Stochastic Transport Equations:

Method of Characteristics versus Scaling Transform Approach

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Contents

1. Introduction

1.1. Brief summary of the problem

In 2015 Viorel Barbu and Michael Röckner developed a technique to prove existence and uniqueness of global solutions to infinite-dimensional stochastic equations of the form

$$\begin{cases} \mathrm{d}X(\cdot,t) = -A(\cdot,t)X(\cdot,t)\,\mathrm{d}t + X(\cdot,t)\,\mathrm{d}\mathbb{W}(\cdot,t) \\ X(\cdot,t) = h(\cdot), \end{cases} \tag{1.1}$$

where A is a nonlinear, monotone, demicontiuous, coercive operator with polynomial growth and $\mathbb{W}$ is a Wiener process on some Hilbert space. In *An operatorial approach to stochastic partial differential equations driven by linear multiplicative noise* [BR15] this technique uses a scaling transform to transfer equation (1.1) into an operator equation of the type

$$\mathscr{B}y + \mathscr{A}y = 0. \tag{1.2}$$

Under the so-called maximal monotonicity of the operators $\mathscr{A},\mathscr{B}$ this kind of equation (1.2) has a unique solution. Consequently there exists also a unique solution to (1.1). But this scaling transform approach is only applicable under the mentioned assumptions of coercivity, demicontinuity and monotonicity. It also yields existence and uniqueness in the case of the stochastic transport equation given by

$$\begin{cases} \mathrm{d}X(x,t) = \sum\limits_{i=1}^{d} \alpha_i(x,t)\dfrac{\partial X(x,t)}{\partial x_i}\,\mathrm{d}t - \beta(x,t)X(x,t)\,\mathrm{d}t \\ \qquad\qquad - \lambda X(x,t)\,|X(x,t)|^{q-2}\,\mathrm{d}t + X(x,t)\,\mathrm{d}\mathbb{W}(x,t) \\ X(x,0) = h(x), \\ X(t) = 0 \text{ on } \Upsilon := \Big\{(x,t) \in \partial\mathbb{O} \times [0,\mathbf{T}] \;\Big|\; -\sum\limits_{i=1}^{d} \alpha_i(x,t)n_i(x) < 0\Big\}, \end{cases} \tag{1.3}$$

for some $\lambda > 0$, $q \geq 2$, $\alpha_i, \beta, i = 1, ..., d$, which are continuous in time and space and α_i are additionally one-times continuously differentiable with respect to space. In the deterministic case the simplest form of a transport equation is given by

$$\frac{\partial u}{\partial t} + c \cdot \nabla u = 0. \tag{1.4}$$

In general, this can be used to model the density of a physical quantity or the transport of a particle in a fluid, such as a solute in a pipe with water. Here one can think of air pollution or a traffic flow problem where for example the density of the vehicles depends on position and time. The basic tool to solve such a kind of transport equation (1.4) is given in almost all literature concerning partial differential equations and known as the method of characteristics. This famous approach is based on the transformation of the partial differential equation into a system of ordinary differential equations. Solving this system, one constructs a solution of the partial differential equation by combining the solutions to the system in a suitable way.

The method of characteristics was published in 1803 by Gaspard Monge in *Mémoire sur la théorie d'une équation aux dérivées partielles du premier ordre* and was developed further, among others by Joseph-Louis Lagrange, Paul Charpit and Sylvestre F. Lacroix (see [HP15]). 1984 Hiroshi Kunita extended this well-known method to stochastic partial differential equations. This approach is called method of stochastic characteristics. In *First order stochastic partial differential equations* [Kun84a] and later in the book *Stochastic flows and stochastic differential equations* [Kun97] the author proves existence and uniqueness of local solutions to stochastic partial differential equations of the form

$$\begin{cases} \mathrm{d}u = F(x, u, \nabla u, \circ \mathrm{d}t), \\ u|_{t=0} = g, \end{cases} \tag{1.5}$$

where F is a semimartingale, which is Hölder continuous and 5-times continuously differentiable with respect to all variables $(x, u, \nabla u)$ and of linear growth in all variables. Obviously, the not so common notation of the semimartingale has to be defined precisely and then a representation result (see Theorem 2.39 below) can be proved. By this it is possible to transform the equation (1.5) into a more convenient type of equations given by

$$\begin{cases} \mathrm{d}u = f_0(x, u, \nabla u, t)\, \mathrm{d}t + \sum_{n\geq 1} f_n(x, u, \nabla u, t) \circ \mathrm{d}W_t^n, \\ u(x, 0) = g(x), \end{cases} \tag{1.6}$$

where $(W_t^n)_{n\geq 1}$ are infinite independent copies of a one-dimensional Brownian motion. Due to the fact that we already know that there exists a unique global solution to the stochastic transport equation (1.3), the question arises if it is possible to solve this equation also by the method of stochastic characteristics due to the main result Theorem 4.5 of [Kun97, Theorem 6.1.5]. The main advantage of the method of characteristics is that in a number of examples one obtains an explicit expression of the solution. The reader might see that the equations (1.3) and (1.6) are given in different settings. [BR15] considers perturbations by general space-dependent Wiener processes and in terms of an Itô integral while [Kun97] works with perturbations by a series of independent Brownian motions and in terms of a Stratonovich integral. Furthermore, the conditions on the coefficient functions are very different.

In this thesis we will elaborate the scaling transform approach in the example of the stochastic transport equation (1.3), as well as the method of stochastic characteristics as given in [Kun97]. During our studies we quickly realized that the main existence and uniqueness result [Kun97, Theorem 6.1.5] as stated by H. Kunita is not applicable in the case of the stochastic transport equation. Therefore we extend the method of stochastic characteristics to a heuristic approach. By direct calculations of the method we end up with an explicit expression of solutions.
In a first step we apply this heuristic approach to Burgers type equations given in the form

$$\mathrm{d}u = h(u) \cdot \nabla u\, \mathrm{d}t + B(u)\, \mathrm{d}W_t \tag{1.7}$$

and for explicitly given coefficient functions $h(u)$. We also generalize the example of Y. Yamato in [Kun84a] to the two-dimensional case. As expected for a heuristic approach we have to verify that all determined candidates for solutions really solve the considered problems. After some successful examples we consider the stochastic transport equation. For the simple reason that an application of Theorem 4.5 is not possible, we firstly determine the solution of the one-dimensional stochastic transport equation with explicitly given coefficient functions and perturbed by a Brownian motion with Stratonovich differentials of the form

$$\begin{cases} \mathrm{d}u(x,t) = \Big(x\,\nabla u(x,t) - \lambda u(x,t)\,|u(x,t)|^{q-2}\Big)\,\mathrm{d}t + u(x,t)\circ \mathrm{d}W_t \\ u(x,0) = x^2. \end{cases} \tag{1.8}$$

In a next step we focus on the perturbation by a general infinite-dimensional Wiener process $\mathbb{W}$. Choosing special orthonormal bases and setting the drift terms to zero, we determine an explicit solution to the simplified stochastic equation

$$\begin{cases} \mathrm{d}u = u\circ \mathrm{d}\mathbb{W} \\ u(x,0) = h(x). \end{cases} \tag{1.9}$$

By using the Itô-Stratonovich formula we end up with the fact that an application of the method of stochastic characteristics to the stochastic transport equation (1.3) is not possible in general. The Itô correction term including the orthonormal basis of the general Wiener process makes an application of the heuristic method of stochastic characteristics impossible.

For the method of stochastic characteristics H. Kunita developed a technique of finding inverse processes. This result is a basic tool in *Fully Nonlinear Stochastic Partial Differential Equations* [DPT96] of the authors Giuseppe Da Prato and Luciano Tubaro, but it is not explicitly stated therein. Therefore we formulate and prove this result (see Lemma 8.5 below) in detail at the end of this thesis.

In his book *Stochastic partial differential equations* [Cho07] Pao-Liu Chow applied the method of stochastic characteristics to solve linear and quasilinear stochastic partial differential equations. In the first two chapters he reproduces many results of [Kun97] in the classical case of Brownian motion. The conditions on the coefficient functions coincide with the conditions for the main result of [Kun97]. One should note that the main tool of the approach is to find an inverse process. With a restriction on the domain using a proper stopping time, the results are given for almost all elements of the probability space and all space and time variables depending on the stopping time. In [Cho07] these restrictions and corresponding stopping times are not given explicitly or have been overlooked, respectively, but the author denotes the solutions as pathwise solutions, which seems to correspond with our notation. Nevertheless the representation formula (see Theorem 2.39 below), which is proved in this thesis, is vaguely stated therein (see [Cho07, Equation (2.13)]).

For the reader's convenience we give an overview in the beginning of each chapter concerning the main results, proofs and contributions.

1.2. Aim of the thesis

The scaling transform approach, as well as the method of stochastic characteristics, both have their own advantages. By the method of stochastic characteristics we get an explicit expression of solutions, provided that we consider explicitly given coefficient functions. On the other hand V. Barbu and M. Röckner prove a general existence and uniqueness result which is valid for a large class of equations and as we will see, also for the stochastic transport equation (1.3). The result includes existence and uniqueness and we know, due to the scaling transform, that the solution is of the form $X(t) = e^{\mathbb{W}}y(t)$, where y solves a certain random partial differential equation (see (7.3) below). We have not an explicit expression, but we obtain the existence of a global solution.

The first aim of this thesis is to reformulate the method of stochastic characteristics in a more convenient and more detailed version. To this end the representation results e.g. Theorem 2.39 below are the most important steps. In [Kun97] these statements are given in vaguely formulated exercises. It is a known fact that there exist different concepts of solutions, like global or local solutions. Hence the kind of solution has to be defined rigorously. Due to the fact that we restrict the domain of the processes to a domain defined for almost all elements ω of the probability space, the solutions are local ones which are defined up to a certain stopping time. In our opinion, this consideration is very important and can easily be overlooked in [Kun97]. Therefore we go into much detail concerning the kind of local solution.

Nevertheless the main task of this thesis is the application of the method of stochastic characteristics to the stochastic transport equation. For this purpose it is necessary to generalize the method of stochastic characteristics to a heuristic approach. It means we have to determine solutions by hand. One should note that for Kunita's main result (see Theorem 4.5 below) and the explicit expression of the solution, one has to solve a system of stochastic differential equation also by hand, as well as in the heuristic approach. We observe that there is only one example given for which Theorem 4.1 in [Kun84a] is applicable. This example is a one-dimensional Burgers equation without drift term. It was done by Y. Yamato in [Kun84a]. We generalize this example to two dimensions. Furthermore, we also consider different kinds of drift terms and observe that the heuristic approach works successfully. Hence we obtain an expression of solutions and therefore existence of the solutions. Furthermore, we give an example which makes the main result of [Kun97] (see Theorem 4.5 below) concrete. Considering the stochastic transport equation (1.3), the Itô-Stratonovich formula has a very important role. The application of the method of stochastic characteristics to the stochastic transport equation perturbed with respect to Stratonovich differential instead of Itô differential is possible in a few situations e.g.

- $\mathrm{d}u(x,t) = \Big(x\,\nabla u(x,t) - \lambda u(x,t)\,|u(x,t)|^{q-2}\Big)\,\mathrm{d}t + u(x,t)\circ \mathrm{d}W_t$

  (see Example 6.4 below)

- $\mathrm{d}u = u \circ \mathrm{d}\mathbb{W} = \sum_{j=1}^{\infty} \sqrt{\frac{2}{\pi}}\,\mu_j\,\sin(jx)\,u\circ \mathrm{d}W_t^j$. (see Example 6.8 below)

If we look at the stochastic transport equation given in the form (1.3) with Itô differential, we have to rewrite it into the Stratonovich setting (see Lemma 6.11) and hence obtain an additional drift term in the differential equation. We demonstrate up to which conditions on the coefficient functions and perturbation the method of stochastic characteristics gives us a local solution. With the detailed preparation of the scaling transform approach in the case of the stochastic transport equation, we compare the three methods, namely the scaling transform approach, the heuristic method of stochastic characteristics and the application of Theorem 4.5 below. The main tool in the proof of Theorem 4.5 is Lemma 4.8 below which states that under certain conditions there exists a process satisfying an inverse property. This result plays an important role in the article [DPT96]. We give a review on the method therein, formulate the theorem to find an inverse process and thus fill a corresponding gap in [DPT96].

1.3. Structure

The thesis is separated in 8 chapters. After a motivating introduction, which includes a repetition of the classical method of characteristics, the second chapter starts with basic definitions to fix the setting. In this part we prove fundamental representation results for stochastic differential equations (see Theorem 2.33, Theorem 2.34 below), as well as for stochastic partial differential equations. The special kind of notation

$$\mathrm{d}u = F(x, u, \nabla u, \circ \mathrm{d}t)$$

used in [Kun97] in the one-dimensional case, as well as in the multidimensional case, can be represented by Brownian motions, which is formulated in Corollary 2.42 below. Furthermore, we repeat some important tools. Chapter 3 contains the derivation of the method of stochastic characteristics which generalizes Subsection 1.5. below of the introduction. The third chapter is written in a nutshell and can be used to apply the heuristic approach of the method of stochastic characteristics. We do not formulate a theorem, but a stepwise derivation. For applications this step-by-step formula is more convenient. Theorem 4.5, considering the existence and uniqueness of solutions to first order stochastic partial differential equations, is the primary part of Chapter 4. In Corollary 4.6 this existence and uniqueness result applied to the case of Brownian motions is formulated. In this chapter also a detailed written proof of the main theorem is included. It follows the fifth chapter working on simple, but precise examples in which we apply the heuristic approach to the case of some stochastic Burgers type equations. Chapter 6 gives the answer to our main initial question. Under certain conditions we solve the stochastic transport equation locally by an application of the heuristic method of stochastic characteristics. Chapter 7 contains the scaling transform approach with a repetition of the main result and a detailed proof in the case of the stochastic transport equation. We finish Chapter 7 with a summarizing diagram which gives a comparison of all methods considered in this thesis. In the last part we work out an application of Lemma 4.8 in the article [DPT96] of G. Da Prato and L. Tubaro.

1.4. Future directions

The method of stochastic characteristics as published in [Kun97] is based on a representation result which we prove in Chapter 2. In applications it might be of interest if other kinds of noises or other local martingale representatives can be used to find solutions of stochastic partial differential equations perturbed by these kinds of processes. In another step it would be useful to generalize the conditions on the coefficient functions, to solve a stochastic partial differential equation by the method of stochastic characteristics. We already know that this method can be applied in many situations although the conditions of [Kun97] are not fulfilled. Furthermore, one could study the application of the method of stochastic characteristics to the random partial differential equation which we obtain by the scaling transform approach.

1.5. The method of characteristics for nonlinear partial differential equations

The method of characteristics is one of the classical approaches to solve quasilinear and also nonlinear partial differential equations of first order locally. It can be found in well-known literature e.g. [Eva08, Chapter 3.2], [Str07, Section 1.2], [Han11, Chapter 2, 2.2], [Smi64, Chapter III, §1], [Gar67, Chapter 2.2] and [Cou68, §3]. The idea of this approach is to transform a partial differential equation into a system of ordinary differential equations. The solutions to these ordinary differential equations and in particular their inverse functions form a solution of the partial differential equation by a smart combination. The transformation itself is based on a coordinate transformation. That means the system of ordinary differential equations is generated by curves - the so-called characteristic curves. The technique is based on the assumption that such curves exist and that we obtain the corresponding system of ordinary differential equations. By solving this system, respectively if we assume that this system is solvable, we construct a solution to the partial differential equation. The geometrical picture behind this is to find a solution to the partial differential equation by constructing a curve lying in the surface of the corresponding graph of the unknown function. In the following we review the method of characteristics for first order nonlinear partial differential equations on $U := \mathbb{R}^d \times (0, \mathbf{T})$ for some $\mathbf{T} > 0$. Let $x = (x_1, ..., x_d, x_{d+1}) \in U$ where x_{d+1} is the time variable and

$$F\colon \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times [0, \mathbf{T}] \to \mathbb{R}$$

be a given smooth function. Let

$$\Gamma := \{x \in \mathbb{R}^{d+1} \,|\, x_{d+1} = 0\} \subset \partial U$$

and $g\colon \Gamma \to \mathbb{R}$ also be a given smooth function. We consider the following Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial x_{d+1}} = F\Big(x_1, ..., x_d, u(x), \dfrac{\partial u}{\partial x_1}(x), ..., \dfrac{\partial u}{\partial x_d}(x), x_{d+1}\Big) \ \forall \ x \in U \\ \quad u|_\Gamma = g. \end{cases} \tag{1.10}$$

We suppose that u solves the partial differential equation (1.10) with boundary condition on Γ and that u is a $\mathcal{C}^2$-function. Let $x \in U$ be fixed. We want to calculate $u(x)$ by finding a curve lying in U and connecting x with an initial value in Γ. Define

$$\Gamma^* := \big\{r^* \in \mathbb{R}^d \,\big|\, (r^*, 0) \in \Gamma\big\}.$$

Let y be a point near Γ with $y = (y_1, ..., y_d, y_{d+1}) \in \mathbb{R}^d \times [0, \mathbf{T}]$. Suppose that y can be reached by a curve, i.e. there exist $r \in \Gamma^*$, $s \in [0, \mathbf{T}]$ and a function

$$\mathbf{x}\colon \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$$

such that $\mathbf{x}(r, s) = y$. That means starting at $r \in \Gamma^*$ with initial condition

$$\mathbf{x}(r, 0) = (r, 0),$$

the i-th component of the curve denoted by $x_i(r, s)$ reaches y_i at time s. Without loss of generality let $y_{d+1} = s$. Then we define with Lagrange's notation ($\frac{\partial u}{\partial x_i} = u_{x_i}$)

$$\begin{aligned} \mathbf{z}(r,s) &:= u(\mathbf{x}(r,s)) = u(x_1(r,s), ..., x_d(r,s), s), \\ p_i(r,s) &:= u_{x_i}(\mathbf{x}(r,s)) = u_{x_i}(x_1(r,s), ..., x_d(r,s), s), \\ \mathbf{p}(r,s) &:= (p_1(r,s), ..., p_d(r,s)). \end{aligned}$$

Additionally to the existence of the curve $\mathbf{x}$ we assume the following:

Assumption 1.1 *The corresponding initial conditions for each $r \in \Gamma^*$ are given by $g\colon \mathbb{R}^{d+1} \to \mathbb{R}$ with*

$$\begin{aligned} \mathbf{z}(r,0) &= g(r,0), \\ p_i(r,0) &= g_{x_i}(r,0) \end{aligned} \tag{1.11}$$

and satisfy

$$p_{d+1}(r,0) - F(x_1(r,0), ..., x_d(r,0), g(r,0), g_{x_1}(r,0), ..., g_{x_d}(r,0), 0) = 0. \tag{1.12}$$

Conditions (1.11) and (1.12) are called **compatibility conditions** and initial conditions satisfying these conditions are called **admissible** (cf. [Eva08, 3.2.3 b.]).

Remark 1.2 *Let $r \in \Gamma^*$. Due to*

$$\begin{aligned} \frac{\partial}{\partial p_{d+1}}\Big[& p_{d+1}(r,0) \\ & - F(x_1(r,0), ..., x_d(r,0), g(r,0), g_{x_1}(r,0), ..., g_{x_d}(r,0), 0)\Big] = 1 \neq 0 \end{aligned} \tag{1.13}$$

another assumption on so-called noncharacteristic initial conditions as written in ([Eva08, §3 - after Lemma 1]) is fulfilled.

Now we rewrite equation (1.10) to obtain

$$p_{d+1}(r,s) - F(x_1(r,s), ..., x_d(r,s), \mathbf{z}(r,s), \mathbf{p}(r,s), x_{d+1}(r,s)) = 0 \tag{1.14}$$

for $r \in \Gamma^*$, $s \in [0, \mathbf{T}]$. By using the notation of Newton's derivative ($\dot{} = \frac{\mathrm{d}}{\mathrm{d}s}$) we have

$$\dot{\mathbf{z}}(r,s) = \sum_{i=1}^{d} \dot{x}_i(r,s) p_i(r,s) + p_{d+1}(r,s).$$

By differentiating (1.14) with respect to s we also know that

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big[p_{d+1}(r,s) - F(x_1(r,s), ..., x_d(r,s), \mathbf{z}(r,s), \mathbf{p}(r,s), x_{d+1}(r,s))\Big] = 0. \tag{1.15}$$

If we apply the classical chain rule, we get that the left hand side of (1.15) is equal to

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\Big[p_{d+1}(r,s) - F(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s))\Big] \\
&= \dot{p}_{d+1}(r,s) - \sum_{i=1}^{d}\dot{x}_i(r,s)F_{x_i}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&\quad - \dot{\mathbf{z}}(r,s)F_{\mathbf{z}}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&\quad - \sum_{i=1}^{d}\dot{p}_i(r,s)F_{p_i}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&\quad - \dot{x}_{d+1}(r,s)F_{x_{d+1}}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&= \dot{p}_{d+1}(r,s) - \sum_{i=1}^{d}\dot{x}_i(r,s)F_{x_i}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&\quad - \Big(\sum_{i=1}^{d}\dot{x}_i(r,s)p_i(r,s) + p_{d+1}(r,s)\Big)F_{\mathbf{z}}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&\quad - \sum_{i=1}^{d}\dot{p}_i(r,s)F_{p_i}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&\quad - F_{x_{d+1}}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&= \dot{p}_{d+1}(r,s) - \sum_{i=1}^{d}\Big(F_{x_i}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&\qquad\qquad + p_i(r,s)F_{\mathbf{z}}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s))\Big)\dot{x}_i(r,s) \\
&\quad - p_{d+1}(r,s)F_{\mathbf{z}}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&\quad - \sum_{i=1}^{d}\dot{p}_i(r,s)F_{p_i}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&\quad - F_{x_{d+1}}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)).
\end{aligned}
$$

If we choose $\dot{x}_i$ and $\dot{p}_i$ such that

$$
\begin{aligned}
\dot{x}_i(r,s) &:= -F_{p_i}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
\dot{p}_i(r,s) &:= F_{x_i}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&\qquad + p_i(r,s)F_{\mathbf{z}}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)),
\end{aligned}
\tag{1.16}
$$

we obtain

$$
\begin{aligned}
\dot{p}_{d+1}(r,s) &- p_{d+1}(r,s)F_{\mathbf{z}}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&\quad - F_{x_{d+1}}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&= F_{x_{d+1}}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&\quad + p_{d+1}(r,s)F_{\mathbf{z}}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&\quad - p_{d+1}(r,s)F_{\mathbf{z}}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&\quad - F_{x_{d+1}}(x_1(r,s),...,x_d(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s),x_{d+1}(r,s)) \\
&= 0
\end{aligned}
$$

and (1.15) holds. Hence we have the following system of **characteristic equations**:

$$
\begin{aligned}
\frac{\mathrm{d}x_i}{\mathrm{d}s} &= -F_{p_i}(x_1,...,x_d,\mathbf{z},\mathbf{p},s), \text{ for } i=1,...,d,\\
\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}s} &= p_{d+1} + \sum_{i=1}^{d} \dot{x}_i p_i\\
&= F(x_1,...,x_d,\mathbf{z},\mathbf{p},s) - \sum_{i=1}^{d} p_i F_{p_i}(x_1,...,x_d,\mathbf{z},\mathbf{p},s),\\
\frac{\mathrm{d}p_i}{\mathrm{d}s} &= F_{x_i}(x_1,...,x_d,\mathbf{z},\mathbf{p},s)\\
&\qquad + F_{\mathbf{z}}(x_1,...,x_d,\mathbf{z},\mathbf{p},s)p_i, \text{ for } i=1,...,d.
\end{aligned}
\tag{CE}
$$

For the sake of simplicity we dropped the parameters (r,s) in the above system of ODEs. One should note that we define by $(\mathbf{x}(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s))$ in total $(2d+3)$ functions, but due to the fact that $\dot{p}_{d+1}$ and $\dot{x}_{d+1}=1$ give no information we end up with a system of $(2d+1)$ differential equations.
Due to Remark 1.2 we apply Lemma 2 of [Eva08, §3]. By this result we know that for any point $r\in\Gamma^*$ there exists a neighborhood such that every point y sufficiently close to Γ can be uniquely determined by a curve $y=\mathbf{x}(r,s)$. This means that the curve starting at point $r=\mathbf{x}(r,0)$ reaches $y=\mathbf{x}(r,s)$ at time s. So, again by Remark 1.2 we invert $\mathbf{x}(r,s)$ near 0, i.e. we find functions

$$R\colon \mathbb{R}^{d+1}\to\Gamma^* \quad\text{and}\quad S\colon\mathbb{R}^{d+1}\to[0,\mathbf{T}]$$

such that $r=R(x)$ and $s=S(x)$ for x sufficiently close to Γ. One obtains this by an application of the inverse mapping theorem (see [Lan96, Chapter XIV, Theorem 1.2]). Hence we get a local solution of our equation by solving the characteristic equations (CE) with initial condition and choosing

$$u(x)=u(\mathbf{x}(r,s))=\mathbf{z}(r,s)=\mathbf{z}(R(x),S(x)) \quad\text{for } x \text{ sufficiently close to } \Gamma. \tag{1.17}$$

The method is based on the assumptions that u solves the Cauchy problem and that we find a curve $\mathbf{x}(r,s)$. Now, one should finally show that the constructed u in (1.17) really solves problem (1.10). For this calculation we refer to [Eva08, Proof of Theorem 2].

Remark 1.3 $(\mathbf{x}(r,s),\mathbf{z}(r,s),\mathbf{p}(r,s))$ *is called characteristic curve or also Monge curves and Monge strips, respectively, in honour of G. Monge. The characteristic equations are also known as Lagrange-Charpit equations in honour of P. Charpit and J.-L. Lagrange.*

2. Preliminaries

In this chapter we recall some basic definitions from [Kun97]. The main aim of the whole chapter is to formulate and prove a representation result for a Stratonovich integral of the form

$$\int_0^t F(\varphi_s, \circ\mathrm{d}s),$$

where F is a semimartingale satisfying some regularity assumptions. To this end we prove a representation result for continuous C-valued local martingales (see Theorem 2.21 below) as well as for Itô integrals based on continuous C-valued local martingales (see Theorem 2.33 below). The latter is redrafted to a more rigorous version as in [Kun97, Exercise 3.2.11]. Due to these results we are able to prove and state a representation result for Stratonovich integrals of the form $\int_0^t F(\varphi_s, \circ\mathrm{d}s)$ based on a continuous semimartingale taking values in a certain space (see Theorem 2.39 below). Theorem 2.39 is based on [Kun97, Exercise 3.3.5], however this exercise is not sufficiently exact concerning the existence of continuous processes $(f_n)_{n\geq 0}$ and the previous representation results. After that the application in the case of Brownian motion is given in Subsection 2.7. below. This can not be found in [Kun97], but it is one famous framework to apply any result of [Kun97]. The reader should note that the presentation form in [Kun97] is constituted by continuous text. Hence the author of this thesis reformulates the necessary definitions and results in a didactic prepared and structured way.

2.1. Basic definitions in the approach of H. Kunita

Let $(\Omega, \mathscr{F}, P)$ be a complete, separable probability space endowed with a normal filtration $(\mathscr{F}_t)_{t\in[0,\mathbf{T}]}$ for the finite time interval $[0,\mathbf{T}]$, $\mathbf{T} > 0$, which is defined in the following way:

Definition 2.1 *A family of sub-σ-fields $(\mathscr{F}_t)_{t\in[0,\boldsymbol{T}]}$ is called a **normal filtration** if $(\mathscr{F}_t)_{t\in[0,\boldsymbol{T}]}$ is right-continuous, i.e. $\mathscr{F}_t = \bigcap_{\varepsilon>0}\mathscr{F}_{t+\varepsilon}$, each $\mathscr{F}_t$ contains all null sets of $\mathscr{F}$ and $\mathscr{F}_s \subseteq \mathscr{F}_t$ for all $s \leq t$.*

In the case of a Brownian motion we define the following:

Definition 2.2 *Let $(W_t)_{t\in[0,\boldsymbol{T}]}$ be a real-valued Brownian motion on a probability space $(\Omega, \mathscr{F}, P)$. For $t \in [0, \boldsymbol{T}]$ define the filtration*

$$\mathscr{F}_t^W := \sigma\big(\{W_s \,|\, 0 \leq s \leq t\}\big).$$

The corresponding filtration

$$\mathscr{F}_{t+}^W := \bigcap_{s>t}\mathscr{F}_s^W, \quad t \in [0, \boldsymbol{T}]$$

is right-continuous. Therefore the normal filtration is given by

$$\mathscr{F}_t := \sigma\big(\mathscr{F}_{t+}^W, \sigma(\{N \in \mathscr{F} \,|\, P(N) = 0\})\big).$$

In the following let $\mathbb{D} \subset \mathbb{R}^d$ be a domain.

Definition 2.3 *A collection of $\mathbb{R}^d$-valued random variables $X(x): \Omega \to \mathbb{R}^d, x \in \mathbb{D}$, is called a **random field** with parameter set $\mathbb{D} \subset \mathbb{R}^d$. If $\mathbb{D} = [0, \boldsymbol{T}]$, then the random field is called a **stochastic process** and is denoted by $(X_t)_{t\in[0,\boldsymbol{T}]}$.*

It is a basic fact that a continuous stochastic process adapted with respect to the normal filtration $(\mathscr{F}_t)_t$ is $(\mathscr{F}_t)_t$-predicable (e.g. [RY05, Chapter IV, (5.1) Proposition]).

Definition 2.4 *A continuous, real-valued, $(\mathscr{F}_t)_t$-adapted (and therefore predictable) stochastic process X_t is called a* ***local martingale*** *if there exists an increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ with $P(\tau_n < \boldsymbol{T}) \xrightarrow{n\to\infty} 0$ and the stopped process $X_{t\wedge\tau_n}$ is a martingale.*

Obviously, each martingale is a local martingale and each continuous local martingale X_t satisfying

$$\mathbb{E}\left[\sup_{s\in[0,\mathbf{T}]} |X_s|\right] < \infty$$

is a martingale as proved e.g. in [Kun97, Theorem 2.1.1].

Definition 2.5 *A stopping time $\tau\colon \mathbb{D}\times\Omega \to [0,\infty]$ is called* ***accessible*** *if there exists a sequence of stopping times $(\tau_n(x))_{n\in\mathbb{N}}$ such that for each $x\in\mathbb{D}$ $\tau_n(x) < \tau(x)$ a.s. and $\lim_{n\to\infty} \tau_n(x) = \tau(x)$ a.s.*

In many references, see for example [Pro15, after Proposition 1], this property is called predictability of a stopping time.

Definition 2.6 *A family of random variables X_t, $t\in[0,\tau)$, is called a* ***local process*** *if τ is an accessible stopping time.*

As stated in [Cho66, 8.1 Proposition] a lower semicontinuous function can be equivalently defined by levelsets. Hence we define the property of a lower semicontinuous stopping time in the following way.

Definition 2.7 *A stopping time $\tau\colon \mathbb{D}\times\Omega\to[0,\infty]$ is called* ***lower semicontinuous****, if one of the following three equivalent conditions holds for almost all ω:*

(i) for all $x_0\in\mathbb{D}$ we have

$$\liminf_{x\to x_0} \tau(x,\omega) \geq \tau(x_0,\omega),$$

(ii) for all $x\in\mathbb{D}$ the levelsets

$$\{\tau(x,\omega)\leq\beta\} \text{ are closed } \forall\ \beta\geq 0,$$

(iii) for all $x\in\mathbb{D}$ the levelsets

$$\{\tau(x,\omega)>\beta\} \text{ are open } \forall\ \beta\geq 0.$$

The above equivalence is formally proved e.g. in [PKY09, Proposition 2.1.3]. Now we extend Definition 2.6 to the case of $\mathbb{R}^d$-valued index sets $\mathbb{D}$.

Definition 2.8 *A family of random variables $X_t(x)$, $x\in\mathbb{D}$, $t\in[0,\tau(x))$ is called a* ***local random field*** *if τ is an accessible and lower semicontinuous stopping time.*

Definition 2.9 *Let $e\in\mathbb{N}$. For given functions $f\colon\mathbb{D}\to\mathbb{R}^e$ and $g\colon\mathbb{D}\times\mathbb{D}\to\mathbb{R}^{e\times e}$ we define the following seminorms for $k\in\mathbb{N}_0$, $0\leq\delta\leq 1$ and $\mathbb{K}\subset\mathbb{D}$ compact:*

$$\|f\|_{k+\delta;\mathbb{K}} := \sup_{x\in\mathbb{K}} \frac{|f(x)|}{1+|x|} + \sum_{1\leq|\alpha|\leq k} \sup_{x\in\mathbb{K}} |D_x^\alpha f(x)| + \sum_{|\alpha|=k} \sup_{\substack{x,y\in\mathbb{K}\\ x\neq y}} \frac{|D_x^\alpha f(x) - D_x^\alpha f(y)|}{|x-y|^\delta}$$

$$\|g\|^{\sim}_{k+\delta;\mathbb{K}} := \sup_{x,y\in\mathbb{K}} \frac{|g(x,y)|}{(1+|x|)(1+|y|)} + \sum_{1\leq|\alpha+\tilde{\alpha}|\leq k} \sup_{x,y\in\mathbb{K}} |D_x^\alpha D_y^{\tilde{\alpha}} g(x,y)|$$

$$+ \sum_{|\alpha+\tilde{\alpha}|=k} \sup_{\substack{x,x',y,y'\in\mathbb{K},\\ x\neq x',y\neq y'}} \frac{|D_x^\alpha D_y^{\tilde{\alpha}} g(x,y) - D_x^\alpha D_y^{\tilde{\alpha}} g(x',y) - D_x^\alpha D_y^{\tilde{\alpha}} g(x,y') + D_x^\alpha D_y^{\tilde{\alpha}} g(x',y')|}{|x-x'|^\delta |y-y'|^\delta},$$

where D_x^α or D_y^α, respectively, denote derivatives in the ordinary sense. Furthermore, we set

$$\|f\|_{k+\delta;\mathbb{D}} := \sup_{x\in\mathbb{D}} \frac{|f(x)|}{1+|x|} + \sum_{1\le|\alpha|\le k} \sup_{x\in\mathbb{D}} |D_x^\alpha f(x)| + \sum_{|\alpha|=k} \sup_{\substack{x,y\in\mathbb{D}\\ x\neq y}} \frac{|D_x^\alpha f(x) - D_x^\alpha f(y)|}{|x-y|^\delta}$$

$$\|g\|_{k+\delta;\mathbb{D}}^{\sim} := \sup_{x,y\in\mathbb{D}} \frac{|g(x,y)|}{(1+|x|)(1+|y|)} + \sum_{1\le|\alpha+\tilde{\alpha}|\le k} \sup_{x,y\in\mathbb{D}} |D_x^\alpha D_y^{\tilde{\alpha}} g(x,y)|$$
$$+ \sum_{|\alpha+\tilde{\alpha}|=k} \sup_{\substack{x,x',y,y'\in\mathbb{D},\\ x\neq x', y\neq y'}} \frac{|D_x^\alpha D_y^{\tilde{\alpha}} g(x,y) - D_x^\alpha D_y^{\tilde{\alpha}} g(x',y) - D_x^\alpha D_y^{\tilde{\alpha}} g(x,y') + D_x^\alpha D_y^{\tilde{\alpha}} g(x',y')|}{|x-x'|^\delta |y-y'|^\delta}.$$

Based on these seminorms we define the following metrics.

Definition 2.10 *Let $k \in \mathbb{N}_0, e \in \mathbb{N}$. Let $\mathcal{C}^k(\mathbb{D}, \mathbb{R}^e)$ denote the set of all k-times continuously differentiable functions mapping the domain $\mathbb{D} \subset \mathbb{R}^d$ into $\mathbb{R}^e$. Let $(\mathbb{K}_i)_{i\in\mathbb{N}}$ be an exhaustion of $\mathbb{D}$ by compact sets. Obviously such an exhaustion of compact sets exists for any open subset of $\mathbb{R}^d$ (see e.g. [KS08, Lemma 1.1]). For all $f_1, f_2 \in \mathcal{C}^k(\mathbb{D}, \mathbb{R}^e)$ define the metric $d_{k+0}(\cdot,\cdot)$ by*

$$d_{k+0}(f_1, f_2) := \sum_{i\in\mathbb{N}} \frac{1}{2^i} \frac{\|f_1 - f_2\|_{k+0;\mathbb{K}_i}}{1 + \|f_1 - f_2\|_{k+0;\mathbb{K}_i}}.$$

Furthermore, let $\mathcal{C}^k(\mathbb{D}\times\mathbb{D}, \mathbb{R}^{e\times e})$ denote the set of all k-times continuously differentiable functions mapping the domain $\mathbb{D}\times\mathbb{D} \subset \mathbb{R}^{2d}$ into $\mathbb{R}^{e\times e}$. For all $g_1, g_2 \in \mathcal{C}^k(\mathbb{D}\times\mathbb{D}, \mathbb{R}^{e\times e})$ we define the metric $d_{k+0}^{\sim}$ by

$$d_{k+0}^{\sim}(g_1, g_2) := \sum_{i\in\mathbb{N}} \frac{1}{2^i} \frac{\|g_1 - g_2\|_{k+0;\mathbb{K}_i}^{\sim}}{1 + \|g_1 - g_2\|_{k+\delta;\mathbb{K}_i}^{\sim}}.$$

These metrics are known as Fréchet metrics. For the proof that the Fréchet metric satisfies the conditions for metrics see e.g. [Alt16, 2.23 (1) Sequence spaces].

Definition 2.11 *Let $k \in \mathbb{N}_0, e \in \mathbb{N}$ and $0 < \delta \le 1$. Define*

$$C^{k,\delta}(\mathbb{D}, \mathbb{R}^e) := \left\{ f \in \mathcal{C}^k(\mathbb{D}, \mathbb{R}^e) \,\middle|\, D_x^\alpha f \text{ is } \delta\text{-Hölder continuous for } |\alpha| = k \right\}.$$

Let $(\mathbb{K}_i)_{i\in\mathbb{N}}$ be an exhaustion of $\mathbb{D}$ by compact sets, then for all $f_1, f_2 \in C^{k,\delta}(\mathbb{D}, \mathbb{R}^e)$ define the metric $d_{k+\delta}(\cdot,\cdot)$ by

$$d_{k+\delta}(f_1, f_2) := \sum_{i\in\mathbb{N}} \frac{1}{2^i} \frac{\|f_1 - f_2\|_{k+\delta;\mathbb{K}_i}}{1 + \|f_1 - f_2\|_{k+\delta;\mathbb{K}_i}}.$$

Furthermore, we define

$$\tilde{C}^{k,\delta}(\mathbb{D}\times\mathbb{D}, \mathbb{R}^{e\times e}) := \left\{ g \in \mathcal{C}^k(\mathbb{D}\times\mathbb{D}, \mathbb{R}^{e\times e}) \,\middle|\, D_x^\alpha D_y^{\hat{\alpha}} g \text{ is } \delta\text{-Hölder continuous for } |\alpha + \hat{\alpha}| = k \right\}$$

and for all $g_1, g_2 \in \tilde{C}^{k,\delta}(\mathbb{D}\times\mathbb{D}, \mathbb{R}^{e\times e})$ the metric $d_{k+\delta}^{\sim}(\cdot,\cdot)$ by

$$d_{k+\delta}^{\sim}(g_1, g_2) := \sum_{i\in\mathbb{N}} \frac{1}{2^i} \frac{\|g_1 - g_2\|_{k+\delta;\mathbb{K}_i}^{\sim}}{1 + \|g_1 - g_2\|_{k+\delta;\mathbb{K}_i}^{\sim}}.$$

As proved in [Alt16, 2.12. Proposition] there exist topologies induced by the above metrics.

Remark 2.12 *$C^{k,\delta}(\mathbb{D},\mathbb{R}^e)$ together with the topology induced by the metric $d_{k+\delta}$ is a Fréchet space. If $\delta = 0$, we write $C^k(\mathbb{D},\mathbb{R}^e)$ instead of $C^{k,0}(\mathbb{D},\mathbb{R}^e)$. Furthermore, if $k = 0$ we write $C(\mathbb{D},\mathbb{R}^e)$ instead of $C^{0,0}(\mathbb{D},\mathbb{R}^e)$.*

The result can be found in [Alt16, 3.3 Continuous functions]. Since we have to work with processes which depend on two parameters, we extend this result to $\mathbb{D} \times \mathbb{D}$.

Remark 2.13 *$\tilde{C}^{k,\delta}(\mathbb{D}\times\mathbb{D},\mathbb{R}^{e\times e})$ together with the topology induced by the metric $d_{\widetilde{k+\delta}}$ is a Fréchet space. If $\delta = 0$, we write $\tilde{C}^k(\mathbb{D}\times\mathbb{D},\mathbb{R}^{e\times e})$ instead of $\tilde{C}^{k,0}(\mathbb{D}\times\mathbb{D},\mathbb{R}^{e\times e})$. If $\delta = 0$, we write $\tilde{C}^k(\mathbb{D}\times\mathbb{D},\mathbb{R}^{e\times e})$ instead of $\tilde{C}^{k,0}(\mathbb{D}\times\mathbb{D},\mathbb{R}^{e\times e})$. Furthermore, if $k = 0$ we write $\tilde{C}(\mathbb{D}\times\mathbb{D},\mathbb{R}^{e\times e})$ instead of $\tilde{C}^{0,0}(\mathbb{D}\times\mathbb{D},\mathbb{R}^{e\times e})$.*

A continuous $(\mathscr{F}_t)_t$-adapted stochastic process $(X_t)_{t\in[0,\mathbf{T}]}$ is called a **continuous semimartingale** if it can be written as the sum $X_t = \mathbf{M}_t + B_t$ of a continuous process of bounded variation B_t and a continuous local martingale $\mathbf{M}_t$ (see e.g. [RY05, Chapter IV, (1.17) Definition]). Next we define a class of specific semimartingales.

Definition 2.14 *A family of continuous $\mathbb{R}^e$–valued semimartingales $F(x,\cdot)$, $x \in \mathbb{D}$, with decomposition $F(x,t) = \boldsymbol{M}(x,t) + B(x,t)$ is called **a family of continuous $C^{k,\delta}(\mathbb{D},\mathbb{R}^e)$ - semimartingales** if*

- *$\boldsymbol{M}(x,t)$ is a continuous $C^{k,\delta}(\mathbb{D},\mathbb{R}^e)$ - local martingale*

  i.e. $\boldsymbol{M}(x,t), t \in [0,\boldsymbol{T}]$, is a local martingale for each $x \in \mathbb{D}$ and $\boldsymbol{M}(\cdot,t)$ is continuous in t a.s. in the space $C^{k,\delta}(\mathbb{D},\mathbb{R}^e)$, hence for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $s \in [0,\boldsymbol{T}]$ with $|t-s| < \delta$ we have

$$d_{k+\delta}(\boldsymbol{M}(\cdot,t),\boldsymbol{M}(\cdot,s)) < \varepsilon \text{ a.s.}$$

- *$B(x,t)$ is a continuous $C^{k,\delta}(\mathbb{D},\mathbb{R}^e)$ - process*

  i.e. $B(\cdot,t)$ is continuous in t a.s. in the space $C^{k,\delta}(\mathbb{D},\mathbb{R}^e)$,

- *$D_x^\alpha B(x,t), t \in [0,\boldsymbol{T}]$, are processes of bounded variation for all $|\alpha| \le k, x \in \mathbb{D}$.*

For $k \in \mathbb{N}$ and $\delta = 0$ we obtain the definition of a family of continuous C^k - semimartingales. Furthermore, if $k = 0$ we write C instead of C^0.

Definition 2.15 *We define the following spaces of continuous processes:*

$$\begin{aligned}\mathscr{M}_c^{loc} &:= \{M = (M_t)_t \,|\, M \text{ is a continuous local martingale}, M_0 = 0\},\\ \mathscr{M}_c &:= \{M = (M_t)_t \,|\, M \text{ is a continuous, square integrable martingale}, M_0 = 0\}.\end{aligned}$$

Let $M, N \in \mathscr{M}_c$, then the inner product and the corresponding norm are given by

$$\begin{aligned}\big(M,N\big)_{\mathscr{M}_c} &:= \sup_{t\in[0,\boldsymbol{T}]} \mathbb{E}[M_t N_t],\\ \|M\|^2_{\mathscr{M}_c} &:= \sup_{t\in[0,\boldsymbol{T}]} \mathbb{E}[|M_t|^2].\end{aligned}$$

As proved e.g. in [Mét82, 16.4 Proposition] the space $\mathscr{M}_c$ with the above inner product is a Hilbert space.

Definition 2.16 *Let $M, N \in \mathscr{M}_c^{\text{loc}}$. The **joint quadratic variation** or also called **covariation** of M, N associated with the partition $\Delta = \{0 = t_0 < t_1 < ... < t_l = \boldsymbol{T}\}$ of $[0,\boldsymbol{T}]$ is defined by*

$$\langle M,N\rangle_t^\Delta := \sum_{k=0}^{l-1}(M_{t\wedge t_{k+1}} - M_{t\wedge t_k})(N_{t\wedge t_{k+1}} - N_{t\wedge t_k}).$$

The following theorem is a classical result and the proof can be found e.g. in [RY05, Chapter IV, (1.9) Theorem].

Theorem 2.17 *Let* $M, N \in \mathscr{M}_c^{\text{loc}}$. $\langle M, N\rangle_t^\Delta$ *converges in probability uniformly in* t *to a uniquely determined continuous process of bounded variation* $\langle M, N\rangle_t$ *as* $|\Delta| \to 0$, *i.e.*

$$P\lim_{|\Delta|\to 0} \sup_{0\le t\le \boldsymbol{T}} \left|\langle M, N\rangle_t^\Delta - \langle M, N\rangle_t\right| = 0.$$

Notation 2.18 *If* $M = N$, *we shortly write*

$$\langle M, M\rangle_t = \langle M\rangle_t.$$

Furthermore, the following result can be found in [Kun97, Theorem 2.3.10].

Theorem 2.19 $\mathscr{M}_c$ *has an orthogonal basis consisting of at most countable elements, provided that* $(\Omega, \mathscr{F}, P)$ *is separable.*

The proof follows the ideas of the proof of [Kun97, Theorem 2.3.10] and is written in a detailed version.

Proof. Consider the following space of square integrable martingales:

$$\mathscr{M} := \{M = (M_t)_t \,|\, M \text{ is a square integrable martingale,} \\ \text{but not necessary continuous in } t, M_0 = 0\}.$$

Define the corresponding norm and inner product by

$$(X, Y)_{\mathscr{M}} := \mathbb{E}[X_{\mathbf{T}} \cdot Y_{\mathbf{T}}],$$
$$\|X\|_{\mathscr{M}} := \mathbb{E}[X_{\mathbf{T}}^2]^{\frac{1}{2}}.$$

The space $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$ is a real Hilbert space (see [Mét82, 17.8 Definition]). Since Ω is separable we conclude that $L^2(\Omega)$ is also separable which is proved in [AF09, 2.21 Theorem]. Therefore it exists a countable dense subset $\{X^k\}_{k\in\mathbb{N}} \subseteq L^2(\Omega)$. Now we want to prove that $\mathscr{M}$ is also separable, i.e. we have to find a countable dense subset in $\mathscr{M}$. For all $t \in [0, \mathbf{T}]$ we define

$$Y_t^k := \mathbb{E}[X^k|\mathscr{F}_t] - \mathbb{E}[X^k|\mathscr{F}_0] \tag{2.1}$$

and show

(i) $(Y_t^k)_{k\in\mathbb{N}}$ is a martingale,
(ii) $Y_t^k \in \mathscr{M}$ for all $k \in \mathbb{N}$,
(iii) $(Y_t^k)_{k\in\mathbb{N}} \subset \mathscr{M}$ is dense, i.e. for an arbitrary $(Y_t)_t \in \mathscr{M}$ there exists a subsequence $(Y_t^{k_m})_{m\in\mathbb{N}} \in \mathscr{M}$ such that $Y_t^{k_m} \xrightarrow{m\to\infty} Y_t$ in $\mathscr{M}$.

ad (i) For fixed $k \in \mathbb{N}$ the martingale property is obviously satisfied, since we have for $s \le t$

$$\begin{aligned}\mathbb{E}[Y_t^k|\mathscr{F}_s] &= \mathbb{E}\big[\mathbb{E}[X^k|\mathscr{F}_t] - \mathbb{E}[X^k|\mathscr{F}_0]\big|\mathscr{F}_s\big] \\ &= \mathbb{E}[X^k|\mathscr{F}_s] - \mathbb{E}[X^k|\mathscr{F}_0] = Y_s^k.\end{aligned}$$

ad (ii) We have to show that $\|Y_t^k\|_{\mathscr{M}} < \infty$ holds for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ be fixed. Then we get by Jensen's inequality

$$\begin{aligned}\|Y^k\|_{\mathscr{M}}^2 &= \mathbb{E}[(Y_{\mathbf{T}}^k)^2] \\ &= \mathbb{E}\big[(\mathbb{E}[X^k|\mathscr{F}_{\mathbf{T}}] - \mathbb{E}[X^k|\mathscr{F}_0])^2\big] \\ &\le \mathbb{E}\big[2\mathbb{E}[X^k|\mathscr{F}_{\mathbf{T}}]^2 + 2\mathbb{E}[X^k|\mathscr{F}_0]^2\big] \\ &\le \mathbb{E}\big[2\mathbb{E}[(X^k)^2|\mathscr{F}_{\mathbf{T}}] + 2\mathbb{E}[(X^k)^2|\mathscr{F}_0]\big] \\ &= 2\mathbb{E}[(X^k)^2] + 2\mathbb{E}[(X^k)^2] < \infty.\end{aligned}$$

ad (iii) To verify the denseness in $\mathscr{M}$ we have to show that for any $(Y_t)_t \in \mathscr{M}$ there exists a subsequence $(Y_t^{k_m})_{m\in\mathbb{N}}$ such that

$$\lim_{m\to\infty} \|Y - Y^{k_m}\|^2_{\mathscr{M}} = 0.$$

Due to the definition of $\mathscr{M}$ we know that Y_t is a martingale and furthermore it is bounded in L^2. Now consider a subsequence $(X^{k_m})_{m\in\mathbb{N}}$ such that X^{k_m} converges to $Y_{\mathbf{T}}$ in L^2 for $m \to \infty$. By using that $\mathbb{E}[Y_{\mathbf{T}}|\mathscr{F}_0] = 0$ we know that

$$\begin{aligned}\mathbb{E}\Big[\big(\mathbb{E}[X^{k_m}|\mathscr{F}_0]\big)^2\Big] &= \mathbb{E}\Big[\big(\mathbb{E}[X^{k_m} - Y_{\mathbf{T}}|\mathscr{F}_0]\big)^2\Big] \\ &\leq \mathbb{E}\big[\mathbb{E}[(X^{k_m} - Y_{\mathbf{T}})^2|\mathscr{F}_0]\big] \\ &= \mathbb{E}\big[(X^{k_m} - Y_{\mathbf{T}})^2\big] \xrightarrow{m\to\infty} 0\end{aligned}$$

holds. Now we conclude

$$\begin{aligned}\|Y - Y^{k_m}\|^2_{\mathscr{M}} &= \mathbb{E}\Big[\big(Y_{\mathbf{T}} - Y^{k_m}_{\mathbf{T}}\big)^2\Big] \\ &= \mathbb{E}\Big[\big(Y_{\mathbf{T}} - \mathbb{E}[X^{k_m}|\mathscr{F}_{\mathbf{T}}] + \mathbb{E}[X^{k_m}|\mathscr{F}_0]\big)^2\Big] \\ &= \mathbb{E}\Big[\mathbb{E}[Y_{\mathbf{T}} - X^{k_m}|\mathscr{F}_{\mathbf{T}}] + \mathbb{E}[X^{k_m}|\mathscr{F}_0]\big)^2\Big] \\ &\leq \mathbb{E}\Big[2\big(\mathbb{E}[Y_{\mathbf{T}} - X^{k_m}|\mathscr{F}_{\mathbf{T}}]\big)^2 + 2\big(\mathbb{E}[X^{k_m}|\mathscr{F}_0]\big)^2\Big] \\ &\leq \mathbb{E}\big[2\mathbb{E}[(Y_{\mathbf{T}} - X^{k_m})^2|\mathscr{F}_{\mathbf{T}}]\big] + 2\mathbb{E}\Big[\big(\mathbb{E}[X^{k_m}|\mathscr{F}_0]\big)^2\Big] \\ &\leq 2\mathbb{E}[(Y_{\mathbf{T}} - X^{k_m})^2] + 2\mathbb{E}\Big[\big(\mathbb{E}[X^{k_m}|\mathscr{F}_0]\big)^2\Big] \\ &\xrightarrow{m\to\infty} 0.\end{aligned}$$

Hence we have $\{Y^k\}_{k\in\mathbb{N}} \subseteq \mathscr{M}$ dense. Due to the fact that $\mathscr{M}_c \subset \mathscr{M}$ is closed (cf. [KS10, Chapter 1, 5.23 Proposition]), we obtain that $\mathscr{M}_c$ is also separable using [AF09, 1.22 Theorem]. Let $\{M^k\}_{k\in\mathbb{N}}$ be a countable dense subset of $\mathscr{M}_c$. By the method of Gram-Schmidt's orthogonalization (see (A.3) in Appendix A) one defines an orthogonal system $\{N^k\}_{k\in\mathbb{N}}$. Therefore it holds that there exists an orthogonal basis of at most countable elements if the probability space is separable. □

Remark 2.20 *The Gram-Schmidt's orthogonalization may end in finite steps. Hence it is possible to obtain an orthogonal basis of finite elements, depending on the dimension of* $\mathscr{M}_c$.

In Appendix A we take a detailed look on the Kunita-Watanabe decomposition (Theorem A.4) which we need for the proof of the following fundamental theorem. Furthermore, the definition of orthogonality of continuous local martingales is reproduced in Definition A.1. Under our assumption that the underlying probability space is separable, Theorem 2.19 guarantees the existence of an orthogonal basis of continuous, square integrable martingales (cf. Definition A.6). The following result is a rigorously reformulated version of [Kun97, Exercise 3.2.10].

Theorem 2.21 *Let* $\{M^n\}_{n\geq 1}$ *be an orthogonal basis of continuous, square integrable martingales. Then the following holds:*

(i) Any continuous $C(\mathbb{D}, \mathbb{R}^e)$ *- local martingale* $\mathbf{M}$ *can be represented for all* $x \in \mathbb{D}$, $t \in [0, \boldsymbol{T})$ *by*

$$\mathbf{M}(x,t) = \sum_{n\geq 1} \int_0^t f_n(x,s)\, \mathrm{d}M^n_s \quad \text{a.s.},$$

where $f_n(x,s)$ *are measurable random fields, predictable in* s *for each* $x \in \mathbb{D}$.

(ii) Let $A(x,y,t)$ be defined for all $x, y \in \mathbb{D}$ by

$$A(x,y,t)_{i,j} := \Big(\langle \mathbf{M}^i(x,\cdot), \mathbf{M}^j(y,\cdot)\rangle_t\Big) \text{ a.s.}$$

for all $i, j = 1, ..., e$, where $\mathbf{M}^i(x,t)$ denotes the i-th component of the $\mathbb{R}^e$ - valued vector $\mathbf{M}(x,t)$. Then there exists a continuous increasing process A_t such that $A(x,y,t)$ is absolutely continuous with respect to $\mathrm{d}A_t$ for all $x, y \in \mathbb{D}$ a.s.

Proof.

(i) The main tool of this proof is the Kunita-Watanabe decomposition given in Theorem A.4. In our situation we have $\mathbf{M} \in \mathscr{M}_c^{\mathrm{loc}}$ and $M^n \in \mathscr{M}_c$ for all $n \in \mathbb{N}$. Hence $M^n \in \mathscr{M}_c^{\mathrm{loc}}$. By Lemma A.3 there exist unique $f_n(x) \in L^2(\langle M\rangle)$, $x \in \mathbb{D}$, $n \in \mathbb{N}$ satisfying

$$\langle \mathbf{M}(x,\cdot), M^n\rangle_t = \int_0^t f_n(x,s)\,\mathrm{d}\langle M^n\rangle_s \tag{2.2}$$

for all $n \in \mathbb{N}$ and hence

$$\sum_{n\geq 1}\langle \mathbf{M}(x,\cdot), M^n\rangle_t = \sum_{n\geq 1}\int_0^t f_n(x,s)\,\mathrm{d}\langle M^n\rangle_s.$$

For each $x \in \mathbb{D}$ we define

$$\mathbf{M}^{(1)}(x,t) := \sum_{n\geq 1}\int_0^t f_n(x,s)\,\mathrm{d}M_s^n,$$
$$\mathbf{M}^{(2)}(x,t) := \mathbf{M}(x,t) - \mathbf{M}^{(1)}(x,t).$$

Then by [Kun97, Theorem 2.3.2], (2.2) and by using the orthogonality of the basis $\{M^n\}_{n\geq 1}$ we have

$$\begin{aligned}\langle \mathbf{M}^{(1)}(x,\cdot), M^n\rangle_t &= \Big\langle \sum_{m\geq 1}\int_0^{\cdot} f_m(x,s)\,\mathrm{d}M_s^m, M^n\Big\rangle_t\\ &= \sum_{m\geq 1}\int_0^t f_m(x,s)\,\mathrm{d}\langle M^m, M^n\rangle_s\\ &= \int_0^t f_n(x,s)\,\mathrm{d}\langle M^n\rangle_s\\ &= \langle \mathbf{M}(x,\cdot), M^n\rangle_t.\end{aligned}$$

We conclude that

$$\begin{aligned}\sum_{n\geq 1}\langle \mathbf{M}^{(2)}(x,\cdot), M^n\rangle_t &= \sum_{n\geq 1}\langle \mathbf{M}(x,\cdot) - \mathbf{M}^{(1)}(x,\cdot), M^n\rangle_t\\ &= \sum_{n\geq 1}\langle \mathbf{M}(x,\cdot), M^n\rangle_t - \sum_{n\geq 1}\langle \mathbf{M}^{(1)}(x,\cdot), M^n\rangle_t\\ &= \sum_{n\geq 1}\langle \mathbf{M}(x,\cdot), M^n\rangle_t - \sum_{n\geq 1}\langle \mathbf{M}(x,\cdot), M^n\rangle_t = 0.\end{aligned}$$

Due to the fact that $\{M^n\}_{n\geq 1}$ is an orthogonal basis, see Definition A.6, we obtain $\mathbf{M}(x,t) = \mathbf{M}^{(1)}(x,t)$ a.s. and this shows the representation

$$\mathbf{M}(x,t) = \sum_{n\geq 1}\int_0^t f_n(x,s)\,\mathrm{d}M_s^n.$$

(ii) By (i) we consider the i-th component of the representation given by

$$\mathbf{M}^i(x,t) = \sum_{n\geq 1} \int_0^t f_n^i(x,s)\,\mathrm{d}M_s^n$$

for each $i = 1, ..., e$. The joint quadratic variation of $\mathbf{M}^i$ and $\mathbf{M}^j$, $i, j = 1, ..., e$, is equal to

$$\begin{aligned}\langle \mathbf{M}^i(x,\cdot), \mathbf{M}^j(y,\cdot)\rangle_t &= \Big\langle \sum_{n\geq 1} \int_0^{\cdot} f_n^i(x,s)\,\mathrm{d}M_s^n, \sum_{n\geq 1} \int_0^{\cdot} f_n^j(y,s)\,\mathrm{d}M_s^n \Big\rangle_t \\ &= \sum_{n\geq 1} \int_0^t f_n^i(x,s) f_n^j(y,s)\,\mathrm{d}\langle M^n, M^n\rangle_s \\ &= \sum_{n\geq 1} \int_0^t f_n^i(x,s) f_n^j(y,s)\,\mathrm{d}\langle M^n\rangle_s,\end{aligned}$$

where we used [Kun97, Theorem 2.3.2]. Now we consider the measure

$$\nu(\mathrm{d}s) := \sum_{n\geq 1} \frac{1}{2^n}\,\mathrm{d}\langle M^n\rangle_s.$$

Consequently there exists also a continuous increasing process A_t defined by

$$A_t := \nu([0,t]) = \int_0^t \sum_{n\geq 1} \frac{1}{2^n}\,\mathrm{d}\langle M^n\rangle_s = \sum_{n\geq 1} \frac{1}{2^n}\big(\langle M^n\rangle_t - \langle M^n\rangle_0\big). \tag{2.3}$$

Obviously, $\nu(\mathrm{d}s)$ is absolutely continuous with respect to $\mathrm{d}\langle M^n\rangle_s$. Hence by Radon-Nikodym theorem (see e.g. [Kle14, Corollary 7.34]) there exists a density ρ_n such that

$$\mathrm{d}\langle M^n\rangle_s = \rho_n(s)\nu(\mathrm{d}s).$$

By applying this construction we obtain for the joint quadratic variation

$$\begin{aligned}\langle \mathbf{M}^i(x,\cdot), \mathbf{M}^j(y,\cdot)\rangle_t &= \sum_{n\geq 1} \int_0^t f_n^i(x,s) f_n^j(y,s)\,\mathrm{d}\langle M^n\rangle_s \\ &= \sum_{n\geq 1} \int_0^t f_n^i(x,s) f_n^j(y,s)\rho_n(s)\,\nu(\mathrm{d}s) \\ &=: \int_0^t a^{ij}(x,y,s)\,\mathrm{d}A_s \quad \text{a.s.}\end{aligned} \tag{2.4}$$

Hence $A(x,y,t)$ is absolutely continuous with respect to $\mathrm{d}A_t$ for all $x, y \in \mathbb{D}$ a.s.

□

Let $F(x,t)$, $x \in \mathbb{D}$, be a family of continuous $C(\mathbb{D}, \mathbb{R}^e)$ - semimartingales with the representation

$$F(x,t,\omega) = \mathbf{M}(x,t,\omega) + B(x,t,\omega),$$

as given in Definition 2.14. The continuous process of bounded variation is absolutely continuous with respect to a continuous increasing measure denoted by $\mathrm{d}\tilde{A}_t$, i.e. it can be written as

$$B(x,t) = \int_0^t b(x,s)\,\mathrm{d}\tilde{A}_s$$

for a family of predictable processes $b(x,t)$, $x \in \mathbb{D}$. The integral on the right hand side is a classical Lebesgue-Stieltjes integral. We obviously find a continuous increasing measure (e.g. $\mathrm{d}A_t + \mathrm{d}\tilde{A}_t$) to which $\mathrm{d}A_t$ and $\mathrm{d}\tilde{A}_t$ are absolutely continuous. Consequently $B(x,t)$ as well as $\langle \mathbf{M}^i(x,\cdot), \mathbf{M}^j(y,\cdot)\rangle_t$ can be written as integrals with respect to this measure. For simplicity let us denote this new measure by $\mathrm{d}A_t$. In the case $A_t = t$ we obtain the classical Lebesgue integral.

Definition 2.22 *The triple*

$$\Big(a(x,y,t), b(x,t), A_t\Big)$$

given by processes $a\colon \mathbb{D} \times \mathbb{D} \times [0, \boldsymbol{T}] \times \Omega \to \mathbb{R}^{e \times e}$, $b\colon \mathbb{D} \times [0, \boldsymbol{T}] \times \Omega \to \mathbb{R}^e$ *and a continuous increasing process* A_t *is called* ***local characteristic*** *if the following conditions are fulfilled:*

(i) $a(x,y,t)$ *is symmetric, i.e.* $a^{ij}(x,y,t) = a^{ji}(y,x,t)$ *holds* P*−a.s. for all* $x, y \in \mathbb{D}$ *and* $i, j = 1, \dots, e$,

(ii) $a(x,y,t)$ *is non-negative definite, i.e.*

$$z^\top a(x,y,t) z = \sum_{i,j=1}^{e} a^{ij}(x,y,t) z_i z_j \geq 0$$

holds P*−a.s. for all* $x, y \in \mathbb{D}$ *and* $z \in \mathbb{R}^e$.

