

Regularisation and Long-Time Behaviour of Random Systems

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ABSTRACT. In this work, we study several different aspects of systems modelled by partial differential equations (PDEs), both deterministic and stochastically perturbed. The thesis is structured as follows:

Chapter I gives a summary of the contents of this work and illustrates the main results and ideas of the rest of the thesis.

Chapter II is devoted to a new model for the flow of an electrically conducting fluid through a porous medium, the tamed magnetohydrodynamics (TMHD) equations. After a survey of regularisation schemes of fluid dynamical equations, we give a physical motivation for our system. We then proceed to prove existence and uniqueness of a strong solution to the TMHD equations, prove that smooth data lead to smooth solutions and finally show that if the onset of the effect of the taming term is deferred indefinitely, the solutions to the tamed equations converge to a weak solution of the MHD equations.

In Chapter III we investigate a stochastically perturbed tamed MHD (STMHD) equation as a model for turbulent flows of electrically conducting fluids through porous media. We consider both the problem posed on the full space \mathbb{R}^3 as well as the problem with periodic boundary conditions. We prove existence of a unique strong solution to these equations as well as the Feller property for the associated semigroup. In the case of periodic boundary conditions, we also prove existence of an invariant measure for the semigroup.

The last chapter deals with the long-time behaviour of solutions to SPDEs with locally monotone coefficients with additive Lévy noise. Under quite general assumptions, we prove existence of a random dynamical system as well as a random attractor. This serves as a unifying framework for a large class of examples, including stochastic Burgers-type equations, stochastic 2D Navier-Stokes equations, the stochastic 3D Leray- α model, stochastic power law fluids, the stochastic Ladyzhenskaya model, stochastic Cahn-Hilliard-type equations, stochastic Kuramoto-Sivashinsky-type equations, stochastic porous media equations and stochastic p -Laplace equations.

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Introduction and Motivation

Random phenomena occur in numerous places in nature and society. They can often be modelled by partial differential equations perturbed by a noise term.

In this thesis, we analyse several different facets of (S)PDE models, ranging from well-posedness and regularity to their long-time behaviour. Many, but not all, of the models considered here have their origin and their main applications in fluid dynamics.

The chapters are written in a self-contained way and can largely be read independently from each other. This chapter serves to give a short overview of the contents of each of the following chapters and some of the main ideas used there.

1. Tamed MHD Equations - Deterministic and Stochastic

The first two chapters of this thesis deal with a regularised version of the magnetohydrodynamics (MHD) equations, which we call the *tamed MHD* (TMHD) *equations* and which are of the form

$$\begin{aligned}\frac{\partial \mathbf{v}}{\partial t} &= \frac{1}{Re} \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v} + S (\mathbf{B} \cdot \nabla) \mathbf{B} + \nabla \left(p + \frac{S |\mathbf{B}|^2}{2} \right) - g_N (|\mathbf{v}, \mathbf{B}|^2) \mathbf{v}, \\ \frac{\partial \mathbf{B}}{\partial t} &= \frac{1}{Rm} \Delta \mathbf{B} - (\mathbf{v} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{v} + \nabla \pi - g_N (|\mathbf{v}, \mathbf{B}|^2) \mathbf{B}.\end{aligned}$$

Here $\mathbf{v} = \mathbf{v}(x, t)$ denotes the velocity field of the fluid, $\mathbf{B} = \mathbf{B}(x, t)$ its magnetic field and $p = p(x, t)$ the pressure, for points x in a domain $\mathbb{D} \subseteq \mathbb{R}^3$. For the appearance of the additional term $\pi = \pi(x, t)$, which arises from introducing the taming term into the equation for the magnetic field and to which we refer to as the “magnetic pressure”, cf. Chapter II, Section 1.3.4. The numbers $Re, Rm > 0$ are the Reynolds numbers of the velocity field and the magnetic field, and $S > 0$ denotes the Lundquist number. The parameter N models the onset of an additional restoring force, i.e. the size of the norm of the fields needed for the additional force to kick in. The function g_N is smooth, equal to zero for small arguments and, after a short onset, for arguments of order N and greater it starts to grow linearly. A more precise definition is given in Chapter II, Section 1.3.6.

This new system of equations is a generalisation of the tamed Navier-Stokes equations of M. Röckner, X.C. Zhang and T.S. Zhang, cf. [194–197, 250], to the MHD case.

The MHD equations (where $g_N \equiv 0$) are an important model in the field of fluid dynamics, which describes the flow of a fluid consisting of electrically conducting particles (e.g. a liquid metal or a plasma). They share many traits with the related Navier-Stokes equations (where $\mathbf{B} = 0$), and the uniqueness and regularity of their solutions are open problems to this day. To address this issue, different *regularisation schemes* have been proposed in the past, starting with the classical work of J. Leray [148] for the Navier-Stokes equations. They aim at making the equations more amenable to analysis by changing them, either by modifying the terms (weakening the influence of the nonlinear convective terms, or strengthening the linear dissipative term) or adding additional regularising terms.

These changes then ensure that the resulting equations become well-posed, and one often can prove that if one makes the effect of the regularisation smaller and smaller, their solutions converge to weak solutions of the non-regularised equations. Moreover, in many models, the long-time behaviour of the regularised and the non-regularised equations can be shown to coincide.

We start in each of the two chapters by giving a relatively detailed (though by no means complete) overview of the existing literature of regularisation schemes for Navier-Stokes as well as MHD equations in the introductions of the chapters, comparing different schemes with each other and embedding our own results into the broader context.

Apart from being a regularised version of the MHD equations, the tamed equations are of their own interest. As a special case of (a magnetohydrodynamical) version of the so-called Brinkman-Forchheimer-extended Darcy model, they describe the flow of an electrically conducting fluid through porous media. The tamed equations have the property that bounded solutions to the MHD equations – if they exist, which has yet to be shown – coincide with solutions to the TMHD equations, a fact that does not necessarily hold for other regularisation schemes.

In Chapter II, we then proceed to study the deterministic tamed MHD equations. After providing the necessary tools in a preparatory section as well as in Appendix A, and defining the notion of weak solution, we prove existence and uniqueness of such weak solutions to the TMHD equations by a Faedo-Galerkin approximation scheme. We then prove that not only uniqueness holds for weak solutions but that for sufficiently smooth initial data, the solutions of the TMHD equations are smooth themselves, i.e. a regularity statement. For the untamed equations, similar regularity results are unknown. Finally, we prove that the solution to the tamed equations converges to a suitable weak solution of the untamed MHD equations as the onset of the taming force is deferred indefinitely, i.e. as N tends to infinity.

In Chapter III, we consider stochastically perturbed TMHD equations, both on the whole space \mathbb{R}^3 and on the torus \mathbb{T}^3 , given by¹

$$\begin{aligned} d\mathbf{v} &= \left[\Delta \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v} + (\mathbf{B} \cdot \nabla) \mathbf{B} + \nabla \left(p + \frac{S|\mathbf{B}|^2}{2} \right) - g_N(|(\mathbf{v}, \mathbf{B})|^2) \mathbf{v} \right] dt \\ &\quad + \sum_{k=1}^{\infty} [(\boldsymbol{\sigma}_k(t) \cdot \nabla) \mathbf{v} + \nabla p_k(t) + \mathbf{h}_k(t, y(t))] dW_t^k + \mathbf{f}_v(t, y(t)) dt, \\ d\mathbf{B} &= [\Delta \mathbf{B} - (\mathbf{v} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{v} + \nabla \pi - g_N(|(\mathbf{v}, \mathbf{B})|^2) \mathbf{B}] dt \\ &\quad + \sum_{k=1}^{\infty} [(\tilde{\boldsymbol{\sigma}}_k(t) \cdot \nabla) \mathbf{B} + \nabla \pi_k(t) + \tilde{\mathbf{h}}_k(t, y(t))] d\tilde{W}_t^k + \mathbf{f}_B(t, y(t)) dt. \end{aligned}$$

We study problems of existence and uniqueness, more precisely, we study existence of a probabilistically weak solution, prove pathwise uniqueness and then conclude that there exists a unique (probabilistically) strong solution to the problem by employing the theorem of Yamada and Watanabe. In the time-homogeneous case, the well-posedness ensures that the solution to our equation is a Markov process and hence we can define an associated operator semigroup. This semigroup is then proven to be a Feller semigroup, which means that it maps the space of bounded and locally uniformly continuous functions to itself. Furthermore, we prove that in the periodic case there exists an invariant measure for the problem.

¹Note that we have set all the parameters Re, Rm, S appearing in the equations to unity.

2. Random Dynamical Systems and Random Attractors

Dynamical systems can exhibit very complicated behaviour on short timescales. On longer timescales, however, their behaviour often simplifies considerably by being confined to a small subset of phase space, called an *attractor*. Although these attractors themselves can still be very intricate objects, they nonetheless provide a simplification in many cases.

For systems subjected to randomness, the notion of a *random attractor* was proposed by H. Crauel and F. Flandoli in their 1994 paper [47], see also [45]. They have since then been studied extensively by many authors – their paper is the most highly cited article of the journal *Probability Theory and Related Fields* since its creation in 1962 – for a variety of different systems described by stochastic evolution equations (a more detailed overview is given in the introduction of Chapter IV). Many of the papers, albeit devoted to particular equations, follow the same patterns in proving existence of random attractors (and also random dynamical systems, the existence of which is a nontrivial task in itself, cf. [78]). In Chapter IV of this thesis we consider equations of the form

$$dX_t = A(X_t)dt + dN_t,$$

with a locally monotone operator $A: V \rightarrow V^*$ and a Lévy process $N_t \in H$. Here, the spaces V, H, V^* form a so-called Gelfand triple, i.e. $V \subset H \subset V^*$ with compact, dense embeddings. The compactness means, in the case that A is a differential operator, that the underlying domain on which our model is defined is bounded.

Typically, for the dynamics of a complex system to settle on a “small” set in phase space, one needs to have some dissipation effects in the system. This dissipativity is provided in our case by the locally monotone operator present in the evolution equation description of the system.

Within the variational or weak solution framework for stochastic equations, the case of monotone operators was treated by B. Gess in [94], who generalised an earlier work the porous medium equations by W.-J. Beyn, B. Gess, P. Lescot and M. Röckner [17]. In a joint work [98] with B. Gess as well as W. Liu, the author has worked on showing existence of random dynamical systems as well as random attractors for equations with locally monotone coefficients driven by additive Lévy noise. We generalised the idea of [94], especially the introduction of a nonlinear Ornstein-Uhlenbeck process generated by the strongly monotone part of the equation, which ensures sufficient regularity to allow for a transformation of the equation into a deterministic PDE with random coefficients, which in turn can be treated pathwise by methods from the deterministic theory of dynamical systems. Then we can undo this transformation and obtain existence of a random dynamical system for the stochastic equation. Random attractors are obtained by using the *a priori* estimates to show there exists an absorbing set. Using the compactness of the flow (which in the variational setting follows immediately from the compactness of the embeddings in the Gelfand triple), we prove that this absorbing set is in fact compact, which is equivalent to the existence of a random attractor, cf. [44].

Our methods apply to a wide range of equations from different areas of science. They include stochastic reaction-diffusion equations, stochastic Burgers-type equations, stochastic 2D Navier-Stokes equations, the stochastic Leray- α model, stochastic power law fluids, the stochastic Ladyzhenskaya model, stochastic Cahn-Hilliard-type equations as well as stochastic Kuramoto-Sivashinsky-type equations. And, of course, those equations with (weakly) monotone coefficients from [94] satisfy the conditions of our theorems and hence we also cover generalised p -Laplace equations, as well as generalised porous media equations. In each of these examples, we can prove the existence of a random dynamical system and a random attractor, in some range of the parameters of each model.

We tried to find the largest possible such range for the models considered. The results were in several cases, but not always, known before, and can of course be improved when one exploits features particular to each model. For most of the examples considered, results in the generality (especially concerning the noise considered here) are new, e.g. for the Burgers-type equations, the 2D Navier-Stokes equations, the hydrodynamical systems of I.D. Chueshov and A. Millet [40, 41] including the Leray- α model, where existence of a random attractor, to the best of the author's knowledge, has not been shown before. The same holds for the Cahn-Hilliard-type equations as well as the Kuramoto-Sivashinsky-type equations. Further discussions of each example can be found in Chapter IV, Section 6. We consider our main contribution to be a unifying framework for dealing with this large class of equations.

Proving the existence of random attractors is only one of several steps towards a full understanding of the long-term dynamics of random dynamical systems. Next steps would include estimating the Lyapunov exponents (i.e. how fast neighbouring trajectories diverge), the exponential decay of the volume element as well as estimating the dimension of the attractor (which can be infinite, as it lies in a space of functions). For more information on this programme, cf. [213]. In this thesis, however, we confine ourselves to proving the existence of random attractors.

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The Deterministic Tamed MHD Equations

ABSTRACT. We study a regularised version of the MHD equations, the tamed MHD (TMHD) equations. They describe the flow of electrically conducting fluids through porous media and have the property that bounded solutions to the MHD system also satisfy the tamed equations. Thus, these (hypothetical) solutions may be studied through the study of the tamed equations. We first give a review of the literature on regularised fluid dynamical equations. Then we prove existence and uniqueness of TMHD on the whole space \mathbb{R}^3 , that smooth data give rise to smooth solutions, and show that solutions to TMHD converge to a suitable weak solution of the MHD equations as the taming parameter N tends to infinity. Furthermore, we adapt a regularity result for the Navier-Stokes equations to the MHD case.

1. Introduction

1.1. Magnetohydrodynamics. The magnetohydrodynamics (MHD) equations describe the dynamic motion of electrically conducting fluids. They combine the equations of motion for fluids (Navier-Stokes equations) with the field equations of electromagnetic fields (Maxwell's equations), coupled via Ohm's law. In plasma physics, the equations are a macroscopic model for plasmas in that they deal with averaged quantities and assume the fluid to be a continuum with frequent collisions. Both approximations are not met in hot plasmas. Nonetheless, the MHD equations provide a good description of the low-frequency, long-wavelength dynamics of real plasmas. In this thesis, we consider the incompressible, viscous, resistive equations with homogeneous mass density, and regularised variants of it. In dimensionless formulation, the MHD equations are of the following form:

$$(1.1) \quad \begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= \frac{1}{Re} \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v} + S (\mathbf{B} \cdot \nabla) \mathbf{B} + \nabla \left(p + \frac{S |\mathbf{B}|^2}{2} \right), \\ \frac{\partial \mathbf{B}}{\partial t} &= \frac{1}{Rm} \Delta \mathbf{B} - (\mathbf{v} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{v} \\ \operatorname{div} \mathbf{v} &= 0, \quad \operatorname{div} \mathbf{B} = 0. \end{aligned}$$

Here, $\mathbf{v} = \mathbf{v}(x, t)$, $\mathbf{B} = \mathbf{B}(x, t)$ denote the velocity and magnetic fields, $p = p(x, t)$ is the pressure, $Re > 0$, $Rm > 0$ are the Reynolds number and the magnetic Reynolds number and $S > 0$ denotes the Lundquist number (all of which are dimensionless constants). The two last equations concerning the divergence-freeness of the velocity and magnetic field are the incompressibility of the flow and Maxwell's second equation. Mathematical treatment of the deterministic MHD equations reaches back to the works of G. Duvaut and J.-L. Lions [65] and M. Sermange and R. Temam [201]. Since then, a large amount of papers have been devoted to the subject. We only mention several interesting regularity criteria [30, 111, 112, 126] and the more recent work on non-resistive MHD equations ($Rm = \infty$) by C.L. Fefferman, D.S. McCormick J.C. Robinson and J.L. Rodrigo on local existence via higher-order commutator estimates [73, 74].

In this chapter, we want to study a regularised version of the MHD equations, which we call the *tamed MHD equations* (TMHD), following M. Röckner and X.C. Zhang [197]. They arise from (1.1) by adding two extra terms (the *taming terms*) that act as restoring forces:

$$\begin{aligned}\frac{\partial \mathbf{v}}{\partial t} &= \frac{1}{Re} \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v} + S(\mathbf{B} \cdot \nabla) \mathbf{B} + \nabla \left(p + \frac{S|\mathbf{B}|^2}{2} \right) - g_N(|(\mathbf{v}, \mathbf{B})|^2) \mathbf{v}, \\ \frac{\partial \mathbf{B}}{\partial t} &= \frac{1}{Rm} \Delta \mathbf{B} - (\mathbf{v} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{v} + \nabla \pi - g_N(|(\mathbf{v}, \mathbf{B})|^2) \mathbf{B}.\end{aligned}$$

The taming terms are discussed in more detail in Section 1.3, and we discuss the results of this chapter in Section 1.4. The extra term $\nabla \pi$, which we call the magnetic pressure, will be explained in Section 1.3.4. However, before we study the tamed equations, we want to give an overview of regularisation schemes for the Navier-Stokes and the MHD equations to put our model into the broader context of the mathematical literature.

We consider both the case of the whole space \mathbb{R}^3 (Cauchy problem) as well as that of a bounded, smooth domain with zero boundary conditions (Dirichlet problem), but treat each case with different methods due to the lack of compactness of embeddings of the associated function spaces in the former case.

1.2. Regularised Fluid Dynamical Equations. Since the question of global well-posedness still remains an open problem for the Navier-Stokes and MHD equations alike, it has been suggested by different authors to regularise the equations to make them more tractable. We consider the following abstract evolution equation-type form of our equations which contains both the case of the Navier-Stokes as well as the MHD equations (more on deriving it in the case of the MHD equations is said in Chapter III, Section 2.1, cf. Equation (2.4)):

$$(1.2) \quad \partial_t y = \mathcal{L}(y) + \mathcal{N}(y, y) + f, \quad \nabla \cdot y = 0.$$

Here, \mathcal{L} is a linear or nonlinear operator (usually related to the Stokes operator $\frac{1}{Re} P \Delta$, with Helmholtz-Leray projection $P: L^2 \rightarrow L^2 \cap \text{div}^{-1}(\{0\})$), \mathcal{N} is a bilinear operator, and f is a forcing term. Usually, the operator \mathcal{N} consists of terms of the form $P[(\varphi \cdot \nabla) \psi]$, where φ, ψ are vectors made of components of y . Using the divergence-freeness constraint, this may be rewritten as $P[\nabla \cdot (\varphi \otimes \psi)]$. The operator $\nabla \cdot y$ has to be understood appropriately.

To be yet more precise, we focus on the following two cases:

(i) *Navier-Stokes equations.* Here $y = \mathbf{v}$ is the velocity field, $\mathcal{L} = \frac{1}{Re} P \Delta$ and $\mathcal{N}(y, y) = \mathcal{N}(\mathbf{v}, \mathbf{v}) = -P(\mathbf{v} \cdot \nabla) \mathbf{v}$, and $\nabla \cdot y := \nabla \cdot \mathbf{v}$.

(ii) *MHD equations.* Here $y = \begin{pmatrix} \mathbf{v} \\ \mathbf{B} \end{pmatrix}$, with velocity field \mathbf{v} and magnetic field \mathbf{B} ,

$$\mathcal{L}y := \begin{pmatrix} \frac{1}{Re} P \Delta \mathbf{v} \\ \frac{1}{Rm} P \Delta \mathbf{B} \end{pmatrix},$$

$$\mathcal{N}(y, y) := P \otimes P \begin{pmatrix} -(\mathbf{v} \cdot \nabla) \mathbf{v} + (\mathbf{B} \cdot \nabla) \mathbf{B} \\ -(\mathbf{v} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{v} \end{pmatrix} = \begin{pmatrix} -P(\mathbf{v} \cdot \nabla) \mathbf{v} + P(\mathbf{B} \cdot \nabla) \mathbf{B} \\ -P(\mathbf{v} \cdot \nabla) \mathbf{B} + P(\mathbf{B} \cdot \nabla) \mathbf{v} \end{pmatrix},$$

and $\nabla \cdot y = 0$ has to be understood in the following sense:

$$\nabla \cdot y = \begin{pmatrix} \nabla \cdot \mathbf{v} \\ \nabla \cdot \mathbf{B} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

There are two classes of regularisations in the literature that we want to consider here:

- (A) *Modify the nonlinear term \mathcal{N} ;*
- (B) *Add regularising terms to the operator \mathcal{L} .*

Note that all of the proposed models are equations different from the original model and their solutions thus in general do not coincide. We can at this stage only show a range of convergence results. These are usually of the form “if the smoothing terms vanish, we get convergence to a weak solution of the original equation” or “if the original equations possess a (weak) solution for all times, then, as $t \rightarrow \infty$, the regularised solution converges to a weak solution of the original equation”.

There are, of course, several other ways (e.g. introducing artificial compressibility) of regularising the equations which we cannot all discuss here. It is also clear that one can combine several schemes with each other, and this has been done in the literature (e.g. there are Brinkman-Forchheimer-Voigt regularisations of the Navier-Stokes equations).

Name	Type	$\mathcal{L}(\mathbf{v}) =$	$\mathcal{N}(\mathbf{v}) =$
Mollifying nonlinearity	(A)	$P\Delta\mathbf{v}$	$-P[(\mathbf{v} * \rho_\kappa) \cdot \nabla] \mathbf{v}$
Leray- α model	(A)	$P\Delta\mathbf{v}$	$-P[(\{I - \alpha^2\Delta\}^{-1}\mathbf{v}) \cdot \nabla] \mathbf{v}$
Linear cutoff-scheme	(A)	$P\Delta\mathbf{v}$	$-\psi_M(\ \Delta^{1/2}\mathbf{u}\ _{L^2})P[(\mathbf{u} \cdot \nabla)\mathbf{u}]$
Globally modified NSE	(A)	$P\Delta\mathbf{v}$	$-\min\left\{1, \frac{N}{\ \Delta^{1/2}\mathbf{u}\ _{L^2}}\right\}P[(\mathbf{u} \cdot \nabla)\mathbf{u}]$
Regularisation by delay	(A)	$P\Delta\mathbf{v}$	$-P[(\mathbf{v}(t - \mu, x) \cdot \nabla)\mathbf{v}(t, x)]$
Hyperviscosity	(B)	$P(\Delta\mathbf{v} - \kappa(-\Delta)^{\ell/2}\mathbf{v})$	$-P(\mathbf{v} \cdot \nabla)\mathbf{v}$
Navier-Stokes-Voigt	(B)	$P(\Delta\mathbf{v} + \alpha^2\Delta\partial_t\mathbf{v})$	$-P(\mathbf{v} \cdot \nabla)\mathbf{v}$
Damped	(B)	$P(\Delta\mathbf{v} - \alpha \mathbf{v} ^{\beta-1}\mathbf{v})$	$-P(\mathbf{v} \cdot \nabla)\mathbf{v}$
Tamed	(B)	$P(\Delta\mathbf{v} - g_N(\mathbf{v} ^2)\mathbf{v})$	$-P(\mathbf{v} \cdot \nabla)\mathbf{v}$

TABLE 1. Overview of several regularisation schemes for the Navier-Stokes equations. We have set the viscosity ν (and hence the Reynolds number Re) to one for simplicity.

We first give a brief survey of strategies of type (A), then of (B). This survey is not aimed at completeness, but just intended to give an overview of the topic of regularised Navier-Stokes and MHD equations. We would also like to draw attention to the survey article by P. Constantin [42], providing a more detailed view on the Navier-Stokes cases of several of the models discussed here. Furthermore, the paper [114] by M. Holst, E. Lunasin and G. Tsogtgerel contains a nice overview as well as a unified framework and many results for many regularised hydrodynamical equations, including MHD.

For ease of presentation, we restrict ourselves to reproducing the formulas only for the Navier-Stokes case and in the MHD case just provide references. The full adaptation to the MHD system can then be found in the references given. Furthermore, we set $Re = 1$ for simplicity.

1.2.1. *Mollifying the Nonlinearity and the Force.* This classical example of a strategy of type (A) was already considered by J. Leray in his seminal 1934 paper [148]. Instead of $\mathcal{N}(\mathbf{v}, \mathbf{v}) := -P(\mathbf{v} \cdot \nabla)\mathbf{v}$, he considered the mollified nonlinearity

$$\begin{aligned}\mathcal{N}_{\text{moll}}(\mathbf{v}, \mathbf{v}) &:= -P [((\mathbf{v} * \rho_\kappa) \cdot \nabla) \mathbf{v}], \\ \mathbf{f}_{\text{moll}} &:= \mathbf{f} * \rho_\kappa,\end{aligned}$$

where $\kappa > 0$, ρ is a smooth, compactly supported function with $\int_{\mathbb{R}^3} \rho dx = 1$ and $\rho_\kappa(x) := \kappa^{-3} \rho(x/\kappa)$, i.e. a mollifier. The operator $\mathbf{u} \mapsto \mathbf{u} * \rho_\kappa$ is sometimes called *filtering operator* (cf. [92, 93] and [107], Sect. 2.3) as it filters out high spatial frequencies and thus maps $(L^1(\mathcal{O}))^3$ to $(C^\infty(\mathcal{O}))^3$, at least for $\mathcal{O} = \mathbb{T}^3$, the torus.

This smoothing of the advection velocity allows one to prove the existence of a unique smooth solution, which converges to a weak solution of the original equation as $\kappa \rightarrow 0$. Moreover, as $t \rightarrow \infty$, for fixed $\kappa > 0$, these solutions converge in a suitable sense toward weak solutions of the Navier-Stokes equations. This result – at least so far – needs the assumption that the initial conditions and forces are “small” in a suitable norm, cf. M. Cannone, G. Karch [29].

1.2.2. *Leray- α Model and Related Models (Clark- α , LANS, ...)* Another way to regularise the nonlinearity consists in applying the smoothing (or filtering) operator $(I - \alpha^2 \Delta)^{-1}$, usually called *Helmholtz filter*, to the first factor of the nonlinearity. Thus

$$\begin{aligned}\mathcal{N}_{\text{Leray-}\alpha}(\mathbf{v}, \mathbf{v}) &:= -P [(((I - \alpha^2 \Delta)^{-1} \mathbf{v}) \cdot \nabla) \mathbf{v}], \\ \mathbf{f}_{\text{Leray-}\alpha} &:= (I - \alpha^2 \Delta)^{-1} \mathbf{f}.\end{aligned}$$

Intuitively, the smoothing operator has the effect of damping high (spatial) Fourier modes¹ k , which correspond to small length scales $\ell \sim k^{-1}$. As this appears in the convection term, it means that convection occurs only by large-scale features, usually called *large eddies*. Thus, the Leray- α model is an example of a large-eddy simulation (LES) model.

The model was proposed by A. Cheskidov, D.D. Holm, E. Olson and E.S. Titi in [39] (cf. [218] for more references). The resulting model for the Navier-Stokes equations is called *Leray- α model*. For this equation there exists a unique global strong solution.

In the case of MHD equations, Leray- α -type models have been studied in several papers. The first paper seems to have been by J.S. Linshiz and E.S. Titi [159], where only the velocity field is filtered and periodic boundary conditions (i.e. posed on \mathbb{T}^3) are applied.

Y.J. Yu and K.T. Li [247] proved existence and uniqueness of a global strong solution in three dimensions for periodic boundary conditions. They proposed to apply the above filtering to all advection terms occurring in the equations. For further results for related MHD α -type models, see [70, 129, 130, 181, 229, 247, 261, 262] and references therein.

Similar models in this direction include the Clark- α model, for more information cf. [102]. There is a rich literature on these so-called α -models, see for example [114] or [146] and the references therein. We consider another example, the Navier-Stokes-Voigt equations, of this class of models in Section 1.2.7.

1.2.3. *A Cutoff Scheme due to Yoshida and Giga.* A third way to modify the nonlinearity of our fluid dynamical equations consists in truncating it. This approach was pioneered by Z. Yoshida and Y. Giga in [243].

¹As its Fourier transform is proportional to $(1 + \alpha^2 |\mathbf{k}|^2)^{-1}$, thus suppressing higher frequencies, i.e. higher $|\mathbf{k}| = k$.

They consider the case of a bounded domain \mathcal{O} , and a truncated operator equal to

$$\mathcal{N}_{YG}(\mathbf{u}) := \psi_M(\|\Delta^{1/2}\mathbf{u}\|_{L^2})P[(\mathbf{u} \cdot \nabla)\mathbf{u}],$$

with the following piecewise linear cutoff function

$$\psi_M(s) := \begin{cases} 1, & 0 \leq s \leq \frac{M}{2}, \\ 2(1 - \frac{s}{M}), & \frac{M}{2} < s < M \\ 0, & s \geq M. \end{cases}$$

This makes the operator $\mathcal{L} + \mathcal{N}_{YG} - cM^4I$ *hyperdissipative* (cf. [133, 134]) in $H := \mathbb{L}^2(\mathcal{O}) := L^2(\mathcal{O}) \cap \operatorname{div}^{-1}(\{0\})$ for sufficiently large constant $c > 0$, and hence one can apply nonlinear semigroup theory to find a unique global-in-time solution to the truncated system. For $d = 2$, this solution coincides with the global-in-time solution of the Navier-Stokes equations. For $d = 3$, this is only true locally in time, unless the initial data are small, in which case it gives a global (in time) solution to the Navier-Stokes equations.

It seems that this approach has not yet been applied to the case of MHD equations and neither do there exist stochastic versions of this equation in the literature.

1.2.4. *Globally Modified Navier-Stokes Equations.* A cutoff scheme similar to the one considered above was introduced by T. Caraballo, P.E. Kloeden and J. Real in [38]. They called the resulting equations the *globally modified Navier-Stokes* (GMNS) equations. They arise by introducing a different (nonlinear) cutoff function in the nonlinear term:

$$\mathcal{N}_{GMNS}(\mathbf{u}) := F_N(\|\Delta^{1/2}\mathbf{u}\|_{L^2})P[(\mathbf{u} \cdot \nabla)\mathbf{u}],$$

with the damping function ($N \in (0, \infty)$) given by

$$F_N(r) := \min \left\{ 1, \frac{N}{r} \right\}.$$

Like Yoshida and Giga, they considered the equation on a bounded domain, but instead of studying mild solutions, they study weak (or variational) solutions. The above mentioned authors prove existence and uniqueness of a weak solution, that this solution is even a strong solution, as well as long-time behaviour and continuous dependence of the solutions and the global attractors on the parameter N . Several further properties of the model have been studied in subsequent papers, cf. [33, 34, 132] and references therein. Global as well as exponential attractors for this system were studied recently by F. Li and B. You [150]. A globally modified version of the MHD equations has not yet been considered.

1.2.5. *Regularisation by Delay.* H. Bessaih, M. Garrido-Atienza and B. Schmalfuss [15] recently suggested regularising the nonlinear term by introducing a *time delay* in the advection velocity, i.e. they consider the nonlinearity

$$\mathcal{N}_{\text{delay}}(\mathbf{v}, \mathbf{v})(x, t) := P[(\mathbf{v}(t - \mu, x) \cdot \nabla)\mathbf{v}(t, x)],$$

where the initial conditions (i.e. the initial velocity \mathbf{v}_0 and the initial delay ϕ) must be sufficiently smooth, e.g.

$$\mathbf{v}_0 \in \dot{\mathbb{H}}^\alpha(\mathbb{T}^3), \phi \in L^2((-\mu, 0); \dot{\mathbb{H}}^{1+\alpha}(\mathbb{T}^3)),$$

for some $\alpha > 1/2$. Here $\dot{\mathbb{H}}^\alpha$ denotes the homogeneous fractional Sobolev space of order α of functions with vanishing divergence (for this notation, see Section 1.5). This time delay again has a smoothing effect on the solution and allows the authors to obtain unique global-in-time weak solutions and for $\alpha \geq 1$ also strong solutions. Letting the delay $\mu \rightarrow 0$, the solutions converge to a weak solution of the Navier-Stokes equations. Their work builds on that of S.M. Guzzo and G. Planas [109, 110] as well as C.J. Niche and G. Planas [183] and W. Varnhorn [217].

T. Caraballo and J. Real [37] seem to have been the first to consider Navier-Stokes equations with delays, proving existence in $d = 2, 3$ and uniqueness for $d = 2$. However, their model contained delay in the forcing term, not in the convective term. To the best of the authors knowledge, MHD equations with delay of this type have not been studied yet.

1.2.6. *Lions' Hyperviscosity Method.* Our first method of type (B) provides another way to add more dissipativity to the model. As opposed to the cutoff scheme of Yoshida and Giga (cf. Section 1.2.3), this is not achieved by weakening the nonlinearity but instead by strengthening the linear operator \mathcal{L} . J.-L. Lions [160] proposed to consider the operator

$$\mathcal{L}_{\text{hyper}}(\mathbf{v}) := P\Delta\mathbf{v} - \kappa P(-\Delta)^{\ell/2}\mathbf{v},$$

where $\ell > 2$ and $\kappa > 0$. For $\ell > 5/2$, he could prove existence of a unique regular solution in a bounded domain. The case of the whole space \mathbb{R}^3 has been treated by N.H. Katz and N. Pavlović in [128]. As in the case of the mollified Navier-Stokes equations (Section 1.2.1), this solution – for small data – converges towards the weak solution of the Navier-Stokes equations as $t \rightarrow \infty$, cf. M. Cannone and G. Karch [29]. Further related problems have been studied, such as optimal control by S.S. Sritharan [205] or the inhomogeneous Navier-Stokes system by D.Y. Fang and R.Z. Zi [72].

In the MHD case, variants of this approach have been studied in three dimensions, e.g. by J.H. Wu, W.R. Yang and Q.S. Jiu in [233, 240] (see the latter for more references).

1.2.7. *Navier-Stokes-Voigt Equations.* The Navier-Stokes-Voigt (sometimes written as Voigt) equations employ the following regularisation of the Stokes operator

$$\mathcal{L}_{\text{NSV}}(\mathbf{v}) := P\Delta\mathbf{v} + \alpha^2 P\Delta\partial_t\mathbf{v}.$$

This regularisation changes the parabolic character of the equations and simulates a property of so-called Kelvin-Voigt fluids, e.g. in polymer solutions, of not immediately reverting back to the original state once external stress is removed. It was pioneered by A.P. Oskolkov [186] in 1973 and has since then been studied by many authors. The Navier-Stokes-Voigt equations were shown to be globally well-posed in [186, 187] as well as in the work of Y.P. Cao, E.M. Lunasin and E.S. Titi, cf. [32]. The long-term behaviour was studied by V.K. Kalantarov and E.S. Titi in [123] and by Kalantarov, Titi and B. Levant in [122]. The case of unbounded domains was treated by C.T. Anh and P.T. Trang in [3, 4].

In the case of the MHD equations, a Voigt regularisation in the inviscid case was studied first by A. Larios and E.S. Titi in a series of papers [144–146]. Consult these for further references.

1.2.8. *Damped Navier-Stokes Equations (or Brinkman-Forchheimer-extended Darcy Models).* Instead of adding a linear dissipative term, one can also add nonlinear, power-type terms that counteract the nonlinearity. This leads to the so-called (*nonlinearly*) *damped Navier-Stokes equations*, where one considers the (nonlinear) operator

$$\mathcal{L}_{\text{damped}}(\mathbf{v}) := P\Delta\mathbf{v} - \alpha P|\mathbf{v}|^{\beta-1}\mathbf{v},$$

with $\alpha > 0$ and $\beta \geq 1$. The damping term $-\alpha|\mathbf{v}|^{\beta-1}\mathbf{v}$ models the resistance to the motion of the flow resulting from physical effects like porous media flow, drag or friction or other dissipative mechanisms (cf. [28] and Section 1.3.1). It represents a restoring force, which for $\beta = 1$ assumes the form of classical, linear damping, whereas $\beta > 1$ means a restoring force that grows superlinearly with the velocity (or magnetic field). X.J. Cai and Q.S. Jiu [28] first proved existence and uniqueness of a global strong solution for $\frac{7}{2} \leq \beta \leq 5$. This range was lowered down to $\beta \in (3, 5]$ by Z.J. Zhang, X.L. Wu and M. Lu in [252].

Furthermore, they considered the case $\beta = 3$ to be critical [252, Remark 3.1]. Y. Zhou in [260] proved the existence of a global solution for all $\beta \in [3, 5]$. For the case $\beta \in [1, 3)$, he established regularity criteria that ensure smoothness. Uniqueness holds for any $\beta \geq 1$ in the class of weak solutions. Existence, decay rates and qualitative properties of weak solutions were also investigated by S.N. Antontsev and H.B. de Oliveira [5].

The Brinkman-Forchheimer-extended Darcy model (cf. Section 1.3.1) is a related model for flow of fluids through porous media and uses the operator

$$\mathcal{L}_{\text{BFED}}(\mathbf{v}) := P\Delta\mathbf{v} - \alpha_0\mathbf{v} - \alpha_1P|\mathbf{v}|\mathbf{v} - \alpha_2P|\mathbf{v}|^2\mathbf{v}.$$

The first problems studied were continuous dependence of the solutions on their parameters, e.g. in F. Franchi, B. Straughan [83]. V.K. Kalantarov and S. Zelik [124] and P.A. Markowich, E.S. Titi and S. Trabelsi [173] proved existence and uniqueness of a weak solution for Dirichlet and periodic boundary conditions, respectively. Long-time behaviour and existence of global attractors have been studied by several authors [188, 216, 224, 246]. An anisotropic version of the equations was studied by H. Bessaih, S. Trabelsi and H. Zorgati [16].

The flow of electrically conducting fluids through porous media, modelled by MHD equations with damping, was studied first by Z. Ye in [241]. He considered the system with nonlinear damping in the equations for both the velocity field (with nonlinear damping parameter α) and the magnetic field (with parameter β) and he proved existence and uniqueness of global strong solutions in the full space case for several ranges of parameters, most interestingly for our purposes for $\alpha, \beta \geq 4$. Z.J. Zhang and X. Yang [253] tried to improve this to $\alpha, \beta > 3$, but apparently made a mistake in their proof ([251, Remark after Equation (9), p. 2]). Z.J. Zhang, C.P. Wu, Z.A. Yao [251] then improved the range to $\alpha \in [3, \frac{27}{8}], \beta \geq 4$. The present chapter, in a way, deals with the ‘‘critical’’ case $\alpha = \beta = 3$, see the discussion of the results below. Furthermore, E.S. Titi and S. Trabelsi [215] proved global well-posedness for an MHD model with nonlinear damping only in the velocity field. They thus avoid the magnetic pressure problem outlined in Section 1.3.4, as opposed to the above papers which seem to have overlooked this issue.

1.3. The Tamed Equations. We first motivate the tamed equations, both from a physical point of view by pointing out situations where similar models arise naturally in applications, as well as from a mathematical point of view. The tamed Navier-Stokes equations are in a sense a variant of the Navier-Stokes equations with damping in the critical case $\beta = 3$, combined with a cutoff.

1.3.1. *Physical Motivation.* Since the tamed equations are closely related to the damped equations of Section 1.2.8, which are much more well-studied, we focus on the occurrence of these in the physics literature.

1. Shallow-Water Systems with Friction on the Bottom. F. Marche derived, at least formally, in [171] a set of equations modelling a shallow-water system with free surface height h including damping terms with $\beta = 1$ and $\beta = 2$ via asymptotic analysis and hydrostatic approximation. The damping terms originate from the kind of boundary conditions imposed on the bottom of the ocean: linear (or laminar) friction of the form

$$(1.3) \quad (\sigma(\mathbf{v}) \cdot \mathbf{n}_b) \cdot \boldsymbol{\tau}_{b_i} = k_{\text{laminar}}(\mathbf{v} \cdot \boldsymbol{\tau}_{b_i}), \quad z = 0$$

leads to a linear damping term of the form $-\alpha_0(h)\mathbf{v}$. Here, σ denotes the total stress tensor, \mathbf{n}_b denotes the outward normal, the vectors $\boldsymbol{\tau}_{b_i}$ form a basis of the tangential surface on the bottom, and k_{laminar} is the laminar friction coefficient.

On the other hand, quadratic friction (associated with turbulence)

$$(1.4) \quad (\boldsymbol{\sigma}(\mathbf{v}) \cdot \mathbf{n}_b) \cdot \boldsymbol{\tau}_{b_i} = k_{\text{turbulent}}(h|\mathbf{v}|\mathbf{v} \cdot \boldsymbol{\tau}_{b_i}), \quad z = 0,$$

implies a damping term of the form $-\alpha_1(h)h|\mathbf{v}|\mathbf{v}$. The full set of equations (cf. [171, Equation (5.20), p. 59]), with both forms of friction, has the form

$$\begin{cases} \partial_t h + \operatorname{div}(h\mathbf{v}) & = 0 \\ \partial_t(h\mathbf{v}) + \operatorname{div}(h\mathbf{v} \otimes \mathbf{v}) + gh\nabla h & = -\alpha_0(h)\mathbf{v} - \alpha_1(h)h|\mathbf{v}|\mathbf{v} + 2\mu \operatorname{div}(hD(\mathbf{v})) \\ & \quad + 2\mu\nabla(h \operatorname{div}(\mathbf{v})) + \beta h\nabla\Delta h - f(h\mathbf{v})^\perp - gh\nabla d, \end{cases}$$

where h denotes the variable height of the body of water, \mathbf{v} the horizontal velocity, averaged vertically, g is the acceleration due to gravity, μ denotes the dynamical viscosity, β a capillary constant, f the Coriolis coefficient and d describes variations in the topography of the bottom. A mathematical analysis can be found in D. Bresch, B. Desjardins [19].

We now make the following simplifying assumptions: $h(x, y, t) = \bar{h} \in (0, \infty)$, constant in space and time, so the first equation reduces to the incompressibility condition. Neglecting gravity, capillary effects and the Coriolis force, i.e. setting $g = 0$, $\beta = 0$ and $f = 0$ and dividing the second equation by h , we heuristically obtain the following equations:

$$\begin{aligned} \operatorname{div}(\mathbf{v}) &= 0, \\ \partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) &= -\alpha_0 \mathbf{v} - \alpha_1 |\mathbf{v}|\mathbf{v} + 2\mu \Delta \mathbf{v}. \end{aligned}$$

These are the damped Navier-Stokes equations with $\beta = 1$ and $\beta = 2$. However, there are several problems with this approach:

- (i) The resulting equations are in two space dimensions only, and this derivation does not cover the case of three dimensions, which is most interesting from a mathematical point of view.
- (ii) The heuristic simplifications above, especially concerning the height, would have to be justified.
- (iii) From this model, we can only get $\beta \in \{1, 2\}$, in particular we cannot get the ‘‘tamed’’ case $\beta = 3$.

They do provide, however, an example of a physically relevant system that includes fluid dynamical systems with damping of a form similar to the one we consider in this work.

