \mid \text{ Relations (1.4) and (1.5)} \\ \text{for all } i, j \text{ with } 1 \leq i < j \leq n-1, \text{ and all } t, s \in T \rangle.$$

$$\mathcal{U}_2(R) = \langle \{e_{ij}(t) \mid t \in T, 2 \leq i < j \leq n\} \mid \text{ Relations (1.4) and (1.5)} \\ \text{for all } i, j \text{ with } 2 \leq i < j \leq n, \text{ and all } t, s \in T \rangle.$$

$$\begin{aligned}\mathcal{U}_3(R) =&\langle \{e_{12}(t), e_{n-1,n}(s) \mid t,s\in T\} \mid [e_{12}(t), e_{n-1,n}(s)] = 1 \text{ for all } t,s\in T,\\ &\text{and relations } (\mathbf{1.5}) \text{ for all } t,s\in T \text{ and } (i,j)\in\{(1,2),(n-1,n)\}\rangle.\end{aligned}$$

$$\begin{aligned}\mathcal{U}_4(R) =&\langle \{e_{13}(p),\, e_{23}(t),\, e_{24}(s) \mid p,t,s\in T\} \mid [e_{ij}(t), e_{kl}(s)] = 1 \text{ for all}\\ &t,s\in T \text{ and } (i,j),(k,l)\in\{(1,3),(2,3),(2,4)\}\,, \text{ and}\\ &\text{relations } (\mathbf{1.5}) \text{ for all } t,s\in T \text{ and } (i,j)\in\{(1,3),(2,3),(2,4)\}\rangle.\end{aligned}$$

The pairwise intersections $\mathcal{U}_i(R)\cap\mathcal{U}_j(R)$ also admit similar presentations by restricting the generators (and corresponding relations) to the indices occurring in both $\mathcal{U}_i(R)$ and $\mathcal{U}_j(R)$. For instance,

$$\mathcal{U}_1(R)\cap\mathcal{U}_2(R) = \begin{pmatrix} 1 & 0 & \cdots & & \cdots & 0 & 0\\ 0 & 1 & * & & \cdots & * & 0\\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots\\ & & 0 & 1 & & * & 0\\ & & & 0 & & 1 & 0\\ 0 & \cdots & & & \cdots & 0 & 1\end{pmatrix} \cong \mathbf{U}_{n-2}(R),$$

with presentation

$$\begin{aligned}\mathcal{U}_1(R)\cap\mathcal{U}_2(R) =&\langle \{e_{ij}(t) \mid t\in T,\, 2\le i<j\le n-1\} \mid \text{ Relations } (\mathbf{1.4}) \text{ and}\\ &(\mathbf{1.5}) \text{ for all } i,j \text{ with } 2\le i<j\le n-1, \text{ and all } t,s\in T\rangle.\end{aligned}$$

Now consider the group U_n defined as follows. As generating set we take

$$\mathcal{X}_n = \{e_{ij}(t),\, e_{kn}(s) \mid 1\le i<j\le n-1,\ 2\le k\le n, \text{ and } t,s\in T\}\,.$$

The set of defining relators $\mathcal{S}_n$ is formed as follows. For all $t,s\in T$ and indices i,j,k,l which are *either all* in $\{1,\dots,n-1\}$ *or all* in $\{2,\dots,n\}$, consider the relations

$$[e_{ij}(t), e_{kl}(s)] = \begin{cases}\prod\limits_u e_{il}(u)^{a_u}, & \text{if } j=k;\\ 1, & \text{if } i\ne l, k\ne j,\end{cases} \tag{3.1}$$

and

$$[e_{12}(t), e_{n-1,n}(s)] = 1, \tag{3.2}$$

where $m(t,s) = \sum\limits_u a_u u \in \bigoplus\limits_{t\in T}\mathbb{Z}t$ is as in Lemma **1.3**. For all $t,s\in T$ and pairs i,j which are *either all* in $\{1,\dots,n-1\}$ *or all* in $\{2,\dots,n\}$, consider additionally the relations

$$\prod_{\ell=1}^{m} e_{ij}(t_\ell)^{a_\ell} = 1 \text{ for each } \sum_{\ell=1}^{m} a_\ell t_\ell\in\mathcal{R}, \tag{3.3}$$

where $\mathcal{R}$ is the fixed set of additive defining relators of $\mathbb{G}_a(R)$ as in Lemma **1.3**. If $n=4$ we need also consider the relations

$$[e_{13}(t), e_{24}(s)] = 1 \tag{3.4}$$

for all pairs $t, s \in T$. We take $\mathcal{S}_n$ to be the set of all relations (**3.1**), (**3.2**), and (**3.3**) (in case $n \geq 5$), and $\mathcal{S}_4$ is the set of all relations (**3.1**) through (**3.4**) above with $n = 4$. We then define U_n by means of the presentation

$$U_n = \langle \mathcal{X}_n \mid \mathcal{S}_n \rangle.$$

Reading off the presentations for the $\mathcal{U}_i(R)$ and for their pairwise intersections, it follows from von Dyck's theorem that $\operatorname{colim} F_{\mathscr{H}_u(n,R)}$ is isomorphic to the group U_n above.

Thus, to finish the proof of the proposition, it suffices to show that $\mathbf{U}_n(R)$ is isomorphic to U_n. To avoid introducing even more symbols and repeating familiar arguments in order to explicitly construct the obvious isomorphism, we proceed as follows. Recall that $\mathbf{U}_n(R)$ admits the presentation $\mathbf{U}_n(R) = \langle \mathcal{Y} \mid \mathcal{S} \rangle$ given in Lemma **1.3**. Abusing notation and comparing the presentations $U_n = \langle \mathcal{X}_n \mid \mathcal{S}_n \rangle$ and $\mathbf{U}_n(R) = \langle \mathcal{Y} \mid \mathcal{S} \rangle$, it suffices to define in U_n the missing generators $e_{1n}(t)$ (for $t \in T$) and also show that all the relations from $\mathcal{S}$ missing from $\mathcal{S}_n$ do hold in U_n. (Inspecting the indices, the missing relations are the ones involving the commutators $[e_{1j}(t), e_{kn}(s)]$ for $j = 2, \ldots, n$ and $k = 1, \ldots, n-1$, and $(j,k) \neq (2, n-1)$.)

For every $t \in T$, define in U_n the element $e_{1n}(t) = [e_{12}(t), e_{2n}(1)]$. With this new commutator at hand, the proof will be concluded once we show that the following equalities hold in U_n.
For all $s, t \in T$ and $j, k \in \{2, \ldots, n-1\}$ with $j \neq k$ and $(j,k) \neq (2, n-1)$,

$$[e_{1j}(t), e_{kn}(s)] = 1. \tag{3.5}$$

For all $t \in T$ and $j \in \{2, \ldots, n-1\}$,

$$[e_{1j}(t), e_{jn}(1)] = [e_{1j}(1), e_{jn}(t)] = e_{1n}(t). \tag{3.6}$$

For all $t, s \in T$ and i, j with $1 \leq i < j \leq n$,

$$[e_{ij}(t), e_{1n}(s)] = 1. \tag{3.7}$$

For all $\sum_{\ell=1}^{m} a_\ell t_\ell \in \mathcal{R}$,

$$\prod_{\ell=1}^{m} e_{1n}(t_\ell)^{a_\ell} = 1. \tag{3.8}$$

Relation (3.5) holds: If $n = 4$ there is nothing to show, since in this case the only equation to verify is $[e_{13}(t), e_{24}(s)] = 1$, which holds by (**3.4**). Assume $n \geq 5$. We first observe that

$$[e_{1j}(t), e_{n-1,n}(s)] = 1 \tag{3.9}$$

for all $t, s \in T$ and $j \in \{3, \dots, n-2\}$ since

$$\begin{aligned} e_{1j}(t)e_{n-1,n}(s) &\overset{(3.1)}{=} e_{12}(t)e_{2j}(1)e_{12}(t)^{-1}e_{2j}(1)^{-1}e_{n-1,n}(s) \\ &\overset{(3.1)}{=} e_{12}(t)e_{2j}(1)e_{12}(t)^{-1}e_{n-1,n}(s)e_{2j}(1)^{-1} \\ &\overset{(3.2)}{=} e_{12}(t)e_{2j}(1)e_{n-1,n}(s)e_{12}(t)^{-1}e_{2j}(1)^{-1} \\ &\overset{(3.1)}{=} e_{12}(t)e_{n-1,n}(s)e_{2j}(1)e_{12}(t)^{-1}e_{2j}(1)^{-1} \\ &\overset{(3.2)}{=} e_{n-1,n}(s)e_{12}(t)e_{2j}(1)e_{12}(t)^{-1}e_{2j}(1)^{-1} \\ &\overset{(3.1)}{=} e_{n-1,n}(s)e_{1j}(t). \end{aligned}$$

Proceeding similarly, we conclude that

$$[e_{12}(t), e_{kn}(s)] = 1 \tag{3.10}$$

for all $t, s \in T$ and $k \in \{3, \dots, n-2\}$. Now suppose $j < k$. Then

$$[e_{1j}(t), e_{kn}(s)] \overset{(3.1)}{=} [e_{1j}(t), [e_{k,n-1}(s), e_{n-1,n}(1)]] = 1$$

because $e_{1j}(t)$ commutes with $e_{n-1,n}(1)$, by **(3.9)**, and with $e_{k,n-1}(s)$, by **(3.1)**. Analogously, if $j > k$, then

$$[e_{1j}(t), e_{kn}(s)] \overset{(3.1)}{=} [[e_{12}(t), e_{2j}(1)], e_{kn}(s)] = 1$$

by **(3.1)** and **(3.10)**. Thus, the relations **(3.5)** hold in U_n.

Relation (3.6) holds: To check **(3.6)** we need Lemma 1.2. First,

$$[e_{12}(t), e_{2n}(1)] \overset{(3.1)}{=} [e_{12}(t), [e_{23}(1), e_{3n}(1)]].$$

Setting $a = e_{12}(t)$, $b = e_{23}(1)$, and $c = e_{3n}(1)$, Hall's identity yields

$$\begin{aligned} 1 &= [cac^{-1}, [b, c]] \cdot [bcb^{-1}, [a, b]] \cdot [aba^{-1}, [c, a]] \\ &\overset{(3.1)}{=} [e_{12}(t), e_{2n}(1)] \cdot [e_{23}(1)e_{3n}(1)e_{23}(1)^{-1}, e_{13}(t)] \\ &\overset{(3.1)}{=} [e_{12}(t), e_{2n}(1)] \cdot [e_{2n}(1)e_{3n}(1), e_{13}(t)] \\ &\overset{(1.3)}{=} [e_{12}(t), e_{2n}(1)] \cdot e_{2n}(1) \cdot [e_{3n}(1), e_{13}(t)] \cdot e_{2n}(1)^{-1} \cdot [e_{2n}(1), e_{13}(t)] \\ &\overset{(3.1)\&(3.5)}{=} [e_{12}(t), e_{2n}(1)] \cdot [e_{3n}(1), e_{13}(t)], \end{aligned}$$

that is, $e_{1n}(t) = [e_{13}(t), e_{3n}(1)]$. On the other hand,

$$[e_{12}(1), e_{2n}(t)] \overset{(3.1)}{=} [e_{12}(1), [e_{23}(t), e_{3n}(1)]].$$

Setting $a = e_{12}(1)$, $b = e_{23}(t)$, and $c = e_{3n}(1)$, Hall's identity and **3.1** yield

$$\begin{aligned}
1 &= [e_{12}(1), e_{2n}(t)] \cdot [e_{23}(t)e_{3n}(1)e_{23}(1)^{-1}, e_{13}(t)] \\
&\overset{(3.1)}{=} [e_{12}(1), e_{2n}(t)] \cdot [e_{2n}(t)e_{3n}(1), e_{13}(t)] \\
&\overset{(1.3)}{=} [e_{12}(1), e_{2n}(t)] \cdot e_{2n}(t) \cdot [e_{3n}(1), e_{13}(t)] \cdot e_{2n}(t)^{-1} \cdot [e_{2n}(t), e_{13}(t)] \\
&\overset{(3.1)\&(3.5)}{=} [e_{12}(1), e_{2n}(t)] \cdot [e_{3n}(1), e_{13}(t)].
\end{aligned}$$

The last product above equals $[e_{12}(1), e_{2n}(t)]e_{1n}(t)^{-1}$ by the previous computations. We have thus proved that

$$e_{1n}(t) \overset{\text{Def.}}{=} [e_{12}(t), e_{2n}(1)] = [e_{12}(1), e_{2n}(t)] = [e_{13}(t), e_{3n}(1)].$$

Since $[e_{12}(1), e_{2n}(t)]$ also equals $[e_{12}(1), [e_{23}(1), e_{3n}(t)]]$, again by **(3.1)**, computations similar to the above also yield $[e_{13}(t), e_{3n}(1)] = [e_{13}(1), e_{3n}(t)]$. Entirely analogous arguments show that

$$[e_{1j}(t), e_{jn}(1)] = [e_{1j}(1), e_{jn}(t)] = e_{1n}(t)$$

for all $j \in \{2, \ldots, n-1\}$.

Relations (3.7) hold: We now prove that the subgroup $Z := \langle \{e_{1n}(t) \mid t \in T\} \rangle \leq U_n$ is central. Let $t, s \in T$ and let i, j be such that $1 \leq i < j \leq n$. We want to show that $e_{1n}(t)$ and $e_{ij}(s)$ commute in U_n. To begin with,

$$[e_{1n}(t), e_{1n}(s)] \overset{(3.6)}{=} [[e_{12}(t), e_{2n}(1)], [e_{13}(s), e_{3n}(1)]] \overset{(3.1)}{=} 1,$$

i.e. Z is abelian. If $i = 1$ and $j \neq n$, then $j \geq 2$ and we can pick $k \in \{2, \ldots, n-1\}$ such that $k \neq j$ because $n \geq 4$, yielding

$$\begin{aligned}
e_{1n}(t)e_{1j}(s) \overset{(3.6)}{=} [e_{1k}(t), e_{kn}(1)]e_{1j}(s) &\overset{(3.5)\&(3.1)}{=} e_{1j}(s)[e_{1k}(t), e_{kn}(1)] \\
&\overset{(3.6)}{=} e_{1j}(s)e_{1n}(t).
\end{aligned}$$

Similarly, if $j = n$ and $i \neq 1$, choose $k \in \{2, \ldots, n-1\}$ such that $k \neq i$. We obtain

$$[e_{1n}(t), e_{in}(s)] \overset{(3.6)}{=} [[e_{1k}(t), e_{kn}(1)], e_{in}(s)] \overset{(3.1)\&(3.5)}{=} 1.$$

It remains to prove $[e_{1n}(t), e_{ij}(s)] = 1$ for $1 < i < j < n$. In this case,

$$\begin{aligned}
e_{1n}(t)e_{ij}(s) &\overset{(3.6)}{=} e_{1i}(1)e_{in}(t)e_{1i}(1)^{-1}e_{in}(t)^{-1}e_{ij}(s) \\
&\overset{(3.1)}{=} e_{1i}(1)e_{in}(t)e_{1i}(1)^{-1}e_{ij}(s)e_{in}(t)^{-1} \\
&\overset{(3.3)\&(3.1)}{=} e_{1i}(1)e_{in}(t)e_{1j}(s)^{-1}e_{ij}(s)e_{1i}(1)^{-1}e_{in}(t)^{-1} \\
&\overset{(3.5)\&(3.1)}{=} e_{1j}(s)^{-1}e_{1i}(1)e_{in}(t)e_{ij}(s)e_{1i}(1)^{-1}e_{in}(t)^{-1} \\
&\overset{(3.1)}{=} e_{1j}(s)^{-1}e_{1i}(1)e_{ij}(s)e_{in}(t)e_{1i}(1)^{-1}e_{in}(t)^{-1} \\
&\overset{(3.3)\&(3.1)}{=} e_{1j}(s)^{-1}e_{1j}(s)e_{ij}(s)e_{1i}(1)e_{in}(t)e_{1i}(1)^{-1}e_{in}(t)^{-1} \\
&\overset{(3.6)}{=} e_{ij}(s)e_{1n}(t).
\end{aligned}$$

Thus, relations **(3.7)** hold for all $t, s \in T$ and i, j with $1 \leq i < j \leq n$.

Relations (3.8) hold: Given any pair $t, s \in T$,

$$\begin{aligned}
[e_{12}(t)e_{12}(s), e_{2n}(1)] &\overset{(1.3)}{=} e_{12}(t)[e_{12}(s), e_{2n}(1)]e_{12}(t)^{-1}[e_{12}(t), e_{2n}(1)] \\
&\overset{(3.6)\&(3.7)}{=} [e_{12}(s), e_{2n}(1)][e_{12}(t), e_{2n}(1)] \\
&\overset{(3.6)\&(3.7)}{=} [e_{12}(t), e_{2n}(1)][e_{12}(s), e_{2n}(1)]. \qquad (3.11)
\end{aligned}$$

Now let $\sum_{\ell=1}^{m} a_\ell t_\ell \in \mathcal{R}$ be an additive defining relator in R—recall that $t_\ell \in T$ and $a_\ell \in \mathbb{Z}$ as in Lemma **1.3**. By induction on $\sum_{\ell=1}^{m} |a_\ell|$ and **(3.11)**, it follows that

$$\prod_{\ell=1}^{m} e_{1n}(t_\ell)^{a_\ell} \overset{\text{Def.}}{=} \prod_{\ell=1}^{m} ([e_{12}(t_\ell), e_{2n}(1)])^{a_\ell} \overset{(3.11)}{=} \left[\prod_{\ell=1}^{m} e_{12}(t_\ell)^{a_\ell}, e_{2n}(1)\right] \overset{(3.1)}{=} 1.$$

Since the relations **(3.5)** through **(3.8)** missing from the presentation for $\mathbf{U}_n(R)$ from Lemma **1.3** also hold in U_n, it follows that U_n and $\mathbf{U}_n(R)$ are isomorphic, as claimed. □

3.2 Remarks on the finiteness lengths of Abels' groups

Recall that Theorem **2.1** guarantees that the finiteness lengths of many soluble linear groups (including Abels groups) do not necessarily increase with the size of their matrices. A follow-up question is: which properties intrinsic to the underlying group scheme $\mathbf{G}$ can affect the finiteness length of $\mathbf{G}(R)$? Theorem **3.2** and the other results mentioned in this section point out that, in order to have better finiteness properties on Abels' groups, the rank $n-2$ of the torus $\mathbf{T}_n \leq \mathbf{A}_n$ should be large enough, which in this case implies that the size of the matrices also grows, by the very definition of the group scheme $\mathbf{A}_n$.

It is in general non-trivial to determine which properties a matrix group $\mathbf{G}$ must have so that $\phi(\mathbf{G}(R)) \gg 0$ for any R. In the S-arithmetic set-up, Abels groups provide series of groups with increasing finiteness properties. The following example is essentially due to Abels and Holz. Since it does not appear in the literature in the general form presented below, we include for convenience a quick proof of the desired properties.

***Example* 3.10.** Let $\mathcal{O}_S$ be a Dedekind domain of arithmetic type and suppose $\mathrm{char}(\mathcal{O}_S) = 0$. Then $\phi(\mathbf{A}_n(\mathcal{O}_S)) = n-2$.

Proof. By the Kneser–Tiemeyer local-global principle [98, Thm. 3.1], we may assume that S contains a single non-archimedean place. By restriction of scalars (see e.g. [69, Lemma 3.1.4]), it suffices to consider the case where $\mathrm{Frac}(\mathcal{O}_S) = \mathbb{Q}$. In this set-up, $\mathcal{O}_S$ is of the form $\mathcal{O}_S = \mathbb{Z}[1/p]$ for some prime number $p \in \mathbb{N}$. Now, $\phi(\mathbf{A}_n(\mathbb{Z}[1/p])) < n-1$, for otherwise it would be of homological type FP_{n-1} and thus of type FP_∞ by [17, Proposition]. In particular, its center $Z(\mathbf{A}_n(\mathbb{Z}[1/p])$ would be finitely generated by [17, Corollary 2]. However, $Z(\mathbf{A}_n(\mathbb{Z}[1/p])$ is the elementary subgroup $\mathbf{E}_{1n}(\mathbb{Z}[1/p]) \cong \mathbb{G}_a(\mathbb{Z}[1/p])$, which is not finitely generated, yielding a contradiction. Finally, $\phi(\mathbf{A}_n(\mathbb{Z}[1/p])) \geq n-2$ by [4, Theorem B] and Brown's criteria (Theorem **3.3** and [29, Proposition 1.1]). ☐

After K. S. Brown's work [29], one can show that the finiteness length of an Abels group $\mathbf{A}_n(R)$ grows by proving that $CC(\mathscr{H}(n,R))$ is highly connected. However, to obtain the example above we made no use of coset complexes whatsoever; see also Witzel's results on generalizations of S-arithmetic Abels groups in characteristic zero [103]. Given the current state of knowledge, there are no general methods to deal with connectivity properties of arbitrary coset complexes. What typically happens is an ad-hoc analysis of the combinatorics of the given complex followed by an application of fiber-type arguments to obtain the desired properties; see [5, 35, 34] for examples.

The following are low-dimensional arithmetic examples that *do* use coset complexes, the first of which is already covered by Example **3.10**.

***Example* 3.11** (Holz)**.** Let $\mathcal{O}_S$ be a Dedekind domain of arithmetic type with $\mathrm{char}(\mathcal{O}_S) = 0$ and suppose $n \leq 5$. Then $\phi(\mathbf{A}_n(\mathcal{O}_S)) \geq n-2$.

Proof. Just as in **3.10** we may assume that $\mathcal{O}_S = \mathbb{Z}[1/p]$ for some prime p. In this case, it follows from [57] and [5, Corollary 3.14] that the coset complex associated to all horospherical subgroups of $\mathbf{A}_n(\mathbb{Z}[1/p])$ is: connected if

$n \geq 3$; simply-connected if $n \geq 4$; and 2-connected if $n = 5$. The cell-stabilizers all have unbounded finiteness lengths, e.g. by [98, Theorems 4.3 and 3.1]. The claim thus follows from [5, Proposition 4.7]. □

***Example* 3.12.** Suppose now that $\mathcal{O}_S$ is a Dedekind domain of arithmetic type with $\mathrm{char}(\mathcal{O}_S) > 0$. If $|S| \leq 3$, then $\phi(\mathbf{A}_n(\mathcal{O}_S)) = \min\{n-2, \phi(\mathbf{B}_2^\circ(\mathcal{O}_S))\}$.

Proof. This follows immediately from Theorem **3.2** and Corollary **2.11**. □

It should be stressed that, if $\mathrm{char}(\mathcal{O}_S) = 0$, then $\phi(\mathbf{B}_2^\circ(\mathcal{O}_S)) = \infty$ by [98, Corollary 4.5]. Theorem **3.2** and the S-arithmetic examples above provide supporting evidence for the following.

