Nonlocal operators
with
symmetric kernels

Universität Bielefeld
Fakultät für Mathematik

Dissertation

zur Erlangung des akademischen Grades
DOKTOR DER MATHEMATIK (DR. MATH.)

vorgelegt von

M.Sc. Tim Schulze
am 5. November 2019

Gedruckt auf alterungsbeständigem Papier DIN ISO 9706

Contents

1. Introduction

The essence

In this thesis we study linear nonlocal operators that involve symmetric integral kernels. The operators under investigation are integro-differential operators of the form

$$\mathcal{L}_k u(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} (u(y) - u(x)) k(x,y) \, \mathrm{d}y, \tag{1.1}$$

where *the kernel* $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ is a measurable function that is symmetric in the sense that $k(x,y) = k(y,x)$ for all $x, y \in \mathbb{R}^d$. Under additional conditions on a pointwise lower bound of the kernel we establish a coercivity estimate in $\dot{H}^{\frac{\alpha}{2}}, \alpha \in (0,2)$, of the bilinear form that is associated to (1.1), and contribute to the regularity theory of weak solutions of the elliptic equation

$$-\mathcal{L}_k u(x) = f(x), \quad x \in \Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^d$ is a domain and f is a function. Furthermore, we prove local boundedness results and deduce Harnack inequalities for weak solutions of the above elliptic equation provided that the kernel k satisfies an integral bound and a pointwise upper bound of the form $c|x-y|^{-d-\alpha}$ for almost all $x, y \in \mathbb{R}^d, x \neq y$ and some uniform constant $c > 0$.

Motivation

Nonlocal equations have a large number of applications in economics (finance [CT04]), mathematical physics (kinetic gas theory [Vil02], peridynamics [Sil00]) and enter also in conformal geometry [CG11]. In mathematics, nonlocal operators lie at the interface between analysis and stochastic processes. One link that connects these two branches is given by semigroup theory. Operators of the form (1.1) appear as infinitesimal generators of semigroups, which correspond to pure-jump symmetric Lévy processes. This connection has proved very fruitful in recent years since it allows the combination of probabilistic methods and analytic techniques in order to prove regularity results as well as Harnack inequalities for solutions of corresponding nonlocal equations.

Let us first fix the terms *local operator* and *nonlocal operator*. Let V, W be function spaces. An operator $\mathcal{L} : V \to W$ is called *a local operator* if for all $f \in V$ it holds $\operatorname{supp}(\mathcal{L}f) \subset \operatorname{supp}(f)$. Otherwise $\mathcal{L}$ is called *a nonlocal operator*.

Let $\alpha \in (0,2)$. A prominent example of a nonlocal operator of the form (1.1) is the *fractional Laplace operator* $(-\Delta)^{\frac{\alpha}{2}}$. This operator may be defined as

$$
\begin{aligned}
&-(-\Delta)^{\frac{\alpha}{2}} : \mathcal{S}(\mathbb{R}^d) \to L^2(\mathbb{R}^d),\\
&-(-\Delta)^{\frac{\alpha}{2}} u(x) = C(d,\alpha) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} (u(y) - u(x))|x-y|^{-d-\alpha} \,\mathrm{d}y, \quad x \in \mathbb{R}^d.
\end{aligned}
$$

Here $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space and $C(d,\alpha)$ is a normalizing constant. We refer the reader to [DPV12] and the survey [Kwa19] for more details concerning this operator. The fractional Laplace operator is the infinitesimal generator of the transition semigroup of the rotationally symmetric α-stable Lévy process, see [Kwa19, Theorem 4.1] and references therein. That is why in our work we use the notation $(-\Delta)^{\frac{\alpha}{2}}$ instead of $(-\Delta)^s$, which carries over to the notation of fractional Sobolev spaces. Here we write $H^{\frac{\alpha}{2}}$ instead of H^s.

As mentioned above, another area of application for operators of the form (1.1) is kinetic gas theory. This thesis is strongly motivated by nonlocal operators that appear in recent contributions to the Boltzmann equation.

This equation models the evolution of the density f of particles in an ideal gas, see the survey by Villani [Vil02] and the original work by Boltzmann [Bol81] for a thoroughly explanation. The function f depends on time $t \in \mathbb{R}$, space $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$ of the particles in the gas. The Boltzmann equation has the form

$$\partial_t f + (v, \nabla_x f) = Q(f,f), \quad t \in \mathbb{R}, x \in \mathbb{R}^d, v \in \mathbb{R}^d.$$

The right-hand side $Q(f,f)$ of the equation is the Boltzmann collision operator. This nonlocal operator depends on the so-called cross section B, which is part of the integrand in Q. As shown in [Vil02], if B satisfies certain properties, then the operator $(f,g) \mapsto Q(f,g)$ can be decomposed into the sum of two terms $Q(f,g) = Q_1(f,g) + Q_2(f,g)$ that satisfy specific properties on which we comment below. This decomposition holds in particular in the physically relevant model derived from inverse power laws. The interesting part now is that for a given function f one can show that

$$Q_1(f,g) = \int_{\mathbb{R}^d} (g(v') - g(v)) k_f(t,v,v') \,\mathrm{d}v', \tag{1.3}$$

see [Sil16, Lemma 4.1]. Above $k_f(t,v,v')$ is an integral kernel that depends also on time t and the solution f of the equation. A key property of this integral kernel is that, under the assumption that physical quantities such as mass, entropy and energy are bounded, the kernel is bounded from below in a cone of directions. That is, $k_f(t,v,v')$ is bounded from below by a constant times $|v'-v|^{-d-\alpha}$ for $\alpha \in (0,2)$, whenever $v'-v$ is in some linear cone $\Xi(v)$ depending on the velocity v, cf. [Sil16, Lemma 4.8] or [IS20, Lemma A.3]. Here a linear cone $\Xi \subset \mathbb{R}^d$ is a set that fulfills $\delta\Xi = \Xi$ for any positive δ. We point out that the kernel k_f is not symmetric in our sense, instead it satisfies $k_f(t,v,v+w) = k_f(t,v,v-w)$ for almost all $t \in \mathbb{R}, v, w \in \mathbb{R}^d$. However, the example of the Boltzmann kernel shows that it is important to derive results for operators with very mild assumptions on the lower bound of the kernel. This is exactly what we do in this work.

The results

1.1. Coercivity in fractional Sobolev spaces

In this part of the thesis we study quadratic functionals on $L^2(\Omega)$ that are associated to (1.1).

For a certain class of integral kernels k we prove an estimate of the form

$$\int_B \int_B (f(x) - f(y))^2 k(x,y)\,\mathrm{d}x\,\mathrm{d}y \;\geq\; c\|f\|^2_{\dot{H}^{\frac{\alpha}{2}}(B)}. \tag{1.4}$$

Here B is any ball in $\mathbb{R}^d$, $f \in L^2(B)$ and $c > 0$ is a constant independent of B and f. The right-hand side refers to the homogeneous fractional Sobolev norm given by

$$\|f\|_{\dot{H}^{\frac{\alpha}{2}}(B)} = \left(\int_B \int_B (f(x) - f(y))^2 |x-y|^{-d-\alpha}\,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{2}}.$$

The above result (1.4) says that the bilinear form

$$\mathcal{E}^k_B(f,g) = \int_B \int_B (f(x) - f(y))(g(x) - g(y)) k(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

is coercive in $\dot{H}^{\frac{\alpha}{2}}(B)$.

Coercivity estimates of the form (1.4) play a key role in the regularity theory of nonlocal operators. They are needed to adapt the De Giorgi-Nash-Moser techniques to the nonlocal setting, which in turn lead to a priori Hölder estimates of corresponding weak solutions of elliptic and parabolic equations that involve the operator (1.1). We refer the reader to the previous works [FK13], [KS14] and [DK20], where coercivity estimates of the form (1.4) appear in the assumptions of the main theorems. In the ensuing section of this introduction, cf. Section 1.2, a version of (1.4) appears in the main result as an assumption as well. Dyda's and Kassmann's article [DK20] can be seen as our initial motivation in order to derive a result of the form (1.4). The reason for this is that the theory established in their article allows to deduce a weak Harnack inequality and regularity results for weak solutions of the elliptic equation (1.2) from (1.4) and additional assumptions. With the main result in this part of the thesis we contribute to the regularity theory derived by Dyda and Kassmann. We will again pick up this topic when we discuss applications of our main result.

Assertions like (1.4) are obviously true for kernels k that fulfill

$$k(x,y) \;\geq\; c|x-y|^{-d-\alpha} \tag{1.5}$$

for almost all $x, y \in \mathbb{R}^d, x \neq y$ and some uniform constant $c > 0$.

In our case the lower bound in (1.5) holds true only if $x - y$ is an element of some cone. Otherwise the pointwise lower bound of $k(x, y)$ is 0.

Let us fix some notation first. By V we denote a double cone in $\mathbb{R}^d$ with apex at $0 \in \mathbb{R}^d$, symmetry axis $v \in \mathbb{R}^d$ and apex angle $\vartheta \in (0, \frac{\pi}{2}]$. Let $\mathcal{V} = (0, \frac{\pi}{2}] \times \mathbb{P}^{d-1}_{\mathbb{R}}$ denote the family of all such double cones. Here $\mathbb{P}^{d-1}_{\mathbb{R}}$ is the real projective space of dimension $d-1$. A shifted double cone is denoted by $V[x] = V + x, x \in \mathbb{R}^d$. A mapping $\Gamma : \mathbb{R}^d \to \mathcal{V}$ is called a configuration. If a configuration Γ enjoys the property that the infimum ϑ over all apex angles of cones in $\Gamma(\mathbb{R}^d)$ is strictly positive, then Γ is called ϑ-bounded. A ϑ-bounded configuration that satisfies

$$\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \,|\, y - x \in \Gamma(x)\} \text{ is a Borel set in } \mathbb{R}^d \times \mathbb{R}^d \tag{M}$$

is called ϑ-admissible. The integral kernels that we consider enjoy for $\alpha \in (0, 2)$ the pointwise estimate

$$k(x, y) \geq \Lambda(\mathbb{1}_{V^\Gamma[x]}(y) + \mathbb{1}_{V^\Gamma[y]}(x))|x - y|^{-d-\alpha} \tag{1.6}$$

for almost all $x, y \in \mathbb{R}^d, x \neq y$, where $d \geq 2$ and $\Lambda > 0$. Above, $V^\Gamma[x] = \Gamma(x) + x$. If Γ is ϑ-admissible, then (M) is natural, since it just says that the lower bound of k in (1.6) is a measurable function.

In the remainder of this section we state our main result, explain the ideas of the proof, discuss similar results and provide applications of the main result.

The main result of this part of the thesis reads as follows.

Theorem (see Theorem 8.1)**.** *Let Γ be a ϑ-admissible configuration and $\alpha \in (0, 2)$. Let $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ be a measurable function satisfying $k(x, y) = k(y, x)$ and (1.6). Then there is a constant $c > 0$ such that for every ball $B \subset \mathbb{R}^d$ and for every $f \in L^2(B)$, the inequality*

$$\int_B \int_B (f(x) - f(y))^2 k(x, y) \,\mathrm{d}x \,\mathrm{d}y \;\geq\; c \int_B \int_B (f(x) - f(y))^2 |x - y|^{-d-\alpha} \,\mathrm{d}x \,\mathrm{d}y \tag{1.7}$$

holds.

The constant c depends on Λ, the dimension d and ϑ. It is independent of k and Γ. For $0 < \alpha_0 \leq \alpha < 2$, the constant c may be chosen to depend on α_0 but not on α.

The idea of the proof of the above theorem is inspired by [HK07, Theorem 2.4]. Here, Husseini and Kassmann use the existence of certain chains, see assumption (B) therein, to establish a comparability result for quadratic forms. Our proof of (1.7) is built on discrete approximations of the quadratic forms involved. We show how one can approximate a given quadratic form on $L^2(B)$ ($B \subset \mathbb{R}^d$ a ball) through a sequence of discrete quadratic forms. Moreover, we provide a discrete analogue of (1.7): We show in Theorem 7.1 that every discrete kernel $w(x,y) : \mathbb{Z}^d \times \mathbb{Z}^d \to [0,\infty]$ that satisfies a suitable version of (1.6) for a ϑ-bounded configuration fulfills

$$\sum_{\substack{x,y\in B_{\kappa R}(x_0)\cap\mathbb{Z}^d\\ |x-y|>R_0}} (f(x)-f(y))^2\omega(x,y) \;\geq\; c \sum_{\substack{x,y\in B_R(x_0)\cap\mathbb{Z}^d\\ |x-y|>R_0}} (f(x)-f(y))^2|x-y|^{-d-\alpha}, \tag{1.8}$$

where the constant $\kappa \geq 1$ is uniform, $c > 0$ depends on the same quantities as the constant in (1.7) and $R_0 > 0$ is some given constant.

One can now deduce a version of the above discrete result for every lattice $h\mathbb{Z}^d$. The continuous result (1.7) follows then by considering the limit $h \to 0$.

The proof of (1.8) is challenging. We first show that any two lattice points x, y can be connected by a path in a graph, where points x, y are directly connected if $x - y$ lies in the cone attached at y or in the cone attached at x. These paths satisfy certain additional properties. An important argument in this proof is that one can pass from the possibly uncountable set of different cones in $\Gamma(\mathbb{R}^d)$ to a finite family of cones. This is only possible since Γ is ϑ-bounded, which enables us to use the induction principle to prove the existence of the above mentioned paths.

Let us comment on similar results like (1.7). In the article [DK20] Dyda and Kassmann treat elliptic equations and provide several examples for kernels and more general families of measures that satisfy (1.4).

Another similar result comes from recent progress regarding the Boltzmann equation. As described in (1.3) for fixed f the map $g \mapsto Q_1(f,g)$ can be understood as a linear operator $g \mapsto \mathcal{L}_v g$ in the velocity variable, whereas $g \mapsto Q_2(f,g)$ turns out to be of lower order. The operator $\mathcal{L}_v$ satisfies

$$\lambda \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(g(v)-g(v'))^2}{|v-v'|^{d+\alpha}}\,\mathrm{d}v\,\mathrm{d}v' \leq -\int_{\mathbb{R}^d} (\mathcal{L}_v g)(v)g(v)\,\mathrm{d}v + \Lambda\|g\|^2_{L^2(\mathbb{R}^d)} \tag{1.9}$$

for any function $g : \mathbb{R}^d \to \mathbb{R}$ supported in B_R, $R > 1$. The constants λ and Λ depend on physical quantities related to f and on the dimension. For the Boltzmann equation the above coercivity condition is well known and can be proved using methods from Fourier analysis, see the references in [IS20]. However, in [IS20, Lemma A.6] Imbert and Silvestre use a result like (1.7) to prove (1.9). The integral kernel derived from the Boltzmann equation enjoys a bound from below that is similar to our lower bound. Yet, the class of cones is different. The cones in [IS20] have the property that they always have an intersection of positive measure. This property is not an artificial assumption, it comes from the Boltzmann equation and the assumptions on the solution. In our case, the intersection of two cones $\Gamma(x) + x$ and $\Gamma(y) + y$ for $x \neq y$ may be the emptyset.

Another similar result was obtained recently by Chaker and Silvestre in [CS19, Theorem 1.2]. Here the authors work under the very weak assumption that the nondegeneracy set $\{y \,|\, k(x,y) \geq c|x-y|^{-d-\alpha}\}$ has some densities in all directions, see [CS19, Assumption

1.1] for a precise statement. Their kernels include our kernels. However, their result is only formulated on all of $\mathbb{R}^d$. They also provide a localized version, but the balls on either side of the inequality are of different radii, cf. [CS19, Theorem 1.3].

Applications of the main result

In the following paragraph we provide applications of the coercivity estimate (1.4). For the applications it is important that we have the coercivity result for every ball $B \subset \mathbb{R}^d$. Our first application concerns function spaces. Let $\Omega \subset \mathbb{R}^d$ be a domain. We denote by

$$H^k(\Omega) = \{f \in L^2(\Omega) \mid \|f\|_{\dot{H}^k(\Omega)} < \infty\}$$

the Hilbert space with seminorm

$$\|f\|_{\dot{H}^k(\Omega)} = \left(\int_\Omega \int_\Omega (f(x) - f(y))^2 k(x,y)\, \mathrm{d}x\, \mathrm{d}y\right)^{\frac{1}{2}}$$

and corresponding norm $\|f\|_{H^k(\Omega)} = \left(\|f\|^2_{L^2(\Omega)} + \|f\|^2_{\dot{H}^k(\Omega)}\right)^{\frac{1}{2}}$. If $k(x,y) = |x-y|^{-d-\alpha}$, then we use the notation $H^k(\Omega) = H^{\frac{\alpha}{2}}(\Omega)$.

Theorem (see Theorem 9.1)**.** *Let k be a kernel that satisfies* (1.4)*. Then $H^k(\mathbb{R}^d) \subset H^{\frac{\alpha}{2}}(\mathbb{R}^d)$.*

In addition, let k satisfy

$$k(x,y) \le B|x-y|^{-d-\alpha} \text{ for almost all } x, y \in \mathbb{R}^d, x \neq y \text{ and } B \ge 1. \tag{1.10}$$

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then the spaces $H^k(\Omega)$ and $H^{\frac{\alpha}{2}}(\Omega)$ coincide. The seminorms $\|\cdot\|_{\dot{H}^k(\Omega)}$ and $\|\cdot\|_{\dot{H}^{\frac{\alpha}{2}}(\Omega)}$ as well as the corresponding norms are comparable on $H^k(\Omega)$. The subspace $C^\infty(\overline{\Omega})$ is dense in $H^k(\Omega)$.

Moreover, $H^k(\mathbb{R}^d) = H^{\frac{\alpha}{2}}(\mathbb{R}^d)$. The seminorms $\|\cdot\|_{\dot{H}^k(\mathbb{R}^d)}$ and $\|\cdot\|_{\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)}$ as well as the corresponding norms are comparable on $H^k(\mathbb{R}^d)$. The subspace $C_c^\infty(\mathbb{R}^d)$ of smooth functions with compact support in $\mathbb{R}^d$ is dense in $H^k(\mathbb{R}^d)$.

The density result gives us another application coming from the theory of Markov jump processes and its connection to Dirichlet forms. We provide the necessary definitions in Chapter 5.

Corollary (see Corollary 9.2)**.** *Let k be a symmetric kernel that satisfies* (1.6) *and* (1.10)*. The Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\mathbb{R}^d)$ with $\mathcal{D}(\mathcal{E}) = H^{\frac{\alpha}{2}}(\mathbb{R}^d)$ and*

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(y)-f(x))\,(g(y)-g(x))\,k(x,y)\,\mathrm{d}x\,\mathrm{d}y\,,$$

is a regular Dirichlet form on $L^2(\mathbb{R}^d)$. There exists a corresponding Hunt process.

As mentioned in the beginning, we contribute to the theory in [DK20]. If k satisfies (1.10) and (1.6), then the result (1.7) says in particular that [DK20, Assumption (A)] is true for $\mu(x, \mathrm{d}y) = k(x,y)\,\mathrm{d}y$. Since the existence of cutoff functions (Assumption (B) in [DK20]) is also guaranteed in our case, we have the following theorem.

Theorem (see Theorem 9.3 and Theorem 9.5)**.** *Let $\Omega \subset \mathbb{R}^d$ be open. Let $\alpha \in (0,2)$ and $f \in L^q(\Omega)$ for $q > \frac{d}{\alpha}$. Assume the kernel k is symmetric, satisfies* (1.6) *and* (1.10)*. Then every weak supersolution $u : \mathbb{R}^d \to \mathbb{R}$ of*

$$-\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} (u(y) - u(x)) k(x,y)\,\mathrm{d}y = f(x), \quad x \in \Omega,$$

satisfies a weak Harnack inequality. If in addition u is a weak subsolution, then u is Hölder regular in the interior of Ω.

The author of this thesis has published the results described in Section 1.1 together with his co-authors in [BKS19]. This thesis contains further details. In particular, we have added some proofs, which were left out in the published version.

1.2. Local boundedness from above and elliptic Harnack inequalities

The second topic we treat in this thesis is the study of the local boundedness of weak solutions of elliptic equations associated to the operator (1.1). In this part the kernel satisfies an integral bound which replaces the pointwise lower bound in (1.5). Weak solutions are defined with the help of bilinear forms governed by (1.1), see Section 4.2. Our studies lead us to the derivation of Harnack inequalities.

The results in this part of the thesis contribute to the question: Which additional properties of the kernel are needed to ensure that a weak Harnack inequality yields a full Harnack inequality? To the best of the author's knowledge, this question is only well understood for kernels satisfying the pointwise lower bound in (1.5). We comment on results related to this question further below.

In the following paragraph we state the assumptions that appear in the main theorems. Fix $\alpha \in (0,2)$. We assume that there is $A > 0$ such that for every ball $B_r(x_0) \subset \mathbb{R}^d$ with $x_0 \in B_1$, $r \in (0,1]$, and each $v \in H^{\frac{\alpha}{2}}(B_r(x_0))$

$$\mathcal{E}^k_{B_r(x_0)}(v,v) \geq A \|v\|^2_{\dot{H}^{\frac{\alpha}{2}}(B_r(x_0))}. \tag{A}$$

Above

$$\mathcal{E}^k_{B_r(x_0)}(v,v) = \int_{B_r(x_0)} \int_{B_r(x_0)} (v(x) - v(y))^2\, k(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Condition (A) is nothing but a localized version of (1.4).

In addition, we always assume that the kernel k is bounded from above in the following way

$$|x-y|^{-d-\alpha} \geq Bk(x,y) \text{ for almost all } x,y \in \mathbb{R}^d, x \neq y, \tag{B}$$

where $B > 0$ is a generic constant.

The combination of (A) and (B) implies a weak Harnack inequality for weak supersolutions corresponding to $\mathcal{L}_k$, see [DK20, Theorem 1.2]. Therefore, a proof of a local boundedness estimate from above immediately implies a *full* Harnack inequality for the operator $\mathcal{L}_k$.

As in the first part of our thesis, we do not assume the lower bound in (1.5) to hold true. Instead, we suppose that the following integral condition, which we refer to as Condition (C), is fulfilled:

There exists a constant $C > 0$ such that for almost all $x \in B_1, y \in \mathbb{R}^d$ with $x \neq y$ and every radius $0 < r \leq \left(\frac{|x-y|}{2} \wedge \frac{1}{4}\right)$ it holds

$$\fint_{B_r(x)} k(z,y)\,\mathrm{d}z \geq Ck(x,y). \tag{C}$$

Condition (C) is not new in the literature. It can be understood as a localized version of the (UJS) assumption that appears for example in [CKW20], where it was used in order to establish equivalent statements to a parabolic Harnack inequality. We comment on this in more detail further below.

We emphasize that, in general, one cannot expect an elliptic Harnack inequality to hold true without any assumptions on the lower bound of the kernel. This was shown in [BS05]. In the latter article Bogdan and Sztonyk present a class of kernels such that weak solutions to (1.2) with $\Omega = B_1$ and $f = 0$ do not satisfy a Harnack inequality. As it turns out, these kernels fulfill (A) and (B) but the lower bound in (1.5) is violated. In our work we give a direct proof that the class of kernels in their example does not satisfy (C).

First, we present the main results in this section. After that, we comment on the proofs and examples of kernels that satisfy all of our assumptions, provide historical context and discuss related results.

Our first main result deals with local boundedness from above. It is stated below.

Theorem (see Theorem 11.10 and Corollary 11.11)**.** *Let $d \geq 2, \alpha \in (0,2)$ and $B_r(x_0) \subset B_1$. Assume $f \in L^q(B_1)$ for some $q > \frac{d}{\alpha}$ and let k be a kernel that satisfies* (A), (B) *and* (C). *Let $u \in V^k(B_1|\mathbb{R}^d) \cap L^1((1+|x|)^{-d-\alpha}\,\mathrm{d}x)$ such that $u \geq 0$ and $\mathcal{E}^k(u,\psi) = (f,\psi)$ for every $\psi \in H^k_{B_1}(\mathbb{R}^d)$. For each $p \in (0,2]$ there exists a constant $c > 0$, depending only on d, α, q, p and the constants from* (A), (B) *and* (C), *such that*

$$\sup_{B_{\frac{r}{8}}(x_0)} u \leq c \left(\fint_{B_{\frac{r}{2}}(x_0)} u^p(x)\,\mathrm{d}x \right)^{\frac{1}{p}} + c r^{\alpha - \frac{d}{q}} \|f\|_{L^q\left(B_{\frac{31}{64}r}(x_0)\right)}.$$

If $\alpha \in [\alpha_0, 2)$, then c may be chosen to depend on α_0 but not on α.

The combination of the local boundedness result and a weak Harnack inequality, which can be deduced from [DK20], yields our second main result, a full Harnack inequality.

Theorem (see Theorem 11.14)**.** *Let $d \geq 2, \alpha \in (0,2)$. Assume $f \in L^q(B_1)$ for $q > \frac{d}{\alpha}$. Let k be a symmetric kernel that satisfies* (A), (B) *and* (C). *Then there exists a positive constant c such that for each $u \in V^k(B_1|\mathbb{R}^d) \cap L^1((1+|x|)^{-d-\alpha}\,\mathrm{d}x)$ that satisfies $u \geq 0$ in B_1 and $\mathcal{E}^k(u,\psi) = (f,\psi)$ for every $\psi \in H^k_{B_1}(\mathbb{R}^d)$, the following inequality holds for every $B_r(x_0) \subset B_1$:*

$$\sup_{B_{\frac{r}{8}}(x_0)} u \leq c \inf_{B_{\frac{r}{8}}(x_0)} u + c \sup_{x \in B_{\frac{15}{16}r}(x_0)} \left(\int_{\mathbb{R}^d \setminus B_1} u_-(y) k(x,y)\,\mathrm{d}y \right) + c r^{\alpha - \frac{d}{q}} \|f\|_{L^q\left(B_{\frac{15}{16}r}(x_0)\right)}.$$

If $\alpha \in [\alpha_0, 2)$, then c can be chosen to depend on α_0 but not on α.

The proof of the above local boundedness result uses tools from [DKP14] and [DKP16] adapted to our setting. In their articles Di Castro, Kuusi and Palatucci prove, amongst other things, a full Harnack inequality for nonlinear operators that involve kernels that are pointwise comparable to the standard kernel of the fractional p-Laplace operator. The proof of this result uses a combination of a weak Harnack type inequality with a local boundedness result for weak solutions u in a domain Ω, which are nonnegative in some ball $B_R \subset \Omega$. The local upper bound is stated in the proof of [DKP14, Lemma 4.2] and it involves an averaged L^p-norm plus some tail term coming from the negative part of the solution. In particular, if u is nonnegative on all of $\mathbb{R}^d$, then the bound consists only of localized quantities. In order to show this local boundedness result, Di Castro, Kuusi and Palatucci combine two ingredients:

- A local boundedness result involving tail terms coming from the positive part of the solution, that is, an estimate of $\sup_{B_{\frac{r}{2}}} u$ in terms of an averaged L^p-norm in the ball $B_r \subset \Omega$ plus a nonlocal tail term, compare [DKP16, Theorem 1.1].
- An estimate of the nonlocal tail term that involves a local (or more precisely localized) term plus some tail term from the negative part of the solution. In fact, the tail function can be estimated by the supremum of u in the ball $B_r \subset B_{\frac{R}{2}}$ plus the tail term from the negative part of the solution, compare [DKP14, Lemma 4.2].

The local boundedness result that we derive in this thesis follows from the two ingredients above adjusted to our setting. An advantage is that we deal with the linear case $p = 2$. This allows us to work with weak solutions that are nonnegative on all of $\mathbb{R}^d$. Thus, in the proofs we do not have any tail terms coming from negative parts of the solution. Later on we can use the linearity of the underlying quadratic form to remove the global nonnegativity assumption. From now on assume $p = 2$ and $\Omega = B_1$. The tail term in

[DKP14] is defined by

$$\mathrm{Tail}(v, x_0, r) = r^{\alpha} \int_{\mathbb{R}^d \setminus B_r(x)} |v(y)||y - x_0|^{-d-\alpha} \, \mathrm{d}y. \tag{1.11}$$

The proof of the tail estimate in [DKP14] explicitly uses the assumption that the kernel under consideration is bounded from below by a constant times the standard kernel. This is also reflected in the definition of the tail function above. The latter was one of the main challenges within this work, since we do not have this pointwise lower bound of the kernel.

A consequence is that we cannot work with the tail term in (1.11). We define a new k-Tail as follows:

$$\mathrm{Tail}_k(v, x_0, r, \lambda, \mu) = r^{\alpha} \sup_{x \in B_\lambda(x_0)} \int_{\mathbb{R}^d \setminus B_\mu(x_0)} |v(y)| k(x, y) \, \mathrm{d}y,$$

where $r > 0, x_0 \in \mathbb{R}^d$ and $0 < \lambda < \mu \leq r$. This k-Tail can then be reproduced by testing with a specific function in the definition of weak solutions.

An equivalent formulation of (C) enables us to replace one of the integrals in the proof of our tail estimate by the kernel, leading to an estimate of the *nonlocal* quantity Tail_k for globally nonnegative weak supersolutions which contains only *localized* terms, see Lemma 11.7.

Using a De Giorgi type iteration and the estimate of Tail_k we then show our local boundedness results Theorem 11.10 and Corollary 11.11.

In this thesis we present examples as well as a counterexample for kernels which satisfy (C). Mainly we focus on two classes of kernels. The first family of examples is given by kernels corresponding to a configuration of cones, as discussed in Section 1.1. Here we provide examples as well as a counterexample.

One of the main motivations for our studies in this part of the thesis was to establish a Harnack inequality for operators with kernels associated to a configuration of cones. For this class of kernels it was not possible to adapt the techniques used in [DKP14]. We prove in this work that Condition (C) is in general neither false nor true for such kernels. The fact that it is true in some cases leads to the following conjecture.

Conjecture. *Let Γ be a ϑ-admissible configuration and k a symmetric integral kernel on* $\mathbb{R}^d \times \mathbb{R}^d$ *satisfying* (1.6) *and* (1.10). *Then weak solutions u to $-\mathcal{L}_k u(x) = f(x), x \in B_1$, which are nonnegative in B_1, enjoy an elliptic Harnack inequality.*

We expect the conjecture to be true for all ϑ-admissible configurations. A careful analysis and new results regarding the conjecture may lead to weaker sufficient conditions for elliptic Harnack inequalities to hold true.

The second family of examples consists of translation invariant kernels. These kernels are of special interest for us. The reason for this is given by the already mentioned article by Bogdan and Sztonyk [BS05]. In this article the authors show that for translation invariant and $(-d-\alpha)$-homogeneous kernels an elliptic Harnack inequality holds for the operator (1.1), if and only if

$$\int_{B_{\frac{1}{2}}(y)} |y-v|^{\alpha-d} k(v)\,\mathrm{d}v \le K \int_{B_{\frac{1}{2}}(y)} k(v)\,\mathrm{d}v, \quad y \in \mathbb{R}^d \setminus B_1, \tag{RK}$$

for some $K > 0$, independent of y, see Appendix B for more precise statements. The condition (RK) is called the *relative Kato condition.* In Appendix B we prove that Condition (C) is sufficient for (RK) to hold true. We also prove that Condition (C) is not necessary for (RK) in the case $d = 3$. We expect that this can be generalized to any $d > 3$, leading to the statement that the combination of (A), (B) and (C) is not a necessary condition in order to have an elliptic Harnack inequality. To the best of the author's knowledge the relation of (RK) and (C) has not been studied in the literature before.

One of our motivations for the studies described in this section was the following conjecture.

Conjecture. *Let* $k, \widetilde{k} : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ *be nonnegative, symmetric kernels. Furthermore, let* $\widetilde{k}$ *be translation invariant and* $(-d-\alpha)$*-homogeneous, that is,* $\widetilde{k}(x, y) = \widetilde{k}(x - y)$ *for all* $x, y \in \mathbb{R}^d$ *and* $\widetilde{k}(t) = |t|^{-d-\alpha}\widetilde{k}(t/|t|)$ *for every* $t \in \mathbb{R}^d \setminus \{0\}$*. Suppose that* $\widetilde{k}$ *satisfies* (RK) *and assume that there is* $c \ge 1$ *such that for every* $x_0 \in B_1$*,* $r \in (0, 1)$ *and each* $v \in H^k(B_r(x_0))$ *it holds*

$$c^{-1}\mathcal{E}^{\widetilde{k}}_{B_r(x_0)}(v, v) \le \mathcal{E}^{k}_{B_r(x_0)}(v, v) \le c\mathcal{E}^{\widetilde{k}}_{B_r(x_0)}(v, v).$$

Then weak solutions u *of*

$$-\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} (u(y) - u(x))^2 k(x, y)\,\mathrm{d}y = 0, \quad x \in B_1,$$

enjoy an elliptic Harnack inequality.

A proof of this conjecture using only the techniques from [DKP14] and [DKP16] was not possible due to the appearance of the Green function in the relative Kato condition.

Let us put the results of this section into historical context and discuss further related results in the literature. We review parts of the regularity theory of elliptic operators and present results that are related to Harnack inequalities, both elliptic and parabolic. We focus on the theory of equations in divergence form since the symmetry assumption of our kernels corresponds to equations of this type.

The De Giorgi-Nash-Moser techniques have their origin in the 1950s. In the celebrated works [De 57] and [Nas58] De Giorgi and Nash prove Hölder estimates for weak solutions

u of second order differential equations in divergence form:

$$\mathrm{div}(A(x)\nabla u(x)) = 0, \quad x \in \Omega. \tag{1.12}$$

Here A is a quadratic matrix, whose entries are measurable functions and satisfy the ellipticity condition $\lambda^{-1}I \leq A(x) \leq \lambda I$ for each $x \in \Omega$, $\lambda \geq 1$. In 1960, Moser gave a new derivation of De Giorgi's result in his note [Mos60]. In [Mos61] Moser proves an elliptic Harnack inequality for solutions of the above equation and, based upon this, derives Hölder estimates. For an extensive introduction to the Harnack inequality with historical details and a discussion of applications in regularity theory and beyond we refer the reader to the survey [Kas07b].

Equations of the above form (1.12) appear as Euler-Lagrange equations for functionals

$$F(w) = \int_\Omega f(x, \nabla w(x))\, \mathrm{d}x, \tag{1.13}$$

where f is at least a C^2-function. Minimizers w of (1.13) are then solutions of (1.12). In [GG82] Giaquinta and Guisti prove a priori Hölder regularity for minimizers of (1.13) and even more general functionals without making use of the Euler-Lagrange equations, that is, without any differentiability condition on f. Their approach was lately generalized by Cozzi in [Coz17] and adapted to the setting of the fractional p-Laplace operator. The author introduces his so-called *fractional De Giorgi classes* and proves a priori regularity for functions that satisfy certain Caccioppoli type inequalities. Cozzi also proves a Harnack inequality using a local boundedness result and a weak Harnack inequality. The proofs of the latter results are similar to the ones in [DKP14] and [DKP16].

In his habilitation thesis [Kas07a] Kassmann achieves Caccioppoli type estimates for $\mathcal{L}_k$-subharmonic functions and proves a local boundedness result from above using the De Giorgi type iteration technique, see Theorem 3.18 therein. However, the structure of the bound from above cannot be used in order to deduce a Harnack inequality.

Apart from the classical works, the latter references treat nonlocal operators. A notable difference between the local and the nonlocal case is the appearance of tail terms in the Harnack inequality. Kassmann showed in [Kas07c] that one cannot obtain a classical version of the Harnack inequality if one reduces the assumption of nonnegativity of the solution in the whole space to nonnegativity in the considered ball, compare [Kas07c, Theorem 1.2]. In [Kas11] Kassmann introduces a new version of the Harnack inequality involving tail terms, which compensate the fact that the solution is not globally nonnegative.

As mentioned earlier, in [DK20] Dyda and Kassmann use the Moser iteration technique in order to obtain a weak Harnack inequality for nonlocal operators. Thus, they can deduce Hölder estimates. The methods of the latter article have recently be adapted to operators with singular anisotropic kernels by Chaker and Kassmann in [CK20].

Relations and characterizations of elliptic Harnack inequalities (EHI) have been studied intensively in the literature. We only state a few references here. The reader may find a more complete list of references therein. We also include results regarding the stability of parabolic Harnack inequalities.

As already mentioned above, Bogdan and Sztonyk provide the equivalence of (RK) and (EHI) in [BS05]. Their result holds for translation invariant and homogeneous kernels that satisfy the upper bound (B) and appear as density functions of certain measures. In these cases the nonlocal operator (1.1) corresponds to a stable Lévy process. The main tool in their proof are estimates of the Green function for the unit ball. In the follow up article [BS07] Bogdan and Sztonyk give a strengthening of their theorem by dropping the upper bound of the kernel and considering more general measures.

In [Bas13] Bass obtained a characterization of (EHI) under the assumption of volume doubling, capacity growth and expected occupation time growth, see [Bas13, Theorem 2.8, Theorem 2.9 and Theorem 7.2]. These assumptions have been relaxed by Barlow and Murugan [BM18] to a volume doubling property at small scales as well as expected occupation time growth at small scales. Their characterization of (EHI) can be found as Theorem 5.15. in [BM18]. We emphasize that in these last two references the authors study (EHI) for *local* Dirichlet forms. A result for symmetric nonlocal Dirichlet forms on metric measure spaces that have the volume doubling property can be found in [CKW19]. In this article Chen, Kumagai and Wang provide criteria that are both necessary and sufficient for an elliptic Harnack inequality to hold true, see Corollary 1.12 therein. However, they work under the assumption that the considered kernel enjoys a pointwise upper and, in particular, a pointwise lower bound in terms of volume of a ball in the considered metric space times some scaling function. In our case, that is the Euclidean case, this would translate to a kernel comparable to $|x-y|^{-d-\alpha}$.

In the recent article [CKW20] Chen, Kumagai and Wang combine probabilistic properties of jump processes with PDE techniques used in [DKP16] in order to obtain stability of parabolic Harnack inequalities for symmetric nonlocal Dirichlet forms on metric measure spaces which enjoy the volume doubling condition. In their main theorem, Theorem 1.20, they establish seven statements that are all pairwise equivalent. One of these statements is a parabolic Harnack inequality. Of particular interest for the author of this thesis is the equivalence of the considered parabolic Harnack inequality and statement number (7) of [CKW20, Theorem 1.20]. For the Euclidean space $\mathbb{R}^d$ this last mentioned equivalence is nothing but the combination of the assumptions that a weak Poincaré inequality, a pointwise upper bound of the kernel and the so-called (UJS) assumption hold true. This result was first proved for continuous time random walks on graphs in [BBK09, Theorem 1.6].

Our Condition (C) is a localized version of the (UJS) assumption. It is localized in the sense that we only require it to hold true for every x in the unit ball. We therefore point out that, if we assume that (A), (B) are satisfied and (C) holds true not only for almost all $x \in B_1$ but for almost all $x \in \mathbb{R}^d$, then the validity of the elliptic Harnack inequality for weak solutions of (1.2) for $\Omega = B_1$ and $f = 0$ is already implied by the result in [CKW20]. However, the method of proof used in this thesis is new. One strength of the proof is that it uses only techniques from PDE-theory. We do not make use of any probabilistic arguments. Another advantage of the proof given in this thesis is that it allows to consider a right-hand side $f \neq 0$ in (11.1).

In [Str19] Strömqvist adapts the methods of [DKP14; DKP16] to the parabolic setting. He works with time dependent kernels that satisfy the condition $k(t,x,y) \asymp |x-y|^{-d-2s}, s \in (0,1)$. He proves tail estimates and uses the method of the De Giorgi iteration to prove local boundedness. As a result, he obtains a parabolic Harnack inequality with negative tail terms. We emphasize that the lower bound of the kernel is crucial for the estimate of the positive tail; otherwise an assertion like [Str19, Lemma 2.7] would not have been possible.

Outline

We finish the introduction with a description of the structure of this thesis.

This thesis consists of three parts. The first part, entitled Basics, contains all the necessary definitions and theorems that we use throughout this work. This part does not contain any proofs. Instead, we refer to the literature. In Chapter 2 we repeat basics from measure and integration theory and review the theory of Lebesgue as well as Sobolev spaces and provide important properties. Chapter 3 is about function spaces corresponding to a symmetric kernel. In Chapter 4 we explain the concept of weak solutions. For didactic purposes we have included a brief chapter about Dirichlet forms and the connection to stochastic processes, see Chapter 5.

In Part II we derive our coercivity result for bilinear forms corresponding to a configuration of cones. We derive a result for discrete quadratic forms in Chapter 7. Our main result is contained in Chapter 8. Applications are discussed in Chapter 9. A more detailed outline is given at the beginning of Part II.

Part III deals with local boundedness of weak solutions of elliptic PDEs. Our main result here is contained in Chapter 11. This chapter also includes definitions of tail functions as well as an estimate of the k-Tail by local(ized) quantities. The reader finds our version of the elliptic Harnack inequality in Section 11.6. For didactic purposes we provide a local version of the local boundedness result in Chapter 10. In Appendix B we explain the relation of Condition (C) to the relative Kato condition. A detailed outline can be found at the beginning of Part III.

Notation

If A, B are two sets, then we write $A \subset B$ if A is a subset of B. If A is a proper subset, then we write $A \subsetneq B$. If X is a linear space and $A \subset X$, then we denote by $A^C = X \setminus A$ the complement of A with respect to X.

By $\mathbb{N}$ we denote the natural numbers. We use the abbreviation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By $\mathbb{Z}$ we denote the set of all integers.

By $\mathbb{R}^d$ we denote the d-dimensional Euclidean space equipped with the Euclidean norm $|\cdot|$, that is, for $x = (x_1, ..., x_d)$ we have $|x| = \left(\sum_{i=1}^d x_i^2\right)^{\frac{1}{2}}$. The uniform norm on $\mathbb{R}^d$ is denoted by $|x|_\infty = \sup_{1 \le i \le d} |x_i|$.

The open ball in $\mathbb{R}^d$ with radius $r > 0$ and center x_0 shall be denoted by $B_r(x_0)$. Unless otherwise clarified, we write $B_r = B_r(0)$.

The $d-1$-dimensional unit sphere shall be denoted by $\mathcal{S}^{d-1}$, that is, $\mathcal{S}^{d-1} = \{x \in \mathbb{R}^d \,|\, |x| = 1\}$.

We write $a \asymp b$ for two quantities a, b if there exists a universal constant $c \ge 1$ so that $c^{-1}a \le b \le ca$. Then a and b are called *comparable* and we refer to c as *comparability constant.*

If M is a subset of $\mathbb{R}^d$, then we shall denote by $\overline{M}$ the closure of M in $\mathbb{R}^d$. The boundary of a set $V \subset \mathbb{R}^d$ is denoted by $\partial V = \overline{V} \cap \overline{V^C}$.

If S is a subset of a topological space, then $\mathcal{B}(S)$ is the Borel σ-algebra of S.

For the Lebesgue measure on $\mathbb{R}^d$ we use the notation λ^d. If $A \subset \mathbb{R}^d$ is measurable, then we write $|A|$ instead of $\lambda^d(A)$. For integrals we use the abbreviation $\lambda^d(\mathrm{d}x) = \mathrm{d}x$. We use the averaged integral notation

$$\fint_B f \,\mathrm{d}x = \frac{1}{|A|} \int_A f \,\mathrm{d}x.$$

If u is a function, then we denote by u_- its negative part defined as $u_-(x) = -\min(0, u(x))$ and by u_+ its positive part $u_+(x) = \max(0, u(x))$. We use the notation $u^p(x)$ instead of $(u(x))^p$.

Unless otherwise specified, the abbreviations sup and inf denote the essential supremum and the essential infimum.

The scalar product on $L^2(\mathbb{R}^d)$ will be denoted by the abbreviated notation

$$(f, g)_{L^2(\mathbb{R}^d)} = (f, g), \quad f, g \in L^2(\mathbb{R}^d).$$

We use several letters (Roman or Greek characters, in upper or lower cases) to denote constants. Sometimes we write the quantities on which constants depend in round brackets. We often use the letter c to denote a general positive constant. We especially point out that the value of c may change between different lines of the proof of the same statement.

For $\beta \in \mathbb{N}_0^d$ let $|\beta| = \beta_1 + \cdots + \beta_d$. Let

$$\partial^\beta = \frac{\partial^{|\beta|}}{\partial^{\beta_1} \ldots \partial^{\beta_d}}.$$

If $\beta = 0 \in \mathbb{N}_0^d$, then we set $\partial^\beta f = f$ for any function f.

Let $\Omega \subset \mathbb{R}^d$ be open and $k \in \mathbb{N}_0$. The following function spaces are used in this thesis:

$$\begin{aligned} C(\Omega) &= \{f : \Omega \to \mathbb{R} \,|\, f \text{ is continuous}\}, \\ C_c(\Omega) &= \{f : \Omega \to \mathbb{R} \,|\, f \text{ is continuous and has compact support in } \Omega\}, \\ C^k(\Omega) &= \{f : \Omega \to \mathbb{R} \,|\, \partial^\beta f \in C(\Omega) \text{ for all } \beta \in \mathbb{N}_0^k, |\beta| \le k\}, \\ C_c^k(\Omega) &= \{f : \Omega \to \mathbb{R} \,|\, \partial^\beta f \in C_c(\Omega) \text{ for all } \beta \in \mathbb{N}_0^k, |\beta| \le k\}. \end{aligned}$$

For $k \in \mathbb{N}_0$ we define

$$C^k(\overline{\Omega}) = \{f \in C^k(\Omega) \,|\, \forall \beta \in \mathbb{N}_0^d \text{ with } |\beta| \le k : \partial^\beta f \text{ has continuous extensions to } \overline{\Omega}\}.$$

Moreover,

$$C^\infty(\Omega) = \bigcap_{m \in N_0} C^m(\Omega).$$

The spaces $C_c^\infty(\Omega)$ and $C^\infty(\overline{\Omega})$ are defined analogously.

Danksagungen

An dieser Stelle möchte ich einigen Menschen danken, die bei der Entstehung dieser Arbeit direkt oder indirekt geholfen haben.

Zuallererst bedanke ich mich bei meinem Betreuer Prof. Dr. Moritz Kaßmann für das in mich investierte Vertrauen und die Möglichkeit diese Arbeit innerhalb seiner Arbeitsgruppe anfertigen zu können. Er hat sich stets mit viel Emphatie meiner Probleme angenommen und war immer gut darin, den *Gegner klein zu reden*, wenn er mal wieder zu groß zu werden drohte. Vielen Dank, es hat Spaß gemacht!

Besonders möchte ich mich auch bei Prof. Dr. Kai-Uwe Bux bedanken, der ohne zu zögern auf unseren Kegelzug aufgesprungen ist und somit zu einem positiven Ausgang dieses Projekts wesentlich beigetragen hat.

Einer weiteren Person, der ich diesbezüglich zu Dank verpflichtet bin, ist Dr. Bartłomiej Dyda. Er hat erste Versionen des Kegelprojekts Korrektur gelesen.

Ich möchte mich bei allen Mitgliedern der (erweiterten) Arbeitsgruppe angewandte Analysis für das angenehme Arbeitsumfeld und die vielen mathematischen sowie nicht mathematischen Diskussionen bedanken. Namentlich hervorheben möchte ich an dieser Stelle Filip Bosnić, Dr. Timothy Candy, Dr. Jamil Chaker, Andrea Nickel und Dr. Vanja Wagner. Insbesondere bedanke ich mich bei Ihnen für die wertvollen Hinweise bezüglich des Textes dieser Arbeit.

Ein großer Dank geht an meine Eltern und an meine Schwester, die mich stets unterstützt haben und die nie daran gezweifelt haben, dass mein Dissertationsvorhaben gelingen wird.

Danke Andrea, danke Anton! Ihr wart und seid Inspiration, Rückhalt und meine größte Motivation. Ohne Euch hätte ich es nicht geschafft.

Abgrenzung des eigenen Beitrags gemäß §10 (2) der Promotionsordnung

Die Resultate aus Part II hat der Autor dieser Arbeit zusammen mit seinen Koautoren Kai-Uwe Bux und Moritz Kaßmann in dem Artikel [BKS19] veröffentlicht. Die Ergebnisse aus Chapter 7 wurden dabei im Wesentlichen von Kai-Uwe Bux und dem Autor dieser Arbeit erarbeitet. An dem Beweis von Theorem 7.20 haben alle drei Autoren mitgewirkt. Die Idee in Chapter 8 mit den gemittelten Kernen zu arbeiten und so das diskrete Resultat zum Beweis der kontinuierlichen Koerzivitätsabschätzung zu nutzen, stammt von Moritz Kaßmann.

Anders als in [BKS19] wird in der vorliegenden Arbeit in den Hauptresultaten aus Chapter 7 und Chapter 8 keine punktweise obere Grenze an den Kern vorausgesetzt, da diese in den Beweisen nicht explizit gebraucht wird. Die Sätze Theorem 9.4 and Theorem 9.5 gehen auf die Arbeit [DK20] zurück und sind in dieser Form nicht in [BKS19] enthalten.

Part I.

Basics

2. Basics from measure theory, integration theory and classical function spaces

In this chapter we review basic concepts from measure and integration theory, which are used constantly in this thesis. We include all theorems, lemmata and propositions that we employ in our work. Another aim of this chapter is to make the reader familiar with the basic notation used in this thesis.

References for this part are [Ama08, Chapter X],[Bau01, Chapter I and II] and [Ler14, Chapter I and III]. The use of additional sources is mentioned at the appropriate place in the text.

Measure spaces

Throughout this chapter we deal with two different kind of spaces. A tuple $(X, \mathcal{A})$ will always denote a measurable space, that is, $\mathcal{A}$ is a σ-algebra on X. A set $A \subset \mathcal{A}$ will be called a *measurable set* or $\mathcal{A}$*-measurable* whenever we want to keep track of the underlying σ-algebra. If $\mathcal{A}$ is a σ-algebra on X and $\mathcal{B}$ is a σ-algebra on Y, then we denote by $\mathcal{A} \times \mathcal{B}$ the smallest σ-algebra that contains the sets $A \times B$, $A \in \mathcal{A}, B \in \mathcal{B}$.

A triple $(X, \mathcal{A}, \mu)$ shall always denote a measure space. Here $\mathcal{A}$ is a σ-algebra on X and $\mu : \mathcal{A} \to [0, \infty]$ is a measure on $\mathcal{A}$. A measure space $(X, \mathcal{A}, \mu)$ is called complete if for every $N \in \mathcal{A}$ with $\mu(N) = 0$ the implication

$$S \subset N \Rightarrow S \in \mathcal{A}$$

holds true. If the measure is normalized, that is $\mu(X) = 1$, then a measure space $(X, \mathcal{A}, \mu)$ shall be called a *probability space*.

By $\mathcal{B}(X)$ we denote the Borel σ-algebra of X when we consider X as a topological space $(X, \mathcal{T})$. This σ-algebra is defined as the smallest σ-algebra that contains all open sets in $\mathcal{T}$. If $B \in \mathcal{B}(X)$, then B shall be referred to as *Borel set.* One fact that we use without further mentioning it is that the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$ equals $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d)$.

Let us consider the case $X = \mathbb{R}^d$ for $d \in \mathbb{N}$ and let λ^d be the Borel-Lebesgue measure acting on the Borel sets of $\mathbb{R}^d$. Then the measure space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda^d)$ is not complete.

However, one can extend the measure space to a complete space. The smallest complete extension is given by $(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d), \overline{\lambda}^d)$. It consists of the d-dimensional Lebesgue σ-algebra $\mathcal{L}(\mathbb{R}^d)$ together with the Lebesgue measure $\overline{\lambda}^d$. In the remainder we will not distinguish between the Lebesgue-Borel measure λ^d and the Lebesgue measure $\overline{\lambda}^d$ and always use the notation λ^d. Also, we usually use the notation $|A| = \lambda^d(A)$ for the d-dimensional Lebesgue measure of a set A belonging to the Lebesgue σ-algebra $\mathcal{L}(\mathbb{R}^d)$.

If $(X, \mathcal{A}, \mu)$ is a measure space, then we shall often say that an inequality or equality holds true μ-almost everywhere on X or for μ-almost all $x \in X$ meaning that it holds true for every $x \in X \setminus N$ where $N \subset X$ is a nullset, i.e. $\mu(N) = 0$. If no confusion can arise, we sometimes drop the explicit mentioning of the measure.

Measurable functions

In the following we give a brief explanation of the notion of measurable functions and provide the definition of the Lebesgue integral. Here we use the notation $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ for a compactification of $\mathbb{R}$. Then the Borel σ-algebra on $\overline{\mathbb{R}}$ is given as

$$\mathcal{B}(\overline{\mathbb{R}}) = \{B, B \cup \{\infty\}, B \cup \{-\infty\}, B \cup \{-\infty, \infty\} \mid B \in \mathcal{B}(\mathbb{R}^d)\}.$$

Definition 2.1. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces. A function $f : X \to Y$ is called $(\mathcal{A}, \mathcal{B})$-measurable (or measurable with respect to $\mathcal{A}$ and $\mathcal{B}$) if $B \in \mathcal{B}$ implies $f^{-1}(B) \in \mathcal{A}$.*
A function $f : X \to \overline{\mathbb{R}}$ is called measurable if it is $(\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))$-measurable or equivalently if for every $t \in \mathbb{R}$ one has

$$\{x \in X \mid f(x) \geq t\} \in \mathcal{A}.$$

Note that, according to the above definition, a real valued function f on $(X, \mathcal{A})$ is measurable if it is $(\mathcal{A}, \mathcal{B}(\mathbb{R}))$-measurable. We shall often say that a function $f : \Omega \to \mathbb{R}$ is measurable for an open subset $\Omega \subset \mathbb{R}$ meaning that f is $(\mathcal{B}(\Omega), \mathcal{B}(\mathbb{R}))$-measurable.

Integrable functions

From now on let $(X, \mathcal{A}, \mu)$ be a complete measure space. A function $f : X \to \overline{\mathbb{R}}$ is called nonnegative if $f \geq 0$ almost everywhere on X.

A simple function $s : X \to \mathbb{R}$ is a nonnegative function that satisfies

$$s = \sum_{k=1}^{m} \lambda_k \mathbb{1}_{A_k},$$

where $m \in \mathbb{N}$, $\lambda_k \geq 0$, $\lambda_i \neq \lambda_j$ for $i \neq j$ and $A_k = s^{-1}(\{\lambda_k\}) \in \mathcal{A}$ for each $k \in \{1, ..., m\}$. Note that

$$\bigcup_{k=1}^{m} A_k = X$$

is a disjoint union. The Lebesgue integral over X of a simple function s is defined as

$$\int_X s(x)\mu(\mathrm{d}x) = \sum_{\substack{1 \leq k \leq m \\ \lambda_k \geq 0}} \lambda_k \, \mu(A_k).$$

If $A \in \mathcal{A}$, then

$$\int_A s(x)\mu(\mathrm{d}x) = \int_X s(x)\mathbb{1}_A(x)\mu(\mathrm{d}x) \tag{2.1}$$

defines the integral of s over a measurable subset of X. Now we are in a position to define the integral over X for every function $f : X \to \overline{\mathbb{R}}$.

