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Did Partisan Voters Spoil the Country? A Randomized-thought Experiment [1]

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Abstract

We study the effect of strategic and partisan voting on electoral outcomes, and on the relative popularity of the victor. Voters are randomly assigned to be partisan or strategic. When all voters are strategic in a plurality election, any equilibrium manipulation of the outcome elects a popular leader. Voting populations with a large proportion of partisan voters are more at risk of electing an unpopular leader: in elections with three candidates, if only one-third of the population is partisan, then the winner of the election may be unpopular with two-thirds of voters. We derive exact bounds for the proportion of the population that benefits from manipulation of the election outcome by strategic voters, for arbitrary numbers of voters, candidates, partisans and strategic voters. The analysis also shows that the unpopularity of the election winner differs between partisan and strategic voters. When most voters are partisan, they may be the vast majority of those who gain from strategic voting.

Keywords: Morality and Politics, Partisan Voting, Winner Unpopularity, Plurality, Impact Heterogeneity, Randomization.

JEL Classification: P16, D72, C7, J15, H41.

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1 Introduction

Should you vote with your head or your heart? In tightly contested elections, this is a question over which many voters agonize endlessly. Voting one's conscience, even if for a candidate or a party with no realistic chance of winning, is commonly viewed as a principled approach to the exercise of one of the most fundamental rights of citizens in a democratic society. But voting one's heart is not without risk: in jurisdictions using the plurality rule, voting for one's preferred candidate often paves the way to the election of an unpopular leader. Voting strategically might result in better outcomes. These observations point to the question which is the central focus of this paper: Should strategic voting be encouraged or deplored? In other words, what is the equilibrium effect of partisan voting on the popularity of the elected leader?

We tackle this problem for plurality elections which involve a minimum of three candidates, and a voting population which is split randomly between *partisans* - who always vote for their preferred candidate - and *strategic voters*, who may decide to vote for one of their less-preferred candidates when this leads to an outcome which they prefer to the outcome which is chosen if they vote in a partisan fashion. Strategic voting generates both winners and losers as compared to the outcome of the electoral process when everyone votes for their preferred candidate. We are the first to show that it is possible to calculate exact bounds on the relative popularity of the elected leader as a function of the proportion of partisan voters in the population. This is given by the proportion of voters who benefit from strategic (as opposed to partisan) voting under the plurality rule. Strikingly, in many settings a clear majority of voters are made better off, especially when the number of partisans is not too high.

Our results sharply highlight the impact of variation in the proportion of highly partisan voters on the distribution of gains from strategic voting. In essence, our analysis can be interpreted as a thought-randomized experiment: if in any particular population Nature randomly assigns voters to either the set of strategic voters, or to the set of partisan voters then, for any of the realized preference profiles, the impact of strategic voting will fall between our minimum and maximum bounds. We show that when partisan voters constitute a large proportion of the electorate, it is far more likely that an unpopular leader will be selected, and spoil the country. In contrast, if no one is a partisan and everyone votes strategically, then whenever an electoral outcome is manipulated it must, in fact, be preferred by a majority of voters to the outcome selected when all voters report their preferences truthfully. In settings where there are no partisans, strategic voting always results in electing a popular leader. In contrast, when most voters are partisans, two key things happen: it is more likely that an unpopular leader is elected as a result of strategic voting, but at the same time the vast majority of those who gain from strategic voting may be partisans.

Why is it important to take account of the fact that voting populations are typically split between strategic and partisan voters when studying the impact of strategic voting under the plurality rule? Empirical investigations of the prevalence of strategic voting suggest that in any given election the vast majority of voters will vote for their preferred candidate (Alvarez and Nagler (2000), Blais et al. (2001), Degan and Merlo (2009)). What is not clear is whether it is strategic voters who are rare, or whether it is only in rare circumstances that voters who prefer candidates who are certain to lose believe that they can influence the outcome of the election by voting strategically (Muller and Page (2015), Kawai and Watanabe (2013), Blais (2002)).

A multitude of factors may in fact explain why some voters are partisans, and *always* vote for their most-preferred candidate (or party). Voting strategically is often portrayed as equivalent to telling a lie — indeed, social choice and political theorists refer to this as 'misreporting your preferences' — and considerable experimental evidence (see, for example, Gneezy (2005), Erat and Gneezy (2012), Vanberg (2008)) suggests that a significant proportion of the population is lie-averse, and will therefore almost certainly always vote truthfully.[2] Moreover, many voters feel tremendous party loyalty: the psychic costs of voting for any party (or candidate) other than the party they love outweigh the gains that would accrue from blocking the election of a party they intensely dislike (Blais (2002), Li and Pique Cebrecos (2016) Shachar (2003)). Regardless of the underlying explanation for the existence of partisan voters, it is critical to take account of the fact that such voters exist, as this is likely to shape the circumstances in which strategic voters can manipulate outcomes.

Our results are also relevant to the literature on mechanism design, which typically takes the view that a well-designed social choice mechanism should implement the outcome that is selected when individuals report their preferences truthfully. There is good reason for wanting to design mechanisms which encourage truthful reporting. As noted by Barbera (2010), the manipulation of social decision procedures is a matter of concern if these procedures select an efficient outcome when voters are truthful, but recommend an inefficient alternative when voters report their preferences strategically; much effort has consequently been directed to designing strategy-proof mechanisms for collective choice. However, ever since the seminal contributions of Gibbard (1973) and Satterthwaite (1975), it has been well understood that the voting mechanisms used in actual

[2]Whereas economists take the view that voting for anyone other than one's preferred candidate is to misrepresent one's preferences, and akin to lying, philosophers disagree. Although there is a moral prohibition on lying, philosophers are of the view that one cannot lie without asserting (verbally or otherwise) a false proposition and no proposition can be asserted by casting a ballot. In general, on a Kantian view, an act is obligatory if and only if it is *always* desirable that every person performs that action; if voting strategically averts a worse outcome than would prevail if voters report their true preferences then the categorical imperative is not violated by either truthful or strategic voting. In contrast, virtue theorists would take the view that voters, *regardless* of their personal preferences, should always cast their ballots for the candidate who would be preferred by a virtuous (that is, moral) voter. What is less clear, however, is whether or not a virtuous voter would determine their preferred candidate by taking account of the likelihood that a ballot cast for that candidate makes it more or less likely that the candidate wins the election. In contrast, consequentialists (such as Brennan (2011)) argue that citizens have a moral duty to vote strategically, when to fail to do so leads to an outcome that is less preferred - either from the viewpoint of themselves as individuals, or for society as a whole. In effect, it is truthful voting, and not strategic voting, that is the morally reprehensible choice.

elections (and, in particular, the plurality or first-past-the-post mechanism, which is by far the most widely used mechanism) are therefore vulnerable to manipulation.

Note, however, that strategic voting affects the election result that is decided using the plurality rule when the outcome selected is the preferred outcome of only a minority of voters, and that the outcome which is selected as a result of strategic voting is Pareto non-comparable to that which is chosen when all voters report their preferences truthfully, but more popular. If there is reason to believe that strategic voting typically leads to a more popular election winner, then it is not clear that the outcome selected by a strategy-proof social choice mechanism should be preferred by many voters - in particular by a majority - to that selected when the election is decided using the plurality rule, even if the electors vote strategically. Our analysis can in some sense be viewed as a cautionary tale against being overly focused on designing collective choice procedures which always select the outcome which prevails when electors vote for their preferred candidate: at the end of the day, this may not be an outcome that is particularly worth championing.[3] The approach taken in this paper points to an alternative metric to apply when comparing the merits of alternative voting mechanisms: if it is true that under one procedure strategic voting typically benefit a large proportion of the voters -the winner is popular- whereas under the other procedure strategic voting typically only advantage a small minority of citizens -the winner is less popular-, then the first procedure should be judged more desirable.

The structure of this paper is as follows. In section 2, below, we first lay out our model, which extends the standard model to voting with partisan voters. In section 3 we study the set of outcomes which satisfy both the standard Gibbard-Satterthwaite definition of strategic vote and are Nash equilibria of the voting game, and determine tight minimum and maximum bounds on the number of voters who benefit from strategic voting as a function of the number of voters, the number of candidates, and the number of partisan voters. Subsequently, we disaggregate the overall effect and determine exact bounds for both partisans and strategic voters, which provides us with additional insight into how these gains are shared. In section 4 we provide simulation results which suggest that strategic voting typically generate gains and losses in the mid-range of the calculated bounds. Section 5 concludes.

2 The Model

We consider a set-up that is, in most essentials, identical to classic political economy models: a set of voters ranks a set of candidates, and the winning candidate is selected using the plurality rule with alphabetical tie-break. We denote the set of candidates by $A = \{a_1, ..., a_m\}$, letting m denote the cardinality of this set; the set of voters is denoted by $N = \{1, ..., n\}$, and has cardinality

[3]Serrano (2018) also expresses reservations about being overly-focused on the outcome selected when all players report preferences truthfully, albeit for different reasons than those presented above.

n. Unlike more traditional models, however, we distinguish between two types of voters: *partisan voters*, who always cast their ballots for their preferred candidate, and *strategic voters*, who vote with their heads rather than their hearts - that is, they will vote for a candidate other than the one they personally prefer whenever this results in an outcome which they prefer to that which prevails if they vote truthfully. Denote the set of (potentially) strategic voters by $S \subseteq N$, and the cardinality of this set by s; the set of partisan voters is consequently $N \backslash S$, and has cardinality $n - s$. We assume that Nature determines whether or not a voter is partisan or strategic; this implies that the assignment does not depend on individual preferences. Note that whether or not a strategic voter chooses, in equilibrium, to vote for a candidate who is not their preferred leader depends upon the particular electoral environment.

The true preference profile of the voting population - that is, each voter's ranking of each of the candidates in declining order of actual preference - is denoted by R^N; an individual voter i's true preference ranking of the candidates is denoted R^i. The set of all possible preference profiles is $\mathcal{R}^N$. As strategic voters may report a ranking that does not correspond to their true preference profile, we denote by Q^N the preference profile which is constructed from the actual reports of each voter. As noted above, the winner of a given election is determined by application of the plurality rule with alphabetical tie-break. That is, denoting by $F\left(a_i, Q^N\right)$ the set of voters who rank $a_i \in A$ as their preferred candidate when the reported ranking is Q^N, then for any $a_i, a_j \in A$, if $\left|F\left(a_i, Q^N\right)\right| = \left|F\left(a_j, Q^N\right)\right|$ and $\left|F\left(a_k, Q^N\right)\right| < \left|F\left(a_i, Q^N\right)\right|$ for any $k \neq i, j$, then candidate a_i is selected as the victor if $i < j$. Let $Pl(Q^N)$ denote the outcome selected by the plurality voting process when the reported preference profile is Q^N.