2. Flows Through Porous Media. Brinkman-Forchheimer-extended Darcy Model. Another system with possibly nonlinear damping is considered as a model for the flow of a fluid through porous media, described for example by the following compressible Euler equations with damping:

$$(1.5) \quad \begin{aligned} \rho_t + \partial_x(\rho v) &= 0, \\ (\rho v)_t + \partial_x(\rho v^2 + p) &= -\alpha \rho v. \end{aligned}$$

The interpretation that this equation models the flow through porous media is in line with the result that as $t \rightarrow \infty$, the density ρ converges to the solution of the porous medium equation (cf. F.M. Huang, R.H. Pan [117]). The momentum, on the other hand, is described in the limit by *Darcy’s law*:

$$\nabla p = -\frac{\mu}{k} \mathbf{v},$$

which represents a simple linear relationship between the flow rate and the pressure drop in a porous medium. Here, k is the permeability of the porous medium and μ is the dynamic viscosity. The velocity \mathbf{v} is called Darcy’s seepage velocity.

In the interface region between a porous medium and a fluid layer, C.T. Hsu and P. Cheng [116, Equation (31), p. 1591] proposed the following equation²:

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0, \\ \partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) &= -\nabla p + \nu \Delta \mathbf{v} - \alpha_0 \mathbf{v} - \alpha_1 |\mathbf{v}| \mathbf{v}, \end{aligned}$$

where \mathbf{v} is the so-called volume-averaged Darcy seepage velocity and p is the volume-averaged pressure. This equation is motivated by a quadratic correction of P. Forchheimer to Darcy's law, called *Forchheimer's law* or *Darcy-Forchheimer law* (cf. for example P.A. Markowich, E.S. Titi, S. Trabelsi [173]):

$$\nabla p = -\frac{\mu}{k} \mathbf{v}_F - \gamma \rho_F |\mathbf{v}_F| \mathbf{v}_F,$$

with the Forchheimer coefficient $\gamma > 0$, the Forchheimer velocity \mathbf{v}_F as well as the density ρ_F . Furthermore, this correction becomes necessary at higher flow rates through porous media, see below for a more detailed discussion.

The question arises whether there are cases where a nonlinear correction of yet higher degree is necessary, i.e. where the flow obeys a *cubic Forchheimer's law*:

$$(1.6) \quad \nabla p = -\frac{\mu}{k} \mathbf{v} - \gamma \rho |\mathbf{v}| \mathbf{v} - \kappa \rho^2 |\mathbf{v}|^2 \mathbf{v}.$$

Indeed, this seems to be the case. P. Forchheimer [81] himself suggested several corrections to Darcy's law at higher flow velocities, one of them being the cubic law (1.6). M. Firdaouss, J.-L. Guermont and P. Le Quéré [77] revisited several historic data sets, amongst them the ones used by Darcy and Forchheimer (who did not correct for Reynolds numbers) and found that the data are actually better described by a linear and cubic Darcy-Forchheimer law (i.e. where $\gamma = 0$), at least in the regime of low to moderate Reynolds numbers, which, as they note³, includes most practical cases:

$$(1.7) \quad \nabla p = -\frac{\mu}{k} \mathbf{v} - \kappa \rho^2 |\mathbf{v}|^2 \mathbf{v}.$$

Concerning the question of when this happens, M. Fourar, G. Radilla, R. Lenormand and C. Moyne [82, p. 670] write: "it is generally admitted that the onset of non-Darcy flow occurs for Re (based on the average velocity and grain size) between 1 and 10."

At higher Reynolds numbers, the correct behaviour seems to be quadratic, i.e. Forchheimer's law, in accordance with numerical simulations, e.g. in the work of M. Fourar *et al.* [82]. The point at which this behaviour changes seems to be dimension-dependent: it occurs much earlier in the numerical simulations of [82, Figure 7] in the 3D case than in the 2D case. Another instance where a cubic Forchheimer law is observed is the high-rate flow in a radial fracture with corrugated walls, cf. M. Buès, M. Panfilov, S. Crosnier and C. Oltean [27, Equation (7.2), p. 54].

Taking into account all nonlinear corrections of Darcy's law, we arrive at the Brinkman-Forchheimer-extended Darcy model

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0, \\ \partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) &= -\nabla p + \nu \Delta \mathbf{v} - \alpha_0 \mathbf{v} - \alpha_1 |\mathbf{v}| \mathbf{v} - \alpha_2 |\mathbf{v}|^2 \mathbf{v}. \end{aligned}$$

²For ease of presentation, we have omitted various physical constants in the formulation of the equations.

³[77, p. 333]: "[T]he most frequent practical applications (for either gas or liquids) involve Reynolds numbers of order 1 or less."

The tamed Navier-Stokes equations model the behaviour of the flow through porous media in the regime of relatively low to moderate Reynolds numbers, assuming that the higher-order behaviour is much more significant than the linear Darcy behaviour. For a more physically accurate model, one should also include the linear damping term, but we want to focus on the nonlinear effects here and thus for simplicity have omitted this term. The fact that the onset of nonlinear behaviour occurs at higher flow rates is modelled by the cutoff function g_N which is nonzero only for sufficiently high velocity. Apart from these physical reasons, there is also a mathematical reason for the form of the taming term.

1.3.2. *Mathematical Motivation.* The tamed Navier-Stokes equations were introduced in [197] by M. Röckner and X.C. Zhang and have the following form:

$$(1.8) \quad \begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= \nu \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v} - g_N(|\mathbf{v}|^2) \mathbf{v} + \nabla p + \mathbf{f} \\ \nabla \cdot \mathbf{v} &= 0 \\ \mathbf{v}(0, x) &= \mathbf{v}_0(x). \end{aligned}$$

The “taming function” allowed them to obtain stronger estimates than for the untamed Navier-Stokes equations, and hence regularity results that are out of reach for the Navier-Stokes equations. Furthermore, they could show that bounded solutions to the Navier-Stokes equations, if they exist, coincide with the solutions to the tamed Navier-Stokes equations, as shown in [197]. This is a feature that most regularisations of the Navier-Stokes equations do not share.

1.3.3. *Review of Results for Tamed Navier-Stokes Equations.* The deterministic case was further studied by X.C. Zhang on uniform C^2 -domains in [250]. In a series of subsequent papers, various properties of the stochastic version of the equations were studied: existence and uniqueness to the stochastic equation as well as ergodicity in [197], Freidlin-Wentzell type large deviations in [195] as well as the case of existence, uniqueness and small time large deviation principles for the Dirichlet problem in bounded domains [194] (both with T.S. Zhang). More recently, there has been resparked interest in the subject, with contributions by Z. Dong and R.R. Zhang [64] (existence and uniqueness for multiplicative Lévy noise) as well as Z. Brzeźniak and G. Dhariwal [23] (existence, uniqueness and existence of invariant measures in the full space \mathbb{R}^3 for a slightly simplified system and by different methods).

The taming function was subsequently simplified by changing the expression of g_N as well as replacing the argument of the function g_N by the square of the spatial L^∞ norm of the velocity, i.e. $g_N(\|\mathbf{v}\|_{\mathbb{L}^\infty}^2)$, see W. Liu and M. Röckner [167, pp. 170 ff]. This leads to simpler assumptions on g_N as well as easier proofs, especially when spatial derivatives are concerned (which then act only on the remaining factor \mathbf{v}). However, this only works within the framework of locally monotone operators which cannot be applied in all settings due to the crucial assumption of compact embeddings. Thus we do not use this simplification in this work.

1.3.4. *The Magnetic Pressure Problem.* From the form of the MHD equations, it would seem like there should also be a “pressure” term $\nabla \pi$ in the equation for the magnetic field. That this is not the case is due to the structure of the nonlinear term in the equation, as was noted already in the work of M. Sermange and R. Temam [201, p. 644]. To make this precise, consider the MHD equations on \mathbb{R}^3

$$(1.9) \quad \frac{\partial \mathbf{v}}{\partial t} = \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v} + (\mathbf{B} \cdot \nabla) \mathbf{B} + \nabla \left(p + \frac{|\mathbf{B}|^2}{2} \right),$$

$$(1.10) \quad \frac{\partial \mathbf{B}}{\partial t} = \Delta \mathbf{B} - (\mathbf{v} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{v},$$

and the associated equation projected on the space of divergence-free functions:

$$\begin{aligned}\frac{\partial \mathbf{v}}{\partial t} &= \Delta \mathbf{v} - P(\mathbf{v} \cdot \nabla) \mathbf{v} + P(\mathbf{B} \cdot \nabla) \mathbf{B}, \\ \frac{\partial \mathbf{B}}{\partial t} &= \Delta \mathbf{B} - P(\mathbf{v} \cdot \nabla) \mathbf{B} + P(\mathbf{B} \cdot \nabla) \mathbf{v}.\end{aligned}$$

Assume that (\mathbf{v}, \mathbf{B}) is a smooth weak solution. Now since the first equation lies in the space of square-integrable divergence-free functions \mathbb{H}^0 , there exists a function $\nabla \tilde{p}$ in the orthogonal complement $(\mathbb{H}^0)^\perp$ such that⁴

$$\frac{\partial \mathbf{v}}{\partial t} = \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v} + (\mathbf{B} \cdot \nabla) \mathbf{B} + \nabla \tilde{p}.$$

Note that

$$(1.11) \quad -(\mathbf{v} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{v} = \nabla \times (\mathbf{v} \times \mathbf{B}),$$

i.e. the nonlinear terms in the magnetic field equation combine to an expression that is manifestly divergence-free. If there existed a magnetic pressure π such that

$$\partial_t \mathbf{B} = \Delta \mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B}) + \nabla \pi,$$

taking the divergence of this equation, observing that $\operatorname{div} \mathbf{B} = 0$, would give

$$\Delta \pi = 0,$$

where $\nabla \pi(t, x) \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3))$, which implies⁵ $\nabla \pi = 0$. Thus, a careful balancing in the two nonlinear terms leads to the “magnetic pressure” being zero. Now, if we introduce further nonlinearities into the equation for the magnetic field, we might offset this cancellation and thus we will get an artificial “magnetic pressure” in our tamed equations. We can show that this pressure converges to zero as $N \rightarrow \infty$, but for the tamed equations, it is undeniably present. We will informally name this phenomenon the *magnetic pressure problem*:

DEFINITION 1.1 (Magnetic Pressure Problem). *Introducing extra terms $\mathcal{N}(y)$ that are not divergence-free into the equation for the magnetic field \mathbf{B} in the MHD equations will lead to the appearance of an artificial, possibly unphysical “magnetic pressure” π , i.e. (1.10) will be of the form*

$$\partial_t \mathbf{B} = \Delta \mathbf{B} - (\mathbf{v} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{v} + \nabla \pi + \mathcal{N}(y).$$

This term does not manifest itself in the weak formulation of the problem, which is most often studied. Our system is no exception here, so when talking about the pointwise form of the equation, we have to include the magnetic pressure term π , as above. This fact is easily overlooked when introducing regularising terms into the equation for the magnetic field. To give an example, in the work of Z.J. Zhang, C.P. Wu and Z.A. Yao [251], the authors introduce a damping term $|\mathbf{B}|^{\beta-1} \mathbf{B}$ into the magnetic field equation, but forgot to include a “magnetic pressure” in the strong form of this equation. Note that in other regularisations of the MHD equations, such as the Leray- α model, this problem is avoided by only introducing terms that preserve the structure of the nonlinearities (1.11).

⁴The function \tilde{p} can then be chosen to have the form $\tilde{p} = p + \frac{|\mathbf{B}|^2}{2}$.

⁵Since $\nabla \pi$ is C^2 by assumption, we find that each of its components $\partial_i \pi$ solves Laplace’s equation $\Delta \partial_i \pi = 0$ and is thus smooth. The integrability condition implies that $\partial_i \pi$ is bounded outside a sufficiently large compact set. The smoothness implies that it is also bounded inside that compact set. Then Liouville’s theorem [100] implies $\pi \equiv \text{const}$, hence $\partial_i \pi \equiv 0$. In the case of bounded domains, we can argue in a similar way using uniqueness.

Ideally, one should thus introduce taming terms for the velocity field only. For mathematical reasons, however, at this point we have to content ourselves with taming terms in both components, for otherwise, in the crucial H^1 -estimate (2.10) we could not cancel all four nonlinearities.

1.3.5. *The Magnetic Field: To Regularise or Not to Regularise?* There seems to be no clear answer, even for schemes which do not introduce magnetic pressure, to the question of whether in the MHD equations the magnetic field should be regularised as well, or whether one should restrict oneself to only regularising the velocity field. A mathematical criticism formulated in J.S. Linshiz and E.S. Titi [159, p. 3] is that regularising the magnetic part as well might add an unnecessary amount of dissipativity to the system. However, for the mathematical reasons discussed in the previous section, we add a taming term to the magnetic field equation as well.

1.3.6. *The Tamed MHD Equations.* We investigate the case of the deterministic version of tamed magnetohydrodynamics (TMHD) equations in this chapter. They can be understood as a model of an electrically conducting fluid in a porous medium at low to moderate Reynolds numbers (cf. P.A. Markowich, E.S. Titi and S. Trabelsi [173]). Following the approach of M. Röckner and X.C. Zhang, we study the following equations:

$$(1.12) \quad \begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v} + (\mathbf{B} \cdot \nabla) \mathbf{B} + \nabla \left(p + \frac{|\mathbf{B}|^2}{2} \right) - g_N(|(\mathbf{v}, \mathbf{B})|^2) \mathbf{v} + \mathbf{f}_v \\ \frac{\partial \mathbf{B}}{\partial t} &= \Delta \mathbf{B} - (\mathbf{v} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{v} + \nabla \pi - g_N(|(\mathbf{v}, \mathbf{B})|^2) \mathbf{B} + \mathbf{f}_B. \end{aligned}$$

For simplicity we have set all the constants appearing in the MHD equations to one:

$$(1.13) \quad S = Rm = Re = 1.$$

If we write $y := (\mathbf{v}, \mathbf{B})$, the equations differ from the “untamed” MHD equations by the taming term

$$-g_N(|y(t, x)|^2)y(t, x),$$

which is a direct generalisation of the term in (1.8). The norm is defined in equation (2.2) below. One could think of other generalisations as well such as adding four taming terms, each tailored to one of the nonlinear terms, so e.g. for the term $-(\mathbf{B} \cdot \nabla) \mathbf{v}$ we could add $-g_N(|\mathbf{B}|^2) \mathbf{v}$ etc. However, this is not necessary. In a sense, the most problematic terms are the ones of the form $(\mathbf{v} \cdot \nabla) \mathbf{X}$, where $\mathbf{X} \in \{\mathbf{v}, \mathbf{B}\}$. The other terms can be dealt with in any case.

The taming function $g_N: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined by

$$(1.14) \quad \begin{cases} g_N(r) := 0, & r \in [0, N], \\ g_N(r) := C_{\text{taming}} \left(r - N - \frac{1}{2} \right), & r \geq N + 1, \\ 0 \leq g'_N(r) \leq C_1, & r \geq 0, \\ |g_N^{(k)}(r)| \leq C_k, & r \geq 0, k \in \mathbb{N}. \end{cases}$$

Here, the constant C_{taming} is defined by

$$C_{\text{taming}} := 2 \max\{Re, Rm\} = 2.$$

For the Navier-Stokes case, M. Röckner and X.C. Zhang in [197] set $C = \frac{1}{\nu} \propto Re$, so the fact that $C_{\text{taming}} \propto Re$ is not surprising. The factor 2 arises from the fact that we need to tame more terms here. The dependency on Rm seems natural as well.

The idea of the taming procedure remains very clear: try to counteract the nonlinear terms of which there are four in the case of the MHD equations. To pinpoint the exact place where the power of the taming function unfolds, see the discussion after Lemma 2.2.

1.4. Results and Structure of This Chapter. We follow the ideas of [197]. However, the proof of the regularity of the solution requires an MHD adaptation of a result from E.B. Fabes, B.F. Jones and N.M. Rivière [69], which the author could not find in the literature. See Appendix A for a discussion and a proof of this result.

Our main results can be summarised as follows:

THEOREM 1.2 (Global well-posedness, cf. Theorems 2.7 and 2.8 below). *Let $y_0 = (\mathbf{v}_0, \mathbf{B}_0) \in \mathcal{H}^1$ and $f = (\mathbf{f}_v, \mathbf{f}_B) \in L^2_{loc}(\mathbb{R}_+; \mathcal{H}^0)$. For any $N > 0$, there exists a unique weak solution y to the TMHD equation in the sense of Definition 2.5, depending continuously on the initial data, such that*

(i) *For all $t \geq 0$,*

$$\|y(t)\|_{\mathcal{H}^0} \leq \|y_0\|_{\mathcal{H}^0} + \int_0^t \|f(s)\|_{\mathcal{H}^0} ds,$$

and

$$\int_0^t \|\nabla y(s)\|_{\mathcal{H}^0} + \|\sqrt{g_N(|y(s)|^2)}|y(s)\|_{L^2}^2 ds \leq \|y_0\|_{\mathcal{H}^0}^2 + 2 \left[\int_0^t \|f(s)\|_{\mathcal{H}^0} ds \right]^2.$$

(ii) *The solution satisfies $y \in C(\mathbb{R}_+; \mathcal{H}^1) \cap L^2_{loc}(\mathbb{R}_+; \mathcal{H}^2)$, $\partial_t y \in L^2_{loc}(\mathbb{R}_+; \mathcal{H}^0)$ and for all $t \geq 0$,*

$$\begin{aligned} & \|y(t)\|_{\mathcal{H}^1}^2 + \int_0^t (\|y(s)\|_{\mathcal{H}^2}^2 + \|\nabla y(s)\|_{L^2}^2) ds \\ & \leq C \left(\|y_0\|_{\mathcal{H}^1}^2 + \int_0^t \|f(s)\|_{\mathcal{H}^0}^2 ds \right) + C(1 + N + t) \left(\|y_0\|_{\mathcal{H}^0}^2 + \left[\int_0^t \|f(s)\|_{\mathcal{H}^0} ds \right]^2 \right). \end{aligned}$$

(iii) *There exist real functions $p(t, x)$ and $\pi(t, x)$, satisfying $\nabla p \in L^2_{loc}(\mathbb{R}_+; L^2(\mathbb{R}^3; \mathbb{R}^3))$, $\nabla \pi \in L^2_{loc}(\mathbb{R}_+; L^2(\mathbb{R}^3; \mathbb{R}^3))$, such that for almost all $t \geq 0$, in $L^2(\mathbb{R}^3; \mathbb{R}^6)$ we have*

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v} + (\mathbf{B} \cdot \nabla) \mathbf{B} + \nabla \left(p + \frac{|\mathbf{B}|^2}{2} \right) - g_N(|(\mathbf{v}, \mathbf{B})|^2) \mathbf{v} + \mathbf{f}_v, \\ \frac{\partial \mathbf{B}}{\partial t} &= \Delta \mathbf{B} - (\mathbf{v} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{v} + \nabla \pi - g_N(|(\mathbf{v}, \mathbf{B})|^2) \mathbf{B} + \mathbf{f}_B. \end{aligned}$$

In the case of smooth data, we can prove smoothness of the solutions to the TMHD equations:

THEOREM 1.3 (Regularity and Strong Solutions, cf. Theorem 2.9 below). *Let $y_0 \in \mathcal{H}^\infty := \bigcap_{m \in \mathbb{N}_0} \mathcal{H}^m$ and $\mathbb{R}_+ \ni t \mapsto f(t) \in \mathcal{H}^m$ be smooth for any $m \in \mathbb{N}_0$. Then there exists a unique smooth velocity field*

$$\mathbf{v}_N \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3; \mathbb{R}^3) \cap C(\mathbb{R}_+; \mathcal{H}^2),$$

a unique smooth magnetic field

$$\mathbf{B}_N \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3; \mathbb{R}^3) \cap C(\mathbb{R}_+; \mathcal{H}^2),$$

and smooth pressure functions

$$p_N, \pi_N \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3; \mathbb{R}),$$

which are defined up to a time-dependent constant. The quadruplet $(\mathbf{v}_N, \mathbf{B}_N, p_N, \pi_N)$ solves the tamed MHD equations (1.12).

Moreover, we have the following estimates: for any $T, N > 0$

$$\begin{aligned} \sup_{t \in [0, T]} \|y_N(t)\|_{\mathcal{H}^0}^2 + \int_0^T \|\nabla y_N\|_{\mathcal{H}^0}^2 ds &\leq C \left(\|y_0\|_{\mathcal{H}^0}^2 + \left[\int_0^T \|f(s)\|_{\mathcal{H}^0} ds \right]^2 \right), \\ \sup_{t \in [0, T]} \|y_N(t)\|_{\mathcal{H}^1}^2 + \int_0^T \|y_N(s)\|_{\mathcal{H}^2}^2 ds &\leq C_{T, y_0, f} \cdot (1 + N), \\ \sup_{t \in [0, T]} \|y_N(t)\|_{\mathcal{H}^2}^2 &\leq C'_{T, y_0, f} + C_{T, y_0, f} \cdot (1 + N^2). \end{aligned}$$

Finally, we have the following convergence result for vanishing taming terms, i.e. in the limit $N \rightarrow \infty$.

THEOREM 1.4 (Convergence to the untamed equations, cf. Theorem 2.10 below). *Let $y_0 \in \mathcal{H}^0$, $f \in L^2([0, T]; \mathcal{H}^0)$, $y_0^N \in \mathcal{H}^1$ such that $\mathcal{H}^0 - \lim_{N \rightarrow \infty} y_0^N = y_0$. Denote by (y_N, p_N, π_N) the unique solutions to the tamed equations (1.12) with initial value y_0^N given by Theorem 1.2.*

Then there is a subsequence $(N_k)_{k \in \mathbb{N}}$ such that y_{N_k} converges to a y in $L^2([0, T]; \mathcal{L}_{loc}^2)$ and p_{N_k} converges weakly to some p in $L^{9/8}([0, T]; L^{9/5}(\mathbb{R}^3))$. The magnetic pressure π_{N_k} converges to zero, weakly in $L^{9/8}([0, T]; L^{9/5}(\mathbb{R}^3))$. Furthermore, (y, p) is a weak solution to (1.1) such that the following generalised energy inequality holds:

$$\begin{aligned} 2 \int_0^T \int_{\mathbb{R}^3} |\nabla y|^2 \phi dx ds &\leq \int_0^T \int_{\mathbb{R}^3} \left[|y|^2 (\partial_t \phi + \Delta \phi) + 2 \langle y, f \rangle \phi \right. \\ &\quad \left. + (|y|^2 - 2p) \langle \mathbf{v}, \nabla \phi \rangle - 2 \langle \mathbf{B}, \mathbf{v} \rangle \langle \mathbf{B}, \nabla \phi \rangle \right] dx ds. \end{aligned}$$

We have been able to extend all the results of [197] as well as [167] to the case of tamed MHD equations. This posed several technical obstacles: we had to extend the regularity result of [69] to the MHD case, which the author could not find in the literature. Moreover, we describe the magnetic pressure problem in regularised MHD equations. Furthermore, our work basically provides the critical case $\alpha = \beta = 3$ of the model considered in [241, 251, 253].

The chapter is organised as follows: we first treat the Cauchy problem (i.e. the equations posed on \mathbb{R}^3) in Section 2. We start in Section 2.1 by introducing the functional framework of the problem. Then we state and prove a number of elementary lemmas regarding estimates as well as (local) convergence results for the operators appearing in the tamed MHD equations. Existence and uniqueness of a weak solution is shown in Section 2.2 via a Faedo-Galerkin approximation procedure. Employing the results of Appendix A, we then show in Section 2.3 that for smooth data the solution to the tamed MHD equations remains smooth. Finally, in Section 2.4 we show that as $N \rightarrow \infty$, the solution to the tamed MHD equations converges to a weak solution of the (untamed) MHD equations.

A publication of the results of this chapter is in preparation, cf. [198].

1.5. Notation. Let $G \subset \mathbb{R}^3$ be a domain and denote the divergence operator by div . We use the following notational hierarchy for L^p and Sobolev spaces:

- (1) For the spaces $L^p(G, \mathbb{R})$ of real-valued integrable (equivalence classes of) functions we use the notation $L^p(G)$ or L^p if no confusion can arise. These are the spaces of the components v^i, B^i of the velocity and magnetic field vector fields.
- (2) We sometimes use the notation $\mathbf{L}^p(G) := L^p(G; \mathbb{R}^3)$ to denote 3-dimensional vector-valued integrable quantities, especially the velocity vector field and magnetic vector field \mathbf{v} and \mathbf{B} .
- (3) The divergence-free and p -integrable vector fields are denoted by \mathbb{L}^p symbols, so $\mathbb{L}^p(G) := \mathbf{L}^p(G) \cap \operatorname{div}^{-1}\{0\}$. Its elements are still denoted by bold-faced symbols \mathbf{v}, \mathbf{B} and they satisfy by definition $\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = 0$, $\operatorname{div} \mathbf{B} = 0$.
- (4) Finally, we denote the space of the combined velocity and magnetic vector fields by \mathcal{L}^p symbols, i.e. $\mathcal{L}^p(G) := \mathbb{L}^p(G) \times \mathbb{L}^p(G)$. It contains elements of the form $y = (\mathbf{v}, \mathbf{B})$, with both \mathbf{v} and \mathbf{B} divergence-free.

For Sobolev spaces, we use the same notational conventions, so for example $\mathbb{H}^k(G) := \mathbf{H}^k(G) \cap \operatorname{div}^{-1}\{0\} := W^{k,2}(G; \mathbb{R}^3) \cap \operatorname{div}^{-1}\{0\}$ etc. Finally, if the domain of the functions is not in \mathbb{R}^3 , in particular if it is a real interval (for the time variable), then we use the unchanged L^p notation.

For brevity, we use the following terminology when discussing the terms on the right-hand side of the tamed MHD equations: the terms involving the Laplace operator are called the *linear terms*, the terms involving the taming function g_N are called *taming terms* and the other terms are called the *nonlinear terms*. Furthermore, we refer to the initial data $y_0 = (\mathbf{v}_0, \mathbf{B}_0)$ and the force $f = (\mathbf{f}_v, \mathbf{f}_B)$ collectively as the *data* of the problem.

2. The Case of the Whole Space

We first consider the equations on an unbounded domain, namely the full space \mathbb{R}^3 . Note that in this case, as the embedding $V \subset H$ is not compact, we cannot apply the local monotonicity framework of W. Liu and M. Röckner, [167], Theorem 5.2.2. Compactness is used heavily in the crucial step of proving that a locally monotone and hemicontinuous operator is pseudo-monotone. Thus we have to prove the claim directly. To this end, will follow the steps in [197]. Starting with stating the main definitions and some important lemmas in Section 2.1, we then move on to prove existence and uniqueness of weak solutions in Section 2.2. For sufficiently smooth data, we show that these are actually strong solutions and prove their regularity in Section 2.3. Finally, we show in Section 2.4 that as $N \rightarrow \infty$, the solution to the TMHD equations converges to a weak solution of the MHD equations.

2.1. Auxiliary Results. We define the following spaces⁶:

$$\mathbf{W}^{m,p} := \overline{C_0^\infty(\mathbb{R}^3; \mathbb{R}^3)}^{\|\cdot\|_{m,p}},$$

the closure with respect to the norm (using the so-called Bessel potentials)

$$\|\mathbf{u}\|_{m,p} := \left(\int_{\mathbb{R}^3} |(I - \Delta)^{m/2} \mathbf{u}|^p dx \right)^{1/p}.$$

⁶As we are in the case of full space as a domain, there are no boundary considerations and hence the spaces $\mathbf{W}_0^{m,p}$ and $\mathbf{W}^{m,p}$ coincide.

This norm is equivalent to the Sobolev norm given by

$$\|\mathbf{u}\|_{W^{m,p}} := \sum_{j=0}^m \|\nabla^j \mathbf{u}\|_{L^p}$$

where $\nabla^j \mathbf{u}$ denotes the j -th total weak derivative of \mathbf{u} of order j . We define the solenoidal spaces by

$$(2.1) \quad \mathbb{H}^m := \{\mathbf{u} \in \mathbf{W}^{m,2} \mid \nabla \cdot \mathbf{u} = 0\},$$

where the divergence is taken in the sense of Schwartz distributions.

To handle the velocity and the magnetic field of the MHD equations at the same time, we will need to define a norm on the space $\mathcal{H}^m := \mathbb{H}^m \times \mathbb{H}^m$. We will define the scalar products in the usual way (see [201], p. 7): for the vector field $y = (\mathbf{v}, \mathbf{B})$ define

$$(2.2) \quad \langle y_1(x), y_2(x) \rangle := \left\langle \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{B}_1 \end{pmatrix} (x), \begin{pmatrix} \mathbf{v}_2 \\ \mathbf{B}_2 \end{pmatrix} (x) \right\rangle := \langle \mathbf{v}_1(x), \mathbf{v}_2(x) \rangle + \langle \mathbf{B}_1(x), \mathbf{B}_2(x) \rangle$$

and similarly, for $y \in \mathbf{H}^m \times \mathbf{H}^m$, we set

$$(2.3) \quad (y_1, y_2)_{\mathcal{H}^m} := (\mathbf{v}_1, \mathbf{v}_2)_{\mathbf{H}^m} + (\mathbf{B}_1, \mathbf{B}_2)_{\mathbf{H}^m},$$

and accordingly for the norms. They behave just like an ℓ^2 -type product norm. In the variational formulation of the problem, we will take the scalar product w.r.t. a test function.

In a similar fashion we define Lebesgue norms by

$$\|y\|_{\mathcal{L}^p} := \left(\int_{\mathbb{R}^d} (|\mathbf{v}|^2 + |\mathbf{B}|^2)^{p/2} dx \right)^{1/p} = \| |y| \|_{L^p(\mathbb{R}^3)}$$

and

$$(2.4) \quad \|y\|_{\mathcal{L}^\infty} := \text{ess sup}_{x \in \mathbb{R}^3} (|\mathbf{v}(x)|^2 + |\mathbf{B}(x)|^2)^{1/2} = \text{ess sup}_{x \in \mathbb{R}^3} |y(x)|.$$

In the following, we will often employ the following Gagliardo-Nirenberg-Sobolev-type interpolation inequality: Let $p, q, r \geq 1$ and $0 \leq j < m$. Assume the following three conditions:

$$m - j - \frac{3}{p} \notin \mathbb{N}_0, \quad \frac{1}{r} = \frac{j}{3} + \alpha \left(\frac{1}{p} - \frac{m}{3} \right) + \frac{1-\alpha}{q}, \quad \frac{j}{m} \leq \alpha \leq 1.$$

Then for any $\mathbf{u} \in W^{m,p} \cap L^q(\mathbb{R}^3; \mathbb{R}^3)$ we have the following estimate:

$$(2.5) \quad \|\nabla^j \mathbf{u}\|_{L^r} \leq C_{m,j,p,q,r} \|\mathbf{u}\|_{m,p}^\alpha \|\mathbf{u}\|_{L^q}^{1-\alpha}.$$

Applying it to each component of the norm for $y = (\mathbf{v}, \mathbf{B})$, the same estimate carries over to yield

$$(2.6) \quad \|\nabla^j y\|_{L^r} \leq C_{m,j,p,q,r} \|y\|_{m,p}^\alpha \|y\|_{L^q}^{1-\alpha}.$$

Define the space of (solenoidal) test functions by

$$\mathcal{V} := \{y = (\mathbf{v}, \mathbf{B}) : \mathbf{v}, \mathbf{B} \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^3), \nabla \cdot \mathbf{v} = 0, \nabla \cdot \mathbf{B} = 0\} \subset C_0^\infty(\mathbb{R}^3; \mathbb{R}^6).$$

We then have the following lemma, the proof of which can be transferred verbatim as it is simply a property of the spaces \mathcal{H}^m and not of the equations.

LEMMA 2.1. *The space \mathcal{V} is dense in \mathcal{H}^m for any $m \in \mathbb{N}$.*

PROOF. See [197], Lemma 2.1. □

Let $P: L^2(\mathbb{R}^3; \mathbb{R}^3) \rightarrow \mathbb{H}^0$ be the Leray-Helmholtz projection. Then P commutes with derivative operators ([192, Lemma 2.9, p. 52]) and can be restricted to a bounded linear operator

$$P|_{H^m}: H^m \rightarrow \mathbb{H}^m.$$

Furthermore, consider the tensorised projection

$$\mathcal{P} := P \otimes P, \quad \mathcal{P}y := (P \otimes P) \begin{pmatrix} \mathbf{v} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} P\mathbf{v} \\ P\mathbf{B} \end{pmatrix}.$$

Then $\mathcal{P}: \mathcal{L}^2 \rightarrow \mathcal{H}^0$ is a bounded linear operator:

$$\begin{aligned} \|\mathcal{P}y\|_{\mathcal{H}^0}^2 &= \|P\mathbf{v}\|_{\mathbb{H}^0}^2 + \|P\mathbf{B}\|_{\mathbb{H}^0}^2 \leq \|P\|_{L^2 \rightarrow \mathbb{H}^0}^2 (\|\mathbf{v}\|_{L^2}^2 + \|\mathbf{B}\|_{L^2}^2) \\ &= \|P\|_{L^2 \rightarrow \mathbb{H}^0}^2 \|y\|_{\mathcal{L}^2}^2. \end{aligned}$$

We define the following operator for the terms on the right-hand side of the TMHD equations, projected on the space of divergence free functions:

$$\mathcal{A}(y) := \mathcal{P}\Delta y - \mathcal{P} \begin{pmatrix} (\mathbf{v} \cdot \nabla)\mathbf{v} - (\mathbf{B} \cdot \nabla)\mathbf{B} \\ (\mathbf{v} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{v} \end{pmatrix} - \mathcal{P}(g_N(|y|^2)y).$$

For $y := (\mathbf{v}, \mathbf{B})$ and a test function $\tilde{y} := (\tilde{\mathbf{v}}, \tilde{\mathbf{B}}) \in \mathcal{H}^1$, consider (using the self-adjointness of the projection \mathcal{P})

$$(2.7) \quad \begin{aligned} \langle \mathcal{A}(y), \tilde{y} \rangle_{\mathcal{H}^0} &= \langle \mathbf{v}, \Delta \tilde{\mathbf{v}} \rangle_{L^2} + \langle \mathbf{B}, \Delta \tilde{\mathbf{B}} \rangle_{L^2} - \langle (\mathbf{v} \cdot \nabla)\mathbf{v}, \tilde{\mathbf{v}} \rangle_{L^2} + \langle (\mathbf{B} \cdot \nabla)\mathbf{B}, \tilde{\mathbf{v}} \rangle_{L^2} \\ &\quad - \langle (\mathbf{v} \cdot \nabla)\mathbf{B}, \tilde{\mathbf{B}} \rangle_{L^2} + \langle (\mathbf{B} \cdot \nabla)\mathbf{v}, \tilde{\mathbf{B}} \rangle_{L^2} - g_N(|y|^2) \langle y, \tilde{y} \rangle. \end{aligned}$$

and for $\tilde{y} \in \mathcal{H}^3$

$$(2.8) \quad \begin{aligned} \langle \mathcal{A}(y), \tilde{y} \rangle_{\mathcal{H}^1} &= \langle \mathcal{A}(y), (I - \Delta)\tilde{y} \rangle_0 \\ &= -\langle \nabla \mathbf{v}, (I - \Delta)\nabla \tilde{\mathbf{v}} \rangle_{L^2} - \langle \nabla \mathbf{B}, \nabla(I - \Delta)\tilde{\mathbf{B}} \rangle_{L^2} \\ &\quad - \langle (\mathbf{v} \cdot \nabla)\mathbf{v}, (I - \Delta)\tilde{\mathbf{v}} \rangle_{L^2} + \langle (\mathbf{B} \cdot \nabla)\mathbf{B}, (I - \Delta)\tilde{\mathbf{v}} \rangle_{L^2} \\ &\quad - \langle (\mathbf{v} \cdot \nabla)\mathbf{B}, (I - \Delta)\tilde{\mathbf{B}} \rangle_{L^2} + \langle (\mathbf{B} \cdot \nabla)\mathbf{v}, (I - \Delta)\tilde{\mathbf{B}} \rangle_{L^2} \\ &\quad - \langle g_N(|y|^2)y, (I - \Delta)\tilde{y} \rangle_{\mathcal{L}^2}. \end{aligned}$$

Let us give names to the linear, nonlinear and taming terms of (2.8):

$$\begin{aligned} \mathcal{A}_1(y, \tilde{y}) &:= -\langle \nabla \mathbf{v}, (I - \Delta)\nabla \tilde{\mathbf{v}} \rangle_{L^2} - \langle \nabla \mathbf{B}, \nabla(I - \Delta)\tilde{\mathbf{B}} \rangle_{L^2}, \\ \mathcal{A}_2(y, \tilde{y}) &:= -\langle (\mathbf{v} \cdot \nabla)\mathbf{v}, (I - \Delta)\tilde{\mathbf{v}} \rangle_{L^2} + \langle (\mathbf{B} \cdot \nabla)\mathbf{B}, (I - \Delta)\tilde{\mathbf{v}} \rangle_{L^2} \\ &\quad - \langle (\mathbf{v} \cdot \nabla)\mathbf{B}, (I - \Delta)\tilde{\mathbf{B}} \rangle_{L^2} + \langle (\mathbf{B} \cdot \nabla)\mathbf{v}, (I - \Delta)\tilde{\mathbf{B}} \rangle_{L^2}, \\ \mathcal{A}_3(y, \tilde{y}) &:= -\langle g_N(|y|^2)y, (I - \Delta)\tilde{y} \rangle_{\mathcal{L}^2}. \end{aligned}$$

The following lemma provides elementary estimates on the terms defined above.

LEMMA 2.2.

(i) For any $y \in \mathcal{H}^1$ and $\tilde{y} \in \mathcal{V}$,

$$|\langle \mathcal{A}(y), \tilde{y} \rangle_{\mathcal{H}^1}| \leq C(1 + \|y\|_{\mathcal{H}^1}^3) \|\tilde{y}\|_{\mathcal{H}^3},$$

i.e. $\langle \mathcal{A}(y), \cdot \rangle_{\mathcal{H}^1}$ can be considered as an element in the dual space $(\mathcal{H}^3)'$ with its norm bounded by $C(1 + \|y\|_{\mathcal{H}^1}^3)$.

(ii) If $y \in \mathcal{H}^1$, then

$$(2.9) \quad \langle \mathcal{A}(y), y \rangle_{\mathcal{H}^0} = -\|\nabla y\|_{\mathcal{H}^0}^2 - \langle g_N(|y|^2)y, y \rangle_{\mathcal{L}^2}.$$

(iii) If $y \in \mathcal{H}^2$, then

$$(2.10) \quad \begin{aligned} \langle \mathcal{A}(y), y \rangle_{\mathcal{H}^1} &\leq -\frac{1}{2} \|y\|_{\mathcal{H}^2}^2 + \|y\|_{\mathcal{H}^0}^2 + 2(N+1) \|\nabla y\|_{\mathcal{H}^0}^2 \\ &\quad - \|\mathbf{v}\| \|\nabla \mathbf{v}\|_{L^2}^2 - \|\mathbf{B}\| \|\nabla \mathbf{B}\|_{L^2}^2 \\ &\quad - \|\mathbf{v}\| \|\nabla \mathbf{B}\|_{L^2}^2 - \|\mathbf{B}\| \|\nabla \mathbf{v}\|_{L^2}^2. \end{aligned}$$

REMARK 2.3. The estimate (2.10) of this lemma lies at the heart of the improved estimates of the tamed equations compared to the untamed equations. Young's inequality allows us to create a minus sign in front of the first two terms on the right-hand side and hence to bring it to the other side of the inequality. The taming term produces the minus signs in the terms of the second and third lines. Thus the otherwise uncontrollable nonlinear term is transformed into a nonpositive term that can be estimated from above by zero and hence made to disappear in subsequent estimates.

PROOF OF LEMMA 2.2. Throughout this proof, let $\varphi, \psi, \theta \in \{\mathbf{v}, \mathbf{B}\}$.

To prove (i), we observe that by Cauchy-Schwarz-Buniakowski for

$$\langle (I - \Delta)^{1/2} \varphi, (I - \Delta)^{1/2} \Delta \psi \rangle \leq C \|\varphi\|_{\mathbb{H}^1} \|\psi\|_{\mathbb{H}^3}$$

and hence

$$\mathcal{A}_1(y, \tilde{y}) \leq C \|y\|_{\mathcal{H}^1} \|\tilde{y}\|_{\mathcal{H}^3}.$$

Similarly, using also the Sobolev embedding theorem we find

$$\begin{aligned} \langle \varphi \otimes \psi, \nabla(I - \Delta)\theta \rangle_{L^2} &\leq \|\varphi \otimes \psi\|_{L^2} \|\nabla(I - \Delta)\theta\|_{\mathbb{H}^0} \\ &\leq C \|\varphi\|_{L^4} \|\psi\|_{L^4} \|\theta\|_{\mathbb{H}^3} \leq C \|\varphi\|_{\mathbb{H}^1} \|\psi\|_{\mathbb{H}^1} \|\theta\|_{\mathbb{H}^3}, \end{aligned}$$

which yields

$$\mathcal{A}_2(y, \tilde{y}) \leq C \|y\|_{\mathcal{H}^1}^2 \|\tilde{y}\|_{\mathcal{H}^3}.$$

For the taming term we use the estimate $g_N(r) \leq Cr$ and the embedding of H^1 into L^6 to find

$$\begin{aligned} \mathcal{A}_3(y, \tilde{y}) &\leq |\langle g_N(|y|^2)y, (I - \Delta)\tilde{y} \rangle_{L^2}| \leq \|g_N(|y|^2)y\|_{L^2} \|(I - \Delta)\tilde{y}\|_{L^2} \\ &\leq C \|y\|_{L^6}^3 \|\tilde{y}\|_{\mathcal{H}^2} \leq C \|y\|_{\mathcal{H}^1}^3 \|\tilde{y}\|_{\mathcal{H}^3}, \end{aligned}$$

to get

$$|\langle \mathcal{A}(y), \tilde{y} \rangle_{\mathcal{H}^0}| \leq C (\|y\|_{\mathcal{H}^1} + \|y\|_{\mathcal{H}^1}^2 + \|y\|_{\mathcal{H}^1}^3) \|\tilde{y}\|_{\mathcal{H}^3},$$

which implies the assertion.