Definition 2.2. *Let $f : X \to \overline{\mathbb{R}}$ be a nonnegative measurable function. Then the Lebesgue (or μ-) integral of f over X is defined as*

$$\int_X f(x)\mu(\mathrm{d}x) = \sup_{\substack{s \text{ simple} \\ 0 \leq s \leq f}} \int_X s(x)\mu(\mathrm{d}x).$$

If $A \in \mathcal{A}$, then $\int_A f(x)\mu(x)$ is defined analogously using (2.1).

If $f : X \to \overline{\mathbb{R}}$ does not satisfy the nonnegativity assumption, then one can write

$$f = f_+ - f_-$$

with $f_+(x) = \max(f(x), 0)$ and $f_-(x) = -\min(f(x), 0)$. The integral of f over $A \in \mathcal{A}$ is then defined as

$$\int_A f(x)\mu(\mathrm{d}x) = \int_A f_+\mu(\mathrm{d}x) - \int_A f_-\mu(\mathrm{d}x).$$

We shall say that f is integrable if f is measurable and $\int_X |f(x)|\mu(\mathrm{d}x)$ is a real number.

In the case where $\mu = \lambda^d$ is the Lebesgue measure we also use the abbreviated notations

$$\int_A f \,\mathrm{d}x = \int_A f(x) \,\mathrm{d}x = \int_A f(x) \, \lambda^d(\mathrm{d}x)$$

in the text of this thesis.

Convergence theorems

We provide three theorems that are used frequently in our work.

Theorem 2.3 (Monotone Convergence Theorem). *Let* (f_n) *be a sequence of nonnegative measurable functions* $f_n : X \to \overline{\mathbb{R}}$ *on a measure space* $(X, \mathcal{A}, \mu)$ *such that* $f_n \leq f_{n+1}$ *almost everywhere on* X *for each* $n \in \mathbb{N}$*. Then*

$$\int_X \lim_{n\to\infty} f_n(x)\mu(\mathrm{d}x) = \lim_{n\to\infty} \int_X f_n(x)\mu(\mathrm{d}x).$$

Theorem 2.4 (Lemma of Fatou). *Let* (f_n) *be a sequence of measurable nonnegative functions* $f : X \to \overline{\mathbb{R}}$ *on some measure space* X *with measure* μ*. Then*

$$\int_X \liminf_{n\to\infty} f_n(x)\mu(\mathrm{d}x) \leq \liminf_{n\to\infty} \int_X f_n(x)\mu(\mathrm{d}x).$$

Theorem 2.5 (Dominated Convergence Theorem). *Let* $(X, \mathcal{A}, \mu)$ *be a measure space. Let* (f_n) *be a sequence of measurable functions* $f_n : X \to \overline{\mathbb{R}}$ *such that* $f_n \to f$ *almost everywhere on* X *for a measurable function* $f : X \to \overline{\mathbb{R}}$*. Suppose that* $g : X \to \overline{\mathbb{R}}$ *is integrable and it holds*

$$\sup_{n\in\mathbb{N}} |f_n| \leq g \quad \mu\text{-almost everywhere.}$$

Then f *and all* f_n *are integrable,*

$$\lim_{n\to\infty} \int_X |f_n(x) - f(x)|\mu(\mathrm{d}x) = 0 \text{ and } \lim_{n\to\infty} \int_X f_n(x)\mu(\mathrm{d}x) = \int_X f(x)\mu(\mathrm{d}x).$$

2.1. Lebesgue spaces

Let $(X, \mathcal{A}, \mu)$ be a measure space and $p \in [1, \infty]$. We denote by $\mathcal{L}^p(X, \mu)$ the set of all measurable functions $f : X \to \mathbb{R}$, so that

$$\|f\|_{L^p(X,\mu)} = \begin{cases} \left(\int_X |f(x)|^p \mu(\mathrm{d}x)\right)^{\frac{1}{p}}, & \text{if } p \in [1, \infty), \\ \inf\{M \in [0, \infty] \,|\, \mu(\{|f| > M\}) = 0\}, & \text{if } p = \infty \end{cases}$$

is finite.

The map $\|\cdot\|_{L^p(X,\mu)}$ is a seminorm on $\mathcal{L}^p(X, \mu)$. The triangle inequality

$$\|f + g\|_{L^p(X,\mu)} \leq \|f\|_{L^p(X,\mu)} + \|g\|_{L^p(X,\mu)}$$

for $f, g \in L^p(X, \mu)$ and $p \in [1, \infty)$ is called *Minkowski inequality.* For $p = \infty$ it becomes obvious.

Note that $\|f\|_{L^p(X,\mu)} = 0$ does not imply $f = 0$ since f can be different from zero on a set of measure zero.

We obtain a normed space from $\mathcal{L}^p(X,\mu)$ in the following way. We identify two functions $f \sim g$ with each other if $f = g$ μ-almost everywhere on X. This gives us an equivalence relation on $\mathcal{L}^p(X,\mu)$.

Definition 2.6. *The factor-space $\mathcal{L}^p(X,\mu)/\sim$ of $\mathcal{L}^p(X,\mu)$ with respect to the above equivalence relation shall be denoted by $L^p(X,\mu)$ and be refered to as* Lebesgue space of integrable functions of order *p.*

We usually identify a representative function f in $\mathcal{L}^p(X,\mu)$ with its equivalence class $[f]$ in $L^p(X,\mu)$ and write, by abuse of notation, $f \in L^p(X,\mu)$. In this sense, we refer to $L^p(X,\mu)$ as the Lebesgue space of all functions integrable of order p. The mapping $\|\cdot\|_{L^p(X,\mu)}$ is a norm on $L^p(X,\mu)$.

Whenever it is clear from the context and no confusion can arise we abbreviate our notation by $L^p(X) = L^p(X,\mu)$. In the case where $\mu = \lambda^d$ is the Lebesgue measure, we use the notation $L^p = L^p(\mathbb{R}^d, \lambda^d)$. If Θ is a measure that is absolutely continuous with respect to the Lebesgue measure λ^d and has the density function θ, then we will also use the notation $L^p(\mathbb{R}^d, \Theta) = L^p(\theta(x)\,\mathrm{d}x)$.

For any open subset Ω of $\mathbb{R}^d$ we also define the space of locally integrable functions $L^p_{\mathrm{loc}}(\Omega)$ of order p as the factor space of

$$\left\{ f : \Omega \to \mathbb{R} \text{ measurable} \;\middle|\; \int_K |f(x)|^p \,\mathrm{d}x < \infty \text{ for every compact subset } K \subset \Omega \right\}$$

with respect to the aforementioned equivalence relation. Let us remark that every function in $L^p(\Omega)$ is in $L^p_{\mathrm{loc}}(\Omega)$.

In the remainder we collect some basic properties of the Lebesgue spaces.

Basic properties of Lebesgue spaces

Theorem 2.7. *The spaces $(L^p(X,\mu), \|\cdot\|_{L^p(X,\mu)})$ are Banach spaces.*

The case $p = 2$ is of special interest. The mapping

$$(\cdot,\cdot)_{L^2(X)} : L^2(X) \times L^2(X) \to \mathbb{R}, \quad (f,g)_{L^2(X)} = \int_X f(x)g(x)\mu(\mathrm{d}x)$$

defines a scalar product on $L^2(X)$. This scalar product induces a norm that is given by $\|\cdot\|_{L^2(X)}$. Together with the theorem above we conclude that $(L^2(X), \|\cdot\|_{L^2(X)})$ is a Hilbert space.

One important tool in our work is the Hölder inequality. Let us impose the convention $\frac{1}{\infty} = 0$. Then the Hölder inequality has the following form.

Theorem 2.8 (Hölder inequality)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $1 \leq p, p' \leq \infty$ be conjugate exponents, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. Suppose $f \in L^p(X)$ and $g \in L^{p'}(X)$. Then the product fg belongs to $L^1(X)$ and we have*

$$\|fg\|_{L^1(X)} \leq \|f\|_{L^p(X)} \|g\|_{L^{p'}(X)}.$$

One can easily deduce the following generalized Hölder inequality from the above theorem.

Corollary 2.9 (Generalized Hölder inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ open. Let $f_1, ..., f_m$ be functions on Ω with $f_i \in L^{p_i}(\Omega)$ for each $1 \leq i \leq m$ and $\sum_{j=1}^m \frac{1}{p_i} = 1$. Then the product $\prod_{i=1}^m f_i$ is a function in $L^1(\Omega)$ and*

$$\left\| \prod_{i=1}^m f_i \right\|_{L^1(\Omega)} \leq \prod_{i=1}^m \|f\|_{L^{p_i}(\Omega)}.$$

In calculations we often change the order of integration or use suitable substitutions to calculate integrals. This is possible due to to the next two theorems.

Theorem 2.10 (Theorem of Fubini)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces. The product $X \times Y$ shall be equipped with the σ-algebra $\mathcal{A} \times \mathcal{B}$. Let $f : X \times Y \to \mathbb{R}$ be a measurable function. Then the following statements hold true.*

1. *If*
$$\int_X \left(\int_Y |f(x, y)| \nu(\mathrm{d}y) \right) \mu(\mathrm{d}x) < \infty,$$
then $f \in L^1(X \times Y)$.

2. *If $f \in L^1(X \times Y)$, then $f(x, \cdot) \in L^1(Y)$ for μ-almost every $x \in X$, $f(\cdot, y) \in L^1(X)$ for ν-almost every $y \in Y$ and*
$$\begin{aligned} \int_X \left(\int_Y f(x, y) \nu(\mathrm{d}y) \right) \mu(\mathrm{d}x) &= \int_Y \left(\int_X f(x, y) \mu(\mathrm{d}x) \right) \nu(\mathrm{d}y) \\ &= \int_{X \times Y} f(x, y) \, (\mu \times \nu)(\mathrm{d}(x, y)). \end{aligned}$$

Theorem 2.11 (Change of variables formula for the Lebesgue measure)**.** *Let U, V be open subsets of $\mathbb{R}^d$, let $\Phi : U \to V$ be a C^1-diffeomorphism and let $f \in L^1(V)$. Then the function*

$$(f \circ \Phi) |\det(J_\Phi)| : U \to \mathbb{R},$$

where $\det(J_\Phi)$ denotes the determinant of the Jacobian matrix J_Φ of Φ, belongs to $L^1(U)$ and

$$\int_V f(y) \, \mathrm{d}y = \int_U f(\Phi(x)) |\det(J_{\Phi(x)})| \, \mathrm{d}x.$$

Differentiation Theorem of Lebesgue

We include another convergence theorem. The following theorem is known as Differentiation Theorem of Lebesgue in the literature. It can be proved with the help of the Hardy-Littlewood maximal function, cf. [SS05, Chapter 3].

Theorem 2.12. *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, then*

$$\lim_{r\to 0} \frac{1}{\lambda^d(B_r(x))} \int_{B_r(x)} f(y)\,\mathrm{d}y = f(x) \quad \text{for almost every } x \in \mathbb{R}^d.$$

We say that a collection of measurable sets $\{U_\alpha\}$ shrinks regularly to x if there is a constant $c > 0$ such that for each U_α there is a ball B with

$$x \in B, \quad U_\alpha \subset B \quad \text{and } \lambda^d(U_\alpha) \geq c\lambda^d(B).$$

Corollary 2.13. *Suppose $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. If $\{U_\alpha\}$ shrinks regularly to x, then*

$$\lim_{\substack{\lambda^d(U_\alpha)\to 0\\ x\in U_\alpha}} \frac{1}{\lambda^d(U_\alpha)} \int_{U_\alpha} f(y)\,\mathrm{d}y = f(x).$$

Here the limit is taken as the volume of the sets U_α containing x goes to 0.

2.2. Sobolev spaces

We define integer Sobolev spaces and Sobolev spaces of fractional order. We review embedding theorems and present dense subsets of the spaces under consideration. A reference for this section is [DD12].

Classical Sobolev spaces

In what follows we use the multi-index notation, that is, $\beta = (\beta_1, \beta_2, ..., \beta_d)$ with $\beta_k \in \mathbb{N}_0, 1 \leq k \leq d$, $|\beta| = \sum_{k=1}^d \beta_k$ and $\partial^\beta = \prod_{k=1}^d \partial_k^{\beta_k}$, where ∂_k denotes the k-th partial derivative.

Let Ω be an open subset of $\mathbb{R}^d$. By $C_c^\infty(\Omega)$ we denote the space of smooth functions on $\mathbb{R}$ that have compact support in Ω. Let us first give the definition of weak derivatives.

Definition 2.14 (weak derivative)**.** *Let $f \in L^1_{\mathrm{loc}}(\Omega)$. A function $g \in L^1_{\mathrm{loc}}(\Omega)$ is called* β-weak derivative (or weak derivative of order $|\beta|$) *of g if*

$$\int_\Omega f(x)\partial^\beta \varphi(x)\,\mathrm{d}x = (-1)^{|\beta|} \int_\Omega g(x)\varphi(x)\,\mathrm{d}x \quad \text{for all } \phi \in C_c^\infty(\Omega).$$

The function g is then denoted as $g = \partial^\beta f$.

Definition 2.15. *Let $1 \leq p \leq \infty$ and $m \in \mathbb{N}$. The Sobolev space $W^{m,p}(\Omega)$ of integer order m consists of all p-integrable functions on Ω whose weak derivatives up to order m exist and are again p-integrable functions. In short,*

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega) \,|\, \forall \beta \in \mathbb{N}_0^d, |\beta| \leq m : \partial^\beta f \in L^p(\Omega)\}.$$

The mapping

$$\|\cdot\|_{W^{m,p}(\Omega)} : W^{m,p}(\Omega) \to [0, \infty),$$
$$\|f\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\sum_{|\beta|\leq m} \|\partial^\beta f\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} & \text{for } p \in [1.\infty), \\ \sum_{|\beta|\leq m} \|\partial^\beta f\|_{L^\infty(\Omega)} & \text{for } p = \infty. \end{cases}$$

defines a norm on the Sobolev space $W^{m,p}(\Omega)$. For every $p \in [1, \infty]$ and every $m \in \mathbb{N}$ the spaces $(W^{m,p}(\Omega), \|\cdot\|_{W^{m,p}(\Omega)})$ are complete. We also define

$$W_0^{m,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{m,p}(\Omega)}}.$$

In the case $p = 2$ we use the notation $W^{m,2}(\Omega) = H^m(\Omega)$. These Sobolev spaces are Hilbert spaces with scalar product given by

$$(f, g)_{H^m(\Omega)} = \sum_{|\beta|\leq m} (\partial^\beta f, \partial^\beta g)_{L^2(\Omega)}.$$

Similar we define

$$H_0^m(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{m,2}(\Omega)}}$$

as the completion of all smooth functions with compact support with respect to the norm $\|\cdot\|_{W^{m,2}(\Omega)}$. Of course, for every m the spaces $(H_0^m(\Omega), \|\cdot\|_{W^{m,2}(\Omega)})$ are again Hilbert spaces.

Theorem 2.16 (classical Poincaré inequality, compare [Pon16, Proposition 5.5])**.** *Let Ω be a bounded open set and $1 \leq p < \infty$. Then there exists a constant $C > 0$ such that for every function $u \in W_0^{1,p}(\Omega)$ one has*

$$\|u\|_{L^q(\Omega)} \leq C\|\nabla u\|_{L^q(\Omega)}.$$

In the following we require $\Omega \subset \mathbb{R}^d$ to be a *Lipschitz open set*. Since we only work with bounded Lipschitz open sets in this thesis, we omit a precise definition of the terminology Lipschitz open set. Instead, we refer the reader to [DD12, Definition 2.65 and Remark 2.67]. In the case that Ω is a bounded Lipschitz open set this definition reduces to the case that the boundary $\partial\Omega$ enjoys the locally Lipschitz property, that is, for every point on the boundary $x \in \partial\Omega$ there is a neighborhood U_x such that $U_x \cap \partial\Omega$ is the graph of a Lipschitz continuous function.

For a proof of the following theorem we refer the reader to [DD12, Theorem 2.72] or [Ada75, Theorem 5.4].

Theorem 2.17 (Sobolev embedding theorem). *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz open set. Let $m \in \mathbb{N}$, $p \in [1, \infty)$ with $d > mp$ and $q \in [p, \frac{dp}{d-mp}]$. Then there exists a constant $c > 0$ such that for every $f \in W^{m,p}(\Omega)$*

$$\|f\|_{L^q(\Omega)} \leq c\|f\|_{W^{m,p}(\Omega)}.$$

Remark. The Sobolev embedding theorem covers even more cases for p. One can find a more comprehensive version in the cited literature.

Applying the classical Poincaré inequality to the above theorem with $m = 1$, we obtain the following inequality, which we refer to as *Sobolev inequality* throughout the rest of this thesis. In the literature it is sometimes called Gagliardo-Nirenberg-Sobolev inequality.

Theorem 2.18 (Sobolev inequality). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz open set. If $1 \leq q < d$, then we have $W_0^{1,p}(\Omega) \subset L^{\frac{dp}{d-p}}(\Omega)$ and there exists a constant $C > 0$ such that for every $f \in W_0^{1,p}(\Omega)$*

$$\|f\|_{L^{\frac{dp}{d-p}}(\Omega)} \leq C\|\nabla f\|_{L^p(\Omega)}.$$

Sobolev spaces of fractional order

Let us also define the Sobolev spaces of fractional order $\frac{\alpha}{2} \in (0, 1)$.

Definition 2.19. *For $\alpha \in (0, 2)$ and $1 \leq p < \infty$ the fractional Sobolev space $W^{\frac{\alpha}{2},p}(\Omega)$ consists of all functions $f \in L^p(\Omega)$ such that*

$$\|f\|_{\dot{W}^{\frac{\alpha}{2},p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{d+(\alpha/2)p}} \,\mathrm{d}x \,\mathrm{d}y\right)^{\frac{1}{p}} < \infty.$$

The mapping $\|\cdot\|_{\dot{W}^{\frac{\alpha}{2},p}(\Omega)}$ defines a seminorm on $W^{\frac{\alpha}{2},p}(\Omega)$. A norm on $W^{\frac{\alpha}{2},p}(\Omega)$ is given by

$$\|f\|_{W^{\frac{\alpha}{2},p}(\Omega)} = \left(\|f\|^p_{L^p(\Omega)} + \|f\|^p_{\dot{W}^{\frac{\alpha}{2},p}(\Omega)}\right)^{\frac{1}{p}}.$$

Using this norm, we may write

$$W^{\frac{\alpha}{2},p}(\Omega) = \{f : \Omega \to \mathbb{R} \text{ measurable } \mid \|f\|_{W^{\frac{\alpha}{2},p}(\Omega)} < \infty\}.$$

Analogous to the integer case we write

$$W^{\frac{\alpha}{2},2}(\Omega) = H^{\frac{\alpha}{2}}(\Omega), \quad \|\cdot\|_{\dot{W}^{\frac{\alpha}{2},2}(\Omega)} = \|\cdot\|_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} \text{ and } \|\cdot\|_{W^{\frac{\alpha}{2},2}(\Omega)} = \|\cdot\|_{H^{\frac{\alpha}{2}}(\Omega)}.$$

The spaces $(H^2(\Omega), \|\cdot\|_{H^{\frac{\alpha}{2}}(\Omega)})$ are Hilbert spaces. The completion of all smooth functions with compact support shall be denoted by

$$H_0^{\frac{\alpha}{2}}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^{\frac{\alpha}{2}}(\Omega)}}.$$

We regularly use embedding theorems. Analogous to the case of the integer Sobolev spaces the fractional Sobolev spaces are contained in the Lebesgue spaces in the following way (see e.g. [DD12, Corollary 4.53] or [Ada75, Theorem 7.57] for a version for more general subsets Ω of $\mathbb{R}^d$).

Theorem 2.20 (fractional Sobolev embedding theorem). *Let $\alpha \in (0,2)$, $p \in (1,\infty)$ with $\alpha p < 2d$ and $q \in [p, \frac{dp}{d-\frac{\alpha}{2}p}]$. Assume $\Omega \subset \mathbb{R}^d$ is a Lipschitz open set. There exists a constant $c > 0$ such that for every $f \in W^{\frac{\alpha}{2},p}(\Omega)$*

$$\|f\|_{L^q(\Omega)} \leq c\|f\|_{W^{\frac{\alpha}{2},p}(\Omega)}.$$

In our thesis we often use a special version of the fractional Sobolev embedding theorem for the case when $\Omega = B_r(x_0)$ is a ball. This embedding can be obtained from Theorem 2.20 in the following way. We apply the above theorem to $\Omega = B_1$ and $q = p = 2$. Then we use a scaling argument (see also Lemma 11.6), which yields the embedding result for $H^{\frac{\alpha}{2}}(B_r(x_0))$ that we state below.

Corollary 2.21 (fractional Sobolev inequality). *Let $d \geq 2$ be a natural number, $R > 0$, $\alpha \in (0,2)$ and $x_0 \in \mathbb{R}^d$. There exists a constant $c_S > 0$ such that for all $r \in (0,R)$ and $f \in H^{\frac{\alpha}{2}}(B_r(x_0))$*

$$\left(\int_{B_r(x_0)} |f(x)|^{\frac{2d}{d-\alpha}} \,\mathrm{d}x \right)^{\frac{d-\alpha}{d}} \leq c_S \int_{B_r(x_0)} \int_{B_r(x_0)} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} \,\mathrm{d}x \,\mathrm{d}y + c_S r^{-\alpha} \int_{B_r(x_0)} |f(x)|^2 \,\mathrm{d}x.$$

If $\alpha \in [\alpha_0, 2)$ for some $\alpha_0 \in (0,2)$, then c_S can be chosen to depend on α_0 but not on α.

Dense subsets

The following two theorems give examples for subspaces of $W^{\frac{\alpha}{2},p}$ that are dense in $W^{\frac{\alpha}{2},p}$. In the literature they can be found in [DD12] as Proposition 4.27 and Proposition 4.52.

Proposition 2.22. *The space $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{\frac{\alpha}{2},p}(\mathbb{R}^d)$.*

Let Ω be a proper subset of $\mathbb{R}^d$. We say that Ω admits an $(\frac{\alpha}{2},p)$-extension if there exists a continuous linear operator Ext (the extension operator) that sends $f \in W^{\frac{\alpha}{2},p}(\Omega)$ to $\mathrm{Ext}(f) \in W^{\frac{\alpha}{2},p}(\mathbb{R}^d)$ such that for $x \in \Omega$ one has $\mathrm{Ext}(f)(x) = f(x)$. It is well known that every Lipschitz open set Ω admits an $(\frac{\alpha}{2},p)$-extension, cf. [DD12, Proposition 4.42]. If Ω admits an $(\frac{\alpha}{2},p)$-extension, then one can use the proposition above and the extension operator in order to prove the following proposition.

Proposition 2.23. *Let Ω be an open set that admits an $(\frac{\alpha}{2},p)$-extension. Then $C_c^\infty(\overline{\Omega})$, the space of restrictions to Ω of functions in $C_c^\infty(\mathbb{R}^d)$, is dense in $W^{\frac{\alpha}{2},p}(\Omega)$.*

3. Generalized function spaces involving a symmetric kernel

In this work a measurable function $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ is called *integral kernel* or just *kernel*. We consider only kernels that satisfy the symmetry assumption $k(x,y) = k(y,x)$ for almost all $x, y \in \mathbb{R}^d$. These kernels are called *symmetric kernels*. Sometimes we will omit the adjective *symmetric*. An integral kernel plays the role of a density function with respect to the Lebesgue measure $\lambda^d \times \lambda^d$. Following the article [FKV15] we use integral kernels to define new function spaces that are custom-made for our purposes.

3.1. Generalization of $H^{\frac{\alpha}{2}}$: The space H^k

The preceding function spaces can be considered as a generalization of $H^{\frac{\alpha}{2}}$.

Definition 3.1. *Let $\Omega \subset \mathbb{R}^d$ open and $k : \mathbb{R}^d \to \mathbb{R}^d \to [0,\infty]$ be a symmetric kernel. The set*

$$H^k(\Omega) = \left\{ f \in L^2(\Omega) \,\middle|\, \|f\|_{\dot{H}^k(\Omega)} < \infty \right\},$$

where

$$\|f\|_{\dot{H}^k(\Omega)} = \left(\int_\Omega \int_\Omega (f(x) - f(y))^2 \, k(x,y) \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}},$$

is a linear space. A norm on $H^k(\Omega)$ is given by

$$\|f\|_{H^k(\Omega)} = \left(\|f\|^2_{L^2(\Omega)} + \|f\|^2_{\dot{H}^k(\Omega)} \right)^{\frac{1}{2}}, \quad f \in H^k(\Omega).$$

Remark. For $k(x,y) = |x-y|^{-d-\alpha}$ we have $H^k = H^{\frac{\alpha}{2}}$.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^d$ be open and $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ be a symmetric kernel. The mapping*

$$(f,g)_{H^k(\Omega)} = (f,g)_{L^2(\Omega)} + \int_\Omega \int_\Omega (f(x) - f(y))(g(x) - g(y)) k(x,y) \, \mathrm{d}x \, \mathrm{d}y,$$

for $f, g \in H^k(\Omega)$, defines a scalar product on $H^k(\Omega)$. It induces the norm $\|\cdot\|_{H^k(\Omega)}$. The normed space $H^k(\Omega)$ is a Hilbert space.

A proof of this result can be obtained similar to the proof of [FKV15, Lemma 2.3], see also [DRV17, Proposition 3.1].

3.2. The spaces $V^k(\Omega|\mathbb{R}^d)$ and $H^k_\Omega(\mathbb{R}^d)$

In this section we provide the definition of the V-spaces corresponding to a given kernel. These spaces will be important later when we define solutions of corresponding nonlocal elliptic equations, see Section 4.2.

Definition 3.3. *Let $\Omega \subset \mathbb{R}^d$ open and let $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ be a symmetric integral kernel. The set*

$$V^k(\Omega|\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{R} \text{ measurable} \,\middle|\, f|_\Omega \in L^2(\Omega), [f,f]_{V^k(\Omega|\mathbb{R}^d)} < \infty \right\},$$

where

$$[f,g]_{V^k(\Omega|\mathbb{R}^d)} = \int_\Omega \int_{\mathbb{R}^d} (f(x) - f(y))(g(x) - g(y)) k(x,y) \,\mathrm{d}x \,\mathrm{d}y, \tag{3.1}$$

is a linear space. A norm on $V^k(\Omega|\mathbb{R}^d)$ is given by

$$\|f\|_{V^k(\Omega|\mathbb{R}^d)} = \left(\|f\|_{L^2(\Omega)} + [f,f]_{V^k(\Omega|\mathbb{R}^d)} \right)^{\frac{1}{2}}.$$

Furthermore, define

$$H^k_\Omega(\mathbb{R}^d) = \left\{ f \in H^k(\mathbb{R}^d) \,\middle|\, f \equiv 0 \text{ almost everywhere in } \mathbb{R}^d \setminus \Omega \right\}.$$

We remark that the normed space $\left(H^k_\Omega(\mathbb{R}^d), \|\cdot\|_{H^k(\mathbb{R}^d)} \right)$ is a separable Hilbert space, see again [FKV15, Lemma 2.3].

From the definition of $V^k(\Omega|\mathbb{R}^d)$ and $H^k_\Omega(\mathbb{R}^d)$ we see that these spaces are not of local type anymore, that is, we have some regularity of the elements of these spaces across the boundary of Ω. This will be important later when we consider nonlocal operators and corresponding weak solutions.

Note that by the symmetry of the kernel and the Theorem of Fubini, $f \in H^k_\Omega(\mathbb{R}^d)$ implies

$$\begin{aligned} \|f\|^2_{H^k(\mathbb{R}^d)} = \|f\|^2_{L^2(\Omega)} &+ \int_\Omega \int_\Omega (f(x) - f(y))^2 k(x,y) \,\mathrm{d}x \,\mathrm{d}y \\ &+ 2 \int_{\mathbb{R}^d \setminus \Omega} \int_\Omega (f(x) - f(y))^2 k(x,y) \,\mathrm{d}x \,\mathrm{d}y, \end{aligned}$$

that is, the double integral can be splitted into a local and a nonlocal part.

4. Elliptic partial differential equations

In parts of our work we are concerned with partial differential equations of the form

$$\begin{cases} -\mathcal{L}u = f & \text{in } \Omega, \\ \quad u = g & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

where f and g are given functions, u is an unknown function (*the solution*) and $\mathcal{L}$ is a linear operator of second order local type in divergence form or of nonlocal type corresponding to some integral kernel. We introduce these operators in the following two sections.

4.1. (Local) Elliptic operators in divergence form

Let $\Omega \subset \mathbb{R}^d$ an open bounded set. For $i, j \in \{1, ..., d\}$ let $a_{ij} : \Omega \to \mathbb{R}$ be measurable functions. A second order operator $\mathcal{L}$ shall be called *operator in divergence form* if

$$\mathcal{L}u(x) = \sum_{1 \le i,j \le d} \partial_j \left(a_{ij}(x) \partial_i u(x) \right), \quad x \in \Omega.$$

We call $\mathcal{L}$ *elliptic operator* if there exists $\Lambda \ge 1$ such that

$$\Lambda^{-1} |\xi|^2 \le \sum_{1 \le i,j \le d} a_{ij}(x) \xi_i \xi_j \le \Lambda |\xi|^2 \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^d.$$

Note that the above line implies $a_{ij} \in L^\infty(\Omega)$ for all $i, j \in \{1, ..., d\}$.

Definition 4.1 (weak subsolution, weak supersolution, weak solution)**.** *Let $f \in L^2_{\text{loc}}(\Omega)$ and $g \in H^1(\Omega)$. A function $u \in H^1(\Omega)$ is called a* weak subsolution *(*weak supersolution*) of* (4.1) *if $u - g \in H^1_0(\Omega)$ and for every nonnegative (nonpositive) $\psi \in H^1_0(\Omega)$*

$$\int_\Omega \sum_{1 \le i,j \le d} a_{ij} \, \partial_i u \, \partial_j \psi \, \mathrm{d}x \le \int_\Omega f \psi \, \mathrm{d}x. \tag{4.2}$$

A function u that is both a weak subsolution and a weak supersolution of (4.1) *is called a* weak solution *of* (4.1)*.*

We remark that, by a density argument, the space of test functions for which (4.2) shall be true could be changed from $H^1_0(\Omega)$ to $C^\infty_c(\Omega)$, the space of smooth functions with compact support in Ω.

Using the Lemma of Lax-Milgram respectively the Riesz representation theorem (see [Alt16, 6.2 respectively 6.1]) one can show that (4.1) has a unique weak solution.

4.2. Nonlocal operators corresponding to an integral kernel

In this section we consider the nonlocal analogue to (4.1) and provide the concept of weak solutions. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain.

Let $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ be a symmetric kernel and $\mathcal{L}$ be a nonlocal operator defined as follows

$$\mathcal{L}_k u(x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} (u(y) - u(x)) k(x, y) \, \mathrm{d}y, \quad x \in \Omega.$$

We now give a definition of weak subsolutions for (4.1) if we replace the local elliptic operator by the nonlocal operator above. The definition requires the following quadratic form. We define

$$\mathcal{E}^k(u, v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))(v(x) - v(y)) k(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

Note that the above quantity is well defined for $u \in V^k(\Omega|\mathbb{R}^d)$ and $v \in H^k_\Omega(\mathbb{R}^d)$ or vice versa. Indeed, by the Hölder inequality,

$$\begin{aligned}
\mathcal{E}^k(u, v) = &\int_\Omega \int_\Omega (u(x) - u(y))(v(x) - v(y)) k(x, y) \, \mathrm{d}x \, \mathrm{d}y \\
&+ 2 \int_{\mathbb{R}^d \setminus \Omega} \int_\Omega (u(x) - u(y)) v(x) k(x, y) \, \mathrm{d}x \, \mathrm{d}y \\
\leq &\left(\int_\Omega \int_\Omega (u(x) - u(y))^2 k(x, y) \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}} \left(\int_\Omega \int_\Omega (v(x) - v(y))^2 k(x, y) \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}} \\
&+ 2 \left(\int_{\mathbb{R}^d \setminus \Omega} \int_\Omega (u(x) - u(y))^2 k(x, y) \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}} \times \\
&\quad \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (v(x) - v(y))^2 k(x, y) \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}}
\end{aligned}$$

and due to the properties of $V^k(\Omega|\mathbb{R}^d)$ and $H^k_\Omega(\mathbb{R}^d)$ all the appearing terms are finite.

Definition 4.2 (weak subsolution, weak supersolution, weak solution)**.** *Let $f \in L^q(\Omega)$ for some $q \geq 2$ and $g \in V^k(\Omega|\mathbb{R}^d)$. A function $u \in V(\Omega|\mathbb{R}^d)$ is called a* weak subsolution *(*weak supersolution*) of* (4.1) *if $u - g \in H^k_\Omega(\mathbb{R}^d)$ and for all nonnegative (nonpositive) $\psi \in H^k_\Omega(\mathbb{R}^d)$*

$$\mathcal{E}^k(u, \psi) \leq (f, \psi)_{L^2}.$$

A function that is both a weak subsolution and a weak supersolution is called a weak solution *of* (4.1).

Remark. In order to have that every classical subsolution (supersolution) is also a weak subsolution (supersolution) one would need to replace $\mathcal{E}^k$ with $\frac{1}{2}\mathcal{E}^k$, but we will ignore the factor $\frac{1}{2}$ in this work.

The existence of weak solutions to (4.1) with an operator $\mathcal{L}$ of nonlocal type as mentioned above has been studied in [FKV15]. Under certain conditions on the kernel k one can prove their existence with the help of the Lax-Milgram lemma, see [FKV15, Section 3]. Let us remark that all the conditions on the kernel for existence and uniqueness of weak solutions are satisfied in the parts of our thesis that deal with weak solutions.

5. Dirichlet forms

One application of the theory developed in Part II of this thesis comes from the theory of Dirichlet forms and its connection to stochastic processes. This short chapter provides the necessary definitions and theorems. As a reference we use the book [FOT94].

Let $(X, \mathcal{B}(X), \mu)$ a σ-finite measure space. Consider a bilinear form $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$, where $\mathcal{D}(\mathcal{E})$ is a dense subspace of $L^2(X, \mu)$, called the *domain* of $\mathcal{E}$.

In what follows we also use the form $\mathcal{E}_1$ that is defined as

$$\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + (u, v)_{L^2(X,\mu)}, \quad u, v \in \mathcal{D}(\mathcal{E}).$$

Definition 5.1. *A bilinear form $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$ with $\mathcal{D}(\mathcal{E}) \subset L^2(X, \mu)$ is called a* (symmetric) Dirichlet form *if the following properties are satisfied.*

1. *The form is symmetric in the sense that $\mathcal{E}(u, v) = \mathcal{E}(v, u)$ for all $u, v \in \mathcal{D}(\mathcal{E})$.*
2. *$\mathcal{E}(u, u) \geq 0$ for every $u \in \mathcal{D}(\mathcal{E})$.*
3. *The form is closed, that is, if $(u_n) \subset \mathcal{D}(\mathcal{E})$ with $\mathcal{E}_1(u_n - u_m, u_n - u_m) \to 0$ for $n, m \to \infty$, then there is $u \in \mathcal{D}(\mathcal{E})$ with $\mathcal{E}_1(u_n - u, u_n - u) \to 0$ for $n \to \infty$. In other words: The subspace $\mathcal{D}(\mathcal{E})$ equipped with the inner product $\mathcal{E}_1$ is a real Hilbert space.*
4. *The unit contraction operates on $\mathcal{E}$, that is, if $u \in \mathcal{D}(\mathcal{E})$ and $v = (0 \vee u) \wedge 1$, then $v \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(v, v) \leq \mathcal{E}(u, u)$.*

If in addition $\mathcal{D}(\mathcal{E}) \cap C_c(X)$ is dense in $\mathcal{D}(\mathcal{E})$ with respect to the norm induced by $\mathcal{E}_1$ and dense in $C_c(X)$ with respect to the uniform norm, then $\mathcal{E}$ is called a regular Dirichlet form.

From the process to the Dirichlet form and back

It is the idea of this short section to describe to the reader some basics of the relation between Dirichlet forms and stochastic processes via semigroup theory. The following paragraphs may be considered as a short reminder for everyone that does not usually work with these objects. In our thesis we do not really use the objects that appear in the remainder of this section. Therefore, it would not be efficient to include all the precise definitions here. We try to be as exact as possible without losing ourselves in details. Sometimes we only give paraphrases instead of precise definitions. Then we refer to the appropriate part in [FOT94].

Formally, a stochastic process M on a measurable space $(S, \mathcal{B}(S))$ with time parameter $\mathcal{T} \subset [0, \infty]$ is given as a quadruple $M = (\Omega, \mathcal{M}, (X_t)_{t \in \mathcal{T}}, \mathbb{P})$, where $X_t : \Omega \to S$ is $(\mathcal{M}, \mathcal{B}(S))$-measurable for every $t \in \mathcal{T}$ and $(\Omega, \mathcal{M}, \mathbb{P})$ is a probability space. We now add to S an isolated single point ∂, which plays the role of the *cemetery*, and replace the probability measure by a family of probability measures $(\mathbb{P}_x)_{x \in S \cup \partial}$. Let us assume that the resulting quadruple $(\Omega, \mathcal{M}, (X_t)_{t \in [0,\infty]}, \mathbb{P}_x)$ is a stochastic process for any $x \in S \cup \partial$ with state space $(S \cup \partial, \mathcal{B}(S)_\partial)$, where $\mathcal{B}(S)_\partial$ is the corresponding Borel σ-algebra. Furthermore, let us assume that the family of probability measures satisfies some measurability condition as well as the following Markov property: There is an admissible filtration $\{\mathcal{M}_t\}_{t \geq 0}$ such that for every $x \in S, t, s \geq 0$ and $E \in \mathcal{B}(S)$ we have $\mathbb{P}_x$-almost surely

$$\mathbb{P}_x(X_{t+s} \in E \,|\, \mathcal{M}_t) = \mathbb{P}_{X_t}(X_s \in E).$$

If we additionally assume that $\mathbb{P}_\partial(X_t = \partial) = 1$ for each $t \geq 0$, then

$$M = (\Omega, \mathcal{M}, (X_t)_{t \in [0,\infty]}, (\mathbb{P}_x)_{x \in S \cup \partial})$$

is called a *Markov process* on $(S, \mathcal{B}(S))$ with time parameter $[0, \infty]$, cf. [FOT94, p. 311] for a precise definition. One object that plays a crucial role in the relation between stochastic processes and Dirichlet forms is the transition function. Let us assume that M is a Markov process. The *transition function* $p_t : [0, \infty] \times \mathcal{B}(S) \to [0, \infty]$ of the process M is defined as

$$p_t(x, E) = \mathbb{P}_x(X_t \in E), \quad x \in S, t \geq 0, E \in \mathcal{B}(S).$$

The Markov property of M implies that

$$p_t p_s u = p_{t+s} u \quad \text{ for all } t, s > 0$$

and all bounded $\mathcal{B}(S)$-measurable functions $u : S \to \mathbb{R}$. Here

$$(p_t u)(x) = \int_S u(y) p_t(x, \mathrm{d}y), \quad x \in S.$$

From now on let $(X, \mathcal{B}(X), \mu)$ be a σ-finite measure space.

We call a Markov process on $(X, \mathcal{B}(X))$ *μ-symmetric* if its transition function p_t is symmetric, that is, p_t satisfies

$$\int_X u(x)(p_t v)(x)\mu(\mathrm{d}x) = \int_X (p_t u)(x) v(x) \mu(\mathrm{d}x)$$

for all nonnegative measurable functions u and v. Such a symmetric transition function enjoys

$$\begin{aligned}((p_t u)(x))^2 &= \left(\int_X u(y) p_t(x, \mathrm{d}y)\right)^2 \\ &\le \int_X p_t(x, \mathrm{d}y) \cdot \int_X u^2(y) p_t(x, \mathrm{d}y) \\ &= (p_t u^2)(x)\end{aligned}$$

for every $x \in X, t > 0$. Therefore, for every bounded and $\mathcal{B}(X)$-measurable function $u : X \to \mathbb{R}$ with $u \in L^2(X, \mu)$ we have

$$\int_X ((p_t u)(x))^2 \, \mu(\mathrm{d}x) \le \int_X u^2(x) \mu(\mathrm{d}x),$$

where we also used the symmetry of p_t.

The above inequality means that p_t can be extended uniquely to a symmetric contractive operator T_t on $L^2(X, \mu)$. We deduce in particular that every T_t is *Markovian*, that is, it holds $0 \le T_t u \le 1$ whenever $u \in L^2(X, \mu)$, $0 \le u \le 1$ μ-almost everywhere. The family $(T_t)_{t>0}$ of operators on $L^2(X, \mu)$ is a semigroup on $L^2(X, \mu)$. That is, each T_t is a symmetric operator with domain $\mathcal{D}(T_t) = L^2(X, \mu)$, the semigroup property $T_t T_s = T_{t+s}$ is satisfied for $t, s > 0$ and the contraction property $(T_t u, T_t u)_{L^2(X,\mu)} \le (u, u)_{L^2(X,\mu)}$ holds for $t > 0, u \in L^2(X, \mu)$. The associated Markovian semigroup $(T_t)_{t>0}$ to the family of transition functions $(p_t)_{t>0}$ is called the *transition semigroup*.

Let us assume that the transition semigroup $(T_t)_{t>0}$ is strongly continuous, that is, for $u \in L^2(X, \mu)$ we have $(T_t u - u, T_t u - u)_{L^2(X,\mu)} \to 0$ for $t \to 0$. Then the *generator* A of the semigroup exists. The generator is defined by

$$A = \lim_{t \to 0} \frac{T_t u - u}{t}, \quad \mathcal{D}(A) = \{u \in L^2(X, \mu) \,|\, Au \text{ exists as a strong limit}\}.$$

The generator of a strongly continuous semigroup is a nonpositive definite self-adjoint operator. The catch is now that there exists a one to one correspondence between the family of nonpositive definite self-adjoint operators and the family of closed symmetric forms on $L^2(X, \mu)$, which can be characterized by

$$\mathcal{D}(A) \subset \mathcal{D}(\mathcal{E}), \quad \mathcal{E}(u, v) = (-Au, v)_{L^2(X,\mu)}, \quad u \in \mathcal{D}(A), v \in \mathcal{D}(\mathcal{E}),$$

cf. [FOT94, Corollary 1.3.1]. According to [FOT94, Theorem 1.4.1] this associated form $\mathcal{E}$ is a Dirichlet form.

From now on let X be a locally compact separable measure space and μ a positive Radon measure on X such that $\operatorname{supp}(\mu) = X$.

A stochastic process M is a *Hunt process* if it is a strong Markov process that is quasi-left continuous with respect to the minimum completed admissible filtration, see [FOT94,

Appendix A] for a precise definition. If M is a Hunt process, then, by [FOT94, Lemma 1.4.3 (i)], the associated semigroup is strongly continuous. Hence, each Hunt process determines a Dirichlet form in the sense of the above paragraph. The reverse statement is also true under the additional assumption that the given Dirichlet form is regular.

Theorem 5.2 ([FOT94, Theorem 7.2.1])**.** *Given a regular Dirichlet form $\mathcal{E}$ on $L^2(X,\mu)$ there exists a μ-symmetric Hunt process M on $(X,\mathcal{B}(X))$ whose Dirichlet form is the given one $\mathcal{E}$.*

Part II.

Coercivity in fractional Sobolev spaces

Detailed Outline of Part II

This part deals with the functional

$$f \mapsto \int_B \int_B (f(x) - f(y))^2 k(x,y) \, \mathrm{d}x \, \mathrm{d}y, \quad f \in L^2(B) \tag{5.1}$$

where $B \subset \mathbb{R}^d$ is a ball and $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ is a symmetric kernel that satisfies $k(x,y) \geq \Lambda |x-y|^{-d-\alpha}$ for some configuration Γ, some constant $\Lambda > 0$ and almost all $x, y \in \mathbb{R}^d, x \neq y$. In Chapter 6 we provide all the necessary definitions. In particular we explain what we mean by the term configuration. As already mentioned in the introduction, our coercivity result requires some assumptions on the configuration regarding measurability, compare (M) in Chapter 1. Chapter 6 contains the reason for this assumption.

We present a result on discrete quadratic forms in Chapter 7. The proof of this result is challenging and involves an induction and a renormalization argument.

Our main result, the coercivity of the bilinear form associated to (5.1) in $\dot{H}^{\frac{\alpha}{2}}(B)$ on every ball $B \subset \mathbb{R}^d$, is proved in Chapter 8. In this chapter we present a general scheme of how to approximate functionals of the type (5.1) by discrete quadratic forms. This result is interesting in itself. As a consequence we can apply our discrete result in order to obtain the mentioned coercivity. A technicality in the proof is that we need to use a covering argument. This is provided in Appendix A.

Applications of the main result are discussed in Chapter 9. We present applications with regard to function spaces, as well as stochastic processes and regularity theory of weak solution to the elliptic PDE with the underlying operator (1.1).

Comment on the notation used in this part

Let us recall some of the notation that we use in this part. For a complete overview we refer to the section about notation after the introduction.

By $|x|$ we denote the Euclidean norm on $\mathbb{R}^d$. With $|x|_\infty$ we denote the supremum $|x|_\infty = \sup_{1 \leq i \leq d} |x_i|$. The notation $B_r(x_0)$ refers to a ball with radius $r > 0$ and center x_0 in the Euclidean metric, precisely $B_r(x_0) = \{x \in \mathbb{R}^d \,|\, |x - x_0| < r\}$. In case $x_0 = 0$, we write $B_r = B_r(0)$.

We write $a \asymp b$ for two quantities a, b if there exists a universal constant $c \geq 1$ so that $c^{-1} a \leq b \leq ca$. Then a and b are called *comparable* and we refer to c as *comparability constant*.

We write $\mathcal{S}^{d-1}$ for the $d-1$-sphere, that is, $\mathcal{S}^{d-1} = \{x \in \mathbb{R}^d \,|\, |x| = 1\}$.

6. Setting and Preliminaries

In this chapter we define and explain the main objects of Part II of this thesis: cones and configurations. First and foremost, this chapter is intended to serve as an introduction to the topic of configurations. Moreover, the chapter includes useful properties that are needed several times in the remainder of Part II. One of these properties is the fact that every ϑ-bounded configuration induces a configuration consisting of only finitely many cone types. This is the key idea on which our coercivity result relies. The statement can be found at the end of Section 6.1, after the main definitions. In Section 6.2 we explain the notion of *ϑ-admissible configurations*. In the last Section 6.3 the reader learns about *favored indices*.

6.1. Cones, ϑ-bounded configurations and reference cones

Definition 6.1 ((double) cone, family of (double) cones, shifted (double) cone, double half-cone)**.** *Given $v \in \mathcal{S}^{d-1}$ and $\vartheta \in (0, \frac{\pi}{2}]$ a* cone *is defined by*

$$\widetilde{V} = \widetilde{V}(v, \vartheta) = \Big\{ h \in \mathbb{R}^d \;\Big|\; h \neq 0, \frac{\langle v, h \rangle}{|h|} > \cos(\vartheta) \Big\}.$$

Let $\widetilde{\mathcal{V}}$ denote the family of all cones. The corresponding double cone *is denoted by V, that is,*

$$V = V(v, \vartheta) = \widetilde{V} \cup (-\widetilde{V}).$$

The set $\mathcal{V}$ of all double cones is simply the manifold $(0, \frac{\pi}{2}] \times \mathbb{P}^{d-1}_{\mathbb{R}}$, where $\mathbb{P}^{d-1}_{\mathbb{R}}$ is the real projective space of dimension $d-1$. For $x \in \mathbb{R}^d$, a shifted cone *shall be defined by $\widetilde{V}[x] = \widetilde{V} + x$ and a* shifted double cone *shall be defined by $V[x] = V + x$.*
For a given cone $\widetilde{V} = \widetilde{V}(v, \vartheta)$ and $r > 0$ we define

$$\widetilde{V}_r = \widetilde{V}_r(v, \vartheta) = \{ y \in \widetilde{V} \mid \overline{B_r(y)} \subset \widetilde{V} \}.$$

For a double cone V we define the set V_r analogously,

$$V_r = V_r(v, \vartheta) = \widetilde{V}_r \cup \left(-\widetilde{V}_r \right),$$

and call it a double half-cone.

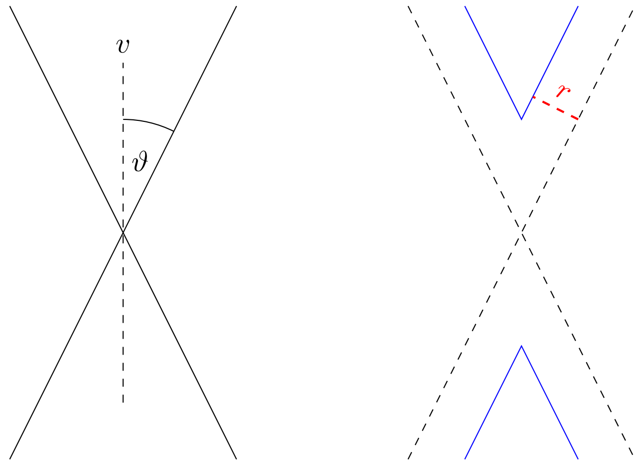

Figure 6.1.: Example of a cone $V(v, \vartheta)$ and a double half-cone $V_r(v, \vartheta)$ for $d = 2$

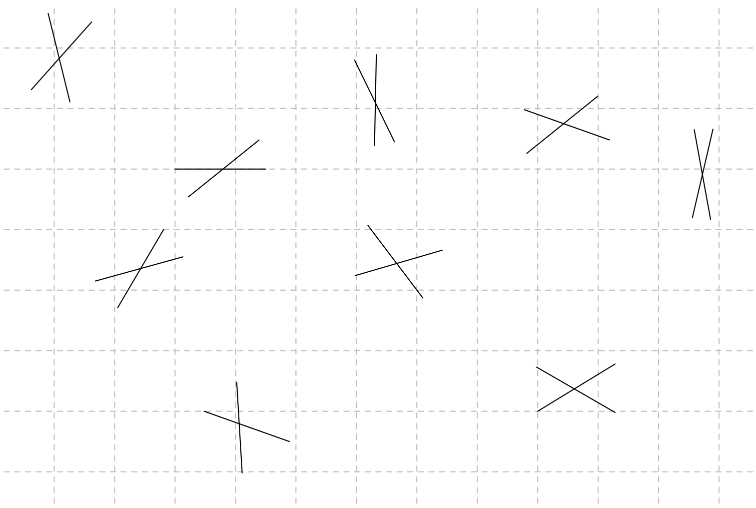

Figure 6.2.: A possible configuration Γ

We are now in the position to define a family of double cones $\Gamma(\mathbb{R}^d)$, which we shall call a *configuration.*

Definition 6.2 ((ϑ-bounded) configuration)**.** *A mapping $\Gamma : \mathbb{R}^d \to \mathcal{V}$ is called a* configuration*. If Γ is a configuration with the property that the infimum ϑ over all apex angles of cones in $\Gamma(\mathbb{R}^d)$ is positive, then Γ is called* ϑ-bounded*. For $x \in \mathbb{R}^d$ and Γ a configuration, we define $V^\Gamma[x] = x + \Gamma(x)$ and analogously for $r > 0$,*

$$V_r^\Gamma[x] = \{y \in V^\Gamma[x] \mid \overline{B_r(y)} \subset V^\Gamma[x]\}.$$

Throughout the rest of this part we sometimes write that two cones have the same type. The precise meaning of this statement is given below.

Definition 6.3 (cone type, points of the same type)**.** *Let* Γ *be a configuration. A point* $x \in \mathbb{R}^d$ *is* of cone type V *if* $V = \Gamma(x)$. *Two points* $x, y \in \mathbb{R}^d$ have the same type *if* $\Gamma(x) = \Gamma(y)$.

The boundedness property of ϑ is crucial in our work. This assumption guarantees that we can pass over from the possible uncountable family of cones $\Gamma(\mathbb{R}^d)$ generated by a ϑ-bounded configuration Γ to a finite family of cones as described in the following lemma. In order to obtain this finite family of cones we just have to cover up the sphere and use a compactness argument.

Lemma 6.4. *Let* Γ *be a* ϑ*-bounded configuration. There are numbers* $L \in \mathbb{N}$ *and* $\theta \in (0, \frac{\pi}{2}]$*, and double cones* $V^1, ..., V^L$ *centered at* 0 *with apex angle* θ *and symmetry axis* $v^1, \dots, v^L \in \mathcal{S}^{d-1}$ *such that*

$$\text{for every } x \in \mathbb{R}^d \text{ there is an index } m \in \{1, \dots, L\} \text{ with } V^m \subset \Gamma(x).$$

The constants L *and* θ *depend on the dimension* d *and* ϑ *but not on* Γ *itself.*

Proof. Obviously

$$\mathcal{S}^{d-1} \subset \bigcup_{v \in \mathcal{S}^{d-1}} V\left(v, \frac{\vartheta}{3}\right).$$

Since $\mathcal{S}^{d-1}$ is compact and the right-hand side is an open cover of $\mathcal{S}^{d-1}$, one can choose finitely many symmetry axis $v^1, ..., v^L \in \mathcal{S}^{d-1}$ such that

$$\mathcal{S}^{d-1} \subset \bigcup_{m=1}^{L} V\left(v^m, \frac{\vartheta}{3}\right).$$

Define $V^m = V\left(v^m, \frac{\vartheta}{3}\right)$ for $m = 1, ..., L$. Now the claim follows with $\theta = \vartheta/3$. ∎

Remark. Note that in the above proof any choice $\theta \in (0, \frac{2}{3}\vartheta)$ would have been possible.

In the following we write $V^m[x]$ instead of $V^m + x$.

Definition 6.5 (reference cones)**.** *The set* $\{V^m | 1 \leq i \leq L\}$ *shall be called a* family of reference cones *associated to* Γ. *Each element is called a* reference cone. *Analogous to Definition 6.1 set*

$$V_r^m = \left\{u \in V^m \,\middle|\, \overline{B_r} \subset V^m\right\}, \quad V_r^m[x] \;=\; V_r^m + x.$$

Obviously a family of reference cones is not unique. There can be uncountably many families of reference cones associated to a ϑ-bounded configuration.

With the help of Lemma 6.4 we can define a new configuration that has useful properties. The following corollary is the key tool for our reasoning in Chapter 7.

Corollary 6.6. *Let Γ be a ϑ-bounded configuration. Then there exists another configuration $\widetilde{\Gamma}$ that fulfills $\#\widetilde{\Gamma}(\mathbb{R}^d) < \infty$ and for every $x \in \mathbb{R}^d$*

$$\widetilde{\Gamma}(x) \subset \Gamma(x).$$

The infimum of apex angles of cones in $\widetilde{\Gamma}(\mathbb{R}^d)$ is ϑ.