Conjecture 3.13. *Suppose R is a finitely generated commutative ring with unity which is* infinitely generated *as a $\mathbb{Z}$-module. Then, for all $n \geq 2$,*

$$\phi(\mathbf{A}_n(R)) = \min\{n-2,\ \phi(\mathbf{B}_2^\circ(R))\}\,.$$

It would be interesting to develop tools to analyze the topology and geometry of the complexes $CC(\mathscr{H}(n,R))$ associated to Abels groups for arbitrary base rings R. (We remark that the construction of the spaces $CC(\mathscr{H}(n,-))$ above, for fixed n, is functorial.) For instance, what can be said about the second homology of $CC(\mathscr{H}(4,R))$ over $\mathbb{Q}$ for arbitrary R? (Example **3.10** shows that $CC(\mathscr{H}(4,\mathcal{O}_S))$ cannot be 2-connected.) Theorem **3.2** can be interpreted as a low-dimensional attempt to obtain general statements about $CC(\mathscr{H}(n,-))$.

3.3 About the proof of Theorem 3.2

Our necessary condition in Theorem **3.2(iii)**—that is, the finite presentability of $\mathbf{B}_2^\circ(R)$—slightly differs from Strebel's necessary condition [94], though one can easily check that they are equivalent. Also, although Strebel's original theorem is slightly more general in that he considers groups of the form

$$A_n(R,Q) := \left\{g \in \mathbf{A}_n(R) \mid \text{the diagonal entries of } g \text{ belong to } Q \leq R^\times\right\},$$

our proof carries over to his case as well as long as one replaces the necessary condition on $\mathbf{B}_2^\circ(R)$ by "the group

$$\{\left(\begin{smallmatrix} \diamond & * \\ 0 & \diamond^{-1} \end{smallmatrix}\right) \in \mathrm{SL}_2(R) \mid * \in R, \diamond \in Q\}$$

is finitely presented."

The author was unable to prove geometrically that the complex $CC(\mathscr{H}(n,R))$ of Section **3.1** is simply-connected. The algebraic proof given here, whose main technical ingredient is Proposition 3.9, is the step whose methods are more similar to those of Strebel's in [94]. There are two key differences between the proofs.

Firstly, under the assumption that $\mathbb{G}_a(R) \rtimes Q$ is finitely presented, Strebel gives a unified construction of *concrete* presentations of the groups $A_n(R, Q)$ above which need the hypothesis $n \geq 4$. Presentations of $\mathbf{A}_n(R)$ using our methods can of course be extracted by using [28, Theorem 1] or more directly by starting with the presentation from Proposition 3.9 and combining it with a presentation of the torus $\mathbf{T}_n(R)$ forming the semi-direct product $\mathbf{U}_n(R) \rtimes \mathbf{T}_n(R)$. However, Strebel's proof has an advantage in that the above mentioned presentations derived from [28, 57] are somewhat cumbersome and none of them is as clean as Strebel's presentation.

Secondly, our proof of Proposition 3.9 drawing from Holz's ideas [57, Anhang] has a slight advantage in that it suggests that there is a K-theoretical phenomenon behind the finiteness length of Abels groups. It is known that $\mathrm{GL}_n(R)$ and $\mathrm{SL}_n(R)$ are typically finitely generated (resp. finitely presented) whenever n is large enough or the base ring has good low-dimensional K-groups [56]. In spirit, a large rank n gives one enough space in $\mathrm{GL}_n(R)$ to work with elementary matrices via commutator calculus. The same happens with the elementary matrices in $\mathbf{A}_n(R)$: while Strebel and Holz need the hypothesis $n \geq 4$ for their results, Holz observes that one can spare some generators and some relations for $\mathbf{A}_n(R)$ in the case $n \geq 5$ in comparison to $\mathbf{A}_4(R)$. This observation is incorporated in our proof and is the reason why the space $CC(\mathscr{H}(n, R))$ used here is 3-dimensional for $n = 4$ but 2-dimensional for $n \geq 5$; compare the proof of Proposition 3.9 for $n = 4$ and $n \geq 5$.

The use of coset complexes here is heavily inspired by the theory of Abels and Holz [2, 57, 3, 5], which in turn was inspired by Tits buildings. In [5, Section III], Abels and Holz show how their constructions generalize buildings associated to BN-pairs and recover, in particular, results on connectivity and amalgams. Proposition 3.9 also has parallels with phenomena occurring for unipotent groups over fields; see, for instance, the introduction of [49] for an overview on recent results. It would be interesting to construct further examples of unitriangular groups over arbitrary rings that admit 'building-like' coset complexes.

Chapter 4

Presentations of parabolic subgroups

In the structure theory of groups of Lie type and algebraic groups, their parabolic subgroups play an important role; see e.g. [99, 48, 68]. The parabolics of classical matrix groups can be characterized differently depending on the base ring [97, 100, 22]. Suppose the base ring is a field and $\mathcal{G}$ is classical. A subgroup $\mathcal{P} \leq \mathcal{G}$ is called parabolic whenever the variety $\mathcal{G}/\mathcal{P}$ is complete. In the theory of buildings, standard parabolic subgroups arise as stabilizers of panels of a fixed fundamental chamber and are intimately related to parabolic subgroups of the corresponding Weyl group. Equivalently, $\mathcal{P}$ is called parabolic if it contains a Borel subgroup. (Over fields, Borel subgroups are precisely the maximal, connected, soluble algebraic subgroups of $\mathcal{G}$.) In particular, parabolic subgroups always contain a maximal split torus.

In this work, we consider analogues of the parabolics above which are most easily described via sets of simple roots. Recall that a classical group $\mathcal{G} \leq \mathrm{GL}_n$ has a corresponding root system Φ with respect to an arbitrary, but fixed, maximal split torus $\mathcal{H} \leq \mathcal{G}$. As pointed out in Chapter 1, if $\mathcal{G} = \mathrm{GL}_n$ itself, the standard torus is $\mathcal{H} = \mathbf{D}_n$, the diagonal subgroup. (Accordingly, if $\mathcal{G} = \mathrm{SL}_n$, then $\mathcal{H} = \mathbf{D}_n \cap \mathrm{SL}_n$.) Given a subset of roots $X \subseteq \Phi$ we let $\Phi_X \subseteq \Phi$ denote the root subsystem generated by X, that is, $\Phi_X = \mathrm{span}_{\mathbb{Z}}(X) \cap \Phi \subseteq \mathbb{R}^{\mathrm{rk}(\Phi)}$. Let us furthermore fix $\Delta \subset \Phi$ a set of simple roots.

Definition 4.1. With the notation above, a *standard parabolic subgroup* $\mathcal{P}(R) = \mathcal{P}_I(R)$ of a classical group $\mathcal{G}(R)$ is a group of the form

$$\mathcal{P}_I(R) = \langle \mathcal{H}(R), \mathfrak{X}_\delta(R) \,:\, \delta \in \Phi^+ \cup \Phi_I \rangle$$

for some subset of *simple* roots $I \subseteq \Delta$. (In particular, $\mathcal{P}_I(R)$ is contained in the elementary subgroup of $\mathcal{G}(R)$; cf. Chapter 1.)

We remark that both the elementary subgroup and the standard Borel subgroup of $\mathcal{G}(R)$ are themselves parabolic subgroups. More precisely, the elementary subgroup is the parabolic $\mathcal{P}_{\Delta}(R)$, whereas the *standard Borel subgroup* is $\mathcal{P}_{\varnothing}(R)$. Both such groups shall also be called the *trivial* standard parabolic subgroups of $\mathcal{G}(R)$. If $\mathcal{G} = \mathrm{GL}_n$ (resp. SL_n), the trivial parabolics are $\mathcal{P}_{\varnothing}(R) = \mathbf{B}_n(R)$ and $\mathcal{P}_{\Delta}(R) = \mathrm{GE}_n(R)$ (resp. $\mathcal{P}_{\varnothing}(R) = \mathbf{B}_n^{\circ}(R)$ and $\mathcal{P}_{\Delta}(R) = E_n(R)$).

The groups from Definition 4.1 generalize parabolics of classical groups—as defined in the beginning of this chapter—in the following sense: if the base ring R is a field, then any $\mathcal{P}_I(R)$ as above is (the group of R-points of) a parabolic subgroup of a classical group and, conversely, any parabolic subgroup of a classical group over the field R is conjugate to an algebraic subgroup of $\mathcal{G}$ the form $\mathcal{P}_I$ for some set I of simple roots; cf. [22, Chapter IV] or [48, Exposé XXVI].

The main result of this chapter is the following lower bound on the finiteness length of standard parabolics.

Theorem 4.2 (Theorem B, restated)**.** *Let $\mathcal{G}$ be a classical group with underlying (reduced, irreducible) root system Φ, let $I \subset \Phi$ be a set of simple roots, and suppose the triple (R, Φ, I) is* **QG***. Given a standard parabolic subgroup $\mathcal{P}_I(R) \leq \mathcal{G}(R)$, let $\mathcal{LE}_I(R)$ denote its extended Levi factor; cf. Definition 4.5. Then $\phi(\mathcal{P}_I(R)) \geq 2$ if and only if $\phi(\mathcal{LE}_I(R)) \geq 2$.*

Theorem 4.2 still needs some explanation. Let (R, Φ, I) be as in the statement. The ring R is said to be 'not very bad' (see e.g. [90, 9, 91]) for the root system Φ—or **NVB** for short—whenever the following holds.

$$\begin{cases} 2 \in R^{\times} & \text{if } \Phi \in \{\mathrm{B}_n, \mathrm{C}_n, \mathrm{F}_4\}, \\ 2, 3 \in R^{\times} & \text{if } \Phi = \mathrm{G}_2. \end{cases}$$

Imposing the **NVB** condition assures that the commutator formulae (1.8) of Chevalley do not degenerate and that the structure constants occurring in the commutators are all invertible. This is an optimal set-up when working with commutator calculus in classical groups. It often happens that dropping the **NVB** condition yields similar—or sometimes the same—results, though in such a case one might have to go through painful case-by-case verifications with degenerate commutator formulae. For our purposes, there are other means to avoid technicalities with structure constants besides the **NVB** condition. We say that the triple (R, Φ, I) is 'quite good'—abbreviated **QG**—if any of the following two conditions hold.

- **i.** $\mathbf{B}_2^{\circ}(R)$ is finitely presented; or
- **ii.** $\mathbf{B}_2^{\circ}(R)$ is finitely generated, R is **NVB** for Φ, and $I \neq \{\alpha\}$ in the case where $\Phi = \mathrm{G}_2$ and $\alpha \in \mathrm{G}_2$ is long.

Imposing finite presentability of $\mathbf{B}_2^\circ(R)$ completely eliminates the problems we would have to deal with regarding characteristic and structure constants, though such condition might be considered too restrictive. The assumption that $\mathbf{B}_2^\circ(R)$ is finitely generated is made to ensure that all parabolics that we consider are finitely generated, which is a necessary assumption for finite presentability. Lastly, the excluded case $\Phi = \mathrm{G}_2$ and $I = \{\alpha\}$ with $\alpha \in \mathrm{G}_2$ long is the only one for which the computations we shall conduct could not be carried out. (We believe, however, that some ingenious calculations might show that such case need not be excluded.) Despite the technical assumptions given by the **QG** condition, we strongly suspect that Theorem **4.2** holds for arbitrary triples (R, Φ, I).

Let us look back at the parabolics $\mathbf{P}_1, \mathbf{P}_2 \leq \mathrm{GL}_{12}$ from the introduction and see how Theorem **4.2** applies.

***Example* 4.3.** We again consider the following parabolic subgroups of $\mathrm{GL}_{12}(\mathbb{Z}[t,t^{-1}])$.

$$\mathbf{P}_1(\mathbb{Z}[t,t^{-1}]) = \begin{pmatrix} \mathbf{1{\times}1} & * & \cdots & * \\ 0 & \mathbf{5{\times}5} & \ddots & \vdots \\ \vdots & \ddots & \mathbf{1{\times}1} & * \\ 0 & \cdots & 0 & \mathbf{5{\times}5} \end{pmatrix}, \quad \mathbf{P}_2(\mathbb{Z}[t,t^{-1}]) = \begin{pmatrix} \mathbf{5{\times}5} & * & \cdots & * \\ 0 & \mathbf{1{\times}1} & * & \vdots \\ \vdots & 0 & \mathbf{1{\times}1} & * \\ 0 & \cdots & 0 & \mathbf{5{\times}5} \end{pmatrix}.$$

Recall from the Introduction that their extended Levi factors (cf. Definition **4.5**) are the subgroups

$$\mathcal{LE}_1(\mathbb{Z}[t,t^{-1}]) = \begin{pmatrix} \mathbf{1\times 1} & 0 & \cdots & 0 \\ 0 & \mathbf{5\times 5} & \ddots & \vdots \\ \vdots & \ddots & \mathbf{1\times 1} & 0 \\ 0 & \cdots & 0 & \mathbf{5\times 5} \end{pmatrix} \leq \mathbf{P}_1(\mathbb{Z}[t,t^{-1}])$$

and

$$\mathcal{LE}_2(\mathbb{Z}[t,t^{-1}]) = \begin{pmatrix} \mathbf{5\times 5} & 0 & \cdots & 0 \\ 0 & \mathbf{1\times 1} & * & \vdots \\ \vdots & 0 & \mathbf{1\times 1} & 0 \\ 0 & \cdots & 0 & \mathbf{5\times 5} \end{pmatrix} \leq \mathbf{P}_2(\mathbb{Z}[t,t^{-1}]).$$

Let us first look at $\mathcal{LE}_1(\mathbb{Z}[t,t^{-1}])$. Since the 5×5 blocks are isomorphic to $\mathrm{GL}_5(\mathbb{Z}[t,t^{-1}])$ by Suslin's theorems [96, Theorem 7.8 and Corollary 7.10], we see via (**1.2**) and (**1.1**) that $\mathcal{LE}_1(\mathbb{Z}[t,t^{-1}])$ is isomorphic to $(\mathrm{GL}_5(\mathbb{Z}[t,t^{-1}]) \times \mathrm{GL}_1(\mathbb{Z}[t,t^{-1}]))^2$. As seen in Example **1.11**, the group $\mathrm{GL}_1(\mathbb{Z}[t,t^{-1}]) = \mathbb{G}_m(\mathbb{Z}[t,t^{-1}]) \cong C_\infty \times C_2$ is finitely presented. The following argument shows that $\mathrm{GL}_5(\mathbb{Z}[t,t^{-1}])$ is finitely presented as well—and thus that $\phi(\mathcal{LE}_1(\mathbb{Z}[t,t^{-1}])) \geq 2$. First of all, the ring $\mathbb{Z}[t,t^{-1}]$ has stable rank at most three [56, Theorem 4.1.11] because it is a localization of $\mathbb{Z}[t]$.

This yields, by Quillen's fundamental theorem [77, Section 6, Corollary to Theorem 8], the following isomorphisms of stable and unstable K-groups.

$$K_{1,5}(\mathbb{Z}[t,t^{-1}]) \cong K_1(\mathbb{Z}) \oplus K_0(\mathbb{Z})$$

and

$$K_{2,5}(\mathbb{Z}[t,t^{-1}]) \cong K_2(\mathbb{Z}) \oplus K_1(\mathbb{Z}).$$

Since the $K_i(\mathbb{Z}[t,t^{-1}])$ for $i \in \{0,1,2\}$ are finitely generated (see e.g. Milnor [72, Section 10] and Rosenberg [81, p. 75]), it follows from [56, Theorem 4.3.25] that $\mathrm{GL}_5(\mathbb{Z}[t,t^{-1}])$ admits a finite presentation, as required. The above arguments—and **(1.2)**—also yield $\phi(\mathcal{LE}_2(\mathbb{Z}[t,t^{-1}])) \geq 1$. However, Theorem **2.1** holds for $\mathcal{LE}_2(\mathbb{Z}[t,t^{-1}])$ because of the retract

$$\mathcal{LE}_2(\mathbb{Z}[t,t^{-1}]) \twoheadrightarrow \begin{pmatrix} \mathbf{1}_5 & 0 & \cdots & 0 \\ 0 & * & * & \vdots \\ \vdots & 0 & * & 0 \\ 0 & \cdots & 0 & \mathbf{1}_5 \end{pmatrix} \cong \mathbf{B}_2(\mathbb{Z}[t,t^{-1}]),$$

whence $\phi(\mathcal{LE}_2(\mathbb{Z}[t,t^{-1}]) = 1$ by [64, Section 4]. It thus follows from Theorem **4.2** that the parabolic subgroups $\mathbf{P}_1(\mathbb{Z}[t,t^{-1}]), \mathbf{P}_2(\mathbb{Z}[t,t^{-1}]) \leq \mathrm{GL}_{12}(\mathbb{Z}[t,t^{-1}])$ are not quasi-isometric.

In Section **4.1** we recall some facts about the structure of parabolics. The construction of the extended Levi factor is also given in Section **4.1** along with some examples and properties. Theorem **4.2** is proved in Section **4.2**. In Section **4.3** we look at applications of Theorem **4.2**. We close the chapter in Section **4.4** with some remarks on Theorem **4.2** and further research directions.

4.1 Structure of parabolics, and the extended Levi factor

It is well-known that a parabolic subgroup $\mathcal{P}_I(R)$ admits a *Levi decomposition* [48, Exposé XXVI, Prop. 1.6], that is, it splits as a semi-direct product $\mathcal{P}_I(R) = \mathcal{U}_I(R) \rtimes \mathcal{L}_I(R)$ with

$$\mathcal{U}_I(R) = \langle \mathfrak{X}_\gamma(R) : \gamma \in \Phi^+ \backslash \Phi_I \rangle \text{ and } \mathcal{L}_I(R) = \langle \mathcal{H}(R), \mathfrak{X}_\alpha(R) : \alpha \in \Phi_I \rangle.$$

(Recall that $\mathcal{H}$ is a maximal standard torus of the classical overgroup.) The normal subgroup $\mathcal{U}_I(R)$ is called the *unipotent radical* of $\mathcal{P}_I(R)$—it is always nilpotent and admits a filtration via levels of roots with respect to the defining subset $I \subseteq \Delta$; see, for instance, [68] for the case of algebraically closed fields or [48, Exposé XXVI, Sec. 2] for the general case. The group $\mathcal{L}_I(R)$ is called the *Levi factor* of $\mathcal{P}_I(R)$. When defined over a field, $\mathcal{L}_I$ is a reductive algebraic group and its derived subgroup will be classical because the overgroup $\mathcal{G} \geq \mathcal{P}_I$ is classical.

***Example* 4.4.** Suppose R is e.g. a field or a semi-local ring. As hinted in the Introduction, the standard (non-trivial) parabolic subgroups of $\mathrm{SL}_n(R) = \mathcal{G}^{sc}_{\mathrm{A}_{n-1}}(R)$ are of the form

$$\mathcal{P}_I(R) = \begin{pmatrix} \mathbf{n_1}\times\mathbf{n_1} & * & * & \cdots & * \\ 0 & \mathbf{n_2}\times\mathbf{n_2} & * & \cdots & \vdots \\ \vdots & & \ddots & & * \\ 0 & \cdots & 0 & & \mathbf{n_k}\times\mathbf{n_k} \end{pmatrix} \leq \mathrm{SL}_n(R),$$

that is, $\mathcal{P}_I(R)$ is a subgroup of block upper triangular matrices. The Levi factor $\mathcal{L}_I(R)$ is the subgroup generated by the diagonal matrices $\mathbf{D}_n(R) \cap \mathrm{SL}_n(R)$ and the square blocks on the diagonal. The condition $\varnothing \neq I \subsetneq \Delta$ (i.e. $\mathcal{P}_I(R)$ is non-trivial) implies that the number k of blocks is at least 2, and that at least one block is a square of size at least two, so that it consists of invertible square matrices with determinant 1. In other words, every block is isomorphic to some $\mathrm{SL}_{n_i}(R)$ if $1 < n_i < n$.
For instance, suppose $n = 6$ and $I = \{\alpha_1, \alpha_3\}$. Then $\mathcal{P}_I(R) = \mathcal{P}_{\{\alpha_1,\alpha_3\}}(R) \leq \mathrm{SL}_6(R)$ is given by

$$\mathcal{P}_{\{\alpha_1,\alpha_3\}}(R) = \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

Its unipotent radical and Levi factor are, respectively,

$$\mathcal{U}_{\{\alpha_1,\alpha_3\}}(R) = \begin{pmatrix} 1 & 0 & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \mathcal{L}_{\{\alpha_1,\alpha_3\}}(R) = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

We observe that the Levi factor $\mathcal{L}_{\{\alpha_1,\alpha_3\}}(R)$ is generated by the following subgroups of $\mathrm{SL}_6(R)$.

$$L_1 = \begin{pmatrix} \mathrm{SL}_2(R) & 0 \\ 0 & \mathbf{1}_4 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & \mathbf{1}_3 \end{pmatrix},$$
$$L_3 = \begin{pmatrix} \mathbf{1}_2 & 0 & 0 \\ 0 & \mathrm{SL}_2(R) & 0 \\ 0 & 0 & \mathbf{1}_2 \end{pmatrix}, \quad H_{4,5} = \begin{pmatrix} \mathbf{1}_3 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

In this case (recall that R is a field or is semi-local), we have the following identifications using the notation from Section **1.1.3**.

$L_1 = \langle \mathfrak{X}_{\alpha_1}(R), \mathfrak{X}_{-\alpha_1}(R) \rangle$, $H_2 = \mathcal{H}_{\alpha_2}(R) \cong \mathbb{G}_m(R)$
$L_3 = \langle \mathfrak{X}_{\alpha_3}(R), \mathfrak{X}_{-\alpha_3}(R) \rangle$, $H_{4,5} = \langle \mathcal{H}_{\alpha_4}(R), \mathcal{H}_{\alpha_5}(R) \rangle \cong \mathbb{G}_m(R)^2$,

and

$$\begin{aligned}\mathcal{U}_{\{\alpha_1,\alpha_3\}}(R) = \langle &\mathfrak{X}_{\alpha_2}(R),\, \mathfrak{X}_{\alpha_4}(R),\, \mathfrak{X}_{\alpha_5}(R),\\ &\mathfrak{X}_{\alpha_1+\alpha_2}(R),\, \mathfrak{X}_{\alpha_2+\alpha_3}(R),\, \mathfrak{X}_{\alpha_3+\alpha_4}(R),\, \mathfrak{X}_{\alpha_4+\alpha_5}(R),\\ &\mathfrak{X}_{\alpha_1+\alpha_2+\alpha_3}(R),\, \mathfrak{X}_{\alpha_2+\alpha_3+\alpha_4}(R),\, \mathfrak{X}_{\alpha_3+\alpha_4+\alpha_5}(R),\\ &\mathfrak{X}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}(R),\, \mathfrak{X}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}(R),\\ &\mathfrak{X}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}(R)\,\rangle.\end{aligned}$$

The decomposition for the Levi factor $\mathcal{L}_{\{\alpha_1,\alpha_3\}}(R)$ from Example **4.4** holds in a more general context. We spell it out below in terms of adjacency relations in the Dynkin diagram of Φ.