For equality (2.9), we note that by the zero divergence conditions on \mathbf{v} and \mathbf{B} ,

$$\langle (\varphi \cdot \nabla) \psi, \psi \rangle_{L^2} = 0.$$

Thus the first and the third nonlinear term will drop out. The remaining two terms cancel since we have (for the same reason) the symmetry condition

$$\langle (\varphi \cdot \nabla) \psi, \theta \rangle_{L^2} = -\langle (\varphi \cdot \nabla) \theta, \psi \rangle_{L^2}.$$

Let us proceed to prove the inequality (2.10). Again, we analyse the linear, nonlinear and taming terms separately. First we find the equality

$$\begin{aligned} \mathcal{A}_1(y, y) &= \langle (\Delta - I + I)\mathbf{v}, (I - \Delta)\mathbf{v} \rangle_{L^2} + \langle (\Delta - I + I)\mathbf{B}, (I - \Delta)\mathbf{B} \rangle_{L^2} \\ &= -\|\mathbf{v}\|_{\mathbb{H}^2}^2 - \|\mathbf{B}\|_{\mathbb{H}^2}^2 + \|\nabla \mathbf{v}\|_{\mathbb{H}^0}^2 + \|\nabla \mathbf{B}\|_{\mathbb{H}^0}^2 \\ &\quad + \|\mathbf{v}\|_{\mathbb{H}^0}^2 + \|\mathbf{B}\|_{\mathbb{H}^0}^2. \end{aligned}$$

The nonlinear terms can be estimated by the Cauchy-Schwarz-Buniakowski and Young inequality:

$$\begin{aligned}
\mathcal{A}_2(y, y) &= -\langle (\mathbf{v} \cdot \nabla) \mathbf{v}, (I - \Delta) \mathbf{v} \rangle_{L^2} + \langle (\mathbf{B} \cdot \nabla) \mathbf{B}, (I - \Delta) \mathbf{v} \rangle_{L^2} \\
&\quad - \langle (\mathbf{v} \cdot \nabla) \mathbf{B}, (I - \Delta) \mathbf{B} \rangle_{L^2} + \langle (\mathbf{B} \cdot \nabla) \mathbf{v}, (I - \Delta) \mathbf{B} \rangle_{L^2} \\
&\leq \|(\mathbf{v} \cdot \nabla) \mathbf{v}\|_{L^2}^2 + \frac{1}{4} \|\mathbf{v}\|_{\mathbb{H}^2}^2 + \|(\mathbf{B} \cdot \nabla) \mathbf{B}\|_{L^2}^2 + \frac{1}{4} \|\mathbf{v}\|_{\mathbb{H}^2}^2 \\
&\quad \|(\mathbf{v} \cdot \nabla) \mathbf{B}\|_{L^2}^2 + \frac{1}{4} \|\mathbf{B}\|_{\mathbb{H}^2}^2 + \|(\mathbf{B} \cdot \nabla) \mathbf{v}\|_{L^2}^2 + \frac{1}{4} \|\mathbf{B}\|_{\mathbb{H}^2}^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{A}_1(y, y) + \mathcal{A}_2(y, y) &\leq -\frac{1}{2} \|\mathbf{v}\|_{\mathbb{H}^2}^2 - \frac{1}{2} \|\mathbf{B}\|_{\mathbb{H}^2}^2 + \|\nabla \mathbf{v}\|_{\mathbb{H}^1}^2 + \|\nabla \mathbf{B}\|_{\mathbb{H}^1}^2 \\
&\quad + \|\mathbf{v}\|_{\mathbb{H}^0}^2 + \|\mathbf{B}\|_{\mathbb{H}^0}^2 \\
&\quad + \|(\mathbf{v} \cdot \nabla) \mathbf{v}\|_{L^2}^2 + \|(\mathbf{B} \cdot \nabla) \mathbf{B}\|_{L^2}^2 \\
&\quad + \|(\mathbf{v} \cdot \nabla) \mathbf{B}\|_{L^2}^2 + \|(\mathbf{B} \cdot \nabla) \mathbf{v}\|_{L^2}^2.
\end{aligned}$$

The taming terms are estimated using integration by parts, the product rule and $g_N(r) \geq C(r - N)$:

$$\begin{aligned}
\mathcal{A}_3(y, y) &= -\langle g_N(|y|^2) y, (I - \Delta) y \rangle_{L^2} \\
&= -\langle g_N(|y|^2) y, y \rangle_{L^2} - \langle \nabla (g_N(|y|^2) y), \nabla y \rangle_{L^2} \\
&\leq -\int_{\mathbb{R}^3} \sum_{i,k=1}^3 \partial_i v^k \partial_i (g_N(|y|^2) v^k) - S \partial_i B^k \partial_i (g_N(|y|^2) B^k) dx \\
&= -\int_{\mathbb{R}^3} \sum_{i,k=1}^3 (\partial_i v^k)^2 g_N(|y|^2) + g'_N(|y|^2) (\partial_i |y|^2) v^k \partial_i v^k dx \\
&\quad - \int_{\mathbb{R}^3} \sum_{i,k=1}^3 (\partial_i B^k)^2 g_N(|y|^2) + g'_N(|y|^2) (\partial_i |y|^2) B^k \partial_i B^k dx \\
&= -\int_{\mathbb{R}^3} g_N(|y|^2) |\nabla y|^2 dx - \int_{\mathbb{R}^3} g'_N(|y|^2) \sum_{i,j,k} \left\{ \partial_i ((v^j)^2 + (B^j)^2) v^k \partial_i v^k \right. \\
&\quad \left. + \partial_i ((v^j)^2 + (B^j)^2) B^k \partial_i B^k \right\} dx \\
&= -\int_{\mathbb{R}^3} g_N(|y|^2) |\nabla y|^2 dx \\
&\quad - 2 \int_{\mathbb{R}^3} g'_N(|y|^2) \sum_{i,j,k} \left\{ v^j \partial_i v^j v^k \partial_i v^k + B^j \partial_i B^j v^k \partial_i v^k \right. \\
&\quad \left. + v^j \partial_i v^j B^k \partial_i B^k + B^j \partial_i B^j B^k \partial_i B^k \right\} dx \\
&= -\int_{\mathbb{R}^3} g_N(|y|^2) |\nabla y|^2 dx \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^3} g'_N(|y|^2) (\langle \nabla |\mathbf{v}|, \nabla |\mathbf{v}| \rangle + 2 \langle \nabla |\mathbf{v}|, \nabla |\mathbf{B}| \rangle + \langle \nabla |\mathbf{B}|, \nabla |\mathbf{B}| \rangle) dx
\end{aligned}$$

$$\begin{aligned}
&= - \int_{\mathbb{R}^3} g_N(|y|^2) |\nabla y|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} g'_N(|y|^2) |\nabla |y|^2| dx \\
&\leq - \int_{\mathbb{R}^3} g_N(|y|^2) |\nabla y|^2 dx,
\end{aligned}$$

as the last term in the penultimate equation is non-negative. Finally, since $g_N(|y|^2) \geq C_{\text{taming}}(|y|^2 - (N + \frac{1}{2}))$ by definition, we get that

$$\begin{aligned}
\mathcal{A}_3(y, y) &\leq -C_{\text{taming}} \int_{\mathbb{R}^3} |y|^2 |\nabla y|^2 dx + C_{\text{taming}} (N + \frac{1}{2}) \|\nabla y\|_{\mathcal{H}^0}^2 \\
&= -C_{\text{taming}} (\|\mathbf{v}\|_{L^2}^2 + \|\mathbf{v}\|_{L^2} \|\nabla \mathbf{B}\|_{L^2}^2 \\
&\quad + \|\mathbf{B}\|_{L^2} \|\nabla \mathbf{v}\|_{L^2}^2 + \|\mathbf{B}\|_{L^2} \|\nabla \mathbf{B}\|_{L^2}^2) + C_{\text{taming}} (N + \frac{1}{2}) \|\nabla y\|_{\mathcal{H}^0}^2.
\end{aligned}$$

Since $C_{\text{taming}} = 2$, we get (2.10) by combining the above three estimates. \square

LEMMA 2.4. *Let $y_n, \tilde{y} \in \mathcal{V}$ and $y \in \mathcal{H}^1$. Let $\Omega := \text{supp}(\tilde{y})$ and assume that*

$$\sup_n \|y_n\|_{\mathcal{H}^1} < \infty, \quad \lim_{n \rightarrow \infty} \|(y_n - y)1_\Omega\|_{\mathcal{L}^2} = 0.$$

Then

$$\lim_{n \rightarrow \infty} \langle \mathcal{A}(y_n), \tilde{y} \rangle_{\mathcal{H}^1} = \langle \mathcal{A}(y), \tilde{y} \rangle_{\mathcal{H}^1}.$$

PROOF. For the linear part we get

$$\begin{aligned}
\lim_{n \rightarrow \infty} |\mathcal{A}_1(y_n, \tilde{y}) - \mathcal{A}_1(y, \tilde{y})| &= \lim_n |\langle y_n - y, \Delta(I - \Delta)\tilde{y} \rangle_{\mathcal{L}^2}| \\
&= \lim_n |\langle (y_n - y)1_\Omega, \Delta(I - \Delta)\tilde{y} \rangle_{\mathcal{L}^2}| \leq C \lim_n \|(y_n - y)1_\Omega\|_{\mathcal{L}^2} \|\tilde{y}\|_{\mathcal{H}^4} = 0.
\end{aligned}$$

The nonlinear part is treated using the trick

$$\nabla \cdot (\phi \otimes \psi) = (\phi \cdot \nabla) \psi.$$

Thus, for example

$$\begin{aligned}
&\lim_{n \rightarrow \infty} |\langle (\mathbf{v}_n \cdot \nabla) \mathbf{B}_n, (I - \Delta) \tilde{\mathbf{B}} \rangle_{L^2} - \langle (\mathbf{v} \cdot \nabla) \mathbf{B}, (I - \Delta) \tilde{\mathbf{B}} \rangle_{L^2}| \\
&= \lim_n |\langle \mathbf{v}_n \otimes \mathbf{B}_n - \mathbf{v} \otimes \mathbf{B}, \nabla(I - \Delta) \tilde{\mathbf{B}} \rangle_{L^2}| \\
&= \lim_n |\langle (\mathbf{v}_n \otimes \mathbf{B}_n - \mathbf{v} \otimes \mathbf{B}) 1_\Omega, \nabla(I - \Delta) \tilde{\mathbf{B}} \rangle_{L^2}| \\
&= 0.
\end{aligned}$$

The other three terms are shown to converge in just the same way.

Using the estimate $|g'_N| \leq C$ as well as the Sobolev embedding theorem, we find for the taming term

$$\begin{aligned}
\lim_n |\mathcal{A}_3(y_n, \tilde{y}) - \mathcal{A}_3(y, \tilde{y})| &= \lim_n |\langle g_N(|y_n|^2)y_n - g_N(|y|^2)y, (I - \Delta)\tilde{y} \rangle| \\
&\leq \lim_n |\langle g_N(|y_n|^2)(y_n - y)1_\Omega, (I - \Delta)\tilde{y} \rangle| \\
&\quad + |\langle g'_N(\theta)(|y_n|^2 - |y|^2)y1_\Omega, (I - \Delta)\tilde{y} \rangle| \\
&\leq C \sup_{x \in \mathbb{R}^3} |(I - \Delta)\tilde{y}(x)| \lim_n \left\{ \|y_n\|_{\mathcal{L}^4}^2 \|(y_n - y)1_\Omega\|_{\mathcal{L}^2} \right. \\
&\quad \left. + \|(y_n - y)1_\Omega\|_{\mathcal{L}^2} (\|y_n\|_{\mathcal{L}^4}^2 + \|y\|_{\mathcal{L}^4}^2) \right\} \\
&\leq C \sup_{x \in \mathbb{R}^3} |(I - \Delta)\tilde{y}(x)| \lim_n \left\{ \|y_n\|_{\mathcal{H}^1}^2 \|(y_n - y)1_\Omega\|_{\mathcal{L}^2} \right. \\
&\quad \left. + \|(y_n - y)1_\Omega\|_{\mathcal{L}^2} (\|y_n\|_{\mathcal{H}^1}^2 + \|y\|_{\mathcal{H}^1}^2) \right\} = 0.
\end{aligned}$$

□

2.2. Existence and Uniqueness of Weak Solutions. In this section we will study the well-posedness of the weak formulation of the TMHD equations. We start by stating our notion of weak solution. We proceed to show uniqueness first and then existence of weak solutions via a Faedo-Galerkin approximation scheme.

DEFINITION 2.5 (Weak solution). *Let $y_0 \in \mathcal{H}^0$, $f \in L^2_{loc}(\mathbb{R}_+; \mathcal{H}^0)$. Let $y = \begin{pmatrix} \mathbf{v} \\ \mathbf{B} \end{pmatrix}$ where \mathbf{v} and \mathbf{B} are measurable vector fields, $\mathbf{v}, \mathbf{B}: \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$. We call y a weak solution of the tamed MHD equations if*

- (i) $\mathbf{v}, \mathbf{B} \in L^\infty_{loc}(\mathbb{R}_+; L^4(\mathbb{R}^3; \mathbb{R}^3)) \cap L^2_{loc}(\mathbb{R}_+; \mathbb{H}^1)$.
- (ii) For all $\tilde{y} \in \mathcal{V}$ and $t \geq 0$,

$$\begin{aligned}
\langle y(t), \tilde{y} \rangle_{\mathcal{H}^0} &= \langle y_0, \tilde{y} \rangle_{\mathcal{H}^0} - \int_0^t \langle \nabla y, \nabla \tilde{y} \rangle_{\mathcal{H}^0} ds \\
&\quad - \int_0^t \langle (\mathbf{v} \cdot \nabla) \mathbf{v}, \tilde{\mathbf{v}} \rangle_{L^2} ds + \int_0^t \langle (\mathbf{B} \cdot \nabla) \mathbf{B}, \tilde{\mathbf{v}} \rangle_{L^2} ds \\
&\quad - \int_0^t \langle (\mathbf{v} \cdot \nabla) \mathbf{B}, \tilde{\mathbf{B}} \rangle_{L^2} ds + \int_0^t \langle (\mathbf{B} \cdot \nabla) \mathbf{v}, \tilde{\mathbf{B}} \rangle_{L^2} ds \\
&\quad - \int_0^t \langle g_N(|y|^2)y, \tilde{y} \rangle_{L^2} ds + \int_0^t \langle f, \tilde{y} \rangle_{\mathcal{H}^0} ds.
\end{aligned} \tag{2.11}$$

- (iii) $\lim_{t \downarrow 0} \|y(t) - y_0\|_{\mathcal{L}^2} = 0$.

This definition deals with purely spatial test functions, but it can be extended to the case of test functions that depend also on time, as the next proposition demonstrates.

PROPOSITION 2.6. Let $y = \begin{pmatrix} \mathbf{v} \\ \mathbf{B} \end{pmatrix}$ be a weak solution and let $T > 0$. Then for all $\tilde{y} \in C^1([0, T]; \mathcal{H}^1)$ such that $\tilde{y}(T) = 0$, we have

$$(2.12) \quad \begin{aligned} \int_0^T \langle y, \partial_t \tilde{y} \rangle_{\mathcal{H}^0} dt &= -\langle y_0, \tilde{y}(0) \rangle_{\mathcal{H}^0} + \int_0^t \langle \nabla y, \nabla \tilde{y} \rangle_{\mathcal{H}^0} ds \\ &+ \int_0^t \langle (\mathbf{v} \cdot \nabla) \mathbf{v}, \tilde{\mathbf{v}} \rangle_{L^2} ds - \int_0^t \langle (\mathbf{B} \cdot \nabla) \mathbf{B}, \tilde{\mathbf{v}} \rangle_{L^2} ds \\ &+ \int_0^t \langle (\mathbf{v} \cdot \nabla) \mathbf{B}, \tilde{\mathbf{B}} \rangle_{L^2} ds - \int_0^t \langle (\mathbf{B} \cdot \nabla) \mathbf{v}, \tilde{\mathbf{B}} \rangle_{L^2} ds \\ &+ \int_0^t \langle g_N(|y|^2) y, \tilde{y} \rangle_{L^2} ds - \int_0^t \langle f, \tilde{y} \rangle_{\mathcal{H}^0} ds. \end{aligned}$$

Moreover, the following energy equality holds:

$$(2.13) \quad \begin{aligned} \|y(t)\|_{\mathcal{H}^0}^2 + 2 \int_0^t \|\nabla y\|_{\mathcal{H}^0}^2 ds + 2 \int_0^t \|\sqrt{g_N(|y|^2)} |y|\|_{L^2}^2 ds \\ = \|y_0\|_{\mathcal{H}^0}^2 + 2 \int_0^t \langle f, y \rangle_{\mathcal{H}^0} ds, \quad \forall t \geq 0. \end{aligned}$$

PROOF. We first show that the right-hand side of (2.12) is well-defined.

Starting with the nonlinear terms, by applying Hölder's inequality for $(p, q) = (4, \frac{4}{3})$ and $(p, q) = (3, \frac{3}{2})$ as well as the Sobolev embedding theorem, we get

$$\begin{aligned} \int_0^T |\langle (\mathbf{v} \cdot \nabla) \mathbf{v}, \tilde{\mathbf{v}} \rangle_{L^2}| dt &\leq \int_0^T \|\tilde{\mathbf{v}}\|_{L^4} \|(\mathbf{v} \cdot \nabla) \mathbf{v}\|_{L^{4/3}} dt \\ &\leq \int_0^T \|\tilde{\mathbf{v}}\|_{L^4} \|\mathbf{v}\| \|\nabla \mathbf{v}\|_{L^{4/3}} dt \\ &\leq \int_0^T \|\tilde{\mathbf{v}}\|_{L^4} \|\mathbf{v}\|_{L^4} \|\nabla \mathbf{v}\|_{L^2} dt \\ &\leq C \sup_{t \in [0, T]} \|\mathbf{v}\|_{L^4} \left(\int_0^T \|\tilde{\mathbf{v}}\|_{H^1} dt \right)^{1/2} \left(\int_0^T \|\mathbf{v}\|_{H^1} dt \right)^{1/2}. \end{aligned}$$

The other three nonlinear terms can be handled in exactly the same way.

For the taming terms, we get, by applying Hölder's inequality (to the spatial integration) and the Sobolev embedding theorem:

$$\begin{aligned} \int_0^T |\langle g_N(|y|^2) y, \tilde{y} \rangle_{L^2}| dt &\leq \int_0^T \int_{\mathbb{R}^3} |g_N(|y|^2)| |\langle y, \tilde{y} \rangle| dx dt \\ &\leq C \int_0^T \int_{\mathbb{R}^3} |y|^3 |\tilde{y}| dx dt \\ &\leq C \int_0^T \|y\|_{\mathcal{L}^4}^3 \|\tilde{y}\|_{\mathcal{L}^4} ds \\ &\leq C \sup_{0 \leq t \leq T} \|y(t)\|_{\mathcal{L}^4}^3 \int_0^T \|\tilde{y}\|_{\mathcal{H}^1} dt. \end{aligned}$$

Thus, the right-hand side of (2.12) is well-defined and if we use the definition of a weak solution, (2.11) and (2.12), the energy equality follows from approximating the solution accordingly, cf. [18, Lemma 2.7, p. 635]. \square

THEOREM 2.7 (Uniqueness of weak solutions). *Let y_1, y_2 be two weak solutions in the sense of Definition 2.5. Then we have $y_1 = y_2$.*

PROOF. We fix a $T > 0$ and set $z(t) := y_1(t) - y_2(t) =: \begin{pmatrix} \mathbf{v} \\ \mathbf{B} \end{pmatrix}$. Then z satisfies the equation

$$\begin{aligned} \langle z(t), \tilde{y} \rangle_{\mathcal{H}^0} &= - \int_0^t \langle \nabla z, \nabla \tilde{y} \rangle_{\mathcal{H}^0} ds - \int_0^t \langle (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_1 - (\mathbf{v}_2 \cdot \nabla) \mathbf{v}_2, \tilde{\mathbf{v}} \rangle_{L^2} ds \\ &\quad + \int_0^t \langle (\mathbf{B}_1 \cdot \nabla) \mathbf{B}_1 - (\mathbf{B}_2 \cdot \nabla) \mathbf{B}_2, \tilde{\mathbf{v}} \rangle_{L^2} ds \\ &\quad - \int_0^t \langle (\mathbf{v}_1 \cdot \nabla) \mathbf{B}_1 - (\mathbf{v}_2 \cdot \nabla) \mathbf{B}_2, \tilde{\mathbf{B}} \rangle_{L^2} ds \\ &\quad + \int_0^t \langle (\mathbf{B}_1 \cdot \nabla) \mathbf{v}_1 - (\mathbf{B}_2 \cdot \nabla) \mathbf{v}_2, \tilde{\mathbf{B}} \rangle_{L^2} ds \\ &\quad - \int_0^t \langle g_N(|y_1|^2) y_1 - g_N(|y_2|^2) y_2, \tilde{y} \rangle_{L^2} ds. \end{aligned}$$

Now, running the same proof as for (2.13) (which essentially amounts to being able to set $\tilde{y} = z$), we get

$$\begin{aligned} \|z(t)\|_{\mathcal{H}^0}^2 &= -2 \int_0^t \|\nabla z\|_{\mathcal{H}^0}^2 ds - 2 \int_0^t \langle \mathbf{v}, (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_1 - (\mathbf{v}_2 \cdot \nabla) \mathbf{v}_2 \rangle_{L^2} ds \\ &\quad + 2 \int_0^t \langle \mathbf{v}, (\mathbf{B}_1 \cdot \nabla) \mathbf{B}_1 - (\mathbf{B}_2 \cdot \nabla) \mathbf{B}_2 \rangle_{L^2} ds \\ &\quad - 2 \int_0^t \langle \mathbf{B}, (\mathbf{v}_1 \cdot \nabla) \mathbf{B}_1 - (\mathbf{v}_2 \cdot \nabla) \mathbf{B}_2 \rangle_{L^2} ds \\ &\quad + 2 \int_0^t \langle \mathbf{B}, (\mathbf{B}_1 \cdot \nabla) \mathbf{v}_1 - (\mathbf{B}_2 \cdot \nabla) \mathbf{v}_2 \rangle_{L^2} ds \\ &\quad - 2 \int_0^t \langle z, g_N(|y_1|^2) y_1 - g_N(|y_2|^2) y_2 \rangle_{L^2} ds \\ &=: I_L(t) + I_{NL}(t) + I_T(t). \end{aligned}$$

We first investigate the linear term and find, using integration by parts and the definition of the norms $\|\cdot\|_{\mathcal{H}^1}$ that

$$\begin{aligned} I_L(t) &= -2 \int_0^t \langle \nabla z, \nabla z \rangle_{\mathcal{H}^0} ds = 2 \int_0^t \langle \Delta z, z \rangle_{\mathcal{H}^0} ds \\ &= -2 \int_0^t \|z\|_{\mathcal{H}^1}^2 ds - 2 \int_0^t \|z\|_{\mathcal{H}^0}^2 ds. \end{aligned}$$

The nonlinear term $I_{NL}(t)$ consists of four terms of the same structure

$$\langle \varphi, (\boldsymbol{\psi}_1 \cdot \nabla) \boldsymbol{\theta}_1 - (\boldsymbol{\psi}_2 \cdot \nabla) \boldsymbol{\theta}_2 \rangle_{L^2} = - \langle \nabla \varphi, \boldsymbol{\psi}_1 \otimes \boldsymbol{\theta}_1 - \boldsymbol{\psi}_2 \otimes \boldsymbol{\theta}_2 \rangle_{L^2}.$$

We estimate them by using Cauchy-Schwarz-Buniakowski and Young's inequality:

$$\langle \nabla \varphi, \boldsymbol{\psi}_1 \otimes \boldsymbol{\theta}_1 - \boldsymbol{\psi}_2 \otimes \boldsymbol{\theta}_2 \rangle_{L^2} \leq \frac{1}{4} \|\nabla \varphi\|_{\mathbb{H}^0}^2 + \|\boldsymbol{\psi}_1 \otimes \boldsymbol{\theta}_1 - \boldsymbol{\psi}_2 \otimes \boldsymbol{\theta}_2\|_{L^2}^2.$$

Thus we need to estimate the latter term which we do by applying Cauchy-Schwarz-Buniakowski, the Sobolev embedding theorem as well as Young's inequality with ε for $(p, q) = (4, 4/3)$:

$$\begin{aligned}
\|\boldsymbol{\psi}_1 \otimes \boldsymbol{\theta}_1 - \boldsymbol{\psi}_2 \otimes \boldsymbol{\theta}_2\|_{L^2}^2 &= \|(\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2) \otimes \boldsymbol{\theta}_1 + \boldsymbol{\psi}_2 \otimes (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2)\|_{L^2}^2 \\
&\leq 2 (\|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_{L^4}^2 \|\boldsymbol{\theta}_1\|_{L^4}^2 + \|\boldsymbol{\psi}_2\|_{L^4}^2 \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_{L^4}^2) \\
&\leq 2C_{1,0,2,2,4} (\|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_{\mathbb{H}^1}^{3/2} \|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_{\mathbb{H}^0}^{1/2} \|\boldsymbol{\theta}_1\|_{L^4}^2 \\
&\quad + \|\boldsymbol{\psi}_2\|_{L^4}^2 \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_{\mathbb{H}^1}^{3/2} \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_{\mathbb{H}^0}^{1/2}) \\
&\leq C_\varepsilon (\|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_{\mathbb{H}^0}^2 \|\boldsymbol{\theta}_1\|_{L^4}^8 + \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_{\mathbb{H}^0}^2 \|\boldsymbol{\psi}_2\|_{L^4}^8) \\
&\quad + \varepsilon \|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_{\mathbb{H}^1}^2 + \varepsilon \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_{\mathbb{H}^1}^2.
\end{aligned}$$

We collect the four terms and use the previous estimates and find (again using the definition of the Sobolev norms)

$$\begin{aligned}
I_{NL}(t) &\leq \int_0^t \|\nabla(y_1 - y_2)\|_{\mathcal{H}^0}^2 ds + 8\varepsilon \int_0^t \|y_1 - y_2\|_{\mathcal{H}^1}^2 ds \\
&\quad + C_\varepsilon \int_0^t (\|y_1 - y_2\|_{\mathcal{H}^0}^2 \|y_1\|_{L^4}^8 + \|y_1 - y_2\|_{\mathcal{H}^0}^2 \|y_2\|_{L^4}^8) ds \\
&\leq \int_0^t \|z\|_{\mathcal{H}^1}^2 ds + 8\varepsilon \int_0^t \|z\|_{\mathcal{H}^1}^2 ds \\
&\quad + C_\varepsilon \int_0^t \|z\|_{\mathcal{H}^0}^2 (1 + \|y_1\|_{L^4}^8 + \|y_2\|_{L^4}^8) ds \\
&\leq (1 + 8\varepsilon) \int_0^t \|z\|_{\mathcal{H}^1}^2 ds + C_\varepsilon M_{y_1, y_2, t} \int_0^t \|z\|_{\mathcal{H}^0}^2 ds,
\end{aligned}$$

where

$$M_{y_1, y_2, t} := \text{ess sup}_{s \in [0, t]} (1 + \|y_1\|_{L^4}^8 + \|y_2\|_{L^4}^8) < \infty$$

by our definition of weak solutions.

Concerning the taming term, $I_T(t)$, we have, using Cauchy-Schwarz-Buniakowski for the scalar product $\langle \cdot, \cdot \rangle$, the mean-value theorem of calculus (and the fact that $|g'| \leq C$), the inequality $g_N(r) \leq Cr$, as well as Cauchy-Schwarz-Buniakowski for the 3D integral, Sobolev's embedding theorem and Young's inequality (for $(p, q) = (4, 4/3)$)

$$\begin{aligned}
&|\langle z, g_N(|y_1|^2)y_1 - g_N(|y_2|^2)y_2 \rangle_{L^2}| \\
&= |\langle z, g_N(|y_1|^2)z \rangle_{L^2} + \langle z, (g_N(|y_1|^2) - g_N(|y_2|^2))y_2 \rangle_{L^2}| \\
&\leq \int_{\mathbb{R}^3} |z|^2 g_N(|y_1|^2) dx + \int_{\mathbb{R}^3} |z| |g_N(|y_1|^2) - g_N(|y_2|^2)| |y_2| dx \\
&\leq C \int_{\mathbb{R}^3} |z|^2 |y_1|^2 + |z| ||y_1|^2 - |y_2|^2| |y_2| dx \\
&\leq C \int_{\mathbb{R}^3} |z|^2 |y_1|^2 + |z|^2 (|y_1| + |y_2|) |y_2| dx \\
&= C \int_{\mathbb{R}^3} |z|^2 (|y_1|^2 + |y_1||y_2| + |y_2|^2) dx \\
&\leq C \int_{\mathbb{R}^3} |z|^2 (|y_1| + |y_2|)^2 dx \\
&\leq C \| |z| (|y_1| + |y_2|) \|_{L^2}^2
\end{aligned}$$

$$\begin{aligned}
&\leq C \|z\|_{\mathcal{L}^4}^2 \| |y_1| + |y_2| \|_{L^4}^2 \\
&\leq C \|z\|_{\mathcal{H}^1}^{3/2} \|z\|_{\mathcal{H}^0}^{1/2} \| |y_1| + |y_2| \|_{L^4}^2 \\
&\leq C_\varepsilon \|z\|_{\mathcal{H}^0}^2 (\|y_1\|_{\mathcal{L}^4}^8 + \|y_2\|_{\mathcal{L}^4}^8) + \varepsilon \|z\|_{\mathcal{H}^1}^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
I_T(t) &= -2 \int_0^t \langle z, g_N(|y_1|^2)y_1 - g_N(|y_2|^2)y_2 \rangle_{\mathcal{L}^2} ds \\
&\leq 2\varepsilon \int_0^t \|z\|_{\mathcal{H}^1}^2 ds + C_\varepsilon \int_0^t \|y\|_{\mathcal{H}^0}^2 (\|y_1\|_{\mathcal{L}^4}^8 + \|y_2\|_{\mathcal{L}^4}^8) ds \\
&\leq 2\varepsilon \int_0^t \|z\|_{\mathcal{H}^1}^2 ds + C_\varepsilon M_{y_1, y_2, t} \int_0^t \|z\|_{\mathcal{H}^0}^2 ds.
\end{aligned}$$

Hence, altogether we have the inequality

$$\begin{aligned}
\|z(t)\|_{\mathcal{H}^0}^2 &\leq -2 \int_0^t \|z(s)\|_{\mathcal{H}^1}^2 ds + (1 + 10\varepsilon) \int_0^t \|z(s)\|_{\mathcal{H}^1}^2 ds \\
&\quad + C_\varepsilon M_{y_1, y_2, t} \int_0^t \|z(s)\|_{\mathcal{H}^0}^2 ds.
\end{aligned}$$

Choosing $\varepsilon = \frac{1}{10}$, we find that

$$\|z(t)\|_{\mathcal{H}^0}^2 \leq C M_{y_1, y_2, t} \int_0^t \|z(s)\|_{\mathcal{H}^0}^2 ds$$

and Gronwall's lemma implies that $z(s) = 0$ for all $s \in [0, t]$. \square

Our next step is to establish existence of weak solutions.

THEOREM 2.8. *Let $y_0 \in \mathcal{H}^1$ and $f \in L_{loc}^2(\mathbb{R}_+; \mathcal{H}^0)$. For any $N > 0$, there exists a weak solution $y = y_N$ in the sense of Definition 2.5 to the TMHD equations such that*

(i) *For all $t \geq 0$,*

$$(2.14) \quad \|y(t)\|_{\mathcal{H}^0} \leq \|y_0\|_{\mathcal{H}^0} + \int_0^t \|f\|_{\mathcal{H}^0} ds$$

and

$$(2.15) \quad \int_0^t \|\nabla y(s)\|_{\mathcal{H}^0} + \|\sqrt{g_N(|y|^2)}|y|\|_{L^2}^2 ds \leq \|y_0\|_{\mathcal{H}^0}^2 + 2 \left[\int_0^t \|f(s)\|_{\mathcal{H}^0} ds \right]^2$$

(ii) *The solution satisfies $y \in C(\mathbb{R}_+; \mathcal{H}^1) \cap L_{loc}^2(\mathbb{R}_+; \mathcal{H}^2)$, $\partial_t y \in L_{loc}^2(\mathbb{R}_+; \mathcal{H}^0)$ and for all $t \geq 0$,*

$$\begin{aligned}
(2.16) \quad &\|y(t)\|_{\mathcal{H}^1}^2 + \int_0^t (\|y\|_{\mathcal{H}^2}^2 + \| |y| |\nabla y| \|_{L^2}^2) ds \\
&\leq C \left(\|y_0\|_{\mathcal{H}^1}^2 + \int_0^t \|f\|_{\mathcal{H}^0}^2 ds \right) \\
&\quad + C(1 + N + t) \left(\|y_0\|_{\mathcal{H}^0}^2 + \left[\int_0^t \|f\|_{\mathcal{H}^0} ds \right]^2 \right) =: C_{t, N, y_0, h}^1.
\end{aligned}$$

(iii) *There exist real functions $p(t, x), \pi(t, x)$, satisfying $\nabla p, \nabla \pi \in L^2_{loc}(\mathbb{R}_+; L^2(\mathbb{R}^3; \mathbb{R}^3))$ such that for almost all $t \geq 0$, in $L^2(\mathbb{R}^3; \mathbb{R}^6)$ we have*

$$\begin{aligned}\frac{\partial \mathbf{v}}{\partial t} &= \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v} + (\mathbf{B} \cdot \nabla) \mathbf{B} + \nabla \left(p + \frac{|\mathbf{B}|^2}{2} \right) - g_N(|(\mathbf{v}, \mathbf{B})|^2) \mathbf{v} \\ \frac{\partial \mathbf{B}}{\partial t} &= \Delta \mathbf{B} - (\mathbf{v} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{v} + \nabla \pi - g_N(|(\mathbf{v}, \mathbf{B})|^2) \mathbf{B}.\end{aligned}$$

PROOF. We use a Faedo-Galerkin approximation on $[0, T]$. Let us construct an orthonormal basis tailored to our needs.

By the separability of \mathcal{H}^3 and the density of $\mathcal{V} \subset \mathcal{H}^3$, there is a linearly independent countable subset $\{\bar{e}_k \mid k \in \mathbb{N}\} \subset \mathcal{V}$ which is dense in \mathcal{H}^3 . To be more precise, by the separability of \mathcal{H}^3 , there exists a countable topological basis $(U_n)_{n \in \mathbb{N}}$ of open subsets of \mathcal{H}^3 . Choose an $\bar{e}_1 \in \mathcal{V} \cap U_1$ (which exists by the density of \mathcal{V}). Now let $\{\bar{e}_1, \dots, \bar{e}_n\}$ be chosen already, and define $S_n := \text{span}\{\bar{e}_1, \dots, \bar{e}_n\}$. Since the dimension of \mathcal{H}^3 is infinite, the finite-dimensional subspace S_n is a proper subspace and thus has empty interior (if it contained an open ball, it would be equal to the whole space). In particular, it does not contain U_{n+1} . Since S_n is closed, $U_{n+1} \setminus S_n = U_{n+1} \cap S_n^c$ is open and thus there exists an $\bar{e}_{n+1} \in \mathcal{V} \cap (U_{n+1} \setminus S_n)$.

Using the Gram-Schmidt orthonormalisation procedure in \mathcal{H}^1 , we can construct an orthonormal basis $\{e_k \mid k \in \mathbb{N}\} \subset \mathcal{V}$ of \mathcal{H}^1 such that $\text{span}\{e_k\}$ is dense in \mathcal{H}^3 . Fix an $n \in \mathbb{N}$. For $z = (z^1, \dots, z^n) \in \mathbb{R}^n$ and $e = (e_1, \dots, e_n) \in \mathcal{V}^n$ set

$$\begin{aligned}z \cdot e &:= \sum_{i=1}^n z^i e_i \in \mathcal{V} \\ b_n(z) &:= (\langle \mathcal{A}(z \cdot e), e_i \rangle_{\mathcal{H}^1})_{i=1}^n \\ f_n(t) &:= (\langle \rho_n * f(t), e_1 \rangle_{\mathcal{H}^1}, \dots, \langle \rho_n * f_n(t), e_n \rangle_{\mathcal{H}^1}),\end{aligned}$$

where the ρ_n are a family of mollifiers such that

$$\|\rho_n * f(t)\|_{\mathcal{H}^0} \leq \|f(t)\|_{\mathcal{H}^0}, \quad \lim_{n \rightarrow \infty} \|\rho_n * f(t) - f(t)\|_{\mathcal{H}^0} = 0.$$

Now we consider the ordinary differential equation

$$(2.17) \quad \begin{cases} \frac{dz_n}{dt}(t) &= b_n(z_n(t)) + f_n(t), \\ z_n(0) &= (\langle y_0, e_i \rangle_{\mathcal{H}^1})_{i=1}^n. \end{cases}$$

Then we have

$$\langle z, b_n(z) \rangle_{\mathbb{R}^n} = \sum_{i=1}^n z^i \langle \mathcal{A}(z \cdot e), e_i \rangle_{\mathcal{H}^1} = \langle \mathcal{A}(z \cdot e), z \cdot e \rangle_{\mathcal{H}^1}.$$

Noting that $z \cdot e \in \mathcal{H}^3 \subset \mathcal{H}^2$, the estimate (2.10) on $\langle \mathcal{A}(y), y \rangle_{\mathcal{H}^1}$ from Lemma 2.2 then yields

$$\langle z, b_n(z) \rangle_{\mathbb{R}^n} = \langle \mathcal{A}(z \cdot e), z \cdot e \rangle_{\mathcal{H}^1} \leq C_{n,N} |z|^2,$$

where the constant $C_{n,N}$ contains the norms of all the e_i for $i = 1, \dots, n$ (as all the terms on the right-hand side of (2.10) are quadratic in y). Moreover, since

$$z \mapsto b_n(z) = (\langle \mathcal{A}(z \cdot e), e_i \rangle_{\mathcal{H}^1})_{i=1}^n \in \mathbb{R}^n$$

is a polynomial in the components of z in each component, it is a smooth map. Hence, the differential equation (2.17) has a unique solution $z_n(t)$ such that

$$z_n(t) = z_n(0) + \int_0^t b_n(z_n(s)) ds + \int_0^t f_n(s) ds, \quad t \geq 0.$$

Now we set

$$\begin{aligned} y_n(t) &:= z_n(t) \cdot e = \sum_{i=1}^n z_n^i(t) e_i \\ \prod_n \mathcal{A}(y_n(t)) &:= \sum_{i=1}^n \langle \mathcal{A}(y_n(t)), e_i \rangle_{\mathcal{H}^1} e_i \\ \prod_n f(t) &:= \sum_{i=1}^n \langle \rho_n * f(t), e_i \rangle_{\mathcal{H}^1} e_i. \end{aligned}$$

Then the function y_n satisfies the differential equation

$$\begin{aligned} \partial_t y_n(t) &= (\partial_t z_n(t)) \cdot e = (b_n(z_n(t)) \cdot e) + (f_n(t) \cdot e) \\ &= \prod_n \mathcal{A}(y_n(t)) + \prod_n f(t) \end{aligned}$$

and for all $n \geq k$

$$\begin{aligned} (2.18) \quad \langle y_n(t), e_k \rangle_{\mathcal{H}^1} &= \langle y_n(0), e_k \rangle_{\mathcal{H}^1} \\ &+ \int_0^t \left\langle \prod_n \mathcal{A}(y_n(s)), e_k \right\rangle_{\mathcal{H}^1} ds + \int_0^t \left\langle \prod_n f(s), e_k \right\rangle_{\mathcal{H}^1} ds \\ &= \langle y_0, e_k \rangle_{\mathcal{H}^1} + \int_0^t \langle \mathcal{A}(y_n(s)), e_k \rangle_{\mathcal{H}^1} ds + \int_0^t \langle \rho_n * f(s), e_k \rangle_{\mathcal{H}^1} ds. \end{aligned}$$

This implies that

$$\|y_n(t)\|_{\mathcal{H}^1}^2 = \|y_0\|_{\mathcal{H}^1}^2 + 2 \int_0^t \langle \mathcal{A}(y_n(s)), y_n(s) \rangle_{\mathcal{H}^1} ds + 2 \int_0^t \langle \rho_n * f(s), y_n(s) \rangle_{\mathcal{H}^1} ds.$$

Using the definition of \mathcal{H}^1 , the self-adjointness of $(I - \Delta)$ and the Cauchy-Schwarz-Buniakowski and Young inequalities for the last term as well as (2.10) for the second term (dropping the nonlinear terms, all of which have negative signs), we find that

$$\begin{aligned} \|y_n(t)\|_{\mathcal{H}^1}^2 &\leq \|y_0\|_{\mathcal{H}^1}^2 - \int_0^t \|y_n\|_{\mathcal{H}^2}^2 + 2\|y_n\|_{\mathcal{H}^0}^2 + 4(N+1)\|\nabla y_n\|_{\mathcal{H}^0}^2 ds \\ &+ 2 \int_0^t \|\rho_n * f(s)\|_{\mathcal{H}^0}^2 ds + \frac{1}{2} \int_0^t \|y_n\|_{\mathcal{H}^2}^2 ds \\ &\leq \|y_0\|_{\mathcal{H}^1}^2 - \int_0^t \frac{1}{2} \|y_n\|_{\mathcal{H}^2}^2 + 2\|y_n\|_{\mathcal{H}^0}^2 + 4(N+1)\|\nabla y_n\|_{\mathcal{H}^0}^2 ds \\ &+ 2 \int_0^t \|f(s)\|_{\mathcal{H}^0}^2 ds. \end{aligned}$$

This implies that

$$(2.19) \quad \|y_n(t)\|_{\mathcal{H}^1}^2 + \int_0^t \|y_n\|_{\mathcal{H}^2}^2 ds \leq C_N \left(\|y_0\|_{\mathcal{H}^1}^2 + \int_0^t \|y_n\|_{\mathcal{H}^1}^2 ds + \int_0^t \|f\|_{\mathcal{H}^0}^2 ds \right).$$

Dropping the second term on the left-hand side and using Gronwall's lemma, we find that

$$\sup_{t \leq T} \|y_n(t)\|_{\mathcal{H}^1}^2 \leq C_{y_0, N, T, f}.$$

Using this information in (2.19), we find that also

$$\int_0^t \|y_n\|_{\mathcal{H}^2}^2 ds \leq C_{y_0, N, T, f}.$$

Now for a fixed $k \in \mathbb{N}$, set $G_n^{(k)}(t) := \langle y_n(t), e_k \rangle_{\mathcal{H}^1}$. Then by the preceding step, the $G_n^{(k)}$ are uniformly bounded on $[0, T]$. Furthermore, they are equi-continuous, as can be seen from

$$\begin{aligned} |G_n^{(k)}(t) - G_n^{(k)}(r)| &= |\langle y_n(t), e_k \rangle_{\mathcal{H}^1} - \langle y_n(r), e_k \rangle_{\mathcal{H}^1}| \\ &= \left| \int_r^t \langle \mathcal{A}(y_n(s), e_k) \rangle_{\mathcal{H}^1} ds + \int_r^t \langle \rho_n * f(s), e_k \rangle_{\mathcal{H}^1} ds \right| \\ &\leq C \int_r^t (1 + \|y_n(s)\|_{\mathcal{H}^1}^3) \|e_k\|_{\mathcal{H}^3} ds + \|e_k\|_{\mathcal{H}^2} \int_r^t \|f(s)\|_{\mathcal{H}^0} ds, \end{aligned}$$

and equation (2.19), where we used Lemma 2.2 (i). Therefore, the theorem of Arzelà-Ascoli implies that $(G_n^{(k)})_{n \in \mathbb{N}}$ is sequentially relatively compact with respect to the uniform topology and hence there is a subsequence such that $(G_{n_i}^{(k)})_{i \in \mathbb{N}}$ converges uniformly to a limit $G^{(k)}$. Now a diagonalisation argument implies that there is a subsequence, which we denote again by $(G_n^{(k)})_{n \in \mathbb{N}}$

$$\forall k \in \mathbb{N} : \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |G_n^{(k)}(t) - G^{(k)}(t)| = 0.$$

Again invoking (2.19), we see that $\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} \|y_n(t)\|_{\mathcal{H}^1}^2 \leq C$. Since closed balls of \mathcal{H}^1 are weakly compact by the Banach-Alaoglu theorem, we find that for almost all $t \in [0, T]$ we have the \mathcal{H}^1 -weak convergence $y_n(t) \rightharpoonup y(t)$ as $n \rightarrow \infty$. To conclude that this holds true for all $t \in [0, T]$ we note that $G^{(k)}$, as the uniform limit of continuous functions, is continuous, and that on the other hand by the weak convergence just mentioned,

$$G_n^{(k)}(t) = \langle y_n(t), e_k \rangle_{\mathcal{H}^1} \rightarrow \langle y(t), e_k \rangle_{\mathcal{H}^1}.$$

Hence, $t \mapsto \langle y(t), e_k \rangle_{\mathcal{H}^1}$ is continuous for all $k \in \mathbb{N}$ and by the density of the $\{e_k\}_{k \in \mathbb{N}}$, we find that $t \mapsto \langle y(t), \tilde{y} \rangle_{\mathcal{H}^1}$ is continuous for all $\tilde{y} \in \mathcal{H}^1$. We can thus conclude that $t \mapsto y(t)$ is weakly continuous in \mathcal{H}^1 , and that for all $\tilde{y} \in \mathcal{H}^1$:

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\langle y_n(t) - y(t), \tilde{y} \rangle_{\mathcal{H}^1}| = 0.$$

This implies (by considering $\tilde{y} = (I - \Delta)^{-1} \tilde{z} \in \mathcal{H}^2 \subset \mathcal{H}^1$ for $z \in \mathcal{H}^0$ and using the formal self-adjointness of $(I - \Delta)$)

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\langle y_n(t) - y(t), \tilde{z} \rangle_{\mathcal{H}^0}| = 0.$$

We next invoke the Helmholtz-Weyl decomposition $\mathcal{L}^2 = \mathcal{H}^0 \oplus (\mathcal{H}^0)^\perp$. Since $y_n(t) - y(t) \in \mathcal{H}^1 \subset \mathcal{H}^0$, this allows us to conclude that

$$(2.20) \quad \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\langle y_n(t) - y(t), \tilde{z} \rangle_{\mathcal{L}^2}| = 0 \quad \forall \tilde{z} \in \mathcal{L}^2.$$

To be more precise, we can use the Helmholtz-Weyl decomposition for $L^2(\mathbb{R}^3; \mathbb{R}^3)$ for both the velocity component and the magnetic field component of $y_n - y$, and putting these two together yields (2.20). Note that this equation implies the component-wise convergence

$$(2.21) \quad \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\langle \phi_{i,n}(t) - \phi_i(t), \tilde{z} \rangle_{L^2}| = 0 \quad \forall z \in L^2(\mathbb{R}^3; \mathbb{R}), \phi \in \{\mathbf{v}, \mathbf{B}\}.$$

This can be seen by taking \tilde{z} of the form $\tilde{z} = (0, \dots, 0, z, 0, \dots, 0)$, where $z \in L^2(\mathbb{R}^3; \mathbb{R})$. Thus, we get weak convergence of each component of the velocity and magnetic field in L^2 .