Proof. Let $V^1, ..., V^L$ be a family of reference cones corresponding to Γ. Define sets

$$\begin{aligned} M_1 &= \{x \in \mathbb{R}^d \mid V^1 \subset \Gamma(x)\} \\ M_2 &= \{x \in \mathbb{R}^d \mid V^2 \subset \Gamma(x)\} \setminus M_1 \\ &\vdots \\ M_L &= \{x \in \mathbb{R}^d \mid V^L \subset \Gamma(x)\} \setminus M_{L-1}. \end{aligned}$$

Now it holds

$$\mathbb{R}^d = \bigcup_{1 \le i \le L} M_i$$

and this union is disjoint. Define $\widetilde{\Gamma} : \mathbb{R}^d \to \mathcal{V}, x \mapsto V^i$ for $x \in M_i$ and arrive at the assertion. ■

We want to achieve that all the above sets $M_1, ..., M_L$ are Lebesgue measurable sets. In general, this does not have to be the case. It depends heavily on the underlying configuration Γ. Therefore, it is necessary to impose another condition on Γ.

6.2. Admissible configurations

In this part we give a sufficient condition on Γ so that the problem of measurability explained above is solved. Recall that we denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel σ-algebra and by $\mathcal{L}(\mathbb{R}^d)$ the Lebesgue σ-algebra. Our aim is to prove the next proposition.

Proposition 6.7. *Let Γ be a configuration. If the set*

$$\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \,|\, y - x \in \Gamma(x)\}$$

is a Borel set in $\mathbb{R}^d \times \mathbb{R}^d$, then for each double cone $V \in \mathcal{V}$ the set $\{x \in \mathbb{R}^d \,|\, V \subset \Gamma(x)\}$ is an element of $\mathcal{L}(\mathbb{R}^d)$.

The theorem below is the basis for the proof of the above proposition. This theorem can be traced back to an article by Debreu [Deb67, (4.4)]. An easier accessible, however more general, version for Suslin spaces[1] can be found in an article by Himmelberg [Him75, Theorem 3.4]. We also refer to [Aum69, Projection Theorem]. However, in the last reference Aumann omits the proof.

[1] A Suslin space is the image of a Polish space under a continuous mapping.

Theorem 6.8. *Let T be a complete measure space with σ-algebra $\mathcal{A}$. Let X be a Polish space[2] with Borel algebra $\mathcal{B}$. If $F \subset T \times X$ is $\mathcal{A} \times \mathcal{B}$-measurable and $B \in \mathcal{B}$, then*

$$F_B = \{t \in T \,|\, (t, b) \in F \text{ for some } b \in B\} \in \mathcal{A}.$$

For a proof we advise the reader to check one of the above mentioned references.

Corollary 6.9. *Let T and X be as in the previous Theorem. If $F \subset T \times X$ is $\mathcal{A} \times \mathcal{B}$-measurable and $B \in \mathcal{B}$, then $F(B) = \{t \in T \,|\, (t, b) \in F \text{ for each } b \in B\} \in \mathcal{A}$.*

Proof. We have

$$\begin{aligned} F(B) &= \{t \in T \,|\, \nexists b \in B : (t, b) \in F^C\} \\ &= \{t \in T \,|\, \exists b \in B : (t, b) \in F^C\}^C \\ &= \left((F^C)_B\right)^C . \end{aligned}$$

We know F^C is $\mathcal{A} \times \mathcal{B}$-measurable since this holds for F. According to Theorem 6.8 this implies measurability of $(F^C)_B$. Then the complement of this set is also measurable. ∎

Proof of Proposition 6.7. Let $G = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \,|\, y - x \in \Gamma(x)\}$ and $F = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \,|\, y \in \Gamma(x)\}$. Then we see that $F = \{(x, y - x) \,|\, (x, y) \in G\}$, that is, F is $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d)$-measurable if and only if G is $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d)$-measurable.

Let us now assume that F is $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d)$-measurable. Choose $T = \mathbb{R}^d$, $\mathcal{A} = \mathcal{L}(\mathbb{R}^d)$ as the Lebesgue σ-algebra and $X = \mathbb{R}^d$, $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$ as the Borel algebra. Then T is a complete measure space (together with the Lebesgue measure λ^d) and X is a polish space. Let $V \in \mathcal{V}$ be any double cone. We have

$$\begin{aligned} F(V) &= \{x \in \mathbb{R}^d \,|\, \forall y \in V : (x, y) \in F\} \\ &= \{x \in \mathbb{R}^d \,|\, \forall y \in V : y \in \Gamma(x)\} \\ &= \{x \in \mathbb{R}^d \,|\, V \subset \Gamma(x)\}. \end{aligned}$$

Since F is $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d)$-measurable, we know in particular that F is $\mathcal{L}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d)$-measurable. By Corollary 6.9 we deduce $F(V) \in \mathcal{L}(\mathbb{R}^d)$ and the proof is finished. ∎

Equipped with this background knowledge we can now define admissible configurations.

Definition 6.10 (ϑ-admissible configuration)**.** *If Γ is a ϑ-bounded configuration and*

$$\Big\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \,\Big|\, y - x \in \Gamma(x)\Big\} \text{ is a Borel set in } \mathbb{R}^d \times \mathbb{R}^d, \tag{M}$$

then Γ is called ϑ-admissible.

[2] A Polish space is a separable completely metrizable topological space.

Remark. Of course, there might be weaker sufficient conditions than (M) so that the assertion of Proposition 6.7 still holds true. In our approach (M) is somehow a natural assumption because it is equivalent to the measurability of the function

$$v : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \quad v(x,y) = \mathbb{1}_{V^{\Gamma}[x]}(y).$$

This function appears later in the lower bound of our integral kernels. It needs to be measurable since we often use it inside of integrals.

6.3. Cubes and favored indices

Let Γ be an admissible configuration and $\{V^1, ..., V^L\}$ a family of reference cones according to Lemma 6.4.

Definition 6.11 (open cube, half-closed cube)**.** *For $h > 0$ and $u = (u_1, ..., u_d) \in \mathbb{R}^d$ let*

$$A_h(u) = \left\{ x \in \mathbb{R}^d \;\middle|\; |x-u|_\infty < \frac{h}{2} \right\}$$

the open cube with center u. The half-closed cube with center u will be denoted by

$$\widetilde{A}_h(u) = \prod_{i=1}^{d} \left[u_i - \frac{h}{2}, u_i + \frac{h}{2} \right).$$

Recall that we denote by $|A| = \lambda^d(A)$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}^d$.

Definition 6.12 (h-favored index)**.** *Given $h > 0, u \in \mathbb{R}^d$ and $m \in \{1, ..., L\}$, we set*

$$A_h^m(u) = \{x \in A_h(u) \,|\, V^m \subset \Gamma(x)\}.$$

An index $m \in \{1, ..., L\}$ is called h-favored by majority at u (or short: h-favored index at u) *if*

$$|A_h^m(u)| = \max_{1 \le i \le L} |A_h^i(u)|.$$

Note that $|A_h^m(u)| \ge L^{-1}|A_h(u)|$ for every h-favored index m at u. This follows directly from

$$A_h(u) = \bigcup_{1 \le i \le L} A_h^i(u)\,.$$

It is clear that the choice of an h-favored index is, in general, not unique.

Now we state an elementary result for the intersection of cones which will be very helpful for us. A consequence of the following lemma is that every intersection of double cones with apex in some cube contains a double half-cone.

Lemma 6.13. *Let $\widetilde{V}$ be a cone with apex angle ϑ. Then:*

1. *For $\ell > 0$ it holds*
$$\widetilde{V}_\ell = \bigcap_{x\in\overline{B_\ell}} \widetilde{V}[\xi].$$

2. *For $h > 0$ the following holds true for each $x \in \mathbb{R}^d$ and every $\xi \in A_h(x)$:*
$$\widetilde{V}_{h\sqrt{d}}[\xi] \subset \widetilde{V}_{\frac{h}{2}\sqrt{d}}[x] \subset \widetilde{V}[\xi].$$
In other words
$$\bigcup_{\xi\in A_h(x)} \widetilde{V}_{h\sqrt{d}}[\xi] \ \subset\ \widetilde{V}_{\frac{h}{2}\sqrt{d}}[x] \ \subset \bigcap_{\xi\in A_h(x)} \widetilde{V}[\xi]\,.$$

Proof. Let $\ell > 0$. Observe
$$\begin{aligned} \zeta \in \bigcap_{\xi\in\overline{B_\ell}} \widetilde{V}[\xi] \ &\Leftrightarrow\ \forall\xi \in \overline{B_\ell} : \zeta - \xi \in \widetilde{V} \\ &\Leftrightarrow\ \zeta - \overline{B_\ell} \subset \widetilde{V} \\ &\Leftrightarrow\ \overline{B_\ell(\zeta)} \subset \widetilde{V} \\ &\Leftrightarrow\ \zeta \in \widetilde{V}_\ell. \end{aligned}$$

This means
$$\widetilde{V}_\ell \ = \bigcap_{\xi\in\overline{B_\ell}} \widetilde{V}[\xi], \tag{6.1}$$
which proves our first claim.

On the other hand, for $\zeta \in \widetilde{V}_{2\ell}$, we have $\overline{B_\ell(\zeta)} \subset \widetilde{V}_\ell$. This is equivalent to
$$\forall\zeta \in \widetilde{V}_{2\ell}\ \forall\xi \in \overline{B_\ell} : \zeta + \xi \in \widetilde{V}_\ell\,.$$

In other words
$$\bigcup_{\xi\in\overline{B_\ell}} \widetilde{V}_{2\ell}[\xi] \ \subset\ \widetilde{V}_\ell\,. \tag{6.2}$$

From (6.1) and (6.2) we conclude for every $\xi \in \overline{B_\ell}$
$$\widetilde{V}_{2\ell}[\xi] \ \subset\ \widetilde{V}_\ell \ \subset\ \widetilde{V}[\xi]\,.$$

Translation by $x \in \mathbb{R}^d$ yields
$$\widetilde{V}_{2\ell}[\xi] \ \subset\ \widetilde{V}_\ell[x] \ \subset\ \widetilde{V}[\xi] \qquad \forall\xi \in \overline{B_\ell(x)}\,.$$

Now set $\ell = \frac{h}{2}\sqrt{d}$ and observe that $A_h(x) \subset \overline{B_\ell(x)}$. ■

7. A result for discrete quadratic forms

The aim of this chapter is to prove the following discrete result.

Theorem 7.1. *Let* Γ *be a* ϑ*-bounded configuration and* $\alpha \in (0,2)$. *Let* $\omega : \mathbb{Z}^d \times \mathbb{Z}^d \to [0,\infty]$ *be a function satisfying* $\omega(x,y) = \omega(y,x)$ *and*

$$\omega(x,y) \;\geq\; \Lambda\big(\mathbb{1}_{V^\Gamma[x]}(y) + \mathbb{1}_{V^\Gamma[y]}(x)\big)|x-y|^{-d-\alpha} \tag{7.1}$$

for all $x, y \in \mathbb{Z}^d$ *with* $|x-y| > R_0$*, where* $R_0, \Lambda > 0$ *are given constants. There exist constants* $\kappa \geq 1, c > 0$ *such that for every* $R > 0$*,* $x_0 \in \mathbb{R}^d$ *and every function* $f : (B_{\kappa R}(x_0) \cap \mathbb{Z}^d) \to \mathbb{R}$*, the inequality*

$$\sum_{\substack{x,y \in B_{\kappa R}(x_0)\cap\mathbb{Z}^d \\ |x-y|>R_0}} (f(x)-f(y))^2\omega(x,y) \;\geq\; c \sum_{\substack{x,y \in B_{R}(x_0)\cap\mathbb{Z}^d \\ |x-y|>R_0}} (f(x)-f(y))^2|x-y|^{-d-\alpha} \tag{7.2}$$

holds.

The constant c *depends on* $\Lambda, \vartheta, R_0, \alpha$ *and on the dimension* d*. It does not depend on* ω *and* Γ*.*

We point out that we have to enlarge the ball on the left-hand side of the inequality (7.2).

In the following short paragraph we explain the idea of the proof. This passage is also meant to explain the structure of this chapter to the reader.

Idea of the proof and structure of this chapter

We define a graph G on $\mathbb{Z}^d$ (cf. Section 7.2) and show that every pair of points $x, y \in B_R(x_0) \cap \mathbb{Z}^d$ can be connected by a path p_{xy} given by

$$x = z_1 - z_2 - \cdots - z_{N-1} - z_N = y$$

not leaving the larger ball $B_{\kappa R}(x_0)$. This is done via an induction argument in Section 7.3. Since we always have to work with lattice points, the proofs are rather technical and the reader may easily get distracted. For this reason, in a short prologe in Section 7.1, we have worked out the induction argument also in $\mathbb{R}^d$.

We proceed by showing that we can find a collection of paths $(p_{xy})_{x,y\in\mathbb{Z}^d}$, which has additional properties: Every path has length bounded by a universal number $N \in \mathbb{N}$ independent of x and y. The distance of neighboring vertices $|x_i - x_{i+1}|$ shall be comparable to $|x-y|$ with comparability constant λ. Every edge of the graph G shall be used in only $M \in \mathbb{N}$ paths of the collection, where M is a universal number. These properties enable us to use the lower bound of $w(x,y)$ in (7.1) in the following way:

$$\begin{aligned}(f(x)-f(y))^2|x-y|^{-d-\alpha} &\leq 2\lambda^{d+\alpha}\sum_{i=1}^{N-1}(f(x_{i+1})-f(x_i))^2|x_{i+1}-x_i|^{-d-\alpha}\\ &\leq 2\Lambda^{-1}\lambda^{d+\alpha}\sum_{i=1}^{N-1}(f(x_{i+1})-f(x_i))^2 w(x_{i+1},x_i).\end{aligned}$$

The technical challenging part of the proof is to show the existence of the collection of paths p_{xy} that have our desired properties. The proof of this result relies on a renormalization argument, which we carry out in Section 7.4. We need to define new objects that we call *blocks and towns*, which are basically subsets and families of subsets of $\mathbb{Z}^d$. On such a town we define a graph (the *favored graph*) with vertex set given by the blocks. Our results corresponding to the connection of any two lattice points can now be carried over to the connection of any two blocks. Finally, in Section 7.5, we show in Theorem 7.20 the existence of the paths with the desired properties. This theorem is one of our main results in this chapter.The proof of Theorem 7.1 follows then as a corollary in Section 7.6.

7.1. Prologe: A chaining argument in $\mathbb{R}^d$

The proof of Theorem 7.1 uses the existence of paths connecting any pair of arbitrary points in $\mathbb{Z}^d$. The main task in Chapter 7 is to prove the existence of these paths. In order to understand better the rather technical proof in the discrete case, we show in this subsection how one can connect two arbitrary points in $\mathbb{R}^d$. The procedure in the continuous setting is less technical than in the discrete setting and a lot of ideas from this subsection can be carried over to the discrete case.

The graph G on $\mathbb{R}^d$

From a configuration $\Gamma\colon \mathbb{R}^d \to \mathcal{V}$ we construct a directed graph G as follows: the vertex set is $\mathbb{R}^d$ and there is a directed edge from x to y if $y \in V^\Gamma[x]$. Note that there are no loops in G as $\Gamma(x)$ is open and does not contain the origin.

We shall be concerned with the question whether G is connected as an *undirected* graph if the underlying configuration is ϑ-bounded. In this case, Corollary 6.6 allows us to assume without loss of generality that the image of Γ contains only a finite number of elements.

Thus, crucial parts of the argument can be proved by induction on the number of cones in $\Gamma(\mathbb{R}^d)$. As it often happens, one needs to strike the right balance and the statement suitable for induction is a little bit stronger (and more technical) than the primary target. We are led to consider subgraphs G_U defined by open subsets $U \subset \mathbb{R}^d$ as follows: the vertex set of G_U is still $\mathbb{R}^d$ and the rule for oriented edges is the same, however, we only put in the edges issuing from vertices in U. Note that vertices outside U still can be used in edge paths since we are interested in undirected connectivity.

In this section we always assume that the configuration Γ is ϑ-bounded.

Our main result is given in the next theorem.

Theorem 7.2. *For any connected open set $U \subset \mathbb{R}^d$, any two points $x, y \in U$ are vertices in the same connected component of G_U.*

For the proof of Theorem 7.2 we need some auxiliary results.

Auxiliary results

Lemma 7.3. *If two points $x, y \in U$ have the same type, then there is an edge path in G_U of length at most two connecting them.*

Proof. Let $V = \Gamma(x) = \Gamma(y)$. Then the translated double cones $V[x]$ and $V[y]$ intersect. We pick a point of intersection (it may lie outside of U). It has an edge incoming from x and another edge incoming from y. These two edges form the desired edge path. ∎

Definition 7.4. *A point x is called* well-connected in U *if there is an open neighborhood W of x that, considered as a set of vertices in G_U, lies entirely in a single connected component of G_U. That is, the point x is connected by edge paths in G_U to all points of an open neighborhood.*

The following lemma lists inter alia some important features of well connected points.

Lemma 7.5. *The following hold:*

(1) For $y \in U$, any point $x \in U \cap V^{\Gamma}[y]$ is well-connected in U.

(2) If $U' \subset U$ is an inclusion of open sets, then any point $x \in U'$ that is well-connected in U' is also well-connected in U.

(3) Any non-empty open set U contains a point that is well-connected in U. In fact, the well-connected points are dense in U.

Proof. For (1) we may choose $U \cap V^\Gamma[y]$ as the open neighborhood. Any two points therein are connected via an edge path of length two with y as the middle vertex. Therefore (1) follows.

Enlarging the open set U' only adds edges to the graph. Hence connectivity can only improve. This proves (2).

For the proof of (3) note that existence of a well-connected point follows from (1). Applying the existence statement to smaller open sets $U' \subset U$, density follows in view of (2). ■

Lemma 7.6. *Consider two points $x, y \in U$ and let $V = \Gamma(y)$ be the cone type of y. Assume that the translated double cone $V[x]$ contains a point z of cone type V. Then x and y are connected.*

Note that we do not assume that $V = \Gamma(x)$. One may also note that in the situation of the lemma, the point x is well-connected in U.

Proof. Since y and z have the same type, they are connected by an edge path of length at most two. Now, $z \in V[x]$ implies $x \in V[z] = V^\Gamma[z]$. Hence, there is an edge from z to x. ■

The induction proof

Proof of Theorem 7.2. According to Lemma 6.4, we may assume that the image of Γ has at most L different elements since Γ is ϑ-bounded. Therefore, we can use induction on the number $\#\Gamma(U)$ of cones realized in U. If there is only a single cone type throughout U, any two points $x, y \in U$ are connected in G_U by an edge path of length at most two. This settles the base of the induction.

For $\#\Gamma(U) > 1$, we start with the following observation:

> There is a constant $\lambda > 0$ depending only on the minimum apex angle ϑ such that for any double cone $V \in \mathcal{V}$ and any two points $x, y \in \mathbb{R}^d$ of distance $|x - y| < \lambda$, the intersection $V[x] \cap V[y]$ contains a point in $B_1(x)$.

Now assume that x is well-connected in U and that the r-ball $B_r(x)$ lies entirely in U. We claim that x is connected to any point $y \in B_{\lambda r}(x)$. Indeed, consider the cone type $V = \Gamma(y)$ of y. If $V[x]$ contains a point of cone type V, the points x and y are connected by Lemma 7.6.

Otherwise, within the open set $U' = U \cap V[x] \neq \emptyset$ the cone type V is not realized. We infer by induction that all points in U' are mutually connected in $G_{U'}$ and hence in G_U. However, $V[y] = V^\Gamma[y]$ intersects $U' \supset B_r(x) \cap V[x]$ by the opening observation. Hence y is connected to a point in U' and therefore to any point in U', which contains points arbitrarily close to x. Since x is well-connected in U, the points y and x are connected in G_U.

It follows that a well-connected point $x \in U$ whose r-neighborhood lies in U is actually connected to any point in its λr-neighborhood. Now density of well-connected points in U (cf. Lemma 7.5 (3)) implies that U is covered by overlapping open well-connected subsets. ∎

7.2. The graph G on the set of lattice points

Similar to the graph with vertex set $\mathbb{R}^d$ defined in the section above, we can define a graph with vertex set $\mathbb{Z}^d$. Every configuration $\Gamma : \mathbb{R}^d \to \mathcal{V}$ induces naturally a mapping $\Gamma|_{\mathbb{Z}^d}$, which we again call configuration and denote by Γ. A configuration defines a directed graph $G = G(\Gamma)$, where the set of vertices is given by $\mathbb{Z}^d$ and there is an oriented edge from x to y if $y \in V^\Gamma[x]$. Our main task is to prove that G is connected as an *undirected* graph provided that the configuration Γ is ϑ-bounded. Therefore, throughout the remainder of Chapter 7, we assume without further notice that Γ is ϑ-bounded. Since in this chapter we deal with questions of connectivity of components of graphs, we may assume without further notice that the image of Γ contains only $L \in \mathbb{N}$ double cones, cf. Corollary 6.6 .

A sequence of points $(x_k)_{k \in I}, I \subset \mathbb{N}$, written as

$$x_1 - x_2 - \cdots - x_k - \ldots,$$

is called a *path of edges (or (edge) path)* in the *undirected* graph G if for each $k \in I$ there is an oriented edge from x_k to x_{k+1} or vice versa. It is essential to keep track of how far an edge path connecting two points might take us away from the end points in question. This will later lead to the constant κ in Theorem 7.1.

7.3. The induction

Preparation for the induction

Given that we consider Γ restricted to $\mathbb{Z}^d$ a technicality is that we always have to work with lattice points. The following lemmata are based on simple geometric properties. The assertions are all needed for the ensuing induction.

Since any closed ball of radius $\frac{\sqrt{d}}{2}$ contains a lattice point, we have:

Lemma 7.7. *Let $\widetilde{V}$ be a cone of apex angle at least ϑ.*

(1) Fix $r > 0$ and assume $R > \frac{r+\sqrt{d}}{\sin(\vartheta)}$. Then, for $x \in \mathbb{R}^d$, the intersection $B_R(x) \cap \widetilde{V}[x]$ contains a lattice point $y \in \mathbb{Z}^d$ with $B_r(y) \subset \widetilde{V}[x]$.

(2) Let $x, y \in \mathbb{R}^d$. Fix $r > |x - y|$ and $R > \frac{r+\sqrt{d}}{\sin(\vartheta)} + r$. Then the intersection

$$B_R(x) \cap \widetilde{V}[x] \cap B_R(y) \cap \widetilde{V}[y]$$

contains a lattice point.

Proof. Let $R > \frac{r+d}{\sin(\vartheta)}$. Within distance $\frac{r+\sqrt{d}/2}{\sin(\vartheta)}$ of x, we find a point z with $B_{r+\sqrt{d}/2}(z) \subset \widetilde{V}[x]$. Within the closed ball of radius $\sqrt{d}/2$ around z, we find the desired lattice point y. Therefore,

$$y \in \overline{B_{\frac{r+(\sqrt{d}/2)}{\sin(\vartheta)}+\frac{\sqrt{d}}{2}}(x)} \subset B_R(x) \text{ and } B_r(y) \subset B_{r+\frac{\sqrt{d}}{2}}(z) \subset \widetilde{V}[x].$$

The second assertion can be seen as another way of looking at the same phenomenon. Let now $R > \frac{r+d}{\sin(\vartheta)} + r$ and let $\varepsilon > 0$ such that $R > \varepsilon + r + \frac{r+\sqrt{d}}{\sin(\vartheta)}$. According to statement *(1)* of this lemma, there is a lattice point $z \in \mathbb{Z}^d$ with $z \in B_{\frac{r+\sqrt{d}}{\sin\vartheta}+\varepsilon}(x) \cap \widetilde{V}[x]$ and $B_r(z) \subset \widetilde{V}[x]$. Now, $z \in \widetilde{V}[y]$ since $\widetilde{V}[y]$ is obtained from $\widetilde{V}[x]$ via translation by a distance less than r. By triangle inequality, $z \in B_R(y)$. ■

A quantitative version of Lemma 7.3 follows immediately.

Corollary 7.8. *Any two lattice points $x, y \in \mathbb{Z}^d$ with $\Gamma(x) = \Gamma(y) = V$ and of distance less than r are connected via a path of two edges of length bounded from above by $R = \frac{r+\sqrt{d}}{\sin(\vartheta)} + r$.*

The following definition matches the definition of well-connected points in $\mathbb{R}^d$.

Definition 7.9. *For $r \leq R$, we call a lattice point $x \in \mathbb{Z}^d$* r-R-connected, *if any lattice point $y \in B_r(x)$ is connected in G to x via an undirected edge path not leaving $B_R(x)$.*

The following lemma is the discrete version of the density of well-connected points (Lemma 7.5 (3)).

Lemma 7.10. *For any $r > 0$, any $R > \frac{r+\sqrt{d}}{\sin(\vartheta)}$, and any lattice point $x \in \mathbb{Z}^d$, there is an r-R-connected lattice point $y \in B_R(x)$.*

Proof. Let $\varepsilon > 0$ so that $R = \varepsilon + \frac{\sqrt{d}+r}{\sin(\vartheta)}$. From Lemma 7.7 we know that $B_R(x) \cap V^\Gamma[x]$ contains a lattice point y whose r-ball $B_r(y)$ lies within the double cone $V^\Gamma[x]$. Thus, any two points in $B_r(y)$ are connected via x, and x has distance less than R from y, which yields the claim. ∎

Our discrete variant of Lemma 7.6 reads as follows:

Lemma 7.11. *Consider two lattice points $x, y \in \mathbb{Z}^d$ of distance less than r. Let $V = \Gamma(y)$ be the cone type of y and let $R > 3r + \frac{2r+\sqrt{d}}{\sin(\vartheta)}$. Assume that $B_r(x) \cap V[x]$ contains a lattice point z of cone type V. Then there is an edge path from y to x not leaving $B_R(x)$.*

Proof. There is a directed edge from z to x with length less than r. Note that the distance of y and z is at most $2r$. Hence with $\widetilde{R} > \frac{2r+\sqrt{d}}{\sin(\vartheta)} + 2r$, we conclude from Lemma 7.7 (2) that

$$B_{\widetilde{R}}(z) \cap V[z] \cap B_{\widetilde{R}}(y) \cap V[y] = B_{\widetilde{R}}(z) \cap V^\Gamma[z] \cap B_{\widetilde{R}}(y) \cap V^\Gamma[y]$$

contains a lattice point. Through this point, y and z are connected. Now we see that the choice of R guarantees that

$$\left(B_{\widetilde{R}}(y) \cap V[y] \cap B_{\widetilde{R}}(z) \cap V[z]\right) \subset \left(B_R(x) \cap V[z] \cap V[y]\right),$$

that is, the edge path from y to x does not leave $B_R(x)$. ∎

Lemma 7.12. *There is a constant $\delta > 0$, depending only on ϑ and the dimension d, such that for any double cone $V \in \Gamma(\mathbb{Z}^d)$ the following condition holds:*

> *If for a lattice point $x \in V$, there is a lattice point in V closer to 0, then there is such a lattice point in $V \cap B_\delta(x)$.*

That is, we can go from x within V to a lattice point of minimum distance to the apex via a chain of jumps (not a path in the graph) each bounded in length from above by δ.

Proof. Alternatively, we may show that we can reach any lattice point in V via a chain of jumps starting at the apex where each jump is bounded in length from above by δ. Let $V \in \Gamma(\mathbb{Z}^d)$ be any double cone. It is enough to prove the result for the single cone $\widetilde{V}$ that has the same symmetry axis as V and the same apex angle. Let $\hat{x} \in \mathbb{Z}^d$ be a point that is contained in $\widetilde{V}$ and has minimal distance to the apex at 0 overall lattice points in $\widetilde{V}$. We know that every point $\lambda\hat{x}$ for $\lambda > 0$ is in $\widetilde{V}$. It is also clear that every $\lambda\hat{x}$ for $\lambda \in \mathbb{N}$ gives us another lattice point in $\widetilde{V}$. Now note that by the properties of the lattice $\mathbb{Z}^d$ itself every point $y \in V \cap \mathbb{Z}^d$ can be obtained by a chain of jumps of length less or equal to $\frac{\sqrt{d}}{2}$ starting from one point on the line $\{\lambda\hat{x} \,|\, \lambda \in \mathbb{N}\}$. Therefore, it is sufficient to choose $\delta > |\hat{x}| + \frac{\sqrt{d}}{2}$. Since $\frac{\sqrt{d}}{2\sin\vartheta} \geq |\hat{x}|$, we can choose $\delta > \frac{\sqrt{d}}{2}\left(1 + \frac{1}{\sin(\vartheta)}\right)$. ∎

The Induction-Theorem

It is our aim to prove that every two lattice points in a given ball of radius r are connected via an edge path that does not leave a larger ball of radius R. Here the radius R shall depend only on r, ϑ and d. In the following lemma we show this for a series of values for r respectively R. The proof uses similar ideas as the proof of the corresponding result in the continuous setting, cf. Theorem 7.2.

In this subsection the constant δ always refers to the constant established in Lemma 7.12.

Theorem 7.13. *There are constants $r_1 \leq \rho_1 \leq R_1, r_2 \leq \rho_2 \leq R_2, \ldots$, depending only on ϑ and d, with $\delta < r_1$ and $r_i < r_{i+1}, \rho_i < \rho_{i+1}, R_i < R_{i+1}$ for every $i \in \mathbb{N}$ such that any lattice point $x \in \mathbb{Z}^d$ is r_k-R_k-connected provided $\#\Gamma(B_{\rho_k}(x) \cap \mathbb{Z}^d) \leq k$.*

Idea of the proof. The proof uses an induction argument, where one inducts on the number of cone types that are realized in the given ball. The key point in the induction step is that we can choose r_k large enough so that $B_{r_k}(x)$ contains a lattice point $\hat{x}$ that is well-connected (meaning here, s-S-connected for certain radii $s < S$ around $\hat{x}$). The task is then to connect any lattice point in $B_{r_k}(x)$ with $\hat{x}$. In this step the induction hypothesis helps. This way x is connected to any other lattice point in $B_{r_k}(x)$ through the well-connected point $\hat{x}$, see Figure 7.1. Keeping track of the length of the edge paths allows to choose R_k.

Proof of Theorem 7.13. We use an induction argument on the number of realized cone types k. The case $k = 1$ follows directly from Corollary 7.8: Choose $\rho_1 = r_1 > \delta$ and $R_1 > \frac{r_1+\sqrt{d}}{\sin(\vartheta)} + r_1$.

For the induction step, assume that constants up to ρ_{k-1}, r_{k-1}, and R_{k-1} have already been found. Choose

$$s > \frac{2\rho_{k-1} + \sqrt{d}}{\sin(\vartheta)} \qquad \text{and} \qquad S > \frac{s + \sqrt{d}}{\sin(\vartheta)}.$$

Note that by Lemma 7.7 (1) for $\hat{x} \in \mathbb{Z}^d$ any set $B_s(\hat{x}) \cap V[\hat{x}]$ contains a lattice point z with $B_{2\rho_{k-1}}(z) \subset V[\hat{x}]$. In particular, $B_{\rho_{k-1}}(z) \subset V[\hat{x}]$. In other words z lies in the double half-cone $V_{\rho_{k-1}}[\hat{x}]$. If $\hat{x}$ is s-S-connected, there is an edge path from $\hat{x}$ to z not leaving $B_S(\hat{x})$. These observations will be important later.

We put $r_k = S$. Let $x \in \mathbb{Z}^d$. By Lemma 7.10, there is an s-S-connected lattice point $\hat{x} \in B_{r_k}(x)$. Consider an arbitrary lattice point $y \in B_{r_k}(x)$. It suffices to choose R_k and ρ_k large enough so that we can ensure the existence of an edge path from $\hat{x}$ to y within $B_{R_k}(x)$. Then, since x is also a lattice point in $B_{r_k}(x)$, x is connected to y through the point $\hat{x}$.

Let $V = \Gamma(y)$ be the cone type of y. The distance of y and $\hat{x}$ is less than $2S$. We are interested in the double half-cone $V_{\rho_{k-1}}[\hat{x}]$. Either apex of the double half-cone is within distance $\frac{\rho_{k-1}}{\sin(\vartheta)} < s < S$ of $\hat{x}$ and thus within distance less than $3S$ of y. Let now $\hat{z} \in \mathbb{R}^d$ be any apex of $\hat{x} + V_{\rho_{k-1}}$. By Lemma 7.7 (2), for any $\hat{s} > \frac{3S+\sqrt{d}}{\sin(\vartheta)} + 4S$ the intersection

$$\left(B_{\hat{s}-S}(\hat{z}) \cap V_{\rho_{k-1}}[\hat{x}] \cap V[y]\right) \subset \left(B_{\hat{s}}(\hat{x}) \cap V_{\rho_{k-1}}[\hat{x}] \cap V[y]\right)$$

contains a lattice point.

Choosing $\rho_k > S + \hat{s} + \rho_{k-1}$, we can use the induction hypothesis as follows. If no lattice point in the region $B_{\hat{s}+\rho_{k-1}}(\hat{x}) \cap V[\hat{x}] \subset B_{\rho_k}(x)$ is of cone type V, then we see that there are at most $k-1$ different cone types realized within $B_{\hat{s}+\rho_{k-1}}(\hat{x}) \cap V[\hat{x}]$. Hence, each lattice point in $B_{\hat{s}}(\hat{x}) \cap V_{\rho_{k-1}}[\hat{x}]$ is r_{k-1}-R_{k-1}-connected. Since $r_{k-1} > \delta$, we get by Lemma 7.12 that all these well-connected balls overlap and are therefore connected to the lattice point z that lies in the double half-cone. Recall that z is within distance s of $\hat{x}$ and that $\hat{x}$ is s-S-connected. Hence, all the lattice points in $B_{\hat{s}}(\hat{x}) \cap V_{\rho_{k-1}}[\hat{x}]$ are connected to $\hat{x}$.

On the other hand, one of these lattice points lies within the double cone $V[y] = V^{\Gamma}[y]$ and is hence directly connected to y. Thus, y is connected to $\hat{x}$. Each edge path used will take us at most S or R_{k-1} outside of $B_{\hat{s}}(\hat{x})$. Thus, we might choose $R_k > 2S + R_{k-1} + \hat{s}$. We might need to increase this number to ensure $\rho_k \leq R_k$, but the increase incurred in treating the remaining case is much worse.

It remains to deal with the possibility that there is a lattice point of cone type V in the region $B_{\hat{s}+\rho_{k-1}}(\hat{x}) \cap V[\hat{x}]$. Since $\hat{x}$ and y are of distance at most $2S < \hat{s}$, Lemma 7.11 applies and we choose $R_k > 3(\hat{s} + \rho_{k-1}) + \frac{2(\hat{s}+\rho_{k-1})+\sqrt{d}}{\sin(\vartheta)}$. ■

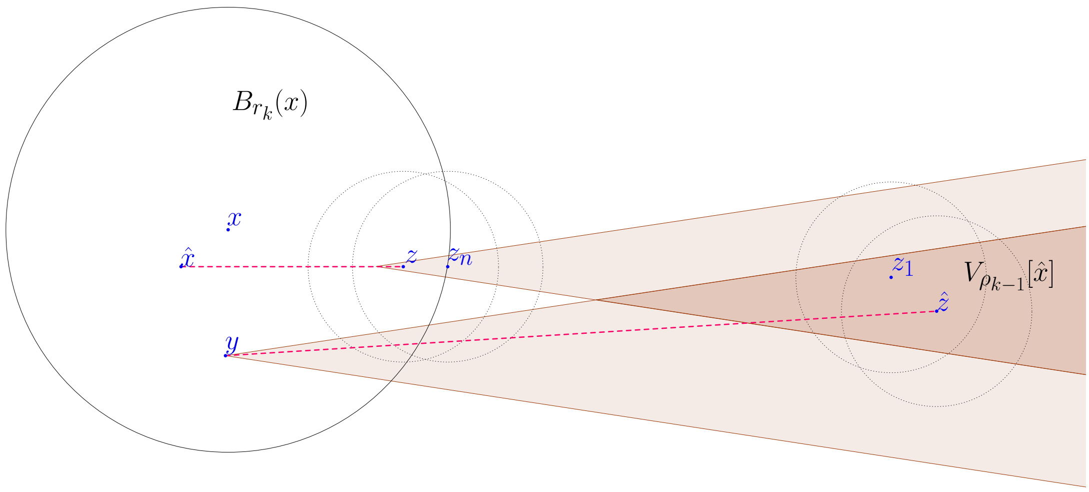

Figure 7.1.: The path $y - \hat{z} - ... - z_1 - ... - z_n - ... - z - \hat{x}$.

Corollary 7.14. *For every $r > 0$ there is $R \geq r$, depending only on r, ϑ and d, such that for any configuration $\Gamma\colon \mathbb{Z}^d \to \mathcal{V}$ with apex angles bounded from below by ϑ any lattice point $x \in \mathbb{Z}^d$ is r-R-connected.*

Proof. By Corollary 6.6 we may and do assume without loss of generality that $\#\Gamma(\mathbb{Z}^d) = L$, where $L \in \mathbb{N}$ is a constant that depends only on ϑ and d. Now the claim follows from Theorem 7.13 and the following observation: If x is r-R-connected, it is r'-R-connected for any $r' \leq r$. ∎

7.4. Renormalization: Blocks and Towns

Since the proof of Theorem 7.1 involves a renormalization argument, it is important to restate Corollary 7.14 for structures at large scale (see Proposition 7.19). To this end, we introduce what we call *blocks* and *towns*. Recall our notation

$$A_\ell(x) \;=\; \left\{ y \in \mathbb{R}^d \;\middle|\; |y - x|_\infty < \frac{\ell}{2} \right\}$$

for cubes.

Lemma 7.15. *For any apex angle ϑ, there is a constant $\delta = \delta(\vartheta) > 0$ such that the following holds for each $\ell > 0$ and any two points $x, y \in \mathbb{R}^d$ of distance at least $\delta\ell$:*

> *If $\widetilde{V}$ is a cone of apex angle $\frac{\vartheta}{2}$ and $y \in \widetilde{V}[x]$, then*
>
> $$A_\ell(y) \;\subset\; \bigcap_{z \in A_\ell(x)} \overline{V}[z]$$
>
> *for the cone $\overline{V}$ with apex angle ϑ and the same symmetry axis as $\widetilde{V}$.*

Proof. Let $\widetilde{V}$ be a cone of apex angle $\frac{\vartheta}{2}$ and symmetry axis v and let $\overline{V}$ be a cone with apex angle ϑ and symmetry axis v. Let $x \in \mathbb{R}^d$ and $\ell > 0$. According to Lemma 6.13 we know

$$\bigcap_{z \in A_\ell(x)} \overline{V}[z] \;\supset\; \overline{V}_{\frac{\ell}{2}\sqrt{d}}[x].$$

It is also known that

$$B_{\frac{\ell}{2}\sqrt{d}}(y) \;\supset\; A_\ell(y)$$

for any $y \in \mathbb{R}^d$.

Therefore, we choose a point $\tilde{y} \in \partial(\widetilde{V}[x])$ on the boundary of $\widetilde{V}[x]$ with the property

$$B_{\frac{\ell}{2}\sqrt{d}}(\tilde{y}) \subset \overline{V}_{\frac{\ell}{2}\sqrt{d}}[x],$$

that is, $B_{\ell\sqrt{d}}(\tilde{y}) \subset \overline{V}[x]$. The claim follows now with $\delta = \frac{|x-\tilde{y}|}{\ell} \geq \frac{\sqrt{d}}{\sin(\vartheta/2)}$. ■

In the remainder we will always refer to the smallest possible δ that satisfies the claim of Lemma 7.15, that is, $\delta = \frac{\sqrt{d}}{\sin(\vartheta/2)}$.

Definition 7.16 (block, town at scale, sparsely populated)**.** *A* block

$$Q_\ell(x) \;=\; \mathbb{Z}^d \cap A_\ell(x)$$

is a collection of lattice points inside a cube.

Let $h, \ell > 0$. The town at scale (h, ℓ) *is the collection*

$$T(h,\ell) \;=\; \left\{ Q_\ell(x) \,\middle|\, x \in h\mathbb{Z}^d \right\}.$$

If the constant δ from Lemma 7.15 is less than $\frac{h}{\ell}$, then the town is called sparsely populated *(or ϑ-sparsely populated when we want to recall that δ depends on ϑ).*

In order to employ geometric language, we implicitly may identify the block $Q_\ell(x)$ with its center x. This way, we think of the distance between two blocks as the distance of their centers. If h is large compared to ℓ, the distance between the centers is a good approximation to any distance between points from the two blocks.

Definition 7.17 (favored by majority)**.** *Let $\Gamma\colon \mathbb{Z}^d \to \mathcal{V}$ be a ϑ-bounded configuration. A double cone $V \in \mathcal{V}$ shall be called* favored by majority *in Q for a block $Q \subset \mathbb{Z}^d$ if the preimage*

$$\Gamma_Q^{-1}(V) = \left\{ x \in Q \,\middle|\, \Gamma(x) = V \right\}$$

has maximal size, that is,

$$\#\Gamma_Q^{-1}(V) \;\geq\; \#\Gamma_Q^{-1}(V') \qquad \textit{for every } V' \in \mathcal{V}.$$

Remark. Given a block Q, the choice of a cone $V \in \mathcal{V}$ that is favored by majority in Q, in general, is not unique.

In the following we want to establish a result similar to the one of Corollary 7.14 but for blocks instead of lattice points. In order to do that we first have to define a graph with vertices given by the blocks. This enables us to study paths in this new graph.

Definition 7.18 (favored graph)**.** *Given a town $T = T(h,\ell)$, a directed graph is defined as follows. The vertices are given by the blocks in T. There is an edge from a block Q to a block P if there is a cone $V \in \mathcal{V}$ favored by majority in Q with*

$$y \in V[x] \qquad \text{for all } x \in Q,\ y \in P.$$

The corresponding undirected graph is called the favored graph.

We derive a connectivity result for the favored graph of a sparsely populated town from Corollary 7.14.

Proposition 7.19. *For any radius $r > 0$ there exists $R \geq r$ depending only on ϑ and d, such that in a ϑ-sparsely populated town T of scale (h,ℓ) any two blocks Q and P within distance hr of some point $z \in h\mathbb{Z}^d$ are connected by an undirected edge path in the favored graph. This path does not pass through blocks farther away from z than hR.*

Proof. Let $r > 0$ and $T = T(h,\ell)$ be a sparsely populated town. Let $Q, P \in T = T(h,\ell)$ be two blocks within distance hr of some point $z \in h\mathbb{Z}^d$. Denote by $W(Q) \in \mathcal{V}$ one of the cones that are favored by majority in Q. Let us show the existence of a path in the favored graph that connects Q and P and does not leave the ball $B_{hR}(z)$. In order to invoke Corollary 7.14, note that

$$\begin{aligned} \mathbb{Z}^d &\longrightarrow T(h,\ell) \\ x &\mapsto Q_\ell(hx) \end{aligned}$$

provides an identification of the town T with the integer lattice $\mathbb{Z}^d$. Denoting by $W_{\frac{1}{2}}(Q)$ the double cone with apex angle $\frac{\vartheta}{2}$ and the same axis as $W(Q)$, let us consider the following configuration:

$$\begin{aligned} \mathbb{Z}^d &\longrightarrow \mathcal{V} \\ x &\mapsto W_{\frac{1}{2}}(Q_\ell(hx)). \end{aligned}$$

If there is an edge from x to y in this configuration, then by Lemma 7.15, there is an edge from the block $Q_\ell(hx)$ to the block $Q_\ell(hy)$ in the favored graph. Indeed, $|x - y| \geq 1$ and thus $|hx - hy| \geq h > \delta\ell$. Then Lemma 7.15 shows that $\widetilde{y} \in W(Q_\ell(hx)) + \widetilde{x}$ for all $\widetilde{x} \in A_\ell(hx), \widetilde{y} \in A_\ell(hy)$ or vice versa with exchanged roles of x and y.

Choose $x, y \in \mathbb{Z}^d$ so that $P = Q_\ell(hx)$ and $Q = Q_\ell(hy)$. Now the claim follows from Corollary 7.14. ■

7.5. Connecting points at scale: The discrete heart

From Corollary 7.14 it is clear that, for any ϑ-bounded configuration $\Gamma\colon \mathbb{Z}^d \to \mathcal{V}$ the associated directed graph $G = G(\Gamma)$ is connected when considered as an undirected graph.

Thus, there is a set of paths, which is large enough to connect every given pair x, y. As described in the introduction of Chapter 7, we want to identify a collection of paths that has additional properties. The aim of this section is to prove quantitative estimates on the length of paths and the number of edges. The following contains our main result in this direction. This theorem is the basis for the proof of Theorem 7.1 and it is the reason for all the effort we invested in the previous sections. That is why we refer to it as *the discrete heart.*

Theorem 7.20. *Let $\Gamma\colon \mathbb{Z}^d \to \mathcal{V}$ be a configuration with apex angles bounded from below by $\vartheta > 0$. Let $R_0 > 0$. There exist positive numbers N and M and a constant $\lambda \geq 1$, all independent of Γ, and a collection $(p_{xy})_{x,y\in\mathbb{Z}^d}$ of unoriented edge paths in G such that the following is true:*

1. *The path p_{xy} starts at x and ends at y.*
2. *Any path p_{xy} has at most N edges.*
3. *Any edge of G is used in at most M paths p_{xy}.*
4. *Any edge in p_{xy} has length strictly bounded from below by R_0. Moreover, each edge is bounded from below by $\lambda^{-1}|x-y|$ and from above by $\lambda|x-y|$.*

Before we state the proof, we first explain the setup and describe the basic idea of the proof.

Setup of the proof. Let us provide the setup of the proof of Theorem 7.20. We pick an odd integer Δ larger than the constant $\max\{\delta, R_0, 1\}$ with the property that

$$\frac{\Delta}{L} \in \mathbb{N}, \tag{7.3}$$

where δ is as in Lemma 7.15 and L is as in Lemma 6.4 and we may and do assume without loss of generality that L is odd (otherwise (7.3) cannot be realized). Hence, the towns $T_n = T(\Delta^n, \Delta^{n-1})$ are all ϑ-sparsely populated so that Proposition 7.19 applies. The distance $|x-y|$ for $x, y \in \mathbb{Z}^d$ lies in exactly one of the intervals

$$[\Delta^0, \Delta^1), \quad [\Delta^1, \Delta^2), \quad [\Delta^2, \Delta^3), \quad \dots$$

Assume $|x-y| \in [\Delta^{n-1}, \Delta^n)$. In this case, we will consider T_n to be the appropriate town for connecting x and y. We call n the *logarithmic scale* of the town T_n.

Assume that $\#\Gamma(\mathbb{Z}^d) \leq L$. Since Δ is an odd integer, each block $Q = Q_{\Delta^{n-1}}(x)$ contains at least $\frac{\Delta^{d(n-1)}}{L}$ lattice points $z \in Q$, where the associated cone $\Gamma(z)$ is favored by majority in Q.

Idea of the proof. We fix a logarithmic scale n and identify one big path in the favored graph that connects all the blocks of T_n that lie in some ball not leaving a larger ball. This big block path gives rise to a collection of paths in the graph G. We then show that any two lattice points in a certain range with distance depending on the scale n can be connected to a certain path in G arising from the big path. In that way we obtain the collection of paths $(p_{xy})_{x,y\in\mathbb{Z}^d}$. With help of the logarithmic scale we can deduce the existence of M and N and prove the assertion on the length of the edges.

Remark. Let us recall that we implicitly always identify the distance of two blocks in T_n with the distance of their center points. Since T_n is sparsely populated the distance of any two points in two blocks of T_n is comparable to the distance of their centers, where the comparability constant only depends on the dimension.

An important step in the construction of p_{xy} is to connect x and y to blocks of T_n. The following lemma deals with this problem. It can be seen as the initial step of the proof of Theorem 7.20.

Lemma 7.21. *There is a constant $R_1 \geq 1$ such that for any point $x \in \mathbb{R}^d$ and any $n \in \mathbb{N}$ there is a block $Q \in T_n$ entirely contained in $B_{\Delta^n R_1}(x) \cap V^\Gamma[x]$. In particular, the radius R_1 can be chosen in such a way that the distance of x and Q is bounded from below by Δ^n times a universal constant, only depending on the dimension.*

Proof. There is a radius r such that for any cone $\widetilde{V}$ of apex at least $\frac{\vartheta}{2}$ and each point $z \in \mathbb{R}^d$, the intersection $B_r(z) \cap \widetilde{V}[z]$ contains a lattice point y with $|z-y| \geq 1$. Now the claim follows by rescaling from Lemma 7.15 applied to $\Delta^n z$ and $\Delta^n y$. As we want to encircle the whole block and not just its center, we choose $R_1 > r + \frac{\sqrt{d}}{2}$. ■

Now we are in the position to prove the main result of this section.

Proof of Theorem 7.20. Let R_1 be the radius from Lemma 7.21, put $r = 1 + R_1$, and let R be the radius resulting with this value from Proposition 7.19. The proof consists of several steps. The radius r is chosen in such a way that one has

$$B_{\Delta^n}(x) \subset B_{\Delta^n R_1}(x) \subset B_{\Delta^n r}(z), \tag{7.4}$$

whenever $x \in B_{\Delta^n}(z) \cap \mathbb{Z}^d$ for some $z \in \mathbb{R}^d$. This property is used in Step 2.

Step 1: Construction of paths in the favored graph for a fixed scale. We fix some logarithmic scale n. For every $z \in \Delta^n\mathbb{Z}^d$, we construct a path $\mathcal{P}_z^n$ in the favored graph that traverses every block of T_n that is a subset of $B_{\Delta^n r}(z)$. By taking the union $\bigcup_{z\in\Delta^n\mathbb{Z}^d} \mathcal{P}_z^n$ we construct paths in the favored graph for a fixed scale.

Let $z \in \Delta^n\mathbb{Z}^d$. Proposition 7.19 allows us to connect every block $Q \in T_n, Q \subset B_{\Delta^n r}(z)$ with every other block $P \in T_n, P \subset B_{\Delta^n r}(z)$ so that the corresponding path traverses not more than $\#(B_R \cap \mathbb{Z}^d) \asymp R^d$ blocks of T_n, which can be chosen to lie in $B_{\Delta^n R}$. If we apply Proposition 7.19 successively to all blocks of T_n, which are subsets of $B_{\Delta^n r}$, then we obtain a path

$$\mathcal{P}_z^n \;=\; Q^1 - Q^2 - \ldots - Q^t \tag{7.5}$$

in the favored graph of blocks of T_n with

$$r^d \;\asymp\; \#(B_{\Delta^n r} \cap \Delta^n \mathbb{Z}^d) \;\leq\; t \;\leq\; \#(B_{\Delta^n r} \cap \Delta^n \mathbb{Z}^d)(B_{\Delta^n R} \cap \Delta^n \mathbb{Z}^d) \;\asymp\; r^d R^d$$

such that the following assertions hold:

1. For each $i \in \{1, \ldots, t\}$ we have $Q^i \subset B_{\Delta^n R}(z)$.
2. The blocks Q^1 and Q^t are subsets of $B_{\Delta^n r}(z)$.
3. If Q is any block of T_n with $Q \subset B_{\Delta^n r}(z)$, then $Q = Q^i$ for some $i \in \{1, \ldots, t\}$.

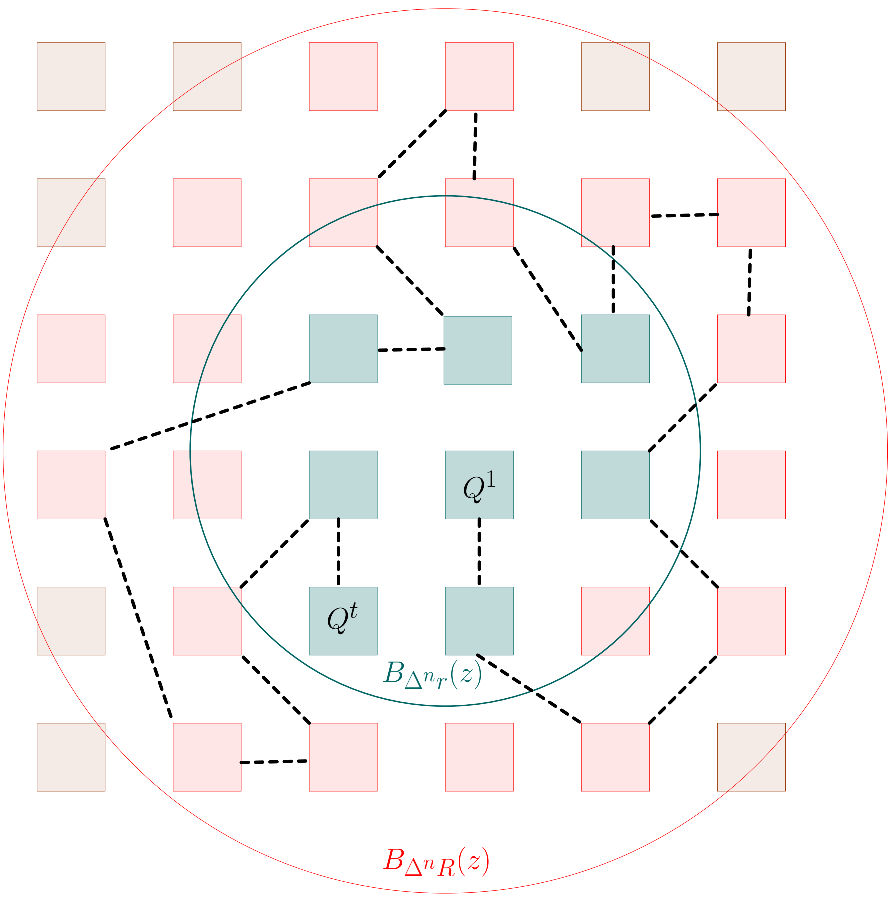

Figure 7.2.: The path $Q_1 - \ldots. - Q_t$

For an illustration of the path (7.5) see Figure 7.2.

Finally, set

$$\mathcal{P}^n = \bigcup_{z \in \Delta^n \mathbb{Z}^d} \mathcal{P}_z^n.$$

Step 2: Construction of paths in the graph G for a fixed scale. For a logarithmic scale n and $x, y \in \mathbb{Z}^d$ with $|x - y| \in [\Delta^{n-1}, \Delta^n)$ we construct a path in the graph G connecting x and y.