Below, we study Nash equilibrium outcomes of the voting game. As the set of equilibrium outcomes typically varies as a function of the membership of the set of strategic voters, we refer to a Nash equilibrium outcome when the set of strategic voters is S as a constrained Nash equilibrium. In the tradition of Gibbard (1973) and Satterthewaite (1975), a strategic vote under the profile of true preferences R^N and the block S of strategic voters is a preference profile Q^N such that (i) a voter $i \in S$ who misreports her preference prefers the outcome under Q^N to the outcome under R^N, that is, $Pl(Q^N) \succ_i Pl(R^N)$, and (ii) all other voters report their preferences truthfully.[4] We denote by $N(S \mid R^N)$ the set of strategic preference profiles that are constrained Nash equilibria under R^N.

In most actual voting situations strategic voters will not gain from misreporting their preferences, and so they will vote for their preferred candidate. However, it is useful to identify preference

[4]Notice that the Gibbard-Satterthwaite definition of strategic voting does not guarantee that all strategic voters are reporting a preference ordering which is a best-response to the preference-orderings reported by the other voters. In particular, strategic voters who are made worse off as a result of a strategic vote by another strategic voter are required to report their preferences honestly, even if they would be able to obtain a better outcome by in turn best responding (in the sense of Gibbard-Satterthewaite) to the profile Q^N. The results presented in this paper are for constrained Nash equilibria which also satisfy the Gibbard-Satterthwaite definition of strategic voting.

profiles for which strategic voting affect the outcome of the election. A preference profile $R^N \in \mathcal{R}^N$ is $S-$unstable if a voter $i \in S$ has an incentive to misreport her true preferences - that is, R^N is $S-$unstable if there is a voter $i \in S$ for whom it is not beneficial to submit a truthful report given that all other voters are truthful. Notice that whether or not a given preference profile is stable will typically depend on the set of strategic voters (and, of course, on its complement - the set of partisan voters). Define by $P(S)$ the set of $S-$unstable profiles of preferences and by $P(S \mid R^N)$ the set of strategic preference profiles given the profile of true preferences R^N. Let $Z(S) \subset P(S)$ denote the subset of $S-$unstable profiles of preferences that admit at least one strategic preference profile that is a constrained Nash equilibrium. Finally, in the event that a strategic voter chooses to misrepresent her preferences at the Nash equilibrium, we need to be able to distinguish the winners and losers as a result of this strategic vote. Given $R^N \in Z(S)$ and $Q^N \in N(S \mid R^N)$, $E(R^N, Q^N)$ is the set of voters - including the partisan voters - who benefit from the strategic vote Q^N.

3 Strategic Voting: Meet the Winners

Our principal objective in this paper is to study the proportion of the population that benefits from strategic voting, in equilibrium, as a function of the population of partisan voters. We derive maximum and minimum bounds for the proportion of citizens, and for the sub-populations of partisan and strategic voters, who benefit from - or are adversely affected by - manipulation of the election outcome as a result of strategic voting, for any number of candidates, and of partisan and strategic voters. There is good reason to think that manipulation of electoral outcomes may benefit large number of voters. Strategic voting, by construction, always benefits those who misrepresent their preferences. However, it also benefits those partisan voters who prefer the candidate selected as a result of the manipulation to the candidate selected when all citizens vote for their preferred candidate. Observe that strategic voting affects the electoral outcome only if the largest voting blocs are of similar size: whenever there is a broad consensus on the preferred leader, strategic voting will not change the electoral outcome. In contrast, if one outcome is selected rather than an alternative simply because of the application of the alphabetical tie-break rule (and particularly in settings with relatively few candidates), then large number of voters may potentially benefit if strategic voting changes which leader (or outcome) is chosen: the floor on the number of citizens who benefit from strategic voting must be at least equal to the number of voters who support the candidate selected when all citizens report their preferences truthfully.

Theorem 1 provides, for any number of voters, and for elections with at least three candidates, exact bounds for the proportion of winners and losers in the population overall.[5] It is important to

[5] As is well known, in elections with only two candidates, it is a dominant strategy for each voter to vote for

study electoral settings for arbitrary numbers of both candidates and voters: elections for national political office typically involve large number of voters, and a small number of candidates, whereas when the members of a National Assembly vote on, say, income tax rates, there are generally relatively small number of voters, and a very large number of potential candidate tax rates.

The bounds also critically depend on the way the population is split between partisan and strategic voters.

What quickly becomes clear is that when *all* voters are strategic, the election winner cannot be unpopular relatively to the status quo. This is explained by the fact that the electoral outcome through strategic voting always benefits a majority of voters. In contrast, when the vast majority of voters are partisans, the proportion of the population which benefits from strategic voting may be much smaller. What this implies is that as voters become increasingly partisan, the leaders who are actually elected are more likely to be unpopular.

Before stating our first substantive result we need to introduce some additional notation. Let $T \subset N$ be a subset of voters. We denote by m^* the minimum proportion of the population - that is, including both partisan and strategic voters - which benefits from strategic voting, i.e.,

$$m^*(m,n,s,T) = \min\{\frac{\left|T \cap E(R^N, Q^N)\right|}{|T|} \mid R^N \in Z(S) \text{ and } Q^N \in P(S \mid R^N) \cap N(S \mid R^N).\}$$

Similarly, M^* denotes the maximum proportion of the population T which benefits from strategic voting, ie.,

$$M^*(m,n,s,T) = \max\{\frac{\left|T \cap E(R^N, Q^N)\right|}{|T|} \mid R^N \in Z(S) \text{ and } Q^N \in P(S \mid R^N) \cap N(S \mid R^N).\}$$

We are particularly interested in three cases; the effect on the whole voting population, that is, when $T = N$; the effect on partisans, that is, when $T = N \backslash S$; and, finally, the effect on non-partisan voters, that is, when $T = S$. Let

$$I^*(m,n,s,T) = [m^*(m,n,s,T), M^*(m,n,s,T)]$$

denote the interval between the maximum and minimum equilibrium proportion of the population T which benefits from strategic voting when there are m candidates n voters and s strategic voters. It follows that $1 - I^*(m,n,s,T)$ can be interpreted as providing exact bounds on the relative unpopularity of the election winner as a function of the proportion of partisan voters, the number of candidates, the population size in a subgroup T of the voters. In this case, $1 - M^*(m,n,s,T)$

their preferred candidate (or policy).

measures the unpopularity of the election winner in the worst case scenario, and $1-m^*(m,n,s,T)$ measures his/her unpopularity in the best case scenario. In this paper, we will be interested in how partisan voting affects leader unpopularity in the whole population ($T=N$) and separately among strategic voters ($T=S$) and partisan voters ($T=N\backslash S$). We can now state our first major result.

Theorem 1 *When an electoral outcome can be manipulated by strategic voters, the proportion of the voting population which prefers the leader selected as a result of strategic voting to the leader who would have been selected as a result of exclusively partisan voting belongs to the interval*

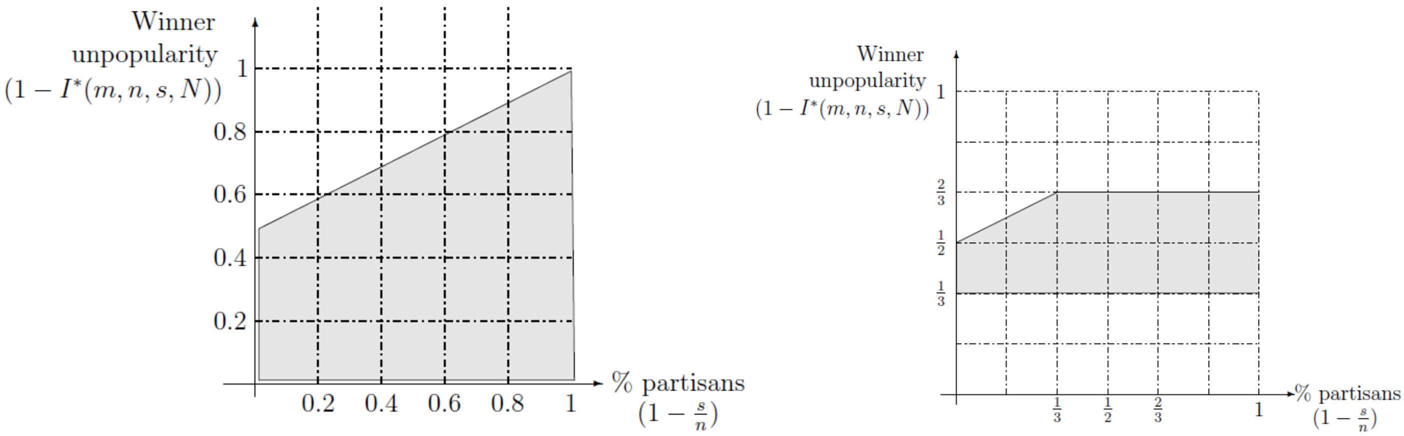
$$I^*(m,n,s,N)=\begin{cases}\left[\frac{1}{n}+\frac{1}{n}\left\lfloor\frac{n}{m}\right\rfloor;1-\frac{1}{n}\left\lceil\frac{n}{m}\right\rceil\right] & \textit{if} \quad s\leq 2\left\lfloor\frac{n}{m}\right\rfloor+1\\ \left[\frac{1}{n}\left\lceil\frac{s}{2}\right\rceil;1-\frac{1}{n}\left\lceil\frac{n}{m}\right\rceil\right] & \textit{if} \quad s>2\left\lfloor\frac{n}{m}\right\rfloor+1\end{cases}$$

In particular, when all voters are strategic voters, then manipulation of electoral outcomes by strategic voters always benefits at least 50% of the population, therefore leading to a winner who is not unpopular.

Proof See Appendix. ■

What does Theorem 1 tell us about the likely number of winners as a result of strategic voting? First, consider a setting where there are at least as many candidates for office (or policies to be selected from) as there are voters, that is, $m\geq n$In this case, the maximum and minimum bounds are independent of m, but vary with n and with s, the number of strategic voters. The first specification of the bound is valid when there is a single strategic voter. In this scenario, the difference between the upper and lower bound becomes very large as the number of voters grows. To understand why the bounds are so large, imagine that each candidate has the support of at most one citizen, so that the victor under truthful voting is selected by alphabetical tie-break. Now consider two extreme cases. First, suppose that the candidate selected under truthful voting is the second-ranked option of all voters except the strategic voter. If the strategic voter manipulates the electoral outcome by voting for the preferred candidate of one of the partisan voters this will make that partisan voter and the strategic voter better off, but all of the other partisan voters will be made worse off. In contrast suppose that the strategic voter's second-ranked candidate is also the second-ranked candidate of all voters (except for the partisan voter for whom this is the preferred candidate). Then strategic voting will make everyone better off with the exception of the one partisan voter whose preferred candidate is no longer selected. In contrast, if there are only 3 voters - and one of these voters is a strategic voter - then the bounds shrink to $[1/3,2/3]$: with small number of voters, both the costs and benefits of strategic voting are attenuated.