Notation 2.23 *From now on, whenever we speak about a family of continuous semimartingales* $F(x,t)$, $x \in \mathbb{D}$, *with local characteristic* (a, b, A_t), *we mean that* F *can be written as* $F(x,t) = B(x,t) + \boldsymbol{M}(x,t)$ *and (2.4) as well as*

$$B(x,t) = \int_0^t b(x,s)\, \mathrm{d}A_s$$

holds.

2.2. Classes of local characteristics

In Chapter 1 we mentioned that H. Kunita considers stochastic partial differential equations with coefficients given in the form $F(x, \mathrm{d}t)$, respectively $F(x, u, p, \circ \mathrm{d}t)$, for some continuous $C^{k,\delta}$-valued semimartingale F. For the main result of Kunita's approach the local characteristics have to fulfill some regularity properties. Therefore we introduce the following classes of local characteristics. In this chapter let (a, b, A_t) be a local characteristic in the sense of Definition 2.22.

Definition 2.24 *We say the pair* (a, A_t), *respectively the process* a, *belongs to the* ***class*** $B^{k,\delta}_{ub}$ *if* $a(\cdot,\cdot,t)$ *is predictable with values in* $\tilde{C}^{k,\delta}(\mathbb{D} \times \mathbb{D}, \mathbb{R}^{e \times e})$ *and the seminorm* $\|a(t)\|^{\sim}_{k+\delta;\mathbb{D}} := \|a(\cdot,\cdot,t)\|^{\sim}_{k+\delta;\mathbb{D}}$ *is uniformly bounded a.s., i.e. there exists* $C > 0$ *such that*

$$\sup_{t \in [0, \boldsymbol{T}]} \|a(t)\|^{\sim}_{k+\delta;\mathbb{D}} \leq C \ \text{a.s.}$$

We say the pair (b, A_t), *respectively the process* b, *belongs to the* ***class*** $B^{k,\delta}_{ub}$ *if* $b(\cdot,t)$ *is predictable with values in* $C^{k,\delta}(\mathbb{D}, \mathbb{R}^e)$ *and the seminorm* $\|b(t)\|_{k+\delta;\mathbb{D}}$ *is uniformly bounded a.s., i.e. there exists* $C > 0$ *such that*

$$\sup_{t \in [0, \boldsymbol{T}]} \|b(t)\|_{k+\delta;\mathbb{D}} \leq C \ \text{a.s.}$$

Definition 2.25 *We say the pair* (a, A_t) *[respectively* (b, A_t)*] belongs to the* ***class*** $B_b^{k,\delta}$ *if the process* $a(\cdot,\cdot,t)$ *[respectively* $b(\cdot,t)$*] is predictable with values in* $\tilde{C}^{k,\delta}(\mathbb{D}\times\mathbb{D},\mathbb{R}^{e\times e})$ *[respectively* $C^{k,\delta}(\mathbb{D},\mathbb{R}^e)$*] and if for almost all* ω *the seminorm* $\|a(t)\|^{\sim}_{k+\delta;\mathbb{D}}$ *[respectively* $\|b(t)\|_{k+\delta;\mathbb{D}}$*] is integrable with respect to the continuous increasing process* A_t*, which means that*

$$\int_0^T \|a(t)\|^{\sim}_{k+\delta;\mathbb{D}}\,\mathrm{d}A_t < \infty \ \ a.s. \quad \left[\text{respectively} \int_0^T \|b(t)\|_{k+\delta;\mathbb{D}}\,\mathrm{d}A_t < \infty \ \ a.s.\right].$$

The next definition is analogously, but here we consider compact subsets of $\mathbb{D}$.

Definition 2.26 *We say the pair* (a, A_t) *[respectively* (b, A_t)*] belongs to the* ***class*** $B^{k,\delta}$ *if the process* $a(\cdot,\cdot,t)$ *[respectively* $b(\cdot,t)$*] is predictable with values in* $\tilde{C}^{k,\delta}(\mathbb{D}\times\mathbb{D},\mathbb{R}^{e\times e})$ *[respectively* $C^{k,\delta}(\mathbb{D},\mathbb{R}^e)$*] and if for almost all* $\omega\in\Omega$ *the seminorm* $\|a(t)\|^{\sim}_{k+\delta;\mathbb{K}}$ *[respectively* $\|b(t)\|_{k+\delta;\mathbb{K}}$*] is integrable with respect to the continuous increasing process* A_t *for all compact sets* $\mathbb{K}\subset\mathbb{D}$*, which means that*

$$\int_0^T \|a(t)\|^{\sim}_{k+\delta;\mathbb{K}}\,\mathrm{d}A_t < \infty \ \ a.s. \quad \left[\text{respectively} \int_0^T \|b(t)\|_{k+\delta;\mathbb{K}}\,\mathrm{d}A_t < \infty \ \ a.s.\right].$$

Notation 2.27 *If* (a, A_t) *belongs to the class* $B^{m,\varepsilon}$ *and* (b, A_t) *belongs to the class* $B^{k,\delta}$ *for some* $k, m \in \mathbb{N}_0$*,* $0\le\delta\le1$ *and* $0\le\varepsilon\le1$*, then we write shortly that the local characteristic* (a, b, A_t) *belongs to the class* $(B^{m,\varepsilon}, B^{k,\delta})$*.*

Lemma 2.28 *We have* $B_{\mathrm{ub}}^{k,\delta}\subset B^{k,\delta}$*, i.e. if a pair* (a, A_t) *[respectively* (b, A_t)*] belongs to the class* $B_{\mathrm{ub}}^{k,\delta}$*, then in particular it belongs to the class* $B^{k,\delta}$*.*

Proof. Let (a, A_t) belong to $B_{\mathrm{ub}}^{k,\delta}$, i.e.

$$\sup_{t\in[0,\mathbf{T}]} \|a(t)\|^{\sim}_{k+\delta,\mathbb{D}} < C \text{ a.s.}$$

By monotonicity of the integral we have

$$\begin{aligned}\int_0^{\mathbf{T}} \|a(t)\|^{\sim}_{k+\delta,\mathbb{K}}\,\mathrm{d}A_t &\le \int_0^{\mathbf{T}} \|a(t)\|^{\sim}_{k+\delta,\mathbb{D}}\,\mathrm{d}A_t \\ &\le \int_0^{\mathbf{T}} C\,\mathrm{d}A_t < \infty \quad \text{a.s. } \forall\ \mathbb{K}\subset\mathbb{D} \text{ compact.}\end{aligned}$$

Consequently (a, A_t) belongs to the class $B^{k,\delta}$. □

2.3. Construction of stochastic integrals

In the previous subsections we defined a particular kind of local martingales and semimartingales taking values in the Fréchet spaces $C^{k,\delta}$ and corresponding local characteristics belonging to some regularity classes. With this knowledge we are now able to give a stepwise construction of an Itô integral based on local martingales and semimartingales.

Definition 2.29 *Let $\boldsymbol{M}(x,t)$, $x \in \mathbb{D}$, be a family of continuous $C^k(\mathbb{D},\mathbb{R})$ - local martingales with local characteristic (a, A_t) belonging to $B^{k,\delta}$ for $k \in \mathbb{N}_0$ and $0 \le \delta \le 1$. Let f_t be a predictable process with values in $\mathbb{D}$ satisfying*

$$\int_0^{\boldsymbol{T}} a(f_r, f_r, r)\,\mathrm{d}A_r < \infty \text{ a.s.}$$

*Then the **Itô integral** $\int_0^t \boldsymbol{M}(f_s, \mathrm{d}s)$ is stepwise defined in the following way:*

① *Let f_t be a simple process with values in $\mathbb{D}$, i.e. there exists a partition $\Delta = \{0 = t_0 < t_1 < ... < t_l = \boldsymbol{T}\}$ of $[0, \boldsymbol{T}]$ such that $f_t = f_{t_k}$ for any $t \in [t_k, t_{k+1})$, $k = 0, ..., l-1$. Then we define*

$$\boldsymbol{M}_t(f) := \int_0^t \boldsymbol{M}(f_r, \mathrm{d}r) := \sum_{k=0}^{l-1} \boldsymbol{M}(f_{t_k \wedge t}, t_{k+1} \wedge t) - \boldsymbol{M}(f_{t_k \wedge t}, t_k \wedge t).$$

② *Now let f_t be a predictable process with values in a compact subset $\mathbb{K} \subset \mathbb{D}$. Then there exists a sequence $(f_t^n)_{n\in\mathbb{N}}$ of simple $(\mathscr{F}_t)_t$-adapted processes with values in $\mathbb{K}$ such that*

$$\int_0^{\boldsymbol{T}} a(f_r^n, f_r^n, r) - 2a(f_r^n, f_r^m, r) + a(f_r^m, f_r^m, r)\,\mathrm{d}A_r \xrightarrow{n,m\to\infty} 0 \text{ a.s.}$$

Then $\langle \boldsymbol{M}_\bullet(f^n) - \boldsymbol{M}_\bullet(f^m)\rangle_{\boldsymbol{T}} \xrightarrow{n,m\to\infty} 0$ a.s. and we obtain due to [Kun97, Theorem 2.2.15] uniform convergence in probability of $\{\boldsymbol{M}_t(f^n)\}_{n\in\mathbb{N}}$ to $\boldsymbol{M}_t(f)$, i.e.

$$P\lim_{n\to\infty} \sup_{0\le t\le \boldsymbol{T}} \Big| \boldsymbol{M}_t(f^n) - \boldsymbol{M}_t(f) \Big| = 0.$$

③ *Let f_t be an arbitrary predictable process satisfying*

$$\int_0^{\boldsymbol{T}} a(f_r, f_r, r)\,\mathrm{d}A_r < \infty \text{ a.s.}$$

Let $(\mathbb{K}_n)_{n\in\mathbb{N}}$ be a sequence of compact subsets of $\mathbb{D}$ such that $\mathbb{K}_n \nearrow \mathbb{D}$. Let $\tilde{f}_t^n$ be a truncation of f_t associated with $\mathbb{K}_n$, $n \in \mathbb{N}$, as reproduced in Definition A.12. Then as in ② the sequence $(\tilde{f}_t^n)_{n\in\mathbb{N}}$ satisfies

$$\int_0^{\boldsymbol{T}} a(\tilde{f}_r^n, \tilde{f}_r^n, r) - 2a(\tilde{f}_r^n, \tilde{f}_r^m, r) + a(\tilde{f}_r^m, \tilde{f}_r^m, r)\,\mathrm{d}A_r \xrightarrow{n,m\to\infty} 0 \text{ a.s.}$$

and therefore we obtain that $\{\boldsymbol{M}_t(\tilde{f}^n)\}_{n\in\mathbb{N}}$ converges uniformly in probability to $\int_0^t \boldsymbol{M}(f_s, \mathrm{d}s)$, i.e.

$$P\lim_{n\to\infty} \sup_{0\le t\le \boldsymbol{T}} \Big| \int_0^t \boldsymbol{M}(\tilde{f}^n, \mathrm{d}s) - \int_0^t \boldsymbol{M}(f_s, \mathrm{d}s) \Big| = 0.$$

Example 2.30 *Let X_t be a continuous one-dimensional local martingale. Consider $\mathbf{M}(x,t) := x \cdot X_t$, $x \in \mathbb{D} \subset \mathbb{R}$, and $\mathbf{M}(x,0) = 0$. Then $\mathbf{M}(x,t)$ is continuous in t a.s. with values in $C(\mathbb{D},\mathbb{R})$, because X_t is continuous in t. Hence we have that $\mathbf{M}(\cdot,t)$ is a*

$C(\mathbb{D},\mathbb{R})$ - local martingale. Let f_t be a predictable process with values in $\mathbb{D}$. Then for any partition $\{0 = t_0 < ... < t_l = \boldsymbol{T}\}$ of $[0, \boldsymbol{T}]$ we have

$$\begin{aligned}\sum_{k=0}^{l-1}\Big(\mathbf{M}(f_{t_k\wedge t}, t_{k+1}\wedge t) - \mathbf{M}(f_{t_k\wedge t}, t_k\wedge t)\Big) &= \sum_{k=0}^{l-1}\Big(f_{t_k\wedge t}\cdot X_{t_{k+1}\wedge t} - f_{t_k\wedge t}\cdot X_{t_k\wedge t}\Big)\\ &= \sum_{k=0}^{l-1} f_{t_k\wedge t}\cdot\Big(X_{t_{k+1}\wedge t} - X_{t_k\wedge t}\Big)\end{aligned}$$

and therefore we obtain

$$\int_0^t \mathbf{M}(f_s, \mathrm{d}s) = \int_0^t f_s\,\mathrm{d}X_s,$$

as one can also find in [Kun97, after Lemma 2.3.1].

Definition 2.31 *Let $F(x,t)$, $x\in\mathbb{D}$, be a family of continuous $C(\mathbb{D},\mathbb{R})$ - semimartingales with local characteristic (a,b,A_t) belonging to $(B^{0,\delta}, B^{0,\delta})$ for some $\delta > 0$ and let f_t be a predictable process with values in $\mathbb{D}$ satisfying*

$$\int_0^{\boldsymbol{T}} a(f_s, f_s, s)\,\mathrm{d}A_s < \infty \quad \text{and} \quad \int_0^{\boldsymbol{T}} |b(f_s, s)|\,\mathrm{d}A_s < \infty \ \text{a.s.} \tag{2.5}$$

*Then the **Itô integral of f_t based on** $F(\cdot,\mathrm{d}t)$ is defined by*

$$\int_0^t F(f_s,\,\mathrm{d}s) := \int_0^t b(f_s, s)\,\mathrm{d}A_s + \int_0^t \boldsymbol{M}(f_s,\,\mathrm{d}s).$$

Remark 2.32 *We defined continuous semimartingales with values in the space $C^{k,\delta}(\Lambda,\mathbb{R}^d)$ for some $\Lambda\subset\mathbb{D}$, $k\in\mathbb{N}_0$ and $0<\delta\le 1$. Furthermore, we introduced the definition of local characteristics belonging to the class $B^{k,\delta}$. These spaces respectively classes include in particular Hölder continuity. Hence, by applying Kolmogorov's continuity theorem [Kun97, Theorem 1.4.1, Theorem 1.4.4] there exists a continuous modification. For example let $\boldsymbol{M}(\lambda,t)$ be a continuous $C^{k,\delta}$ - local martingale for some $k\ge 1$ and $0<\delta\le 1$ and let $\tilde{\boldsymbol{M}}(\lambda,t)$ be a continuous modification. That means for all $\lambda\in\Lambda$ there exists Ω_λ such that $P(\Omega_\lambda) = 1$ and*

$$\boldsymbol{M}(\lambda,\cdot,\omega) = \tilde{\boldsymbol{M}}(\lambda,\cdot,\omega) \quad \forall\,\omega\in\Omega_\lambda.$$

Now we define

$$\Omega_{\mathbb{Q}^d\cap\Lambda} := \bigcap_{\lambda\in\mathbb{Q}^d\cap\lambda}\Omega_\lambda.$$

Then we conclude that $P(\Omega_{\mathbb{Q}^d\cap\Lambda}) = 1$ and

$$\boldsymbol{M}(\lambda,\cdot,\omega) = \tilde{\boldsymbol{M}}(\lambda,\cdot,\omega) \quad \forall\,\omega\in\Omega_{\mathbb{Q}^d\cap\Lambda},$$

which is equal to

$$\boldsymbol{M}(\lambda,\cdot,\omega) = \tilde{\boldsymbol{M}}(\lambda,\cdot,\omega) = \lim_{n\to\infty}\tilde{\boldsymbol{M}}(\lambda_n,\cdot,\omega)$$

for every sequence $(\lambda_n)_{n\ge 0}\subset\mathbb{Q}^d\cap\Lambda$ with $\lambda_n \xrightarrow{n\to\infty} \lambda$ due to the continuity of $\lambda\mapsto\tilde{\boldsymbol{M}}(\lambda,\cdot,\omega)$. So rigorously, we obtain in the situation of Definition 2.29

$$\int_0^t \tilde{\boldsymbol{M}}(\lambda,\mathrm{d}s) = \lim_{\lambda_n\to\lambda}\int_0^t \tilde{\boldsymbol{M}}(\lambda_n,\mathrm{d}s)$$

$$= \lim_{\lambda_n \to \lambda} \lim_{|\Delta| \to 0} \tilde{\boldsymbol{M}}(\lambda_n, t_{k+1} \wedge t) - \tilde{\boldsymbol{M}}(\lambda_n, t_k \wedge t).$$

Hence considering $\tilde{\boldsymbol{M}}(\lambda)$ we are working with a double limit procedure. One should note that a direct construction on the space $C^{k,\delta}(\Lambda)$ with the help of UMD-spaces is not possible, since $C^{k,\delta}(\Lambda, \mathbb{R}^e)$ is not UMD as shown in a counterexample by M. Yor.

2.4. Representation results for Itô integrals

The following statement can be found as an exercise (cf. [Kun97, Exercise 3.2.11]). Now we state this representation result rigorously and prove it in detail.

Theorem 2.33 *Let $\{M^n\}_{n\geq 1}$ be an orthogonal basis of continuous, square integrable martingales. Let $\mathbf{M}(x,t)$, $x \in \mathbb{D}$, be continuous $C(\mathbb{D}, \mathbb{R})$ - local martingales with the representation*

$$\mathbf{M}(x,t) = \sum_{n\geq 1} \int_0^t f_n(x,s) \, \mathrm{d}M_s^n, \tag{2.6}$$

where $f_n(x,s)$, $n \geq 1$, are measurable random fields, predictable in s for each $x \in \mathbb{D}$. Let (a, A_t) be the local characteristic belonging to the class $B^{k,\delta}$ for some $k \in \mathbb{N}_0$ and $0 \leq \delta \leq 1$. Then we have

$$\int_0^t \mathbf{M}(\varphi_s, \mathrm{d}s) = \sum_{n\geq 1} \int_0^t f_n(\varphi_s, s) \, \mathrm{d}M_s^n \tag{2.7}$$

for any continuous predictable $\mathbb{D}$-valued process φ_t.

Proof. Due to Definition 2.29 we prove (2.7) by using the stepwise construction of the stochastic integral:
① *Simple functions φ_t*
Let φ_t be a simple process, i.e. there exists a partition $\Delta = \{0 = t_0 < t_1 < ... < t_l = \mathbf{T}\}$ of $[0, \mathbf{T}]$ respectively for a $t \in [0, \mathbf{T}]$ we consider $\Delta = \{0 = t_0 < t_1 < ... < t_l = t\}$ such that $\varphi_s = \varphi_{t_k}$ for all $s \in [t_k, t_{k+1})$. By Definition 2.29 and the representation (2.6) we obtain

$$\begin{aligned}
\int_0^t \mathbf{M}(\varphi_s, \mathrm{d}s) &= \sum_{k=0}^{l-1} \mathbf{M}(\varphi_{t_k \wedge t}, t_{k+1} \wedge t) - \mathbf{M}(\varphi_{t_k \wedge t}, t_k \wedge t) \\
&= \sum_{k=0}^{l-1} \Big(\sum_{n\geq 1} \int_0^{t_{k+1}\wedge t} f_n(\varphi_{t_k \wedge t}, s) \, \mathrm{d}M_s^n - \sum_{n\geq 1} \int_0^{t_k \wedge t} f_n(\varphi_{t_k\wedge t}, s) \, \mathrm{d}M_s^n \Big) \\
&= \sum_{k=0}^{l-1} \Big(\sum_{n\geq 1} \Big(\int_0^{t_{k+1}\wedge t} f_n(\varphi_{t_k \wedge t}, s) \, \mathrm{d}M_s^n - \int_0^{t_k \wedge t} f_n(\varphi_{t_k\wedge t}, s) \, \mathrm{d}M_s^n \Big) \Big) \\
&= \sum_{k=0}^{l-1} \Big(\sum_{n\geq 1} \Big(\int_{t_k \wedge t}^{t_{k+1}\wedge t} f_n(\varphi_{t_k \wedge t}, s) \, \mathrm{d}M_s^n \Big) \\
&= \sum_{n\geq 1} \Big(\sum_{k=0}^{l-1} \Big(\int_{t_k \wedge t}^{t_{k+1}\wedge t} f_n(\varphi_{t_k \wedge t}, s) \, \mathrm{d}M_s^n \Big) \\
&= \sum_{n\geq 1} \int_0^t f_n(\varphi_s, s) \, \mathrm{d}M_s^n,
\end{aligned}$$

where we used $\varphi_s = \varphi_{t_k}$ for all $s \in [t_k, t_{k+1})$.

② *Predictable processes on compact subsets*
Let φ_t be a predictable process with values in $\mathbb{K} \subset \mathbb{D}$ compact. By construction there exists a sequence $(\varphi_t^j)_{j\geq 1}$ of simple processes with values in $\mathbb{K}$ such that

$$\int_0^{\mathbf{T}} a(\varphi_t^i, \varphi_t^i, t) - 2a(\varphi_t^i, \varphi_t^j, t) + a(\varphi_t^j, \varphi_t^j, t) \, \mathrm{d}A_t \xrightarrow{i,j\to\infty} 0 \text{ a.s.}$$

We have to show that

$$\langle \mathbf{M}_\cdot(\varphi_\cdot^i) - \mathbf{M}_\cdot(\varphi_\cdot^j) \rangle_{\mathbf{T}} \xrightarrow{i,j\to\infty} 0 \text{ a.s.}$$

holds. For simple processes we know that for all $j \in \mathbb{N}$

$$\int_0^t \mathbf{M}(\varphi_t^j, \mathrm{d}s) = \sum_{n\geq 1} \int_0^t f_n(\varphi_s^j, s) \, \mathrm{d}M_s^n$$

is valid. Therefore we have for $i, j \in \mathbb{N}$

$$\begin{aligned}
\langle \mathbf{M}_\cdot(\varphi_\cdot^i) - \mathbf{M}_\cdot(\varphi_\cdot^j) \rangle_{\mathbf{T}} &= \Big\langle \int_0^\cdot \mathbf{M}(\varphi_s^i, \mathrm{d}s) - \int_0^\cdot \mathbf{M}(\varphi_s^j, \mathrm{d}s) \Big\rangle_{\mathbf{T}} \\
&= \Big\langle \sum_{n\geq 1} \int_0^\cdot f_n(\varphi_s^i, s) \, \mathrm{d}M_s^n - \sum_{n\geq 1} \int_0^\cdot f_n(\varphi_s^j, s) \, \mathrm{d}M_s^n \Big\rangle_{\mathbf{T}} \\
&= \Big\langle \sum_{n\geq 1} \int_0^\cdot \big(f_n(\varphi_s^i, s) - f_n(\varphi_s^j, s) \big) \, \mathrm{d}M_s^n \Big\rangle_{\mathbf{T}}.
\end{aligned}$$

By using [Kun97, Corollary 2.3.3.] and the fact that $\{M^n\}_{n\geq 1}$ is an orthogonal basis we obtain

$$\begin{aligned}
\langle \mathbf{M}_\cdot(\varphi_\cdot^i) - \mathbf{M}_\cdot(\varphi_\cdot^j) \rangle_{\mathbf{T}} &= \sum_{n\geq 1} \int_0^{\mathbf{T}} \Big(\big(f_n(\varphi_s^i, s) - f_n(\varphi_s^j, s) \big)^2 \, \mathrm{d}\langle M^n \rangle_s \\
&= \sum_{n\geq 1} \int_0^{\mathbf{T}} \big(f_n(\varphi_s^i, s)^2 - 2 f_n(\varphi_s^i, s) f_n(\varphi_s^j, s) + f_n(\varphi_s^j, s)^2 \big) \, \mathrm{d}\langle M^n \rangle_s \\
&= \sum_{n\geq 1} \int_0^{\mathbf{T}} f_n(\varphi_s^i, s)^2 \, \mathrm{d}\langle M^n \rangle_s - \sum_{n\geq 1} \int_0^{\mathbf{T}} 2 f_n(\varphi_s^i, s) f_n(\varphi_s^j, s) \, \mathrm{d}\langle M^n \rangle_s \\
&\quad + \sum_{n\geq 1} \int_0^{\mathbf{T}} f_n(\varphi_s^j, s)^2 \, \mathrm{d}\langle M^n \rangle_s.
\end{aligned}$$

Now we make use of (2.4) to receive

$$\begin{aligned}
&\langle \mathbf{M}_\cdot(\varphi_\cdot^i) - \mathbf{M}_\cdot(\varphi_\cdot^j) \rangle_{\mathbf{T}} \\
&= \int_0^{\mathbf{T}} a(\varphi_s^i, \varphi_s^i, s) \, \mathrm{d}A_s - \int_0^{\mathbf{T}} 2a(\varphi_s^i, \varphi_s^j, s) \, \mathrm{d}A_s + \int_0^{\mathbf{T}} a(\varphi_s^j, \varphi_s^j, s) \, \mathrm{d}A_s \\
&= \int_0^{\mathbf{T}} a(\varphi_s^i, \varphi_s^i, s) - 2a(\varphi_s^i, \varphi_s^j, s) + a(\varphi_s^j, \varphi_s^j, s) \, \mathrm{d}A_s \xrightarrow{i,j\to\infty} 0 \text{ a.s.}
\end{aligned}$$

Due to [Kun97, Theorem 2.2.15] $\mathbf{M}_t(\varphi_t^n)$ converges uniformly in probability to $\mathbf{M}_t(\varphi_t)$.
③ *Arbitrary predictable processes*
Now let φ_t be an arbitrary predictable process satisfying the integrability condition

$$\int_0^{\mathbf{T}} a(\varphi_t, \varphi_t, t)\, \mathrm{d}A_t < \infty \text{ a.s.}$$

Let $(\mathbb{K}_j)_{j\in\mathbb{N}}$ be a sequence of compact subsets of $\mathbb{D}$ such that $\mathbb{K}_j \subset \mathbb{D}$ and $\mathbb{K}_j \nearrow \mathbb{D}$ for $j \to \infty$. Define the following truncation of φ_t:

$$\tilde{\varphi}_t^j := \begin{cases} \varphi_t, & \text{if } \varphi_t \in \mathbb{K}_j \\ x_0 \in K_j, & \text{if } \varphi_t \notin \mathbb{K}_j. \end{cases}$$

By construction we have

$$\int_0^{\mathbf{T}} a(\tilde{\varphi}_t^i, \tilde{\varphi}_t^i, t) - 2a(\tilde{\varphi}_t^i, \tilde{\varphi}_t^j, t) + a(\tilde{\varphi}_t^j, \tilde{\varphi}_t^j, t)\, \mathrm{d}A_t \xrightarrow{i,j\to\infty} 0.$$

By an application of [Kun97, Corollary 2.3.3.] and (2.4), we obtain for $i, j \in \mathbb{N}$ as in step ②

$$\begin{aligned}
\langle \mathbf{M}_\cdot(\tilde{\varphi}_\cdot^i) - \mathbf{M}_\cdot(\tilde{\varphi}_\cdot^j)\rangle_{\mathbf{T}} &= \Big\langle \int_0^{\cdot} \mathbf{M}(\tilde{\varphi}_s^i, \mathrm{d}s) - \int_0^{\cdot} \mathbf{M}(\tilde{\varphi}_s^j, \mathrm{d}s)\Big\rangle_{\mathbf{T}} \\
&= \Big\langle \sum_{n\ge1} \int_0^{\cdot} f_n(\tilde{\varphi}_s^i, s)\, \mathrm{d}M_s^n - \sum_{n\ge1} \int_0^{\cdot} f_n(\tilde{\varphi}_s^j, s)\, \mathrm{d}M_s^n\Big\rangle_{\mathbf{T}} \\
&= \Big\langle \sum_{n\ge1} \int_0^{\cdot} \big(f_n(\tilde{\varphi}_s^i, s) - f_n(\tilde{\varphi}_s^j, s)\big)\, \mathrm{d}M_s^n\Big\rangle_{\mathbf{T}} \\
&= \sum_{n\ge1} \int_0^{\mathbf{T}} \big(f_n(\tilde{\varphi}_s^i, s)^2 - 2f_n(\tilde{\varphi}_s^i, s)f_n(\tilde{\varphi}_s^j, s) + f_n(\tilde{\varphi}_s^j, s)^2\big)\, \mathrm{d}\langle M^n\rangle_s \\
&= \sum_{n\ge1} \int_0^{\mathbf{T}} f_n(\tilde{\varphi}_s^i, s)^2\, \mathrm{d}\langle M^n\rangle_s - \sum_{n\ge1} \int_0^{\mathbf{T}} 2f_n(\tilde{\varphi}_s^i, s)f_n(\tilde{\varphi}_s^j, s)\, \mathrm{d}\langle M^n\rangle_s \\
&\quad + \sum_{n\ge1} \int_0^{\mathbf{T}} f_n(\tilde{\varphi}_s^j, s)^2\, \mathrm{d}\langle M^n\rangle_s \\
&= \int_0^{\mathbf{T}} a(\tilde{\varphi}_s^i, \tilde{\varphi}_s^i, s)\, \mathrm{d}A_s - \int_0^{\mathbf{T}} 2a(\tilde{\varphi}_s^i, \tilde{\varphi}_s^j, s)\, \mathrm{d}A_s + \int_0^{\mathbf{T}} a(\tilde{\varphi}_s^j, \tilde{\varphi}_s^j, s)\, \mathrm{d}A_s \\
&= \int_0^{\mathbf{T}} a(\tilde{\varphi}_s^i, \tilde{\varphi}_s^i, s) - 2a(\tilde{\varphi}_s^i, \tilde{\varphi}_s^j, s) + a(\tilde{\varphi}_s^j, \tilde{\varphi}_s^j, s)\, \mathrm{d}A_s \xrightarrow{i,j\to\infty} 0 \text{ a.s.}
\end{aligned}$$

Then by [Kun97, Theorem 2.2.15] $\mathbf{M}_t(\tilde{\varphi}_t^n)$ converges uniformly in probability to $\int_0^t \mathbf{M}(\varphi_s, \mathrm{d}s)$. □

As a conclusion we obtain the following representation result for stochastic Itô integrals based on semimartingales.

Theorem 2.34 *Let $\{M^n\}_{n\geq 1}$ be an orthogonal basis of continuous, square integrable martingales. Let $F(x,t)$, $x \in \mathbb{D}$, be a family of continuous $C(\mathbb{D},\mathbb{R})$ - semimartingales with local characteristic*

$$\big(a(x,y,t), b(x,t), A_t\big)$$

belonging to the class $(B^{0,\delta}, B^{0,\delta})$ for some $0 < \delta \leq 1$. Let φ_t be a predictable process with values in $\mathbb{D}$ and let condition (2.5) be fulfilled i.e.

$$\int_0^{\boldsymbol{T}} a(\varphi_s, \varphi_s, s)\, \mathrm{d}A_s < \infty \quad \text{and} \quad \int_0^{\boldsymbol{T}} |b(\varphi_s, s)|\, \mathrm{d}A_s < \infty \ a.s.$$

Then the Itô integral based on $F(\cdot, \mathrm{d}t)$ can be represented as

$$\int_0^t F(\varphi_s, \mathrm{d}s) = \int_0^t f_0(\varphi_s, s)\, \mathrm{d}A_s + \sum_{n\geq 1} \int_0^t f_n(\varphi_s, s)\, \mathrm{d}M_s^n, \tag{2.8}$$

where $f_n(x,s)$, $n \geq 1$, are measurable random fields, predictable in s for each $x \in \mathbb{D}$.

Proof. Due to Definition 2.31 and Theorem 2.33, the representation formula (2.8) is valid in the following sense:

$$\begin{aligned}
\int_0^t F(\varphi_s, \mathrm{d}s) &= \int_0^t B(\varphi_s, \mathrm{d}s) + \int_0^t \mathbf{M}(\varphi_s, \mathrm{d}s) \\
&= \int_0^t b(\varphi_s, s)\, \mathrm{d}A_s + \int_0^t \mathbf{M}(\varphi_s, \mathrm{d}s) \\
&= \int_0^t f_0(\varphi_s, s)\, \mathrm{d}A_s + \sum_{n\geq 1} \int_0^t f_n(\varphi_s, s)\, \mathrm{d}M_s^n,
\end{aligned}$$

where $b(x,t) =: f_0(x,t)$. □

2.5. Itô-Stratonovich formula

The main advantage of working with Stratonovich integrals is the applicability of the chain rule. Similarly to the classical chain rule one obtains the fundamental theorem of calculus. Let W_t be a standard one-dimensional Brownian motion, then

$$\int_0^t W_s \circ \mathrm{d}W_s = \frac{1}{2}(W_t)^2 - \frac{1}{2}(W_0)^2$$

holds and for any smooth function $f : \mathbb{R} \to \mathbb{R}$ with first derivative f' we have

$$\int_0^t f'(W_s) \circ \mathrm{d}W_s = f(W_t) - f(W_0)$$

(see [KS10, 2.29]). We use these tools of the Stratonovich integral in applications (e.g. Chapter 5 below). Of course if we want to calculate and solve systems of stochastic differential equations, the application of the chain rule also for stochastic integrals is very helpful. Additionally, it is a well-known result that if we want to rewrite an Itô integral into a Stratonovich integral we have to add a correction term, the so-called Itô correction term. One can find this Itô-Stratonovich formula for example in [KP91, p. 316].

Let $(X_t)_{t\in[0,\mathbf{T}]}$ be a d-dimensional Itô process which, under appropriate assumptions on $b\colon \mathbb{R}^d \times [0,\mathbf{T}] \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \times [0,\mathbf{T}] \to \mathbb{R}^{d\times m}$, satisfies for an m-dimensional Brownian motion $W_t = (W_t^1, ..., W_t^m)$ the following equation

$$\begin{aligned} X_t &= X_0 + \int_0^t b(X_s, s)\,\mathrm{d}s + \int_0^t \sigma(X_s, s)\,\mathrm{d}W_s \\ &= X_0 + \int_0^t b(X_s, s)\,\mathrm{d}s + \sum_{n=1}^m \int_0^t \sigma_{\cdot n}(X_s, s)\,\mathrm{d}W_s^n \end{aligned} \tag{2.9}$$

for all $t \in [0,\mathbf{T}]$. Equation (2.9) can be written equivalently as a Stratonovich stochastic differential equation:

$$\begin{aligned} X_t = X_0 &+ \int_0^t b(X_s, s)\,\mathrm{d}s + \sum_{n=1}^m \int_0^t \sigma_{\cdot n}(X_s, s) \circ \mathrm{d}W_s^n \\ &- \int_0^t \frac{1}{2}\sum_{n=1}^m \sum_{j=1}^d \sigma_{jn}(X_s, s)\frac{\partial \sigma_{\cdot n}}{\partial x_j}(X_s, s)\,\mathrm{d}s. \end{aligned} \tag{2.10}$$

The following theorem gives us this relation in our setting and can be found including the proof in [Kun97, Theorem 3.2.5].

Theorem 2.35 *Assume that $F(x,t)$, $x \in \mathbb{D}$, is a family of continuous $C^1(\mathbb{D},\mathbb{R}^d)$-semimartingales with local characteristic (a, b, A_t) belonging to $(B^{2,\delta}, B^{1,0})$ for some $0 < \delta \le 1$. Furthermore, let φ_t be a continuous semimartingale. Then the* ***Stratonovich integral*** *is well-defined and related to the Itô integral by*

$$\int_0^t F(\varphi_s, \circ\mathrm{d}s) = \int_0^t F(\varphi_s, \mathrm{d}s) + \frac{1}{2}\sum_{j=1}^d \Big\langle \int_0^{\bullet} \frac{\partial F}{\partial x_j}(\varphi_s, \mathrm{d}s), \varphi_{\bullet}^j \Big\rangle_t.$$

Lemma 2.36 *By applying the representation result Theorem 2.34 componentwise, the above results (2.10) and Theorem 2.35 are equivalent for an m-dimensional Brownian motion.*

Proof. Let $W_t = (W_t^1, ..., W_t^m)$ be an m-dimensional Brownian motion. We rewrite the stochastic differential equations into the same notation for drift and diffusion terms. We consider the cases $b(x,s) = f_0(x,s)$ and $\sigma(x,s) = (f_{ij}(x,s))_{\substack{i=1,\ldots,d \\ j=1,\ldots,m}}$. The d-dimensional Itô process φ_t which solves

$$\int_0^t F(\varphi_s, \mathrm{d}s) = \varphi_t$$

is given in the j-th component due to the representation result Theorem 2.34 by

$$\varphi_t^j = \int_0^t f_0^j(\varphi_s, s)\,\mathrm{d}s + \int_0^t \sum_{n=1}^m f_{jn}(\varphi_s, s)\,\mathrm{d}W_s^n, \tag{2.11}$$

provided $\varphi_0^j = 0$ for simplicity.
Now we prove the equivalence using Theorem 2.35 and Theorem 2.33, i.e. we get

$$\int_0^t F(\varphi_s, \circ\mathrm{d}s) = \int_0^t F(\varphi_s, \mathrm{d}s) + \frac{1}{2}\sum_{j=1}^d \Big\langle \int_0^{\bullet} \frac{\partial F}{\partial x_j}(\varphi_s, \mathrm{d}s), \varphi_{\bullet}^j \Big\rangle_t$$

$$
\begin{aligned}
&= \int_0^t B(\varphi_s, \mathrm{d}s) + \int_0^t \mathbf{M}(\varphi_s, \mathrm{d}s) \\
&\quad + \frac{1}{2}\sum_{j=1}^d \Big\langle \int_0^\bullet \frac{\partial [B+\mathbf{M}]}{\partial x_j}(\varphi_s, \mathrm{d}s), \varphi^j_\bullet \Big\rangle_t \\
&= \int_0^t B(\varphi_s, \mathrm{d}s) + \int_0^t \mathbf{M}(\varphi_s, \mathrm{d}s) + \frac{1}{2}\sum_{j=1}^d \Big\langle \int_0^\bullet \frac{\partial \mathbf{M}}{\partial x_j}(\varphi_s, \mathrm{d}s), \varphi^j_\bullet \Big\rangle_t,
\end{aligned}
$$

where we use that $D^\alpha_x B(x,t)$, $x \in \mathbb{D}$, $t \in [0,\mathbf{T}]$, are processes of bounded variation (see Definition 2.14). By the representation of F and (2.11) we conclude

$$
\begin{aligned}
\int_0^t F(\varphi_s, \circ\mathrm{d}s) &= \int_0^t f_0(\varphi_s, s)\,\mathrm{d}s + \int_0^t \sum_{n=1}^m f_{\cdot n}(\varphi_s, s)\,\mathrm{d}W^n_s \\
&\quad + \frac{1}{2}\sum_{j=1}^d \Big\langle \int_0^\bullet \frac{\partial [\sum_{n=1}^m f_{\cdot n}]}{\partial x_j}(\varphi_s, s)\,\mathrm{d}W^n_s, \varphi^j_\bullet \Big\rangle_t \\
&= \int_0^t f_0(\varphi_s, s)\,\mathrm{d}s + \int_0^t \sum_{n=1}^m f_{\cdot n}(\varphi_s, s)\,\mathrm{d}W^n_s \\
&\quad + \frac{1}{2}\sum_{j=1}^d \Big\langle \int_0^\bullet \frac{\partial [\sum_{n=1}^m f_{\cdot n}]}{\partial x_j}(\varphi_s, s)\,\mathrm{d}W^n_s, \int_0^\bullet f^j_0(\varphi_s, s)\,\mathrm{d}s + \int_0^\bullet \sum_{n=1}^m f_{jn}(\varphi_s, s)\,\mathrm{d}W^n_s \Big\rangle_t \\
&= \int_0^t f_0(\varphi_s, s)\,\mathrm{d}s + \int_0^t \sum_{n=1}^m f_{\cdot n}(\varphi_s, s)\,\mathrm{d}W^n_s \\
&\quad + \frac{1}{2}\sum_{j=1}^d \Big\langle \int_0^\bullet \sum_{n=1}^m \frac{\partial f_{\cdot n}}{\partial x_j}(\varphi_s, s)\,\mathrm{d}W^n_s, \int_0^\bullet f^j_0(\varphi_s, s)\,\mathrm{d}s \Big\rangle \\
&\quad + \frac{1}{2}\sum_{j=1}^d \Big\langle \int_0^\bullet \sum_{n=1}^m \frac{\partial f_{\cdot n}}{\partial x_j}(\varphi_s, s)\,\mathrm{d}W^n_s, \int_0^\bullet \sum_{n=1}^m f_{jn}(\varphi_s, s)\,\mathrm{d}W^n_s \Big\rangle_t \\
&= \int_0^t f_0(\varphi_s, s)\,\mathrm{d}s + \int_0^t \sum_{n=1}^m f_{\cdot n}(\varphi_s, s)\,\mathrm{d}W^n_s \\
&\quad + \frac{1}{2}\int_0^t \sum_{n=1}^m \sum_{j=1}^d \frac{\partial f_{\cdot n}}{\partial x_j}(\varphi_s, s) f_{jn}(\varphi_s, s)\langle \mathrm{d}W^n_\bullet, \mathrm{d}W^n_\bullet\rangle_s \\
&= \int_0^t f_0(\varphi_s, s)\,\mathrm{d}s + \int_0^t \sum_{n=1}^m f_{\cdot n}(\varphi_s, s)\,\mathrm{d}W^n_s + \frac{1}{2}\int_0^t \sum_{n=1}^m \sum_{j=1}^d \frac{\partial f_{\cdot n}}{\partial x_j}(\varphi_s, s) f_{jn}(\varphi_s, s)\,\mathrm{d}s.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
\int_0^t F(\varphi_s, \mathrm{d}s) &= \int_0^t F(\varphi_s, \circ\mathrm{d}s) - \frac{1}{2}\int_0^t \sum_{n=1}^m \sum_{j=1}^d \frac{\partial f_{\cdot n}}{\partial x_j}(\varphi_s, s) f_{jn}(\varphi_s, s)\mathrm{d}s \\
&= \int_0^t f_0(\varphi_s, s)\,\mathrm{d}s + \int_0^t \sum_{n=1}^m f_{\cdot n}(\varphi_s, s)\,\mathrm{d}W^n_s \\
&= \varphi_t - \varphi_0.
\end{aligned} \tag{2.12}
$$

□

Remark 2.37 *If we consider $\varphi_t = x \in \mathbb{D}$ in the situation of Theorem 2.35, the Itô correction term vanishes and we obtain*

$$\int_0^t F(x, \circ \mathrm{d}s) = \int_0^t F(x, \mathrm{d}s)$$

which is due to the decomposition

$$\int_0^t F(x, \mathrm{d}s) = \int_0^t \boldsymbol{M}(x, \mathrm{d}s) + \int_0^t B(x, \mathrm{d}s).$$

By using the construction of the Itô integral (see Definition 2.29) we get for every partition $\{0 = t_0 < \ldots < t_l = t\}$ of $[0, t]$

$$\begin{aligned}\sum_{k=0}^{l-1} (\boldsymbol{M}(x, t_{k+1} \wedge t) - \boldsymbol{M}(x, t_k \wedge t)) + \sum_{k=0}^{l-1} (B(x, t_{k+1} \wedge t) - B(x, t_k \wedge t))& \\ = M(x, t) - M(x, 0) + B(x, t) - B(x, 0)& \\ = \big(\boldsymbol{M}(x, t) + B(x, t)\big) - \big(\boldsymbol{M}(x, 0) + B(x, 0)\big)& \\ = F(x, t) - F(x, 0).&\end{aligned}$$

Hence we conclude

$$\int_0^t F(x, \circ \mathrm{d}s) = \int_0^t F(x, \mathrm{d}s) = F(x, t) - F(x, 0).$$

2.6. Representation results for Stratonovich integrals

To formulate a representation result for Stratonovich integrals based on semimartingales we start with the formal definition of a Stratonovich integral as in [Kun97, before Theorem 3.2.5].

Definition 2.38 *Let $F(x, t)$, $x \in \mathbb{D}$, be a family of continuous $C(\mathbb{D}, \mathbb{R})$ - semimartingales and let φ_t be a continuous process with values in $\mathbb{D}$. For a partition $\Delta = \{0 = t_0 < \ldots < t_l = \boldsymbol{T}\}$ we define*

$$\begin{aligned}F_t^{\Delta}(\varphi) := \sum_{k=0}^{l-1} \frac{1}{2}\Big(&F(\varphi_{t_{k+1} \wedge t}, t_{k+1} \wedge t) + F(\varphi_{t_k \wedge t}, t_{k+1} \wedge t) \\ &- F(\varphi_{t_{k+1} \wedge t}, t_k \wedge t) - F(\varphi_{t_k \wedge t}, t_k \wedge t)\Big).\end{aligned}$$

If the sequence $(F_t^{\Delta_m}(\varphi))_{m \in \mathbb{N}}$ converges in probability uniformly in t for any sequence of partitions Δ_m such that $|\Delta_m| \to 0$, then the limit $\int_0^t F(\varphi_s, \circ \mathrm{d}s)$ is called Stratonovich integral of φ_t based on $F(x, t)$, i.e.

$$P \lim_{|\Delta_m| \to 0} \sup_{0 \le t \le \boldsymbol{T}} \left| F_t^{\Delta_m}(\varphi) - \int_0^t F(\varphi_s, \circ \mathrm{d}s) \right| = 0.$$

We follow the idea of [Kun97, Exercise 3.3.5] to formulate the representation result in the case of Stratonovich integrals rigorously.

Theorem 2.39 *Let $\{M_t^n\}_{n\geq1}$ be an orthogonal basis of continuous, square integrable martingales. Let $F(x,t)$, $x \in \mathbb{D}$, be a family of continuous $C^1(\mathbb{D},\mathbb{R})$ - semimartingales with local characteristic (a,b,A_t) belonging to $(B^{2,\delta},B^{1,0})$ for some $0<\delta\leq 1$. Assume that $F(x,t)$ can be represented by continuous C^2 - processes and C^1 - semimartingales $(\tilde{f}_n)_{n\geq0}$ with local characteristics belonging to the class $(B^{1,0},B^{1,0})$ as in Theorem 2.34. Then we find $(f_n(x,t))_{n\geq0}$ continuous C^2 - processes and C^1 - semimartingales such that for every continuous semimartingale φ_t with values in $\mathbb{D}$ we have*

$$\int_0^t F(\varphi_s,\circ\mathrm{d}s) = \int_0^t f_0(\varphi_s,s)\,\mathrm{d}A_s + \sum_{n\geq1}\int_0^t f_n(\varphi_s,s)\circ\mathrm{d}M_s^n. \tag{2.13}$$

Proof. Due to the assumption on $F(x,t)$ we are able to apply Theorem 2.34. By Definition 2.31 we have for any continuous semimartingale φ_t

$$\int_0^t F(\varphi_s,\mathrm{d}s) = \int_0^t \tilde{f}_n(\varphi_s,s)\,\mathrm{d}A_s + \sum_{n\geq1}\int_0^t \tilde{f}_n(\varphi_s,s)\,\mathrm{d}M_s^n.$$

The integrability condition (2.5) of the definition is fulfilled for the continuous semimartingale, since $B^{1,0}\subset B^{0,\delta}$ as shown in Corollary A.8. First we prove

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^d\Big\langle\int_0^\bullet\sum_{n\geq1}\frac{\partial\tilde{f}_n}{\partial x_i}(\varphi_s,s)\,\mathrm{d}M_s^n,\varphi_\bullet^i\Big\rangle_t \\ = \frac{1}{2}\sum_{n\geq1}\Big\langle\tilde{f}_n(\varphi_\bullet,\bullet),M_\bullet^n\Big\rangle_t - \frac{1}{2}\sum_{n\geq1}\Big\langle\int_0^\bullet\tilde{f}_n(\varphi_s,\mathrm{d}s),M_\bullet^n\Big\rangle_t.\end{aligned} \tag{2.14}$$

By using [Kun97, Theorem 2.3.2] we have

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^d\Big\langle\int_0^\bullet\sum_{n\geq1}\frac{\partial\tilde{f}_n}{\partial x_i}(\varphi_s,s)\,\mathrm{d}M_s^n,\varphi_\bullet^i\Big\rangle_t &= \frac{1}{2}\sum_{n\geq1}\sum_{i=1}^d\int_0^t\frac{\partial\tilde{f}_n}{\partial x_i}(\varphi_s,s)\,\mathrm{d}\langle\varphi_\bullet^i,M_\bullet^n\rangle_s \\ &= \frac{1}{2}\sum_{n\geq1}\Big\langle\sum_{i=1}^d\int_0^\bullet\frac{\partial\tilde{f}_n}{\partial x_i}(\varphi_s,s)\,\mathrm{d}\varphi_s^i,M_\bullet^n\Big\rangle_t\end{aligned}$$

Now we add some proper terms, which are in particular of bounded variation, and hence their joint quadratic variations with the orthogonal basis $\{M_t^n\}_{n\geq1}$ are zero. We conclude by the generalized d-dimensional Itô formula in Theorem A.13

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^d\Big\langle\int_0^\bullet\sum_{n\geq1}\frac{\partial\tilde{f}_n}{\partial x_i}(\varphi_s,s)\,\mathrm{d}M_s^n,\varphi_\bullet^i\Big\rangle_t \\ &= \frac{1}{2}\sum_{n\geq1}\Big\langle\sum_{i=1}^d\int_0^\bullet\frac{\partial\tilde{f}_n}{\partial x_i}(\varphi_s,s)\,\mathrm{d}\varphi_s^i,M_\bullet^n\Big\rangle_t + \frac{1}{2}\sum_{n\geq1}\Big\langle\tilde{f}_n(\varphi_0,0),M_\bullet^n\Big\rangle_t \\ &\quad+\frac{1}{2}\sum_{n\geq1}\Big\langle\frac{1}{2}\sum_{i,j=1}^d\int_0^\bullet\frac{\partial^2\tilde{f}_n}{\partial x_i\partial x_j}(\varphi_s,s)\,\mathrm{d}\langle\varphi_\bullet^i,\varphi_\bullet^j\rangle_s,M_\bullet^n\Big\rangle_t \\ &\quad+\frac{1}{2}\sum_{n\geq1}\Bigg\langle\sum_{i=1}^d\Big\langle\int_0^\bullet\frac{\partial\tilde{f}_n}{\partial x_i}(\varphi_s,\mathrm{d}s),\varphi_\bullet^i\Big\rangle_\bullet,M_\bullet^n\Bigg\rangle_t \\ &\quad+\frac{1}{2}\sum_{n\geq1}\Big\langle\int_0^\bullet\tilde{f}_n(\varphi_s,\mathrm{d}s),M_\bullet^n\Big\rangle_t - \frac{1}{2}\sum_{n\geq1}\Big\langle\int_0^\bullet\tilde{f}_n(\varphi_s,\mathrm{d}s),M_\bullet^n\Big\rangle_t\end{aligned}$$

$$= \frac{1}{2}\sum_{n\geq 1}\Big\langle \tilde{f}_n(\varphi_0,0) + \int_0^{\bullet} \tilde{f}_n(\varphi_s,\mathrm{d}s) + \sum_{i=1}^d \int_0^{\bullet} \frac{\partial \tilde{f}_n}{\partial x_i}(\varphi_s,s)\,\mathrm{d}\varphi_s^i$$

$$+ \frac{1}{2}\sum_{i,j=1}^d \int_0^{\bullet} \frac{\partial^2 \tilde{f}_n}{\partial x_i \partial x_j}(\varphi_s,s)\,\mathrm{d}\langle \varphi_\bullet^i, \varphi_\bullet^j\rangle_s + \sum_{i=1}^d \Big\langle \int_0^{\bullet} \frac{\partial \tilde{f}_n}{\partial x_i}(\varphi_s,s)\,\mathrm{d}\varphi_s^i, \varphi_\bullet^i\Big\rangle_\bullet, M_\bullet^n\Big\rangle_t$$

$$- \frac{1}{2}\sum_{n\geq 1}\Big\langle \int_0^{\bullet} \tilde{f}_n(\varphi_s,\mathrm{d}s), M_\bullet^n\Big\rangle_t$$

$$= \frac{1}{2}\sum_{n\geq 1}\Big\langle \tilde{f}_n(\varphi_\bullet,\bullet), M_\bullet^n\Big\rangle_t - \frac{1}{2}\sum_{n\geq 1}\Big\langle \int_0^{\bullet} \tilde{f}_n(\varphi_s,\mathrm{d}s), M_\bullet^n\Big\rangle_t.$$

We know that $(\tilde{f}_n)_{n\geq 0}$ are semimartingales so hence they also can be represented due to Theorem 2.34 by

$$\int_0^t \tilde{f}_n(\varphi_s,\mathrm{d}s) = \int_0^t \tilde{f}_{n,0}(\varphi_s,s)\,\mathrm{d}A_s + \sum_{m\geq 1}\int_0^t \tilde{f}_{n,m}(\varphi_s,s)\,\mathrm{d}M_s^m.$$

By applying as before [Kun97, Theorem 2.3.2] and using that $\{M_t^n\}_{n\geq 1}$ is an orthogonal basis (see Notation 2.23 above) we obtain for the second term on the right hand side of (2.14)

$$\frac{1}{2}\sum_{n\geq 1}\Big\langle \int_0^{\bullet} \tilde{f}_n(\varphi_s,\mathrm{d}s), M_\bullet^n\Big\rangle_t$$

$$= \frac{1}{2}\sum_{n\geq 1}\Big\langle \int_0^{\bullet} \tilde{f}_{n,0}(\varphi_s,s)\,\mathrm{d}A_s, M_\bullet^n\Big\rangle_t + \frac{1}{2}\sum_{n\geq 1}\sum_{m\geq 1}\int_0^t \tilde{f}_{n,m}(\varphi_s,s)\,\mathrm{d}\langle M_\bullet^m, M_\bullet^n\rangle_s$$

$$= \frac{1}{2}\sum_{n\geq 1}\int_0^t \tilde{f}_{n,n}(\varphi_s,s)\,\mathrm{d}A_s$$

Finally we achieve for (2.14) equivalently

$$\frac{1}{2}\sum_{i=1}^d \Big\langle \int_0^{\bullet} \sum_{n\geq 1} \frac{\partial \tilde{f}_n}{\partial x_i}(\varphi_s,s)\,\mathrm{d}M_s^n, \varphi_\bullet^i\Big\rangle_t$$
$$= \frac{1}{2}\sum_{n\geq 1}\Big\langle \tilde{f}_n(\varphi_\bullet,\bullet), M_\bullet^n\Big\rangle_t - \frac{1}{2}\sum_{n\geq 1}\int_0^t \tilde{f}_{n,n}(\varphi_s,s)\,\mathrm{d}A_s. \tag{2.15}$$

Now we are able to prove the claimed representation result. We apply Theorem 2.35 and equation (2.15) to get

$$\int_0^t F(\varphi_s,\circ \mathrm{d}s) = \int_0^t F(\varphi_s,\mathrm{d}s) + \frac{1}{2}\sum_{i=1}^d \Big\langle \int_0^{\bullet} \frac{\partial F}{\partial x_i}(\varphi_s,\mathrm{d}s), \varphi_\bullet^i\Big\rangle_t$$

$$= \int_0^t \tilde{f}_0(\varphi_s,s)\,\mathrm{d}A_s + \sum_{n\geq 1}\int_0^t \tilde{f}_n(\varphi_s,s)\,\mathrm{d}M_s^n$$

$$+ \frac{1}{2}\sum_{i=1}^d \Big\langle \int_0^{\bullet} \sum_{n\geq 1} \frac{\partial \tilde{f}_n}{\partial x_i}(\varphi_s,s)\,\mathrm{d}M_s^n, \varphi_\bullet^i\Big\rangle_t$$

$$= \int_0^t \tilde{f}_0(\varphi_s, s)\,\mathrm{d}A_s + \sum_{n\geq 1} \int_0^t \tilde{f}_n(\varphi_s, s)\,\mathrm{d}M_s^n$$

$$+ \frac{1}{2}\sum_{n\geq 1}\left\langle \tilde{f}_n(\varphi_\bullet, \bullet), M_\bullet^n \right\rangle_t - \frac{1}{2}\sum_{n\geq 1}\int_0^t \tilde{f}_{n,n}(\varphi_s, s)\,\mathrm{d}A_s$$

$$= \int_0^t \left(\tilde{f}_0(\varphi_s, s) - \frac{1}{2}\sum_{n\geq 1}\int_0^t \tilde{f}_{n,n}(\varphi_s, s) \right) \mathrm{d}A_s$$

$$+ \sum_{n\geq 1}\left(\int_0^t \tilde{f}_n(\varphi_s, s)\,\mathrm{d}M_s^n + \frac{1}{2}\langle \tilde{f}_n(\varphi_\bullet, \bullet), M_\bullet^n\rangle_t \right)$$

By using [Kun97, Theorem 2.3.5] we conclude with a suitable definition of $(f_n)_{n\geq 0}$

$$\int_0^t F(\varphi_s, \circ \mathrm{d}s) = \int_0^t f_0(\varphi_s, s)\,\mathrm{d}A_s + \sum_{n\geq 1}\int_0^t f_n(\varphi_s, s) \circ \mathrm{d}M_s^n$$

□

2.7. Results in the case of Brownian motion

The representation results of stochastic integrals as in the previous subsections are based on the orthogonal basis of continuous, square integrable martingales and the corresponding local characteristics. In this subsection we choose finite or infinite independent copies of a standard Brownian motion $(W_t^n)_{n\geq 1}$ as the orthogonal basis and use that the quadratic variation of the Brownian motion is given by $\langle W_\bullet, W_\bullet \rangle_t = t$ respectively $\langle W_\bullet^i, W_\bullet^j \rangle_t = \delta_{ij} t$.

Notation 2.40 *If we consider infinite independent copies of a standard Brownian motion $(W_t^n)_{n\geq 1}$, the local characteristic of a family of continuous $C^{k,\delta}(\mathbb{D}, \mathbb{R})$ - semimartingales $F(x,t)$, $x \in \mathbb{D}$, are given in an explicit form. Remembering Theorem 2.21, we obtain for (2.3) by the geometric series*

$$\begin{aligned} A_t &= \sum_{n\geq 1} \frac{1}{2^n}\left(\langle W_\cdot^n \rangle_t - \langle W_\cdot^n \rangle_0 \right) \\ &= \sum_{n\geq 1} \frac{1}{2^n}\left(t - 0 \right) \\ &= \sum_{n\geq 0} \frac{1}{2^{n+1}} t = t. \end{aligned}$$

Consequently we have equivalently to (2.4)

$$a(x,y,t) = \sum_{n\geq 1} f_n(x,t) f_n(y,t)$$

and so the local characteristic is given by

$$\left(\sum_{n\geq 1} f_n(x,t) f_n(y,t), f_0(x,t), t \right)$$

as mentioned in [Kun97, Example after Lemma 3.4.4, p.106].

By applying Theorem 2.34 in the case of Brownian motion we obtain the following result:

Corollary 2.41 *Let $(W_t^n)_{n\geq 1}$ be infinite independent copies of a one-dimensional standard Brownian motion. Let $F(x,t)$, $x \in \mathbb{D}$, be a family of continuous $C(\mathbb{D},\mathbb{R})$ - semimartingales with local characteristic*

$$\big(a(x,y,t), b(x,t), A_t\big) = \Big(\sum_{n\geq 1} f_n(x,t) f_n(y,t), f_0(x,t), t\Big)$$

belonging to the class $(B^{0,\delta}, B^{0,\delta})$ for some $\delta > 0$. Let φ_t be a predictable process with values in $\mathbb{D}$ and satisfying condition (2.5) with

$$\int_0^T \sum_{n\geq 1} f_n^2(\varphi_s, s)\,\mathrm{d}s < \infty \quad \text{and} \quad \int_0^T |f_0(\varphi_s, s)|\,\mathrm{d}s < \infty \ \text{a.s.}$$

Then the Itô stochastic integral is represented by

$$\begin{aligned} \int_0^t F(\varphi_s, \mathrm{d}s) &= \int_0^t f_0(\varphi_s, s)\,\mathrm{d}s + \sum_{n\geq 1} \int_0^t f_n(\varphi_s, s)\,\mathrm{d}W_s^n \\ &= \sum_{n\geq 0} \int_0^t f_n(\varphi_s, s)\,\mathrm{d}W_s^n \end{aligned} \tag{2.16}$$

provided that $W_t^0 := t$.

In the case of Stratonovich integrals we obtain the following application of Theorem 2.39:

Corollary 2.42 *Let $(W_t^n)_{n\geq 1}$ be infinite independent copies of a standard Brownian motion. Let $F(x,t)$, $x \in \mathbb{D}$, be a family of continuous $C^1(\mathbb{D},\mathbb{R})$ - semimartingales with local characteristic (a, b, A_t) belonging to $(B^{2,\delta}, B^{1,0})$ for some $0 < \delta \leq 1$. Assume that $F(x,t)$ can be represented by continuous C^2 - processes and C^1 - semimartingales $(\tilde{f}_n)_{n\geq 0}$ with local characteristics belonging to the class $(B^{1,0}, B^{1,0})$. Then we find $(f_n(x,t))_{n\geq 0}$ continuous C^2 - processes and C^1 - semimartingales such that for every continuous semimartingale φ_t with values in $\mathbb{D}$ we have*

$$\begin{aligned} \int_0^t F(\varphi_s, \circ\mathrm{d}s) &= \sum_{n\geq 0} \int_0^t f_n(\varphi_s, s) \circ \mathrm{d}W_s^n \\ &= \int_0^t f_0(\varphi_s, s)\,\mathrm{d}s + \sum_{n\geq 1} \int_0^t f_n(\varphi_s, s) \circ \mathrm{d}W_s^n \end{aligned} \tag{2.17}$$

provided $W_t^0 := t$.

3. The method of stochastic characteristics

In this chapter we derive the method of stochastic characteristics in a nutshell, that means we summarize the whole method. Compared to the classical method of characteristics as remembered in Subsection 1.5., we obtain for almost all ω and all space and time variables up to a certain stopping time similar definitions of stochastic characteristic curves, the associated stochastic characteristic equations and finally a solution to a stochastic partial differential equation. This abstract of the method can be used to apply it as a heuristic approach. Under the assumption that the characteristic curves exist, one determines a candidate for the solution, provided that an explicit problem is given. At the end obviously, one has to verify that the candidate of the solution really solves the problem. Furthermore, the author of this thesis discusses two questions in Subsection 3.2. below:

- Why do solutions to the stochastic characteristic equations exist?
- What are the corresponding assumptions to Assumption 1.1 and of noncharacteristic initial data (see Remark 1.2)?

Concerning the first question some results of [Kun97, Chapter 4] has to be recalled, in particular the framework of stochastic flows and the concept of local processes. As written in Remark 3.14 below we clarify that the results are given for almost all ω and all space and time pairs (x,t) with t up to a certain stopping time depending on x and ω. These stopping times play an important role in finding the inverse process and finally obtain the solution to the stochastic partial differential equation. This concept is sketched in this chapter.

3.1. Derivation of the method in a nutshell

In the previous chapter all definitions and results were given and proved on a domain $\mathbb{D} \subset \mathbb{R}^d$, respectively for space variables in $\mathbb{R}^d$. For working with stochastic partial differential equations of the form

$$du = F(x, u, \nabla u, \circ dt)$$

we have to consider the multivariable case for $(x, u, \nabla u) \in \mathbb{R}^{2d+1}$. Instead of $\mathbb{R}^d$ respectively $\mathbb{D} \subset \mathbb{R}^d$ all results can be extended easily to the space $\mathbb{R}^{2d+1}$. Hence in this chapter we consider families of continuous semimartingales $F(x,u,p,t)$, $(x,u,p) \in \mathbb{R}^{2d+1}$, as in Definition 2.14. Obviously, the Fréchet spaces (cf. Definition 2.11) and classes of local characteristics (cf. Definition 2.24 - Definition 2.26) are also defined for indices $(x,u,p) \in \mathbb{R}^{2d+1}$.