From this equation as well as (2.19) and Fatou's lemma, we get

$$\int_0^T \|y(s)\|_{\mathcal{H}^2}^2 ds \leq \liminf_{n \rightarrow \infty} \int_0^T \|y_n(s)\|_{\mathcal{H}^2}^2 ds < \infty.$$

Next we want to show that y is indeed a solution of the tamed MHD equations (1.8).

To this end, recall first the following Friedrichs' inequality⁷ (see e.g. [141]): let $Q \subset \mathbb{R}^3$ be a bounded cuboid. Then for all $\varepsilon > 0$ there is a $K_\varepsilon \in \mathbb{N}$ and functions $h_i^\varepsilon \in L^2(G)$, $i = 1, \dots, K_\varepsilon$ such that for all $w \in W_0^{1,2}(G)$

$$(2.22) \quad \int_Q |w(x)|^2 dx \leq \sum_{i=1}^{K_\varepsilon} \left(\int_Q w(x) h_i^\varepsilon(x) dx \right)^2 + \varepsilon \int_Q |\nabla w(x)|^2 dx.$$

Now let $G \subset \bar{G} \subset Q$ and choose a smooth cutoff function ρ such that $1 \geq \rho \geq 0$, $\rho \equiv 1$ on G and $\text{supp } \rho \subset Q$. Then we have for all $j = 1, 2, 3$ and $\phi \in \{\mathbf{v}, \mathbf{B}\}$ that $\rho(\phi_{j,n}(t, \cdot) - \phi_j(t, \cdot)) \in W_0^{1,2}(Q)$ and hence by applying Friedrichs' inequality, we find

$$\begin{aligned} & \int_G |y_n(t, x) - y(t, x)|^2 dx \\ &= \sum_{j=1}^3 \int_G |v_{j,n}(t, x) - v_j(t, x)|^2 dx + \int_G |B_{j,n}(t, x) - B_j(t, x)|^2 dx \\ &\leq \sum_{j=1}^3 \int_Q \rho^2(x) |v_{j,n}(t, x) - v_j(t, x)|^2 dx + \int_Q \rho^2(x) |B_{j,n}(t, x) - B_j(t, x)|^2 dx \\ &\leq \sum_{j=1}^3 \sum_{i=1}^{K_\varepsilon} \left(\int_Q (v_{j,n} - v_j) \rho h_i^\varepsilon dx \right)^2 + \varepsilon \sum_{j=1}^3 \int_Q |\nabla [\rho(v_{j,n} - v_j)]|^2 dx \\ &\quad + \sum_{j=1}^3 \sum_{i=1}^{K_\varepsilon} \left(\int_Q (B_{j,n} - B_j) \rho h_i^\varepsilon dx \right)^2 + \varepsilon \sum_{j=1}^3 \int_Q |\nabla [\rho(B_{j,n} - B_j)]|^2 dx. \end{aligned}$$

The first and third terms in the last two lines vanish in the limit $n \rightarrow \infty$ by (2.21), since $\rho h_i^\varepsilon \in L^2(\mathbb{R}^3)$. To the second and fourth term (those proportional to ε), we apply the product rule for weak derivatives (see e.g. [68, Theorem 5.2.3.1 (iv), pp. 261 f.]) and (2.19) to see that the integrals are bounded. As $\varepsilon > 0$ is arbitrary, we find

$$(2.23) \quad \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \int_G |y_n(t, x) - y(t, x)|^2 dx = 0.$$

Now let, for $k \in \mathbb{N}$, $\text{supp}(e_k) \subset G_k$ for bounded sets G_k . If we fix $s \in [0, t]$, then by (2.19) and (2.23) we get

$$\sup_n \|y_n(s, \cdot)\|_{\mathcal{H}^1} < \infty \quad \text{and} \quad \lim_n \|(y_n(s, \cdot) - y(s, \cdot))1_{G_k}\|_{L^2} = 0.$$

Thus an application of Lemma 2.4 and Lebesgue's dominated convergence theorem yields

$$\int_0^t \langle \mathcal{A}(y_n(s)), e_k \rangle_{\mathcal{H}^1} ds \rightarrow \int_0^t \langle \mathcal{A}(y(s)), e_k \rangle_{\mathcal{H}^1} ds.$$

⁷The first use of the inequality in the context of the Navier-Stokes equations seems to be in E. Hopf [115, p. 230]. Hopf uses the inequality and cites R. Courant's and D. Hilbert's book [43]. The inequality and a proof can be found in Chapter VII, Paragraph 3, Section 1, Satz 1, p. 489. Hopf also notes that the statement is not true for arbitrary bounded domains. For a more modern presentation, cf. J.C. Robinson, J.L. Rodrigo and W. Sadowski [192, Exercises 4.2–4.9, pp. 107 ff.].

Having established this convergence, we can take limits $n \rightarrow \infty$ in (2.18) to find

$$\langle y(t), e_k \rangle_{\mathcal{H}^1} = \langle y_0, e_k \rangle_{\mathcal{H}^1} + \int_0^t \langle \mathcal{A}(y(s)), e_k \rangle_{\mathcal{H}^1} ds + \int_0^t \langle f(s), (I - \Delta)e_k \rangle_{\mathcal{H}^0} ds.$$

As this equation is linear in e_k , it holds for linear combinations and since $\text{span}\{e_k\}$ forms a dense subset in \mathcal{H}^3 , we conclude

$$\langle y(t), \tilde{y} \rangle_{\mathcal{H}^1} = \langle y_0, \tilde{y} \rangle_{\mathcal{H}^1} + \int_0^t \langle \mathcal{A}(y(s)), \tilde{y} \rangle_{\mathcal{H}^1} ds + \int_0^t \langle f(s), (I - \Delta)\tilde{y} \rangle_{\mathcal{H}^0} ds.$$

Now, letting $\tilde{y} := (I - \Delta)^{-1}\bar{y}$ for $\bar{y} \in \mathcal{H}^3$,

$$\langle y(t), \bar{y} \rangle_{\mathcal{H}^0} = \langle y_0, \bar{y} \rangle_{\mathcal{H}^0} + \int_0^t \langle \mathcal{A}(y(s), \bar{y}) \rangle_{\mathcal{H}^0} ds + \int_0^t \langle f(s), \bar{y} \rangle_{\mathcal{H}^0} ds,$$

that is, Equation (2.11).

We are left to prove (i) - (iii). We will start with (iii). At first it might seem difficult to extract two pointwise 3-D equations from one weak 1-D equation, as [214, Chapter I, Section 1, Proposition 1.1., p. 14] only works when the number of components of the solution equals the space dimension. Note, however, that our weak formulation holds for a larger class of test functions. In particular, we note that if $\tilde{\mathbf{v}} \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^3)$ with $\nabla \cdot \tilde{\mathbf{v}} = 0$, then $(\tilde{\mathbf{v}}, 0) \in \mathcal{V}$. Thus in equation (2.11) we set $\tilde{y} = (\tilde{\mathbf{v}}, 0)$ and we find for almost all $t \geq 0$ that

$$(2.24) \quad \frac{\partial \mathbf{v}}{\partial t} = \Delta \mathbf{v} - \mathcal{P}[(\mathbf{v} \cdot \nabla) \mathbf{v} + (\mathbf{B} \cdot \nabla) \mathbf{B}] - \mathcal{P}[g_N(|y|^2)\mathbf{v}] + \mathbf{f}_1(t)$$

and infer from Proposition 1.1. of [214] the existence of a function \bar{p} with $\nabla \bar{p} \in L_{\text{loc}}^2(\mathbb{R}_+; L^2(\mathbb{R}^3))$ such that (for almost all $t \geq 0$)

$$\frac{\partial \mathbf{v}}{\partial t} = \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v} + (\mathbf{B} \cdot \nabla) \mathbf{B} - g_N(|y|^2)\mathbf{v} + \nabla \bar{p} + \mathbf{f}_1(t).$$

Now we *define* the pressure p by

$$\bar{p} = p + \frac{|\mathbf{B}|^2}{2},$$

and we observe that since $\nabla \frac{|\mathbf{B}|^2}{2} = (\mathbf{B} \cdot \nabla) \mathbf{B} \in L_{\text{loc}}^2(\mathbb{R}_+; L^2(\mathbb{R}^3; \mathbb{R}^3))$ due to (2.16), and by the regularity of \bar{p} , p also satisfies the right regularity condition $\nabla p \in L_{\text{loc}}^2(\mathbb{R}_+; L^2(\mathbb{R}^3))$.

In the same way (cf. Section 1.3.4), by testing against $\tilde{y} = (0, \tilde{\mathbf{B}}) \in \mathcal{V}$ to get

$$(2.25) \quad \frac{\partial \mathbf{B}}{\partial t} = \Delta \mathbf{B} - \mathcal{P}(\mathbf{v} \cdot \nabla) \mathbf{B} + \mathcal{P}(\mathbf{B} \cdot \nabla) \mathbf{v} - \mathcal{P}[g_N(|(\mathbf{v}, \mathbf{B})|^2)\mathbf{B}] + \mathbf{f}_2,$$

we can find a π such that $\nabla \pi \in L_{\text{loc}}^2(\mathbb{R}_+; L^2(\mathbb{R}^3))$ (note that this function in general cannot be expected to be equal to zero for our equation, as we argued in Section 1.3.4) such that for almost all $t \geq 0$

$$\frac{\partial \mathbf{B}}{\partial t} = \Delta \mathbf{B} - (\mathbf{v} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{v} + \nabla \pi - g_N(|(\mathbf{v}, \mathbf{B})|^2)\mathbf{B} + \mathbf{f}_2.$$

These two statements imply (iii).

Next we want to prove (i). We take the scalar product in \mathcal{H}^0 of the system (2.24), (2.25) with $y(t)$ as well as (2.9) to find

$$(2.26) \quad \begin{aligned} \langle \partial_t y(t), y(t) \rangle_{\mathcal{H}^0} &= \langle \mathcal{A}(y(t)), y(t) \rangle_{\mathcal{H}^0} + \langle f(t), y(t) \rangle_{\mathcal{H}^0} \\ &= -\|\nabla y(t)\|_{\mathcal{H}^0}^2 - \|\sqrt{g_N(|y(t)|^2)}|y(t)\|_{L^2}^2 + \langle f(t), y(t) \rangle_{\mathcal{H}^0} \\ &\leq -\|\nabla y(t)\|_{\mathcal{H}^0}^2 - \|\sqrt{g_N(|y(t)|^2)}|y(t)\|_{L^2}^2 + \|f(t)\|_{\mathcal{H}^0} \|y(t)\|_{\mathcal{H}^0}, \end{aligned}$$

which yields (as $\langle \partial_t y(t), y(t) \rangle_{\mathcal{H}^0} = \frac{1}{2} \frac{d}{dt} \|y(t)\|_{\mathcal{H}^0}^2 = \|y(t)\|_{\mathcal{H}^0} \frac{d}{dt} \|y(t)\|_{\mathcal{H}^0}$)

$$\frac{d}{dt} \|y(t)\|_{\mathcal{H}^0} \leq \|f(t)\|_{\mathcal{H}^0}.$$

Integrating this inequality gives

$$(2.27) \quad \|y(t)\|_{\mathcal{H}^0} \leq \|y_0\|_{\mathcal{H}^0} + \int_0^t \|f(s)\|_{\mathcal{H}^0} ds,$$

and integrating (2.26), we find

$$(2.28) \quad \begin{aligned} & \int_0^t \|\nabla y(s)\|_{\mathcal{H}^0}^2 + \|\sqrt{g_N(|y(s)|^2)}|y(s)\|_{L^2}^2 ds \\ & \leq \frac{1}{2} \|y_0\|_{\mathcal{H}^0}^2 + \int_0^t \|f(s)\|_{\mathcal{H}^0} \|y(s)\|_{\mathcal{H}^0} ds \\ & \leq \frac{1}{2} \|y_0\|_{\mathcal{H}^0}^2 + \int_0^t \|f(s)\|_{\mathcal{H}^0} \left(\|y_0\|_{\mathcal{H}^0} + \int_0^s \|f(r)\|_{\mathcal{H}^0} dr \right) ds \\ & \leq \frac{1}{2} \|y_0\|_{\mathcal{H}^0}^2 + \|y_0\|_{\mathcal{H}^0} \int_0^t \|f(s)\|_{\mathcal{H}^0} ds + \left[\int_0^t \|f(s)\|_{\mathcal{H}^0} ds \right]^2 \\ & \leq \|y_0\|_{\mathcal{H}^0}^2 + \frac{3}{2} \left[\int_0^t \|f(s)\|_{\mathcal{H}^0} ds \right]^2. \end{aligned}$$

Thus we have shown (i).

For (ii), we note that

$$\mathcal{H}^2 \hookrightarrow \mathcal{H}^1 \hookrightarrow \mathcal{H}^0$$

forms a Gelfand triple and thus by [214, Chapter III, Section 1, Lemma 1.2, p. 260 f.] and (2.24), (2.25) we get the equality

$$\|y(t)\|_{\mathcal{H}^1}^2 = \|y_0\|_{\mathcal{H}^1}^2 + 2 \int_0^t \langle \mathcal{A}(y), y \rangle_{\mathcal{H}^1} ds + 2 \int_0^t \langle f, y \rangle_{\mathcal{H}^1} ds.$$

The right-hand side is continuous in t and thus together with the weak continuity of $t \mapsto y(t) \in \mathcal{H}^1$ by [248, Proposition 21.23 (d), p. 258] we get that $t \mapsto y(t) \in \mathcal{H}^1$ is strongly continuous. We then apply (2.10), (i) and Young's inequality to find

$$\begin{aligned} & \|y(t)\|_{\mathcal{H}^1}^2 \\ & \leq \|y_0\|_{\mathcal{H}^1}^2 - \int_0^t \|y\|_{\mathcal{H}^2}^2 ds + 2 \int_0^t \|y\|_{\mathcal{H}^0}^2 ds + 2(N+1) \int_0^t \|\nabla y\|_{\mathcal{H}^0}^2 ds \\ & \quad - 2 \int_0^t (\|\mathbf{v}\|\nabla \mathbf{v}\|_{L^2}^2 - \|\mathbf{B}\|\nabla \mathbf{B}\|_{L^2}^2 - \|\mathbf{v}\|\nabla \mathbf{B}\|_{L^2}^2 - \|\mathbf{B}\|\nabla \mathbf{v}\|_{L^2}^2) ds \\ & \quad + 2 \int_0^t \|f\|_{\mathcal{H}^0} \|y\|_{\mathcal{H}^2} ds \\ & \leq C \left(\|y_0\|_{\mathcal{H}^1}^2 + \int_0^t \|f\|_{\mathcal{H}^0} ds \right) + C(1+N+t) \left(\|y_0\|_{\mathcal{H}^0}^2 + \left[\int_0^t \|f\|_{\mathcal{H}^0} ds \right]^2 \right) \\ & \quad - 2 \int_0^t (\|\mathbf{v}\|\nabla \mathbf{v}\|_{L^2}^2 - \|\mathbf{B}\|\nabla \mathbf{B}\|_{L^2}^2 - \|\mathbf{v}\|\nabla \mathbf{B}\|_{L^2}^2 - \|\mathbf{B}\|\nabla \mathbf{v}\|_{L^2}^2) ds \\ & \quad - \frac{1}{2} \int_0^t \|y\|_{\mathcal{H}^2}^2 ds. \end{aligned}$$

Hence we can conclude that

$$\begin{aligned} & \|y(t)\|_{\mathcal{H}^1}^2 + \frac{1}{2} \int_0^t \|y\|_{\mathcal{H}^2}^2 ds \\ & + 2 \int_0^t \left(\|\mathbf{v}\|_{L^2} \|\nabla \mathbf{v}\|_{L^2}^2 + \|\mathbf{B}\|_{L^2} \|\nabla \mathbf{B}\|_{L^2}^2 + \|\mathbf{v}\|_{L^2} \|\nabla \mathbf{B}\|_{L^2}^2 + \|\mathbf{B}\|_{L^2} \|\nabla \mathbf{v}\|_{L^2}^2 \right) ds \\ & \leq C \left(\|y_0\|_{\mathcal{H}^1}^2 + \int_0^t \|f\|_{\mathcal{H}^0} ds \right) + C(1 + N + t) \left(\|y_0\|_{\mathcal{H}^0}^2 + \left[\int_0^t \|f\|_{\mathcal{H}^0} ds \right]^2 \right), \end{aligned}$$

which implies (2.16). \square

2.3. Existence, Uniqueness and Regularity of a Strong Solution. In this section, we will show that for smooth initial data, the TMHD equations admit a smooth solution. To prove this, we have to prove their regularity, which is done via the regularity result of Appendix A. Our main result in this section is the following.

THEOREM 2.9 (Strong Solutions). *Let $y_0 \in \mathcal{H}^\infty := \bigcap_{m \in \mathbb{N}_0} \mathcal{H}^m$ and $\mathbb{R}_+ \ni t \mapsto f(t) \in \mathcal{H}^m$ be smooth for any $m \in \mathbb{N}_0$. Then there exists a unique smooth velocity field*

$$\mathbf{v}_N \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3; \mathbb{R}^3) \cap C(\mathbb{R}_+; \mathbb{H}^2),$$

a unique smooth magnetic field

$$\mathbf{B}_N \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3; \mathbb{R}^3) \cap C(\mathbb{R}_+; \mathbb{H}^2)$$

and pressure functions

$$p_N, \pi_N \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3; \mathbb{R})$$

which are defined up to a time-dependent constant. The triple $(\mathbf{v}_N, \mathbf{B}_N, p_N)$ solves the tamed MHD equations, (1.12).

Moreover, we have the following estimates: for any $T, N > 0$

$$(2.29) \quad \sup_{t \in [0, T]} \|y_N(t)\|_{\mathcal{H}^0}^2 + \int_0^T \|\nabla y_N\|_{\mathcal{H}^0}^2 ds \leq C \left(\|y_0\|_{\mathcal{H}^0}^2 + \left[\int_0^T \|f(s)\|_{\mathcal{H}^0} ds \right]^2 \right)$$

and

$$(2.30) \quad \sup_{t \in [0, T]} \|y_N(t)\|_{\mathcal{H}^1}^2 + \int_0^T \|y_N(s)\|_{\mathcal{H}^2}^2 ds \leq C_{T, y_0, f} \cdot (1 + N),$$

$$(2.31) \quad \sup_{t \in [0, T]} \|y_N(t)\|_{\mathcal{H}^2}^2 \leq C'_{T, y_0, f} + C_{T, y_0, f} \cdot (1 + N^2).$$

To be precise, the constant $C_{T, y_0, f}$ depends on $\|y_0\|_{\mathcal{H}^1}$ and $\int_0^T \|f\|_{\mathcal{H}^0} ds$ and goes to zero as both these quantities tend towards zero. The constant $C'_{T, y_0, f}$ depends on T , $\|y_0\|_{\mathcal{H}^2}$ and $\sup_{t \in [0, T]} \|f(t)\|_{\mathcal{H}^0}$ as well as $\int_0^T \|\partial_s f\|_{\mathcal{H}^0}^2 ds$.

We use the notation from Appendix A. We denote the space-time L^p norms by $\|y\|_{\mathcal{L}^p(S_T)}$. Let $(\mathcal{F}_t)_{t \geq 0}$ be the Gaussian heat semigroup on \mathbb{R}^3 . We define its action on a function by the space-convolution

$$\mathcal{F}_t h(x) := \frac{1}{(4\pi t)^{3/2}} \int_{\mathbb{R}^3} e^{-\frac{|x-z|^2}{4t}} h(z) dz.$$

We can thus rewrite the operator $\bar{\mathcal{B}}$ as

$$\bar{\mathcal{B}}(u, u)_i(t, x) = \sum_{j=1}^3 \int_0^t (D_{x_j} \mathcal{F}_{t-s}) \left[u^j(s) u_i(s) - \sum_k R_i R_k u^k(s) u^j(s) \right] (x) ds.$$

Then by Appendix A, Lemma A.4 and Theorem A.5, the weak solution y constructed in Theorem 2.8 satisfies the integral equation

$$y(t, x) = f_N(t, x) - \mathcal{B}(y, y)(t, x),$$

where

$$\begin{aligned} f_N(t, x) &:= \mathcal{F}_t y_0 - \Gamma \bar{\otimes} (1_{\mathbb{R}_+} \mathcal{P}(g_N(|y|^2)y - f))(t, x) \\ &:= \mathcal{F}_t y_0 - \int_0^t \mathcal{F}_{t-s} (\mathcal{P}(g_N(|y(s)|^2)y(s) - f(s)))(x) ds. \end{aligned}$$

The Riesz projection term vanishes here because the Helmholtz-Leray projection \mathcal{P} ensures that the divergence of the taming term is zero, and the forcing term has zero divergence by assumption.

PROOF OF THEOREM 2.9. We first prove the regularity statement. To this end, we show the following for all $k \in \mathbb{N}$:

$$(2.32) \quad y \in \mathcal{L}^{10 \cdot (\frac{5}{3})^{k-1}}(S_T), D_x^\alpha D_t^j y \in \mathcal{L}^{2 \cdot (\frac{5}{3})^k}(S_T), \quad |\alpha| + 2j \leq 2.$$

We use a proof by induction.

$k = 1$. First, by the Sobolev embedding theorem, since

$$\frac{0}{3} + \frac{1}{5} \left(\frac{1}{2} - \frac{2}{3} \right) + \frac{4}{6} = \frac{3}{30} = \frac{1}{10},$$

we have

$$\begin{aligned} \|y\|_{\mathcal{L}^{10}(S_T)}^{10} &= \int_0^T \int_{\mathbb{R}^3} |y(s, x)|^{10} dx ds \\ &\leq C_{2,0,2,6,10}^{10} \int_0^T \left(\|y\|_{\mathcal{H}^2}^{1/5} \|y\|_{\mathcal{L}^6}^{4/5} \right)^{10} ds \\ &= C_{2,0,2,6,10}^{10} \int_0^T \|y\|_{\mathcal{H}^2}^2 \|y\|_{\mathcal{L}^6}^8 ds \\ &\leq C \int_0^T \|y\|_{\mathcal{H}^2}^2 \|y\|_{\mathcal{H}^1}^8 ds \\ &\leq C \sup_{t \in [0, T]} \|y(t)\|_{\mathcal{H}^1}^8 \int_0^T \|y(s)\|_{\mathcal{H}^2}^2 ds < \infty. \end{aligned}$$

Hence we find that

$$g_N(|y|^2)y \in \mathcal{L}^{10/3}(S_T).$$

Now, as y_0 and f are smooth, by Lemma A.2, we find that

$$D_x^\alpha D_t^j f_N \in \mathcal{L}^{10/3}(S_T), \quad |\alpha| + 2j \leq 2.$$

An application of Theorem A.7 then yields

$$D_x^\alpha D_t^j y \in \mathcal{L}^{10/3}(S_T), \quad |\alpha| + 2j \leq 2.$$

$k \rightarrow k + 1$. Assume (2.32). We want to apply the Sobolev embedding theorem, which is justified as

$$\frac{0}{3} + \frac{1}{5} \left(\frac{1}{2 \cdot (\frac{5}{3})^k} - \frac{2}{3} \right) + \frac{1 - 1/5}{6} = \frac{1}{10 \cdot (\frac{5}{3})^k}.$$

Therefore,

$$\begin{aligned} \|y\|_{\mathcal{L}^{10 \cdot (\frac{5}{3})^k}}^{10 \cdot (\frac{5}{3})^k} &\leq C_{2,0,2 \cdot (\frac{5}{3})^k, 6, 10 \cdot (\frac{5}{3})^k}^{10 \cdot (\frac{5}{3})^k} \int_0^T \left(\|y\|_{2,2 \cdot (\frac{5}{3})^k}^{1/5} \|y\|_{\mathcal{L}^6}^{4/5} \right)^{10 \cdot (\frac{5}{3})^k} ds \\ &\leq C \int_0^T \|y\|_{2,2 \cdot (\frac{5}{3})^k}^{2 \cdot (\frac{5}{3})^k} \|y\|_{\mathcal{H}^1}^{8 \cdot (\frac{5}{3})^k} ds \\ &\leq C \sup_{t \in [0, T]} \|y(t)\|_{\mathcal{H}^1}^{8 \cdot (\frac{5}{3})^k} \int_0^T \|y\|_{2,2 \cdot (\frac{5}{3})^k}^{2 \cdot (\frac{5}{3})^k} ds < \infty, \end{aligned}$$

which implies

$$g_N(|y|^2)y \in \mathcal{L}^{2 \cdot (\frac{5}{3})^{k+1}}(S_T)$$

and by another application of Lemma A.2, this yields

$$D_x^\alpha D_t^j f_N \in L^{2 \cdot (\frac{5}{3})^{k+1}}(S_T), \quad |\alpha| + 2j \leq 2,$$

and hence, by Theorem A.7,

$$D_x^\alpha D_t^j y \in L^{2 \cdot (\frac{5}{3})^{k+1}}(S_T), \quad |\alpha| + 2j \leq 2.$$

We have thus shown that

$$D_x^\alpha D_t^j f_N \in \bigcap_{q>1} L^q(S_T), \quad |\alpha| + 2j \leq 2.$$

The next step of the proof consists of another induction on the number of derivatives. Namely we want to show that

$$D_x^\alpha D_t^j f_N \in \bigcap_{q>1} L^q(S_T), \quad |\alpha| + 2j \leq m.$$

We have shown the base case $m = 2$ already. So we are left to show the induction step $m \rightarrow m + 1$.

We have to consider two cases:

(a) There is at least one spatial derivative, i.e. we have

$$D_x^\alpha D_t^j f_N = D_{x_k} D_x^\beta D_t^j f_N, \quad |\beta| = |\alpha| - 1 > 0, \quad |\beta| + 1 + 2j = m + 1.$$

In this case we have

$$\begin{aligned} &\|D_{x_k} D_x^\beta D_t^j f_N\|_{L^q(S_T)} \\ &= \left\| D_{x_k} D_x^\beta D_t^j \left(\mathcal{F}_t y_0 - \int_0^t \mathcal{F}_{t-s} (\mathcal{P}(g_N(|y(s)|^2)y(s) - f(s))) (x) ds \right) \right\|_{L^q}. \end{aligned}$$

Applying linearity and the triangle inequality, we see that the term containing the initial condition y_0 is bounded. For the other term we get the upper bound

$$\begin{aligned} & \left\| D_{x_k} D_x^\beta D_t^j \int_0^t \mathcal{F}_{t-s} (\mathcal{P}(g_N(|y(s)|^2)y(s) - f(s))) ds \right\|_{L^q} \\ & \leq \left\| D_t^{j-1} D_{x_k} D_x^\beta \mathcal{P}(g_N(|y(t)|^2)y(t) - f(t)) \right\|_{L^q} \\ & + \left\| \int_0^t (D_{x_k} \mathcal{F}_{t-s}) D_x^\beta D_t^j (\mathcal{P}(g_N(|y(s)|^2)y(s) - f(s))) ds \right\|_{L^q}. \end{aligned}$$

The first term is bounded by the induction hypothesis, since

$$|\beta| + 1 + 2(j-1) = |\alpha| - 1 + 2j = m.$$

The second term is bounded by Young's convolution inequality and the fact that $D_{x_k} \mathcal{F}_t \in L^1(S_T)$ just like in the proof of Lemma A.1.

(b) There are only derivatives with respect to time, i.e.

$$D_x^\alpha D_t^j f_N = D_t^j f_N, \quad 2j = m + 1.$$

The term containing the initial condition is again not a problem. In a similar way as before we find

$$\begin{aligned} & \left\| D_t^j \int_0^t \mathcal{F}_{t-s} (\mathcal{P}(g_N(|y(s)|^2)y(s) - f(s))) ds \right\|_{L^q} \\ & \leq \left\| D_t^{j-1} \mathcal{P}(g_N(|y(t)|^2)y(t) - f(t)) \right\|_{L^q} \\ & + \left\| \int_0^t (D_t \mathcal{F}_{t-s}) D_t^{j-1} (\mathcal{P}(g_N(|y(s)|^2)y(s) - f(s))) ds \right\|_{L^q} \\ & = \left\| D_t^{j-1} \mathcal{P}(g_N(|y(t)|^2)y(t) - f(t)) \right\|_{L^q} \\ & + \left\| \int_0^t (\Delta \mathcal{F}_{t-s}) D_t^{j-1} (\mathcal{P}(g_N(|y(s)|^2)y(s) - f(s))) ds \right\|_{L^q}, \end{aligned}$$

where we have used that \mathcal{F}_t solves the heat equation. Now by using integration by parts, in the last term, we transfer one spatial derivative from the Laplacian to the second factor and since $2j-1 = m$, we conclude the boundedness as before by Young's convolution inequality and the L^1 -boundedness of $D_{x_k} \mathcal{F}_{t-s}$.

By the Sobolev embedding theorem we now find that y is smooth. Thus we get for every $t \geq 0$ that

$$(2.33) \quad \begin{aligned} \partial_t \mathbf{v}(t) &= \Delta \mathbf{v}(t) - \mathcal{P}((\mathbf{v} \cdot \nabla) \mathbf{v}) + \mathcal{P}((\mathbf{B} \cdot \nabla) \mathbf{B}) - \mathcal{P}(g_N(|\mathbf{v}|^2) \mathbf{v}) + \mathbf{f}_v(t) \\ \partial_t \mathbf{B}(t) &= \Delta \mathbf{B}(t) - \mathcal{P}((\mathbf{v} \cdot \nabla) \mathbf{B}) + \mathcal{P}((\mathbf{B} \cdot \nabla) \mathbf{v}) - \mathcal{P}(g_N(|\mathbf{B}|^2) \mathbf{B}) + \mathbf{f}_B(t) \end{aligned}$$

We take this equation and apply 3 different inner products to both sides of the equations:

- (a) $\langle \cdot, \partial_t y(t) \rangle_{\mathcal{H}^0}$, which will lead to an estimate for $\int_0^T \|\partial_t y\|_{\mathcal{H}^0}^2 dt$
- (b) First apply ∂_t , then apply $\langle \cdot, \partial_t y(t) \rangle_{\mathcal{H}^0}$. This will lead to an estimate for $\|\partial_t y\|_{\mathcal{H}^0}^2$.
- (c) $\langle \cdot, y(t) \rangle_{\mathcal{H}^1}$, which gives an estimate for $\|y(t)\|_{\mathcal{H}^2}^2$.

(a) We find by using Young's inequality

$$\begin{aligned}
\|\partial_t y\|_{\mathcal{H}^0}^2 &= \langle \Delta y, \partial_t y(t) \rangle_{\mathcal{H}^0} - \langle (\mathbf{v} \cdot \nabla) \mathbf{v}, \partial_t \mathbf{v} \rangle_{L^2} + \langle (\mathbf{B} \cdot \nabla) \mathbf{B}, \partial_t \mathbf{v} \rangle_{L^2} \\
&\quad - \langle (\mathbf{v} \cdot \nabla) \mathbf{B}, \partial_t \mathbf{B} \rangle_{L^2} + \langle (\mathbf{B} \cdot \nabla) \mathbf{v}, \partial_t \mathbf{B} \rangle_{L^2} - \langle g_N(|y|^2)y, \partial_t y \rangle_{L^2} \\
&\quad + \langle f, \partial_t y \rangle_{\mathcal{H}^0} \\
&= -\langle \nabla y, \partial_t \nabla y(t) \rangle_{\mathcal{H}^0} - \langle (\mathbf{v} \cdot \nabla) \mathbf{v}, \partial_t \mathbf{v} \rangle_{L^2} + \langle (\mathbf{B} \cdot \nabla) \mathbf{B}, \partial_t \mathbf{v} \rangle_{L^2} \\
&\quad - \langle (\mathbf{v} \cdot \nabla) \mathbf{B}, \partial_t \mathbf{B} \rangle_{L^2} + \langle (\mathbf{B} \cdot \nabla) \mathbf{v}, \partial_t \mathbf{B} \rangle_{L^2} - \int g_N(|y|^2) \frac{1}{2} \partial_t |y|^2 dx \\
&\quad + \langle f, \partial_t y \rangle_{\mathcal{H}^0} \\
&= -\frac{1}{2} \frac{d}{dt} \|\nabla y\|_{\mathcal{H}^0}^2 - \langle (\mathbf{v} \cdot \nabla) \mathbf{v}, \partial_t \mathbf{v} \rangle_{L^2} + \langle (\mathbf{B} \cdot \nabla) \mathbf{B}, \partial_t \mathbf{v} \rangle_{L^2} \\
&\quad - \langle (\mathbf{v} \cdot \nabla) \mathbf{B}, \partial_t \mathbf{B} \rangle_{L^2} + \langle (\mathbf{B} \cdot \nabla) \mathbf{v}, \partial_t \mathbf{B} \rangle_{L^2} - \frac{1}{2} \frac{d}{dt} \|G_N(|y|^2)\|_{L^1} \\
&\quad + \langle f, \partial_t y \rangle_{\mathcal{H}^0} \\
&\leq -\frac{1}{2} \frac{d}{dt} \|\nabla y\|_{\mathcal{H}^0}^2 + \frac{1}{8} (2\|\partial_t \mathbf{v}(t)\|_{\mathcal{H}^0}^2 + 2\|\partial_t \mathbf{B}(t)\|_{\mathcal{H}^0}^2) \\
&\quad + 2(\|\mathbf{v}\| \|\nabla \mathbf{v}\|_{L^2}^2 + \|\mathbf{B}\| \|\nabla \mathbf{B}\|_{L^2}^2 + \|\mathbf{v}\| \|\nabla \mathbf{B}\|_{L^2}^2 + \|\mathbf{B}\| \|\nabla \mathbf{v}\|_{L^2}^2) \\
&\quad + \frac{1}{4} \|\partial_t y(t)\|_{\mathcal{H}^0}^2 + \|f(t)\|_{\mathcal{H}^0}^2 - \frac{1}{2} \frac{d}{dt} \|G_N(|y|^2)\|_{L^1} \\
&= -\frac{1}{2} \frac{d}{dt} \|\nabla y\|_{\mathcal{H}^0}^2 + \frac{1}{2} \|\partial_t y(t)\|_{\mathcal{H}^0}^2 + 2\| |y| \cdot |\nabla y| \|_{L^2}^2 \\
&\quad + \|f(t)\|_{\mathcal{H}^0}^2 - \frac{1}{2} \frac{d}{dt} \|G_N(|y|^2)\|_{L^1}.
\end{aligned}$$

Here, we denote by G_N a primitive function of g_N . Since $g_N(r) \leq 2r$, we find $0 \leq G_N(r) := \int_0^r g_N(s) ds \leq 2 \frac{r^2}{2} = r^2$. Integrating from 0 to T yields – estimating the nonpositive terms by zero – the following:

$$\begin{aligned}
&\frac{1}{2} \int_0^T \|\partial_t y(t)\|_{\mathcal{H}^0}^2 dt \\
&\leq \frac{1}{2} \|\nabla y(0)\|_{\mathcal{H}^0}^2 + \int_0^T (2\| |y| \cdot |\nabla y| \|_{L^2}^2 + \|f\|_{\mathcal{H}^0}^2) dt \\
(2.34) \quad &\quad + \frac{1}{2} \|G_N(|y(0)|^2)\|_{L^1} \\
&\leq \frac{1}{2} \|\nabla y(0)\|_{\mathcal{H}^0}^2 + 2TC_{T,N,y_0,f}^{(1)} + \int_0^T \|f\|_{\mathcal{H}^0}^2 dt + \frac{1}{2} \|y_0\|_{\mathcal{L}^4}^4 \\
&=: C_{T,N,y_0,f}^{(2)}.
\end{aligned}$$

(b) We first differentiate Equation (2.33) with respect to t and then take the inner product with $\partial_t y$ in \mathcal{H}^0 . Note that for $\boldsymbol{\theta}, \boldsymbol{\phi}, \boldsymbol{\psi} \in \{\mathbf{v}, \mathbf{B}\}$, we get

$$(2.35) \quad \langle \partial_t((\boldsymbol{\theta} \cdot \nabla) \boldsymbol{\phi}), \partial_t \boldsymbol{\psi} \rangle_{L^2} = \langle (\partial_t \boldsymbol{\theta} \cdot \nabla) \boldsymbol{\phi}, \partial_t \boldsymbol{\psi} \rangle_{L^2} + \langle (\boldsymbol{\theta} \cdot \nabla) \partial_t \boldsymbol{\phi}, \partial_t \boldsymbol{\psi} \rangle_{L^2}.$$

By the (anti-)symmetry of the nonlinear terms, if $\boldsymbol{\phi} = \boldsymbol{\psi}$, the second term vanishes, which accounts for the $(\mathbf{v} \cdot \nabla) \mathbf{v}$ and $(\mathbf{B} \cdot \nabla) \mathbf{B}$ terms. The other two nonlinear terms cancel each other, so we are left with four variations of the first term of the right-hand side of Equation (2.35), which can be simplified further

using the divergence-freeness to yield

$$\langle (\partial_t \boldsymbol{\theta} \cdot \nabla) \boldsymbol{\phi}, \partial_t \boldsymbol{\psi} \rangle_{L^2} = \langle \nabla \cdot (\partial_t \boldsymbol{\theta} \otimes \boldsymbol{\phi}), \partial_t \boldsymbol{\psi} \rangle_{L^2} = -\langle \partial_t \boldsymbol{\theta} \otimes \boldsymbol{\phi}, \partial_t \nabla \boldsymbol{\psi} \rangle_{L^2}.$$