A generic setting that describes most elections for political office is when $m<n$, and where the number of candidates, m, is much smaller than n. Notice that in both settings, the bounds



(a) Case $n < m$

(b) Case $n = 300$ and $m = 3$

Figure 1: Impact of partisan voting on the unpopularity of the election winner

on the proportion of beneficiaries from strategic voting shrink as the proportion of partisan voters decrease: when the vast majority of voters are partisans, there is a real risk that strategic voting will lead to the election of an unpopular leader; see Figure 1. In contrast, and just as in the setting with many candidates or policy alternatives, when $n = s$, the lower bound on the proportion of the population which benefits from strategic voting is never less than 50%, and so the elected leader will be popular; see Figure 1 (b).

Notice that, for all values of m, n and s, it is always the case that the outcome selected when all voters report their preferences truthfully is Pareto non-comparable to the outcome that results when some electors vote strategically. When $m > n$, the election winner at the strategic preferences profile may nonetheless be Pareto inefficient: with a large number of candidates, and an alphabetical tie-break rule, the only possible strategic preference profiles may be ones which select a candidate listed earlier in the alphabet, although all voters might prefer a candidate listed towards the end of the alphabet. This can be true even when $s = n$, and strategic voting benefits at least 50% of the population; indeed, it is straightforward to construct preference profiles at which at least one strategic voting leads to an election winner who is preferred by $n - 1$ voters to the outcome that is selected when preferences are reported truthfully, even though this outcome is Pareto inefficient.

In contrast, when $m < n$, it is always the case that strategic voting leads to a Pareto efficient outcome: a strategic vote affects the electoral outcome only if there are at least two candidates who benefit from a level of support which differs by at most one vote when all voters report their preferences truthfully. This means that the leader selected as a result of strategic voting must be the preferred candidate of at least some voters, and consequently the outcome will be Pareto efficient. Although Theorem 1 establishes that the bounds on the minimum and maximum proportion of winners (and losers) depends on the share of partisan voters in the overall voting population, efficiency considerations cannot be invoked as a reason to prefer the electoral outcome when voters report their preferences truthfully to the outcome selected when some electors vote strategically.

What policy implications can be drawn from this discussion? Observe that, holding constant the number of partisans and of strategic voters, practices which are intended to facilitate entry into candidacy for political office may have the unintended consequence of increasing uncertainty about the quality - or at least the popularity - of the selected leader: a larger m widens the bounds on the proportion of the population that may benefit from strategic voting. Equally crucially, however, widespread partisan voting *also* increases the likelihood of electing an unpopular leader: if all voters could be persuaded to vote strategically they might not select the candidate who would be chosen under purely partisan voting, but at least the leader who was selected would command widespread support. Arguably, the fact that there are often barriers to standing as a candidate (nomination papers requiring a minimum number of signatures, deposit fees, etc.) in elections which are decided using the plurality rule should increase the likelihood that the elected representative will be a popular leader, at least locally, and may be a factor underlying the so-called incumbency effect. Finally, policies which ensure that the number of candidates is smaller than the number of voters also ensure that the outcome will always be Pareto efficient, regardless of whether it is manipulated or not.

4 Why Partisan Voters Stand to Gain from Strategic Voting

Strategic voting creates both winners and losers. Importantly, many of the winners may in fact be partisans. In particular, consider an election in which, were everyone to vote for their preferred candidate, the selected leader would receive support only from strategic voters. Moreover, imagine that there is only one strategic voter who voted for a losing candidate, and that all the remaining voters are partisans, and prefer any of the other candidates to the leader selected when everyone votes for their preferred leader. Now let the one strategic voter who voted for a losing candidate switch their vote to a different losing candidate whose platform is preferred by this elector to that of the winning candidate, with the result that there is a change in the electoral outcome, and a new leader is selected. In such a setting, strategic voting benefits all partisan voters, and only one strategic voter - and partisan voters therefore have a strong incentive to encourage strategic voting.

The next theorem provides exact bounds on the proportion of partisan voters who benefit from strategic voting. When there are relatively few partisan voters, these bounds are very large, and strategic voting may be of benefit to no partisan voters, or to all partisans. In contrast, when there are relatively few strategic voters, then the bounds become tighter: strategic voting is certain to benefit at least some partisans, but not all partisan voters will be advantaged by strategic voting.

Theorem 2 *In equilibrium, the proportion of partisan voters who benefit from a strategic vote belongs to the interval*

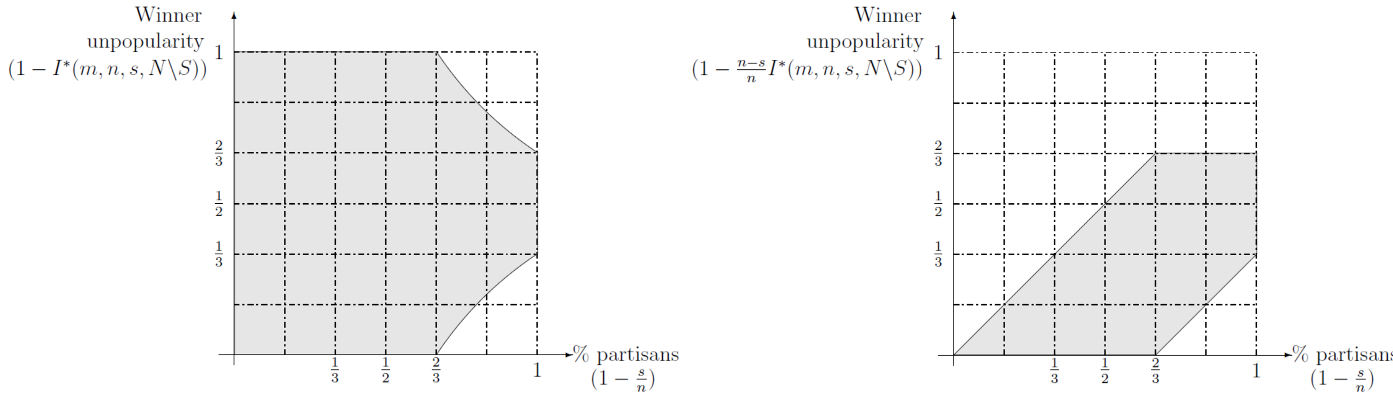
$$I^*(m,n,s,N\backslash S) = \begin{cases} [0;1] & \text{if} \quad s-1 \geq \lceil \frac{n}{m} \rceil \\ \left[\frac{1}{n-s}(1+\lfloor \frac{n}{m} \rfloor - s); \frac{1}{n-s}(n - \lceil \frac{n}{m} \rceil - 1)\right] & \text{if} \quad s-1 < \lceil \frac{n}{m} \rceil \end{cases}$$

Proof See Appendix. ■

As above, it is useful to consider how these bounds vary as the number of candidates, partisans, and strategic voters change relative to the size of the population. Once again, we first consider a setting where there are more candidates (or policy options) than voters, that is, $m > n$. The first statement of the bound applies when there are two or more strategic voters; the second statement when there is a single strategic voter. Note, however, that when $m > n$ then the second statement of the bound simplifies to $\left[0, \frac{n-2}{n-1}\right]$. As discussed above in relation to Theorem 1, when there are large numbers of candidates relative to the number of voters, and voter preferences are sufficiently heterogeneous so that each candidate attracts the support of at most one voter, then strategic voting benefits all partisans if the candidate selected at the strategic preferences profile is the most or second-most preferred candidate of all partisan voters, whereas it adversely affects all partisans if the candidate propelled to office by strategic voting (for example, a candidate earlier in the alphabet) is viewed less favorably by all partisans than the candidate who would have been elected had all voters supported their preferred leader. It is of interest to note that these results are independent of the actual number of partisan voters: the heavy lifting is being done by the fact that the number of candidates exceeds the number of voters.

What is more interesting to consider are scenarios where $n > m$; this is illustrated by Figure 2. When n is large relative to m, and the number of strategic voters exceeds the minimum number required to potentially elect a winning candidate, then the first statement of the bounds applies: a variation on the scenario discussed above, when $m > n$ explains why all partisans may benefit, or alternatively may be adversely impacted. In contrast, when the vast majority of voters are partisan, and there is no scenario under which strategic voters can, on their own, elect the leader, then both the lower and upper bounds shrink: partisans must be among both the voters who benefit from strategic voting.

There is an interesting tension between settings in which strategic voting seems more likely to benefit the overall population, and those which are most likely to benefit partisans. In the preceding section, we showed that as the proportion of strategic voters increases, that this increases the floor on the proportion of the overall population which benefits from strategic voting, and if there are



(a) Ratio on partisans (b) Ratio on the whole population

Figure 2: Impact of partisan voting on the unpopularity of the election winner within the subgroup of partisans: case $m = 3$.

no partisans, then the strategic voting must benefit a majority of voters. In contrast, it is precisely when there are lots of strategic voters that we calculate the widest possible bounds in terms of the potential impact on partisan voters: strategic voting may not benefit any partisans, or it may benefit all partisans. In contrast, when there are relatively few strategic voters, and when there are relatively few candidates, then there will always be partisans among those who benefit from strategic voting, as well as partisans who are disadvantaged. Consequently, when strategic voters are a large proportion of the population, there may be many preference profiles with respect to which partisans have a vested interest in actively discouraging strategic voting even though it might be to the advantage of a majority of the population. Or, equivalently, if a voter were behind a veil of ignorance, and knew merely that they were going to be a partisan voter, they might prefer to find themselves in an electoral setting in which most voters are partisans - even if they knew that this meant that strategic voting could harm a majority of the population - rather than an electoral setting where most electors are strategic voters, since strategic voting may adversely affect all partisans when partisans are scarce. Moreover, as the tightest bounds are associated with a small number of candidates, it is clear that partisan voters may very much favour policies which limit the number of candidates for electoral office.