Let $\mathbf{T} > 0$ and consider the time interval $[0, \mathbf{T}]$. Let $F(x,u,p,t)$ be a continuous $C^{k,\delta}(\mathbb{R}^{2d+1}, \mathbb{R})$ - semimartingale in the sense of Corollary 2.42 with local characteristic belonging to the class $(B^{k+1,\delta}, B^{k,\delta})$ for some $k \geq 5$ and $0 < \delta \leq 1$. Hence we can find $(f_n(x,t))_{n\geq 0}$ continuous C^2 - processes and C^1 - semimartinagales with local characteristics belonging to $(B^{1,0}, B^{1,0})$ as shown in Theorem 2.39. Furthermore, let $g \in C^{k,\delta}(\mathbb{R}^d, \mathbb{R})$. We consider the nonlinear stochastic partial differential equation of first order given by

$$\begin{cases} du = f_0(x,u,\nabla u,t)\, dt + \sum_{n\geq 1} f_n(x,u,\nabla u,t) \circ dW_t^n \\ u|_{t=0} = g, \end{cases} \tag{3.1}$$

and perturbed by infinite independent copies of the one-dimensional Brownian motion $(W_t^n)_{n\geq 1}$. Obviously, one can also consider the finite-dimensional case of the form

$$\begin{cases} \mathrm{d}u = f_0(x,u,\nabla u,t)\,\mathrm{d}t + \sum_{n=1}^{m} f_n(x,u,\nabla u,t)\circ \mathrm{d}W_t^n, \\ u|_{t=0} = g, \end{cases} \tag{3.2}$$

where $W_t = (W_t^1,...,W_t^m)$ is an m-dimensional standard Brownian motion as studied in [Kun84a]. As proved in Theorem 2.39 the above equations are equivalent to the following expression of the Cauchy problem

$$\begin{cases} \mathrm{d}u = F(x,u,\nabla u,\circ\mathrm{d}t), \\ u = g \quad \text{on} \quad \Gamma := \{x \in \mathbb{R}^d \times [0,\mathbf{T}] \,|\, x = (x_1,...,x_d,t), t = 0\}. \end{cases} \tag{3.3}$$

Since we consider partial differential equations with perturbations by Brownian motion we get an ω - dependence in the solution. Therefore the solution to equation (3.3) is denoted by $u(x,t,\omega)$, but for short notation we only write $u(x,t)$. Suppose u is a solution to (3.3) and at least one-times continuously differentiable with respect to space and time for fixed $\omega \in \Omega$. Furthermore, we assume that there exists a curve $\xi_s(r)$ which maps the point $r \in \Gamma$ to a point of a neighborhood in Γ at time s. Additionally, we assume $\xi_0(x) = x$ for all $x \in \mathbb{R}^d$ as the initial condition. Due to these assumptions we define the following functions, now for fixed ω, $r \in \mathbb{R}^d$ and $s \in [0,\mathbf{T}]$:

$$\begin{aligned} (\xi_s(r,\omega),s) &:= \big(\xi_s^1(r,\omega),...,\xi_s^d(r,\omega),s\big), \\ \eta_s(r,\omega) &:= u(\xi_s(r,\omega),s), \\ \chi_s^i(r,\omega) &:= u_{\xi_s^i}(\xi_s(r,\omega),s), \\ \chi_s(r,\omega) &:= \big(\chi_s^1(r,\omega),...,\chi_s^d(r,\omega)\big), \end{aligned} \tag{3.4}$$

where $\xi_t^i(r)$ denotes the i-th component of $\xi_t(r)$. In the next step we combine (3.3) with equations (3.4) and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[u(\xi_t(r),t) - u(\xi_0(r),0) - \int_0^t F(\xi_s^1(r),...,\xi_s^d(r),\eta_s(r),\chi_s^1(r),...,\chi_s^d(r),\circ\mathrm{d}s)\right] = 0.$$

By similar calculations as in Chapter 1 (cf. (1.15)) we receive a system of $(2d+1)$ stochastic differential equations (cf. (CE)), but now in the sense of Stratonovich:

$$\begin{cases} \mathrm{d}\xi_t = -F_{\chi_t}(\xi_t^1,...,\xi_t^d,\eta_t,\chi_t^1,...,\chi_t^d,\circ\mathrm{d}t), \\ \mathrm{d}\eta_t = F(\xi_t^1,...,\xi_t^d,\eta_t,\chi_t^1,...,\chi_t^d,\circ\mathrm{d}t) - \chi_t\cdot F_{\chi_t}(\xi_t^1,...,\xi_t^d,\eta_t,\chi_t^1,...,\chi_t^d,\circ\mathrm{d}t) \\ \mathrm{d}\chi_t = F_{\xi_t}(\xi_t^1,...,\xi_t^d,\eta_t,\chi_t^1,...,\chi_t^d,\circ\mathrm{d}t) + F_{\eta_t}(\xi_t^1,...,\xi_t^d,\eta_t,\chi_t^1,...,\chi_t^d,\circ\mathrm{d}t)\chi_t. \end{cases} \tag{SCE}$$

Equivalently we can write (SCE) componentwise for all $i = 1,...,d$, as

$$\begin{cases} \mathrm{d}\xi_t^i = -F_{\chi_t^i}(\xi_t^1,...,\xi_t^d,\eta_t,\chi_t^1,...,\chi_t^d,\circ\mathrm{d}t), \\ \mathrm{d}\eta_t = F(\xi_t^1,...,\xi_t^d,\eta_t,\chi_t^1,...,\chi_t^d,\circ\mathrm{d}t) - \sum_{i=1}^{d}\chi_t^i F_{\chi_t^i}(\xi_t^1,...,\xi_t^d,\eta_t,\chi_t^1,...,\chi_t^d,\circ\mathrm{d}t) \\ \mathrm{d}\chi_t^i = F_{\xi_t^i}(\xi_t^1,...,\xi_t^d,\eta_t,\chi_t^1,...,\chi_t^d,\circ\mathrm{d}t) + F_{\eta_t}(\xi_t^1,...,\xi_t^d,\eta_t,\chi_t^1,...,\chi_t^d,\circ\mathrm{d}t)\chi_t^i. \end{cases} \tag{3.5}$$

The above stochastic differential equations (SCE) are called **stochastic characteristic equations**. Given a point $x \in \mathbb{R}^d$ and assuming that there exist unique solutions to (SCE) starting from x at time $t = 0$, these solutions solve the corresponding integral equation with initial function g:

$$\xi_t(x) = x - \int_0^t F_{\chi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ \mathrm{d}s)$$

$$\eta_t(x) = g(x) - \int_0^t \chi_s \cdot F_{\chi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ \mathrm{d}s) + \int_0^t F(\xi_s(x), \eta_s(x), \chi_s(x), \circ \mathrm{d}s)$$

$$\chi_t(x) = \nabla g(x) + \int_0^t F_{\xi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ \mathrm{d}s) + \int_0^t F_{\eta_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ \mathrm{d}s)\chi_s.$$

Let us assume that the solutions $(\xi_t(x), \eta_t(x), \chi_t(x))$ exist up to a stopping time $T(x)$. Furthermore, let the inverse process ξ_t^{-1} of ξ_t exist up to some stopping time $\sigma(x)$. Then we define for almost all ω and for all (x,t) with $t < \sigma(x, \omega)$

$$u(x,t) := \eta_t(\xi_t^{-1}(x)). \tag{3.6}$$

In Chapter 4 we show (cf. Theorem 4.5 below) that (3.6) is indeed the solution of the stochastic partial differential equation (3.1) respectively (3.3).

3.2. Existence of solutions to stochastic characteristic equations

If we compare the stochastic method with the classical one, two questions arise. First, why should such solutions to (SCE) exist and second, what are the corresponding assumptions to Assumption 1.1 and of noncharacteristic initial data (see Remark 1.2). To answer these questions we use the 1-to-1-correspondence between stochastic flows and solutions of stochastic differential equations. In [Kun97, Chapter 4] we find results considering the following two cases:

- Given a stochastic flow φ_t (of special type), there exists a unique continuous semimartingale F such that $\varphi_t = x + \int_0^t F(\varphi_s, \mathrm{d}s)$ (see e.g. [Kun97, Theorem 4.4.1]).
- Given a semimartingale $F(x,t)$ with corresponding local characteristic belonging to a certain class, there exists a stochastic flow φ_t (see e.g. [Kun97, Theorem 4.6.5, Theorem 4.7.3]).

The stochastic characteristic equations (SCE) are stochastic differential equations in the sense of Stratonovich with the following type of solutions:

Definition 3.1 *Let $F(x,t)$, $x \in \mathbb{R}^d$, be a family of continuous $C(\mathbb{R}^d, \mathbb{R}^d)$ - semimartingales with local characteristic (a, b, A_t) belonging to $(B^{2,\delta}, B^{1,0})$ for some $0 < \delta \le 1$. Let σ_∞ be a stopping time and $x_0 \in \mathbb{R}^d$. A continuous local semimartingale φ_t, $t \in [0, \sigma_\infty)$, with values in $\mathbb{R}^d$ is called a **local solution** of the Stratonovich stochastic differential equation*

$$\varphi_t = x_0 + \int_0^t F(\varphi_s, \circ \mathrm{d}s) \tag{3.7}$$

if

$$\varphi_{t \wedge \sigma_N} = x_0 + \int_0^{t \wedge \sigma_N} F(\varphi_{s \wedge \sigma_N}, \circ \mathrm{d}s) \ \textit{a.s.}$$

is satisfied for any $N \in \mathbb{N}$*, where* $(\sigma_N)_{N\in\mathbb{N}}$ *is a localizing sequence, i.e.* $\sigma_N < \sigma_\infty$ *for any* $N \in \mathbb{N}$ *and* $\sigma_N \nearrow \sigma_\infty$ *for* $N \to \infty$*. That means* σ_∞ *is accessible.*
If

$$\lim_{t\nearrow\sigma_\infty} \varphi_t = \infty \text{ holds on } \{\sigma_\infty < \boldsymbol{T}\},$$

where ∞ *denotes the Alexandrov point in* $\mathbb{R}^d$*, then* φ_t *is called* ***maximal solution*** *and* σ_∞ *is called the* ***explosion time****.*

Hence a maximal solution is defined up to a stopping time, the so-called explosion time, which we formally define next.

Definition 3.2 *Let* X_t*,* $t \in [0,\tau)$*, be a local process. The stopping time* τ *is called* ***terminal time*** *of the local process* X_t*. If*

$$\lim_{t\nearrow\tau} |X_t| = \infty,$$

then τ *is called* ***explosion time****.*

Now we present results which ensure the existence and uniqueness of solutions to (SCE) under the condition that $F(x,u,p,t)$ is a continuous $C^{k,\delta}(\mathbb{R}^{2d+1},\mathbb{R})$- semimartingale with local characteristic belonging to $(B^{k+1,\delta}, B^{k,\delta})$ for some $k \geq 5$ and $0 < \delta \leq 1$.

We start with an existence and uniqueness result of maximal solutions (see [Kun97, Theorem 3.4.5]) in the sense of Itô. In line with Definition 3.1 we define a maximal solution to an Itô SDE in the following way.

Definition 3.3 *Let* $F(x,t)$*,* $x \in \mathbb{R}^d$*, be a family of continuous* $C(\mathbb{R}^d,\mathbb{R}^d)$ *- semimartingales with local characteristic* (a,b,A_t) *belonging to* $(B^{0,\delta}, B^{0,\delta})$ *for some* $\delta > 0$*. Let* σ_∞ *be a stopping time and* $x_0 \in \mathbb{R}^d$*. A continuous local process* φ_t*,* $t \in [0,\sigma_\infty)$*, with values in* $\mathbb{R}^d$ *and adapted to* $(\mathscr{F}_t)_t$ *is called a* ***local solution*** *of the Itô stochastic differential equation*

$$\varphi_t = x_0 + \int_0^t F(\varphi_s, \mathrm{d}s) \tag{3.8}$$

if

$$\varphi_{t\wedge\sigma_N} = x_0 + \int_0^{t\wedge\sigma_N} F(\varphi_{s\wedge\sigma_N}, \mathrm{d}s) \text{ a.s.}$$

is satisfied for any $N \in \mathbb{N}$*, where* $(\sigma_N)_{N\in\mathbb{N}}$ *is a localizing sequence, i.e.* $\sigma_N < \sigma_\infty$ *for any* $N \in \mathbb{N}$ *and* $\sigma_N \nearrow \sigma_\infty$ *for* $N \to \infty$*. If*

$$\lim_{t\nearrow\sigma_\infty} \varphi_t = \infty \text{ holds on } \{\sigma_\infty < \boldsymbol{T}\},$$

where ∞ *denotes the Alexandrov point in* $\mathbb{R}^d$*, then* φ_t *is called* ***maximal solution*** *and again* σ_∞ *is called the* ***explosion time****.*

Theorem 3.4 *Let* $F(x,t)$*,* $x \in \mathbb{R}^d$*, be a family of continuous semimartingales with values in* $C(\mathbb{R}^d,\mathbb{R}^d)$ *and local characteristic belonging to* $(B^{0,1}, B^{0,1})$*. Then for each* $t_0 \in [0,\boldsymbol{T}]$ *and* $x_0 \in \mathbb{R}^d$ *the Itô stochastic differential equation given by*

$$\varphi_t = x_0 + \int_{t_0}^t F(\varphi_s, \mathrm{d}s)$$

has a unique maximal solution φ_t*,* $t \in [t_0,\sigma_\infty)$*, where* σ_∞ *is the explosion time of* φ_t*.*

For the proof see [Kun97, Theorem 3.4.5]. Based on the representation result Corollary 2.41 we apply the above theorem to the case of infinite independent copies of a one-dimensional Brownian motion.

Corollary 3.5 *Let $(W_t^n)_{n\geq 1}$ be infinite independent copies of a one-dimensional standard Brownian motion. Let f_n, $n \geq 0$, be measurable and predictable random fields. Let $F(x,t)$, $x \in \mathbb{R}^d$, be a family of continuous semimartingales with values in $C(\mathbb{R}^d, \mathbb{R}^d)$ and local characteristic*

$$\left(\left(\sum_{n\geq 1} f_n^i(x,t) f_n^j(y,t)\right)_{i,j=1,\dots,d}, f_0(x,t), t\right)$$

belonging to $(B^{0,1}, B^{0,1})$. Then for any $t_0 \in [0, \boldsymbol{T}]$ and $x_0 \in \mathbb{R}^d$ the Itô stochastic differential equation given by

$$\varphi_t = x_0 + \int_{t_0}^{t} f_0(\varphi_s, s)\, \mathrm{d}s + \sum_{n\geq 1} \int_{t_0}^{t} f_n(\varphi_s, s)\, \mathrm{d}W_s^n$$

has a unique maximal solution φ_t, $t \in [t_0, \sigma_\infty)$, with explosion time σ_∞ of φ_t.

Remark 3.6 *The finite-dimensional version of Corollary 3.5 is a consequence of the classical existence and uniqueness result for SDEs as presented for example in [Oks07, Theorem 5.2.1] or [Kun97, Theorem 3.4.1]. The class $B_{\mathrm{ub}}^{0,1}$ implies that the drift term f_0 and the diffusion terms $f_1, f_2, \dots$ are uniformly Lipschitz continuous and of uniformly linear growth. We use [Kun97, Remark after Theorem 3.2.4] to extend the result to $B_{\mathrm{b}}^{0,1}$. By truncation as formulated in the proof of Theorem 3.4.5 in [Kun97] the result is also valid for the class $B^{0,1}$.*

Since the equations (SCE) are given in the sense of Stratonovich, we have to make use of the Itô-Stratonovich formula as stated in Theorem 2.35. Then we extend the above result to the setting of Stratonovich as proved in [Kun97, Theorem 3.4.7].

Theorem 3.7 *Let $F(x,t)$, $x \in \mathbb{R}^d$, be a family of continuous $C^1(\mathbb{R}^d, \mathbb{R}^d)$ - semimartingales with local characteristic (a, b, A_t) belonging to $(B^{2,\delta}, B^{1,0})$ for some $0 < \delta \leq 1$. Then for each $t_0 \in [0, \boldsymbol{T}]$ and $x_0 \in \mathbb{R}^d$ the Stratonovich equation given by*

$$\varphi_t = x_0 + \int_{t_0}^{t} F(\varphi_s, \circ \mathrm{d}s) \tag{3.9}$$

has a unique maximal solution φ_t, $t \in [t_0, \sigma_\infty)$, in the sense of Definition 3.1.

Due to [Kun97, Theorem 4.7.3] such maximal solutions can be characterized as stochastic flows which are defined in the following sense. Let here $\circ$ denote the composition of two functions.

Definition 3.8 *Let $\varphi_{s,t}(x)$, $s, t \in [0, \boldsymbol{T}]$, $x \in \mathbb{R}^d$, be a continuous random field on $(\Omega, \mathscr{F}, P)$. Then for almost all ω, $\varphi_{s,t}(\cdot, \omega) = \varphi_{s,t}(\omega) \colon \mathbb{R}^d \to \mathbb{R}^d$ defines a family of continuous maps for all $s, t \in [0, \boldsymbol{T}]$. $(\varphi_{s,t}(\omega))_{s,t\in[0,\boldsymbol{T}]}$ is called a* ***stochastic flow of homeomorphisms*** *if there exists a null set $N \subset \Omega$ such that for all $\omega \in N^{\mathsf{C}}$ the family $(\varphi_{s,t}(\omega))_{s,t\in[0,\boldsymbol{T}]}$ defines a flow of homeomorphisms, i.e. it satisfies:*

(i) $\varphi_{s,u}(\omega) = \varphi_{t,u}(\omega) \circ \varphi_{s,t}(\omega)$ *for all* $0 \leq s \leq t \leq u \leq \boldsymbol{T}$,

(ii) $\varphi_{s,s}(\cdot, \omega) = \mathrm{Id}(\cdot)$ *for all* $s \in [0, \boldsymbol{T}]$, *where* Id *is the identity map,*

(iii) $\varphi_{s,t}(\omega)\colon \mathbb{R}^d \to \mathbb{R}^d$ *is a homeomorphism for all* $s,t \in [0, \boldsymbol{T}]$.

Consider the set of all homeomorphisms on $\mathbb{R}^d$ defined by

$$G := \{f\colon \mathbb{R}^d \to \mathbb{R}^d \,|\, f \text{ is bijective, continuous and } f^{-1} \text{ is continuous}\}$$

and let the product of $\Psi_1, \Psi_2 \in G$ be the composite function $\Psi_1 \circ \Psi_2$. Then $(G, \circ)$ becomes obviously a group as stated e.g. in [Fis10, 2.1.4, Satz]. By defining a metric d_G on G with

$$d_G(\Psi_1, \Psi_2) := \sum_{i=1}^{\infty} \frac{1}{2^i}\left(\frac{\sup\limits_{|x|\le i} |\Psi_1(x) - \Psi_2(x)|}{1 + \sup\limits_{|x|\le i} |\Psi_1(x) - \Psi_2(x)|} + \frac{\sup\limits_{|x|\le i} |\Psi_1^{-1}(x) - \Psi_2^{-1}(x)|}{1 + \sup\limits_{|x|\le i} |\Psi_1^{-1}(x) - \Psi_2^{-1}(x)|} \right),$$

we obtain that G is a complete topological group (cf. [Kun97, Chapter 4, 4.1 Preliminaries]). In other words, a stochastic flow of homeomorphisms is a continuous random field with values in G satisfying properties (i) and (ii) of Definition 3.8. Now we consider the subgroup G_k of G which consists of all C^k- diffeomorphisms. Define

$$G^k := \{f\colon \mathbb{R}^d \to \mathbb{R}^d \,|\, f, f^{-1} \text{ are } k\text{-times continuously differentiable}\}$$

and let

$$\begin{aligned} d_k(\Psi_1, \Psi_2) := &\sum_{|\alpha|\le k} \sum_{i=1}^{\infty} \frac{1}{2^i}\left(\frac{\sup\limits_{|x|\le i} |D_x^\alpha \Psi_1(x) - D_x^\alpha \Psi_2(x)|}{1 + \sup\limits_{|x|\le i} |D_x^\alpha \Psi_1(x) - D_x^\alpha \Psi_2(x)|} \right) \\ &+ \sum_{|\alpha|\le k} \sum_{i=1}^{\infty} \frac{1}{2^i}\left(\frac{\sup\limits_{|x|\le i} |D_x^\alpha \Psi_1^{-1}(x) - D_x^\alpha \Psi_2^{-1}(x)|}{1 + \sup\limits_{|x|\le i} |D_x^\alpha \Psi_1^{-1}(x) - D_x^\alpha \Psi_2^{-1}(x)|} \right) \end{aligned}$$

be the corresponding metric. Then (G^k, d_k) is a complete separable metric space.

Definition 3.9 *A continuous random field* $\varphi_{s,t}(x)$, $s,t \in [0, \boldsymbol{T}]$, $x \in \mathbb{R}^d$, *is called a* ***stochastic flow with values in*** G^k *if* $\varphi_{s,t}$ *takes values in* G^k *and if properties (i) and (ii) of Definition 3.8 are fulfilled.*

Definition 3.10 *Let* $\varphi_{s,t}$, $s,t \in [0, \boldsymbol{T}]$, *be a stochastic flow with values in* G^k. *If we define* $\mathcal{N} := \{A \in \mathscr{F} \,|\, P(A) = 0\}$ *and*

$$\tilde{\mathscr{F}}_{s,t} := \bigcap_{\varepsilon>0} \sigma(\varphi_{u,v} \mid s - \varepsilon \le u, v \le t - \varepsilon),$$

the filtration $\mathscr{F}_{s,t} := \sigma(\tilde{\mathscr{F}}_{s,t} \cup \mathcal{N})$ *is a filtration depending on two parameters and is called* ***filtration generated by the flow*** $\varphi_{s,t}$.

Definition 3.11 *Let* $\varphi_{s,t}$, $s,t \in [0, \boldsymbol{T}]$, *be a stochastic flow with values in* G^k *for some* $k \in \mathbb{N}_0$. *Let* $(\mathscr{F}_{s,t})_{0\le s\le t\le \boldsymbol{T}}$ *be the filtration generated by* $\varphi_{s,t}$. *The forward part* $\varphi_{s,t}$, $0 \le s \le t \le \boldsymbol{T}$, *is called* ***forward*** $C^{k,\delta}$ ***- semimartingale flow****, if for every* $s \in [0, \boldsymbol{T}]$ *the stochastic flow* $\varphi_{s,t}$, $t \in [s, \boldsymbol{T}]$, *is a continuous* $C^{k,\delta}(\mathbb{R}^d, \mathbb{R}^d)$ *- semimartingale adapted to* $(\mathscr{F}_{s,t})_{t\in[s,\boldsymbol{T}]}$.

It follows by Definition 3.11 that semimartingale flows are in particular semimartingales and can be characterized by local characteristics (e.g. [Kun97, Theorem 4.4.1]).
Furthermore, we have the following important embeddings of the classes of local characteristics. Due to $C^{k+1} \subset C^k$ for $k \ge 2$ one can prove for $k \ge 2$ and some $0 < \delta \le 1$ that

- $(B^{k+1,\delta}, B^{k,\delta}) \subseteq (B^{2,\delta}, B^{1,0})$
- $(B^{k,\delta}, B^{k-1,\delta}) \subseteq (B^{2,\delta}, B^{1,0})$

holds. For the reader's convenience the proofs are given in Appendix A, see Lemma A.10.

Definition 3.12 *Let $F(x,t)$, $t \in [0,\tau(x))$, $x \in \mathbb{R}^d$, be a local random field. If for the domain*

$$\mathbb{D}_t(\omega) := \{x \in \mathbb{R}^d \mid \tau(x,\omega) > t\}$$

and for almost all ω the map $F(\cdot,t,\omega)\colon \mathbb{D}_t(\omega) \to \mathbb{R}$ is a $C^{k,\delta}$-function for any t, then $F(x,t)$ is called a ***local $C^{k,\delta}$-process****.*

By Definition 2.8 we know that $\tau(x)$ is lower semicontinuous. Hence $\mathbb{D}_t(\omega)$ is open in $\mathbb{R}^d$.

Definition 3.13 *Let $F(x,t)$, $x \in \mathbb{R}^d$, $t \in [0,\tau(x))$, be a continuous local $C^{k,\delta}$-process and $(\tau_n(x))_{n\geq 1}$ be an associated sequence of stopping times increasing to $\tau(x)$. Then $F(x,t)$ is called a continuous* ***local $C^{k,\delta}$-semimartingale*** *if the stopped processes $D_x^\alpha F(x,t\wedge\tau_n(x))$, $x \in \mathbb{R}^d$, $|\alpha| \leq k$, $n \in \mathbb{N}$, are all continuous semimartingales.*

Remark 3.14 *In the previous definitions we change the domain of the corresponding processes and name them local. For almost all ω we consider pairs (x,t) such that $x \in \mathbb{D}_t(\omega)$. For continuous local processes we obtain results and equations which hold pathwise, i.e. for almost all ω and all*

$$(x,t) \in \left\{(\tilde{x},\tilde{t}) \in \mathbb{R}^d \times [0,\boldsymbol{T}] \,\middle|\, \tau(\tilde{x},\omega) > \tilde{t}\right\}$$

the results and equations are satisfied. One should note that we get statements almost surely, but $\tau(x,\omega)$ could be very small and hence $\mathbb{D}_t(\omega)$ could be a very small set.

The next result ([Kun97, Theorem 4.7.3]) shows that maximal solutions of Stratonovich equation (3.9) are in particular stochastic flows. It is one of the basic results concerning the 1-to-1-correspondence between stochastic flows and solutions to SDEs.

Theorem 3.15 *Let $F(x,t)$, $x \in \mathbb{R}^d$, be a family of continuous $C(\mathbb{R}^d,\mathbb{R}^d)$-semimartingales with local characteristic belonging to $(B^{k+1,\delta}, B^{k,\delta})$ for some $k \geq 1$, $0 < \delta \leq 1$. Then the system of maximal solutions (which exists due to Theorem 3.7) of Stratonovich equation (3.9) defines a forward stochastic flow of local C^k-diffeomorphisms. Furthermore, it is a continuous local $C^{k,\varepsilon}$-semimartingale flow for any $\varepsilon < \delta$.*

The proof is a consequence of [Kun97, Theorem 4.7.2]. Now we return to our system (SCE) given by

$$\begin{cases} \mathrm{d}\xi_t = -F_{\chi_t}(\xi_t,\eta_t,\chi_t,\circ\mathrm{d}t), \\ \mathrm{d}\eta_t = F(\xi_t,\eta_t,\chi_t,\circ\mathrm{d}t) - \chi_t \cdot F_{\chi_t}(\xi_t,\eta_t,\chi_t,\circ\mathrm{d}t) \\ \mathrm{d}\chi_t = F_{\xi_t}(\xi_t,\eta_t,\chi_t,\circ\mathrm{d}t) + F_{\eta_t}(\xi_t,\eta_t,\chi_t,\circ\mathrm{d}t)\chi_t. \end{cases}$$

In the underlying situation $F(x,u,p,t)$, $(x,u,p) \in \mathbb{R}^{2d+1}$, is a family of continuous $C^{k,\delta}(\mathbb{R}^{2d+1},\mathbb{R})$-semimartingales for some $k \geq 5$, $0 < \delta \leq 1$ with local characteristic (a,b,A_t) belonging to $(B^{k+1,\delta}, B^{k,\delta})$. Theorem C.2 ensures that the Stratonovich integral can be differentiate with respect to the parameters $(x,u,p) \in \mathbb{R}^{2d+1}$. By Definition 2.11 of the Fréchet space $C^{k,\delta}$ we know that the k-th derivative of F is in particular δ-Hölder continuous, hence the partial derivatives F_x, F_u, F_p of $F(x,u,p,t)$ considered in (SCE) are continuous $C^{k-1,\delta}$- semimartingales. The same argumentation offers that the corresponding local characteristics of the partial derivatives of F belong to $(B^{k,\delta}, B^{k-1,\delta})$ (cf. [Kun97, Theorem 4.6.5 and the proof]). Since $C^{k-1,\delta} \subset C^1$ and $(B^{k,\delta}, B^{k-1,\delta}) \subseteq (B^{2,\delta}, B^{1,0})$ hold for $k \geq 2$ and $0 \leq \delta \leq 1$, we are in the situation of Theorem 3.7 and therefore we obtain existence

and uniqueness of maximal solutions to (SCE). Therefore the first question, namely why should such solutions to (SCE) exist, is answered.
One should note that there exist maximal solutions $(\xi_t(x), \eta_t(x), \chi_t(x))$ for almost all ω and all (x,t) with $t < T(x)$, where $T(x)$ denotes the explosion time of the maximal solutions. In Chapter 1 we have seen that Assumption 1.1 and Remark 1.2 on noncharacteristic initial data are necessary to be able to apply the inverse function theorem. In the stochastic case we compensate Assumption 1.1 and Remark 1.2 by using stopping times and a restriction to a proper domain. Fix $\omega \in \Omega$. Let us consider one of the maximal solutions to (SCE) namely

$$\xi_t(\cdot,\omega) \colon \{x \in \mathbb{R}^d \,|\, T(x,\omega) > t\} \to \mathbb{R}^d.$$

Due to Theorem 3.15 we conclude that ξ_t defines a forward stochastic flow of local C^{k-1}-diffeomorphisms and in particular it is a continuous local $C^{k-1,\varepsilon}$- semimartingale flow for $\varepsilon < \delta$. Furthermore, the explosion time $T(x)$ is by Definition 3.12 and Definition 2.8 a lower semicontinuous stopping time, hence the domain $\{x \in \mathbb{R}^d \,|\, T(x,\omega) > t\}$ is an open set. Le us consider the Jacobian matrix of $\xi_t(x)$. The Jacobian matrix $D\xi_t(x)$ could be singular, i.e.

$$\det D\xi_t(x,\omega) = 0$$

for some $t < T(x,\omega)$. So the solution $\xi_t(\cdot,\omega)$ would not be a diffeomorphism. Of course, if $\det D\xi_t(x) \neq 0$ for all $t < T(x)$, we are able to find ξ_t^{-1}. Therefore we define the following stopping times

$$\begin{aligned} \tau_{\text{inv}}(x) &:= \inf\{t \in (0,\mathbf{T}] \,|\, \det D\xi_t(x) = 0\} \\ \tau(x) &:= \tau_{\text{inv}}(x) \wedge T(x), \end{aligned} \tag{3.10}$$

for $x \in \mathbb{R}^d$. From time t up to $\tau_{\text{inv}}(x)$ the inverse function of $\xi_t(x)$ exists. The stopping times $\tau(x)$, $x \in \mathbb{R}^d$, are accessible and lower semicontinuous (cf. Definition 2.5 and Definition 2.7) as proved in Lemma B.1 for the reader's convenience. By the definition of $\tau(x)$ we have

$$\lim_{t \nearrow \tau(x)} \det D\xi_t(x) = 0$$

if $\tau(x) < T(x)$ for $x \in \mathbb{R}^d$. By restricting ξ_t to

$$\xi_t|_{\{\tau>t\}}(\cdot,\omega) \colon \{x \in \mathbb{R}^d \,|\, \tau(x) > t\} \to \mathbb{R}^d,$$

$\xi_t(\cdot,\omega)$ becomes a diffeomorphism and the inverse function ξ_t^{-1} exists. Similarly one introduces an adjoint stopping time for the inverse process ξ_t^{-1} to ensure that the inverse process takes values in the certain domain of the process ξ_t. Let us recall the domains and codomains of ξ_t and ξ_t^{-1}, respectively,

$$\begin{aligned} \xi_t \colon \left\{x \in \mathbb{R}^d \,\middle|\, \tau(x) > t\right\} &\to \left\{\xi_t(x) \in \mathbb{R}^d \,\middle|\, x \in \{z \,|\, \tau(z) > t\}\right\} \\ \xi_t^{-1} \colon \left\{y \in \mathbb{R}^d \,\middle|\, y \in \xi_t(\{x \in \mathbb{R}^d \,|\, \tau(x) > t\})\right\} &\to \left\{x \in \mathbb{R}^d \,\middle|\, \tau(x) > t\right\}. \end{aligned}$$

Hence for all fixed t the curve $\xi_t(x)$ defined on $\{x \,|\, \tau(x) > t\}$ has an inverse process. Now we define

$$\sigma(y) : = \inf\left\{t \geq 0 \,\middle|\, y \notin \xi_t(\{x \,|\, \tau(x) > t\})\right\},$$

as the first time when y is no longer an element of $\xi_t(\{x \,|\, \tau(x) > t\})$. Consequently $(\xi_t^{-1})_{t\in[0,\sigma)}$ is well-defined and maps $\{y \in \mathbb{R}^d \,|\, \sigma(y) > t\}$ into $\{x \in \mathbb{R}^d \,|\, \tau(x) > t\}$. The stopping time σ is also called adjoint stopping time.
To get an idea of the construction of ξ_t^{-1}, the terminology of an inverse process is convenient, but not precise. The aim is to define a local process $(\psi_t)_t$ satisfying the properties for every

t to be the inverse of ξ_t. Hence for all $y \in \mathbb{R}^d$ we define local processes $\psi_t(y)$, $t \in [0, \hat{\sigma}(y))$, which satisfy

$$\psi_t(\xi_t(x)) = x \quad \text{and} \quad \xi_t(\psi_t(x)) = x,$$

and $\hat{\sigma}$ denotes its explosion time. As detailed written in the next chapter (cf. Lemma 4.8 below), we can prove that $\sigma = \hat{\sigma}$ a.s. for all $y \in \mathbb{R}^d$. Therefore the inverse process $\xi_t^{-1}(x) := \psi_t(x)$ exists for $t < \sigma(x)$.

The local solution to (3.1) respectively (3.3) can be defined by (3.6) for almost all ω and all (x,t) with $t < \sigma(x,\omega)$. Therefore the method of stochastic characteristics is applicable. The corresponding stochastic characteristic equations (SCE) are of the same type as in the classical method. For the existence of solutions to (SCE) we have to assume that the semimartingales takes values in C^1 and that the local characteristic belongs at least to $(B^{2,\delta}, B^{1,\delta})$ for some $0 < \delta \leq 1$. For the main theorem, which we will prove in the next chapter, this regularity assumption is not enough (cf. Theorem 4.5 below).

4. The existence and uniqueness result of H. Kunita

In this chapter we state the main existence and uniqueness result of H. Kunita under rigorous conditions and prove it in detail (see Theorem 4.5 below). Herein, in particular a continuous $C^{k,\delta}$-valued semimartingale for some $k \geq 5$ and $0 < \delta \leq 1$ is necessary. The main tool for the proof is Lemma 4.8 below, which is also proved in detail. As discussed also in Chapter 3 we give the results always for almost all ω and all pairs (x,t) for t up to a certain stopping time. From our point of view this is necessary since otherwise the solution is not well-defined.

4.1. The main theorem

In this part we recall the existence and uniqueness result based on the method of stochastic characteristics as given in the book *Stochastic flows and stochastic differential equations* of H. Kunita [Kun97] in a detailed way. We start with the following deterministic equation which is of the same type as (1.10) in Chapter 1:

$$\begin{cases} \dfrac{\partial u}{\partial t} = F(x,u,u_x,t), \\ u|_{t=0} = g. \end{cases} \tag{4.1}$$

By a generalization of the method of characteristics to the stochastic setting, we want to look at a similar type of Cauchy problem. Therefore we consider the equation

$$\begin{cases} \mathrm{d}u = F(x,u,\nabla u,\circ \mathrm{d}t), \\ u|_{t=0} = g. \end{cases} \tag{4.2}$$

Equation (4.2) is equivalent to the following nonlinear stochastic partial differential equation of first order given in integral form by

$$u(x,t) = g(x) + \int_0^t F\big(x, u(x,r), \nabla u(x,r), \circ \mathrm{d}r\big). \tag{4.3}$$

As proved in Theorem 2.39, respectively Corollary 2.42, we find continuous C^2 - processes and C^1 - semimartingales $(f_n(x,u,p,t))_{n\geq 0}$ such that

$$\begin{aligned} F(x,u,p,\circ \mathrm{d}t) &= B(x,u,p,\mathrm{d}t) + \mathbf{M}(x,u,p,\circ \mathrm{d}t) \\ &= f_0(x,u,p,t)\,\mathrm{d}A_t + \sum_{n\geq 1} f_n(x,u,p,t) \circ \mathrm{d}M_t^n. \end{aligned}$$

Hence in this special case equation (4.2) corresponds to

$$\begin{cases} \mathrm{d}u = f_0(x,u,\nabla u,t)\,\mathrm{d}A_t + \sum_{n\geq 1} f_n(x,u,\nabla u,t) \circ \mathrm{d}M_t^n, \\ u(x,0) = g(x). \end{cases} \tag{4.4}$$

In the special case of Brownian motion with $A_t = t$, $M_t^n = W_t^n$ for all $n \geq 1$ and for a family of semimartingales $F(x,u,p,t)$, $(x,u,p) \in \mathbb{R}^{2d+1}$, we find a decomposition as before by

$$F(x,u,p,\circ \mathrm{d}t) = f_0(x,u,p,t)\,\mathrm{d}t + \sum_{n\geq 1} f_n(x,u,p,t) \circ \mathrm{d}W_t^n.$$

for $(x,u,p) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$. Hence equation (4.2) corresponds to

$$\begin{cases} \mathrm{d}u = f_0(x,u,\nabla u,t)\,\mathrm{d}t + \sum_{n\geq 1} f_n(x,u,\nabla u,t)\circ \mathrm{d}W_t^n, \\ u(x,0) = g(x). \end{cases} \tag{4.5}$$

To study the existence of solutions we first give the definition of a local solution to equation (4.3).

Definition 4.1 *Let $T\colon \mathbb{R}^d \times \Omega \to [0,\boldsymbol{T}]$ be a stopping time such that $T(x)$ is accessible and lower semicontinuous. A local $\mathbb{R}$ - valued random field $u(x,t)$ is called* ***local solution of (4.3)*** *with initial function $g(\cdot) \in C^1(\mathbb{R}^d,\mathbb{R})$ if $u(x,t)$, $0 \leq t < T(x)$, is a local $C^{1,\varepsilon}$ - semimartingale for some $\varepsilon > 0$ and for almost every ω and all (x,t) with $t < T(x,\omega)$ the equation*

$$\begin{aligned} u(x,t) &= g(x) + \int_0^t F\big(x,u(x,r),\nabla u(x,r),\circ \mathrm{d}r\big) \\ &= g(x) + \int_0^t f_0(x,u(x,r),\nabla u(x,r),r)\,\mathrm{d}A_r \\ &\qquad + \sum_{n\geq 1}\int_0^t f_n(x,u(x,r),\nabla u(x,r),r)\circ \mathrm{d}M_r^n \end{aligned} \tag{4.6}$$

is satisfied.

Definition 4.2 *The stochastic characteristic system (cf. (SCE)) associated with (4.3) is given by*

$$\begin{aligned} \mathrm{d}\xi_t &= -F_{\chi_t}(\xi_t,\eta_t,\chi_t,\circ \mathrm{d}t) \\ &= -\frac{\partial f_0}{\partial \chi_t}(\xi_t,\eta_t,\chi_t,t)\,\mathrm{d}A_t - \sum_{n\geq 1}\frac{\partial f_n}{\partial \chi_t}(\xi_t,\eta_t,\chi_t,t)\circ \mathrm{d}M_t^n, \\ \mathrm{d}\eta_t &= F(\xi_t,\eta_t,\chi_t,\circ \mathrm{d}t) - \chi_t\cdot F_{\chi_t}(\xi_t,\eta_t,\chi_t,\circ \mathrm{d}t) \\ &= \Big(f_0(\xi_t,\eta_t,\chi_t,t) - \chi_t\cdot\frac{\partial f_0}{\partial \chi_t}(\xi_t,\eta_t,\chi_t,t)\Big)\,\mathrm{d}A_t \\ &\qquad + \sum_{n\geq 1}\Big(f_n(\xi_t,\eta_t,\chi_t,t) - \chi_t\cdot\frac{\partial f_n}{\partial \chi_t}(\xi_t,\eta_t,\chi_t,t)\Big)\circ \mathrm{d}M_t^n, \\ \mathrm{d}\chi_t &= F_{\xi_t}(\xi_t,\eta_t,\chi_t,\circ \mathrm{d}t) + F_{\eta_t}(\xi_t,\eta_t,\chi_t,\circ \mathrm{d}t)\chi_t \\ &= \Big(\frac{\partial f_0}{\partial \xi_t}(\xi_t,\eta_t,\chi_t,t) + \frac{\partial f_0}{\partial \eta_t}(\xi_t,\eta_t,\chi_t,t)\chi_t\Big)\,\mathrm{d}A_t \\ &\qquad + \sum_{n\geq 1}\Big(\frac{\partial f_n}{\partial \xi_t}(\xi_t,\eta_t,\chi_t,t) + \frac{\partial f_n}{\partial \eta_t}(\xi_t,\eta_t,\chi_t,t)\chi_t\Big)\circ \mathrm{d}M_t^n, \end{aligned} \tag{4.7}$$

where (ξ_t,η_t,χ_t) are processes given in Chapter 3 (see equations (3.4) and (SCE)).

Remark 4.3 *Equation (4.7) forms a $(2d+1)$-dimensional system of stochastic differential equations, i.e. it can be rewritten in the form*

$$\begin{pmatrix} \mathrm{d}\xi_t \\ \mathrm{d}\eta_t \\ \mathrm{d}\chi_t \end{pmatrix} = \begin{pmatrix} \mathrm{d}\xi_t^1 \\ \vdots \\ \mathrm{d}\xi_t^d \\ \mathrm{d}\eta_t \\ \mathrm{d}\chi_t^1 \\ \vdots \\ \mathrm{d}\chi_t^d \end{pmatrix} = \begin{pmatrix} -F_{\chi_t^1}(\xi_t,\eta_t,\chi_t,\circ \mathrm{d}t) \\ \vdots \\ -F_{\chi_t^d}(\xi_t,\eta_t,\chi_t,\circ \mathrm{d}t) \\ F(\xi_t,\eta_t,\chi_t,\circ \mathrm{d}t) - \chi_t\cdot F_{\chi_t}(\xi_t,\eta_t,\chi_t,\circ \mathrm{d}t) \\ F_{\xi_t^1}(\xi_t,\eta_t,\chi_t,\circ \mathrm{d}t) + F_{\eta_t}(\xi_t,\eta_t,\chi_t,\circ \mathrm{d}t)\chi_t^1 \\ \vdots \\ F_{\xi_t^d}(\xi_t,\eta_t,\chi_t,\circ \mathrm{d}t) + F_{\eta_t}(\xi_t,\eta_t,\chi_t,\circ \mathrm{d}t)\chi_t^d \end{pmatrix}.$$

As introduced in Chapter 3 the solutions to the stochastic characteristic equations (4.7) solve the corresponding integral equations

$$\begin{aligned}
\xi_t(x) &= x - \int_0^t F_{\chi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ \mathrm{d}s), \\
\eta_t(x) &= g(x) - \int_0^t \chi_s \cdot F_{\chi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ \mathrm{d}s) \\
&\qquad + \int_0^t F(\xi_s(x), \eta_s(x), \chi_s(x), \circ \mathrm{d}s), \qquad (4.8) \\
\chi_t(x) &= \nabla g(x) + \int_0^t F_{\xi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ \mathrm{d}s) \\
&\qquad + \int_0^t F_{\eta_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ \mathrm{d}s) \chi_s,
\end{aligned}$$

with initial function $g \in C^{k,\delta}(\mathbb{R}^d, \mathbb{R})$. Under the assumption that these integral equations are solvable for almost all ω and (x,t) with $0 \leq t < T(x,\omega)$ we obtain the **stochastic characteristic curve** (ξ_t, η_t, χ_t).

If $F(x,u,p,t)$, $(x,u,p) \in \mathbb{R}^{2d+1}$, is a family of continuous $C^{k,\delta}$ - semimartingales with local characteristic belonging to the class $(B^{k+1,\delta}, B^{k,\delta})$ for some $k \geq 5$ and $0 < \delta \leq 1$, then there exist maximal solutions to (4.8) as we have seen in Subsection 3.2. Next, we define stopping times as in Chapter 3 in a formal way.

Definition 4.4 *Let $T(x)$ be the explosion time of the maximal solutions (ξ_t, η_t, χ_t). Then we define for all $x, y \in \mathbb{R}^d$ the stopping times*

$$\begin{aligned}
\tau_{\mathrm{inv}}(x) &:= \inf\left\{t > 0 \,\middle|\, \det D\xi_t(x) = 0\right\}, \\
\tau(x) &:= \tau_{\mathrm{inv}}(x) \wedge T(x), \\
\sigma(y) &:= \inf\left\{t > 0 \,\middle|\, y \notin \xi_t(\{x \in \mathbb{R}^d \,|\, \tau(x) > t\})\right\},
\end{aligned}$$

where $D\xi_t$ denotes the Jacobian matrix.

Now we state the main result Theorem 6.1.5 of [Kun97] in the following rigorous version.

Theorem 4.5 *Let $(\Omega, \mathscr{F}, P)$ be a separable and complete probability space. Let $F(x,u,p,t)$, $(x,u,p) \in \mathbb{R}^{2d+1}$, be a family of continuous $C^{k,\delta}(\mathbb{R}^{2d+1}, \mathbb{R})$ - semimartingales with local characteristic (a, b, A_t) belonging to the class $(B^{k+1,\delta}, B^{k,\delta})$ for some $k \geq 5$ and $0 < \delta \leq 1$. Let g be a function in $C^{k,\delta}(\mathbb{R}^d, \mathbb{R})$. Let (ξ_t, η_t, χ_t) be the stochastic characteristic curve solving (4.8). Then $u(x,t)$ defined for almost all ω and all (x,t) such that $t \in [0, \sigma(x,\omega))$ by*

$$u(x,t) := \eta_t(\xi_t^{-1}(x)) \qquad (4.9)$$

is a unique local solution of (4.3). Furthermore, $u(x,t)$ is a continuous local $C^{k-1,\varepsilon}$ - semimartingale for some $\varepsilon > 0$.

Obviously, one can formulate the main theorem applied to the classical case of Brownian motion (4.5) using Theorem 2.39 in the following way.

Corollary 4.6 *Let $(\Omega, \mathscr{F}, P)$ be a separable and complete probability space. Let $(W_t^n)_{n\geq 1}$ be infinite independent copies of a standard Brownian motion. Let $F(x,u,p,t)$, $(x,u,p) \in \mathbb{R}^{2d+1}$, be a family of continuous $C^{k,\delta}(\mathbb{R}^{2d+1}, \mathbb{R})$ - valued semimartingales with local characteristic (a, b, A_t) belonging to the class $(B^{k+1,\delta}, B^{k,\delta})$ for some $k \geq 5$ and $0 < \delta \leq 1$. Let $(f_n(x,u,p,t))_{n\geq 0}$ be measurable predictable $C^{k,\delta}$ - processes such that*

$$(a, b, A_t) = \left(\Big(\sum_{n\geq 1} f_n(x,u,p,t) f_n(\tilde{x}, \tilde{u}, \tilde{p}, t) \Big), f_0(x,u,p,t), t \right).$$

That means we assume

BM-HP (i) $\sum_{n\geq 0} f_n(\cdot,\cdot,\cdot,t)$ *is continuous in t with values in $C^{k,\delta}(\mathbb{R}^{2d+1}, \mathbb{R})$,*

BM-HP (ii) $\Big(\sum_{n\geq 1} f_n(x,u,p,t) f_n(\tilde{x}, \tilde{u}, \tilde{p}, t) \Big)$ *is non-negative definite and symmetric,*

BM-HP (iii) $\Big(\sum_{n\geq 1} f_n(x,u,p,t) f_n(\tilde{x}, \tilde{u}, \tilde{p}, t) \Big)$ *has a modification which is a predictable process with values in $\tilde{C}^{k+1,\delta}(\mathbb{R}^{2d+1} \times \mathbb{R}^{2d+1}, \mathbb{R})$,*

BM-HP (iv) *for all compact subsets $\mathbb{K} \subset \mathbb{R}^{2d+1} \times \mathbb{R}^{2d+1}$*
$\int_0^T \Big\| \sum_{n\geq 1} f_n(t) f_n(t) \Big\|^{\sim}_{(k+1)+\delta, \mathbb{K}} \mathrm{d}t < \infty$ *holds a.s.,*

BM-HP (v) $f_0(x,u,p,t,\omega)$ *has a modification which is a predictable process with values in $C^{k,\delta}(\mathbb{R}^{2d+1}, \mathbb{R})$,*

BM-HP (vi) *for all compact subsets $\mathbb{K} \subset \mathbb{R}^{2d+1}$*
$\int_0^T \Big\| f_0(t) \Big\|_{k+\delta, \mathbb{K}} \mathrm{d}t < \infty$ *holds a.s.*

Let $g(\cdot) \in C^{k,\delta}(\mathbb{R}^d, \mathbb{R})$ and $(\xi_t(x), \eta_t(x), \chi_t(x))$ be the system of maximal solutions solving (4.8). Then the unique local solution $u(x,t)$ to the stochastic partial differential equation

$$\begin{cases} \mathrm{d}u = f_0(x, u(x,t), \nabla u(x,t), t)\, \mathrm{d}t + \sum_{n\geq 1} f_n(x, u(x,t), \nabla u(x,t), t) \circ \mathrm{d}W_t^n \\ u(x,0) = g(x) \end{cases}$$

is defined for almost all ω and all (x,t) with $t \in [0, \sigma(x,\omega))$ by

$$u(x,t) := \eta_t(\xi_t^{-1}(x)).$$

Further it is a continuous local $C^{k-1,\varepsilon}$ - semimartingale for some $\varepsilon > 0$.

4.2. Tools for the proof

One of the main tools for the proof of Theorem 4.5 is the following generalized Itô formula stated as Theorem 3.3.2 in [Kun97].

Theorem 4.7 *Let $F(x,t)$, $x = (x_1, ..., x_d) \in \mathbb{R}^d$, be a family of continuous $C^3(\mathbb{R}^d, \mathbb{R}^d)$ - processes and continuous $C^2(\mathbb{R}^d, \mathbb{R}^d)$ - semimartingales with local characteristic belonging to the class $(B^{2,\delta}, B^{1,0})$ for some $0 < \delta \leq 1$. Let g_t be a continuous $\mathbb{R}^d$ - valued semimartingale. Then the formula*

$$F(g_t, t) - F(g_0, 0) = \int_0^t F(g_s, \circ \mathrm{d}s) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(g_s, s) \circ \mathrm{d}g_s^i$$

is satisfied, where g_t^i denotes the i-th component of $g_t = (g_t^1, ..., g_t^d)$.

The proof is given in Appendix C. As we have seen in Theorem 4.5 the solution (4.9) contains an inverse process denoted by ξ_t^{-1}. The name inverse process could be misinterpreted, therefore we want to call attention that ξ_t^{-1} is a process which satisfies for almost all ω and (x,t) with $t < \tau(x)$ the condition to be the right inverse and left inverse to $\xi_t(x)$. To be precise we signify this process by ψ_t and define $\xi_t^{-1} := \psi_t$. Nevertheless, we call ψ_t the inverse map as in [Kun97]. The following result can be found in [Kun97, Lemma 6.1.1]. We follow the ideas of the proof therein and formulate it in a detailed way.

Lemma 4.8

(i) The map $\xi_t\colon \{x \in \mathbb{R}^d \,|\, \tau(x,\omega) > t\} \to \mathbb{R}^d$ *is a* C^{k-1} *- diffeomorphism for every* t *a.s.*

(ii) The inverse map $\xi_t^{-1}(y)$, $t < \sigma(y)$, *is a continuous local* C^{k-1} *- process and a local* $C^{k-2,\varepsilon}$ *- semimartingale for some* $\varepsilon > 0$ *and satisfies for almost all* ω *and all* (y,t) *with* $t < \sigma(y,\omega)$

$$\begin{cases} \mathrm{d}\xi_t^{-1} = \big(D\xi_t(\xi_t^{-1}(y))\big)^{-1} F_{\chi_t}\big(y, \eta_t(\xi_t^{-1}(y)), \chi_t(\xi_t^{-1}(y)), \circ \mathrm{d}t\big) \\ \xi_0^{-1}(y) = y. \end{cases} \tag{4.10}$$

(iii) $\sigma(y)$ *is an accessible, lower semicontinuous stopping time such that if* $\sigma(y) < \boldsymbol{T}$*, we have*

$$\lim_{t \nearrow \sigma(y)} |\det D\xi_t^{-1}(y)| = \infty \quad \text{or} \quad \lim_{t \nearrow \sigma(y)} \xi_t^{-1}(y) \notin \{x \,|\, T(x) > \sigma(y)\}.$$

Proof. Let $\{M_t^n\}_{n\geq 1}$ be an orthogonal basis of continuous, square integrable martingales. We separate the proof into 7 steps:

Step 1: Definition of G *and* ψ_t

We consider a Stratonovich equation based on the following function:

$$G(x,t) := \int_0^t \big(D\xi_s(x)\big)^{-1} F_{\chi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ \mathrm{d}s). \tag{4.11}$$

By using the representation result Theorem 2.39 we are able to rewrite $G(x,t)$ using Remark 2.37 as

$$\begin{aligned} \int_0^t G(x, \circ \mathrm{d}s) &= G(x,t) - G(x,0) \\ &= \int_0^t \big(D\xi_s(x)\big)^{-1} F_{\chi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ \mathrm{d}s) \\ &= \int_0^t \big(D\xi_s(x)\big)^{-1} \frac{\partial f_0}{\partial \chi_s}(\xi_s(x), \eta_s(x), \chi_s(x), s)\, \mathrm{d}A_s \\ &\quad + \int_0^t \big(D\xi_s(x)\big)^{-1} \sum_{n\geq 1} \frac{\partial f_n}{\partial \chi_s}(\xi_s(x), \eta_s(x), \chi_s(x), s) \circ \mathrm{d}M_s^n. \end{aligned}$$

For $y \in \mathbb{R}^d$ we consider the following stochastic differential equation in the sense of Stratonovich

$$\psi_t(y) = y + \int_0^t G(\psi_s, \circ \mathrm{d}s), \tag{4.12}$$

which is equal to

$$\begin{aligned}
\psi_t(y) &= y + \int_0^t G(\psi_s(y), \circ \mathrm{d}s) \\
&= y + \int_0^t \big(D\xi_s(\psi_s(y))\big)^{-1} F_{\chi_s}(\xi_s(\psi_s(y)), \eta_s(\psi_s(y)), \chi_s(\psi_s(y)), \circ \mathrm{d}s) \\
&= y + \int_0^t \big(D\xi_s(\psi_s(y))\big)^{-1} \frac{\partial f_0}{\partial \chi_s}(\xi_s(\psi_s(y)), \eta_s(\psi_s(y)), \chi_s(\psi_s(y)), s)\, \mathrm{d}A_s \\
&\quad + \int_0^t \big(D\xi_s(\psi_s(y))\big)^{-1} \sum_{n\geq 1} \frac{\partial f_n}{\partial \chi_s}(\xi_s(\psi_s(y)), \eta_s(\psi_s(y)), \chi_s(\psi_s(y)), s) \circ \mathrm{d}M_s^n.
\end{aligned}$$

Step 2: ψ_t is a local $C^{k-2,\varepsilon}$ - semimartingale
The underlying $F(x,u,p,t)$ is $C^{k,\delta}(\mathbb{R}^{2d+1}, \mathbb{R})$ - valued, hence $F_p(x,u,p,t)$ is $C^{k-1,\delta}$ - valued. For the maximal solution ξ_t itself there exists also a modification which is a $C^{k-1,\varepsilon}$ - valued semimartingale (cf. Chapter 3, Theorem 3.15), hence the Jacobian matrix $(D\xi_t)^{-1}$ is a $C^{k-2,\varepsilon}$ - semimartingale. The considered G in (4.11) is again a $C^{k-2,\varepsilon}$ - semimartingale. By Theorem 3.7 we obtain the existence of a unique maximal solution for almost all ω and all (y,t), $t \in [0, \hat{\sigma}(y))$ denoted by $\psi_t(y)$ such that $\psi_0(y) = y$ and $\psi_t(y) \in \{x \,|\, \tau(x) > t\}$, since G is in particular a continuous C^1 - semimartingale. Here $\hat{\sigma}(y)$ denotes the explosion time of ψ_t. $\hat{\sigma}(y)$ is in particular an accessible and lower semicontinuous stopping time by definition. Due to Theorem 3.15 the solution ψ_t is a local $C^{k-2,\varepsilon}$ - semimartingale.

Step 3: Study of stopping times
Based on Definition 4.4 we should remember the underlying situation:

- $\xi_t(x)$ is a maximal solution to (4.7) defined for almost all ω and all (x,t) with $t < T(x,\omega)$, up to an explosion time $T(x)$ such that
$$\lim_{t \nearrow T(x)} \xi_t(x) = \infty, \quad \text{if } T(x) < \mathbf{T}.$$
- $\psi_t(y)$ is a maximal solution to (4.12) defined for almost all ω and all (y,t) with $t < \hat{\sigma}(y,\omega)$, up to an explosion time $\hat{\sigma}(y)$ such that
$$\lim_{t \nearrow \hat{\sigma}(y)} \psi_t(y) = \infty, \quad \text{if } \hat{\sigma}(y) < \mathbf{T}.$$

Let us fix ω. For reasons of notation we drop the ω-dependence in each process and stopping time. Now we want to ask, what can happen if t goes to $\hat{\sigma}(y)$. Due to the property

$$\psi_t(y) \in \{x \,|\, \tau(x) > t\} \;\; \forall\, t \in [0, \hat{\sigma}(y))$$

of solution ψ_t as written in Step 2 we observe for $t \nearrow \hat{\sigma}(y)$ two possible cases: On the one hand $\tau_{\text{inv}}(x)$ could coincide with $\hat{\sigma}$ and hence we conclude

$$\lim_{t \nearrow \hat{\sigma}(y)} |\det D\xi_t(\psi_t(y))| = 0.$$

On the other hand $\hat{\sigma}$ could be the explosion time $T(x)$ and hence we have in particular

$$\lim_{t \nearrow \hat{\sigma}(y)} \psi_t(y) \notin \big\{x \,\big|\, T(x) > \hat{\sigma}(y)\big\}.$$

In summary we have for all $y \in \mathbb{R}^d$

$$\lim_{t \nearrow \hat{\sigma}(y)} |\det D\xi_t(\psi_t(y))| = 0 \quad \textbf{or} \quad \lim_{t \nearrow \hat{\sigma}(y)} \psi_t(y) \notin \{x \,|\, T(x) > \hat{\sigma}(y)\}. \tag{4.13}$$

Step 4: ψ_t is right inverse

Remember that $\xi_t(x)$ is one of the maximal solutions solving

$$\xi_t(x) = x - \int_0^t F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s). \tag{4.14}$$

By using the notation $\xi_t(r) = \xi(r,t)$ equation (4.14) can be rewritten as

$$\int_0^t \xi(x, \circ \mathrm{d}s) = \xi_t(x) - \xi_0(x) = -\int_0^t F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s). \tag{4.15}$$

By an application of the generalized Itô formula (Theorem 4.7) we obtain for $\xi_t(x)$

$$\xi_t(\psi_t) - \xi_0(\psi_0) = \int_0^t \xi(\psi_r, \circ \mathrm{d}r) + \sum_{i=1}^d \int_0^t \frac{\partial \xi_r}{\partial x_i}(\psi_r) \circ \mathrm{d}\psi_r^i.$$

and by (4.12) this leads to

$$\begin{aligned} \xi_t(\psi_t) - \xi_0(\psi_0) &= -\int_0^t F_{\chi_r}(\xi_r(\psi_r), \eta_r(\psi_r), \chi_r(\psi_r), \circ \mathrm{d}r) + \sum_{i=1}^d \int_0^t \frac{\partial \xi_r}{\partial x_i}(\psi_r) \circ \mathrm{d}\psi_r^i \\ &= -\int_0^t F_{\chi_r}(\xi_r(\psi_r), \eta_r(\psi_r), \chi_r(\psi_r), \circ \mathrm{d}r) + \sum_{i=1}^d \int_0^t \frac{\partial \xi_r}{\partial x_i}(\psi_r)\; G^i(\psi_r, \circ \mathrm{d}r). \end{aligned}$$

By using the definition (4.11) we finally get

$$\begin{aligned} \xi_t(\psi_t) - \xi_0(\psi_0) = &-\int_0^t F_{\chi_r}(\xi_r(\psi_r), \eta_r(\psi_r), \chi_r(\psi_r), \circ \mathrm{d}r) \\ &+ \int_0^t D\xi_r(\psi_r)(D\xi_r(\psi_r))^{-1} F_{\chi_r}(\xi_r(\psi_r), \eta_r(\psi_r), \chi_r(\psi_r), \circ \mathrm{d}r) \\ = &-\int_0^t F_{\chi_r}(\xi_r(\psi_r), \eta_r(\psi_r), \chi_r(\psi_r), \circ \mathrm{d}r) \\ &+ \int_0^t F_{\chi_r}(\xi_r(\psi_r), \eta_r(\psi_r), \chi_r(\psi_r), \circ \mathrm{d}r) = 0. \end{aligned}$$

As a conclusion we obtain

$$\xi_t(\psi_t) = \xi_0(\psi_0),$$

and receive that ψ_t is the right inverse to ξ_t, since

$$\xi_t(\psi_t(y)) = \xi_0(\psi_0(y)) = \xi_0(y) = y.$$

holds for almost all ω and all (y,t) with $t < \hat{\sigma}(y, \omega)$.

Step 5: ψ_t is left inverse
The Jacobian matrix $D\xi_t(\psi_t(y))$ is non-singular for $t < \hat{\sigma}(y)$. By the implicite function theorem [Lan96, Chapter XIV, Theorem 1.2] we conclude that ψ_t is a continuous C^{k-1}-process. Now we define the stopping time

$$\hat{\tau}(x) := \inf\left\{t > 0 \,\middle|\, \xi_t(x) \notin \{y \,|\, \hat{\sigma}(y) > t\} \text{ or } |\det D\psi_t(\xi_t(x))| = \infty\right\} \wedge \tau(x). \tag{4.16}$$

The aim is to show that ψ_t is also the left inverse for almost all ω and (x,t) with $t < \hat{\tau}(x,\omega)$ i.e.

$$\psi_t(\xi_t(x)) = x.$$

If we differentiate $\xi_t(\psi_t(y)) = y$, we obtain by the classical chain rule

$$\mathbb{I} = D\xi_t(\psi_t(y))D\psi_t(y).$$

By plugging in $\xi_t(x)$ and building the inverse we obviously obtain

$$\mathbb{I} = \big(D\psi_t(\xi_t(x))\big)^{-1}\big(D\xi_t(\psi_t(\xi_t(x)))\big)^{-1}.$$

Hereby equation (4.14) can be written as

$$\begin{aligned} \mathrm{d}\xi_t &= -F_{\chi_t}(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t) \\ &= -\mathbb{I}\, F_{\chi_t}(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t) \\ &= -\big(D\psi_t(\xi_t(x))\big)^{-1}\big(D\xi_t(\psi_t(\xi_t(x)))\big)^{-1} F_{\chi_t}(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t). \end{aligned} \tag{4.17}$$

By application of the generalized Itô formula (Theorem 4.7) to $\psi_t(\xi_t)$ we obtain

$$\psi_t(\xi_t) - \psi_0(\xi_0) = \int_0^t \psi(\xi_s, \circ\mathrm{d}s) + \sum_{i=1}^d \int_0^t \frac{\partial \psi_s}{\partial x_i}(\xi_s) \circ \mathrm{d}\xi_s^i.$$

By using (4.12) and (4.17) we receive

$$\begin{aligned} &\psi_t(\xi_t(x)) - \psi_0(\xi_0(x)) \\ &= \int_0^t \big(D\xi_s(\psi_s(\xi_s(x)))\big)^{-1} \\ &\qquad \cdot F_{\chi_s}(\xi_s(\psi_s(\xi_s(x))), \eta_s(\psi_s(\xi_s(x))), \chi_s(\psi_s(\xi_s(x))), \circ\mathrm{d}s) \\ &\quad - \int_0^t \big(D\psi_s(\xi_s(x))\big)\big(D\psi_s(\xi_s(x))\big)^{-1} \\ &\qquad \cdot \big(D\xi_s(\psi_s(\xi_s(x)))\big)^{-1} F_{\chi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ\mathrm{d}s) \\ &= \int_0^t \big(D\xi_s(\psi_s(\xi_s(x)))\big)^{-1} \\ &\qquad \cdot F_{\chi_s}(\xi_s(x), \eta_s(\psi_s(\xi_s(x))), \chi_s(\psi_s(\xi_s(x))), \circ\mathrm{d}s) \\ &\quad - \int_0^t \big(D\xi_s(\psi_s(\xi_s(x)))\big)^{-1} F_{\chi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ\mathrm{d}s) \end{aligned} \tag{4.18}$$

The above stochastic differential equation for $\nu_t(x) := \psi_t(\xi_t(x))$ has a unique solution given for almost all ω and all (x,t), $t < \hat{\tau}(x,\omega)$ by

$$\nu_t(x) = \psi_t(\xi_t(x)) = x. \tag{4.19}$$

Details are written in Lemma C.1. Equation (4.19) shows that ψ_t is also the left inverse of ξ_t up to the stopping time $\hat{\tau}$.