Taking this into account and applying Young's inequality, we find

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_t y(t)\|_{\mathcal{H}^0}^2 \\ &= \langle \Delta \partial_t y(t), \partial_t y(t) \rangle_{\mathcal{H}^0} + \langle \partial_t f, \partial_t y \rangle_{\mathcal{H}^0} \\ &+ \langle \partial_t \mathbf{v} \otimes \mathbf{v}, \partial_t \nabla \mathbf{v} \rangle_{L^2} - \langle \partial_t \mathbf{B} \otimes \mathbf{B}, \partial_t \nabla \mathbf{v} \rangle_{L^2} \\ &+ \langle \partial_t \mathbf{v} \otimes \mathbf{B}, \partial_t \nabla \mathbf{B} \rangle_{L^2} - \langle \partial_t \mathbf{B} \otimes \mathbf{v}, \partial_t \nabla \mathbf{B} \rangle_{L^2} \\ &- \langle g_N(|y|^2) \partial_t y, \partial_t y \rangle_{L^2} - \langle g'_N(|y|^2) y \partial_t |y|^2, \partial_t y \rangle_{L^2} \\ &= -\|\partial_t \nabla y(t)\|_{\mathcal{H}^0}^2 + \langle \partial_t f, \partial_t y \rangle_{\mathcal{H}^0} \\ &+ \langle \partial_t \mathbf{v} \otimes \mathbf{v}, \partial_t \nabla \mathbf{v} \rangle_{L^2} - \langle \partial_t \mathbf{B} \otimes \mathbf{B}, \partial_t \nabla \mathbf{v} \rangle_{L^2} \\ &+ \langle \partial_t \mathbf{v} \otimes \mathbf{B}, \partial_t \nabla \mathbf{B} \rangle_{L^2} - \langle \partial_t \mathbf{B} \otimes \mathbf{v}, \partial_t \nabla \mathbf{B} \rangle_{L^2} \\ &- \|\sqrt{g_N(|y|^2)} |\partial_t y|\|_{L^2}^2 - \|\sqrt{g'_N(|y|^2)} |\partial_t |y|^2|\|_{L^2}^2 \\ &\leq -\|\partial_t \nabla y(t)\|_{\mathcal{H}^0}^2 + \frac{1}{4} \|\partial_t f\|_{\mathcal{H}^0}^2 + \|\partial_t y\|_{\mathcal{H}^0}^2 \\ &+ \|\|\mathbf{v}\| \|\partial_t \mathbf{v}\|_{L^2}^2 + \|\|\mathbf{B}\| \|\partial_t \mathbf{B}\|_{L^2}^2 + \|\|\mathbf{v}\| \|\partial_t \mathbf{B}\|_{L^2}^2 + \|\|\mathbf{B}\| \|\partial_t \mathbf{v}\|_{L^2}^2 \\ &+ \frac{1}{4} (2\|\partial_t \nabla \mathbf{v}\|_{\mathcal{H}^0}^2 + 2\|\partial_t \nabla \mathbf{B}\|_{\mathcal{H}^0}^2) \\ &- \|\sqrt{g_N(|y|^2)} |\partial_t y|\|_{L^2}^2 \\ &\leq -\frac{1}{2} \|\partial_t \nabla y(t)\|_{\mathcal{H}^0}^2 + \|\|y\| |\partial_t y|\|_{L^2}^2 + \frac{1}{4} \|\partial_t f\|_{\mathcal{H}^0}^2 + \|\partial_t y\|_{\mathcal{H}^0}^2 \\ &- \|\sqrt{2(|y|^2 - (N + \frac{1}{2}))} |\partial_t y|\|_{L^2}^2 \\ &= -\frac{1}{2} \|\partial_t \nabla y(t)\|_{\mathcal{H}^0}^2 - \|\|y\| |\partial_t y|\|_{L^2}^2 + \frac{1}{4} \|\partial_t f\|_{\mathcal{H}^0}^2 + 2(N + 1) \|\partial_t y\|_{\mathcal{H}^0}^2. \end{aligned}$$

Integrating from 0 to $t \leq T$ then gives (again estimating non-positive terms by zero)

$$\begin{aligned} & \|\partial_t y(t)\|_{\mathcal{H}^0}^2 \\ & \leq \|\partial_t y(0)\|_{\mathcal{H}^0}^2 + 4(N + 1) \int_0^T \|\partial_s y(s)\|_{\mathcal{H}^0}^2 ds + \frac{1}{2} \int_0^T \|\partial_s f(s)\|_{\mathcal{H}^0}^2 ds \\ (2.36) \quad & \leq C(1 + \|y_0\|_{\mathcal{H}^2}^6 + \|f(0)\|_{\mathcal{H}^0}^2) + 8(N + 1) C_{T,N,y_0,f}^{(2)} + \frac{1}{2} \int_0^T \|\partial_s f\|_{\mathcal{H}^0}^2 ds \\ & =: C_{T,N,y_0,f}^{(3)}. \end{aligned}$$

Here we used (2.34) as well as the following estimate for the time derivative of the initial condition: since (2.33) holds for all t , we can set $t = 0$ there to and take the \mathcal{H}^0 -norm to find

$$\begin{aligned} \|\partial_t y_0\|_{\mathcal{H}^0}^2 &\leq C \left(\|\Delta y_0\|_{\mathcal{H}^0}^2 + \|y_0\| \|\nabla y_0\|_{L^2}^2 + \|y_0\|_{\mathcal{H}^0}^6 + \|f(0)\|_{\mathcal{H}^0}^2 \right) \\ &\leq C \left(\|y_0\|_{\mathcal{H}^2}^2 + \|y_0\|_{\mathcal{H}^0}^2 \|\nabla y_0\|_{\mathcal{H}^0}^2 + \|y_0\|_{\mathcal{H}^2}^6 + \|f(0)\|_{\mathcal{H}^0}^2 \right) \\ &\leq C \left(\|y_0\|_{\mathcal{H}^2}^2 + \|y_0\|_{\mathcal{H}^0}^4 + \|\nabla y_0\|_{\mathcal{H}^0}^4 + \|y_0\|_{\mathcal{H}^2}^6 + \|f(0)\|_{\mathcal{H}^0}^2 \right) \\ &\leq C \left(1 + \|y_0\|_{\mathcal{H}^2}^6 + \|f(0)\|_{\mathcal{H}^0}^2 \right). \end{aligned}$$

(c) Finally, we take the \mathcal{H}^1 inner product with $y(t)$ and use Equation (2.10):

$$\begin{aligned} &\langle \partial_t y(t), y(t) \rangle_{\mathcal{H}^1} \\ &= \langle \mathcal{A}y(t), y(t) \rangle_{\mathcal{H}^1} + \langle f(t), y(t) \rangle_{\mathcal{H}^1} \\ &\leq -\frac{1}{2} \|y\|_{\mathcal{H}^2}^2 + \|y\|_{\mathcal{H}^0}^2 + 2(N+1) \|\nabla y\|_{\mathcal{H}^0}^2 - \|\mathbf{v}\| \|\nabla \mathbf{v}\|_{L^2}^2 - \|\mathbf{B}\| \|\nabla \mathbf{B}\|_{L^2}^2 \\ &\quad - \|\mathbf{v}\| \|\nabla \mathbf{B}\|_{L^2}^2 - \|\mathbf{B}\| \|\nabla \mathbf{v}\|_{L^2}^2 + \frac{1}{4} \|y\|_{\mathcal{H}^2}^2 + \|f\|_{\mathcal{H}^0}^2, \end{aligned}$$

which implies

$$\begin{aligned} &\|y(t)\|_{\mathcal{H}^2}^2 \\ &\leq 4\|y(t)\|_{\mathcal{H}^0}^2 + 8(N+1) \|\nabla y(t)\|_{\mathcal{H}^0}^2 + 4\|f(t)\|_{\mathcal{H}^0}^2 + 4|\langle \partial_t y(t), y(t) \rangle_{\mathcal{H}^1}| \\ &\leq 4\|y(t)\|_{\mathcal{H}^0}^2 + 8(N+1) \|\nabla y(t)\|_{\mathcal{H}^0}^2 + 4\|f(t)\|_{\mathcal{H}^0}^2 + 8\|\partial_t y(t)\|_{\mathcal{H}^0}^2 + \frac{1}{2} \|y(t)\|_{\mathcal{H}^2}^2, \end{aligned}$$

and hence, using (2.36) and (2.29),

$$\begin{aligned} &\sup_{t \in [0, T]} \|y(t)\|_{\mathcal{H}^2}^2 \\ &\leq 16(N+1) \sup_{t \in [0, T]} \|\nabla y(t)\|_{\mathcal{H}^0}^2 + 8 \sup_{t \in [0, T]} \|y(t)\|_{\mathcal{H}^0}^2 \\ &\quad + 8 \sup_{t \in [0, T]} \|f(t)\|_{\mathcal{H}^0}^2 + 16 \sup_{t \in [0, T]} \|\partial_t y(t)\|_{\mathcal{H}^0}^2 \\ &\leq C(N+1) C_{T, N, y_0, f}^{(1)} + 4 \left[\|y_0\|_{\mathcal{H}^0} + \int_0^T \|f\|_{\mathcal{H}^0} ds \right]^2 \\ &\quad + 8 \sup_{t \in [0, T]} \|f(t)\|_{\mathcal{H}^0}^2 + 16 C_{T, N, y_0, f}^{(3)} \\ &= C'_{T, N, y_0, f} + C_{T, N, y_0, f} (1 + N^2), \end{aligned}$$

i.e. (2.31). Equation (2.29) follows from (2.15), and Equation (2.30) follows from (2.16). This concludes the proof. \square

2.4. Convergence to the Untamed MHD Equations. In this section we stress the dependence of the solution to the tamed equation on N by writing y_N . We will prove that as $N \rightarrow \infty$, the solutions to the tamed equations converge to weak solutions of the untamed equations. The precise statement is given in the following theorem.

THEOREM 2.10 (Convergence to the untamed equations). *Let the data $y_0 \in \mathcal{H}^0$, $f \in L^2([0, T]; \mathcal{H}^0)$, $y_0^N \in \mathcal{H}^1$ be given such that $\mathcal{H}^0 - \lim_{N \rightarrow \infty} y_0^N = y_0$. Denote by (y_N, p_N, π_N) the unique solutions to the tamed equations (1.12) with initial value y_0^N given by Theorem 2.8.*

Then there is a subsequence $(N_k)_{k \in \mathbb{N}}$ such that y_{N_k} converges to a y in $L^2([0, T]; \mathcal{L}_{loc}^2)$ and p_{N_k} converges weakly to some p in $L^{9/8}([0, T]; L^{9/5}(\mathbb{R}^3))$. The magnetic pressure π_{N_k} converges to a zero, weakly in $L^{9/8}([0, T]; L^{9/5}(\mathbb{R}^3))$. Furthermore, (y, p) is a weak solution to the (untamed) MHD equations (1.1) such that the following generalised energy inequality holds: for any non-negative $\phi \in C_0^\infty((0, T) \times \mathbb{R}^3)$,

$$(2.37) \quad 2 \int_0^T \int_{\mathbb{R}^3} |\nabla y|^2 \phi dx ds \leq \int_0^T \int_{\mathbb{R}^3} \left[|y|^2 (\partial_t \phi + \Delta \phi) + 2 \langle y, f \rangle \phi + (|y|^2 - 2p) \langle \mathbf{v}, \nabla \phi \rangle - 2 \langle \mathbf{B}, \mathbf{v} \rangle \langle \mathbf{B}, \nabla \phi \rangle \right] dx ds.$$

REMARK 2.11. *Compared to the Navier-Stokes case, we get a different type of term in the inequality, namely the last one on the right-hand side of the inequality.*

Note also that the "magnetic pressure" disappears as $N_k \rightarrow \infty$.

PROOF. The proof follows along the same lines as that of Theorem 1.2 in [197].

Let $y_0^N \in \mathcal{H}^1$ with $y_0^N \rightarrow y_0$ in \mathcal{H}^0 and (y_N, p_N) be the associated unique strong solution given by Theorem 2.8. Combining (2.27) with (2.28) yields

$$(2.38) \quad \sup_{t \in [0, T]} \|y_N(t)\|_{\mathcal{H}^0}^2 + \int_0^T \|y_N\|_{\mathcal{H}^1}^2 + \|\sqrt{g_N(|y|^2)}|y|\|_{L^2}^2 ds \leq C_{y_0, f, T}.$$

For $q \in [2, \infty)$, $r \in (2, 6]$ such that

$$\frac{3}{r} + \frac{2}{q} = \frac{3}{2},$$

by an application of the Sobolev embedding (2.6) and (2.38) we find

$$(2.39) \quad \int_0^T \|y_N\|_{\mathcal{L}^r}^q dt \leq C_{1,0,2,2,r}^q \int_0^T \|y_N\|_{\mathcal{H}^1}^2 \|y_N\|_{\mathcal{H}^0}^{q-2} dt \leq C_{y_0, f, T, r, q}.$$

Employing the Arzelà-Ascoli theorem and the Helmholtz-Weyl decomposition in the same way as in the proof of Theorem 2.8, we find a subsequence y_{N_k} (again denoted by y_N) and a $y = (\mathbf{v}, \mathbf{B}) \in L^\infty([0, T]; \mathcal{H}^0) \cap L^2([0, T]; \mathcal{H}^1)$ such that for all $\tilde{y} = (\tilde{\mathbf{v}}, \tilde{\mathbf{B}}) \in \mathcal{L}^2$

$$(2.40) \quad \lim_{N \rightarrow \infty} \sup_{t \in [0, T]} |\langle y_N(t) - y(t), \tilde{y} \rangle_{\mathcal{L}^2}| = 0.$$

In fact, we can even prove that for every bounded open set $G \subset \mathbb{R}^3$, we have

$$(2.41) \quad \lim_{N \rightarrow \infty} \int_0^T \int_G |y_N(t, x) - y(t, x)|^2 dx dt = 0.$$

To this end, let $G \subset \bar{G} \subset Q$ for a cuboid Q , and ρ be a smooth, non-negative cutoff function with $\rho \equiv 1$ on G , $\rho \equiv 0$ on $\mathbb{R}^3 \setminus Q$, and by Friedrichs' inequality (2.22)

$$\begin{aligned}
& \int_0^T \int_G |y_N(t, x) - y(t, x)|^2 dx dt \\
&= \int_0^T \int_G |\mathbf{v}_N(t, x) - \mathbf{v}(t, x)|^2 dx dt + \int_0^T \int_G |\mathbf{B}_N(t, x) - \mathbf{B}(t, x)|^2 dx dt \\
&\leq \int_0^T \int_Q |\mathbf{v}_N(t, x) - \mathbf{v}(t, x)|^2 \rho^2(x) dx dt + \int_0^T \int_Q |\mathbf{B}_N(t, x) - \mathbf{B}(t, x)|^2 \rho^2(x) dx dt \\
&\leq \sum_{i=1}^{K_\varepsilon} \int_0^T \left(\int_Q (\mathbf{v}_N(t, x) - \mathbf{v}(t, x)) \rho(x) h_i^\varepsilon(x) dx \right)^2 dt \\
&+ \sum_{i=1}^{K_\varepsilon} \int_0^T \left(\int_Q (\mathbf{B}_N(t, x) - \mathbf{B}(t, x)) \rho(x) h_i^\varepsilon(x) dx \right)^2 dt \\
&+ \varepsilon \int_0^T \int_Q |\nabla((\mathbf{v}_N - \mathbf{v})\rho)(x)|^2 dx dt + \varepsilon \int_0^T \int_Q |\nabla((\mathbf{B}_N - \mathbf{B})\rho)(x)|^2 dx dt \\
&=: I_1(N, \varepsilon) + I_2(N, \varepsilon) + I_3(N, \varepsilon) + I_4(N, \varepsilon).
\end{aligned}$$

The terms $I_1(N, \varepsilon)$, $I_2(N, \varepsilon)$ vanish for $N \rightarrow \infty$ as using (2.40) we get

$$\lim_{N \rightarrow \infty} I_1(N, \varepsilon) \leq T \sum_{i=1}^{K_\varepsilon} \lim_{N \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_{\mathbb{R}^3} (\mathbf{v}_N(t, x) - \mathbf{v}(t, x)) \rho(x) h_i^\varepsilon(x) 1_Q(x) dx \right|^2 = 0,$$

and an analogous computation yields $\lim_{N \rightarrow \infty} I_2(N, \varepsilon) = 0$.

The other two terms can be bounded by

$$I_3(N, \varepsilon) + I_4(N, \varepsilon) \leq \varepsilon \cdot C_\rho \int_0^T (\|y_N(t)\|_{\mathcal{H}^1}^2 + \|y(t)\|_{\mathcal{H}^1}^2) dt \leq C_{\rho, y_0, T, f} \cdot \varepsilon,$$

and the arbitrariness of $\varepsilon > 0$ implies the claim.

Next we prove that for any $\tilde{y} \in \mathcal{V}$

$$\lim_{N \rightarrow \infty} \int_0^t \langle g_N(|y_N(s)|^2) y_N(s), \tilde{y} \rangle_{\mathcal{H}^0} ds = 0.$$

This can be seen as follows:

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \int_0^t \langle g_N(|y_N(s)|^2) y_N(s), \tilde{y} \rangle_{\mathcal{H}^0} ds \\
&\leq \|\tilde{y}\|_{L^\infty} \cdot \limsup_{N \rightarrow \infty} \int_0^t \int_{\mathbb{R}^3} |y_N(s)|^3 \cdot 1_{\{|y_N(s, x)|^2 \geq N\}} dx ds \\
&\leq \|\tilde{y}\|_{L^\infty} \cdot \limsup_{N \rightarrow \infty} \left(\int_0^t \|y_N(s)\|_{\mathcal{L}^{10/3}}^{10/3} ds \right)^{9/10} \cdot \left(\int_0^t \int_{\mathbb{R}^3} 1_{\{|y_N(s, x)|^2 \geq N\}} dx ds \right)^{1/10} \\
&\leq C_{\tilde{y}, y_0, T, f} \cdot \limsup_{N \rightarrow \infty} \left(\frac{1}{N} \int_0^t \|y_N(s)\|_{\mathcal{H}^0}^2 ds \right)^{1/10} = 0,
\end{aligned}$$

where we have used (2.39) and Chebychev's inequality.

As in the proof of Theorem 2.8, there are $p_N, \pi_N \in L^2([0, T]; L^2_{\text{loc}}(\mathbb{R}^3))$ such that $\nabla p_N, \nabla \pi_N \in L^2([0, T]; L^2(\mathbb{R}^3; \mathbb{R}^3))$, and we have for almost all $t \geq 0$ that

$$(2.42) \quad \frac{\partial \mathbf{v}_N}{\partial t} = \Delta \mathbf{v}_N - (\mathbf{v}_N \cdot \nabla) \mathbf{v}_N + (\mathbf{B}_N \cdot \nabla) \mathbf{B}_N + \nabla \left(p_N + \frac{|\mathbf{B}_N|^2}{2} \right) - g_N(|(\mathbf{v}_N, \mathbf{B}_N)|^2) \mathbf{v}_N + \mathbf{f}_1$$

$$(2.43) \quad \frac{\partial \mathbf{B}_N}{\partial t} = \Delta \mathbf{B}_N - (\mathbf{v}_N \cdot \nabla) \mathbf{B}_N + (\mathbf{B}_N \cdot \nabla) \mathbf{v}_N + \nabla \pi_N - g_N(|(\mathbf{v}_N, \mathbf{B}_N)|^2) \mathbf{B}_N + \mathbf{f}_2.$$

To derive the generalised energy inequality, we take a non-negative $\phi \in C_0^\infty((0, T) \times \mathbb{R}^3)$ and then take the inner products with $2y_N \phi$ in \mathcal{H}^0 of this equation. Let us use the abbreviation $\iint = \int_0^T \int_{\mathbb{R}^3} dx dt$. Then we get

$$(2.44) \quad \iint \partial_t \mathbf{v}_N \cdot 2\mathbf{v}_N \phi + \iint \partial_t \mathbf{B}_N \cdot 2\mathbf{B}_N \phi$$

$$(2.45) \quad = \iint \Delta \mathbf{v}_N \cdot 2\mathbf{v}_N \phi + \iint \Delta \mathbf{B}_N \cdot 2\mathbf{B}_N \phi$$

$$(2.46) \quad - \iint (\mathbf{v}_N \cdot \nabla) \mathbf{v}_N \cdot 2\mathbf{v}_N \phi + \iint (\mathbf{B}_N \cdot \nabla) \mathbf{B}_N \cdot 2\mathbf{v}_N \phi$$

$$(2.47) \quad - \iint (\mathbf{v}_N \cdot \nabla) \mathbf{B}_N \cdot 2\mathbf{B}_N \phi + \iint (\mathbf{B}_N \cdot \nabla) \mathbf{v}_N \cdot 2\mathbf{B}_N \phi$$

$$(2.48) \quad - 2 \iint g_N(|y_N|^2) |\mathbf{v}_N|^2 \phi - 2 \iint g_N(|y_N|^2) |\mathbf{B}_N|^2 \phi$$

$$(2.49) \quad + 2 \iint \nabla p_N \cdot \mathbf{v}_N \phi + 2 \iint \nabla \pi_N \cdot \mathbf{B}_N \phi + 2 \iint \langle f, y_N \rangle \phi.$$

Let us discuss this equation line by line. The first line (2.44) is a simple application of integration by parts (with respect to the time variable):

$$\begin{aligned} \iint \partial_t y_N \cdot y_N \phi &= - \int \int y_N \cdot \partial_t (y_N \phi) \\ &= - \iint y_N \cdot (\partial_t y_N) \phi - \iint |y_N|^2 \partial_t \phi \end{aligned}$$

which in turn yields

$$2 \iint \partial_t y_N \cdot y_N \phi = - \iint |y_N|^2 \partial_t \phi.$$

For the second line (2.45), we proceed along similar lines, this time with respect to the space variable. To avoid confusion, we will write the equation in components:

$$\begin{aligned} &2 \iint \Delta \mathbf{v}_N \cdot \mathbf{v}_N \phi + 2 \iint \Delta \mathbf{B}_N \cdot \mathbf{B}_N \phi \\ &= 2 \iint \sum_{i,k} (\partial_i \partial_i \mathbf{v}_N^k) \mathbf{v}_N^k \phi + 2 \iint \sum_{i,k} (\partial_i \partial_i \mathbf{B}_N^k) \mathbf{B}_N^k \phi \\ &= -2 \iint \sum_{i,k} (\partial_i \mathbf{v}_N^k) \partial_i (\mathbf{v}_N^k \phi) - 2 \iint \sum_{i,k} (\partial_i \mathbf{B}_N^k) \partial_i (\mathbf{B}_N^k \phi). \end{aligned}$$

We will focus on the velocity terms, the magnetic field works in exactly the same way.

$$-2 \iint \sum_{i,k} (\partial_i \mathbf{v}_N^k) \partial_i (\mathbf{v}_N^k \phi) = -2 \iint \sum_{i,k} |\partial_i \mathbf{v}_N^k|^2 \phi - 2 \iint (\partial_i \mathbf{v}_N^k) \mathbf{v}_N^k \partial_i \phi.$$

The last term on the right-hand side equals, after another application of integration by parts,

$$\begin{aligned} & -2 \iint (\partial_i \mathbf{v}_N^k) \mathbf{v}_N^k \partial_i \phi = 2 \iint \mathbf{v}_N^k \partial_i (\mathbf{v}_N^k \partial_i \phi) \\ & = 2 \iint (\partial_i \mathbf{v}_N^k) \mathbf{v}_N^k \partial_i \phi + 2 \iint \mathbf{v}_N^k \mathbf{v}_N^k (\partial_i^2 \phi), \end{aligned}$$

and thus

$$-2 \iint (\partial_i \mathbf{v}_N^k) \mathbf{v}_N^k \partial_i \phi = \iint |\mathbf{v}_N|^2 \Delta \phi.$$

Hence, (2.45) can be rewritten as

$$2 \iint \Delta \mathbf{y}_N \cdot \mathbf{y}_N \phi = - \iint |\nabla y_N|^2 \phi + \iint |y_N|^2 \Delta \phi.$$

The third line, (2.46) will be dealt with term by term. By the incompressibility condition

$$-2 \iint (\mathbf{v}_N \cdot \nabla) \mathbf{v}_N \cdot \mathbf{v}_N \phi = 2 \iint |\mathbf{v}_N|^2 \nabla \cdot (\mathbf{v}_N \phi) = 2 \iint |\mathbf{v}_N|^2 \mathbf{v}_N \cdot \nabla \phi.$$

The second term, in a similar fashion, becomes (again using the divergence-freeness)

$$\begin{aligned} & 2 \iint (\mathbf{B}_N \cdot \nabla) \mathbf{B}_N \cdot \mathbf{v}_N \phi = 2 \sum_{i,k} \iint \mathbf{B}_N^i (\partial_i \mathbf{B}_N^k) \mathbf{v}_N^k \phi \\ & = -2 \sum_{i,k} \iint \mathbf{B}_N^i \mathbf{B}_N^k \partial_i (\mathbf{v}_N^k \phi) \\ & = -2 \sum_{i,k} \iint \mathbf{B}_N^i \mathbf{B}_N^k (\partial_i \mathbf{v}_N^k) \phi - 2 \sum_{i,k} \iint \mathbf{B}_N^i \mathbf{B}_N^k \mathbf{v}_N^k \partial_i \phi \\ & = -2 \iint (\mathbf{B}_N \cdot \nabla) \mathbf{v}_N \cdot \mathbf{B}_N \phi - 2 \iint (\mathbf{B}_N \cdot \mathbf{v}_N) (\mathbf{B}_N \cdot \nabla \phi). \end{aligned}$$

The first term of the last line here cancels with the second term of (2.47). Thus we only have to deal with the first term of (2.47):

$$\begin{aligned} & -2 \iint (\mathbf{v}_N \cdot \nabla) \mathbf{B}_N \cdot \mathbf{B}_N \phi = -2 \sum_{i,k} \iint \mathbf{v}_N^i (\partial_i \mathbf{B}_N^k) \mathbf{B}_N^k \phi \\ & = 2 \sum_{i,k} \iint \mathbf{B}_N^k \partial_i (\mathbf{v}_N^i \mathbf{B}_N^k \phi) = 2 \sum_{i,k} \iint \mathbf{v}_N^i \mathbf{B}_N^k \partial_i (\mathbf{B}_N^k \phi) \\ & = 2 \sum_{i,k} \iint \mathbf{v}_N^i \mathbf{B}_N^k (\partial_i \mathbf{B}_N^k) \phi + 2 \sum_{i,k} \iint \mathbf{v}_N^i \mathbf{B}_N^k \mathbf{B}_N^k \partial_i \phi \\ & = 2 \iint (\mathbf{v}_N \cdot \nabla) \mathbf{B}_N \cdot \mathbf{B}_N \phi + 2 \iint |\mathbf{B}_N|^2 \mathbf{v}_N \cdot \nabla \phi, \end{aligned}$$

and therefore

$$-2 \iint (\mathbf{v}_N \cdot \nabla) \mathbf{B}_N \cdot \mathbf{B}_N \phi = \iint |\mathbf{B}_N|^2 \mathbf{v}_N \cdot \nabla \phi.$$

The last terms that we have to treat are the pressure terms of (2.49). For the first term, we find, again by integration by parts and the incompressibility

$$2 \iint \nabla p_N \cdot \mathbf{v}_N \phi = -2 \iint p_N \mathbf{v}_N \cdot \nabla \phi,$$

and similarly for the second term

$$2 \iint \nabla \pi_N \cdot \mathbf{B}_N \phi = -2 \iint \pi_N \mathbf{B}_N \cdot \nabla \phi.$$

Thus, altogether we find that

$$(2.50) \quad \begin{aligned} & 2 \int_0^T \int_{\mathbb{R}^3} |\nabla y_N|^2 \phi \, dx ds + 2 \int_0^T \int_{\mathbb{R}^3} g_N(|y_N|^2) |y_N|^2 \phi \, dx ds \\ &= \int_0^T \int_{\mathbb{R}^3} \left[|y_N|^2 (\partial_t \phi + \Delta \phi) + 2 \langle y_N, f \rangle \phi - 2 \pi_N \langle \mathbf{B}_N, \nabla \phi \rangle_{\mathbb{R}^3} \right. \\ & \quad \left. + (|y_N|^2 - 2p_N) \langle \mathbf{v}_N, \nabla \phi \rangle_{\mathbb{R}^3} - 2 \langle \mathbf{B}_N, \mathbf{v}_N \rangle_{\mathbb{R}^3} \langle \mathbf{B}_N, \nabla \phi \rangle_{\mathbb{R}^3} \right] dx ds. \end{aligned}$$

Since $0 \leq \phi \in C_c^\infty$, it acts as a density, and thus, by [53, Theorem 1.2.1], the map $y \mapsto \int_0^T \int_{\mathbb{R}^3} |\nabla y|^2 \phi \, dx ds$ is lower semi-continuous in $L^2([0, T]; \mathcal{H}^0)$. Thus, the limit of the left-hand side of (2.50) as $N \rightarrow \infty$ is greater than or equal to

$$\begin{aligned} & \liminf_{N \rightarrow \infty} \int_0^T \int_{\mathbb{R}^3} |\nabla y_N|^2 \phi \, dx ds + 2 \int_0^T \int_{\mathbb{R}^3} g_N(|y_N|^2) |y_N|^2 \phi \, dx ds \\ & \geq \liminf_{N \rightarrow \infty} \int_0^T \int_{\mathbb{R}^3} |\nabla y_N|^2 \phi \, dx ds \geq 2 \int_0^T \int_{\mathbb{R}^3} |\nabla y|^2 \phi \, dx ds. \end{aligned}$$

On the other hand, the limit of the right-hand side as $N \rightarrow \infty$ consists of four terms, which we treat individually. We denote $G := \text{supp } \phi$.

For the first term, by Cauchy-Schwarz-Buniakowski, (2.38) and (2.41), we find

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^3} (|y_N|^2 - |y|^2) (\partial_t \phi + \Delta \phi) \, dx ds \\ & \leq \int_0^T \int_G |y_N - y| (|y_N| + |y|) (\partial_t \phi + \Delta \phi) \, dx ds \\ & \leq C_\phi \left(\int_0^T \int_G |y_N - y|^2 dx ds \right)^{1/2} \left(\int_0^T \int_G 2(|y_N|^2 + |y|^2) dx ds \right)^{1/2} \\ & \leq C_{\phi, y_0, T, f} \left(\int_0^T \int_G |y_N - y|^2 dx ds \right)^{1/2} \xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

The second term can be treated in a similar fashion:

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^3} 2 \langle y_N - y, f \rangle \phi \, dx ds \\ & \leq \left(\int_0^T \int_G |y_N - y|^2 dx ds \right)^{1/2} \left(\int_0^T \int_G |f|^2 \phi^2 dx ds \right)^{1/2} \\ & \leq C_\phi \|f\|_{L^2([0, T]; \mathcal{H}^0)} \left(\int_0^T \int_G |y_N - y|^2 dx ds \right)^{1/2} \xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

For the term

$$\int_0^T \int_{\mathbb{R}^3} |y_N|^2 \langle \mathbf{v}_N, \nabla \phi \rangle_{\mathbb{R}^3} dx ds = \int_0^T \int_G |y_N|^2 \left\langle \mathbf{v}_N, 1_G \frac{\nabla \phi}{\phi} \right\rangle_{\mathbb{R}^3} \phi dx ds,$$

we note that since $\phi \in C_c^\infty((0, T) \times \mathbb{R}^3)$, (2.41) implies convergence in measure for the finite measure $\mu := \phi dx \otimes ds$. This can be seen as follows: Let $\varepsilon > 0$. Define $X_N := 1_G |y_N|^2 \left\langle \mathbf{v}_N, \frac{\nabla \phi}{\phi} \right\rangle_{\mathbb{R}^3}$. By the binomial formula and Young's inequality

$$\begin{aligned} & \left| |y_N|^2 \left\langle \mathbf{v}_N, 1_G \frac{\nabla \phi}{\phi} \right\rangle_{\mathbb{R}^3} - |y|^2 \left\langle \mathbf{v}, 1_G \frac{\nabla \phi}{\phi} \right\rangle_{\mathbb{R}^3} \right| \\ & \leq \frac{3}{2} |y_N - y| (|y_N|^2 + |y|^2) 1_G \left| \frac{\nabla \phi}{\phi} \right|. \end{aligned}$$

Set $A := \left\{ \frac{3}{2} |y_N - y| (|y_N|^2 + |y|^2) 1_G \left| \frac{\nabla \phi}{\phi} \right| > \varepsilon \right\}$. Fix an arbitrary $\delta > 0$. By the above computation we find

$$\begin{aligned} \mu(|X_N - X| > \varepsilon) & \leq \mu \left(\frac{3}{2} |y_N - y| (|y_N|^2 + |y|^2) 1_G \left| \frac{\nabla \phi}{\phi} \right| > \varepsilon \right) \\ & = \mu(A \cap \{|y_N - y| > \delta\}) + \mu(A \cap \{|y_N - y| \leq \delta\}) \\ & \leq \mu(|y_N - y| > \delta) + \mu(A \cap \{|y_N - y| \leq \delta\}). \end{aligned}$$

The first term converges to zero by the Chebychev inequality and (2.41):

$$\begin{aligned} \mu(|y_N - y| > \delta) & \leq \frac{1}{\delta^2} \int \int_G |y_N - y|^2 \phi dx ds \\ & \leq \frac{C_\phi}{\delta^2} \int \int_G |y_N - y|^2 dx ds \xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

The second term can be bounded by the Markov inequality and (2.38):

$$\begin{aligned} \mu(A \cap \{|y_N - y| \leq \delta\}) & \leq \mu \left(\frac{3}{2} \delta (|y_N|^2 + |y|^2) 1_G \left| \frac{\nabla \phi}{\phi} \right| > \varepsilon \right) \\ & \leq \delta \cdot \frac{3}{2\varepsilon} \int \int_G (|y_N|^2 + |y|^2) 1_G \left| \frac{\nabla \phi}{\phi} \right| \phi dx ds \\ & \leq \delta \cdot \frac{3}{2\varepsilon} \int \int_G (|y_N|^2 + |y|^2) |\nabla \phi| dx ds \\ & \leq \delta \cdot \frac{3}{2\varepsilon} C_{\phi, y_0, T, f}. \end{aligned}$$

The claim $\lim_{N \rightarrow \infty} \mu(|X_N - X| > \varepsilon) = 0$ now follows since $\delta > 0$ was arbitrary.

Furthermore, the family $(X_N)_N$ lies in $L^1([0, T] \times \mathbb{R}^3, \mu)$ since by (2.39) for $r = q = 10/3$

$$\begin{aligned} \|X_N\|_{L^1(\mu)} & \leq \int_0^T \int_{\mathbb{R}^3} |y_N|^3 |\nabla \phi| dx ds \\ & \leq \left(\int_0^T \int_{\mathbb{R}^3} |y_N|^{10/3} dx ds \right)^{9/10} \left(\int_0^T \int_{\mathbb{R}^3} |\nabla \phi|^{10} dx ds \right)^{1/10} \\ & \leq C_{y_0, T, f} \cdot C_\phi < \infty. \end{aligned}$$

Finally, using (2.39) in a similar way, we see that the family $(X_N)_N$ is μ -uniformly integrable:

$$\begin{aligned}
& \limsup_{c \rightarrow \infty} \sup_N \int \int_{\{|X_N| \geq c\}} |X_N| \mu(dx, ds) = \limsup_{c \rightarrow \infty} \sup_N \int \int 1_{\{|X_N| \geq c\}} |X_N| \phi dx ds \\
& = \limsup_{c \rightarrow \infty} \sup_N \int \int 1_{\{|X_N| \geq c\}} |y_N|^2 |\langle \mathbf{v}_N, \nabla \phi \rangle_{\mathbb{R}^3}| dx ds \\
& \leq \limsup_{c \rightarrow \infty} \sup_N \int \int 1_{\{|X_N| \geq c\}} |y_N|^3 |\nabla \phi| dx ds \\
& \leq \limsup_{c \rightarrow \infty} \sup_N \left(\int \int |y_N|^{10/3} dx ds \right)^{9/10} \left(\int \int 1_{\{|X_N| \geq c\}} |\nabla \phi| dx ds \right)^{1/10} \\
& \leq C_{y_0, T, f} \limsup_{c \rightarrow \infty} \sup_N \left(\int \int 1_{\{|X_N| \geq c\}} |\nabla \phi| dx ds \right)^{1/10} = 0.
\end{aligned}$$

Thus, by the generalised Lebesgue dominated convergence theorem we get

$$\int_0^T \int_{\mathbb{R}^3} |y_N|^2 \langle \mathbf{v}_N, \nabla \phi \rangle_{\mathbb{R}^3} dx ds \xrightarrow{N \rightarrow \infty} \int_0^T \int_{\mathbb{R}^3} |y|^2 \langle \mathbf{v}, \nabla \phi \rangle_{\mathbb{R}^3} dx ds.$$

Moving on with the energy inequality, the last term of (2.50) can be treated in the same way as just discussed. We are left with the pressure term. As in [197], pp. 547 f., we take the divergence of (2.42) to find

$$\begin{aligned}
(2.51) \quad \Delta p_N &= \operatorname{div} \left((\mathbf{v}_N \cdot \nabla) \mathbf{v}_N - (\mathbf{B}_N \cdot \nabla) \mathbf{B}_N - \nabla \frac{|\mathbf{B}_N|^2}{2} + g_N(|y_N|^2) \mathbf{v}_N \right) \\
&= \operatorname{div} \left((\mathbf{v}_N \cdot \nabla) \mathbf{v}_N - (\mathbf{B}_N \cdot \nabla) \mathbf{B}_N - \mathbf{B}_N \cdot (\nabla \mathbf{B}_N) + g_N(|y_N|^2) \mathbf{v}_N \right).
\end{aligned}$$

Similarly, we take the divergence of (2.43) and obtain⁸

$$(2.52) \quad \Delta \pi_N = \operatorname{div} (g_N(|y_N|^2) \mathbf{B}_N).$$

We note that for N sufficiently large

$$(g_N(r))^{9/8} \cdot r^{9/16} \leq C g_N(r) \cdot r.$$

This is obviously true on the set $\{r \mid g_N(r) = 0\}$. If $r > 0$ is such that $g_N(r) > 0$ (which implies $r \geq 1$), we have

$$\begin{aligned}
& (g_N(r))^{9/8} \cdot r^{9/16} \leq g_N(r) (g_N(r))^{1/8} \cdot r^{9/16} \\
& \leq g_N(r) 2^{1/8} r^{1/8} \cdot r^{9/16} \leq g_N(r) 2^{1/8} \cdot r^{11/16} \leq 2^{1/8} g_N(r) \cdot r,
\end{aligned}$$

where the factor of 2 appears due to the definition of the taming function. Using this inequality and (2.38), we find

$$(2.53) \quad \int_0^T \int_{\mathbb{R}^3} |g_N(|y|^2) y|^{9/8} dx dt \leq C \int_0^T \int_{\mathbb{R}^3} g_N(|y|^2) \cdot |y|^2 dx dt \leq C_{T, y_0, f}.$$

⁸Noting that $(\mathbf{v}_N \cdot \nabla) \mathbf{B}_N - (\mathbf{B}_N \cdot \nabla) \mathbf{v}_N = \nabla \times (\mathbf{v} \times \mathbf{B})$, which is divergence-free, cf. Section 1.3.4.

For the first three nonlinear terms on the right-hand side of (2.51), we note that by Hölder's inequality (first for the product measure $dx \otimes dt$ and with $p = 16/7$, $q = 16/9$, then for dt with $p = 14/6$, $q = 14/8$) and the Sobolev embedding (2.6) we have

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^3} |(\mathbf{v}_N \cdot \nabla) \mathbf{v}_N|^{9/8} dx dt \leq \int_0^T \int_{\mathbb{R}^3} |\mathbf{v}_N|^{9/8} |\nabla \mathbf{v}_N|^{9/8} dx dt \\
& \leq \left(\int_0^T \|\mathbf{v}_N\|_{L^{18/7}}^{18/7} dt \right)^{\frac{7}{16}} \cdot \left(\int_0^T \|\mathbf{v}_N\|_{\mathbb{H}^1}^2 dt \right)^{\frac{9}{16}} \\
& \leq \left(\int_0^T \|y_N\|_{\mathcal{L}^{18/7}}^{18/7} dt \right)^{\frac{7}{16}} \cdot \left(\int_0^T \|y_N\|_{\mathcal{H}^1}^2 dt \right)^{\frac{9}{16}} \\
(2.54) \quad & \leq C_{y_0, T, f} \left(C_{2,0,2,2,18/7}^{18/7} \int_0^T \left(\|y_N\|_{\mathcal{H}^1}^{1/3} \|y_N\|_{\mathcal{L}^2}^{2/3} \right)^{18/7} dt \right)^{7/16} \\
& = C_{y_0, T, f} C_{2,0,2,2,18/7}^{9/8} \left(\int_0^T \|y_N\|_{\mathcal{H}^1}^{6/7} \|y_N\|_{\mathcal{L}^2}^{12/7} dt \right)^{7/16} \\
& \leq C_{y_0, T, f} C_{2,0,2,2,18/7}^{9/8} \left(\left(\int_0^T \|y_N\|_{\mathcal{H}^1}^2 dt \right)^{6/14} \left(\int_0^T \|y_N\|_{\mathcal{L}^2}^3 dt \right)^{8/14} \right)^{7/16} \\
& \leq C_{T, y_0, f}.
\end{aligned}$$

The other terms can be bounded in the same way.

Again using the interpolation inequality (2.6), this time with $j = 0$, $m = 1$, $p = 9/8$, $r = 9/5$ and $\alpha = 1$, we find

$$(2.55) \quad \int_0^T \|p_N\|_{L^{9/5}}^{9/8} dt \leq C_{1,0,9/8,9/8,9/5}^{9/8} \int_0^T \|p_N\|_{1,9/8}^{9/8} dt.$$

Recall that $\Delta p_N = \operatorname{div} \mathcal{R}_N$, where \mathcal{R}_N is defined by (2.51). Then we have

$$\begin{aligned}
\|p_N\|_{1,9/8} &= \|(I - \Delta)^{1/2} \Delta^{-1} \Delta p_N\|_{L^{9/8}} \\
&= \|(I - \Delta)^{1/2} \Delta^{-1} \operatorname{div} \mathcal{R}_N\|_{L^{9/8}} \\
&= \|(I - \Delta)^{1/2} (\operatorname{div} \nabla)^{-1} \operatorname{div} \mathcal{R}_N\|_{L^{9/8}} \\
&= \|(I - \Delta)^{1/2} \nabla^{-1} \operatorname{div}^{-1} \operatorname{div} \mathcal{R}_N\|_{L^{9/8}} \\
&= \|(I - \Delta)^{1/2} \nabla^{-1} \mathcal{R}_N\|_{L^{9/8}} \leq \|\mathcal{R}_N\|_{L^{9/8}}.
\end{aligned}$$

In the last step, we used the L^p theory for singular integrals, e.g. [207, Chapter V.3.2, Lemma 2, p. 133 f.]. By (2.53) and (2.54) it follows that the right-hand side of (2.55) is uniformly bounded in N .