Fix a logarithmic scale n. Let $z \in \Delta^n \mathbb{Z}^d$. Choose for every block in (7.5) a favored cone and call the corresponding set of points in the block where this cone is associated a *majority set*. Each majority set contains at least

$$a = \frac{\Delta^{d(n-1)}}{L} \in \mathbb{N}$$

points. Without loss of generality, we may and do assume that every majority set contains exactly a different elements. Then we identify a block in (7.5) with its majority set, that is, if Q^k is the k-th block in (7.5), then

$$Q^k = (q_i^k)_{1 \leq i \leq a}.$$

Starting from (7.5) we now fix certain paths in the graph G, which then give rise to the collection (p_{xy}). Let $i \in \{1, ..., a\}$. Without loss of generality we may and do assume that t is an even number (for odd t just erase the last edge in the following scheme). We construct a set M of paths in the graph G. Let us first fix $i \in \{1, ..., a\}$. Consider the following paths in G:

$$\begin{array}{ccccccccccc} q_i^1 & - & q_i^2 & - & q_i^3 & - & q_i^4 & - & \dots & - & q_i^t \\ q_i^1 & - & q_{i+1}^2 & - & q_i^3 & - & q_{i+1}^4 & - & \dots & - & q_{i+1}^t \\ q_i^1 & - & q_{i+2}^2 & - & q_i^3 & - & q_{i+2}^4 & - & \dots & - & q_{i+2}^t \\ q_i^1 & - & q_{i+3}^2 & - & q_i^3 & - & q_{i+3}^4 & - & \dots & - & q_{i+3}^t \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ q_i^1 & - & q_{i+a-1}^2 & - & q_i^3 & - & q_{i+a-1}^4 & - & \dots & - & q_{i+a-1}^t, \end{array} \tag{7.6}$$

where the lower index is to be read modulo a, that is, $k + a = k$ for every k. The scheme (7.6) gives us a different paths in G. Let us collect them in the set M_i. Then we set

$$M = \bigcup_{i=1}^{a} M_i.$$

The set M consists of a^2 paths.

Now we associate to every pair of lattice points $(x, y) \in A$ with

$$A = \Big\{(x, y) \in (B_{\Delta^n}(z) \cap \mathbb{Z}^d) \times (B_{\Delta^n}(z) \cap \mathbb{Z}^d) \;\Big|\; |x - y| \in [\Delta^{n-1}, \Delta^n)\Big\}$$

one path of M. Since the number $\#A$ can be bounded from above by Ka^2, where $K \geq 1$ is a universal constant independent of the logarithmic scale n (but depending on Δ, L), this can be realized by a function

$$\phi_z : A \to M$$

with

$$\#\phi_z^{-1}(p) \;\leq\; K \quad \text{for every } p \in M \tag{7.7}$$

where, K is as above and does not dependent on p. In order to use the path $\phi_z(x,y)$ to connect x and y, it remains to make sure that x and y are both connected in G to one element in $\phi_z(x,y)$ respectively. This follows from (7.4) combined with Lemma 7.21, which guarantees that every $x \in B_{\Delta^n}(z)$ is connected to every point in some block Q^k of $\mathcal{P}_z^n$. Note in particular that, by the supplement in Lemma 7.21, the distance of Q^k and x is bounded from below by Δ^n times some universal constant. In this way the path $\phi_z(x,y)$ induces a path in G that starts in x and ends in y (cf. Figure 7.3).

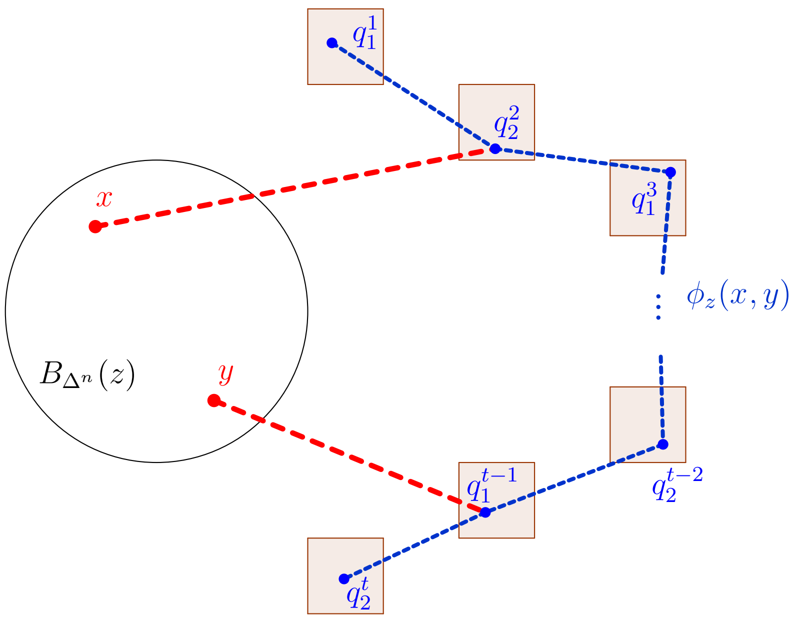

Figure 7.3.: Construction of a path using $\phi_z(x,y)$

Using this construction scheme, we have constructed a path for each pair of lattice points $(x,y) \in B_{\Delta^n}(z) \times B_{\Delta^n}(z)$ with $|x-y| \in [\Delta^{n-1}, \Delta^n)$. Let M_z^n be the set of all these paths. We can carry out this principle of construction of the paths for every $z \in \Delta^n\mathbb{Z}^d$.

Step 3: Construction of p_{xy}. Note that the whole construction process of Step 2 has been performed for an arbitrary $n \in \mathbb{N}$. We define p_{xy} for $x, y \in \mathbb{Z}^d$ as follows. Choose

$n \in \mathbb{N}$ such that $|x-y| \in [\Delta^{n-1}, \Delta^n)$. Next, choose any $z \in \Delta^n \mathbb{Z}^d$ such that $\phi_z(x,y)$ represents a path connecting x and y. In this way,

$$p_{xy} \in \bigcup_{z \in \Delta^n \mathbb{Z}^d} M_z^n.$$

Step 4: Bounds of the length of each edge path. The second claim of Theorem 7.20 follows immediately from $t \leq \#(B_r \cap \mathbb{Z}^d) \cdot \#(B_R \cap \mathbb{Z}^d)$.

Step 5: Bounds of the length of each edge. By construction, all edges used in p_{xy} for some $x, y \in \mathbb{Z}^d$ with $|x-y| \in [\Delta^{n-1}, \Delta^n)$ have lengths bounded from below by $\Delta^{n-1} > R_0$ and from above by $2\Delta^n R$. Ergo, the fourth claim follows with $\lambda = 2R\Delta$.

Step 6: Bounds of the multiplicity of edges. According to Step 5, it is enough to proof the third claim of Theorem 7.20 for one fixed logarithmic scale. Therefore, we fix n. Assume e is an edge of length in $[\Delta^{n-1}, 2\Delta^n R)$. Then there exists a point $z \in \Delta^n \mathbb{Z}^d$ so that $e \subset B_{\Delta^n R}(z)$. Since the number of lattice points in B_{2R} bounds from above the number of block centers $z \in \Delta^n \mathbb{Z}^d$ for which $B_{\Delta^n R}(z)$ contains e, it is enough to bound the number of times e is used by paths belonging to a fixed z. But now by construction (cf. Step 1) for every edge in $B_{\Delta^n R}(z)$ the usage of paths that start in some point x and end in some other point y with $x, y \in B_{\Delta^n}(z)$ so that $|x-y| \in [\Delta^{n-1}, \Delta^n)$, is bounded by K and this number is independent of the scale. ∎

7.6. Proof of Theorem 7.1

Finally, we are in the position to prove Theorem 7.1. The proof is just an easy consequence of Theorem 7.20.

Proof. Let $R > 0$ and $x_0 \in \mathbb{R}^d$. For $x, y \in B_R(x_0) \cap \mathbb{Z}^d$ denote by

$$x = z_1 - z_2 - \dots - z_{N-1} - z_N = y$$

the path p_{xy} that satisfies properties (1)-(4) of Theorem 7.20. For the sake of readability we suppress the dependence on (x,y) in the notation of the path. The reader should keep in mind that $z_i = z_i^{(x,y)}$ for each i. For simplicity we assume that every path in (p_{xy}) is of length N. Then with use of the properties (1)-(4) of Theorem 7.20 and of (7.1) we find:

$$\sum_{\substack{x,y \in B_R(x_0) \cap \mathbb{Z}^d \\ |x-y| > R_0}} (f(x) - f(y))^2 |x-y|^{-d-\alpha}$$

$$
\begin{aligned}
&\leq 2\lambda^{d+\alpha} \sum_{\substack{x,y\in B_R(x_0)\cap\mathbb{Z}^d \\ |x-y|>R_0}} \sum_{i=1}^{N-1} (f(z_{i+1}) - f(z_i))^2 |z_{i+1} - z_i|^{-d-\alpha} \\
&\leq 2\lambda^{d+\alpha} \sum_{\substack{x,y\in B_R(x_0)\cap\mathbb{Z}^d \\ |x-y|>R_0}} (N-1) \max_{i\in\{1,\ldots,N-1\}} \Big[(f(z_{i+1}) - f(z_i))^2 |z_{i+1} - z_i|^{-d-\alpha} \Big] \\
&\leq 2\Lambda^{-1}\lambda^{d+\alpha} \sum_{\substack{x,y\in B_R(x_0)\cap\mathbb{Z}^d \\ |x-y|>R_0}} (N-1) \max_{i\in\{1,\ldots,N-1\}} \big[(f(z_{i+1}) - f(z_i))^2 \omega(z_{i+1}, z_i) \big] \\
&\leq 2\Lambda^{-1}\lambda^{d+\alpha}(N-1)M \sum_{\substack{x,y\in B_{(N-1)\lambda R}(x_0)\cap\mathbb{Z}^d \\ |x-y|>R_0}} (f(x) - f(y))^2 \omega(x, y).
\end{aligned}
$$

Therefore, we choose $c = \big(2\Lambda^{-1}\lambda^{d+\alpha}(N-1)M\big)^{-1}$ and $\kappa = (N-1)\lambda$. ∎

8. The coercivity result

The purpose of this chapter is to prove the following coercivity estimate.

Theorem 8.1. *Let Γ be a ϑ-admissible configuration and $\alpha \in (0,2)$. Let $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ be a measurable function satisfying $k(x,y) = k(y,x)$ and*

$$k(x,y) \;\geq\; \Lambda\big(\mathbb{1}_{V^\Gamma[x]}(y) + \mathbb{1}_{V^\Gamma[y]}(x)\big)|x-y|^{-d-\alpha} \tag{8.1}$$

for almost all $x, y \in \mathbb{R}^d, x \neq y$, where $\Lambda > 0$ is some constant. Then there is a constant $c > 0$ such that for every ball $B \subset \mathbb{R}^d$ and for every $f \in L^2(B)$, the inequality

$$\int_B \int_B (f(x) - f(y))^2 k(x,y) \,\mathrm{d}x \,\mathrm{d}y \;\geq\; c \int_B \int_B (f(x) - f(y))^2 |x-y|^{-d-\alpha} \,\mathrm{d}x \,\mathrm{d}y \tag{8.2}$$

holds.

The constant c depends on Λ, the dimension d and ϑ. It is independent of k and Γ. For $0 < \alpha_0 \leq \alpha < 2$, the constant c may be chosen to depend on α_0 but not on α.

We remark that $f \in L^2(B)$ does not imply that any of the two terms in (8.2) is finite. The result in particular says that the term on the left-hand side is infinite if the term on the right-hand side is infinite.

Idea of the proof and structure of this chapter

In Section 8.1 we first derive a version of the discrete result on every lattice $h\mathbb{Z}^d$ for $h > 0$. The idea is then to consider a discrete version of the kernel k that can be plugged into our discrete result. The claim follows by considering the limit $h \to 0$ on each side of the inequality.

The discrete version of the kernel is introduced in Section 8.2. In this section we also show that this discrete version satisfies the conditions of the discrete result Theorem 7.1.

The limiting argument and the proof of Theorem 8.1 are presented in Section 8.3. One issue is that the balls in either side of (7.2) in Theorem 7.1 are of different size. This carries over to the limit. With the help of a Whitney-type covering argument one can obtain the same ball on either side of the resulting inequality. In order not to disturb the flow of reading, we explain this covering result in the Appendix, see Lemma A.2.

8.1. A result in $h\mathbb{Z}^d$

By scaling we can deduce the following $h\mathbb{Z}^d$-Version from Theorem 7.1.

Corollary 8.2. *Let Γ be a ϑ-bounded configuration and let $h > 0$. Let $\omega : h\mathbb{Z}^d \times h\mathbb{Z}^d \to [0, \infty]$ be a function satisfying $\omega(x, y) = \omega(y, x)$ and*

$$\omega(x,y) \;\geq\; \Lambda\big(\mathbb{1}_{V^{\Gamma}[x]}(y) + \mathbb{1}_{V^{\Gamma}[y]}(x)\big)|x-y|^{-d-\alpha} \tag{8.3}$$

for $x, y \in h\mathbb{Z}^d$ with $|x - y| > R_0 h$, where $R_0, \Lambda > 0$ are some constants. There exist constants $\kappa \geq 1$ and $c > 0$, such that for every $R > 0$, every $x_0 \in \mathbb{R}^d$, and every function $f : (B_{\kappa R} \cap h\mathbb{Z}^d) \to \mathbb{R}$, the inequality

$$\sum_{\substack{x,y \in B_{\kappa R} \cap h\mathbb{Z}^d \\ |x-y| > R_0 h}} (f(x) - f(y))^2 \omega(x,y) \;\geq\; c \sum_{\substack{x,y \in B_R \cap h\mathbb{Z}^d \\ |x-y| > R_0 h}} (f(x) - f(y))^2 |x-y|^{-d-\alpha}$$

holds. The constant c depends on Λ, ϑ, R_0 and on the dimension d. It does not depend on ω, Γ and h.

Proof. Let

$$M \;=\; \left\{ \omega : h\mathbb{Z}^d \times h\mathbb{Z}^d \to [0,\infty] \;\middle|\; \begin{array}{l} \omega(x,y) = \omega(y,x) \text{ and} \\ (8.3) \text{ for some configuration } \Gamma \text{ with } \vartheta > 0 \end{array} \right\},$$

$$N \;=\; \left\{ \omega : \mathbb{Z}^d \times \mathbb{Z}^d \to [0,\infty] \;\middle|\; \begin{array}{l} \omega(x,y) = \omega(y,x) \text{ and} \\ (7.1) \text{ for some configuration } \Gamma \text{ with } \vartheta > 0 \end{array} \right\}.$$

Every element $\omega \in M$ is of the form

$$h^{-d-\alpha}\tilde{\omega}(h^{-1}x, h^{-1}y) \text{ for some } \tilde{\omega} \in N.$$

If $R > 0, x_0 \in \mathbb{R}^d$ and $f : B_{\kappa R}(x_0) \cap h\mathbb{Z}^d \to \mathbb{R}$ is a function, we define a function on $\mathbb{Z}^d$ by

$$g : B_{\kappa R}(x_0) \cap \mathbb{Z}^d \to \mathbb{R}, \quad g(x) = f(hx).$$

Then with use of Theorem 7.1:

$$\begin{aligned} & c \sum_{\substack{x,y \in B_R(x_0) \cap h\mathbb{Z}^d \\ |x-y| > R_0 h}} (f(x) - f(y))^2 |x-y|^{-d-\alpha} \\ = {} & c \sum_{\substack{x,y \in B_R(x_0) \cap \mathbb{Z}^d \\ |x-y| > R_0}} (g(x) - g(y))^2 h^{-d-\alpha} |x-y|^{-d-\alpha} \\ \leq {} & \sum_{\substack{x,y \in B_{\kappa R}(x_0) \cap \mathbb{Z}^d \\ |x-y| > R_0}} (g(x) - g(y))^2 h^{-d-\alpha} \tilde{\omega}(x,y) \end{aligned}$$

$$= \sum_{\substack{x,y\in B_{\kappa R}(x_0)\cap h\mathbb{Z}^d \\ |x-y|>R_0h}} (f(x)-f(y))^2 h^{-d-\alpha}\tilde{\omega}(h^{-1}x,h^{-1}y)$$

$$= \sum_{\substack{x,y\in B_{\kappa R}(x_0)\cap h\mathbb{Z}^d \\ |x-y|>R_0h}} (f(x)-f(y))^2 \omega(x,y).$$

This proves the claim. ■

8.2. The discrete version of the kernel

In the remainder we always assume that Γ is a fixed ϑ-admissible configuration and $\{V^m\}_{1\le m\le L}$ is an associated family of reference cones. We always denote the symmetry axis of a reference cone V^m by v^m for $m \in \{1,...,L\}$.

For $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ a nonnegative measurable function and $h > 0$, we define

$$\omega_h^k : h\mathbb{Z}^d \times h\mathbb{Z}^d \to [0,\infty] \text{ by } \omega_h^k(x,y) \;=\; h^{-2d}\int_{A_h(x)}\int_{A_h(y)} k(s,t)\,\mathrm{d}s\,\mathrm{d}t.$$

Note that $\omega_h^k(x,y)$ may be infinite for x and y from neighboring cubes.

We want to apply Corollary 8.2 to $\omega = \omega_h^k$. Therefore, we need to make sure that the function ω_h^k satisfies (8.3). The idea is to show this claim first for the case $h = 1$ and then deduce the assertion for any $h > 0$ from a scaling argument.

The next three technical lemmas are custom-made for the proof of (8.3) in the case $h = 1$.

Lemma 8.3. *For all $x, y \in \mathbb{Z}^d$, all 1-favored indices m at x and n at y, all $t \in A_1^n(y)$, and all $s \in A_1^m(x)$, the inequality*

$$\mathbb{1}_{V^m_{\sqrt{d}/2}[x]}(t) + \mathbb{1}_{V^n_{\sqrt{d}/2}[y]}(s) \;\ge\; \mathbb{1}_{V^m_{\sqrt{d}}[x]}(y) + \mathbb{1}_{V^n_{\sqrt{d}}[y]}(x)$$

holds.

Proof. Let $x, y \in \mathbb{Z}^d$ and let m be a 1-favored index at x. Assume $y \in V^m_{\sqrt{d}}[x]$. Then, $B_{\sqrt{d}/2}(y) \subset V^m_{\sqrt{d}/2}[x]$. Therefore

$$A_1^n(y) \;\subset\; A_1(y) \;\subset\; B_{\sqrt{d}/2}(y) \;\subset\; V^m_{\sqrt{d}/2}[x]$$

and the claim follows. ■

Lemma 8.4. *For every $h > 0$, all $x, y \in h\mathbb{Z}^d$ with $|x - y| > \sqrt{d}h$ and all $s \in A_h(x), t \in A_h(y)$, the following holds:*

$$\frac{1}{2\sqrt{d}}|x-y| < |s-t| < 2\sqrt{d}|x-y|.$$

Proof. This is about comparing the Euclidean norm to the maximum norm $|\cdot|_\infty$ on $\mathbb{R}^d$. Note that for any vector $v \in \mathbb{R}^d$ we have

$$|v|_\infty \leq |v| \leq \sqrt{d}|v|_\infty. \tag{8.4}$$

Let $h = 1$ and $x, y \in \mathbb{Z}^d$ with $|x - y| > \sqrt{d}$. Since the maximum norm takes only integer values on lattice points and $|x - y| > \sqrt{d}$, it follows that $|x - y|_\infty \geq 2$. As a consequence of the triangle inequality, we have for $s \in A_1(x)$ and $t \in A_1(y)$

$$\frac{1}{2}|x-y|_\infty \leq |x-y|_\infty - 1 < |s-t|_\infty < |x-y|_\infty + 1 \leq 2|x-y|_\infty.$$

Using (8.4), we conclude:

$$\begin{aligned} |s-t| &\leq \sqrt{d}|s-t|_\infty < 2\sqrt{d}|x-y|_\infty \leq 2\sqrt{d}|x-y|, \\ |x-y| &\leq \sqrt{d}|x-y|_\infty < 2\sqrt{d}|s-t|_\infty \leq 2\sqrt{d}|s-t|. \end{aligned}$$

The general case for arbitrary $h > 0$ follows by a scaling argument. ■

Lemma 8.5. *Let $r > 0$ and let V be a cone of apex angle $\vartheta > 0$. There is an apex angle $\theta > 0$, depending only on d, ϑ and r, such that for some cone $V(\theta)$ of angle θ we have*

$$\left(V(\theta) \cap \mathbb{Z}^d\right) \subset \left(V_r \cap \mathbb{Z}^d\right).$$

Proof. Let V be any cone of apex angle ϑ and $\widetilde{V}$ be the cone that has the same axis as V but the apex angle is $\vartheta/2$. We start with the following observation: There exists a radius $R > 0$, depending only on ϑ and r, such that $\widetilde{V} \cap (\mathbb{R}^d \setminus B_R) \subset V$. Indeed, one can choose $R = \frac{r}{\sin(\vartheta/2)}$. The ball B_R contains only finitely many lattice points. Let us now consider all these lattice points in B_R as vectors and refer to them as B_R-vectors. We choose an apex angel $\varepsilon > 0$ such that 2ε is smaller than every angle between two arbitrary B_R-vectors. If now $V(\varepsilon)$ is a cone with apex angel such that $V(\varepsilon) \cap B_R$ contains a lattice point z, then every other lattice point in $V(\varepsilon) \cap B_R$ lies on the line through z and the origin. Thus, if we consider any cone with apex angle $\varepsilon/3$, then the intersection with this cone and B_R cannot contain a lattice point. Setting $\theta = \min(\vartheta, \varepsilon/3)$ we obtain the claim of the lemma. ■

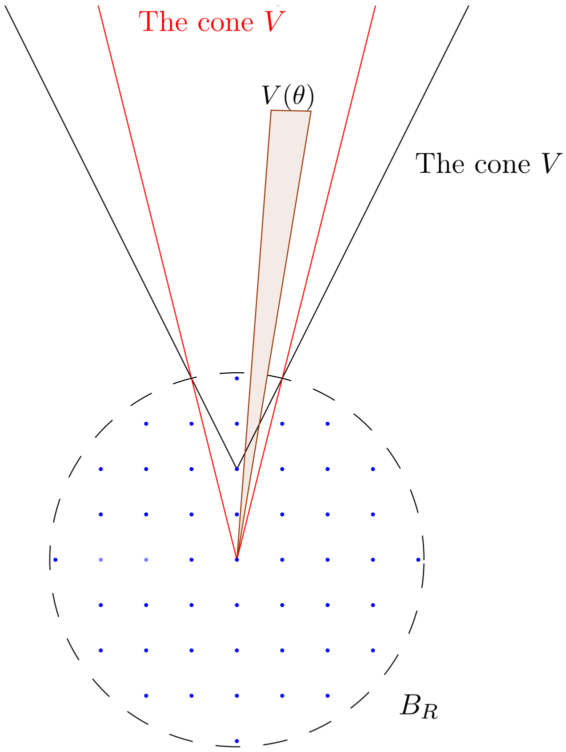

Figure 8.1.: Construction of V_θ

Remark. The above lemma in particular yields the following: For every $m \in \{1, ..., L\}$ there exists an axis $v(m) \in \mathbb{R}^d$ so that

$$\left(V(v(m), \theta) \cap \mathbb{Z}^d \right) \; \subset \; \left(V_r^m \cap \mathbb{Z}^d \right),$$

where θ is as above. In this way we use the lemma in the upcoming proposition.

We are now in a position to prove (8.3) for ω_h^k. We start by considering the case $h = 1$. A technicality is that we need to modify the configuration Γ, which also leads to a possibly smaller infimum ϑ' of all apex angles of the cones in the new configuration. However, this new constant ϑ' depends only on ϑ.

Proposition 8.6. *Let $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ be a symmetric and measurable function satisfying (8.1) for a ϑ-admissible configuration Γ. Then there are constants $C = C(d, \vartheta) > 0$ and $\vartheta' \in (0, \frac{\pi}{2}]$ and a ϑ'-bounded configuration Γ' such that for all $x, y \in \mathbb{Z}^d$ with $|x - y| > \sqrt{d}$ it holds*

$$\omega_1^k(x, y) \; \geq \; C\Lambda \left(\mathbb{1}_{V^{\Gamma'}[x]}(y) + \mathbb{1}_{V^{\Gamma'}[y]}(x) \right) |x - y|^{-d-\alpha}.$$

The angle ϑ' does only depend on the infimum ϑ of the apex angles of all cones in Γ. There is no further dependence on Γ.

Proof. Note that

$$\omega_1^k(x,y) \;\geq\; \Lambda \int_{A_1(x)} \int_{A_1(y)} \Big[\mathbb{1}_{V^{\Gamma(s)}}(t) + \mathbb{1}_{V^{\Gamma(t)}}(s)\Big]\, |t-s|^{-d-\alpha}\, \mathrm{d}s\, \mathrm{d}t.$$

Therefore, we just need to concentrate on the integral. Let m be a 1-favored index at x and n be a 1-favored index at y. Then, using Lemma 6.13, Lemma 8.3, Lemma 8.5, and Lemma 8.4, we estimate

$$\begin{aligned}
&\int_{A_1(x)} \int_{A_1(y)} \Big[\mathbb{1}_{V^{\Gamma}[s]}(t) + \mathbb{1}_{V^{\Gamma}[t]}(s)\Big]\, |t-s|^{-d-\alpha}\, \mathrm{d}s\, \mathrm{d}t \\
\geq\;& \int_{A_1^m(x)} \int_{A_1^n(y)} \big[\mathbb{1}_{V^m[s]}(t) + \mathbb{1}_{V^n[t]}(s)\big]\, |t-s|^{-d-\alpha}\, \mathrm{d}s\, \mathrm{d}t \\
\geq\;& \int_{A_1^m(x)} \int_{A_1^n(y)} \Big[\mathbb{1}_{V^m_{\sqrt{d}/2}[x]}(t) + \mathbb{1}_{V^n_{\sqrt{d}/2}[y]}(s)\Big]\, |t-s|^{-d-\alpha}\, \mathrm{d}s\, \mathrm{d}t \\
\geq\;& \frac{1}{(2\sqrt{d})^{d+\alpha}}\, |A_1^m(x) \times A_1^n(y)|\, \Big[\mathbb{1}_{V^m_{\sqrt{d}}[x]}(y) + \mathbb{1}_{V^n_{\sqrt{d}}[y]}(x)\Big]\, |x-y|^{-d-\alpha} \\
\geq\;& \frac{1}{(2\sqrt{d})^{d+\alpha}}\, |A_1^m(x) \times A_1^n(y)|\, \big[\mathbb{1}_{V(v(m),\theta)[x]}(y) + \mathbb{1}_{V(v(n),\theta)[y]}(x)\big]\, |x-y|^{-d-\alpha}.
\end{aligned}$$

Here θ is the constant from Lemma 8.5, which does only depend on ϑ and the dimension d. Now the claim follows with $C = \frac{1}{(2\sqrt{d})^{d+2}\cdot L^2} \leq \frac{1}{(2\sqrt{d})^{d+\alpha}}\, |A_1^m(x) \times A_1^n(y)|$ and some appropriate choice of Γ'. We can choose $\vartheta' = \theta$. ■

Corollary 8.7. *Let $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ be a symmetric and measurable function satisfying* (8.1) *for a ϑ-admissible configuration Γ. Then there are $\vartheta' > 0$ and $C > 0$ so that for each $h > 0$ there is a configuration Γ^h on $\mathbb{R}^d$ with the following properties:*

(i) The infimum of the apex angles of all cones in $\Gamma^h(\mathbb{R}^d)$ equals ϑ'.

(ii) For all $x, y \in h\mathbb{Z}^d$ with $|x-y| > \sqrt{d}h$, the inequality

$$\omega_h^k(x,y) \;\geq\; C\Big(\mathbb{1}_{V^{\Gamma^h}[x]}(y) + \mathbb{1}_{V^{\Gamma^h}[y]}(x)\Big)\, |x-y|^{-d-\alpha} \tag{8.5}$$

holds.

Proof. For $h > 0$ define a new configuration Γ_h on $\mathbb{R}^d$ by $\Gamma_h(x) = \Gamma(hx)$. Note that the infimum of the apex angles of all cones in $\Gamma_h(\mathbb{R}^d)$ is the same as the infimum of the apex angles of all cones in $\Gamma(\mathbb{R}^d)$. It does not depend on h. Note also that (M) holds true for

Γ if and only if (M) holds true for Γ_h. Therefore, Γ_h is a ϑ-admissible configuration. Define $k_h : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ via $k_h(x,y) = k(hx,hy)h^{d+\alpha}$. Since k satisfies (8.1), we also have for almost all $x, y \in \mathbb{R}^d$ the inequality

$$k(hx,hy)h^{d+\alpha} \geq \Lambda \left(\mathbb{1}_{V^\Gamma[hx]}(hy) + \mathbb{1}_{V^\Gamma[hy]}(hx) \right) |x-y|^{-d-\alpha}. \tag{8.6}$$

Fix some $h > 0$. We note that for all $x, y \in \mathbb{R}^d$ the statement $hy \in V^\Gamma[hx]$ is equivalent to $y \in V^{\Gamma_h}[x]$. Indeed,

$$hy \in V^\Gamma[hx] \Leftrightarrow h(y-x) \in \Gamma(hx) \Leftrightarrow y - x \in \Gamma(hx) \Leftrightarrow y \in V^{\Gamma_h}[x].$$

This together with (8.6) shows that (8.1) is satisfied if we plug in $k = k_h$ and $\Gamma = \Gamma_h$. Therefore, we can apply Proposition 8.6 to $\Gamma = \Gamma_h$ and $k = k_h$. We obtain a configuration $(\Gamma_h)'$ with a positive infimum of the apex angles of all cones ϑ' and some constant $C > 0$ such that for all $x, y \in \mathbb{Z}^d$ with $|x-y| > \sqrt{d}$, we have

$$\omega_1^{k_h}(x,y) \;\geq\; C\Lambda \left(\mathbb{1}_{V^{(\Gamma_h)'}[x]}(y) + \mathbb{1}_{V^{(\Gamma_h)'}[y]}(x) \right) |x-y|^{-d-\alpha}. \tag{8.7}$$

Note that ϑ' does only depend on the infimum of the apex angles of all cones in Γ. We define a new configuration $(\Gamma_h)'_{h^{-1}}$ via $(\Gamma_h)'_{h^{-1}}(x) = (\Gamma_h)'(h^{-1}x)$. The infimum of the apex angles of all cones in this new configuration is obviously still ϑ'. Since for all $x, y \in \mathbb{Z}^d$

$$y \in V^{(\Gamma_h)'}[x] \;\;\Leftrightarrow\;\; hy \in V^{(\Gamma_h)'_{h^{-1}}}[hx]\,,$$

inequality (8.7) is equivalent to

$$\omega_1^{k_h}(x,y) \;\geq\; C\Lambda \left(\mathbb{1}_{V^{(\Gamma_h)'_{h^{-1}}}[hx]}(hy) + \mathbb{1}_{V^{(\Gamma_h)'_{h^{-1}}}[hy]}(hx) \right) |x-y|^{-d-\alpha} \tag{8.8}$$

for all $x, y \in \mathbb{Z}^d$ with $|x-y| > \sqrt{d}$.
Now let $x, y \in h\mathbb{Z}^d$ with $|x-y| > \sqrt{d}h$. Then $h^{-1}x, h^{-1}y \in \mathbb{Z}^d$ with $|h^{-1}x - h^{-1}y| > \sqrt{d}$. With use of (8.8) and a change of variables we obtain

$$\begin{aligned}
&C\Lambda \left(\mathbb{1}_{V^{(\Gamma_h)'_{h^{-1}}}[x]}(y) + \mathbb{1}_{V^{(\Gamma_h)'_{h^{-1}}}[y]}(x) \right) |x-y|^{-d-\alpha} \\
&= C\Lambda \left(\mathbb{1}_{V^{(\Gamma_h)'_{h^{-1}}}[h(h^{-1}x)]}(h(h^{-1}y)) \right. \\
&\qquad\qquad \left. + \mathbb{1}_{V^{(\Gamma_h)'_{h^{-1}}}[h(h^{-1}y)]}(h(h^{-1}x)) \right) |h^{-1}x - h^{-1}y|^{-d-\alpha} h^{-d-\alpha} \\
&\leq \omega_1^{k_h}(h^{-1}x, h^{-1}y) h^{-d-\alpha} \\
&= \omega_h^k(x,y).
\end{aligned}$$

Now one may rename the constant $C\Lambda$ to C. The claim follows with $\Gamma^h = (\Gamma_h)'_{h^{-1}}$. ■

8.3. The limiting argument and the proof of the main result

Lemma 8.8. *Let $B \subset \mathbb{R}^d$ be a ball. Let $\alpha \in (0,2)$ and Γ be a ϑ-admissible configuration. Let $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ be a measurable function satisfying $k(x,y) = k(y,x)$ and (8.1). Then $H^k(B) \subset H^{\frac{\alpha}{2}}(B)$. Furthermore,*

$$\|f\|_{\dot{H}^k(B)} \;\geq\; c\|f\|_{\dot{H}^{\frac{\alpha}{2}}(B)} \quad \text{for } f \in H^k(B),$$

where $c > 0$ is a constant which is independent of the ball B. In the case $0 < \alpha_0 \leq \alpha < 2$ the constant may be chosen to depend on α_0 but not on α.

Proof. Let $R > 0$, $x_0 \in \mathbb{R}^d$ and κ as in Corollary 8.2. Troughout the proof we use the notation $B = B_R(x_0)$ and $B^* = B_{\kappa R}(x_0)$. Let $f \in H^k(B^*)$. For $h \in (0,1)$ we consider the following piecewise constant approximation of f. We define for $x \in h\mathbb{Z}^d \cap B^*$ a function f_h via

$$f_h(x) \;=\; h^{-d} \int_{A_h(x)\cap B^*} f(s)\,\mathrm{d}s\,.$$

Because of Corollary 8.7, there is a constant $C > 0$ and a configuration Γ_h with $\vartheta' > 0$ such that for all $x, y \in h\mathbb{Z}^d$ with $|x-y| > \sqrt{d}h$ the inequality

$$\omega_h^k(x,y) \;\geq\; C\left(\mathbb{1}_{V^{\Gamma_h}[x]}(y) + \mathbb{1}_{V^{\Gamma_h}[y]}(x)\right)|x-y|^{-d-\alpha}$$

holds. Thus, $\omega = \omega_h^k$ together with $\Gamma = \Gamma_h$ fulfill (8.3) for $R_0 = \sqrt{d}$ and $\Lambda = C$. Corollary 8.2 implies the existence of $c > 0$, independent of f, R, α and h, so that

$$\sum_{\substack{x,y\in B^*\cap h\mathbb{Z}^d\\ |x-y|>\sqrt{d}h}} (f_h(x)-f_h(y))^2\omega(x,y) \;\geq\; c \sum_{\substack{x,y\in B\cap h\mathbb{Z}^d\\ |x-y|>\sqrt{d}h}} (f_h(x)-f_h(y))^2|x-y|^{-d-\alpha}.$$

Using Lemma 8.4 we obtain

$$\begin{aligned}
&\sum_{\substack{x,y\in B^*\cap h\mathbb{Z}^d\\ |x-y|>\sqrt{d}h}} (f_h(x)-f_h(y))^2 \int_{A_h(x)}\int_{A_h(y)} k(s,t)\,\mathrm{d}s\,\mathrm{d}t \\
&\geq\; c \sum_{\substack{x,y\in B\cap h\mathbb{Z}^d\\ |x-y|>\sqrt{d}h}} (f_h(x)-f_h(y))^2 \int_{A_h(x)}\int_{A_h(y)} |s-t|^{-d-\alpha}\,\mathrm{d}s\,\mathrm{d}t, \qquad (8.9)
\end{aligned}$$

for a constant $c > 0$ that differs from the one above by a factor only depending on the dimension d.

For technical reasons we need the property that every $x \in \mathbb{R}^d$ is contained in some cube. Therefore, we consider half-closed cubes. Given $x = (x_1, ..., x_d) \in \mathbb{R}^d$ and $h \in (0,1)$, we recall the notation

$$\widetilde{A}_h(x) \;=\; \prod_{i=1}^{d}\Big[x_i - \frac{h}{2}, x_i + \frac{h}{2}\Big).$$

For $h \in (0,1)$, we define a function $g_h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ via

$$g_h(s,t) = \sum_{x,y \in h\mathbb{Z}^d} \Big[(f_h(x) - f_h(y))^2 k(s,t) \, \mathbb{1}_{\widetilde{A}_h(x) \times \widetilde{A}_h(y)}(s,t) \, \mathbb{1}_{\{x,y \in B^* | \sqrt{d}h < |x-y|\}}(x,y) \Big]$$

and claim that g_h converges for $h \to 0$ almost everywhere to the function $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ with

$$g(s,t) = (f(s) - f(t))^2 k(s,t) \mathbb{1}_{B^* \times B^*}(s,t).$$

Indeed, $g_h(s,t) = (f_h(x_h) - f_h(y_h))^2 k(s,t)$ for appropriate points x_h and y_h, whenever $s \neq t$ and h is sufficiently small. We conclude with the help of Lemma A.3, $g_h(s,t) \to g(s,t)$ for almost every $(s,t) \in B^* \times B^*$. In the same way we can show that the function $\widetilde{g}_h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ with

$$\begin{aligned} \widetilde{g}_h(s,t) = \sum_{x,y \in h\mathbb{Z}^d} \Big[& (f_h(x) - f_h(y))^2 |s-t|^{-d-\alpha} \, \mathbb{1}_{\widetilde{A}_h(x) \times \widetilde{A}_h(y)}(s,t) \\ & \times \mathbb{1}_{\{x,y \in B | \sqrt{d}h < |x-y|\}}(x,y) \Big] \end{aligned}$$

converges for $h \to 0$ pointwise a.e. to

$$\begin{aligned} & \widetilde{g} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \\ & \widetilde{g}(s,t) = (f(s) - f(t))^2 |s-t|^{-d-\alpha} \mathbb{1}_{B \times B}(s,t). \end{aligned}$$

For the left-hand side in (8.9) this implies with the help of the Dominated Convergence Theorem, see Theorem 2.5,

$$\begin{aligned} & \sum_{\substack{x,y \in B^* \cap h\mathbb{Z}^d \\ |x-y| > \sqrt{d}h}} (f_h(x) - f_h(y))^2 \int_{\widetilde{A}_h(x)} \int_{\widetilde{A}_h(y)} k(s,t) \, \mathrm{d}s \, \mathrm{d}t \\ & = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_h(s,t) \, \mathrm{d}s \, \mathrm{d}t \overset{h \to 0}{\longrightarrow} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s,t) \, \mathrm{d}s \, \mathrm{d}t \end{aligned}$$

Note that g_h is dominated by the integrable function $g + \mathbb{1}_{B^* \times B^*} \in H^k(B^*)$ for sufficient small h.

With regard to the right-hand side of (8.9), note that the Lemma of Fatou (see Theorem 2.4) implies

$$\begin{aligned} & \liminf_{h \to 0} \sum_{\substack{x,y \in B \cap h\mathbb{Z}^d \\ |x-y| > \sqrt{d}h}} (f_h(x) - f_h(y))^2 \int_{\widetilde{A}_h(x)} \int_{\widetilde{A}_h(y)} |s-t|^{-d-\alpha} \, \mathrm{d}s \, \mathrm{d}t \\ & = \liminf_{h \to 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widetilde{g}_h(s,t) \, \mathrm{d}s \, \mathrm{d}t \ \geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widetilde{g}(s,t) \, \mathrm{d}s \, \mathrm{d}t. \end{aligned}$$

In conclusion, we have shown that the discrete inequality (8.9) yields the continuous version

$$\|f\|_{\dot{H}^k(B^*)} \;\geq\; c\|f\|_{\dot{H}^{\frac{\alpha}{2}}(B)} \quad \text{for all } f \in H^k(B^*).$$

This is true for every ball B since c is independent of B. Using Lemma A.2, we conclude for each ball $B \subset \mathbb{R}^d$ and each $f \in H^k(B)$

$$\|f\|_{\dot{H}^k(B)} \;\geq\; c^*\|f\|_{\dot{H}^{\frac{\alpha}{2}}(B)}$$

for some $c^* > 0$, independent of the ball B. This proves the claim of the lemma. ∎

Proof of Theorem 8.1. The inequality (8.2) is obviously true if the left-hand side is infinite. Hence, we can restrict ourselves to functions $f \in H^k(B)$. The above Lemma 8.8 yields the existence of $c > 0$ such that $\|\cdot\|_{\dot{H}^k(B)} \geq c\|\cdot\|_{\dot{H}^{\frac{\alpha}{2}}(B)}$ on $H^k(B)$, which implies (8.2). The proof is complete. ∎

9. Applications of the main result

In this chapter we provide statements that are based on our main result Theorem 8.1 respectively on Lemma 8.8.

Function Spaces

Theorem 9.1. *Let k be a kernel as in Theorem 8.1. Then $H^k(\mathbb{R}^d) \subset H^{\frac{\alpha}{2}}(\mathbb{R}^d)$.*

In addition, let k satisfy

$$k(x,y) \leq C|x-y|^{-d-\alpha} \text{ for almost all } x,y \in \mathbb{R}^d, x \neq y \text{ and } C \geq 1. \tag{9.1}$$

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then the spaces $H^k(\Omega)$ and $H^{\frac{\alpha}{2}}(\Omega)$ coincide. The seminorms $\|\cdot\|_{\dot{H}^k(\Omega)}$ and $\|\cdot\|_{\dot{H}^{\frac{\alpha}{2}}(\Omega)}$ as well as the corresponding norms are comparable on $H^k(\Omega)$. The subspace $C^\infty(\overline{\Omega})$ is dense in $H^k(\Omega)$.

Moreover, $H^k(\mathbb{R}^d) = H^{\frac{\alpha}{2}}(\mathbb{R}^d)$. The seminorms $\|\cdot\|_{\dot{H}^k(\mathbb{R}^d)}$ and $\|\cdot\|_{\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)}$ as well as the corresponding norms are comparable on $H^k(\mathbb{R}^d)$. The subspace $C_c^\infty(\mathbb{R}^d)$ of smooth functions with compact support in $\mathbb{R}^d$ is dense in $H^k(\mathbb{R}^d)$.

Proof of Theorem 9.1. The constant c in Lemma 8.8 is independent of the radius R of the respective ball. Thus the result for the whole space is obtained in the limit $R \to \infty$ using the Monotone Convergence Theorem, compare Theorem 2.3.

Now let Ω be a bounded Lipschitz domain. In view of Lemma 8.8 and Lemma A.2 we conclude

$$\|\cdot\|_{\dot{H}^k(\Omega)} \;\geq\; c\|\cdot\|_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} \text{ on } H^k(\Omega)$$

for a constant $c > 0$, which leads to $H^k(\Omega) \subset H^{\frac{\alpha}{2}}(\Omega)$. Since the inclusion $H^{\frac{\alpha}{2}}(\Omega) \subset H^k(\Omega)$ is obvious by the properties of k, one obtains $H^k(\Omega) = H^{\frac{\alpha}{2}}(\Omega)$. For the assertions concerning density of smooth functions, we note that $C^\infty(\overline{\Omega})$ is a dense subset of $H^{\frac{\alpha}{2}}(\Omega)$, see Proposition 2.23. Furthermore, $C_c^\infty(\mathbb{R}^d)$ is a dense subset of $H^{\frac{\alpha}{2}}(\mathbb{R}^d)$, see Proposition 2.22. ∎

Stochastic Processes

Corollary 9.2. *Let k be as in Theorem 8.1 and, additionally, let k satisfy the pointwise upper bound* (9.1)*. The Dirichlet form $(\mathcal{E}^k, \mathcal{D}(\mathcal{E}^k))$ on $L^2(\mathbb{R}^d)$ with $\mathcal{D}(\mathcal{E}^k) = H^{\frac{\alpha}{2}}(\mathbb{R}^d)$ and*

$$\mathcal{E}^k(f,g) \;=\; \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(y)-f(x))\,(g(y)-g(x))\,k(x,y)\,\mathrm{d}x\,\mathrm{d}y\,,$$

is a regular Dirichlet form on $L^2(\mathbb{R}^d)$. There exists a corresponding Hunt process.

Proof. It is obvious that $\mathcal{E}^k$ satisfies the first two conditions in Definition 5.1. The inner product $\mathcal{E}_1^k(\cdot,\cdot)$ induces the norm $\|\cdot\|_{H^k(\mathbb{R}^d)}$, which is comparable to $\|\cdot\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}$ by Theorem 9.1. As mentioned in Section 2.2, the space $\left(H^{\frac{\alpha}{2}}(\mathbb{R}^d), \|\cdot\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}\right)$ is a Hilbert space. Together this shows that the third condition in Definition 5.1 is true. An easy case analysis shows that condition 4 in Definition 5.1 holds true.

Since the norm induced by $\mathcal{E}_1^k$ is comparable to $\|\cdot\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}$, we conclude from Proposition 2.22 that $\overline{C_c^\infty(\mathbb{R}^d)}^{\|\cdot\|_{H^k(\mathbb{R}^d)}} = \overline{(\mathcal{D}(\mathcal{E}^k)\cap C_c(\mathbb{R}^d))}^{\|\cdot\|_{H^k(\mathbb{R}^d)}} = H^{\frac{\alpha}{2}}(\mathbb{R}^d)$. It is well known that $C_c^\infty(\mathbb{R}^d)$ is dense in $C_c(\mathbb{R}^d)$ with respect to uniform convergence.

The above reasoning shows that $(\mathcal{E}^k, \mathcal{D}(\mathcal{E}^k))$ is a regular Dirichlet form on $L^2(\mathbb{R}^d)$. The existence of the Hunt process follows from Theorem 5.2. ∎

Weak Harnack inequality and regularity theory for corresponding weak (super-)solutions

A very strong consequence of our coercivity result Theorem 8.1 is that we can apply the regularity theory developed in [DK20] and obtain a weak Harnack inequality as well as Hölder estimates for weak solutions, provided that we assume an upper bound of the kernel as in (9.1). First, let us mention that the results in [DK20] are very general and apply for a wider class of measures. Let $k:\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty]$ be a kernel and $\alpha\in(0,2)$. Dyda and Kassmann require two assumptions to hold true in order to have a weak Harnack inequality:

(A) Comparability of seminorms on every scale locally in the unit ball, that is,

$$\|v\|_{\dot{H}^k(B_r(x_0))} \asymp \|v\|_{\dot{H}^{\frac{\alpha}{2}}(B_r(x_0))}$$

for every $r\in(0,1), x_0\in B_1$ and each $v\in H^{\frac{\alpha}{2}}(B_r(x_0))$, where the comparability constant is independent of v, r and x_0, cf. [DK20, Assumption (A)].

(B) The existence of cutoff functions, cf. [DK20, Assumption (B)].

For symmetric kernels that satisfy (8.1) and (9.1) we know that (A) holds true by Theorem 8.1. In our case (B) does not impose an additional restriction because we can always choose the standard cutoff function given on page 4 of [DK20]. Therefore, we conclude from [DK20, Theorem 1.2] combined with a scaling argument (see also Lemma 11.6) the following weak Harnack inequality using the notations *inf* respectively *sup* for the essential infimum respectively essential supremum.

Theorem 9.3 (Weak Harnack inequality)**.** *Let $r > 0, x_0 \in \mathbb{R}^d$. Assume $\alpha_0 > 0$, $\alpha \in [\alpha_0, 2)$, and the kernel k is as in Theorem 8.1 and satisfies (9.1). Suppose $f \in L^q(B_r(x_0))$ for some $q > \frac{d}{\alpha}$. Let $u \in V^k(B_r(x_0)|\mathbb{R}^d)$, $u \geq 0$ in $B_r(x_0)$ and assume u satisfies $\mathcal{E}^k(u,\psi) \geq (f,\psi)$ for every nonnegative $\psi \in H^k_{B_r(x_0)}(\mathbb{R}^d)$. Then*

$$\inf_{B_{\frac{r}{4}}(x_0)} u \geq c \left[\left(\fint_{B_{\frac{r}{2}}(x_0)} u(x)^{p_0} \, \mathrm{d}x \right)^{\frac{1}{p_0}} - \sup_{x \in B_{\frac{15}{16}r}(x_0)} \int_{\mathbb{R}^d} u_-(z) k(x,z) \, \mathrm{d}z - r^{\alpha - \frac{d}{q}} \|f\|_{L^q(B_{\frac{15}{16}r}(x_0))} \right], \tag{9.2}$$

with constants $p_0 \in (0,1), c > 0$ depending only on d, α_0, Λ and ϑ.

The aim now is to show that the weak Harnack inequality already implies the Hölder continuity of corresponding weak solutions. In order to do that, we need a result on the decay of the oscillation, which we state below in a rather general form. It is a version of [DK20, Theorem 1.5] adapted to our setup. In Theorem 1.5 of Dyda's and Kassmann's article the authors only treat the situation where the right-hand side of the equation f is equal to 0. For this reason we give a full prove of the oscillation estimate including a right-hand side f. Nevertheless, the proof is just an adaptation of [CK20, Theorem 4.1] to our setting.

Theorem 9.4. *Let $\alpha \in (0,2), x_0 \in \mathbb{R}^d$, $r_0 > 0$. Suppose $1 < \theta < \lambda < \Theta$. Assume k is a symmetric kernel so that the weak Harnack inequality holds true in $B_r(x_0)$ with constants θ, λ, Θ, that is, there are constants $p \in (0,1), c_H > 0$ such that the following holds: If $0 < r \leq r_0$, $f \in L^q(B_r(x_0))$ for $q > \frac{d}{\alpha}$, and $u \in V^k(B_r(x_0)|\mathbb{R}^d)$ satisfies $u \geq 0$ in $B_r(x_0)$ as well as $\mathcal{E}^k(u,\psi) = (f,\psi)$ for every $\psi \in H^k_{B_r(x_0)}(\mathbb{R}^d)$, then*

$$\left(\fint_{B_{\frac{r}{\lambda}}(x_0)} u^p(x) \, \mathrm{d}x \right)^{\frac{1}{p}} \leq c_H \left(\inf_{B_{\frac{r}{\Theta}}(x_0)} u + r^\alpha \sup_{x \in B_{\frac{r}{\theta}}(x_0)} \int_{\mathbb{R}^d} u_-(z) \, k(x,z) \, \mathrm{d}z \right) + c_H r^{\alpha - \frac{d}{q}} \|f\|_{L^q\left(B_{\frac{r}{\theta}}\right)(x_0)}. \tag{9.3}$$

Then there exists $\beta \in (0,1)$ such that for $0 < r \leq r_0$, $u \in V^k(B_r(x_0)|\mathbb{R}^d)$ satisfying

$\mathcal{E}^k(u,\psi) = (f,\psi)$ for every $\psi \in H^k_{B_r(x_0)}(\mathbb{R}^d)$, it holds

$$\operatorname*{osc}_{B_\rho(x_0)} u \le 2\Theta^\beta \|u\|_{L^\infty} \left(\frac{\rho}{r}\right)^\beta + c\left(\frac{\rho}{r}\right)^\beta r^{\alpha-\frac{d}{q}} \|f\|_{L^q\left(B_{\frac{r}{\theta}}(x_0)\right)} \quad (0<\rho\le r),$$

where $\operatorname{osc}_B u = \sup_B u - \inf_B u$ for $B \subset \mathbb{R}^d$.

Proof. We follow the lines of the proof of [CK20, Theorem 4.1], which is a modification of the proof of [DK20, Theorem 1.5]. In the following we may and do assume $x_0 = 0$. We abbreviate our notation by $B_r = B_r(0)$ for $r > 0$ and $F = \|f\|_{L^q(B_{\frac{r}{\theta}})}$. We remark that we may assume $\|u\|_{L^\infty} < \infty$, because otherwise the assertion is obviously true. Let c_H and p be the constants from the weak Harnack inequality (9.3). We set $\kappa = (2c_H 2^{\frac{1}{p}})^{-1}$ and choose $\beta \le \frac{-\ln\left(1-\frac{2}{\kappa}\right)}{\ln(\Theta)}$, which implies

$$1 - \frac{\kappa}{2} \le \Theta^{-\beta}. \tag{9.4}$$

The last inequality will be important in the remainder.

Suppose $0 < r \le r_0$ and $u \in V^k(B_r|\mathbb{R}^d)$ satisfies $u \ge 0$ in B_r and $\mathcal{E}^k(u,\psi) = (f,\psi)$ for every $\psi \in H^k_{B_r}(\mathbb{R}^d)$. Set

$$\widetilde{u} = \frac{u}{\|u\|_{L^\infty} + \frac{2}{\kappa} r^{\alpha-\frac{d}{q}} F}.$$

Furthermore, we set

$$M_0 = \|\widetilde{u}\|_{L^\infty}, \quad m_0 = \inf_{x\in\mathbb{R}^d} \widetilde{u}, \quad M_{-n} = M_0 \quad \text{and } m_{-n} = m_0.$$

Claim (♡): There is an increasing sequence (m_n) and a decreasing sequence (M_n) such that for all $n \in \mathbb{Z}$

$$\begin{cases} m_n \le \widetilde{u} \le M_n & \text{almost everywhere on } B_{r\Theta^{-n}}, \\ M_n - m_n \le 2\Theta^{-n\beta}. \end{cases} \tag{9.5}$$

Let us first show that (9.5) implies the claim of the theorem. Let $0 < \rho \le r$. We can find $j \in \mathbb{N}_0$ such that $r\Theta^{-j-1} \le \rho \le r\Theta^{-j}$, which implies $\Theta^{-j} \le \rho\Theta/r$. With (9.5) we estimate

$$\operatorname*{osc}_{B_\rho} \widetilde{u} \le \operatorname*{osc}_{B_{r\Theta^{-j}}} \widetilde{u} \le M_j - m_j \le 2\Theta^{-\beta j} \le 2\Theta^\beta \left(\frac{\rho}{r}\right)^\beta.$$

After plugging in the definition of $\widetilde{u}$, we obtain the desired decay of oscillation.

It remains to prove Claim (♡). The idea is to use the method of complete induction. Assume there exists $i \in \mathbb{N}$ and there are M_n, m_n such that (9.5) hols true for $n \le i-1$. Then we need to choose M_k and m_k in a way so that (9.5) is satisfied for $n = i$.

For $z \in \mathbb{R}^d$ set
$$v(z) = \left(\widetilde{u}(z) - \frac{M_{i-1} + m_{i-1}}{2}\right)\Theta^{\beta(i-1)}.$$
Then $|v(z)| \le 1$ for almost every $z \in B_{r\Theta^{-(i-1)}}$. The definition of v implies also that
$$\mathcal{E}^k(v, \psi) = (\widetilde{f}, \psi)$$
for every $\psi \in H^k_{B_{r\Theta^{-(i-1)}}}(\mathbb{R}^d)$, where
$$\widetilde{f}(x) = \frac{\Theta^{(i-1)\beta}}{\|u\|_{L^\infty} + \frac{2}{\kappa} r^{\alpha - \frac{d}{q}} F}\, f(x), \quad x \in B_{r\Theta^{-(i-1)}}. \tag{9.6}$$
Our aim is to show that (9.3) implies either $v \le 1 - \kappa$ or $v \ge \kappa - 1$ on $B_{r\Theta^{-i}}$. This will enable us to choose m_i and M_i.