Step 6: $\tau = \hat{\tau}$
Now we analyse the connection between the stopping time $\hat{\tau}(x)$ given by (4.16) and $\tau(x)$ as in Definition 4.4. In this step we show that for all $x \in \mathbb{R}^d$

$$\tau(x) = \hat{\tau}(x) \text{ a.s.}$$

holds. Let ω be fixed. If $\hat{\tau}(x) \geq T(x)$, it is clear that

$$\tau(x) = \tau_{\text{inv}}(x) \wedge T(x) = T(x)$$

and therefore $\hat{\tau}(x) = \tau(x)$ is valid.
If $\hat{\tau}(x) < T(x)$, we consider the two cases in the definition of $\hat{\tau}$ separately. First, let

$$\begin{aligned}\hat{\tau}(x) &= \inf\left\{t > 0 \,\middle|\, \xi_t(x) \notin \{y \,|\, \hat{\sigma}(y) > t\}\right\}\\ &= \inf\left\{t > 0 \,\middle|\, \xi_t(x) \in \{y \,|\, \hat{\sigma}(y) \leq t\}\right\}.\end{aligned}$$

If $\xi_t(x) \in \{y \,|\, \hat{\sigma}(y) \leq t\}$, then we also know by (4.13) that

$$\begin{aligned}\xi_t(x) \in &\Big(\left\{y \,|\, \hat{\sigma}(y) \leq t\right\} \cap \left\{y \,|\, \lim_{s \nearrow \hat{\sigma}(y)} |\det D\xi_s(\psi_s(y))| = 0\right\}\Big)\\ &\cup \Big(\left\{y \,|\, \hat{\sigma}(y) \leq t\right\} \cap \left\{y \,|\, \lim_{s \nearrow \hat{\sigma}(y)} \psi_s(y) \in \left\{z \,\middle|\, T(z) \leq \hat{\sigma}(y)\right\}\right\}\Big).\end{aligned}$$

Considering the first intersection we conclude $\tau_{\text{inv}}(x) \leq t$ or due to the second intersection $T(x) \leq t$. That means $\tau(x) \leq \hat{\tau}(x)$. On the other hand we have by definition $\hat{\tau}(x) \leq \tau(x)$. Consequently, $\tau(x) = \hat{\tau}(x)$ is proved.

Let us consider the other case. If $\hat{\tau}(x) = \inf\left\{t > 0 \,\middle|\, |\det D\psi_t(\xi_t(x))| = \infty\right\}$, then we obtain

$$\lim_{t \nearrow \hat{\tau}(x)} \det D\xi_t(x) = 0,$$

since by $D\psi_t(\xi_t(x))D\xi_t(x) = \mathbb{I}$ we know

$$\lim_{t \nearrow \hat{\tau}(x)} \frac{1}{\det D\xi_t(x)} = \lim_{t \nearrow \hat{\tau}(x)} \frac{\det \mathbb{I}}{\det D\xi_t(x)} = \lim_{t \nearrow \hat{\tau}(x)} \det D\psi_t(\xi_t(x)) = \infty.$$

And therefore $\hat{\tau}(x) = \tau_{\text{inv}}(x) \geq \tau(x) \geq \hat{\tau}(x)$. Summarizing we proved $\tau(x) = \hat{\tau}(x)$.

Step 7: Conclusion and formal proofs of the statements (i), (ii), (iii)
Suppose

$$\xi_t(x) = \xi_t(x') \text{ for } x, x' \in \left\{\tilde{x} \,|\, \tau(\tilde{x}) > t\right\}.$$

Since $\psi_t(\xi_t(x)) = x$ on $\{\tilde{x} \,|\, \tau(\tilde{x}) > t\}$ we obtain $x = x'$. Therefore $\xi_t(x)$ is injective. Due to the implicite function theorem [Lan96, Chapter XIV, Theorem 1.2] and by using the fact that the Jacobian matrix $D\xi_t$ is non-singular, we know that $\xi_t(x)$ is a C^{k-1}-diffeomorphism. Hence the first claim **(i)** of the Lemma is proved.
The results of part **(ii)**, which are that $\psi_t(y)$, $t < \sigma(y)$, is a continuous local C^{k-1}-process and a local $C^{k-2,\varepsilon}$-semimartingale for $\varepsilon > 0$, are shown in particular in Step 2. Hereby equation (4.10) correlates with (4.12). The properties of ψ_t to be the right inverse as well as the left inverse to ξ_t were proved in Step 4 and Step 5.

For the last claim we consider the explosion time $\hat{\sigma}$ of ψ_t. In Step 3 we proved the analogous assertions of part (iii) but for stopping time $\hat{\sigma}$. Now we have to show it for $\sigma(y)$, where

$$\sigma(y) = \inf\left\{t > 0 \,\middle|\, y \notin \xi_t(\{x \in \mathbb{R}^d \,|\, \tau(x) > t\})\right\}$$

as defined in Definition 4.4. Therefore we will prove $\sigma(y) = \hat{\sigma}(y)$ a.s. Due to part (i) and Steps 4, 5 we have

$$\xi_t(\{x \,|\, \tau(x) > t\}) \subseteq \{y \,|\, \hat{\sigma}(y) > t\}, \tag{4.20}$$

since $\psi_t(y)$ is well-defined for almost all ω and all $y \in \xi_t(\{x \,|\, \tau(x,\omega) > t\})$. Due to the definition of ψ_t we know that the following relation

$$\psi_t \colon \{y \,|\, \hat{\sigma}(y) > t\} \to (\{x \,|\, \tau(x) > t\})$$

holds and we receive as written in Step 2

$$\psi_t(\{y \,|\, \hat{\sigma}(y) > t\}) \subseteq \{x \,|\, \tau(x) > t\}.$$

Now we conclude that

$$\xi_t(\psi_t(\{y \,|\, \hat{\sigma}(y) > t\})) \subseteq \xi_t(\{x \,|\, \tau(x) > t\}).$$

Since $\xi_t(\psi_t(y)) = y$ for all $y \in \{\tilde{y} \,|\, \hat{\sigma}(\tilde{y}) > t\}$, it follows

$$\{y \,|\, \hat{\sigma}(y) > t\} \subseteq \xi_t(\{x \,|\, \tau(x) > t\}). \tag{4.21}$$

Due to (4.20) and (4.21) we have

$$\xi_t(\{x \,|\, \tau(x) > t\}) = \{y \,|\, \hat{\sigma}(y) > t\}.$$

Consequently by the definition of $\sigma(y)$ we conclude that

$$\sigma(y) = \hat{\sigma}(y).$$

Thereby claim **(iii)** is proved. □

The third important tool for the proof of Theorem 4.5 is the chain rule for the stochastic characteristic curve as written in [Kun97, Lemma 6.1.3]. We follow the ideas of the proof therein.

Lemma 4.9 *For the inverse function ξ_t^{-1} the relation*

$$\frac{\partial}{\partial x_i}\left[\eta_t(\xi_t^{-1})\right] = \chi_t^i(\xi_t^{-1}) \tag{4.22}$$

holds for $i = 1, \ldots, d$.

Proof. Our first step is to show

$$\frac{\partial \eta_t}{\partial x_i} = \chi_t \cdot \frac{\partial \xi_t}{\partial x_i}. \tag{4.23}$$

So, we define

$$\theta_t^i := \frac{\partial \eta_t}{\partial x_i} - \chi_t \cdot \frac{\partial \xi_t}{\partial x_i} \tag{4.24}$$

and prove $\theta_t^i = 0$ for all $i = 1, \ldots, d$. To this end let us consider the stochastic differential equation which generates θ_t^i and observe the stochastic characteristic equation (4.7) of η_t

$$\mathrm{d}\eta_t = F(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t) - \chi_t \cdot F_{\chi_t}(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t) \quad \text{with} \quad \eta_0(x) = g(x).$$

Therefore we obtain

$$\frac{\partial \eta_t}{\partial x_i} - \frac{\partial g}{\partial x_i} = \frac{\partial}{\partial x_i}\left[g + \int_0^t F(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) - \int_0^t \chi_s F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)\right] - \frac{\partial g}{\partial x_i}$$
$$= \frac{\partial}{\partial x_i}\left[\int_0^t F(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)\right] - \frac{\partial}{\partial x_i}\left[\int_0^t \chi_s F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)\right].$$

Now we apply Theorem C.2 to the second term which admits us to interchange integration and differentiation

$$\frac{\partial \eta_t}{\partial x_i} - \frac{\partial g}{\partial x_i} = \int_0^t \frac{\partial \xi_s}{\partial x_i}\frac{\partial F}{\partial \xi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) + \int_0^t \frac{\partial \eta_s}{\partial x_i}\frac{\partial F}{\partial \eta_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)$$
$$+ \int_0^t \frac{\partial \chi_s}{\partial x_i}\frac{\partial F}{\partial \chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) - \frac{\partial}{\partial x_i}\left[\int_0^t \chi_s F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)\right].$$

By applying the classical product rule and Theorem C.2 to the last term we obtain

$$\frac{\partial \eta_t}{\partial x_i} - \frac{\partial g}{\partial x_i} = \int_0^t \frac{\partial \xi_s}{\partial x_i} F_{\xi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) + \int_0^t \frac{\partial \eta_s}{\partial x_i} F_{\eta_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)$$
$$+ \int_0^t \frac{\partial \chi_s}{\partial x_i} F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) - \int_0^t \frac{\partial \chi_s}{\partial x_i} F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)$$
$$- \int_0^t \chi_s \frac{\partial}{\partial x_i}\left[F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)\right]$$
$$= \int_0^t \frac{\partial \xi_s}{\partial x_i} F_{\xi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) + \int_0^t \frac{\partial \eta_s}{\partial x_i} F_{\eta_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)$$
$$- \int_0^t \chi_s \frac{\partial}{\partial x_i}\left[F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)\right].$$

Consequently we have on the one hand

$$\frac{\partial \eta_t}{\partial x_i} - \frac{\partial g}{\partial x_i} = \int_0^t \frac{\partial \xi_s}{\partial x_i} F_{\xi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) + \int_0^t \frac{\partial \eta_s}{\partial x_i} F_{\eta_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)$$
$$- \int_0^t \chi_s \frac{\partial}{\partial x_i}\left[F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)\right]. \tag{4.25}$$

Under the initial conditions $\chi_0(x) = \frac{\partial g}{\partial x}(x)$ and $\xi_0(x) = x$ we consider

$$\chi_t \frac{\partial \xi_t}{\partial x_i} - \frac{\partial g}{\partial x_i} = \chi_t \frac{\partial \xi_t}{\partial x_i} - \chi_0 \frac{\partial \xi_0}{\partial x_i} = \int_0^t \mathrm{d}\left[\chi_s \frac{\partial \xi_s}{\partial x_i}\right].$$

By applying Itô's product rule (see [RY05, Chapter IV, 3.1 Proposition]) we get

$$\chi_t \frac{\partial \xi_t}{\partial x_i} - \frac{\partial g}{\partial x_i} = \int_0^t \chi_s \,\mathrm{d}\left[\frac{\partial \xi_s}{\partial x_i}\right] + \int_0^t \frac{\partial \xi_s}{\partial x_i}\,\mathrm{d}\chi_s + \left\langle \chi_\cdot, \frac{\partial \xi_\cdot}{\partial x_i}\right\rangle_t$$

$$= \int_0^t \chi_s \,\mathrm{d}\Big[\frac{\partial \xi_s}{\partial x_i}\Big] + \frac{1}{2}\Big\langle \chi_\cdot, \frac{\partial \xi_\cdot}{\partial x_i}\Big\rangle_t + \int_0^t \frac{\partial \xi_s}{\partial x_i}\,\mathrm{d}\chi_s + \frac{1}{2}\Big\langle \chi_\cdot, \frac{\partial \xi_\cdot}{\partial x_i}\Big\rangle_t.$$

In the next step we use the classical Itô-Stratonovich formula (see [Kun97, Theorem 2.3.5]) to receive

$$\chi_t \frac{\partial \xi_t}{\partial x_i} - \frac{\partial g}{\partial x_i} = \int_0^t \chi_s \circ \mathrm{d}\Big[\frac{\partial \xi_s}{\partial x_i}\Big] + \int_0^t \frac{\partial \xi_s}{\partial x_i} \circ \mathrm{d}\chi_s.$$

By an application of [Kun97, Theorem 2.3.6 (ii)] we conclude by (4.7)

$$\begin{aligned}
\chi_t \frac{\partial \xi_t}{\partial x_i} - \frac{\partial g}{\partial x_i} &= -\int_0^t \chi_s \frac{\partial}{\partial x_i}\Big[F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)\Big] + \int_0^t \frac{\partial \xi_s}{\partial x_i} \circ \mathrm{d}\chi_s \\
&= -\int_0^t \chi_s \frac{\partial}{\partial x_i}\Big[F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)\Big] \qquad (4.26) \\
&\quad + \int_0^t \frac{\partial \xi_s}{\partial x_i} F_{\xi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) + \int_0^t \frac{\partial \xi_s}{\partial x_i} \chi_s F_{\eta_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s).
\end{aligned}$$

By using (4.24), (4.25) and adding and subtracting $\frac{\partial g}{\partial x_i}$ we obtain

$$\begin{aligned}
\theta_t^i &= \frac{\partial \eta_t}{\partial x_i} - \frac{\partial g}{\partial x_i} - \Big(\chi_t \cdot \frac{\partial \xi_t}{\partial x_i} - \frac{\partial g}{\partial x_i}\Big) \\
&= \int_0^t \frac{\partial \xi_s}{\partial x_i} F_{\xi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) + \int_0^t \frac{\partial \eta_s}{\partial x_i} F_{\eta_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) \\
&\quad - \int_0^t \chi_s \frac{\partial}{\partial x_i}\Big[F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)\Big] - \Big(\chi_t \cdot \frac{\partial \xi_t}{\partial x_i} - \frac{\partial g}{\partial x_i}\Big).
\end{aligned}$$

By applying (4.26) we get

$$\begin{aligned}
\theta_t^i &= \int_0^t \frac{\partial \xi_s}{\partial x_i} F_{\xi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) + \int_0^t \frac{\partial \eta_s}{\partial x_i} F_{\eta_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) \\
&\quad - \int_0^t \chi_s \frac{\partial}{\partial x_i}\Big[F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)\Big] + \int_0^t \chi_s \frac{\partial}{\partial x_i}\Big[F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)\Big] \\
&\quad - \int_0^t \frac{\partial \xi_s}{\partial x_i} F_{\xi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) - \int_0^t \chi_s \frac{\partial \xi_s}{\partial x_i} F_{\eta_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) \\
&= \int_0^t \frac{\partial \eta_s}{\partial x_i} F_{\eta_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) - \int_0^t \chi_s \frac{\partial \xi_s}{\partial x_i} F_{\eta_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) \\
&= \int_0^t \Big(\frac{\partial \eta_s}{\partial x_i} - \chi_s \frac{\partial \xi_s}{\partial x_i}\Big) \cdot F_{\eta_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) \\
&= \int_0^t \theta_s^i \cdot F_{\eta_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s).
\end{aligned}$$

The unique solution of the linear stochastic differential equation

$$\begin{aligned} d\theta_t^i &= \theta_t^i F_{\eta_t}(\xi_t, \eta_t, \chi_t, \circ dt) \quad \text{with} \\ \theta_0^i &= \frac{\partial \eta_0}{\partial x_i} - \chi_0 \frac{\partial \xi_0}{\partial x_i} = \frac{\partial g}{\partial x_i} - \nabla g \cdot e_i = \frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial x_i} = 0, \end{aligned}$$

where $e_i = (0,..,1,..0)$ denotes the unit vector, is $\theta_t^i = 0$. Here we make use of the facts that the initial condition is 0 and that the solution is of the form $\theta_t^i = \theta_0^i \cdot \exp(...)$, as in [Oks07, Exercise 5.3.*]. Therefore we have

$$\theta_t^i = 0 = \frac{\partial \eta_t}{\partial x_i} - \chi_t \cdot \frac{\partial \xi_t}{\partial x_i}$$

which is equivalent to

$$\frac{\partial \eta_t}{\partial x^i} = \chi_t \cdot \frac{\partial \xi_t}{\partial x^i}.$$

Hence we have shown equation (4.23). Furthermore, we have

$$\nabla(\eta_t(\xi_t^{-1})) = \nabla \eta_t(\xi_t^{-1}) \Big(D\xi_t(\xi_t^{-1}) \Big)^{-1} \tag{4.27}$$

by the classical chain rule and the Theorem of the inverse function [Rud64, 9.24 Theorem]. Due to equations (4.23) and (4.27) we obtain

$$\begin{aligned} \nabla\big(\eta_t(\xi_t^{-1}(y)\big) &= \chi_t(\xi_t^{-1}(y)) \cdot D\xi_t(\xi_t^{-1}(y)) \Big(D\xi_t(\xi_t^{-1}(y)) \Big)^{-1} \\ &= \chi_t(\xi_t^{-1}(y)) \end{aligned} \tag{4.28}$$

and we have componentwise

$$\frac{\partial}{\partial x_i} \Big[\eta_t(\xi_t^{-1}) \Big] = \chi_t^i(\xi_t^{-1})$$

which proves the statement of the Lemma. □

4.3. Proof of the main theorem

The proof of the main Theorem 4.5 follows the ideas of Theorem 6.1.2 and Theorem 6.1.4 in [Kun97].

Proof. In the theorem we define for almost all ω and all (x,t) with $t < \sigma(x,\omega)$

$$u(x,t) := \eta_t(\xi_t^{-1}(x)).$$

Let ω be fixed. Due to the fact that η_t is a continuous C^{k-1}- process and ξ_t^{-1} is a continuous C^{k-2}- process, we have that u is also a continuous local C^{k-2}- process. Furthermore,

$$u_{x_i} := \frac{\partial u}{\partial x_i}$$

is a local $C^{k-2,\varepsilon}$- semimartingale by Lemma 4.9, since

$$u_{x_i}(x,t) = \frac{\partial \big[\eta_t(\xi_t^{-1}(x)) \big]}{\partial x_i} = \chi_t^i(\xi_t^{-1}(x)) \tag{4.29}$$

holds for almost all ω and all $(x,t), t < \sigma(x,\omega)$. By integration we obtain $(k-1)$-differentiability instead of a $(k-2)$-differentiability, therefore $u(x,t)$ is a continuous local

$C^{k-1,\varepsilon}$- semimartingale. By application of the generalized Itô formula in Theorem 4.7 we receive

$$\mathrm{d}\big[\eta_t(\xi_t^{-1})\big] = \mathrm{d}\eta_t(\xi_t^{-1}) + \sum_{i=1}^{d} \frac{\partial \eta_t}{\partial x_i}(\xi_t^{-1}) \circ \mathrm{d}\big[(\xi_t^{-1})^i\big]. \tag{4.30}$$

Let us consider the first term of (4.30) and make use of (4.7)

$$\begin{aligned}\mathrm{d}\eta_t(\xi_t^{-1}) &= F(\xi_t(\xi_t^{-1}), \eta_t(\xi_t^{-1}), \chi_t(\xi_t^{-1}), \circ \mathrm{d}t) - \chi_t(\xi_t^{-1})\ F_{\chi_t}(\xi_t(\xi_t^{-1}), \eta_t(\xi_t^{-1}), \chi_t(\xi_t^{-1}), \circ \mathrm{d}t)\\ &= F(\cdot, \eta_t(\xi_t^{-1}), \chi_t(\xi_t^{-1}), \circ \mathrm{d}t) - \chi_t(\xi_t^{-1})\ F_{\chi_t}(\cdot, \eta_t(\xi_t^{-1}), \chi_t(\xi_t^{-1}), \circ \mathrm{d}t).\end{aligned}$$

By using (4.10) the second term of (4.30) is equal to

$$\begin{aligned}\sum_{i=1}^{d} \frac{\partial \eta_t}{\partial x_i}(\xi_t^{-1}) \circ \mathrm{d}\big[(\xi_t^{-1})^i\big] &= \nabla \eta_t(\xi_t^{-1}) \circ \mathrm{d}\xi_t^{-1}\\ &= \nabla \eta_t(\xi_t^{-1})\big(D\xi_t(\xi_t^{-1})\big)^{-1}\ F_{\chi_t}(\cdot, \eta_t(\xi_t^{-1}), \chi_t(\xi_t^{-1}), \circ \mathrm{d}t).\end{aligned}$$

Adding both terms together we obtain for (4.30) by using (4.27) and (4.28)

$$\begin{aligned}\mathrm{d}\big[\eta_t(\xi_t^{-1})\big] &= F(\cdot, \eta_t(\xi_t^{-1}), \chi_t(\xi_t^{-1}), \circ \mathrm{d}t) - \chi_t(\xi_t^{-1})\ F_{\chi_t}(\cdot, \eta_t(\xi_t^{-1}), \chi_t(\xi_t^{-1}), \circ \mathrm{d}t)\\ &\quad + \nabla(\eta_t(\xi_t^{-1}))\big(D\xi_t(\xi_t^{-1})\big)^{-1}\ F_{\chi_t}(\cdot, \eta_t(\xi_t^{-1}), \chi_t(\xi_t^{-1}), \circ \mathrm{d}t)\\ &= F(\cdot, \eta_t(\xi_t^{-1}), \chi_t(\xi_t^{-1}), \circ \mathrm{d}t) - \chi_t(\xi_t^{-1})\ F_{\chi_t}(\cdot, \eta_t(\xi_t^{-1}), \chi_t(\xi_t^{-1}), \circ \mathrm{d}t)\\ &\quad + \chi_t(\xi_t^{-1})\ F_{\chi_t}(\cdot, \eta_t(\xi_t^{-1}), \chi_t(\xi_t^{-1}), \circ \mathrm{d}t)\\ &= F(\cdot, \eta_t(\xi_t^{-1}), \chi_t(\xi_t^{-1}), \circ \mathrm{d}t).\end{aligned}$$

Therefore we have locally

$$u(\cdot, t) = \eta_t(\xi_t^{-1}(\cdot)) = g(\cdot) + \int_0^t F(\cdot, \eta_r(\xi_r^{-1}), \chi_r(\xi_r^{-1}), \circ \mathrm{d}r)$$

and due to (4.29) u solves

$$u(x,t) = g(x) + \int_0^t F(x, u(x,r), \nabla u(x,r), \circ \mathrm{d}r)$$

for almost all ω and all (x,t) such that $t < \sigma(x,\omega)$. To show the uniqueness of the solution let $\tilde{u}(x,t)$ be another solution to equation (4.3) satisfying (4.6). The aim is to show that for almost all ω and all (x,t) with $t < T(x) \wedge \sigma(x)$

$$\tilde{u}(\xi_t(x), t) = \eta_t(x)$$

holds. We have seen that $\tilde{u}$ is in particular a continuous local $C^{k-1,\varepsilon}$-semimartingale for some $k \geq 5$. That means the local characteristic belongs at least to the class $(B^{4,\varepsilon}, B^{4,\varepsilon})$. By applying the generalized Itô formula in Theorem 4.7, since we have at least C^3-processes, we obtain

$$\begin{aligned}&\tilde{u}(\xi_t, t) - \tilde{u}(\xi_0, 0)\\ &= \int_0^t \tilde{u}(\xi_s, \circ \mathrm{d}s) + \sum_{i=1}^{d} \int_0^t \frac{\partial \tilde{u}}{\partial x_i}(\xi_s, s) \circ \mathrm{d}\xi_s^i\\ &= \int_0^t F(\xi_s, \tilde{u}(\xi_s, s), \nabla \tilde{u}(\xi_s, s), \circ \mathrm{d}s) - \sum_{i=1}^{d} \int_0^t \frac{\partial \tilde{u}}{\partial x_i}(\xi_s, s) F_{\chi_s^i}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)\end{aligned}$$

$$= \int_0^t F(\xi_s, \tilde{u}(\xi_s, s), \nabla\tilde{u}(\xi_s, s), \circ \mathrm{d}s) - \int_0^t \nabla\tilde{u}(\xi_s, s) \cdot F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s).$$

Due to (4.8) we obtain for the difference

$$\begin{aligned}
&\big(\tilde{u}(\xi_t, t) - \tilde{u}(\cdot, 0)\big) - \big(\eta_t - \eta_0\big) \\
&= \int_0^t F(\xi_s, \tilde{u}(\xi_s, s), \nabla\tilde{u}(\xi_s, s), \circ \mathrm{d}s) - \int_0^t \nabla\tilde{u}(\xi_s, s) \cdot F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) \\
&\quad + \int_0^t \chi_s \cdot F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) - \int_0^t F(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) \\
&= \int_0^t F(\xi_s, \tilde{u}(\xi_s, s), \nabla\tilde{u}(\xi_s, s), \circ \mathrm{d}s) - \int_0^t F(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) \\
&\quad - \int_0^t \Big(\nabla\tilde{u}(\xi_s, s) - \chi_s\Big) \cdot F_{\chi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s).
\end{aligned}$$

By using essentially that $\nabla\tilde{u}$ is at least a C^3-process we can apply Theorem 4.7 to $\nabla\tilde{u}$ and ξ_t and receive

$$\begin{aligned}
&\nabla\tilde{u}(\xi_t, t) - \nabla\tilde{u}(\xi_0, 0) \\
&= \int_0^t \nabla\tilde{u}(\xi_s, \circ \mathrm{d}s) + \sum_{i=1}^d \int_0^t \frac{\partial[\nabla\tilde{u}]}{\partial x_i}(\xi_s, s) \circ \mathrm{d}\xi_s^i \\
&= \int_0^t \nabla F(\xi_s, \tilde{u}(\xi_s, s), \nabla\tilde{u}(\xi_s, s), \circ \mathrm{d}s) - \sum_{i=1}^d \int_0^t \frac{\partial[\nabla\tilde{u}]}{\partial x_i}(\xi_s, s) F_{\chi_s^i}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) \\
&= \int_0^t F_{\xi_s}(\xi_s, \tilde{u}(\xi_s, s), \nabla\tilde{u}(\xi_s, s), \circ \mathrm{d}s) + \int_0^t \nabla\tilde{u}(\xi_s, s) F_{\eta_s}(\xi_s, \tilde{u}(\xi_s, s), \nabla\tilde{u}(\xi_s, s), \circ \mathrm{d}s) \\
&\quad + \sum_{i=1}^d \int_0^t \frac{\partial[\nabla\tilde{u}]}{\partial x_i}(\xi_s, s) F_{\chi_s^i}(\xi_s, \tilde{u}(\xi_s, s), \nabla\tilde{u}(\xi_s, s), \circ \mathrm{d}s) \\
&\quad - \sum_{i=1}^d \int_0^t \frac{\partial[\nabla\tilde{u}]}{\partial x_i}(\xi_s, s) F_{\chi_s^i}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s).
\end{aligned}$$

As before we obtain for the difference

$$\begin{aligned}
&\big(\nabla\tilde{u}(\xi_t, t) - \nabla\tilde{u}(\cdot, 0)\big) - \big(\chi_t - \chi_0\big) \\
&= \int_0^t F_{\xi_s}(\xi_s, \tilde{u}(\xi_s, s), \nabla\tilde{u}(\xi_s, s), \circ \mathrm{d}s) + \int_0^t \nabla\tilde{u}(\xi_s, s) F_{\eta_s}(\xi_s, \tilde{u}(\xi_s, s), \nabla\tilde{u}(\xi_s, s), \circ \mathrm{d}s) \\
&\quad + \sum_{i=1}^d \int_0^t \frac{\partial[\nabla\tilde{u}]}{\partial x_i}(\xi_s, s) F_{\chi_s^i}(\xi_s, \tilde{u}(\xi_s, s), \nabla\tilde{u}(\xi_s, s), \circ \mathrm{d}s) \\
&\quad - \sum_{i=1}^d \int_0^t \frac{\partial[\nabla\tilde{u}]}{\partial x_i}(\xi_s, s) F_{\chi_s^i}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) - \int_0^t F_{\xi_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s) \\
&\quad - \int_0^t F_{\eta_s}(\xi_s, \eta_s, \chi_s, \circ \mathrm{d}s)\chi_s
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t F_{\xi_s}(\xi_s, \tilde{u}(\xi_s, s), \nabla\tilde{u}(\xi_s, s), \circ\mathrm{d}s) + \int_0^t \nabla\tilde{u}(\xi_s, s) F_{\eta_s}(\xi_s, \tilde{u}(\xi_s, s), \nabla\tilde{u}(\xi_s, s), \circ\mathrm{d}s) \\
&\quad + \sum_{i=1}^d \int_0^t \frac{\partial[\nabla\tilde{u}]}{\partial x_i}(\xi_s, s)\Big(F_{\chi_s^i}(\xi_s, \tilde{u}(\xi_s, s), \nabla\tilde{u}(\xi_s, s), \circ\mathrm{d}s) - F_{\chi_s^i}(\xi_s, \eta_s, \chi_s, \circ\mathrm{d}s)\Big) \\
&\quad - \int_0^t F_{\xi_s}(\xi_s, \eta_s, \chi_s, \circ\mathrm{d}s) - \int_0^t F_{\eta_s}(\xi_s, \eta_s, \chi_s, \circ\mathrm{d}s)\chi_s.
\end{aligned}
$$

Next, we consider the following systems of stochastic differential equations for $(\tilde{u}(\xi_t, t) - \eta_t)$

$$
\begin{cases}
\mathrm{d}[\tilde{u}(\xi_t, t) - \eta_t] = F(\xi_t, \tilde{u}(\xi_t, t), \nabla\tilde{u}(\xi_t, t), \circ\mathrm{d}t) - F(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t) \\
\qquad\qquad - \Big(\nabla\tilde{u}(\xi_t, t) - \chi_t\Big) \cdot F_{\chi_t}(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t) \\
\tilde{u}(\xi_0, 0) - \eta_0 = 0
\end{cases} \tag{4.31}
$$

and

$$
\begin{cases}
\mathrm{d}[\nabla\tilde{u}(\xi_t, t) - \chi_t] = F_{\xi_t}(\xi_t, \tilde{u}(\xi_t, t), \nabla\tilde{u}(\xi_t, t), \circ\mathrm{d}t) \\
\qquad\qquad + \nabla\tilde{u}(\xi_t, t) F_{\eta_t}(\xi_t, \tilde{u}(\xi_t, t), \nabla\tilde{u}(\xi_t, t), \circ\mathrm{d}t) \\
\qquad\qquad + \sum_{i=1}^d \frac{\partial[\nabla\tilde{u}](\xi_t, t)}{\partial x_i}\Big(F_{\chi_t^i}(\xi_t, \tilde{u}(\xi_t, t), \nabla\tilde{u}(\xi_t, t), \circ\mathrm{d}t) \\
\qquad\qquad\qquad - F_{\chi_t^i}(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t)\Big) \\
\qquad\qquad - F_{\xi_t}(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t) - F_{\eta_t}(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t)\chi_t \\
\tilde{u}(\xi_0, 0) - \eta_0 = 0.
\end{cases} \tag{4.32}
$$

Due to Theorem 3.7 the systems have unique solutions given by

$$
\begin{aligned}
&\tilde{u}(\xi_t(x), t) - \eta_t(x) = 0 \\
\text{and} \quad &\nabla\tilde{u}(\xi_t(x), t) - \chi_t(x) = 0.
\end{aligned}
$$

Consequently we proved that (4.9) defines a unique solution to equation (4.3). □

4.4. Application to an example in the linear case

In this subsection we apply the existence and uniqueness result of H. Kunita as stated in Corollary 4.6 above to a linear stochastic partial differential equation. The result includes that the solution is given as composite function of the solutions to the system of stochastic characteristics equations (4.8). Hence we separate the following example into two parts. First, we have to verify that the corresponding local characteristic as defined in Definition 2.22 belongs to the class $(B^{k+1,\delta}, B^{k,\delta})$ for some $k \geq 5$ and $0 < \delta \leq 1$. Due to the representation result Theorem 2.39 we know that we are able to rewrite the one-dimensional problem in terms of a semimartingale $F(x, u, p, t)$ for $(x, u, p) \in \mathbb{R}^3$. Hence we obtain existence and uniqueness of the solution. Hereafter we determine the solution by solving the system of stochastic characteristic equations and finding the inverse process. Obviously, we do not have to check that the solution solves the equation.

Example 4.10 *Let W_t be a standard Brownian motion. We assume that $\tilde{\phi}, \tilde{\psi} \colon [0, \boldsymbol{T}] \to \mathbb{R}$ are continuous functions and $h \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$. We consider the following linear stochastic partial differential equation for $x \in \mathbb{R}$*

$$
\begin{cases}
\mathrm{d}u(x, t) = \Big(\tilde{\phi}(t)\, x\, \nabla u(x, t) - \tilde{\psi}(t)\, u(x, t) - u(x, t)\Big)\, \mathrm{d}t + \mathrm{d}W_t \\
u(x, 0) = h(x).
\end{cases} \tag{4.33}
$$

By applying the Itô-Stratonovich formula the equation (4.33) is obviously equivalent to

$$\begin{cases} \mathrm{d}u(x,t) = \left(\tilde{\phi}(t)\, x\, \nabla u(x,t) - \tilde{\psi}(t)\, u(x,t) - u(x,t)\right) \mathrm{d}t + \circ \mathrm{d}W_t \\ u(x,0) = h(x). \end{cases} \tag{4.34}$$

Due to the representation result the corresponding semimartingale is given for almost all ω and all $(x,y,z) \in \mathbb{R}^3$ by

$$\int_0^t F(x,y,z,\circ \mathrm{d}s) = \int_0^t \left(\tilde{\phi}(s)\, x\, z - \tilde{\psi}(s)\, y - y\right) \mathrm{d}t + \int_0^t \circ \mathrm{d}W_s$$

Hence the local characteristic (a,b,A_t) is defined for all $(x,y,z) \in \mathbb{R}^3$ and $(\bar{x},\bar{y},\bar{z}) \in \mathbb{R}^3$ by

$$\begin{aligned} a(x,y,z,\bar{x},\bar{y},\bar{z},t) &:= 1, \\ b(x,y,z,t) &:= \tilde{\phi}(t)\, x\, z - \tilde{\psi}(t)\, y - y, \\ A_t &:= t. \end{aligned}$$

Let us verify BM-HP (i) - BM-HP (vi): BM-HP (ii) is fulfilled, since $a = 1$ is a constant function and obviously symmetric and non-negative definite. BM-HP (iii) and BM-HP (v) are clearly satisfied. Consider for any $\mathbb{K} \subset \mathbb{R}^3$ compact

$$\begin{aligned} \int_0^{\mathbf{T}} \|a(t)\|^{\sim}_{(k+1)+\delta;\mathbb{K}}\, \mathrm{d}t = &\int_0^{\mathbf{T}} \sup_{\substack{(x,y,z)\in\mathbb{K} \\ (x',y',z')\in\mathbb{K}}} \frac{1}{(1+|(x,y,z)|)(1+|(x',y',z')|)}\, \mathrm{d}t + \int_0^{\mathbf{T}} 1\, \mathrm{d}t \\ &+ \int_0^{\mathbf{T}} \sup_{\substack{(x,y,z),(x',y',z')\in\mathbb{K} \\ (\bar{x},\bar{y},\bar{z}),(\bar{x}',\bar{y}',\bar{z}')\in\mathbb{K} \\ (x,y,z)\neq(\bar{x},\bar{y},\bar{z}) \\ (x',y',z')\neq(\bar{x}',\bar{y}',\bar{z}')}} \frac{0}{|(x,y,z)-(\bar{x},\bar{y},\bar{z})|^{\delta}|(x',y',z')-(\bar{x}',\bar{y}',\bar{z}')|^{\delta}}\, \mathrm{d}t \\ < \infty & \end{aligned}$$

Hence BM-HP (iv) is fulfilled. Concerning BM-HP (vi) we have

$$\begin{aligned} \int_0^{\mathbf{T}} \|b(t)\|_{k+\delta;\mathbb{K}}\, \mathrm{d}t = &\int_0^{\mathbf{T}} \sup_{(x,y,z)\in\mathbb{K}} \frac{|\tilde{\phi}(t)\, x\, z - \tilde{\psi}(t)\, y - y|}{(1+|(x,y,z)|)}\, \mathrm{d}t \\ &+ \int_0^{\mathbf{T}} \sup_{(x,y,z)\in\mathbb{K}} |\tilde{\phi}(t)\, z + \tilde{\psi}(t) - 1 + \tilde{\phi}(t)\, x + \tilde{\phi}(t) + \tilde{\phi}(t)|\, \mathrm{d}t + \int_0^{\mathbf{T}} 0\, \mathrm{d}t \\ \leq &\int_0^{\mathbf{T}} |\tilde{\phi}(t)| \sup_{(x,z)\in\mathbb{K}} \sqrt{|x|\,|z|}\, \mathrm{d}t + \int_0^{\mathbf{T}} (|\tilde{\psi}(t)| + 1)\, \mathrm{d}t \\ &+ \int_0^{\mathbf{T}} \sup_{(x,y,z)\in\mathbb{K}} |\tilde{\phi}(t)| \left(|z| + 2 + |x|\right) + |\tilde{\psi}(t)| + 1\, \mathrm{d}t < \infty \end{aligned}$$

By the same arguments BM-HP (i) is satisfied. Therefore the local characteristic (a,b,A_t) belongs to the class $(B^{k+1,\delta}, B^{k,\delta})$ for some $k \geq 5$ and $\delta > 0$. By Corollary 4.6 we know that there exists a unique solution. To find this solution we have to solve the system of stochastic characteristic equations for

$$F(\xi_t, \eta_t, \chi_t, \circ \mathrm{d}t) = \left(\tilde{\phi}(t)\, \xi_t\, \chi_t - \tilde{\psi}(t)\, \eta_t - \eta_t\right) \mathrm{d}t + \circ \mathrm{d}W_t$$

given by

$$\mathrm{d}\xi_t = -\tilde{\phi}(t)\,\xi_t\,\mathrm{d}t$$
$$\mathrm{d}\eta_t = \big(-\tilde{\psi}(t)\,\eta_t - \eta_t\big)\,\mathrm{d}t + 1 \circ \mathrm{d}W_t.$$

As proved in Lemma C.3 the solutions are given with initial values $\xi_0(x) = x, \eta_t(x) = h(x)$ for almost all ω and all (x,t) with t up to the explosion time by

$$\xi_t(x) = \exp\left(x - \int_0^t \tilde{\phi}(s)\,\mathrm{d}s\right)$$
$$\eta_t(x) = \frac{h(x) + \int_0^t \exp\left(\int_0^s \tilde{\psi}(r)\,\mathrm{d}r + s\right) \circ \mathrm{d}W_s}{\exp\left(\int_0^t \tilde{\psi}(s)\,\mathrm{d}s + t\right)}. \tag{4.35}$$

Obviously, the inverse process of ξ_t is given for almost all ω and all (x,t), $t \in [0, \sigma(x,\omega))$ by

$$\xi_t^{-1}(x) = \ln(x) + \int_0^t \tilde{\phi}(s)\,\mathrm{d}s.$$

Hence the unique local solution to equation (4.33) is given for almost all ω and all (x,t), t up to a stopping time $\sigma(x,\omega)$ by

$$u(x,t) = \frac{h\left(\ln(x) + \int_0^t \tilde{\phi}(s)\,\mathrm{d}s\right) + \int_0^t \exp\left(\int_0^s \tilde{\psi}(r)\,\mathrm{d}r + s\right) \circ \mathrm{d}W_s}{\exp\left(\int_0^t \tilde{\psi}(s)\,\mathrm{d}s + t\right)}.$$

5. Application to stochastic Burgers equations

In Chapter 3 we introduced the heuristic method of stochastic characteristics. Under the assumptions that a solution of the considered problem exist and that there exists a stochastic characteristic curve as defined by (3.4), we obtain stochastic characteristic equations (SCE) which have to be solved. In this chapter we state some examples concerning the new heuristic approach of the method of stochastic characteristics. For this purpose we look at different stochastic Burgers equations, find possible candidates for solutions under the above assumptions and hence have to verify if they really solve the problems. Due to these examples we see the main advantage of the method of stochastic characteristics, namely to receive an explicit expression of the solution. We will start with a generalization of Yamato's example to two dimensions (see Example 5.1 below). After that we extend Yamato's example by adding an drift term (see Example 5.3 below). Due to the fact that we want to study stochastic Burgers equations with Itô differential we formulate and prove in Lemma 5.4 below an application of the Itô-Stratonovich formula. By this we achieve a tool for solving different Burgers type equations and further determine explicit solutions (see Example 5.5 and Example 5.6 below). In Example 5.7 below we solve by the heuristic method of stochastic characteristics a stochastic transport equation with coefficient functions of polynomial growth. Obviously these functions do not fulfill the conditions of the main Theorem 4.5 in Chapter 4.

Yamato's example ([Kun84a, Example after Theorem 4.1.]) for the quasilinear stochastic partial differential equation in one dimension is given in the following form

$$\begin{cases} \mathrm{d}u(x,t) = u(x,t)\dfrac{\partial u}{\partial x}(x,t) \circ \mathrm{d}W_t \\ u(x,0) = g(x) \end{cases} \tag{5.1}$$

for $x \in \mathbb{R}$, $t \in [0, \mathbf{T}]$ and with initial function $g(x) = x$ and $g(x) = x^2$, respectively. The first step is to extend problem (5.1) to two dimensions. Let us consider the 2-dimensional version of problem (5.1).

Example 5.1 *Let $W_t = (W_t^1, W_t^2)$ be a 2-dimensional Brownian motion on a complete and separable probability space $(\Omega, \mathscr{F}, P)$. We consider the following equation for $x \in \mathbb{R}^2$, $t \in [0, \boldsymbol{T}]$*

$$\begin{cases} \mathrm{d}u(x,t) = u(x,t)\nabla u(x,t) \circ \mathrm{d}W_t \\ \qquad\qquad = \displaystyle\sum_{i=1}^{2} u(x,t)\frac{\partial u}{\partial x_i}(x,t) \circ \mathrm{d}W_t^i \\ u(x,0) = g(x),\ g \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}). \end{cases} \tag{5.2}$$

By using the representation result Theorem 2.39 this equation is equivalent to

$$\begin{cases} \mathrm{d}u(x,t) = F(x, u(x,t), \nabla u(x,t), \circ\mathrm{d}t) \\ u(x,0) = g(x), \end{cases}$$

where

$$F(x_1, x_2, u, p_1, p_2, \circ\mathrm{d}t) := \sum_{i=1}^{2} u\, p_i \circ \mathrm{d}W_t^i$$

for all $(x_1, x_2, u, p_1, p_2) \in \mathbb{R}^5$. We concentrate on the heuristic method of stochastic characteristics, hence we will not look at the corresponding local characteristic or formal conditions concerning the results in Chapter 4. As written in Chapter 3 the heuristic approach will *give us a candidate for a solution. Let us assume that u solves (5.2) and that we have a stochastic curve* (ξ_t, η_t, χ_t) *defined as in Chapter 3 (see equation (3.4)). The associated stochastic characteristic equations (SCE) for*

$$F(\xi_t, \eta_t, \chi_t, \circ \mathrm{d}t) = \sum_{i=1}^{2} \eta_t \chi_t^i \circ \mathrm{d}W_t^i$$

are given for $i = 1, 2$ *by*

$$\begin{aligned} \mathrm{d}\xi_t^i &= -F_{\chi_t^i}(\xi_t, \eta_t, \chi_t, \circ \mathrm{d}t) = -\frac{\partial[\eta_t \chi_t^i]}{\partial \chi_t^i} \circ \mathrm{d}W_t^i = -\eta_t \circ \mathrm{d}W_t^i \\ \mathrm{d}\eta_t &= F(\xi_t, \eta_t, \chi_t, \circ \mathrm{d}t) - \sum_{i=1}^{2} F_{\chi_t^i}(\xi_t, \eta_t, \chi_t, \circ \mathrm{d}t) \chi_t^i \\ &= \sum_{i=1}^{2} \eta_t \chi_t^i \circ \mathrm{d}W_t^i - \sum_{i=1}^{2} \chi_t^i \eta_t \circ \mathrm{d}W_t^i = 0. \end{aligned} \tag{5.3}$$

By considering the initial conditions $\eta_0(x) = g(x)$ *and* $\xi_0(x) = x$ *for* $x \in \mathbb{R}^2$ *we obtain on the one hand the solution*

$$\eta_t(x) = \eta_0(x) = g(x),$$

and on the other hand componentwise for $i = 1, 2$ *with* $x = (x_1, x_2)$

$$\begin{aligned} \xi_t^i(x) &= \xi_0^i(x) - \int_0^t g(x) \circ \mathrm{d}W_s^i \\ &= x_i - g(x) W_t^i. \end{aligned}$$

The above solution can be rewritten as

$$\xi_t(x) = \Big(x_1 - g(x) W_t^1 \, , x_2 - g(x) W_t^2 \Big).$$

Let us assume that (ξ_t, η_t) *exist up to an explosion time* T*. Similarly to Lemma 4.8 we consider stopping times for the inverse process* ξ_t^{-1} *as in Definition 4.4*

$$\begin{aligned} \tau_{\mathrm{inv}}(x) &:= \inf\{t > 0 \,|\, \det D\xi_t(x) = 0\}, \\ \tau(x) &:= \tau_{\mathrm{inv}}(x) \wedge T(x), \\ \sigma(x) &:= \inf\big\{ t > 0 \,\big|\, x \notin \xi_t(\{\tilde{x} \in \mathbb{R}^d \,|\, \tau(\tilde{x}) > t\}) \big\}. \end{aligned}$$

For an explicit initial function $g(x) = |x|^2 = \sum_{k=1}^{2} x_k^2$ *we get*

$$\begin{aligned} \xi_t^1(x) &= x_1 - (x_1^2 + x_2^2) W_t^1, \\ \xi_t^2(x) &= x_2 - (x_1^2 + x_2^2) W_t^2. \end{aligned} \tag{5.4}$$

The corresponding Jacobian matrix is given by

$$D\xi_t(x) = \begin{pmatrix} (1 - 2x_1 W_t^1) & (-2x_1 W_t^2) \\ (-2x_2 W_t^1) & (1 - 2x_2 W_t^2) \end{pmatrix}.$$

Hence the determinant of the Jacobian matrix can be identified by

$$\det D\xi_t(x) = 1 - 2x_1 W_t^1 - 2x_2 W_t^2 = 1 - 2(x \cdot W_t). \tag{5.5}$$

Up to the first time $\tau_{\text{inv}}(x)$ for which $1-2(x\cdot W_t)=0$ the inverse process ξ_t^{-1} exists. The stopping time $\tau_{\text{inv}}(x)$ is given for $x\in\mathbb{R}^2$ by

$$\tau_{\text{inv}}(x)=\inf\Big\{t>0\Big|\sum_{j=1}^{2}x_jW_t^j=\frac{1}{2}\Big\}.$$

As computed in Lemma D.1 for the reader's convenience, the inverse process ξ_t^{-1} for the explicit initial function g is given by

$$\begin{aligned}
&\xi_t^{-1}(x_1,x_2)\\
&=\Bigg(\frac{2x_1(W_t^2)^2+W_t^1-2x_2W_t^1W_t^2}{2(W_t^1)^2+2(W_t^2)^2}\\
&\qquad-\frac{W_t^1\sqrt{(1-4x_2^2(W_t^1)^2-x_1W_t^1(4-8x_2W_t^2)-4x_1^2(W_t^2)^2-4x_2W_t^2)}}{2(W_t^1)^2+2(W_t^2)^2},\\
&\quad\frac{W_t^1W_t^2-2x_1(W_t^1)^2W_t^2+2x_2(W_t^1)^3}{2(W_t^1)^3+2(W_t^2)^2}\\
&\qquad-\frac{W_t^1W_t^2\sqrt{(1-4x_2^2(W_t^1)^2+8x_1x_2W_t^1W_t^2-4x_1W_t^1-4x_1^2(W_t^2)^2-4x_2W_t^2)}}{2(W_t^1)^3+2(W_t^2)^2}\Bigg)
\end{aligned}\tag{5.6}$$

for almost all ω and all (x,t), $t<\hat\sigma(x,\omega)$, where $\hat\sigma(x)$ is the explosion time of $\xi_t^{-1}(x)$. At this point we will ignore that $\xi_0^{-1}(x)$ is not well-defined. Since we assumed that there is a stochastic characteristic curve (ξ_t,η_t) which prepare the system of SDEs (5.3), we obtain a candidate for a local solution to (5.2). Hence we set for almost all ω and all (x,t), $t\in[0,\hat\sigma(x,\omega))$

$$\begin{aligned}
u(x,t)&:=\eta_t(\xi_t^{-1}(x))\\
&=\frac{2x_1^2+2x_2^2}{1-2x_1W_t^1-2x_2W_t^2+Z},
\end{aligned}\tag{5.7}$$

where

$$Z:=\sqrt{1-4x_2W_t^2-4x_1^2(W_t^2)^2-4x_1W_t^1+8x_1x_2W_t^1W_t^2-4x_2^2(W_t^1)^2}.$$

For $t=0$ with $W_0^1=W_0^2=0$ the initial condition of (5.2) is fulfilled, since for all $x=(x_1,x_2)$

$$u(x,0)=\frac{2(x_1^2+x_2^2)}{2}=|x|^2$$

holds. Now we have to verify that u as defined in (5.7) is a solution to (5.2). To this end we use the differential equation and the notation of Newton's derivative of the Brownian motion given by $\dot W_t^1$ and $\dot W_t^2$, respectively. We have to show

$$\frac{\mathrm{d}u}{\mathrm{d}t}(x,t)=u(x,t)\,\frac{\partial u}{\partial x_1}(x,t)\,\dot W_t^1+u(x,t)\,\frac{\partial u}{\partial x_2}(x,t)\,\dot W_t^2.\tag{5.8}$$

Calculating the left hand side of (5.8) we get by classical rules of differential calculus

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t}(x,t)&=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{2x_1^2+2x_2^2}{1-2x_1W_t^1-2x_2W_t^2+Z}\right]\\
&=\frac{1}{Z(1-2x_1W_t^1-2x_2W_t^2+Z)^2}\Big(-(2x_1^2+2x_2^2)\Big[Z(-2x_1\dot W_t^1-2x_2\dot W_t^2)\\
&\quad+(-2x_2\dot W_t^2-4x_1^2W_t^2\dot W_t^2-2x_1\dot W_t^1+4x_1x_2\dot W_t^1W_t^2+4x_1x_2W_t^1\dot W_t^2-4x_2^2\dot W_t^1W_t^1)\Big]\Big)
\end{aligned}$$

$$= \frac{(2x_1^2+2x_2^2)}{Z(1-2x_1W_t^1-2x_2W_t^2+Z)^2}\Big((-1)\Big[Z(-2x_1\dot W_t^1-2x_2\dot W_t^2)$$
$$+(-2x_2\dot W_t^2-4x_1^2W_t^2\dot W_t^2-2x_1\dot W_t^1+4x_1x_2\dot W_t^1W_t^2+4x_1x_2W_t^1\dot W_t^2-4x_2^2\dot W_t^1W_t^1)\Big]\Big).$$

Hence we obtain

$$\frac{\mathrm{d}u}{\mathrm{d}t}(x,t)=u(x,t)\Bigg(\frac{(-1)}{Z(1-2x_1W_t^1-2x_2W_t^2+Z)}$$
$$\cdot\Big[Z(-2x_1\dot W_t^1-2x_2\dot W_t^2)+(-2x_2\dot W_t^2-4x_1^2W_t^2\dot W_t^2-2x_1\dot W_t^1 \tag{5.9}$$
$$+4x_1x_2\dot W_t^1W_t^2+4x_1x_2W_t^1\dot W_t^2-4x_2^2\dot W_t^1W_t^1)\Big]\Bigg).$$

The partial derivative with respect to x_1 is given by

$$\frac{\partial u}{\partial x_1}(x,t)=\frac{\partial}{\partial x_1}\left[\frac{2x_1^2+2x_2^2}{1-2x_1W_t^1-2x_2W_t^2+Z}\right]$$
$$=\frac{4x_1(1-2x_1W_t^1-2x_2W_t^2+Z)-(2x_1^2+2x_2^2)\left[-2W_t^1+\frac{-8x_1(W_t^2)^2-4W_t^1+8x_2W_t^1W_t^2}{2Z}\right]}{(1-2x_1W_t^1-2x_2W_t^2+Z)^2}$$
$$=\frac{4x_1Z(1-2x_1W_t^1-2x_2W_t^2+Z)}{Z(1-2x_1W_t^1-2x_2W_t^2+Z)^2}$$
$$-\frac{(2x_1^2+2x_2^2)[(-2W_t^1)Z-4x_1(W_t^2)^2-2W_t^1+4x_2W_t^1W_t^2]}{Z(1-2x_1W_t^1-2x_2W_t^2+Z)^2}.$$

Now we make use of the following equality by adding and subtracting the term $4x_1(W_t^1)^2$:

$$[(-2W_t^1)Z-4x_1(W_t^2)^2-2W_t^1+4x_2W_t^1W_t^2]$$
$$=(1-2x_1W_t^1-2x_2W_t^2+Z)\,(-2W_t^1)-4x_1\big((W_t^1)^2+(W_t^2)^2\big).$$

Hence the partial derivative with respect to x_1 is finally given by

$$\frac{\partial u}{\partial x_1}(x,t)=\frac{1}{Z(1-2x_1W_t^1-2x_2W_t^2+Z)^2}\Big(4x_1Z(1-2x_1W_t^1-2x_2W_t^2+Z)$$
$$-(2x_1^2+2x_2^2)\cdot\big[(1-2x_1W_t^1-2x_2W_t^2+Z)\,(-2W_t^1)-4x_1\big((W_t^1)^2+(W_t^2)^2\big)\big]\Big).$$

The partial derivative with respect to x_2 can be calculated in a similar way by

$$\frac{\partial u}{\partial x_2}(x,t)=\frac{\partial}{\partial x_2}\left[\frac{2x_1^2+2x_2^2}{1-2x_1W_t^1-2x_2W_t^2+Z}\right]$$
$$=\frac{4x_2(1-2x_1W_t^1-2x_2W_t^2+Z)-(2x_1^2+2x_2^2)\left[-2W_t^2+\frac{-4W_t^2+8x_1W_t^1W_t^2-8x_2(W_t^1)^2}{2Z}\right]}{(1-2x_1W_t^1-2x_2W_t^2+Z)^2}$$
$$=\frac{1}{Z(1-2x_1W_t^1-2x_2W_t^2+Z)^2}\cdot\Big(4x_2Z(1-2x_1W_t^1-2x_2W_t^2+Z)$$
$$-(2x_1^2+2x_2^2)[(-2W_t^2)Z-2W_t^2+4x_1W_t^1W_t^2-4x_2(W_t^1)^2]\Big).$$

As before we add and subtract the matching term $4x_2(W_t^2)^2$ and get the following equality

$$[(-2W_t^2)Z-2W_t^2+4x_1W_t^1W_t^2-4x_2(W_t^1)^2]$$

$$= (1 - 2x_1W_t^1 - 2x_2W_t^2 + Z)\,(-2W_t^2) - 4x_2\big((W_t^1)^2 + (W_t^2)^2\big).$$

Hence we receive

$$\frac{\partial u}{\partial x_2}(x,t) = \frac{1}{Z(1 - 2x_1W_t^1 - 2x_2W_t^2 + Z)^2} \cdot \Big(4x_2 Z(1 - 2x_1W_t^1 - 2x_2W_t^2 + Z) - (2x_1^2 + 2x_2^2)\big[(1 - 2x_1W_t^1 - 2x_2W_t^2 + Z)\,(-2W_t^2) - 4x_2\big((W_t^1)^2 + (W_t^2)^2\big)\big]\Big).$$

Now we are able to verify (5.8). Due to (5.9) it is enough to prove that

$$\begin{aligned}
&\frac{\partial u}{\partial x_1}(x,t)\cdot \dot W_t^1 + \frac{\partial u}{\partial x_2}(x,t)\cdot \dot W_t^2 \\
&= \frac{(-1)}{Z(1 - 2x_1W_t^1 - 2x_2W_t^2 + Z)} \cdot \Big(Z(-2x_1\dot W_t^1 - 2x_2\dot W_t^2) \\
&\quad + \big(-2x_2\dot W_t^2 - 4x_1^2W_t^2\dot W_t^2 - 2x_1\dot W_t^1 + 4x_1x_2\dot W_t^1 W_t^2 \\
&\qquad + 4x_1x_2W_t^1\dot W_t^2 - 4x_2^2\dot W_t^1 W_t^1\big)\Big).
\end{aligned} \tag{5.10}$$

Let us start on the left hand side of (5.10)

$$\begin{aligned}
&\frac{\partial u}{\partial x_1}(x,t)\,\dot W_t^1 + \frac{\partial u}{\partial x_2}(x,t)\,\dot W_t^2 \\
&= \frac{4x_1\dot W_t^1 Z(1 - 2x_1W_t^1 - 2x_2W_t^2 + Z)}{Z(1 - 2x_1W_t^1 - 2x_2W_t^2 + Z)^2} \\
&\quad - \frac{(2x_1^2 + 2x_2^2)\big[(1 - 2x_1W_t^1 - 2x_2W_t^2 + Z)\,(-2W_t^1) - 4x_1\big((W_t^1)^2 + (W_t^2)^2\big)\big]\,\dot W_t^1}{Z(1 - 2x_1W_t^1 - 2x_2W_t^2 + Z)^2} \\
&\quad + \frac{4x_2\dot W_t^2 Z(1 - 2x_1W_t^1 - 2x_2W_t^2 + Z)}{Z(1 - 2x_1W_t^1 - 2x_2W_t^2 + Z)^2} \\
&\quad - \frac{(2x_1^2 + 2x_2^2)\big[(1 - 2x_1W_t^1 - 2x_2W_t^2 + Z)\,(-2W_t^2) - 4x_2\big((W_t^1)^2 + (W_t^2)^2\big)\big]\,\dot W_t^2}{Z(1 - 2x_1W_t^1 - 2x_2W_t^2 + Z)^2}.
\end{aligned}$$

With the help of a second short notation $Y := (1 - 2x_1W_t^1 - 2x_2W_t^2 + Z)$ *we get*

$$\begin{aligned}
&\frac{\partial u}{\partial x_1}(x,t)\,\dot W_t^1 + \frac{\partial u}{\partial x_2}(x,t)\,\dot W_t^2 \\
&= \frac{4x_1\dot W_t^1 ZY}{ZY^2} - \frac{(2x_1^2 + 2x_2^2)\big[Y\,(-2W_t^1) - 4x_1\big((W_t^1)^2 + (W_t^2)^2\big)\big]\,\dot W_t^1}{ZY^2} \\
&\quad + \frac{4x_2\dot W_t^2 ZY}{ZY^2} - \frac{(2x_1^2 + 2x_2^2)\big[Y\,(-2W_t^2) - 4x_2\big((W_t^1)^2 + (W_t^2)^2\big)\big]\,\dot W_t^2}{ZY^2}.
\end{aligned}$$

Now we use the following result

$$\begin{aligned}
&4(x_1^2 + x_2^2)\big((W_t^1)^2 + (W_t^2)^2\big) \\
&= 4x_2^2(W_t^2)^2 + 4x_1^2(W_t^1)^2 + 4x_1^2(W_t^2)^2 + 4x_2^2(W_t^1)^2 \\
&= \big(1 - 4x_2W_t^2 + 4x_2^2(W_t^2)^2 - 4x_1W_t^1 + 8x_1x_2W_t^1W_t^2 + 4x_1^2(W_t^1)^2\big) \\
&\quad - \big(1 - 4x_2W_t^2 - 4x_1^2(W_t^2)^2 - 4x_1W_t^1 + 8x_1x_2W_t^1W_t^2 - 4x_2^2(W_t^1)^2\big) \\
&= (1 - 2x_1W_t^1 - 2x_2W_t^2)^2 - Z^2 \\
&= \big((1 - 2x_1W_t^1 - 2x_2W_t^2) + Z\big)\big((1 - 2x_1W_t^1 - 2x_2W_t^2) - Z\big) \\
&= Y\,\big((1 - 2x_1W_t^1 - 2x_2W_t^2) - Z\big)
\end{aligned}$$

to achieve

$$
\begin{aligned}
&\frac{\partial u}{\partial x_1}(x,t)\,\dot W_t^1 + \frac{\partial u}{\partial x_2}(x,t)\,\dot W_t^2\\
&= \frac{4x_1\dot W_t^1 ZY}{ZY^2} + \frac{(2x_1^2+2x_2^2)\left[(2YW_t^1)+4x_1\big((W_t^1)^2+(W_t^2)^2\big)\right]\dot W_t^1}{ZY^2}\\
&\quad + \frac{4x_2\dot W_t^2 ZY}{ZY^2} + \frac{(2x_1^2+2x_2^2)\left[(2YW_t^2)+4x_2\big((W_t^1)^2+(W_t^2)^2\big)\right]\dot W_t^2}{ZY^2}\\
&= \frac{4x_1\dot W_t^1 ZY}{ZY^2} + \frac{(2x_1^2+2x_2^2)(2YW_t^1\dot W_t^1)+4(x_1^2+x_2^2)\big((W_t^1)^2+(W_t^2)^2\big)\cdot 2x_1\dot W_t^1}{ZY^2}\\
&\quad + \frac{4x_2\dot W_t^2 ZY}{ZY^2} + \frac{(2x_1^2+2x_2^2)(2YW_t^2\dot W_t^2)+4(x_1^2+x_2^2)\big((W_t^1)^2+(W_t^2)^2\big)\cdot 2x_2\dot W_t^2}{ZY^2}\\
&= \frac{4x_1 Z\dot W_t^1 Y + (2x_1^2+2x_2^2)\cdot 2W_t^1\dot W_t^1 Y + 2x_1\dot W_t^1 Y(1-2x_1W_t^1-2x_2W_t^2-Z)}{ZY^2}\\
&\quad + \frac{4x_2 Z\dot W_t^2 Y + (2x_1^2+2x_2^2)\cdot 2W_t^2\dot W_t^2 Y + 2x_2\dot W_t^2 Y(1-2x_1W_t^1-2x_2W_t^2-Z)}{ZY^2}\\
&= \frac{4x_1 Z\dot W_t^1 + (2x_1^2+2x_2^2)\cdot 2W_t^1\dot W_t^1 + 2x_1\dot W_t^1(1-2x_1W_t^1-2x_2W_t^2-Z)}{ZY}\\
&\quad + \frac{4x_2 Z\dot W_t^2 + (2x_1^2+2x_2^2)\cdot 2W_t^2\dot W_t^2 + 2x_2\dot W_t^2(1-2x_1W_t^1-2x_2W_t^2-Z)}{ZY}\\
&= \frac{4x_1 Z\dot W_t^1 + 4x_2 Z\dot W_t^2 + (2x_1^2+2x_2^2)[2W_t^1\dot W_t^1+2W_t^2\dot W_t^2]}{ZY}\\
&\quad + \frac{(2x_1\dot W_t^1+2x_2\dot W_t^2)(1-2x_1W_t^1-2x_2W_t^2) - 2x_1Z\dot W_t^1 - 2x_2Z\dot W_t^2}{ZY}\\
&= \frac{2x_1 Z\dot W_t^1 + 2x_2 Z\dot W_t^2 + (2x_1^2+2x_2^2)[2W_t^1\dot W_t^1+2W_t^2\dot W_t^2]}{ZY}\\
&\quad + \frac{(1-2x_1W_t^1-2x_2W_t^2)(2x_1\dot W_t^1+2x_2\dot W_t^2)}{ZY}\\
&= \frac{Z(2x_1\dot W_t^1+2x_2\dot W_t^2) + 4x_1^2W_t^1\dot W_t^1 + 4x_1^2W_t^2\dot W_t^2 + 4x_2^2W_t^1\dot W_t^1 + 4x_2^2W_t^2\dot W_t^2}{ZY}\\
&\quad + \frac{2x_1\dot W_t^1 + 2x_2\dot W_t^2 - 4x_1^2W_t^1\dot W_t^1 - 4x_1x_2W_t^1\dot W_t^2 - 4x_1x_2W_t^2\dot W_t^1 - 4x_2^2W_t^2\dot W_t^2}{ZY}\\
&= \frac{(-1)}{Z(1-2x_1W_t^1-2x_2W_t^2+Z)}\cdot\Big(Z(-2x_1\dot W_t^1-2x_2\dot W_t^2)\\
&\quad + \big(-2x_2\dot W_t^2 - 4x_1^2W_t^2\dot W_t^2 - 2x_1\dot W_t^1 + 4x_1x_2\dot W_t^1W_t^2 + 4x_1x_2W_t^1\dot W_t^2 - 4x_2^2\dot W_t^1W_t^1\big)\Big)
\end{aligned}
$$

which is equal to the right hand side of (5.10). We finally proved that u as defined in (5.7) is a local solution to the 2-dimensional problem (5.2).