Therefore, by the Eberlein-Šmuljan theorem (cf. [248, Theorem 21.D, p. 255]), there is a subsequence $(p_{N_k})_k$ and a function

$$p \in L^{9/8}([0, T]; L^{9/5}(\mathbb{R}^3; \mathbb{R}^3))$$

such that for $k \rightarrow \infty$

$$(2.56) \quad p_{N_k} \rightarrow p \quad \text{weakly in } L^{9/8}([0, T]; L^{9/5}(\mathbb{R}^3; \mathbb{R}^3)).$$

Finally, by another application of (2.39), with $q = 12$, $r = 9/4$ (so $\frac{3}{9/4} + \frac{2}{12} = \frac{12}{9} + \frac{2}{12} = \frac{54}{36} = \frac{3}{2}$), we find

$$\int_0^T \|y_N\|_{\mathcal{L}^{9/4}}^{12} dt \leq C_{T, y_0, f}.$$

Thus, in the same way as above, we can employ the generalised Lebesgue dominated convergence theorem to conclude that for $\phi \in C_0^\infty((0, T) \times \mathbb{R}^3)$

$$\begin{aligned} & \lim_{N \rightarrow \infty} \left| \int_0^T \int_{\mathbb{R}^3} \langle p_N \mathbf{v}_N - p \mathbf{v}, \nabla \phi \rangle_{\mathbb{R}^3} dx dt \right| \\ & \leq \lim_{N \rightarrow \infty} \left| \int_0^T \int_{\mathbb{R}^3} (p_N - p) \langle \mathbf{v}, \nabla \phi \rangle_{\mathbb{R}^3} dx dt \right| \\ & \quad + \left| \int_0^T \int_{\mathbb{R}^3} p_N \langle \mathbf{v}_N - \mathbf{v}, \nabla \phi \rangle_{\mathbb{R}^3} dx dt \right| \\ & \leq \lim_{N \rightarrow \infty} \left(\|p_N\|_{L^{9/5}}^{9/8} dt \right)^{8/9} \left(\int_0^T \|\mathbf{v}_N - \mathbf{v}\| \cdot \|\nabla \phi\|_{L^{9/4}}^9 dt \right)^{1/9} = 0. \end{aligned}$$

In exactly the same way we find a subsequence $(N_k)_{k \in \mathbb{N}}$ such that

$$\pi_{N_k} \rightarrow \pi \quad \text{weakly in } L^{9/8}([0, T]; L^{9/5}(\mathbb{R}^3; \mathbb{R}^3)).$$

The limit π satisfies the equation

$$\Delta \pi = 0,$$

which, combined with the integrability property of π yields $\pi \equiv 0$, thus eliminating the "magnetic pressure" from the resulting weak equation as well as the generalised energy inequality. Hence we have shown that the solutions to the tamed MHD equations converge to suitable weak solutions to the MHD equations. \square

REMARK 2.12. *It is to be expected that existence and uniqueness in the case of a bounded domain $\mathbb{D} \subset \mathbb{R}^3$ can be shown in a similar way as for the tamed Navier-Stokes equations, as in W. Liu and M. Röckner [167, p. 170 ff.]. However, their method uses an inequality [167, Equation (5.61), p. 166], sometimes called Xie's inequality, for the L^∞ -norm of a function in terms of the L^2 -norms of the gradient and the Laplacian (more precisely, the Stokes operator). This inequality holds for Dirichlet boundary conditions on quite general domains (cf. R.M. Brown, Z.W. Shen [20, Equation (0.2), p. 1184] for Lipschitz boundaries). If we were to use method of [167], we would need to have a similar inequality for the magnetic field as well. Unfortunately, to the best of the author's knowledge, such an inequality has not yet been established for the boundary conditions*

$$\mathbf{B} \cdot \boldsymbol{\nu} = 0, \quad (\nabla \times \mathbf{B}) \times \boldsymbol{\nu} \quad \text{on } \partial \mathbb{D}$$

of the magnetic field (which mean that the boundary is perfectly conducting, cf. [201, Equation (1.3), p. 637]). Here, $\boldsymbol{\nu}$ is the outward unit normal vector of the boundary of the domain. If such an inequality could be shown, the rest of the proof of Liu and Röckner should follow in exactly the same way.

CHAPTER III

The Stochastic Tamed MHD Equations

ABSTRACT. We study the tamed magnetohydrodynamics equations perturbed by multiplicative Wiener noise of transport type on the whole space \mathbb{R}^3 and on the torus \mathbb{T}^3 . In a first step, existence of a unique strong solution are established by constructing a weak solution, proving that pathwise uniqueness holds and using the Yamada-Watanabe theorem. We then study the associated Markov semigroup and prove that it has the Feller property. Finally, existence of an invariant measure of the equation is shown for the case of the torus.

1. Introduction

In this chapter, we consider a randomly perturbed version of the tamed MHD (TMHD) equations of the previous chapter. This aims at modelling the turbulent behaviour of a flow of electrically conducting fluids through porous media. To be precise, we study existence and uniqueness of strong solutions, as well as existence of invariant measures of the following system of equations:

$$\begin{aligned}
 (1.1) \quad d\mathbf{v} &= \left[\Delta \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v} + (\mathbf{B} \cdot \nabla) \mathbf{B} + \nabla \left(p + \frac{|\mathbf{B}|^2}{2} \right) - g_N(|(\mathbf{v}, \mathbf{B})|^2) \mathbf{v} \right] dt \\
 &\quad + \sum_{k=1}^{\infty} [(\boldsymbol{\sigma}_k(t) \cdot \nabla) \mathbf{v} + \nabla p_k(t) + \mathbf{h}_k(t, y(t))] dW_t^k + \mathbf{f}_v(t, y(t)) dt, \\
 d\mathbf{B} &= \left[\Delta \mathbf{B} - (\mathbf{v} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{v} + \nabla \pi - g_N(|(\mathbf{v}, \mathbf{B})|^2) \mathbf{B} \right] dt \\
 &\quad + \sum_{k=1}^{\infty} [(\bar{\boldsymbol{\sigma}}_k(t) \cdot \nabla) \mathbf{B} + \nabla \pi_k(t) + \bar{\mathbf{h}}_k(t, y(t))] d\bar{W}_t^k + \mathbf{f}_B(t, y(t)) dt, \\
 \operatorname{div}(\mathbf{v}) &= 0, \quad \operatorname{div}(\mathbf{B}) = 0.
 \end{aligned}$$

Here, $\mathbf{v} = \mathbf{v}(t, x)$ denotes the velocity field of the fluid, $\mathbf{B} = \mathbf{B}(t, x)$ is the magnetic field, $p = p(t, x)$ is the pressure, $\pi = \pi(t, x)$ is the ‘‘magnetic pressure’’ (cf. Chapter II, Section 1.3.4), g_N denotes the taming function, whereas $\mathbf{f}_v = \mathbf{f}_v(t, x, y(t))$ and $\mathbf{f}_B = \mathbf{f}_B(t, x, y(t))$ are forces acting on the fluid. The form of the noise term will be discussed below. For simplicity, we have set all the constants appearing in the equations to one. For the assumptions on the coefficients, see Section 2.1.

We do not repeat the motivation for studying these equations, cf. Chapter II, Section 1.3, but note that these equations can be seen as a model for the flow of electrically conducting fluids through porous media at low to moderate Reynolds numbers.

The stochastic MHD equations were first studied by S.S. Sritharan and P. Sundar in [206] who proved existence of martingale solutions in the two- and three-dimensional case. The paper [211] by Z. Tan, D.H. Wang and H.Q. Wang contains results on existence of a global strong solution in 3D for small initial data as well as existence and uniqueness of a local solution. Their paper, however, was later retracted due to allegations of plagiarism.

Questions of ergodicity in two dimensions were studied by V. Barbu, G. Da Prato [11] for Wiener noise, and for α -stable noise by T.L. Shen and J.H. Huang [203]. K. Yamazaki [237] proved ergodicity in the case of random forcing only by a few modes. In three dimensions, he also proved ergodicity of a Faedo-Galerkin approximation of the MHD equations for degenerate noise in [236].

The asymptotic behaviour of the SMHD equations was studied in the 2D additive white noise case by W.Q. Zhao and Y.R. Li [257], and in the 2D fractional case by J.H. Huang and T.L. Shen [118]. H.Q. Wang [227] studied the system's exponential behaviour. Furthermore, S.H. Wang and Y.R. Li [228] proved long-time robustness of the associated random attractor.

Jump-type and fractional noises have been studied by P. Sundar [209]. U. Manna and M.T. Mohan [169], as well as E. Motyl [182] studied the jump noise case, and the latter author provided a nice and very general framework for 3D hydrodynamic-type equations with Lévy noise on unbounded domains, generalising the 2D framework of I.D. Chueshov and A. Millet [40]. Chueshov and Millet proved large deviations principles as well, and in [41] also a Wong-Zakai approximation and a support theorem.

The existence of solutions to the non-resistive MHD equations with Lévy noise was investigated by U. Manna, M.T. Mohan and S.S. Sritharan [170].

K. Yamazaki proved existence of global martingale solutions for the nonhomogeneous system [234]. The case of non-Newtonian electrically conducting fluids and their long-time behaviour was studied in a paper by P.A. Razafimandimby and M. Sango [190].

In modelling the noise, we follow the approach of R. Mikulevicius and B.L. Rozovskii [178–180] who proposed a multiplicative noise of transport-type for the Navier-Stokes equations, motivated by the turbulence theory of R.H. Kraichnan [135], which was further developed by K. Gawedzki and co-authors in [90, 91]. Transport-type noise was studied by several other authors as well, e.g. Z. Brzeźniak, M. Capiński and F. Flandoli [21, 22], as well as in Flandoli and D. Gatarek [79, Section 3.3, pp. 378 f.]. More recently, M. Hofmanová, J.-M. Leahy and T. Nilssen [113] studied the problem via rough path methods.

We note that Mikulevicius and Rozovskii consider the case of only Hölder continuous σ as being important to Kraichnan's turbulent velocity model, but for simplicity, we restrict ourselves to the case of differentiable σ (see Assumption (H2) below). We would also like to point out that we have chosen Itô noise instead of the Stratonovich noise considered in [179].

Existence and uniqueness as well as ergodicity for the stochastic tamed Navier-Stokes equation were studied by M. Röckner and X.C. Zhang in [197]. The study of Freidlin-Wentzell-type large deviations was carried out by M. Röckner, T.S. Zhang and X.C. Zhang in [195]. The case of existence, uniqueness and small time large deviation principles for the Dirichlet problem in bounded domains can be found in the work of M. Röckner and T.S. Zhang [194]. More recently, there has been resparked interest in the subject, with contributions by Z. Dong and R.R. Zhang [64] (existence and uniqueness for multiplicative Lévy noise), as well as Z. Brzeźniak and G. Dhariwal [23] (existence, uniqueness and existence of invariant measures in the full space \mathbb{R}^3 by different methods).

From a physical point of view, the fact that our model is most appropriate for low to moderate Reynolds numbers raises the question of whether or not a stochastic model for turbulence (which is commonly associated with high Reynolds numbers) is appropriate in this setting. It is, nevertheless, an interesting mathematical problem that we want to address in this chapter.

Note also that there are instances of apparently turbulent flows at low Reynolds numbers, such as elastic turbulence (which can even occur at arbitrarily low Reynolds numbers), found in non-Newtonian fluids like polymer solutions and described by A. Groisman and V. Steinberg in [103]. Another example occurs in microfluidics, as discovered by G.R. Wang, F. Yang and W. Zhao [226]. Here, the Reynolds number at which the flow exhibits turbulent behaviour is of order $Re \approx 1$. However, these are more “exotic” examples and not relatable to our case.

1.1. Regularisation of Fluid Dynamical Equations. In this section, we give a short overview of other regularisations of fluid dynamical equations. For additional regularisations, cf. Section 1.2 of Chapter II. As in that section, we consider fluid dynamical equations as abstract evolution equations

$$(1.2) \quad \partial_t y = \mathcal{L}(y) + \mathcal{N}(y, y) + f, \quad \nabla \cdot y = 0,$$

where \mathcal{L} is a linear or nonlinear operator, \mathcal{N} is a bilinear operator, and f is a forcing term. We limit our presentation of the regularised terms to the Navier-Stokes case, where $y = \mathbf{v}$, $\mathcal{L}(y) := \mathcal{L}(\mathbf{v}) := \frac{1}{Re} P \Delta \mathbf{v}$, and $\mathcal{N}(y, y) := \mathcal{N}(\mathbf{v}, \mathbf{v}) = -P [(\mathbf{v} \cdot \nabla) \mathbf{v}]$. Here we denote the Helmholtz-Leray projection by $P: L^2 \rightarrow L^2 \cap \text{div}^{-1}(\{0\})$. For the related MHD models, we give references for each model.

1.1.1. *Leray- α Model.* The Leray- α model for the deterministic Navier-Stokes equations was introduced by A. Cheskidov, D.D. Holm, E. Olson and E.S. Titi in [39] (cf. [218] for more references). Here, the nonlinearity is regularised by applying the smoothing (or filtering) operator $(I - \alpha^2 \Delta)^{-1}$ to the first factor of the nonlinearity. Thus

$$\begin{aligned} \mathcal{N}_{\text{Leray-}\alpha}(\mathbf{v}, \mathbf{v}) &:= -P [(\{ (I - \alpha^2 \Delta)^{-1} \mathbf{v} \} \cdot \nabla) \mathbf{v}], \\ f_{\text{Leray-}\alpha} &:= (I - \alpha^2 \Delta)^{-1} f. \end{aligned}$$

In the stochastic case, the model was first studied by G. Deugoué and M. Sango in a series of papers [58–62]. The works [58–60] establish (probabilistically) weak and strong solutions for Wiener noise and prove convergence to a weak solution of the stochastic Navier-Stokes equations. [61] investigates the convergence of a numerical scheme to these equations, whereas [62] is devoted to proving the existence and uniqueness of strong solutions in the case of Lévy noise. The case of the Euler equations (i.e. with zero viscosity, $\nu = 0$, or equivalently $Re = \infty$) was carried out by D. Barbato, H. Bessaih and B. Ferrario [10]. More recently, the less regularising Leray- α model with fractional dissipation (akin to the hyperviscosity scheme introduced below, but for $\ell < 2$ instead of $\ell > 2$, i.e. the “hypoviscous” case) was studied by L. Debbi [54] who proved well-posedness on bounded domains and on the torus \mathbb{T}^3 , and also by S.H. Li, L. Wei and Y.C. Xie in [151–153] where they prove large deviation principles, ergodicity and exponential mixing, respectively. Finally, the existence of random dynamical systems and random attractors for additive Lévy noise was studied by B. Gess, W. Liu and the author in [98], and is discussed in greater detail in Chapter IV, Section 6.3.

The stochastic 3D MHD Leray- α model was introduced by G. Deugoué, M. Sango and P.A. Razafimandimby in [57], where both weak and strong solutions are studied. Deugoué [56] considered the long-time behaviour for $\alpha \rightarrow 0$. Furthermore, R.R. Zhang [249] proved backwards uniqueness for these equations. Moreover, a related model for the MHD equations was investigated by N.E. Wilson in [231, 232].

1.1.2. *Globally Modified Navier-Stokes Equations.* The *globally modified Navier-Stokes equations* (GMNS) were introduced by T. Caraballo, P.E. Kloeden and J. Real in [38]. Here, the nonlinearity is modified through a nonlinear damping factor:

$$\mathcal{N}_{\text{GMNS}}(\mathbf{u}) := F_N(\|\Delta^{1/2}\mathbf{u}\|_{L^2})P[(\mathbf{u} \cdot \nabla)\mathbf{u}],$$

with a damping function ($N \in (0, \infty)$) given by

$$F_N(r) := \min\left\{1, \frac{N}{r}\right\}.$$

In the deterministic case, further properties of the model have been studied in a series of subsequent papers, cf. [33, 34, 132] and references therein. The stochastic Navier-Stokes case has first been treated quite recently by G. Deugoué and T. Tachim Medjo [63], who studied existence, uniqueness and convergence as $N \rightarrow \infty$. To the best of the author's knowledge, the MHD case is still completely open.

1.1.3. *Regularisation by Delay.* Another way of regularising the nonlinear term, studied recently by H. Bessaih, M. Garrido-Atienza and B. Schmalfuss [15], consists in introducing a *time delay* in the advection velocity, i.e. in considering the nonlinearity

$$\mathcal{N}_{\text{delay}}(\mathbf{v}, \mathbf{v})(t, x) := P[(\mathbf{v}(t - \mu, x) \cdot \nabla)\mathbf{v}(t, x)].$$

This time delay again has a smoothing effect on the solution and allows the authors to obtain unique global-in-time weak solutions and, for a certain parameter range in the regularity of the initial conditions, also strong solutions. For $\mu \rightarrow 0$, their solutions converge to a weak solution of the Navier-Stokes equations. Their work builds on that of S.M. Guzzo and G. Planas [109, 110] as well as C.J. Niche and G. Planas [183] and W. Varnhorn [217].

In the stochastic Navier-Stokes case, X.C. Gao and H.J. Gao [89] showed existence and uniqueness of weak solutions for the above model with linear multiplicative noise. The stochastic MHD equations have not been studied yet.

1.1.4. *Lions' Hyperviscosity Method.* Another way to add more dissipativity to the model was proposed by J.-L. Lions [160], who considered the operator

$$\mathcal{L}_{\text{hyper}}(\mathbf{v}) := \nu_0 P \Delta \mathbf{v} - \nu_1 P (-\Delta)^{\ell/2} \mathbf{v},$$

where $\ell > 2$, $\nu_0 \geq 0$ and $\nu_1 > 0$. For $\ell > 5/2$, he could prove existence of a unique regular solution in a bounded domain. In the stochastic case, it was first considered in 2D by J.C. Mattingly and coauthors in [174, 175] where $\nu_0 = 0$ and by S.S. Sritharan in the setting of stochastic control (in 2D and 3D) in [205]. It was then further studied by B. Ferrario [75] who proved well-posedness in 3D and in [76] characterised the law via a Girsanov transformation for the vorticity form of the equation for $\ell > 3$. Furthermore, F.Y. Wang and L. Xu [225] derived a Bismut-type derivative formula for the semigroup associated with the stochastic hyperdissipative equations on the torus for $\ell > 5$.

To the best of our knowledge, the stochastic MHD equations with this regularisation have not yet been studied mathematically.

1.1.5. *Navier-Stokes-Voigt Equations.* The Navier-Stokes-Voigt (sometimes written as Voigt) equations employ the following regularisation of the Stokes operator

$$\mathcal{L}_{\text{NSV}}(\mathbf{v}) := \nu P \Delta \mathbf{v} + \alpha^2 P \Delta \partial_t \mathbf{v}.$$

This regularisation changes the parabolic character of the equations and simulates a property of so-called Kelvin-Voigt fluids, e.g. polymer solutions, of not immediately reverting back to the original state once external stress is removed. It was pioneered by A.P. Osolkov [186] in 1973 and has since then been studied by many authors.

The stochastic case was studied by H. Gao and C. Sun in [88, 208], where the existence and uniqueness of weak solutions as well as questions regarding the system's random attractor and its Hausdorff dimension were posed and answered, see also C.T. Anh and N.V. Thanh [2]. Q.B. Tang [212] proved existence and upper semicontinuity of random attractors for unbounded domains and H. Liu and C. Sun first proved large deviation principles in [163].

The stochastic MHD-Voigt case has not been studied so far.

1.1.6. *Damped Navier-Stokes Equations (or Brinkman-Forchheimer-extended Darcy Models)*. Instead of adding a linear dissipative term, one can also add nonlinear, power-type terms that counteract the nonlinearity. This leads to the so-called (*nonlinearly*) *damped Navier-Stokes equations*. Thus, we consider the (nonlinear) operator

$$\mathcal{L}_{\text{damped}}(\mathbf{v}) := P\Delta\mathbf{v} - \alpha P|\mathbf{v}|^{\beta-1}\mathbf{v},$$

where $\alpha > 0$ and $\beta \geq 1$. The damping term $-\alpha|\mathbf{v}|^{\beta-1}\mathbf{v}$ models the resistance to the motion of the flow resulting from physical effects like porous media flow, drag or friction, or other dissipative mechanisms (cf. [28]). It represents a restoring force, which for $\beta = 1$ assumes the form of classical, linear damping, whereas $\beta > 1$ means a restoring force that grows superlinearly with the velocity (or magnetic field). X.J. Cai and Q.S. Jiu [28] first proved existence and uniqueness of a global strong solution for $\frac{7}{2} \leq \beta \leq 5$. This range was lowered down to $\beta \in (3, 5]$ by Z.J. Zhang, X.L. Wu and M. Lu in [252] and considered this the case $\beta = 3$ to be critical [252, Remark 3.1]. Y. Zhou in [260] proved the existence of a global solution for all $\beta \in [3, 5]$. For the case $\beta \in [1, 3)$, he established regularity criteria that ensure smoothness. Uniqueness holds for any $\beta \geq 1$ in the class of weak solutions.

The first problems studied in the stochastic damped Navier-Stokes case were related to the inviscid limit of the damped equations for $\beta = 1$ in 2D, cf. H. Bessaih and B. Ferrario [14] and N. Glatt-Holtz, V. Šverák and V. Vicol [101]. B. You [244] proved existence of a random attractor under the assumption of well-posedness for additive noise for $\beta \in (3, 5]$ (notably leaving out the critical, or tamed, case $\beta = 3$). K. Yamazaki [235] proved a Lagrangian formulation and extended Kelvin's circulation theorem to the partially damped case (i.e. only a few components are damped, but the damping there is much stronger, e.g. $\beta_k = 9$ in two components $k = 3, 4$ in the 4D case). Z. Brzeźniak and B. Ferrario [24] showed existence of stationary solutions on the whole space \mathbb{R}^3 for $\beta = 1$. H. Liu and H.J. Gao [161] proved existence and uniqueness of an invariant measure and a random attractor for $\beta \in [3, 5]$. The same authors in [86] proved a small-time large deviation principle for the same parameter range. Furthermore, again for the same range, for multiplicative noise, exponential convergence of the weak solutions in L^2 to the stationary solution as well as stabilisation were proved by H. Liu, L. Lin, C.F. Sun and Q.K. Xiao [162]. Finally, for jump noise, H. Liu and H.J. Gao showed well-posedness and existence of invariant measures in [87].

To the best of the author's knowledge, the stochastic damped MHD equations have not been considered so far. This work is a first step in this direction.

1.2. Results and Structure of This Chapter. Here are our main results for this chapter. First, we prove existence and uniqueness of a strong solution to the tamed MHD equations. To be precise, we show this for the evolution equation form (2.4).

THEOREM 1.1 (Existence and uniqueness, Theorems 3.9, 3.10 and 3.11 below). *Let the coefficients $f := (\mathbf{f}_v, \mathbf{f}_B)$, $\Sigma := (\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}})$ and $H := (\mathbf{h}, \bar{\mathbf{h}})$ satisfy Assumptions (H1)–(H3) and let $y_0 \in \mathcal{H}^1$. Then there exists a unique strong solution y to the stochastic tamed MHD equations, in the sense of Definition 3.7, with the following properties:*

(i) $y \in L^2(\Omega, P; C([0, T]; \mathcal{H}^1)) \cap L^2(\Omega, P; L^2([0, T]; \mathcal{H}^2))$ for all $T > 0$ and

$$(1.3) \quad \mathbb{E} \left[\sup_{t \in [0, T]} \|y(t)\|_{\mathcal{H}^1}^2 \right] + \int_0^t \mathbb{E} [\|y(s)\|_{\mathcal{H}^2}^2] ds \leq C_{T, f, H} (1 + \|y_0\|_{\mathcal{H}^1}^2) \cdot N.$$

(ii) In \mathcal{H}^0 , the following equation holds:

$$(1.4) \quad y(t) = \int_0^t [\mathcal{A}(y(s)) + \mathcal{P}f(s, y(s))] ds + \sum_{k=1}^{\infty} \int_0^t \mathcal{B}_k(s, y(s)) dW_s^k \quad \forall t \geq 0, P - \text{a.s.}$$

By the preceding theorem, for time-homogeneous coefficients, the solution process $y = (y(t; y_0))_{t \geq 0}$ is a strong Markov process. For $t \geq 0$, we can define the associated Markov semigroup on the space $BC_{\text{loc}}(\mathcal{H}^1)$ of bounded, locally uniformly continuous functions on the divergence-free Sobolev space \mathcal{H}^1 , by

$$T_t \phi(y_0) := \mathbb{E} [\phi(y(t; y_0))], \quad \phi \in BC_{\text{loc}}(\mathcal{H}^1), y_0 \in \mathcal{H}^1.$$

Then, under a slightly stronger assumption on the coefficients of the noise, this semigroup satisfies the Feller property.

THEOREM 1.2 (Feller property, Theorem 4.2 below). *Under the Assumptions (H1), (H2) and (H3)', for every $t \geq 0$, T_t maps $BC_{\text{loc}}(\mathcal{H}^1)$ into itself, i.e. it is a Feller semigroup on $BC_{\text{loc}}(\mathcal{H}^1)$.*

In the case of a periodic domain $\mathbb{D} = \mathbb{T}^3$, we prove the existence of an invariant measure.

THEOREM 1.3 (Existence of an invariant measure, Theorem 4.3 below). *Under the hypotheses (H1), (H2), (H3)', in the periodic case $\mathbb{D} = \mathbb{T}^3$, there exists an invariant measure $\mu \in \mathcal{P}(\mathcal{H}^1)$ associated to $(T_t)_{t \geq 0}$, i.e. a measure μ such that for every $t \geq 0$, $\phi \in BC_{\text{loc}}(\mathcal{H}^1)$*

$$\int_{\mathcal{H}^1} T_t \phi(y_0) d\mu(y_0) = \int_{\mathcal{H}^1} \phi(y_0) d\mu(y_0).$$

The goal of this chapter is to generalise the results of M. Röckner and X.C. Zhang [196] for the stochastic tamed Navier-Stokes equations to the stochastic tamed MHD case. In doing so, we have to prove MHD versions of several technical tools they use, in particular the estimates of the Lemmas 2.3, 2.4 and 2.5, which are more difficult in our setting, the tightness criterion Lemma 2.6 and the *a priori* estimates 3.11.

The chapter is organised as follows: we state our assumptions in Section 2.1 as well as auxiliary results and estimates on the coefficients of our equation in Section 2.2. Since our proof of existence involves a tightness argument, we provide a tightness criterion in Section 2.3. Existence and uniqueness of a strong solution is then proven in Section 3. We start by defining weak and strong solutions in 3.1. The next section, 3.2 is then devoted to proving pathwise uniqueness.

Existence of a weak solution is proved in Section 3.3 via the by now classical strategy of proving *a priori* estimates for the Faedo-Galerkin approximation to the equation and using them to infer tightness of the sequence of laws, Skorokhod coupling to obtain almost sure convergence, and concluding by proving uniform moment estimates and convergence in probability. This allows us to obtain – using the Yamada-Watanabe theorem – that there exists a unique strong solution. The Feller property of the semigroup as well as existence of invariant measures are then shown in Section 4.

A publication of the results of this chapter is in preparation, cf. [199].

2. Preliminaries

In this section, we provide the basic tools needed later. After establishing our notation and the assumptions the coefficients, we prove some elementary estimates for the operators associated to the coefficients. We then provide a tightness criterion for later use.

2.1. Notation and Assumptions. We define the spaces L^q , $W^{m,p}$, \mathcal{L}^q , \mathcal{H}^k , etc. as in Chapter II, Section 2.1 and follow the notational conventions established in Chapter II, Section 1.5. By ℓ^2 , we denote the space of square-summable sequences.

We make the following assumptions on our coefficients:

(H1) For any $T > 0$, the function $f = (\mathbf{f}_v, \mathbf{f}_B)$ with $\mathbf{f}_v, \mathbf{f}_B: [0, T] \times \mathbb{D} \times \mathbb{R}^6 \rightarrow \mathbb{R}^3$ satisfies: there is a constant $C_{T,f} > 0$ and a function $F_f(t, x) \in L^\infty([0, T]; L^1(\mathbb{D}))$

$$|\partial_{x^j} f(t, x, y)|^2 + |f(t, x, y)|^2 \leq C_{T,f} |y|^2 + F_f(t, x), \quad j = 1, 2, 3, x \in \mathbb{D}, y \in \mathbb{R}^6,$$

$$|\partial_{y^l} f(t, x, y)| \leq C_{T,f}, \quad l = 1, \dots, 6, x \in \mathbb{D}, y \in \mathbb{R}^6.$$

(H2) For any $T > 0$, for the function $\Sigma = (\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}})$ with $\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}: [0, T] \times \mathbb{D} \rightarrow \ell^2(\mathbb{R}^3)$, there are constants $C_{\boldsymbol{\sigma}, T}, C_{\bar{\boldsymbol{\sigma}}, T} > 0$ such that for $j = 1, 2, 3$

$$\sup_{t \in [0, T], x \in \mathbb{D}} \|\partial_{x^j} \boldsymbol{\sigma}(t, x)\|_{\ell^2} \leq C_{\boldsymbol{\sigma}, T}, \quad \sup_{t \in [0, T], x \in \mathbb{D}} \|\partial_{x^j} \bar{\boldsymbol{\sigma}}(t, x)\|_{\ell^2} \leq C_{\bar{\boldsymbol{\sigma}}, T},$$

as well as

$$\sup_{t \in [0, T], x \in \mathbb{D}} \|\Sigma(t, x)\|_{\ell^2}^2 \leq \frac{1}{36}.$$

(H3) For any $T > 0$, for the function $H = (\mathbf{h}, \bar{\mathbf{h}})$ with $\mathbf{h}, \bar{\mathbf{h}}: [0, T] \times \mathbb{D} \times \mathbb{R}^6 \rightarrow \ell^2(\mathbb{R}^3)$, there exists a constant $C_{T,H} > 0$ and $F_H(t, x) \in L^\infty([0, T]; L^1(\mathbb{D}))$ such that for any $0 \leq t \leq T$, $x \in \mathbb{D}$, $y \in \mathbb{R}^6$ and $j = 1, 2, 3$, $l = 1, 2, \dots, 6$

$$\|\partial_{x^j} H(t, x, y)\|_{\ell^2}^2 + \|H(t, x, y)\|_{\ell^2}^2 \leq C_{T,H} |y|^2 + F_H(t, x),$$

$$\|\partial_{y^l} H(t, x, y)\|_{\ell^2} \leq C_{T,H}.$$

REMARK 2.1. *The origin of the constant $36 = 4 \cdot 3^2$ in Assumption (H2) lies in the fact that in the places where we need the numerical value, i.e. in the proof of (2.13) of Lemma 2.5 as well as in the proof of Lemma 4.1, we estimate the homogeneous second-order Sobolev norm against the Bessel potential norm via Lemma 2.2 below, giving an additional factor of $9 = 3^2$. We do not claim that this value for Σ is sharp, but we wanted to give an explicit bound that suffices to make all the calculations work.*

The integrability conditions $F_f, F_H \in L^\infty([0, T]; L^1(\mathbb{D}))$ are used in proving continuity in time, as we want to estimate

$$\int_s^t \|F_{f,H}(r)\|_{L^1(\mathbb{D})} dr \leq C|t - s|$$

in the proof of Lemma 3.12.

One could also model the equations in a way that the terms \mathbf{f}_v and \mathbf{h} in the equations of \mathbf{v} depend only on \mathbf{v} instead of y . However, this case is included in our assumptions, which are more symmetric this way.

For definiteness, we want to state the following very elementary relationship between the classical homogeneous second-order Sobolev norm and the norm we use in this work (which is defined via Bessel potentials).

LEMMA 2.2. *Let $\mathbb{D} \in \{\mathbb{R}^3, \mathbb{T}^3\}$ and $y \in \mathcal{H}^2(\mathbb{D})$. Then the following estimate holds:*

$$(2.1) \quad \|y\|_{\dot{W}^{2,2}(\mathbb{D})}^2 := \sum_{i,j=1}^d \|\partial_{x^i} \partial_{x^j} y\|_{\mathcal{L}^2(\mathbb{D})}^2 \leq 3^2 \|y\|_{\mathcal{H}^2(\mathbb{D})}^2.$$

PROOF. Using Plancherel's theorem and Young's inequality, we find

$$\begin{aligned} \sum_{i,j=1}^3 \|\partial_{x^i} \partial_{x^j} y\|_{\mathcal{L}^2}^2 &= \sum_{i,j=1}^3 \|\xi_i \xi_j \hat{y}\|_{\mathcal{L}^2}^2 = \sum_{i,j=1}^3 \left(\|\xi_i \xi_j \hat{y}\|_{\mathcal{L}^2(\{|\xi| \leq 1\})}^2 + \|\xi_i \xi_j \hat{y}\|_{\mathcal{L}^2(\{|\xi| > 1\})}^2 \right) \\ &\leq \sum_{i,j=1}^3 \left(\|\hat{y}\|_{\mathcal{L}^2}^2 + \left\| \frac{(\xi_i^2 + \xi_j^2)}{2} \hat{y} \right\|_{\mathcal{L}^2(\{|\xi| > 1\})}^2 \right) \leq 3^2 \|y\|_{\mathcal{L}^2}^2 + 3 \|\xi^2 \hat{y}\|_{\mathcal{L}^2(\{|\xi| > 1\})}^2 \\ &\leq 3^2 (\|y\|_{\mathcal{L}^2}^2 + \|\Delta y\|_{\mathcal{L}^2}^2) \leq 3^2 (\|y\|_{\mathcal{L}^2}^2 + 2\|\nabla y\|_{\mathcal{L}^2}^2 + \|\Delta y\|_{\mathcal{L}^2}^2) = 3^2 \|(I - \Delta)y\|_{\mathcal{L}^2}^2 \\ &= 3^2 \|y\|_{\mathcal{H}^2}^2. \end{aligned}$$

□

We define the set of solenoidal test functions as

$$\mathcal{V} := \{y = (\mathbf{v}, \mathbf{B}) \mid \mathbf{v}, \mathbf{B} \in C_c^\infty(\mathbb{D}; \mathbb{R}^3), \operatorname{div}(\mathbf{v}) = \operatorname{div}(\mathbf{B}) = 0\}.$$

As in Chapter II, Lemma 2.1, \mathcal{V} is dense in \mathcal{H}^k for any $k \in \mathbb{N}$. Let $P: L^2(\mathbb{D}; \mathbb{R}^3) \rightarrow \mathbb{H}^0$ be the Leray-Helmholtz projection. In the case of $\mathbb{D} \in \{\mathbb{T}, \mathbb{R}^3\}$, P commutes with derivative operators ([192, Lemma 2.9, p. 52]) and can be restricted to a bounded linear operator

$$P|_{H^m}: H^m \rightarrow \mathbb{H}^m.$$

Furthermore, consider the tensorised projection

$$\mathcal{P} := P \otimes P, \quad \mathcal{P}y := (P \otimes P) \begin{pmatrix} \mathbf{v} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} P\mathbf{v} \\ P\mathbf{B} \end{pmatrix}.$$

Then $\mathcal{P}: \mathcal{L}^2 \rightarrow \mathcal{H}^0$ is a bounded linear operator:

$$\begin{aligned} \|\mathcal{P}y\|_{\mathcal{H}^0}^2 &= \|P\mathbf{v}\|_{\mathbb{H}^0}^2 + \|P\mathbf{B}\|_{\mathbb{H}^0}^2 \leq \|P\|_{L^2 \rightarrow \mathbb{H}^0}^2 (\|\mathbf{v}\|_{L^2}^2 + \|\mathbf{B}\|_{L^2}^2) \\ &= \|P\|_{L^2 \rightarrow \mathbb{H}^0}^2 \|y\|_{\mathcal{L}^2}^2. \end{aligned}$$

We now define operators

$$\mathcal{A}(y) := \mathcal{P}\Delta y - \mathcal{P} \begin{pmatrix} (\mathbf{v} \cdot \nabla)\mathbf{v} - (\mathbf{B} \cdot \nabla)\mathbf{B} \\ (\mathbf{v} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{v} \end{pmatrix} - \mathcal{P}(g_N(|y|^2)y),$$

$$(2.2) \quad \langle \mathcal{A}(y), \tilde{y} \rangle_{\mathcal{H}^1} = \langle \mathcal{A}(y), (I - \Delta)\tilde{y} \rangle_{\mathcal{H}^0} = \mathcal{A}_1(y, \tilde{y}) + \mathcal{A}_2(y, \tilde{y}) + \mathcal{A}_3(y, \tilde{y}),$$

where

$$\begin{aligned}\mathcal{A}_1(y, \tilde{y}) &:= \langle \mathcal{P}\Delta y, (I - \Delta)\tilde{y} \rangle_{\mathcal{H}^0}, \\ \mathcal{A}_2(y, \tilde{y}) &:= - \left\langle \mathcal{P} \begin{pmatrix} (\mathbf{v} \cdot \nabla)\mathbf{v} - (\mathbf{B} \cdot \nabla)\mathbf{B} \\ (\mathbf{v} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{v} \end{pmatrix}, (I - \Delta)\tilde{y} \right\rangle_{\mathcal{H}^0}, \\ \mathcal{A}_3(y, \tilde{y}) &:= - \langle \mathcal{P}g_N(|y|^2)y, (I - \Delta)\tilde{y} \rangle_{\mathcal{H}^0}.\end{aligned}$$

For the noise terms, we define $\Sigma(t, x) := \begin{pmatrix} \boldsymbol{\sigma}_k(t, x) \\ \bar{\boldsymbol{\sigma}}_k(t, x) \end{pmatrix}_{k \in \mathbb{N}} \in \ell^2(\mathbb{R}^6)$ and for $y = \begin{pmatrix} \mathbf{v} \\ \mathbf{B} \end{pmatrix}$

$$(\Sigma_k(t, x) \cdot \nabla) y := \begin{pmatrix} (\boldsymbol{\sigma}_k(t, x) \cdot \nabla)\mathbf{v} \\ (\bar{\boldsymbol{\sigma}}_k(t, x) \cdot \nabla)\mathbf{B} \end{pmatrix}.$$

Similarly, we define $H_k(t, y) := \begin{pmatrix} \mathbf{h}_k(t, y) \\ \bar{\mathbf{h}}_k(t, y) \end{pmatrix}$, and

$$\mathcal{B}_k(t, x, y) := \mathcal{P}((\Sigma_k(t, x) \cdot \nabla)y) + \mathcal{P}H_k(t, x, y).$$

Finally, let $\{W_t^k \mid t \in \mathbb{R}_+, k \in \mathbb{N}\}, \{\tilde{W}_t^k \mid t \in \mathbb{R}_+, k \in \mathbb{N}\}$ be two independent sequences of independent Brownian motions, and define

$$\mathcal{W}_t^k := \begin{pmatrix} W_t^k \\ \tilde{W}_t^k \end{pmatrix},$$

as well as

$$\int_0^t \mathcal{B}_k(s, x, y) d\mathcal{W}_s^k := \begin{pmatrix} \int_0^t P(\boldsymbol{\sigma}_k(s, x) \cdot \nabla)\mathbf{v} + P\mathbf{h}_k(s, x, y) dW_s^k \\ \int_0^t P(\bar{\boldsymbol{\sigma}}_k(s, x) \cdot \nabla)\mathbf{B} + P\bar{\mathbf{h}}_k(s, x, y) d\tilde{W}_s^k \end{pmatrix}.$$

The Brownian motions W and \tilde{W} can be understood as independent cylindrical Brownian motions on the space ℓ^2 . Similarly, \mathcal{W} is a cylindrical Brownian motion on the space $\ell^2 \times \ell^2$. For $y \in \mathcal{H}^m$, $m = 1, 2$, $\mathcal{B}(t, x, y(x))$ can be understood as a linear operator

$$\mathcal{B}(t, \cdot, y) : \ell^2 \times \ell^2 \rightarrow \mathcal{H}^{m-1}.$$

To make this clear, we note that if we take the canonical basis of ℓ^2 , i.e. orthonormal basis consisting of the sequences $e_1 := (1, 0, 0, \dots), e_2 := (0, 1, 0, 0, \dots), \dots$, then the system

$$\left\{ \begin{pmatrix} e_k \\ 0 \end{pmatrix} \in \ell^2 \times \ell^2 \mid k \in \mathbb{N} \right\} \cup \left\{ \begin{pmatrix} 0 \\ e_k \end{pmatrix} \in \ell^2 \times \ell^2 \mid k \in \mathbb{N} \right\}$$

forms an orthonormal basis of $\ell^2 \times \ell^2$. Then we define

$$\mathcal{B}(t, \cdot, y) \begin{pmatrix} e_k \\ 0 \end{pmatrix} (x) := \begin{pmatrix} P(\boldsymbol{\sigma}_k(t, x) \cdot \nabla)\mathbf{v}(x) + P\mathbf{h}_k(t, x, y(x)) \\ 0 \end{pmatrix} \in \mathcal{H}^{m-1},$$

and

$$(2.3) \quad \mathcal{B}(t, \cdot, y) \begin{pmatrix} 0 \\ e_k \end{pmatrix} (x) := \begin{pmatrix} 0 \\ P(\bar{\boldsymbol{\sigma}}_k(t, x) \cdot \nabla)\mathbf{v}(x) + P\bar{\mathbf{h}}_k(t, x, y(x)) \end{pmatrix} \in \mathcal{H}^{m-1}.$$

It turns out that \mathcal{B} is even a Hilbert-Schmidt operator, i.e. $\mathcal{B}(t, \cdot, y) \in L_2(\ell^2 \times \ell^2; \mathcal{H}^{m-1})$, as will be proven below in Lemma 2.5. Hence we are in the usual framework of stochastic analysis on Hilbert spaces, cf. [52] or [167].

We can now formulate Equation (1.1) in the following abstract form as an evolution equation:

$$(2.4) \quad \begin{aligned} dy(t) &= [\mathcal{A}(y(t)) + \mathcal{P}f(t, y(t))]dt + \sum_{k=1}^{\infty} \mathcal{B}_k(t, y(t))d\mathcal{W}_t^k, \\ y(0) &= y_0 \in \mathcal{H}^1. \end{aligned}$$

2.2. Estimates on the Operators \mathcal{A} and \mathcal{B} . In this section, we will prove important but elementary estimates on the operators \mathcal{A} and \mathcal{B} . These play an important role in deriving the *a priori* estimates for the STMHD equations. The first theorem is concerned with estimates for the case of testing with the solution y itself.