5 The Costs and Benefits of Strategic Voting for Strategic Voters

It remains to examine the costs and benefits of strategic voting for strategic voters. By construction, the floor on benefits for strategic voters is strictly bounded away from zero: a strategic voter misreports their preferences only when this leads to the selection of a leader whom they prefer to

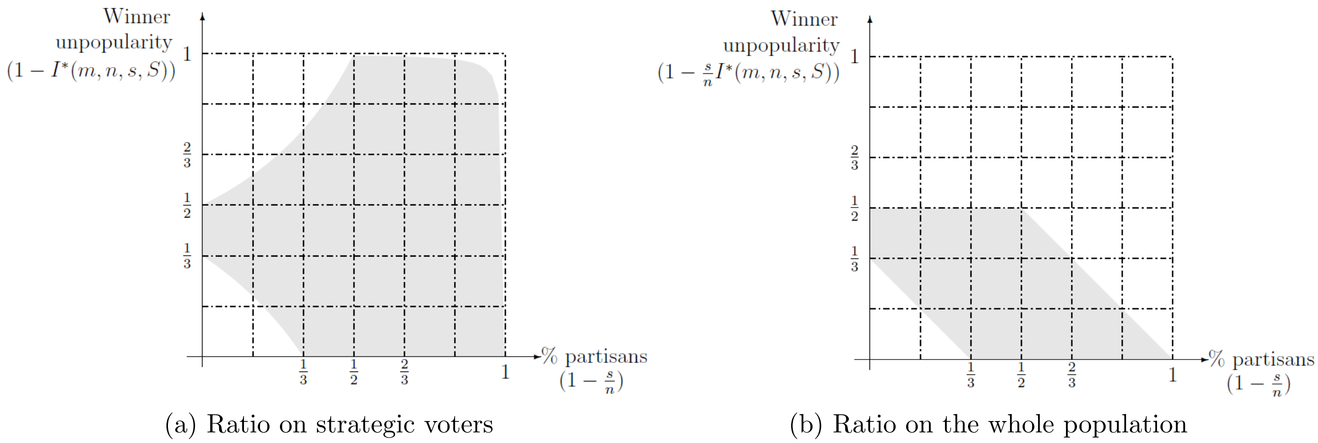


Figure 3: Impact of partisan voting on the unpopularity of the election winner within the subgroup of strategic voters: case $m = 3$.

the candidate who is selected when they vote for their most preferred candidate. In many settings, however, it can be expected that relatively few (or possibly no) strategic voters will be in a position to affect the electoral outcome by voting for a less-preferred candidate, and so there will be little evidence of strategic voting in actual elections. Below, we derive exact bounds for the proportion of the non partisan voting population which benefits from strategic voting.

Theorem 3 *In equilibrium, the proportion of strategic voters who benefit from manipulation of the electoral outcome belongs to the interval*

$$I^*(m,n,s,S) = \begin{cases} \left[\frac{1}{s}(s - \lfloor\frac{n}{2}\rfloor); \frac{1}{s}(n - \lceil\frac{n}{m}\rceil)\right] & \textit{if} \quad s > n - \lceil\frac{n}{m}\rceil \\ \left[\frac{1}{s}(s - \lfloor\frac{n}{2}\rfloor); 1\right] & \textit{if} \quad \lfloor\frac{n}{2}\rfloor + 2 \leq s \leq n - \lceil\frac{n}{m}\rceil \\ \left[\frac{1}{s}; 1\right] & \textit{if} \quad s \leq \lfloor\frac{n}{2}\rfloor + 1 \end{cases}$$

Proof See Appendix. ■

To interpret these bounds, as above, we first consider situations with large numbers of candidates, that is $m > n$. With large numbers of potential candidates (or policy alternatives), the first statement of the bound applies only when $s = n$, that is, all voters are strategic. Observe that the lower bound on the proportion of non-partisan which actually benefit from strategic voting does not fall below 50%, even with a large number of candidates. This result is consistent with Theorem 1: when there is no partisan voter, then whenever strategic voting occurs, it must lead to an outcome which most voters prefer to the outcome which prevails when all voters report their preferences truthfully. Of course, in such a setting, there must be at least one strategic voter whose preferred candidate is no longer selected, so the upper bound is not equal to 1, but if the voting population is very large, then the upper bound on the proportion of voters who benefit from strategic voting gets very close to 100%.

The second statement of the bound applies when strategic voters make up more than half (but not 100%) of the entire voting population, whereas the third statement applies when a majority of voters are partisans. With partisan voters in the mix, both the upper and lower bounds widen. The fact that the upper bound is equal to one is not surprising: as before, it suffices that no candidate is preferred by more than one voter, and that the candidate selected when all voters report their preferences truthfully be the preferred candidate of a partisan voter, whereas the leader selected as a result of the strategic voting is the second-most preferred candidate of all strategic voters. Similarly, it is not surprising that the lower bound is equal to $1/s$ when strategic voters are in the minority: this reflects situations where the candidate selected as a result of strategic voting is preferred to the candidate who would otherwise be elected by the one strategic voter who manipulates the electoral outcome, but this candidate is less highly-ranked by all other strategic voters. What is striking, however, is that the lower bound is strictly greater than $1/s$, albeit less than the $1/2$ threshold that applies when $s = n$, whenever strategic voters are in the majority: the fact that there are many other voters who are willing to vote strategically limits the capacity of any one strategic voter, at the equilibrium, to affect the electoral outcome, if doing so disadvantage a majority of voters.

Suppose now that $m < n$, that is, there are more voters than there are candidates; this is illustrated by Figure 3. The first statement of the bound applies when strategic voters are a clear majority: there is no preference profile in which a leader is elected without the support of at least one voter who is not a partisan - which also means that there must be at least one strategic voter who prefers the candidate elected when all voters report their preferences truthfully to the candidate elected at the strategic preferences profile.

Notice that the width of the second and third statements of the bounds are independent of the number of candidates'; moreover, although the ceiling on the maximum number of strategic voters in the population is lower when $m < n$ than when the number of candidates exceeds the number of voters, the interpretation of these bounds, and of the conditions under which the limits on the bounds are actually attained, is exactly as above. What emerges is a suggestive pattern: the larger the proportion of the population that is made up of strategic voters, the higher the floor on the proportion of the population, and on the proportion of non-partisan population, that benefits from strategic voting. If voters could choose between being partisans or being strategic voters, it would be in the general interest if all voters were strategic voters. In effect, the best protection against the potentially adverse impact of strategic voting is a population of strategic voters.

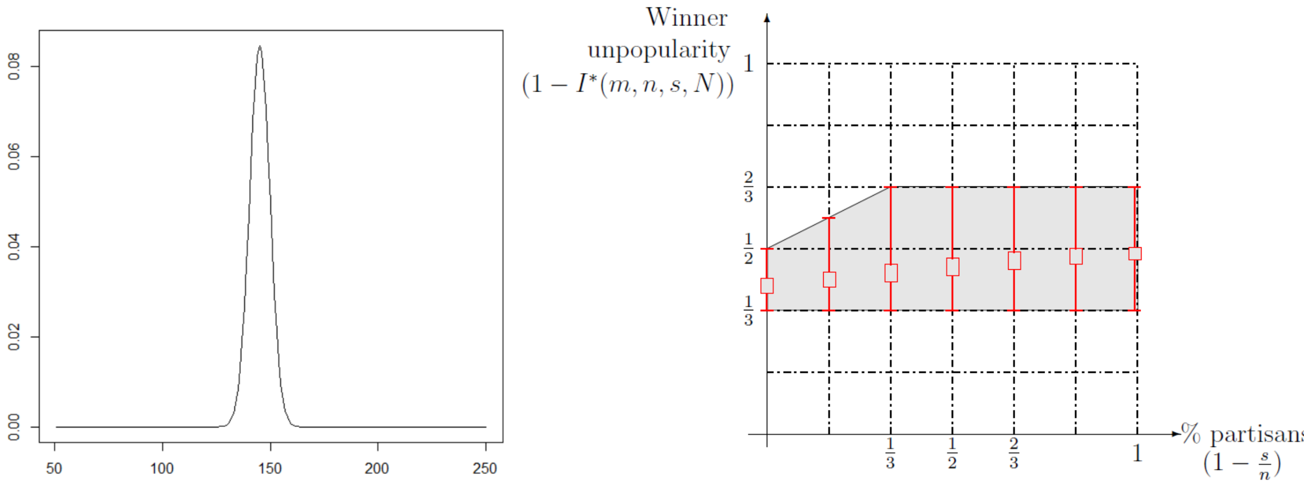
6 Strategic Voting for the Greater Good: A Statistical Exploration

Above, we derived exact bounds for the proportion of the population that benefits from (and is adversely impacted by) strategic voting. These are theoretical results, and as such are perfectly general, but it is natural to wonder whether in practice, for a given specification of m, n and s, there are as many preference profiles which generate gains and losses at the upper and lower limits of these bounds as there are preference profiles which generate distributions of gains and losses that fall in the middle of these intervals. Or, in contrast, will it typically be the case that strategic voting almost always benefits a substantial share of the voting population? In other words, is it almost always the case that the election winner at a strategic preference profile is more popular than the election winner at the profile of true preferences? In this section, we undertake a simulation analysis, and explore elections with 3 candidates (a, b and c) and 300 voters. For all possible preference profiles, and all possible variations in the number of strategic and partisan voters, we calculate the winning candidate when all voters report their preferences truthfully. A subset of these outcomes can be manipulated; for these outcomes we count the number of voters who benefit from strategic voting. For this particular specification of the electoral environment, we find that it is almost always the case that strategic voting benefits a clear majority of the population.