Given $z \in \mathbb{R}^d \setminus B_{r\Theta^{-(i-1)}}$, there is $j \in \mathbb{N}$ such that $z \in B_{r\Theta^{-(i-j-1)}} \setminus B_{r\Theta^{-(i-j)}}$. For such j and z we conclude, using the induction hypothesis and the properties of the sequences,
$$\begin{aligned} v(z) &\le \left(M_{i-j-1} - m_{i-j-1} + m_{i-j-1} - \frac{M_{i-1} + m_{i-1}}{2}\right)\Theta^{\beta(i-1)} \\ &\le \left(M_{i-j-1} - m_{i-j-1} + \frac{m_{i-1} - M_{i-1}}{2}\right)\Theta^{\beta(i-1)} \\ &\le (M_{i-j-1} - m_{i-j-1})\,\Theta^{\beta(i-1)} - v(z) \\ &\le 2\Theta^{\beta j} - v(z). \end{aligned}$$
This implies
$$v(z) \le \Theta^{j\beta} \le \Theta^{j\beta} - 1 + \Theta^{j\beta} \le 2\Theta^{j\beta} - 1. \tag{9.7}$$
In the same way we get $v(z) \ge 1 - 2\Theta^{\beta j}$.

We distinguish two cases:

Case 1: $|\{x \in B_{r\Theta^{-(i-1)}/\lambda} \,|\, v(x) \le 0\}| \ge \frac{1}{2}|B_{r\Theta^{-(i-1)}/\lambda}|$

In this case we show
$$v(z) \le 1 - \kappa \quad \text{for almost every } z \in B_{r\Theta^{-i}}. \tag{9.8}$$
This is sufficient for (9.5) to hold true. Indeed, recalling (9.4),
$$\begin{aligned} \widetilde{u}(z) &= \Theta^{-(i-1)\beta} v(z) + \frac{M_{i-1} + m_{i-1}}{2} \\ &\le \Theta^{-(i-1)\beta}(1 - \kappa) + \frac{M_{i-1} - m_{i-1}}{2} + m_{i-1} \\ &\le m_{i-1} + \left(1 - \frac{\kappa}{2}\right) 2\Theta^{-(i-1)\beta} \end{aligned}$$

$$\leq m_{i-1} + 2\Theta^{-i\beta}.$$

Choosing $m_i = m_{i-1}$ and $M_i = m_{i-1} + 2\Theta^{-i\beta}$, we obtain, by the induction hypothesis, $u(z) \geq m_{i-1} = m_i$ and, by the above calculation, $u(z) \leq M_i$.

It remains to prove (9.8). Let $w = 1 - v \in V^k(B_r|\mathbb{R}^d)$. Then w satisfies $w \geq 0$ in $B_{r\Theta^{-(i-1)}}$ as well as $\mathcal{E}^k(w, \psi) = (-\widetilde{f}, \psi)$ for every $\psi \in H^k_{B_{r\Theta^{-(i-1)}}}(\mathbb{R}^d)$. We may apply the weak Harnack inequality to w for the radius $r_1 = r\Theta^{-(i-1)}$. This gives us

$$\left(\fint_{B_{\frac{r_1}{\lambda}}} w^p(x) \,\mathrm{d}x \right) \leq c_H \left(\inf_{B_{\frac{r_1}{\Theta}}} w + r_1^\alpha \sup_{x \in B_{\frac{r_1}{\theta}}} \int_{\mathbb{R}^d} w_-(x) k(x,z) \,\mathrm{d}z \right) + c_H r_1^{\alpha - \frac{d}{q}} \|\widetilde{f}\|_{L^q\left(B_{\frac{r_1}{\theta}}\right)}.$$

The left-hand side can be estimated using the assumption of Case 1 in the following way

$$\left(\fint_{B_{\frac{r_1}{\lambda}}} w^p(x) \right)^{\frac{1}{p}} \geq \left(\frac{|\{x \in B_{r_1/\lambda} \,|\, v(x) \leq 0\}|}{|B_{r_1/\lambda}|} \right)^{\frac{1}{p}} \geq 2^{-\frac{1}{p}}. \tag{9.9}$$

If we choose $\beta \leq \alpha - \frac{d}{q}$, then

$$r_1^{\alpha - \frac{d}{q}} \|\widetilde{f}\|_{L^q\left(B_{\frac{r_1}{\theta}}\right)} \leq \frac{\|f\|_{L^q\left(B_{\frac{r_1}{\theta}}\right)} \Theta^{(i-1)(\beta - \alpha + \frac{d}{q})}}{\frac{2}{\kappa} F} \leq \frac{\kappa}{2}. \tag{9.10}$$

Let us also take a closer look at the tail term. We deduce with the help of (9.7) and

$$\frac{|z|}{|x-z|} \leq 1 + \frac{r\Theta^{-i+1}/\theta}{r\Theta^{-i+j} - r\Theta^{-i+1}/\theta} \leq 1 + \frac{1}{\Theta^{j-1}(\theta - 1)} \leq 1 + \frac{1}{\theta - 1} = c(\theta)$$

for every $x \in B_{r\Theta^{-(i-1)}/\theta}$ and $z \in \mathbb{R}^d \setminus B_{r\Theta^{-i+j}}$ the estimate

$$\begin{aligned}
\int_{\mathbb{R}^d \setminus B_{r\Theta^{-(i-1)}}} w_-(z) k(x,z) \,\mathrm{d}z &= \sum_{j=1}^\infty \int_{B_{r\Theta^{-(i-j-1)}} \setminus B_{r\Theta^{-(i-j)}}} (1 - v(z))^- k(x,z) \,\mathrm{d}z \\
&\leq \sum_{j=1}^\infty (2\Theta^{j\beta} - 2) \int_{\mathbb{R}^d \setminus B_{r\Theta^{-(i-j)}}} k(x,z) \,\mathrm{d}z \\
&\leq \sum_{j=1}^\infty (\Theta^{j\beta} - 1) c(\theta, d, \alpha) \int_{\mathbb{R}^d \setminus B_{r\Theta^{-(i-j)}}} |z|^{-d-\alpha} \,\mathrm{d}z \\
&\leq \sum_{j=1}^\infty (\Theta^{j\beta} - 1) c(\theta, d, \alpha) (r\Theta^{-i+j})^{-\alpha}
\end{aligned}$$

for every $x \in B_{r\Theta^{-(i-1)}/\theta}$, where the constant $c(\theta, d, \alpha)$ may differ from line to line. Thus, for every $\ell \in \mathbb{N}$

$$
\begin{aligned}
(r\Theta^{-i+1})^{\alpha} \int_{\mathbb{R}^d} w_-(z)k(x,z)\,\mathrm{d}z &\le c\sum_{j=1}^{\ell}(\Theta^{j\beta}-1)(\Theta^{-j+1})^{\alpha} + 2c\sum_{j=\ell+1}^{\infty}\Theta^{j\beta}(\Theta^{-j+1})^{\alpha} \\
&= I_1 + I_2,
\end{aligned}
$$

where the constant $c > 0$ depends only on θ, d and α. First, we assume $\beta < \frac{\alpha}{2}$. Second, we choose ℓ sufficiently large so that $I_2 \le \frac{\kappa}{4}$. Third, we decrease β in such a way that $I_1 \le \frac{\kappa}{4}$. The constant β may be chosen to depend only on $\vartheta, \Theta, \alpha$ and d. This shows

$$
\sup_{x \in B_{r\Theta^{-(i-1)}}} (r\Theta^{-i+1})^{\alpha} \int_{\mathbb{R}^d} w_-(z)k(x,z)\,\mathrm{d}z \le \frac{\kappa}{2} \tag{9.11}
$$

for arbitrary i. If we combine (9.9), (9.10) and (9.11), then we obtain from the weak Harnack inequality for w on $B_{\frac{r_1}{\Theta}}$ the estimate

$$
2^{-\frac{1}{p}} \le c_H \left(\inf_{B_{\frac{r_1}{\Theta}}} w + \kappa \right) \le c_H w + c_H \kappa,
$$

which implies, recalling the definition of κ, $1 - v = w \ge \kappa$ on $B_{\frac{r_1}{\Theta}} = B_{r\Theta^{-i}}$. The proof of (9.8) is finished.

Case 2: $|\{x \in B_{r\Theta^{-(i-1)}/\lambda} \,|\, v(x) > 0\}| \ge \frac{1}{2}|B_{r\Theta^{-(i-1)}/\lambda}|$

In this case one can show that

$$
v(z) \ge \kappa - 1 \quad \text{for almost every } z \in B_{r\Theta^{-i}}. \tag{9.12}
$$

The proof of the above inequality is similar to the proof of (9.8) in Case 1. Here one has to use the auxiliary function $w = 1 + v$. We omit the details.

It remains to show that (9.12) implies Claim (♡). For almost every $z \in B_{r\Theta^{-i}}$ we deduce

$$
\begin{aligned}
\widetilde{u}(z) &= \frac{v(z)}{\Theta^{(i-1)\beta}} + \frac{M_{i-1} + m_{i-1}}{2} \\
&\ge \frac{\kappa - 1}{\Theta^{(i-1)\beta}} + M_{i-1} - \frac{M_{i-1} - m_{i-1}}{2} \\
&\ge M_{i-1} - 2\Theta^{-i\beta}.
\end{aligned}
$$

We choose $M_i = M_{i-1}$ and $m_i = M_{i-1} - 2\Theta^{-i\beta}$. Then, by the induction hypothesis and the previous calculation for almost every $z \in B_{r\Theta^{-i}}$, we find

$$
m_i \le \widetilde{u}(z) \le M_i.
$$

The proof is finished. ■

Remark. Without studying the proof carefully one may wonder why the above theorem is formulated for solutions u instead of supersolutions as in the previous weak Harnack inequality. The reason is the case analysis in the proof. In Case 1 one needs to consider the function $w = 1 - v$ and if v was only a supersolution, then w would be a subsolution.

From Theorem 9.4 we can now deduce our Hölder regularity result. The proof uses a standard covering argument as it was also used in the proof of [DK20, Corollary 5.2].

Theorem 9.5. *Let k be a kernel as in Theorem 8.1 and assume k satisfies* (9.1). *Let $B_r = B_r(x_0)$ be a ball in $\mathbb{R}^d$. Suppose $f \in L^q(B_r)$ for some $q > \frac{d}{\alpha}$. Assume $u \in V^k(B_r|\mathbb{R}^d)$ fulfills*

$$\mathcal{E}^k(u, \psi) = (f, \psi) \quad \text{for every } \psi \in H^k_{B_r}(\mathbb{R}^d).$$

Then there are constants $c \geq 1$ and $\beta \in (0,1)$, independent of u, such that for almost all $x, y \in B_{\delta r}$, $\delta \in (0,1)$, it holds

$$|u(x) - u(y)| \leq c \left(\frac{|x-y|}{r - \delta r} \right)^{\beta} \left(\|u\|_{L^\infty} + r^{\alpha - \frac{d}{q}} \|f\|_{L^q\left(B_{\frac{15}{16}r}\right)} \right).$$

Proof. In the proof we abbreviate our notation of balls by writing $B_r = B_r(x_0)$. Let $x, y \in B_{\delta r}$. If $|x-y| \geq (r - \delta r)/4$, then the claim of the lemma is obviously true. Let us therefore consider the case $|x - y| < (r - \delta r)/4$. The idea now is to use a covering argument. There exists a number $\rho \in (0, (r-\delta r)/8)$ such that $\rho \leq |x-y| \leq 2\rho$. Then $r - \delta r > 4|x-y| \geq 4\rho$, which implies $y \in B_{r-4\rho}$. We cover $B_{r-4\rho} \subset B_{\delta r}$ by a countable family (B_n) of balls with radius ρ. Each B_n may be chosen as a subset of $B_{r-(7/2)\rho}$. Denote by $3B_n$ the ball with the same center as B_n but of radius 3ρ. Then $B_n \subset 3B_n \subset B_{r-\rho}$.

We conclude from Theorem 9.3 in combination with Theorem 9.4 that there is $\beta \in (0,1)$ and $c \geq 1$ so that

$$\begin{aligned} \operatorname*{osc}_{3B_n} u &\leq c \left(\frac{3\rho}{r - \rho} \right)^{\beta} \left(\|u\|_{L^\infty} + r^{\alpha - \frac{d}{q}} \|f\|_{L^q\left(B_{\frac{15}{16}r}\right)} \right) \\ &\leq c(r - \delta r)^{-\beta} |x-y|^{\beta} \left(\|u\|_{L^\infty} + r^{\alpha - \frac{d}{q}} \|f\|_{L^q\left(B_{\frac{15}{16}r}\right)} \right). \end{aligned}$$

We observe that $x, y \in 3B_n$ for some $n \in \mathbb{N}$. Therefore, $|u(x) - u(y)| \leq \operatorname{osc}_{3B_n} u$. Hence, the claim of the theorem is proved. ■

Part III.

Local boundedness from above and elliptic Harnack inequalities

Detailed outline of Part III

This part deals with weak solutions u of the equation

$$-\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d\setminus B_\varepsilon(x)} (u(y)-u(x))k(x,y)\,\mathrm{d}y = f(x), \quad x\in B_1.$$

Here k is a given symmetric kernel. We propose additional assumptions on k (a coercivity estimate, an upper bound and a specific mean value integral bound, compare Section 11.2) that allow to prove a local boundedness result of corresponding weak solutions. In order to do that, we modify techniques from [DKP14] and [DKP16], which enable us to bound nonlocal tail terms by local quantities.

The purpose of Chapter 10 is to make the reader familiar with the technique of the De Giorgi iteration. Here we consider local elliptic operators in divergence form and prove local boundedness. The following Chapter 11 contains the main part. Here we prove the tail estimate and the local boundedness of weak solutions of the above equation. Applying a result from [DK20] we conclude that weak solutions enjoy a Harnack inequality. In Chapter 12 we discuss examples of kernels that satisfy our assumptions. Here we also provide kernels that are used in Part II of this thesis.

Appendix B pertains to this part of the thesis. This chapter deals with a connection of our result to a condition of Bogdan and Sztonyk given in [BS05]. In the latter article they derive an equivalent statement to the elliptic Harnack inequality for nonlocal operators of (1.1) where the measure $k(x,y)\,\mathrm{d}y$ is replaced by a homogeneous Lévy measure. We consider translation invariant and homogeneous kernels and show directly, under additional assumptions, that our Condition (C) implies the so called *relative Kato* condition, see Section B.2. Moreover, we prove that the combination of all our assumptions is only a sufficient condition for the elliptic Harnack inequality in the case $d=3$, see Section B.3.

Comment on notation used in this part

Let us recall some of the notation we need in this part. For an overview we refer to the section on notation at the end of Chapter 1. By $B_r(x)$ we denote the ball with center $x\in\mathbb{R}^d$ and radius $r>0$ with respect to the norm $|\cdot|$. If nothing else is said, $B_r = B_r(0)$. If u is a function, then we denote by u_- its negative part defined as $u_-(x) = -\min(0,u(x))$ and by u_+ its positive part $u_+(x) = \max(0,u(x))$. Whenever we use the abbreviations *sup* and *inf*, we mean the essential supremum and the essential infimum.

We use several letters (Roman or Greek characters, in upper or lower cases) to denote constants. Sometimes we write the quantities, on which constants depend, in round brackets. We often use the letter c to denote a general positive constant. We especially point out that the value of c may change between different lines of the proof of the same statement.

10. A local prelude: Local boundedness for solutions of local elliptic PDEs

In this chapter we study solutions of local elliptic PDEs in divergence form and obtain a local boundedness result of corresponding solutions, where the bound depends on the L^p-norm of the solution and the right-hand side of the equation. This chapter serves as a motivation to the local boundedness result for solutions of nonlocal PDEs, which will be established in the next chapter. Moreover, it satisfies didactic purposes as we want to make the reader familiar with the technique of the so-called De Giorgi type iteration. This iteration method is also used in the nonlocal setting, however, because of nonlocal terms, the proofs are more technical. We emphasize that the results in this section are well known.

The De Giorgi iteration technique is part of the so-called De Giorgi-Nash-Moser techniques. As mentioned in the introduction of this thesis, these techniques were originally introduced in [De 57; Nas58; Mos60] to study regularity of solutions of elliptic equations with rough coefficients. For further reading on this subject, we refer the reader to the overview [Vas16]. Here the De Giorgi iteration is contained in the proof of [Vas16, Lemma 5] and the local boundedness result is stated in [Vas16, Corollary 6].

Our local boundedness result is a bit more technical than [Vas16, Corollary 6]. We include a parameter $\delta \in (0,1]$, which allows interpolation between the L^p norm of u and the L^q norm of f. In the local case this parameter is only a technicality, which is not very momentous. However, when passing to the case of nonlocal operators, this interpolation parameter is crucial since it allows to interpolate between local and nonlocal terms. That is why we also include it in the local case.

The results of this chapter are based on the monograph [HL11, Chapter 4].

Toolbox

Since we do not want to disturb the flow of reading, we first collect some auxiliary results, which will be important in the remainder.

The following iteration lemma can be traced back to [GG82, Lemma 1.1]

Lemma 10.1. *Let $0 \leq T_0 \leq T_1$ and $f : [T_0, T_1] \to \mathbb{R}$ be a nonnegative bounded function. Assume there are nonnegative constants A, B, ν, θ with $\theta < 1$ such that for $T_0 \leq t < s \leq T_1$*

one has

$$f(t) \leq \frac{A}{(s-t)^{\nu}} + B + \theta f(s).$$

Then there exists a constant c, depending only on ν and θ such that for every ρ, R with $T_0 \leq \rho < R \leq T_1$ it holds

$$f(\rho) \leq c\frac{A}{(R-\rho)^{\nu}} + cB. \tag{10.1}$$

Proof. Let $\theta^{\frac{1}{\nu}} < \varepsilon < 1$. Consider the sequence $\{\tau_i\}_{i\in\mathbb{N}_0}$ given inductively by

$$\tau_0 = \rho, \quad \tau_{i+1} - \tau_i = (1-\varepsilon)\varepsilon^i(R-\rho).$$

By induction we see that for every $n \in \mathbb{N}_0$ it holds

$$f(\tau_0) \leq \theta^n f(\tau_n) + \left(\frac{A}{(R-\rho)^{\nu}} + B\right)(1-\varepsilon)^{-\nu}\sum_{k=0}^{n-1}\left(\frac{\theta}{\varepsilon^{\nu}}\right)^k.$$

Taking the limit $n \to \infty$ we recover (10.1) with $c = \frac{\varepsilon^{\nu}}{(\varepsilon^{\nu}-\theta)(1-\varepsilon)^{\nu}}$. ■

Lemma 10.2. *Let $\varepsilon, \kappa > 0, C > 1$ and $(A_j)_{j\in\mathbb{N}_0}$ be a sequence of nonnegative integers. If A_j satisfies for each $j \in \mathbb{N}_0$ the inequality*

$$\frac{A_{j+1}}{\overline{\ell}} \leq \kappa C^j \left(\frac{A_j}{\overline{\ell}}\right)^{1+\varepsilon},$$

and $\overline{\ell} \geq A_0\kappa^{\frac{1}{\varepsilon}}C^{\frac{1}{\varepsilon^2}}$, then $A_j \to 0$ for $j \to \infty$.

Proof. By induction we see that for each $j \in \mathbb{N}_0$ we have

$$\frac{A_j}{\overline{\ell}} \leq \kappa^{-\frac{1}{\varepsilon}}C^{-\frac{1}{\varepsilon^2}-\frac{j}{\varepsilon}}.$$

Since $C > 1$, we deduce

$$C^{-\frac{j}{\varepsilon}} \to 0 \text{ for } j \to \infty,$$

which implies the claim of the lemma. ■

Proof of the local boundedness result

We consider an elliptic operator in divergence form and the corresponding equation on the unit ball, that is,

$$-\sum_{1\le i,j\le d} \partial_j(a_{ij}(x)\partial_i u(x)) = f(x), \quad x \in B_1, \tag{10.2}$$

As explained in Section 4.1 the coefficients a_{ij} satisfy

$$\Lambda^{-1}|\xi|^2 \le \sum_{i,j} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \quad x \in B_1, \xi \in \mathbb{R}^d \tag{10.3}$$

for some constant $\Lambda \ge 1$, and (10.3) implies that $2\Lambda^{-1} \le a_{ij}(x) \le 2\Lambda$ for each pair i, j and every $x \in B_1$.

Our aim is to prove that weak subsolutions of (10.2) are locally bounded and satisfy an (L^∞-L^p)-estimate in the following sense.

Theorem 10.3. *Let $f \in L^q(B_1)$ for $q > \frac{d}{2}$. Assume $u \in H^1(B_1), u \ge 0$, satisfies for every nonnegative $\eta \in H_0^1(B_1)$*

$$\int_{B_1} \sum_{i,j} a_{ij}\partial_i u\, \partial_j \eta \,\mathrm{d}x \le \int_{B_1} f\eta \,\mathrm{d}x. \tag{10.4}$$

Let $p \in (0,2]$. There exists a constant $C > 0$, independent of u, such that for every ball $B_r(x_0) \subset B_1$

$$\sup_{B_{\frac{r}{4}}(x_0)} u \le C \left(\fint_{B_r(x_0)} |u(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}} + r^{2-\frac{d}{q}} \|f\|_{L^q(B_r(x_0))}.$$

Remark. The constant $\frac{1}{4}$ appearing in the radius of the ball in the left-hand side of the inequality is arbitrary. One may choose any constant less than 1. Of course C blows up if the constant tends to 1.

The proof of Theorem 10.3 consists of the following steps:

- *Caccioppoli inequality:* A reverse Poincaré inequality, which enables us to give a priori estimates of the L^2-norm of the derivatives of a weak subsolution u in terms of the L^2-norm of u.
- *L^2-upper-level-set inequality:* An estimate of the L^2-norm of a weak subsolution on a certain upper level set by the L^2-norm of the same weak subsolution on a larger upper level set. This inequality is the starting point of our iteration procedure.

- $(L^\infty$-$L^2)$-*estimate*: We take the L^2-upper-level-set inequality and plug in sequences for the radii and the different cutoff levels and consider the limit. Here the so called De Giorgi iteration takes place. As a result, we obtain a local bound for weak solutions that consists of an L^2-term of u plus some norm of f.
- $(L^\infty$-$L^p)$-*estimate*: By a covering argument and a standard iteration lemma we recover the claim of the theorem starting from the $(L^\infty$-$L^2)$-estimate.

Lemma 10.4 (Caccioppoli inequality). *Let $f \in L^2_{\text{loc}}(B_1)$. Let $u \in H^1(B_1)$ satisfy (10.4) for each nonnegative $\eta \in H^1_0(B_1)$. Then there exists a constant $c > 0$, independent of u, such that for every ball $B_r(x_0) \subset B_1$, each $\ell > 0$ and every $\phi \in C_c^\infty(B_r(x_0))$*

$$\int_{B_r(x_0)} |\nabla(w\phi)|^2 \,\mathrm{d}x \le c\left(\int_{B_r(x_0)} w^2|\nabla\phi|^2 \,\mathrm{d}x + \int_{B_r(x_0)} f w\phi^2 \,\mathrm{d}x\right), \tag{10.5}$$

where $w(x) = (u(x) - \ell)_+$.

Proof. We abbreviate our notation by writing $B_r = B_r(x_0)$. Let $\phi \in C_c^\infty(B_r)$. We plug $\eta = w\phi^2$ as a test function into the definition of a weak subsolution (10.4). This yields

$$\int_{B_1} \sum_{i,j} a_{ij}\, \partial_i u \left[\partial_j w\, \phi^2 + 2\phi w\, \partial_j \phi\right] \,\mathrm{d}x \le \int_{B_1} f w\phi^2 \,\mathrm{d}x.$$

Note that $\nabla u = \nabla w$ almost everywhere in B_1 for $u \ge \ell$ and $w = 0, \nabla w = 0$ almost everywhere in $B_1 \cap \{u \le \ell\}$. That is, in the above inequality we integrate over the set $B_1 \cap \{u \ge \ell\}$. Using this and the ellipticity condition (10.3) the above inequality yields

$$\begin{aligned}
\int_{B_1} \sum_{i,j} & a_{ij}\, \partial_i u\, [\partial_j w\, \phi^2 + 2\phi\, w\, \partial_j\phi] \,\mathrm{d}x \\
&\ge \Lambda^{-1} \int_{B_1} |\nabla w|^2\, \phi^2 \,\mathrm{d}x - 4d^2\Lambda \int_{B_1} |\nabla w|\, |\nabla\phi|\, |\phi| w \,\mathrm{d}x \\
&\ge \frac{\Lambda^{-1}}{2} \int_{B_1} |\nabla w|^2 \phi^2 \,\mathrm{d}x - 2d^2\Lambda\varepsilon^{-1} \int_{B_1} w^2 |\nabla\phi|^2 \,\mathrm{d}x,
\end{aligned}$$

where we applied the Young inequality with $\varepsilon = \frac{\Lambda^{-1}}{4d^2\Lambda}$ in the last line, that is,

$$4d^2\Lambda\, (|\nabla w||\nabla\phi||\phi|w) \le 4d^2\Lambda \left(\frac{|\nabla w|^2\phi^2}{2}\varepsilon + \frac{w^2|\nabla\phi|^2}{2\varepsilon}\right).$$

Hence, we obtain

$$\int_{B_1} |\nabla w|^2 \phi^2 \,\mathrm{d}x \le c\left(\int_{B_1} w^2 |\nabla\phi|^2 \,\mathrm{d}x + \int_{B_1} f w\phi^2 \,\mathrm{d}x\right)$$

from which the inequality

$$\int_{B_1} |\nabla(w\phi)|^2 \,\mathrm{d}x \le c\left(\int_{B_1} w^2|\nabla\phi|^2 \,\mathrm{d}x + \int_{B_1} f w\phi^2 \,\mathrm{d}x\right)$$

follows. Note that the value of the constant c changes between the above two inequalities. Recalling that $\operatorname{supp}(\phi) \subset B_r$, we obtain the assertion. ∎

Lemma 10.5 (L^2-upper-level-set inequality)**.** *Let $f \in L^q(B_1)$ for some $q > \frac{d}{2}$ and let $u \in H^1(B_1)$ satisfy (10.4) for each nonnegative $\eta \in H_0^1(B_1)$. There are constants $c, \varepsilon > 0$, independent of u, such that for all $0 < \lambda < \mu \le 1$ and all $L > \ell \ge \sqrt{|B_1|}\|u\|_{L^2(B_1)}$*

$$\|(u-L)_+\|_{L^2(B_\lambda)} \le c\left(\frac{1}{\mu-\lambda} + \frac{\|f\|_{L^q(B_1)}}{L-\ell}\right)\frac{1}{(L-\ell)^\varepsilon}\|(u-\ell)_+\|^{1+\varepsilon}_{L^2(B_\mu)}.$$

Proof. Here we use the notation $2^* = \frac{2d}{d-2}$ for $d > 2$ and 2^* may be any number greater than 2 if $d = 2$. In the following computations we focus on the case $d > 2$. The case $d = 2$ can be treated analogously. Let w be as in Lemma 10.4 for $\ell > 0$.

Recall the Sobolev inequality, see Theorem 2.18, for $w\phi \in H_0^1(B_1)$:

$$\left(\int_{B_1} (w\phi)^{2^*} \,\mathrm{d}x\right)^{\frac{2}{2^*}} \le c\int_{B_1} |\nabla(w\phi)|^2 \,\mathrm{d}x.$$

If $\phi \in C_c^\infty(B_1)$ and $\phi \le 1$, then we deduce with the help of the Hölder inequality, the Sobolev inequality and the Young inequality with $\delta > 0$:

$$\begin{aligned}\int_{B_1} f w\phi^2 \,\mathrm{d}x &\le \left(\int_{B_1} |f|^q \,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_{B_1} |w\phi|^{2^*} \,\mathrm{d}x\right)^{\frac{1}{2^*}} |\{w\phi \ne 0\}|^{1-\frac{1}{2^*}-\frac{1}{q}} \\ &\le c\|f\|_{L^q(B_1)}\|\nabla(w\phi)\|_{L^2(B_1)}\,|\{w\phi \ne 0\}|^{\frac{1}{2}+\frac{1}{d}-\frac{1}{q}} \\ &\le \delta\|\nabla(w\phi)\|^2_{L^2(B_1)} + c(\delta)\|f\|^2_{L^q(B_1)}\,|\{w\phi \ne 0\}|^{1+\frac{2}{d}-\frac{2}{q}}. \end{aligned} \tag{10.6}$$

Let us assume

$$|\{w\phi \ne 0\}| \le 1 \tag{10.7}$$

in the following calculations. Later on, we will show that we can choose the level $\ell > 0$ sufficiently large so that (10.7) holds true.

Combining (10.5) with (10.6) and using $1+\frac{2}{d}-\frac{2}{q}>1-\frac{1}{q}$ for $q>\frac{d}{2}$ gives

$$\int_{B_1} |\nabla(w\phi)|^2 \,\mathrm{d}x \le c\left(\int_{B_1} w^2|\nabla\phi|^2 \,\mathrm{d}x + \|f\|^2_{L^q(B_1)}|\{w\phi\neq 0\}|^{1-\frac{1}{q}}\right). \tag{10.8}$$

We obtain by the Hölder inequality and by the Sobolev inequality

$$\begin{aligned}\int_{B_1} (w\phi)^2 \,\mathrm{d}x &\le \left(\int_{B_1} (w\phi)^{2^*} \,\mathrm{d}x\right)^{\frac{2}{2^*}} |\{w\phi\neq 0\}|^{1-\frac{2}{2^*}} \\ &\le c\int_{B_1} |\nabla(w\phi)|^2 \,\mathrm{d}x\, |\{w\phi\neq 0\}|^{\frac{2}{d}}.\end{aligned}$$

Thus, we have by (10.8) with $F=\|f\|_{L^q(B_1)}$

$$\int_{B_1} (w\phi)^2 \,\mathrm{d}x \le c\left(\int_{B_1} w^2|\nabla\phi|^2 \,\mathrm{d}x\, |\{w\phi\neq 0\}|^{\frac{2}{d}} + F^2|\{w\phi\neq 0\}|^{1+\frac{2}{d}-\frac{1}{q}}\right).$$

Since $|\{w\phi\neq 0\}|\le 1$, we conclude that there exists $\varepsilon>0$ such that

$$\int_{B_1} (w\phi)^2 \,\mathrm{d}x \le c\left(\int_{B_1} w^2|\nabla\phi|^2 \,\mathrm{d}x\, |\{w\phi\neq 0\}|^{\varepsilon} + F^2|\{w\phi\neq 0\}|^{1+\varepsilon}\right). \tag{10.9}$$

Now we specify the choice of the cutoff function. For fixed $0<\lambda<\mu\le 1$ choose $\phi\in C_c^\infty(B_\mu)$ such that $\phi=1$ in B_λ and $0\le\phi\le 1$. Furthermore, we require $|\nabla\phi|\le\frac{c}{\mu-\lambda}$ in B_1. We introduce the *upper level set* notation

$$A(\ell,r)=\{x\in B_r \,|\, u\ge\ell\},$$

which we will use frequently throughout the rest of this part. Note

$$|A(\ell,\mu)| \le \frac{1}{\ell}\int_{A(\ell,\mu)} u \,\mathrm{d}x \le \frac{\sqrt{|B_1|}}{\ell}\|u\|_{L^2(B_1)}.$$

Hence, we conclude that (10.7) is satisfied if $\ell\ge\sqrt{|B_1|}\|u\|_{L^2(B_1)}$. In particular we have $|A(\ell,\mu)|\le 1$ if $\ell\ge\ell_0=\sqrt{|B_1|}\|u\|_{L^2(B_1)}$. Furthermore, we use the following properties. If $L>\ell>0$, then obviously $A(L,r)\subset A(\ell,r)$ for every $0<r\le 1$. Hence, we have for each $0<r\le 1$

$$\int_{A(L,r)} (u-L)^2 \,\mathrm{d}x \le \int_{A(\ell,r)} (u-\ell)^2 \,\mathrm{d}x,$$

and

$$|A(L,r)| = |B_r\cap\{u-\ell\ge L-\ell\}| \le \frac{1}{(L-\ell)^2}\int_{A(\ell,r)} (u-\ell)^2 \,\mathrm{d}x.$$

We go on from (10.9) and obtain for all $0<\lambda<\mu\le 1$ and $\ell\ge\sqrt{|B_1|}\|u\|_{L^2(B_1)}$ the estimate

$$\int_{A(\ell,\lambda)} (u-\ell)^2 \,\mathrm{d}x \le c\left(\frac{|A(\ell,\mu)|^\varepsilon}{(\mu-\lambda)^2}\int_{A(\ell,\mu)} (u-\ell)^2 \,\mathrm{d}x + F^2|A(\ell,\mu)|^{1+\varepsilon}\right). \tag{10.10}$$

Using the above mentioned properties of the upper level sets we conclude from (10.10) for all $0 < \lambda < \mu \leq 1$ and $L > \ell \geq \sqrt{|B_1|}\|u\|_{L^2(B_1)}$

$$\begin{aligned}\int_{A(L,\lambda)} (u-L)^2 \,\mathrm{d}x &\leq c\left(\frac{1}{(\mu-\lambda)^2}\int_{A(L,\mu)} (u-L)^2\,\mathrm{d}x + F^2|A(L,\mu)|\right)|A(L,\mu)|^{\varepsilon}\\ &\leq c\left(\frac{1}{(\mu-\lambda)^2}+\frac{F^2}{(L-\ell)^2}\right)\int_{A(\ell,\mu)}(u-\ell)^2\,\mathrm{d}x\,|A(L,\mu)|^{\varepsilon}\\ &\leq c\left(\frac{1}{(\mu-\lambda)^2}+\frac{F^2}{(L-\ell)^2}\right)\frac{1}{(L-\ell)^{2\varepsilon}}\left(\int_{A(\ell,\mu)}(u-\ell)^2\,\mathrm{d}x\right)^{1+\varepsilon},\end{aligned}$$

which implies our claim. ■

Theorem 10.6. *(local (L^∞-L^2)-estimate) Suppose that $u \in H^1(B_1), u \geq 0$ is a weak subsolution of* (10.2)*, that is, u satisfies* (10.4) *for every $\eta \in H_0^1(B_1)$. If $f \in L^q(B_1)$ for some $q > \frac{d}{2}$ and $B_r(x_0) \subset B_1$, then for every $\delta \in (0,1]$*

$$\sup_{B_{\frac{r}{2}}(x_0)} u \leq C\left[\delta^{-\frac{1}{\varepsilon}} r^{-\frac{d}{2}}\|u\|_{L^2(B_r(x_0))} + r^{2-\frac{d}{q}}\delta\|f\|_{L^q(B_r(x_0))}\right], \tag{10.11}$$

where $C = C(d,\Lambda)$ is a positive constant, independent of u, and $\varepsilon > 0$ is as in Lemma 10.5.

Proof. First, we prove the assertion for $r = 1$ and $x_0 = 0$. Define for $j \in \mathbb{N}_0$

$$\ell_0 = \sqrt{|B_1|}\|u\|_{L^2(B_1)}, \quad \ell_j = \ell_0 + \overline{\ell}\left(1-2^{-j}\right) \text{ for some } \overline{\ell} \text{ to be specified later},$$
$$r_j = \frac{1}{2}(1+2^{-j}), \quad A_j = \|(u-\ell_j)_+\|_{L^2(B_{r_j})}.$$

We have

$$\ell_{j+1} - \ell_j = 2^{-j-1}\overline{\ell}, \quad r_j - r_{j+1} = \frac{1}{2^{j+2}}.$$

We use Lemma 10.5 with

$$L = \ell_{j+1}, \quad \ell = \ell_j, \quad \mu = r_j, \quad \lambda = r_{j+1},$$

and set $F = \|f\|_{L^q(B_1)}$ in order to get for $j \in \mathbb{N}_0$

$$A_{j+1} \leq c\left(2^j + \frac{2^{j+1}F}{\overline{\ell}}\right)\frac{2^{\varepsilon(j+1)}}{\overline{\ell}^{\varepsilon}} A_j^{1+\varepsilon}. \tag{10.12}$$

For a given $\delta \in (0,1]$ we specify the choice of the constant $\overline{\ell}$. We choose

$$\overline{\ell} \geq \delta F. \tag{10.13}$$

Then (10.12) implies for $j \in \mathbb{N}_0$

$$\begin{aligned}\frac{A_{j+1}}{\bar{\ell}} &\le c2^{j(2+\varepsilon)}\delta^{-1}\left(\frac{A_j}{\bar{\ell}}\right)^{1+\varepsilon}\\ &\le C^j c\delta^{-1}\left(\frac{A_j}{\bar{\ell}}\right)^{1+\varepsilon}\end{aligned}$$

for $C = 2^{2+\varepsilon}$, where we again point out that the value of c changes between the lines of this proof. Now we are in a position to use Lemma 10.2, which tells us that we get $A_j \to 0$ for $j \to \infty$ if

$$\bar{\ell} \ge A_0\delta^{-\frac{1}{\varepsilon}}c^{\frac{1}{\varepsilon}}C^{\frac{1}{\varepsilon^2}}.$$

Since we need to make sure that also (10.13) is satisfied, we choose

$$\bar{\ell} = \delta F + \widetilde{C}A_0\delta^{-\frac{1}{\varepsilon}} \le \delta F + \delta^{-\frac{1}{\varepsilon}}\widetilde{C}\|u\|_{L^2(B_1)},$$

where $\widetilde{C} = c^{\frac{1}{\varepsilon}}C^{\frac{1}{\varepsilon^2}}$. From $A_j \to 0$ we conclude

$$\sup_{B_{\frac{1}{2}}} u \le \widetilde{C}\delta^{-\frac{1}{\varepsilon}}\|u\|_{L^2(B_1)} + \ell_0 + \delta F \le C\delta^{-\frac{1}{\varepsilon}}\|u\|_{L^2(B_1)} + C\delta F,$$

for some appropriate $C \ge 1$. This finishes the proof for the case $r = 1$ and $x_0 = 0$.

The claim for arbitrary $B_r(x_0) \subset B_1$ follows now from a scaling argument. Let $J : B_1 \to B_r(x_0), J(x) = rx + x_0$. Define for $x \in B_1$

$$\widetilde{u}(x) = u(J(x)), \quad \widetilde{a}_{ij}(x) = a_{ij}(J(x)), \quad \widetilde{f}(x) = r^2 f(J(x)).$$

Then one can easily see that $\widetilde{u}$ is a weak subsolution of

$$-\sum_{i,j}\partial_j(\widetilde{a}_{ij}(x)\partial_i\widetilde{u}(x)) = \widetilde{f}(x), \quad x \in B_1,$$

and that (10.3) holds also if we replace a_{ij} with $\widetilde{a}_{ij}$. We apply what we have just proved and rewrite the result in terms of u and f. This leads to

$$\sup_{B_{\frac{r}{2}}(x_0)} u \le C\left[\delta^{-\frac{1}{\varepsilon}}r^{-\frac{d}{2}}\|u\|_{L^2(B_r(x_0))} + r^{2-\frac{d}{q}}\delta\|f\|_{L^q(B_r(x_0))}\right],$$

what we claimed. ∎

Proof of Theorem 10.3. We apply Theorem 10.6 and choose $\delta = 1$ in (10.11). Let $\frac{1}{2} \le t < s \le 1$. Observe that this implies $s - t \le t$. Then

$$\sup_{B_{t\frac{r}{2}}(x_0)} u \le C\left[\left(\frac{1}{tr}\right)^{\frac{d}{2}}\|u\|_{L^2(B_{tr}(x_0))} + (tr)^{2-\frac{d}{q}}\|f\|_{L^q(B_{tr}(x_0))}\right]$$

$$
\begin{aligned}
&\leq C\left[\left(\frac{2s}{tr}\right)^{\frac{d}{2}}\|u\|_{L^2(B_{sr}(x_0))}+r^{2-\frac{d}{q}}\|f\|_{L^q(B_r(x_0))}\right]\\
&\leq C\left[\left(\frac{1}{s-t}\right)^{\frac{d}{2}}\left(\fint_{B_{sr}(x_0)}u^2\,\mathrm{d}x\right)^{\frac{1}{2}}+r^{2-\frac{d}{q}}\|f\|_{L^q(B_r(x_0))}\right].
\end{aligned}
\tag{10.14}
$$

For $p\in(0,2)$ we have by the Young inequality

$$
\begin{aligned}
\sup_{B_{t\frac{r}{2}}(x_0)}u &\leq C\left[\left(\sup_{B_{sr}(x_0)}u\right)^{\frac{2-p}{p}}\frac{1}{(s-t)^{\frac{d}{2}}}\left(\fint_{B_{sr}(x_0)}u^p\,\mathrm{d}x\right)^{\frac{1}{2}}+r^{2-\frac{d}{q}}\|f\|_{L^q(B_r(x_0))}\right]\\
&\leq \frac{1}{2}\sup_{B_{\frac{r}{2}}(x_0)}u+C\left[\frac{1}{(s-t)^{\frac{d}{p}}}\left(\fint_{B_r(x_0)}u^p\,\mathrm{d}x\right)^{\frac{1}{p}}+r^{2-\frac{d}{q}}\|f\|_{L^q(B_r(x_0))}\right].
\end{aligned}
\tag{10.15}
$$

Note that in the last line the constant C depends on p. We remark that (10.15) holds obviously true for $p=2$ by (10.14). Now we are in the position to apply Lemma 10.1 and conclude

$$
\sup_{B_{\frac{r}{4}}(x_0)}u\leq C\left[\left(\fint_{B_{r(x_0)}}u^p\,\mathrm{d}x\right)^{\frac{1}{p}}+r^{2-\frac{d}{q}}\|f\|_{L^q(B_r(x_0))}\right],
$$

which is the assertion. ■

11. Local boundedness and Harnack inequalities for nonlocal operators

In this chapter we study properties of weak solutions u of the nonlocal equation

$$-\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d\setminus B_\varepsilon(x)}(u(y)-u(x))k(x,y)\,\mathrm{d}y = f(x), \quad x\in B_1, \tag{11.1}$$

cf. Section 4.2, where we always assume that u is equal to some arbitrary function on the boundary ∂B_1. Here the function $k:\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty]$ is a symmetric kernel.

Let us recall that the quadratic form $\mathcal{E}^k$ is defined as

$$\mathcal{E}^k(u,v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x)-u(y))(v(x)-v(y))k(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

which is finite for $u\in V^k(B_1|\mathbb{R}^d)$ and $v\in H^k_{B_1}(\mathbb{R}^d)$, see Definition 3.3 for the definition of these spaces.

Toolbox

As in the previous chapter we collect some auxiliary results in order to allow a better flow of reading.

Lemma 11.1. *Let $a,b\geq 0$. For each $\varepsilon>0$ it holds*

$$b^2-a^2\leq \varepsilon b^2+\frac{(b-a)^2}{\varepsilon}.$$

Proof. Assuming $0\leq a\leq b$ we have

$$b^2-a^2=(b-a)(b+a)\leq 2b(b-a).$$

Now we use the Young inequality with $\varepsilon>0$ to deduce

$$2(b-a)b\leq 2\left(\frac{b-a}{\varepsilon^{\frac{1}{2}}}\right)b\varepsilon^{\frac{1}{2}}\leq \varepsilon^{-1}(b-a)^2+b^2\varepsilon. \tag{11.2}$$

On the other hand, if $0\leq b<a$, then the assertion of the lemma is obvious. ∎

Lemma 11.2. *Let $a, b, c, d \in \mathbb{R}$. Then*

$$(ab - cd)^2 \leq 2(a^2 + c^2)(b - d)^2 + 2(a - c)^2(b^2 + d^2).$$

Proof. We have

$$\begin{aligned}(ab - cd)^2 &= (ab - cb + cb - cd)^2 \\ &\leq 2(a - c)^2 b^2 + 2(b - d)^2 c^2 \\ &\leq 2(a - c)^2(b^2 + d^2) + 2(b - d)^2(a^2 + c^2),\end{aligned}$$

which is what we wanted to prove. ■

11.1. Nonlocal tail functions

An object that plays a crucial role in the proof of the local boundedness of solutions is the ensuing tail function. Following [DKP16] we define this purely nonlocal function as follows:

$$\mathrm{Tail}(v, x_0, r) = r^\alpha \int_{\mathbb{R}^d \setminus B_r(x_0)} \frac{|v(x)|}{|x - x_0|^{d+\alpha}} \,\mathrm{d}x,$$

whenever the quantity is finite.

Lemma 11.3. *Let $r > 0$ and $x_0 \in \mathbb{R}^d$. If $v \in L^1((1 + |x|)^{-d-\alpha} \,\mathrm{d}x)$, then* $\mathrm{Tail}(v, x_0, r) < \infty$.

Proof. An easy application of the Hölder inequality combined with the facts

$$1 + |x - x_0| \leq \left(\frac{1}{r} + 1\right) |x - x_0| \quad \text{for every } x \in \mathbb{R}^d \setminus B_r(x_0),$$

and

$$1 + |x + x_0| \leq (1 + |x_0|)(1 + |x|) \text{ for all } x, x_0 \in \mathbb{R}^d,$$

shows that

$$\begin{aligned}\int_{\mathbb{R}^d \setminus B_r(x_0)} \frac{|v(x)|}{|x - x_0|^{d+\alpha}} \,\mathrm{d}x &\leq c(r) \int_{\mathbb{R}^d \setminus B_r(x_0)} \frac{|v(x)|}{(1 + |x - x_0|)^{d+\alpha}} \,\mathrm{d}x \\ &\leq c(r) \int_{\mathbb{R}^d \setminus B_r(0)} \frac{|v(x + x_0)|}{(1 + |x|)^{d+\alpha}} \,\mathrm{d}x \\ &\leq c(r, x_0) \int_{\mathbb{R}^d} \frac{|v(x)|}{(1 + |x|)^{d+\alpha}} \,\mathrm{d}x,\end{aligned}$$

that is, $\mathrm{Tail}(v, x_0, r)$ is finite for every $r > 0$ and $x_0 \in \mathbb{R}$, whenever $v \in L^1((1 + |x|)^{-d-\alpha}\,\mathrm{d}x)$. ■

In [DKP16] the authors show in the case of the nonlinear fractional p-Laplace operator how the *nonlocal* tail can be bounded from above by a *local* term, precisely the essential supremum of weak supersolutions over some ball. This is the key ingredient in their proof of the Harnack inequality for nonnegative solutions.

In our work we consider only the case where the nonlocal operator is linear. However, our integral kernels are allowed to be from a wider class of functions. The main difference is that the kernels are not pointwise almost everywhere comparable to the standard kernel $k(x,y) = |x-y|^{-d-\alpha}$. A consequence is that we need to introduce a different kind of *nonlocal* tail, which we will call k-Tail. The idea is now to modify the proofs in [DKP14; DKP16] in order to show a local boundedness result of the k-Tail. It turns out that this bound requires additional assumptions on the kernel. These we state after the definition of the k-Tail.

Definition 11.4 (k-Tail)**.** *Let $\alpha \in (0,2)$ and $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ be a kernel. Let $r > 0$, $x_0 \in \mathbb{R}^d$, $0 < \lambda < \mu \le r$ and $v \in L^1(1 + |x|^{-d-\alpha})\,\mathrm{d}x)$. The function*

$$\mathrm{Tail}_k(v, x_0, r, \lambda, \mu) = r^\alpha \sup_{x \in B_\lambda(x_0)} \int_{\mathbb{R}^d \setminus B_\mu(x_0)} |v(y)| k(x,y)\,\mathrm{d}y$$

is called the nonlocal k-Tail.

Of course, if $k(x,y) \le c|x-y|^{-d-\alpha}$ for some $c > 0$, then

$$\mathrm{Tail}_k(v, x_0, r, \lambda, \mu) \le c(\mu, \lambda)\,\mathrm{Tail}(v, x_0, \mu)$$

for all $0 < \lambda < \mu \le r$, that is, the function $\mathrm{Tail}_k(\cdot, x_0, r, \lambda, \mu) : L^1((1 + |x|)^{-d-\alpha}\,\mathrm{d}x) \to \mathbb{R}$ is well defined.

11.2. Assumptions on the kernel

In this part we state the assumptions on the considered integral kernels. The different conditions are listed below for a given $\alpha \in (0,2)$. Throughout this section we assume that $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ is a symmetric integral kernel.

Condition (A) (local coercivity estimate on every scale)**.** *There exists a constant $A > 0$ such that for every ball $B_r(x_0) \subset \mathbb{R}^d$ with $x_0 \in B_1, 0 < r \le 1$ and each $v \in H^{\frac{\alpha}{2}}(B_r(x_0))$ it holds*

$$\mathcal{E}^k_{B_r(x_0)}(v,v) \ge A \|v\|^2_{\dot{H}^{\frac{\alpha}{2}}(B_r(x_0))}. \tag{A}$$

Condition (B) (pointwise upper bound)**.** *The kernel k is pointwise bounded from above by a constant times $|x-y|^{-d-\alpha}$, that is, there exists a constant $B>0$ such that*

$$|x-y|^{-d-\alpha} \geq B\, k(x,y) \qquad \text{for almost all } x,y \in \mathbb{R}^d. \tag{B}$$

The local coercivity assumption Condition (A) is the central assumption in this part. It guarantees that we can make use of functional inequalities, e.g., the Sobolev inequality.

The second assumption, Condition (B), will prove necessary for us because later on we want to separate nonlocal integrals in $\mathbb{R}^d \times \mathbb{R}^d$ into two separate integrals in $\mathbb{R}^d$ in order to make an iteration process work.

We remark that the combination of Condition (A) and Condition (B) implies that the main assumptions in [DK20, Theorem 1.2] hold true. Provided that f satisfies specific assumptions, this immediately leads to a weak Harnack inequality for weak supersolutions of (11.1).

We show that Condition (A) and Condition (B) together imply a local boundedness result for solutions of the underlying elliptic equation (11.1). However, the bound on the solution here depends on *nonlocal* quantities such as a nonlocal tail function as defined in the previous Section 11.1.

We aim to prove a Harnack inequality for solutions of the elliptic equation (11.1). In general we cannot expect that such a result holds true if we only presuppose that Condition (A) and Condition (B) hold true. A well known counterexample is given in [BS05, p. 148]. Therefore, we have to propose another condition on our kernel concerning its lower bound. This new assumption replaces the pointwise lower bound in [DKP14] as well as [Coz17]. In this way our result is more general than the result for the linear case in [DKP14]; however, the method used in our proof is the same.

The pointwise lower bound on the kernel in [DKP14] is essential to bound the nonlocal tail only by localized quantities. The authors use this lower bound in order to bound a double integral with nonlocal parts from below by the nonlocal tail times the volume of some ball. Our Condition (C) enables us to copy exactly this behavior of the decomposition of the double integral.

The assumption reads as follows.

Condition (C) (averaged integral bound on every scale)**.** *There exists a constant $C>0$ such that for almost all $x \in B_1, y \in \mathbb{R}^d$ with $x \neq y$ and every radius $0 < r \leq \left(\frac{|x-y|}{2} \wedge \frac{1}{4}\right)$ it holds*

$$\fint_{B_r(x)} k(z,y)\,\mathrm{d}z \geq C k(x,y). \tag{C}$$

We often say that k satisfies (X) for $\mathrm{X} \in \{\mathrm{A},\mathrm{B},\mathrm{C}\}$ meaning that Condition (X) holds for k.

Remark. For some time we worked with an equivalent formulation of Condition (C), see Proposition 11.5 below. We realized that these assumptions were equivalent after discovering the article [CKW20]. In this article the authors use a similar assumption as Condition (C) in order to prove a parabolic Harnack inequality, which they call (UJS) assumption. Our Condition (C) is nothing but a localized version of (UJS) (compare also with the remark after the scaled version of Condition (C), see Section 11.3). After our discovery, we decided to use also the analogous formulation of (UJS) since it seems to be the most accessible version of the equivalent statements. To the best of the author's knowledge, the (UJS) assumption can be traced back to [BBK09], where it first appeared in the setup of discrete graphs.

Remark. We stress that Condition (C) is neither necessary nor sufficient for the validity of Condition (A). Indeed, from Part II of this thesis we know that any quadratic form belonging to a kernel generated by an admissible configuration of cones satisfies Condition (A). But we can easily construct a configuration of cones such that Condition (C) does not hold true. This is done in Section 12.1. It shows that Condition (C) is not necessary for Condition (A) to hold true. On the other hand, the kernel $k(x,y) = \mathbb{1}_{B_5 \setminus B_4}(x-y)|x-y|^{-d-\alpha}$ satisfies, according to Lemma 12.4, Condition (C). To show that Condition (A) does not hold for this kernel, take a function $u \in C_c^\infty(\mathbb{R}^d)$ such that $u(x) = x$ on B_1. Then

$$\int_{B_1}\int_{B_1} (u(x)-u(y))^2 k(x,y) \, \mathrm{d}x \, \mathrm{d}y = 0.$$

But

$$\begin{aligned} \int_{B_1}\int_{B_1} (u(x)-u(y))^2 |x-y|^{-d-\alpha} \, \mathrm{d}x \, \mathrm{d}y &\geq \int_{B_{\frac{1}{2}}} \int_{B_{\frac{1}{2}}(y)} |x-y|^{2-d-\alpha} \, \mathrm{d}x \, \mathrm{d}y \\ &= c\left(\frac{1}{2}\right)^{2-\alpha} > 0. \end{aligned}$$

Equivalent formulations of Condition (C)

The following proposition provides equivalent formulations of Condition (C). The second version of the condition is especially important since it is used later on to prove the localized estimate of the k-Tail.

Proposition 11.5. *The following statements are equivalent.*

1. *Condition (C).*

2. *There exists a constant $C > 0$ such that for each $x_0 \in B_1$ and every pair of radii (λ, μ) with $0 < \lambda < \mu \leq \frac{1}{2}$ the following holds true for almost every $y \in \mathbb{R}^d \setminus B_\mu(x_0)$:*

$$\int_{B_{\frac{\mu+\lambda}{2}}(x_0)} k(z,y) \, \mathrm{d}z \geq C|\mu-\lambda|^d \sup_{x \in B_\lambda(x_0)} k(x,y). \tag{11.3}$$

3. *There exists a constant* $C > 0$ *such that for each* $x_0 \in B_1$ *and every pair of radii* (λ, μ) *with* $0 < \lambda < \mu \leq \frac{1}{2}$ *the following holds true for almost every* $y \in \mathbb{R}^d \setminus B_\mu(x_0)$ *and almost every* $x \in B_\lambda(x_0)$*:*

$$\int_{B_\mu(x_0)} k(z,y)|y-z|^{d+\alpha}\,\mathrm{d}z \geq C|\mu-\lambda|^d k(x,y)|x-y|^{d+\alpha}. \tag{11.4}$$

Remark. We intentionally use the representation (11.3) with the larger term on the left-hand side of the inequality because this is exactly the direction in which we use the estimate later on in our localized estimate of the tail. This Proposition 11.5 is also the reason why we chose to put the larger term on the left-hand side in (C).