LEMMA 2.3 (Estimates for \mathcal{A}). *Let $y \in \mathcal{H}^2$. Then the following estimates hold true*

$$(2.5) \quad \|\mathcal{A}(y)\|_{\mathcal{H}^0} \leq C(1 + \|y\|_{\mathcal{H}^0}^6 + \|y\|_{\mathcal{H}^2}^2),$$

$$(2.6) \quad \langle \mathcal{A}(y), y \rangle_{\mathcal{H}^0} = -\|\nabla y\|_{\mathcal{H}^0}^2 - \|\sqrt{g_N(|y|^2)}|y|\|_{\mathcal{L}^2}^2$$

$$(2.7) \quad \leq -\|\nabla y\|_{\mathcal{H}^0}^2 - \|y\|_{\mathcal{L}^4}^2 + CN\|y\|_{\mathcal{H}^0}^2,$$

$$(2.8) \quad \begin{aligned} \langle \mathcal{A}(y), y \rangle_{\mathcal{H}^1} &\leq -\frac{1}{2}\|y\|_{\mathcal{H}^2}^2 - \left(\|\mathbf{v} \cdot |\nabla \mathbf{v}|\|_{L^2}^2 + \|\mathbf{B} \cdot |\nabla \mathbf{B}|\|_{L^2}^2 \right. \\ &\quad \left. + \|\mathbf{v} \cdot |\nabla \mathbf{B}|\|_{L^2}^2 + \|\mathbf{B} \cdot |\nabla \mathbf{v}|\|_{L^2}^2 \right) \\ &\quad + (2N + 1)\|\nabla y\|_{\mathcal{H}^0}^2 + \|y\|_{\mathcal{H}^0}^2. \end{aligned}$$

PROOF. For the first inequality, we have

$$\begin{aligned} \|\mathcal{A}(y)\|_{\mathcal{H}^0} &\leq \|\mathcal{P}(I - \Delta)y + \mathcal{P}y\|_{\mathcal{H}^0} + \left\| \mathcal{P} \begin{pmatrix} (\mathbf{v} \cdot \nabla)\mathbf{v} - (\mathbf{B} \cdot \nabla)\mathbf{B} \\ (\mathbf{v} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{v} \end{pmatrix} \right\|_{\mathcal{H}^0} \\ &\quad + \|\mathcal{P}g_N(|y|^2)y\|_{\mathcal{H}^0} \\ &\leq C_N(\|y\|_{\mathcal{H}^2} + \|y\|_{\mathcal{H}^0} + \|P(\mathbf{v} \cdot \nabla)\mathbf{v}\|_{\mathbb{H}^0} + \dots + \|P(\mathbf{B} \cdot \nabla)\mathbf{v}\|_{\mathbb{H}^0} + \|y\|_{\mathcal{L}^6}^3). \end{aligned}$$

The nonlinear terms can be dealt with as follows: applying first the Sobolev embedding and then Young's inequality twice, we find

$$\begin{aligned} \|P(\mathbf{v} \cdot \nabla)\mathbf{B}\|_{\mathcal{H}^0} &\leq C\|(\mathbf{v} \cdot \nabla)\mathbf{B}\|_{L^2} \leq C\|\mathbf{v}\|_{L^4}\|\nabla \mathbf{B}\|_{L^4} \\ &\leq C\left(\|\mathbf{v}\|_{H^1}^{3/4}\|\mathbf{v}\|_{L^2}^{1/4}\|\mathbf{v}\|_{H^2}^{3/4}\|\mathbf{v}\|_{H^1}^{1/4}\right) \\ &\leq C\left(\|\mathbf{v}\|_{H^1}^{3/2}\|\mathbf{v}\|_{L^2}^{1/2} + \|\mathbf{v}\|_{H^2}^{3/2}\|\mathbf{v}\|_{H^1}^{1/2}\right) \\ &\leq C\left(\|\mathbf{v}\|_{H^1}^2 + \|\mathbf{v}\|_{L^2}^2 + \|\mathbf{v}\|_{H^2}^2 + \|\mathbf{v}\|_{H^1}^2\right) \\ &\leq C(1 + \|y\|_{\mathcal{H}^2}^2), \end{aligned}$$

and similarly for the other terms. The last term can be estimated by the Sobolev embedding theorem and Young's inequality by

$$\|y\|_{\mathcal{L}^6}^3 \leq C\|y\|_{\mathcal{H}^2}^{3/2}\|y\|_{\mathcal{H}^0}^{3/2} \leq C(\|y\|_{\mathcal{H}^2}^2 + \|y\|_{\mathcal{H}^0}^6).$$

Combining these estimates yields (2.5).

Equation (2.6) follows in exactly the same way as Chapter II, Equation 2.9. Equation (2.7) can then be obtained by using the definition of g_N .

The proof of (2.8) is exactly the one of Chapter II, Equation (2.8). \square

The next lemma deals with the case of testing the right-hand side of our equation with another, arbitrary, compactly supported test function.

LEMMA 2.4 (Estimates for \mathcal{A} and \mathcal{B}). *Let $\tilde{y} \in \mathcal{V}$ with compact support $\text{supp}(\tilde{y}) \subset \mathcal{O} := \{x \in \mathbb{D} \mid |x| \leq m\}$ for some $m \in \mathbb{N}$. Let $T > 0$. Then for any $y, y' \in \mathcal{H}^2$ and $t \in [0, T]$, we have*

$$(2.9) \quad |\langle \mathcal{A}(y), \tilde{y} \rangle_{\mathcal{H}^1}| \leq C_{\tilde{y}} \left(1 + \|y\|_{L^3(\mathcal{O})}^3\right),$$

$$(2.10) \quad \|\langle \mathcal{B}(t, y), \tilde{y} \rangle_{\mathcal{H}^1}\|_{\ell^2}^2 \leq C_{\tilde{y}, \Sigma, H, T} \left(\|F_H(t)\|_{L^1(\mathbb{D})} + \|y\|_{L^2(\mathcal{O})}^2\right),$$

and

$$(2.11) \quad |\langle \mathcal{A}(y) - \mathcal{A}(y'), \tilde{y} \rangle_{\mathcal{H}^1}| \leq C_{\tilde{y}, N} \|y - y'\|_{\mathcal{L}^2} \left(1 + \|y\|_{\mathcal{H}^1}^2 + \|y'\|_{\mathcal{H}^1}^2\right).$$

PROOF. The first inequality follows easily from the following calculations:

$$\mathcal{A}_1(y, \tilde{y}) = \langle y, (I - \Delta)\Delta\tilde{y} \rangle_{\mathcal{H}^0} \leq \|y\|_{L^2(\mathcal{O})} \|(I - \Delta)\Delta\tilde{y}\|_{\mathcal{H}^0} \leq C_{\tilde{y}} \|y\|_{L^3(\mathcal{O})} \|\tilde{y}\|_{\mathcal{H}^4},$$

where the constant $C_{\tilde{y}}$ depends on the domain \mathcal{O} , and hence on \tilde{y} . For the next term we find

$$\begin{aligned} \mathcal{A}_2(y, \tilde{y}) &= -\langle (\mathbf{v} \cdot \nabla)\mathbf{v}, (I - \Delta)\tilde{\mathbf{v}} \rangle_{L^2} + \langle (\mathbf{B} \cdot \nabla)\mathbf{B}, (I - \Delta)\tilde{\mathbf{v}} \rangle_{\mathcal{H}^0} \\ &\quad - \langle (\mathbf{v} \cdot \nabla)\mathbf{B}, (I - \Delta)\tilde{\mathbf{B}} \rangle_{L^2} + \langle (\mathbf{B} \cdot \nabla)\mathbf{v}, (I - \Delta)\tilde{\mathbf{B}} \rangle_{L^2} \\ &= \langle \mathbf{v} \otimes \mathbf{v}, \nabla(I - \Delta)\tilde{\mathbf{v}} \rangle_{L^2} - \langle \mathbf{B} \otimes \mathbf{B}, \nabla(I - \Delta)\tilde{\mathbf{v}} \rangle_{L^2} \\ &\quad + \langle \mathbf{v} \otimes \mathbf{B}, \nabla(I - \Delta)\tilde{\mathbf{B}} \rangle_{L^2} - \langle \mathbf{B} \otimes \mathbf{v}, \nabla(I - \Delta)\tilde{\mathbf{B}} \rangle_{L^2} \\ &\leq \left(\left(\|\mathbf{v}\|_{L^2(\mathcal{O})}^2 + \|\mathbf{B}\|_{L^2(\mathcal{O})}^2 \right) \cdot \sup_{x \in \mathbb{D}} |\nabla(I - \Delta)\tilde{\mathbf{v}}(x)| \right. \\ &\quad \left. + 2\|\mathbf{v}\|_{L^2(\mathcal{O})} \|\mathbf{B}\|_{L^2(\mathcal{O})} \cdot \sup_{x \in \mathbb{D}} |\nabla(I - \Delta)\tilde{\mathbf{B}}(x)| \right) \\ &\leq 2\|y\|_{\mathcal{L}^2(\mathcal{O})}^2 \cdot \sup_{x \in \mathbb{D}} |\nabla(I - \Delta)\tilde{\mathbf{y}}(x)| \\ &\leq C_{\tilde{y}} \|y\|_{\mathcal{L}^3(\mathcal{O})}^2 \cdot \sup_{x \in \mathbb{D}} |\nabla(I - \Delta)\tilde{\mathbf{y}}(x)|. \end{aligned}$$

Finally,

$$\begin{aligned} \mathcal{A}_3(y, \tilde{y}) &= -\langle g_N(|y|^2)y, (I - \Delta)\tilde{y} \rangle_{L^2} \leq \|g_N(|y|^2)y\|_{\mathcal{L}^1(\mathcal{O})} \cdot \sup_{x \in \mathbb{D}} |(I - \Delta)\tilde{y}(x)| \\ &\leq C \|y\|_{\mathcal{L}^3(\mathcal{O})}^3 \sup_{x \in \mathbb{D}} |(I - \Delta)\tilde{y}(x)|. \end{aligned}$$

Combining the above estimates yields (2.9).

For the second estimate (2.10), we have by the boundedness of the Leray-Helmholtz projections

$$\begin{aligned} \|\langle \mathcal{B}(t, y), \tilde{y} \rangle_{\mathcal{H}^1}\|_{\ell^2}^2 &\leq \|\langle (\Sigma \cdot \nabla)y, (I - \Delta)\tilde{y} \rangle_{\mathcal{L}^2} + \langle H(t, y), (I - \Delta)\tilde{y} \rangle_{\mathcal{L}^2}\|_{\ell^2}^2 \\ &\leq 2 \left(\sum_k |\langle (\Sigma_k \cdot \nabla)y, (I - \Delta)\tilde{y} \rangle_{\mathcal{L}^2}|^2 + |\langle H_k(t, y), (I - \Delta)\tilde{y} \rangle_{\mathcal{L}^2}|^2 \right). \end{aligned}$$

The first term can be estimated using the definitions of the scalar products and norms involved, integration by parts, the product rule and Jensen's inequality:

$$\begin{aligned}
& \sum_k \left| \langle (\Sigma_k \cdot \nabla) y, (I - \Delta) \tilde{y} \rangle_{\mathcal{L}^2} \right|^2 \\
&= \sum_k \left| \int_{\mathbb{D}} \begin{pmatrix} (\boldsymbol{\sigma}_k \cdot \nabla) \mathbf{v} \\ (\bar{\boldsymbol{\sigma}}_k \cdot \nabla) \mathbf{B} \end{pmatrix} \cdot \begin{pmatrix} (I - \Delta) \tilde{\mathbf{v}} \\ (I - \Delta) \tilde{\mathbf{B}} \end{pmatrix} dx \right|^2 \\
&= \sum_k \left| \int_{\mathbb{D}} \langle (\boldsymbol{\sigma}_k \cdot \nabla) \mathbf{v}, (I - \Delta) \tilde{\mathbf{v}} \rangle_{\mathbb{R}^3} + \langle (\bar{\boldsymbol{\sigma}}_k \cdot \nabla) \mathbf{B}, (I - \Delta) \tilde{\mathbf{B}} \rangle_{\mathbb{R}^3} dx \right|^2 \\
&= \sum_k \left| \int_{\mathbb{D}} \sum_{j,l=1}^3 \sigma_k^j (\partial_{x^j} v^l) (I - \Delta) \tilde{v}^l + \bar{\sigma}_k^j (\partial_{x^j} B^l) (I - \Delta) \tilde{B}^l dx \right|^2 \\
&= \sum_k \left| - \int_{\mathbb{D}} \sum_{j,l=1}^3 v^l \partial_{x^j} (\sigma_k^j (I - \Delta) \tilde{v}^l) + B^l \partial_{x^j} (\bar{\sigma}_k^j (I - \Delta) \tilde{B}^l) dx \right|^2 \\
&= \sum_k \left| \int_{\mathcal{O}} \sum_{j,l=1}^3 v^l (\partial_{x^j} \sigma_k^j) (I - \Delta) \tilde{v}^l + v^l \sigma_k^j (I - \Delta) \partial_{x^j} \tilde{v}^l \right. \\
&\quad \left. + B^l (\partial_{x^j} \bar{\sigma}_k^j) (I - \Delta) \tilde{B}^l + B^l \bar{\sigma}_k^j (I - \Delta) \partial_{x^j} \tilde{B}^l dx \right|^2 \\
&\leq 36\lambda(\mathcal{O}) \sum_k \int_{\mathcal{O}} \sum_{j,l=1}^3 |v^l (\partial_{x^j} \sigma_k^j) (I - \Delta) \tilde{v}^l|^2 + |v^l \sigma_k^j (I - \Delta) \partial_{x^j} \tilde{v}^l|^2 \\
&\quad + |B^l (\partial_{x^j} \bar{\sigma}_k^j) (I - \Delta) \tilde{B}^l|^2 + |B^l \bar{\sigma}_k^j (I - \Delta) \partial_{x^j} \tilde{B}^l|^2 dx \\
&\leq 36\lambda(\mathcal{O}) \left(\sup_{t \in [0, T], x \in \mathbb{D}} \|\nabla_x \boldsymbol{\sigma} \cdot (t, x)\|_{\ell^2}^2 \sup_{x \in \mathbb{D}} |(I - \Delta) \tilde{\mathbf{v}}(x)|^2 \|\mathbf{v}\|_{L^2(\mathcal{O})}^2 \right. \\
&\quad + \sup_{t \in [0, T], x \in \mathbb{D}} \|\boldsymbol{\sigma} \cdot (t, x)\|_{\ell^2}^2 \sup_{x \in \mathbb{D}} |\nabla(I - \Delta) \tilde{\mathbf{v}}(x)|^2 \|\mathbf{v}\|_{L^2(\mathcal{O})}^2 \\
&\quad + \sup_{t \in [0, T], x \in \mathbb{D}} \|\nabla_x \bar{\boldsymbol{\sigma}} \cdot (t, x)\|_{\ell^2}^2 \sup_{x \in \mathbb{D}} |(I - \Delta) \tilde{\mathbf{B}}(x)|^2 \|\mathbf{B}\|_{L^2(\mathcal{O})}^2 \\
&\quad \left. + \sup_{t \in [0, T], x \in \mathbb{D}} \|\bar{\boldsymbol{\sigma}} \cdot (t, x)\|_{\ell^2}^2 \sup_{x \in \mathbb{D}} |\nabla(I - \Delta) \tilde{\mathbf{B}}(x)|^2 \|\mathbf{B}\|_{L^2(\mathcal{O})}^2 \right) \\
&\leq 36\lambda(\mathcal{O}) \left(\sup_{t \in [0, T], x \in \mathbb{D}} \|\nabla_x \Sigma \cdot (t, x)\|_{\ell^2}^2 \sup_{x \in \mathbb{D}} |(I - \Delta) \tilde{y}(x)|^2 \right. \\
&\quad \left. + \sup_{t \in [0, T], x \in \mathbb{D}} \|\Sigma \cdot (t, x)\|_{\ell^2}^2 \sup_{x \in \mathbb{D}} |\nabla(I - \Delta) \tilde{y}(x)|^2 \right) \|y\|_{\mathcal{L}^2(\mathcal{O})}^2
\end{aligned}$$

which we estimate further by using Assumption (H2), arriving at

$$\sum_k \left| \langle (\Sigma_k \cdot \nabla) y, (I - \Delta) \tilde{y} \rangle_{\mathcal{L}^2} \right|^2 \leq C_{\Sigma, T, \tilde{y}} \|y\|_{\mathcal{L}^2(\mathcal{O})}^2.$$

Similarly, for the second term we find

$$\begin{aligned}
& \sum_k |\langle H_k(t, y), (I - \Delta)\tilde{y} \rangle_{\mathcal{L}^2}|^2 \\
&= \sum_k \left| \int_{\mathbb{D}} (\mathbf{h}_k(t, x, y), \bar{\mathbf{h}}_k(t, x, y)), \begin{pmatrix} (I - \Delta)\tilde{\mathbf{v}} \\ (I - \Delta)\tilde{\mathbf{B}} \end{pmatrix} dx \right|^2 \\
&= \sum_k \left| \int_{\mathcal{O}} \langle \mathbf{h}_k(t, x, y), (I - \Delta)\tilde{\mathbf{v}}(x) \rangle_{\mathbb{R}^3} + \langle \bar{\mathbf{h}}_k(t, x, y), (I - \Delta)\tilde{\mathbf{B}}(x) \rangle_{\mathbb{R}^3} dx \right|^2 \\
&\leq 2\lambda(\mathcal{O}) \sum_k \int_{\mathcal{O}} |\langle \mathbf{h}_k(t, x, y), (I - \Delta)\tilde{\mathbf{v}}(x) \rangle_{\mathbb{R}^3}|^2 + |\langle \bar{\mathbf{h}}_k(t, x, y), (I - \Delta)\tilde{\mathbf{B}}(x) \rangle_{\mathbb{R}^3}|^2 dx \\
&\leq 2\lambda(\mathcal{O}) \sup_{x \in \mathbb{D}} |(I - \Delta)\tilde{\mathbf{v}}(x)|^2 \int_{\mathcal{O}} \|\mathbf{h}_k(t, x, y)\|_{\ell^2}^2 + \sup_{x \in \mathbb{D}} |(I - \Delta)\tilde{\mathbf{B}}(x)|^2 \|\bar{\mathbf{h}}_k(t, x, y)\|_{\ell^2}^2 dx \\
&\leq 2\lambda(\mathcal{O}) C_{T,H} \sup_{x \in \mathbb{D}} |(I - \Delta)\tilde{y}(x)|^2 \int_{\mathcal{O}} |y(x)|^2 + F_H(t, x) dx \\
&\leq C_{H,T,\tilde{y}} \left(\|y\|_{\mathcal{L}^2(\mathcal{O})}^2 + \|F_H(t)\|_{L^1(\mathbb{D})} \right).
\end{aligned}$$

Thus, altogether we have proven (2.10).

For estimate (2.11), we look at each of the three terms of \mathcal{A} separately:

$$\begin{aligned}
|\mathcal{A}_1(y, \tilde{y}) - \mathcal{A}_1(y', \tilde{y})| &= |\langle (y - y'), 1_{\mathcal{O}}(I - \Delta)\Delta\tilde{y} \rangle_{\mathcal{H}^0}| \\
&\leq \sup_{x \in \mathbb{D}} |(I - \Delta)\Delta\tilde{y}(x)| \|y - y'\|_{\mathcal{L}^2(\mathcal{O})} = C_{\tilde{y}} \|y - y'\|_{\mathcal{L}^2(\mathcal{O})}.
\end{aligned}$$

The nonlinear terms are handled in the usual way

$$\begin{aligned}
& |\mathcal{A}_2(y, \tilde{y}) - \mathcal{A}_2(y', \tilde{y})| \\
&\leq \left| \langle (\mathbf{v} \cdot \nabla)\mathbf{v} - (\mathbf{v}' \cdot \nabla)\mathbf{v}', (I - \Delta)\tilde{\mathbf{v}} \rangle_{L^2(\mathcal{O})} \right| \\
&\quad + \left| \langle (\mathbf{B} \cdot \nabla)\mathbf{B} - (\mathbf{B}' \cdot \nabla)\mathbf{B}', (I - \Delta)\tilde{\mathbf{v}} \rangle_{L^2(\mathcal{O})} \right| \\
&\quad + \left| \langle (\mathbf{v} \cdot \nabla)\mathbf{B} - (\mathbf{v}' \cdot \nabla)\mathbf{B}', (I - \Delta)\tilde{\mathbf{B}} \rangle_{L^2(\mathcal{O})} \right| \\
&\quad + \left| \langle (\mathbf{B} \cdot \nabla)\mathbf{v} - (\mathbf{B}' \cdot \nabla)\mathbf{v}', (I - \Delta)\tilde{\mathbf{B}} \rangle_{L^2(\mathcal{O})} \right|.
\end{aligned}$$

We treat the third term only, the other terms work in a similar manner:

$$\begin{aligned}
& \left| \langle (\mathbf{v} \cdot \nabla)\mathbf{B} - (\mathbf{v}' \cdot \nabla)\mathbf{B}', (I - \Delta)\tilde{\mathbf{B}} \rangle_{L^2(\mathcal{O})} \right| \\
&= \left| \langle (\mathbf{v} - \mathbf{v}') \cdot \nabla \mathbf{B} - (\mathbf{v}' \cdot \nabla)(\mathbf{B}' - \mathbf{B}), (I - \Delta)\tilde{\mathbf{B}} \rangle_{L^2(\mathcal{O})} \right| \\
&= \left| \langle (\mathbf{v} - \mathbf{v}') \otimes \mathbf{B} - \mathbf{v}' \otimes (\mathbf{B}' - \mathbf{B}), \nabla(I - \Delta)\tilde{\mathbf{B}} \rangle_{L^2(\mathcal{O})} \right| \\
&\leq \sup_{x \in \mathbb{D}} |\nabla(I - \Delta)\tilde{\mathbf{B}}(x)| (\|\mathbf{v} - \mathbf{v}'\|_{L^2(\mathcal{O})} \|\mathbf{B}\|_{L^2(\mathcal{O})} + \|\mathbf{B} - \mathbf{B}'\|_{L^2(\mathcal{O})} \|\mathbf{v}'\|_{L^2(\mathcal{O})}) \\
&\leq C_{\tilde{y}} \|y - y'\|_{\mathcal{L}^2(\mathcal{O})} (\|y\|_{\mathcal{L}^2(\mathcal{O})} + \|y'\|_{\mathcal{L}^2(\mathcal{O})}),
\end{aligned}$$

and thus we obtain

$$\begin{aligned} |\mathcal{A}_2(y, \tilde{y}) - \mathcal{A}_2(y', \tilde{y})| &\leq C_{\tilde{y}} \|y - y'\|_{\mathcal{L}^2(\mathcal{O})} (\|y\|_{\mathcal{L}^2(\mathcal{O})} + \|y'\|_{\mathcal{L}^2(\mathcal{O})}) \\ &\leq C_{\tilde{y}} \|y - y'\|_{\mathcal{L}^2(\mathcal{O})} (\|y\|_{\mathcal{L}^4(\mathcal{O})} + \|y'\|_{\mathcal{L}^4(\mathcal{O})}). \end{aligned}$$

For the taming term, we find

$$\begin{aligned} |\mathcal{A}_3(y, \tilde{y}) - \mathcal{A}_3(y', \tilde{y})| &= \left| \langle g_N(|y|^2)y - g_N(|y'|^2)y', (I - \Delta)\tilde{y} \rangle_{\mathcal{L}^2(\mathcal{O})} \right| \\ &= \left| \langle g_N(|y|^2)(y - y') - (g_N(|y'|^2) - g_N(|y|^2))y', (I - \Delta)\tilde{y} \rangle_{\mathcal{L}^2(\mathcal{O})} \right| \\ &= \left| \langle g_N(|y|^2)(y - y') - g'_N(\theta)(|y|^2 - |y'|^2)y', (I - \Delta)\tilde{y} \rangle_{\mathcal{L}^2(\mathcal{O})} \right| \\ &\leq \sup_{x \in \mathbb{D}} |(I - \Delta)\tilde{y}(x)| \int_{\mathcal{O}} |y - y'| [g_N(|y|^2) + g'_N(\theta)(|y| + |y'|)] |y'| dx \\ &\leq C_{\tilde{y}} C_N \int_{\mathcal{O}} |y - y'| [|y|^2 + (|y| + |y'|)|y'|] dx \\ &\leq C_{\tilde{y}} C_N \int_{\mathcal{O}} |y - y'| [|y|^2 + |y'|^2] dx \\ &\leq C_{\tilde{y}, N} \|y - y'\|_{\mathcal{L}^2(\mathcal{O})} (\|y\|_{\mathcal{L}^4(\mathcal{O})}^2 + \|y'\|_{\mathcal{L}^4(\mathcal{O})}^2). \end{aligned}$$

Collecting the terms, we get

$$\begin{aligned} |\langle \mathcal{A}(y) - \mathcal{A}(y'), \tilde{y} \rangle_{\mathcal{H}^1}| \\ \leq C_{\tilde{y}, N} \|y - y'\|_{\mathcal{L}^2(\mathcal{O})} \left(1 + \|y\|_{\mathcal{L}^4(\mathcal{O})} + \|y'\|_{\mathcal{L}^4(\mathcal{O})} + \|y\|_{\mathcal{L}^4(\mathcal{O})}^2 + \|y'\|_{\mathcal{L}^4(\mathcal{O})}^2 \right), \end{aligned}$$

which implies (2.11) by Sobolev embedding. \square

The estimates of the next lemma appear often in expressions for the quadratic variation of the Itô term when we apply Itô's formula.

LEMMA 2.5 (Estimates for \mathcal{B}). *For any $T > 0$, $0 \leq t \leq T$ and $y \in \mathcal{H}^2$, the following estimates hold:*

$$(2.12) \quad \|\mathcal{B}(t, y)\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)}^2 \leq \frac{1}{2} \|y\|_{\mathcal{H}^1}^2 + C_{T, H} \|y\|_{\mathcal{H}^0}^2 + \|F_H(t)\|_{L^1(\mathbb{D})},$$

$$(2.13) \quad \|\mathcal{B}(t, y)\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^1)}^2 \leq \frac{1}{2} \|y\|_{\mathcal{H}^2}^2 + C_{T, H, \Sigma} \|y\|_{\mathcal{H}^1}^2 + C \|F_H(t)\|_{L^1(\mathbb{D})}.$$

PROOF. To calculate the Hilbert-Schmidt norm (cf. [167, Definition B.0.5, p. 217]), we take the orthonormal basis of $\ell^2 \times \ell^2$ from Equation (2.3) and enumerate it as a set $\{\hat{e}_l\}_{l \in \mathbb{N}}$. Then by definition, for $m = 0, 1$,

$$\begin{aligned} \|\mathcal{B}(t, y)\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^m)}^2 &= \sum_{l \in \mathbb{N}} \|\mathcal{B}(t, y)\hat{e}_l\|_{\mathcal{H}^m}^2 = \sum_{k \in \mathbb{N}} \left\| \mathcal{B}(t, y) \begin{pmatrix} e_k \\ 0 \end{pmatrix} \right\|_{\mathcal{H}^m}^2 + \sum_{k \in \mathbb{N}} \left\| \mathcal{B}(t, y) \begin{pmatrix} 0 \\ e_k \end{pmatrix} \right\|_{\mathcal{H}^m}^2 \\ &= \sum_{k \in \mathbb{N}} \|\mathcal{B}_k(t, y)\|_{\mathcal{H}^m}^2. \end{aligned}$$

The first inequality follows from Assumptions (H2) and (H3):

$$\begin{aligned}
\|\mathcal{B}(t, y)\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)}^2 &= \sum_k \int_{\mathbb{D}} |\mathcal{B}_k(t, x, y)|^2 dx \\
&= \sum_k \int_{\mathbb{D}} |(\Sigma_k(t, x) \cdot \nabla)y + H_k(t, x, y)|^2 dx \\
&\leq 2 \int_{\mathbb{D}} \|\Sigma(t, x)\|_{\ell^2}^2 |\nabla y|^2 + \|H_k(t, x, y)\|_{\ell^2}^2 dx \\
&\leq 2 \left(\sup_{t \in [0, T], x \in \mathbb{D}} \|\Sigma(t, x)\|_{\ell^2}^2 \right) \int_{\mathbb{D}} |\nabla y|^2 + 2C_{T, H} |y|^2 + F_H(t, x) dx \\
&\leq \frac{1}{2} \|y\|_{\mathcal{H}^1}^2 + (2C_{T, H} - \frac{1}{2}) \|y\|_{\mathcal{H}^0}^2 + \|F_H(t)\|_{L^1(\mathbb{D})}.
\end{aligned}$$

For the second inequality, we note that

$$\|\mathcal{B}(t, y)\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^1)}^2 = \|\mathcal{B}(t, y)\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)}^2 + \|\nabla \mathcal{B}(t, y)\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)}^2,$$

and, using the commutativity of derivatives and \mathcal{P} as well as the chain rule, we find

$$\begin{aligned}
\partial_{x^j} \mathcal{B}_k(t, x, y) &= \partial_{x^j} \mathcal{P} [(\Sigma_k(t, x) \cdot \nabla)y + H_k(t, x, y)] = \mathcal{P} \partial_{x^j} [(\Sigma_k(t, x) \cdot \nabla)y + H_k(t, x, y)] \\
&= \mathcal{P} [(\partial_{x^j} \Sigma_k(t, x) \cdot \nabla)y + (\Sigma_k(t, x) \cdot \nabla) \partial_{x^j} y + (\partial_{x^j} H_k)(t, x, y) + \sum_{i=1}^6 \partial_{y^i} H_k(t, x, y) \partial_{x^j} y^i].
\end{aligned}$$

Therefore, employing Assumptions (H2), (H3), as well as Equation (2.1), we find

$$\begin{aligned}
\|\nabla \mathcal{B}(t, y)\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)}^2 &\leq \sum_k \sum_{j=1}^3 \int_{\mathbb{D}} |(\partial_{x^j} \Sigma_k(t, x) \cdot \nabla)y + (\Sigma_k(t, x) \cdot \nabla) \partial_{x^j} y \\
&\quad + (\partial_{x^j} H_k)(t, x, y) + \sum_{i=1}^6 \partial_{y^i} H_k(t, x, y) \partial_{x^j} y^i|^2 dx \\
&\leq \sum_k \sum_{j=1}^3 \int_{\mathbb{D}} 2|(\Sigma_k(t, x) \cdot \nabla) \partial_{x^j} y|^2 + 2|(\partial_{x^j} \Sigma_k(t, x) \cdot \nabla)y + \\
&\quad + (\partial_{x^j} H_k)(t, x, y) + \sum_{i=1}^6 \partial_{y^i} H_k(t, x, y) \partial_{x^j} y^i|^2 dx \\
&\leq 2 \int_{\mathbb{D}} \sum_{i, j=1}^3 \|\Sigma(t, x)\|_{\ell^2}^2 |\partial_{x^i} \partial_{x^j} y|^2 dx + 6 \int_{\mathbb{D}} \sum_{j=1}^3 \|\partial_{x^j} \Sigma(t, x)\|_{\ell^2}^2 |\nabla y|^2 \\
&\quad + \|(\partial_{x^j} H_k)(t, x, y)\|_{\ell^2}^2 + 6 \sum_{i=1}^6 \|\partial_{y^i} H(t, x, y)\|_{\ell^2}^2 |\partial_{x^j} y^i|^2 dx \\
&\leq 2 \cdot 3^2 \sup_{t \in [0, T], x \in \mathbb{D}} \|\Sigma(t, x)\|_{\ell^2}^2 \|y\|_{\mathcal{H}^2}^2 + 6 \left(\sup_{t \in [0, T], x \in \mathbb{D}} \|\partial_{x^j} \Sigma(t, x)\|_{\ell^2}^2 \|\nabla y\|_{\mathcal{H}^0}^2 \right. \\
&\quad \left. + C_{T, H} \|y\|_{\mathcal{H}^0}^2 + \|F_H(t)\|_{L^1(\mathbb{D})} + C_{T, H} \|\nabla y\|_{\mathcal{H}^0}^2 \right) \\
&\leq \frac{1}{2} \|y\|_{\mathcal{H}^2}^2 + C_{T, H, \Sigma} (\|\nabla y\|_{\mathcal{H}^0}^2 + \|y\|_{\mathcal{H}^0}^2 + \|F_H(t)\|_{L^1(\mathbb{D})}),
\end{aligned}$$

which, together with (2.12), implies (2.13). \square

2.3. A Tightness Criterion. In this section we establish a tightness criterion, which is a version of a corresponding result in [196, Lemma 2.7]. As the case of periodic boundary conditions can be treated in the same way, we will focus on the case of the full space \mathbb{R}^3 .

Let $\mathbb{D} = \mathbb{R}^3$. We endow the space $\mathcal{H}_{\text{loc}}^0$ of locally L^2 -integrable and divergence-free vector fields with the following Fréchet metric: for $y, z \in \mathcal{H}_{\text{loc}}^0$

$$\rho(y, z) := \sum_{m \in \mathbb{N}} 2^{-m} \left(\left[\int_{|x| \leq m} |y(x) - z(x)|^2 dx \right]^{1/2} \wedge 1 \right).$$

Then the space $(\mathcal{H}_{\text{loc}}^0, \rho)$ is a Polish space and $\mathcal{H}^0 \subset \mathcal{H}_{\text{loc}}^0$.

Now set $\mathcal{X} := C(\mathbb{R}_+; \mathcal{H}_{\text{loc}}^0)$ and endow it with the metric

$$\rho_{\mathcal{X}}(y, z) := \sum_{m=1}^{\infty} 2^{-m} \left(\sup_{t \in [0, m]} \rho(y(t), z(t)) \wedge 1 \right).$$

Similar to the considerations in the proof of Chapter II, Theorem 2.8, we fix a complete orthonormal basis $\mathcal{E} := \{e_k \mid k \in \mathbb{N}\} \subset \mathcal{V}$ of \mathcal{H}^1 such that $\text{span}\{\mathcal{E}\}$ is a dense subset of \mathcal{H}^3 and - in the case $\mathbb{D} = \mathbb{T}^3$ - such that it is also an orthogonal basis of \mathcal{H}^0 . Given $y \in \mathcal{H}^0, z \in \mathcal{H}^2$, the inner product $\langle y, z \rangle_{\mathcal{H}^1}$ is understood in the generalised sense:

$$(2.14) \quad \langle y, z \rangle_{\mathcal{H}^1} = \langle y, (I - \Delta)z \rangle_{\mathcal{H}^0}.$$

This works in accordance to our choice of evolution triple

$$\mathcal{H}^2 \subset \mathcal{H}^1 \subset \mathcal{H}^0.$$

The next lemma provides conditions for a set of \mathcal{X} to be relatively compact, similar to the classical theorem of Arzelá and Ascoli.

LEMMA 2.6 (Compactness in \mathcal{X}). *Let $K \subset \mathcal{X}$ satisfy the following conditions: for every $T > 0$*

- (i) $\sup_{y \in K} \sup_{s \in [0, T]} \|y(s)\|_{\mathcal{H}^1} < \infty$,
- (ii) $\lim_{\delta \rightarrow 0} \sup_{y \in K} \sup_{s, t \in [0, T], |t-s| < \delta} |\langle y(t) - y(s), e \rangle_{\mathcal{H}^1}| = 0$ for all $e \in \mathcal{E}$.

Then K is relatively compact in \mathcal{X} .

PROOF. We follow the proof of [196, Lemma 2.6]. It suffices to prove that K is relatively compact in $C([0, T]; \mathcal{H}_{\text{loc}}^0)$ for every $T > 0$. Then for $T = 1$, we can find a convergent subsequence $(y_n^{(1)})_n \rightarrow y^{(1)}$ in $C([0, 1]; \mathcal{H}_{\text{loc}}^0)$. This sequence then has a subsequence converging in $C([0, 2]; \mathcal{H}_{\text{loc}}^0)$ to $y^{(2)}$. By uniqueness of limits, we find that $y^{(2)}|_{[0, 1]} = y^{(1)}$. Continuing inductively, we find a function $\bar{y} \in C(\mathbb{R}_+; \mathcal{H}_{\text{loc}}^0)$ by defining $\bar{y}(t) := y^{(m)}(t)$ for $t \in [0, m]$. The diagonal sequence $\{y_n^{(m)} \mid m \in \mathbb{N}\} \subset K$ then converges to \bar{y} in \mathcal{X} .

Thus we fix a $T > 0$ in the following. Let $\{y_n \mid n \in \mathbb{N}\}$ be any sequence in K . For $e \in \mathcal{E}$, define

$$G_n^e(t) := \langle y_n(t), e \rangle_{\mathcal{H}^1} \quad (= \langle y_n(t), (I - \Delta)e \rangle_{\mathcal{H}^0}).$$

The two assumptions immediately yield that the sequence of functions $\{t \mapsto G_n^e(t) \mid n \in \mathbb{N}\}$ is uniformly bounded and equicontinuous on $[0, T]$. By the Arzelà-Ascoli theorem, we can find a subsequence $(G_{n_k}^e)$ and a continuous function G^e such that

$$\lim_{k \rightarrow \infty} \sup_{t \in [0, T]} |G_{n_k}^e(t) - G^e(t)| = 0.$$

Since \mathcal{E} is a countable set, we enumerate it as $\{e_n \mid n \in \mathbb{N}\}$ and apply the above observation to e_1 to find $(G_{n_k}^{e_1})$ converging uniformly to G^{e_1} .

We take the associated $(y_{n_k})_k \subset K$ and apply the same procedure with e_2 to find a subsequence $(y_{n_{k_l}})_l$ such that $(G_{n_{k_l}}^{e_i})$ converges uniformly to G^{e_i} , $i = 1, 2$. We continue again inductively and consider the diagonal sequence, which we denote by $(y_n)_n$ such that for every $e \in \mathcal{E}$

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |G_n^e(t) - G^e(t)| = 0.$$

On the other hand, by Assumption (i), for every t , the sequence $\{y_n(t) \mid n \in \mathbb{N}\}$ is bounded in \mathcal{H}^1 . By the Eberlein-Šmuljan theorem (cf. [248, Theorem 21.D, p. 255]), closed balls in \mathcal{H}^1 are weakly compact and thus there is a function $y \in L^\infty([0, T]; \mathcal{H}^1)$ such that for any $e \in \mathcal{E}$

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\langle y_n(t) - y(t), e \rangle_{\mathcal{H}^1}| = 0.$$

By definition of \mathcal{E} , we can conclude that for all $z \in \mathcal{H}^1$

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\langle y_n(t) - y(t), z \rangle_{\mathcal{H}^1}| = 0.$$

For $\tilde{z} \in \mathcal{H}^0$, we have $(I - \Delta)^{-1} \tilde{z} \in \mathcal{H}^2 \subset \mathcal{H}^1$ and thus we also find

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\langle y_n(t) - y(t), \tilde{z} \rangle_{\mathcal{H}^0}| = \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\langle y_n(t) - y(t), (I - \Delta)^{-1} \tilde{z} \rangle_{\mathcal{H}^1}| = 0.$$

Finally, using the Helmholtz-Weyl decomposition for both components \mathbf{v} , \mathbf{B} , we find that even for $z \in \mathcal{L}^2$

$$(2.15) \quad \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\langle y_n(t) - y(t), z \rangle_{\mathcal{L}^2}| = 0.$$

We use this to show the convergence

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \rho(y_n(t), y(t)) = 0,$$

which, by definition of ρ , follows if we can show that for any $m \in \mathbb{N}$

$$(2.16) \quad \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \int_{\{|x| \leq m\}} |y_n(t) - y(t)|^2 dx = 0.$$

To this end, recall first the following Friedrich's inequality (see e.g. [141] and Chapter II, Equation (2.22)): let $G \subset \mathbb{R}^3$ be a bounded set. Then for all $\varepsilon > 0$ there is a $K_\varepsilon \in \mathbb{N}$ and functions $h_i^\varepsilon \in L^2(G)$, $i = 1, \dots, K_\varepsilon$ such that for all $w \in W_0^{1,2}(G; \mathbb{R}^3) \times W_0^{1,2}(G; \mathbb{R}^3)$

$$\int_G |w(x)|^2 dx \leq \sum_{i=1}^{K_\varepsilon} \left(\int_G w(x) h_i^\varepsilon(x) dx \right)^2 + \varepsilon \int_G |\nabla w(x)|^2 dx.$$

Fix an $\varepsilon > 0$. We choose $G := \{|x| \leq m\} \subset \mathbb{R}^3$, $w_n := \zeta_m(y_n(t) - y(t))$ where $0 \leq \zeta_m \leq 1$ is a smooth cutoff function such that $\text{supp } \zeta_m \subset G$ and $\zeta_m \equiv 1$ on an open set $G' \subset\subset G$.

We find, similar to the calculations in the proof of Chapter II, Theorem 2.8, that

$$\begin{aligned}
& \int_{\{|x| \leq m\}} |y_n(t) - y(t)|^2 dx \\
& \leq \sum_{i=1}^{K_\varepsilon} (\langle y_n(t) - y(t), \zeta_m \cdot h_i^\varepsilon \rangle_{\mathcal{L}^2})^2 + \varepsilon \int_G |[\nabla \zeta_m](y_n - y)(t) + \zeta_m \nabla(y_n - y)(t)|^2 dx \\
& \leq \sum_{i=1}^{K_\varepsilon} (\langle y_n(t) - y(t), \zeta_m \cdot h_i^\varepsilon \rangle_{\mathcal{L}^2})^2 \\
& \quad + 2\varepsilon (\|\nabla \zeta_m\|_{L^\infty}^2 \vee \|\zeta_m\|_{L^\infty}^2) \int_{\mathbb{R}^3} |y_n(t) - y(t)|^2 + |\nabla(y_n(t) - y(t))|^2 dx \\
& = \sum_{i=1}^{K_\varepsilon} (\langle y_n(t) - y(t), \zeta_m \cdot h_i^\varepsilon \rangle_{\mathcal{L}^2})^2 + 2\varepsilon (\|\nabla \zeta_m\|_{L^\infty}^2 \vee \|\zeta_m\|_{L^\infty}^2) \|y_n(t) - y(t)\|_{\mathcal{H}^1}^2 \\
& \leq \sum_{i=1}^{K_\varepsilon} (\langle y_n(t) - y(t), \zeta_m \cdot h_i^\varepsilon \rangle_{\mathcal{L}^2})^2 + 2\varepsilon (\|\nabla \zeta_m\|_{L^\infty}^2 \vee \|\zeta_m\|_{L^\infty}^2) C_T,
\end{aligned}$$

where we used Assumption (i). Thus, by (2.15)

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \int_{\{|x| \leq m\}} |y_n(t) - y(t)|^2 dx \leq 2\varepsilon (\|\nabla \zeta_m\|_{L^\infty}^2 \vee \|\zeta_m\|_{L^\infty}^2) C_T,$$

which, since $\varepsilon > 0$ was arbitrary, implies (2.16) and concludes the proof. \square

This compactness statement can be turned into a tightness condition, as we demonstrate in the next lemma.

LEMMA 2.7 (Tightness in \mathcal{X}). *Let $(\mu_n)_{n \in \mathbb{N}}$ be a family of probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and assume that*

(i) *For all $T > 0$*

$$\lim_{R \rightarrow \infty} \sup_{n \in \mathbb{N}} \mu_n \left\{ y \in \mathcal{X} \mid \sup_{s \in [0, T]} \|y(s)\|_{\mathcal{H}^1} > R \right\} = 0.$$

(ii) *For all $e \in \mathcal{E}$ and any $\varepsilon, T > 0$*

$$\lim_{\delta \downarrow 0} \sup_{n \in \mathbb{N}} \mu_n \left\{ y \in \mathcal{X} \mid \sup_{s, t \in [0, T], |s-t| \leq \delta} |\langle y(t) - y(s), e \rangle_{\mathcal{H}^1}| > \varepsilon \right\} = 0.$$

Then $(\mu_n)_{n \in \mathbb{N}}$ is tight on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$.