Recall that the root system Φ has an arbitrary, but fixed, choice of (totally ordered) simple roots Δ. Viewing Δ as the set of vertices of its Dynkin diagram $\mathcal{D}_\Delta$, if I is a subset of simple roots, we write $\mathrm{Adj}(I)$ for the set of simple roots *not in* I that are adjacent to some (not necessarily the same) element of I. In symbols,

$$\begin{aligned}\mathrm{Adj}(I) := \{\delta \in \Delta\backslash I \mid \exists \alpha \in I \text{ for which there is an edge in } \mathcal{D}_\Delta\\ \text{connecting } \delta \text{ to } \alpha\}.\end{aligned}$$

Now, let $\varnothing \neq I \subsetneq \Delta$. This subset I of simple roots generates a subdiagram $\mathcal{I}$ in the Dynkin diagram $\mathcal{D}_\Delta$. Denote by $I_1,\ldots,I_k$ the (pairwise disjoint) subsets of I that span the connected components of $\mathcal{I}$ in $\mathcal{D}_\Delta$. We observe that

$$\Phi_I = \Phi_{I_1} \cup \Phi_{I_2} \cup \cdots \cup \Phi_{I_k}.$$

It then follows from Chevalley's formulae (**1.8**) and Steinberg's relations (**1.9**) that the Levi factor $\mathcal{L}_I(R) \leq \mathcal{P}_I(R) \leq \mathcal{G}(R)$ is in fact an extension of a direct product of elementary groups, of rank smaller than $\mathrm{rk}(\Phi)$, by a torus. For instance, if $\mathcal{G} = \mathcal{G}_\Phi^{\mathrm{sc}}$, then

$$\begin{aligned}\mathcal{L}_I(R) &= E_{\Phi_I}^{sc}(R) \rtimes \langle \mathcal{H}_\alpha(R) \,:\, \alpha \in \Delta\backslash I\rangle\\ &= \left(\prod_{j=1}^{k} E_{\Phi_{I_j}}^{sc}(R)\right) \rtimes \langle \mathcal{H}_\alpha(R) \,:\, \alpha \in \Delta\backslash I\rangle.\end{aligned}$$

In case $\mathcal{G} = \mathrm{GL}_n$, we have

$$\mathcal{L}_I(R) = \left(\prod_{j=1}^{k} \mathrm{GE}_{n_j}(R)\right) \rtimes \mathbb{G}_m(R)^{n-n_1-\ldots-n_k}.$$

If $I = \varnothing$, then $\mathcal{L}_I(R)$ is just the standard torus $\mathcal{H}(R)$. From the above description of $\mathcal{L}_I(R)$ and Corollary **1.14**, it follows that the torus—whence the group of units $\mathbb{G}_m(R)$—must be finitely generated if $\mathcal{P}_I(R)$ is to be finitely presented.

We define the *non-adjacent* roots of I to be the complement

$$\mathrm{NAdj}(I) = \Delta\backslash(I \cup \mathrm{Adj}(I)) = \{\alpha \in \Delta\backslash I \mid \alpha \text{ is adjacent to no element of } I\}\,,$$

and the *extension* of the given set I is defined as $\mathrm{Ext}(I) = \Delta\backslash\,\mathrm{Adj}(I) = I \cup \mathrm{NAdj}(I)$. With the adjacency terminology, the extended Levi factor is defined as follows.

Definition 4.5. Let $\mathcal{P}_I(R)$ be a standard parabolic subgroup of a classical group $\mathcal{G}(R)$. The *extended Levi factor* of $\mathcal{P}_I(R)$, denoted by $\mathcal{LE}_I(R)$, is the subgroup generated by the standard torus $\mathcal{H}(R)$ and either a single root subgroup $\mathfrak{X}_\alpha(R)$ with $\alpha \in \Delta$ the first long root, in case I is empty, or by the root subgroups $\mathfrak{X}_\alpha(R)$ for $\alpha \in \Phi_I$ together with the non-adjacent positive root subgroups $\mathfrak{X}_\beta(R)$ with $\beta \in \Phi^+_{\mathrm{NAdj}(I)}$, in case $I \neq \varnothing$. In symbols,

$$\mathcal{LE}_I(R) := \begin{cases} \langle \mathcal{H}(R),\, \mathfrak{X}_\alpha(R) \,:\, \alpha \text{ is the first long root of } \Delta\rangle, & \text{if } I = \varnothing; \\ \langle \mathcal{H}(R),\, \mathfrak{X}_\alpha(R),\, \mathfrak{X}_\beta(R) \,:\, \alpha \in \Phi_I,\ \beta \in \Phi^+_{\mathrm{NAdj}(I)}\rangle & \text{otherwise.} \end{cases}$$

We usually reserve the notation $\mathcal{LE}_I(R)$ for the case $I \neq \varnothing$ and write $\mathcal{LE}_\varnothing(R)$ otherwise. By the very definition the extended Levi factor is unique when $I \neq \varnothing$. In contrast, $\mathcal{LE}_\varnothing(R)$ depends a priori on the given ordering on the set of simple roots Δ. However, by the Weyl relations **(1.10)**, different orderings yield conjugate (hence isomorphic) extended Levi factors.

We remark that $\mathcal{LE}_I(R)$ contains $\mathcal{L}_I(R)$, though this containment might not be proper. In fact, one has the following *split* short exact sequences.

$$\mathfrak{X}_\alpha(R) \hookrightarrow \mathcal{LE}_\varnothing(R) \twoheadrightarrow \mathcal{L}_\varnothing(R) = \mathcal{H}(R) \cong \mathcal{LE}_n(R)/\mathfrak{X}_\alpha(R)$$

and

$$\langle \mathfrak{X}_\beta(R) \,:\, \beta \in \Phi^+_{\mathrm{NAdj}(I)}\rangle \hookrightarrow \mathcal{LE}_I(R) \twoheadrightarrow \mathcal{L}_I(R) \cong \frac{\mathcal{LE}_I(R)}{\langle \mathfrak{X}_\beta(R) \,:\, \beta \in \Phi^+_{\mathrm{NAdj}(I)}\rangle}.$$

Thus, by Lemma **1.13**, if an extended Levi factor is finitely presented, then so is the Levi factor itself.

In the language of Example **0.7** from the Introduction, in the case where $I \neq \varnothing$, the root subgroups $\mathfrak{X}_\beta(R)$ with $\beta \in \mathrm{NAdj}(I)$ are the generators of the *triangular blocks* of $\mathcal{P}_I(R)$, whereas $\mathcal{L}_I(R)$ is generated by both the *square blocks*—generated by the $\mathfrak{X}_\alpha(R)$ with $\alpha \in \Phi_I$—and the torus $\mathcal{H}(R)$.

***Example* 4.6.** We describe the extended Levi factor for the parabolic subgroup $\mathcal{P}_{\{\alpha_1,\alpha_3\}}(R) \leq \mathrm{SL}_6(R)$ from Example **4.4** (R a field, or semi-local). As observed before, we have two subgroups which are generated by root subgroups with roots from $\Phi_{\{\alpha_1,\alpha_3\}}$, namely $L_1 = \langle \mathfrak{X}_{\alpha_1}(R),\, \mathfrak{X}_{-\alpha_1}(R)\rangle$ and $L_3 = \langle \mathfrak{X}_{\alpha_3}(R),\, \mathfrak{X}_{-\alpha_3}(R)\rangle$. Those are precisely the square blocks from $\mathcal{P}_{\{\alpha_1,\alpha_3\}}(R)$. Now, the only simple root which is not adjacent to

$I = \{\alpha_1, \alpha_3\}$ is the last one, α_5. So $\mathrm{NAdj}(I)$ is the singleton $\{\alpha_5\}$ and one has $\Phi_{\mathrm{NAdj}(I)} = \{\pm\alpha_5\}$ and $\Phi^+_{\mathrm{NAdj}(I)} = \{\alpha_5\}$, whence the triangular block of $\mathcal{P}_{\{\alpha_1,\alpha_3\}}(R)$ is just the root subgroup $\mathfrak{X}_{\alpha_5}(R)$. Pictorially,

$$L_1 = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; L_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \mathfrak{X}_{\alpha_5}(R) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus,

$$\mathcal{LE}_{\{\alpha_1,\alpha_3\}}(R) = \left(\begin{array}{cc|cc|cc} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{array}\right).$$

Of course, we still have to show that the extended Levi factor $\mathcal{LE}_I(R)$ fits in our framework of retracts with respect to $\mathcal{P}_I(R)$.

Proposition 4.7. *Suppose $I \neq \varnothing$. There is a retract $r : \mathcal{P}_I(R) \twoheadrightarrow \mathcal{LE}_I(R)$ with kernel*

$$\mathcal{K}_I(R) = \langle \mathfrak{X}_\gamma(R) \,:\, \gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)} \rangle.$$

If $I = \varnothing$, then one has a retract $r : \mathcal{P}_\varnothing(R) = \mathcal{B}_\Phi(R) \twoheadrightarrow \mathcal{LE}_\varnothing(R)$ with kernel

$$\mathcal{K}_\varnothing(R) = \langle \mathfrak{X}_\gamma(R) \,:\, \gamma \in \Phi^+ \backslash \{\alpha\} \rangle.$$

Proof. Assume first that $I = \varnothing$. By Chevalley's formulae (**1.8**) and Steinberg's relations (**1.9**), we see that $\mathcal{K}_\varnothing(R) \trianglelefteq \mathcal{B}_\Phi(R)$ and $\mathcal{LE}_\varnothing(R) = \mathfrak{X}_\alpha(R) \rtimes \mathcal{H}(R) \cong \mathcal{B}_\Phi(R)/\mathcal{K}_\varnothing(R)$. Suppose $I \neq \varnothing$. Again from (**1.8**) and (**1.9**), it follows that $\mathcal{K}_I(R) \trianglelefteq \mathcal{P}_I(R)$. Since $\mathrm{Ext}(I)$ is the disoint union of I and $\mathrm{NAdj}(I)$, we have that the unipotent root subgroups that generate $\mathcal{LE}_I(R)$ do not involve any $\mathfrak{X}_\gamma(R) \leq \mathcal{K}_I(R)$. Furthermore, such unipotent root subgroups are partitioned into blocks which pairwise have only non-adjacent roots between them. Hence, $\mathcal{LE}_I(R)$ is a complement of $\mathcal{K}_I(R)$ in $\mathcal{P}_I(R)$ with trivial intersection, so that the natural projection $\mathcal{P}_I(R) \twoheadrightarrow \mathcal{P}_I(R)/\mathcal{K}_I(R)$ yields the desired retract. □

It follows from Lemma **1.13** and Proposition **4.7** that the finite presentability of $\mathcal{LE}_I(R)$ (resp. $\mathcal{LE}_\varnothing(R)$) is a necessary condition for the finite presentability of the whole parabolic $\mathcal{P}_I(R)$ (resp. $\mathcal{P}_\varnothing(R)$). Proposition **4.7** also implies that we can make use of the usual presentation of a semi-direct product to build a presentation for $\mathcal{P}_I(R)$ out of presentations of $\mathcal{LE}_I(R)$ and $\mathcal{K}_I(R)$. For the Borel subgroup $\mathcal{P}_\varnothing(R)$, we will instead make use of a presentation for its unipotent radical $\mathcal{U}_\varnothing(R) = \langle \mathfrak{X}_\gamma(R) : \gamma \in \Phi^+ \rangle$. We shall therefore need convenient presentations for $\mathcal{U}_\varnothing(R)$ and $\mathcal{K}_I(R)$, to be described in the sequel, for the proof of Theorem **4.2**.

Recall that the unipotent radical $\mathcal{U}_I(R)$ admits a well-known presentation obtained as follows. The unipotent root elements $x_\gamma(r)$, with γ running over $\Phi^+\backslash\Phi_I$ and $r \in R$, form the generating set. The defining relators are given by the commutator formulae **(1.8)** and the additive condition $x_\gamma(r) \cdot x_\gamma(s) = x_\gamma(r+s)$ for all $\gamma \in \Phi^+\backslash\Phi_I$, $r, s \in R$. This is precisely the analogue of Lemma **1.3** in the more general context of parabolics in classical groups. We note, in particular, that if the classical overgroup $\mathcal{G}$ equals GL_n or SL_n, then $\mathcal{U}_\varnothing(R) = \mathbf{U}_n(R)$, the subgroup of upper unitriangular matrices.

As seen in Section **1.1.2**, commutator calculus yields concrete presentations for $\mathcal{U}_I(R)$. In what follows we fix the notation for a presentation of $(R,+) = \mathbb{G}_a(R) = (\bigoplus_{t\in T}\mathbb{Z}t)/\langle\mathcal{R}\rangle$ just as in Section **1.1.2**. In particular, $T \subseteq R$ is a subset of additive generators containing 1, the set $\mathcal{R} \subseteq \bigoplus_{t\in T}\mathbb{Z}t$ is a set of additive defining relators, and we have a function $m : (\bigoplus_{t\in T}\mathbb{Z}t) \times (\bigoplus_{t\in T}\mathbb{Z}t) \to (\bigoplus_{t\in T}\mathbb{Z}t)$ for which the image of $m(r,s)$ in R equals the product of the images of r and s in R under the given natural projection. We observe that $m(-,-)$ can be used to represent products of powers of additive expressions as follows. Given natural numbers $k, \ell \in \mathbb{N}$ and $r, s \in (\bigoplus_{t\in T}\mathbb{Z}t)$, define $p_{k,\ell}(-,-)$ recursively as

$$p_{k,\ell}(r,s) := m(\underbrace{m(r,\ldots m(r,r))\cdots)}_{k \text{ occurrences of } r},\ \underbrace{m(s,\ldots m(s,s))\cdots)}_{\ell \text{ occurrences of } s}).$$

Lemma 4.8. *Let R be a commutative ring with unity and let T, $\mathcal{R}$, and $m(r,s) = \sum_u a_u u$ be as in Section* **1.1.2**. *The unipotent radical $\mathcal{U}_I(R)$ of a parabolic subgroup $\mathcal{P}_I(R)$ of a classical group $\mathcal{G}(R)$ admits a presentation $\mathcal{U}_I(R) = \langle \mathcal{Y} \mid \mathcal{S}\rangle$ with generating set*

$$\mathcal{Y} = \left\{x_\gamma(t) \mid t \in T, \gamma \in \Phi^+\backslash\Phi_I\right\}$$

and defining relators $\mathcal{S}$ given as follows. For all $\gamma, \eta \in \Phi^+\backslash\Phi_I$ and $t, s \in T$,

$$[x_\gamma(t), x_\eta(s)] = \begin{cases} \displaystyle\prod_{m\gamma+n\eta\in\Phi^+} \left(\prod_u x_{m\gamma+n\eta}(u)^{a_u}\right)^{C^{\gamma,\eta}_{m,n}}, & \text{if } \gamma+\eta \in \Phi; \\ 1 \quad \text{otherwise}, \end{cases} \tag{4.1}$$

where $\sum_u a_u u \in \bigoplus_{t\in T}\mathbb{Z}t$ is the value of $p_{m,n}(t,s)$ as above, and

$$\prod_\lambda x_\gamma(t_\lambda)^{a_\lambda} = 1 \text{ for all } \sum_\lambda a_\lambda t_\lambda \in \mathcal{R} \text{ as in Section 1.1.2.} \tag{4.2}$$

Proof. This is a straightforward generalization of Lemma **1.3**, which itself covers the case $I = \varnothing$ and $\mathcal{G} \in \{\mathrm{GL}_n, \mathrm{SL}_n\}$. In fact, an analogous proof applies. To see this, recall that $\mathcal{U}_I(R)$ is also nilpotent by **(1.8)** and that the factors of its lower central series are direct products of $\mathbb{G}_a(R)$; see e.g. [93, Chapter 3] and [9, Theorem 2.a, see also closing remark 1]. Recasting Lemma **1.3** with Chevalley's formulae (**1.8**), which generalize the commutator relations (**1.1**), just repeat the same steps of the proof of Lemma **1.3** to obtain the above presentation for $\mathcal{U}_I(R)$. □

Our next remark is that $\mathcal{K}_I(R)$ admits a presentation very similar to that of $\mathcal{U}_I(R)$ described above. The proof is analogous to that of Lemma **1.3**. Since the filtration of $\mathcal{K}_I(R)$ via a central series might not be clear a priori, we outline below how to imitate the arguments from Lemma **1.3**. In particular, the reader unfamiliar with the full proof of Lemma **4.8** might just adapt the proof of the lemma below to the set-up of **4.8**.

Lemma 4.9. *Let R be a commutative ring with unity and let T, $\mathcal{R}$, and $m(r,s) = \sum_u a_u u$ be as in Section* **1.1.2**. *Suppose $I \neq \varnothing$. Then the kernel $\mathcal{K}_I(R)$ of the retraction $r : \mathcal{P}_I(R) \twoheadrightarrow \mathcal{LE}_I(R)$ of Proposition* **4.7** *admits a presentation $\mathcal{K}_I(R) = \langle \mathcal{Y} \mid \mathcal{S} \rangle$ with generating set*

$$\mathcal{Y} = \left\{ x_\gamma(t) \mid t \in T, \gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)} \right\}$$

and a set of defining relators $\mathcal{S}$ given by the same formulae (**4.1**) *and* (**4.2**) *of Lemma* **4.8**, *but now for all $\gamma, \eta \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)}$.*

Sketch of proof following Lemma **1.3**. Given a positive root $\alpha \in \Phi^+$, we can write (uniquely)

$$\alpha = \sum_{\delta \in \mathrm{Ext}(I)} p_\delta \delta + \sum_{\gamma \in \mathrm{Adj}(I)} q_\gamma \gamma, \text{ with } p_\delta, q_\gamma \in \mathbb{Z}_{\geq 0}.$$

We define the *adjacency level of* α, denoted by $\mathrm{alvl}(\alpha)$, to be the integer $\mathrm{alvl}(\alpha) = \sum_{\gamma \in \mathrm{Adj}(I)} q_\gamma$ from the equation above. Let

$$\mathcal{K}_j = \langle \mathfrak{X}_\gamma(R) : \gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)} \text{ has } \mathrm{alvl}(\gamma) \geq j \rangle.$$

By the commutator formuale (**1.8**), we see that each $\mathcal{K}_j$ is normal in $\mathcal{K}_I(R)$ and each factor $\mathcal{K}_j / \mathcal{K}_{j+1}$ is canonically isomorphic to

$$\prod_{\substack{\gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)} \\ \mathrm{alvl}(\gamma) = j}} \mathfrak{X}_\gamma(R) \cong \prod_{\substack{\gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)} \\ \mathrm{alvl}(\gamma) = j}} \mathbb{G}_a(R).$$

Hence, the subgroups $\mathcal{K}_j$ yield a terminating central series for $\mathcal{K}_I(R)$—though it might not coincide with the lower central series.

Now let $\widetilde{\mathcal{K}}_I(R)$ be the group defined by the presentation stated in the lemma, with the decoration $\widetilde{\ }$ above the elements of the generating set; e.g. $\widetilde{x}_\gamma(t)$ instead of $x_\gamma(t)$. Define analogously

$$\widetilde{\mathcal{K}}_j = \langle \left\{ \widetilde{x}_\gamma(t) \mid t \in T \text{ and } \gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)} \text{ of } \mathrm{alvl}(\gamma) \geq j \right\} \rangle.$$

By von Dyck's theorem, the obvious map

$$f : \mathcal{Y} \longrightarrow \mathcal{K}_I(R)$$

$$\widetilde{x}_\gamma(t) \mapsto x_\gamma(t)$$

induces a surjection $\widetilde{\mathcal{K}}_I(R) \twoheadrightarrow \mathcal{K}_I(R)$—which we also call f by abuse of notation—because the defining relations **(4.1)** and **(4.2)** hold in $\mathcal{K}_I(R)$ and each $\mathfrak{X}_\gamma(R) \cong \mathbb{G}_a(R)$ is generated by the $\{x_\gamma(t) \mid t \in T\}$. By **(4.1)** we also see that $\widetilde{\mathcal{K}}_j \trianglelefteq \widetilde{\mathcal{K}}_I(R)$ for every j, and f restricts to surjections $\widetilde{\mathcal{K}}_j \twoheadrightarrow \mathcal{K}_j$. Furthermore, relations **(4.1)** and **(4.2)** imply—analogously to Lemma **1.3**—that each factor $\widetilde{K}_j/\widetilde{K}_{j+1}$ is isomorphic to

$$\prod_{\substack{\gamma\in\Phi^+\setminus\Phi_{\mathrm{Ext}(I)} \\ \mathrm{alvl}(\gamma)=j}} \widetilde{\mathfrak{X}}_\gamma \cong \prod_{\substack{\gamma\in\Phi^+\setminus\Phi_{\mathrm{Ext}(I)} \\ \mathrm{alvl}(\gamma)=j}} \mathbb{G}_a(R),$$

where the $\widetilde{\mathfrak{X}_\gamma}$ are defined in the obvious way. Such maps are all induced by f. Thus, $\widetilde{\mathcal{K}}_I(R)$ and $\mathcal{K}_I(R)$ are nilpotent groups with isomorphic (terminating) central series via isomorphisms induced by the same surjection. It then follows by induction on the nilpotency class that f is an isomorphism. □

4.2 Proof of Theorem 4.2

By Proposition **4.7**, the extended Levi factor $\mathcal{LE}_I(R)$ is a retract of its parabolic overgroup $\mathcal{P}_I(R)$, whence $\phi(\mathcal{P}_I(R)) \leq \phi(\mathcal{LE}_I(R))$ by Corollary **1.14**. Thus, to prove Theorem **4.2**, it remains to settle the following.

Theorem 4.10. *Let $\mathcal{P}_I(R)$ be a standard parabolic subgroup of a classical group $\mathcal{G}(R)$ and suppose (R, Φ, I) is* **QG**. *If the extended Levi factor $\mathcal{LE}_I(R)$ admits a finite presentation, i.e. if $\phi(\mathcal{LE}_I(R)) \geq 2$, then $\phi(\mathcal{P}_I(R)) \geq 2$.*

We shall prove Theorem **4.10** only in the cases where the classical overgroup $\mathcal{G}$ is a universal Chevalley–Demazure group scheme $\mathcal{G}_\Phi^{\mathrm{sc}}$. The reason for this is that the proof given here carries over almost verbatim to the case where $\mathcal{G} = \mathrm{GL}_n$ after recasting the proof steps in the language of elementary and diagonal matrices. (Recall from Example **1.6** that the unipotent root subgroups of $\mathrm{GL}_n(R)$ are the $\mathbf{E}_{ij}(R)$ and the standard maximal torus is the diagonal subgroup $\mathbf{D}_n(R)$.) Thus, we **assume for the remainder of Section 4.2 that the classical overgroup $\mathcal{G} \geq \mathcal{P}_I$ is a universal Chevalley–Demazure group scheme $\mathcal{G} = \mathcal{G}_\Phi^{\mathrm{sc}}$.**

The strategy to prove Theorem **4.10** is as follows. We first analyze the structure of the base ring R under our standing assumptions. Now, recall that a semi-direct product $G = N \rtimes_\varphi Q$, with $N = \langle Y \mid S\rangle$, $Q = \langle X \mid R\rangle$

and φ the homomorphism $Q \xrightarrow{\varphi} \mathrm{Aut}(N)$ determining the action, admits the following presentation.

$$G = \langle X \cup Y \mid S \cup R \cup \{xyx^{-1}\varphi(y) \mid x \in X,\ y \in Y\} \rangle. \qquad (*)$$

We then consider the two cases, $I = \varnothing$ and $I \neq \varnothing$. In the former, $\mathcal{P}_I(R)$ is the standard Borel subgroup $\mathcal{B}_\Phi(R) = \mathcal{U}_\varnothing(R) \rtimes \mathcal{H}(R)$. Since $\mathcal{H}(R)$ is a finitely generated abelian group, we can take a *finite* presentation for $\mathcal{H}(R)$ with finitely many semi-simple root elements as generators. Combining this with the presentation for $\mathcal{U}_\varnothing(R)$ from Lemma **4.8** and Steinberg's relations **(1.9)**, we get a canonical (in general infinite) presentation for $\mathcal{B}_\Phi(R)$ as in **(*)**. On the other hand, using the fact that the extended Levi factor $\mathcal{LE}_\varnothing(R)$ is finitely presented, Theorem **2.1** implies that each subgroup $\mathfrak{X}_\gamma(R) \rtimes \mathcal{H}(R) \leq \mathcal{B}_\Phi(R)$ for $\gamma \in \Phi^+$ has a finite presentation. We fix such presentations with convenient generating sets, and we appropriately add unipotent root elements and Chevalley and Steinberg relations to construct a finitely presented group $\widetilde{\mathcal{B}}_\Phi(R)$. Finally, we apply von Dyck's theorem twice to show that $\widetilde{\mathcal{B}}_\Phi(R)$ is isomorphic to $\mathcal{B}_\Phi(R)$.