Proof of Proposition 11.5. Suppose Condition (C) holds true. Let $0 < \lambda < \mu \leq \frac{1}{2}$ be given and let $x_0 \in B_1$. Assume $y \in \mathbb{R}^d \setminus B_\mu(x_0)$ and $x \in B_\lambda(x_0)$ so that $k(x,y) > 0$. Then

$$\left(\frac{|x-y|}{2} \wedge \frac{1}{4}\right) \geq \frac{\mu-\lambda}{2},$$

and thus (C) yields

$$\int_{B_{\frac{\mu-\lambda}{2}}(x)} k(z,y)\,\mathrm{d}z \geq C(\mu-\lambda)^d k(x,y).$$

Taking into account that $B_{\frac{\mu-\lambda}{2}}(x) \subset B_{\frac{\mu+\lambda}{2}}(x_0)$ we deduce that (11.3) holds true.

It is easy to see that (11.3) implies (11.4) . Indeed, we have $|x-y| \asymp |z-y|$ for $x \in B_\lambda(x_0), z \in B_{\frac{\mu+\lambda}{2}}(x_0)$ and $y \in \mathbb{R}^d \setminus B_\mu(x_0)$. Thus, we have for almost every $x \in B_\lambda(x_0)$

$$\begin{aligned}\int_{B_\mu(x_0)} k(z,y)|z-y|^{d+\alpha}\,\mathrm{d}z &\geq \int_{B_{\frac{\mu+\lambda}{2}}(x_0)} k(z,y)|z-y|^{d+\alpha}\,\mathrm{d}z \\ &\asymp \int_{B_{\frac{\mu+\lambda}{2}}(x_0)} k(z,y)|x-y|^{d+\alpha}\,\mathrm{d}z \\ &\geq C|\mu-\lambda|^d|x-y|^{d+\alpha}k(x,y).\end{aligned}$$

For the remaining direction assume (11.4) holds true and take $x \in B_1, y \in \mathbb{R}^d$ with $x \neq y$ and $k(x,y) > 0$. Let $0 < r \leq \left(\frac{|x-y|}{2} \wedge \frac{1}{4}\right)$ be given. Choose $x_0 = x, \lambda = \frac{r}{2}$ and $\mu = r$. Then we have $y \in \mathbb{R}^d \setminus B_\mu(x_0) = \mathbb{R}^d \setminus B_\mu(x)$. Thus, by (11.4)

$$\int_{B_r(x)} k(z,y)|z-y|^{d+\alpha}\,\mathrm{d}z \geq 2^d C r^d |x-y|^{d+\alpha} k(x,y).$$

Condition (C) follows now after using $|x-y| \asymp |z-y|$. The proof is complete. ∎

11.3. Behavior of the conditions under scaling

Let us give the analog version of Condition (A) for solutions on arbitrary balls.

Condition (A, ξ_0, R). *Given $\xi_0 \in \mathbb{R}^d, R > 0$, we say that the kernel k satisfies (A, ξ_0, R) if: There exists $A > 0$ such that for every ball $B_r(x_0)$ with $0 < r \leq R, x_0 \in B_R(\xi_0)$ and every $v \in H^{\frac{\alpha}{2}}(B_r(\xi_0))$ we have*

$$\mathcal{E}^k_{B_r(x_0)}(v,v) \geq A\|v\|_{\dot{H}^{\frac{\alpha}{2}}(B_r(x_0))}. \qquad (A, \xi_0, R)$$

A scaled version of Condition (C) looks as follows.

Condition (C, ξ_0, R). *Given $\xi_0 \in \mathbb{R}^d$ and $R > 0$ we say that the kernel k satisfies (C, ξ_0, R) if the following holds: There exists a constant $C > 0$ such that for almost all $x \in B_R(\xi_0), y \in \mathbb{R}^d$ with $x \neq y$ and every radius $0 < r \leq R\left(\frac{|x-y|}{2} \wedge \frac{1}{4}\right)$ we have*

$$\fint_{B_r(x)} k(z,y)\,\mathrm{d}z \geq Ck(x,y). \qquad (C, \xi_0, R)$$

Remark. If we assume validity of (C, ξ_0, R) for every $R > 0$, then we end up with the (UJS) assumption as in [CKW20]. That is why, for some fixed R, Condition (C, ξ_0, R) can be seen as a localized version of (UJS).

The next lemma shows the behavior of the underlying operator with respect to rescaled functions.

Lemma 11.6. *Suppose $\xi_0 \in \mathbb{R}^d$, $R > 0$ and $f \in L^2_{\mathrm{loc}}(B_R(\xi_0))$. Let k be a symmetric kernel and let $u \in V^k(B_R(\xi_0)|\mathbb{R}^d) \cap L^1((1+|x|)^{-d-\alpha}\,\mathrm{d}x)$ fulfill $\mathcal{E}^k(u,\psi) = (f,\psi)$ for every $\psi \in H^k_{B_R(\xi_0)}(\mathbb{R}^d)$. Let $J : \mathbb{R}^d \to \mathbb{R}^d, J(x) = Rx + \xi_0$. Define rescaled versions of u, f and k on B_1 by*

$$\widetilde{u}(x) = u(J(x)), \quad \widetilde{f}(x) = R^\alpha f(J(x)) \quad \text{and} \quad \widetilde{k}(x,y) = R^{\alpha+d}k(J(x),J(y)).$$

Then the following holds.

1. *The function $\widetilde{u}$ is an element of $V^{\widetilde{k}}(B_1|\mathbb{R}^d) \cap L^1((1+|x|)^{-d-\alpha}\,\mathrm{d}x)$ and satisfies for all $\psi \in H^{\widetilde{k}}_{B_1}(\mathbb{R}^d)$ the following equality*

$$\mathcal{E}^{\widetilde{k}}(\widetilde{u},\psi) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\widetilde{u}(x) - \widetilde{u}(y))(\psi(x) - \psi(y))\widetilde{k}(x,y)\,\mathrm{d}x\,\mathrm{d}y = (\widetilde{f},\psi).$$

2. *If k satisfies (A, ξ_0, R) for some $\xi_0 \in \mathbb{R}^d, R > 0, \alpha \in (0,2)$ and some $A > 0$, then $\widetilde{k}$ satisfies (A) with the same constant A.*

3. *If k satisfies (B) for some $\alpha \in (0,2)$ and some $B > 0$, then $\widetilde{k}$ satisfies (B) with the same constant B.*

4. *If k satisfies (C,ξ_0,R) for some $\xi_0 \in \mathbb{R}^d, R>0$ and some $C>0$, then $\widetilde{k}$ satisfies* (C) *with the same constant C.*

Proof. Let us first show that $\widetilde{u} \in V^{\widetilde{k}}(B_1|\mathbb{R}^d)$. According to the change of variables formula (cf. Theorem 2.11) we have

$$\int_{B_1}\int_{\mathbb{R}^d} (\widetilde{u}(x)-\widetilde{u}(y))^2\widetilde{k}(x,y)\,\mathrm{d}x\,\mathrm{d}y = R^{\alpha-d}\int_{B_R(\xi_0)}\int_{\mathbb{R}^d}(u(x)-u(y))^2k(x,y)\,\mathrm{d}x\,\mathrm{d}y < \infty$$

because $u \in V^k(B_R(\xi_0)|\mathbb{R}^d)$. It is obvious that $\widetilde{u}$ is measurable and that $\widetilde{u}|_{B_1}$ is an L^2-function. Also,

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{|\widetilde{u}(x)|}{(1+|x|)^{d+\alpha}}\,\mathrm{d}x &= R^{-d}\int_{\mathbb{R}^d}\frac{|u(x)|}{(1+|J^{-1}(x)|)^{d+\alpha}}\,\mathrm{d}x\\ &\le R^{-d}c(R,d,\alpha,|\xi_0|)\int_{\mathbb{R}^d}\frac{|u(x)|}{(1+|x|)^{d+\alpha}}\,\mathrm{d}x\\ &<\infty,\end{aligned}$$

where we made use of

$$\frac{1+|x|}{1+R^{-1}|x-\xi_0|} \le \frac{1+|x-\xi_0|+|\xi_0|}{1+R^{-1}|x-\xi_0|} \le 1+|\xi_0|+R.$$

Now let $\psi \in H^{\widetilde{k}}_{B_1}(\mathbb{R}^d)$. Define $\psi_{J^{-1}} \in H^k_{B_R(\xi_0)}(\mathbb{R}^d)$ by $\psi_{J^{-1}} = \psi \circ J^{-1}$. We conclude again by a change of variables

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\widetilde{u}(x)-\widetilde{u}(y))(\psi(x)-\psi(y))\widetilde{k}(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad= R^{\alpha+d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(J(x))-u(J(y)))(\psi_{J^{-1}}(J(x))-\psi_{J^{-1}}(J(y)))k(J(x),J(y))\,\mathrm{d}x\,\mathrm{d}y\\ &\quad= R^{\alpha-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x)-u(y))(\psi_{J^{-1}}(x)-\psi_{J^{-1}}(y))k(x,y))\,\mathrm{d}x\,\mathrm{d}y\\ &\quad= R^{\alpha-d}\int_{\mathbb{R}^d} f(x)\psi_{J^{-1}}(x)\,\mathrm{d}x\\ &\quad= R^{\alpha}\int_{\mathbb{R}^d} f(J(x))\psi(x)\,\mathrm{d}x\\ &\quad= \int_{\mathbb{R}^d}\widetilde{f}(x)\psi(x)\,\mathrm{d}x.\end{aligned}$$

Let us now prove that $\widetilde{k}$ satisfies (A) provided that k satisfies (A,ξ_0,R). Let $x_0 \in B_1$ and $0<r\le 1$. Then $Rx_0+\xi_0 \in B_R(\xi_0)$ for $R>0$. Let $v \in H^{\frac{\alpha}{2}}(B_r(x_0))$. Then the function $\hat{v}: B_{Rr}(Rx_0+\xi_0) \to \mathbb{R}^d$ with $\hat{v} = v\circ J^{-1}$ is an element of $H^{\frac{\alpha}{2}}(B_{Rr}(R\xi_0+x_0))$. In the following computation we use the abbreviations $B_r = B_r(x_0)$ and $B_{Rr} = B_{Rr}(Rx_0+\xi_0)$. Using the change of variables formula and (A,ξ_0,R) we arrive at

$$\mathcal{E}^{\widetilde{k}}_{B_r}(v,v) = \int_{B_r}\int_{B_r}(v(y)-v(x))^2\,\widetilde{k}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$
\begin{aligned}
&= R^{\alpha+d} \int_{B_r} \int_{B_r} (\hat{v}(J(x)) - \hat{v}(J(y))^2 \, k(J(x), J(y)) \, \mathrm{d}x \, \mathrm{d}y \\
&= R^{\alpha-d} \int_{B_{Rr}} \int_{B_{Rr}} (\hat{v}(x) - \hat{v}(y))^2 \, k(x,y) \, \mathrm{d}x \, \mathrm{d}y \\
&\geq AR^{\alpha-d} \int_{B_{Rr}} \int_{B_{Rr}} \frac{(\hat{v}(x) - \hat{v}(y))^2}{|x-y|^{d+\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\
&= AR^{-2d} \int_{B_{Rr}} \int_{B_{Rr}} \frac{(v(J^{-1}(x)) - v(J^{-1}(y)))^2}{|J^{-1}(x) - J^{-1}(y)|^{d+\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\
&= A \int_{B_r} \int_{B_r} \frac{(v(x) - v(y))^2}{|x-y|^{d+\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\
&= A \|v\|_{\dot{H}^{\frac{\alpha}{2}}(B_r)}.
\end{aligned}
$$

Thus, we have shown that (A) holds with the same constant.

Suppose k satisfies (C, ξ_0, R) for some $\xi_0 \in \mathbb{R}^d, R > 0$ and $C > 0$. Consider $x \in B_1, y \in \mathbb{R}^d$. We have $J(x) \in B_R(\xi_0)$. If $0 < r \leq \left(\frac{|x-y|}{2} \wedge \frac{1}{4}\right)$, then by (C, ξ_0, R)

$$
\begin{aligned}
\int_{B_r(x)} \widetilde{k}(z,y) \, \mathrm{d}z &= R^{\alpha+d} \int_{B_r(x)} k(J(z), J(y)) \, \mathrm{d}z \\
&= R^{\alpha} \int_{B_{Rr}(J(x))} k(z, J(y)) \, \mathrm{d}z \\
&\geq CR^{\alpha}(Rr)^d k(J(x), J(y)) \\
&= Cr^d \, \widetilde{k}(x,y),
\end{aligned}
$$

which proves the fourth item of our assertion.

The third claim is obviously true. ∎

Remark. If $R > 0$ and $\xi_0 \in \mathbb{R}^d$ are given so that $B_R(\xi_0) \subset B_1$ and a kernel k satisfies (A) and (C), then k obviously satisfies (A, ξ_0, R) and (C, ξ_0, R). In this case both k and $\widetilde{k}$ satisfy (A) and (C).

11.4. A local tail estimate

The statement of the following lemma is oriented on [DKP14, Lemma 4.2]. The aim is to bound the nonlocal k-Tail by localized quantities coming from supersolutions. The proof of the lemma uses the same split up idea of the double integral as it was used in the proof of Lemma 4.2 in [DKP14]. Furthermore, it relies heavily on assumption (C). In fact, this lemma is the reason for establishing (C) in the first place.

Lemma 11.7. *Assume $\alpha \in (0,2)$ and $f \in L^q(B_1)$ for some $q > \frac{d}{\alpha}$. Let (λ, μ) be a pair of radii with $\frac{1}{4} \le \lambda < \mu \le \frac{1}{2}$. Suppose k is a symmetric kernel that satisfies* (A), (B) *and* (C). *Assume $u \in V^k(B_1|\mathbb{R}^d) \cap L^1((1+|x|)^{-d-\alpha}\,\mathrm{d}x)$ is a nonnegative function satisfying $\mathcal{E}^k(u,\psi) \ge (f,\psi)$ for every nonnegative $\psi \in H^k_{B_1}(\mathbb{R}^d)$. Then there exists a constant $c > 0$, depending only on the dimension d, α, q and the constants from* (A), (B) *and* (C), *such that*

$$\mathrm{Tail}_k\left(u, 0, \frac{1}{2}, \lambda, \mu\right) \le c(\mu-\lambda)^{-2d-\alpha}\left(\sup_{B_\mu} u + \|f\|_{L^q\left(B_{\frac{3\mu+\lambda}{4}}\right)}\right).$$

If $\alpha \in [\alpha_0, 2)$ for some $\alpha_0 \in (0,2)$, then c can be chosen to depend on α_0 but not on α.

Proof. We borrow ideas of [DKP14, Lemma 4.2]. Let us emphasize that in the computations we often use the letter c to denote a generic positive constant. The value of c may change between different lines of the proof.

Choose some (λ, μ) with $\frac{1}{4} \le \lambda < \mu \le \frac{1}{2}$. The claim of the lemma is obvious if $\sup_{B_\mu} u = \infty$. Thus, we may assume $\sup_{B_\mu} u < \infty$. Set

$$\rho = \frac{\mu+\lambda}{2}, \quad \widetilde{\rho} = \frac{\rho+\mu}{2}.$$

Let $\phi \in C_c^\infty(B_{\widetilde{\rho}})$ with $0 \le \phi \le 1$, $\phi = 1$ in B_ρ and $|D\phi| \le c|\widetilde{\rho}-\rho|^{-1} = 4c(\mu-\lambda)^{-1}$. In the case $\sup_{B_\mu} u > 0$ we consider the following nonpositive test function

$$\eta = (u - 2S)\phi^2,$$

where $S = \sup_{B_\mu} u$. We have

$$\begin{aligned}
(f,\eta) &\ge \int_{B_\mu}\int_{B_\mu} (u(x)-u(y))(\eta(x)-\eta(y))k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 2\int_{\mathbb{R}^d\setminus B_\mu}\int_{B_\mu} (u(x)-u(y))(u(x)-2S)\phi^2(x)k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= I_1 + 2I_2.
\end{aligned}$$

We treat the integrals separately. We start with I_2:

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^d\setminus B_\mu}\int_{B_\mu} (u(x)-u(y))(u(x)-2S)\phi^2(x)\cdot \mathbb{1}_{\{u(y)\ge S\}}(y)\,k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^d\setminus B_\mu}\int_{B_\mu} (u(x)-u(y))(u(x)-2S)\phi^2(x)\cdot \mathbb{1}_{\{u(y)< S\}}(y)\,k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\ge \int_{\mathbb{R}^d\setminus B_\mu}\int_{B_\mu} (u(y)-u(x))(2S-u(x))\phi^2(x)\cdot \mathbb{1}_{\{u(y)\ge S\}}(y)\,k(x,y)\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^d \setminus B_\mu} \int_{B_\mu} 2S(u(x) - u(y))_+ \phi^2(x) \cdot \mathbb{1}_{\{u(y)<S\}}(y)\, k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\geq \int_{\mathbb{R}^d \setminus B_\mu} \int_{B_\mu} (u(y) - S) S \phi^2(x) \cdot \mathbb{1}_{\{u(y)\geq S\}}(y)\, k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \int_{\mathbb{R}^d \setminus B_\mu} \int_{B_\mu} 2S(u(x) - u(y))_+ \phi^2(x) \cdot \mathbb{1}_{\{u(y)<S\}}(y)\, k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= I_{2,1} - I_{2,2}.
\end{aligned}
$$

Let us further estimate $I_{2,1}$. For the calculation we use the fact that $\frac{1}{4} \leq \mu \leq \frac{1}{2}$ and

$$
\frac{|y|}{|x-y|} \leq \left(1 + \frac{\widetilde{\rho}}{\mu - \widetilde{\rho}}\right) = \left(\frac{\mu}{\mu - \widetilde{\rho}}\right) \leq c(\mu - \lambda)^{-1} \text{ for } x \in B_{\widetilde{\rho}},\ y \in \mathbb{R}^d \setminus B_\mu. \tag{11.5}
$$

Using these estimates together with (B),(C) and Proposition 11.5, we obtain

$$
\begin{aligned}
I_{2,1} &\geq S \int_{\mathbb{R}^d \setminus B_\mu} \int_{B_\rho} u(y)\, k(x,y)\,\mathrm{d}x\,\mathrm{d}y - B^{-1} S^2 \int_{\mathbb{R}^d \setminus B_\mu} \int_{B_{\widetilde{\rho}}} |x-y|^{-d-\alpha}\,\mathrm{d}x\,\mathrm{d}y \\
&\geq S \int_{\mathbb{R}^d \setminus B_\mu} u(y) \underbrace{\left(\int_{B_\rho} k(x,y)\,\mathrm{d}x\right)}_{\geq C|\mu-\lambda|^{d+\alpha} \sup_{x \in B_\lambda} k(x,y)} \mathrm{d}y - (\mu - \lambda)^{-d-\alpha} c S^2 |B_{\widetilde{\rho}}| \frac{\mu^{-\alpha}}{\alpha} \\
&\geq cS|\mu - \lambda|^d \operatorname{Tail}_k\left(u, 0, \frac{1}{2}, \lambda, \mu\right) - (\mu - \lambda)^{-d-\alpha} c S^2 |B_\mu| \frac{\mu^{-\alpha}}{\alpha} \\
&\geq cS(\mu - \lambda)^d \operatorname{Tail}_k\left(u, 0, \frac{1}{2}, \lambda, \mu\right) - (\mu - \lambda)^{-d-\alpha} c S^2.
\end{aligned}
$$

Next, we estimate $I_{2,2}$. With use of (11.5) we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d \setminus B_\mu} \int_{B_\mu} 2S(u(x) - u(y))_+ \phi^2(x) \cdot \mathbb{1}_{\{u(y)<S\}}(y)\, k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq 2cS^2 \int_{\mathbb{R}^d \setminus B_\mu} \int_{B_{\widetilde{\rho}}} |x-y|^{-d-\alpha}\,\mathrm{d}x\,\mathrm{d}y \\
&\leq cS^2 (\mu - \lambda)^{-d-\alpha}.
\end{aligned}
$$

Let us now treat the localized integral I_1. Let $w(x) = u(x) - 2S$. Then

$$
-2S \leq w(x) \leq -S \tag{11.6}
$$

for every $x \in B_\mu$. We note that the following algebraic inequality holds:

$$
(a-b)(a\alpha^2 - b\beta^2) = (b\beta - a\alpha)^2 - ba(\alpha - \beta)^2 \geq -ba(\alpha - \beta)^2, \quad a, b, \alpha, \beta \in \mathbb{R}.
$$

Choosing $a = w(x), b = w(y), \alpha = \phi(x)$ and $\beta = \phi(y)$ for $(x, y) \in B_\mu \times B_\mu$, we obtain using the algebraic inequality and (11.6)

$$
\begin{aligned}
(w(x) - w(y))&(w(x)\phi^2(x) - w(y)\phi^2(y)) \\
&\geq -|w(x)||w(y)|(\phi(x) - \phi(y))^2 \\
&\geq -4S^2(\phi(x) - \phi(y))^2.
\end{aligned}
$$

With the help of (B) this directly implies

$$
\begin{aligned}
I_1 &\geq -cS^2 \int_{B_\mu}\int_{B_\mu} |\phi(x) - \phi(y)|^2 |x - y|^{-d-\alpha} \,\mathrm{d}x\,\mathrm{d}y \\
&\geq -cS^2(\widetilde{\rho} - \rho)^{-2} \int_{B_\mu}\int_{B_\mu} |x - y|^{2-\alpha-d} \,\mathrm{d}x\,\mathrm{d}y \\
&\geq -cS^2(\widetilde{\rho} - \rho)^{-2}\mu^{2-\alpha}|B_\mu| \\
&\geq -cS^2(\mu - \lambda)^{-2}
\end{aligned}
$$

Putting these three estimates together and using the fact that u is a supersolution we get

$$
\begin{aligned}
(f, \eta) \geq &- cS^2\left((\mu - \lambda)^{-2} + (\mu - \lambda)^{-d-\alpha}\right) \\
&+ cS(\mu - \lambda)^d \operatorname{Tail}_k\left(u, 0, \frac{1}{2}, \lambda, \mu\right).
\end{aligned}
$$

We obtain by the Hölder inequality with $q' = \frac{q}{q-1}$ the inequality

$$
(f, \eta) = \int_{B_{\widetilde{\rho}}} f(x)(u(x) - 2S)\phi^2(x) \,\mathrm{d}x \leq cS \,\|f\|_{L^q(B_{\widetilde{\rho}})}\mu^{\frac{d}{q'}}.
$$

Therefore,

$$
\begin{aligned}
&\operatorname{Tail}_k\left(u, 0, \frac{1}{2}, \lambda, \mu\right) \\
&\quad\leq cS\left[(\mu - \lambda)^{-2} + (\mu - \lambda)^{-d-\alpha}\right](\mu - \lambda)^{-d} \\
&\qquad + c(\mu - \lambda)^{-d}\|f\|_{L^q(B_{\frac{3\mu+\lambda}{4}})} \\
&\quad\leq c(\mu - \lambda)^{-2d-\alpha}\left(S + \|f\|_{L^q(B_{\frac{3\mu+\lambda}{4}})}\right),
\end{aligned}
$$

which is what we wanted to prove.

In the case $S = 0$ the claim of the lemma follows easily if we test with $\eta = \phi$, where ϕ is defined as above. ■

11.5. Local boundedness of solutions

The aim of this section is to prove that weak solutions of corresponding elliptic equations are locally bounded. We use the same methods as we did in the local setting. The statements and the computations are a little more complicated because of the nonlocal terms. For the readers convenience we proceed in the same order as we did in the local case.

A Caccioppoli type estimate

The following theorem is a version of [DKP16, Theorem 1.4] in our setup. The proof is also adapted from there. It is the nonlocal version of Lemma 10.4.

Theorem 11.8 (Caccioppoli inequality)**.** *Assume $f \in L^2_{\mathrm{loc}}(B_1)$ and let k be a symmetric kernel. Let $u \in V^k(B_1|\mathbb{R}^d)$ so that $\mathcal{E}^k(u,\psi) \le (f,\psi)$ for each nonnegative $\psi \in H^k_{B_1}(\mathbb{R}^d)$. Let $w = (u-\ell)_+$ for $\ell \in \mathbb{R}$. There exists a constant $c > 0$, independent of u and ℓ, such that the following inequality holds true for every $B_r(x_0) \subset B_1$ and each $\phi \in C_c^\infty(B_r(x_0))$:*

$$\begin{aligned}\int_{B_r(x_0)}\int_{B_r(x_0)} &|w(x)\phi(x) - w(y)\phi(y)|^2 k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\le c\int_{B_r(x_0)}\int_{B_r(x_0)} \left(w^2(x)+w^2(y)\right)|\phi(x)-\phi(y)|^2 k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + c\int_{\mathbb{R}^d\setminus B_r(x_0)}\int_{B_r(x_0)} w(y)w(x)\phi^2(x)k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + (f, w\phi^2).\end{aligned} \tag{11.7}$$

Proof. Let $B_r(x_0) \subset B_1$ and $u \in V^k(B_1|\mathbb{R}^d)$ satisfy $\mathcal{E}^k(u,\psi) \le (f,\psi)$ for every admissible nonnegative test function ψ. We plug $\psi = w\phi^2$ into the definition of a weak subsolution. Here $\phi \in C_c^\infty(B_r(x_0))$ is a nonnegative function. We obtain with $B_r = B_r(x_0)$ the estimate

$$\begin{aligned}(f, w\phi^2) &\ge \int_{B_r}\int_{B_r}(u(x)-u(y))(w(x)\phi^2(x) - w(y)\phi^2(y))k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + 2\int_{\mathbb{R}^d\setminus B_r}\int_{B_r}(u(x)-u(y))w(x)\phi^2(x)k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= I_1 + 2I_2.\end{aligned} \tag{11.8}$$

Let us further investigate the nonlocal term I_2. We see that

$$I_2 \ge -\int_{\mathbb{R}^d\setminus B_r}\int_{B_r}(u(y)-u(x))_+(u(x)-\ell)_+\phi^2(x)k(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$\geq -\int_{\mathbb{R}^d\setminus B_r}\int_{B_r}(u(y)-\ell)_+(u(x)-\ell)_+\phi^2(x)k(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= -\int_{\mathbb{R}^d\setminus B_r}\int_{B_r} w(y)w(x)\phi^2(x)k(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{11.9}$$

Next, let us consider the localized term I_1. An easy case analysis shows that

$$(u(x)-u(y))(w(x)\phi^2(x)-w(y)\phi^2(y)) \geq (w(x)-w(y))(w(x)\phi^2(x)-w(y)\phi^2(y)).$$

We obtain by Lemma 11.1 for $\varepsilon > 0$ the estimate

$$\phi^2(x) \geq \phi^2(y)(1-\varepsilon) - \frac{1}{\varepsilon}(\phi(x)-\phi(y))^2.$$

Assuming $w(x) > w(y)$ and plugging in $\varepsilon = \frac{1}{2}\frac{w(x)-w(y)}{w(x)}$ gives

$$\begin{aligned}&(w(x)-w(y))w(x)\phi^2(x)\\ &\qquad\geq (w(x)-w(y))w(x)\phi^2(y) - \frac{1}{2}(w(x)-w(y))^2\phi^2(y)\\ &\qquad\quad - 2w^2(x)(\phi(x)-\phi(y))^2.\end{aligned} \tag{11.10}$$

It is also obviously true that

$$\begin{aligned}&(w(x)-w(y))w(x)\phi^2(x)\\ &\qquad\geq (w(x)-w(y))w(x)\phi^2(x) - \frac{1}{2}(w(x)-w(y))^2\phi^2(y)\\ &\qquad\quad - 2w^2(x)(\phi(x)-\phi(y))^2.\end{aligned} \tag{11.11}$$

Therefore, combining (11.10) and (11.11) leads to

$$\begin{aligned}&(w(x)-w(y))w(x)\phi^2(x)\\ &\qquad\geq (w(x)-w(y))w(x)\max\{\phi(x),\phi(y)\}^2\\ &\qquad\quad - \frac{1}{2}(w(x)-w(y))^2\max\{\phi(x),\phi(y)\}^2\\ &\qquad\quad - 2w^2(x)(\phi(x)-\phi(y))^2,\end{aligned} \tag{11.12}$$

for $w(x) > w(y)$. Note that (11.12) is obvious for $w(x) = w(y)$. Hence, we may assume $w(x) \geq w(y)$ in the following computation. From (11.12) we obtain

$$\begin{aligned}&(w(x)-w(y))(w(x)\phi^2(x)-w(y)\phi^2(y))\\ &\quad\geq (w(x)-w(y))w(x)\max\{\phi(x),\phi(y)\}^2 - (w(x)-w(y))w(y)\max\{\phi(x),\phi(y)\}^2\\ &\qquad - \frac{1}{2}(w(x)-w(y))^2\max\{\phi(x),\phi(y)\}^2 - 2w^2(x)(\phi(x)-\phi(y))^2\\ &\quad= \frac{1}{2}(w(x)-w(y))^2\max\{\phi(x),\phi(y)\}^2 - 2w^2(x)(\phi(x)-\phi(y))^2\end{aligned}$$

$$\geq \frac{1}{4}(w(x)-w(y))^2(\phi^2(x)+\phi^2(y)) - 2(w^2(x)+w^2(y))(\phi(x)-\phi(y))^2, \qquad (11.13)$$

whenever $w(x) \geq w(y)$. Note that the inequality (11.13) is symmetric in x and y, i.e., we can exchange the roles of x and y to obtain the estimate (11.13) for the case $w(y) > w(x)$. In conclusion,

$$\begin{aligned} I_1 \geq & \int_{B_r}\int_{B_r} \frac{1}{4}(w(x)-w(y))^2(\phi^2(x)+\phi^2(y))k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ & - \int_{B_r}\int_{B_r} 2(w^2(x)+w^2(y))(\phi(x)-\phi(y))^2 k(x,y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \qquad (11.14)$$

By Lemma 11.2 we observe that

$$\begin{aligned} (w(x)\phi(x)-w(y)\phi(y))^2 \leq\ & 2(w(x)-w(y))^2(\phi^2(x)+\phi^2(y)) \\ & + 2(w^2(x)+w^2(y))(\phi(x)-\phi(y))^2 \end{aligned}$$

or equivalently

$$\begin{aligned} \frac{1}{4}(w(x)-w(y))^2(\phi^2(x)+\phi^2(y)) \geq\ & \frac{1}{8}(w(x)\phi(x)-w(y)\phi(y))^2 \\ & - \frac{1}{4}(w^2(x)+w^2(y))(\phi(x)-\phi(y))^2. \end{aligned}$$

Combining the last estimate with (11.14) we obtain the following lower bound for the localized double integral

$$\begin{aligned} I_1 \geq & \int_{B_r}\int_{B_r} \frac{1}{8}(w(x)\phi(x)-w(y)\phi(y))^2 k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ & - \int_{B_r}\int_{B_r} \frac{9}{4}(w^2(x)+w^2(y))(\phi(x)-\phi(y))^2 k(x,y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \qquad (11.15)$$

Putting together (11.8),(11.9) and (11.15) gives

$$\begin{aligned} (f, w\phi^2) \geq & \int_{B_r}\int_{B_r} \frac{1}{8}(w(x)\phi(x)-w(y)\phi(y))^2 k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ & - \int_{B_r}\int_{B_r} \frac{9}{4}(w^2(x)+w^2(y))(\phi(x)-\phi(y))^2 k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ & - 2\int_{\mathbb{R}^d\setminus B_r}\int_{B_r} w(y)w(x)\phi^2(x)k(x,y)\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

which yields the claim of the theorem. ■

An L^2-upper-level-set inequality

The following lemma is the starting point of our iteration. It is the nonlocal analogue of Lemma 10.5.

Lemma 11.9. *Let $d \geq 2, \alpha \in (0,2)$. Assume $f \in L^q(B_1)$ for some $q > \frac{d}{\alpha}$. Let k be a symmetric kernel that satisfies* (A) *and* (B). *Suppose $u \in V^k(B_1|\mathbb{R}^d) \cap L^1((1+|x|)^{-d-\alpha}\,\mathrm{d}x)$ is nonnegative and fulfills $\mathcal{E}^k(u,\psi) \leq (f,\psi)$ for every nonnegative $\psi \in H^k_{B_1}(\mathbb{R}^d)$. Then there are constants $c > 0$, depending only on d, α, q and the constants from* (A)*and* (B) *and $\varepsilon \in (0,1)$, depending only on d, α and q, such that the following estimate holds true for all radii $\frac{1}{2} \leq \kappa < \lambda < \mu \leq 1$ and all constants $L > \widetilde{\ell} > \ell > \sqrt{|B_1|}\|u\|_{L^2(B_1)}$:*

$$
\begin{aligned}
&(\widetilde{\ell} - \ell)^{\varepsilon} \|(u - L)_+\|_{L^2(B_\kappa)} \\
&\leq c \left[(\lambda - \kappa)^{-2} + (\mu - \lambda)^{-d-\alpha} \frac{\operatorname{Tail}\left(u, 0, \frac{1}{2}\right)}{\widetilde{\ell} - \ell} \right. \\
&\qquad \left. + \frac{\|f\|^2_{L^q(B_\lambda)}}{(\widetilde{\ell} - \ell)^2} \right]^{\frac{1}{2}} \|(u - \ell)_+\|^{1+\varepsilon}_{L^2(B_\mu)}.
\end{aligned} \tag{11.16}
$$

If in addition k satisfies (C) *and u fulfills $\mathcal{E}^k(u,\psi) = (f,\psi)$ for every $\psi \in H^k_{B_1}(\mathbb{R}^d)$, then for all $\frac{1}{4} \leq \kappa < \lambda < \mu \leq \frac{1}{2}$ and all constants $L > \widetilde{\ell} > \ell > \sqrt{|B_{\frac{1}{2}}|}\|u\|_{L^2(B_{\frac{1}{2}})}$:*

$$
\begin{aligned}
&(\widetilde{\ell} - \ell)^{\varepsilon} \|(u - L)_+\|_{L^2(B_\kappa)} \\
&\leq c \left[(\lambda - \kappa)^{-2} + 1 + \frac{(\mu - \lambda)^{-2d-\alpha}}{\widetilde{\ell} - \ell} \left(\sup_{B_{\frac{1}{2}}} u + \|f\|_{L^q\left(B_{\frac{3\mu+\lambda}{4}}\right)} \right) \right. \\
&\qquad \left. + \frac{\|f\|^2_{L^q(B_\lambda)}}{(\widetilde{\ell} - \ell)^2} \right]^{\frac{1}{2}} \|(u - \ell)_+\|^{1+\varepsilon}_{L^2(B_\mu)},
\end{aligned} \tag{11.17}
$$

where c also depend on the constant from (C). *If $\alpha \in [\alpha_0, 2)$, then the constants c and ε in* (11.16) *respectively* (11.17) *may be chosen to depend on α_0 but not on α.*

Proof. Let $\widetilde{\ell} > 0$ and $0 < \mu \leq 1$ and $\phi \in C_c^\infty(B_\mu)$ with $0 \leq \phi \leq 1$. We use the notation $w = (u - \widetilde{\ell})_+$ and $2^* = \frac{2d}{d-\alpha}$. The Sobolev embedding, compare Corollary 2.21, and the Caccioppoli inequality, see Theorem 11.8, yield

$$
\begin{aligned}
\left(\int_{B_\mu} |w(x)\phi(x)|^{2^*} \right)^{\frac{2}{2^*}} &\leq c \int_{B_\mu} \int_{B_\mu} (w(x)\phi(x) - w(y)\phi(y))^2 k(x,y) \,\mathrm{d}x \,\mathrm{d}y \\
&\quad + c \int_{B_\mu} |w(x)\phi(x)|^2 \,\mathrm{d}x \\
&\leq c \int_{B_\mu} \int_{B_\mu} w^2(x)(\phi(x) - \phi(y))^2 k(x,y) \,\mathrm{d}x \,\mathrm{d}y
\end{aligned}
$$

$$+ c \int_{B_\mu} \int_{\mathbb{R}^d \setminus B_\mu} w(x)\phi^2(x) w(y) k(x,y) \,\mathrm{d}y \,\mathrm{d}x$$
$$+ c \int_{B_\mu} w^2(x)\phi^2(x) \,\mathrm{d}x + c \int_{B_\mu} f(x) w(x) \phi^2(x) \,\mathrm{d}x. \qquad (11.18)$$

Using the Hölder inequality and the Young inequality with $\delta \in (0,1)$ we estimate the last term on the right-hand side in the following way

$$\int_{B_\mu} f(x)w(x)\phi^2(x)\,\mathrm{d}x \le \|f\|_{L^q(\mathrm{supp}\phi)} \left(\int_{B_\mu} |w(x)\phi(x)|^{2^*} \right)^{\frac{1}{2^*}} |w\phi \neq 0|^{1-\frac{1}{2^*}-\frac{1}{q}}$$
$$\le \delta \left(\int_{B_\mu} |w(x)\phi(x)|^{2^*} \right)^{\frac{2}{2^*}}$$
$$+ c(\delta) \|f\|^2_{L^q(\mathrm{supp}\phi)} |w\phi \neq 0|^{2(1-\frac{1}{2^*}-\frac{1}{q})}. \qquad (11.19)$$

Combining (11.18) and (11.19) we receive

$$\left(\int_{B_\mu} |w(x)\phi(x)|^{2^*} \right)^{\frac{2}{2^*}} \le c \Bigg[\int_{B_\mu} \int_{B_\mu} w^2(x)(\phi(x)-\phi(y))^2 k(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int_{B_\mu} w(x)\phi^2(x) \int_{\mathbb{R}^d \setminus B_\mu} w(y)k(x,y)\,\mathrm{d}y\,\mathrm{d}x \qquad (11.20)$$
$$+ \int_{B_\mu} w^2(x)\phi^2(x)\,\mathrm{d}x + \|f\|^2_{L^q(\mathrm{supp}\phi)} |w\phi \neq 0|^{2(1-\frac{1}{2^*}-\frac{1}{q})}. \Bigg]$$

Now (11.20) implies

$$\int_{B_\mu} (w(x)\phi(x))^2 \,\mathrm{d}x \le \left(\int_{B_\mu} |w(x)\phi(x)|^{2^*} \,\mathrm{d}x \right)^{\frac{2}{2^*}} |w\phi \neq 0|^{1-\frac{2}{2^*}}$$
$$\le c \Bigg[\int_{B_\mu} \int_{B_\mu} w^2(x)(\phi(x)-\phi(y))^2 k(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int_{B_\mu} w(x)\phi^2(x) \int_{\mathbb{R}^d \setminus B_\mu} w(y)k(x,y)\,\mathrm{d}y\,\mathrm{d}x \qquad (11.21)$$
$$+ \int_{B_\mu} w^2(x)\phi^2(x)\,\mathrm{d}x + \|f\|^2_{L^q(\mathrm{supp}\phi)} |w\phi \neq 0|^{1+\frac{\alpha}{d}-\frac{2}{q}} \Bigg] |w\phi \neq 0|^{\frac{\alpha}{d}}.$$

From now on let us assume

$$|w\phi \neq 0| \le 1, \qquad (11.22)$$

which will be verified later. Note that $q > \frac{d}{\alpha}$ is equivalent to $1 + \frac{\alpha}{d} - \frac{2}{q} > 1 - \frac{1}{q}$. By (11.22) we therefore have

$$|w\phi \neq 0|^{1+\frac{\alpha}{d}-\frac{2}{q}} |w\phi \neq 0|^{\frac{\alpha}{d}} \le |w\phi \neq 0|^{1+\frac{\alpha}{d}-\frac{1}{q}}.$$

Also, $q > \frac{d}{\alpha}$ implies $\frac{\alpha}{d} - \frac{1}{q} > 0$. Thus, there exists $\varepsilon \in (0,1)$ such that

$$|w\phi \neq 0|^{1+\frac{\alpha}{d}-\frac{2}{q}} |w\phi \neq 0|^{\frac{\alpha}{d}} \leq |w\phi \neq 0|^{1+\varepsilon}$$

and

$$|w\phi \neq 0|^{\frac{\alpha}{d}} \leq |w\phi \neq 0|^{\varepsilon}.$$

Our estimate (11.21) becomes

$$\begin{aligned}\int_{B_\mu} (w(x)\phi(x))^2 \,\mathrm{d}x \leq c\Big[&\int_{B_\mu}\int_{B_\mu} w^2(x)(\phi(x)-\phi(y))^2 k(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &+ \int_{B_\mu} w(x)\phi^2(x) \int_{\mathbb{R}^d\setminus B_\mu} w(y)k(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &+ \int_{B_\mu} w(x)\phi^2(x)\,\mathrm{d}x + \|f\|^2_{L^q(\mathrm{supp}\phi)} |w\phi \neq 0| \Big] |w\phi \neq 0|^{\varepsilon} \\ &= c|w\phi \neq 0|^{\varepsilon}\left[I_1 + I_2 + I_3 + F^2 |w\phi \neq 0|\right], \end{aligned} \tag{11.23}$$

where $F = \|f\|_{L^q(\mathrm{supp}\phi)}$.

Let now $\frac{1}{2} \leq \kappa < \lambda < \mu \leq 1$. We specify our choice of the cutoff function ϕ. We take $\phi \in C_c^\infty(B_\mu)$ so that $\mathrm{supp}(\phi) \subset B_\lambda, 0 \leq \phi \leq 1, \phi = 1$ in B_κ and $D\phi \leq \frac{C}{\lambda-\kappa}$. Let us introduce the notation

$$A(\ell, r) = \{x \in B_r | u \geq \ell\} \quad \text{for } \ell, r > 0.$$

The next step is to bound the terms in the right-hand side of (11.23). By applying Condition (B) we estimate

$$\begin{aligned} I_1 &\leq \frac{c}{(\lambda-\kappa)^2} \int_{A(\mu,\widetilde{\ell})}\int_{B_\mu} \left(u(x)-\widetilde{\ell}\right)^2 |x-y|^{-d-\alpha+2}\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \frac{c}{(\lambda-\kappa)^2}\frac{\mu^{2-\alpha}}{2-\alpha} \int_{A(\mu,\widetilde{\ell})} (u(x)-\widetilde{\ell})^2\,\mathrm{d}x \\ &\leq \frac{c}{2-\alpha}(\lambda-\kappa)^{-2} \int_{A(\mu,\widetilde{\ell})} (u(x)-\widetilde{\ell})^2\,\mathrm{d}x \end{aligned} \tag{11.24}$$

and

$$I_2 \leq \int_{A(\mu,\widetilde{\ell})} (u(x)-\widetilde{\ell}) \sup_{x\in B_\lambda} \int_{\mathbb{R}^d\setminus B_\mu} u(y)|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x$$

Now note that for $x \in B_\lambda, y \in \mathbb{R}^d \setminus B_\mu$

$$\frac{|y|}{|x-y|} \leq \frac{|x-y|+|x|}{|x-y|} \leq 1 + \frac{\lambda}{\mu-\lambda}$$

$$\Rightarrow |x-y|^{-d-\alpha} \leq \left(\frac{\mu}{\mu-\lambda}\right)^{d+\alpha} |y|^{-d-\alpha}. \tag{11.25}$$

Thus, we have

$$\begin{aligned} I_2 &\leq \left(\frac{\mu}{\mu-\lambda}\right)^{d+\alpha} \int_{A(\mu,\widetilde{\ell})} (u(x)-\widetilde{\ell})\,\mathrm{d}x \int_{\mathbb{R}^d\setminus B_\mu} u(y)|y|^{-d-\alpha}\,\mathrm{d}y \\ &\leq (\mu-\lambda)^{-d-\alpha}\,\mathrm{Tail}\left(u,0,\frac{1}{2}\right) \int_{A(\mu,\widetilde{\ell})} (u(x)-\widetilde{\ell})\,\mathrm{d}x. \end{aligned} \tag{11.26}$$

Obviously

$$I_3 \leq \int_{A(\mu,\widetilde{\ell})} (u(x)-\widetilde{\ell})^2\,\mathrm{d}x. \tag{11.27}$$

The left-hand side of (11.23) can be estimated by

$$\int_{A(\mu,\widetilde{\ell})} (u(x)-\widetilde{\ell})^2\phi^2(x)\,\mathrm{d}x \geq \int_{A(\kappa,\widetilde{\ell})} (u(x)-\widetilde{\ell})^2\,\mathrm{d}x. \tag{11.28}$$

The combination of (11.23) with (11.24), (11.26), (11.27) and (11.28) yields

$$\begin{aligned} \int_{A(\kappa,\widetilde{\ell})} (u(x)-\widetilde{\ell})^2\,\mathrm{d}x \leq c\Big[&\big((\lambda-\kappa)^{-2}+1\big) \int_{A(\mu,\widetilde{\ell})} (u(x)-\widetilde{\ell})^2\,\mathrm{d}x \\ &+ (\mu-\lambda)^{-d-\alpha}\,\mathrm{Tail}\left(u,0,\frac{1}{2}\right) \int_{A(\mu,\widetilde{\ell})} (u(x)-\widetilde{\ell})\,\mathrm{d}x \\ &+ F^2|A(\lambda,\widetilde{\ell})|\Big]\,|A(\lambda,\widetilde{\ell})|^{\varepsilon}. \end{aligned} \tag{11.29}$$

Let now $L > \widetilde{\ell} > \ell$, where a lower bound for ℓ will be given later. Then we have

$$\begin{aligned} &|A(\lambda,\widetilde{\ell})| \leq |A(\mu,\widetilde{\ell})|, \\ &\int_{A(\kappa,L)} (u(x)-L)^2\,\mathrm{d}x \leq \int_{A(\kappa,\widetilde{\ell})} (u(x)-\widetilde{\ell})^2\,\mathrm{d}x, \\ &\int_{A(\mu,\widetilde{\ell})} (u(x)-\widetilde{\ell})\,\mathrm{d}x \leq \int_{A(\mu,\ell)} (u(x)-\ell)\,\mathrm{d}x, \\ &\int_{A(\mu,\widetilde{\ell})} (u(x)-\widetilde{\ell})^2\,\mathrm{d}x \leq \int_{A(\mu,\ell)} (u(x)-\ell)^2\,\mathrm{d}x, \\ &\int_{A(\mu,\widetilde{\ell})} (u(x)-\widetilde{\ell})\,\mathrm{d}x \leq \frac{1}{\widetilde{\ell}-\ell} \int_{A(\mu,\ell)} (u(x)-\ell)^2\,\mathrm{d}x, \end{aligned}$$

and

$$|A(\mu,\widetilde{\ell})| = |B_\mu \cap \{u \geq \widetilde{\ell}\}|$$

$$
\begin{aligned}
&= |B_\mu \cap \{u - \ell \geq \widetilde{\ell} - \ell\}| \\
&\leq \frac{1}{(\widetilde{\ell} - \ell)^2} \int_{A(\mu,\ell)} (u(x) - \ell)^2 \,\mathrm{d}x.
\end{aligned}
$$

From (11.29) we obtain

$$
\begin{aligned}
\int_{A(\kappa,L)} (u(x) - L)^2 \,\mathrm{d}x \leq c \Bigg[(\lambda - \kappa)^{-2} + 1 + (\mu - \lambda)^{-d-\alpha} \frac{\mathrm{Tail}\left(u, 0, \frac{1}{2}\right)}{\widetilde{\ell} - \ell} \\
+ \frac{F^2}{(\widetilde{\ell} - \ell)^2} \Bigg] \frac{1}{(\widetilde{\ell} - \ell)^{2\varepsilon}} \left(\int_{A(\mu,\ell)} (u(x) - \ell)^2 \,\mathrm{d}x \right)^{1+\varepsilon} \qquad (11.30)
\end{aligned}
$$

or

$$
\begin{aligned}
\|(u - L)_+\|_{L^2(B_r)} \leq c \Bigg[(\lambda - \kappa)^{-2} + 1 + (\mu - \lambda)^{-d-\alpha} \frac{\mathrm{Tail}\left(u, 0, \frac{1}{2}\right)}{\widetilde{\ell} - \ell} \\
+ \frac{F^2}{(\widetilde{\ell} - \ell)^2} \Bigg]^{\frac{1}{2}} \frac{1}{(\widetilde{\ell} - \ell)^{\varepsilon}} \|(u - \ell)_+\|^{1+\varepsilon}_{L^2(B_\mu)}. \qquad (11.31)
\end{aligned}
$$

We recall that (11.31) only holds true if (11.22) is satisfied. But

$$
|w\phi \neq 0| = |A(\lambda, \widetilde{\ell})| \leq \frac{1}{\widetilde{\ell}} \int_{B_\lambda} u(x) \,\mathrm{d}x \leq \frac{\sqrt{|B_1|}}{\ell} \|u\|_{L^2(B_\mu)}.
$$

Therefore, we choose in (11.31)

$$
L > \widetilde{\ell} > \ell > \sqrt{|B_1|} \|u\|_{L^2(B_1)}. \qquad (11.32)
$$

This finishes the proof of (11.16). The proof of (11.17) works analogously, we just need to modify the estimate of the term I_2. Here we we take $\frac{1}{4} \leq \kappa < \lambda < \mu \leq \frac{1}{2}$ and start from the inequality (11.23).

We obtain the estimate

$$
\begin{aligned}
I_2 &\leq \int_{A(\mu,\widetilde{\ell})} (u(x) - \widetilde{\ell}) \,\mathrm{d}x \sup_{x \in B_\lambda} \int_{\mathbb{R}^d \setminus B_\mu} u(y) k(x, y) \,\mathrm{d}y \\
&\leq \int_{A(\mu,\widetilde{\ell})} (u(x) - \widetilde{\ell}) \,\mathrm{d}x \, \mathrm{Tail}_k \left(u, 0, \frac{1}{2}, \lambda, \mu \right)
\end{aligned}
$$

From the first assertion of the lemma we already know $\sup_{B_{\frac{1}{2}}} u < \infty$. With the help of Lemma 11.7 we compute

$$
I_2 \leq \int_{A(\mu,\widetilde{\ell})} (u(x) - \widetilde{\ell}) \,\mathrm{d}x \, \mathrm{Tail}_k \left(u, 0, \frac{1}{2}, \lambda, \mu \right)
$$

$$\leq c(\mu-\lambda)^{-2d-\alpha}\left[\sup_{B_{\frac{1}{2}}} u + \|f\|_{L^q\left(B_{\frac{3\mu+\lambda}{4}}\right)}\right]\int_{A(\mu,\widetilde{\ell})}(u(x)-\widetilde{\ell})\,\mathrm{d}x.$$

Since the estimates of I_1, I_3 stay the same, we get analogously to the derivation of (11.31) for $\frac{1}{4} \leq \kappa < \lambda < \mu \leq \frac{1}{2}$ and $L > \widetilde{\ell} > \ell > \sqrt{|B_{\frac{1}{2}}|}\|u\|_{L^2(B_{\frac{1}{2}})}$ the inequality

$$\begin{aligned}\|(u-L)_+\|_{L^2(B_\kappa)} &\\ &\leq c\Bigg[(\lambda-\kappa)^{-2}+1+(\mu-\lambda)^{-2d-\alpha}\frac{\sup_{B_{\frac{1}{2}}} u + \|f\|_{L^q\left(B_{\frac{3\mu+\lambda}{4}}\right)}}{\widetilde{\ell}-\ell} \\ &\qquad + \frac{F^2}{(\widetilde{\ell}-\ell)^2}\Bigg]^{\frac{1}{2}}\frac{\|(u-\ell)_+\|^{1+\varepsilon}_{L^2(B_\mu)}}{(\widetilde{\ell}-\ell)^{\varepsilon}}.\end{aligned}$$

This implies (11.17). ∎

Control of the L^∞-norm

The L^2-upper-level-set inequality given in Lemma 11.9 is the starting point for our iteration procedure. This De Giorgi type iteration is done in the proof of the next theorem below. Note that the upcoming estimates contain an interpolation parameter $\delta \in (0,1]$. This δ is important because it allows us later to reabsorb the supremum on the right-hand side of inequality (11.34). The local analogue of this result is stated in Theorem 10.6. For the nonlinear setting of the fractional p-Laplace operator, the analogous result can be found in [DKP16, Theorem 1.1]. The idea of our proof is taken from there.

Theorem 11.10. *Let $d \geq 2$, $\alpha \in (0,2)$. Assume $f \in L^q(B_1)$ for some $q > \frac{d}{\alpha}$. Let k be a symmetric kernel that satisfies* (A) *and* (B)*. Suppose $u \in V^k(B_1|\mathbb{R}^d) \cap L^1((1+|x|)^{-d-\alpha}\,\mathrm{d}x)$ is nonnegative and satisfies $\mathcal{E}^k(u,\psi) \leq (f,\psi)$ for every nonnegative $\psi \in H^k_{B_1}(\mathbb{R}^d)$. Then there is a constant $c > 0$, depending only on d, α, q and the constants from* (A) *and* (B)*, and $\varepsilon \in (0,1)$, depending only on d, α and q, such that the following estimate holds true for each $B_r(x_0) \subset B_1$ and every $\delta \in (0,1]$:*

$$\begin{aligned}\sup_{B_{\frac{r}{2}}(x_0)} u \leq c\delta^{-\frac{1}{\varepsilon}}\left(⨍_{B_r(x_0)} u^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}} &+ c\delta\,\mathrm{Tail}(u, x_0, \tfrac{r}{2}) \\ &+ c\delta r^{\alpha-\frac{d}{q}}\|f\|_{L^q\left(B_{\frac{7}{8}r}(x_0)\right)}.\end{aligned} \qquad (11.33)$$

If in addition k satisfies (C) *and u fulfills $\mathcal{E}^k(u,\psi) = (f,\psi)$ for every $\psi \in H^k_{B_1}(\mathbb{R}^d)$, then*

$$\sup_{B_{\frac{r}{4}}(x_0)} u \le c\delta^{-\frac{1}{\varepsilon}} \left(\fint_{B_{\frac{r}{2}}(x_0)} |u(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} + c\delta \sup_{B_{\frac{r}{2}}(x_0)} u + c\delta r^{\alpha-\frac{d}{q}} \|f\|_{L^q\left(B_{\frac{31}{64}r}(x_0)\right)}, \quad (11.34)$$

where the constant c also depends on the constant from (C)*. If $\alpha \in [\alpha_0, 2)$, then c and ε in* (11.33) *resp.* (11.34) *can be chosen to depend on α_0 but not on α.*

Proof. Let us first prove (11.33) and (11.34) for the case $r = 1, x_0 = 0$. We start with the proof of (11.33). Let

$$r_j = \frac{1}{2}\left(1 + \frac{1}{2^j}\right), \quad \ell_j = \ell_0 + \left(1 - \frac{1}{2^j}\right)\overline{\ell}$$

with $\ell_0 = c(d)\|u\|_{L^2(B_1)}$ and $\overline{\ell} > 0$ to be specified later. We want to apply Lemma 11.9 with

$$L = \ell_{j+1}, \quad \widetilde{\ell} = \frac{\ell_{j+1} + \ell_j}{2}, \quad \ell = \ell_j,$$

and

$$\kappa = r_{j+1}, \quad \lambda = \frac{r_{j+1} + r_j}{2}, \quad \mu = r_j.$$

This gives us

$$\widetilde{\ell} - \ell = c2^{-j}\overline{\ell}, \quad \lambda - \kappa = \mu - \lambda = \frac{1}{4}2^{-j}.$$

Thus, by (11.16) with $A_j = \|(u - \ell_j)_+\|_{L^2(B_{r_j})}$ and $\varepsilon \in (0,1)$ as in Lemma 11.9

$$\begin{aligned} 2^{-j\varepsilon}\overline{\ell}^{\varepsilon} A_{j+1} &\le c\left[2^{2j} + 2^{j(d+\alpha)+j}\overline{\ell}^{-1}\,\mathrm{Tail}\left(u, 0, \frac{1}{2}\right) + 2^{2j}\overline{\ell}^{-2}\|f\|_{L^q(B_\lambda)}\right]^{\frac{1}{2}} A_j^{1+\varepsilon} \\ &\le c2^{j(d+\alpha+1)}\left[1 + \overline{\ell}^{-1}\,\mathrm{Tail}\left(u, 0, \frac{1}{2}\right) + \overline{\ell}^{-2}\|f\|^2_{L^q(B_{\frac{7}{8}})}\right]^{\frac{1}{2}} A_j^{1+\varepsilon}. \end{aligned} \quad (11.35)$$

Let $\delta \in (0,1]$ be given. Choose

$$\overline{\ell} > \delta\,\mathrm{Tail}\left(u, 0, \frac{1}{2}\right) + \delta\|f\|_{L^q(B_{\frac{7}{8}})}. \quad (11.36)$$

Then (11.35) implies

$$\overline{\ell}^{\varepsilon} A_{j+1} \le c2^{j(d+\alpha+1+\varepsilon)}\delta^{-1} A_j^{1+\varepsilon},$$

or with $C = 2^{(d+\alpha+1+\varepsilon)}$

$$\left(\frac{A_{j+1}}{\overline{\ell}}\right) \le cC^j\delta^{-1}\left(\frac{A_j}{\overline{\ell}}\right)^{1+\varepsilon}.$$

We want to argue that $\overline{\ell}$ can be chosen in such a way that (11.36) holds and that $A_j \to 0$ for $j \to \infty$.