In this special case, with 3 candidates, and 300 voters, Theorem 1 tells that the ratio of voters who benefit from strategic voting belongs to the interval

$$I^*(m=3, n=300, s, N) = \begin{cases} \left[\frac{1}{300} + \frac{1}{3}; \frac{2}{3}\right] & \text{if} \quad s \leq 201 \\ \left[\frac{1}{n}\left\lceil\frac{s}{2}\right\rceil; \frac{2}{3}\right] & \text{if} \quad s > 201. \end{cases}$$

We are interested in the distribution of the number (ratio) of voters who benefit from (or equivalently, voters who are adversely impacted by) strategic voting: given $r \in \{100, \cdots, 199\}$ and any $s \in \{1, \cdots, 300\}$, what is the relative number of profile of true preferences R^N for which there exists a strategic profile of preference $Q^N = (Q^i, R^{-i})$ which is a Nash equilibrium, and where the number of voters who benefit from strategic voting is exactly r? For simplicity, we only consider scenarios where the elected candidate at R^N is $Pl(R^N) = a$ and, the elected candidate at the strategic profile $Q^N = (Q^i, R^{-i})$ is $Pl(Q^N) = b$. In Figure 4(a), given any $r > 0$, we represent the relative number of profiles of true preferences that admit at least one unilateral deviation (strategic preference profile) that is a Nash equilibrium and in which the exact number of voters adversely affected by strategic voting is r. (See Proposition 7 in page 29 for details.) In Figure 4(b), the distribution of the number of voters adversely affected by strategic voting is provided for different values of the proportion $1 - \frac{s}{n}$ of partisan voters. For each value of ratio



(a) Case $m = 3$, $n = 300$ and $s = 50$

(b) Boxplot: $n = 300$ and $m = 3$

Figure 4: Exploration: impact of partisan voting on the unpopularity of the election winner

$1 - \frac{s}{n}$, the represented box contains 90% of the observations. The gray area represents the domain where the number of voters who are adversely affected by strategic voting lies, in settings with 3 candidates, and sufficiently many voters. We observe for each value of s that the number of voters adversely affected by strategic voting tends to be normally distributed (see Figure 4(a)) around a central value which increases with the ratio $1 - \frac{s}{n}$ (see Figure 4(b)): the more there are partisans, higher is the ratio of voters for whom the election winner is unpopular. Moreover, the likelihood that no more than 50% voters will be adversely affected by strategic voting is greater than 0.58, and decreases with the proportion $1 - \frac{s}{n}$ of partisan voters. If less than two third of the voting population is made of partisans, then the latter likelihood is greater than 0.98. This implies that independently of how the voting population is split between partisans and strategic voters, and assuming that all preference profiles are equally likely, a risk neutral social planner who wishes to maximize the expected popularity of the winning candidate in an election has no reason to prefer a strategy-proof social choice mechanism to the plurality rule.

7 Conclusions

Whereas the social choice literature traditionally grades collective choice mechanisms with respect to their capacity to deliver the outcome that would be selected if all participants report their preferences truthfully, the analysis conducted in this paper explores an alternative metric for judging voting mechanisms, namely, the maximum and the minimum bounds on the popularity of the election winner if electors vote strategically, relatively to the election winner when all electors vote in a partisan fashion. This is the first paper to calculate such bounds. The salience of this alternative metric is undeniable given that the mechanisms the most widely used in actual elections are all known to be vulnerable to strategic voting. Voting procedures which are likely to select more popular winner when some voters vote strategically are arguably better mechanisms than ones which are more likely to generate less popular winner. One of the lessons to be drawn from this analysis is that - at least under the plurality rule - strategic voting may often be a virtue, rather than a vice, and citizens should indeed be encouraged to vote with their heads, rather than their hearts.

The importance of this alternative metric is well illustrated by the most recent US election cycle. Arguably, the key factor underlying Donald Trump's success in the Republican primaries was that the anti-Trump vote was split for too long among too many candidates, and that many of the voters who supported Rubio, Cruz and Kasich persisted in voting truthfully for their preferred candidate, rather than coalescing around a single opponent, thereby clearing a path for Trump to victory. Moreover, once Trump was confirmed as the Republican Presidential candidate, it was arguably crystal clear to those anti-Trump voters who preferred a third-party candidate to Hillary Clinton that it was risky to vote truthfully. Whether or not these warnings were heeded is of course a matter of speculation, but it is worth noting that in many of the so-called swing states which ended up favouring Trump (including Michigan, Wisconsin, and Pennsylvania), the number of votes separating Trump and Clinton was smaller than the vote totals which accrued to Jill Stein (of the Green Party). In this particular election, the plurality voting mechanism arguably delivered the outcome that would have been selected had all voters cast their ballots truthfully. However, had more voters been persuaded that it was acceptable to vote strategically (either at the initial primary stage, or later in the Presidential election) it is certainly possible that a President who was ultimately selected would have commanded broader support.

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8 Appendices

8.1 Proof of Theorem 1

Proposition 1, below, establishes an exact upper bound on the proportion of the overall voting population which can benefit from a strategic vote.

Proposition 1 *In equilibrium, the maximum proportion of voters who benefit from a strategic voting is*

$$M^*(m,n,s,N) = 1 - \frac{1}{n}\left\lceil \frac{n}{m} \right\rceil$$

To prove Proposition 1, our first step is to relate the set of voters who rank the successful candidate, a_j, first when the reported preference profile is Q^N and the set of voters - including those who always vote truthfully - who benefit from the strategic voting, that is, the set $E(R^N, Q^N)$. This relationship, together with the fact that under the plurality rule the floor on the number of voters who support the winning candidate can be determined as the number of voters divided by the number of candidates, helps us to start to calculate the number of winners and losers as a result of a strategic vote.

Lemma 1 *Let $R^N \in P(S)$ and $Q^N \in P(N \mid R^N)$. Suppose that $Pl(R^N) = a_l$ and $Pl(Q^N) = a_j$. Then $F(a_j, Q^N) \subseteq E(R^N, Q^N)$, $\left|F(a_j, Q^N)\right| > \frac{n}{m}$ and therefore $\left|E(R^N, Q^N)\right| > \frac{n}{m}$.*

Proof The first inclusion is an immediate consequence of the fact that Q^N is a profile of strategic preferences under R^N; the first inequality follows as a result of the plurality rule. ■

We also find it useful to establish two intermediate lemmata. The overall strategy of the proof is to first construct a (unstable) preference profile R^N for which there exists a strategic preference profile Q^N which is also a constrained Nash equilibrium. By calculating the proportion of voters who benefit from this strategic vote, it follows that the upper bound on the proportion of voters benefiting from a strategic vote is not less than this proportion. Then, in the proof of the proposition, we calculate the minimum proportion of voters that are adversely affected by any strategic vote. Since the sum of these numbers is equal to one, it follows that the upper bound which is calculated in proving the lemma must be exact.

Lemma 2 *For any number of voters, $n \geq 2$, and any number of candidates, $m \geq 3$, there exists two preference profiles $R^N, Q^N \in \mathcal{R}^N$ such that $\left|E(R^N, Q^N)\right| = n - \left\lceil \frac{n}{m} \right\rceil$ and $Q^N \in P(\{2\} \mid R^N) \cap N(S \mid R^N)$ for all S such that $2 \in S$, that is a strategic profile which coincides with the true preference profile except for the report of voter 2 and that is a constrained Nash equilibrium for all set S of strategic voters such that $2 \in S$.*

Proof For any $m \geq 3,\ n \geq 2$, let $q \geq 0$ and $r \in \{0, 1, ..., m-1\}$ so that $n = q \cdot m + r$. We now construct a R^N and Q^N which satisfy the statement of the lemma; note that the R^N, Q^N which are constructed for this purpose depend upon the particular values of q and r.

Case 1 : $r = 0$. Consider a partition $\{N_1, N_2, ..., N_m\}$ of N in m subsets, each of cardinality q, and such that voter 2 belongs to N_2. Let R^N be a profile such that

$$\forall i \in N_3, R^i = a_3 \cdots \text{ and,}$$
$$\forall i \in N_k, R^i = a_k a_3 \cdots \text{ for } k \neq 3$$

that is, the true preference profile is such that for all voters belonging to N_3, candidate a_3 is ranked first (and all other candidates can be ranked in any order) whereas for all other voters in partition N_k, candidate a_k is ranked first, candidate a_3 is ranked second (and all other candidates can be ranked in any order). We next construct a strategic preference profile Q^N which is also a constrained Nash equilibrium under R^N. Thus, consider the (mis)report for voter 2, $Q^2 = a_3 a_2 ...$ so that $Q^N = (Q^2, R^{-2})$. Observe that $Pl(R^N) = a_1$. Since there are now $q+1$ voters who rank a_3 as their preferred candidate, $Pl(Q^N) = a_3$. Moreover, since voter 2 (and, indeed, all voters except those in partition N_1) prefers a_3 to a_1, this misreport satisfies the requirement of a strategic preference profile - that is, the voter mis-representing his preferences prefers the outcome under Q^N to the outcome under R^N-. Finally, Q^N is a constrained Nash equilibrium voting profile. Indeed no voter who does not belong to N_3 can profitably change the outcome of the voting process by mis-reporting their true preferences given the reports of the other voters, no voter in N_3 wishes to misreport, and there is no alternative strategy which improves the payoff of voter 2.

Case 2 : $q = 0$. Notice that this implies that $n < m$. Define the profile R^N as follows:

$$R^1 = a_2 a_1 a_3 ... \tag{1}$$
$$\text{and} \quad R^i = a_{i+1} a_1 a_2 ... \text{ for all } i \neq 1. \tag{2}$$

Observe that with this preference profile there are n candidates who receive one vote when all voters report their preference truthfully, $m - n$ candidates who receive no votes, and $Pl(R^N) = a_2$. Now let $Q^2 = a_1 \cdots$ and consider $Q^N = (Q^2, R^{-2})$. As when all voters cast their ballots truthfully, no candidate receives more than one vote. However, $Pl(Q^N) = a_1$. Notice that, with the exception of voter 1, candidate a_1 is preferred by all voters to candidate a_2. As voter 2 prefers a_1 to a_2, Q^N meets the requirements of a strategic preference profile. Moreover, Q^N is a Nash equilibrium (and therefore a constrained Nash equilibrium): given that a_1 is the second-most-preferred outcome for all voters - including voter 1 - none of these other voters can obtain an outcome they prefer to the outcome a_1 by misreporting their true preference.

Case 3 : $r \geq 1$ and $q \geq 1$. We now allocate voters into blocks of either q or $q+1$ voters. Consider a partition $\{N_1, N_2, ..., N_m\}$ of N into m subsets such that voter 2 belongs to N_3, $|N_k| = q+1$ if $k \in \{2, 3, ..., r+1\}$ and $|N_k| = q$ if $k \in \{1, r+2, r+3, ...m\}$. Let R^N be the profile defined by

$$\forall i \in N_1, R^i = a_1 a_2 ... a_m \tag{3}$$
$$\forall i \in N_2, R^i = a_2 a_1 a_3 \cdots a_m \text{ for all } k \neq 1 \tag{4}$$
$$\forall i \in N_k, R^i = a_k a_1 a_2 \cdots a_m \text{ for all } k \notin \{1, 2\} \tag{5}$$

that is, the true preference profile is such that all voters in N_1 rank a_1 first (and all other candidates in alphabetical order), whereas all other voters in N_k rank candidate a_k first, and candidate a_1 second (and all other candidates in alphabetical order). Observe that in the partition $\{N_1, N_2, ..., N_m\}$ there are r blocks with $q+1$ voters, and that one of these blocks is N_2, so that $Pl(R^N) = a_2$. Now let $Q^2 = a_1 \cdots$ and $Q^N = (Q^2, R^{-2})$. Notice that $Pl(Q^N) = a_1$, and that a_1 is preferred by all voters, except for those in N_2, to the outcome under R^N. Q^N therefore meets the requirements of a strategic preference profile under R^N, as only voter 2 misreports, and voter 2 prefers the outcome under Q^N to the outcome under R^N. Moreover, Q^N is a Nash equilibrium under R^N: for all voters in blocks $N_k, k \neq 2$, a_1 is preferred to any outcome other than a_k but by misreporting their preference they cannot obtain a_k elected and so voting a_k is a best response to Q^N; voters in N_2 cannot misreport and obtain a_2, and a_1 is voter 2's best response to R^{-2}. This completes Case 3.