For $I \neq \varnothing$, we start by taking a finite presentation for the extended Levi factor $\mathcal{LE}_I(R)$ with generating set given by appropriately chosen unipotent and semi-simple root elements. The canonical presentation **(*)** for $\mathcal{P}_I(R)$ is given by the chosen presentation for $\mathcal{LE}_I(R)$ together with the presentation of $\mathcal{K}_I(R)$ from Lemma **4.9** with the addition of Chevalley and Steinberg relations. We then break down the proof into two cases given by the **QG** condition: If $\mathbf{B}_2^\circ(R)$ is finitely presented, we get finite presentations for each $\mathfrak{X}_\gamma(R) \rtimes \mathcal{H}(R)$ for $\gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)}$ and proceed similarly to the Borel case; If R is **NVB** for Φ, we construct a finitely presented group $\widetilde{\mathcal{P}}_I(R)$ from $\mathcal{LE}_I(R)$ adding just the obvious generators from $\mathcal{K}_I(R)$ as a normal subgroup of $\mathcal{P}_I(R)$ and the necessary Chevalley and Steinberg relations, then proceed to prove $\widetilde{\mathcal{P}}_I(R) \cong \mathcal{P}_I(R)$ via commutator computations.

Unless stated otherwise, the arguments given here are for arbitrary, but fixed, total orders on the set of simple roots $\Delta \subset \Phi$—which extend to total orders on Φ. The above mentioned finite presentations thus use the same given, fixed ordering. But the reader can work right from the start with presentations which are independent of choice of ordering—just add to the given group presentation the (finitely many) copies of relations listed here that correspond to all possible choices of ordering.

Results in this section distinguish as little as possible the different types of root systems. We therefore warn the reader to be armed with patience to face the lengthy notation battle ahead.

Let us begin with the following elementary results on root systems and root subgroups that will be needed in the sequel.

Lemma 4.11 ([58, 10.2 A]). *Given a non-simple root $\gamma \in \Phi^+ \backslash \Delta$, there exist a simple root $\alpha \in \Delta$ and a positive root $\beta \in \Phi^+$ such that $\alpha + \beta = \gamma$.*

Lemma 4.12. *If $\varnothing \neq I \subsetneq \Delta$ and $\alpha \in \mathrm{Adj}(I)$, then there exist $\widetilde{\alpha} \in \Phi_I$ and $\widetilde{\beta} \in \Phi^+$ such that $\widetilde{\alpha} + \widetilde{\beta} = \alpha$.*

Proof. Take $\delta \in I$ adjacent to α. Then $\widetilde{\beta} := \delta + \alpha \in \Phi^+$ and the claim follows for $\widetilde{\alpha} := -\delta$. □

Lemma 4.13. *Let $\mathfrak{X}_\alpha(R), \mathfrak{X}_\beta(R) \leq \mathcal{G}_\Phi^{\mathrm{sc}}(R)$ be root subgroups with α and β linearly independent. There exist a one-dimensional subtorus $H_{\alpha,\beta}(R) \leq \mathcal{H}(R)$, given by the assignment $h : \mathbb{G}_m(R) \xrightarrow{\cong} H_{\alpha,\beta}$, $u \in R^\times \mapsto h(u)$, and an integer $n = n(\alpha, \beta) \neq 0$ such that $H_{\alpha,\beta}(R)$ centralizes $\mathfrak{X}_\alpha(R)$ and $h(u)x_\beta(r)h(u)^{-1} = x_\beta(u^n r)$ for all $x_\beta(r) \in \mathfrak{X}_\beta(R)$ and all $h(u) \in H_{\alpha,\beta}(R)$.*

Proof. We may assume α and β to be simple by **(1.10)**. If α is orthogonal to β, define $H_{\alpha,\beta}$ to be the semi-simple root subgroup $\mathcal{H}_\beta(R)$ and the claim follows. The case $\mathrm{rk}(\Phi) \geq 3$ is also easy: choose another simple root γ which is adjacent to β and non-adjacent to α and set $H_{\alpha,\beta} = \mathcal{H}_\gamma(R)$. For the general case, pick integers $p, q \in \mathbb{Z} \setminus \{0\}$ such that $2p - q \cdot (\alpha, \beta) = 0$ and set $h(u) := h_\beta(u)^{-q} h_\alpha(u)^p$ and $H_{\alpha,\beta}(R) := \langle \{h(u) \mid u \in R^\times\} \rangle \leq \mathcal{H}(R)$. Then $H_{\alpha,\beta}(R)$ centralizes $\mathfrak{X}_\alpha(R)$ by Steinberg's relations **(1.9)**. As the Cartan integers lie in $\{\pm 1, \pm 2, \pm 3\}$, one easily checks that $n(\alpha, \beta) := p \cdot (\beta, \alpha) - 2q$ must be non-zero, and the result follows again from **(1.9)**. □

From the proof of Lemma **4.13** one can take $n(\alpha, \beta)$ to be ± 1 or ± 2 in many cases. Furthermore, the torus $H_{\alpha,\beta}(R)$ needs not be unique, thus the integers $n(\alpha, \beta)$ may vary. Clearly, $n(-\alpha, \beta) = -n(\alpha, \beta)$.

Definition 4.14. Given a commutative ring R with unity and two roots $\alpha, \beta \in \Phi$, we define their *toral constant* to be

$$c_{\alpha,\beta}(R) = \min \{|n(\alpha, \beta)| \,:\, H_{\alpha,\beta}(R) \text{ and } n(\alpha, \beta) \text{ are as in Lemma } \mathbf{4.13}\}.$$

The *toral constant of* Φ *and* R is defined as

$$c_\Phi(R) = \max \{c_{\alpha,\beta}(R) \mid \alpha, \beta \in \Phi\}.$$

In the **next few pages** we establish a great deal of notation and remarks necessary to construct the presentations for the proof of Theorem **4.10**. We kindly ask the reader to bear with us during this task.

The first step is to establish notation concerning the base ring R, keeping in mind the standing assumptions of Theorem **4.10**. By the **QG** condition, the Borel subgroup $\mathbf{B}_2^\circ(R)$ of rank one is always finitely generated. Hence, the torus $\mathcal{H}(R)$ of $\mathcal{G}_\Phi^{\mathrm{sc}}(R)$ is finitely generated as well; see Section **4.1**. We may thus fix $A \subseteq \mathbb{G}_m(R) = R^\times$ a finite generating set for the multiplicative (abelian) group of units of R, say $A = \{v_1, \ldots, v_\xi\}$. Now, $\mathbf{B}_2^\circ(R)$ is isomorphic to the semi-direct product $\mathbb{G}_a(R) \rtimes \mathbb{G}_m(R)$ with action given by

$$\begin{aligned} \mathbb{G}_m(R) \times \mathbb{G}_a(R) &\longrightarrow \mathbb{G}_a(R) \\ R^\times \times R \ni (u, r) &\longmapsto u^2 r. \end{aligned}$$

We may therefore choose a finite set $T_0 = \{x_0 = 1, x_1, \ldots, x_\nu\} \subseteq R$ such that $R^\times \cdot T_0 := \{ux_i \mid 0 \le i \le \nu \text{ and } u \in \langle A \rangle\}$ additively generates R. Given a positive integer $c \in \mathbb{N}$, let $A^{[c]}$ denote the set of monomials $\left\{v_1^{\varepsilon_1} \cdots v_\xi^{\varepsilon_\xi} \mid -c \le \varepsilon_j \le c \ \forall j\right\}$ over A with powers of the letters bounded by $\pm c$; notice that $1 \in A^{[c]}$. Using the action of $R^\times$ on R given above, we have that $A^{[2]} \cdot T_0$ generates R as a $\mathbb{Z}[R^\times]$-module. Setting $c_\Phi := c_\Phi(R) \in \mathbb{N}$ the toral constant of Φ and R, one has that $\widetilde{T} := A^{[c_\Phi]} \cdot T_0$ is also a (*finite*) generating set for R as a $\mathbb{Z}[R^\times]$-module. Hence, the (possibly infinite) set $T := \langle A^{[c_\Phi]} \rangle \cdot T_0 = R^\times \cdot T_0$ additively generates the ring R.

Put differently, the set T_0 above freely generates the free $\mathbb{Z}[R^\times]$-module

$$M = \bigoplus_{\ell=0}^{\nu} \mathbb{Z}[R^\times] \cdot x_\ell,$$

and $T = \langle A^{[c_\Phi]} \rangle \cdot T_0$ and $\widetilde{T} = A^{[c_\Phi]} \cdot T_0$ are, respectively, a generating set for M as a $\mathbb{Z}$-module and a finite generating set for M as a $\mathbb{Z}[R^\times]$-module. We let furthermore $\mathcal{A} \subseteq M$ denote an arbitrary, but fixed, set of additive defining relators for $(R, +)$. That is, we fix an epimorphism $\pi : M \twoheadrightarrow \mathbb{G}_a(R)$ of *abelian groups* which maps every $ux_i \in R^\times \cdot T_0 \subset M$ to its copy ux_i in R and fix a set $\mathcal{A} \subset M$ such that $\ker(\pi) = \mathrm{span}_{\mathbb{Z}}(\mathcal{A}) \subseteq M$.

What now follows is a detailed adaptation of what we did before in Sections **1.1.2** and **4.1**. For every pair $x_i, x_j \in T_0$, fix an expression $m(x_i, x_j) \in M$ representing the product $x_i x_j = x_j x_i$ in the ring R. That is, $m(x_i, x_j) \in M$ is chosen so that $\pi(m(x_i, x_j)) = x_i x_j$ in R. (We are abusing notation since we are looking at $T_0 = \{x_0, \ldots, x_\nu\}$ both as a subset of R and as a free basis of M.) Furthermore, we choose the expressions $m(x_i, x_j)$ so that $m(x_i, 1) = x_i = m(1, x_i)$ and $m(x_i, x_j) = m(x_j, x_i)$ for all $x_i, x_j \in T_0$.

Now define a family $\{p_{m,n}\}_{(m,n) \in \mathbb{N} \times \mathbb{N}}$ of maps $p_{m,n} : M \times M \to M$ recursively and $\mathbb{Z}$-linearly as follows. Firstly, given $t, s \in T = R^\times \cdot T_0$ with $t = ux_i$ and $s = vx_j$, where $x_i, x_j \in T_0$ and $u, v \in R^\times$, we set

$$p_{1,1}(t, s) := p(t, s) = p(ux_i, vx_j) := uv \cdot m(x_i, x_j),$$

where $m(x_i, x_j) \in M$ is as in the previous paragraph. Having defined $p(-,-)$ over $T \times T$, we extend it $\mathbb{Z}$-linearly to $M \times M$, which is possible because M is free abelian with basis T. The $p(-,-)$ thus induce maps from $M \times M$ to M, which we also call $p(-,-)$, as follows. Given elements $r = \sum_{i=0}^{\nu} a_i u_i x_i$ and $s = \sum_{j=0}^{\nu} b_j v_j x_j$ in M, where $a_i, b_j \in \mathbb{Z}$ and $u_i x_i, v_j x_j \in T$, set

$$p_{1,1}(r, s) := p(r, s) := \sum_{i=0}^{\nu} \sum_{j=0}^{\nu} a_i b_j \cdot p(u_i x_i, v_j x_j).$$

Secondly, with the maps $p : M \times M \to M$ at hand and and given $m, n \in$

$\mathbb{N} \times \mathbb{N}$, we recursively define $p_{m,n}(-,-)$, also first on $T \times T$, by setting

$$p_{m,n}(t,s) := p(\underbrace{p(t,\ldots p(t,t))\cdots)}_{m \text{ occurrences of } t},\ \underbrace{p(s,\ldots p(s,s))\cdots)}_{n \text{ occurrences of } s}).$$

For example, $p_{3,2}(t,s) = p(p(t,p(t,t)),p(s,s))$. Finally, since the maps $\{p_{m,n}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ are defined on the basis T of M as a free abelian group, they extend $\mathbb{Z}$-linearly like in the previous case of $p(-,-)$ to maps which we also call $p_{m,n}$, defined on the whole of $M \times M$.

Each map $p_{m,n}(-,-) : M \times M \to M$ has the following properties. Given $t = ux_i$ and $s = vx_j$ in T, one has

$$p_{m,n}(t,s) = p_{m,n}(ux_i, vx_j) = u^m v^n \cdot p_{m,n}(x_i, x_j). \tag{4.3}$$

In particular, if we have $t = wux_i$ and $s = zvx_j$ for $w,z \in R^\times$ and $ux_i, vx_j \in \widetilde{T} = A^{[c_\Phi]} \cdot T_0$, then $p_{m,n}(t,s)$ **does *not* depend** on the chosen representations of t and s as products $t = wux_i$ and $s = zvx_j$ in $T = \langle A^{[c_\Phi]}\rangle \cdot T_0 = R^\times \cdot \widetilde{T}$. And, given arbitrary elements $r = \sum_{i=0}^{\nu} a_i u_i x_i$ and $s = \sum_{j=0}^{\nu} b_j v_j x_j$ in M with $a_i, b_j \in \mathbb{Z}$ and $u_i x_i, v_j x_j \in T$, one has

$$p_{m,n}(r,s) = \sum_{i=0}^{\nu}\sum_{j=0}^{\nu} u_i^m v_j^n \cdot p_{m,n}(x_i,x_j). \tag{4.4}$$

The previous equations imply, in particular, the following relations among the $\mathbb{Z}$- and $R^\times$-coefficients occurring in an expression $p_{m,n}(t,s)$. Given arbitrary elements $u,v \in R^\times$ and $x_i, x_j \in T_0$, write

$$p_{m,n}(ux_i, vx_j) = \sum_{\ell=0}^{\nu} a_\ell w_\ell x_\ell \quad\text{and}\quad p_{m,n}(x_i,x_j) = \sum_{\ell=0}^{\nu} b_\ell z_\ell x_\ell,$$

where the coefficients a_ℓ and b_ℓ lie in $\mathbb{Z}$ and $w_\ell, z_\ell \in R^\times$. Then, for all $\ell = 0,\ldots,\nu$, the following relations hold.

$$a_\ell = b_\ell \quad\text{and}\quad w_\ell = u^m v^n z_\ell. \tag{4.5}$$

The above are simple consequences of **(4.3)** and the fact that M is free with basis T_0 over the group ring $\mathbb{Z}[R^\times]$, which itself is free abelian over $R^\times$.

Furthermore, much like the expressions $m(-,-)$ chosen previously, the maps $p_{m,n}(-,-)$ yield representations in M of products of elements in the ring R. More precisely, for all expressions $r,s \in M$,

$$\pi(p_{m,n}(r,s)) = \pi(r)^m \pi(s)^n \in R,$$

where π is the given projection of M onto R as abelian groups.

During the proof of Theorem **4.10**, it will be convenient to represent every unit in R as a product of the form some k-th power in R times some unit from the generating set $A^{[c_\Phi]} \subseteq R$—the power k is typically taken to be two. Such representations will allow us to manipulate relations by using the action of the torus $\mathcal{H}(R)$ on unipotent root subgroups.

Recall that $\widetilde{T} \subset T$ is the finite set $\widetilde{T} = A^{[c_\Phi]} \cdot T_0$. Every element of the (finitely generated) abelian group $R^\times = \langle A^{[c_\Phi]} \rangle$ can be written as a product $w^k u$ where $w \in R^\times$ and $u \in A^{[c_\Phi]}$, whence every $t \in T$ can be written as $t = w^k u x_i$ with $u x_i \in \widetilde{T}$ and $w \in R^\times$. Such a representation is by no means unique. However, if $w^k u x_i$ and $z^k v x_j$ are two distinct representations of the same $t \in T$, then $x_i = x_j$ and thus the relation $w^k u = z^k v$ holds in $R^\times$ because M is a free $\mathbb{Z}[R^\times]$-module on the basis T_0.

With this in mind, we can manipulate the coefficients of the expressions $p_{m,n}(-,-)$ using squares as follows. Given $t, s \in T$, let us write

$$p_{m,n}(t,s) = \sum_{\ell=0}^{\nu} a_\ell(m,n,t,s) g_\ell(m,n,t,s)^2 \mathfrak{o}_\ell(m,n,t,s) x_\ell, \tag{4.6}$$

where $a_\ell(m,n,t,s) \in \mathbb{Z}$, $g_\ell(m,n,t,s) \in R^\times$, $\mathfrak{o}_\ell(m,n,t,s) \in A^{[c_\Phi]}$ and $x_\ell \in T_0$. If any representations $t = w^2 u x_i = \overline{w} x_i$, $s = z^2 v x_j = \overline{z}$ are chosen with $w, z, \overline{w}, \overline{z} \in R^\times$, $u, v \in A^{[c_\Phi]}$ and $x_i, x_j \in T_0$, then **(4.3)** yields the following relations between coefficients of $p_{m,n}(t,s)$, $p_{m,n}(ux_i, vx_j)$ and $p_{m,n}(x_i, x_j)$.

$$\begin{aligned} a_\ell(m,n,t,s) &= a_\ell(m,n,ux_i,vx_j) = a_\ell(m,n,x_i,x_j) \quad \text{and} \\ g_\ell(m,n,t,s)^2 \mathfrak{o}_\ell(m,n,t,s) &= (w^{2m} z^{2n}) g_\ell(m,n,ux_i,vx_j)^2 \mathfrak{o}_\ell(m,n,ux_i,vx_j) \\ &= (\overline{w}^m \overline{z}^n) g_\ell(m,n,x_i,x_j)^2 \mathfrak{o}_\ell(m,n,x_i,x_j) \\ &= (w^2 u)^m (z^2 v)^n g_\ell(m,n,x_i,x_j)^2 \mathfrak{o}_\ell(m,n,x_i,x_j). \end{aligned} \tag{4.7}$$

(We keep in mind that the coefficients g_ℓ and $\mathfrak{o}_\ell$ are not uniquely determined, but rather that $p_{m,n}(t,s)$ does not depend on such choices.)

With the notation established—**which is fixed throughout the present section**—we now show how certain Chevalley- and Steinberg-like relations can be derived using only finitely many commutator and conjugation relations. In fact, the next few lemmata represent the core of the present chapter. The upcoming proofs are rather simple once the reader is comfortable with the notation spelled out above, so we urge them to *consult the previous pages* whenever needed. Recall that $R^\times = \langle A \rangle$.

***Remark* 4.15.** Let J be a finite set and let G be a group containing the (finitely many) symbols $X = \{h_\alpha(a) \mid \alpha \in J,\ a \in A\}$. For every $\alpha \in J$, define the subgroup $H_\alpha = \langle \{h_\alpha(a) \mid a \in A\} \rangle \leq G$. If the subgroup $\langle X \rangle \leq G$ is abelian and if for all $\alpha \in I$ one has

$$\prod_{i=1}^{k} h(a_i)^{\varepsilon_i} = 1 \text{ in } G \text{ whenever } a_1^{\varepsilon_1} \cdots a_k^{\varepsilon_k} = 1 \text{ in } R^\times = \langle A \rangle,$$

then each $H_\alpha \leq G$ is a homomorphic image of $\mathbb{G}_m(R) = R^\times$ and the direct product $\prod_{\alpha \in J} H_\alpha$ surjects onto $\langle X \rangle \leq G$. In this set-up, given $\alpha \in J$ and a unit $u = a_1^{\varepsilon_1} \cdots a_k^{\varepsilon_k} \in R^\times$, we denote

$$h_\alpha(u) := \prod_{i=1}^{k} h(a_i)^{\varepsilon_i},$$

in which case $h_\alpha(u) = h_\alpha(v)$ in G whenever $u = v$ in $R^\times$.

Proof. This is an obvious consequence of von Dyck's theorem, for all relations from $\mathbb{G}_m(R) = R^\times$ hold in each H_α and the whole of $\langle X \rangle$ is abelian. □

To ease computations in the remainder of this section, we shall write $a \bullet b = aba^{-1}$ to denote the **conjugation** of b by a in a group $G \ni a, b$.

Lemma 4.16. *Let $\Psi \subseteq \Phi$ be a (reduced, irreducible) root subsystem of rank two and let G be a group containing the (finitely many) symbols*

$$X = \left\{ x_\gamma(ux_i), h_\alpha(a) \mid \gamma, \alpha \in \Psi,\ a \in A,\ ux_i \in \widetilde{T} = A^{[c_\Phi]} \cdot T_0 \right\}.$$

Given $t = w^2 ux_i \in T = R^\times \cdot \widetilde{T}$ and $\gamma \in \Psi$, define an element $x_\gamma(t)$ in G as

$$x_\gamma(t) := h_\gamma(w) \bullet x_\gamma(ux_i).$$

*Now suppose the hypotheses from Remark 4.15 hold for G and assume further that the following (*finitely many*) relations hold in G:*
for all $\alpha, \beta, \gamma \in \Psi$, $x_i \in T_0$, $m \in \mathbb{Z}$, $a, b \in A$ and $u, v \in A^{[c_\Phi]}$ such that $a^{(\gamma,\alpha)} u = b^{m(\gamma,\beta)} v$ in $R^\times = \langle A^{[c_\Phi]} \rangle$, one has

$$h_\alpha(a) \bullet x_\gamma(ux_i) = h_\beta(b)^m \bullet x_\gamma(vx_i). \tag{4.8}$$

Then the following hold true:

i. *For all $x_i \in T_0$, $w, z \in R^\times$, $\alpha, \beta, \gamma \in \Psi$, $u, v \in A^{[c_\Phi]}$ and $m \in \mathbb{Z}$ such that $w^{(\gamma,\alpha)} u = z^{m(\gamma,\beta)} v$, one has*

$$h_\alpha(w) \bullet x_\gamma(ux_i) = h_\beta(z)^m \bullet x_\gamma(vx_i).$$

ii. *For any $t \in T$ and $\gamma \in \Psi$, the element $x_\gamma(t)$ as above is well-defined, that is, it does not depend on the chosen representation of t as a product of the form $t = w^2 ux_i$ with $w \in R^\times$, $u \in A^{[c_\Phi]}$ and $x_i \in T_0$. Moreover, relations between units hold inside the $x_\gamma(-)$, that is, $x_\gamma(wx_i) = x_\gamma(zx_i)$ for all $w, z \in R^\times$ with $w = z$ in $R^\times$.*

iii. *The Steinberg relations hold in G, that is, for all $t \in T, w \in R^\times$ and $\alpha, \gamma \in \Psi$, one has $h_\alpha(w) \bullet x_\gamma(t) = x_\gamma(w^{(\gamma,\alpha)} t)$.*

Proof. First a clarification. There exist only finitely many relations of the form (**4.8**) because the sets Ψ, T_0 and A are all finite and the powers occurring in the elements of the subset

$$A^{[c_\Phi]} = \left\{ v_1^{\varepsilon_1} \cdots v_\xi^{\varepsilon_\xi} \mid v_j \in A, -c_\Phi \leq \varepsilon_j \leq c_\Phi \ \forall j \right\}$$

of the (finitely generated) abelian group $R^\times$ are bounded.