If

$$\overline{\ell} \geq A_0 \delta^{-\frac{1}{\varepsilon}} c^{\frac{1}{\varepsilon}} C^{-\frac{1}{\varepsilon^2}} + \delta \operatorname{Tail}\left(u, 0, \frac{1}{2}\right) + \delta \|f\|_{L^q(B_{\frac{7}{8}})}, \tag{11.37}$$

then we obtain from Lemma 10.2 that $A_j \to 0$ for $j \to \infty$.

By the Dominated Convergence Theorem, see Theorem 2.5, and continuity of the maximum function we get for almost every $x \in B_{\frac{1}{2}}$

$$0 = \lim_{j\to\infty} (u - \ell_j)_+ = (u - (\overline{\ell} + \ell_0))_+,$$

that is

$$\sup_{B_{\frac{1}{2}}} u \leq c\delta^{-\frac{1}{\varepsilon}} \left(\int_{B_1} u^2(x) \, \mathrm{d}x \right)^{\frac{1}{2}} + \delta \operatorname{Tail}\left(u, 0, \frac{1}{2}\right) + \delta \|f\|_{L^q(B_{\frac{7}{8}})}. \tag{11.38}$$

The proof of (11.34) works now analogously. Here we choose

$$r_j = \frac{1}{4}\left(1 + \frac{1}{2^j}\right), \quad \ell_j = \ell_0 + \left(1 - \frac{1}{2^j}\right)\overline{\ell}$$

with $\ell_0 = c(d)\|u\|_{L^2(B_{\frac{1}{2}})}$ and $\overline{\ell} > 0$ to be specified later. We apply Lemma 11.9 with

$$L = \ell_{j+1}, \quad \widetilde{\ell} = \frac{\ell_{j+1} + \ell_j}{2}, \quad \ell = \ell_j,$$

and

$$\kappa = r_{j+1}, \quad \lambda = \frac{r_{j+1} + r_j}{2}, \quad \mu = r_j.$$

Then $\widetilde{\ell} - \ell = c2^{-j}\overline{\ell}, \lambda - \kappa = \mu - \lambda = \frac{2^j}{8}$ and (11.17) with $A_j = \|(u - \ell_j)_+\|_{L^2(B_{r_j})}$ gives us

$$\frac{A_{j+1}}{\overline{\ell}} \leq c 2^{j(2d+\alpha+1+\varepsilon)} \left[\frac{\sup_{B_{1/2}} u + \|f\|_{L^q(B_{31/64})}}{\overline{\ell}} + \frac{\|f\|^2_{L^q(B_{31/64})}}{\overline{\ell}^2} \right]^{\frac{1}{2}} \left(\frac{A_j}{\overline{\ell}}\right)^{1+\varepsilon}. \tag{11.39}$$

For a given $\delta \in (0, 1]$ we now choose

$$\overline{\ell} \geq \delta \sup_{B_{\frac{1}{2}}} u + \delta \|f\|_{L^q\left(B_{\frac{31}{64}}\right)}.$$

Note that this choice is possible since by the first assertion (11.16) we know $\sup_{B_{\frac{1}{2}}} u < \infty$. Plugging this into (11.39) leads to

$$\frac{A_{j+1}}{\overline{\ell}} \leq c\delta^{-1} 2^{j(2d+\alpha+1+\varepsilon)} \left(\frac{A_j}{\overline{\ell}}\right)^{1+\varepsilon}. \tag{11.40}$$

By Lemma 10.2 and the same reasoning as in the proof of (11.33) we conclude

$$\sup_{B_{\frac{1}{4}}} u \leq c\delta^{-\frac{1}{\varepsilon}} \left(\int_{B_{\frac{1}{2}}} u^2(x)\,\mathrm{d}x \right)^{\frac{1}{2}} + \delta \sup_{B_{\frac{1}{2}}} u + \delta \|f\|_{L^q\left(B_{\frac{31}{64}}\right)}, \tag{11.41}$$

which finishes the proof for the case $r = 1$ and $x_0 = 0$.

Let now $B_r(x_0) \subset B_1$. Consider $J : B_1 \to B_r(x_0)$, $J(x) = rx + x_0$ and set

$$\widetilde{u}(x) = u(J(x)), \quad \widetilde{f}(x) = r^{\alpha} f(J(x)) \quad \text{and} \quad \widetilde{k}(x,y) = r^{\alpha+d} k(J(x), J(y)).$$

Note that since u is a weak subsolution in B_1, it in particular is a weak subsolution in $B_r(x_0) \subset B_1$. Therefore, we can apply Lemma 11.6. This together with (11.38) and (11.41) gives us

$$\begin{aligned}
\sup_{B_{\frac{r}{2}}(x_0)} u &= \sup_{B_{\frac{1}{2}}} \widetilde{u} \\
&\leq c\delta^{-\frac{1}{\varepsilon}} \left(\int_{B_1} \widetilde{u}^2(x)\,\mathrm{d}x \right)^{\frac{1}{2}} + \delta \operatorname{Tail}\left(\widetilde{u}, 0, \frac{1}{2}\right) + \delta \|\widetilde{f}\|_{L^q(B_{\frac{7}{8}})} \\
&= c\delta^{-\frac{1}{\varepsilon}} \left(\fint_{B_r(x_0)} u^2(x)\,\mathrm{d}x \right)^{\frac{1}{2}} + \delta \operatorname{Tail}\left(u, x_0, \frac{r}{2}\right) + \delta r^{\alpha - \frac{d}{q}} \|f\|_{L^q(B_{\frac{7}{8}r}(x_0))}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{B_{\frac{r}{4}}(x_0)} u &= \sup_{B_{\frac{1}{4}}} \widetilde{u} \\
&\leq c\delta^{-\frac{1}{\varepsilon}} \left(\int_{B_{\frac{1}{2}}} \widetilde{u}^2(x)\,\mathrm{d}x \right)^{\frac{1}{2}} + \delta \sup_{B_{\frac{1}{2}}} \widetilde{u} + \delta \|\widetilde{f}\|_{L^q\left(B_{\frac{31}{64}}\right)} \\
&= c\delta^{-\frac{1}{\varepsilon}} \left(\fint_{B_{\frac{r}{2}}(x_0)} u^2(x)\,\mathrm{d}x \right)^{\frac{1}{2}} + \delta \sup_{B_{\frac{r}{2}}(x_0)} u + \delta r^{\alpha - \frac{d}{q}} \|f\|_{L^q\left(B_{\frac{31}{64}r}(x_0)\right)},
\end{aligned}$$

which finishes the proof of the theorem. ■

In the next corollary we derive our desired local boundedness result, the counterpart of the weak Harnack inequality. The idea of the proof is to choose the free parameter δ in the estimate (11.34) in the right way, so that well known interpolation tools lead to an inequality of the form mentioned in the iteration Lemma 10.1. The proof is analogous to the version in the local setting, see Theorem 10.3 and it is also analogous to parts of the proof of [DKP14, Theorem 1.1], see the paragraph *Proof of the nonlocal Harnack inequality* in Section 4 of [DKP14].

Corollary 11.11. *Let $d \geq 2, \alpha \in (0,2)$ and $B_r(x_0) \subset B_1$. Assume $f \in L^q(B_1)$ for some $q > \frac{d}{\alpha}$ and let k be a kernel that satisfies* (A), (B) *and* (C). *Let $u \in V^k(B_1|\mathbb{R}^d) \cap L^1((1+|x|)^{-d-\alpha}\,\mathrm{d}x)$ such that $u \geq 0$ and $\mathcal{E}^k(u,\psi) = (f,\psi)$ for every $\psi \in H^k_{B_1}(\mathbb{R}^d)$. For each $p \in (0,2]$ there exists a constant $c > 0$, depending only on d, α, q, p and the constants from* (A), (B), (C), *such that*

$$\sup_{B_{\frac{r}{8}}(x_0)} u \leq c \left(\fint_{B_{\frac{r}{2}}(x_0)} u^p(x)\,\mathrm{d}x \right)^{\frac{1}{p}} + c r^{\alpha - \frac{d}{q}} \|f\|_{L^q\left(B_{\frac{31}{64}r}(x_0)\right)}.$$

If $\alpha \in [\alpha_0, 2)$, then c can be chosen to depend on α_0 but not on α.

Proof. Let $\frac{1}{2} \leq t < s \leq 1$, which in particular implies $s - t < t$. Then we know from the first assertion of Theorem 11.10 that

$$\sup_{B_{\frac{s}{2}r}(x_0)} u \leq \sup_{B_{\frac{r}{2}}(x_0)} u < \infty.$$

From the second assertion of Theorem 11.10 we conclude with $\gamma = \frac{1}{\varepsilon}$

$$\begin{aligned}
\sup_{B_{t\frac{r}{4}}(x_0)} u &\leq c\delta^{-\gamma} \left(\fint_{B_{t\frac{r}{2}}(x_0)} u^2 \right)^{\frac{1}{2}} + c\delta \sup_{B_{t\frac{r}{2}}(x_0)} u + c\delta r^{\alpha-\frac{d}{q}} \|f\|_{L^q\left(B_{\frac{31}{64}tr}(x_0)\right)} \\
&\leq c\delta^{-\gamma} \left(\frac{2s}{2t}\right)^{\frac{d}{2}} \left(\fint_{B_{s\frac{r}{2}}(x_0)} u^2\,\mathrm{d}x \right)^{\frac{1}{2}} + c\delta \sup_{B_{s\frac{r}{2}}(x_0)} u + c\delta r^{\alpha-\frac{d}{q}} \|f\|_{L^q\left(B_{\frac{31}{64}tr}(x_0)\right)} \\
&\leq c\delta^{-\gamma} \left(\frac{1}{s-t}\right)^{\frac{d}{2}} \left(\fint_{B_{s\frac{r}{2}}(x_0)} u^2\,\mathrm{d}x \right)^{\frac{1}{2}} + c\delta \sup_{B_{s\frac{r}{2}}(x_0)} u + c\delta r^{\alpha-\frac{d}{q}} \|f\|_{L^q\left(B_{\frac{31}{64}tr}(x_0)\right)} \\
&\leq c\frac{\delta^{-\gamma}}{(s-t)^{\frac{d}{2}}} \left(\sup_{B_{s\frac{r}{2}}(x_0)} u \right)^{\frac{2-p}{2}} \left(\fint_{B_{s\frac{r}{2}}(x_0)} u^p(x)\,\mathrm{d}x \right)^{\frac{1}{2}} \\
&\quad + c\delta \sup_{B_{s\frac{r}{2}}(x_0)} u + c\delta r^{\alpha-\frac{d}{q}} \|f\|_{L^q\left(B_{\frac{31}{64}tr}(x_0)\right)},
\end{aligned} \tag{11.42}$$

where $p \in (0,2]$. If $p \in (0,2)$ we can choose $\delta = \frac{1}{4c}$ and apply the Young inequality with $\widetilde{p} = \frac{2}{2-p}$ and $\widetilde{q} = \frac{2}{p}$ to obtain

$$\begin{aligned}
\sup_{B_{t\frac{r}{4}}(x_0)} u &\leq \frac{1}{2} \sup_{B_{s\frac{r}{2}}(x_0)} u + \frac{1}{4} r^{\alpha-\frac{d}{q}} \|f\|_{L^q\left(B_{\frac{31}{64}tr}(x_0)\right)} + \frac{c(p)}{(s-t)^{\frac{d}{p}}} \left(\fint_{B_{s\frac{r}{2}}(x_0)} u^p(x)\,\mathrm{d}x \right)^{\frac{1}{p}} \\
&\leq \frac{1}{2} \sup_{B_{s\frac{r}{2}}(x_0)} u + \frac{1}{4} r^{\alpha-\frac{d}{q}} \|f\|_{L^q\left(B_{\frac{31}{64}r}(x_0)\right)} + \frac{c(p)}{(s-t)^{\frac{d}{p}}} \left(\fint_{B_{\frac{r}{2}}(x_0)} u^p(x)\,\mathrm{d}x \right)^{\frac{1}{p}}.
\end{aligned} \tag{11.43}$$

Note that the constant in front of the averaged L^p-norm depends on p. If $p = 2$, then (11.43) follows from (11.42) by choosing $\delta = \frac{1}{4c}$.

Now we can apply Lemma 10.1 with

$$T_0 = \frac{r}{8}, \quad T_1 = \frac{r}{4}, \quad f(t) = \sup_{B_{t\frac{r}{4}}(x_0)} u, \quad \nu = \frac{d}{p},$$

$$B = \frac{1}{4} r^{\alpha - \frac{d}{q}} \|f\|_{L^q\left(B_{\frac{31}{64}r}(x_0)\right)}, \quad A = c\left(\fint_{B_{\frac{r}{2}}} u^p(x)\,\mathrm{d}x\right)^{\frac{1}{p}}$$

in order to get

$$\sup_{B_{\frac{r}{8}}(x_0)} u \le c\left(\fint_{B_{\frac{r}{2}}(x_0)} u^p(x)\,\mathrm{d}x\right)^{\frac{1}{p}} + cr^{\alpha-\frac{d}{q}}\|f\|_{L^q\left(B_{\frac{31}{64}r}(x_0)\right)},$$

which is what we claimed. ■

11.6. Harnack inequalities

In this section we prove a full Harnack inequality for weak solutions. The proof combines our local boundedness result with the weak Harnack inequality established in [DK20], which we also used in Part II of this thesis. For our own convenience we recall the weak Harnack inequality using the notation *inf* for the essential infimum.

Theorem 11.12 (weak Harnack inequality)**.** *Let $d \ge 2, \alpha \in (0,2)$. Assume $f \in L^q(B_1)$ for $q > \frac{d}{\alpha}$. Let k be a symmetric kernel that satisfies* (A) *and* (B)*. Suppose $u \in V^k(B_1|\mathbb{R}^d)$ is a nonnegative function that fulfills $\mathcal{E}^k(u,\psi) \ge (f,\psi)$ for each nonnegative $\psi \in H^k_{B_1}(\mathbb{R}^d)$. Then there exist constants $c > 0$ and $p_0 \in (0,1)$, independent of u, such that for every $B_r(x_0) \subset B_1$*

$$\inf_{B_{\frac{r}{4}}(x_0)} u \ge c\left(\fint_{B_{\frac{r}{2}}(x_0)} |u(x)|^{p_0}\,\mathrm{d}x\right)^{\frac{1}{p_0}} - cr^{\alpha-\frac{d}{q}}\|f\|_{L^q(B_{\frac{15}{16}r}(x_0))}.$$

Proof. The claim follows from [DK20, Theorem 4.1] and a scaling argument. ■

Theorem 11.13. *Let $d \ge 2, \alpha \in (0,2)$. Assume $f \in L^q(B_1)$ for $q > \frac{d}{\alpha}$. Let k be a symmetric kernel that satisfies* (A), (B) *and* (C)*. Then there exists a positive constant* *c such that for each nonnegative* $u \in V^k(B_1|\mathbb{R}^d) \cap L^1((1+|x|)^{-d-\alpha}\,\mathrm{d}x)$ *that has the property* $\mathcal{E}^k(u,\psi) = (f,\psi)$ *for every* $\psi \in H^k_{B_1}(\mathbb{R}^d)$*, the following inequality holds for every* $B_r(x_0) \subset B_1$*:*

$$\sup_{B_{\frac{r}{8}}(x_0)} u \le c \inf_{B_{\frac{r}{8}}(x_0)} u + c r^{\alpha - \frac{d}{q}} \|f\|_{L^q\left(B_{\frac{15}{16}r}(x_0)\right)}. \tag{11.44}$$

If $\alpha \in [\alpha_0, 2)$*, then the constant* c *may be chosen to depend on* α_0 *but not on* α*.*

Proof. We write $B_r = B_r(x_0)$. The combination of Corollary 11.11 with Theorem 11.12 gives us

$$\begin{aligned}\sup_{B_{\frac{r}{8}}} u &\le c \left(\fint_{B_{\frac{r}{2}}} |u(x)|^{p_0}\,\mathrm{d}x \right)^{\frac{1}{p_0}} + c r^{\alpha-\frac{d}{q}} \|f\|_{L^q\left(B_{\frac{31}{64}r}\right)} \\ &\le c \inf_{B_{\frac{r}{4}}} u + c r^{\alpha-\frac{d}{q}} \|f\|_{L^q\left(B_{\frac{15}{16}r}\right)} \\ &\le c \inf_{B_{\frac{r}{8}}} u + c r^{\alpha-\frac{d}{q}} \|f\|_{L^q\left(B_{\frac{15}{16}r}\right)},\end{aligned}$$

what we wanted to prove. ∎

Note that (11.44) reduces to the classical Harnack inequality if we consider the case $f = 0$.

Let us now drop the assumption that u is nonnegative almost everywhere on $\mathbb{R}^d$ and replace it with a nonnegative assumption on the unit ball.

Theorem 11.14. *Let* $d \ge 2, \alpha \in (0,2)$*. Assume* $f \in L^q(B_1)$ *for* $q > \frac{d}{\alpha}$*. Let* k *be a symmetric kernel that satisfies* (A), (B) *and* (C)*. Then there exists a positive constant* c *such that for each* $u \in V^k(B_1|\mathbb{R}^d) \cap L^1((1+|x|)^{-d-\alpha}\,\mathrm{d}x)$ *that satisfies* $u \ge 0$ *in* B_1 *and* $\mathcal{E}^k(u,\psi) = (f,\psi)$ *for every* $\psi \in H^k_{B_1}(\mathbb{R}^d)$*, the following inequality holds for every* $B_r(x_0) \subset B_1$*:*

$$\sup_{B_{\frac{r}{8}}(x_0)} u \le c \inf_{B_{\frac{r}{8}}(x_0)} u + c \sup_{x \in B_{\frac{15}{16}r}(x_0)} \left(\int_{\mathbb{R}^d \setminus B_1} u_-(y) k(x,y)\,\mathrm{d}y \right) + c r^{\alpha-\frac{d}{q}} \|f\|_{L^q\left(B_{\frac{15}{16}r}(x_0)\right)}. \tag{11.45}$$

If $\alpha \in [\alpha_0, 2)$*, then* c *can be chosen to depend on* α_0 *but not on* α*.*

Proof. We prove the claim for $B_r(x_0) = B_1$. The assertion then follows by a scaling argument. Set $u = u_+ - u_-$, where $u_+ = \max(0,u)$ and $u_- = -\min(0,u)$. Then

$$\mathcal{E}^k(u_+,\phi) = \mathcal{E}^k(u_-,\phi) + (f,\phi)$$

$$= \int_{B_1} f(x)\phi(x)\,\mathrm{d}x - 2\int_{B_1}\int_{\mathbb{R}^d\setminus B_1} u_-(y)\phi(x)k(x,y)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_{B_1} \phi(x)\left[f(x) - 2\int_{\mathbb{R}^d\setminus B_1} u_-(y)k(x,y)\,\mathrm{d}y\right]\mathrm{d}x$$

for every $\phi \in H^k_{B_1}(\mathbb{R}^d)$. Set

$$\hat{f}:\mathbb{R}^d \to \mathbb{R}, \quad \hat{f}(x) = f(x) - 2\int_{\mathbb{R}^d\setminus B_1} u_-(y)k(x,y)\,\mathrm{d}y.$$

Suppose that $\hat{f} \in L^q(B_1)$. Then the function u_+ satisfies all conditions of Theorem 11.13 with f replaced by $\hat{f}$. On the other hand, if $\sup_{B_{\frac{15}{16}}} \hat{f} = \infty$, then the assertion of Theorem 11.13 is obvious for u_+. In any case we can deduce

$$\sup_{B_{\frac{1}{8}}} u = \sup_{B_{\frac{1}{8}}} u_+ \le c \inf_{B_{\frac{1}{8}}} u_+ + c\|\hat{f}\|_{L^q\left(B_{\frac{15}{16}}\right)}$$
$$\le c\inf_{B_{\frac{1}{8}}} u + c \sup_{B_{\frac{15}{16}}} \left(\int_{\mathbb{R}^d\setminus B_1} u_-(y)k(x,y)\,\mathrm{d}y\right) + c\|f\|_{L^q\left(B_{\frac{15}{16}}\right)}.$$

The statement of the theorem follows now by scaling. ∎

Remark. Using Lemma 11.6 one can replace the unit ball B_1 in all the assertions of Section 11.6 by some arbitrary ball $B_R(\xi_0) \subset \mathbb{R}^d$ and the assumptions (A), (C) by (A, ξ_0, R) and (C, ξ_0, R). Then one will recover a full Harnack inequality with tail terms for weak solutions in $B_R(\xi_0)$. The constant c in (11.45) however stays the same as in the case of the unit ball. We emphasize in particular that there is no dependence of c on R.

At the end of this section we provide a more classical version of the Harnack inequality for nonnegative solutions to equations without a right-hand side.

Theorem 11.15. *Let $d \ge 2, \alpha \in (0,2)$ and $B_R(\xi_0) \subset \mathbb{R}^d$ any ball. Let k be a symmetric kernel that satisfies (A, ξ_0, R),* (B) *and (C, ξ_0, R). Assume $u \in V^k(B_R(\xi_0)|\mathbb{R}^d) \cap L^1((1+|x|)^{-d-\alpha})$ is nonnegative and satisfies $\mathcal{E}^k(u,\psi) = 0$ for every $\psi \in H^k_{B_R(\xi_0)}(\mathbb{R}^d)$. Suppose $\Omega \subset \mathbb{R}^d$ and $\overline{\Omega} \subset B_R(\xi_0)$. There is a constant $c > 0$, independent of u, such that*

$$\sup_{\Omega} u \le c \inf_{\Omega} u. \tag{11.46}$$

Proof. The proof uses a standard covering argument. Set $\rho = \operatorname{dist}(\Omega, B_R(\xi_0)) > 0$. Cover up Ω with a finite family of balls $B_i \subset B_R(x_0)$ of radius $\frac{\varrho}{8}$ so that for every ball B_i there is another ball B_j with $B_i \cap B_j \neq \emptyset$. From Theorem 11.13 and a scaling argument we know $\sup_{B_i} \le c \inf_{B_i}$ for every i. If now $x, y \in \Omega$, then we can construct a chain of points

$x = x_1, x_2, ..., x_n = y$ so that for every $i \in \{1, ..., n\}$, x_i and x_{i+1} are in the same ball B_i of the family $\{B_i\}$. Now

$$u(x) \leq \sup_{B_1} u(x) \leq c \inf_{B_1} u(x) \leq \cdots \leq c^{n-1} \inf_{B_{n-1}} u(x) \leq c^n u(y).$$

The assertion of the theorem follows. ∎

12. Examples and a counterexample

In this chapter we provide examples of kernels that satisfy our Condition (A), Condition (B) and Condition (C). We refer to these examples as *positive examples*. A consequence from the previous chapter is that corresponding weak solutions satisfy a Harnack inequality.

We also present a class of kernels that does not satisfy Condition (C).

In the following we often consider d-regular sets.

Definition 12.1. *A nonempty set $F \subset \mathbb{R}^d$ is called d-regular if there exists $\gamma \geq 1$ such that for all $x \in \overline{F}$ and all $0 < r \leq 1$*

$$\gamma^{-1} r^d \leq |F \cap B_r(x)| \leq \gamma r^d. \tag{12.1}$$

Remark. A reference for further reading on d-regular sets is [JW84]. Here the authors first define d-measures on nonempty closed sets F. Then F is called a d-set if such a d-measure exists on F. In our definition we simplify the situation by taking always the Lebesgue measure on F as d-measure, that is, F is a d-set (in the sense of [JW84, Chapter II.1] with d-measure $\mathbb{1}_F(x)\,\mathrm{d}x$.

12.1. Kernels corresponding to a configuration

This section deals with kernels corresponding to admissible configurations Γ as defined in Part II of this thesis. As in Part II we use the notation $V^\Gamma[x] = \Gamma(x) + x$. For $\alpha \in (0,2)$ we consider kernels of the following form

$$k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty], \quad k(x,y) \asymp \Big(\mathbb{1}_{V^\Gamma[x]}(y) + \mathbb{1}_{V^\Gamma[y]}(x)\Big)\,|x-y|^{-d-\alpha}.$$

Obviously, Condition (B) is satisfied for k. From Theorem 8.1 it follows that Condition (A) is also fulfilled. Therefore, we only need to investigate wheter Condition (C) holds true or not.

A positive example. Let us assume that $\mathbb{R}^d$ is split by a hyperplane H into two regions H_1 and H_2. Let Γ be an admissible configuration such that $\Gamma(\mathbb{R}^d) = \{V^1, V^2\}$ and $\Gamma(H_1) = \{V^1\}, \Gamma(H_2) = \{V^2\}$. Let $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ be measurable with

$$k(x,y) \asymp \Big(\mathbb{1}_{V^\Gamma[x]}(y) + \mathbb{1}_{V^\Gamma[y]}(x)\Big)\,|x-y|^{-d-\alpha}. \tag{12.2}$$

Since V^1 and V^2 are both d-regular sets, we can choose a constant $\gamma \geq 1$ such that (12.1) holds for $F = V^1$ respectively $F = V^2$ with the same constant γ. Then γ depends on the minimum of both apex angles of the cones. Let us emphasize that the radius in (12.1) needs not to be bounded from above by 1, that is, (12.1) holds true for all radii $r > 0$.

A consequence of the ensuing lemma is that the kernels above enjoy Condition (C).

Lemma 12.2. *Let k be an integral kernel as in* (12.2)*. There is a constant $c > 0$ such that for every $x, y \in \mathbb{R}^d$ and every radius $0 < r \leq \frac{|x-y|}{2}$ it holds*

$$\fint_{B_r(x)} \left(\mathbb{1}_{V^\Gamma[z]}(y) + \mathbb{1}_{V^\Gamma[y]}(z)\right) \mathrm{d}z \geq c\left(\mathbb{1}_{V^\Gamma[x]}(y) + \mathbb{1}_{V^\Gamma[y]}(x)\right).$$

Proof. Let $x, y \in \mathbb{R}^d$ and $0 < r \leq \frac{|x-y|}{2}$. Suppose $\mathbb{1}_{V^\Gamma[x]}(y) + \mathbb{1}_{V^\Gamma[y]}(x) \geq 1$. Then we can distinguish two cases. First, we assume $\mathbb{1}_{V^\Gamma[y]}(x) = 1$, that is, $x \in V^\Gamma[y]$. Since a double cone is d-regular, this implies $|B_r(x) \cap V^\Gamma[y]| \geq \widetilde{\gamma} r^d$, where the constant $\widetilde{\gamma} > 0$ depends only on the dimension d and on the apex angle of $\Gamma(y)$. Therefore

$$\begin{aligned}
\int_{B_r(x)} \left(\mathbb{1}_{V^\Gamma[z]}(y) + \mathbb{1}_{V^\Gamma[y]}(z)\right) \mathrm{d}z &\geq \int_{B_r(x)} \mathbb{1}_{V^\Gamma[y]}(z)\,\mathrm{d}z \\
&= |B_r(x) \cap V^\Gamma[y]| \\
&\geq \widetilde{\gamma} r^d \\
&\geq \widetilde{\gamma} r^d \left(\mathbb{1}_{V^\Gamma[x]}(y) + \mathbb{1}_{V^\Gamma[y]}(x)\right).
\end{aligned}$$

Now let us consider the second case. We assume $x, y \in \mathbb{R}^d$ are given such that $\mathbb{1}_{V^\Gamma[x]}(y) = 1$ and $\mathbb{1}_{V^\Gamma[y]}(x) = 0$. This implies $y \in V^\Gamma[x]$ and $\Gamma(x) \neq \Gamma(y)$.

We may and do assume $\Gamma(x) = V^1$ and $\Gamma(y) = V^2$.

For simplicity we restrict ourselves to the case that $V_1[z] \cap H_1$ is a connected set for every $z \in H$. The other case can be treated similarly.

We easily find a single cone $\widetilde{V}$ that satisfies the following property: If $y \in H_2$, then $\widetilde{V}[z] \subset V^1[y]$ for every $z \in H$, see Figure 12.1. The cone $\widetilde{V}$ does only depend on the configuration Γ. As a consequence, the set $\widetilde{V}$ is d-regular and the constant $\gamma \geq 1$ in (12.1) depends only on Γ.

Assume now $\mathbb{1}_{V^\Gamma[x]}(y) = \mathbb{1}_{V^1[x]}(y) = 1$. Then $x \in V^1[y]$ and there exists $z \in H$ such that $x \in \widetilde{V}[z]$. By the d-set property of $\widetilde{V}[z]$ we conclude

$$\int_{B_r(x)} \left(\mathbb{1}_{V^\Gamma[z]}(y) + \mathbb{1}_{V^\Gamma[y]}(z)\right) \mathrm{d}z \geq |B_r(x) \cap V^1[y]| \geq |B_r(x) \cap \widetilde{V}[z]| \geq \gamma^{-1} r^d.$$

The claim of the lemma follows with $c = \widetilde{\gamma} \wedge \gamma^{-1}$, which depends only on Γ. ∎

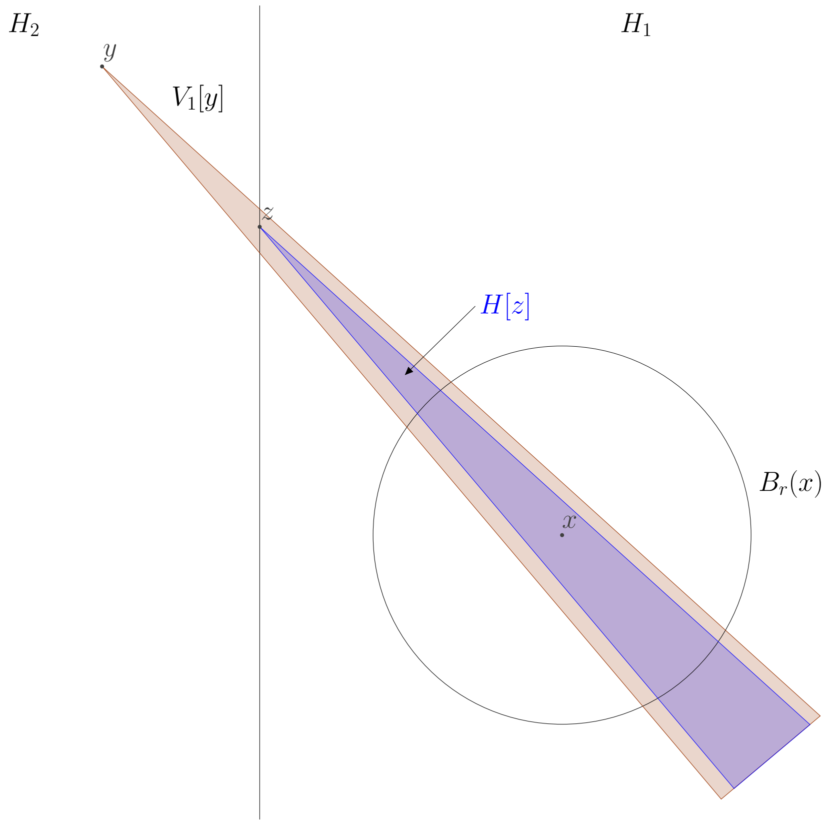

Figure 12.1.: The set $H[z]$ (in blue)

If now $x \in B_1, y \in \mathbb{R}^d$ and $z \in B_r(x)$ for $0 \le r \le \left(\frac{|x-y|}{2} \wedge \frac{1}{4}\right)$, then $|x-y| \asymp |z-y|$. This yields the upcoming corollary.

Corollary 12.3. *Let k be an integral kernel as in* (12.2). *Then k satisfies Condition (C).*

A counterexample. In general, we cannot expect Condition (C) to hold true for kernels arising from configurations. In the following paragraph we give a counterexample. The verification of this counterexample seems rather technical but the idea behind it can easily be understood. We consider a configuration that separates $\mathbb{R}^d$ into two regions. The catch is that our Condition (C) does not allow the boundary of any of the regions to differ to much from a hyperplane. Our counterexample even includes corner points.

We consider $d = 2$. We look at the admissible configuration Γ with $\Gamma(\mathbb{R}^2) = \{V^1, V^2\}$ and

$$H_1 = \Gamma^{-1}(V^1) = \{(x, y) \,|\, x_1 \ge 0, x_2 \ge 0\}, \quad H_2 = \mathbb{R}^2 \setminus H_1.$$

Denote by $\{e_1, e_2\}$ the standard basis of $\mathbb{R}^2$. The double cone V^2 shall have symmetry axis $v_2 = \lambda e_1$ and a very small apex angle $\vartheta_2 \ll \frac{\pi}{4}$. The double cone V^1 shall have symmetry axis $v_1 = e_1 \cos\left(\frac{5}{8}\pi\right) + e_2 \sin\left(\frac{5}{8}\pi\right)$ and angle $\vartheta_1 \leq \frac{\pi}{8}$.

Before we state the formal proof we want to give a brief explanation of the phenomenon behind this counterexample. Note that we can take a point $y \in \partial V^1$ so that $|(V^1 + y) \cap H_1| = 0$, see Figure 12.2. Moreover, since ϑ_2 is small, y can be chosen so that $y \notin V^2 + z$ for any $z \in B_r(x)$, where $0 < r \leq \frac{|x-y|}{2}$. This already gives a hint that Condition (C) cannot hold in this case. Since we exclude nullsets in Condition (C), we have to work a little bit more to prove that Condition (C) is violated. Our *hint* from Figure 12.2 can be understood as the limit case for $n \to \infty$ in the formal proof that follows below.

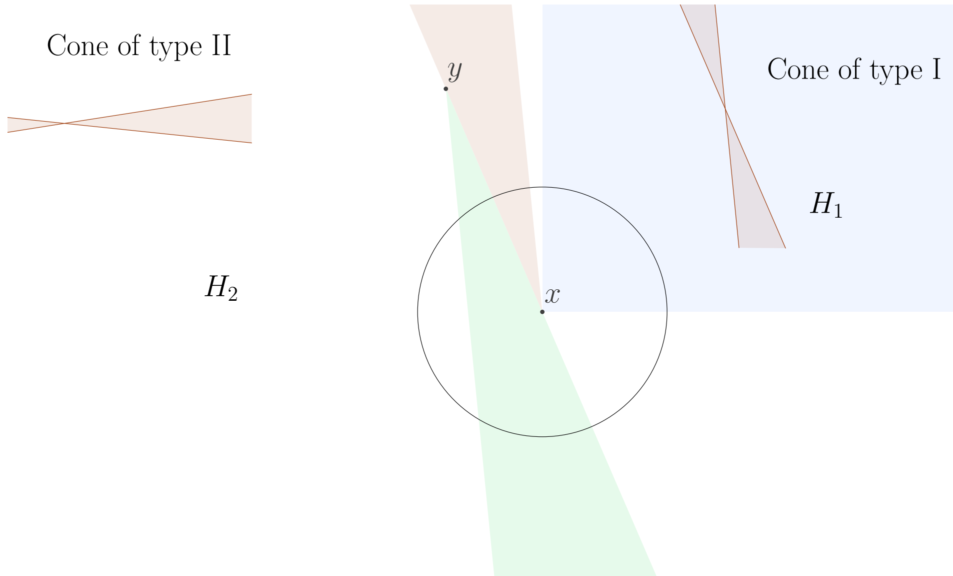

Figure 12.2.: Counterexample for kernels generated by a configuration

We aim at showing the following assertion.

For every $C > 0$ there are $0 < \lambda < \mu \leq \frac{1}{2}, x_0 \in B_1$ and sets of positive measure $M_1 \subset B_\lambda(x_0), M_2 \subset \left(\mathbb{R}^2 \setminus B_\mu(x_0)\right)$ such that for each $x \in M_1$ and every $y \in M_2$ it holds

$$\int_{B_{\frac{\mu+\lambda}{2}}(x_0)} \left(\mathbb{1}_{V^\Gamma[x]}(y) + \mathbb{1}_{V^\Gamma[y]}(x)\,\mathrm{d}x\right) < C|\mu - \lambda|^2 \left(\mathbb{1}_{V^\Gamma[x]}(y) + \mathbb{1}_{V^\Gamma[y]}(x)\right). \tag{12.3}$$

Then it follows from Proposition 11.5 that Condition (C) cannot hold for k.

For the present example we can even show that there are universal constants λ, μ and a universal $x_0 \in B_1$ such that (12.3) holds for all $C > 0$. Take $x_0 = 0, \mu = \frac{1}{2}, \lambda = \frac{1}{4}$. Let

$n \in \mathbb{N}$. We consider the cube $A_{\frac{1}{n}}(\frac{1}{2n}, \frac{1}{2n})$ with center $(\frac{1}{2n}, \frac{1}{2n})$ and side length $\frac{1}{n}$ and set $M_1 = A_{\frac{1}{n}}(\frac{1}{2n}, \frac{1}{2n})$. The latter is a subset of $B_\lambda(x_0)$ provided that n is large enough. Now we construct the set M_2. Let $V = \bigcap_{x \in M_1}(V^1 + x)$. We know from Lemma 6.13 that V contains the double half cone $V^1_{\frac{\sqrt{2}}{2n}} + (\frac{1}{2n}, \frac{1}{2n})$. We only consider a subset of this double half cone, namely

$$\widetilde{V} = V^1_{\frac{\sqrt{2}}{2n}} + \left(\frac{1}{2n}, \frac{1}{2n}\right) \cap \{(x,y)\,|y \geq 0\}.$$

Then $\widetilde{V}$ is a (single) cone with apex at some point $\widetilde{x} \in \mathbb{R}^2$ and symmetry axis v_1. We can write the straight line $\{\lambda v_1 + \widetilde{x} \,|\, \lambda \in \mathbb{R}\}$ as a graph of some function g. For $\varepsilon > 0$ consider the set $\widetilde{V}(\varepsilon)$ of all points in $\widetilde{V}$ that have distance at most ε to the boundary of $\widetilde{V}$ and that lie below the straight line $\{\lambda v_1 + \widetilde{x} \,|\, \lambda \in \mathbb{R}\}$, that is,

$$\widetilde{V}(\varepsilon) = \{x \in \widetilde{V} \,|\, \mathrm{dist}(x, \partial\widetilde{V}) < \varepsilon\} \cap \{(x_1, x_2) \in \widetilde{V} \,|\, x_2 < g(x_1)\}.$$

Now set

$$M_2 = M_2(n) = \left(\mathbb{R}^2 \setminus B_1\right) \cap \widetilde{V}\left(\left(\frac{1}{n} - \frac{1}{2n}\right)\sqrt{2}\right) = \left(\mathbb{R}^2 \setminus B_1\right) \cap \widetilde{V}\left(\frac{\sqrt{2}}{2n}\right).$$

This construction guarantees that every $y \in M_2$ is in particular an element of $V^1_{\frac{\sqrt{2}}{2n}} + \left(\frac{1}{2n}, \frac{1}{2n}\right)$ but not of $V^1_{\frac{\sqrt{2}}{n}} + \left(\frac{1}{2n}, \frac{1}{2n}\right)$. Thus, we have for each $y \in M_2$ the property

$$M_1 = A_{\frac{1}{n}}\left(\frac{1}{2n}, \frac{1}{2n}\right) \subset V_1 + y \subset A_{\frac{2}{n}}\left(\frac{1}{2n}, \frac{1}{2n}\right). \tag{12.4}$$

Suppose now that $C > 0$ is some given number. We see that the right-hand side of (12.3) is equal to $C4^{-2}$ for $x \in M_1$ and $y \in M_2$. We note that $\mathbb{1}_{V^\Gamma[y]}(x) = 0$ for all $x \in M_1, y \in M_2$ by construction. For the left-hand side we conclude from (12.4)

$$\int_{B_{\frac{3}{8}}(x_0)} \left(\mathbb{1}_{V^\Gamma[z]}(y) + \mathbb{1}_{V^\Gamma[y]}(z)\,\mathrm{d}z\right) \leq \int_{B_{\frac{3}{8}}(x_0)} \mathbb{1}_{V^\Gamma[z]}(y)\,\mathrm{d}z \leq \frac{c}{n^2}, \tag{12.5}$$

whenever $x \in M_1, y \in M_2$. Here c is a constant depending only on ϑ_1. If we choose n sufficiently large, then we see that (12.3) is satisfied.

This counterexample can easily be generalized to any dimension greater than two.

Remark. The above example is important. In contrast to the assertions made in [CKW20, Example 1.3] it shows that the parabolic Harnack inequality for the corresponding kernel does not hold true.

12.2. Translation invariant kernels

A function f on the product space $\mathbb{R}^d \times \mathbb{R}^d$ is called translation invariant if the value of f at the point $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ depends only on the difference $x - y \in \mathbb{R}^d$. In that case,

f can be considered as a function on $\mathbb{R}^d$ and by a slight abuse of notation we may write $f(x,y) = f(x-y)$.

In this short subsection we consider translation invariant kernels.

Positive examples

For $\alpha \in (0,2)$ we deal with kernels of the form

$$k(x) \asymp \mathbb{1}_S(x)|x|^{-d-\alpha}, \tag{12.6}$$

where the set S is symmetric in the sense $S = -S$.

Lemma 12.4. *If k is a kernel of the form* (12.6)*, where $S \in \mathcal{B}(\mathbb{R}^d)$ is symmetric, then k satisfies Condition (C) if and only if there is $C > 0$ so that for almost every $x \in \mathbb{R}^d, x \neq 0$ and all $0 < r \leq \left(\frac{|x|}{2} \wedge \frac{1}{4}\right)$:*

$$\textit{If } x \in S, \textit{ then } r^d \leq C|B_r(x) \cap S|. \tag{12.7}$$

Proof. Let $x \in B_1, y \in \mathbb{R}^d$ with $x \neq y$. Assume $0 < r \leq \left(\frac{|x-y|}{2} \wedge \frac{1}{4}\right)$. Note that $|x-y|$ is comparable to $|z-y|$ for all $z \in B_r(x)$. Thus, we have for some $C > 0$

$$\begin{aligned}(C) &\Leftrightarrow r^d \mathbb{1}_S(x-y) \leq C \int_{B_r(x)} \mathbb{1}_S(z-y)\,\mathrm{d}z \\ &\Leftrightarrow r^d \mathbb{1}_S(x-y) \leq C|B_r(x-y) \cap S|.\end{aligned}$$

Renaming $x-y$ by x leads to (12.7). Starting from (12.7) one can use the display above to deduce Condition (C) for k. ■

Remark. We point out that (12.7) trivially holds true if S is d-regular.

Condition (B) is obviously satisfied for kernels of the form (12.6). In order to have condition (A) satisfied, we can apply recent results from [DK20, Theorem 1.6 and Proposition 6.13]. Putting everything together we obtain the following corollary.

Corollary 12.5. *Let S be symmetric and assume S satisfies* (12.7)*. Furthermore, suppose that S fulfills one of the following two conditions:*

1. *The measure $\nu_*(\mathrm{d}z) = \mathbb{1}_S(z)|z|^{-d-\alpha}\mathrm{d}z$ is nondegenerate and satisfies $\nu_*(rB) = r^{-\alpha}\nu_*(B)$ for any measurable set $B \subset \mathbb{R}^d$.*

2. *There exist $a > 1$, a positive constant C and $x_n \in (a^{-n}\xi_n, a^{-n+1}\xi_n)$ for $\xi_n \in \mathcal{S}^{d-1}, n \in \mathbb{N}_0$, such that $\bigcup_{n\in\mathbb{N}_0} B_{Ca^{-n}}(x_n) \subset S$.*

Then the kernel k with $k(z) \asymp \mathbb{1}_S(z)|z|^{-d-\alpha}$ satisfies (A)*,* (B) *and* (C)*.*

Remark. The notation for the measure ν_* is chosen such that it is consistent with the notation in [DK20].

Note that the condition (12.7) is automatically fulfilled if S is equal to the the symmetrized union of balls as described in item 2. This is the statement of the next corollary.

Corollary 12.6. *Let $a > 0$ and $C > 0$. Let (x_n) be a sequence with $x_n = C_n\xi_n$ for some $\xi_n \in \mathcal{S}^{d-1}$, $C_n \in (a^{-n+1}, a^{-n})$, $n = 0, 1, ...$ Assume $S = \bigcup_{n\in\mathbb{N}_0} B_{Ca^{-n}}(x_n)$. Then every kernel*

$$k(z) \asymp \mathbb{1}_{S\cup(-S)}(z)|z|^{-d-\alpha}$$

satisfies (A)*,* (B) *and* (C)*.*

Proof. By definition the kernel k is symmetric. Let $x \in S \cup (-S)$. Then there is $n \in \mathbb{N}_0$ so that $x \in B_{Ca^{-n}}(x_n)$. It follows

$$|B_r(x) \cap (S \cup (-S))| \geq |B_r(x) \cap B_{Ca^{-n}}(x_n)| \geq cr^d$$

for every $0 < r < Ca^{-n}$ and some constant $c > 0$, depending only on d. But $|x| \leq |x_n| + |x - x_n| \leq 2Ca^{-n}$. Therefore, the above estimate holds in particular true for $0 < r \leq \frac{|x|}{2}$. This shows, that (12.7) is valid. By construction of the sequence we immediately see that item 2 holds. The claim follows from Corollary 12.6. ∎

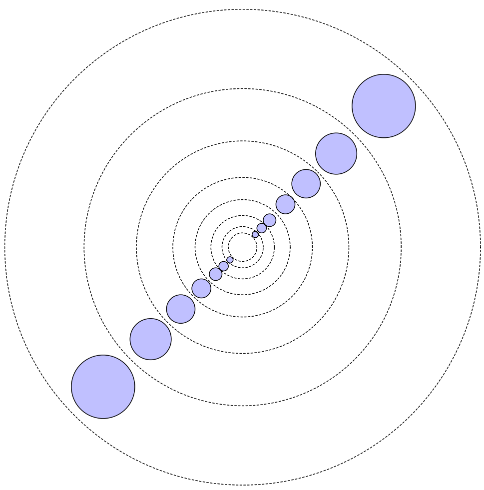

Figure 12.3.: Example for $S \cup (-S)$ with $a = \frac{3}{2}$ and $C = \frac{1}{5}$

A. Auxiliary results for the coercivity estimate

First, let us recall the Whitney decomposition technique for open sets. The following proposition is taken from [Gra14, Appendix J]

Proposition A.1. *Let Ω be an open nonempty proper subset of $\mathbb{R}^d$. Then there exists a family of closed dyadic cubes $\{Q_j\}_j$ (called the* Whitney cubes *of Ω) such that*

1. $\bigcup_j Q_j = \Omega$ *and* $int(Q_i) \cap int(Q_j) = \emptyset$ *for* $i \neq j$*, where* int *denotes the interior.*
2. $\sqrt{d}(\text{diameter } Q_j) \leq \operatorname{dist}(Q_j, \Omega^C) \leq 4\sqrt{d}(\text{diameter } Q_j)$*. Thus* $10\sqrt{d}\, Q_j$ *meets* Ω^c*.*
3. *If the boundaries of Q_j and Q_k touch, then*
$$\frac{1}{4} \leq \frac{\text{diameter } Q_j}{\text{diameter } Q_k} \leq 4.$$
4. *For a given cube Q_j there exist at most $12^d - 4^d$ cubes Q_k that touch Q_j.*

The next lemma is a version of [DK20, Lemma 6.9] that matches our integral kernels. Note that [DK20, Lemma 6.9] is concerned with translation invariant expressions. The proof also applies to our case.

Lemma A.2. *Let $\alpha \in (0,2)$ and $\kappa \geq 1$. For $B = B_R(x_0), R > 0, x_0 \in \mathbb{R}^d$ we set $B^* = B_{\kappa R}(x_0)$. Let $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a symmetric kernel that satisfies* (8.1)*. Suppose that for some $c > 0$*

$$c \int_B \int_B (f(x) - f(y))^2 |x-y|^{-d-\alpha} \,\mathrm{d}x \,\mathrm{d}y \ \leq \int_{B^*} \int_{B^*} (f(x) - f(y))^2 k(x,y) \,\mathrm{d}x \,\mathrm{d}y$$

for every ball $B \subset \mathbb{R}^d$ and every $f \in H^k(B^*)$. Then for every bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ there exists a constant $\widetilde{c} = \widetilde{c}(d, \kappa, \alpha, \Omega) > 0$ such that for every $f \in H^k(\Omega)$

$$\widetilde{c}c \int_\Omega \int_\Omega (f(x) - f(y))^2 |x-y|^{-d-\alpha} \,\mathrm{d}x \,\mathrm{d}y \ \leq \int_\Omega \int_\Omega (f(x) - f(y))^2 k(x,y) \,\mathrm{d}x \,\mathrm{d}y.$$

The constant $\widetilde{c}$ depends on the domain Ω only up to scaling. In particular, if Ω is a ball, the constant can be chosen independently of Ω. For $0 < \alpha_0 \leq \alpha < 2$, the constant $\widetilde{c}$ may be chosen to depend on α_0 but not on α.

Proof. Let Ω be a bounded Lipschitz domain. The Whitney decomposition technique provides a family $\mathcal{B}$ of balls with the following properties.

(i) There exists a constant $c = c(d)$ such that for every $x, y \in \Omega$ with $|x - y| < c\,\mathrm{dist}(x, \partial\Omega)$ there exists a ball $B \in \mathcal{B}$ with $x, y \in B$.

(ii) For every $B \in \mathcal{B}$, $B^* \subset \Omega$.

(iii) The family $\{B^*\}_{B\in\mathcal{B}}$ has the finite overlapping property, that is, each point of Ω belongs to at most $M = M(d)$ balls B^*.

Thus, we have for each $f \in H^k(\Omega)$,

$$\begin{aligned}
&\int_\Omega \int_\Omega (f(x) - f(y))^2 k(x, y)\,\mathrm{d}x\,\mathrm{d}y \\
&\geq \frac{1}{M^2} \sum_{B\in\mathcal{B}} \int_{B^*} \int_{B^*} (f(x) - f(y))^2 k(x, y)\,\mathrm{d}x\,\mathrm{d}y \\
&\geq \frac{c}{M^2} \sum_{B\in\mathcal{B}} \int_{B} \int_{B} (f(x) - f(y))^2 |x-y|^{-d-\alpha}\,\mathrm{d}x\,\mathrm{d}y \\
&\geq \frac{c\tilde{c}}{M^2} \int_\Omega \int_\Omega (f(x) - f(y))^2 |x-y|^{-d-\alpha}\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned} \tag{A.1}$$

where we applied inequality (13) in [Dyd06, proof of Theorem 1] to derive the last inequality, see also [PS17, Theorem 1.6]. For a scaled version of Ω we can scale all balls in the family $\mathcal{B}$ by the same factor and arrive at the same constant $\widetilde{c}$. The constant stays bounded when $\alpha \in [\alpha_0, 2)$ for $\alpha_0 > 0$. ∎

The next lemma follows from the Differentiation Theorem of Lebesgue, see Theorem 2.12.

Lemma A.3. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be locally integrable. The following holds for almost every $s \in \mathbb{R}^d$. If $(x_h)_{h>0}$ is a sequence in $h\mathbb{Z}^d$ such that $s \in \widetilde{A}_h(x_h)$ for every $h > 0$, then*

$$\frac{1}{|A_h(x_h)|} \int_{A_h(x_h)} \varphi(t)\,\mathrm{d}t \overset{h\to 0}{\longrightarrow} \varphi(s).$$

Proof. The cube $\widetilde{A}_h(x_h)$ is contained in the ball $B_{2h\sqrt{d}}(s)$ and we know $|\widetilde{A}_h(x_h)| = c|B_{2h\sqrt{d}}(s)|$ for a constant c only depending on the dimension d. Thus, it follows from the Differentiation Theorem of Lebesgue for almost every $s \in \mathbb{R}^d$

$$\frac{1}{|A_h(x_h)|} \int_{A_h(x_h)} |\varphi(t) - \varphi(s)|\,\mathrm{d}s \leq \frac{1}{|B_{2h\sqrt{d}}(s)|} \int_{B_{2h\sqrt{d}}(s)} |\varphi(t) - \varphi(s)|\,\mathrm{d}s \overset{h\to 0}{\longrightarrow} 0.$$

This implies our claim. ∎

B. The relative Kato condition, the Harnack inequality and Condition (C)

In [BS05] Bogdan and Sztonyk present a condition that is equivalent to the elliptic Harnack inequality. They call this condition the *relative Kato* condition. In the following short paragraph we present the setup of [BS05] and state and explain the relative Kato condition. In the remainder of the paragraph we give a direct proof of the statement that the relative Kato condition holds, provided that Condition (C) is satisfied, the kernel is as in [BS05] and satisfies a pointwise comparability assumption. Moreover, we also explain that (C) is not necessary for the relative Kato condition to hold true at least in the case $d = 3$.

B.1. Preliminaries

We provide definitions of the Hausdorff and the spherical measure, which are only needed in this part of the thesis. After that we introduce the relative Kato condition, derive equivalent formulations and explain the relationship to the Harnack inequality.