To complete the proof of the lemma, it suffices to observe that, in each case, the only voters who do not benefit from the strategic voting are those $\lceil \frac{n}{m} \rceil$ voters (or, in case 2, voter 1) whose preferred candidate is selected under truthful voting. Consequently $\left|E(R^N, Q^N)\right| = n - \lceil \frac{n}{m} \rceil$. ■

We can now prove our first proposition, which follows straightforwardly from our preceding lemmata.

Proof of Proposition 1.

It follows from Lemma 2 that $M^*(m, n, s, N) \geq 1 - \frac{1}{n} \lceil \frac{n}{m} \rceil$. In addition, if $Pl(R^N) = a_j$, then $\left|E(a_j, R^N)\right| \geq \lceil \frac{n}{m} \rceil$. This implies that at least $\lceil \frac{n}{m} \rceil$ voters suffer from strategic voting if it happens. So $n \cdot M^*(m, n, s, N) \leq n - \lceil \frac{n}{m} \rceil$. ■

Proposition 1 provides the exact upper bound on the proportion of individuals who may, in equilibrium, benefit from a strategic vote; in particular, when the number of candidates is larger than the number of voters, this may be almost the entirety of the population. Such a scenario might arise, for example, if a small group of voters is tasked with selecting a successful applicant from a large pool of job seekers. This result may nonetheless appear to be of limited interest, because in any specific setting there is no reason to expect that the actual preference profile will be similar to the particular preference profile used to construct the proof. Rather, what may seem more pertinent is clearer insight into the minimum proportion of the population which benefits from the strategic voting. This is the focus of our next Proposition.

Proposition 2 *The minimum proportion of the whole population which benefits from a strategic voting is*

$$m^*(m,n,s,N) = \begin{cases} \frac{1}{n} + \frac{1}{n}\left\lfloor \frac{n}{m} \right\rfloor & \text{if } s \leq 2\left\lfloor \frac{n}{m} \right\rfloor + 1 \\ \frac{1}{n}\left\lceil \frac{s}{2} \right\rceil & \text{if } s > 2\left\lfloor \frac{n}{m} \right\rfloor + 1 \end{cases}$$

As a first step towards establishing this result, we first establish a minor lemma.

Lemma 3 *If the number of strategic voters s is greater than $2\left\lfloor \frac{n}{m} \right\rfloor + 1$, then $\frac{s}{2} > \frac{n-s}{m-2}$.*

Proof Assuming that $s \geq 2\left\lfloor \frac{n}{m} \right\rfloor + 2$ we have that $\frac{s}{2} > \frac{n}{m}$. Cross-multiplying and adding mn from both sides, we observe that this is equivalent to $\frac{n}{m} > \frac{n-s}{m-2}$. ■

It is now possible to calculate the lower bound on the number of voters who benefit from a strategic voting.

Proof of Proposition 2. Assume that voter 2 belongs to the set of strategic voters, S. We proceed in 2 cases:

Case 1: $s \leq 2\left\lfloor \frac{n}{m} \right\rfloor + 1$

Let $R^N \in Z(S)$, and let Q^N be a strategic preference profile under R^N. Suppose that $Pl(R^N) = a_l$ and $Pl(Q^N) = a_j$. From Lemma 1, we have that $\left|E(a_j, Q^N)\right| > \frac{n}{m}$ and that $\left|E(R^N, Q^N)\right| \geq 1 + \left\lfloor \frac{n}{m} \right\rfloor$. Consequently, $m^*(m,n,s,N) \geq \frac{1}{n} + \frac{1}{n}\left\lfloor \frac{n}{m} \right\rfloor$. We now show that there exist possible preference profiles for which $m^*(m,n,s,N) \leq \frac{1}{n} + \frac{1}{n}\left\lfloor \frac{n}{m} \right\rfloor$. As above, let $n = q \cdot m + r$ where q is the largest non-negative integer such that $q \cdot m \leq n$.

Case 1.1 Assume that $r = 0$, and consider a partition $\{N_1, N_2, ..., N_m\}$ of N into m subsets with equal number of voters, i.e., such that $|N_j| = q$ for all $j = 1, ..., m$. Without loss of generality, assume that voter $2 \in N_3$ and, moreover, that the set of strategic voters, less voter 2, are all members of N_2 if there are no more than $q+1$ strategic voters, that is, $S\backslash\{2\} \subset N_2$ if $|S| \leq q+1$, or alternatively they are all members of the first two blocks of the latter partition, that is, $N_2 \subset S\backslash\{2\}$ and $S\backslash\{2\} \subset N_2 \cup N_1$ if $q+1 < |S| \leq 2q+1$. Now consider a possible true preference profile R^N such that voters in N_1 rank a_1 first, a_2 second and all other candidates in any order; voter 2 ranks a_3 first, a_2 second, and all other candidates in any order; and voters in $N_k \backslash \{2\}, k \neq 1$, rank a_k first, a_1 second, and all other candidates in any order. Notice that each alternative is top ranked by exactly q voters, so $Pl\left(R^N\right) = a_1$. Now suppose that voter 2 were to mis-report his preferences by reporting $Q^2 = a_2 a_3 \cdots$ in which a_2 is ranked first, a_3 is ranked second, and all other candidates in any order. There are now $q+1$ voters ranking a_2 as the preferred candidate, and so $Pl\left(Q^2, R^{-2}\right) = a_2$. Since a_2 is preferred by voter 2 to a_1, Q^N satisfies the requirement of a strategic preference profile, and we have $R^N \in P(S)$ and $Q^N = (Q^2, R^{-2}) \in P(S \mid R^N)$. Voters of $N_2 \cup \{2\}$ benefit from the strategic vote. In contrast, voters of $S \backslash (N_2 \cup \{2\})$ rank a_1 first and a_2 second in the profile Q^N. However, it is not possible to any of voters to profitably change the outcome at Q^N by misrepresenting his preferences. Consequently, Q^N is a constrained Nash equilibrium. We have $\left|E(R^N, Q^N)\right| = q+1$ voters who benefit, and so the minimum proportion of the population which benefits from a strategic voting, $m^*(m,n,s,N)$, is less than or equal to $\frac{1}{n} + \frac{1}{n}\left\lfloor \frac{n}{m} \right\rfloor$.

Case 1.2 Now consider the case where $q = 0$. By construction, the set of strategic voters is therefore a singleton, that is, $S = \{2\}$ and $n < m$. Now consider a possible true preference profile R^N such that voter 1 ranks a_2 first and all other candidates in any order, voter 2 ranks a_{n+1} first, a_1 second, and all other candidates in any order, and all voters $i \notin \{1,2\}$ rank a_i first, a_2 second, and all other candidates in any order. If all voters report their true preferences, then $Pl(R^N) = a_2$. Now suppose that the unique strategic voter, voter 2, reports a preference $Q^2 = a_1 \cdots$ where candidate a_1 is ranked first and all other candidates in any order. Then, the elected candidate at the profile $Q^N = (Q^2, R^{-2})$ is a_1. The profile Q^N is trivially a strategic preference profile, and a constrained Nash equilibrium under R^N. We have $\left|E(R^N, Q^N)\right| = 1$ and therefore the minimum proportion of the population which benefits from strategic voting, $m^*(m,n,s,N)$, is less than or equal to $\frac{1}{n}$ which equals $\frac{1}{n} + \frac{1}{n}\left\lfloor \frac{n}{m} \right\rfloor$ since $n < m$.

Case 1.3 Finally, suppose that $r \geq 1$ and $q \geq 1$. Consider a partition $\{N_1, N_2, ..., N_m\}$ of N in m blocks such that voter 2 belongs to N_3, and such that there are $q+1$ voters in blocks N_2 to N_{r+1}, and q voters in all of the other blocks in this partition. Additionally, assume that all of the strategic voters, with the exception of voter 2, are assigned to N_1 if there are no more than $q+1$ members of S, or are assigned to N_1 and N_2 if $q+1 < |S| \leq 2q+1$. Now consider a possible true preference profile R^N such that voters in N_1 rank a_1 first and all other candidates in any order; voters in N_2 rank a_2 first, a_1 second, and all other candidates in any order; voter 2 ranks a_3 first and a_1 second and all other candidates in any order, while voters in $N_k \backslash \{2\}, k \notin \{1,2\}$ rank a_k first, a_2 second, and all other candidates in any order. Then a_1 and a_2 are top ranked by exactly q voters and $q+1$ voters respectively. Since $|N_k| \leq q+1$ for all $k \in \{1,2,...,m\}$, $Pl\left(R^N\right) = a_2$. Next, notice that the only voters who benefit from a switch to a_1 rather than a_2 when the true preference profile is R^N are the members of $N_1 \cup \{2\}$. Now suppose that voter 2 chooses to strategically misreport, and announces $Q^2 = a_1 a_3 ...$ so that we have $Q^N = (Q^2, R^{-2})$. The candidate chosen by the plurality rule is now $Pl(Q^N) = a_1$, and Q^N satisfies the definition of strategic preference profile. Moreover, Q^N is also a constrained Nash equilibrium since no strategic voter can improve his payoff by changing his reported preferences. We have $\left|E(R^N, Q^N)\right| = q+1$

voters who benefit, and so the minimum proportion of the population which benefits from the strategic voting, $m^*(n,m,s,N)$, is less than or equal to $\frac{1}{n}+\frac{q}{n}=\frac{1}{n}+\frac{1}{n}\left\lfloor\frac{n}{m}\right\rfloor$. This establishes our claim for the case where $s\leq 2\left\lfloor\frac{n}{m}\right\rfloor+1$, that is, when there are relatively few strategic voters.