An obvious but useful observation is the following. If $v_j \in A$ and $\overline{u} = v_1^{\varepsilon_1} \cdots v_\xi^{\varepsilon_\xi}$ is an element of $A^{[c_\Phi]}$ for which $-c_\Phi \leq m(\gamma,\beta) + \varepsilon_j \leq c_\Phi$ for some $m \in \mathbb{Z}$ and $\gamma, \beta \in \Psi$, then Relation (**4.8**) implies

$$h_\beta(v_i)^m \bullet x_\gamma(\overline{u}x_i) = x_\gamma(ux_i), \tag{4.9}$$

where $u = v_1^{\varepsilon_1} \cdots v_j^{m(\gamma,\beta)+\varepsilon_j} \cdots v_\xi^{\varepsilon_\xi} \in A^{[c_\Phi]}$.

Proof of (i): Write $w = a_1 \cdots a_f$ as a product of generators $a_i \in A$. Following Remark **4.15**, we have $h_\alpha(w) = h_\alpha(a_1) \cdots h_\alpha(a_f)$. Applying Relation (**4.8**) f times and again Remark **4.15**, we obtain $m_1, \ldots, m_f \in \mathbb{Z}$, $b_1, \ldots b_f \in A$ and $\overline{v} \in A^{[c_\Phi]}$ such that

$$h_\alpha(w) \bullet x_\gamma(ux_i) = h_\beta(b_1^{m_1} \cdots b_f^{m_f}) \bullet x_\gamma(\overline{v}x_i) \text{ in } G \quad \text{and}$$
$$(b_1^{m_1} \cdots b_f^{m_f})^{(\gamma,\beta)}\overline{v} = w^{(\gamma,\alpha)}u = z^{(\gamma,\beta)}v \text{ in } R^\times.$$

In particular, one has $(z^{-m}b_1^{m_1} \cdots b_f^{m_f})^{(\gamma,\beta)}\overline{v} = v \in A^{[c_\Phi]}$.

Since $R^\times = \langle A \rangle$ and by definition of $A^{[c_\Phi]}$, we may write

$$z^{-m}b_1^{m_1} \cdots b_f^{m_f} = v_1^{\varepsilon_1} \cdots v_\xi^{\varepsilon_\xi} \quad \text{and} \quad \overline{v} = v_1^{\delta_1} \cdots v_\xi^{\delta_\xi}$$

such that the powers ϵ_j, δ_j satisfy $-c_\Phi \leq \varepsilon_j(\gamma,\beta) + \delta_j \leq c_\Phi$ for all $j = 1, \ldots, \xi$. In this way, we also have

$$v = v_1^{\varepsilon_1(\gamma,\beta)+\delta_1} \cdots v_\xi^{\varepsilon_\xi(\gamma,\beta)+\delta_\xi}. \tag{4.10}$$

Again using Remark **4.15**, we obtain

$$\begin{aligned} h_\alpha(w) \bullet x_\gamma(ux_i) &= h_\beta(b_1^{m_1} \cdots b_f^{m_f}) \bullet x_\gamma(\overline{v}x_i) \\ &= (h_\beta(z)^m h_\beta(z)^{-m} h_\beta(b_1^{m_1} \cdots b_f^{m_f})) \bullet x_\gamma(\overline{v}x_i) \\ &= (h_\beta(z)^m h_\beta(v_1^{\varepsilon_1} \cdots v_\xi^{\varepsilon_\xi})) \bullet x_\gamma(\overline{v}x_i) \\ &= (h_\beta(z)^m h_\beta(v_1)^{\varepsilon_1} \cdots h_\beta(v_\xi)^{\varepsilon_\xi}) \bullet x_\gamma(v_1^{\delta_1} \cdots v_\xi^{\delta_\xi} x_i). \end{aligned}$$

But we can apply (**4.9**) multiple times to the last equation above. In other words, by induction on $\sum_{j=1}^{\xi} |\varepsilon_j|$, it follows from (**4.9**) and (**4.10**) that

$$(h_\beta(v_1)^{\varepsilon_1} \cdots h_\beta(v_\xi)^{\varepsilon_\xi}) \bullet x_\gamma(v_1^{\delta_1} \cdots v_\xi^{\delta_\xi} x_i) = x_\gamma(vx_i).$$

Thus $h_\alpha(w) \bullet x_\gamma(ux_i) = h_\beta(z)^m \bullet x_\gamma(vx_i)$, as desired.

Proof of (ii): This follows from Remark **4.15** and the previous item. Indeed, suppose $t = w^2ux_i = \overline{w}^2\overline{u}x_i$ in $R^\times$. Then $u = (w^{-1}\overline{w})^2\overline{u}$ and thus

$$h_\gamma(\overline{w}) \bullet x_\gamma(\overline{u}x_i) = (h_\gamma(ww^{-1}\overline{w})) \bullet x_\gamma(\overline{u}x_i) = h_\gamma(w) \bullet (h_\gamma(w^{-1}\overline{w}) \bullet x_\gamma(\overline{u}x_i))$$
$$\overset{\textbf{(i)}}{=} h_\gamma(w) \bullet x_\gamma(ux_i).$$

The second claim follows from the above with **(i)** and Remark **4.15**.

Proof of (iii): We first remark that the notation $x_\gamma(w^{(\gamma,\alpha)}t)$ makes sense, for the product of $w^{(\gamma,\alpha)} \in R^\times$ with $t \in T = R^\times \cdot \widetilde{T}$ lies in T. Now let $t = w_0^2ux_i$ be any representation of $t \in T$ with $w_0 \in R^\times$, $u \in A^{[c\Phi]}$ and $x_i \in T_0$. Let furthermore $z \in R^\times$, $v \in A^{[c\Phi]}$ and $m \in \mathbb{Z}$ be such that $w^{(\gamma,\alpha)}u = z^{2m}v$ in $R^\times$. In particular, one has that

$$(w_0z^m)^2vx_i = w^{(\gamma,\alpha)}w_0^2ux_i = w^{(\gamma,\alpha)}t \text{ in } T,$$

i.e. the product $(w_0z^m)^2vx_i$ represents the element $w^{(\gamma,\alpha)}t$ in $R^\times$. Thus,

$$h_\alpha(w) \bullet x_\gamma(t) \overset{\textbf{(ii)}}{=} h_\alpha(w) \bullet (h_\gamma(w_0) \bullet x_\gamma(ux_i)) \overset{\textbf{4.15}}{=} h_\gamma(w_0) \bullet (h_\alpha(w) \bullet x_\gamma(ux_i))$$
$$\overset{\textbf{(i)}}{=} h_\gamma(w_0) \bullet (h_\gamma(z)^m \bullet x_\gamma(vx_i)) \overset{\textbf{4.15\&(ii)}}{=} x_\gamma(w^{(\gamma,\alpha)}t),$$

as claimed. □

A group G for which the hypotheses of **4.15** and **4.16** hold also admits an analogue of Lemma **4.13**, as the following shows.

Lemma 4.17. *Let $\Psi \subseteq \Phi$ be a (reduced, irreducible) root subsystem of rank two and let G be a group containing the (finitely many) symbols*

$$X = \left\{x_\gamma(ux_i), h_\alpha(a) \mid \gamma, \alpha \in \Psi,\ a \in A,\ ux_i \in \widetilde{T} = A^{[c\Phi]} \cdot T_0\right\}.$$

Suppose all the hypotheses of Lemma **4.16** *hold for G. For any given $\gamma \in \Psi$, let $\mathfrak{X}_\gamma$ denote the following (finitely generated) subgroup of G.*

$$\mathfrak{X}_\gamma = \left\langle \left\{ h_\alpha(a) \bullet x_\gamma(t) \middle| t \in \widetilde{T}, a \in A, \alpha \in \Psi \right\} \right\rangle \leq G.$$

The following holds: Given two linearly independent roots $\alpha, \beta \in \Psi$, there exist non-zero integers p, q such that the subgroup $H_{\alpha,\beta} := \langle \{h(u) \mid u \in R^\times\} \rangle \leq G$ centralizes $\mathfrak{X}_\alpha$ and $h(u) \bullet x_\beta(t) = x_\beta(u^nt)$ for all $u \in R^\times, t \in T$, where $h(u) := h_\beta(u)^{-q}h_\alpha(u)^p$ and $n := p(\beta,\alpha) - 2q \neq 0$.

Proof. As in the proof of Lemma **4.13**, take any pair $p, q \in \mathbb{Z}\setminus\{0\}$ for which $2p - q \cdot (\alpha,\beta) = 0$. Again we have that $n = p(\beta,\alpha) - 2q$ must be non-zero. The equation from the statement, with n as above, follows immediately from Lemma **4.16(iii)**. And also by Lemma **4.16(iii)** one has

$$h(u) \bullet x_\alpha(t) = x_\alpha(u^{2p-q\cdot(\alpha,\beta)}t) = x_\alpha(u^0t) = x_\alpha(t) \text{ for all } t \in T \text{ and } u \in R^\times,$$

which finishes the proof. □

So far we derived torus- and Steinberg-like relations from certain controlled sets. Next we consider commutators. Suppose once again we have a group G containing the symbols $\left\{x_\gamma(t) \,\middle|\, \gamma \in \Psi,\ t \in T = R^\times \cdot \widetilde{T}\right\}$, where $\Psi \subseteq \Phi$ is a (reduced, irreducible) root subsystem of rank two. Given $t, s \in T$ and $(m,n) \in \mathbb{N} \times \mathbb{N}$, let us write the expression for $p_{m,n}(t,s)$ as

$$p_{m,n}(t,s) = \sum_{\ell=0}^{\nu} a_\ell(m,n,t,s) g_\ell(m,n,t,s)^2 \mathfrak{o}_\ell(m,n,t,s) x_\ell \in \mathbb{Z}[R^\times]$$

with $a_\ell := a_\ell(m,n,t,s) \in \mathbb{Z}$, $g_\ell := g_\ell(m,n,t,s) \in R^\times$, $\mathfrak{o}_\ell := \mathfrak{o}_\ell(m,n,t,s) \in A^{[c_\Phi]}$, $x_\ell \in T_0$. With this notation, given $\gamma, \eta \in \Psi$, define in G the expression

$$\zeta_{m,n}(\gamma,\eta,t,s) = \prod_{\ell=0}^{\nu} x_{m\gamma+n\eta}(g_\ell^2 \mathfrak{o}_\ell x_\ell)^{a_\ell} \tag{4.11}$$

whenever the (integer) linear combination $m\gamma + n\eta$ lies in Ψ. Such an expression might a priori depend on the choice of squares $g_\ell^2 \in R^\times$ and elements $\mathfrak{o}_\ell \in A^{[c_\Phi]}$ representing the units that show up as coefficients of $p_{m,n}(t,s)$. But if G also contains the symbols $\{h_\alpha(a) \mid \alpha \in \Psi,\ a \in R^\times\}$ then, **as long as the hypotheses of Lemma 4.16 hold**, the expression $\zeta_{m,n}(\gamma,\eta,t,s)$ is well-defined. This is because $x_{m\gamma+n\eta}(g_\ell^2 \mathfrak{o}_\ell x_\ell)$ can be written as

$$x_{m\gamma+n\eta}(g_\ell^2 \mathfrak{o}_\ell x_\ell) = h_{m\gamma+n\eta}(g_\ell) \bullet x_{m\gamma+n\eta}(\mathfrak{o}_\ell x_\ell)$$

and $R^\times$ acts on the elements $x_{m\gamma+n\eta}(-)$ via its images $\langle\{h_\alpha(u) \mid u \in R^\times,\ \alpha \in \Psi\}\rangle \leq G$. Moreover $x_{m\gamma+n\eta}(g_\ell^2 \mathfrak{o}_\ell x_\ell) = x_{m\gamma+n\eta}(\overline{g}_\ell^2 \overline{\mathfrak{o}}_\ell x_\ell)$ whenever $g_\ell^2 \mathfrak{o}_\ell = \overline{g}_\ell^2 \overline{\mathfrak{o}}_\ell$ in $R^\times$. Thus, the relations showing that $p_{m,n}(t,s)$ is well-defined in the $\mathbb{Z}[R^\times]$-module M have, by Remark **4.15** and Lemma **4.16**, analogues in G which imply that $\zeta_{m,n}(\gamma,\eta,t,s)$ is well-defined. We can now show that the commutator formulae also derive from controlled sets of relations.

Lemma 4.18. *Let $\Psi \subseteq \Phi$ be a (reduced, irreducible) root subsystem of rank two and let G be a group containing the (finite) set of symbols*

$$X = \left\{x_\gamma(ux_i), h_\alpha(a) \,\middle|\, \gamma, \alpha \in \Psi,\ a \in A,\ ux_i \in \widetilde{T} = A^{[c_\Phi]} \cdot T_0\right\}.$$

Suppose all hypotheses of Lemma **4.16** *hold for G and assume furthermore that the following (*finitely many*) relations hold in G:*
for all $\gamma, \eta \in \Psi$ linearly independent *roots and all $ux_i, vx_j \in \widetilde{T}$,*

$$[x_\gamma(ux_i), x_\eta(vx_j)] = \begin{cases} \displaystyle\prod_{\substack{m\gamma+n\eta\in\Psi \\ m,n>0}} \zeta_{m,n}(\gamma,\eta,ux_i,vx_j)^{C^{\gamma,\eta}_{m,n}}, & \text{if } \gamma+\eta \in \Psi; \\ 1 & \text{otherwise,} \end{cases} \tag{4.12}$$

where the expressions $\zeta_{m,n}(-)$ are defined as in (**4.11**).

Now define in G elements $h_\alpha(u)$ and $x_\gamma(t)$ as in **4.15** *and* **4.16***, with $u \in R^\times$ and $t \in T = R^\times \cdot \widetilde{T}$. Then Chevalley's commutator formula for linearly independent roots holds in G. That is, for all $t, s \in T$ and all $\gamma, \eta \in \Psi$ linearly independent, one has*

$$[x_\gamma(t), x_\eta(s)] = \begin{cases} \prod\limits_{\substack{m\gamma+n\eta\in\Psi \\ m,n>0}} \zeta_{m,n}(\gamma,\eta,t,s)^{C^{\gamma,\eta}_{m,n}}, & \textit{if } \gamma+\eta \in \Psi; \\ 1 & \textit{otherwise.} \end{cases} \tag{4.13}$$

Proof. The main trick here is Lemma **4.17**. It explains why the commutator relations between the $x_\gamma(t)$ and $x_\eta(s)$ over the *infinite* set $T \ni t, s$ can be extracted from the given commutator formulae over the *finite* set $\widetilde{T}$.

Let $\gamma, \eta \in \Psi$ be linearly independent. Following Lemma **4.17**, there exist $p_1, p_2, q_1, q_2 \in \mathbb{Z} \setminus \{0\}$ for which

$$2p_1 - q_1(\eta,\gamma) = 0 = 2p_2 - q_2(\gamma,\eta) \quad \text{and} \tag{4.14}$$

$$p_1(\gamma,\eta) - 2q_1 \neq 0 \neq p_2(\eta,\gamma) - 2q_2. \tag{4.15}$$

Let now $t, s \in T$ be arbitrary. Since $T = R^\times \widetilde{T} = R^\times A^{[c_\Phi]} T_0$, we can find w, z in the (finitely generated) abelian group $R^\times$ and $ux_i, vx_j \in \widetilde{T}$ such that

$$t = w^{p_1(\gamma,\eta)-2q_1} ux_i \quad \text{and} \quad s = z^{p_2(\eta,\gamma)-2q_2} vx_j. \tag{4.16}$$

Using the above, Remark **4.15**, and Lemmata **4.16** and **4.17**, we have that

$$\begin{aligned} [x_\gamma(t), x_\eta(s)] &= [(h_\gamma(w)^{-q_1} h_\eta(w)^{p_1}) \bullet x_\gamma(ux_i), (h_\eta(z)^{-q_2} h_\gamma(z)^{p_2}) \bullet x_\eta(vx_j)] \\ &= (h_\gamma(w)^{-q_1} h_\eta(w)^{p_1} h_\eta(z)^{-q_2} h_\gamma(z)^{p_2}) \bullet [x_\gamma(ux_i), x_\eta(vx_j)]. \end{aligned} \tag{4.17}$$

If $\gamma + \eta \notin \Psi$, then $x_\gamma(ux_i)$ and $x_\eta(vx_j)$ commute by **(4.12)** and thus so do $x_\gamma(t)$ and $x_\eta(s)$ by **(4.17)**.

Now suppose $\gamma + \eta \in \Psi$. Substituting **(4.12)** in **(4.17)**, the proof of the lemma will be finished once we conclude that

$$(h_\gamma(w)^{-q_1} h_\eta(w)^{p_1} h_\eta(z)^{-q_2} h_\gamma(z)^{p_2}) \bullet \zeta_{m,n}(\gamma,\eta,ux_i,vx_j) = \zeta_{m,n}(\gamma,\eta,t,s)$$

for every pair m, n for which $m\gamma + n\eta \in \Psi$. By the definition **(4.11)** of $\zeta_{m,n}$, one has that $\zeta_{m,n}(\gamma,\eta,ux_i,vx_j)$ is equal to the product

$$\prod_{\ell=0}^{\nu} x_{m\gamma+n\eta}(g_\ell(m,n,ux_i,vx_j)^2 \mathfrak{o}_\ell(m,n,ux_i,vx_j)x_\ell)^{a_\ell(m,n,ux_i,vx_j)},$$

where the coefficients $a_\ell(m,n,ux_i,vx_j) \in \mathbb{Z}$, $g_\ell(m,n,ux_i,vx_j) \in R^\times$ and $\mathfrak{o}_\ell(m,n,ux_i,vx_j) \in A^{[c_\Phi]}$ are as in **(4.11)**. By Lemma **4.16**, one has

$$\begin{aligned} &(h_\gamma(z)^{p_2}) \bullet x_{m\gamma+n\eta}(g_\ell(m,n,ux_i,vx_j)^2 \mathfrak{o}_\ell(m,n,ux_i,vx_j)x_\ell) = \\ &= x_{m\gamma+n\eta}(z^{p_2(m\gamma+n\eta,\gamma)} g_\ell(m,n,ux_i,vx_j)^2 \mathfrak{o}_\ell(m,n,ux_i,vx_j)x_\ell). \end{aligned}$$

But $(m\gamma+n\eta,\gamma)=2m+n(\eta,\gamma)$, which yields

$$z^{p_2(m\gamma+n\eta,\gamma)}g_\ell(m,n,ux_i,vx_j)^2\mathfrak{o}_\ell(m,n,ux_i,vx_j)= \\ =(z^{2p_2})^m g_\ell(m,n,ux_i,z^{p_2(\eta,\gamma)}vx_j)^2\mathfrak{o}_\ell(m,n,ux_i,z^{p_2(\eta,\gamma)}vx_j)$$

by **(4.7)**. Thus,

$$(h_\gamma(z)^{p_2})\bullet x_{m\gamma+n\eta}(g_\ell(m,n,ux_i,vx_j)^2\mathfrak{o}_\ell(m,n,ux_i,vx_j)x_\ell)= \\ =x_{m\gamma+n\eta}((z^{2p_2})^m g_\ell(m,n,ux_i,z^{p_2(\eta,\gamma)}vx_j)^2\mathfrak{o}_\ell(m,n,ux_i,z^{p_2(\eta,\gamma)}vx_j)x_\ell).$$

Similarly,

$$(h_\eta(z)^{-q_2})\bullet \\ \bullet x_{m\gamma+n\eta}((z^{2p_2})^m g_\ell(m,n,ux_i,z^{p_2(\eta,\gamma)}vx_j)^2\mathfrak{o}_\ell(m,n,ux_i,z^{p_2(\eta,\gamma)}vx_j)x_\ell)= \\ =x_{m\gamma+n\eta}\Big((z^{-q_2(\gamma,\eta)+2p_2})^m g_\ell(m,n,ux_i,z^{-2q_2+p_2(\eta,\gamma)}vx_j)^2\cdot \\ \cdot\mathfrak{o}_\ell(m,n,ux_i,z^{-2q_2+p_2(\eta,\gamma)}vx_j)x_\ell\Big)= \\ =x_{m\gamma+n\eta}(g_\ell(m,n,ux_i,s)^2\mathfrak{o}_\ell(m,n,ux_i,s)x_\ell)$$

by **(4.14)**, **(4.15)** and **(4.16)**. Analogously, we conclude that

$$(h_\gamma(w)^{-q_1}h_\eta(w)^{p_1})\bullet x_{m\gamma+n\eta}(g_\ell(m,n,ux_i,s)^2\mathfrak{o}_\ell(m,n,ux_i,s)x_\ell)= \\ =x_{m\gamma+n\eta}(g_\ell(m,n,t,s)^2\mathfrak{o}_\ell(m,n,t,s)x_\ell).$$

Since $a_\ell(m,n,ux_i,vx_j)=a_\ell(m,n,t,s)$ by **(4.7)** as well, it follows that

$$(h_\gamma(w)^{-q_1}h_\eta(w)^{p_1}h_\eta(z)^{-q_2}h_\gamma(z)^{p_2})\bullet\zeta_{m,n}(\gamma,\eta,ux_i,vx_j)= \\ =(h_\gamma(w)^{-q_1}h_\eta(w)^{p_1}h_\eta(z)^{-q_2}h_\gamma(z)^{p_2})\bullet \\ \bullet\prod_{\ell=0}^{\nu}x_{m\gamma+n\eta}(g_\ell(m,n,ux_i,vx_j)^2\mathfrak{o}_\ell(m,n,ux_i,vx_j)x_\ell)^{a_\ell(m,n,ux_i,vx_j)}= \\ =\prod_{\ell=0}^{\nu}\big((h_\gamma(w)^{-q_1}h_\eta(w)^{p_1}h_\eta(z)^{-q_2}h_\gamma(z)^{p_2})\bullet x_{m\gamma+n\eta}\big(g_\ell(m,n,ux_i,vx_j)^2\cdot \\ \cdot\mathfrak{o}_\ell(m,n,ux_i,vx_j)x_\ell)\big)^{a_\ell(m,n,ux_i,vx_j)}= \\ =\prod_{\ell=0}^{\nu}x_{m\gamma+n\eta}(g_\ell(m,n,t,s)^2\mathfrak{o}_\ell(m,n,t,s)x_\ell)^{a_\ell(m,n,t,s)}=\zeta_{m,n}(\gamma,\eta,t,s),$$

as desired. □

4.2.1 Proof of Theorem 4.10 for $I=\varnothing$

The case $I=\varnothing$ concerns the Borel subgroup of the given Chevalley–Demazure group, denoted by $\mathcal{B}_\Phi(R)=\mathcal{P}_\varnothing(R)\le\mathcal{G}_\Phi^{\mathrm{sc}}(R)$. The Levi decomposition for such group is just $\mathcal{B}_\Phi(R)=\mathcal{U}_\varnothing(R)\rtimes\mathcal{H}(R)$, where $\mathcal{H}(R)=\langle\mathcal{H}_\alpha(R)\,:\,\alpha\in\Phi\rangle$ is the standard torus and $\mathcal{U}_\varnothing(R)=\langle\mathfrak{X}_\gamma(R)\,:\,\gamma\in\Phi^+\rangle$.