Remark 5.2 *Obviously, it is possible to extend the above Example 5.2 to three or more dimensions, but then the calculations and expressions will become much longer and a challenge for reading and writing.*

Due to the definition of (SCE) we can see that the system of stochastic differential equations is simplified in the case where the diffusion term depends on the gradient of the unknown function. In the notation of the stochastic characteristic curve the dependence on χ_t leads to $\mathrm{d}\eta_t = 0$. The next example is an extension of Yamato's example to the case where the diffusion term coincides with the drift term.

Example 5.3 *Let W_t be a one-dimensional Brownian motion on a complete and separable probability space $(\Omega, \mathscr{F}, P)$. Consider the following equation for $x \in \mathbb{R}$, $t \in [0, \boldsymbol{T}]$,*

$$\begin{cases} \mathrm{d}u(x,t) = u(x,t)\dfrac{\partial u}{\partial x}(x,t)\ \mathrm{d}t + u(x,t)\dfrac{\partial u}{\partial x}(x,t) \circ \mathrm{d}W_t \\ \ u(x,0) = g(x). \end{cases} \tag{5.11}$$

Similarly to Example 5.1 the stochastic characteristic equations, provided that u solves (5.11) and that there exists the stochastic characteristic curve (ξ_t, η_t), are given by

$$\begin{aligned} \mathrm{d}\xi_t &= -\eta_t\ \mathrm{d}t - \eta_t \circ \mathrm{d}W_t \\ \mathrm{d}\eta_t &= \big(\eta_t\chi_t - \chi_t\eta_t\big)\ \mathrm{d}t + (\eta_t\chi_t - \chi_t\eta_t) \circ \mathrm{d}W_t = 0. \end{aligned} \tag{5.12}$$

With initial condition $\eta_0(x) = g(x) = x^2$ the equation $\mathrm{d}\eta_t = 0$ has the solution

$$\eta_t(x) = x^2. \tag{5.13}$$

Hence we have to study

$$\mathrm{d}\xi_t = -x^2\ \mathrm{d}t - x^2 \circ \mathrm{d}W_t,$$

which is equivalent to

$$\begin{aligned} \xi_t(x) &= \xi_0(x) - \int_0^t x^2\ \mathrm{d}s - \int_0^t x^2 \circ \mathrm{d}W_s \\ &= x - x^2 t - x^2 W_t. \end{aligned}$$

Up to its explosion time $\hat{\sigma}(x)$ the inverse process ξ_t^{-1} is given by

$$\xi_t^{-1}(x) = \frac{2x}{1 + \sqrt{1 - 4x(t + W_t)}}$$

for almost all ω and all (x,t) such that $t < \hat{\sigma}(x,\omega)$ as proved in Lemma D.2 for the reader's convenience. Hence we set for almost all ω and all (x,t) such that $t \in [0, \hat{\sigma}(x,\omega))$

$$u(x,t) := \eta_t(\xi_t^{-1}(x)) = \left(\frac{2x}{1 + \sqrt{1 - 4x(t + W_t)}}\right)^2. \tag{5.14}$$

The calculations of the partial derivatives with respect to x and t are given in Lemma D.3. Now we verify that (5.14) is really a solution to (5.11). To this end we plug (5.14) into (5.11) and prove that (5.14) solves the differential equation written as

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t}(x,t) &= u(x,t)\frac{\partial u}{\partial x}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t)\dot{W}_t \\ &= u(x,t)\frac{\partial u}{\partial x}(x,t) \cdot \big(1 + \dot{W}_t\big), \end{aligned}$$

where we deal with the notation $\frac{\circ \mathrm{d}W_t}{\mathrm{d}t} =: \dot{W}_t$. By using the fact that

$$\frac{\mathrm{d}u}{\mathrm{d}t}(x,t) = u(x,t) \cdot \frac{4x(1 + \dot{W}_t)}{\sqrt{1 - 4x(t + W_t)}\big(1 + \sqrt{1 - 4x(t + W_t)}\big)},$$

as shown in Lemma D.3, it is enough to prove

$$\frac{\partial u}{\partial x}(x,t) = \frac{4x}{\sqrt{1 - 4x(t + W_t)}\big(1 + \sqrt{1 - 4x(t + W_t)}\big)}.$$

The above equality is valid, since

$$\begin{aligned}\frac{\partial u}{\partial x}(x,t) &= \frac{8x\big(1+\sqrt{1-4x(t+W_t)}-2xt-2xW_t\big)}{\sqrt{1-4x(t+W_t)}(1+\sqrt{1-4x(t+W_t)})^3}\\ &= \frac{4x\big(2+2\sqrt{1-4x(t+W_t)}-4xt-4xW_t\big)}{\sqrt{1-4x(t+W_t)}(1+\sqrt{1-4x(t+W_t)})^3}\\ &= \frac{4x\big(1+\sqrt{1-4x(t+W_t)}\big)^2}{\sqrt{1-4x(t+W_t)}(1+\sqrt{1-4x(t+W_t)})^3}\\ &= \frac{4x}{\sqrt{1-4x(t+W_t)}(1+\sqrt{1-4x(t+W_t)})}.\end{aligned}$$

In the previous Examples 5.1 and Example 5.3 we look at problems dealing with the Stratonovich integral. The next extension is to solve Burgers type equations dealing with a classical Itô integral. We consider stochastic Burgers equations as given in [LR15]. Remark 5.19 in [LR15] includes without consideration of the assumptions that the equation

$$\mathrm{d}u = (\Delta u + h(u)\cdot\nabla u)\,\mathrm{d}t + B(u)\,\mathrm{d}\mathbb{W}$$

is called stochastic generalized Burgers equation if Δ is the Laplace operator, $\mathbb{W}$ is a cylindrical Wiener process on a Hilbert space and $h=(h_1,...,h_d)\colon \mathbb{R}\to\mathbb{R}^d$ are Lipschitz functions. Due to the fact that we want to apply the heuristic method of stochastic characteristics, we can only look at first order stochastic generalized Burgers equations in one dimension perturbed by a standard Brownian motion W_t. Hence we consider equations of the form

$$\mathrm{d}u = h(u)\cdot\nabla u\,\mathrm{d}t + B(u)\,\mathrm{d}W_t. \tag{5.15}$$

To get a better view of the calculations we firstly solve the so-called stochastic Burgers equation for dimension one with $h(u)=u$. Furthermore, we choose a special diffusion term $B(u)=\sqrt{2}\,u$. Since the perturbation is not given by a Stratonovich differential, we have to apply the Itô-Stratonovich formula. Before we come to a concrete example namely Example 5.5 below, we need the following result.

Lemma 5.4 *Let W_t be a standard one-dimensional Brownian motion and $c>0$. Then the equation*

$$\mathrm{d}u(x,t) = u(x,t)\,\frac{\partial u}{\partial x}(x,t)\,\mathrm{d}t + c\,u(x,t)\,\mathrm{d}W_t \tag{5.16}$$

is equivalent to

$$\mathrm{d}u(x,t) = \Big(1-\frac{c^2}{2}\Big)\,u(x,t)\,\frac{\partial u}{\partial x}(x,t)\,\mathrm{d}t + c\,u(x,t)\circ\mathrm{d}W_t \tag{5.17}$$

Proof. The result is a classical application of Theorem 2.35 and in particular (2.12), since

$$\begin{aligned}u(x,t)-u(x,0) &= \int_0^t u(x,s)\frac{\partial u}{\partial x}(x,s)\,\mathrm{d}s + \int_0^t c\,u(x,s)\,\mathrm{d}W_s\\ &= \int_0^t u(x,s)\frac{\partial u}{\partial x}(x,s)\,\mathrm{d}s + \int_0^t c\,u(x,s)\circ\mathrm{d}W_s\\ &\quad - \frac{1}{2}\int_0^t \frac{\partial[c\,u]}{\partial x}(x,s)\cdot c\,u(x,s)\,\mathrm{d}s\end{aligned}$$

$$= \int_0^t \Big(1 - \frac{c^2}{2}\Big) u(x,s) \frac{\partial u}{\partial x}(x,s)\,\mathrm{d}s + \int_0^t c\,u(x,s) \circ \mathrm{d}W_s.$$

Hence we obtain (5.17) in terms of the Stratonovich integral. □

Example 5.5 *Let W_t be a standard one-dimensional Brownian motion. Consider the following equation for $x \in \mathbb{R}$, $t \in [0, \boldsymbol{T}]$:*

$$\begin{cases} \mathrm{d}u(x,t) = u(x,t)\dfrac{\partial u}{\partial x}(x,t)\,\mathrm{d}t + \sqrt{2}\,u(x,t)\,\mathrm{d}W_t \\ u(x,0) = x^2. \end{cases} \tag{5.18}$$

Due to Lemma 5.4 we know that (5.18) is equivalent to

$$\begin{cases} \mathrm{d}u(x,t) = \Big(1 - \dfrac{(\sqrt{2})^2}{2}\Big)u(x,t)\dfrac{\partial u}{\partial x}(x,t)\,\mathrm{d}t + \sqrt{2}\,u(x,t) \circ \mathrm{d}W_t \\ \qquad\quad = \sqrt{2}\,u(x,t) \circ \mathrm{d}W_t \\ u(x,0) = x^2. \end{cases} \tag{5.19}$$

Let us assume that u solves equation (5.19) and that the stochastic characteristic curve (ξ_t, η_t) as stated in Chapter 3 exists. Then the stochastic characteristic equations considering

$$F(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t) = \sqrt{2}\,\eta_t \circ \mathrm{d}W_t$$

are given by

$$\mathrm{d}\xi_t = 0 \tag{5.20}$$

and

$$\mathrm{d}\eta_t = \sqrt{2}\,\eta_t \circ \mathrm{d}W_t. \tag{5.21}$$

With initial condition $\eta_0(x) = x^2$ the equation $\mathrm{d}\eta_t = \sqrt{2}\,\eta_t \circ \mathrm{d}W_t$ has the solution

$$\eta_t(x) = x^2 \exp\big(\sqrt{2}\,W_t\big), \tag{5.22}$$

where we applied Lemma D.4. Furthermore, we obtain for (5.20) the solution

$$\xi_t(x) = x.$$

Obviously, the inverse process of $\xi_t(x)$ is given by

$$\xi_t^{-1}(x) = x.$$

Hence we set for almost all ω and all (x,t) with $t < \hat{\sigma}(x,\omega)$

$$u(x,t) := x^2 \exp\big(\sqrt{2}\,W_t\big) \tag{5.23}$$

to obtain a candidate for a solution to (5.19). We finally have to verify that u is really a local solution to (5.19). Due to the chain rule for Stratonovich integrals we receive

$$u(x,0) + \int_0^t \sqrt{2}\,u(x,s) \circ \mathrm{d}W_s = x^2 + \int_0^t \sqrt{2}\,x^2 \exp\big(\sqrt{2}\,W_s\big) \circ \mathrm{d}W_s$$

$$= x^2 + \sqrt{2}\,x^2 \int_0^t \exp\big(\sqrt{2}\,W_s\big) \circ \mathrm{d}W_s$$

$$
\begin{aligned}
&= x^2 + \sqrt{2}\,x^2 \left[\frac{1}{\sqrt{2}} \exp\left(\sqrt{2}\,W_t\right) - \frac{1}{\sqrt{2}} \exp(0)\right] \\
&= x^2 + \sqrt{2}\,x^2 \,\frac{1}{\sqrt{2}} \exp\left(\sqrt{2}\,W_t\right) - \sqrt{2}\,x^2 \,\frac{1}{\sqrt{2}} \\
&= x^2 \exp\left(\sqrt{2}\,W_t\right) \\
&= u(x,t).
\end{aligned}
$$

Now we extend Example 5.5 to the case of $B(u) = u$.

Example 5.6 *Let W_t be a standard one-dimensional Brownian motion. Consider the following equation for $x \in \mathbb{R}$, $t \in [0, \boldsymbol{T}]$:*

$$
\begin{cases}
\mathrm{d}u(x,t) = u(x,t)\dfrac{\partial u}{\partial x}(x,t)\,\mathrm{d}t + u(x,t)\,\mathrm{d}W_t \\
\;u(x,0) = x^2.
\end{cases}
\tag{5.24}
$$

Due to Lemma 5.4 we know that (5.24) is equivalent to

$$
\begin{cases}
\mathrm{d}u(x,t) = \left(1 - \dfrac{1}{2}\right) u(x,t)\dfrac{\partial u}{\partial x}(x,t)\,\mathrm{d}t + u(x,t) \circ \mathrm{d}W_t \\
\qquad\qquad = \dfrac{1}{2} u(x,t)\dfrac{\partial u}{\partial x}(x,t)\,\mathrm{d}t + u(x,t) \circ \mathrm{d}W_t \\
\;u(x,0) = x^2.
\end{cases}
\tag{5.25}
$$

Let us assume that u solves (5.25) and that the stochastic characteristic curve (ξ_t, η_t) as stated in Chapter 3 exists. Then the stochastic characteristic equations considering

$$
F(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t) = \frac{1}{2}\eta_t \chi_t\,\mathrm{d}t + \eta_t \circ \mathrm{d}W_t
$$

are given by

$$
\mathrm{d}\xi_t = -\frac{1}{2}\eta_t\,\mathrm{d}t \tag{5.26}
$$

and

$$
\mathrm{d}\eta_t = \eta_t \circ \mathrm{d}W_t. \tag{5.27}
$$

As before we apply Lemma D.4 to equation (5.27) with initial condition $g(x) = x^2$, $c = 1$ and obtain the solution

$$
\eta_t(x) = x^2 \exp\left(W_t\right). \tag{5.28}
$$

Now we plug (5.28) into (5.26) and get the integral equation

$$
\xi_t(x) = x - \frac{1}{2}x^2 \int_0^t \exp(W_s)\,\mathrm{d}s.
$$

Under the condition that the solutions ξ_t, η_t exist up to the explosion time T we can find the inverse process. Up to another explosion time $\hat{\sigma}$ we obtain for almost all ω and all (x,t) with $t < \hat{\sigma}(x,\omega)$

$$
\xi_t^{-1}(x) = \frac{2x}{1 + \sqrt{1 - 2x\left(\int_0^t \exp(W_s)\,\mathrm{d}s\right)}}.
$$

For the reader's convenience the details can be found in Lemma D.5. Hence we define for almost all ω and all (x,t) such that $t < \hat{\sigma}(x,\omega)$

$$u(x,t) := \frac{4x^2 \exp(W_t)}{\left(1 + \sqrt{1 - 2x\left(\int_0^t \exp(W_s)\,\mathrm{d}s\right)}\right)^2} \tag{5.29}$$

to be a candidate for a solution to (5.25). Due to the assumptions we finally have to verify that (5.29) is really a local solution to (5.25). So we will show that u fulfills

$$\frac{\mathrm{d}u}{\mathrm{d}t}(x,t) = \frac{1}{2}u(x,t)\frac{\partial u}{\partial x}(x,t) + u(x,t)\dot{W}_t. \tag{5.30}$$

Detailed calculations concerning the partial derivatives are given in Lemma D.6. The partial derivative of u with respect to time t is given in terms of a short notation for

$$Z := \sqrt{1 - 2x\left(\int_0^t e^{W_s}\,\mathrm{d}s\right)}$$

by

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t}(x,t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{4x^2 e^{W_t}}{(1+Z)^2}\right] \\
&= \frac{4x^2 e^{W_t}\left[Z(1+Z)^2\dot{W}_t + 2xe^{W_t} + 2xe^{W_t}Z\right]}{Z(1+Z)^4} \\
&= \frac{4x^2 e^{W_t}\left[2xe^{W_t} + \dot{W}_t Z(1+Z)\right]}{Z(1+Z)^3}.
\end{aligned}$$

Furthermore, we have for the right hand side of (5.30)

$$\begin{aligned}
&\frac{1}{2}u(x,t)\frac{\mathrm{d}u}{\mathrm{d}x}(x,t) + u(x,t)\dot{W}_t \\
&= \frac{1}{2}\frac{4x^2e^{W_t}}{(1+Z)^2}\frac{\left(Z(1+Z)^2\cdot 8xe^{W_t} + 8x^2e^{W_t}\left(\int_0^t e^{W_s}\,\mathrm{d}s\right)(1+Z)\right)}{Z(1+Z)^4} + \frac{4x^2e^{W_t}\dot{W}_t}{(1+Z)^2} \\
&= \frac{2x^2e^{W_t}}{(1+Z)^2}\frac{\left(Z(1+Z)^2\cdot 8xe^{W_t} + 8x^2e^{W_t}\left(\int_0^t e^{W_s}\,\mathrm{d}s\right)(1+Z)\right)}{Z(1+Z)^4} + \frac{4x^2e^{W_t}\dot{W}_t}{(1+Z)^2} \\
&= \frac{2x^2e^{W_t}\left(Z(1+Z)^2\cdot 8xe^{W_t} + 8x^2e^{W_t}\left(\int_0^t e^{W_s}\,\mathrm{d}s\right)(1+Z)\right)}{Z(1+Z)^6} + \frac{4x^2e^{W_t}\dot{W}_t}{(1+Z)^2} \\
&= \frac{2x^2e^{W_t}\left(Z(1+Z)\cdot 8xe^{W_t} + 8x^2e^{W_t}\left(\int_0^t e^{W_s}\,\mathrm{d}s\right)\right)(1+Z)}{Z(1+Z)^6} + \frac{4x^2e^{W_t}\dot{W}_t}{(1+Z)^2} \\
&= \frac{2x^2e^{W_t}\left(Z(1+Z)\cdot 8xe^{W_t} + 8x^2e^{W_t}\left(\int_0^t e^{W_s}\,\mathrm{d}s\right)\right)}{Z(1+Z)^5} + \frac{4x^2e^{W_t}\dot{W}_t}{(1+Z)^2} \\
&= \frac{16x^3e^{2W_t}Z(1+Z) + 16x^4e^{2W_t}\left(\int_0^t e^{W_s}\,\mathrm{d}s\right) + 4x^2e^{W_t}\dot{W}_tZ(1+Z)^3}{Z(1+Z)^5} \\
&= \frac{4x^2e^{W_t}\left[4xe^{W_t}Z(1+Z) + 4x^2e^{W_t}\left(\int_0^t e^{W_s}\,\mathrm{d}s\right) + \dot{W}_tZ(1+Z)^3\right]}{Z(1+Z)^5}.
\end{aligned}$$

By using

$$4xe^{W_t}Z(1+Z) = 4xe^{W_t}Z + 4xe^{W_t} - 8x^2e^{W_t}\Big(\int_0^t e^{W_s}\,\mathrm{d}s\Big)$$

we receive

$$\begin{aligned}
&\frac{1}{2}u(x,t)\frac{\mathrm{d}u}{\mathrm{d}x}(x,t) + u(x,t)\dot{W}_t \\
&= \frac{4x^2e^{W_t}\Big[4xe^{W_t}Z + 4xe^{W_t} - 4x^2e^{W_t}\big(\int_0^t e^{W_s}\,\mathrm{d}s\big) + \dot{W}_tZ(1+Z)^3\Big]}{Z(1+Z)^5} \\
&= \frac{4x^2e^{W_t}\Big[2xe^{W_t}\big(2+2Z-2x\big(\int_0^t e^{W_s}\,\mathrm{d}s\big)\big) + \dot{W}_tZ(1+Z)^3\Big]}{Z(1+Z)^5} \\
&= \frac{4x^2e^{W_t}\Big[2xe^{W_t}(1+Z)^2 + \dot{W}_tZ(1+Z)^3\Big]}{Z(1+Z)^5} \\
&= \frac{4x^2e^{W_t}\Big[2xe^{W_t} + \dot{W}_tZ(1+Z)\Big]}{Z(1+Z)^3}.
\end{aligned}$$

Hence (5.25) holds and u defined by (5.29) is a local solution.

Let us consider (5.15) again. If we choose $h(u) = u^3$ and $B(u) = u^2$, the coefficient functions and hence the local characteristic are not of linear growth. Consequently, an application of the existence result in Theorem 4.5 is not possible so far. But by using the heuristic method introduced in Chapter 3 we have a tool to find a candidate for a solution.

Example 5.7 *Let W_t be a one-dimensional standard Brownian motion. Let us consider by using the Itô-Stratonovich formula*

$$\begin{cases}
\mathrm{d}u(x,t) = u(x,t)^3\,\dfrac{\partial u}{\partial x}(x,t)\,\mathrm{d}t + u(x,t)^2\,\mathrm{d}W_t \\
\qquad = u(x,t)^3\,\dfrac{\partial u}{\partial x}(x,t)\,\mathrm{d}t + u(x,t)^2\circ\mathrm{d}W_t - \dfrac{1}{2}\cdot u(x,t)^2\cdot 2u(x,t)\cdot\dfrac{\partial u}{\partial x}(x,t)\,\mathrm{d}t \\
\qquad = \Big(u(x,t)^3\dfrac{\partial u}{\partial x}(x,t) - u(x,t)^3\dfrac{\partial u}{\partial x}(x,t)\Big)\,\mathrm{d}t + u(x,t)^2\circ\mathrm{d}W_t \\
\qquad = u(x,t)^2\circ\mathrm{d}W_t \\
u(x,0) = x^2.
\end{cases} \tag{5.31}$$

Under the assumption that u solves (5.31) and that the stochastic characteristic curve (ξ_t, η_t, χ_t) as defined in (3.4) exists, we obtain for

$$F(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t) = \eta_t^2\circ\mathrm{d}W_t$$

the system of stochastic characteristic equations given by

$$\begin{aligned}
\mathrm{d}\xi_t &= 0 \\
\mathrm{d}\eta_t &= \eta_t^2\circ\mathrm{d}W_t
\end{aligned}$$

with initial values $\xi_0(x) = x$ and $\eta_0(x) = x^2$. The solutions are given for almost all ω and all (x,t) with t up to exposition time $T(x,\omega)$ by

$$\xi_t(x) = x \quad \text{and} \quad \eta_t(x) = \frac{x^2}{1-x^2W_t}.$$

Hence we obtain a candidate for a solution for almost all ω *and all* (x,t), t *up to explosion time* $T(x,\omega)$ *by*

$$u(x,t) := \frac{x^2}{1 - x^2 W_t}. \tag{5.32}$$

By using Newton's derivative one can easily determine the partial derivative of u *with respect to time to verify (5.31), since*

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \left(\frac{x^2}{1 - x^2 W_t}\right)^2 \dot{W}_t.$$

Hence (5.32) solves (5.31) locally.

6. Application of the method of stochastic characteristics to stochastic transport equations

The main component of this thesis can be found in this chapter in which we apply the method of stochastic characteristics to the stochastic transport equation as given in *An operatorial approach to stochastic partial differential equations driven by linear multiplicative noise* [BR15] of V. Barbu and M. Röckner. If one compares the conditions on the coefficient functions of the stochastic transport equation with the assumptions in Theorem 4.5 above, it is obvious that a direct application of Theorem 6.1.2 in [Kun97] is not possible. Hence we try to solve the problem by using the new heuristic approach of the method of stochastic characteristics. We do this step-by-step. That means we discuss different cases of (6.1) below to see under which assumptions an application is possible or not. As we have seen in the previous Chapter 4 the main advantage of the method is the explicit expression of the local solution, provided that the coefficient functions are given. Therefore, in each subsection we consider a different example with explicit coefficient functions satisfying continuity and continuous differentiability as in [BR15]. In Subsection 6.1. below we take a look at the stochastic transport equation with Stratonovich integral in dimension one and with perturbation by a standard Brownian motion (see Lemma 6.3 and Example 6.4 below). This subsection will end with an extended result for d-dimensional space variables and perturbation by a series of independent copies of standard Brownian motions, see Lemma 6.5 below. In Subsection 6.2. below we discuss the dependence on the general infinite-dimensional Wiener process as defined below in Definition 6.1. Hence, we look at an equation without drift terms, i.e. an equation of the form

$$\mathrm{d}u = u \circ \mathrm{d}\mathbb{W}.$$

For this kind of equations we formulate and prove an existence result in Theorem 6.6 below and give a detailed derivation. Additionally, we consider explicit orthonormal bases, namely the trigonometrical bases on $L^2([0,1])$ as well as on $L^2([0,\pi])$, see Example 6.8 and Example 6.9 below. We will see (cf. Example 6.10 below) that we are not able to combine the examples of Subsection 6.1. with our result in Subsection 6.2. In the third subsection we look at the original stochastic partial differential equation (6.1) below with Itô differential and rewrite it in terms of the Stratonovich differential by using the Itô-Stratonovich formula. Here an application of the heuristic method of stochastic characteristics is not possible, too (see Example 6.12 below). The crucial point is in the Itô-Stratonovich dilemma as we will see later.

Let $\mathbb{O} \subset \mathbb{R}^d$ be an open and bounded set with smooth boundary $\partial\mathbb{O}$. Let $\phi_i, \psi \colon \bar{\mathbb{O}} \times [0, \mathbf{T}] \to \mathbb{R}$ be continuous functions with $\nabla_x \phi_i \in \mathcal{C}(\bar{\mathbb{O}} \times [0,\mathbf{T}], \mathbb{R}^d)$ for $i = 1, ..., d$. The stochastic transport equation is a stochastic first order hyperbolic equation on $\mathbb{O} \times [0, \mathbf{T})$ of the form

$$\begin{cases} \mathrm{d}u(x,t) = \sum_{i=1}^{d} \phi_i(x,t) \dfrac{\partial u}{\partial x_i}(x,t)\,\mathrm{d}t - \psi(x,t)u(x,t)\,\mathrm{d}t \\ \qquad\qquad - \lambda u(x,t)\,|u(x,t)|^{q-2}\,\mathrm{d}t + u(x,t)\,\mathrm{d}\mathbb{W}(x,t) \\ u(x,0) = h(x). \end{cases} \tag{6.1}$$

for $\lambda > 0$, $q \geq 2$ and initial function $h \in \mathcal{C}^2(\mathbb{O})$. $\mathbb{W}$ is given as in the next definition.

Definition 6.1 *A Wiener process* $\mathbb{W}$ *on a real separable Hilbert space* H *is defined by*

$$\mathbb{W}(x,t) := \sum_{j=1}^{\infty} \mu_j e_j(x) W_t^j, \tag{6.2}$$

for all $x \in \mathbb{O}$, $t \geq 0$, *where for all* $j = 1, 2, ...$

- W_t^j *is an independent system of real-valued Brownian motions on* $(\Omega, \mathscr{F}, P)$ *with normal filtration* $(\mathscr{F}_t)_{t\geq 0}$
- $e_j \in \mathcal{C}^2(\bar{\mathbb{O}}, \mathbb{R}) \cap H$ *is an orthonormal basis in* H *and*
- $\mu_j \in \mathbb{R}$.

We denote by $\|\cdot\|_\infty$ the sup norm.

Assumption 6.2 *We assume that there exist* $\tilde{\gamma}_j \in [1, \infty)$, $j = 1, 2, ...$, *such that for all* $y \in H$

$$\|ye_j\|_H \leq \tilde{\gamma}_j \|e_j\|_\infty \|y\|_H$$

and

$$\sum_{j=1}^{\infty} \mu_j^2 \tilde{\gamma}_j^2 \|e_j\|_\infty^2 < \infty.$$

Let

$$\mu := \frac{1}{2} \sum_{j=1}^{\infty} \mu_j^2 e_j^2$$

be a multiplier in V *and a symmetric multiplier in* H.

One of the main questions in this thesis deals with the comparability of the methods, namely the scaling transform approach and the method of stochastic characteristics applied to (6.1). In Chapter 7 we will show that the stochastic transport equation has a unique (global) solution using the scaling transform approach. Hence we will have a closer look on the general setting and further conditions on (6.1).

6.1. The stochastic transport equation with Stratonovich differential of standard Brownian motion

We consider $H = \mathbb{R}$ and a standard Brownian motion as a special case of a Wiener process given in Definition 6.1. Hence we solve problem (6.1) in the case of $\mu_1 := 1$, $\mu_k := 0$ for all $k > 1$ and $e_1(x) = 1$, $e_k(x) = 0$ for all $k > 1$. Let W_t be a standard Brownian motion, hence we obtain for (6.2)

$$\mathbb{W}(t) = W_t.$$

Furthermore, we look at the Stratonovich differential instead of the Itô differential as written in (6.1).

Lemma 6.3 *Let* $\lambda > 0$ *and* $q \geq 2$. *Consider the one-dimensional stochastic transport equation given by*

$$\begin{cases} \mathrm{d}u(x,t) = \phi_1(x,t)\nabla u(x,t)\,\mathrm{d}t - \psi(x,t)u(x,t)\,\mathrm{d}t \\ \qquad\qquad - \lambda u(x,t)\,|u(x,t)|^{q-2}\,\mathrm{d}t + u(x,t)\circ \mathrm{d}W_t \\ u(x,0) = h(x) \end{cases} \tag{6.3}$$

for $x \in \mathbb{O} \subset \mathbb{R}$ *and* $t \in [0, \boldsymbol{T}]$, *where* W_t *is a standard Brownian motion in one dimension. Then the solutions to the stochastic characteristic equations (SCE) are the solutions to the following integral equations for almost all* ω *and* $x \in \mathbb{O}$, $t \in [0, T(x,\omega))$

$$\begin{aligned} \xi_t(x) &= x - \int_0^t \phi_1(\xi_s(x), s)\,\mathrm{d}s, \\ \eta_t(x) &= \frac{e^{W_t} e^{-\int_0^t \psi(\xi_r(x),r)\,\mathrm{d}r}}{\left(|h(x)|^{-(q-2)} + \lambda(q-2)\int_0^t e^{(q-2)\left[W_s - \int_0^s \psi(\xi_r(x),r)\,\mathrm{d}r\right]}\,\mathrm{d}s\right)^{\frac{1}{q-2}}}. \end{aligned} \tag{6.4}$$

Proof. We assume that u solves (6.3) and that there exists the stochastic characteristic curve (ξ_t, η_t, χ_t) defined by (3.4) in Chapter 3. The corresponding stochastic characteristic equations (SCE) of the semimartingale

$$F(\xi_t, \eta_t, \chi_t, \circ \mathrm{d}t) = \big(\phi_1(\xi_t, t)\chi_t - \psi(\xi_t, t)\eta_t - \lambda\eta_t|\eta_t|^{q-2}\big)\,\mathrm{d}t + \eta_t \circ \mathrm{d}W_t$$

are given by

$$\begin{aligned}
\mathrm{d}\xi_t &= -F_{\chi_t}(\xi_t, \eta_t, \chi_t, \circ \mathrm{d}t)\\
&= -\phi_1(\xi_t, t)\,\mathrm{d}t\\
\mathrm{d}\eta_t &= F(\xi_t, \eta_t, \chi_t, \circ \mathrm{d}t) - \chi_t \cdot F_{\chi_t}(\xi_t, \eta_t, \chi_t, \circ \mathrm{d}t)\\
&= \big[\phi_1(\xi_t, t)\chi_t - \psi(\xi_t, t)\eta_t - \lambda\eta_t|\eta_t|^{q-2} - \phi_1(\xi_t, t)\chi_t\big]\,\mathrm{d}t + \eta_t \circ \mathrm{d}W_t\\
&= \big(- \psi(\xi_t, t)\eta_t - \lambda\eta_t|\eta_t|^{q-2}\big)\,\mathrm{d}t + \eta_t \circ \mathrm{d}W_t,
\end{aligned} \tag{6.5}$$

where we make use of the representation result Theorem 2.39. The corresponding initial conditions are

$$\begin{aligned}
\xi_0(x) &= x,\\
\eta_0(x) &= h(x).
\end{aligned}$$

Under the assumption on continuity of ϕ_1 Theorem 1.3 in [CL55] yields that there exists a solution given by an integral equation. Furthermore, $\nabla_x\phi_1 \in \mathcal{C}$ implies Lipschitz continuity since we are acting on $(\bar{\mathbb{O}} \times [0, \mathbf{T}])$. Theorem 2.2 in [CL55] shows that the stochastic characteristic equation $\mathrm{d}\xi_t = -\phi_1(\xi_t, t)\,\mathrm{d}t$ with initial condition $\xi_0(x) = x$ has a unique solution given by the integral equation

$$\xi_t(x) = x - \int_0^t \phi_1(\xi_s(x), s)\,\mathrm{d}s.$$

Now we solve the system of stochastic differential equations (6.5) up to an explosion time $T(x)$. We achieve by an equivalence transformation

$$\frac{\mathrm{d}\eta_t}{\mathrm{d}t} = \big(- \psi(\xi_t, t)\eta_t - \lambda\eta_t|\eta_t|^{q-2}\big) + \eta_t\frac{\circ \mathrm{d}W_t}{\mathrm{d}t}.$$

Rewritten in Newton's notation ($\dot{W}_t = \frac{\circ \mathrm{d}W_t}{\mathrm{d}t}$) for the time derivative we conclude

$$\dot{\eta}_t = \big(- \psi(\xi_t, t)\eta_t - \lambda\eta_t|\eta_t|^{q-2}\big) + \eta_t\dot{W}_t,$$

which is equivalent to

$$\frac{1}{\eta_t}\dot{\eta}_t = -\psi(\xi_t, t) - \lambda|\eta_t|^{q-2} + \dot{W}_t$$

and we finally obtain

$$\frac{1}{\eta_t}\dot{\eta}_t - \dot{W}_t + \psi(\xi_t, t) = -\lambda|\eta_t|^{q-2}. \tag{6.6}$$

Now we define

$$\nu_t := \ln(\eta_t) - W_t + \int_0^t \psi(\xi_s, s)\,\mathrm{d}s, \tag{6.7}$$

hence

$$\dot{\nu}_t = \frac{1}{\eta_t}\dot{\eta}_t - \dot{W}_t + \psi(\xi_t, t)$$

is equal to the left hand side of equation (6.6). It is clear that

$$\begin{aligned} \nu_0 &= \ln(|h(x)|), \\ |\eta_t| &= \exp\Big(\nu_t + W_t - \int_0^t \psi(\xi_s, s)\,\mathrm{d}s\Big) \end{aligned} \tag{6.8}$$

hold. By substitution (6.7) we achieve

$$\dot{\nu}_t = -\lambda|\eta_t|^{q-2} = -\lambda e^{(q-2)\nu_t} e^{(q-2)W_t} e^{-(q-2)\int_0^t \psi(\xi_s,s)\,\mathrm{d}s},$$

which is equivalent to

$$-(q-2)\dot{\nu}_t e^{-(q-2)\nu_t} = \lambda(q-2)e^{(q-2)W_t} e^{-(q-2)\int_0^t \psi(\xi_s,s)\,\mathrm{d}s}$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}[e^{-(q-2)\nu_t}] = \lambda(q-2)e^{(q-2)W_t} e^{-(q-2)\int_0^t \psi(\xi_s,s)\,\mathrm{d}s}.$$

Due to the fundamental theorem of calculus the above equation is equivalent to

$$e^{-(q-2)\nu_t} = e^{-(q-2)\nu_0} + \lambda(q-2)\int_0^t e^{(q-2)W_s} e^{-(q-2)\int_0^s \psi(\xi_r,r)\,\mathrm{d}r}\,\mathrm{d}s.$$

Combining the definition of ν_t and (6.8) this leads to

$$\begin{aligned} &|\eta_t|^{-(q-2)} e^{(q-2)W_t} e^{-(q-2)\int_0^t \psi(\xi_r,r)\,\mathrm{d}r} \\ &\qquad = |h(x)|^{-(q-2)} + \lambda(q-2)\int_0^t e^{(q-2)W_s} e^{-(q-2)\int_0^s \psi(\xi_r,r)\,\mathrm{d}r}\,\mathrm{d}s \end{aligned}$$

and hence

$$\begin{aligned} |\eta_t|^{-(q-2)} &= e^{-(q-2)W_t} e^{(q-2)\int_0^t \psi(\xi_r,r)\,\mathrm{d}r} \\ &\quad \cdot \Big(|h(x)|^{-(q-2)} + \lambda(q-2)\int_0^t e^{(q-2)W_s} e^{-(q-2)\int_0^s \psi(\xi_r,r)\,\mathrm{d}r}\,\mathrm{d}s\Big). \end{aligned}$$

Finally, we get for almost all ω and $x \in \mathbb{O}$, $t \in [0, T(x,\omega))$,

$$\eta_t(x) = \frac{e^{W_t} e^{-\int_0^t \psi(\xi_r(x),r)\,\mathrm{d}r}}{\Big(|h(x)|^{-(q-2)} + \lambda(q-2)\int_0^t e^{(q-2)[W_s - \int_0^s \psi(\xi_r(x),r)\,\mathrm{d}r]}\,\mathrm{d}s\Big)^{\frac{1}{q-2}}},$$

where $T(x)$ is the explosion time defined by

$$T(x) := \inf\left\{ t \in [0, \mathbf{T}) \,\middle|\, \lambda(q-2)\int_0^t e^{(q-2)[W_s - \int_0^s \psi(\xi_r(x),r)\,\mathrm{d}r]}\,\mathrm{d}s = -|h(x)|^{-(q-2)} \right\}.$$

□

One advantage of the method of stochastic characteristics is that one obtains an explicit solution provided that proper coefficient functions and initial conditions are given. Therefore, we look at a one-dimensional stochastic transport equation with simple but concrete drift functions ϕ_1, ψ, initial function $h(x) = x^2$ and standard one-dimensional Brownian motion W_t. Let $\mathbb{O} = [0,1]$ and ϕ_1 be the identity map.

Example 6.4 *In the special case of $\phi_1(x,t) = x$ and $\psi(x,t) = 0$ there exists a solution to the following one-dimensional stochastic transport equation given by*

$$\begin{cases} \mathrm{d}u(x,t) = \Big(x\,\nabla u(x,t) - \lambda u(x,t)\,|u(x,t)|^{q-2}\Big)\,\mathrm{d}t + u(x,t)\circ \mathrm{d}W_t \\ u(x,0) = x^2. \end{cases} \tag{6.9}$$

By separation of variables we determine the solution $\xi_t(x) = xe^{-t}$, which solves the stochastic differential equation

$$\frac{\mathrm{d}\xi_t}{\mathrm{d}t}(x) = -\xi_t(x).$$

Obviously, the inverse process is given for all $x \in [0,1]$, $t \in [0, T(x))$ by

$$\xi_t^{-1}(x) = x\,e^t,$$

where

$$T(x) := \inf\Big\{t > 0\,\Big|\, - x\,e^t = 0\Big\} \wedge \boldsymbol{T} = \boldsymbol{T}.$$

Due to Lemma 6.3 we additionally obtain

$$\eta_t(x) = \Big(\exp(W_t)\Big)\left(\left(x^2\right)^{-(q-2)} + \lambda(q-2)\int_0^t e^{(q-2)W_s}\,\mathrm{d}s\right)^{-\frac{1}{q-2}}.$$

Hence we define for almost all ω and $x \in [0,1]$, $t \in [0, T(x,\omega))$

$$u(x,t) = \Big(\exp(W_t)\Big)\left(\left(x^2e^{2t}\right)^{-(q-2)} + \lambda(q-2)\int_0^t e^{(q-2)W_s}\,\mathrm{d}s\right)^{-\frac{1}{q-2}} \tag{6.10}$$

to be a candidate for a local solution to (6.9). The initial condition for $t = 0$ is fulfilled, since

$$u(x,0) = \Big(\exp(W_0)\Big)\left(\left(x^2\right)^{-(q-2)} + \lambda(q-2)\int_0^0 e^{(q-2)W_s}\,\mathrm{d}s\right)^{-\frac{1}{q-2}} = x^2.$$

Now we have to verify that (6.10) is really a solution to (6.9). For the reader's convenience we calculate the partial derivatives in Lemma D.7. Due to these results with

$$N := \left(x^2e^{2t}\right)^{-(q-2)} + \lambda(q-2)\int_0^t e^{(q-2)W_s}\,\mathrm{d}s$$

we have

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t}(x,t) &= u(x,t)\,\dot W_t + u(x,t)\left(\frac{2(x^2e^{2t})^{-(q-2)} - \lambda e^{(q-2)W_t}}{N}\right) \\
&= u(x,t)\left(\frac{2(x^2e^{2t})(x^2e^{2t})^{-(q-1)} - \lambda e^{(q-2)W_t}}{N}\right) + u(x,t)\,\dot W_t \\
&= x\,u(x,t)\left(\frac{2(xe^{2t})(x^2e^{2t})^{-(q-1)}}{N}\right) - \lambda\,u(x,t)\left(\frac{e^{W_t}}{N^{\frac{1}{q-2}}}\right)^{q-2} + u(x,t)\,\dot W_t \\
&= x\,u(x,t)\left(\frac{(2xe^{2t})(x^2e^{2t})^{-(q-1)}}{N}\right) - \lambda\,u(x,t)\,|u(x,t)|^{q-2} + u(x,t)\,\dot W_t \\
&= x\,\nabla u(x,t) - \lambda\,u(x,t)\,|u(x,t)|^{q-2} + u(x,t)\,\dot W_t
\end{aligned}$$

and hence that u solves equation (6.9).

For the sake of completeness we extend Lemma 6.3 to the d-dimensional case and perturbation by a series of independent copies of a Brownian motion. We note that we are no longer in the situation of Definition 6.1.

Lemma 6.5 *Consider the d-dimensional problem of (6.3) given by*

$$\begin{cases} \mathrm{d}u(x,t) = \sum_{i=1}^{d} \phi_i(x,t) \frac{\partial u}{\partial x_i}(x,t)\,\mathrm{d}t - \psi(x,t)u(x,t)\,\mathrm{d}t \\ \qquad\qquad - \lambda u(x,t)\,|u(x,t)|^{q-2}\,\mathrm{d}t + \sum_{j=1}^{\infty} u(x,t) \circ \mathrm{d}W_t^j \\ u(x,0) = h(x), \\ \qquad\qquad \text{for } x \in \mathbb{O}. \end{cases} \tag{6.11}$$

Under the assumption that u solves (6.11) and that there exists the stochastic characteristic curve (ξ_t, η_t, χ_t), given as in equation (3.4), the solutions to the stochastic characteristic equations have to satisfy the following integral equations for almost all ω and $x \in \mathbb{O}$ with $x = (x_1, ..., x_d)$ and $t \in [0, T(x,\omega))$

$$\begin{aligned} \xi_t^1(x) &= x_1 - \int_0^t \phi_1(\xi_s(x), s)\,\mathrm{d}s, \\ &\vdots \\ \xi_t^1(x) &= x_d - \int_0^t \phi_d(\xi_s(x), s)\,\mathrm{d}s, \\ \eta_t(x) &= \frac{\exp\Big(\sum_{j=1}^{\infty} W_t^j\Big)\exp\Big(\int_0^t \psi(\xi_r(x), r)\,\mathrm{d}r\Big)}{\left(|h(x)|^{-(q-2)} + \lambda(q-2)\int_0^t e^{(q-2)\left[\sum_{j=1}^{\infty} W_s^j - \int_0^s \psi(\xi_r(x),r)\,\mathrm{d}r\right]}\,\mathrm{d}s\right)^{\frac{1}{q-2}}}. \end{aligned} \tag{6.12}$$

Proof. By the same technique as in Lemma 6.3 we obtain that the corresponding stochastic characteristic equations for

$$F(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t) = \Big(\sum_{i=1}^{d} \phi_i(\xi_t, t)\chi_t^i - \psi(\xi_t, t)\,\eta_t - \lambda\eta_t\,|\eta_t|^{q-2}\Big)\,\mathrm{d}t + \sum_{j=1}^{\infty} \eta_t \circ \mathrm{d}W_t^j$$

are given by

$$\begin{aligned} \mathrm{d}\xi_t^1 &= -F_{\chi_t^1}(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t) = -\phi_1(\xi_t, t)\,\mathrm{d}t \\ &\vdots \\ \mathrm{d}\xi_t^d &= -F_{\chi_t^d}(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t) = -\phi_d(\xi_t, t)\,\mathrm{d}t \\ \mathrm{d}\eta_t &= F(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t) - \sum_{i=1}^{d} \chi_t^i \cdot F_{\chi_t^i}(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t) \\ &= \big(-\psi(\xi_t, t)\eta_t - \lambda\eta_t|\eta_t|^{q-2}\big)\,\mathrm{d}t + \sum_{j=1}^{\infty} \eta_t \circ \mathrm{d}W_t^j. \end{aligned} \tag{6.13}$$

Under the assumption on continuity of ϕ_i and Lipschitz continuity the results Theorem 1.3 and Theorem 2.2. in [CL55] are also applicable for systems as written in [CL55, Chapter 1, Section 5]. And we conclude that the stochastic characteristic equations $\mathrm{d}\xi_t^i = -\phi_i(\xi_t, t)\,\mathrm{d}t$

with initial condition $\xi_0^i(x) = x_i$ have unique solutions given by the integral equations

$$\xi_t^i(x) = x_i - \int_0^t \phi_i(\xi_s(x), s)\,\mathrm{d}s, \text{ for } i = 1, ..., d.$$

By a similar calculation as in the proof of Lemma 6.3 we end up with

$$\eta_t(x) = \frac{\exp\Big(\sum_{j=1}^{\infty} W_t^j\Big)\exp\Big(\int_0^t \psi(\xi_r(x), r)\,\mathrm{d}r\Big)}{\left(|h(x)|^{-(q-2)} + \lambda(q-2)\int_0^t e^{(q-2)\left[\sum_{j=1}^{\infty} W_s^j - \int_0^s \psi(\xi_r(x), r)\,\mathrm{d}r\right]}\mathrm{d}s\right)^{\frac{1}{q-2}}}.$$

□

6.2. Application to general infinite-dimensional Wiener processes

Up to now we considered series of standard Brownian motions. Now we investigate what happens if we perturb an equation by a general infinite-dimensional Wiener process on a Hilbert space H as given in Definition 6.1. Let us look at the following differential equation

$$\begin{cases} \mathrm{d}u = u \circ \mathrm{d}\mathbb{W} \\ u(x, 0) = h(x) \end{cases} \tag{6.14}$$

for $x \in \mathbb{O}$ and initial function h on $\mathbb{O}$ with values in $\mathbb{R}$. Due to Definition 6.1 the problem is equivalent to

$$\begin{cases} \mathrm{d}u = \sum_{j=1}^{\infty} \mu_j\, e_j(x)\, u \circ \mathrm{d}W_t^j \\ u(x, 0) = h(x). \end{cases} \tag{6.15}$$

Equation (6.14) has no drift term which simplifies the corresponding stochastic differential equations if we apply the heuristic method of stochastic characteristics as we will see in the proof of the following existence result.

Theorem 6.6 *Let $\mathbb{W}$ be a general Wiener process in the sense of Definition 6.1 satisfying Assumption 6.2. Then a local solution to (6.14) and (6.15), respectively, is given for almost all ω and all $x \in \mathbb{O}$ with $t \in [0, T(x, \omega))$ by*

$$u(x, t) = h(x)\exp\left(\sum_{j=1}^{\infty} \mu_j e_j(x) W_t^j\right). \tag{6.16}$$

Before we prove the theorem, let us see how to deduce the expression of the solution (6.16).

Derivation
Obviously, we apply the heuristic method of stochastic characteristics as introduced in Chapter 3. Under the assumption that u solves (6.15) and that (ξ_t, η_t, χ_t) is the stochastic characteristic curve, we obtain due to the representation result Theorem 2.39

$$F(\xi_t, \eta_t, \chi_t, \circ\mathrm{d}t) = \sum_{j=1}^{\infty} \mu_j\, e_j(\xi_t)\, \eta_t \circ \mathrm{d}W_t^j.$$

Consequently, we look at the corresponding stochastic characteristic equations with initial conditions $\xi_0(x) = x$ *and* $\eta_0(x) = h(x)$*, which are given by*

$$\begin{aligned} &\mathrm{d}\xi_t = 0 \\ \textit{and} \quad &\mathrm{d}\eta_t = \sum_{j=1}^{\infty} \mu_j e_j(\xi_t)\eta_t \circ \mathrm{d}W_t^j \\ &\quad = \sum_{j=1}^{\infty} \mu_j e_j(x)\eta_t \circ \mathrm{d}W_t^j. \end{aligned}$$

Evidently, we receive the trivial solution $\xi_t(x) = x$*. By using Newton's derivative we determine by an equivalence transformation*

$$\begin{aligned} & \dot{\eta}_t &=& \sum_{j=1}^{\infty} \mu_j e_j(x)\eta_t \dot{W}_t^j \\ \Leftrightarrow \quad & \frac{1}{\eta_t}\dot{\eta}_t &=& \sum_{j=1}^{\infty} \mu_j e_j(x)\dot{W}_t^j \\ \Leftrightarrow \quad & \frac{1}{\eta_t}\dot{\eta}_t - \sum_{j=1}^{\infty} \mu_j e_j(x)\dot{W}_t^j &=& 0 \\ \Leftrightarrow \quad & \frac{\mathrm{d}}{\mathrm{d}t}\left[\ln(\eta_t) - \sum_{j=1}^{\infty} \mu_j e_j(x)W_t^j\right] &=& 0 \\ \Leftrightarrow \quad & \ln(\eta_t) &=& \sum_{j=1}^{\infty} \mu_j e_j(x)W_t^j + \ln(h(x)) \\ \Leftrightarrow \quad & \eta_t(x) &=& h(x)\exp\left(\sum_{j=1}^{\infty} \mu_j e_j(x)W_t^j\right). \end{aligned}$$

The inverse process ξ_t^{-1} *of* $\xi_t(x) = x$ *is naturally equal to*

$$\xi_t^{-1}(x) = x.$$

Hence a candidate for a local solution to (6.15) can be defined for almost all ω *and* $x \in \mathbb{O}$ *with* $t \in [0, T(x,\omega))$ *by*

$$\begin{aligned} u(x,t) &= \eta_t(\xi_t^{-1}(x)) \\ &= h(x)\exp\left(\sum_{j=1}^{\infty} \mu_j e_j(x)W_t^j\right), \end{aligned} \tag{6.17}$$

where $T(x)$ *is the explosion time of* η_t*. The stopping times as defined in Definition 4.4 are given by*

$$\begin{aligned} \tau_{\text{inv}}(x) &= \inf\{t > 0 \mid \det D\xi_t(x) = 0\} \\ &= \inf\{t > 0 \mid \det \mathbb{I} = 0\} = \infty \\ \tau(x) &= T(x). \end{aligned}$$

Now we have to prove that (6.17) is a solution to (6.15), i.e. we have to show that

$$\frac{\mathrm{d}u}{\mathrm{d}t}(x,t) = \sum_{j=1}^{\infty} \mu_j e_j(x)u(x,t)\dot{W}_t^j \tag{6.18}$$

holds.

Proof of Theorem 6.6 We have

$$\frac{\mathrm{d}u}{\mathrm{d}t}(x,t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[h(x)\exp\left(\sum_{j=1}^{\infty} \mu_j e_j(x)W_t^j\right)\right]$$

$$= h(x)\left(\sum_{j=1}^{\infty}\mu_j e_j(x)\dot{W}_t^j\right)\exp\left(\sum_{j=1}^{\infty}\mu_j e_j(x)W_t^j\right)$$
$$= \sum_{j=1}^{\infty}\mu_j e_j(x)\left(h(x)\exp\left(\sum_{j=1}^{\infty}\mu_j e_j(x)W_t^j\right)\right)\dot{W}_t^j,$$

and therefore the statement is valid.

□

Now we apply Theorem 6.6 to the stochastic transport equation (6.1) acting on $H = L^2(\mathbb{O})$. Let us consider the one-dimensional case with $\mathbb{O} \subset \mathbb{R}$ and the following orthonormal bases which are in particular continuously differentiable.

Theorem 6.7 *The following collections are orthonormal bases on $L^2(\mathbb{O})$:*

$$\left\{\sqrt{\frac{2}{\pi}}\sin(jx)\right\}_{j\geq 1} \quad \textit{for } \mathbb{O} = [0,\pi], \tag{6.19}$$

$$\left\{\sqrt{2}\sin(j\pi x)\right\}_{j\geq 1} \quad \textit{for } \mathbb{O} = [0,1]. \tag{6.20}$$

Statement (6.19) is stated and proved in [RY08, Theorem 3.56] and result (6.20) including the proof corresponds to Theorem 4.1 in [HW96]. For $\mu_j \in \mathbb{R}$, $j \geq 1$, satisfying

$$\sum_{j=1}^{\infty}\frac{4}{\pi^2}\mu_j^2 < \infty \tag{6.21}$$

Assumption 6.2 is fulfilled for the orthonormal basis (6.19). For the reader's convenience it is proved in Lemma D.8. Under the assumption

$$\sum_{j=1}^{\infty} 8\,\mu_j^2 < \infty \tag{6.22}$$

on $\mu_j \in \mathbb{R}$, $j \geq 1$, the analogous result is stated in Lemma D.9 for the orthonormal basis (6.20).

Example 6.8 *Let $\mathbb{W}$ be a general infinite-dimensional Wiener process on $L^2([0,\pi])$ with orthonormal basis $e_j(x)$, $j \geq 1$, given by (6.19). Let $\mu_j \in \mathbb{R}$, $j \geq 1$, satisfy (6.21). Then a local solution to*

$$\begin{cases} \mathrm{d}u = u \circ \mathrm{d}\mathbb{W} = \sum_{j=1}^{\infty}\sqrt{\frac{2}{\pi}}\,\mu_j\,\sin(jx)\,u \circ \mathrm{d}W_t^j \\ u(x,0) = h(x) \end{cases}$$

is given for almost all ω and $x \in [0,\pi]$ with $t \in [0, T(x,\omega))$ by

$$u(x,t) = h(x)\exp\left(\sum_{j=1}^{\infty}\sqrt{\frac{2}{\pi}}\,\mu_j\,\sin(jx)\,W_t^j\right). \tag{6.23}$$

Obviously, the above example as well as the following one are applications of Theorem 6.6. $T(x)$ denotes the explosion time of η_t as written in the proof of Theorem 6.6.

Example 6.9 *Let $\mathbb{W}$ be a general infinite-dimensional Wiener process on $L^2([0,1])$ with orthonormal basis $e_j(x)$, $j \geq 1$, given by (6.20). Let $\mu_j \in \mathbb{R}$, $j \geq 1$, satisfy (6.22). Then a local solution to*

$$\begin{cases} \mathrm{d}u = u \circ \mathrm{d}\mathbb{W} = \sum_{j=1}^{\infty}\sqrt{2}\,\mu_j\,\sin(j\pi x)\,u \circ \mathrm{d}W_t^j \\ u(x,0) = h(x) \end{cases}$$

is given for almost all ω *and* $x \in [0,1]$ *with* $t \in [0, T(x,\omega))$ *by*

$$u(x,t) = h(x) \exp\left(\sum_{j=1}^{\infty} \sqrt{2}\, \mu_j \, \sin(j\pi x)\, W_t^j \right). \tag{6.24}$$

In the previous part we have seen some examples where the method of stochastic characteristics is applicable and leads to an explicit expression of solutions. Now we try to combine Example 6.4 and Example 6.9.

Example 6.10 *Let* $\mathbb{W}$ *be a general infinite-dimensional Wiener process on* $L^2([0,1])$ *with orthonormal basis* $e_j(x)$, $j \geq 1$, *given by (6.20). Let* $\mu_j \in \mathbb{R}$, $j \geq 1$, *satisfy (6.22). Consider for* $x \in [0,1]$

$$\begin{cases} \mathrm{d}u = \left(x \,\nabla u - \lambda\, u|u|^{q-2} \right)\, \mathrm{d}t + u \circ \mathrm{d}\mathbb{W} \\ \quad\;\; = \left(x \,\nabla u - \lambda\, u|u|^{q-2} \right)\, \mathrm{d}t + \sum_{j=1}^{\infty} \sqrt{2}\, \mu_j \, \sin(j\, \pi\, x) u \circ \mathrm{d}W_t^j \\ u(x,0) = h(x). \end{cases} \tag{6.25}$$

Assume that u *is a solution to (6.25) and that there exists the stochastic characteristic curve* (ξ_t, η_t, χ_t). *Let us define*

$$F(\xi_t, \eta_t, \chi_t, \circ \mathrm{d}t) := \left(\xi_t \, \chi_t - \lambda \, \eta_t |\eta_t|^{q-2} \right)\, \mathrm{d}t + \sum_{j=1}^{\infty} \sqrt{2}\, \mu_j \, \sin(j\, \pi\, \xi_t) \eta_t \circ \mathrm{d}W_t^j.$$

Then the corresponding stochastic characteristic equations are given by

$$\begin{aligned} \mathrm{d}\xi_t &= -\xi_t \,\mathrm{d}t, \\ \xi_0(x) &= x \end{aligned} \tag{6.26}$$

and

$$\begin{aligned} \mathrm{d}\eta_t &= \left(-\lambda\, \eta_t |\eta_t|^{q-2} \right) \mathrm{d}t + \sum_{j=1}^{\infty} \sqrt{2}\, \mu_j \, \sin(j\, \pi\, \xi_t) \eta_t \circ \mathrm{d}W_t^j, \\ \eta_0(x) &= h(x). \end{aligned} \tag{6.27}$$

Due to Example 6.4 we know that the solution to (6.26) is given for all $x \in [0,1]$, $t \in [0, \boldsymbol{T})$ *by*

$$\xi_t(x) = x\, e^{-t}.$$

Now we try to determine the solution to (6.27) in the same way as before and obtain by using Newton's derivative

$$\frac{1}{\eta_t} \dot{\eta}_t = -\lambda |\eta_t|^{q-2} + \sum_{j=1}^{\infty} \sqrt{2}\, \mu_j \, \sin\left(j\, \pi\, x\, e^{-t} \right) \dot{W}_t^j,$$

which is equivalent to

$$\frac{1}{\eta_t} \dot{\eta}_t - \sum_{j=1}^{\infty} \sqrt{2}\, \mu_j \, \sin\left(j\, \pi\, x\, e^{-t} \right) \dot{W}_t^j = -\lambda |\eta_t|^{q-2}.$$

With our techniques there is no chance to find the primitive function of the term $\sin\left(j\, \pi\, x\, e^{-t} \right) \dot{W}_t^j$. *Hence the method of stochastic characteristics is not applicable in the mixed Example 6.10 at the moment.*

In the previous example we have seen that we are not able to solve the stochastic transport equation (6.25) by the method of stochastic characteristics. In particular a time-dependence of solution ξ_t causes problems. To see this we look at the general case of the one-dimensional stochastic transport equation with Stratonovich differential given by

$$\begin{cases} \mathrm{d}u = \left(\phi_1(x,t)\nabla u - \psi(x,t)u - \lambda u\,|u|^{q-2}\right)\,\mathrm{d}t + \sum_{j=1}^{\infty} \mu_j\, e_j(x)\, u \circ \mathrm{d}W_t^j \\ u(x,0) = h(x). \end{cases} \tag{6.28}$$

Provided that u solves (6.28) and that there exists the stochastic characteristic curve (ξ_t, η_t, χ_t), we end up with the stochastic characteristic equations

$$\begin{aligned} \mathrm{d}\xi_t &= -\phi_1(\xi_t,t)\,\mathrm{d}t \\ \mathrm{d}\eta_t &= \left(-\psi(\xi_t,t)\eta_t - \lambda\eta_t\,|\eta_t|^{q-2}\right)\,\mathrm{d}t + \sum_{j=1}^{\infty} \mu_j\, e_j(\xi_t)\, \eta_t \circ \mathrm{d}W_t^j \end{aligned} \tag{6.29}$$

as we have seen before. This system of stochastic differential equations (6.29) can only be solved if $\mathrm{d}\xi_t = -\phi_1(\xi_t,t)\,\mathrm{d}t$ generates a time-independent solution. Due to the initial condition $\xi_0(x) = x$ the only time-independent solution is $\xi_t(x) = x$ which implies $\phi_1(x,t) = 0$. In this special case we are able to determine the solution of

$$\begin{aligned} \mathrm{d}\eta_t &= \left(-\psi(\xi_t,t)\eta_t - \lambda\eta_t\,|\eta_t|^{q-2}\right)\,\mathrm{d}t + \sum_{j=1}^{\infty} \mu_j\, e_j(\xi_t)\, \eta_t \circ \mathrm{d}W_t^j \\ &= \left(-\psi(x,t)\eta_t - \lambda\eta_t\,|\eta_t|^{q-2}\right)\,\mathrm{d}t + \sum_{j=1}^{\infty} \mu_j\, e_j(x)\, \eta_t \circ \mathrm{d}W_t^j. \end{aligned}$$

As seen many times before the equation is equivalent to

$$\frac{1}{\eta_t}\dot{\eta}_t + \psi(x,t) - \sum_{j=1}^{\infty} \mu_j\, e_j(x)\dot{W}_t^j = -\lambda\,|\eta_t|^{q-2}.$$

In particular we find a substitution for the term $\sum_{j=1}^{\infty} \mu_j\, e_j(x)\dot{W}_t^j$ as in the proof of Theorem 6.6. This is not the case if we have to solve

$$\frac{1}{\eta_t}\dot{\eta}_t + \psi(\xi_t,t) - \sum_{j=1}^{\infty} \mu_j\, e_j(\xi_t)\dot{W}_t^j = -\lambda\,|\eta_t|^{q-2},$$

where the stochastic characteristic curve ξ_t depends on time t.

6.3. The stochastic transport equation with Itô differential

Up to now we considered the stochastic transport equation with Stratonovich differential. In [BR15] the equation is given with Itô differential and for general Wiener processes as defined in Definition 6.1. Due to the fact that the method of stochastic characteristics is only working with Stratonovich differential we have to rewrite (6.1) by applying the Itô-Stratonovich formula.

Lemma 6.11 *The stochastic partial differential equation (6.1) is equivalent to*

$$\begin{cases} \mathrm{d}u(x,t) = \sum_{i=1}^{d} \phi_i(x,t)\frac{\partial u}{\partial x_i}(x,t)\,\mathrm{d}t - \psi(x,t)u(x,t)\,\mathrm{d}t \\ \qquad - \lambda u(x,t)\,|u(x,t)|^{q-2}\,\mathrm{d}t \\ \qquad - \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{\infty} \mu_j^2 e_j^2(x)u(x,t)\frac{\partial u}{\partial x_i}(x,t)\,\mathrm{d}t \\ \qquad - \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{\infty} \mu_j^2 e_j(x)u^2(x,t)\frac{\partial e_j}{\partial x_i}(x)\,\mathrm{d}t \\ \qquad + \sum_{j=1}^{\infty} \mu_j e_j(x)u(x,t)\circ \mathrm{d}W_t^j \\ u(x,0) = h(x) \end{cases} \tag{6.30}$$

for all $x \in \mathbb{O}$ *and* $t \in [0, \boldsymbol{T}]$.