The Hausdorff and the spherical measure

In this part of our thesis we work with the spherical measure. Let us first give a formal definition.

Definition B.1 (Hausdorff measure, spherical measure, compare [Ama08, p. 29 and p. 36])**.** *Let (X, ρ) be a separable metric space with induced topology $\mathcal{T}$. For any set $O \subset X$ let*

$$\operatorname{diam}(O) = \sup\{\rho(x, y) \,|\, x, y \in O\}.$$

For $s > 0, \varepsilon > 0$ and $A \subset X$ define

$$\mathcal{H}^s_\varepsilon(A) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(O_i)^s \,|\, A \subset \bigcup_{i=1}^{\infty} O_i, O_i \in \mathcal{T}, \operatorname{diam}(O_i) < \varepsilon \textit{ for all } i\right\}$$

and set

$$\mathcal{H}^s(A) = \sup_{\varepsilon > 0} \mathcal{H}^s_\varepsilon(A).$$

Then $\mathcal{H}^s$ is a measure on the Borel σ-algebra of X. It is called the s-dimensional Hausdorff measure.

For $X = \mathcal{S}^{d-1}$ the d-dimensional unit sphere with respect to the Euclidean metric in $\mathbb{R}^d$, let

$$\sigma(A) = \mathcal{H}^{d-1}(A)$$

for any $A \in \mathcal{B}(\mathcal{S}^{d-1})$. This measure is called spherical measure.

The relative Kato condition

We start with a short description of the result of Bogdan and Sztonyk. In the following we only consider kernels k that have the ensuing properties. Our first assumption is that $k \neq 0$ is translation invariant. In this case k can be considered as a function on $\mathbb{R}^d$. Second, we assume that k is $(-d-\alpha)$-homogeneous for $\alpha \in (0,2)$, that is,

$$k(z) = |z|^{-d-\alpha} k\left(\frac{z}{|z|}\right), \quad z \in \mathbb{R}^d \setminus \{0\}.$$

Definition B.2. *A kernel k satisfies the* relative Kato condition *if there exists a constant $K > 0$ such that for every $y \in \mathbb{R}^d \setminus B_1$*

$$\int_{B_{\frac{1}{2}}(y)} |y-v|^{\alpha-d} k(v)\,\mathrm{d}v \leq K \int_{B_{\frac{1}{2}}(y)} k(v)\,\mathrm{d}v. \tag{RK}$$

Note that the reverse inequality to (RK) is always true if $d \geq 2$.

The authors show in [BS05, Theorem 1] the following result using tools from potential theory.

Theorem B.3 (compare [BS05, Theorem 1])**.** *The kernel k satisfies the relative Kato condition* (RK) *if and only if there exists a constant $C = C(\alpha, k)$ such that for every nonnegative harmonic function u in B_1 the following Harnack inequality holds for all $x_1, x_2 \in B_{\frac{1}{2}}$:*

$$u(x_1) \leq C u(x_2). \tag{B.1}$$

The next lemma provides equivalent formulations of (RK), which will be useful later on. It includes a spherical version of (RK).

Equivalent statements to (RK)

Theorem B.4. *Let $d \in \mathbb{N}, d > 2, \alpha \in (0,2)$. Let $k \neq 0$ be a symmetric kernel that is $(-d-\alpha)$-homogeneous and satisfies $k(z) \asymp \kappa(\frac{z}{|z|})|z|^{-d-\alpha}$ for a bounded function κ on*

$\mathcal{S}^{d-1}$. Denote by σ the spherical measure on $\mathcal{S}^{d-1}$ and by $B_{\xi,r} = B_r(\xi) \cap \mathcal{S}^{d-1}$ the ball of radius $r > 0$ and center $\xi \in \mathcal{S}^{d-1}$ intersected with the unit sphere. Let $\widetilde{\sigma}(\mathrm{d}\eta) = \kappa(\eta)\sigma(\mathrm{d}\eta)$. The following statements are pairwise equivalent:

1. *The kernel k satisfies* (RK).
2. *There are constants $K_1 > 0$ and $r_0 \in (0,1)$ such that for each $\xi \in \mathcal{S}^{d-1}$ and every $r \in (0, r_0)$*
$$\int_{B_{\xi,r}} \left(\frac{|\eta - \xi|}{r} \right)^{\alpha-(d-1)} \widetilde{\sigma}(\mathrm{d}\eta) \leq K_1 \, \widetilde{\sigma}(B_{\xi,r}). \tag{B.2}$$
3. *There are constants K_2 and $r_0 \in (0,1)$ such that for each $\xi \in \mathcal{S}^{d-1}$ and every $r \in (0, r_0)$*
$$\int_0^1 \frac{\widetilde{\sigma}(B_{\xi,rs})}{s^{d-\alpha}} \,\mathrm{d}s \leq K_2 \, \widetilde{\sigma}(B_{\xi,r}). \tag{B.3}$$

Proof. The equivalence of the first two assertions is part of [BS05, pp. 146]. It remains to prove the equivalence of the last two statements. This follows from the ensuing equality:

$$\begin{aligned}\int_{B_{\xi,r}} \left(\frac{|\eta - \xi|}{r} \right)^{\alpha-d+1} \widetilde{\sigma}(\mathrm{d}\eta) &= \int_{B_{\xi,r}} \left(1 + (d-\alpha-1) \int_{\frac{|\eta-\xi|}{r}}^1 s^{\alpha-d} \mathrm{d}s \right) \widetilde{\sigma}(\mathrm{d}\eta) \\ &= \left(\widetilde{\sigma}(B_{\xi,r}) + (d-\alpha-1) \int_0^1 \frac{\widetilde{\sigma}(B_{\xi,sr})}{s^{d-\alpha}} \,\mathrm{d}s \right).\end{aligned}$$

In the last line we used the theorem of Fubini, which is allowed since the integrand is nonnegative. ■

B.2. Condition (C) implies the relative Kato condition

We note that under the homogeneity assumption on the kernel Condition (C) expands to the whole $\mathbb{R}^d$.

Lemma B.5. *Let k be a translation invariant, $(-d-\alpha)$-homogeneous kernel. Then Condition (C) is equivalent to: There exists a constant $C > 0$ so that for almost all $x, y \in \mathbb{R}^d$ with $x \neq y$ and every $0 < r \leq \frac{|x-y|}{2}$*

$$⨍_{B_r(x)} k(z,y) \,\mathrm{d}z \geq C k(x,y).$$

The proof uses a simple substitution combined with the homogeneity of k. We omit it here.

Now let us assume that k fulfills in addition Condition (A) and Condition (B). Then we know from Theorem B.3 and Theorem 11.13 that the following holds true:

$$\text{Condition (C)} \Rightarrow \text{Harnack inequality} \Leftrightarrow \text{(RK)}.$$

Therefore, it would be nice to have a direct proof of the fact that Condition (C) implies (RK). In the case were k satisfies some comparability assumption, this is done in the next proposition.

Theorem B.6. *Let M be a Borel set that is symmetric in the sense $M = -M$. If k is a $(-d-\alpha)$-homogeneous kernel that is translation invariant, satisfies $k \neq 0$,* (C) *and*

$$k(z) \asymp \mathbb{1}_M(z)|z|^{-d-\alpha} \text{ for almost all } z \in \mathbb{R}^d, \tag{B.4}$$

then k satisfies the relative Kato condition.

Proof. For nullsets M the assertion is trivial. Therefore, we assume $|M| > 0$. Let $y \in \mathbb{R}^d \setminus B_1$.

Since $|z| \asymp |y|$ for each $z \in B_{\frac{1}{2}}(y)$ it is sufficient to prove the existence of $K > 0$ independent of y so that

$$\int_{B_{\frac{1}{2}}(y)} |y-z|^{\alpha-d}\mathbb{1}_M(z)\,\mathrm{d}z \leq K \int_{B_{\frac{1}{2}}(y)} \mathbb{1}_M(z)\,\mathrm{d}z. \tag{B.5}$$

We derive from Condition (C) in its equivalent formulation (11.4) with $\lambda = \frac{1}{4}, \mu = \frac{1}{2}$ and $x_0 = 0$ the inequality

$$4^{-d}C \sup_{x \in B_{\frac{1}{4}}} \mathbb{1}_M(x-y) \leq \int_{B_{\frac{1}{2}}} \mathbb{1}_M(z-y)\,\mathrm{d}z,$$

where above and in the remainder of this proof *sup* denotes the essential supremum. Using the symmetry of M we see that the above line is equivalent to

$$4^{-d}C \sup_{x \in B_{\frac{1}{4}}(y)} \mathbb{1}_M(x) \leq \int_{B_{\frac{1}{2}}(y)} \mathbb{1}_M(z)\,\mathrm{d}z. \tag{B.6}$$

Denote the essential distance dist^* of y and M by

$$\mathrm{dist}^*(y, M) = \inf\left\{\ell > 0 \,|\, |B_\ell(y) \cap M| > 0\right\}.$$

First, we consider the case $\operatorname{dist}^*(y, M) < \frac{1}{4}$. Then $|B_{\frac{1}{4}}(y) \cap M| > 0$, which gives

$$\sup_{x \in B_{\frac{1}{4}}(y)} \mathbb{1}_M(x) = 1.$$

Therefore, by (B.6)

$$4^{-d}C = 4^{-d}C \sup_{x \in B_{\frac{1}{4}}(y)} \mathbb{1}_M(x) \leq \int_{B_{\frac{1}{2}}(y)} \mathbb{1}_M(z)\,\mathrm{d}z. \tag{B.7}$$

Also, for some $c > 0$ depending only on the dimension d

$$\begin{aligned}\int_{B_{\frac{1}{2}}(y)} |y-z|^{\alpha-d}\mathbb{1}_M(z)\,\mathrm{d}z &\leq \int_{B_{\frac{1}{2}}(y)} |y-z|^{\alpha-d}\,\mathrm{d}z \\ &\leq \frac{c}{\alpha}2^{-\alpha}. \end{aligned}\tag{B.8}$$

We can choose $K_1 > 0$ so that

$$\frac{c}{\alpha}2^{-\alpha} \leq 4^{-d}CK_1 \tag{B.9}$$

holds. Then putting together (B.7) and (B.8), we arrive at the estimate (B.5) with $K = K_1$.

Now suppose $\operatorname{dist}^*(y, M) \geq \frac{1}{4}$. Then we have $|B_{\frac{1}{8}}(y) \cap M| = 0$ and by a decomposition of the integral we simply get

$$\begin{aligned}\int_{B_{\frac{1}{2}}(y)} |y-z|^{\alpha-d}\mathbb{1}_M(z)\,\mathrm{d}z &= \int_{B_{\frac{1}{8}}(y)\cap M} |y-z|^{\alpha-d}\,\mathrm{d}z + \int_{B_{\frac{1}{2}}(y)\setminus B_{\frac{1}{8}}(y)} |y-z|^{\alpha-d}\mathbb{1}_M(z)\,\mathrm{d}z \\ &\leq 8^{d-\alpha}\int_{B_{\frac{1}{2}}(y)} \mathbb{1}_M(z)\,\mathrm{d}z.\end{aligned}$$

In this case, we can take $K_2 = 8^{d-\alpha}$ and arrive at (B.5) with $K = K_2$. Now the claim follows if we choose K as the maximum of K_1 and K_2. ■

Remark. In [BS05, p. 148] Bogdan and Sztonyk provide a family of examples of kernels for which the relative Kato condition fails. The support of these kernels contains a countably infinite union of cones, where the apex angle degenerates at some point. With Theorem B.6 we provide a (more or less) direct proof that Condition (C) also fails for these kernels.

B.3. The relative Kato condition does not imply Condition (C)

We want to show that our Condition (C) does not follow from the relative Kato condition. We provide a counterexample of a class of kernels where each kernel satisfies (RK), (A) and (B), but violates (C) in the case $\alpha \in (1,2)$. It follows from Theorem B.3 that weak solutions of the corresponding PDE enjoy a Harnack inequality.

We work in three dimensions. The idea is to construct a set $C \subset \mathbb{R}^3$ that involves a cusp. Then the corresponding kernel $k(x,y) = \mathbb{1}_C(x-y)|x-y|^{-d-\alpha}$ cannot satisfy Condition (C).

The counterexample. Consider the two dimensional set

$$D = \{(x_1,x_2)\,|\,0 < x_1 < 1, 0 < x_2 < x_1^2\}.$$

We can map it to the upper unit sphere $\big(\mathcal{S}^2 \cap \{x_3 > 0\}\big) \subset \mathbb{R}^3$ via the map

$$\Phi : (x_1,x_2) \mapsto (x_1,x_2,1) \mapsto (x_1,x_2,1)/|(x_1,x_2,1)|. \tag{B.10}$$

We set $\mathfrak{D} = \Phi(D) \cup (-\Phi(D))$, which is a symmetric set on the sphere $\mathcal{S}^2$. Let k be a translation invariant, $(-d-\alpha)$-homogeneous kernel that satisfies $k(\frac{x}{|x|}) \asymp \mathbb{1}_{\mathfrak{D}}(\frac{x}{|x|})$ for each $x \in \mathbb{R}^3 \setminus \{0\}$, with a comparability constant independent of x.

Our idea now is to use (B.3) in order to verify the relative Kato condition for this specific kernels. Therefore, we set from now on

$$\widetilde{\sigma}(\mathrm{d}\eta) = \mathbb{1}_{\mathfrak{D}}(\eta)\sigma(\mathrm{d}\eta).$$

We aim at the following statement.

Proposition B.7. *The measure $\widetilde{\sigma}$ satisfies* (B.3) *for $\alpha \in (1,2)$. In particular, every translation invariant and $(-d-\alpha)$-homogeneous kernel k with $k(x) \asymp \mathbb{1}_{\mathfrak{D}}(\frac{x}{|x|})|x|^{-d-\alpha}$ has the property* (RK) *provided that $\alpha \in (1,2)$.*

Reducing the claim of the proposition to a two-dimensional problem

In order to prove the claim we only need to work in two dimensions and afterwards use the properties of the map Φ. The following statement yields Proposition B.7. A proof follows after the proof of the lemma.

Lemma B.8. *Let $\nu(\mathrm{d}z) = \mathbb{1}_D(z)\mathrm{d}z$. There exist constants $C, r_0, s_0 > 0$ such that for all $r \in (0,r_0)$, $s \in (0,s_0)$ and $x \in \mathbb{R}^2$*

$$\nu(B_{sr}(x)) \leq Cs\,\nu(B_r(x)). \tag{B.11}$$

Proof. The proof consists of three different cases. In the proof we estimate the volume of balls by the volume of cubes. Therefore, we first recall our notation for cubes

$$A_\ell(x) = \left\{ y \in \mathbb{R}^2 \,\middle|\, |x-y|_\infty < \frac{\ell}{2} \right\}, \quad \ell > 0, \quad x \in \mathbb{R}^2.$$

Furthermore, we denote by

$$D^* = \left\{ (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, 0 < x_1 < \frac{1}{2}, 0 < x_2 < x_1^2 \right\} \subset D.$$

The set D^* has the property that $y = (y_1, y_2) \in D^*$ implies $(y_1 + \frac{r}{2}, y_2) \in D$ for each $r \in (0,1)$. We use this property in the Case 3. Let $r_0 = \frac{1}{2}$ and let $r \in (0, \frac{1}{2})$. Let $x \in \mathbb{R}^2$.

Case 1: $\operatorname{dist}(x, D) > \frac{r}{2}$. In this case we always have $\nu(B_{sr}(x)) = 0$ for $s \in (0, \frac{1}{2})$. Thus, the assertion is obviously true.

Case 2: There exists $y \in B_{\frac{r}{2}}(x) \cap (D \setminus D^*)$. Since $D \setminus D^*$ is a 2-set, we know $\nu(B_{\frac{r}{2}}(y)) \geq cr^2$ for a constant $c > 0$ independent of y and r. In this case we have $\nu(B_r(x)) \geq \nu(B_{\frac{r}{2}}(y)) \geq cr^2$ and $\nu(B_{sr}(x)) \leq |B_{sr}(x)| \leq \widetilde{c} s^2 r^2$ for any $s \in (0,1)$ and $\widetilde{c} > 0$ depending only on the dimension d. Thus,

$$\frac{\nu(B_{sr}(x))}{\nu(B_r(x))} \leq c_1 s^2 \leq c_1 s$$

for c_1 depending only on the dimension d and on D^*.

Case 3: Let us assume that Case 1 and Case 2 are not satisfied. First, we consider any $y \in \overline{D^*}$ and find a lower bound for $\nu(B_{\frac{r}{2}}(y))$ and an upper bound for $\nu(B_{sr}(y))$ for $s \in (0, s_0)$, where s_0 is to be determined later.

We use the fact that every ball contains a cube of side length comparable to $\frac{r}{2}$. Therefore, $\nu(B_{\frac{r}{2}}(y)) \geq \nu(A_{\delta\frac{r}{2}}(y))$ for $\delta \in \big(0, (\sqrt{2})^{-1}\big)$. In the following calculation we use the quantities h_1 and h_2. In order to understand these quantities we refer the reader to Figure B.1 and Figure B.2. Note that the definition of h_2 depends on the upper edge of the cube and its relation to the set D. Now

$$\begin{aligned} \nu(B_{\frac{r}{2}}(y)) &\geq \nu(A_{\delta\frac{r}{2}}(y)) \\ &\geq c(\delta) h_1 \frac{r}{2} + \widetilde{c}(\delta) \Bigg(\underbrace{\left[\frac{1}{6} \left(\frac{r}{2}\right)^3 + y_1 \left(\frac{r}{2}\right)^2 \right]}_{=:\mathscr{A}_1} \wedge \underbrace{\left[\frac{1}{2} h_2 \left(\frac{r}{2}\right) \right]}_{=:\mathscr{A}_2} \Bigg) \qquad \text{(B.12)} \\ &\geq c(\delta) r \left(\frac{h_1}{2} + \left[\frac{1}{48} r^2 + \frac{y_1}{4} r \right] \wedge \frac{h_2}{4} \right), \end{aligned}$$

where we used the notation $a \wedge b = \min\{a, b\}$. Let us explain the estimate in (B.12). Let $\widetilde{y} = (y_1, y_1^2)$. The term $\mathscr{A}_1$ is the arithmetic mean of the following two numbers:

1. The area inside the cube between the translated normal parabola with apex $\widetilde{y}$ and the line $f(x) = y_1^2$, see the red dashed area in Figure B.1.

2. The area inside the cube between the tangent in $\widetilde{y}$ and the line $f(x) = y_1^2$, see the green dashed area in Figure B.1.

The term $\mathscr{A}_2$ takes into account that the upper edge of the cube can lie below the graph of x^2. It describes the area of the green dashed triangle in Figure B.2.

Also we have for any $s \in (0, \frac{1}{2\delta})$

$$\begin{aligned}\nu(B_{sr}(y)) &\leq \nu(A_{2sr}(y)) \\ &\leq 2h_1 sr + \big(\underbrace{\big[(y_1 + rs)r^2s^2\big]}_{=:\mathscr{A}_3} \wedge [h_2 rs]\big) \\ &= sr\left(2h_1 + \left(y_1 rs + r^2s^2\right) \wedge h_2\right).\end{aligned} \tag{B.13}$$

The above values of h_1 and h_2 correspond to the cube $A_{2sr}(y)$. But, since $A_{2sr}(y) \subset A_{\delta\frac{r}{2}}(y)$ for $s \in (0, \frac{1}{2\delta})$, we may estimate them by the corresponding values for h_1 and h_2 for $A_{\delta\frac{r}{2}}(y)$, that is, h_1 and h_2 have the same value in (B.12) and (B.13). The term $\mathscr{A}_3$ describes the area of the triangle with edge points $\widetilde{y}$, $(y_1 + sr, y_1^2)$ and $(y_1 + sr, f(y_1 + sr))$. Here f is the linear function with slope $2(y_1 + sr)$ and $\widetilde{y} \in \text{graph}(f)$, that is, the tangent at $(y_1 + sr, (y_1 + sr)^2)$ translated such that $\widetilde{y} \in \text{graph}(f)$.

Let us now again turn to $x \in \mathbb{R}^2$. Since the first two cases are not satisfied, the intersection $B_{\frac{r}{2}}(x) \cap \overline{D}$ contains only elements in $\overline{D^*}$. Since $\nu(B_{rs}(x)) = 0$ for $\text{dist}(x, D) \geq sr$, we only need to consider the case where $\text{dist}(x, D) < sr$. In this case we find $y \in D^* \cap B_{sr}(x)$ and $B_{sr}(x) \subset B_{2sr}(y)$. Then we have for $s \in (0, \frac{1}{4\delta})$

$$\nu(B_{sr}(x)) \leq \nu(B_{2sr}(y)) \leq 2sr(2h_1 + (2y_1 rs + 4r^2s^2) \wedge h_2) \tag{B.14}$$

and

$$\nu(B_r(x)) \geq \nu(B_{\frac{r}{2}}(y)) \geq c(\delta)r\left(\frac{h_1}{2} + \left[\frac{1}{48}r^2 + \frac{y_1}{4}r\right] \wedge \frac{h_2}{4}\right). \tag{B.15}$$

Now from (B.14) and (B.15) we deduce (B.11).

In the end we define C as the maximum of both constants that we found in Case 2 and in Case 3. The claim follows then with $r_0 = \frac{1}{2}$ and $s_0 = \frac{1}{4\delta}$. ∎

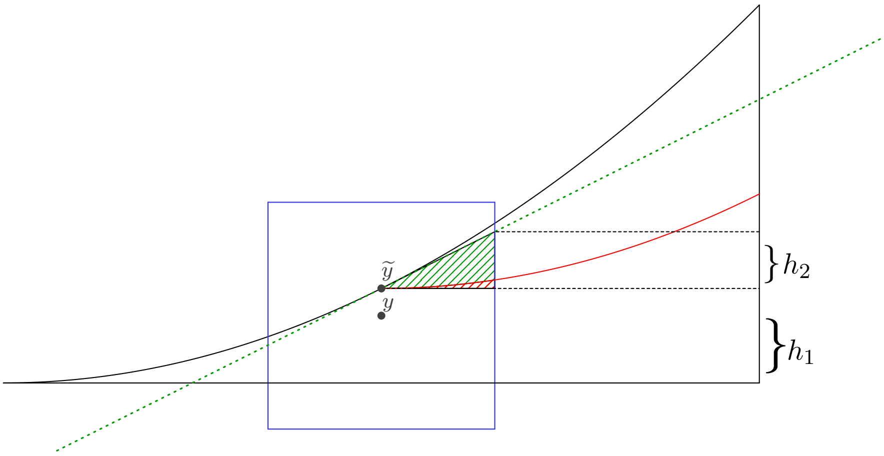

Figure B.1.: The cube $A_{\delta\frac{r}{2}}(y)$, situation 1.

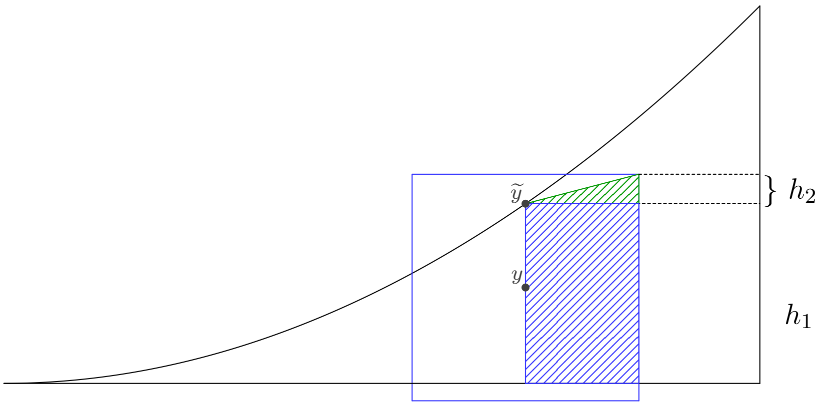

Figure B.2.: The cube $A_{\delta\frac{r}{2}}(y)$, situation 2.

Proof of Proposition B.7. The claim follows from the above lemma combined with properties of the map Φ defined in (B.10). Because of the symmetry, it is sufficient to prove the claim for the measure $\mathbb{1}_{\Phi(D)}(\eta)\sigma(\mathrm{d}\eta)$ on $\mathcal{S}^2 \cap \{x_3 > 0\}$, which we will, by a slight abuse of notation, again denote by $\widetilde{\sigma}(\mathrm{d}\eta)$. From [BBI01, Theorem 5.5.5] we know for any Borel set $A \subset \mathbb{R}^2$

$$\widetilde{\sigma}(\Phi(A)) = \int_A \mathbb{1}_D(x) \mathrm{Jac}_\Phi(x)\,\mathrm{d}x, \tag{B.16}$$

where $\mathrm{Jac}_\Phi(x) = \mathrm{Jac}_\Phi(x_1,x_2) = \sqrt{\det\left[(J_\Phi(x_1,x_2))^T J_\Phi(x_1,x_2)\right]}$ and J_Φ denotes the Jacobian matrix. We have

$$(J_\Phi(x_1,x_2))^T J_\Phi(x_1,x_2) = \frac{1}{(x_1^2+x_2^2+1)^2}\begin{pmatrix} x_2^2+1 & -x_1x_2 \\ -x_1x_2 & x_1^2+1 \end{pmatrix},$$

which yields

$$\mathrm{Jac}_\Phi(x_1, x_2) = \frac{1}{(x_1^2 + x_2^2 + 1)^{\frac{3}{2}}}.$$

Thus, $\frac{1}{C} \leq J_\Phi(x) \leq C$ for $x \in \Omega$ for any bounded domain $\Omega \subset \mathbb{R}^2$ and $C > 0$ depending only on Ω. From (B.16) we conclude for measurable sets $A \subset \Omega$

$$\widetilde{\sigma}(\Phi(A)) \asymp |A \cap D|, \tag{B.17}$$

where the comparability constant is given by the above C.

In the remainder of the proof we use the notation $\widetilde{B}_r(x)$ for balls in $\mathbb{R}^2$ and $B_r(x)$ for balls in $\mathbb{R}^3$ and since we restrict ourselves to the upper sphere we write $B_{\xi,r} = B_r(\xi) \cap S_2 \cap \{x_3 > 0\}$.

By continuity of Φ we know that the preimage of any disc $B_{\xi,r}$ on the upper sphere contains an open set. Note that the set

$$\overline{D}_1 = \overline{\bigcup_{x \in D} \widetilde{B}_1(x)}$$

is compact. Therefore, Φ is uniformly continuous on $\overline{D}_1$. Choose $\rho_0 \in (0,1)$ small enough so that $|B_{\xi,\rho_0} \cap \Phi(D)| > 0$ implies $B_{\xi,\rho_0} \subset \Phi(\overline{D}_1)$. For $r \in (0, \rho_0)$ we deduce the existence of a constant $c \geq 1$ such that for any spherical ball $B_{\xi,r}$ that satisfies $|B_{\xi,r} \cap \Phi(D)| > 0$ we have the following assertion:

$$\Phi(\widetilde{B}_{c^{-1}r}(\Phi^{-1}(\xi))) \subset B_{\xi,r} \subset \Phi(\widetilde{B}_{cr}(\Phi^{-1}(\xi))). \tag{B.18}$$

Simple geometric observations show that c can be chosen independently of r. Thus, the constant c depends only on the set D.

Let r_0, s_0 be as in Lemma B.8. We additionally assume that $cr_0 \leq \rho_0$, which may be realized, if necessary, by decreasing r_0. If $\xi \in \mathcal{S}^2$ and $r \in (0, cr_0)$ are chosen so that $|B_{\xi,r} \cap \Phi(D)| > 0$, then we have by Lemma B.8, (B.17) with $\Omega = \overline{D}_1$, and (B.18) for all $s \in (0, \frac{s_0}{c^2})$

$$\begin{aligned}
\widetilde{\sigma}(B_{\xi,sr}) &\leq C|\widetilde{B}_{csr}(y) \cap D| \\
&= C|\widetilde{B}_{c^2 sc^{-1}r}(y) \cap D| \\
&\leq Cc^2 s|\widetilde{B}_{c^{-1}r}(y) \cap D| \\
&\leq \widetilde{c}s\widetilde{\sigma}(B_{\xi,r}),
\end{aligned}$$

where $y = \Phi^{-1}(\xi)$ and $\widetilde{c} > 0$ does not depend on r or ξ. On the other hand, if $|B_{\xi,r} \cap \Phi(D)| = 0$ for every $r \in (0, cr_0)$, then $\widetilde{\sigma}(B_{\xi,sr}) \leq \widetilde{c}s\widetilde{\sigma}(B_{\xi,r})$ is trivially true for each $s \in (0,1)$. Renaming the constants cr_0 to r_0 and s_0/c^2 to s_0 we conclude: There are constants $C > 0, r_0 \in (0,1)$ and $s_0 \in (0,1)$ such that for all $\xi \in \mathcal{S}^2 \cap \{x_3 > 0\}, s \in (0, s_0), r \in (0, r_0)$

$$\widetilde{\sigma}(B_{\xi,sr}) \leq Cs\,\widetilde{\sigma}(B_{\xi,r}).$$

Thus,

$$\begin{aligned}\int_0^1 \frac{\widetilde{\sigma}(B_{\xi,sr})}{s^{2-\alpha}}\,\mathrm{d}s &\leq C\int_0^{s_0} \frac{s\widetilde{\sigma}(B_{\xi,r})}{s^{2-\alpha}}\,\mathrm{d}s + \int_{s_0}^1 \frac{\widetilde{\sigma}(B_{\xi,r})}{s^{2-\alpha}}\,\mathrm{d}s \\ &\leq C\left(\int_0^{s_0} s^{\alpha-1}\,\mathrm{d}s + \int_{s_0}^1 s^{\alpha-2}\,\mathrm{d}s\right)\widetilde{\sigma}(B_{\xi,r}).\end{aligned}$$

The claim follows with $K_2 = C\left(\int_0^{s_0} s^{\alpha-1}\,\mathrm{d}s + \int_{s_0}^1 s^{\alpha-2}\,\mathrm{d}s\right) < \infty$ for $\alpha \in (1,2)$. ■

Remark. The inequality (B.11) is crucial for the range of α in Proposition B.7. It is possible that more precise estimates in the proof of Lemma B.8 allow for a larger range of α. One could also try to modify the set D by changing the cusp at the origin in order to find a larger range of α. Since we were only interested in finding an example of a kernel that satisfies (RK) and violates Condition (C), we will not investigate this topic further.

We remark that kernels of the type $k(z) \asymp \mathbb{1}_{\mathfrak{D}}(\frac{z}{|z|})|z|^{-d-\alpha}$ satisfy Condition (A) by [DK20, Theorem 1.6] and, obviously, Condition (B), but they do not satisfy Condition (C), as shown in the next proposition.

Proposition B.9. *Every translation invariant and $(-d-\alpha)$-homogeneous kernel k with $k(\frac{x}{|x|}) \asymp \mathbb{1}_{\mathfrak{D}}(\frac{x}{|x|})$ for $x \in \mathbb{R}^3 \setminus \{0\}$ violates Condition (C).*

The proof uses a contradiction argument. Since Condition (C) is formulated with respect to nullsets, the proof becomes rather technical. The reader may keep the following idea in mind while reading the proof. Let

$$\mathfrak{C} = \mathbb{R}_+\mathfrak{D} = \{\lambda\xi \mid \lambda > 0, \xi \in \mathfrak{D}\}.$$

Take $x = (0,0,1) \in \mathfrak{D}$. Then

$$|B_r(x) \cap \mathfrak{C}| \asymp r^4,$$

by the construction of D as a set below the graph of $z \mapsto z^2$. This leads to a contradiction to $|B_r(x) \cap \mathfrak{C}|C \geq r^3$ for $C > 0$ if we consider the limit $r \to 0$. But, provided that (C) is satisfied, according to Lemma 12.4 the last inequality holds true (at least almost everywhere) for sufficiently small r.

Proof of Proposition B.9. In this proof we work with balls in $\mathbb{R}^2$ as well as balls in $\mathbb{R}^3$. In order to avoid any confusion, we denote by $\widetilde{B}_r(x)$ the ball with respect to the Euclidean metric in $\mathbb{R}^2$ and use the notation $B_r(x)$ for the ball with respect to the metric in $\mathbb{R}^3$.

Assume k satisfies (C). Similar to the proof of Lemma 12.4 we derive from Lemma B.5 the assertion: There exists a constant $C > 0$ such that for almost every $x \in \mathbb{R}^3, x \neq 0$ and all $0 < r \leq \frac{|x|}{2}$:

$$\left(\frac{x}{|x|} \in \mathfrak{D} \Rightarrow r^3 \leq C|B_r(x) \cap \mathfrak{C}|\right). \tag{B.19}$$

Now let $C > 0$ be any given constant. We construct a set $A_\varepsilon \subset \mathfrak{C}$ of positive measure and a sequence (r_ε) of radii, so that (B.19) is false for $\varepsilon > 0$ depending on C.

Let $\varepsilon \in (0, \frac{1}{3})$ and consider the set $D_\varepsilon = \{x \in \mathbb{R}^2 \,|\, 0 < x_1 < \varepsilon, 0 < x_2 < x_1^2\}$. Set $\mathfrak{D}_\varepsilon = \Phi(D_\varepsilon)$, where Φ is as in (B.10). Similar to the reasoning leading to (B.18) we find a constant $c_\Phi \geq 1$ so that for every $\xi \in \mathfrak{D}_\varepsilon$ and sufficiently small $r > 0$ there is $y \in D_\varepsilon$ with

$$B_{\xi,r} \subset \Phi(\widetilde{B}_{c_\Phi r}(y)). \tag{B.20}$$

Choose $r_\varepsilon = 2(c_\Phi)^{-1}\varepsilon\sqrt{1+\varepsilon^2}$. Then $\frac{c_\Phi r_\varepsilon}{2}$ is large enough so that for every $x \in D_\varepsilon$ we have $|\widetilde{B}_{\frac{c_\Phi r_\varepsilon}{2}}(x) \cap D| \asymp \varepsilon^3$ with a comparability constant independent of x and ε. With the same reasoning as in the proof of Proposition B.7 we obtain for each $x \in D_\varepsilon$

$$\widetilde{\sigma}(\Phi(\widetilde{B}_{\frac{c_\Phi r_\varepsilon}{2}}(x))) \asymp |\widetilde{B}_{\frac{c_\Phi r_\varepsilon}{2}}(x) \cap D| \asymp \varepsilon^3, \tag{B.21}$$

where all comparability constants are independent of x and ε.

Set $A_\varepsilon = \{\lambda\xi \,|\, \xi \in \mathfrak{D}_\varepsilon, 2 < \lambda < 3\}$. If $x \in A_\varepsilon$, then $\frac{x}{|x|} \in \mathfrak{D}_\varepsilon$. Now it follows from (B.20) and (B.21) for ε small enough that there is $c \geq 1$, independent of x and ε, such that

$$\begin{aligned}
|B_{r_\varepsilon}(x) \cap \mathfrak{C}| &= |x|^3 \left| B_{\frac{r_\varepsilon}{|x|}}\left(\frac{x}{|x|}\right) \cap \mathfrak{C} \right| \\
&\leq |x|^3 \left| [1 - r_\varepsilon, 1 + r_\varepsilon] \cdot \left(B_{\frac{r_\varepsilon}{|x|}}\left(\frac{x}{|x|}\right) \cap \mathfrak{D} \right) \right| \\
&\leq c r_\varepsilon \widetilde{\sigma}\left(B_{\frac{x}{|x|}, \frac{r_\varepsilon}{2}} \right) \\
&\leq c r_\varepsilon \widetilde{\sigma}(\Phi(\widetilde{B}_{\frac{c_\Phi r_\varepsilon}{2}}(y)) \\
&\leq c r_\varepsilon \varepsilon^3.
\end{aligned}$$

Since $r_\varepsilon \leq 1 < \frac{|x|}{2}$, we have by (B.19) the inequality $r_\varepsilon^3 \leq C r_\varepsilon \varepsilon^3$, that is,

$$\frac{4}{(c_\Phi)^2}\varepsilon^2(1+\varepsilon^2) \leq C\varepsilon^3 \Leftrightarrow \frac{1}{\varepsilon} + \varepsilon \leq \frac{C(c_\Phi)^2}{4}.$$

But this cannot hold true if we choose ε small enough. ∎

Corollary B.10. *Let $k : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty]$ be a symmetric kernel that has the properties* (A) *and* (B)*. Then Condition (C) is sufficient but not necessary for the validity of the elliptic Harnack inequality* (B.1).

Remark. We expect that the results of this last section and especially the statement from the above corollary can be generalized to any $d > 3$.

Bibliography

[Ada75] Robert A. Adams. *Sobolev spaces*. Pure and Applied Mathematics, Vol. 65. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975, pp. xviii+268 (cit. on pp. 32, 34).

[Alt16] Hans Wilhelm Alt. *Linear Functional Analysis*. eng. Universitext. London: Springer London, 2016, XII, 435 p. 19 illus. ISBN: 978-1-4471-7280-2. URL: `https://link.springer.com/book/10.1007%2F978-1-4471-7280-2` (cit. on p. 38).

[Ama08] Herbert Amann. *Analysis III*. ger. Ed. by Joachim Escher. Zweite Auflage. Grundstudium Mathematik. Basel: Birkhäuser Basel, 2008. ISBN: 978-3-7643-8884-3. URL: `https://link.springer.com/book/10.1007%2F978-3-7643-8884-3` (cit. on pp. 25, 149).

[Aum69] Robert J. Aumann. "Measurable utility and the measurable choice theorem". In: *La Décision, 2: Agrégation et Dynamique des Ordres de Préférence (Actes Colloq. Internat., Aix-en-Provence, 1967)*. Éditions du Centre Nat. Recherche Sci., Paris, 1969, pp. 15–26 (cit. on p. 52).

[BBK09] Martin T. Barlow, Richard F. Bass, and Takashi Kumagai. "Parabolic Harnack inequality and heat kernel estimates for random walks with long range jumps". In: *Math. Z.* 261.2 (2009), pp. 297–320. ISSN: 0025-5874. DOI: `10.1007/s00209-008-0326-5`. URL: `https://doi.org/10.1007/s00209-008-0326-5` (cit. on pp. 17, 113).

[BM18] Martin T. Barlow and Mathav Murugan. "Stability of the elliptic Harnack inequality". In: *Ann. of Math. (2)* 187.3 (2018), pp. 777–823. ISSN: 0003-486X. DOI: `10.4007/annals.2018.187.3.4`. URL: `https://doi.org/10.4007/annals.2018.187.3.4` (cit. on p. 17).

[Bas13] Richard F. Bass. "A stability theorem for elliptic Harnack inequalities". In: *J. Eur. Math. Soc. (JEMS)* 15.3 (2013), pp. 857–876. ISSN: 1435-9855. DOI: `10.4171/JEMS/379`. URL: `https://doi.org/10.4171/JEMS/379` (cit. on p. 17).

[Bau01] Heinz Bauer. *Measure and integration theory*. eng. Vol. 26. De Gruyter studies in mathematics ; 26. Berlin [u.a.]: de Gruyter, 2001, XVI, 230 S. ISBN: 3-11-016719-0 (cit. on p. 25).

[BS07] Krzysztof Bogdan and Paweł Sztonyk. “Estimates of the potential kernel and Harnack’s inequality for the anisotropic fractional Laplacian”. In: *Studia Math.* 181.2 (2007), pp. 101–123. ISSN: 0039-3223. DOI: 10.4064/sm181-2-1. URL: https://doi.org/10.4064/sm181-2-1 (cit. on p. 17).

[BS05] Krzysztof Bogdan and Paweł Sztonyk. “Harnack’s inequality for stable Lévy processes”. In: *Potential Anal.* 22.2 (2005), pp. 133–150. ISSN: 0926-2601. DOI: 10.1007/s11118-004-0590-x. URL: https://doi.org/10.1007/s11118-004-0590-x (cit. on pp. 12, 15, 17, 97, 112, 149–151, 153).

[Bol81] Ludwig Boltzmann. *Vorlesungen ueber Gastheorie.* Erw. Nachdr. d. 1896 - 1898 bei Barth in Leipzig ersch. Ausg. Vol. 1. Gesamtausgabe / Boltzmann, Ludwig ; 1. Graz: Akad. Druck- u. Verlagsanst., 1981. ISBN: 3-201-01168-1, 3-528-08550-9 (cit. on p. 6).

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry.* Vol. 33. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001, pp. xiv+415. ISBN: 0-8218-2129-6. DOI: 10.1090/gsm/033. URL: https://doi.org/10.1090/gsm/033 (cit. on p. 157).

[BKS19] Kai-Uwe Bux, Moritz Kassmann, and Tim Schulze. “Quadratic forms and Sobolev spaces of fractional order”. In: *Proc. Lond. Math. Soc. (3)* 119.3 (2019), pp. 841–866. ISSN: 0024-6115. DOI: 10.1112/plms.12246. URL: https://doi.org/10.1112/plms.12246 (cit. on pp. 11, 22).

[CK20] Jamil Chaker and Moritz Kassmann. “Nonlocal operators with singular anisotropic kernels”. In: *Comm. Partial Differential Equations* 45.1 (2020), pp. 1–31. ISSN: 0360-5302. DOI: 10.1080/03605302.2019.1651335. URL: https://doi.org/10.1080/03605302.2019.1651335 (cit. on pp. 16, 89, 90).

[CS19] Jamil Chaker and Luis Silvestre. “Coercivity estimates for integro-differential operators”. In: *arXiv* (2019). eprint: arXiv:1904.13014 (cit. on pp. 9, 10).

[CG11] Sun-Yung Alice Chang and María del Mar González. “Fractional Laplacian in conformal geometry”. In: *Adv. Math.* 226.2 (2011), pp. 1410–1432. ISSN: 0001-8708. DOI: 10.1016/j.aim.2010.07.016. URL: https://doi.org/10.1016/j.aim.2010.07.016 (cit. on p. 5).

[CKW19] Zhen-Qing Chen, Takashi Kumagai, and Jian Wang. “Elliptic Harnack inequalities for symmetric non-local Dirichlet forms”. In: *J. Math. Pures Appl. (9)* 125 (2019), pp. 1–42. ISSN: 0021-7824. DOI: 10.1016/j.matpur.2017.10.011. URL: https://doi.org/10.1016/j.matpur.2017.10.011 (cit. on p. 17).

[CKW20] Zhen-Qing Chen, Takashi Kumagai, and Jian Wang. “Stability of parabolic Harnack inequalities for symmetric non-local Dirichlet forms”. In: *J. Eur. Math. Soc. (JEMS)* (to appear in 2020) (cit. on pp. 12, 17, 113, 115, 143).

[CT04] Rama Cont and Peter Tankov. *Financial modelling with jump processes.* Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004, pp. xvi+535. ISBN: 1-5848-8413-4 (cit. on p. 5).

[Coz17] Matteo Cozzi. "Regularity results and Harnack inequalities for minimizers and solutions of nonlocal problems: a unified approach via fractional De Giorgi classes". In: *J. Funct. Anal.* 272.11 (2017), pp. 4762–4837. ISSN: 0022-1236. DOI: 10.1016/j.jfa.2017.02.016. URL: https://doi.org/10.1016/j.jfa.2017.02.016 (cit. on pp. 16, 112).

[De 57] Ennio De Giorgi. "Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari". In: *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)* 3 (1957), pp. 25–43 (cit. on pp. 15, 99).

[Deb67] Gerard Debreu. "Integration of correspondences". In: *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 1.* Univ. California Press, Berkeley, Calif., 1967, pp. 351–372 (cit. on p. 52).

[DD12] Françoise Demengel and Gilbert Demengel. *Functional spaces for the theory of elliptic partial differential equations.* eng. Universitext. London: Springer [u.a.], 2012, XVIII, 465 S. ISBN: 978-1-4471-2806-9, 978-2-7598-0698-0 (cit. on pp. 31, 32, 34).

[DKP16] Agnese Di Castro, Tuomo Kuusi, and Giampiero Palatucci. "Local behavior of fractional p-minimizers". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 33.5 (2016), pp. 1279–1299. ISSN: 0294-1449. DOI: 10.1016/j.anihpc.2015.04.003. URL: https://doi.org/10.1016/j.anihpc.2015.04.003 (cit. on pp. 13, 15–18, 97, 110, 111, 121, 129).

[DKP14] Agnese Di Castro, Tuomo Kuusi, and Giampiero Palatucci. "Nonlocal Harnack inequalities". In: *J. Funct. Anal.* 267.6 (2014), pp. 1807–1836. ISSN: 0022-1236. DOI: 10.1016/j.jfa.2014.05.023. URL: https://doi.org/10.1016/j.jfa.2014.05.023 (cit. on pp. 13–16, 18, 97, 111, 112, 117, 118, 132).

[DPV12] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. "Hitchhiker's guide to the fractional Sobolev spaces". In: *Bull. Sci. Math.* 136.5 (2012), pp. 521–573. ISSN: 0007-4497. DOI: 10.1016/j.bulsci.2011.12.004. URL: https://doi.org/10.1016/j.bulsci.2011.12.004 (cit. on p. 6).

[DRV17] Serena Dipierro, Xavier Ros-Oton, and Enrico Valdinoci. "Nonlocal problems with Neumann boundary conditions". In: *Rev. Mat. Iberoam.* 33.2 (2017), pp. 377–416. ISSN: 0213-2230. DOI: 10.4171/RMI/942. URL: https://doi.org/10.4171/RMI/942 (cit. on p. 35).

[Dyd06] Bartłomiej Dyda. "On comparability of integral forms". In: *J. Math. Anal. Appl.* 318.2 (2006), pp. 564–577. ISSN: 0022-247X. DOI: 10.1016/j.jmaa.2005.06.021. URL: http://dx.doi.org/10.1016/j.jmaa.2005.06.021 (cit. on p. 148).

[DK20] Bartłomiej Dyda and Moritz Kassmann. "Regularity estimates for elliptic nonlocal operators". In: *Analysis & PDE* (to appear in 2020) (cit. on pp. 7, 9, 11–13, 16, 22, 88–90, 94, 97, 112, 134, 144, 145, 147, 159).

[FK13] Matthieu Felsinger and Moritz Kassmann. "Local regularity for parabolic nonlocal operators". In: *Comm. Partial Differential Equations* 38.9 (2013), pp. 1539–1573. ISSN: 0360-5302. DOI: 10.1080/03605302.2013.808211. URL: https://doi.org/10.1080/03605302.2013.808211 (cit. on p. 7).

[FKV15] Matthieu Felsinger, Moritz Kassmann, and Paul Voigt. "The Dirichlet problem for nonlocal operators". In: *Mathematische Zeitschrift* 279.3 (Apr. 2015), pp. 779–809. ISSN: 1432-1823. DOI: 10.1007/s00209-014-1394-3. URL: https://doi.org/10.1007/s00209-014-1394-3 (cit. on pp. 35, 36, 39).

[FOT94] Masatoshi Fukushima, Yoichi Oshima, and Masayoshi Takeda. *Dirichlet forms and symmetric Markov processes*. Vol. 19. De Gruyter Studies in Mathematics. Walter de Gruyter & Co., Berlin, 1994, pp. x+392. ISBN: 3-11-011626-X. DOI: 10.1515/9783110889741. URL: https://doi.org/10.1515/9783110889741 (cit. on pp. 41–44).

[GG82] Mariano Giaquinta and Enrico Giusti. "On the regularity of the minima of variational integrals". In: *Acta Math.* 148 (1982), pp. 31–46. DOI: 10.1007/BF02392725. URL: https://doi.org/10.1007/BF02392725 (cit. on pp. 16, 99).

[Gra14] Loukas Grafakos. *Classical Fourier Analysis*. eng. 3rd ed. 2014. Graduate Texts in Mathematics ; 249. New York, NY: Springer New York, 2014, XVII, 638 p. 14 illus., 2 illus. in color. ISBN: 978-1-4939-1194-3. URL: https://link.springer.com/book/10.1007%2F978-1-4939-1194-3 (cit. on p. 147).

[HL11] Qing Han and Fanghua Lin. *Elliptic partial differential equations*. eng. Second edition. Vol. 1. Courant lecture notes ; 1. New York, New York: Courant Institute of Mathematical Sciences, 2011, 147 Seiten. ISBN: 9781470411367. URL: http://www.ams.org/books/cln/001/ (cit. on p. 99).

[Him75] C. Himmelberg. "Measurable relations". eng. In: *Fundamenta Mathematicae* 87.1 (1975), pp. 53–72. URL: http://eudml.org/doc/214809 (cit. on p. 52).

[HK07] Ryad Husseini and Moritz Kassmann. "Markov chain approximations for symmetric jump processes". In: *Potential Anal.* 27.4 (2007), pp. 353–380. ISSN: 0926-2601. DOI: 10.1007/s11118-007-9060-6. URL: https://doi.org/10.1007/s11118-007-9060-6 (cit. on p. 8).

[IS20] Cyril Imbert and Luis Silvestre. "The weak Harnack inequality for the Boltzmann equation without cut-off". In: *J. Eur. Math. Soc. (JEMS)* 22.2 (2020), pp. 507–592. ISSN: 1435-9855. DOI: 10.4171/jems/928. URL: https://doi.org/10.4171/jems/928 (cit. on pp. 6, 9).

[JW84] Alf Jonsson and Hans Wallin. "Function spaces on subsets of $\mathbb{R}^n$". In: *Math. Rep.* 2.1 (1984), pp. xiv+221. ISSN: 0275-7214 (cit. on p. 139).

[Kas07a] Moritz Kassmann. "Analysis of symmetric Markov jump processes. A localization technique for non-local operators". Habilitationsschrift. Universität Bonn, 2007 (cit. on p. 16).

[Kas11] Moritz Kassmann. "Harnack inequalities and Hölder regularity estimates for nonlocal operators revisited". In: *SFB 701 preprint No. 11015* (2011), pp. 1–25 (cit. on p. 16).

[Kas07b] Moritz Kassmann. "Harnack inequalities: an introduction". In: *Bound. Value Probl.* (2007), Art. ID 81415, 21. ISSN: 1687-2762. DOI: 10.1155/2007/81415. URL: https://doi.org/10.1155/2007/81415 (cit. on p. 16).

[Kas07c] Moritz Kassmann. "The classical Harnack inequality fails for nonlocal operators". In: *SFB 601 preprint No. 360* (2007), pp. 1–7 (cit. on p. 16).

[KS14] Moritz Kassmann and Russell W. Schwab. "Regularity results for nonlocal parabolic equations". In: *Riv. Math. Univ. Parma (N.S.)* 5.1 (2014), pp. 183–212. ISSN: 0035-6298 (cit. on p. 7).

[Kwa19] Mateusz Kwaśnicki. "Fractional Laplace operator and its properties". In: *Handbook of fractional calculus with applications. Vol. 1.* De Gruyter, Berlin, 2019, pp. 159–193 (cit. on p. 6).

[Ler14] Nicolas Lerner. *A Course on Integration Theory.* eng. Basel: Springer Basel, 2014, XVIII, 492 p. 15 illus., 3 illus. in color. ISBN: 978-3-0348-0694-7. URL: https://link.springer.com/book/10.1007%2F978-3-0348-0694-7 (cit. on p. 25).

[Mos60] Jürgen Moser. "A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations". In: *Comm. Pure Appl. Math.* 13 (1960), pp. 457–468. ISSN: 0010-3640. DOI: 10.1002/cpa.3160130308. URL: https://doi.org/10.1002/cpa.3160130308 (cit. on pp. 16, 99).

[Mos61] Jürgen Moser. "On Harnack's theorem for elliptic differential equations". In: *Communications on Pure and Applied Mathematics* 14.3 (1961), pp. 577–591. DOI: 10.1002/cpa.3160140329. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.3160140329. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/cpa.3160140329 (cit. on p. 16).

[Nas58] J. Nash. "Continuity of solutions of parabolic and elliptic equations". In: *Amer. J. Math.* 80 (1958), pp. 931–954. ISSN: 0002-9327. DOI: 10.2307/2372841. URL: https://doi.org/10.2307/2372841 (cit. on pp. 15, 99).

[Pon16] Augusto C. Ponce. *Elliptic PDEs, measures and capacities.* Vol. 23. EMS Tracts in Mathematics. From the Poisson equations to nonlinear Thomas-Fermi problems. European Mathematical Society (EMS), Zürich, 2016, p. 453. ISBN: 978-3-03719-140-8. DOI: 10.4171/140. URL: https://doi.org/10.4171/140 (cit. on p. 32).

[PS17] Martí Prats and Eero Saksman. "A T(1) Theorem for Fractional Sobolev Spaces on Domains". In: *J. Geom. Anal.* 27.3 (2017), pp. 2490–2538. ISSN: 1050-6926. DOI: 10.1007/s12220-017-9770-y. URL: http://dx.doi.org/10.1007/s12220-017-9770-y (cit. on p. 148).

[Sil00] S.A. Silling. "Reformulation of elasticity theory for discontinuities and long-range forces". In: *Journal of the Mechanics and Physics of Solids* 48.1 (2000), pp. 175–209. ISSN: 0022-5096. DOI: https://doi.org/10.1016/S0022-5096(99)00029-0. URL: http://www.sciencedirect.com/science/article/pii/S0022509699000290 (cit. on p. 5).

[Sil16] Luis Silvestre. "A new regularization mechanism for the Boltzmann equation without cut-off". In: *Comm. Math. Phys.* 348.1 (2016), pp. 69–100. ISSN: 0010-3616. DOI: 10.1007/s00220-016-2757-x. URL: https://doi.org/10.1007/s00220-016-2757-x (cit. on p. 6).

[SS05] Elias M. Stein and Rami Shakarchi. *Real analysis.* eng. Vol. 3. Princeton lectures in analysis / Elias M. Stein & Rami Shakarchi ; 3. Princeton [u.a.]: Princeton Univ. Press, 2005, XIX, 402 S. : graph. Darst. ISBN: 978-0-691-11386-9, 0-691-11386-6 (cit. on p. 31).

[Str19] Martin Strömqvist. "Harnack's inequality for parabolic nonlocal equations". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 36.6 (2019), pp. 1709–1745. ISSN: 0294-1449. DOI: 10.1016/j.anihpc.2019.03.003. URL: https://doi.org/10.1016/j.anihpc.2019.03.003 (cit. on p. 18).

[Vas16] Alexis F. Vasseur. "The De Giorgi method for elliptic and parabolic equations and some applications". In: *Lectures on the analysis of nonlinear partial differential equations. Part 4.* Vol. 4. Morningside Lect. Math. Int. Press, Somerville, MA, 2016, pp. 195–222 (cit. on p. 99).

[Vil02] Cédric Villani. "A review of mathematical topics in collisional kinetic theory". In: *Handbook of mathematical fluid dynamics, Vol. I.* North-Holland, Amsterdam, 2002, pp. 71–305. DOI: 10.1016/S1874-5792(02)80004-0. URL: https://doi.org/10.1016/S1874-5792(02)80004-0 (cit. on pp. 5, 6).