Consider the following sets of relations. For all $\alpha \in \Phi$, $\gamma \in \Phi^+$, $v \in A$, and $t \in T$,

$$h_\alpha(v)x_\gamma(t)h_\alpha(v)^{-1} = x_\gamma(v^{\langle \gamma,\alpha \rangle}t). \tag{4.18}$$

For all $t_1, t_2 \in T$ and $\gamma, \eta \in \Phi^+$,

$$[x_\gamma(t_1), x_\eta(t_2)] = \begin{cases} \prod\limits_{m\gamma+n\eta \in \Phi^+} \zeta_{m,n}(\gamma,\eta,t_1,t_2), & \text{if } \gamma+\eta \in \Phi; \\ 1 & \text{otherwise,} \end{cases} \tag{4.19}$$

where the expression $\zeta_{m,n}(\gamma,\eta,t_1,t_2)$ is defined as in (4.11). And recalling that $\mathcal{A}$ is the kernel of the natural projection $\pi : M \twoheadrightarrow \mathbb{G}_a(R)$, we also consider the following relations: for all $a = \sum_{l=0}^{\nu} a_l w_l x_l \in \mathcal{A}$—where $a_l \in \mathbb{Z}$, $w_l \in R^\times$ and $x_l \in T_0$—and $\gamma \in \Phi^+$,

$$\prod_{l=0}^{\nu} x_\gamma(w_l x_l)^{a_l} = 1. \tag{4.20}$$

Let $\mathcal{S}_\mathcal{B}$ be the set of all relations (4.18), (4.19) and (4.20) given above.

Since $\mathcal{H}(R)$ is finitely generated abelian, we may fix a presentation

$$\mathcal{H}(R) \cong \langle \{h_\alpha(v) \mid \alpha \in \Phi, v \in A\} \mid \mathcal{T} \rangle$$

with $\mathcal{T}$ finite. Combining this with Lemma 4.8 and the given descriptions of R, $\mathcal{A}$ and $\mathcal{S}_\mathcal{B}$, we obtain as in (*) a standard presentation

$$\mathcal{B}_\Phi(R) \cong \langle \{h_\alpha(v), x_\gamma(t) \mid \alpha \in \Phi, \gamma \in \Phi^+, v \in A, t \in T\} \mid \mathcal{T} \cup \mathcal{S}_\mathcal{B} \rangle. \tag{4.21}$$

Suppose the extended Levi factor $\mathcal{LE}_\varnothing(R)$ of $\mathcal{B}_\Phi(R)$ is finitely presented. By definition, $\mathcal{LE}_\varnothing(R)$ consists of a single unipotent root subgroup acted upon non-trivially by the standard torus. Hence every subgroup $\mathfrak{X}_\gamma(R) \rtimes \mathcal{H}(R) \leq \mathcal{B}_\Phi(R)$ for $\gamma \in \Phi^+$ is finitely presented, not just $\mathcal{LE}_\varnothing(R) = \mathfrak{X}_\alpha(R) \rtimes \mathcal{H}(R)$—this follows immediately from Theorem 2.1 and the following observation: given a positive root γ, there exist a simple root $\eta \in \Delta$ and an isomorphism $w : \mathfrak{X}_\gamma(R) \rtimes \mathcal{H}(R) \xrightarrow{\cong} \mathfrak{X}_\eta(R) \rtimes \mathcal{H}(R)$. Indeed, due to the action of the Weyl group W of Φ, we can find $\beta \in \Phi$ and a simple root $\eta \in \Phi$ such that $r_\beta(\gamma) = \eta$, where $r_\beta \in W$ is the reflection associated to β. By the reflection relations (1.10) we can thus take w to be the conjugation in $\mathcal{G}_\Phi^{\mathrm{sc}}(R)$ by w_β, where w_β is the element of (the image of) W corresponding to r_β as defined in Section 1.1.3.

For each $\gamma \in \Phi^+$, let then $\mathfrak{X}_\gamma(R) \rtimes \mathcal{H}(R) = \langle \mathcal{X}_\gamma \mid \mathcal{S}_\gamma \rangle$ be a finite presentation with generating set

$$\mathcal{X}_\gamma = \left\{ h_\alpha(v), x_\gamma(t) \,\middle|\, v \in A, t \in \widetilde{T}, \alpha \in \Phi \right\}.$$

Let $\widetilde{\mathcal{Y}}$ be the *finite* set of generators

$$\widetilde{\mathcal{Y}} = \left\{ \widetilde{x}_\gamma(t) \,\middle|\, t \in \widetilde{T}, \gamma \in \Phi^+ \right\}.$$

We define the following *finite* sets of relations. For all $a, b \in A$, $ux_i, vx_i \in \widetilde{T}$, $m \in \mathbb{Z}$ and $\alpha, \beta \in \Phi, \gamma \in \Phi^+$ for which $a^{(\gamma,\alpha)}u = b^{m(\gamma,\beta)}v$ in $R^\times$,

$$h_\alpha(a)x_\gamma(ux_i)h_\alpha(a)^{-1} = h_\beta(b)^m x_\gamma(vx_i)h_\beta(b)^{-m}. \tag{4.22}$$

(The relations above are finite in number as explained e.g. in the proof of Lemma **4.16**.) For all $t_1, t_2 \in \widetilde{T}, \gamma, \eta \in \Phi^+$,

$$[\widetilde{x}_\gamma(t_1), \widetilde{x}_\eta(t_2)] = \begin{cases} \prod\limits_{m\gamma+n\eta\in\Phi^+} \widetilde{\zeta}_{m,n}(\gamma, \eta, t_1, t_2), & \text{if } \gamma + \eta \in \Phi; \\ 1 & \text{otherwise,} \end{cases} \tag{4.23}$$

where $\widetilde{\zeta}_{m,n}$ is obtained from the $\zeta_{m,n}$ from **(4.11)** by formally replacing $x_{m\gamma+n\eta}(-)$ by $\widetilde{x}_{m\gamma+n\eta}(-)$.
Finally, we let $\widetilde{\mathcal{S}}_\mathcal{B}$ denote the union of the sets $\mathcal{S}_\gamma$ (with γ running over Φ^+) and the sets of all relations **(4.22)** and **(4.23)** given above. Notice that we did not add any defining relators coming from the underlying additive group $\mathbb{G}_a(R)$, except for those possibly contained in the $\mathcal{S}_\gamma$.

Let $\widetilde{\mathcal{B}}_\Phi(R)$ be the group given by the presentation

$$\widetilde{\mathcal{B}}_\Phi(R) \cong \langle \{h_\alpha(v) \mid \alpha \in \Phi, v \in A\} \cup \widetilde{\mathcal{Y}} \mid \mathcal{T} \cup \widetilde{\mathcal{S}}_\mathcal{B} \rangle. \tag{4.24}$$

By construction, $\widetilde{\mathcal{B}}_\Phi(R)$ is finitely presented. We claim that $\widetilde{\mathcal{B}}_\Phi(R) \cong \mathcal{B}_\Phi(R)$. Consider the obvious map $h_\alpha(v) \mapsto h_\alpha(v)$, $\widetilde{x}_\gamma(t) \mapsto x_\gamma(t)$ from $\widetilde{\mathcal{B}}_\Phi(R)$ to $\mathcal{B}_\Phi(R)$. Since (the images of) all the relations in $\mathcal{T} \cup \widetilde{\mathcal{S}}_\mathcal{B}$ hold in $\mathcal{B}_\Phi(R)$ and because $\mathcal{B}_\Phi(R)$ is generated by $\mathcal{H}(R)$ and the $x_\gamma(t)$ for $t \in \widetilde{T}$ and $\gamma \in \Phi^+$, we get a natural epimorphism $\widetilde{\mathcal{B}}_\Phi(R) \twoheadrightarrow \mathcal{B}_\Phi(R)$ by von Dyck's theorem. To prove that this is also injective, let F be the free group on the generating set $\{h_\alpha(v), x_\gamma(t) \mid \alpha \in \Phi, \gamma \in \Phi^+, v \in A, t \in T\}$ of **(4.21)** and consider the homomorphism f given by

$$\begin{aligned} f : F &\longrightarrow \widetilde{\mathcal{B}}_\Phi(R) \\ h_\alpha(v) &\longmapsto h_\alpha(v) \\ x_\gamma(t) &\longmapsto h_\gamma(w)\widetilde{x}_\gamma(ux_i)h_\gamma(w)^{-1}, \end{aligned}$$

where $w \in \langle A^{[c_\Phi]} \rangle$ and $ux_i \in \widetilde{T}$ are such that $t = w^2ux_i$. Since all relations from the torus $\mathcal{H}(R)$ hold in its image in $\widetilde{\mathcal{B}}_\Phi(R)$ and because the relations **(4.22)** are contained in the given presentation for $\widetilde{\mathcal{B}}_\Phi(R)$, it follows from Remark **4.15** and Lemma **4.16** that f is well-defined. It remains to show that the set of relations $\mathcal{S}_\mathcal{B}$ given in **(4.21)** is contained in $\ker(f)$.

The relations (4.18) follow immediately from Lemma 4.16(iii). We now want to show that

$$f([x_\gamma(t_1), x_\gamma(t_2)]) = 1$$

for all $t_1, t_2 \in T$. This is essentially trivial, for f restricted to $\langle \mathfrak{X}_\gamma(R), \mathcal{H}(R)\rangle \le \mathcal{B}_\Phi(R)$ yields a surjection onto

$$\langle \widetilde{\mathfrak{X}}_\gamma, \mathcal{H}_\gamma(R)\rangle = \left\langle \left\{ \widetilde{x}_\gamma(t), h_\gamma(v) \,\middle|\, t \in \widetilde{T}, v \in A \right\} \right\rangle$$

by definition of f and because the set $\widetilde{\mathcal{S}}_{\mathcal{B}}$ from (4.24) contains (the copy of) $\mathcal{S}_\gamma$. As for the remaining commutator relations, which only involve linearly independent roots, we simply observe that all hypotheses from Lemma 4.18 hold for the group $\widetilde{\mathcal{B}}_\Phi(R)$. It then follows that all relations (4.19) are contained in $\ker(f)$. Similarly, we see that the relations (4.20) are in $\ker(f)$. Indeed, whenever $a = \sum_{l=0}^{\nu} a_l w_l x_l \in \mathcal{A}$ then f maps any $\prod_{l=0}^{\nu} x_\gamma(w_l x_l)^{a_l}$ to 1 in $\widetilde{\mathcal{B}}_\Phi(R)$ because this holds in $\langle \mathfrak{X}_\gamma(R), \mathcal{H}(R)\rangle$, which surjects onto $\langle \widetilde{\mathfrak{X}}_\gamma, \mathcal{H}(R)\rangle \le \widetilde{\mathcal{B}}_\Phi(R)$. This concludes the proof.

4.2.2 Proof of Theorem 4.10 for $I \neq \varnothing$

Recall from Section 4.1 that $\mathcal{P}_I(R) = \mathcal{K}_I(R) \rtimes \mathcal{LE}_I(R) \le E_\Phi^{\mathrm{sc}}(R)$ with

$$\mathcal{K}_I(R) = \langle \mathfrak{X}_\gamma(R) \,:\, \gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)} \rangle \quad \text{and}$$

$$\mathcal{LE}_I(R) = \langle \mathcal{H}_\eta(R),\, \mathfrak{X}_\alpha(R) \,:\, \eta \in \Phi \text{ and } \alpha \in \Phi_I \cup \Phi^+_{\mathrm{NAdj}(I)} \rangle.$$

By hypothesis, we may fix a presentation $\mathcal{LE}_I(R) = \langle \mathcal{X} \mid \mathcal{R} \rangle$ with (finite) generating set

$$\mathcal{X} = \left\{ h_\beta(v), x_\alpha(t) \,\middle|\, \beta \in \Phi, \alpha \in \Phi_I \cup \Phi^+_{\mathrm{NAdj}(I)}, v \in A \text{ and } t \in \widetilde{T} \right\}$$

and $\mathcal{R}$ finite. Now consider the following sets of relations. For all $\beta \in \Phi$, $\gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)}$, $v \in A$, and $t \in T$,

$$h_\beta(v) x_\gamma(t) h_\alpha(v)^{-1} = x_\gamma(v^{\langle \gamma, \beta \rangle} t). \tag{4.25}$$

For all $t \in \widetilde{T}$, $t_1 \in T$, $\alpha \in \Phi_I \cup \Phi^+_{\mathrm{NAdj}(I)}$, and $\gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)}$,

$$[x_\alpha(t), x_\gamma(t_1)] = \begin{cases} \displaystyle\prod_{\substack{m,n>0 \\ m\alpha + n\gamma \in \Phi}} \zeta_{m,n}(\alpha, \gamma, t, t_1), & \text{if } \alpha + \gamma \in \Phi; \\ 1 & \text{otherwise,} \end{cases} \tag{4.26}$$

where the expression $\zeta_{m,n}(\gamma, \eta, t, t_1)$ is defined as in (4.11).
For all $t_1, t_2 \in T$ and $\gamma, \eta \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)}$,

$$[x_\gamma(t_1), x_\eta(t_2)] = \begin{cases} \displaystyle\prod_{m\gamma + n\eta \in \Phi^+} \zeta_{m,n}(\gamma, \eta, t_1, t_2), & \text{if } \gamma + \eta \in \Phi; \\ 1 & \text{otherwise,} \end{cases} \tag{4.27}$$

where the expression $\zeta_{m,n}(\gamma, \eta, t_1, t_2)$ is defined as in **(4.11)**.
For all $a = \sum_{l=0}^{\nu} a_l w_l x_l \in \mathcal{A}$ and $\gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)}$,

$$\prod_{l=0}^{\nu} x_\gamma(w_l x_l)^{a_l} = 1. \tag{4.28}$$

Let $\mathcal{S}_I$ be the set of all relations **(4.25)**, **(4.26)**, **(4.27)** and **(4.28)**. Then

$$\mathcal{P}_I(R) \cong \langle \mathcal{X} \cup \{x_\gamma(t) \mid \gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)}, t \in T\} \mid \mathcal{R} \cup \mathcal{S}_I \rangle \tag{4.29}$$

is a presentation for the parabolic subgroup $\mathcal{P}_I(R) \leq \mathcal{G}_\Phi^{\mathrm{sc}}(R)$ as in **(*)**.

To prove Theorem **4.10** for $I \neq \varnothing$ we consider two cases according to the **QG** condition. Recall that the triple (R, Φ, I) is **QG** whenever $\mathbf{B}_2^\circ(R)$ is finitely presented or $\mathbf{B}_2^\circ(R)$ is finitely generated, R is **NVB** for Φ, and $I \neq \{\alpha\}$ in the case where $\Phi = \mathrm{G}_2$ and $\alpha \in \mathrm{G}_2$ is long.

We shall thus list two finite presentations according to whether $\mathbf{B}_2^\circ(R)$ is finitely presented or not, and show that $\mathcal{P}_I(R)$ is isomorphic to them in the given cases. Both presentations have a lot in common—in particular the following (finite) generating set. Let

$$\widetilde{\mathcal{Y}} = \left\{ \widetilde{x}_\gamma(t) \,\middle|\, t \in \widetilde{T}, \gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)} \right\}$$

and define furthermore the following (finite) sets of relations.
For all $a, b \in A$, $ux_i, vx_i \in \widetilde{T}$, $m \in \mathbb{Z}$, $\alpha, \beta \in \Phi$ and $\gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)}$ for which $a^{(\gamma,\alpha)} u = b^{m(\gamma,\beta)} v$ in $R^\times$,

$$h_\alpha(a) \widetilde{x}_\gamma(ux_i) h_\alpha(a)^{-1} = h_\beta(b)^m \widetilde{x}_\gamma(vx_i) h_\beta(b)^{-m}. \tag{4.30}$$

For all $t, s \in \widetilde{T}$, $\alpha \in \Phi_I \cup \Phi^+_{\mathrm{NAdj}(I)}$ and $\gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)}$,

$$[x_\alpha(t), x_\gamma(s)] = \begin{cases} \displaystyle\prod_{\substack{m,n>0 \\ m\alpha+n\gamma \in \Phi}} \widetilde{\zeta}_{m,n}(\alpha, \gamma, t, s), & \text{if } \alpha + \gamma \in \Phi; \\ 1 & \text{otherwise,} \end{cases} \tag{4.31}$$

where $\widetilde{\zeta}_{m,n}$ is obtained from the $\zeta_{m,n}$ from **(4.11)** by formally replacing $x_{m\gamma+n\eta}(-)$ by $\widetilde{x}_{m\gamma+n\eta}(-)$. For all $t_1, t_2 \in \widetilde{T}$ and $\gamma, \eta \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)}$,

$$[\widetilde{x}_\gamma(t_1), \widetilde{x}_\eta(t_2)] = \begin{cases} \displaystyle\prod_{m\gamma+n\eta \in \Phi^+} \widetilde{\zeta}_{m,n}(\gamma, \eta, t_1, t_2), & \text{if } \gamma + \eta \in \Phi; \\ 1 & \text{otherwise,} \end{cases} \tag{4.32}$$

where $\widetilde{\zeta}_{m,n}$ is as above. The finite presentations to be used in the remainder of this proof have $\widetilde{\mathcal{Y}}$ as generating set and contain all relations **(4.30)**, **(4.31)** and **(4.32)** besides a few additional, case-dependent relations.

Case 1 – $\mathbf{B}_2^\circ(R)$ is finitely presented

Similarly to the previous case **4.2.1**, we fix a *finite* presentation

$$\mathbf{B}_2^\circ(R) = \left\langle \left\{ h_{\alpha_0}(v), x_{\alpha_0}(t) \mid v \in A, t \in \widetilde{T} \right\} \mid \mathcal{S}_0 \right\rangle \leq E_{A_1}^{sc}(R)$$

and, for each $\gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)}$, we let $\mathcal{S}_\gamma$ be the set obtained from $\mathcal{S}_0$ by formally replacing α_0 by γ wherever α_0 occurs. Define $\widetilde{S}_{\mathcal{B},I}$ to be the (not necessarily disjoint) union $\bigcup_{\gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)}} \mathcal{S}_\gamma$ together with the sets of all relations (**4.30**), (**4.31**) and (**4.32**) from the end of the previous section. We claim that the finitely presented group

$$\widetilde{\mathcal{P}}_I(R) = \langle \mathcal{X} \cup \widetilde{\mathcal{Y}} \mid \mathcal{R} \cup \widetilde{\mathcal{S}}_{\mathcal{B},I} \rangle \tag{4.33}$$

is isomorphic to the parabolic group $\mathcal{P}_I(R)$.

It is clear that the natural map $h_\alpha(v) \mapsto h_\alpha(v), \widetilde{x}_\gamma(t) \mapsto x_\gamma(t)$ from $\widetilde{\mathcal{P}}_I(R)$ to $\mathcal{P}_I(R)$ induces an epimorphism $\widetilde{\mathcal{P}}_I(R) \twoheadrightarrow \mathcal{P}_I(R)$. Let F be the free group on the generating set $\mathcal{X} \cup \left\{ x_\gamma(t) \mid \gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)}, t \in T \right\}$ of (**4.29**) and consider the homomorphism $f : F \to \widetilde{\mathcal{P}}_I(R)$ given by $\mathcal{X} \ni x \mapsto x$ and $x_\gamma(t) \mapsto h_\gamma(w)\widetilde{x}_\gamma(ux_i)h_\gamma(w)^{-1}$, where $w \in \langle A^{[c_\Phi]} \rangle$ and $ux_i \in \widetilde{T}$ are such that $t = w^2 ux_i$. As in **4.2.1**, the map f is well-defined. Proving that f induces a left-inverse of $\widetilde{\mathcal{P}}_I(R) \twoheadrightarrow \mathcal{P}_I(R)$ is essentially a reprise of the previous case **4.2.1**. In effect, the proof that the relations (**4.25**) are contained in $\ker(f)$ is exactly the one given in **4.2.1**. And since $\widetilde{\mathcal{S}}_I$ contains copies of the $\mathcal{S}_\gamma$ that define $\mathbf{B}_2^\circ(R)$, the commutativity between $\widetilde{x}_\alpha(t_1)$ and $\widetilde{x}_\gamma(t_2)$ is also dealt with exactly like in **4.2.1**. The relations (**4.26**) and (**4.27**) for $\gamma \neq \eta$ belong to $\ker(f)$ by Lemma **4.18**. Finally, the additive relations (**4.28**) belong to $\ker(f)$ because $\langle \mathfrak{X}_\gamma(R), \mathcal{H}_\gamma(R) \rangle \cong \mathbf{B}_2^\circ(R)$ surjects onto $\langle \widetilde{\mathfrak{X}}_\gamma, \mathcal{H}_\gamma(R) \rangle \leq \widetilde{\mathcal{P}}_I(R)$ via f. Thus $\mathcal{S}_I \subseteq \ker(f)$ and we are done.

Case 2 – R is not very bad

The assumptions imposed by the **NVB** condition imply that *the structure constants of the commutator formulae are all invertible*. Thus, for this section we **assume them to be in the generating set** $A^{[c_\Phi]}$ of the group of units $R^\times$. In particular, $(C_{m,n}^{\gamma,\eta})^{\pm 1} \cdot ux_i \in \widetilde{T}$ for all $x_i \in T_0$, $\gamma, \eta \in \Phi_I \cup \Phi^+$ and $u \in R^\times$ for which the product $(C_{m,n}^{\gamma,\eta})^{\pm 1} u$ belongs to $A^{[c_\Phi]}$.

Now let $\widetilde{\mathcal{S}}_I$ be the set of all relations (**4.30**), (**4.31**) and (**4.32**) from Section **4.2.2** together with the following (finitely many) relations (**4.34**) and (**4.35**) regarding the structure constants.

$$\widetilde{x}_\delta((C_{m,n}^{\gamma,\eta})^{\mp 1} ux_i)^{\pm C_{m,n}^{\gamma,\eta}} = \widetilde{x}_\delta(ux_i) \text{ whenever } (C_{m,n}^{\gamma,\eta})^{\mp 1} u \in A^{[c_\Phi]}, \text{ and} \tag{4.34}$$

$$h_\alpha(C_{m,n}^{\gamma,\eta})^{\pm 1} \widetilde{x}_\delta(ux_i)^{\mp(\delta,\alpha) C_{m,n}^{\gamma,\eta}} h_\alpha(C_{m,n}^{\gamma,\eta})^{\mp 1} = \widetilde{x}_\delta(ux_i). \tag{4.35}$$

We shall prove that the finitely presented group

$$\widetilde{\mathcal{P}}_I(R) = \langle \mathcal{X} \cup \widetilde{\mathcal{Y}} \mid \mathcal{R} \cup \widetilde{\mathcal{S}}_I \rangle$$

is isomorphic to $\mathcal{P}_I(R)$. The set-up is the same as in the previous sections, the goal being to show that the relations $\mathcal{S}_I$ live in $\ker(f)$. Following the previous cases, most of the relations in $\mathcal{S}_I$ were already dealt with. For the commutator relations we need only show that

$$f([x_\gamma(t), x_\gamma(s)]) = 1$$

for all $t, s \in T$ and $\gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)}$ because the remaining commutator relations already follow from Lemma **4.18**, as done twice before.