Proof. We use the Itô-Stratonovich formula in d dimensions. The application of Theorem 2.35 provides

$$\begin{aligned} \mathrm{d}u(x,t) &= \Big(\sum_{i=1}^{d}\phi_i(x,t)\frac{\partial u}{\partial x_i}(x,t) - \psi(x,t)u(x,t) - \lambda u(x,t)\,|u(x,t)|^{q-2}\Big)\,\mathrm{d}t \\ &\quad + \sum_{j=1}^{\infty}\mu_j e_j(x)u(x,t)\,\mathrm{d}W_t^j \\ &= \Big(\sum_{i=1}^{d}\phi_i(x,t)\frac{\partial u}{\partial x_i}(x,t) - \psi(x,t)u(x,t) - \lambda u(x,t)\,|u(x,t)|^{q-2}\Big)\,\mathrm{d}t \\ &\quad + \sum_{j=1}^{\infty}\mu_j e_j(x)u(x,t)\circ\mathrm{d}W_t^j - \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{\infty}\mu_j e_j(x)u(x,t)\frac{\partial[u(x,t)\mu_j e_j(x)]}{\partial x_i}\,\mathrm{d}t \\ &= \Big(\sum_{i=1}^{d}\phi_i(x,t)\frac{\partial u}{\partial x_i}(x,t) - \psi(x,t)u(x,t) - \lambda u(x,t)\,|u(x,t)|^{q-2}\Big)\,\mathrm{d}t \\ &\quad + \sum_{j=1}^{\infty}u(x,t)\mu_j e_j(x)\circ\mathrm{d}W_t^j \\ &\quad - \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{\infty}\Big(\mu_j e_j(x)u(x,t)\Big[\frac{\partial u(x,t)}{\partial x_i}\mu_j e_j(x) + u(x,t)\mu_j\frac{\partial e_j(x)}{\partial x_i}\Big]\Big)\,\mathrm{d}t \\ &= \Big(\sum_{i=1}^{d}\phi_i(x,t)\frac{\partial u}{\partial x_i}(x,t) - \psi(x,t)u(x,t) - \lambda u(x,t)\,|u(x,t)|^{q-2} \\ &\quad - \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{\infty}\mu_j^2 e_j^2(x)u(x,t)\frac{\partial u}{\partial x_i}(x,t) - \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{\infty}\mu_j^2 e_j(x)u^2(x,t)\frac{\partial e_j}{\partial x_i}(x)\Big)\,\mathrm{d}t \\ &\quad + \sum_{j=1}^{\infty}\mu_j e_j(x)u(x,t)\circ\mathrm{d}W_t^j. \end{aligned}$$

□

Example 6.12 *Let* $\mathbb{W}$ *be a general infinite-dimensional Wiener process as in Definition 6.1 on* $H = L^2(\mathbb{O})$ *and suppose that Assumption 6.2 is satisfied. For simplicity we consider* *the stochastic transport equation (6.1) in dimension one with* $\phi_1 = \psi = 0$ *given by*

$$\begin{cases} \mathrm{d}u(x,t) = -\lambda\, u(x,t)\; |u(x,t)|^{q-2}\; \mathrm{d}t + u(x,t)\; \mathrm{d}\mathbb{W} \\ \qquad\qquad = -\lambda\, u(x,t)\, |u(x,t)|^{q-2}\; \mathrm{d}t + \sum_{j=1}^{\infty} \mu_j\, e_j(x)\, u(x,t)\; \mathrm{d}W_t^j \\ u(x,0) = h(x). \end{cases} \tag{6.31}$$

By applying Lemma 6.11 equation (6.31) is equivalent to

$$\begin{cases} \mathrm{d}u(x,t) = -\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{\infty} \mu_j^2 \left(e_j(x)^2\, u(x,t)\, \frac{\partial u}{\partial x_i}(x,t) + e_j(x)\, u^2(x,t)\, \frac{\partial e_j}{\partial x_i}(x) \right) \mathrm{d}t \\ \qquad\qquad - \lambda\, u(x,t)\; |u(x,t)|^{q-2}\; \mathrm{d}t + \sum_{j=1}^{\infty} \mu_j\, e_j(x)\, u(x,t) \circ \mathrm{d}W_t^j \\ u(x,0) = h(x). \end{cases} \tag{6.32}$$

Under the assumption that u solves (6.32) and (ξ_t, η_t, χ_t) *is the corresponding stochastic characteristic curve as defined in Chapter 3 the associated system of stochastic characteristic equations for*

$$\begin{aligned} F(\xi_t, \eta_t, \chi_t, \circ \mathrm{d}t) = {} & -\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{\infty} \mu_j^2 \left(e_j(\xi_t)^2\, \eta_t\, \chi_t^i + e_j(\xi_t)\, (\eta_t)^2\, \frac{\partial e_j}{\partial x_i}(\xi_t) \right) \mathrm{d}t \\ & - \lambda\, \eta_t\; |\eta_t|^{q-2}\; \mathrm{d}t + \sum_{j=1}^{\infty} \mu_j\, e_j(\xi_t)\, \eta_t \circ \mathrm{d}W_t^j \end{aligned}$$

are given by

$$\begin{aligned} \mathrm{d}\xi_t^i &= -F_{\chi_t^i}(\xi_t, \eta_t, \chi_t, \circ \mathrm{d}t) \\ &= \frac{1}{2}\sum_{j=1}^{\infty} \mu_j^2 e_j(\xi_t)^2\, \eta_t\, \mathrm{d}t \quad \textit{for } i = 1, ..., d, \end{aligned} \tag{6.33}$$

and

$$\begin{aligned} \mathrm{d}\eta_t &= F(\xi_t, \eta_t, \chi_t, \circ \mathrm{d}t) - \sum_{i=1}^{d} \chi_t^i F_{\chi_t^i}(\xi_t, \eta_t, \chi_t, \circ \mathrm{d}t) \\ &= \left(-\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{\infty} e_j(x)\, (\eta_t)^2\, \frac{\partial e_j}{\partial x_i}(\xi_t) - \lambda \eta_t\, |\eta_t|^{q-2} \right) \mathrm{d}t \\ &\quad + \sum_{j=1}^{\infty} \mu_j\, e_j(\xi_t)\, \eta_t \circ \mathrm{d}W_t^j. \end{aligned} \tag{6.34}$$

Obviously, (6.33) does not generate a time-independent solution. Hence we are not able to find the solutions (ξ_t, η_t) *and finally solve (6.31) by the method of stochastic characteristics.*

Due to Example 6.12 we conclude that at the moment we have no technique to solve (6.1) by the method of stochastic characteristics.

Remark 6.13 *In subsection 6.2 we have shown that equation (6.14) given by*

$$\begin{cases} \mathrm{d}u = u \circ \mathrm{d}\mathbb{W} \\ \quad\;\; = \sum_{j=1}^{\infty} \mu_j\, e_j\, u \circ \mathrm{d}W_t^j \\ u(x,0) = h(x) \end{cases}$$

for $x \in \mathbb{O} \subset \mathbb{R}^d$ is solvable by the method of stochastic characteristics. Furthermore, a local solution is given for almost all ω and $x \in \mathbb{O}$ with $t \in [0, T(x,\omega))$ by

$$u(x,t) = h(x) \exp\left(\sum_{j=1}^{\infty} \mu_j \, e_j(x) \, W_t^j \right),$$

where $T(x)$ denotes the explosion time. Due to Lemma 6.11 the solution u solves also locally

$$\begin{aligned} \mathrm{d}u &= \left(\frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{\infty} \mu_j^2 \left(e_j^2 \, u \, \frac{\partial u}{\partial x_i} + e_j \, u^2 \, \frac{\partial e_j}{\partial x_i} \right) \right) \mathrm{d}t \\ &\quad + \sum_{j=1}^{\infty} \mu_j \, e_j(x) \, u \, \mathrm{d}W_t^j \\ &= \left(\frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{\infty} \mu_j^2 \left(e_j^2 \, u \, \frac{\partial u}{\partial x_i} + e_j \, u^2 \, \frac{\partial e_j}{\partial x_i} \right) \right) \mathrm{d}t \\ &\quad + u \, \mathrm{d}\mathbb{W}. \end{aligned} \tag{6.35}$$

If we compare equation (6.35) given by

$$\begin{aligned} \mathrm{d}u(x,t) = &\left(\frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{\infty} \mu_j^2 \left(e_j(x)^2 \, u(x,t) \, \frac{\partial u}{\partial x_i}(x,t) + e_j(x) \, u^2(x,t) \, \frac{\partial e_j}{\partial x_i}(x) \right) \right) \mathrm{d}t \\ &+ u(x,t) \, \mathrm{d}\mathbb{W}(x,t) \end{aligned}$$

with equation (6.1)

$$\begin{aligned} \mathrm{d}u(x,t) = &\left(\sum_{i=1}^{d} \phi_i(x,t) \frac{\partial u}{\partial x_i}(x,t) - \psi(x,t) u(x,t) - \lambda u(x,t) \, |u(x,t)|^{q-2} \right) \mathrm{d}t \\ &+ u(x,t) \, \mathrm{d}\mathbb{W}(x,t), \end{aligned}$$

one could come up with the idea to find a proper choose of the coefficient functions ϕ_i, ψ such that (6.35) can be written in terms of (6.1). But this is not possible because in particular the coefficients ϕ_i do not depend on u. Accordingly, we see no chance to solve the general stochastic transport equation (6.1) by the method of stochastic characteristics.

Remark 6.14 *In general a direct application of the main result given in Corollary 4.6 to the stochastic transport equation is not possible. In [BR15] we have the assumptions that the coefficient functions ϕ_i are continuously differentiable in space and ψ is a continuous function. In [Kun97] the coefficient functions have to be at least 5-times continuously differentiable in all variables $(x, u, \nabla u)$, Hölder continuous and of linear growth. In particular the term $\lambda u|u|^{q-2}$ for $q > 2$ does not fulfill these conditions. In very special and simplified cases an application is possible, as seen in Example 4.10. In this case we take $q = 2$ and consider space-independent orthonormal bases. Hence the critical term $\lambda u|u|^{q-2}$ for $q > 2$ of the stochastic transport equation as written in [BR15] has been dropped in Example 4.10.*

7. The scaling transform approach

In this chapter we repeat the operatorial approach developed by V. Barbu and M. Röckner in *An operatorial approach to stochastic partial differential equations driven by linear multiplicative noise* [BR15]. We do this in the case of the stochastic transport equation, despite the fact that the conditions of the main existence and uniqueness result Theorem 3.1 in [BR15] are not exactly satisfied. Nevertheless it is possible to apply the operatorial approach as written in [BR15, Section 6.3.]. We reproduce this in a detailed and restructured way to see at the end that the scaling transform solves the problem. Furthermore, we illustrate in a summarizing diagram the three considered methods in the case of the stochastic transport equation, see Subsection 7.3. below.

The scaling transform approach is applicable to general infinite-dimensional stochastic partial differential equations for $t \in [0, \mathbf{T}]$ of the form

$$\begin{cases} \mathrm{d}X(\cdot,t) = -A(\cdot,t)X(\cdot,t)\,\mathrm{d}t + X(\cdot,t)\,\mathrm{d}\mathbb{W}(\cdot,t) \\ \quad X(\cdot,t) = h(\cdot), \end{cases} \tag{7.1}$$

where A is a monotone-like operator, $\mathbb{W}$ is a Wiener process and h is some initial function. We consider this type of equations on a Gelfand triple

$$V \subset H \cong H^* \subset V^*.$$

The idea of the scaling transform approach is to multiply the equation with $e^{-\mathbb{W}(t)}$, i.e. we consider that solutions are of the form

$$X(t) = e^{\mathbb{W}(t)}y(t) \tag{7.2}$$

and transform SPDE (7.1) into a random differential equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}y(t) = -e^{-\mathbb{W}(t)}A(t)\big(e^{\mathbb{W}(t)}y(t)\big) - \mu\, y(t), \quad t \in [0,\mathbf{T}] \\ \quad y(0) = h, \end{cases} \tag{7.3}$$

where μ depends on the representation of the Wiener process. Then we rewrite equation (7.3) into an equation of the form

$$\mathscr{B}y + \mathscr{T}y + \mathscr{A}y = 0, \tag{7.4}$$

where $\mathscr{A}, \mathscr{B}$ and $\mathscr{T}$ are maximal monotone operators on a new defined Gelfand triple

$$\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*.$$

Finally we apply an existence and uniqueness result for such kind of operator equations and hence we obtain existence and uniqueness of solutions to (7.1).

7.1. The existence and uniqueness result of V. Barbu and M. Röckner

Let $\mathbf{T} > 0$ and $[0, \mathbf{T}]$ be the underlying time interval. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with normal filtration $(\mathscr{F}_t)_{t\geq 0}$. Let $(V, \|\cdot\|_V)$ be a reflexive Banach space continuously and densely embedded into a real separable Hilbert space $(H, \|\cdot\|_H)$. By the Riesz isomorphism (cf. [Bré11, Theorem 5.5. - Chapter 5.2]) H is isometric isomorph to its dual space H^* and the dual pairing fulfills ${}_{V^*}\langle v,w\rangle_V = \langle v,w\rangle_H$ for all $v \in H, w \in V$. The dual space is again continuously and densely embedded into the dual space of V denoted by V^*. The above construction forms a Gelfand triple (cf. [Emm04, Definition 8.1.7]) written as

$$V \subset H \cong H^* \subset V^*.$$

Let $\mathbb{O} \subset \mathbb{R}^d$ be a bounded and open subset with smooth boundary $\partial\mathbb{O}$. We consider a stochastic differential equation with multiplicative noise in H of the form

$$\begin{cases} \mathrm{d}X(t) = -A(t)X(t)\,\mathrm{d}t + X(t)\,\mathrm{d}\mathbb{W}(t) \\ X(0) = h, \end{cases} \tag{7.5}$$

where $h \in H$. The perturbation $\mathbb{W}$ is a Wiener process on H in the following sense as already defined in Chapter 6. Let us recall the definition.

Definition 6.1 *A Wiener process $\mathbb{W}$ on a real separable Hilbert space H is defined by*

$$\mathbb{W}(x,t) := \sum_{j=1}^{\infty} \mu_j e_j(x) W_t^j, \tag{7.6}$$

for all $x \in \mathbb{O}$, $t \geq 0$, where for all $j = 1, 2, ...$

- *W_t^j is an independent system of real-valued Brownian motions on $(\Omega, \mathscr{F}, P)$ with normal filtration $(\mathscr{F}_t)_{t\geq 0}$*
- *$e_j \in \mathcal{C}^2(\bar{\mathbb{O}}, \mathbb{R}) \cap H$ is an orthonormal basis in H and*
- *$\mu_j \in \mathbb{R}$.*

Furthermore, let Assumption 6.2 be fulfilled.

Definition 7.1 *A $(\mathscr{F}_t)_{t\geq 0}$-adapted process $X\colon [0, \boldsymbol{T}] \to H$ with continuous sample paths is called a* ***solution*** *to (7.5) for initial value $h \in H$, if it satisfies for some $1 < q < \infty$*

$$X \in L^\infty((0,\boldsymbol{T}), L^2(\Omega, H)),$$
$$X(t) = h - \int_0^t A(s)X(s)\,\mathrm{d}s + \int_0^t X(s)\,\mathrm{d}\mathbb{W}(s),\ t \in [0,\boldsymbol{T}],$$
$$AX \in L^{\frac{q}{q-1}}((0,\boldsymbol{T}) \times \Omega, V^*) \quad \textit{and} \quad X \in L^q((0,\boldsymbol{T}) \times \Omega, V).$$

In [BR15] the method of scaling and transforming a stochastic partial differential equation into a random differential equation is used to prove an existence and uniqueness result for equation (7.1) under certain conditions. We state this result in Theorem 7.2 below. The random differential equation corresponds to an equation with maximal monotone operators. Even if one does not know the validity of these conditions, it is possible to apply the method. To call attention to the scaling transform approach we state the main result as well as the underlying conditions **HP(i) - HP(iv)**.

HP(i) $V \subset H \subset V^*$ is a Gelfand triple with a separable, reflexive Banach space V and separable real Hilbert space H. By Asplund's Theorem [Bar10, Theorem 1.1.] the spaces V and V^* are strictly convex with respect to an equivalent norm on V.

HP(ii) The operator $A\colon [0,\mathbf{T}] \times V \times \Omega \to V^*$ is progressively measurable, i.e. for every $t \in [0,\mathbf{T}]$ the restricted operator $A|_{[0,t]\times V\times\Omega}$ is $\mathcal{B}([0,t]) \otimes \mathcal{B}(V) \otimes \mathscr{F}_t$-measurable, where $\mathcal{B}$ denotes the Borel-σ-algebra on the corresponding spaces.

HP(iii) There is $\delta \geq 0$ such that, for each $t \in [0, \mathbf{T}]$ and $\omega \in \Omega$, the operator

$$\begin{aligned} G\colon V &\longrightarrow V^* \\ u = u(\cdot, t, \omega) &\mapsto G(u) := \delta u + A(t, \omega)u = \delta u(\cdot, t, \omega) + A(t, \omega)u(\cdot, t, \omega) \end{aligned}$$

is monotone, demicontinuous and for some $1 < q < \infty$ there are constants $\gamma_1 > 0, \gamma_2, \gamma_3 \in \mathbb{R}, \bar{\gamma}_i \in \mathbb{R}$ with $i = 1, 2, 3$ such that

$$_{V^*}\langle A(t)u, u\rangle_V \geq \gamma_1 \|u\|_V^q + \gamma_2 \|u\|_H^2 + \gamma_3 \qquad \forall u \in V, t \in [0, \mathbf{T}], \tag{7.7}$$

$$\|A(t)u\|_{V^*} \leq \bar{\gamma}_1 \|u\|_V^{q-1} + \bar{\gamma}_2 + \bar{\gamma}_3 \|u\|_H, \quad \forall u \in V, t \in [0, \mathbf{T}] \tag{7.8}$$

P−a.s.

HP(iv) $e^{\pm \mathbb{W}(t)}$ is, for each t, a multiplier in V and a symmetric multiplier in H such that there exists a $(\mathscr{F}_t)_t$-adapted, $\mathbb{R}_+$-valued process $Z(t)$, $t \in [0, \mathbf{T}]$, with

$$\mathbb{E}\Big[\sup_{t \in [0, \mathbf{T}]} |Z(t)|^r\Big] < \infty$$

for all $1 \leq r < \infty$ and

$$\|e^{\pm \mathbb{W}(t)} y\|_V \leq Z(t)\, \|y\|_V \qquad \forall\, t \in [0, \mathbf{T}],\ \forall\, y \in V, \tag{7.9}$$

$$\|e^{\pm \mathbb{W}(t)} y\|_H \leq Z(t)\, \|y\|_H \qquad \forall\, t \in [0, \mathbf{T}],\ \forall\, y \in H, \tag{7.10}$$

$$\left\langle e^{\pm \mathbb{W}(t)} x, y \right\rangle_H = \left\langle x, e^{\pm \mathbb{W}(t)} y \right\rangle_H \quad \forall\, t \in [0, \mathbf{T}],\ \forall\, x, y \in H \tag{7.11}$$

P-a.s. and $t \mapsto e^{\pm \mathbb{W}(t)} \in H$ is continuous for fixed ω.

For the reader's convenience the definitions of monotonicity and demicontinuity are mentioned in Appendix E, namely in Definition E.1 and Definition E.2, respectively.

Theorem 7.2 *Under hypotheses* **HP(i)**- **HP(iv)**, *for each $h \in H$, equation (7.5) has a unique solution X. Moreover, the function $t \mapsto e^{-\mathbb{W}(t)} X(t)$ is V^*- absolutely continuous on $[0, \boldsymbol{T}]$ and*

$$\mathbb{E}\left[\int_0^{\boldsymbol{T}} \left\| e^{\mathbb{W}(t)} \frac{\mathrm{d}}{\mathrm{d}t}\big[e^{-\mathbb{W}(t)} X(t)\big]\right\|_{V^*}^{\frac{q}{q-1}} \mathrm{d}t\right] < \infty$$

holds.

Details can be found in [BR15]. In this chapter we focus on the stochastic transport equation. Therefore we study this equation explicitly and finally apply the following main result concerning the existence and uniqueness of a solution to an operator equation of the form (7.4).

Proposition 7.3 *Let $\mathscr{A}, \mathscr{B}$ be maximal monotone operators in $\mathcal{V} \times \mathcal{V}^*$. Let $\mathscr{T} : D(\mathscr{T}) \subset \mathcal{V} \to \mathcal{V}^*$ be a maximal monotone operator such that $\mathscr{B} + \mathscr{T}$ is maximal monotone in $\mathcal{V} \times \mathcal{V}^*$. Then, for any $f \in \mathcal{V}^*$, there is a unique solution y to the equation*

$$\mathscr{B}y + \mathscr{T}y + \mathscr{A}y = f. \tag{7.12}$$

The proof can be found in [BR15, Proposition 4.4.]. The strategy for proving the existence and uniqueness result for the stochastic transport equation is to apply the above Proposition 7.3 to well-chosen operators. We will not apply Theorem 7.2 directly. The basic tool of the operatorial approach is the following Definition 7.4 of a maximal monotone operator (cf. [Bar10, Definition 2.1.]).

Definition 7.4 *Let V, V^* be two real Banach spaces. Any operator $A\colon V \to V^*$ can be identified with its graph*

$$\text{graph}\{(u, Au) \in V \times V^*\}.$$

Therefore, we define for $A \subset V \times V^*$

$$\begin{aligned} Au &:= \{v \in V^* \,|\, (u,v) \in A\}, \\ D(A) &:= \{u \in V \,|\, Au \neq \varnothing\}, \\ R(A) &:= \bigcup_{u \in D(A)} Au. \end{aligned}$$

*A set $A \subset V \times V^*$ is called **monotone**, if*

$${}_{V^*}\langle v_1 - v_2, u_1 - u_2 \rangle_V \geq 0 \text{ for all } (u_i, v_i) \in A, i = 1, 2.$$

*A monotone set $A \subset V \times V^*$ is called **maximal monotone** if it is not properly contained in any other monotone subset of $V \times V^*$.*

Further necessary tools are the following theorems of [Bar10] which give in particular equivalent statements to maximal monotonicity.

Theorem 7.5 *Let J be the duality mapping of V. Let V and V^* be reflexive and strictly convex and let $A \subset V \times V^*$ be a monotone set. Then A is maximal monotone in $V \times V^*$ if and only if, for each $\lambda > 0$ and $q > 0$*

$$\bigcup_{u \in D(A + \lambda J \|\cdot\|_V^{q-1})} \left(Au + \lambda J(u) \|u\|_V^{q-1} \right) = V^*$$

holds.

The above result including the proof can be found in [Bar10, Theorem 2.3]. In Appendix E we give the definition of a hemicontinuous operator, see Definition E.3. Furthermore, we remember Theorem 2.4 from [Bar10]:

Theorem 7.6 *Let V be a reflexive Banach space and $A\colon V \to V^*$ be a monotone and hemicontinuous operator. Then A is maximal monotone in $V \times V^*$.*

7.2. Application to stochastic transport equations

Let $\mathbb{O} \subset \mathbb{R}^d$ be a bounded and open subset with smooth boundary $\partial\mathbb{O}$ and $q \geq 2$. Consider the Gelfand triple

$$L^q(\mathbb{O}, \mathbb{R}) \quad \subset \quad L^2(\mathbb{O}, \mathbb{R}) \quad \subset \quad L^{\frac{q}{q-1}}(\mathbb{O}, \mathbb{R}). \tag{7.13}$$

We study the hyperbolic stochastic partial differential equation of first order given by

$$\begin{cases} \mathrm{d}X(x,t) = \displaystyle\sum_{i=1}^d \alpha_i(x,t) \frac{\partial X(x,t)}{\partial x_i} \,\mathrm{d}t - \beta(x,t) X(x,t) \,\mathrm{d}t \\ \qquad\qquad - \lambda X(x,t) \,|X(x,t)|^{q-2} \,\mathrm{d}t + X(x,t) \,\mathrm{d}\mathbb{W}(x,t) \\ X(x,0) = h(x), \\ \quad X(t) = 0 \text{ on } \Upsilon := \left\{ (x,t) \in \partial\mathbb{O} \times [0, \mathbf{T}] \,\middle|\, -\displaystyle\sum_{i=1}^d \alpha_i(x,t) n_i(x) < 0 \right\}, \end{cases} \tag{STE}$$

where we assume

$$\lambda > 0, \tag{7.14}$$
$$h \in H^1(\mathbb{O},\mathbb{R}) \cap L^q(\mathbb{O},\mathbb{R}), h = 0 \text{ on } \Upsilon, \tag{7.15}$$
$$\alpha_i, \beta \in \mathcal{C}(\bar{\mathbb{O}} \times [0,\mathbf{T}],\mathbb{R}) \;\forall\; i = 1,...,d, \tag{7.16}$$
$$\nabla_x \alpha_i = \Big(\frac{\partial \alpha_i}{\partial x_1},\dots,\frac{\partial \alpha_i}{\partial x_d}\Big) \in \mathcal{C}(\bar{\mathbb{O}} \times [0,\mathbf{T}],\mathbb{R}^d), \tag{7.17}$$
$$(n_i)_{i=1,\dots,d} \text{ is the normal vector to } \partial\mathbb{O}, \tag{7.18}$$
$$\frac{1}{2}\operatorname{div}_x \alpha(x,t) + \beta(x,t) > \nu, \tag{7.19}$$

with $\nu := \sum_{j\geq 1} \mu_j^2 \tilde{\gamma}_j^2 \|e_j\|_\infty^2$ and $\tilde{\gamma}_j \in [1,\infty)$, $j \geq 1$, as given in Assumption 6.2. For short notation we define

$$\alpha := (\alpha_1,...,\alpha_d) \colon \bar{\mathbb{O}} \times [0,\mathbf{T}] \to \mathbb{R}^d, \tag{7.20}$$
$$B(u) := -u|u|^{q-2}. \tag{7.21}$$

Remark 7.7 *In the case $V = L^q(\mathbb{O},\mathbb{R})$ for $q > 2$, it is convenient to estimate*

$$\mathbb{E}\big[\|e^{\mathbb{W}(t)}\|_V^q\big] \quad \textit{respectively} \quad \mathbb{E}\big[\|e^{\mathbb{W}(t)}\|_H^q\big]$$

and consequently verify **HP(iv)**. *We conclude this from Fernique's Theorem [DPZ14, Theorem 2.6.]. The statement of Fernique's Theorem is that on a separable Banach space a Gaussian random variable has exponential tails. Applied to the exponential series, we can show that* **HP(iv)** *is fulfilled and we obtain*

$$\exp\Big(\sup_{0\leq t\leq \boldsymbol{T}} |\mathbb{W}(t)|_\infty\Big) \in L^q(\Omega).$$

Now we reformulate the existence and uniqueness result of solutions to the stochastic transport equation as published in [BR15, Section 6.3].

Theorem 7.8 *Under the conditions (7.14) - (7.19) there exists a unique solution X to the stochastic transport equation (STE).*

Due to the strategy of applying Proposition 7.3 the following proof and calculations are given in a backward direction to see that the scaling transform (7.2) mentioned at the beginning of this chapter is useful. The proof follows the ideas of the proof of [BR15, Theorem 3.1, Proposition 3.3]

Proof. In a first step we define analogously to [BR15, Section 4] the following Gelfand triple

$$\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^* \tag{7.22}$$

with

- $\mathcal{V} \equiv$ Banach space of all $L^q(\mathbb{O},\mathbb{R})$-valued, $(\mathscr{F}_t)_t$-adapted processes $y\colon [0,\mathbf{T}] \to L^q(\mathbb{O},\mathbb{R})$ with norm

$$\|y\|_{\mathcal{V}} := \left(\mathbb{E}\Big[\int_0^{\mathbf{T}} \left\|e^{\mathbb{W}(t)} y(t)\right\|_{L^q}^q \,\mathrm{d}t\Big]\right)^{\frac{1}{q}} < \infty.$$

- $\mathcal{H} \equiv$ Hilbert space of all $L^2(\mathbb{O},\mathbb{R})$-valued, $(\mathscr{F}_t)_t$-adapted processes $y\colon [0,\mathbf{T}] \to L^2(\mathbb{O},\mathbb{R})$ with norm

$$\|y\|_{\mathcal{H}} := \left(\mathbb{E}\Big[\int_0^{\mathbf{T}} \left\|e^{\mathbb{W}(t)}y(t)\right\|_{L^2}^2 \,\mathrm{d}t\Big]\right)^{\frac{1}{2}} < \infty$$

and inner product

$$\langle y,z\rangle_{\mathcal{H}} := \mathbb{E}\Big[\int_0^{\mathbf{T}} \langle e^{\mathbb{W}(t)}y(t), e^{\mathbb{W}(t)}z(t)\rangle_{L^2} \,\mathrm{d}t\Big] < \infty.$$

- $\mathcal{V}^* \equiv$ Banach space of all $L^{q'}(\mathbb{O},\mathbb{R})$-valued, $(\mathscr{F}_t)_t$-adapted processes $y\colon [0,\mathbf{T}] \to L^{q'}(\mathbb{O},\mathbb{R})$ with norm

$$\|y\|_{\mathcal{V}^*} := \left(\mathbb{E}\Big[\int_0^{\mathbf{T}} \left\|e^{\mathbb{W}(t)}y(t)\right\|_{L^{q'}}^{q'} \,\mathrm{d}t\Big]\right)^{\frac{1}{q'}} < \infty,\ q' = \frac{q}{q-1}.$$

For fixed initial function $h \in H^1(\mathbb{O},\mathbb{R}) \cap L^q(\mathbb{O},\mathbb{R})$ we define the operators

$$\begin{aligned}
\mathscr{A} &: \mathcal{V} \longrightarrow \mathcal{V}^* \\
&y(t) \mapsto (\mathscr{A}y)(t) := \lambda e^{(q-2)\mathbb{W}(t)}(y(t)+h)\,|y(t)+h|^{q-2} \\
&\qquad\qquad - \sum_{i=1}^{d} \alpha_i(t) e^{-\mathbb{W}(t)} \frac{\partial[e^{\mathbb{W}(t)}h]}{\partial x_i} + (\beta(t)+\mu)h
\end{aligned} \tag{7.23}$$

$$\begin{aligned}
\mathscr{B}&\colon D(\mathscr{B}) \subset \mathcal{V} \longrightarrow \mathcal{V}^* \\
&y(t) \mapsto (\mathscr{B}y)(t) := \frac{\mathrm{d}y}{\mathrm{d}t}(t) + (\mu+\nu)y(t)
\end{aligned} \tag{7.24}$$

$$\begin{aligned}
\mathscr{T}&\colon D(\mathscr{T}) \subset \mathcal{V} \longrightarrow \mathcal{V}^* \\
&y(t) \mapsto (\mathscr{T}y)(t) := -\sum_{i=1}^{d} \alpha_i(t) e^{-\mathbb{W}(t)} \frac{\partial[e^{\mathbb{W}(t)}y(t)]}{\partial x_i} \\
&\qquad\qquad + (\beta(t)-\nu)y(t),
\end{aligned} \tag{7.25}$$

with corresponding domains

$$\begin{aligned}
D(\mathscr{B}) &:= \Big\{ y \in \mathcal{V} \,\Big|\, y \in \mathcal{AC}([0,\mathbf{T}], L^{q'}(\mathbb{O},\mathbb{R})) \cap \mathcal{C}([0,\mathbf{T}], L^2(\mathbb{O},\mathbb{R}))\ P\text{-a.s.}, \\
&\qquad \frac{\mathrm{d}y}{\mathrm{d}t} \in \mathcal{V}^*, y(0) = 0 \Big\}, \\
D(\mathscr{T}) &:= \left\{ y \in \mathcal{V} \,\middle|\, \sum_{i=1}^{d} e^{-\mathbb{W}} \alpha_i \frac{\partial[e^{\mathbb{W}}y]}{\partial x_i} \in \mathcal{V}^*, y = 0 \text{ on } \Upsilon \right\},
\end{aligned}$$

where $\mathcal{AC}([0,\mathbf{T}], L^{q'}(\mathbb{O},\mathbb{R}))$ denotes the space of all absolutely continuous $L^{q'}$-valued functions on $[0,\mathbf{T}]$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ is defined as in Definition E.5. We consider the following integral and integrate by parts

$$\begin{aligned}
&-\int_{\mathbb{O}} \sum_{i=1}^{d} \alpha_i (e^{\mathbb{W}}y) \frac{\partial[e^{\mathbb{W}}y]}{\partial x_i} \,\mathrm{d}x \\
&\quad = -\int_{\partial\mathbb{O}} \sum_{i=1}^{d} \alpha_i n_i (e^{\mathbb{W}}y)^2 \,\mathrm{d}x + \int_{\mathbb{O}} \sum_{i=1}^{d} (e^{\mathbb{W}}y) \frac{\partial[\alpha_i e^{\mathbb{W}}y]}{\partial x_i} \,\mathrm{d}x
\end{aligned}$$

$$= -\int_{\partial\mathbb{O}} \sum_{i=1}^{d} \alpha_i n_i (e^{\mathbb{W}} y)^2 \, \mathrm{d}x + \int_{\mathbb{O}} \sum_{i=1}^{d} (e^{\mathbb{W}} y)\Big(\alpha_i \frac{\partial [e^{\mathbb{W}} y]}{\partial x_i} + (e^{\mathbb{W}} y)\frac{\partial \alpha_i}{\partial x_i}\Big) \, \mathrm{d}x$$

$$= -\int_{\partial\mathbb{O}} \sum_{i=1}^{d} \alpha_i n_i (e^{\mathbb{W}} y)^2 \, \mathrm{d}x + \int_{\mathbb{O}} \sum_{i=1}^{d} \alpha_i (e^{\mathbb{W}} y)\frac{\partial [e^{\mathbb{W}} y]}{\partial x_i} \, \mathrm{d}x + \int_{\mathbb{O}} \sum_{i=1}^{d} \frac{\partial \alpha_i}{\partial x_i}(e^{\mathbb{W}} y)^2 \, \mathrm{d}x.$$

Hence we obtain

$$\begin{aligned} &-\int_{\mathbb{O}} \sum_{i=1}^{d} \alpha_i (e^{\mathbb{W}} y)\frac{\partial [e^{\mathbb{W}} y]}{\partial x_i} \, \mathrm{d}x \\ &\qquad = -\frac{1}{2}\int_{\partial\mathbb{O}} \sum_{i=1}^{d} \alpha_i n_i (e^{\mathbb{W}} y)^2 \, \mathrm{d}x + \frac{1}{2}\int_{\mathbb{O}} \sum_{i=1}^{d} \frac{\partial \alpha_i}{\partial x_i}(e^{\mathbb{W}} y)^2 \, \mathrm{d}x. \end{aligned} \tag{7.26}$$

Under assumption (7.19) given by

$$\frac{1}{2}\operatorname{div}_x \alpha(x,t) + \beta(x,t) - \nu > 0,$$

we can show that the operator $\mathscr{T}$ as defined in (7.25) is monotone, since by Riesz isomorphism we receive

$$\begin{aligned} {}_{\mathcal{V}^*}\langle \mathscr{T}(y), y\rangle_{\mathcal{V}} &= \mathbb{E}\Big[\int_0^{\mathbf{T}} \langle e^{\mathbb{W}(t)}(\mathscr{T}y)(t), e^{\mathbb{W}(t)} y(t)\rangle_{L^2} \, \mathrm{d}t\Big] \\ &= \mathbb{E}\Big[\int_0^{\mathbf{T}}\int_{\mathbb{O}} e^{\mathbb{W}(t)}\Big(-\sum_{i=1}^{d} \alpha_i(x,t) e^{-\mathbb{W}(t)} \frac{\partial [e^{\mathbb{W}(t)} y(t)]}{\partial x_i} + (\beta(x,t)-\nu) y(t)\Big) \\ &\qquad\qquad \cdot \Big(e^{\mathbb{W}(t)} y(t)\Big) \, \mathrm{d}x \, \mathrm{d}t\Big] \\ &= \mathbb{E}\Big[\int_0^{\mathbf{T}}\int_{\mathbb{O}} -e^{\mathbb{W}(t)} \sum_{i=1}^{d} \alpha_i(x,t) e^{-\mathbb{W}(t)} \frac{\partial [e^{\mathbb{W}(t)} y(t)]}{\partial x_i}\Big(e^{\mathbb{W}(t)} y(t)\Big) \\ &\qquad\qquad + e^{\mathbb{W}(t)}(\beta(x,t)-\nu) y(t)\Big(e^{\mathbb{W}(t)} y(t)\Big) \, \mathrm{d}x \, \mathrm{d}t\Big] \\ &= \mathbb{E}\Big[\int_0^{\mathbf{T}}\int_{\mathbb{O}} \sum_{i=1}^{d} -\alpha_i(x,t) \frac{\partial [e^{\mathbb{W}(t)} y(t)]}{\partial x_i}(e^{\mathbb{W}(t)} y(t)) \, \mathrm{d}x \, \mathrm{d}t\Big] \\ &\qquad + \mathbb{E}\Big[\int_0^{\mathbf{T}}\int_{\mathbb{O}} (\beta(x,t)-\nu)(e^{\mathbb{W}(t)} y(t))^2 \, \mathrm{d}x \, \mathrm{d}t\Big]. \end{aligned}$$

Now we make use of (7.26) to obtain

$$\begin{aligned} {}_{\mathcal{V}^*}\langle \mathscr{T}(y), y\rangle_{\mathcal{V}} &= \mathbb{E}\Big[\int_0^{\mathbf{T}} \Big(\int_{\partial\mathbb{O}} -\frac{1}{2}\sum_{i=1}^{d} \alpha_i n_i (e^{\mathbb{W}} y)^2 \, \mathrm{d}x + \int_{\mathbb{O}} \frac{1}{2}\frac{\partial \alpha_i}{\partial x_i}(e^{\mathbb{W}} y) \, \mathrm{d}x\Big) \, \mathrm{d}t\Big] \\ &\qquad + \mathbb{E}\Big[\int_0^{\mathbf{T}}\int_{\mathbb{O}} (\beta(x,t)-\nu)(e^{\mathbb{W}(t)} y(t))^2 \, \mathrm{d}x \, \mathrm{d}t\Big] \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\Bigg[\int_0^{\mathbf{T}}\int_{\partial\mathbb{O}} -\frac{1}{2}\sum_{i=1}^d \alpha_i n_i (e^{\mathbb{W}}y)^2 \,\mathrm{d}x\,\mathrm{d}t\Bigg] + \mathbb{E}\Bigg[\int_0^{\mathbf{T}}\int_{\mathbb{O}} \frac{1}{2}\operatorname{div}_x \alpha (e^{\mathbb{W}}y)^2 \,\mathrm{d}x\,\mathrm{d}t\Bigg]\\
&\quad + \mathbb{E}\Bigg[\int_0^{\mathbf{T}}\int_{\mathbb{O}} (\beta(x,t)-\nu)(e^{\mathbb{W}(t)}y(t))^2 \,\mathrm{d}x\,\mathrm{d}t\Bigg]\\
&= \mathbb{E}\Bigg[\int_0^{\mathbf{T}}\int_{\partial\mathbb{O}} -\frac{1}{2}\sum_{i=1}^d \alpha_i n_i (e^{\mathbb{W}}y)^2 \,\mathrm{d}x\,\mathrm{d}t\Bigg]\\
&\quad + \mathbb{E}\Bigg[\int_0^{\mathbf{T}}\int_{\mathbb{O}} \Big(\frac{1}{2}\operatorname{div}_x \alpha + \beta(x,t)-\nu\Big)(e^{\mathbb{W}(t)}y(t))^2 \,\mathrm{d}x\,\mathrm{d}t\Bigg]\\
&= \mathbb{E}\Bigg[\int_{\Upsilon} -\frac{1}{2}\sum_{i=1}^d \alpha_i n_i (e^{\mathbb{W}}y)^2 \,\mathrm{d}(x,t)\Bigg] + \mathbb{E}\Bigg[\int_{\partial\mathbb{O}\times[0,\mathbf{T}]\setminus\Upsilon} -\frac{1}{2}\sum_{i=1}^d \alpha_i n_i (e^{\mathbb{W}}y)^2 \,\mathrm{d}(x,t)\Bigg]\\
&\quad + \mathbb{E}\Bigg[\int_0^{\mathbf{T}}\int_{\mathbb{O}} \Big(\frac{1}{2}\operatorname{div}_x \alpha + \beta(x,t)-\nu\Big)(e^{\mathbb{W}(t)}y(t))^2 \,\mathrm{d}x\,\mathrm{d}t\Bigg].
\end{aligned}
$$

Due to assumption (7.19), the definition of the domain $D(\mathscr{T})$ and the definition of Υ we conclude

$$
{}_{\mathcal{V}^*}\langle \mathscr{T}(y), y\rangle_{\mathcal{V}} \geq 0.
$$

Furthermore, we prove that the operator $\mathscr{A}$ given by (7.23) is maximal monotone in $\mathcal{V}\times\mathcal{V}^*$. To this end we use Theorem 7.6 which states that monotonicity and hemicontinuity of an operator imply maximal monotonicity. We obtain monotonicity since by Riesz isomorphism we have for $y, \hat{y} \in \mathcal{V}$

$$
\begin{aligned}
&{}_{\mathcal{V}^*}\langle \mathscr{A}(y)-\mathscr{A}(\hat{y}), y-\hat{y}\rangle_{\mathcal{V}}\\
&= \mathbb{E}\Big[\int_0^{\mathbf{T}}\int_{\mathbb{O}} \Big(e^{\mathbb{W}}\lambda\, e^{(q-2)\mathbb{W}}(y+h)\Big|(y+h)\Big|^{q-2}\\
&\qquad - e^{\mathbb{W}}\lambda\, e^{(q-2)\mathbb{W}}(\hat{y}+h)\Big|\hat{y}+h\Big|^{q-2}\Big)\Big(e^{\mathbb{W}}y - e^{\mathbb{W}}\hat{y}\Big)\,\mathrm{d}x\,\mathrm{d}t\Big]\\
&\quad + \mathbb{E}\Big[\int_0^{\mathbf{T}}\int_{\mathbb{O}} \Big(-\sum_{i=1}^d e^{\mathbb{W}}\alpha_i e^{-\mathbb{W}}\frac{\partial[e^{\mathbb{W}}h]}{\partial x_i} + e^{\mathbb{W}}(\beta+\mu)h\\
&\qquad + \sum_{i=1}^d e^{\mathbb{W}}\alpha_i e^{-\mathbb{W}}\frac{\partial[e^{\mathbb{W}}h]}{\partial x_i} - e^{\mathbb{W}}(\beta+\mu)h\Big)\Big(e^{\mathbb{W}}y - e^{\mathbb{W}}\hat{y}\Big)\,\mathrm{d}x\,\mathrm{d}t\Big]\\
&= \mathbb{E}\Big[\int_0^{\mathbf{T}}\int_{\mathbb{O}} \Bigg(\lambda\Big(e^{\mathbb{W}}(y+h)\Big)\Big|\Big(e^{\mathbb{W}}(y+h)\Big)\Big|^{q-2}\\
&\qquad - \lambda\Big(e^{\mathbb{W}}(\hat{y}+h)\Big)\Big|\Big(e^{\mathbb{W}}(\hat{y}+h)\Big)\Big|^{q-2}\Bigg)\Big((e^{\mathbb{W}}(y+h) - (e^{\mathbb{W}}(\hat{y}+h)\Big)\,\mathrm{d}x\,\mathrm{d}t\Big]\\
&\geq 0,
\end{aligned}
$$

since $s \mapsto s|s|^{q-2}$ is increasing for $q \geq 2$. Furthermore, hemicontinuity of $\mathscr{A}$ is fulfilled since again by Riesz isomorphism

$$
\begin{aligned}
&\lim_{\kappa\to 0} {}_{\mathcal{V}^*}\langle \mathscr{A}(y+\kappa v) - \mathscr{A}(y), z\rangle_{\mathcal{V}} \\
&= \lim_{\kappa\to 0} \mathbb{E}\Bigg[\int_0^{\mathbf{T}} \Big\langle e^{\mathbb{W}} \Big[\lambda e^{(q-2)\mathbb{W}} (y+\kappa v + h)\big|y+\kappa v+h\big|^{q-2} - \sum_{i=1}^d \alpha_i e^{-\mathbb{W}} \frac{\partial[e^{\mathbb{W}}h]}{\partial x_i} + (\beta+\mu)h \\
&\qquad - \lambda e^{(q-2)\mathbb{W}}(y+h)\big|y+h\big|^{q-2} + \sum_{i=1}^d \alpha_i e^{-\mathbb{W}} \frac{\partial[e^{\mathbb{W}}h]}{\partial x_i} - (\beta+\mu)h\Big], e^{\mathbb{W}}z\Big\rangle_{L^2} \,\mathrm{d}t\Bigg] \\
&= \lim_{\kappa\to 0} \mathbb{E}\Bigg[\int_0^{\mathbf{T}} \Big\langle e^{\mathbb{W}} \Big[\lambda e^{(q-2)\mathbb{W}} \Big((y+\kappa v+h)\big|y+\kappa v+h\big|^{q-2} - (y+h)\big|y+h\big|^{q-2}\Big)\Big], e^{\mathbb{W}}z\Big\rangle_{L^2} \,\mathrm{d}t\Bigg].
\end{aligned}
$$

Similarly as in [PR07, Example 4.1.5] we use Hölder's inequality to obtain

$$
\begin{aligned}
&\lim_{\kappa\to 0} {}_{\mathcal{V}^*}\Big|\langle \mathscr{A}(y+\kappa v) - \mathscr{A}(y), z\rangle_{\mathcal{V}}\Big| \\
&\leq \lim_{\kappa\to 0} \mathbb{E}\Bigg[\int_0^{\mathbf{T}} \Big\| e^{\mathbb{W}} \Big[\lambda e^{(q-2)\mathbb{W}} \Big((y+\kappa v+h)\big|y+\kappa v+h\big|^{q-2} - (y+h)\big|y+h\big|^{q-2}\Big)\Big]\Big\|_{L^{q'}} \\
&\qquad \cdot \big\| e^{\mathbb{W}} z\big\|_{L^q} \,\mathrm{d}t\Bigg] \\
&\leq \lim_{\kappa\to 0} \mathbb{E}\Bigg[\int_0^{\mathbf{T}} \Big\| e^{\mathbb{W}} \Big[\lambda e^{(q-2)\mathbb{W}} \Big((y+\kappa v+h)\big|y+\kappa v+h\big|^{q-2} - (y+h)\big|y+h\big|^{q-2}\Big)\Big]\Big\|_{L^{q'}}^{q'} \,\mathrm{d}t\Bigg]^{\frac{1}{q'}} \\
&\qquad \cdot \mathbb{E}\Bigg[\int_0^{\mathbf{T}} \big\| e^{\mathbb{W}} z\big\|_{L^q}^q \,\mathrm{d}t\Bigg]^{\frac{1}{q}} \\
&= \lim_{\kappa\to 0} \Big\| \lambda e^{(q-2)\mathbb{W}} \Big((y+\kappa v+h)\big|y+\kappa v+h\big|^{q-2} - (y+h)\big|y+h\big|^{q-2}\Big)\Big\|_{\mathcal{V}^*} \|z\|_{\mathcal{V}} = 0
\end{aligned}
$$

with Lebesgue's dominated convergence theorem. Now we apply the existence and uniqueness result Proposition 7.3 for operator equations of the type (7.12) with $f = 0$, a maximal monotone operator $\mathscr{T}\colon D(\mathscr{T}) \subset \mathcal{V} \to \mathcal{V}^*$ and maximal monotone sum $\mathscr{B} + \mathscr{T}$ in $\mathcal{V} \times \mathcal{V}^*$. We have shown that $\mathscr{T}$ is monotone. But in fact it is not maximal monotone, so we know that $\mathscr{B} + \mathscr{T}$ is also not maximal monotone. Hence we are not able to apply Proposition 7.3 directly. Therefore we use [BB69] and [Lio69] to show that $\mathscr{B} + \mathscr{T}$ is closable in $L^q((0,\mathbf{T}) \times \mathbb{O}, \mathbb{R})$ for fixed $\omega \in \Omega$ and the closure $\overline{\mathscr{B} + \mathscr{T}}$ is maximal monotone in $L^q((0,\mathbf{T}) \times \mathbb{O}, \mathbb{R}) \times L^{q'}((0,\mathbf{T}) \times \mathbb{O}, \mathbb{R})$. By applying Theorem 7.5 the equation

$$
(\overline{\mathscr{B} + \mathscr{T}})y + e^{-\mathbb{W}} F(e^{-\mathbb{W}} y) = 0 \quad \text{in } (0,\mathbf{T}) \times \mathbb{O}
$$

has a unique solution $y \in D(\overline{\mathscr{B} + \mathscr{T}})$, where F is the corresponding dual mapping; for details see Lemma E.6. As written in Lemma E.7 it is also true, that $\overline{\mathscr{B} + \mathscr{T}}$ is maximal monotone in $\mathcal{V} \times \mathcal{V}^*$. All in all we are able to apply Proposition 7.3 to the new operator equation

$$
(\overline{\mathscr{B} + \mathscr{T}})y + \mathscr{A} y = 0,
$$

since both operators are maximal monotone. Therefore there exists a unique solution $y \in D(\overline{\mathscr{B} + \mathscr{T}})$ i.e. there exists $(y_n)_{n\in\mathbb{N}} \subset D(\mathscr{B} + \mathscr{T})$ with $y_n \to y$ in $\mathcal{V}$ for $n \to \infty$ and $(\mathscr{B} + \mathscr{T})y_n + \mathscr{A} y_n \to 0$ in $\mathcal{V}$ for $n \to \infty$.

Now we use the definitions of the operators (7.23), (7.24) and (7.25) to verify that y is the solution of the random differential equation (7.3) in the sense of [BR15, Definition 3.2.]. Let $y \in D(\overline{\mathscr{B}+\mathscr{T}})$, then we have

$$\begin{aligned}
0 &= \mathscr{B}y + \mathscr{T}y + \mathscr{A}y \\
&= \frac{\mathrm{d}y}{\mathrm{d}t} + (\mu+\nu)y - \sum_{i=1}^{d} \alpha_i e^{-\mathbb{W}} \frac{\partial[e^{\mathbb{W}}y]}{\partial x_i} + (\beta-\nu)y \\
&\qquad + \lambda e^{(q-2)\mathbb{W}}(y+h)\,|y+h|^{q-2} - \sum_{i=1}^{d} \alpha_i e^{-\mathbb{W}} \frac{\partial[e^{\mathbb{W}}h]}{\partial x_i} + (\beta+\mu)h \\
&= \frac{\mathrm{d}y}{\mathrm{d}t} - \sum_{i=1}^{d} \alpha_i e^{-\mathbb{W}} \frac{\partial[e^{\mathbb{W}}y]}{\partial x_i} + \beta y + \beta h + \mu y + \mu h \\
&\qquad + \lambda e^{(q-2)\mathbb{W}}(y+h)\,|y+h|^{q-2} - \sum_{i=1}^{d} \alpha_i e^{-\mathbb{W}} \frac{\partial[e^{\mathbb{W}}h]}{\partial x_i} \\
&= \frac{\mathrm{d}y}{\mathrm{d}t} - \sum_{i=1}^{d} \alpha_i e^{-\mathbb{W}} \frac{\partial[e^{\mathbb{W}}y]}{\partial x_i} + \big(\beta+\mu\big)(y+h) \\
&\qquad + \lambda e^{(q-2)\mathbb{W}}(y+h)\,|y+h|^{q-2} - \sum_{i=1}^{d} \alpha_i e^{-\mathbb{W}} \frac{\partial[e^{\mathbb{W}}h]}{\partial x_i}.
\end{aligned}$$

Finally we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sum_{i=1}^{d} \alpha_i e^{-\mathbb{W}} \frac{\partial[e^{\mathbb{W}}(y+h)]}{\partial x_i} - \big(\beta+\mu\big)(y+h) - \lambda e^{(q-2)\mathbb{W}}(y+h)\,|y+h|^{q-2}.$$

By shifting y to $y-h$ we get the following equivalent random differential equation:

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t}(x,t) = \displaystyle\sum_{i=1}^{d} \alpha_i(x,t) e^{-\mathbb{W}(x,t)} \frac{\partial[e^{\mathbb{W}(x,t)}y(x,t)]}{\partial x_i} - \big(\beta(x,t)+\mu(x)\big)y(x,t) \\ \qquad\qquad - \lambda e^{(q-2)\mathbb{W}(x,t)} y(x,t)\,|y(x,t)|^{q-2}, \\ y(x,0) = h(x). \end{cases}$$

We conclude that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sum_{i=1}^{d} \alpha_i e^{-\mathbb{W}} \frac{\partial[e^{\mathbb{W}}y]}{\partial x_i} - \big(\beta+\mu\big)y - \lambda e^{(q-2)\mathbb{W}} y\,|y|^{q-2},$$

which is equivalent to

$$\begin{aligned}
\mathrm{d}y &= \sum_{i=1}^{d} \alpha_i e^{-\mathbb{W}} \frac{\partial[e^{\mathbb{W}}y]}{\partial x_i}\,\mathrm{d}t - \big(\beta+\mu\big)y\,\mathrm{d}t \\
&\qquad - \lambda e^{(q-2)\mathbb{W}}(e^{\mathbb{W}}e^{-\mathbb{W}})y\,|y|^{q-2}\,\mathrm{d}t \\
&= \sum_{i=1}^{d} \alpha_i e^{-\mathbb{W}} \frac{\partial[e^{\mathbb{W}}y]}{\partial x_i}\,\mathrm{d}t - e^{-\mathbb{W}}\big(\beta+\mu\big)(e^{\mathbb{W}}y)\,\mathrm{d}t \\
&\qquad - e^{-\mathbb{W}}\lambda(e^{\mathbb{W}}y)\,\big|e^{\mathbb{W}}y\big|^{q-2}\,\mathrm{d}t.
\end{aligned}$$

If we multiply with $e^{\mathbb{W}}$, we obtain

$$e^{\mathbb{W}}\,\mathrm{d}y = \sum_{i=1}^{d} \alpha_i \frac{\partial[e^{\mathbb{W}}y]}{\partial x_i}\,\mathrm{d}t - \big(\beta+\mu\big)(e^{\mathbb{W}}y)\,\mathrm{d}t - \lambda(e^{\mathbb{W}}y)\,\big|e^{\mathbb{W}}y\big|^{q-2}\,\mathrm{d}t$$

$$= \sum_{i=1}^{d} \alpha_i \frac{\partial[e^{\mathbb{W}} y]}{\partial x_i} \,\mathrm{d}t - \beta(e^{\mathbb{W}} y) \,\mathrm{d}t - \mu(e^{\mathbb{W}} y) \,\mathrm{d}t - \lambda(e^{\mathbb{W}} y) \left|e^{\mathbb{W}} y\right|^{q-2} \,\mathrm{d}t$$

and therefore

$$e^{\mathbb{W}} \,\mathrm{d}y + \mu(e^{\mathbb{W}} y) \,\mathrm{d}t = \sum_{i=1}^{d} \alpha_i \frac{\partial[e^{\mathbb{W}} y]}{\partial x_i} \,\mathrm{d}t - \beta(e^{\mathbb{W}} y) \,\mathrm{d}t - \lambda(e^{\mathbb{W}} y) \left|e^{\mathbb{W}} y\right|^{q-2} \,\mathrm{d}t.$$

By adding $(e^{\mathbb{W}} y) \,\mathrm{d}\mathbb{W}$ on both sides we get

$$\begin{aligned} &e^{\mathbb{W}} \,\mathrm{d}y + \mu(e^{\mathbb{W}} y) \,\mathrm{d}t + (e^{\mathbb{W}} y) \,\mathrm{d}\mathbb{W} \\ &\quad = \sum_{i=1}^{d} \alpha_i \frac{\partial[e^{\mathbb{W}} y]}{\partial x_i} \,\mathrm{d}t - \beta(e^{\mathbb{W}} y) \,\mathrm{d}t - \lambda(e^{\mathbb{W}} y) \left|e^{\mathbb{W}} y\right|^{q-2} \,\mathrm{d}t + (e^{\mathbb{W}} y) \,\mathrm{d}\mathbb{W}. \end{aligned} \tag{7.27}$$

It is suggestive to apply Itô's product rule (see [RY05, Chapter IV, 3.1 Proposition]) to $e^{\mathbb{W}} y$ which leads to

$$e^{\mathbb{W}(t)} y(t) = e^{\mathbb{W}(0)} y(0) + \int_0^t e^{\mathbb{W}(s)} \,\mathrm{d}y(s) + \int_0^t y(s) \,\mathrm{d}(e^{\mathbb{W}(s)}) + \langle e^{\mathbb{W}(\cdot)}, y(\cdot) \rangle_t.$$

Now we use Lemma E.4 and obtain

$$e^{\mathbb{W}(t)} y(t) = y(0) + \int_0^t e^{\mathbb{W}(s)} \,\mathrm{d}y(s) + \int_0^t e^{\mathbb{W}(s)} y(s) \,\mathrm{d}\mathbb{W}(s) + \int_0^t \mu e^{\mathbb{W}(s)} y(s) \,\mathrm{d}s. \tag{7.28}$$

In terms of differentials equation (7.28) is equivalent to

$$\mathrm{d}[e^{\mathbb{W}} y] = e^{\mathbb{W}} \,\mathrm{d}y + \mu e^{\mathbb{W}} y \,\mathrm{d}t + e^{\mathbb{W}} y \,\mathrm{d}\mathbb{W}.$$

By using the right hand side of (7.27) this leads to

$$\mathrm{d}[e^{\mathbb{W}} y] = \sum_{i=1}^{d} \alpha_i \frac{\partial[e^{\mathbb{W}} y]}{\partial x_i} \,\mathrm{d}t - \beta(e^{\mathbb{W}} y) \,\mathrm{d}t - \lambda(e^{\mathbb{W}} y) \left|e^{\mathbb{W}} y\right|^{q-2} \,\mathrm{d}t + (e^{\mathbb{W}} y) \,\mathrm{d}\mathbb{W}. \tag{7.29}$$

Now the right hand side of (7.29) is equal to the right hand side of (STE) for $(e^{\mathbb{W}} y)$. Therefore we have that

$$X := e^{-\mathbb{W}} y \tag{7.30}$$

is the unique solution to equation (STE). In summary, we have that the scaling transform approach leads to the existence and uniqueness of the solution $X(t) = e^{\mathbb{W}(t)} y(t)$ to (STE).

□

7.3. Summarizing diagram

Stochastic transport equation

$$\begin{cases} \mathrm{d}X(x,t) = \sum_{i=1}^{d} \alpha_i(x,t) \frac{\partial X(x,t)}{\partial x_i} \, \mathrm{d}t - \beta(x,t) X(x,t) \, \mathrm{d}t \\ \qquad - \lambda X(x,t) \, |X(x,t)|^{q-2} \, \mathrm{d}t + X(x,t) \, \mathrm{d}\mathbb{W}(x,t) \\ X(x,0) = h(x) \end{cases}$$

and $\Upsilon := \left\{ (x,t) \in \partial\mathbb{O} \times [0,\mathbf{T}] \,\middle|\, -\sum_{i=1}^{d} \alpha_i(x,t) n_i(x) < 0 \right\}$

	Scaling transform approach	Method of stochastic characteristics - Theorem 4.5 -	Method of stochastic characteristics - heuristic approach -
Setting	• Gelfand triple: $L^q(\mathbb{O}) \subset L^2(\mathbb{O}) \subset L^{\frac{q}{q-1}}(\mathbb{O})$	• Representation in terms of semimartingales with values in $C^{k,\delta}(\mathbb{R}^d, \mathbb{R})$	• Representation in terms of semimartingales
Condition on the boundary	• $\partial\mathbb{O}$ smooth, $X(t) = 0$ on Υ	• do not need Υ	• do not need Υ
Kind of perturbation	• general Wiener process on $L^2(\mathbb{O})$ • orthonormal bases in $\mathcal{C}^2(\mathbb{O}) \cap L^2(\mathbb{O})$ • Itô integral	• series of independent copies of standard Brownian motions • Stratonovich integral	• series of independent copies of standard Brownian motions • Stratonovich integral
Conditions on coefficient terms for existence	• $\alpha_i(x,t)$ are continuous in x,t • α_i in $\mathcal{C}^1(\mathbb{O},\mathbb{R})$ • β is continuous in x,t	• $\alpha_i(x,t), \beta(x,t)$ in $\mathcal{C}^5(\mathbb{O},\mathbb{R})$ • $D_x^\gamma \alpha_i, D_x^\gamma \beta$ are δ - Hölder continuous for $\lvert\gamma\rvert = 5$ • α_i, β are of linear growth	• $\alpha_i(x,t) = 0$ • $\beta(x,t)$ is given explicitly
kind of solution	• global solution for all $t \in [0,\mathbf{T}]$ with values in $L^\infty((0,\mathbf{T}), L^2(\Omega, L^2(\mathbb{O})))$ and $L^q((0,\mathbf{T}) \times \Omega, L^q(\mathbb{O}))$	• local solution for almost all ω and for all (x,t), $t < \sigma(x,\omega)$ • never get global solution	• local solution for almost all ω and for all (x,t), $t < \sigma(x,\omega)$
advantages	• functional analysis • existence and uniqueness for a large class of equations	• explicit representation of the solution	• explicit representation of the solution

8. An application of Lemma 4.8 to [DPT96]

In the proof of Theorem 4.5 we essentially use Lemma 4.8 to determine for a given process another process which satisfies the property to be the right and left inverse. This so-called inverse process is given again for almost all ω and all time and space variables up to a certain stopping time. In *Fully Nonlinear Stochastic Partial Differential Equations* [DPT96] the authors G. Da Prato and L. Tubaro use this result to solve a second order nonlinear stochastic partial differential equation under proper conditions. Herein the authors declare that such an inverse process exists and refer to a first publication of [Kun97]. Due to our knowledge regarding the book [Kun97] we work up this important tool namely Lemma 8.5 below and prove it in detail. Additionally, we prove the Itô-Wentzell formula as stated in Theorem 8.3 below by using [Kun97]. In [DPT96] this formula is not given explicitly, but it is refered to [Tub88]. The latter states the formula in a different framework and applies results of Kunita's lecture notes [Kun84b] and [Kun84a] in the proof. We reproduce the whole framework of [DPT96] and give rigorous derivations to the considered equations. One should note that all result are also given for almost all ω and all parameters depending on ω.