Case 2: $s>2\left\lfloor\frac{n}{m}\right\rfloor+1$

As above, we first establish that the proportion of the population which benefits from the strategic voting is at least equal to $\frac{1}{n}\left\lceil\frac{s}{2}\right\rceil$. Consider an unstable profile $R^N\in Z(S)$ and let $Q^N=(Q^i,R^{-i})$ be a strategic preference profile that is a constrained Nash equilibrium under R^N. The proof is by contradiction. Let $Pl(R^N)=a$, $Pl(Q^N)=b$ and suppose that the number of voters who prefer outcome b to outcome a under the true preference profile R^N is strictly less than $\left\lceil\frac{s}{2}\right\rceil$. Denote by $L(b,a,R^N)=\backslash E(R^N,Q^N)$ the set of participants who prefer a to b when the true preference profile is R^N. Then it must be true that $\left|L(b,a,R^N)\cap S\right|>\left\lceil\frac{s}{2}\right\rceil$ which implies that

$$S\cap L(b,a,R^N)\cap(N\backslash F(a,R^N))\neq\emptyset.$$

That is, there exists at least one strategic voter j who prefers candidate a to candidate b and who did not ranked a first in the profile R^N and therefore in the strategic preference profile Q^N as well. Let $R'^j=a\cdots$ be a strategic preference of voter j in which a is ranked first and all other candidates are ranked in any order. Let $R'^N=(R'^j,Q^{-j})=(R'^j,Q^i,R^{-i,j})$. From R^N to R'^N, candidate a receives one additional vote while b receives at most one additional vote, and other candidates do not receive any additional vote. As $Pl(R^N)=a$, it must be that $Pl(R'^N)=a$. As voter j prefers a to b, the profile R'^N is a profitable deviation from Q^N. A contradiction arises since Q^N is an constrained Nash equilibrium. We conclude that $\left|L(a,b,R^N)\right|\geq\left\lceil\frac{s}{2}\right\rceil$. That is $m^*(n,m,s,N)\geq\frac{1}{n}\left\lceil\frac{s}{2}\right\rceil$.

We now construct a true preference profile $R^N\in Z(S)$ and a strategic preference profile $Q^N\in N(S\mid R^N)$ such that $\left|E(R^N,Q^N)\right|=\left\lceil\frac{s}{2}\right\rceil$. Choose k,r,p and q such that $s=2k+r$ with $r\in\{0,-1\}$ and $n-s=(m-2)q+p$ with $p\in\{0,...,m-3\}$. By construction, from Lemma 3, we have $k\geq q+1$. Now construct a partition $\{N_1,N_2,...,N_m\}$ of N such that $|N_1|=k-1$, $|N_2|=k$, $|N_j|=q+1$ for $j=3,...p+3+r$, and $|N_j|=q$ for $j>p+3+r$. Moreover, suppose that voter $2\in N_3$ and let $S\subset N_1\cup N_2\cup\{2\}$.[6] Now consider the possible true preference profile R^N such that

[6]Note that N_1 or N_2 may contain one voter not in S. This is the case when $s=2k-1$.

$$R^i = a_1 a_2 a_3 \cdots \text{ for all } i \in N_1$$
$$R^i = a_2 a_1 \cdots \text{ for all } i \in N_2; \quad \text{and}$$
$$R^2 = a_3 a_1 a_2 \cdots$$
$$R^i = a_l a_2 a_1 \cdots \text{for all } i \in N_j \backslash \{2\}, j \geq 3.$$

By construction, we have $Pl(R^N) = a_2$. However, the voters belonging to $N_1 \cup \{2\}$ would prefer that candidate a_1 prevail. Consider, now, a possible misreport by voter 2, such that $Q^2 = a_1 a_3 a_2 \cdots$ and let $Q^N = (Q^2, R^{-2})$. Observe that $Pl(Q^N) = a_1$, and Q^N is a constrained Nash equilibrium: with the exception of voter 2, all of the strategic voters are voting for their preferred outcome, and cannot improve their payoff by changing their vote. We have $\left|E(R^N, Q^N)\right| = k = \left\lceil \frac{s}{2} \right\rceil$. That is $m^* (n, m, s) \leq \frac{1}{n} \left\lceil \frac{s}{2} \right\rceil$. We have therefore shown that $\frac{1}{n} \left\lceil \frac{s}{2} \right\rceil \leq m^* (n, m, s) \leq \frac{1}{n} \left\lceil \frac{s}{2} \right\rceil$ which completes our claim. ■

8.2 Proof of Theorem 2

The benefits of strategic voting will typically be shared between both partisans and strategic voters, and although there must be at least some proportion of the strategic voting population which benefits - for otherwise a strategic voter has no incentive to manipulate the truthful voting outcome - there is no reason *a priori* to expect that these gains will accrue primarily to either the partisans or to the strategic voters. For this reason, it is useful to derive exact bounds for the maximum and minimum proportion of partisan voters who benefit from a manipulation of the voting outcome.

We proceed in two steps. We first derive the upper bound on the proportion of partisan voters who benefit from strategic voting.

Proposition 3 *Suppose that the proportion of strategic voters is strictly less than 1. Then the maximum proportion of partisan voters who benefit from a strategic voting is given by*

$$M_1^*(m, n, s, N \backslash S) = \begin{cases} 1 \ \textit{if} \ s - 1 > \left\lceil \frac{n}{m} \right\rceil \\ \frac{1}{n-s}(n - \left\lceil \frac{n}{m} \right\rceil - 1) \ \textit{if} \ s - 1 \leq \left\lceil \frac{n}{m} \right\rceil \end{cases}$$

Proof We first consider the case when the number of strategic voters is one more than the minimum winning coalition size, i.e., $s - 1 > \left\lceil \frac{n}{m} \right\rceil$. Consider the true preference profile R^N and the strategic preference profile Q^N used earlier in the proof of Lemma 2. Now rename the voters so that $N \backslash E(R^N, Q^N) \subset S \backslash \{2\}$. We then have that $Q^N \in P(S \mid R^N) \cap N(S \mid R^N)$ and $N \backslash S \subset E(R^N, Q^N)$.

This means that $M_1^*(m,n,s,N\backslash S) \geq \frac{1}{n-s}(n-s)$. That is $M_1^*(m,n,s,N\backslash S) = 1$: all partisan voters benefit from the strategic voting.

The second case arises when the number of strategic voters is less than or equal to the minimum winning coalition size, i.e., $s-1 \leq \left\lceil \frac{n}{m} \right\rceil$. From Proposition 1, we know that at most $n - \left\lceil \frac{n}{m} \right\rceil$ voters benefit from strategic voting. Since only a voter belonging to S may choose to vote strategically, at most $n - \left\lceil \frac{n}{m} \right\rceil - 1$ voters of $N\backslash S$ can benefit from a strategic voting. That is $M_1^*(m,n,s,N\backslash S) \leq \frac{1}{n-s}(n - \left\lceil \frac{n}{m} \right\rceil - 1)$. To complete the proof, consider once again the true preference profile R^N and the strategic preference profile Q^N used to establish Lemma 2. Rename the voters so that $S\backslash\{2\} \subset N\backslash E(R^N, Q^N)$. We have $Q^N \in P(S \mid R^N) \cap N(S \mid R^N)$ and $E(R^N, Q^N) \subset (N\backslash S) \cup \{2\}$. Since $\left|E(R^N, Q^N)\right| = n - \left\lceil \frac{n}{m} \right\rceil$, we have $M_1^*(m,n,s,N\backslash S) \geq \frac{1}{n-s}(n - \left\lceil \frac{n}{m} \right\rceil - 1)$. ■

Of course, whilst it is interesting to know that - at least in some circumstances - it is the entire population of partisan voters who benefit from strategic voting, there is no reason to believe, *a priori*, that such circumstances arise with any particular frequency. What is of greater importance is to get some sense of the difference between the maximum and minimum proportion of partisan voters who benefit from a strategic voting. Proposition 4, below, provides an exact bound on the minimum proportion of partisan voters who benefit from a strategic voting. Whereas Proposition 3 takes an optimistic view of the impact of a strategic voting on partisan voters, Proposition 4 is pessimistic; it is derived under the assumption that the composition of the group of voters adversely affected by the strategic voting is comprised primarily (and possibly exclusively) of partisan voters. What this means in practice is that when strategic voters comprise a large share of the overall voting population, then none of the partisan voters may end up benefiting from the strategic voting; as the proportion of strategic voters in the overall population falls, then this makes it more likely that partisan voters end up in the group of voters which are positively impacted by the strategic voting.

Proposition 4 $m_1^*(m,n,s,N\backslash S) = \begin{cases} 0 \ \textit{if} \ s-1 \geq \left\lfloor \frac{n}{m} \right\rfloor \\ \frac{1}{n-s}(1 + \left\lfloor \frac{n}{m} \right\rfloor - s) \ \textit{if} \ s-1 < \left\lfloor \frac{n}{m} \right\rfloor \end{cases}$

Proof This result follows directly from Proposition 2. ■

8.3 Proof of Theorem 3

To complete our analysis, we derive exact bounds for the maximum and minimum proportion of strategic voters who benefit from strategic voting. These results largely echo those established above: we show that in some settings, strategic voting may in fact benefit all strategic voters, and that the minimum proportion of this population which profits from a strategic vote is bounded strictly away from zero.

Proposition 5 $M_2^*(m,n,s,S) = \begin{cases} 1 \text{ } \textit{if } s \leq n - \left\lceil \frac{n}{m} \right\rceil \\ \frac{1}{s}(n - \left\lceil \frac{n}{m} \right\rceil) \text{ } \textit{if } s > n - \left\lceil \frac{n}{m} \right\rceil \end{cases}$

Proof Follows obviously from Proposition 1. ■

Proposition 6 $m_2^*(m,n,s,S) = \begin{cases} \frac{1}{s} & \textit{if } s-1 \leq \left\lfloor \frac{n}{2} \right\rfloor \\ \frac{1}{s}(s - \left\lfloor \frac{n}{2} \right\rfloor) & \textit{if } s-1 > \left\lfloor \frac{n}{2} \right\rfloor \end{cases}$

Proof We first construct a profile $R^N \in \mathcal{R}^N$ of true preferences which is unstable, a strategic preference profile $Q^N \in N(S|R^N)$ which is a constrained Nash equilibrium, and such that the proportion of voters who benefit from strategic voting equals $m^*(m,n,s.S)$. Assume that voter 2 belongs to S. Consider a partition $\{N_1, N_2, \{2\}\}$ of N, and a true preference profile $R^N \in \mathcal{R}^N$ and a preference $Q^2 \in \mathcal{R}$ such that:

If $n = 2k$, then $|N_1| + 1 = |N_2| = k$, and preferences are as follows

$$\begin{aligned} R^i &= a_1 a_2 ... \text{ for all } i \in N_1, \\ R^i &= a_2 a_1 \cdots \text{ for all } i \in N_2, \\ R^2 &= a_3 a_1 a_2 ... \text{ and} \\ Q^2 &= a_1 a_3 a_2 \cdots . \end{aligned}$$