We remind the reader that, since the torus $\mathcal{H}(R)$ is contained in the extended Levi factor $\mathcal{LE}_I(R)$—whose presentation is included in that of $\widetilde{\mathcal{P}}_I(R)$— and all hypotheses for Lemma **4.18** hold for the group $\widetilde{\mathcal{P}}_I(R)$, we may (and do) make free use in $\widetilde{\mathcal{P}}_I(R)$ of any notation and relations between root elements, as discussed on and after Remark **4.15**. In particular, if we define the homomorphism $f : F \to \widetilde{\mathcal{P}}_I(R)$ via $\mathcal{X} \ni x \mapsto x$ and $x_\gamma(t) \mapsto h_\gamma(w)\widetilde{x}_\gamma(ux_i)h_\gamma(w)^{-1}$, where $w \in \langle A^{[c_\Phi]} \rangle$ and $ux_i \in \widetilde{T}$ are such that $t = w^2 ux_i$, then f is well-defined by Remark **4.15** and Lemma **4.16**, and furthermore we can define $\widetilde{x}_\gamma(t) := f(x_\gamma(t))$ for all $t \in T$.

In the sequel we simplify the computations by making use of more explicit Chevalley commutator formulae. Such formulae might depend on the chosen ordering of roots. Nevertheless, working with them imposes no loss of generality since the proofs are analogous if the ordering of the roots (and thus the formulae) change. We refer the reader to classic references (e.g. [48, 93, 39]) for explicit formulae and structure constants, with the usual warning that the explicit structure constants vary with the ordering of roots. We shall often simplify the notation on the structure constants using capital letters, writing e.g. $A, B, C, D, E, \ldots$ instead of $C^{\gamma,\eta}_{m,n}, C^{\alpha,\beta}_{a,b}, \ldots$, and so on.

Let $t, s \in T$ and $\gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)}$. We first want to show that all $\widetilde{x}_\gamma(t)$ and $\widetilde{x}_\gamma(s)$ commute, that is,

$$1 = f([x_\gamma(t), x_\gamma(s)]) = \left[h_\gamma(w)\widetilde{x}_\gamma(ux_i)h_\gamma(w)^{-1}, h_\gamma(z)\widetilde{x}_\gamma(vx_j)h_\gamma(z)^{-1}\right] \quad (**)$$

where $t = w^2 ux_i$ and $s = z^2 vx_j$. Relations **(4.32)** already give us $[\widetilde{x}_\gamma(t), \widetilde{x}_\gamma(s)] = 1$ for all $t, s \in \widetilde{T}$. Since $\gamma \in \Phi^+ \backslash \Phi_{\mathrm{Ext}(I)}$, we may choose $\alpha \in \Phi_I \cup \Phi^+$ and $\beta \in \Phi^+$ such that $\alpha + \beta = \gamma$, which exist either by Lemma **4.11** or by Lemma **4.12**. Now, for each $s \in \widetilde{T}$ and using the commutator relations already at our disposal together with **(4.34)**, we obtain from the explicit commutator formulae (regardless of ordering of roots) some equations for $\widetilde{x}_\gamma(s)$ of the following forms, which depend on the type of the subsystem $\Phi_{\{\alpha,\beta\}}$. (The reader familiar with Chevalley's formulae might rightfully protest that there are other possible equations for $\widetilde{x}_\gamma(s)$, but the ones below suffice for our purposes.)

$$\widetilde{x}_\gamma(s) = \begin{cases} [\widetilde{x}_\alpha(s), \widetilde{x}_\beta(A^{-1})]^{\pm 1}, \text{ if } (m\alpha + n\beta \in \Phi_{\{\alpha,\beta\}} \iff m = n = 1); \\ [\widetilde{x}_\alpha(s), \widetilde{x}_\beta(B^{\pm 1})]\widetilde{\zeta}_{1,2}(\alpha, \beta, s, B^{\pm 1})^{\pm C}, \text{ if } \gamma, \gamma+\beta \in \Phi_{\{\alpha,\beta\}} = \mathrm{B}_2; \\ [\widetilde{x}_\alpha(s), \widetilde{x}_\beta(D^{\pm 1})]\widetilde{\zeta}_{2,1}(\alpha, \beta, s, D^{\pm 1})^{\pm E}\widetilde{\zeta}_{1,2}(\alpha, \beta, s, D^{\pm 2})^{\pm F}, \text{ if } \gamma \text{ is} \\ \text{short and } \Phi_{\{\alpha,\beta\}} \text{ is of type } \mathrm{G}_2 \text{ with both } \alpha, \beta \text{ short.} \end{cases} \tag{4.36}$$

In the above, $A, B, C, ...$ are shortenings for the appropriate structure constants involved in each type. (Warning: Here we are slightly misusing notation. Indeed—recalling the definition of the generating set $\mathcal{X} \cup \widetilde{\mathcal{Y}}$ for our presentation of $\widetilde{\mathcal{P}}_I(R)$—if the root α lies in $\Phi_I \cup \Phi^+_{\mathrm{NAdj}(I)}$, then the symbol $\widetilde{x}_\alpha(-)$ does not exist a priori, but rather $x_\alpha(-)$ does. We then just simply define $\widetilde{x}_\alpha(-)$ to be $x_\alpha(-)$ in this case. Similarly for $\widetilde{x}_\beta$.) We now prove that **(**)** holds using the slightly more explicit equalities from **(4.36)**.

Case 1: Suppose $\gamma = \alpha + \beta$ is the only linear combination of α and β in $\Phi_{\{\alpha,\beta\}}$. Then $\widetilde{x}_\gamma(-)$ is simply a commutator of $\widetilde{x}_\alpha$ and $\widetilde{x}_\beta$ so that the first equation of **(4.36)** applies. We then have

$$f([x_\gamma(t), x_\gamma(s)]) = [\widetilde{x}_\gamma(t), \widetilde{x}_\gamma(s)] \overset{(4.36)}{=} [\widetilde{x}_\gamma(t), [\widetilde{x}_\alpha(s), \widetilde{x}_\beta(A^{-1})]^{\pm 1}] \overset{4.18}{=} 1$$

by Lemma **4.18** since $\gamma + \alpha \notin \Phi_{\{\alpha,\beta\}} \not\ni \gamma + \beta$.

Case 2: Suppose $\Phi_{\{\alpha,\beta\}} = \mathrm{B}_2$ with $\gamma, \gamma + \beta \in \Phi_{\{\alpha,\beta\}}$. Here, $\widetilde{x}_\gamma(-)$ appears as the first term of the formula for $[\widetilde{x}_\alpha(-), \widetilde{x}_\beta(-)]$. Thus

$$f([x_\gamma(t), x_\gamma(s)]) \overset{(4.36)}{=} \Big[\widetilde{x}_\gamma(t), \big[\widetilde{x}_\alpha(s), \widetilde{x}_\beta(B^{\pm 1})\big]\, \widetilde{\zeta}_{1,2}(\alpha, \beta, s, B^{\pm 1})^{\pm C}\Big].$$

Now, both $\gamma + (\alpha + 2\beta)$ and $\gamma + \alpha$ do not lie in $\Phi_{\{\alpha,\beta\}}$, so that any term $\widetilde{x}_\gamma(-)$ commutes with any terms $\widetilde{x}_\alpha(-)$ and $\widetilde{\zeta}_{1,2}(\alpha, \beta, -, -)$ by Lemma **4.18** (and the definition **(4.11)** of $\widetilde{\zeta}$). But Lemma **4.18** also yields equations

$$\begin{aligned} &\widetilde{x}_\gamma(t)\widetilde{x}_\beta(B^{\pm 1}) = \widetilde{x}_\beta(B^{\pm 1})\widetilde{x}_\gamma(t)\widetilde{\zeta}_{1,1}(\gamma, \beta, t, B^{\pm 1})^{\pm G} \quad \text{and} \\ &\widetilde{x}_\gamma(t)\widetilde{x}_\beta(B^{\pm 1})^{-1} = \widetilde{x}_\beta(B^{\pm 1})^{-1}\widetilde{x}_\gamma(t)\widetilde{\zeta}_{1,1}(\gamma, \beta, t, B^{\pm 1})^{\mp G}. \end{aligned}$$

Since $\gamma + \beta = \alpha + 2\beta \in \Phi_{\{\alpha,\beta\}} = \mathrm{B}_2$, it follows from Lemma **4.18** that any $\widetilde{\zeta}_{1,1}(\gamma, \beta, -, -)$ term commutes with the $\widetilde{x}_\gamma(-)$, $\widetilde{x}_\beta(-)$ and $\widetilde{x}_\alpha(-)$ terms. Thus $\widetilde{x}_\gamma(t)$ also commutes with $[\widetilde{x}_\alpha(s), \widetilde{x}_\beta(B^{\pm 1})]$ because

$$\begin{aligned} &\widetilde{x}_\gamma(t)[\widetilde{x}_\alpha(s), \widetilde{x}_\beta(B^{\pm 1})] = \widetilde{x}_\gamma(t)\widetilde{x}_\alpha(s)\widetilde{x}_\beta(B^{\pm 1})\widetilde{x}_\alpha(s)^{-1}\widetilde{x}_\beta(B^{\pm 1})^{-1} = \\ &= \widetilde{x}_\alpha(s)\widetilde{x}_\beta(B^{\pm 1})\widetilde{\zeta}_{1,1}(\gamma, \beta, t, B^{\pm 1})^{\pm G}\widetilde{x}_\alpha(s)^{-1}\widetilde{x}_\beta(B^{\pm 1})^{-1}\widetilde{\zeta}_{1,1}(\gamma, \beta, t, B^{\pm 1})^{\mp G} \cdot \\ &\cdot \widetilde{x}_\gamma(t) = [\widetilde{x}_\alpha(s), \widetilde{x}_\beta(B^{\pm 1})] \left(\widetilde{\zeta}_{1,1}(\gamma, \beta, t, B^{\pm 1})\widetilde{\zeta}_{1,1}(\gamma, \beta, t, B^{\pm 1})^{-1}\right)^{\pm G} \widetilde{x}_\gamma(t) = \\ &= [\widetilde{x}_\alpha(s), \widetilde{x}_\beta(B^{\pm 1})]\widetilde{x}_\gamma(t). \end{aligned}$$

Thus $\left[\widetilde{x}_\gamma(t), \left[\widetilde{x}_\alpha(s), \widetilde{x}_\beta(B^{\pm 1})\right] \widetilde{\zeta}_{1,2}(\alpha, \beta, s, B^{\pm 1})^{\pm C}\right] = 1$, as required.

Case 3: Assume $\Phi_{\{\alpha,\beta\}}$ to be of type G_2. If γ is long, then we may assume without loss of generality that $\{\alpha, \beta\}$ span a subsystem of type $\mathrm{A}_2 \subset \mathrm{G}_2$ (all roots long)—here, the proof is the same as in Case 1.

Suppose then that γ is short. Because (R, Φ, I) is **QG** and $\Phi = \mathrm{G}_2$, the given parabolic $\mathcal{P}_I(R)$ is of the form $\mathcal{P}_{\{\delta\}}(R)$ where $\delta \in \mathrm{G}_2$ is a short root. For the moment let us denote by $\{\delta, \eta\}$ the simple roots of G_2 with η being the long one. Since $\gamma \in \Phi^+ \setminus \mathrm{Ext}(I) = \mathrm{G}_2 \setminus \{\delta\}$, and after analyzing all possibilities for the short root γ as a sum of two roots α, β, we may assume both α and β to be short. In this case, the last equation from **(4.36)** holds for $\widetilde{x}_\gamma(s)$. To show that $f([x_\gamma(t), x_\gamma(s)]) = 1$ we proceed as in Case 2.

Now, γ is a short root which is a sum of two short roots, whence $\gamma+\alpha$ and $\gamma+\beta$ are the only linear combinations of γ, α and γ, β in $\Phi_{\{\alpha,\beta\}}$, respectively. Moreover $\alpha + (\gamma + \beta) = \alpha + (\alpha + 2\beta) \notin \Phi_{\{\alpha,\beta\}}$ because no root in G_2 equals two times a sum of short roots. And $\alpha + (\gamma + \alpha) \notin \Phi_{\{\alpha,\beta\}}$ as well since $\alpha + (\gamma + \alpha) = 3\alpha + \beta$ does not occur in type G_2 with both α and β short.

Combining the above with Lemma **4.18**, we obtain the equalities

$$\begin{aligned}
&\widetilde{x}_\gamma(t)\widetilde{x}_\alpha(s) = \widetilde{x}_\alpha(s)\widetilde{x}_\gamma(t)\widetilde{\zeta}_{1,1}(\gamma, \alpha, t, s)^{\pm H},\\
&\widetilde{x}_\gamma(t)\widetilde{x}_\alpha(s)^{-1} = \widetilde{x}_\alpha(s)^{-1}\widetilde{x}_\gamma(t)\widetilde{\zeta}_{1,1}(\gamma, \alpha, t, s)^{\mp H},\\
&\widetilde{x}_\gamma(t)\widetilde{x}_\beta(D^{\pm 1}) = \widetilde{x}_\beta(D^{\pm 1})\widetilde{x}_\gamma(t)\widetilde{\zeta}_{1,1}(\gamma, \beta, t, D^{\pm 1})^{\pm J} \text{ and}\\
&\widetilde{x}_\gamma(t)\widetilde{x}_\beta(D^{\pm 1})^{-1} = \widetilde{x}_\beta(D^{\pm 1})^{-1}\widetilde{x}_\gamma(t)\widetilde{\zeta}_{1,1}(\gamma, \beta, t, D^{\pm 1})^{\mp J}.
\end{aligned}$$

Furthermore, the expressions $\widetilde{\zeta}_{1,1}(\gamma, \alpha, -, -)$ commute with the terms $\widetilde{x}_\alpha(-)$, $\widetilde{x}_\beta(-)$ and $\widetilde{\zeta}_{1,1}(\gamma, \beta, -, -)$, and the $\widetilde{\zeta}_{1,1}(\gamma, \beta, -, -)$ themselves also commute with the terms $\widetilde{x}_\alpha(-)$ and $\widetilde{x}_\beta(-)$. Expanding the commutator in the product $\widetilde{x}_\gamma(t) \cdot [\widetilde{x}_\alpha(s), \widetilde{x}_\beta(D^{\pm 1})]$ and moving $\widetilde{x}_\gamma(t)$ to the right as in Case 2, it follows that $\widetilde{x}_\gamma(t)$ commutes with the whole commutator $[\widetilde{x}_\alpha(s), \widetilde{x}_\beta(D^{\pm 1})]$. Since $\widetilde{x}_\gamma(-)$ also commutes with the expressions $\widetilde{\zeta}_{2,1}(\alpha, \beta, -, -)$ and $\widetilde{\zeta}_{1,2}(\alpha, \beta, -, -)$ by the properties of linear combinations of γ, α, β from the previous paragraphs, it follows that $\widetilde{x}_\gamma(t)$ commutes with

$$[\widetilde{x}_\alpha(s), \widetilde{x}_\beta(D^{\pm 1})]\widetilde{\zeta}_{2,1}(\alpha, \beta, s, D^{\pm 1})^{\pm E}\widetilde{\zeta}_{1,2}(\alpha, \beta, s, D^{\pm 2})^{\pm F} = \widetilde{x}_\gamma(s).$$

Lemma **4.18** and Cases 1, 2, 3 show that Relations **(4.27)** lie in $\ker(f)$.

It remains to prove that the additive relations **(4.28)** also belong to $\ker(f)$. It suffices to verify this for the simple roots in $\Phi^+ \setminus \Phi_{\mathrm{Ext}(I)}$. Such a simple root, say β, is necessarily adjacent to an $\alpha \in I$, whence by Lemma **4.12** we may assume that the root subgroup $\mathfrak{X}_\beta(R) \leq \mathcal{P}_I(R)$ lies in the unipotent radical of a parabolic subgroup in type $\Phi_{\{\alpha,\beta\}} = \mathrm{A}_2$, B_2 or G_2 whose Levi factor is generated by $\mathfrak{X}_\alpha(R)$, $\mathfrak{X}_{-\alpha}(R)$ and $\mathcal{H}_\beta(R)$. Assume without loss of generality that $\{\alpha, \beta\}$ is a basis for $\Phi_{\{\alpha,\beta\}}$.

From now on we make use of the full commutator relations in $\widetilde{\mathcal{P}}_I(R)$ without further references, for they have just been shown to hold. In particular, we have proved that the root subgroups $\widetilde{\mathfrak{X}}_\gamma(R) := \langle\{\widetilde{x}_\gamma(t) \mid t \in T\}\rangle$ of the unipotent radical of the finitely presented group $\widetilde{\mathcal{P}}_I(R)$ are all abelian, and in fact $\mathbb{Z}[R^\times]$-modules via the action of the torus $\mathcal{H}(R) \leq \widetilde{\mathcal{P}}_I(R)$. This action is compatible with the $R^\times$-action by multiplication on the abelian group $(R,+)$, which is generated (over $\mathbb{Z}$) by T. Proving the relations **(4.28)** lie in $\ker(f)$ means showing that each $\widetilde{\mathfrak{X}}_\gamma(R)$ is isomorphic to $(R,+)$.

Recall from Lemma **1.2** that the equation $[ab,c] = a[b,c]a^{-1}[a,c]$ holds in any group. In particular, if a_i commutes with $[a_j,c]$ and $[a_i,c]$ commutes with $[a_j,c]$ for all i,j, one has

$$\left[\prod_{i=1}^{d} a_i, c\right] = \prod_{i=1}^{d}[a_i,c]. \tag{4.37}$$

Now let $a = \sum_{\ell=0}^{\nu} a_\ell t_\ell \in \mathcal{A}$ be an arbitrary additive relation of R, with $a_\ell \in \mathbb{Z}$ and $t_\ell = w_\ell^2 u_\ell x_\ell \in T = R^\times \widetilde{T}$. We have to show that $\widetilde{x}_\beta(a) := \prod_{\ell=0}^{\nu} \widetilde{x}_\beta(t_\ell)^{a_\ell} = 1$. Suppose first that either $\Phi_{\{\alpha,\beta\}} = \mathrm{A}_2$ or that the given $\alpha \in I$ is a short root with $\Phi_{\{\alpha,\beta\}} = \mathrm{B}_2$ or G_2. In such cases, there exists $\eta \in \Phi_{\{\alpha,\beta\}}$ such that β is the only linear combination either of α,η or of $-\alpha,\eta$. From the commutator relations and invertibility of the structure constants **(4.34)**, one has—for some structure constant C—that

$$\widetilde{x}_\beta(a) = \prod_{\ell=0}^{\nu} \widetilde{x}_\beta(t_\ell)^{a_\ell} = \prod_{\ell=0}^{\nu} [x_{\pm\alpha}(t_\ell), \widetilde{x}_\eta(C^{\mp 1})]^{a_\ell} \overset{(4.37)}{=} \left[\prod_{\ell=0}^{\nu} x_{\pm\alpha}(t_\ell)^{a_\ell}, \widetilde{x}_\eta(C^{\mp 1})\right]$$

But $x_{\pm\alpha}(a) := \prod_{\ell=0}^{\nu} x_{\pm\alpha}(t_\ell)^{a_\ell} = 1$ for this holds in (the image of) the subgroup $\mathcal{LE}_I(R)$ of the finite presentation $\widetilde{\mathcal{P}}_I(R)$. Thus, $\widetilde{x}_\beta(a) = 1$.

Now, since the case where $\Phi_{\{\alpha,\beta\}} = \mathrm{G}_2$ with $\alpha \in I$ long is excluded by the **QG** condition, it remains only to check $\widetilde{x}_\beta(a) = 1$ in the case where $\Phi_{\{\alpha,\beta\}} = \mathrm{B}_2$ and α is long (and thus β is short). We have $\Phi_{\{\alpha,\beta\}} = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha+2\beta)\}$. This time we shall use commutator formulae for type B_2 with more explicit structure constants (e.g. from [93, Chapter 10]) to avoid complicated notations with arbitrary symbols for structure constants. In the given subgroup $\langle \widetilde{\mathfrak{X}}_\beta(R), \mathfrak{X}_{\pm\alpha}(R), \mathcal{H}_\beta(R)\rangle \leq \widetilde{\mathcal{P}}_I(R)$ we have in particular the following equations for all $t \in T$.

$$\begin{cases} [x_\alpha(t), \widetilde{x}_\beta(1)] = \widetilde{x}_{\alpha+\beta}(t)^{\pm 1} \widetilde{x}_{\alpha+2\beta}(t)^{\pm 1} \\ [x_{-\alpha}(t), \widetilde{x}_{\alpha+\beta}(1)] = \widetilde{x}_\beta(t)^{\pm 1} \widetilde{x}_{\alpha+2\beta}(t)^{\pm 1} \\ [\widetilde{x}_{\alpha+\beta}(t), \widetilde{x}_\beta(1)] = \widetilde{x}_{\alpha+2\beta}(t)^{\pm 2}. \end{cases} \tag{4.38}$$

From the second equation, and employing commutator formulae, one has

$$\begin{aligned}
\widetilde{x}_\beta(a) &\overset{\text{Def.}}{=} \prod_{\ell=0}^{\nu} \widetilde{x}_\beta(t_\ell)^{a_\ell} = \prod_{\ell=0}^{\nu} \left([x_{-\alpha}(t_\ell), \widetilde{x}_{\alpha+\beta}(1)]\widetilde{x}_{\alpha+2\beta}(t_\ell)^{\mp 1}\right)^{a_\ell} \\
&\overset{4.18}{=} \left(\prod_{\ell=0}^{\nu}[x_{-\alpha}(t_\ell), \widetilde{x}_{\alpha+\beta}(1)]^{a_\ell}\right)\left(\prod_{\ell=0}^{\nu}\widetilde{x}_{\alpha+2\beta}(t_\ell)^{a_\ell}\right)^{\mp 1} \\
&\overset{(4.37)}{=} \left[\prod_{\ell=0}^{\nu} x_{-\alpha}(t_\ell)^{a_\ell}, \widetilde{x}_{\alpha+\beta}(1)\right]\left(\prod_{\ell=0}^{\nu}\widetilde{x}_{\alpha+2\beta}(t_\ell)^{a_\ell}\right)^{\mp 1} \\
&= \left(\prod_{\ell=0}^{\nu}\widetilde{x}_{\alpha+2\beta}(t_\ell)^{a_\ell}\right)^{\mp 1} =: (\widetilde{x}_{\alpha+2\beta}(a))^{\mp 1}
\end{aligned}$$

because $x_{-\alpha}(a) = \prod_{\ell=0}^{\nu} x_{-\alpha}(t_\ell)^{a_\ell} = 1$ in (the image of) $\mathcal{LE}_I(R)$ in $\widetilde{\mathcal{P}}_I(R)$.