Let us consider second order nonlinear stochastic partial differential equations of the form

$$\begin{cases} \mathrm{d}u(t,\cdot) = L(t,\cdot,u,Du,D^2u)\,\mathrm{d}t + \langle b(t,\cdot)Du + h(t,\cdot)u, \mathrm{d}W_t\rangle_{\mathbb{R}^{d_1}} \\ u(0,\cdot) = u_0(\cdot), \end{cases} \tag{8.1}$$

where

$$\begin{aligned} L\colon [0,\infty)\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^{d\times d} &\to \mathbb{R} \\ b\colon [0,\mathbf{T}]\times\mathbb{R}^d &\to \mathbb{R}^{d_1\times d} \\ h\colon [0,\mathbf{T}]\times\mathbb{R}^d &\to \mathbb{R}^{d_1} \\ u_0\colon \mathbb{R}^d &\to \mathbb{R} \end{aligned}$$

satisfy some conditions mentioned below and W_t is a $\mathbb{R}^{d_1}$ - valued standard Brownian motion on a probability space $(\Omega,\mathscr{F},P)$ and adapted to a normal filtration $(\mathscr{F}_t)_{t\geq 0}$. The idea of [DPT96] is to transform SPDE (8.1) into a deterministic partial differential equation for fixed ω

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t}(t,\cdot) = \Lambda(t,\cdot,v,Dv,D^2v) \\ v(0,\cdot) = u_0(\cdot) \end{cases} \tag{8.2}$$

and hence to obtain an equivalence between these problems. Furthermore, the authors prove an existence result of solutions to (8.2) in a maximal time interval. The drift and diffusion terms L, b and h have to fulfill the following conditions:

Assumption 8.1

(i) For some $\alpha,\beta\in(0,1)$ the map

$$L\colon [0,\infty)\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^{d\times d}\to\mathbb{R}$$

and its partial derivatives $D_x^hD_u^kD_p^lD_q^mL$ with $|h|+|k|+|l|+|m|\leq 2$ are α-Hölder continuous in time t, β-Hölder continuous in space x and locally Lipschitz continuous in u,p,q uniformly. For example in the case of the drift operator L that means for any $\boldsymbol{T}>0$, $r>0$ there exists a constant $M_{\boldsymbol{T},r}$ such that

$$\begin{aligned} &|L(t,x,u,p,q) - L(s,x',u',p',q')| \\ &\qquad \leq M_{\boldsymbol{T},r}\Big(|t-s|^\alpha + |x-x'|^\beta + |u-u'| + |p-p'| + |q-q'|\Big) \end{aligned}$$

holds for all $t, s \in [0, \boldsymbol{T}]$, $x, x' \in \mathbb{R}^d$ *and* u, u', p, p', q, q' *with* $|u|, |u'| \le r$, $|p|, |p'| \le r$, $|q|, |q'| \le r$.

(ii) *There exists* $\varepsilon > 0$ *such that, for any* $r > 0$, *there is* $C_r > 0$ *satisfying*

$$|L(0, x, u, p, q) - L(0, x', u, p, q)| \le C_r |x - x'|^{\beta + \varepsilon}$$

for all $x, x' \in \mathbb{R}^d$ *and* u, p, q *with* $|u|, |p|, |q| \le r$.

(iii) *The drift operator* L *and its partial derivatives* $D_x^h D_u^k D_p^l D_q^m L$ *with* $|h|+|k|+|l|+|m| = 3$ *are continuous with respect to all variables and, for any* $\boldsymbol{T} > 0$ *and* $r > 0$, *there exists a constant* $N_{\boldsymbol{T},r}$ *such that*

$$|L(t, x, u, p, q)| \le N_{\boldsymbol{T},r}$$

holds for all $t \in [0, \boldsymbol{T}]$, $x \in \mathbb{R}^d$ *and* u, p, q *with* $|u|, |p|, |q| \le r$.

(iv) *The diffusion terms* b, h *and their partial derivatives* $D_x^k b, D_u^k b, D_x^k h, D_u^k h$ *with* $|k| \le 4$ *are uniformly continuous and bounded in* $[0, \boldsymbol{T}] \times \mathbb{R}^d$.

(v) *The partial derivatives* $D_x^k b, D_u^k b, D_x^k h, D_u^k h$ *with* $|k| \le 4$ *are of class* $\mathcal{C}^1$ *in time, uniformly in* x, *i.e.* $D_x^k b(\cdot, x) \in \mathcal{C}^1([0, \boldsymbol{T}], \mathbb{R})$, *that means in particular that they are globally Lipschitz continuous in time (since* $\mathcal{C}^1$ *implies Lipschitz continuity) and locally Lipschitz continuous in space. For* $D_x^k b, D_x^k h$ *it holds that*

$$|D_x^k b(t, x) - D_x^k b(s, x)| \le K|t - s| \quad \forall\ x \in \mathbb{R}^d, t, s \in [0, \boldsymbol{T}]$$

and

$$|D_x^k b(t, x) - D_x^k b(t, y)| \le K|x - y| \quad \forall\ x, y \in \mathbb{K} \subset \mathbb{R}^d \text{ compact}, t \in [0, \boldsymbol{T}].$$

(vi) *For all* $\boldsymbol{T}, r > 0$, *there exists* $\nu_{\boldsymbol{T},r} > 0$ *such that for the transposed matrix* $b^\top(t, x)$ *of* $b(t, x)$ *and the identity matrix* $\mathbb{I}$

$$\frac{\partial L}{\partial q}(t, x, u, p, q) - \frac{1}{2} b(t, x) b^\top(t, x) \ge \nu_{\boldsymbol{T},r} \mathbb{I}$$

holds for all $t \in [0, \boldsymbol{T}]$, $x \in \mathbb{R}^d$ *and* u, p, q *with* $|u|, |p|, |q| \le r$.

Now we define the following operator.

Definition 8.2 *Consider a mapping*

$$\alpha \colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$$
$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \mapsto \alpha(x) = \begin{pmatrix} \alpha_1(x) \\ \vdots \\ \alpha_N(x) \end{pmatrix}.$$

For the partial derivatives $\partial_i \alpha_j(x) = \frac{\partial \alpha_j}{\partial x_i}(x)$ *and gradient* $D\alpha(x)$ *we define*

$$\mathbb{R}^N \ni \mathrm{TR}[D\alpha(x) \cdot \alpha(x)] = \begin{pmatrix} \sum_{i=1}^N \partial_1 \alpha_i(x) \cdot \alpha_i(x) \\ \vdots \\ \sum_{i=1}^N \partial_N \alpha_i(x) \cdot \alpha_i(x) \end{pmatrix}.$$

For matrix valued functions $A(x) \in \mathbb{R}^{M \times N}$ *with* $x \in \mathbb{R}^N$ *and* $A(x) = \big(a_{ij}(x)\big)_{\substack{i=1,\dots,M \\ j=1,\dots,N}}$ *this operator is defined by*

$$\mathrm{TR}[DA(x) \cdot A^\top(x)] := \begin{pmatrix} \mathrm{trace}[D[A(x) \cdot e_1] \cdot A^\top(x)] \\ \vdots \\ \mathrm{trace}[D[A(x) \cdot e_N] \cdot A^\top(x)] \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{M} \sum_{j=1}^{N} \partial_j a_{i1}(x) \cdot a_{ij}(x) \\ \vdots \\ \sum_{j=1}^{N} \sum_{i=1}^{M} \partial_j a_{iN}(x) \cdot a_{ij}(x) \end{pmatrix},$$

where $A^\top$ denotes the transposed matrix of A.

In the following we have a closer look on the transformation in which Kunita's result of finding an inverse process plays an important role. The whole transformation can be separated into 7 steps based on the Itô-Wentzell formula applied to (8.1) and a well-chosen SDE (8.18) below. But first of all we give an **overview:**

Define

$$\tilde{L}(t,\xi,u,Du,D^2u) := L(t,\xi,u,Du,D^2u) - \frac{1}{2}\operatorname{trace}[b^\top \cdot b \cdot D^2u] \ (\textbf{Step 1}).$$

Consider (8.1) given by

$$\mathrm{d}u = L(t,x,u,Du,D^2u)\,\mathrm{d}t + \langle b \cdot Du + h \cdot u, \mathrm{d}W_t\rangle_{\mathbb{R}^{d_1}}.$$

Solve

$$\boxed{\mathrm{d}\xi = \mathrm{TR}[Db(t,\xi)\cdot b^\top(t,\xi)]\,\mathrm{d}t - b^\top(t,\xi)\,\mathrm{d}W_t} \quad (\textbf{Step 2}).$$

Let $\xi(t,x)$ be the solution and determine

$$\xi^{-1}(t,x) =: \eta(t,x). \quad (\textbf{Step 3})$$

It holds $\eta(t,\xi(t,x)) = x$. Then we apply the Itô-Wentzell formula to $\mathrm{d}[u(t,\xi(t,x))]$ and obtain

$$\begin{aligned} \mathrm{d}[u(t,\xi)] = \tilde{L}(t,\xi,u,Du,D^2u)\,\mathrm{d}t - \langle b \cdot Du, h\rangle_{\mathbb{R}^{d_1}}\,\mathrm{d}t \\ - \operatorname{trace}\left[Dh \cdot b^\top\right] u\,\mathrm{d}t + u\,\langle h, \mathrm{d}W_t\rangle_{\mathbb{R}^{d_1}}. \end{aligned} \tag{8.3}$$

Let $y(t,x) := u(t,\xi(t,x))$ be the solution (**Step 4**) and set $u(t,x) := y(t,\eta(t,x))$. Consider

$$\boxed{\mathrm{d}\varrho = |h|^2\varrho\,\mathrm{d}t - \varrho\langle h, \mathrm{d}W_t\rangle_{\mathbb{R}^{d_1}}} \quad (\textbf{Step 5}).$$

Let $\varrho(t,x)$ be the solution and define $v(t,x) = \varrho(t,x)\cdot y(t,x)$ (**Step 6**). Now we show $u(t,x) = \frac{v(t,\eta(t,x))}{\varrho(t,\eta(t,x))}$. (**Step 7**)

Calculating $Du(t,x)$, $D^2u(t,x)$ as well as $Du(t,\xi(t,x)), D^2u(t,\xi(t,x))$ we finally receive

$$\frac{\partial v}{\partial t}(t,x) = \frac{\partial}{\partial t}[\varrho(t,x)u(t,\xi(t,x))] = \Lambda(t,x,v,Dv,D^2v),$$

which is equivalent to

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t} = \Lambda(t,\cdot,v,Dv,D^2v) \\ v(0) = u_0. \end{cases}$$

The basic idea of this approach is the application of the **Itô-Wentzell formula** (cf. Theorem A.13), but given in the following version similar to [Tub88, Proposition 2]. From now on we use the short notation ∂_i for $\frac{\partial}{\partial x_i}$.

Theorem 8.3 *Let $(W_t^n)_{n=1,...,d_1}$ be a d_1-dimensional Brownian motion. Let $f_0, f_1, ..., f_{d_1}$ be continuous $\mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$ -functions such that*

$$\left(\sum_{n=1}^{d_1} f_n(t,x) f_n(t,y), f_0(t,x), t\right)$$

belong to the class $(B^{1,0}, B^{1,0})$. Consider the solution $u(t,x)$ of the stochastic differential equation

$$\mathrm{d}u(t,x) = f_0(t,x)\,\mathrm{d}t + \sum_{n=1}^{d_1} f_n(t,x)\,\mathrm{d}W_t^n.$$

Let $\kappa(t,x)$ be a continuous semimartingale with values in $\mathbb{D}$, i.e. it is represented componentwise for all $i = 1, ..., d$ by

$$\mathrm{d}\kappa_i(t,x) = \rho_i(t,x)\,\mathrm{d}t + \sum_{n=1}^{d_1} \sigma_{in}(t,x)\,\mathrm{d}W_t^n,$$

where ρ_i, σ_{in}, $i = 1, ..., d$, $n = 1, ..., d_1$ are continuous functions. Then the process $v(t,x) = u(t, \kappa(t,x))$ solves the following stochastic differential equation

$$\begin{aligned}
\mathrm{d}v(t,x) = f_0(t, \kappa(t,x))\,\mathrm{d}t &+ \sum_{n=1}^{d_1} f_n(t, \kappa(t,x))\,\mathrm{d}W_t^n \\
&+ \langle Du(t, \kappa(t,x)), \rho(t,x)\rangle_{\mathbb{R}^d}\,\mathrm{d}t \\
&+ \sum_{n=1}^{d_1} \langle Du(t, \kappa(t,x)), \sigma_{\cdot n}(t,x)\rangle_{\mathbb{R}^d}\,\mathrm{d}W_t^n \\
&+ \frac{1}{2}\sum_{i,j=1}^{d} \partial_i\partial_j u(t, \kappa(t,x)) \cdot \sum_{n=1}^{d_1} \sigma_{in}(t,x) \cdot \sigma_{jn}(t,x)\,\mathrm{d}t \\
&+ \sum_{i=1}^{d}\sum_{n=1}^{d_1} \partial_i f_n(t, \kappa(t,x)) \cdot \sigma_{in}(t,x)\,\mathrm{d}t
\end{aligned}$$

The proof follows the ideas of the proof of Theorem 3.3.1 in [Kun97].

Proof. Due to Theorem A.13 and Theorem 2.34 we know that for some F, g fulfilling

$$\begin{aligned}
F(x, \mathrm{d}t) &= f_0(t,x)\,\mathrm{d}t + \sum_{n=1}^{d_1} f_n(t,x)\,\mathrm{d}W_t^n, \\
g(x, \mathrm{d}t) &= \rho(t,x)\,\mathrm{d}t + \sum_{n=1}^{d_1} \sigma_{\cdot n}(t,x)\,\mathrm{d}W_t^n,
\end{aligned}$$

we have

$$\begin{aligned}
F(g(x,t),t) - F(g(x,0),0) &= \int_0^t \mathrm{d}F(g(x,s),s) \\
&= \int_0^t F(g(x,s), \mathrm{d}s) + \sum_{i=1}^{d} \int_0^t \partial_i F(g(x,s),s)\,\mathrm{d}g_i(x,s) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{d} \int_0^t \partial_i\partial_j F(g(x,s),s)\,\mathrm{d}\langle g_i(x,\bullet), g_j(x,\bullet)\rangle_s \\
&\quad + \sum_{i=1}^{d} \left\langle \int_0^{\bullet} \partial_i F(g(x,s), \mathrm{d}s), g_i(x,\bullet)\right\rangle_t .
\end{aligned} \tag{8.4}$$

One should note that $\partial_i F(g(x,s),s)$ means that we evaluate the partial derivative of F with respect to x_i at $(g(x,s),s)$. We obtain by Remark 2.37

$$F(x,s) - F(x,0) = \int_0^t F(x,\mathrm{d}s) = \int_0^t f_0(s,x)\,\mathrm{d}s + \sum_{n=1}^{d_1} \int_0^t f_n(s,x)\,\mathrm{d}W_s^n$$

By applying (8.4) to $\mathrm{d}u(t,x) = u(\mathrm{d}t,x)$ instead of F and $\mathrm{d}\kappa(t,x) = \kappa(\mathrm{d}t,x)$ instead of g we receive

$$\begin{aligned}
v(t,x) - v(0,x) &= u(t,\kappa(t,x)) - u(0,\kappa(0,x)) \\
&= \int_0^t f_0(s,\kappa(s,x))\,\mathrm{d}s + \int_0^t \sum_{n=1}^{d_1} f_n(s,\kappa(s,x))\,\mathrm{d}W_s^n \\
&\quad + \int_0^t \sum_{i=1}^{d} \partial_i u(s,\kappa(s,x))\,\mathrm{d}\kappa_i(s,x) \\
&\quad + \frac{1}{2}\int_0^t \sum_{i,j=1}^{d} \partial_i\partial_j u(s,\kappa(s,x))\,\mathrm{d}\left\langle \kappa_i(\bullet,x),\kappa_j(\bullet,x)\right\rangle_s \\
&\quad + \sum_{i=1}^{d} \left\langle \int_0^\bullet \partial_i u(\mathrm{d}s,\kappa(s,x)),\kappa_i(\bullet,x)\right\rangle_t \\
&= \int_0^t f_0(s,\kappa(s,x))\,\mathrm{d}s + \int_0^t \sum_{n=1}^{d_1} f_n(s,\kappa(s,x))\,\mathrm{d}W_s^n \\
&\quad + \int_0^t \sum_{i=1}^{d} \partial_i u(s,\kappa(s,x))\cdot\rho_i(s,x)\,\mathrm{d}s \\
&\quad + \int_0^t \sum_{i=1}^{d}\sum_{n=1}^{d_1} \partial_i u(s,\kappa(s,x))\cdot\sigma_{in}(s,x)\,\mathrm{d}W_s^n \\
&\quad + \frac{1}{2}\int_0^t \sum_{i,j=1}^{d} \partial_i\partial_j u(s,\kappa(s,x))\,\mathrm{d}\left\langle \kappa_i(\bullet,x),\kappa_j(\bullet,x)\right\rangle_s \\
&\quad + \sum_{i=1}^{d} \left\langle \int_0^\bullet \partial_i u(\mathrm{d}s,\kappa(s,x)),\kappa_i(\bullet,x)\right\rangle_t .
\end{aligned}$$

By using the classical fact that the quadratic variation vanishes if one element is of bounded variation, we conclude

$$\begin{aligned}
v(t,x) - v(0,x) &= \int_0^t f_0(s,\kappa(s,x))\,\mathrm{d}s + \int_0^t \sum_{n=1}^{d_1} f_n(s,\kappa(s,x))\,\mathrm{d}W_s^n \\
&+ \int_0^t \sum_{i=1}^{d} \partial_i u(s,\kappa(s,x))\cdot\rho_i(s,x)\,\mathrm{d}s + \int_0^t \sum_{n=1}^{d_1}\sum_{i=1}^{d} \partial_i u(s,\kappa(s,x))\cdot\sigma_{in}(s,x)\,\mathrm{d}W_s^n \\
&+ \frac{1}{2}\int_0^t \sum_{i,j=1}^{d} \partial_i\partial_j u(s,\kappa(s,x))\cdot\sum_{n=1}^{d_1}\sigma_{in}(s,x)\cdot\sigma_{jn}(s,x)\,\mathrm{d}s \\
&+ \sum_{i=1}^{d} \left\langle \int_0^\bullet \partial_i u(\mathrm{d}s,\kappa(s,x)),\kappa_i(\bullet,x)\right\rangle_t
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t f_0(s,\kappa(s,x))\,\mathrm{d}s + \int_0^t \sum_{n=1}^{d_1} f_n(s,\kappa(s,x))\,\mathrm{d}W_s^n \\
&\quad + \int_0^t \langle Du(s,\kappa(s,x)),\rho(s,x)\rangle_{\mathbb{R}^d}\,\mathrm{d}s \\
&\quad + \int_0^t \sum_{n=1}^{d_1} \langle Du(s,\kappa(s,x)),\sigma_{\cdot n}(s,x)\rangle_{\mathbb{R}^d}\,\mathrm{d}W_s^n \\
&\quad + \frac{1}{2}\int_0^t \sum_{i,j=1}^{d} \partial_i\partial_j u(s,\kappa(s,x))\cdot \sum_{n=1}^{d_1}\sigma_{in}(s,x)\cdot\sigma_{jn}(s,x)\,\mathrm{d}s \\
&\quad + \int_0^t \sum_{i=1}^{d}\sum_{n=1}^{d_1} \partial_i f_n(s,\kappa(s,x))\cdot\sigma_{in}(s,x)\,\mathrm{d}s
\end{aligned}
$$

□

Due to Assumption 8.1 (iii) and (iv) the operator L, the diffusion terms $b \cdot Du$ and $h \cdot u$ are $\mathcal{C}^2$ - functions and continuous. Consider (8.1) given by

$$
\begin{aligned}
\mathrm{d}u &= L(t,x,u,Du,D^2u)\,\mathrm{d}t + \langle b(t,x)Du + h(t,x)u, \mathrm{d}W_t\rangle_{\mathbb{R}^{d_1}} \\
&= L(t,x,u,Du,D^2u)\,\mathrm{d}t + \sum_{k=1}^{d_1}\sum_{i=1}^{d} b_{ki}(t,x)\cdot\partial_i u(t,x) + h_k(t,x)\cdot u(t,x)\,\mathrm{d}W_t^k
\end{aligned}
$$

and an arbitrary stochastic differential equation given by

$$
\begin{aligned}
\mathrm{d}\xi &= \xi^{\text{drift}}\,\mathrm{d}t + \xi^{\text{diffu}}\,\mathrm{d}W_t \\
&= \xi^{\text{drift}}\,\mathrm{d}t + \sum_{k=1}^{d_1}\xi^{\text{diffu}}_{\cdot k}\,\mathrm{d}W_t^k \\
&= \begin{pmatrix}\xi^{\text{drift}}_1\\ \vdots \\ \xi^{\text{drift}}_d\end{pmatrix}\mathrm{d}t + \begin{pmatrix}\sum_{k=1}^{d_1}\xi^{\text{diffu}}_{1k}\,\mathrm{d}W_t^k\\ \vdots \\ \sum_{k=1}^{d_1}\xi^{\text{diffu}}_{dk}\,\mathrm{d}W_t^k\end{pmatrix},
\end{aligned}
\tag{8.5}
$$

where ξ^{drift} is $\mathbb{R}^d$-valued and $\xi^{\text{diffu}} = (\xi^{\text{diffu}}_{\cdot 1},\ldots,\xi^{\text{diffu}}_{\cdot d_1})$ is $\mathbb{R}^{d\times d_1}$-valued. By applying Theorem 8.3 to (8.1) and (8.5) we obtain

$$
\begin{aligned}
\mathrm{d}u(t,\xi) = L(t,\xi,u(t,\xi),Du(t,\xi),D^2u(t,\xi))\,\mathrm{d}t& \\
+ \sum_{k=1}^{d_1}\left(\sum_{i=1}^{d} b_{ki}(t,\xi)\partial_i u(t,\xi) + h_k(t,\xi)\cdot u(t,\xi)\right)\mathrm{d}W_t^k& \\
+ \langle Du(t,\xi),\xi^{\text{drift}}\rangle_{\mathbb{R}^d}\,\mathrm{d}t + \sum_{k=1}^{d_1}\langle Du(t,\xi),\xi^{\text{diffu}}_{\cdot k}\rangle_{\mathbb{R}^d}\,\mathrm{d}W_t^k& \\
+ \frac{1}{2}\sum_{i,j=1}^{d}\partial_i\partial_j u(t,\xi)\sum_{k=1}^{d_1}\xi^{\text{diffu}}_{ik}\cdot\xi^{\text{diffu}}_{jk}\,\mathrm{d}t& \\
+ \sum_{i=1}^{d}\sum_{k=1}^{d_1}\sum_{j=1}^{d}\partial_i\left[b_{kj}(t,\xi)\cdot\partial_j u(t,\xi)\right]\cdot\xi^{\text{diffu}}_{ik}\,\mathrm{d}t&
\end{aligned}
$$

$$+\sum_{i=1}^{d}\sum_{k=1}^{d_1}\partial_i\left[h_k(t,\xi)\cdot u(t,\xi)\right]\cdot\xi_{ik}^{\text{diffu}}\ \mathrm{d}t.$$

Hence we have finally

$$\begin{aligned}
du(t,\xi) &= L(t,\xi,u(t,\xi),Du(t,\xi),D^2u(t,\xi))\ \mathrm{d}t && (8.6)\\
&+\frac{1}{2}\sum_{i,j=1}^{d}\partial_i\partial_j u(t,\xi)\sum_{k=1}^{d_1}\xi_{ik}^{\text{diffu}}\cdot\xi_{jk}^{\text{diffu}}\ \mathrm{d}t && (8.7)\\
&+\langle Du(t,\xi),\xi^{\text{drift}}\rangle_{\mathbb{R}^d}\ \mathrm{d}t && (8.8)\\
&+\sum_{i=1}^{d}\sum_{k=1}^{d_1}\sum_{j=1}^{d}\partial_i b_{kj}(t,\xi)\cdot\partial_j u(t,\xi)\cdot\xi_{ik}^{\text{diffu}}\ \mathrm{d}t && (8.9)\\
&+\sum_{i=1}^{d}\sum_{k=1}^{d_1}\sum_{j=1}^{d} b_{kj}(t,\xi)\cdot\partial_i\partial_j u(t,\xi)\cdot\xi_{ik}^{\text{diffu}}\ \mathrm{d}t && (8.10)\\
&+\sum_{i=1}^{d}\sum_{k=1}^{d_1}\partial_i h_k(t,\xi)\cdot u(t,\xi)\cdot\xi_{ik}^{\text{diffu}}\ \mathrm{d}t && (8.11)\\
&+\sum_{i=1}^{d}\sum_{k=1}^{d_1} h_k(t,\xi)\cdot\partial_i u(t,\xi)\cdot\xi_{ik}^{\text{diffu}}\ \mathrm{d}t && (8.12)\\
&+\sum_{k=1}^{d_1}\left(\sum_{i=1}^{d} b_{ki}(t,\xi)\partial_i u(t,\xi)+h_k(t,\xi)\cdot u(t,\xi)\right)\mathrm{d}W_t^k && (8.13)\\
&+\sum_{k=1}^{d_1}\langle Du(t,\xi),\xi_{\cdot k}^{\text{diffu}}\rangle_{\mathbb{R}^d}\ \mathrm{d}W_t^k. && (8.14)
\end{aligned}$$

Now we summarize the diffusion parts (8.13) and (8.14)

$$\begin{aligned}
&\sum_{k=1}^{d_1}\left(\sum_{i=1}^{d} b_{ki}(t,\xi)\partial_i u(t,\xi)+h_k(t,\xi)u(t,\xi)\right)\mathrm{d}W_t^k+\sum_{k=1}^{d_1}\langle Du(t,\xi),\xi_{\cdot k}^{\text{diffu}}\rangle_{\mathbb{R}^N}\ \mathrm{d}W_t^k\\
&=\sum_{k=1}^{d_1}\left(\sum_{i=1}^{d} b_{ki}(t,\xi)\partial_i u(t,\xi)+h_k(t,\xi)u(t,\xi)\right)+\langle Du(t,\xi),\xi_{\cdot k}^{\text{diffu}}\rangle_{\mathbb{R}^d}\ \mathrm{d}W_t^k\\
&=\sum_{k=1}^{d_1}\sum_{i=1}^{d} b_{ki}(t,\xi)\partial_i u(t,\xi)+h_k(t,\xi)u(t,\xi)+\partial_i u(t,\xi)\xi_{ik}^{\text{diffu}}\ \mathrm{d}W_t^k\\
&=\sum_{k=1}^{d_1}\sum_{i=1}^{d} b_{ki}(t,\xi(t,x))\partial_i u(t,\xi)+\partial_i u(t,\xi)\xi_{ik}^{\text{diffu}}(t,x)\ \mathrm{d}W_t^k+\sum_{k=1}^{d_1} h_k(t,\xi)u(t,\xi)\ \mathrm{d}W_t^k.
\end{aligned}$$

We choose $\xi_{ik}^{\text{diffu}}(t,x)$ such that

$$\Big(b_{ki}(t,\xi(t,x))+\xi_{ik}^{\text{diffu}}(t,x)\Big)=0.$$

Hence by defining

$$\boxed{\xi_{ik}^{\text{diffu}}(t,x):=-b_{ki}(t,\xi(t,x))=-b_{ik}^{\top}(t,\xi(t,x))}$$

we obtain

$$\begin{aligned}
&\sum_{k=1}^{d_1}\sum_{i=1}^{d}\partial_i u(t,\xi)\Big(b_{ki}(t,\xi)-b_{ki}(t,\xi)\Big)\ \mathrm{d}W_t^k+u(t,\xi)\cdot\sum_{k=1}^{d_1} h_k(t,\xi)\ \mathrm{d}W_t^k\\
&=u(t,\xi)\cdot\sum_{k=1}^{d_1} h_k(t,\xi)\ \mathrm{d}W_t^k
\end{aligned}$$

$$= u(t,\xi)\cdot\langle h(t,\xi),\mathrm{d}W_t\rangle_{\mathbb{R}^{d_1}}$$
$$= y(t,x)\cdot\langle h(t,\xi),\mathrm{d}W_t\rangle_{\mathbb{R}^{d_1}},$$

where we set $y(t,x) := u(t,\xi(t,x))$. Therefore we have

$$\boxed{\xi^{\mathrm{diffu}}(t,x) = -b^\top(t,\xi(t,x)) \in \mathbb{R}^{d\times d_1}} \tag{8.15}$$

Next, we consider the sum of (8.8), (8.9), (8.11) and (8.12)

$$\begin{aligned}
&\langle Du(t,\xi),\xi^{\mathrm{drift}}\rangle_{\mathbb{R}^d} + \sum_{i,j=1}^{d}\sum_{k=1}^{d_1}\Bigg(\partial_i b_{kj}(t,\xi)\cdot\partial_j u(t,\xi)\cdot\xi^{\mathrm{diffu}}_{ik} \\
&\quad + \partial_i h_k(t,\xi)\cdot u(t,\xi)\cdot\xi^{\mathrm{diffu}}_{ik} + h_k(t,\xi)\cdot\partial_i u(t,\xi)\cdot\xi^{\mathrm{diffu}}_{ik}\Bigg)\,\mathrm{d}t \\
&= \langle Du(t,\xi),\xi^{\mathrm{drift}}\rangle_{\mathbb{R}^d} + \sum_{i,j=1}^{d}\sum_{k=1}^{d_1}\Bigg(\partial_i b_{kj}(t,\xi)\cdot\partial_j u(t,\xi)\cdot(-b_{ki}(t,\xi)) \\
&\quad + \partial_i h_k(t,\xi)\cdot u(t,\xi)\cdot(-b_{ki}(t,\xi)) + h_k(t,\xi)\cdot\partial_i u(t,\xi)\cdot(-b_{ki}(t,\xi))\Bigg)\,\mathrm{d}t \\
&= \sum_{j=1}^{d}\partial_j u(t,\xi)\cdot\xi^{\mathrm{drift}}_j + \sum_{i,j=1}^{d}\sum_{k=1}^{d_1}\Big(-\partial_i b_{kj}(t,\xi)\cdot\partial_j u(t,\xi)\cdot b_{ki}(t,\xi)\Big) \\
&\quad - \partial_i h_k(t,\xi)\cdot u(t,\xi)\cdot b_{ki}(t,\xi) - h_k(t,\xi)\cdot\partial_i u(t,\xi)\cdot b_{ki}(t,\xi)\,\mathrm{d}t \\
&= \sum_{j=1}^{d}\partial_j u(t,\xi)\Bigg(\xi^{\mathrm{drift}}_j - \sum_{i=1}^{d}\sum_{k=1}^{d_1}\partial_i b_{kj}(t,\xi)\cdot b_{ki}(t,\xi)\Bigg) \\
&\quad - \sum_{k=1}^{d_1}\sum_{i=1}^{d}\partial_i h_k(t,\xi)\cdot u(t,\xi)\cdot b_{ki}(t,\xi) - h_k(t,\xi)\cdot\partial_i u(t,\xi)\cdot b_{ki}(t,\xi)\,\mathrm{d}t.
\end{aligned}$$

We choose $\xi^{\mathrm{drift}}_j(t,x)$ such that

$$\Big(\xi^{\mathrm{drift}}_j - \sum_{k=1}^{d_1}\sum_{i=1}^{d}\partial_i b_{kj}(t,\xi)\cdot b_{ki}(t,\xi)\Big) = 0.$$

Hence by defining

$$\boxed{\xi^{\mathrm{drift}}_j(t,x) := \mathrm{trace}[D(b(t,\xi(t,x))\cdot e_j)\cdot b^\top(t,\xi(t,x))]}$$

we obtain by Definition 8.2

$$\begin{aligned}
&-\sum_{i=1}^{d}\sum_{k=1}^{d_1}h_k(t,\xi)\cdot\partial_i u(t,\xi)\cdot b_{ki}(t,\xi) - \sum_{i=1}^{d}\sum_{k=1}^{d_1}\partial_i h_k(t,\xi)\cdot u(t,\xi)\cdot b_{ki}(t,\xi)\,\mathrm{d}t \\
&= -\langle b(t,\xi)Du(t,\xi),h(t,\xi)\rangle_{\mathbb{R}^{d_1}}\,\mathrm{d}t - \mathrm{trace}[Dh(t,\xi)b^\top(t,\xi)]u(t,\xi)\,\mathrm{d}t \\
&= -\langle b(t,\xi)Dy(t,x),h(t,\xi)\rangle_{\mathbb{R}^{d_1}}\,\mathrm{d}t - \mathrm{trace}[Dh(t,\xi)b^\top(t,\xi)]y(t,x)\,\mathrm{d}t.
\end{aligned}$$

Therefore we have

$$\boxed{\begin{aligned}\xi^{\mathrm{drift}}(t,x) &= \begin{pmatrix}\mathrm{trace}[(D(b(t,\xi(t,x))\cdot e_1))\cdot b^\top(t,\xi(t,x))]\\ \vdots \\ \mathrm{trace}[(D(b(t,\xi(t,x))\cdot e_d))\cdot b^\top(t,\xi(t,x))]\end{pmatrix}\\ &= \mathrm{TR}[Db(t,\xi(t,x))\cdot b^\top(t,\xi(t,x))]\end{aligned}} \tag{8.16}$$

Now we determine the sum of (8.6), (8.7) and (8.10)

$$
\begin{aligned}
& L(t,\xi,u(t,\xi),Du(t,\xi),D^2u(t,\xi))\,\mathrm{d}t + \frac{1}{2}\sum_{i,j=1}^{d}\partial_i\partial_j u(t,\xi)\sum_{k=1}^{d_1}\xi_{ik}^{\text{diffu}}\xi_{jk}^{\text{diffu}}\,\mathrm{d}t \\
&\quad + \sum_{i,j=1}^{d}\sum_{k=1}^{d_1} b_{kj}(t,\xi(t,x))\cdot\partial_i\partial_j u(t,\xi)\cdot\xi_{ik}^{\text{diffu}}\,\mathrm{d}t \\
&= L(t,\xi,u(t,\xi),Du(t,\xi),D^2u(t,\xi))\,\mathrm{d}t \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{d}\partial_i\partial_j u(t,\xi)\sum_{k=1}^{d_1}(-b_{kj}(t,\xi))(-b_{ki}(t,\xi))\,\mathrm{d}t \\
&\quad - \sum_{i,j=1}^{N}\sum_{k=1}^{d_1} b_{kj}(t,\xi)\cdot\partial_i\partial_j u(t,\xi)\cdot b_{ki}(t,\xi)\,\mathrm{d}t \\
&= L(t,\xi,u(t,\xi),Du(t,\xi),D^2u(t,\xi))\,\mathrm{d}t + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{d_1}\partial_i\partial_j u(t,\xi)b_{kj}(t,\xi)b_{ki}(t,\xi)\,\mathrm{d}t \\
&\quad - \sum_{i,j=1}^{d}\sum_{k=1}^{d_1}\partial_i\partial_j u(t,\xi)b_{kj}(t,\xi)b_{ki}(t,\xi)\,\mathrm{d}t \\
&= L(t,\xi,u(t,\xi),Du(t,\xi),D^2u(t,\xi))\,\mathrm{d}t - \frac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{d_1}\partial_i\partial_j u(t,\xi)b_{kj}(t,\xi)b_{ki}(t,\xi)\,\mathrm{d}t \\
&= L(t,\xi,u(t,\xi),Du(t,\xi),D^2u(t,\xi))\,\mathrm{d}t - \frac{1}{2}\operatorname{trace}[b^\top(t,\xi)b(t,\xi)D^2u(t,\xi)]\,\mathrm{d}t.
\end{aligned}
$$

Step 1: New operator
Define a new elliptic nonlinear operator by

$$
\tilde{L}(t,x,u,p,q) := L(t,x,u,p,q) - \frac{1}{2}\operatorname{trace}[b^\top(t,x)\cdot b(t,x)\cdot q] \tag{8.17}
$$

Step 2: New SDE
Consider the system of stochastic differential equations for the paramatrized space variable $\xi(t,x)$

$$
\begin{cases}
\mathrm{d}\xi = \mathrm{TR}[Db(t,\xi)\cdot b^\top(t,\xi)]\,\mathrm{d}t - b^\top(t,\xi)\,\mathrm{d}W_t \\
\xi(0) = x.
\end{cases} \tag{8.18}
$$

The function $b(t,x)$ satisfies in particular Condition 8.1 (v), which is necessary to apply the next Lemma.

Lemma 8.4 *Under Assumption 8.1 there exists a unique solution $\xi(t,x)$ to the stochastic differential equation (8.18).*

Proof. We apply [LR15, Theorem 3.1.1.] to obtain an existence and uniqueness result to equation (8.18). The conditions which we have to verify are formally an integrability condition, the local weak monotonicity and weak coercivity. As explained in [KRZ99, Remark 1.3.], it follows that the above assumptions are satisfied if

$$
\int_0^{\mathbf{T}} \|-b^\top(t,0)\|^2 - |\mathrm{TR}[Db(t,0)\cdot b^\top(t,0)]|\,\mathrm{d}t < \infty \tag{8.19}
$$

and the global Lipschitz condition in space

$$
\|-b^\top(t,x)+b^\top(t,y)\|^2 + |\mathrm{TR}[Db(t,x)\cdot b^\top(t,x)] - \mathrm{TR}[Db(t,y)\cdot b^\top(t,y)]| \le K|x-y|
$$

holds for all $t \in [0, \mathbf{T}]$, $x, y \in \mathbb{R}^d$ and a constant $K > 0$. Condition (8.19) contains the integrability in point 0 of the corresponding norms for the drift and diffusion term. Due to the classical result that a uniformly continuous function is integrable, this condition is fulfilled by Assumption 8.1 (iv). The global Lipschitz condition is satisfied due to Assumption 8.1 (v). □

The solution $\xi(t,x)$ is defined up to an explosion time $T(x,\omega)$, i.e. for almost all ω and all (x,t) with $t \in [0, T(x,\omega))$. Since $\xi(t,x)$ is not a diffeomorphism in general, we restrict this map to a domain for which the determinant of the Jacobian matrix is not singular. Therefore we define as before

$$\tau(x) := \inf\left\{ t > 0 \,\middle|\, \det D\xi(t,x) = 0 \right\} \wedge T(x).$$

The corresponding adjoint stopping time is

$$\sigma(y) := \inf\left\{ t > 0 \,\middle|\, y \notin \xi(t, \{x \,|\, \tau(x) > t\}) \right\}$$

i.e. up to this time a point y is in the codomain of $\{x \,|\, \tau(x) > t\}$.

Step 3: Inverse function

We want to find the inverse function of the solution. By using the approach of H. Kunita, similarly to [Kun84a, Lemma 3.1.], we are able to prove an existence result.

Lemma 8.5 *Let $\xi(t,x)$ be the solution of equation (8.18). Then*

a) the map $\xi(t,\cdot)\colon \{x \in \mathbb{R}^d \,|\, \tau(x,\omega) > t\} \to \mathbb{R}^d$ is a diffeomorphism and

b) for almost all ω and for all (x,t) such that $t \in [0, \sigma(x))$ the inverse function $\eta(t,x)$ satisfies the following stochastic differential equation

$$\begin{cases} \mathrm{d}\eta = \frac{1}{2}(D\xi(\eta))^{-1}\,\mathrm{TR}[Db(t,\xi(\eta))b^\intercal(t,\xi(\eta))]\,\mathrm{d}t \\ \qquad\qquad + (D\xi(\eta))^{-1} b^\intercal(t,\xi(\eta)) \circ \mathrm{d}W_t \\ \eta(0,x) = x. \end{cases} \tag{8.20}$$

Proof. The Ansatz is to use the Itô-Wentzell formula similarly to the proof Lemma 4.8 above.

Step A: Transformation into Stratonovich setting

As shown in Theorem 2.35 a multidimensional Itô stochastic differential equation can be written equivalently in terms of Stratonovich differentials. Carried over to the stochastic differential equation of $\mathrm{d}\xi$ we obtain

$$\begin{aligned}
\mathrm{d}\xi &= \mathrm{TR}[Db(t,\xi)\cdot b^\intercal(t,\xi)]\,\mathrm{d}t - b^\intercal(t,\xi)\,\mathrm{d}W_t \\
&= \begin{pmatrix} \sum_{k=1}^{d_1}\sum_{i=1}^{d} \frac{\partial b_{k1}}{\partial x_i}(t,\xi) b_{ki}(t,\xi) \\ \vdots \\ \sum_{k=1}^{d_1}\sum_{i=1}^{d} \frac{\partial b_{kd}}{\partial x_i}(t,\xi) b_{ki}(t,\xi) \end{pmatrix} \mathrm{d}t - \sum_{k=1}^{d_1} \begin{pmatrix} b_{k1}(t,\xi) \\ \vdots \\ b_{kd}(t,\xi) \end{pmatrix} \mathrm{d}W_t^k \\
&= \begin{pmatrix} \sum_{k=1}^{d_1}\sum_{i=1}^{d} \frac{\partial b_{k1}}{\partial x_i}(t,\xi) b_{ki}(t,\xi) - \frac{1}{2}\sum_{k=1}^{d_1}\sum_{i=1}^{d} \frac{\partial b_{k1}}{\partial x_i}(t,\xi) b_{ki}(t,\xi) \\ \vdots \\ \sum_{k=1}^{d_1}\sum_{i=1}^{d} \frac{\partial b_{kd}}{\partial x_i}(t,\xi) b_{ki}(t,\xi) - \frac{1}{2}\sum_{k=1}^{d_1}\sum_{i=1}^{d} \frac{\partial b_{k1}}{\partial x_i}(t,\xi) b_{ki}(t,\xi) \end{pmatrix} \mathrm{d}t - \sum_{k=1}^{d_1} \begin{pmatrix} b_{k1}(t,\xi) \\ \vdots \\ b_{kd}(t,\xi) \end{pmatrix} \circ \mathrm{d}W_t^k
\end{aligned}$$

$$= -\frac{1}{2}\begin{pmatrix} \sum_{k=1}^{d_1}\sum_{i=1}^{d} \frac{\partial b_{k1}}{\partial x_i}(t,\xi)b_{ki}(t,\xi) \\ \vdots \\ \sum_{k=1}^{d_1}\sum_{i=1}^{d} \frac{\partial b_{kd}}{\partial x_i}(t,\xi)b_{ki}(t,\xi) \end{pmatrix} \mathrm{d}t - \sum_{k=1}^{d_1}\begin{pmatrix} b_{k1}(t,\xi) \\ \vdots \\ b_{kd}(t,\xi)\end{pmatrix} \circ \mathrm{d}W_t^k$$

$$= -\frac{1}{2}\,\mathrm{TR}[Db(t,\xi)\cdot b^\top(t,\xi)]\,\mathrm{d}t - b^\top(t,\xi)\circ \mathrm{d}W_t \tag{8.21}$$

Step B: The stochastic differential equation of the inverse process
Let $\eta(t,x)$ be the solution to

$$\begin{cases} \mathrm{d}\eta = \frac{1}{2}(D\xi(\eta))^{-1}\,\mathrm{TR}[Db(t,\xi(\eta))b^\top(t,\xi(\eta))]\,\mathrm{d}t \\ \qquad\quad + (D\xi(\eta))^{-1}b^\top(t,\xi(\eta))\circ \mathrm{d}W_t \\ \eta(0,x) = x \end{cases}$$

Step C: Application of the Itô-Wentzell formula
By using the generalized Itô formula of Theorem 4.7 we show that $\mathrm{d}[\xi(\eta)] = \mathrm{d}[\xi(t,\eta(t,x))] = 0$. With the initial condition $\xi(0,\eta(0,x)) = \xi(0,x) = x$ we obtain that η is the right inverse function of ξ.

$$\begin{aligned}
\mathrm{d}[\xi(\eta)] &= \mathrm{d}\xi(t,\eta) + \sum_{i=1}^{d}\partial_i\xi(t,\eta)\circ \mathrm{d}\eta^i \\
&= -\frac{1}{2}\,\mathrm{TR}[Db(t,\xi(t,\eta))\cdot b^\top(t,\xi(t,\eta))]\,\mathrm{d}t - b^\top(t,\xi(t,\eta))\circ \mathrm{d}W_t \\
&\quad + \frac{1}{2}\sum_{i=1}^{d}\partial_i\xi(t,\eta)(D\xi(t,\eta))^{-1}\,\mathrm{TR}[Db(t,\xi(t,\eta))b^\top(t,\xi(t,\eta))]\,\mathrm{d}t \\
&\quad + \sum_{i=1}^{d}\partial_i\xi(t,\eta)(D\xi(t,\eta))^{-1}b^\top(t,\xi(\eta))\circ \mathrm{d}W_t \\
&= -\frac{1}{2}\,\mathrm{TR}[Db(t,\xi(t,\eta))\cdot b^\top(t,\xi(t,\eta))]\,\mathrm{d}t - b^\top(t,\xi(t,\eta))\circ \mathrm{d}W_t \\
&\quad + \frac{1}{2}(D\xi(t,\eta))\cdot(D\xi(t,\eta))^{-1}\,\mathrm{TR}[Db(t,\xi(t,\eta))b^\top(t,\xi(t,\eta))]\,\mathrm{d}t \\
&\quad + (D\xi(t,\eta))\cdot(D\xi(t,\eta))^{-1}b^\top(t,\xi(t,\eta))\circ \mathrm{d}W_t \\
&= -\frac{1}{2}\,\mathrm{TR}[Db(t,\xi(t,\eta))\cdot b^\top(t,\xi(t,\eta))]\,\mathrm{d}t - b^\top(t,\xi(t,\eta))\circ \mathrm{d}W_t \\
&\quad + \frac{1}{2}\,\mathrm{TR}[Db(t,\xi(t,\eta))b^\top(t,\xi(t,\eta))]\,\mathrm{d}t \\
&\quad + b^\top(t,\xi(t,\eta))\circ \mathrm{d}W_t \\
&= 0.
\end{aligned}$$

In particular we proved that the right inverse function exists for almost all ω and for all (t,x) with $t < \sigma(x,\omega)$, hence

$$\xi(t,\eta(t,x)) = x$$

holds.
Step D: Definition of stopping times
Now we define for the explosion time $\hat{\sigma}$ of η

$$\hat{\tau}(x) = \inf\Big\{t > 0 \,\Big|\, \xi(t,x) \notin \{y \,|\, \hat{\sigma}(y,\omega) > t\} \text{ or } |\det D\eta(t,\xi(t,x)| = \infty\Big\} \wedge \tau(x)$$

and show that the property of η to be the left inverse is also fulfilled by proving

$$d[\eta(t,\xi(t,x))] = 0 \quad \text{if } t < \hat{\tau}(x).$$

Since $\xi(t,\eta(t,x)) = x$ as shown before, we obtain by an application of the chain rule

$$D\xi(t,\eta(t,x)) \cdot D\eta(t,x) = D[\xi(t,\eta(t,x))] = Dx = \mathbb{I}. \tag{8.22}$$

Taking the inverse and evaluating at $\xi(t,\cdot)$ we get

$$\Big(D\eta(t,\xi(t,\cdot)\Big)^{-1} \cdot \Big(D\xi(t,\eta(t,\xi(t,\cdot))\Big)^{-1} = \mathbb{I}.$$

In the next step we rewrite equation (8.21) and use of equation (8.22)

$$\begin{aligned} d\xi &= -\frac{1}{2}\,\mathrm{TR}[Db(t,\xi)\cdot b^\top(t,\xi)]\,dt - b^\top(t,\xi)\circ dW_t \\ &= -\frac{1}{2}\,\mathrm{TR}[(D\eta(t,\xi(t,\cdot))^{-1}\cdot(D\xi(t,\eta(t,\xi(t,\cdot)))^{-1}\cdot Db(t,\xi)\cdot b^\top(t,\xi)]\,dt \\ &\quad -(D\eta(t,\xi(t,\cdot))^{-1}\cdot(D\xi(t,\eta(t,\xi(t,\cdot)))^{-1}\cdot b^\top(t,\xi)\circ dW_t. \end{aligned}$$

Then we apply again the generalized Itô formula and obtain

$$d[\eta(\xi)] = d[\eta(t,\xi(t,x))] = 0.$$

Step E: Prove that $\hat{\tau} = \tau$ and that ξ_t is a diffeomorphism
By the same arguments as in the proof of Lemma 4.8 we can show that $\hat{\tau}(x) = \tau(x)$ a.s. Suppose $\xi(t,x) = \xi(t,x')$ holds for $x,x' \in \{\tilde{x}\,|\,\tau(\tilde{x}) > t\}$. Since $\eta(t,\xi(t,x)) = x$ holds for almost all ω and $t < \tau(x,\omega)$, we obtain

$$\eta(t,\xi(t,x)) = \eta(t,\xi(t,x')) \Rightarrow x = x'.$$

So $\xi(t,x)|_{\{\tau>t\}}$ is one-to-one (injective). By using the inverse mapping theorem (see [Lan96, Chapter XIV, Theorem 1.2]) we obtain that ξ is a diffeomorphism. □

Step 4: Composition of solutions
We define

$$y(t,x) := u(t,\xi(t,x)). \tag{8.23}$$

Obviously we obtain for the inverse function

$$y(t,\eta(t,x)) = u(t,\xi(t,\eta(t,x))) = u(t,x).$$

Hence a solution to (8.1) is given in the form

$$u(t,x) = y(t,\eta(t,x)).$$

Step 5: New SDE by multiplication with a process
We set

$$v(t,x) := \varrho(t,x)\cdot y(t,x), \tag{8.24}$$

where $\varrho(t,x)$ is the solution of

$$\begin{cases} d\varrho = |h(t,x)|^2\varrho(t,x)\,dt - \varrho(t,x)\langle h(t,x), dW_t\rangle_{\mathbb{R}^{d_1}} \\ \varrho(0) = 1. \end{cases} \tag{8.25}$$

As claimed in [DPT96] the solution is given by

$$\varrho(t,x) = \exp\left[\frac{1}{2}\int_0^t |h(s,\xi(s,x))|^2 \,\mathrm{d}s - \int_0^t \langle h(s,\xi(s,x)), \mathrm{d}W_s\rangle_{\mathbb{R}^{d_1}}\right]. \tag{8.26}$$

Here we have to use Itô's product rule [RY05, Chapter IV, 3.1 Proposition]. A detailed derivation of (8.25) can be found in Appendix G. Calculating the partial derivative with respect to time t of $v(t,x)$, we receive

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t}(t,x) = \varrho(t,x)\tilde{L}(t,\xi,u(t,\xi),Du(t,\xi),D^2u(t,\xi)) - \varrho(t,x)\langle b(t,\xi)Du, h\rangle_{\mathbb{R}^{d_1}} \\ \qquad\qquad - \operatorname{trace}[Dh\cdot b]v(t,x) \\ v(0) = u_0. \end{cases} \tag{8.27}$$

Step 6: Combination of u, v, ϱ, η
Plugging $y(t,\eta(t,x)) = u(t,x)$ into (8.24), we get an expression of solution to (8.1) in terms of the three solutions v, η, ϱ by

$$u(t,x) = y(t,\eta(t,x)) = \frac{\varrho(t,\eta(t,x))}{v(t,\eta(t,x))}.$$

Step 7: Solution of problem (8.2)
By computing Du, D^2u and by plugging them into (8.27) we obtain (8.2).

Definition 8.6 *Let T be a $(\mathscr{F}_t)_{t\geq 0}$ - stopping time. A **strong solution** of problem (8.1) in $[0,T]$ is a mapping*

$$u\colon [0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$$

such that the following hold:

(i) $u(t,\cdot)$ is $\mathscr{F}_t$-Bochner-measurable for all $t\geq 0$, i.e. $u(t,\cdot)$ is the a.s.-limit of simple random variables with values in $\mathcal{C}^{2,\beta}(\mathbb{R}^d,\mathbb{R})$

(ii) For all $x\in\mathbb{R}^d$ the real-valued stochastic process $u(\cdot,x)$ is such that

$$L(t,x,u,Du,D^2u) \in L^1([0,T]\times\Omega),\mathbb{R})$$
$$b(\cdot,x)Du \in L^2([0,T]\times\Omega),\mathbb{R}^{d_1})$$
$$h(\cdot,x)u \in L^2([0,T]\times\Omega),\mathbb{R}^{d_1}).$$

(iii) For almost all ω and all (x,t) with $t\in[0,T(x,\omega)]$

$$u(t,\cdot) = u_0 + \int_0^t L(s,\cdot,u,Du,D^2u)\,\mathrm{d}s + \int_0^t \langle b(s,\cdot)Du(s,\cdot) + h(s,\cdot)u(s,\cdot), \mathrm{d}W_s\rangle_{\mathbb{R}^{d_1}}$$

holds.

As found in the Appendix A of [DPT96] we also define

Definition 8.7 *Let X, Y be Banach spaces such that $X\subset Y$ continuously. Let $F\colon[0,\infty)\times X\to Y$ be an operator and $J\subset[0,\infty)$ be an interval such that $\min J = 0$. A function v is called a **strict solution** of*

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t} = F(t,v(t)) \\ v(0) = v_0\in X, \end{cases} \tag{8.28}$$

if for a fixed $\theta\in(0,1)$ the following hold:

(i) $v \in \mathcal{C}^{1,\theta}(J_1, Y) \cap \mathcal{C}^{0,\theta}(J_1, X)$ *for any closed and bounded subinterval* $J_1 \subset J$
(ii) $\frac{\mathrm{d}v}{\mathrm{d}t}(t) = F(t, v(t))$, $t \in J$, *and* $v(0) = v_0$ *is satisfied.*

The following result [DPT96, Proposition 2.2] established the equivalence relation between the stochastic partial differential equation (8.1) and the transformed problem (8.2).

Theorem 8.8 *Let* $u_0 \in \mathcal{C}^{2,\beta}(\mathbb{R}^d)$ *and let* $\tau(x) \leq \boldsymbol{T}$ *be a stopping time with respect to the filtration* $(\mathscr{F}_t)_{t\geq 0}$.

- *If* $u(\cdot, x)$ *is a strong solution of (8.1) in* $[0, \tau(x)]$, *then the function* $v(\cdot,\cdot) = \varrho(\cdot,\cdot)u(\cdot, \xi(\cdot,\cdot))$ *is a strict solution of (8.2).*
- *If* $v(\cdot, x) \in \mathcal{C}([0, \tau(x)], \mathcal{C}^{2,\beta}(\mathbb{R}^d)) \cap \mathcal{C}^1([0, \tau(x)], \mathcal{C}^{\beta}(\mathbb{R}^d))$ *for almost all* ω *and* v *is a strict solution of (8.2) such that* $v(t, \cdot)$ *is* $(\mathscr{F}_t)$*-Bochner-measurable for any* $t \geq 0$, *then* $u(\cdot,\cdot) = \frac{v(\cdot,\eta(\cdot,\cdot))}{\varrho(\cdot,\eta(\cdot,\cdot))}$ *is a strong solution of (8.1).*

The existence result of strict solutions to (8.28) are also written in the Appendix of [DPT96, Theorem A.2], but we will not go into detail, since the aim of this chapter was an application of Kunita's approach with Lemma 8.5.

A. Appendix to the preliminaries

First of all we reproduce the basic definition of orthogonality.

Definition A.1 *Let $M, N \in \mathscr{M}_c^{\text{loc}}$. If $\langle M, N\rangle_t = 0$ for all $t \in [0, \boldsymbol{T}]$, then M, N are called* ***orthogonal*** *$(M \perp N)$. Let $M, N \in \mathscr{M}_c$. Then M, N are orthogonal if and only if*

$$\mathbb{E}\Big[(M_t - M_s)(N_t - N_s)\Big|\mathscr{F}_s\Big] = 0$$

holds for all $s, t \in [0, \boldsymbol{T}]$, $s < t$.

Definition A.2 *Let $L^2(\langle M\rangle)$ be the set of all predictable processes f_t such that*

$$\int_0^{\boldsymbol{T}} |f_s|^2 \,\mathrm{d}\langle M\rangle_s < \infty \text{ a.s.}$$

Let $M \in \mathscr{M}_c^{\text{loc}}$. Define

$$\mathscr{L}(\langle M\rangle) := \Big\{ \int_0^t f_s \,\mathrm{d}M_s \,\Big|\, f \in L^2(\langle M\rangle), t \in [0, \boldsymbol{T}]\Big\}.$$

In this section we reproduce the result concerning an orthogonal decomposition of local martingales as in [Kun97, after Theorem 2.3.6] which is also known as the Kunita-Watanabe decomposition. To this end we first look at the following lemma.

Lemma A.3 *Let $M, N \in \mathscr{M}_c^{\text{loc}}$. Then there exists a unique $f \in L^2(\langle M\rangle)$ satisfying*

$$\langle M, N\rangle_t = \int_0^t f_s \,\mathrm{d}\langle M\rangle_s. \tag{A.1}$$

In particular if $N \in \mathscr{M}_c$, then we have

$$\mathbb{E}\Big[\int_0^{\boldsymbol{T}} |f_s|^2 \,\mathrm{d}\langle M\rangle_s\Big] < \infty.$$

The proof is given in [Kun97, Lemma 2.3.7]. Now we state the famous Kunita-Watanabe decomposition as in [Kun97, Theorem 2.3.8].

Theorem A.4 *Let $M, N \in \mathscr{M}_c^{\text{loc}}$. Then there exists a unique $N^{(1)} \in \mathscr{L}(\langle M\rangle)$ and a unique $N^{(2)} \in \mathscr{M}_c^{\text{loc}}$ such that $N^{(2)}$ is orthogonal on $\mathscr{L}(\langle M\rangle)$ and the decomposition $N = N^{(1)} + N^{(2)}$ holds.*

The proof follows the idea of the proof of [Kun97, Theorem 2.3.8]. For the reader's convenience we check all the details.

Proof.
Existence of the decomposition
Due to Lemma A.3 there exists a unique $f \in L^2(\langle M\rangle)$ such that (A.1) holds. Define

$$N_t^{(1)} := \int_0^t f_s \,\mathrm{d}M_s \quad\text{and}\quad N_t^{(2)} := N_t - N_t^{(1)}.$$

Then we obtain with [Kun97, Theorem 2.3.2]

$$\langle N^{(1)}, M\rangle_t = \Big\langle \int_0^{\cdot} f_s \,\mathrm{d}M_s, M\Big\rangle_t$$

$$= \int_0^t f_s \, \mathrm{d}\langle M, M\rangle_s$$
$$= \int_0^t f_s \, \mathrm{d}\langle M\rangle_s.$$

Hence by using (A.1)

$$\langle N^{(1)}, M\rangle_t = \langle N, M\rangle_t \tag{A.2}$$

Therefore we get by definition and linearity of the joint quadratic variation (see [Kun97, Theorem 2.2.13])

$$\begin{aligned}\langle N^{(2)}, M\rangle_t &= \langle N - N^{(1)}, M\rangle_t \\ &= \langle N, M\rangle_t - \langle N^{(1)}, M\rangle_t \\ &= \langle N, M\rangle_t - \langle N, M\rangle_t = 0.\end{aligned}$$

Hence orthogonality of $N^{(2)}$ on $\mathscr{L}(\langle M\rangle)$ is fulfilled, since for an arbitrary $\tilde{M} \in \mathscr{L}(\langle M\rangle)$ given by $\int_0^t \tilde{f}_s \, \mathrm{d}M_s$ we have

$$\langle N^{(2)}, \int_0^{\cdot} \tilde{f}_s \, \mathrm{d}M_s\rangle_t = \int_0^t \tilde{f}_s \, \mathrm{d}\langle N^{(2)}, M\rangle_s = 0.$$

Hence we have shown the existence of an orthogonal decomposition with

$$N = N^{(1)} + N^{(2)}.$$

Uniqueness of the decomposition
Suppose that $N = \hat{N}^{(1)} + \hat{N}^{(2)}$ with $\hat{N}^{(1)} \in \mathscr{L}(\langle M\rangle)$ and $\hat{N}^{(2)} \in \mathscr{M}_c^{\mathrm{loc}}$ orthogonal on $\mathscr{L}(\langle M\rangle)$. Then

$$\begin{aligned}\hat{N}^{(2)} - N^{(2)} &= N - \hat{N}^{(1)} - N + N^{(1)} \\ &= N^{(1)} - \hat{N}^{(1)} \in \mathscr{L}(\langle M\rangle).\end{aligned}$$

Since $M \in \mathscr{M}_c^{\mathrm{loc}}$, we obtain with (A.2)

$$\begin{aligned}\langle \hat{N}^{(2)} - N^{(2)}, M\rangle_t &= \langle N^{(1)} - \hat{N}^{(1)}, M\rangle_t \\ &= \langle N^{(1)}, M\rangle_t - \langle \hat{N}^{(1)}, M\rangle_t \\ &= \langle N, M\rangle_t - \langle N, M\rangle_t = 0.\end{aligned}$$

Hence $\hat{N}_t^{(2)} - N_t^{(2)}$ is orthogonal to M and on $\mathscr{L}(\langle M\rangle)$. We conclude $\hat{N}_t^{(2)} = N_t^{(2)}$ and $\hat{N}_t^{(1)} = N_t^{(1)}$. □

Remark A.5 *Now we denote $N^{(1)}$ by $\mathscr{P}_{\mathscr{L}(\langle M\rangle)}[N]$ the orthogonal projection of N to $\mathscr{L}(\langle M\rangle)$. Due to Theorem A.4 we have*

$$\begin{aligned}N = N^{(1)} + N^{(2)} &= \mathscr{P}_{\mathscr{L}(\langle M\rangle)}[N] + N^{(2)} \\ \textit{and} \quad N^{(2)} &= N - \mathscr{P}_{\mathscr{L}(\langle M\rangle)}[N].\end{aligned}$$

By the Gram-Schmidt orthogonalization (see e.g. [Gre75, 10.8 The Schmidt-orthogonalization]) we construct an orthogonal basis of elements in $\mathscr{M}_c^{\mathrm{loc}}$. Let $M_t^{(1)}, ..., M_t^{(n)} \in \mathscr{M}_c^{\mathrm{loc}}$. Define

$$\begin{aligned}N_t^{(1)} &:= M_t^{(1)} \\ N_t^{(2)} &:= M_t^{(2)} - \mathscr{P}_{\mathscr{L}(\langle N^{(1)}\rangle)}[M_t^{(2)}] \\ &\vdots \\ N_t^{(n)} &:= M_t^{(n)} - \sum_{k=1}^{n-1} \mathscr{P}_{\mathscr{L}(\langle N^{(k)}\rangle)}[M_t^{(n)}].\end{aligned} \tag{A.3}$$

Then $N_t^{(1)},...,N_t^{(n)}$ is an orthogonal system and $M^{(k)} \in \mathscr{L}(\langle N^{(1)}\rangle) \oplus ... \oplus \mathscr{L}(\langle N^{(n)}\rangle)$ for all $k = 1,...,n$. Furthermore, if $M^{(k)} \in \mathscr{M}_c$ then $N^{(k)} \in \mathscr{M}_c$ (see [Kun97, Theorem 2.3.8]) for all $k = 1,...,n$.

Definition A.6 *Let $\{M^{(n)}\}_{n\in\mathbb{N}}$ be an orthogonal system in $\mathscr{M}_c$ and $M \in \mathscr{M}_c$. If $\langle M^{(n)}, M\rangle_t = 0$ for all $n \in \mathbb{N}$ implies $M = 0$, then the system is called an* ***orthogonal basis****.*

The following result can be found in [Kun97, Theorem 2.3.9].

Theorem A.7 *Let $\{M^{(n)}\}_{n\geq 1}$ be an orthogonal system. It is an orthogonal basis if and only if any $M \in \mathscr{M}_c$ is expanded as*

$$M = \sum_{k\geq 1} \mathscr{P}_{\mathscr{L}(\langle M^{(k)}\rangle)}[M].$$

For the reader's convenience we prove the inclusion of classes of local characteristics as used in the proof of Theorem 2.39.