If $n = 2k+1$, then $|N_1| = |N_2| = k$, and preferences are as follows

$$\begin{aligned} R^i &= a_2 a_1 ... \text{ for all } i \in N_1, \\ R^i &= a_1 a_2 \cdots \text{ for all } i \in N_2, \\ R^2 &= a_3 a_2 a_1 ..., \text{ and} \\ Q^2 &= a_2 a_3 a_1 \cdots . \end{aligned}$$

Pose $Q^N = (Q^2, R^{-2})$. It is easy to check that Q^N is a strategic preference profile, that the set $E(R^N, Q^N)$ of voters who benefit from strategic voting equals $N_1 \cup \{2\}$, and that the profile Q^N is a Nash equilibrium given R^N: no unilateral deviation can be profitable, were all voters non partisan. Now rename voters of $N \backslash \{2\}$ such that $S \backslash \{2\} \subseteq N_2$ if $s - 1 \leq k$ or $N_2 \subseteq S \backslash \{2\}$ otherwise. In this case we have $\left| E(R^N, Q^N) \cap S \right| = |N_1 \cap S| + 1$. It follows that

$$\left| E(R^N, Q^N) \cap S \right| = \begin{cases} 1 \text{ if } s - 1 \leq k \\ s - \left\lfloor \frac{n}{2} \right\rfloor \text{ otherwise.} \end{cases}$$

We deduce that

$$m_2^*(m,n,s,S) \leq \begin{cases} \frac{1}{s} & \text{if } s-1 \leq \left\lfloor \frac{n}{2} \right\rfloor \\ \frac{1}{s}(s - \left\lfloor \frac{n}{2} \right\rfloor) & \text{if } s-1 > \left\lfloor \frac{n}{2} \right\rfloor . \end{cases}$$

Since only members of S can play strategically, we have $m_2^*(m,n,s,S) \geq \frac{1}{s}$. It is left to show that $m_2^*(m,n,s,S)$ can not be strictly less than $\frac{1}{s}(s - \lfloor \frac{n}{2} \rfloor)$ for values of s such that $s - 1 > \lfloor \frac{n}{2} \rfloor$. Assume that there exists an unstable profile $R^N \in Z(S)$ and a profile $Q^N = (Q^i, Q^{-i}) \in N(S \mid R^N) \cap P(S \mid R^N)$ that is a constrained Nash equilibrium such that $\left|E(R^N, Q^N) \cap S\right| < s - \lfloor \frac{n}{2} \rfloor$. Let $S_1 = S \backslash E(R^N, Q^N)$ be the set of strategic voters adversely affected by the strategic voting, $Pl(R^N) = a$ be the elected candidate at the profile R^N, and $Pl(Q^N) = b$ be the elected candidate at Q^N. We have $S_1 = S \cap E(b, a, R^N)$ and $|S_1| > \lfloor \frac{n}{2} \rfloor$. (Recall that $E(b, a, R^N) = N \backslash E(R^N, Q^N)$.) This means that there exists at least one strategic voter $j \in S_1$ who prefers candidate a to b according to the profile of true preferences R^N, but who did not ranked a first in the profile R^N and therefore in the strategic preference profile Q^N as well. (In the other case, candidate a is ranked first by more than half of the voters in the profile of true preferences R^N, and therefore is not vulnerable to manipulation.) Let R'^j be a strategic preference of voter j in which a is ranked first and all other candidates are ranked in any order. Let $R'^N = (R'^j, Q^{-j}) = (R'^j, Q^i, R^{-i,j})$ be a deviation from Q^N by voter j. From R^N to R'^N, both candidates a and b receive one additional vote while other candidates do not receive any additional vote. As the elected candidate at the profile R^N is $Pl(R^N) = a$, it must be that the elected candidate at the profile R'^N is $Pl(R'^N) = a$. As voter j prefers a to b, the profile R'^N is a profitable deviation from Q^N. A contradiction arises since Q^N is an constrained Nash equilibrium. We conclude that if $s - 1 > \lfloor \frac{n}{2} \rfloor$, then $\left|E(R^N, Q^N) \cap S\right| \geq s - \lfloor \frac{n}{2} \rfloor$. ■

8.4 Complement on Statistical Exploration

Assume that there are only 3 candidates a, b, and c, and s voters. Let $r > 0$. What is the number of unstable profiles R^N that admit at least one deviation Q^N that is a Nash equilibrium and such that (1) The elected candidate at R^N is a, (2) the elected candidate at Q^N is b, and where (3) the number of voters who are adversely affected by strategic voting is equal to r? Denote the later number by $y(a, b, r, s)$. Proposition 7 provides the exact value of $y(a, b, r, s)$ given that n is a multiple of 6.

Proposition 7 *The number of profiles of true preference R^N with winner a that admit a manipulation Q^N which is a Nash equilibrium with winner b, and where the number of voters adversely impacted by the manipulation is r is given by*

$$y(a, b, r, s) = \sum_{p_0 \leq p \leq p_1} \sum_{q_0 \leq q \leq q_1} \sum_{\overline{p}_0 \leq \overline{p} \leq \overline{p}_1} C_s^p C_{n-s}^q C_{s-p}^{\overline{p}} C_{n-s-q}^{p+q-\overline{p}} C_{n-s-2q-p+\overline{p}}^{r-p-q} 2^{2(p+q)}$$

where

$p_0 = \max\{s - 4k, 0\}$, $p_1 = \min\{r, s - 1, 3k - 1\}$, $q_0 = \max\{0, 2k - p\}$, $q_1 = \min\{r - p, 3k - 1 - p, n - s\}$, $\overline{p}_0 = \max\{0, p + 2q - n + s, r - n + s + q\}$ $\overline{p}_1 = \min\{s - p - 1, p + q\}$, *and* $n = 6k$.

Proof of Proposition 7 It must hold that the true preference of the deviator, voter i, is $R^i = cba$, and $N \backslash E(R^N, Q^N) \subseteq F(a, R^N) \cup (N \backslash S)$. Indeed voters who are adversely affected by strategic voting either prefer candidate a to any other candidate or, are partisans. Now decompose the set of voters who are adversely impacted by the deviation in three subsets: (1) The set $\left[(N \backslash S) \backslash F(a, R^N)\right] \cap \left[N \backslash E(R^N, Q^N)\right]$ of partisan voters who suffer from strategic voting and whose most preferred candidate is c. The true preference of a voter of this set is cab. Let t be the cardinality of the latter set of voters. (2) The set $F(a, R^N) \cap S$ of strategic voters whose most prefered candidate is a. The true preference of a voter of this set is either abc or acb. Let p be the cardinality of $F(a, R^N) \cap S$. (3) The set $F(a, R^N) \cap [N \backslash S]$ of partisan voters whose most prefered candidate is a. The preference of a voter of this set is either abc or acb. Let q be the cardinality of $F(a, R^N) \cap [N \backslash S]$. The number of voters adversely impacted by the strategic voting (from R^N to Q^N) is $p + q + t$.

Assume that $r > 0$ is the number of voters adversely affected by the Strategic voting. What is the number of profile R^N that are unstable and that admit at least one deviation Q^N that is a Nash equilibrium and where the exact number of voters who are adversely affected by strategic voting is equal to r? Write $r = p + q + t$ where p, q, and t are as above described.

Choosing p. Obviously, we have (1) $p \leq r$ as $p + q + t = r$. We also have (2) $p \leq s - 1$ as $F(a, R^N) \cap S$ is a part of S and the potential deviator is a member of S who benefit from the deviation, and is therefore not a member of the block $F(a, R^N)$. Candidate a is elected at R^N, and candidate b is elected at Q^N. This implies that $|F(a, R^N)| + 1 = |F(a, Q^N)| + 1 = |F(b, Q^N)|$. It follows that $p + q \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, and thereafter that (3) $p \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. The inequality (4) $p + q \geq \left\lceil \frac{n}{3} \right\rceil$ also hold because we have three candidates, and the elected candidate at the profile R^N received only $p + q$ votes. Accounting the fact that $q \leq n - s$, Inequality (4) implies that (5) $\left\lceil \frac{n}{3} \right\rceil - (n - s) \leq p$. In conclusion, p is such that $\max\{s - 4k, 0\} \leq p \leq \min\{r, s - 1, 3k - 1\}$.

Choosing q**.** In a similar fashion we can choose the range for q. Firstly, $q \leq r - p$ as $p + q + t = r \geq 0$. Secondly, $2k - p \leq q \leq 3k - 1 - p$ as $p + q \in \left[\left\lceil \frac{n}{3} \right\rceil ; \left\lfloor \frac{n-1}{2} \right\rfloor\right]$. Thirdly, $q \leq n - s$. It follows that $\max\{0, 2k - p\} \leq q \leq \min\{r - p, 3k - 1 - p, n - s\}$.

Given r, p, and q, t is given by $t = r - p - q$. Up to now we have constructed the set of players who will be adversely affected by the strategic voting. Let's complete the construction and obtain the profile R^N. In the profile Q^N, exactly $p + q + 1$ voters rank candidate b first. This implies that $|F(b, R^N)| = p + q$. Let $\overline{p}$ be the number of strategic voters whose most preferred candidate is b. There are exactly $p + q - \overline{p}$ partisan voters whose most preferred candidate is b. Voters who are not adversely impacted by strategic voting and whose most preferred candidate is not b have cba as true preference.

Choosing $\overline{p}$. Obviously, we have that (1) $\overline{p} \leq s - p - 1$. The reason is that the most preferred candidate of p strategic voters is candidate a, and the potential deviator (from R^N) has preference of the form cba. As no more than $p + q$ voters rank candidate b first in the profile R^N, we have that $\overline{p} \leq p+q$. We also have $p+q-\overline{p} \leq n-s-q$. Indeed $p+q-\overline{p}$ partisan voters rank b first in the profile R^N while q partisan voters rank s fisrt in the profile R^N. Finally, we have that $r-(p+q) \leq n-s-q-(p+q-\overline{p})$. This condition ensures that candidate b is effectively a potential winner. It follows that $\max\{0, p+2q-n+s, r-n+s+q\} \leq \overline{p} \leq \min\{s-p-1, p+q\}$. It is easy to check that given $r \in I^*(m=3, n=6k, s, N)$ (as given in Theorem 1), any choice of $(p, q, \overline{p})$ in the ranges above determines a set of profiles of true preferences which admits at least one manipulation which is a strong Nash equilibrium, and where exactly r voters are adversely affected by the manipulation. This concludes the proof of Proposition 7. ■