By the first equation from **(4.38)** and an analogous computation, one has that $\widetilde{x}_{\alpha+\beta}(a) = \widetilde{x}_{\alpha+2\beta}(a)^{\pm 1}$ and, also using **(4.37)**, we obtain from the third equation in **(4.38)** that $[\widetilde{x}_{\alpha+\beta}(a), \widetilde{x}_\beta(1)] = \widetilde{x}_{\alpha+2\beta}(a)^{\pm 2}$. Since the structure constants are invertible and contained in our generating set for $R^\times$, we can apply relation **(4.35)** (together with Remark 4.15 and Lemma 4.16) repeatedly to the expression $\widetilde{x}_{\alpha+2\beta}(a)^{\pm 2}$ to obtain the equality $\widetilde{x}_{\alpha+2\beta}(a)^{\pm 2} = h_{\alpha+2\beta}(\pm 2)\widetilde{x}_{\alpha+2\beta}(a)h_{\alpha+2\beta}(\pm 2)^{-1}$.

Combining all of the above gives

$$\begin{aligned}
\widetilde{x}_\beta(a) &= (\widetilde{x}_{\alpha+2\beta}(a))^{\mp 1} = h_{\alpha+2\beta}(\pm 2)^{-1}[\widetilde{x}_{\alpha+\beta}(a), \widetilde{x}_\beta(1)]^2 h_{\alpha+2\beta}(\pm 2) \\
&= h_{\alpha+2\beta}(\pm 2)^{-1}[\widetilde{x}_{\alpha+2\beta}(a)^{\pm 1}, \widetilde{x}_\beta(1)]^2 h_{\alpha+2\beta}(\pm 2) = 1
\end{aligned}$$

because $\widetilde{x}_{\alpha+2\beta}(-)$ commues with $\widetilde{x}_\beta(-)$ by Lemma 4.18.

Thus $\mathcal{S}_I \subseteq \ker(f)$, which concludes the proof of the theorem. □

4.3 Special cases: simply-laced, and S-arithmetic

Below we state as corollary a special case of Theorem 4.2 for ease of reference and to illustrate its reach. Recall that a standard parabolic subgroup of a classical group is *maximal* if the only standard parabolic properly containing it is the whole elementary subgroup of the given classical group.

Corollary 4.19. *Let $\mathcal{G}(R)$ be a classical group with underlying (reduced, irreducible)* simply-laced *root system Φ of rank at least two, and suppose all of its standard parabolic subgroups are finitely generated. Then a standard, non-trivial, maximal parabolic subgroup of $\mathcal{G}(R)$ is finitely presented if and only if its Levi factor is finitely presented.*

Proof. Let $\mathcal{P}_I(R) \leq \mathcal{G}(R)$ be a parabolic subgroup as in the statement. Since $\mathcal{P}_I(R)$ is maximal, every simple root is adjacent to some element of

I, whence the extended Levi factor of $\mathcal{P}_I(R)$ coincides with its Levi factor. Since all standard parabolics of $\mathcal{G}(R)$ are finitely generated, then so is $\mathbf{B}_2^\circ(R)$ by Theorem **2.1**. Because Φ is simply-laced, the triple (R, Φ, I) is certainly **QG** because the **NVB** condition imposes no restrictions on the base ring R in this case. The claim thus follows from Theorem **4.2**. □

The punchline is that the Levi factor does have a 'strong enough action' on the unipotent radical in the cases above. If $\mathbf{B}_2^\circ(R)$ is finitely generated, then Corollary **4.19** and an inductive argument yield a characterization of finite presentability—in terms of the block diagonal—for the parabolics

$$\begin{pmatrix} \mathbf{n_1 \times n_1} & * & \cdots & * \\ 0 & \mathbf{n_2 \times n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \mathbf{n_k \times n_k} \end{pmatrix} \quad \text{in} \quad \mathrm{GL}_n(R) \quad (\text{or} \quad \mathrm{SL}_n(R))$$

fow which no consecutive pairs $(n_i, n_{i+1}) = (1, 1)$ occur. (Recall that such a result cannot hold true in this generality if all n_i are equal to one, as Corollary **2.11** shows.)

An important application of Theorem **4.2** is the following contribution to the theory of S-arithmetic groups. To better state the result we introduce some notation. Let $\mathbf{G}$ be a split reductive linear algebraic group defined over a field $\mathbb{K}$. We say that a parabolic subgroup $\mathbf{P} \leq \mathbf{G}$ has *root gaps* if there exists a $\mathbb{K}$-retract of $\mathbf{P}$ onto a connected $\mathbb{K}$-subgroup $\mathbf{H} \leq \mathbf{G}$ of type (R) with soluble geometric fibers.

Theorem 4.20. *Let $\mathbf{G}$ be a split, connected, reductive, linear algebraic group defined over a global field $\mathbb{K}$. Let $\mathbf{P} \leq \mathbf{G}$ be a proper parabolic subgroup with Levi factor $\mathbf{L} \leq \mathbf{P}$ and suppose $\Gamma \leq \mathbf{P}$ is an S-arithmetic subgroup. Assume further that $|S| > 1$ if $\mathrm{char}(\mathbb{K}) > 0$. The following hold.*

- **i.** *If $\mathrm{char}(\mathbb{K}) = 0$, then Γ is finitely presented;*
- **ii.** *If $\mathrm{char}(\mathbb{K}) > 0$ and $\mathbf{P}$ has root gaps, then Γ is finitely presented if and only if $|S| \geq 3$;*
- **iii.** *If $\mathbf{P}$ has* no *root gaps, $\mathrm{char}(\mathbb{K}) > 0$, and $\mathbb{K}$ is* **NVB** *for the underlying root system of $\mathbf{G}$, then Γ is finitely presented if and only if any S-arithmetic subgroup of $\mathbf{L}$ is so.*

It should be stressed that Theorem **4.20** also holds for the exceptional parabolic in type G_2 excluded from Theorem **4.2** by the **QG** condition.

Part (**i**) above is far from new. Though not formally proved by Abels in [3], it is an immediate consequence of [3, Theorems 5.6.1 and 6.2.3], the Kneser–Tiemeyer local-global principle [60], and cocompactness of parabolics [25, Proposition 9.3]. It was also known to P. Abramenko, via geometric methods. The first published proof of **4.20**(**i**) was Tiemeyer's theorem [98, Corollary 4.5], which also relies on [25, Proposition 9.3].

Part (**ii**) establishes finite presentability of S-arithmetic parabolics in new cases while including the low-dimensional version of Corollary **2.11**—this time with a new proof, independent of [16].

The results of **4.20**(**iii**) for non-minimal proper parabolics over function fields were, to the best of our knowledge, unknown. Finite presentations for small classes of parabolics in the Kac–Moody set-up (over finite fields) have been obtained; see e.g. [49, Corollary 1.2] and references therein. Part (**iii**) also gives examples of 'strong action' of the Levi factor.

Before proving Theorem **4.20**, we recall the following.

Lemma 4.21. *Let $r : \mathbf{H} \twoheadrightarrow \mathbf{G}$ be a $\mathbb{K}$-retract of connected linear algebraic groups. If $\Gamma \leq \mathbf{G}$ is S-arithmetic, then $\phi(\Gamma) \geq \phi(\Lambda)$ for any S-arithmetic subgroup $\Lambda \leq \mathbf{H}$.*

Proof. Since $r : \mathbf{H} \twoheadrightarrow \mathbf{G}$ is a $\mathbb{K}$-retract, we may (and do) identify $\mathbf{G}$ with a $\mathbb{K}$-closed subgroup of $\mathbf{H}$. Without loss of generality, fix a $\mathbb{K}$-embedding $\mathbf{H} \hookrightarrow \mathrm{GL}_n$ for some n, so that both $\mathbf{H}$ and its subgroup $\mathbf{G}$ are seen as $\mathbb{K}$-closed subgroups of the same GL_n. Because S-arithmetic subgroups are commensurable, we may restrict ourselves to the S-arithmetic subgroup $\Gamma := \mathbf{G} \cap \mathrm{GL}_n(\mathcal{O}_S)$ of $\mathbf{G}$, by Lemma **1.9**. Now let $\Lambda \leq \mathbf{H}$ be an arbitrary S-arithmetic subgroup of $\mathbf{H}$ and let $\Lambda_0 := r^{-1}(\Gamma) \leq \mathbf{H}$ be the full pre-image of Γ under r. Since $r : \mathbf{H} \twoheadrightarrow \mathbf{G}$ is a $\mathbb{K}$-retract, it restricts to an ordinary group retract $r|_{\Lambda_0} : \Lambda_0 \twoheadrightarrow \Gamma$. In particular, we have that $\phi(\Gamma) \geq \phi(\Lambda_0)$ by Corollary **1.14**. On the other hand, [69, Lemma 3.1.3(a)] implies that Λ_0 is commensurable with Λ. Thus, $\phi(\Lambda_0) = \phi(\Lambda)$, whence the lemma. □

Proof of Theorem **4.20**. We first reduce the problem to Chevalley–Demazure case. Let $\mathbf{G}$ be a split, connected, reductive, linear algebraic group defined over a global field $\mathbb{K}$, and let $\mathbf{P}$ be a proper parabolic subgroup. Since every parabolic in $\mathbf{G}$ is conjugate to a standard one, we may restrict ourselves to the standard parabolic subgroups with respect to an arbitrary, but fixed, maximal split torus. Proceeding exactly as in the beginning of the proof of Corollary **2.11**, we may assume $\mathbf{G}$ to be a universal Chevalley–Demazure group scheme and we may restrict ourselves to the S-arithmetic subgroup $\mathbf{P}(\mathcal{O}_S)$ of $\mathbf{P}$. Finally, the assumption $|S| > 1$ if $\mathbb{K}$ is a function field guarantees that the parabolics of $\mathbf{P}(\mathcal{O}_S)$ are finitely generated by O'Meara's structure theorem [74, Theorem 23.2].

Part (i). If $\mathrm{char}(\mathbb{K}) = 0$, Abels' theorem [3] implies that $\mathbf{B}_2^\circ(\mathcal{O}_S)$ is always finitely presented, whence $\mathcal{O}_S$ satisfies the **QG** condition. Now, the extended Levi factor is an extension, by a torus, of a direct product of Borel subgroups or reductive groups; cf. Section **4.1**. From Abels' theorem and [24, Theorem 6.2], the extended Levi factors in characteristic zero are always finitely presented. Thus, Part (**i**) follows from Theorem **4.2**.

Part (ii). From now on we assume $\text{char}(\mathbb{K}) > 0$. Since $\mathbf{P}$ has root gaps, it follows from Lemma **4.21** and Theorem **2.1** that $\mathbf{B}_2^\circ(\mathcal{O}_S)$ must be finitely presented if Γ is so, whence $|S| \geq 3$ by Corollary **2.11**. Conversely, if $|S| \geq 3$, then $\mathbf{B}_2^\circ(\mathcal{O}_S)$ is finitely presented by Corollary **2.11**. In particular, regardless of root system Φ and subset of simple roots I, the triple $(\mathcal{O}_S, \Phi, I)$ is always **QG**. Furthermore, since $|S| \geq 3$, the reductive part of an extended Levi factor will always be finitely presented by the low-dimensional Rank Theorem of Behr [14]. Thus, Γ is finitely presented by Theorem **4.2**.

Part (iii). Still assuming $\text{char}(\mathbb{K}) > 0$, suppose further that $\mathbb{K}$ is not very bad for the underlying root system of $\mathbf{G}$. Then the subring $\mathcal{O}_S$ is also **NVB** since it contains the (finite) prime field by [74, 23.1 and 23.2]. Recall that $\mathbf{P} = \mathcal{U} \rtimes \mathcal{L}$ is a standard parabolic subgroup of the simply-connected, semi-simple group $\mathbf{G}$, where $\mathcal{U}$ is the unipotent radical and $\mathcal{L}$ the Levi factor. We then have $\mathcal{L} = (\prod_i \mathcal{G}_{\Phi_i}) \rtimes \mathcal{H}$, where $\mathcal{H}$ is a torus and each $\mathcal{G}_{\Phi_i}$ is a Chevalley–Demazure group scheme; see Section **4.1**. Furthermore, $\mathcal{U}(\mathcal{O}_S)$ admits a presentation as given in Lemma **4.9**.

Suppose $\text{rk}(\mathcal{G}_{\Phi_i}) \geq 2$ for all i. In this case we know from [70, Cor. 4.6] and [10, Theorem 14.1] that each $\mathcal{G}_{\Phi_i}(\mathcal{O}_S)$ equals its elementary subgroup $E_{\Phi_i}(\mathcal{O}_S)$, whence $\mathbf{P}(\mathcal{O}_S)$ has the form given in Definition **4.1**. Thus, Theorem **4.2** applies directly, and we are done.

Assume then $\text{rk}(\mathcal{G}_{\Phi_i}) = 1$ for some i—this might even include the case excluded from Theorem **4.2** where $\mathbf{P}(\mathcal{O}_S) = \mathcal{P}_{\{\alpha\}}(\mathcal{O}_S) \leq \mathcal{G}_{\mathrm{G}_2}(\mathcal{O}_S)$ with α long. Here, $\mathbf{P}(\mathcal{O}_S)$ is finitely presented only if so is $\mathcal{G}_{\Phi_i}(\mathcal{O}_S)$, which in turn is finitely presented if and only if its derived subgroup $\mathcal{G}'_{\Phi_i}(\mathcal{O}_S)$ is so. But the latter is isomorphic to $\text{SL}_2(\mathcal{O}_S)$, which is finitely presented only if $|S| \geq 3$ by Behr's theorem [14]. In this case, the Borel subgroup $\mathbf{B}_2^\circ(\mathcal{O}_S)$ is also finitely presented by Corollary **2.11**, so that Theorem **4.2** holds in this case, too. This concludes the proof of Theorem **4.20**. □

4.4 Concluding remarks, and future directions

As mentioned, we believe Theorem **4.2** still holds without the **QG** condition. The excluded parabolic in G_2 can likely be dealt with by choosing the right ordering of roots. Until then, as a test case, one may consider:

Problem 4.22. Prove that the parabolic $\mathcal{P}_{\{\alpha\}}(\mathbb{F}_5[t, t^{-1}]) \leq \mathcal{G}_{\mathrm{G}_2}^{sc}(\mathbb{F}_5[t, t^{-1}])$, with α long, is finitely presented if and only if its Levi factor is finitely presented (even though none of them admits a finite presentation).

Theorem **4.2** might be strengthened by proving that the finite presentability of the Borel subgroup $\mathbf{B}_2^\circ(R)$ of rank one implies that of (any) universal elementary Chevalley–Demazure group $E_\Phi^{\text{sc}}(R)$. This is true over Dedekind rings of arithmetic type by Borel–Serre and Behr and was often used in the proof of Theorem **4.20**. Whether this is true in general is likely well-known to specialists, though we were unable to find a reference.

Question 4.23. Is there a commutative ring R with unity for which $\mathbf{B}_2^\circ(R) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ is finitely presented but the elementary subgroup $E_2(R) \leq \mathrm{SL}_2(R)$ is not? Equivalently, is the kernel of the natural map $St(A_1, R) \twoheadrightarrow E_2(R)$ finitely generated as a normal subgroup whenever $\mathbf{B}_2^\circ(R)$ is finitely presented? Here, $St(\Phi, R)$ denotes the Steinberg group of type Φ over R.

Besides the restrictions on the characteristic of the base ring, the only missing piece for a full characterization of finite presentability of *all* S-arithmetic parabolics in terms of root gaps and Levi factors is the case where $\mathrm{char}(\mathcal{O}_S) > 0$ and $|S| = 1$. Here, we think of $\mathcal{O}_S$ simply as the ring of polynomials over a finite field. Proceeding e.g. as in [89] or [7], it can be shown that a *maximal* parabolic subgroup of $\mathrm{GL}_n(\mathbb{F}_q[t])$ or $\mathrm{SL}_n(\mathbb{F}_q[t])$ is finitely presented if and only if its Levi factor is so. This likely extends to non-maximal parabolics with no root gaps as well as to other simply-laced root systems. Whether this holds for root systems of types B, C, F and G, however, is unknown to us.

Theorem **4.20(iii)** uses H. Behr's low-dimensional Rank Theorem [14]. Applying his result explicitly, Case **(iii)** of Theorem **4.20** becomes precise:

- Suppose $\mathrm{char}(\mathbb{K}) > 0$ and that $\mathbb{K}$ is **NVB**;
- The semi-simple part of the Levi factor $\mathbf{L} \leq \mathbf{P}$ is covered via a central isogeny by a direct product of (finitely many) classical groups $\mathcal{G}_{\Phi_i}$ of types Φ_i;
- Each such algebraic group has global rank $d_i = |S| \cdot \mathrm{rk}(\Phi_i)$;
- Setting $d = \min_i\{d_i\}$, it follows from **4.20(iii)** and [14] that an S-arithmetic subgroup $\Lambda \leq \mathbf{L}$ (and hence Γ) is finitely presented if and only if $d \geq 3$.

The proofs of the full Rank Theorem [36] and Bux's equality [33] are geometric and independent of type of roots and of characteristic of the base ring. Moreover, they also deal with higher finiteness properties. The following conjecture was posed by Kai-Uwe Bux in the case of maximal parabolics. In face of Theorem **4.2** and appropriately generalizing the notion of extended Levi factor, we extend Bux's conjecture as follows.

Conjecture 4.24. *Let $\mathbf{G} \leq \mathrm{GL}_n$ be a reductive linear algebraic group defined over a global field $\mathbb{K}$. Suppose $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ and let $\mathcal{LE} \leq \mathbf{P}$ denote its extended Levi factor. If $\Gamma \leq \mathbf{P}$ and $\Lambda \leq \mathcal{LE}$ are S-arithmetic, then the equality $\phi(\Gamma) = \phi(\Lambda)$ holds.*

We remind the reader that Conjecture **4.24** holds in characteristic zero by Tiemeyer's theorem [98, Corollary 4.5], with $\phi(\Gamma) = \phi(\Lambda) = \infty$, and also in the split soluble case in positive characteristic by Corollary **2.11**, with $\phi(\Gamma) = \phi(\Lambda) = |S| - 1$.

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List of symbols

$\mathbf{1}_n$	The $n \times n$ identity matrix.	7
$\mathrm{Adj}(I)$	Set of simple roots in $\Delta \setminus I$ that are adjacent to I.	66
$\mathbf{A}_n$	An Abels $\mathbb{Z}$-group scheme ($n \geq 2$).	5
$\mathbf{B}_2^{\circ}(R)$	The standard Borel subgroup of rank one, that is, $\mathbf{B}_2^{\circ}(R) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \leq \mathrm{SL}_2(R)$.	3
$\mathbf{B}_n(R)$	Subgroup of upper triangular matrices of $\mathrm{GL}_n(R)$.	15
$\mathbf{B}_n^{\circ}(R)$	Subgroup of upper triangular matrices of $\mathrm{SL}_n(R)$.	15
$\mathcal{B}_{\Phi}(R)$	The standard Borel subgroup of the universal Chevalley–Demazure group $\mathcal{G}_{\Phi}^{\mathrm{sc}}(R)$.	25
$CC(\mathscr{H})$	The coset complex associated to the family $\mathscr{H}$ of subgroups of a given group G.	46
$\mathrm{char}(\mathbb{K})$	Characteristic of the field $\mathbb{K}$.	2
C_n	The cyclic group of order n.	4
Δ	Subset of simple roots of the root system Φ.	32
Diag	Diagonal matrix of GL_n, that is, $\mathrm{Diag}(u_1, \ldots, u_n) \in \mathrm{GL}_n(R)$ is the matrix whose diagonal entries are $u_1, \ldots, u_n \in R^{\times}$ and all other entries are zero.	14
$D_j(R)$	Subgroup of diagonal matrices of $\mathrm{GL}_n(R)$ whose only diagonal entries $\neq 1$ are in the j-th position.	15
$\mathbf{D}_n(R)$	Subgroup of diagonal matrices of $\mathrm{GL}_n(R)$.	15
$\mathbf{E}_{ij}(R)$	The subgroup of $\mathrm{GL}_n(R)$ generated by all elementary matrices $e_{ij}(r)$ in the same position ij.	14
$\mathbf{E}_k(R)$	The k-th term of the lower central series of $\mathbf{U}_n(R)$.	16
$E_n(R)$	Subgroup of $\mathrm{SL}_n(R)$ generated by all elementary matrices.	14
E_{Φ}^{P}	Elementary Chevalley–Demazure group of type (Φ, P).	23

$\mathrm{Ext}(I)$	Extension of the set of simple roots I, that is, the union of I with its non-adjacent roots $\mathrm{NAdj}(I)$.	67
$\mathbb{G}_a$	The additive affine $\mathbb{Z}$-group scheme, that is, $\mathbb{G}_a(R) = (R,+) \cong \left(\begin{smallmatrix}1 & * \\ 0 & 1\end{smallmatrix}\right) \leq \mathrm{GL}_2(R)$.	14
$\mathbb{G}_m$	The multiplicative affine $\mathbb{Z}$-group scheme, that is, $\mathbb{G}_m(R) = (R^\times, \cdot) = \mathrm{GL}_1(R)$, the group of units of R.	15
$\mathcal{G}_\Phi^P$	Chevalley–Demazure group scheme of type (Φ, P), defined over $\mathbb{Z}$.	23
$\mathcal{G}_\Phi^{\mathrm{sc}}$	Universal Chevalley–Demazure group scheme of type Φ, defined over $\mathbb{Z}$.	23
$\mathcal{H}$	Maximal split torus of a classical group $\mathcal{G}$, defined over $\mathbb{Z}$.	61
$\mathcal{H}_\alpha(R)$	Semi-simple root subgroup of $\mathcal{G}_\Phi^P(R)$ attached to $\alpha \in \Phi$.	23
$\mathscr{H}(n,R)$	A family of horospherical subgroups of $\mathbf{A}_n(R)$.	47
$\mathcal{LE}_I(R)$	The extended Levi factor of a standard parabolic $\mathcal{P}_I(R)$.	67
$\mathcal{L}_I(R)$	Levi factor of a parabolic $\mathcal{P}_I(R)$.	64
$\mathrm{NAdj}(I)$	Simple roots of $\Delta \backslash I$ that are not adjacent to I.	67
$N(\{X_\lambda\}_{\lambda\in\Lambda})$	The nerve complex of the covering $\{X_\lambda\}_{\lambda\in\Lambda}$ of a set X.	46
$\mathcal{O}_S$	A Dedekind ring of arithmetic type associated to the finite set of places S of its fraction field.	26
$\mathcal{P}_\varnothing(R)$	The standard Borel subgroup of a classical group $\mathcal{G}(R)$.	62
Φ	A root system, usually assumed in this work to be reduced and irreducible.	3
$\phi(G)$	The finiteness length of the group G.	10
Φ_X	The root subsystem generated by the subset $X \subseteq \Phi$, that is, $\Phi_X = \mathrm{span}_{\mathbb{Z}}(X) \cap \Phi$.	61
$\mathcal{P}_I(R)$	The standard parabolic subgroup of a classical group $\mathcal{G}(R)$, associated to a subset I of simple roots of $\mathcal{G}$.	61
R	An arbitrary commutative ring with unity.	2
$\mathcal{U}_I(R)$	Unipotent radical of a parabolic $\mathcal{P}_I(R)$.	64
$\mathbf{U}_n(R)$	Subgroup of upper unitriangular matrices of $\mathrm{GL}_n(R)$.	16
$\mathfrak{X}_\alpha(R)$	Unipotent root subgroup of $\mathcal{G}_\Phi^P(R)$ attached to $\alpha \in \Phi$.	23

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