Corollary A.8 *We have $B^{1,0} \subset B^{0,\delta}$ for every $0 \leq \delta \leq 1$.*

Proof. Let (b, A_t) belong to the class $B^{1,0}$, i.e. for all compact sets $\mathbb{K} \subset \mathbb{D}$ we have

$$\int_0^t \|b(s)\|_{1+0;\mathbb{K}} \,\mathrm{d}A_s < \infty \text{ a.s.}$$

This means in particular

$$\int_0^t \sup_{x\in\mathbb{K}} \frac{|b(x,s)|}{1+|x|} \,\mathrm{d}A_s < \infty \text{ a.s.,} \tag{A.4}$$

$$\int_0^t \sup_{x\in\mathbb{K}} |D_x^\alpha b(x,s)| \,\mathrm{d}A_s < \infty \text{ a.s.,} \tag{A.5}$$

$$\begin{aligned} &\int_0^t \sup_{x,y\in\mathbb{K}} \frac{|D_x^\alpha b(x,s) - D_x^\alpha b(y,s)|}{|x-y|^0} \,\mathrm{d}A_s \\ &\quad = \int_0^t \sup_{x,y\in\mathbb{K}} |D_x^\alpha b(x,s) - D_x^\alpha b(y,s)| \,\mathrm{d}A_s < \infty \text{ a.s.} \end{aligned} \tag{A.6}$$

for all $|\alpha| \leq 1$. Let $\delta \in [0,1]$ be arbitrary. Obviously, we have for all compact sets $\mathbb{K}$ also

$$\begin{aligned} \int_0^t \|b(s)\|_{0+\delta;\mathbb{K}} \,\mathrm{d}A_s &= \int_0^t \sup_{x\in\mathbb{K}} \frac{|b(x,s)|}{1+|x|} \,\mathrm{d}A_s + \int_0^t \sup_{x\in\mathbb{K}} \sum_{1\leq|\alpha|\leq 0} |D_x^\alpha b(x,s)| \,\mathrm{d}A_s \\ &\quad + \int_0^t \sup_{x,y\in\mathbb{K}} \frac{|D_x^0 b(x,s) - D_x^0 b(y,s)|}{|x-y|^\delta} \,\mathrm{d}A_s \\ &= \int_0^t \sup_{x\in\mathbb{K}} \frac{|b(x,s)|}{1+|x|} \,\mathrm{d}A_s \\ &\quad + \int_0^t \sup_{x,y\in\mathbb{K}} \frac{|b(x,s) - b(y,s)|}{|x-y|^\delta} \,\mathrm{d}A_s < \infty \text{ a.s.} \end{aligned}$$

since the first term is finite a.s. by (A.5) and furthermore (A.6) implies Lipschitz continuity and Hölder continuity. Obviously, each one-times continuously differentiable function with bounded derivative is Lipschitz continuous. □

Corollary A.9 $B^{0,1}_{\mathrm{ub}} \subset B^{0,1}_{\mathrm{b}}$.

Proof. Let f belong to the class $B^{0,1}_{\mathrm{ub}}$. By Definition 2.25 and the fact that each bounded function is integrable on a bounded domain, we easily conclude that f also belongs to $B^{0,1}_{\mathrm{b}}$. □

Lemma A.10 *If a local characteristic* (a, b, A_t) *belongs to* $(B^{k+1,\delta}, B^{k,\delta})$ *for some* $k \geq 5$ *and* $0 < \delta \leq 1$, *then it also belongs to the subclass* $(B^{2,\delta}, B^{1,0})$.

Proof. For $k \geq 5$ it is clear, that $C^{k+1} \subset C^1$. Therefore a local characteristic (a, A_t) belonging to $B^{k+1,\delta}$ is also in the class $B^{2,\delta}$ for each Hölder exponent $\delta > 0$. Now we consider the local characteristic (b, A_t) belonging to $B^{k,\delta}$. By definition and linearity of the integral we have for every $\mathbb{K} \subset \mathbb{D}$ compact:

$$\begin{aligned}
&\int_0^t \sup_{x\in\mathbb{K}} \frac{|b(x,t)|}{1+|x|}\,\mathrm{d}A_s < \infty \text{ a.s.},\\
&\int_0^t \sum_{1\leq|\alpha|\leq k} \sup_{x\in\mathbb{K}} |D_x^\alpha b(x,t)|\,\mathrm{d}A_s < \infty \text{ a.s. and}\\
&\int_0^t \sum_{|\alpha|=k} \sup_{\substack{x,y\in\mathbb{K}\\ x\neq y}} \frac{|D_x^\alpha b(x,t) - D_x^\alpha b(y,t)|}{|x-y|^\delta}\,\mathrm{d}A_s < \infty \text{ a.s.}
\end{aligned} \tag{A.7}$$

Since every Hölder continuous function is uniformly continuous and bounded on a compact set, we obtain for $\delta = 0$:

$$\begin{aligned}
\int_0^t \|b(t)\|_{1+0;\mathbb{K}}\,\mathrm{d}A_s &= \int_0^t \sup_{x\in\mathbb{K}} \frac{|b(x,t)|}{1+|x|}\,\mathrm{d}A_s + \int_0^t \sum_{1\leq|\alpha|\leq 1} \sup_{x\in\mathbb{D}} |D_x^\alpha b(x,t)|\,\mathrm{d}A_s\\
&\quad + \int_0^t \sum_{|\alpha|=1} \sup_{\substack{x,y\in\mathbb{K}\\ x\neq y}} \frac{|D_x^\alpha b(x,t) - D_x^\alpha b(y,t)|}{|x-y|^0}\,\mathrm{d}A_s\\
&\leq \int_0^t \sup_{x\in\mathbb{K}} \frac{|b(x,t)|}{1+|x|}\,\mathrm{d}A_s + \int_0^t \sum_{1\leq|\alpha|\leq 1} \sup_{x\in\mathbb{D}} |D_x^\alpha b(x,t)|\,\mathrm{d}A_s\\
&\quad + \int_0^t 2 \sum_{|\alpha|=1} \sup_{x\in\mathbb{K}} |D_x^\alpha b(x,t)|\,\mathrm{d}A_s < \infty \text{ a.s.}
\end{aligned}$$

which is finite by (A.7). □

In the same way one proves the following inclusion.

Corollary A.11 *If a local characteristic* (a, b, A_t) *belongs to* $(B^{k,\delta}, B^{k-1,\delta})$ *for some* $k \geq 5$ *and* $0 < \delta \leq 1$, *then it also belongs to the subclass* $(B^{2,\delta}, B^{1,0})$.

For the sake of completeness we state the definition of a truncation.

Definition A.12 *A **truncation** of a function* $f\colon [0, \boldsymbol{T}] \to \mathbb{R}^d$ *associated with a compact set* $\mathbb{K} \subset \mathbb{R}^d$ *is defined by*

$$\tilde{f}_t = \begin{cases} f_t, & \text{if } f_t \in \mathbb{K},\\ x_0 \in \mathbb{K} \text{ fixed}, & \text{if } f_t \notin \mathbb{K}. \end{cases}$$

As proved in [Kun97, Theorem 3.3.1] the following so-called **generalized Itô formula** is valid.

Theorem A.13 *Let $F(x,t)$, $x \in \mathbb{D}$, be a family of continuous C^2 - processes and continuous, C^1 - semimartingales with local characteristic belonging to the class $(B^{1,0}, B^{1,0})$. Let g_t be a continuous $\mathbb{D}$−valued semimartingale. Then $F(g_t,t)$ is a continuous semimartingale and*

$$\begin{aligned} F(g_t,t) - F(g_0,0) = &\int_0^t F(g_s, \mathrm{d}s) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(g_s, s)\, \mathrm{d}g_s^i \\ &+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(g_s, s)\, \mathrm{d}\langle g_\cdot^i, g_\cdot^j \rangle_s \\ &+ \sum_{i=1}^d \left\langle \int_0^\cdot \frac{\partial F}{\partial x_i}(g_s, \mathrm{d}s), g_\cdot^i \right\rangle_t \end{aligned}$$

is satisfied.

B. Appendix to the method of stochastic characteristics

Lemma B.1 *The stopping time $\tau(x)$ given by*

$$\begin{aligned}\tau_{\text{inv}}(x) &:= \inf\{t \in (0, \boldsymbol{T}] \mid \det D\xi_t(x) = 0\}\\ \tau(x) &:= \tau_{\text{inv}}(x) \wedge T(x).\end{aligned}$$

is accessible and lower semicontinuous.

Proof. Since ξ_t is the maximal solution of (SCE) and in particular a continuous local semimartingale, it is defined up to the explosion time T. We know by Definition 2.8 that T is accessible and lower semicontinuous. Furthermore, we know that the minimum of two accessible stopping times is accessible (cf. [Del72, Chapter III, T45]). Hence we have to show that also τ_{inv} is accessible. Define the sequence

$$\tau_{\text{inv}}^{(n)}(x) := \inf\left\{t \in (0, \mathbf{T}] \,\middle|\, \det D\xi_t(x) \leq \frac{1}{n}\right\}.$$

The $(\mathscr{F}_t)_t$-adapted process ξ_t is in particular continuous and continuously differentiable w.r.t. space (cf. [Kun97, Theorem 3.3.4]), that means $\det D\xi_t$ is also continuous and adapted. By Début Theorem (cf. [Del72, Chapter III, T23]) we get that for each $n \in \mathbb{N}$ $\tau_{\text{inv}}^{(n)}$ is a stopping time. By definition we have

$$\begin{aligned}\tau_{\text{inv}}^{(n)}(x) &= \inf\left\{t \in (0, \mathbf{T}] \,\middle|\, \det D\xi_t(x) \leq \frac{1}{n}\right\}\\ &\leq \inf\left\{t \in (0, \mathbf{T}] \,\middle|\, \det D\xi_t(x) \leq \frac{1}{n+1}\right\}\\ &= \tau_{\text{inv}}^{(n+1)}(x).\end{aligned}$$

Moreover for all $x \in \mathbb{R}^d$

$$\tau_{\text{inv}}^{(n)}(x) < \tau_{\text{inv}}(x) \quad \text{a.s.} \tag{B.1}$$

holds, since if we assume $\tau_{\text{inv}}^{(n)}(x) = \tau_{\text{inv}}(x)$, we conclude by definition of $\tau_{\text{inv}}^{(n)}(x)$ that for all $\varepsilon > 0$

$$\det D\xi_{\tau_{\text{inv}}(x)-\varepsilon}(x) > \frac{1}{n}$$

holds, which contradicts the continuity of $\det D\xi_t(x)$. Obviously,

$$\lim_{n\to\infty} \tau_{\text{inv}}^{(n)}(x) = \tau_{\text{inv}}(x)$$

holds. So τ_{inv} is accessible.

Next we prove that τ is lower semicontinuous. To this end let $x_0 \in \mathbb{R}^d$. Let $B_\varepsilon(x_0)$ denote the open ball with radius ε centered at x_0 in $\mathbb{R}^d$, i.e.

$$B_\varepsilon(x_0) := \left\{y \in \mathbb{R}^d \,\middle|\, |y - x_0| < \varepsilon\right\}.$$

By using the lower semicontinuity of T we obtain

$$\liminf_{x\to x_0} T(x) \geq T(x_0) \geq \tau(x_0).$$

Now we have to show

$$\liminf_{x\to x_0} \inf\Big\{t \in (0,\mathbf{T}] \,\Big|\, \det D\xi_t(x) = 0\Big\} \geq \inf\Big\{t \in (0,\mathbf{T}] \,\Big|\, \det D\xi_t(x_0) = 0\Big\}. \tag{B.2}$$

Due to the fact that $\xi_t(x)$ is continuous in t with values in $C^{k-1,\delta}$, we know in particular that $\det D\xi_t(x)$ is continuous in x. Let us assume that

$$\det D\xi_t(x_0) > 0,$$

then there exists an $\varepsilon > 0$ such that $\det D\xi_t(x) > 0$ for all $x \in B_\varepsilon(x_0)$. By definition of τ_{inv} we conclude that

$$\tau_{\text{inv}}(x) \geq t \quad \forall x \in B_\varepsilon(x_0).$$

Hence we have

$$\liminf_{x\to x_0} \tau_{\text{inv}}(x) \geq t \tag{B.3}$$

Let us assume the contraposition of (B.2), i.e.

$$\liminf_{x\to x_0} \inf\{t \in (0,\mathbf{T}] \,|\, \det D\xi_t(x) = 0\} < \inf\{t \in (0,\mathbf{T}] \,|\, \det D\xi_t(x_0) = 0\}.$$

Then there exists a $t \in (0,\mathbf{T}]$ such that $\tau_{\text{inv}}(x_0) \geq t$ holds and also

$$\liminf_{x\to x_0} \tau_{\text{inv}}(x) < t$$

which contradicts (B.3). Hence (B.2) is proved. □

C. Appendix to the existence and uniqueness result of H. Kunita

Theorem 4.7 *Let $F(x,t)$, $x = (x_1, ..., x_d) \in \mathbb{R}^d$, be a family of continuous $C^3(\mathbb{R}^d, \mathbb{R}^d)$-processes and continuous $C^2(\mathbb{R}^d, \mathbb{R}^d)$-semimartingales with local characteristic belonging to the class $(B^{2,\delta}, B^{1,0})$ for some $0 < \delta \le 1$. Let g_t be a continuous $\mathbb{R}^d$-valued semimartingale. Then the formula*

$$F(g_t, t) - F(g_0, 0) = \int_0^t F(g_s, \circ \mathrm{d}s) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(g_s, s) \circ \mathrm{d}g_s^i \tag{C.1}$$

is satisfied, where g_t^i denotes the i-th component of $g_t = (g_t^1, ..., g_t^d)$.

The proof follows the ideas of the proof of Theorem 3.3.2 in [Kun97].

Proof. We apply the generalized Itô's formula Theorem A.13 to

$$\tilde{F}(g_t, t) := \frac{\partial F}{\partial x_i}(g_t, t) \tag{C.2}$$

to obtain the assertion that $\tilde{F}(g_t, t)$ is a continuous semimartingale. The assumptions on F are satisfied since F is a C^3-process and a continuous C^2-semimartingale, hence $\frac{\partial F}{\partial x_i} \in C^2$ and a continuous C^1-semimartingale. Therefore $\tilde{F}(g_t, t)$ is a continuous semimartingale and due to the generalized Itô formula we know

$$\begin{aligned}\tilde{F}(g_t, t) - \tilde{F}(g_0, 0) &= \int_0^t \tilde{F}(g_s, \mathrm{d}s) + \sum_{i=1}^d \int_0^t \frac{\partial \tilde{F}}{\partial x_i}(g_s, s)\, \mathrm{d}g_s^i \\ &+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 \tilde{F}}{\partial x_i \partial x_j}(g_s, s)\, \mathrm{d}\langle g_\cdot^i, g_\cdot^j \rangle_s + \sum_{i=1}^d \left\langle \int_0^\cdot \frac{\partial \tilde{F}}{\partial x_i}(g_s, \mathrm{d}s), g_\cdot^i \right\rangle_t\end{aligned} \tag{C.3}$$

holds. By the Itô-Stratonovich formula [Kun97, Theorem 2.3.5] and the fact that only the first term and second term on the right hand side of (C.3) is not of bounded variation, we get

$$\begin{aligned}\sum_{i=1}^d \int_0^t \tilde{F}(g_s, s) \circ \mathrm{d}g_s^i &= \sum_{i=1}^d \int_0^t \tilde{F}(g_s, s)\, \mathrm{d}g_s^i + \frac{1}{2} \sum_{i=1}^d \langle \tilde{F}(g_\cdot, \cdot), g_\cdot^i \rangle_t \\ &= \sum_{i=1}^d \int_0^t \tilde{F}(g_s, s)\, \mathrm{d}g_s^i + \frac{1}{2} \sum_{i=1}^d \left\langle \tilde{F}(g_0, 0) + \int_0^\cdot \tilde{F}(g_s, \mathrm{d}s) + \sum_{i=1}^d \int_0^\cdot \frac{\partial \tilde{F}}{\partial x_i}(g_s, s)\, \mathrm{d}g_s^i, g_\cdot^i \right\rangle_t \\ &= \sum_{i=1}^d \int_0^t \tilde{F}(g_s, s)\, \mathrm{d}g_s^i + \frac{1}{2} \sum_{i=1}^d \left\langle \int_0^\cdot \tilde{F}(g_s, \mathrm{d}s), g_\cdot^i \right\rangle_t + \frac{1}{2} \sum_{i=1}^d \left\langle \sum_{i=1}^d \int_0^\cdot \frac{\partial \tilde{F}}{\partial x_i}(g_s, s)\, \mathrm{d}g_s^i, g_\cdot^i \right\rangle_t .\end{aligned}$$

Now we use (C.2) to obtain

$$\begin{aligned}\sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(g_s, s) \circ \mathrm{d}g_s^i &= \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(g_s, s)\, \mathrm{d}g_s^i + \frac{1}{2} \sum_{i=1}^d \left\langle \int_0^\cdot \frac{\partial F}{\partial x_i}(g_s, \mathrm{d}s), g_\cdot^i \right\rangle_t \\ &+ \frac{1}{2} \sum_{i=1}^d \left\langle \sum_{j=1}^d \int_0^\cdot \frac{\partial^2 F}{\partial x_i \partial x_j}(g_s, s)\, \mathrm{d}g_s^j, g_\cdot^i \right\rangle_t .\end{aligned}$$

By applying [Kun97, Theorem 2.3.2] we receive

$$\sum_{i=1}^{d}\int_0^t \frac{\partial F}{\partial x_i}(g_s,s)\circ \mathrm{d}g_s^i = \sum_{i=1}^{d}\int_0^t \frac{\partial F}{\partial x_i}(g_s,s)\,\mathrm{d}g_s^i + \frac{1}{2}\sum_{i=1}^{d}\Big\langle \int_0^{\cdot} \frac{\partial F}{\partial x_i}(g_s,\mathrm{d}s), g_\cdot^i\Big\rangle_t + \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(g_s,s)\,\mathrm{d}\langle g_\cdot^i, g_\cdot^j\rangle_s.$$

Starting on the right hand side of (C.1) we obtain by an application of Theorem 2.35 to the first term

$$\begin{aligned}
&\int_0^t F(g_s,\circ\mathrm{d}s) + \sum_{i=1}^{d}\int_0^t \frac{\partial F}{\partial x_i}(g_s,s)\circ \mathrm{d}g_s^i \\
&= \int_0^t F(g_s,\mathrm{d}s) + \frac{1}{2}\sum_{i=1}^{d}\Big\langle \int_0^{\cdot} \frac{\partial F}{\partial x_i}(g_s,\mathrm{d}s), g_\cdot^i\Big\rangle_t \\
&\quad + \sum_{i=1}^{d}\int_0^t \frac{\partial F}{\partial x_i}(g_s,s)\,\mathrm{d}g_s^i + \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(g_s,s)\,\mathrm{d}\langle g_\cdot^i, g_\cdot^j\rangle_s \\
&\quad + \frac{1}{2}\sum_{i=1}^{d}\Big\langle \int_0^{\cdot} \frac{\partial F}{\partial x_i}(g_s,\mathrm{d}s), g_\cdot^i\Big\rangle_t \\
&= \int_0^t F(g_s,\mathrm{d}s) + \sum_{i=1}^{d}\int_0^t \frac{\partial F}{\partial x_i}(g_s,s)\,\mathrm{d}g_s^i + \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(g_s,s)\,\mathrm{d}\langle g_\cdot^i, g_\cdot^j\rangle_s \\
&\quad + \sum_{i=1}^{d}\Big\langle \int_0^{\cdot} \frac{\partial F}{\partial x_i}(g_s,\mathrm{d}s), g_\cdot^i\Big\rangle_t.
\end{aligned}$$

Finally we apply the generalized Itô formula Theorem A.13 to receive

$$\int_0^t F(g_s,\circ\mathrm{d}s) + \sum_{i=1}^{d}\int_0^t \frac{\partial F}{\partial x_i}(g_s,s)\circ \mathrm{d}g_s^i = F(g_t,t) - F(g_0,0).$$

□

Lemma C.1 *The unique solution to equation (4.18) is given by*

$$\psi_t(\xi_t(x)) = x.$$

Proof. We have to verify that (4.19) solves the integral equation (4.18). Define $\nu_t(x) = \psi_t(\xi_t(x))$. Hence consider for $\nu_t(x) = x$

$$\begin{aligned}
&\nu_0(x) + \int_0^t \big(D\xi_s(\nu_s(x))\big)^{-1}\cdot F_{\chi_s}(\xi_s(x),\eta_s(\nu_s(x)),\chi_s(\nu_s(x)),\circ\mathrm{d}s) \\
&\qquad - \int_0^t \big(D\xi_s(\nu_s(x))\big)^{-1} F_{\chi_s}(\xi_s(x),\eta_s(x),\chi_s(x),\circ\mathrm{d}s) \\
&\quad = x + \int_0^t \big(D\xi_s(x)\big)^{-1}\cdot F_{\chi_s}(\xi_s(x),\eta_s(x),\chi_s(x),\circ\mathrm{d}s)
\end{aligned}$$

$$- \int_0^t \big(D\xi_s(x)\big)^{-1} F_{\chi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ \mathrm{d}s)$$

$$= x$$

Uniqueness follows by Theorem 3.7 and the same arguments as written in Step 2 in the proof of Lemma 4.8. □

Theorem C.2 *Let $F(x,t)$, $x \in \mathbb{D}$, be a family of continuous $C^{m,\delta}$-semimartingale with local characteristic belonging to the class $(B^{m+1,\delta}, B^{m,\delta})$ for some $m \geq 1$ and $\delta > 0$. For a family of continuous $\mathbb{D}$-valued $C^{k,\gamma}$-semimartingales $f(\lambda,t)$, $\lambda \in \Lambda \subset \mathbb{R}^e$ domain, for some $k \geq 2$ and $\gamma > 0$, we define $\mathring{L}(\lambda,t) := \int_0^t F(f(\lambda,s), \circ \mathrm{d}s)$. Then*

$$\frac{\partial \mathring{L}}{\partial \lambda_i}(\lambda,t) = \sum_{l=1}^{d} \int_0^t \frac{\partial f^l}{\partial \lambda_i}(\lambda,s) \frac{\partial F}{\partial x_l}(f(\lambda,s), \circ \mathrm{d}s)$$

holds, which is equivalent to

$$\frac{\partial}{\partial \lambda_i}\Big(\int_0^t F(f(\lambda,s), \circ \mathrm{d}s)\Big) = \sum_{l=1}^{d} \int_0^t \frac{\partial f^l}{\partial \lambda_i}(\lambda,s) \frac{\partial F}{\partial x_l}(f(\lambda,s), \circ \mathrm{d}s).$$

The proof is given in [Kun97, Theorem 3.3.4].

Lemma C.3 *By using the technique of an integrating factor the following equations*

$$\begin{aligned} \xi_t(x) &= \exp\left(x - \int_0^t \tilde{\phi}_1(s)\,\mathrm{d}s \right) \\ \eta_t(x) &= \frac{h(x) + \int_0^t \exp\left(\int_0^s \tilde{\psi}(r)\,\mathrm{d}r + s \right) \circ \mathrm{d}W_s}{\exp\left(\int_0^t \tilde{\psi}(s)\,\mathrm{d}s + t \right)} \end{aligned} \tag{C.4}$$

solve the system of stochastic differential equations

$$\mathrm{d}\xi_t = -\tilde{\phi}_1(t)\xi_t\,\mathrm{d}t \tag{C.5}$$

$$\mathrm{d}\eta_t = \big(-\tilde{\psi}(t)\eta_t - \eta_t \big)\,\mathrm{d}t + 1 \circ \mathrm{d}W_t, \tag{C.6}$$

with $\xi_0(x) = x$ and $\eta_0(x) = h(x)$.

Proof. We easily determine the partial derivatives w.r.t. time t to verify that the stochastic differential equation is fulfilled

$$\begin{aligned} \frac{\mathrm{d}\eta_t}{\mathrm{d}t}(x) = {} & \exp\left(\int_0^t \tilde{\psi}(s)\,\mathrm{d}s + t \right) \cdot \exp\left(\int_0^t \tilde{\psi}(r)\,\mathrm{d}r + t \right) \cdot \dot{W}_t \cdot \exp\left(\int_0^t \tilde{\psi}(s)\,\mathrm{d}s + t \right)^{-2} \\ & - \exp\left(\int_0^t \tilde{\psi}(s)\,\mathrm{d}s + t \right)^{-2} \cdot \left[\left(h(x) + \int_0^t \exp\left(\int_0^s \tilde{\psi}(r)\,\mathrm{d}r + s \right) \circ \mathrm{d}W_s \right) \right. \\ & \left. \cdot \left(\tilde{\psi}(t) + 1 \right) \cdot \exp\left(\int_0^t \tilde{\psi}(s)\,\mathrm{d}s + t \right) \right] \cdot \end{aligned}$$

$$= \dot{W}_t - \left(\tilde{\psi}(t) + 1\right) \cdot \eta_t$$

Analogously we receive by the fundamental theorem of calculus and chain rule

$$\begin{aligned} \frac{\mathrm{d}\xi_t}{\mathrm{d}t}(x) &= -\tilde{\phi}_1(t) \cdot \exp\left(x - \int_0^t \tilde{\phi}_1(s)\,\mathrm{d}s \right) \\ &= -\tilde{\phi}_1(t) \cdot \xi_t. \end{aligned}$$

Hence (C.4) are the solutions to (C.5) and (C.6). □

D. Appendix to the application to stochastic Burgers equations and stochastic transport equations

Lemma D.1 *The inverse process in the setting of Example 5.1 is given by*

$$\begin{aligned}
&\xi_t^{-1}(x_1, x_2) \\
&= \Bigg(\frac{2x_1 (W_t^2)^2 + W_t^1 - 2x_2 W_t^1 W_t^2}{2(W_t^1)^2 + 2(W_t^2)^2} \\
&\qquad - \frac{W_t^1 \sqrt{(1 - 4x_2^2 (W_t^1)^2 - x_1 W_t^1 (4 - 8x_2 W_t^2) - 4x_1^2 (W_t^2)^2 - 4x_2 W_t^2)}}{2(W_t^1)^2 + 2(W_t^2)^2}, \\
&\quad \frac{W_t^1 W_t^2 - 2x_1 (W_t^1)^2 W_t^2 + 2x_2 (W_t^1)^3}{2(W_t^1)^3 + 2(W_t^2)^2} \\
&\qquad - \frac{W_t^1 W_t^2 \sqrt{(1 - 4x_2^2 (W_t^1)^2 + 8x_1 x_2 W_t^1 W_t^2 - 4x_1 W_t^1 - 4x_1^2 (W_t^2)^2 - 4x_2 W_t^2)}}{2(W_t^1)^3 + 2(W_t^2)^2} \Bigg).
\end{aligned}$$

Proof. Obviously, we have to prove that for any $a, b \in \mathbb{R}$

$$\xi_t^{-1}\Big(a - (a^2 + b^2) W_t^1, b - (a^2 + b^2) W_t^2\Big) = (a, b) \tag{D.1}$$

is satisfied. Let us start with the first component:

$$\begin{aligned}
&\frac{-bW_t^1 W_t^2 + a(W_t^2)^2 + \frac{W_t^1}{2} - \frac{W_t^1 \sqrt{\left(1 + 4a^2 (W_t^1)^2 + 8abW_t^1 W_t^2 - 4aW_t^1 + 4b^2 (W_t^2)^2 - 4bW_t^2\right)}}{2}}{(W_t^1)^2 + (W_t^2)^2} \\
&= \frac{-2bW_t^1 W_t^2 + 2a(W_t^2)^2 + W_t^1 - W_t^1 \sqrt{\left(1 - 2aW_t^1 - 2bW_t^2\right)^2}}{2\big((W_t^1)^2 + (W_t^2)^2\big)} \\
&= \frac{-2bW_t^1 W_t^2 + 2a(W_t^2)^2 + W_t^1 - W_t^1 + 2a(W_t^1)^2 + 2bW_t^1 W_t^2}{2\big((W_t^1)^2 + (W_t^2)^2\big)} \\
&= \frac{2a\big((W_t^1)^2 + (W_t^2)^2\big)}{2\big((W_t^1)^2 + (W_t^2)^2\big)} = a.
\end{aligned}$$

For the second component we get

$$\begin{aligned}
&\frac{b(W_t^1)^3 - a(W_t^1)^2 W_t^2 + \frac{W_t^1 W_t^2}{2} - \frac{W_t^2 \sqrt{(W_t^1)^2 \left(1 + 4a^2 (W_t^1)^2 + 8abW_t^1 W_t^2 - 4aW_t^1 + 4b^2 (W_t^2)^2 - 4bW_t^2\right)}}{2}}{W_t^1\big((W_t^1)^2 + (W_t^2)^2\big)} \\
&= \frac{2b(W_t^1)^2 - 2aW_t^1 W_t^2 + W_t^2 - W_t^2 \sqrt{(1 - 2aW_t^1 - 2bW_t^2)^2}}{2\big((W_t^1)^2 + (W_t^2)^2\big)} \\
&= \frac{2b(W_t^1)^2 - 2aW_t^1 W_t^2 + W_t^2 - W_t^2 + 2aW_t^1 W_t^2 + 2b(W_t^2)^2}{2\big((W_t^1)^2 + (W_t^2)^2\big)} \\
&= \frac{2b\big((W_t^1)^2 + (W_t^2)^2\big)}{2\big((W_t^1)^2 + (W_t^2)^2\big)} = b.
\end{aligned}$$

□

Lemma D.2 *The inverse process in Example 5.3 is given for almost all ω and all (x,t) with $t < \hat{\sigma}(x,\omega)$ by*

$$\xi_t^{-1}(x) = \frac{2x}{1+\sqrt{1-4x(t+W_t)}}.$$

Proof. Obviously, we have to prove that for any $x \in \mathbb{R}$

$$\xi_t^{-1}\big(\xi_t(x)\big) = x$$

is true for $\xi_t(x) = x - x^2 t - x^2 W_t$. This property is fulfilled, since

$$\begin{aligned}
\xi_t^{-1}\big(\xi_t(x)\big) &= \xi_t^{-1}\big(x - x^2t - x^2W_t\big) \\
&= \frac{2(x - x^2t - x^2W_t)}{1+\sqrt{1-4(x-x^2t-x^2W_t)(t+W_t)}} \\
&= \frac{2(x - x^2t - x^2W_t)}{1+\sqrt{1-4(xt+xW_t-x^2t^2-x^2tW_t-x^2tW_t-x^2(W_t)^2)}} \\
&= \frac{2x(1-xt-xW_t)}{1+\sqrt{1-4xt-4xW_t+4x^2t^2+8x^2tW_t+4x^2(W_t)^2}} \\
&= \frac{2x(1-xt-xW_t)}{1+\sqrt{(1-2xt-2xW_t)^2}} \\
&= \frac{2x(1-xt-xW_t)}{1+1-2xt-2xW_t} \\
&= \frac{2x(1-xt-xW_t)}{2(1-xt-xW_t)} = x.
\end{aligned}$$

Furthermore, ξ_t^{-1} is the right inverse to ξ_t since

$$\begin{aligned}
\xi_t\big(\xi_t^{-1}(x)\big) &= \xi_t\left(\frac{2x}{1+\sqrt{1-4x(t+W_t)}}\right) \\
&= \frac{2x}{1+\sqrt{1-4x(t+W_t)}} - \left(\frac{2x}{1+\sqrt{1-4x(t+W_t)}}\right)^2 t \\
&\qquad - \left(\frac{2x}{1+\sqrt{1-4x(t+W_t)}}\right)^2 W_t \\
&= \frac{2x}{1+\sqrt{1-4x(t+W_t)}} - \frac{4x^2t}{(1+\sqrt{1-4x(t+W_t)})^2} \\
&\qquad - \frac{4x^2W_t}{(1+\sqrt{1-4x(t+W_t)})^2} \\
&= \frac{2x(1+\sqrt{1-4x(t+W_t)})}{(1+\sqrt{1-4x(t+W_t)})^2} \\
&\qquad - \frac{4x^2t}{(1+\sqrt{1-4x(t+W_t)})^2} - \frac{4x^2W_t}{(1+\sqrt{1-4x(t+W_t)})^2} \\
&= \frac{2x+2x\sqrt{1-4x(t+W_t)}-4x^2t-4x^2W_t}{(1+\sqrt{1-4x(t+W_t)})^2} \\
&= \frac{x(1+\sqrt{1-4x(t+W_t)})^2}{(1+\sqrt{1-4x(t+W_t)})^2} = x.
\end{aligned}$$

□

Lemma D.3 *The partial derivatives of u defined by (5.14) are given for almost all ω and all (x,t) with $t < \hat{\sigma}(x,\omega)$ by*

$$\frac{\mathrm{d}u}{\mathrm{d}t}(x,t) = u(x,t) \cdot \frac{4x(1+\dot{W}_t)}{\sqrt{1-4x(t+W_t)}\big(1+\sqrt{1-4x(t+W_t)}\big)}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}(x,t) = \frac{8x\big(1+\sqrt{1-4x(t+W_t)}-2xt-2xW_t\big)}{\sqrt{1-4x(t+W_t)}(1+\sqrt{1-4x(t+W_t)})^3}.$$

Proof. Due to the quotient rule of differential calculus we obtain

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t}(x,t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{4x^2}{2+2\sqrt{1-4x(t+W_t)}-4xt-4xW_t}\right] \\
&= \frac{-4x^2\Big(-2(4x+4x\dot{W}_t)\frac{1}{2}\frac{1}{\sqrt{1-4x(t+W_t)}}-(4x+4x\dot{W}_t)\Big)}{(1+\sqrt{1-4x(t+W_t)})^4} \\
&= \frac{4x^2(4x+4x\dot{W}_t)\big(1+\sqrt{1-4x(t+W_t)}\big)}{\sqrt{1-4x(t+W_t)}(1+\sqrt{1-4x(t+W_t)})^4} \\
&= \frac{16x^2(x+x\dot{W}_t)}{\sqrt{1-4x(t+W_t)}(1+\sqrt{1-4x(t+W_t)})^3} \\
&= u(x,t)\cdot\frac{4x(1+\dot{W}_t)}{\sqrt{1-4x(t+W_t)}\big(1+\sqrt{1-4x(t+W_t)}\big)}.
\end{aligned}$$

Analogously we determine the partial derivative with respect to space variable x

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x}(x,t) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{4x^2}{2+2\sqrt{1-4x(t+W_t)}-4xt-4xW_t}\right] \\
&= \frac{8x(1+\sqrt{1-4x(t+W_t)})^2-4x^2\Big(-\frac{4(t+W_t)\big(1+\sqrt{1-4x(t+W_t)}\big)}{\sqrt{1-4x(t+W_t)}}}{(1+\sqrt{1-4x(t+W_t)})^4} \\
&= \frac{8x(1+\sqrt{1-4x(t+W_t)})^2+16x^2(t+W_t)\big(1+\sqrt{1-4x(t+W_t)}\big)}{\sqrt{1-4x(t+W_t)}(1+\sqrt{1-4x(t+W_t)})^4} \\
&= \frac{8x}{\sqrt{1-4x(t+W_t)}(1+\sqrt{1-4x(t+W_t)})^2}+\frac{16x^2(t+W_t)}{\sqrt{1-4x(t+W_t)}(1+\sqrt{1-4x(t+W_t)})^3} \\
&= \frac{8x\big(1+\sqrt{1-4x(t+W_t)}-2xt-2xW_t\big)}{\sqrt{1-4x(t+W_t)}(1+\sqrt{1-4x(t+W_t)})^3}.
\end{aligned}$$

□

Lemma D.4 *Let $c > 0$. The local solution to the stochastic differential equation*

$$\begin{cases} \mathrm{d}\eta_t = c\,\eta_t \circ \mathrm{d}W_t \\ \eta_0(x) = g(x) \end{cases}$$

is given by

$$\eta_t(x) = g(x)\exp(c\,W_t)$$

for almost all ω and all x,t such that $t < T(x,\omega)$, where $T(x)$ is the explosion time.

Proof. By using Newton's derivative $\frac{\circ \mathrm{d}W_t}{\mathrm{d}t} =: \dot{W}_t$ we have to verify that $\eta_t(x)$ solves

$$\frac{\mathrm{d}\eta_t}{\mathrm{d}t}(x) = c\,\eta_t(x)\dot{W}_t.$$

Determine the partial derivative, we obtain

$$\begin{aligned}\frac{\mathrm{d}\eta_t}{\mathrm{d}t}(x) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[g(x)\exp(c\,W_t)\right]\\ &= g(x)\,c\,\dot{W}_t\,\exp(c\,W_t)\\ &= c\,\eta_t(x)\,\dot{W}_t.\end{aligned}$$

□

Lemma D.5 *The inverse process of*

$$\xi_t(x) = x - \frac{1}{2}x^2\int_0^t \exp(W_s)\,\mathrm{d}s$$

is given for almost all ω and all (x,t) with $t < \hat{\sigma}(x,\omega)$ by

$$\xi_t^{-1}(x) = (2x)\cdot\left(1+\sqrt{1-2x\left(\int_0^t e^{W_s}\,\mathrm{d}s\right)}\right)^{-1}.$$

Proof. We have to prove that for any $x \in \mathbb{R}$

$$\xi_t^{-1}\big(\xi_t(x)\big) = x$$

is true. This property is fulfilled, since

$$\begin{aligned}\xi_t^{-1}\big(\xi_t(x)\big) &= \xi_t^{-1}\left(x - \frac{1}{2}x^2\int_0^t e^{W_s}\,\mathrm{d}s\right)\\ &= \frac{2\left(x - \frac{1}{2}x^2\int_0^t e^{W_s}\,\mathrm{d}s\right)}{1+\sqrt{1-2\left(x-\frac{1}{2}x^2\int_0^t e^{W_s}\,\mathrm{d}s\right)\left(\int_0^t e^{W_s}\,\mathrm{d}s\right)}}\\ &= \frac{2x - x^2\left(\int_0^t e^{W_s}\,\mathrm{d}s\right)}{1+\sqrt{1-2x\left(\int_0^t e^{W_s}\,\mathrm{d}s\right) - x^2\left(\int_0^t e^{W_s}\,\mathrm{d}s\right)^2}}\\ &= \frac{2x - x^2\left(\int_0^t e^{W_s}\,\mathrm{d}s\right)}{1+\sqrt{\left(1-x\left(\int_0^t e^{W_s}\,\mathrm{d}s\right)\right)^2}}\\ &= \frac{x\left(2-x\left(\int_0^t e^{W_s}\,\mathrm{d}s\right)\right)}{2-x\left(\int_0^t e^{W_s}\,\mathrm{d}s\right)} = x.\end{aligned}$$

□

Lemma D.6 *The partial derivatives of u defined by (5.29) are given by*

$$\frac{\mathrm{d}u}{\mathrm{d}t}(x,t) = \frac{4x^2 e^{W_t}\big[Z(1+Z)^2\dot{W}_t + 2xe^{W_t} + 2xe^{W_t}Z\big]}{Z\cdot(1+Z)^4},$$
$$\frac{\mathrm{d}u}{\mathrm{d}x}(x,t) = \frac{Z(1+Z)^2\cdot 8xe^{W_t} + 8x^2e^{W_t}\big(\int_0^t e^{W_s}\,\mathrm{d}s\big)(1+Z)}{Z(1+Z)^4},$$

where we make use of the short notation

$$Z := \sqrt{1-2x\Big(\int_0^t e^{W_s}\,\mathrm{d}s\Big)}.$$

Proof. Due to the quotient rule of differential calculus we obtain

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t}(x,t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{4x^2e^{W_t}}{(1+Z)^2}\right]\\ &= \frac{Z(1+Z)^2\cdot 4x^2\dot{W}_te^{W_t} - 4x^2e^{W_t}\big(-2xe^{W_t}\big)(1+Z)}{Z(1+Z)^4}\\ &= \frac{Z(1+Z)^2\cdot 4x^2\dot{W}_te^{W_t} + 8x^3e^{2W_t}(1+Z)}{Z(1+Z)^4}\\ &= \frac{4x^2e^{W_t}\big[Z(1+Z)^2\dot{W}_t + 2xe^{W_t} + 2xe^{W_t}Z\big]}{Z\cdot(1+Z)^4}.\end{aligned}$$

Additionally, we get for the partial derivative with respect to x

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}x}(x,t) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{4x^2e^{W_t}}{(1+Z)^2}\right]\\ &= \frac{(1+Z)^2\cdot 8xe^{W_t} - 4x^2e^{W_t}\big(-2\int_0^t e^{W_s}\,\mathrm{d}s\big)\frac{(1+Z)}{Z}}{(1+Z)^4}\\ &= \frac{Z(1+Z)^2\cdot 8xe^{W_t} + 8x^2e^{W_t}\big(\int_0^t e^{W_s}\,\mathrm{d}s\big)(1+Z)}{Z(1+Z)^4}.\end{aligned}$$

□

Lemma D.7 *The partial derivative with respect to x and t of (6.10) are given by*

$$\frac{\mathrm{d}u}{\mathrm{d}t}(x,t) = u(x,t)\,\dot{W}_t + u(x,t)\left(\frac{2(x^2e^{2t})^{-(q-2)} - \lambda e^{(q-2)W_t}}{N}\right),$$
$$\frac{\mathrm{d}u}{\mathrm{d}x}(x,t) = u(x,t)\left(\frac{(2xe^{2t})(x^2e^{2t})^{-(q-1)}}{N}\right),$$

where we use the short notation

$$N := \big(x^2e^{2t}\big)^{-(q-2)} + \lambda(q-2)\int_0^t e^{(q-2)W_s}\,\mathrm{d}s.$$

Proof. Due to classical derivation rules we have

$$\frac{\mathrm{d}u}{\mathrm{d}t}(x,t) = \frac{N^{\frac{1}{q-2}}\,\dot{W}_t\,e^{W_t}}{N^{\frac{2}{q-2}}}$$

$$- \frac{e^{W_t} \Big(\frac{1}{q-2} N^{\frac{1}{q-2}} N^{-1} \Big(-(q-2) \, (x^2 e^{2t})^{-(q-1)} \, (2x^2 e^{2t}) - \lambda (q-2) e^{(q-2)W_t} \Big) \Big)}{N^{\frac{2}{q-2}}}$$
$$= \frac{e^{W_t}}{N^{\frac{1}{q-2}}} \dot{W}_t + \frac{e^{W_t}}{N^{\frac{1}{q-2}}} \left(\frac{2(x^2 e^{2t})^{-(q-2)} - \lambda e^{(q-2)W_t}}{N} \right)$$

and

$$\frac{\mathrm{d}u}{\mathrm{d}x}(x,t) = - \frac{e^{W_t} \Big(\frac{1}{q-2} N^{\frac{1}{q-2}} N^{-1} \Big(-(q-2) \, (x^2 e^{2t})^{-(q-1)} \, (2x e^{2t}) \Big) \Big)}{N^{\frac{2}{q-2}}}$$
$$= \frac{e^{W_t}}{N^{\frac{1}{q-2}}} \left(\frac{(2x e^{2t})(x^2 e^{2t})^{-(q-1)}}{N} \right).$$

□

Lemma D.8 *Let $\mu_j \in \mathbb{R}$. If (6.21) given by*

$$\sum_{j=1}^{\infty} \frac{4}{\pi^2} \mu_j^2 < \infty$$

is satisfied, the orthonormal basis (6.19), i.e.

$$\left\{ \sqrt{\frac{2}{\pi}} \sin(jx) \right\}_{j \geq 1}$$

fulfills Assumption 6.2.

Proof. Let $f \in L^2([0,\pi])$, then we conclude

$$\begin{aligned} \| f \cdot e_j \|_{L^2}^2 &= \int_0^{\pi} \left| f(x) \cdot \sqrt{\frac{2}{\pi}} \cdot \sin(jx) \right|^2 \mathrm{d}x \\ &= \frac{2}{\pi} \int_0^{\pi} |f(x) \cdot \sin(jx)|^2 \, \mathrm{d}x \\ &= \frac{2}{\pi} \int_0^{\pi} |f(x)|^2 \, |\sin(jx)|^2 \, \mathrm{d}x \\ &\leq \frac{2}{\pi} \cdot \sup_{x \in [0,\pi]} |\sin(jx)|^2 \cdot \| f \|_{L^2}^2 . \end{aligned}$$

Hence we choose $\tilde{\gamma}_j = \sqrt{\frac{2}{\pi}}$. Under Assumption (6.21) it follows that

$$\sum_{j=1}^{\infty} \mu_j^2 \frac{2}{\pi} \left\| \sqrt{\frac{2}{\pi}} \sin(jx) \right\|_{\infty}^2 = \sum_{j=1}^{\infty} \frac{4}{\pi^2} \mu_j^2 \, \| \sin(jx) \|_{\infty}^2 = \sum_{j=1}^{\infty} \frac{4}{\pi^2} \mu_j^2 < \infty.$$

Due to the fact that $\mathcal{C}^{\infty}([0,\pi]) \subset L^q([0,\pi])$ dense for $1 \leq q < \infty$, it is obvious that

$$\mu = \frac{1}{2} \sum_{j=1}^{\infty} \mu_j^2 \Big(\frac{2}{\pi} \cdot \sin^2(jx) \Big) = \frac{1}{\pi} \sum_{j=1}^{\infty} \mu_j^2 \cdot \sin^2(jx)$$

is a multiplier in $L^q([0,\pi])$, $q \geq 2$, and a symmetric one in $L^2([0,\pi])$. □

Lemma D.9 *Let $\mu_j \in \mathbb{R}$. If (6.22) given by*

$$\sum_{j=1}^{\infty} 8\,\mu_j^2 < \infty$$

is satisfied, the orthonormal basis (6.20), i.e.

$$\left\{\sqrt{2}\sin(j\pi x)\right\}_{j\geq 1}$$

fulfills Assumption 6.2.

Proof. Let $f \in L^2([0,1])$, then we conclude

$$\begin{aligned}
\|f \cdot e_j\|_{L^2}^2 &= \int_0^1 \left|f(x)\cdot\sqrt{2}\cdot\sin(j\pi x)\right|^2 \,\mathrm{d}x \\
&= 2\int_0^1 |f(x)\cdot\sin(j\pi x)|^2 \,\mathrm{d}x \\
&= 2\int_0^1 |f(x)|^2\,|\sin(j\pi x)|^2 \,\mathrm{d}x \\
&\leq 2\cdot \sup_{x\in[0,1]} |\sin(j\pi x)|^2 \cdot \|f\|_{L^2}^2.
\end{aligned}$$

Hence we choose $\tilde{\gamma}_j = 2$. Under Assumption (6.22)

$$\sum_{j=1}^{\infty} \mu_j^2\, 4\left\|\sqrt{2}\,\sin(j\pi x)\right\|_{\infty}^2 = \sum_{j=1}^{\infty} 8\,\mu_j^2\,\|\sin(j\pi x)\|_{\infty}^2 = \sum_{j=1}^{\infty} 8\,\mu_j^2 < \infty$$

follows. Due to the fact that $\mathcal{C}^\infty([0,\pi]) \subset L^q([0,\pi])$ dense for $1 \leq q < \infty$, it is obvious that

$$\mu = \frac{1}{2}\sum_{j=1}^{\infty} \mu_j^2\big(2\,\sin(j\pi x)\big) = \sum_{j=1}^{\infty} \mu_j^2 \sin^2(j\pi x)$$

is a multiplier in $L^q([0,\pi])$, $q \geq 2$, and a symmetric one in $L^2([0,\pi])$. □

E. Appendix to the scaling transform approach

The following definition is borrowed from [Bar10, Definition 2.1.].

Definition E.1 *An operator $A\colon V \to V^*$ is called monotone, if*

$$ {}_{V^*}\langle A(u) - A(v), u - v\rangle_V \geq 0 $$

for all $u, v \in V$.

Definition E.2 *Let V be a reflexive, real Banach space and $A\colon V \to V^*$ be an operator. Then A is called demicontinuous if and only if strong convergence in V implies weak convergence in V^*, i.e.*

$$ u_n \xrightarrow{n\to\infty} u \text{ in } V \Rightarrow A(u_n) \rightharpoonup A(u) \text{ in } V^*. $$

The above definition is taken from [Ruz04, Definition 1.3.].

Definition E.3 *Let V be a real Banach space and $A\colon V \to V^*$ be an operator. Then A is called hemicontinuous if for all $v, w \in V$ and $\varphi \in V$*

$$ \lim_{\lambda\to 0} {}_{V^*}\langle A(v + \lambda w), \varphi\rangle_V = {}_{V^*}\langle A(v), \varphi\rangle_V $$

holds.

Lemma E.4 *The following equation holds true*

$$ \mathrm{d}\Big[e^{\mathbb{W}(t)}\Big] = e^{\mathbb{W}(t)}\,\mathrm{d}\mathbb{W}(t) + \mu e^{\mathbb{W}(t)}\,\mathrm{d}t. $$

Proof. By an application of Itô formula (see [Oks07, Theorem 4.1.2]) to the exponential function we obtain for all $t \in [0, \mathbf{T}]$ and $x \in \mathbb{O}$

$$ \begin{aligned} e^{\mathbb{W}(x,t)} &= e^{\mathbb{W}(x,0)} + \int_0^t e^{\mathbb{W}(x,s)}\,\mathrm{d}\mathbb{W}(x,s) + \frac{1}{2}\int_0^t e^{\mathbb{W}(x,s)}\,\mathrm{d}\langle\mathbb{W}(x,\cdot)\rangle_s \\ &= e^0 + \int_0^t e^{\mathbb{W}(x,s)}\,\mathrm{d}\mathbb{W}(x,s) + \frac{1}{2}\int_0^t e^{\mathbb{W}(x,s)}\,\mathrm{d}\Big\langle \sum_{j=1}^\infty \mu_j e_j(x) W_\cdot^j, \sum_{j=1}^\infty \mu_j e_j(x) W_\cdot^j\Big\rangle_s \\ &= 1 + \int_0^t e^{\mathbb{W}(x,s)}\,\mathrm{d}\mathbb{W}(x,s) + \frac{1}{2}\sum_{j=1}^\infty \mu_j^2 e_j^2(x) \int_0^t e^{\mathbb{W}(x,s)}\,\mathrm{d}\langle W_\cdot^j, W_\cdot^j\rangle_s \\ &= 1 + \int_0^t e^{\mathbb{W}(x,s)}\,\mathrm{d}\mathbb{W}(x,s) + \frac{1}{2}\sum_{j=1}^\infty \mu_j^2 e_j^2(x) \int_0^t e^{\mathbb{W}(x,s)}\,\mathrm{d}s \\ &= 1 + \int_0^t e^{\mathbb{W}(x,s)}\,\mathrm{d}\mathbb{W}(x,s) + \mu(x) \int_0^t e^{\mathbb{W}(x,s)}\,\mathrm{d}s, \end{aligned} $$

where we define

$$ \mu(x) := \frac{1}{2}\sum_{j=1}^\infty \mu_j^2 e_j^2(x) \tag{E.1} $$

for all $x \in \mathbb{O}$. □

Definition E.5 *Let $y(t), t \in [0, \boldsymbol{T}]$, be an H-valued, $(\mathscr{F}_t)_t$-adapted process with continuous sample paths and let it be V^*-absolutely continuous on $[0, \boldsymbol{T}]$, $\mathbb{P}$-a.s. The function $\frac{\mathrm{d}y}{\mathrm{d}t}$ is defined by*

$$y(t,\omega) = y(0,\omega) + \int_0^t \frac{\mathrm{d}y}{\mathrm{d}s}(s,\omega)\ \mathrm{d}s, \quad \forall\ t \in [0, \boldsymbol{T}], \omega \in \Omega$$

and $(e^{\mathbb{W}}\frac{\mathrm{d}y}{\mathrm{d}t}) \in L^{q'}((0,\boldsymbol{T}) \times \Omega, V^*)$.

This definition is taken from [BR15, after Definition 3.2.]

Lemma E.6 *Let the assumptions of Theorem 7.8 be fulfilled. Let $J : \mathcal{V} \to \mathcal{V}^*$ be the duality mapping on $\mathcal{V}$ and F be defined by*

$$\begin{aligned} F &: \mathcal{V} \to \mathcal{V}^* \\ &y \mapsto F(y)(t) = e^{\mathbb{W}(t)} J(e^{\mathbb{W}(t)} y(t)) |e^{\mathbb{W}(t)} y(t)|_V^{q-2}\ \ \forall\ t \in [0, \boldsymbol{T}]. \end{aligned}$$

Then the equation

$$(\overline{\mathscr{B} + \mathscr{T}})y + e^{-\mathbb{W}} F(e^{-\mathbb{W}} y) = 0$$

has a unique solution.

Proof. Let us define $G(z) := J(z)\|z\|^{q-2}$. The operator $(\overline{\mathscr{B} + \mathscr{T}})$ is maximal monotone in $L^q((0,\mathbf{T}) \times \mathbb{O}, \mathbb{R}) \times L^{q'}((0,\mathbf{T}) \times \mathbb{O}, \mathbb{R})$, hence by [Bar10, Theorem 2.3] with $\lambda = 1$ and for $q > 2$ we obtain

$$L^{q'}((0,\mathbf{T}) \times \mathbb{O}, \mathbb{R}) = \bigcup_{y \in D(\overline{\mathscr{B}+\mathscr{T}}+G)} \Big((\overline{\mathscr{B} + \mathscr{T}})y + G(y) \Big).$$

With the help of [Bar10, Theorem 2.1, Lemma 2.2.] there exists a unique $\hat{y} \in D(\overline{\mathscr{B} + \mathscr{T}})$ such that for $0 \in L^{q'}((0,\mathbf{T}) \times \mathbb{O}, \mathbb{R})$

$$\begin{aligned} 0 &= (\overline{\mathscr{B} + \mathscr{T}})\hat{y} + G(\hat{y}) \\ &= (\overline{\mathscr{B} + \mathscr{T}})\hat{y} + J(\hat{y})\|\hat{y}\|^{q-2} \\ &= (\overline{\mathscr{B} + \mathscr{T}})\hat{y} + e^{\mathbb{W}} e^{-\mathbb{W}} J(e^{\mathbb{W}} e^{-\mathbb{W}} \hat{y}) \|e^{\mathbb{W}} e^{-\mathbb{W}} \hat{y}\|_{L^q}^{q-2} \\ &= (\overline{\mathscr{B} + \mathscr{T}})\hat{y} + e^{-\mathbb{W}} F(e^{-\mathbb{W}} \hat{y}). \end{aligned}$$

□

Lemma E.7 *Let the assumptions of Theorem 7.8 be fulfilled. The operator $\overline{\mathscr{B} + \mathscr{T}}$ is maximal monotone in $\mathcal{V} \times \mathcal{V}^*$.*

We follow the ideas of the proof of Theorem 2.2. in [Bar10].

Proof. We assume that $(\overline{\mathscr{B} + \mathscr{T}})$ is not maximal monotone in $\mathcal{V} \times \mathcal{V}^*$, i.e. there exists $(x_0, y_0) \in \mathcal{V} \times \mathcal{V}^*$ such that

$$(x_0, y_0) \notin (\overline{\mathscr{B} + \mathscr{T}}) \tag{E.2}$$

and

$$_{\mathcal{V}^*}\langle y - y_0, x - x_0 \rangle_{\mathcal{V}} \geq 0 \quad \forall (x,y) \in (\overline{\mathscr{B} + \mathscr{T}}). \tag{E.3}$$

Let $J : \mathcal{V} \to \mathcal{V}^*$ be the duality mapping on $\mathcal{V}$. We define $G(z) := J(z)\|z\|^{q-2}$. We can show as in [BR15, Lemma 4.2] that

$$R((\overline{\mathscr{B} + \mathscr{T}}) + \lambda G) = \mathcal{V}^*,$$

where $R(A)$ denotes the union $\bigcup_{u \in D(A)} Au$ for any operator A with domain $D(A)$ (see Definition 7.4). Therefore there exists $(x_1, y_1) \in (\overline{\mathscr{B} + \mathscr{T}})$ with

$$\lambda G(x_1) + y_1 = \lambda G(x_0) + y_0,$$

which is equivalent to

$$\lambda G(x_0) - \lambda G(x_1) = y_1 - y_0.$$

So, by (E.3) we get

$$_{\mathcal{V}^*}\langle G(x_0) - G(x_1), x_1 - x_0\rangle_{\mathcal{V}} = {}_{\mathcal{V}^*}\langle y - y_0, x - x_0\rangle_{\mathcal{V}} \geq 0.$$

Due to the definition of the duality mapping given by

$$J(z) := \{z^* \in \mathcal{V}^* \mid \langle z^*, z\rangle = \|z\|^2\}$$

as stated in [Bar10, equation (1.1)] we obtain for $z \in \mathcal{V}$

$$\begin{aligned} {}_{\mathcal{V}^*}\langle G(z), z\rangle_{\mathcal{V}} &= {}_{\mathcal{V}^*}\langle J(z)\|z\|^{q-2}, z\rangle_{\mathcal{V}} \\ &= \|z\|^{q-2}{}_{\mathcal{V}^*}\langle J(z), z\rangle_{\mathcal{V}} \\ &= \|z\|^{q-2}\|z\|^2 = \|z\|^q. \end{aligned}$$

Hence we have

$$\begin{aligned} 0 &\leq {}_{\mathcal{V}^*}\langle G(x_0) - G(x_1), x_1 - x_0\rangle_{\mathcal{V}} \\ &= {}_{\mathcal{V}^*}\langle G(x_0), x_1\rangle_{\mathcal{V}} - {}_{\mathcal{V}^*}\langle G(x_0), x_0\rangle_{\mathcal{V}} - {}_{\mathcal{V}^*}\langle G(x_1), x_1\rangle_{\mathcal{V}} + {}_{\mathcal{V}^*}\langle G(x_1), x_0\rangle_{\mathcal{V}} \\ &= {}_{\mathcal{V}^*}\langle G(x_0), x_1\rangle_{\mathcal{V}} - \|x_0\|^q - \|x_1\|^q + {}_{\mathcal{V}^*}\langle G(x_1), x_0\rangle_{\mathcal{V}}, \end{aligned}$$

which is equivalent to

$$\|x_0\|^q + \|x_1\|^q \leq {}_{\mathcal{V}^*}\langle G(x_0), x_1\rangle_{\mathcal{V}} + {}_{\mathcal{V}^*}\langle G(x_1), x_0\rangle_{\mathcal{V}}.$$

Next, we consider

$$\begin{aligned} {}_{\mathcal{V}^*}\langle G(x_1) - G(x_0), x_1 - x_0\rangle_{\mathcal{V}} &= \|x_1\|^q + \|x_0\|^q - \Big({}_{\mathcal{V}^*}\langle G(x_0), x_1\rangle_{\mathcal{V}} + {}_{\mathcal{V}^*}\langle G(x_1), x_0\rangle_{\mathcal{V}}\Big) \\ &\leq 0. \end{aligned}$$

Due to (E.3) we conclude ${}_{\mathcal{V}^*}\langle G(x_1) - G(x_0), x_1 - x_0\rangle_{\mathcal{V}} = 0$. Consequently we have

$$\|x_0\|^q = \|x_1\|^q = {}_{\mathcal{V}^*}\langle G(x_0), x_1\rangle_{\mathcal{V}} = {}_{\mathcal{V}^*}\langle G(x_1), x_0\rangle_{\mathcal{V}}$$

which shows

$$G(x_0) = G(x_1).$$

As written in [Bar10, Section 1.1] the duality mapping J^{-1} of $\mathcal{V}^*$ is single-valued, since $\mathcal{V}$ is strictly convex, we obtain

$$x_1 = x_0.$$

Therefore we obtain $(x_0, y_0) = (x_1, y_1) \in (\overline{\mathscr{B} + \mathscr{T}})$, which contradicts (E.2), and show that $(\overline{\mathscr{B} + \mathscr{T}})$ is maximal monotone. □

F. Appendix to an application of Lemma 4.8 to [DPT96]

We apply Itô's product rule (see [RY05, Chapter IV, 3.1 Proposition]) to deduce equation (8.25).

Derivation *We apply Itô's product rule to the equation (8.3)*

$$\mathrm{d}y = \Big(\tilde{L}(t,\xi,y,Dy,D^2y) - \langle b\cdot Dy, h\rangle_{\mathbb{R}^d} - \mathrm{trace}[Dh\cdot b^\top]\Big)\,\mathrm{d}t + y\,\langle h, \mathrm{d}W_t\rangle_{\mathbb{R}^{d_1}}$$

and an arbitrary stochastic differential equation given by

$$\mathrm{d}\varrho = \varrho^{drift}\,\mathrm{d}t + \sum_{k=1}^{d_1}\varrho_k^{diff}\,\mathrm{d}W_t^k.$$

and obtain

$$\begin{aligned}\mathrm{d}\varrho\ \mathrm{d}y = \Big[\varrho\cdot\Big(\tilde{L}(t,\xi,y,Dy,D^2y) - \langle b\cdot Dy,h\rangle_{\mathbb{R}^d} - \mathrm{trace}[Dh\cdot b]\Big)\Big]\,\mathrm{d}t \quad &(\mathrm{F.1})\\ + \varrho\cdot y\cdot\langle h,\mathrm{d}W_t\rangle_{\mathbb{R}^{d_1}} \quad &(\mathrm{F.2})\\ + y\cdot\varrho^{drift}\,\mathrm{d}t \quad &(\mathrm{F.3})\\ + \sum_{k=1}^{d_1} y\cdot\varrho_k^{diff}\,\mathrm{d}W_t^k \quad &(\mathrm{F.4})\\ + \Big\langle \sum_{k=1}^{d_1}\varrho_k^{diff}\,\mathrm{d}W_\cdot^k, \sum_{k=1}^{d_1} y\cdot h_k\,\mathrm{d}W_\cdot^k\Big\rangle_t. \quad &(\mathrm{F.5})\end{aligned}$$

As before we consider the diffusion term and drift term separately to determine ϱ^{drift} and ϱ^{diff}. Let us start with the diffusion terms, i.e. the sum of (F.2) and (F.4)

$$\begin{aligned}\varrho\cdot y\cdot\langle h,\mathrm{d}W_t\rangle_{\mathbb{R}^{d_1}} + \sum_{k=1}^{d_1} y\cdot\varrho_k^{diff}\,\mathrm{d}W_t^k &= \sum_{k=1}^{d_1}\Big(\varrho\cdot y\cdot h_k + y\cdot\varrho_k^{diff}\Big)\,\mathrm{d}W_t^k\\ &= \sum_{k=1}^{d_1} y\cdot\Big(\varrho\cdot h_k + \varrho^{diff_k}\Big)\,\mathrm{d}W_t^k.\end{aligned}$$

Hence we define

$$\boxed{\varrho_k^{diff} := -\varrho\cdot h_k}$$

Next, we plug ϱ_k^{diff} into the covariation term (F.5) to get

$$\begin{aligned}\Big\langle \sum_{k=1}^{d_1}\varrho^{diff}\,\mathrm{d}W_\cdot^k, \sum_{k=1}^{d_1} y\cdot h_k\,\mathrm{d}W_\cdot^k\Big\rangle_t &= \Big\langle \sum_{k=1}^{d_1}(-\varrho\cdot h_k)\,\mathrm{d}W_\cdot^k, \sum_{k=1}^{d_1} y\cdot h_k\,\mathrm{d}W_\cdot^k\Big\rangle_t\\ &= \sum_{k=1}^{d_1} -\varrho\cdot h_k\cdot y\cdot h_k\,\mathrm{d}\langle W_\cdot^k, W_\cdot^k\rangle_t\\ &= -\sum_{k=1}^{d_1}\varrho\cdot y\cdot h_k^2\,\mathrm{d}t\\ &= -\varrho\cdot y\cdot\Big(\sum_{k=1}^{d_1} h_k^2\Big)\,\mathrm{d}t\\ &= -\varrho\cdot y\cdot|h|^2\,\mathrm{d}t. \qquad (\mathrm{F.6})\end{aligned}$$

Let us consider the sum of the drift terms (F.1), (F.3) and (F.6)

$$\begin{aligned}\Big(\varrho\cdot\big(\tilde{L}(t,\xi,y,Dy,D^2y)-\langle b\cdot Dy,h\rangle_{\mathbb{R}^d}-\mathrm{trace}[Dh\cdot b]\big)+y\cdot\varrho^{drift}-\varrho\cdot y\cdot|h|^2\Big)\,\mathrm{d}t\\ =\Big(\varrho\cdot y+y\cdot\varrho^{drift}-\varrho\cdot y\cdot|h|^2\Big)\,\mathrm{d}t\\ =(y\cdot\varrho)\,\mathrm{d}t+y\cdot\Big(\varrho^{drift}-\varrho\cdot|h|^2\Big)\,\mathrm{d}t.\end{aligned}$$

To get

$$y\cdot(\varrho^{drift}-\varrho\cdot|h|^2)=0$$

we have to choose

$$\boxed{\varrho^{drift}:=\varrho\cdot|h|^2}$$

Finally we obtain the stochastic differential equation (8.25) given by

$$\boxed{\begin{aligned}\mathrm{d}\varrho&=|h|^2\cdot\varrho\,\mathrm{d}t-\varrho\cdot\sum_{k=1}^{d_1}h_k\,\mathrm{d}W_t^k\\ &=|h|^2\cdot\varrho\,\mathrm{d}t-\varrho\cdot\langle h,\mathrm{d}W_t\rangle_{\mathbb{R}^{d_1}}\end{aligned}}$$

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