PROOF. Fix $\eta > 0$. For any given $l \in \mathbb{N}$, Assumption (i) guarantees that for $R_l > 0$ sufficiently large

$$\sup_{n \in \mathbb{N}} \mu_n \left\{ y \in \mathcal{X} \mid \sup_{s \in [0, l]} \|y(s)\|_{\mathcal{H}^1} > R_l \right\} \leq \frac{\eta}{2^l}.$$

For any $k, l \in \mathbb{N}$, and $e_i \in \mathcal{E}$, Assumption (ii) guarantees that for $\delta_{k, i, l} > 0$ sufficiently small

$$\sup_{n \in \mathbb{N}} \mu_n \left\{ y \in \mathcal{X} \mid \sup_{s, t \in [0, l], |s-t| \leq \delta_{k, i, l}} |\langle y(t) - y(s), e_i \rangle_{\mathcal{H}^1}| > \frac{1}{k} \right\} \leq \frac{\eta}{2^{k+i+l}}.$$

Now we set

$$K_1 := \bigcap_{l \in \mathbb{N}} \left\{ y \in \mathcal{X} : \sup_{s \in [0, l]} \|y(s)\|_{\mathcal{H}^1} \leq R_l \right\},$$

$$K_2 := \bigcap_{k, i, l \in \mathbb{N}} \left\{ y \in \mathcal{X} \mid \sup_{s, t \in [0, l], |s-t| \leq \delta_{k, i, l}} |\langle y(t) - y(s), e_i \rangle_{\mathcal{H}^1}| \leq \frac{1}{k} \right\}.$$

The set $K := K_1 \cap K_2$ is relatively compact by Lemma 2.6 and hence the closure \bar{K} is compact. Then, using the definitions of the sets K_1, K_2 , we find

$$\mu_n(\bar{K}^c) \leq \mu_n(K^c) \leq \mu_n(K_1^c) + \mu_n(K_2^c) \leq \sum_{l=1}^{\infty} \frac{\eta}{2^l} + \sum_{k, i, l=1}^{\infty} \frac{\eta}{2^{k+i+l}} \leq \eta.$$

Since $\eta > 0$ was arbitrary, we conclude that the sequence $(\mu_n)_{n \in \mathbb{N}}$ is tight on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. \square

3. Existence and Uniqueness of Strong Solutions

In this section, we prove the main results of this chapter, namely that there exists a unique strong solution to the TMHD equations. After defining the notions of weak solution, strong solutions and unique strong solution and stating the celebrated Yamada-Watanabe theorem, we prove that pathwise uniqueness holds for the STMHD equations. In proving the existence of a strong solution, we employ the classical Faedo-Galerkin approximation scheme and show existence of a weak solution via the solution of the corresponding martingale problems. Together with the pathwise uniqueness, the Yamada-Watanabe theorem implies existence of a unique strong solution.

3.1. Weak and Strong Solutions. For a metric space \mathcal{U} , we denote the set of all probability measures on \mathcal{U} by $\mathcal{P}(\mathcal{U})$.

DEFINITION 3.1. Equation (2.4) is said to possess a weak solution with initial law $\vartheta \in \mathcal{P}(\mathcal{H}^1)$ if there exists a stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$, an \mathcal{H}^1 -valued, $(\mathcal{F}_t)_t$ -adapted process y and two independent infinite sequences of independent standard (\mathcal{F}_t) -Brownian motions $\left\{ \mathcal{W}^k(t) = \begin{pmatrix} W^k(t) \\ \bar{W}^k(t) \end{pmatrix} \mid t \geq 0, k \in \mathbb{N} \right\}$ such that

- (i) $y(0)$ has law ϑ in \mathcal{H}^1 ;
- (ii) For P -a.e. $\omega \in \Omega$ and every $T > 0$, $y(\cdot, \omega) \in C([0, T]; \mathcal{H}^1) \cap L^2([0, T]; \mathcal{H}^2)$;
- (iii) it holds that in \mathcal{H}^0

$$y(t) = y_0 + \int_0^t [\mathcal{A}(y(t)) + \mathcal{P}f(t, y(t))] dt + \sum_{k=1}^{\infty} \int_0^t \mathcal{B}_k(t, y(t)) d\mathcal{W}_t^k,$$

for all $t \geq 0$, P -a.s.

The solution is denoted by $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0}; \mathcal{W}; y)$.

REMARK 3.2. Under the Assumptions (H1)–(H3) in this section, the above integrals are well-defined by Equation (2.5).

For weak solutions, there are several notions of uniqueness. In this work, we are concerned mostly with pathwise uniqueness.

DEFINITION 3.3. *We say that pathwise uniqueness holds for Equation (2.4) if, whenever we are given two weak solutions on the same stochastic basis with the same Brownian motion,*

$$(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0}; \mathcal{W}; y), (\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0}; \mathcal{W}; y'),$$

the condition $P\{y(0) = y'(0)\} = 1$ implies $P\{y(t) = y'(t), \forall t \geq 0\} = 1$.

The following proposition links existence of weak solutions to existence of a solution to a corresponding martingale problem.

PROPOSITION 3.4. *Let \mathcal{E} and \mathcal{X} be given as in Section 2.3. For $\vartheta \in \mathcal{P}(\mathcal{H}^1)$, the following are equivalent:*

- (i) *There exists a weak solution to Equation (2.4) with initial law ϑ ;*
- (ii) *There exists a measure $P_\vartheta \in \mathcal{P}(\mathcal{X})$ with the following property: for P_ϑ -almost all $y \in \mathcal{X}$ and any $T > 0$, the real-valued process defined by*

$$(3.1) \quad y \in L^\infty([0, T]; \mathcal{H}^1) \cap L^2([0, T]; \mathcal{H}^2),$$

and for any $\varphi \in C_0^\infty(\mathbb{R})$ and any $e \in \mathcal{E}$

$$\begin{aligned} M_e^\varphi(t, y) &:= \varphi(\langle y(t), e \rangle_{\mathcal{H}^1}) - \varphi(\langle y(0), e \rangle_{\mathcal{H}^1}) \\ &\quad - \int_0^t \varphi'(\langle y(s), e \rangle_{\mathcal{H}^1}) \cdot \langle \mathcal{A}(y(s)), e \rangle_{\mathcal{H}^1} ds \\ &\quad - \int_0^t \varphi'(\langle y(s), e \rangle_{\mathcal{H}^1}) \cdot \langle f(s, y(s)), e \rangle_{\mathcal{H}^1} ds \\ &\quad - \frac{1}{2} \int_0^t \varphi''(\langle y(s), e \rangle_{\mathcal{H}^1}) \cdot \|\langle \mathcal{B}(s, y(s)), e \rangle_{\ell^2} ds \end{aligned}$$

is a continuous local P_ϑ -martingale with respect to $(\mathcal{B}_t(\mathcal{X}))_t$, where $\mathcal{B}_t(\mathcal{X})$ denotes the sub σ -algebra of \mathcal{X} up to time t .

PROOF. We follow the proof of [196], Section 6.1, pp. 257 ff.

(i) \Rightarrow (ii) follows from Itô's formula. For (ii) \Rightarrow (i), we consider the e -th ‘‘component’’

$$M_e(t, y) := \langle (y(t) - y(0)), e \rangle_{\mathcal{H}^1} - \int_0^t \langle \mathcal{A}(y(s)), e \rangle_{\mathcal{H}^1} ds - \int_0^t \langle f(s, y(s)), e \rangle_{\mathcal{H}^1} ds.$$

Then by [191, Chapter VII, Section 2, (2.2) Proposition, p. 295], $(M_e(t, y))_{t \geq 0}$ is a continuous local P_ϑ -martingale with respect to $\mathcal{B}_t(\mathcal{X})$, and its quadratic variation process is given by

$$[M_e](t, y) = \int_0^t \|\langle \mathcal{B}(s, y(s)), e \rangle_{\mathcal{H}^1}\|_{\ell^2}^2 ds.$$

Now we consider the process

$$(3.2) \quad M(t, y) := \sum_{j=1}^{\infty} M_{e_j}(t, y) e_j.$$

As it involves a limit, we need to show that it is well-defined. More precisely, we show that $t \mapsto M(t, y)$ is an \mathcal{H}^1 -valued continuous local P_ϑ -martingale with respect to $(\mathcal{B}_t(\mathcal{X}))_t$. To this end, let $R > 0$ and define the stopping time

$$\tau_R := \tau_R(y) := \inf\{t \geq 0 \mid \int_0^t \|\mathcal{B}(s, y(s))\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^1)}^2 ds \geq R\}.$$

By (2.13), (3.1) and Assumption (H3), we find for P_ϑ -almost every $y \in \mathcal{X}$ that

$$\begin{aligned} \int_0^T \|\mathcal{B}(s, y(s))\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^1)}^2 ds &\leq \int_0^T \frac{1}{2} \|y(s)\|_{\mathcal{H}^2}^2 + C_{T,H} \|y(s)\|_{\mathcal{H}^1}^2 + C \|F_H(s)\|_{L^1(\mathbb{D})} ds \\ &\leq \frac{1}{2} \|y\|_{L^2([0,T]; \mathcal{H}^2)} + TC_{T,H} \|y\|_{L^\infty([0,T]; \mathcal{H}^1)}^2 + C \|F_H\|_{L^1([0,T] \times \mathbb{D})} < \infty, \end{aligned}$$

and therefore we have that $(\tau_R(y))_R$ is a localising sequence for the local martingale, i.e.

$$\tau_R(y) \uparrow \infty, \quad P_\vartheta - a.a. \ y, \quad \text{as } R \rightarrow \infty.$$

Now we define the approximations

$$M^{R,n}(t, y) := \sum_{j=1}^n M_{e_j}(t \wedge \tau_R, y) e_j.$$

As a finite sum of martingales, $(M^{R,n}(t, y))_t$ is an \mathcal{H}^1 -valued continuous martingale with square-variation (some authors use the terminology tensor-quadratic process, cf. [176, pp. 11-13])

$$\begin{aligned} \ll M^{R,n} \gg_{\mathcal{H}^1}(t, y) &= \sum_{i,j=1}^n [M_{e_i}, M_{e_j}](t \wedge \tau_R, y) \cdot e_i \otimes e_j \\ &= \sum_{i,j=1}^n \int_0^{t \wedge \tau_R} \langle \langle \mathcal{B}(s, y(s)), e_i \rangle_{\mathcal{H}^1}, \langle \mathcal{B}(s, y(s)), e_j \rangle_{\mathcal{H}^1} \rangle_{\ell^2} \cdot e_i \otimes e_j ds. \end{aligned}$$

Burkholder's inequality (cf. [172, Theorem 1.1]) applied to the \mathcal{H}^1 -valued martingale $N_t := M^{R,n}(t, y) - M^{R,m}(t, y)$ implies that for any $T > 0$

$$\begin{aligned} &\mathbb{E}^{P_\vartheta} \left[\sup_{t \in [0, T]} \|M^{R,n}(t, y) - M^{R,m}(t, y)\|_{\mathcal{H}^1}^2 \right] \\ &\leq C \mathbb{E}^{P_\vartheta} [\langle M^{R,n} - M^{R,m} \rangle(T, y)] \\ &= C \mathbb{E}^{P_\vartheta} [\text{Tr} \ll M^{R,n} - M^{R,m} \gg_{\mathcal{H}^1}(T, y)] \\ &= C \mathbb{E}^{P_\vartheta} \left[\text{Tr} \ll \sum_{j=m}^n M_{e_j}(t \wedge \tau_R, y) e_j \gg_{\mathcal{H}^1}(T, y) \right] \\ &= C \mathbb{E}^{P_\vartheta} \left[\sum_{i,j=m}^n \text{Tr} \int_0^{T \wedge \tau_R} \left(\sum_{k=1}^{\infty} \langle \mathcal{B}_k(s, y(s)), e_i \rangle_{\mathcal{H}^1} \langle \mathcal{B}_k(s, y(s)), e_j \rangle_{\mathcal{H}^1} \right) \cdot (e_i \otimes e_j) ds \right] \\ &= C \mathbb{E}^{P_\vartheta} \left[\sum_{e_l \in \mathcal{E}} \sum_{i,j=m}^n \int_0^{T \wedge \tau_R} \left(\sum_{k=1}^{\infty} \langle \mathcal{B}_k(s, y(s)), e_i \rangle_{\mathcal{H}^1}, \langle \mathcal{B}_k(s, y(s)), e_j \rangle_{\mathcal{H}^1} \right) \langle (e_i \otimes e_j) e_l, e_l \rangle_{\mathcal{H}^1} ds \right] \\ &= C \mathbb{E}^{P_\vartheta} \left[\sum_{e_l \in \mathcal{E}} \sum_{i,j=m}^n \int_0^{T \wedge \tau_R} \left(\sum_{k=1}^{\infty} \langle \mathcal{B}_k(s, y(s)), e_i \rangle_{\mathcal{H}^1}, \langle \mathcal{B}_k(s, y(s)), e_j \rangle_{\mathcal{H}^1} \right) \delta_{j,l} \delta_{i,l} ds \right] \\ &= C \sum_{j=m}^n \mathbb{E}^{P_\vartheta} \left[\int_0^{T \wedge \tau_R} \|\langle \mathcal{B}(s, y(s)), e_j \rangle_{\mathcal{H}^1}\|_{\ell^2}^2 ds \right] \rightarrow 0, \end{aligned}$$

as $n, m \rightarrow \infty$. Here we have used the equality

$$\mathbb{E}[\langle M \rangle_t] = \mathbb{E}[\text{Tr} \ll M \gg_t]$$

for Hilbert space-valued martingales, cf. [176, p. 9 and Equation (2.3.7)]. This proves convergence of the series of (3.2) in $C([0, T]; \mathcal{H}^1)$, P_ϑ -a.s. and hence, $M^R(t, y) := M(t \wedge \tau_R, y)$ is an \mathcal{H}^1 -valued continuous square-integrable martingale with tensor-quadratic process

$$\begin{aligned} \ll M^R \gg_{\mathcal{H}^1}(t, y) &= \sum_{i,j=1}^{\infty} [M_{e_i}, M_{e_j}](t \wedge \tau_R, y) \cdot e_i \otimes e_j \\ &= \sum_{i,j=1}^{\infty} \int_0^{t \wedge \tau_R} \langle \langle \mathcal{B}(s, y(s)), e_i \rangle_{\mathcal{H}^1}, \langle \mathcal{B}(s, y(s)), e_j \rangle_{\mathcal{H}^1} \rangle_{\ell^2} \cdot e_i \otimes e_j ds. \end{aligned}$$

Letting $R \rightarrow \infty$, we obtain that $M(t, y)$ is an \mathcal{H}^1 -valued continuous local martingale. In particular, we find that the following semimartingale representation holds in \mathcal{H}^0 :

$$y(t) = y(0) + \int_0^t \mathcal{A}(y(s)) ds + \int_0^t f(s, y(s)) ds + M(t, y), \quad P_\vartheta - a.s.$$

Now, applying Itô's formula to $\|y(t)\|_{\mathcal{H}^1}^2$, we find that $t \mapsto \|y(t)\|_{\mathcal{H}^1}^2$ and hence also $t \mapsto \|y(t)\|_{\mathcal{H}^1}$ are P_ϑ -a.s. continuous functions. Considering the evolution triple

$$\mathcal{H}^2 \subset \mathcal{H}^1 \subset \mathcal{H}^0,$$

with the semimartingale representation of $y(t)$ from above, we see that $t \mapsto y(t)$ is P_ϑ -almost surely continuous in \mathcal{H}^0 and hence weakly continuous in \mathcal{H}^1 . By [248, Proposition 21.23 (d), p. 258], weak continuity and continuity of the norm imply strong continuity in \mathcal{H}^1 , thus

$$P_\vartheta(C([0, T]; \mathcal{H}^1)) = 1.$$

As any continuous martingale can be represented as the stochastic integral of some Brownian motion \mathcal{W} (by means of the martingale representation theorem, applied on the space \mathcal{H}^1 , cf. [52, Theorem 8.2, p. 220 f.]), we get the existence of a weak solution $(\mathcal{X}, \mathcal{B}(\mathcal{X}), P_\vartheta, (\mathcal{B}_t(\mathcal{X}))_{t \geq 0}; \mathcal{W}; y)$. \square

To be able to define the notion of strong solutions to Equation (2.4), we need a canonical realisation of an infinite sequence of independent standard Brownian motions on a Polish space. To this end, consider the space $C(\mathbb{R}_+, \mathbb{R})$ of continuous functions on \mathbb{R}_+ with the metric

$$\tilde{\rho}(w, w') := \sum_{k=1}^{\infty} 2^{-k} \left(\sup_{t \in [0, k]} |w(t) - w'(t)| \wedge 1 \right).$$

We define the product space $\mathbb{W} := \prod_{j=1}^{\infty} C(\mathbb{R}_+; \mathbb{R})$ and endow it with the metric

$$\rho_{\mathbb{W}}(w, w') := \sum_{j=1}^{\infty} 2^{-j} (\tilde{\rho}(w^j, w'^j) \wedge 1), \quad w = (w^1, w^2, \dots), \quad w' = (w'^1, w'^2, \dots).$$

The space $(\mathbb{W}, \rho_{\mathbb{W}})$ is a Polish space. We denote the σ -algebra up to time t by $\mathcal{B}_t(\mathbb{W}) \subset \mathcal{B}(\mathbb{W})$ and endow $(\mathbb{W}, \mathcal{B}(\mathbb{W}))$ with the Wiener measure \mathbb{P} such that the coordinate process

$$w(t) := (w^1(t), w^2(t), \dots)$$

is an infinite sequence of independent standard $(\mathcal{B}_t(\mathbb{W}))_t$ -Brownian motions on the probability space $(\mathbb{W}, \mathcal{B}(\mathbb{W}), \mathbb{P})$.

To cater for the two noise terms present in the stochastic MHD equations, we take two copies of \mathbb{W} and take their product $\mathfrak{W} := \mathbb{W} \times \mathbb{W}$, with the metric

$$\rho_{\mathfrak{W}}((w, \bar{w}), (w', \bar{w}')) := \rho_{\mathbb{W}}(w, w') + \rho_{\mathbb{W}}(\bar{w}, \bar{w}').$$

Then $(\mathfrak{W}, \rho_{\mathfrak{W}})$ is also a Polish space. In the same way as above, we introduce the filtration $\mathcal{B}_t(\mathfrak{W}) \subset \mathcal{B}(\mathfrak{W})$, and endow $(\mathfrak{W}, \mathcal{B}(\mathfrak{W}))$ with the product $\mathfrak{P} := \mathbb{P} \otimes \mathbb{P}$ of the two Wiener measures. Then the coordinate process

$$\mathcal{W}(t) := \begin{pmatrix} W(t) \\ \bar{W}(t) \end{pmatrix}$$

consists of two independent infinite sequences of independent standard Brownian motions. For simplicity, in the following we will sometimes refer to the process \mathcal{W} as a Brownian motion.

Now consider the space of continuous, \mathcal{H}^1 -valued paths that are square-integrable in time with values in \mathcal{H}^2 , $\mathfrak{B} := C(\mathbb{R}_+; \mathcal{H}^1) \cap L_{\text{loc}}^2(\mathbb{R}_+; \mathcal{H}^2)$ with the metric

$$\rho_{\mathfrak{B}}(y, y') := \sum_{k \in \mathbb{N}} 2^{-k} \left(\sup_{t \in [0, k]} \|y(t) - y'(t)\|_{\mathcal{H}^1} + \int_0^k \|y(t) - y'(t)\|_{\mathcal{H}^2}^2 dt \right) \wedge 1.$$

The σ -algebra up to time t of this space is denoted by $\mathcal{B}_t(\mathfrak{B}) \subset \mathcal{B}(\mathfrak{B})$. For any measure space $(S, \mathcal{S}, \lambda)$, we denote the completion of the σ -algebra \mathcal{S} with respect to the measure λ by $\bar{\mathcal{S}}^\lambda$.

DEFINITION 3.5. *Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0}; \mathcal{W}; y)$ be a weak solution of Equation (2.4) with initial distribution $\vartheta \in \mathcal{P}(\mathcal{H}^1)$. If there exists a $\overline{\mathcal{B}(\mathcal{H}^1) \times \mathcal{B}(\mathfrak{W})}^{\vartheta \otimes P} / \mathcal{B}(\mathfrak{B})$ -measurable function $F_\vartheta: \mathcal{H}^1 \times \mathfrak{W} \rightarrow \mathfrak{B}$ such that*

- (i) *For every $t > 0$, F_ϑ is $\hat{\mathcal{B}}_t / \mathcal{B}_t(\mathfrak{B})$ -measurable, where $\hat{\mathcal{B}}_t := \overline{\mathcal{B}(\mathcal{H}^1) \times \mathcal{B}_t(\mathfrak{W})}^{\vartheta \otimes P}$,*
- (ii) *$y(\cdot) = F_\vartheta(y(0), \mathcal{W}(\cdot))$, P -a.s.,*

then we call (\mathcal{W}, y) a strong solution.

REMARK 3.6. *The function F_ϑ is a “machine” that turns an initial value $y_0 \in \mathcal{H}^1$ and a Brownian motion $\mathcal{W} \in \mathfrak{W}$ into a solution via $y = F_\vartheta(y_0, \mathcal{W})$. The first property of F_ϑ is a type of adaptedness.*

Our next definition serves the purpose to clarify what we mean by a unique solution.

DEFINITION 3.7. *Equation (2.4) is said to have a unique strong solution associated to $\vartheta \in \mathcal{P}(\mathcal{H}^1)$ if there exists a function $F_\vartheta: \mathcal{H}^1 \times \mathfrak{W} \rightarrow \mathfrak{B}$ as in Definition 3.5 such that also the following two conditions are satisfied:*

- (i) *for any two independent copies of infinite sequence of independent standard Brownian motions $\{\mathcal{W}(t) \mid t \geq 0\}$ on the stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$ and any \mathcal{H}^1 -valued, \mathcal{F}_0 -measurable random variable y_0 with distribution $P \circ y_0^{-1} = \vartheta$,*

$$(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0}; \mathcal{W}; F_\vartheta(y_0, \mathcal{W}(\cdot))) \text{ is a weak solution of Equation (2.4);}$$

- (ii) *for any weak solution $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0}; \mathcal{W}; y)$ of Equation (2.4) with initial law ϑ ,*

$$y(\cdot) = F_\vartheta(y(0), \mathcal{W}(\cdot)), \quad P - \text{a.s.}$$

The following Yamada-Watanabe theorem shows the relationship between the different definitions introduced in this section and gives a way to show existence of a unique strong solution to our equation.

THEOREM 3.8 (Yamada-Watanabe). *If there exists a weak solution to Equation (2.4) and pathwise uniqueness holds, then there exists a unique strong solution to Equation (2.4).*

PROOF. We want to apply the results of M. Röckner, B. Schmulland and X.C. Zhang [193, Theorem 2.1]. In the setting of that paper, we have $V := \mathcal{H}^2$, $H := \mathcal{H}^1$, $E := \mathcal{H}^0$ such that $V \subset H \subset E$. Furthermore, $U := \ell^2 \times \ell^2$. We define the operators

$$\begin{aligned} b(t, Y)(x) &:= \mathcal{A}(Y(t, x)) + f(t, x, Y(t, x)), \quad t \in [0, T], Y \in \mathfrak{B}, \\ \sigma(t, Y)(x) &:= \mathcal{B}(t, x, Y(t, x)), \quad t \in [0, T], Y \in \mathfrak{B}, \end{aligned}$$

so our equation assumes the form

$$dY = b(t, Y)dt + \sigma(t, Y)d\mathcal{W}(t), \quad t \in \mathbb{R}_+.$$

Then one can easily check that our assumptions imply the assumptions on the coefficients b and σ in [193]. In particular, the integrability conditions

$$\int_0^T \|b(s, Y(\omega))\|_{\mathcal{H}^0} ds + \int_0^T \|\sigma(s, Y(\omega))\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^1)}^2 ds < \infty, \quad P - a.e. \omega \in \Omega$$

follow from Equation (2.5) of Lemma 2.3, and Equation (2.13) in Lemma 2.5. Therefore we can apply the Yamada-Watanabe theorem [193, Theorem 2.1], which concludes the proof. \square

3.2. Pathwise Uniqueness. In this section we prove that pathwise uniqueness holds for the tamed MHD equations, following the ideas of [196].

THEOREM 3.9 (pathwise uniqueness). *Let the Assumptions (H1)–(H3) be satisfied. Then pathwise uniqueness holds for (2.4).*

PROOF. Let y_1, y_2 belong to two weak solutions defined on the same stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$ and with the same Brownian motion \mathcal{W} and same initial condition y_0 . Let $T > 0$, $R > 0$ and define the stopping time

$$\tau_R := \inf\{t \in [0, T] \mid \|y_1(t)\|_{\mathcal{H}^1} \vee \|y_2(t)\|_{\mathcal{H}^1} \geq R\} \wedge T.$$

By Assumption (ii) of the definition of a weak solution, we know that for P -a.e. ω $y_i(\cdot, \omega) \in C([0, T]; \mathcal{H}^1)$, $i = 1, 2$, and thus

$$\tau_R \uparrow T, \quad \text{as } R \rightarrow \infty, \quad P - a.s.$$

So τ_R is a localising sequence. Now we set $z(t) := y_1(t) - y_2(t)$. z satisfies the equation

$$\begin{aligned} dz(t) &= [\mathcal{A}(y_1(t)) - \mathcal{A}(y_2(t)) + \mathcal{P}(f(t, y_1(t)) - f(t, y_2(t)))] dt \\ &\quad + \sum_{k=1}^{\infty} (\mathcal{B}_k(t, y_1(t)) - \mathcal{B}_k(t, y_2(t))) d\mathcal{W}_t^k, \\ z(0) &= 0. \end{aligned}$$

Thus by Itô's formula for $\|z\|_{\mathcal{H}^0}^2$, we find (noting the self-adjointness of the projection \mathcal{P} with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}^0} = \langle \cdot, \cdot \rangle_{\mathcal{L}^2}$)

$$\begin{aligned}
(3.3) \quad \|z(t)\|_{\mathcal{H}^0}^2 &= 2 \int_0^t \langle \mathcal{A}(y_1(s)) - \mathcal{A}(y_2(s)), z(s) \rangle_{\mathcal{H}^0} ds \\
&\quad + 2 \int_0^t \langle f(s, y_1(s)) - f(s, y_2(s)), z(s) \rangle_{\mathcal{H}^0} ds \\
&\quad + 2 \sum_{k=1}^{\infty} \int_0^t \langle \mathcal{B}_k(s, y_1(s)) - \mathcal{B}_k(s, y_2(s)), z(s) \rangle_{\mathcal{H}^0} d\mathcal{W}_s^k \\
&\quad + \sum_{k=1}^{\infty} \int_0^t \|\mathcal{B}_k(s, y_1(s)) - \mathcal{B}_k(s, y_2(s))\|_{\mathcal{H}^0}^2 ds \\
&=: I_1(t) + I_2(t) + I_3(t) + I_4(t).
\end{aligned}$$

We stop with τ_R and estimate each term separately.

The first term $I_1(t \wedge \tau_R)$ can be rewritten and estimated using integration by parts and Young's inequality:

$$\begin{aligned}
I_1(t \wedge \tau_R) &= -2 \int_0^{t \wedge \tau_R} \|\nabla z(s)\|_{\mathcal{H}^0}^2 ds \\
&\quad + 2 \int_0^{t \wedge \tau_R} \left\langle \left(\begin{aligned} &(\mathbf{v}_1 \otimes \mathbf{v}_1 - \mathbf{v}_2 \otimes \mathbf{v}_2) - (\mathbf{B}_1 \otimes \mathbf{B}_1 - \mathbf{B}_2 \otimes \mathbf{B}_2) \\ &(\mathbf{v}_1 \otimes \mathbf{B}_1 - \mathbf{v}_2 \otimes \mathbf{B}_2) - (\mathbf{B}_1 \otimes \mathbf{v}_1 - \mathbf{B}_2 \otimes \mathbf{v}_2) \end{aligned} \right), \nabla z \right\rangle_{\mathcal{L}^2} ds \\
&\quad - 2 \int_0^{t \wedge \tau_R} \langle g_N(y_1)y_1 - g_N(y_2)y_2, z \rangle_{\mathcal{L}^2} ds \\
&\leq - \int_0^{t \wedge \tau_R} \|\nabla z(s)\|_{\mathcal{H}^0}^2 ds \\
&\quad + 2 \int_0^{t \wedge \tau_R} \left[\|\mathbf{v}_1 \otimes \mathbf{v}_1 - \mathbf{v}_2 \otimes \mathbf{v}_2\|_{\mathbb{L}^2}^2 + \|\mathbf{B}_1 \otimes \mathbf{B}_1 - \mathbf{B}_2 \otimes \mathbf{B}_2\|_{\mathbb{L}^2}^2 \right. \\
&\quad \left. + \|\mathbf{v}_1 \otimes \mathbf{B}_1 - \mathbf{v}_2 \otimes \mathbf{B}_2\|_{\mathbb{L}^2}^2 + \|\mathbf{B}_1 \otimes \mathbf{v}_1 - \mathbf{B}_2 \otimes \mathbf{v}_2\|_{\mathbb{L}^2}^2 \right] ds \\
&\quad - 2 \int_0^{t \wedge \tau_R} \langle g_N(y_1)y_1 - g_N(y_2)y_2, z \rangle_{\mathcal{L}^2} ds =: J_1(t \wedge \tau_R) + J_2(t \wedge \tau_R) + J_3(t \wedge \tau_R).
\end{aligned}$$

The terms of $J_2(t \wedge \tau_R)$ are of the general form

$$\|\boldsymbol{\theta}_1 \otimes \boldsymbol{\psi}_1 - \boldsymbol{\theta}_2 \otimes \boldsymbol{\psi}_2\|_{\mathbb{L}^2}^2, \quad \boldsymbol{\theta}_i, \boldsymbol{\psi}_i \in \{\mathbf{v}, \mathbf{B}\}, i = 1, 2,$$

and can be estimated as follows:

$$\begin{aligned}
&\|\boldsymbol{\theta}_1 \otimes \boldsymbol{\psi}_1 - \boldsymbol{\theta}_2 \otimes \boldsymbol{\psi}_2\|_{\mathbb{L}^2}^2 \\
&= \|\boldsymbol{\theta}_1 \otimes (\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2) + (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2) \otimes \boldsymbol{\psi}_2\|_{\mathbb{L}^2}^2 \\
&\leq 2 \left(\|\boldsymbol{\theta}_1\|_{\mathbb{L}^4}^2 \|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_{\mathbb{L}^4}^2 + \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_{\mathbb{L}^4}^2 \|\boldsymbol{\psi}_2\|_{\mathbb{L}^4}^2 \right) \\
&\leq 2C_{1,4}^2 \left(\|\boldsymbol{\theta}_1\|_{\mathbb{H}^1}^2 \|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_{\mathbb{H}^1}^{3/2} \|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_{\mathbb{H}^0}^{1/2} + \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_{\mathbb{H}^1}^{3/2} \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_{\mathbb{H}^0}^{1/2} \|\boldsymbol{\psi}_2\|_{\mathbb{H}^1}^2 \right) \\
&\leq 2C_{1,4}^2 R^2 \left(\|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_{\mathbb{H}^1}^{3/2} \|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_{\mathbb{H}^0}^{1/2} + \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_{\mathbb{H}^1}^{3/2} \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_{\mathbb{H}^0}^{1/2} \right) \\
&\leq \frac{1}{16} \|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_{\mathbb{H}^1}^2 + \frac{1}{16} \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_{\mathbb{H}^1}^2 + C_R \left(\|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_{\mathbb{H}^0}^2 + \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_{\mathbb{H}^0}^2 \right).
\end{aligned}$$

Counting all possible combinations of θ and ψ and combining the \mathbf{v} and \mathbf{B} -norms into the corresponding norms for y , we find that

$$J_2(t \wedge \tau_R) \leq \frac{1}{4} \int_0^t \|\nabla z(s)\|_{\mathcal{H}^0}^2 ds + C_R \int_0^t \|z(s)\|_{\mathcal{H}^0}^2 ds.$$

Since $|g_N(r) - g_N(r')| \leq 2|r - r'|$, we see, using a short calculation similar to the one in Lemma 2.4 as well as the Sobolev embedding from above and Young's inequality, that

$$\begin{aligned} J_3(t \wedge \tau_R) &\leq 8 \int_0^{t \wedge \tau_R} \| |z| \cdot (|y_1| + |y_2|) \|_{L^2}^2 ds \leq 16 \int_0^{t \wedge \tau_R} \|z\|_{\mathcal{L}^4}^2 (\|y_1\|_{\mathcal{L}^4}^2 + \|y_2\|_{\mathcal{L}^4}^2) ds \\ &\leq 16C_{1,4}^2 \int_0^{t \wedge \tau_R} \|\nabla z\|_{\mathcal{H}^0}^{3/2} \|z\|_{\mathcal{H}^0}^{1/2} (\|y_1\|_{\mathcal{H}^1}^2 + \|y_2\|_{\mathcal{H}^1}^2) ds \\ &\leq 16C_{1,4}^2 R^2 \int_0^{t \wedge \tau_R} \|\nabla z\|_{\mathcal{H}^0}^{3/2} \|z\|_{\mathcal{H}^0}^{1/2} ds \\ &\leq \frac{1}{4} \int_0^{t \wedge \tau_R} \|\nabla z\|_{\mathcal{H}^0}^2 ds + C_R \int_0^{t \wedge \tau_R} \|z\|_{\mathcal{H}^0}^2 ds. \end{aligned}$$

Thus, altogether we find that

$$I_1(t \wedge \tau_R) \leq -\frac{1}{2} \int_0^{t \wedge \tau_R} \|\nabla z(s)\|_{\mathcal{H}^0}^2 ds + C_R \int_0^{t \wedge \tau_R} \|z(s)\|_{\mathcal{H}^0}^2 ds.$$

By Cauchy-Schwarz-Buniakowski and Assumption (H1), we find for $I_2(t \wedge \tau_R)$

$$2 \int_0^t \langle f(s, y_1(s)) - f(s, y_2(s)), z(s) \rangle_{\mathcal{H}^0} ds \leq C_{T,F} \int_0^{t \wedge \tau_R} \|z(s)\|_{\mathcal{H}^0}^2 ds.$$

The term $I_3(t \wedge \tau_R)$ is a martingale and thus killed upon taking expectations.

For $I_4(t \wedge \tau_R)$ we have by Assumptions (H2) and (H3)

$$\begin{aligned} I_4(t \wedge \tau_R) &= \sum_{k=1}^{\infty} \int_0^{t \wedge \tau_R} \|\mathcal{B}_k(s, y_1(s)) - \mathcal{B}_k(s, y_2(s))\|_{\mathcal{H}^0}^2 ds \\ &= \sum_{k=1}^{\infty} \int_0^{t \wedge \tau_R} \|\mathcal{P}((\Sigma^k \cdot \nabla)z(s)) + \mathcal{P}(H^k(s, y_1(s)) - H^k(s, y_2(s)))\|_{\mathcal{H}^0}^2 ds \\ &\leq \sup_{t \in [0, T], x \in \mathbb{D}} \|\Sigma(t, x)\|_{\ell^2}^2 \int_0^{t \wedge \tau_R} \|\nabla z(s)\|_{\mathcal{H}^0}^2 ds + C_{T,H} \int_0^{t \wedge \tau_R} \|z(s)\|_{\mathcal{H}^0}^2 ds \\ &\leq \frac{1}{4} \int_0^{t \wedge \tau_R} \|\nabla z(s)\|_{\mathcal{H}^0}^2 ds + C_{T,H} \int_0^{t \wedge \tau_R} \|z(s)\|_{\mathcal{H}^0}^2 ds. \end{aligned}$$

Hence, if we stop (3.3) with τ_R and take expectations, we find, using the previous estimates, that

$$\begin{aligned} \mathbb{E} [\|z(t \wedge \tau_R)\|_{\mathcal{H}^0}^2] &\leq C_{R,T} \mathbb{E} \left[\int_0^{t \wedge \tau_R} \|z(s)\|_{\mathcal{H}^0}^2 ds \right] \\ &\leq C_{R,T,f,H} \int_0^t \mathbb{E} [\|z(s \wedge \tau_R)\|_{\mathcal{H}^0}^2] ds. \end{aligned}$$

Applying Gronwall's lemma yields that for any $t \in [0, T]$

$$\mathbb{E} [\|z(t \wedge \tau_R)\|_{\mathcal{H}^0}^2] = 0.$$

Finally, we employ Fatou's lemma to find

$$\mathbb{E} [\|z(t)\|_{\mathcal{H}^0}^2] \leq \liminf_{R \rightarrow \infty} \mathbb{E} [\|z(t \wedge \tau_R)\|_{\mathcal{H}^0}^2] = 0.$$

Thus, $z(t) = 0$ for all $t \geq 0$, P -a.s., i.e. pathwise uniqueness holds. \square

3.3. Existence of Martingale Solutions. The main result of this section is the following

THEOREM 3.10. *Under the Assumptions (H1)–(H3), for any initial law $\vartheta \in \mathcal{P}(\mathcal{H}^1)$, there exists a weak solution for Equation (2.4) in the sense of Definition 3.1.*

The proof proceeds in the usual fashion, analogously to the proof of [196, Theorem 3.8] by considering Faedo-Galerkin approximations of our equation. To be precise, fix a stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$ and two independent infinite sequences of independent standard $(\mathcal{F}_t)_t$ -Brownian motions $\{\mathcal{W}^k(t) \mid t \geq 0, k \in \mathbb{N}\}$ and an \mathcal{F}_0 -measurable random variable y_0 with initial law $\vartheta \in \mathcal{P}(\mathcal{H}^1)$.

The set $\mathcal{E} = \{e_i \mid i \in \mathbb{N}\} \subset \mathcal{V}$ was chosen as a complete orthonormal basis of \mathcal{H}^1 . We consider the finite-dimensional subspaces

$$\mathcal{H}_n^1 := \text{span}\{e_i \mid i = 1, \dots, n\}$$

and consider the projections onto \mathcal{H}_n^1 , i.e. for $y \in \mathcal{H}^0$, we define (recalling our convention (2.14))

$$\Pi_n y := \sum_{i=1}^n \langle y, e_i \rangle_{\mathcal{H}^1} e_i = \sum_{i=1}^n \langle y, (I - \Delta)e_i \rangle_{\mathcal{H}^0} e_i.$$

We want to study the following finite-dimensional stochastic ordinary differential equations in \mathcal{H}_n^1 as approximations for our infinite-dimensional equation:

$$\begin{cases} dy_n(t) &= [\Pi_n \mathcal{A}(y_n(t)) + \Pi_n f(t, y_n(t))]dt + \sum_k \Pi_n \mathcal{B}_k(t, y_n(t))d\mathcal{W}_t^k, \\ y_n(0) &= \Pi_n y_0. \end{cases}$$

Using the Lemmas 2.4 and 2.5, there exists a constant $C_{n,N}$ such that for any $y \in \mathcal{H}_n^1$ the following growth conditions holds:

$$\begin{aligned} \langle y, \Pi_n \mathcal{A}(y) + \Pi_n f(t, y) \rangle_{\mathcal{H}_n^1} &\leq C_{n,N} \left(\|y\|_{\mathcal{H}_n^1}^2 + 1 \right) \\ \|\Pi_n \mathcal{B}(t, y)\|_{\ell^2 \times \mathcal{H}_n^1} &\leq C_{n,N} \left(\|y\|_{\mathcal{H}_n^1}^2 + 1 \right). \end{aligned}$$

This can be seen as follows: by using linearity, the definition (2.2) of $\langle \mathcal{A}(y), \cdot \rangle_{\mathcal{H}^1}$ and inequality (2.8) of Lemma 2.3 (where we drop all terms with negative sign)

$$\begin{aligned} \langle y, \Pi_n \mathcal{A}(y) \rangle_{\mathcal{H}_n^1} &= \left\langle y, \sum_{i=1}^n \langle \mathcal{A}(y), e_i \rangle_{\mathcal{H}^1} e_i \right\rangle_{\mathcal{H}_n^1} = \langle \mathcal{A}(y), \sum_{i=1}^n \langle y, e_i \rangle_{\mathcal{H}^1} e_i \rangle_{\mathcal{H}^1} \\ &= \langle \mathcal{A}(y), y \rangle_{\mathcal{H}^1} = \langle \mathcal{A}(y), (I - \Delta)y \rangle_{\mathcal{H}^0} \\ &= \langle \mathcal{A}(y), y \rangle_{\mathcal{H}^1} \leq (2N + 1) \|\nabla y\|_{\mathcal{H}^0}^2 + \|y\|_{\mathcal{H}^0}^2 \leq 2N \|y\|_{\mathcal{H}_n^1}^2, \end{aligned}$$

and similarly for the other terms.

Furthermore, our Assumptions (H1)–(H3) ensure that the maps

$$\begin{aligned} \mathcal{H}_n^1 \ni y &\mapsto \Pi_n \mathcal{A}(y) + \Pi_n f(t, y) \in \mathcal{H}_n^1, \\ \mathcal{H}_n^1 \ni y &\mapsto \Pi_n \mathcal{B}(y) \in L_2(\ell^2 \times \ell^2; \mathcal{H}_n^1), \end{aligned}$$

are locally Lipschitz continuous: by Equation (2.11)

$$\begin{aligned} \|\Pi_n (\mathcal{A}(y_1) - \mathcal{A}(y_2))\|_{\mathcal{H}_n^1} &= \left\| \sum_{i=1}^n \langle \mathcal{A}(y_1) - \mathcal{A}(y_2), (I - \Delta)e_i \rangle_{\mathcal{H}^0} e_i \right\|_{\mathcal{H}_n^1} \\ &\leq \sum_{i=1}^n |\langle \mathcal{A}(y_1) - \mathcal{A}(y_2), e_i \rangle_{\mathcal{H}^1}| \|e_i\|_{\mathcal{H}_n^1} \leq \sum_{i=1}^n C_i \|y_1 - y_2\|_{\mathcal{L}^2(\mathcal{O}_i)} \left(1 + \|y_1\|_{\mathcal{H}_n^1}^2 + \|y_2\|_{\mathcal{H}_n^1}^2\right), \end{aligned}$$

where $\mathcal{O}_i := \text{supp}(e_i)$, and similarly by Assumption (H1)

$$\begin{aligned} \|\Pi_n (f(t, y_1) - f(t, y_2))\|_{\mathcal{H}_n^1} &\leq \sum_{i=1}^n |\langle f(t, y_1) - f(t, y_2), (I - \Delta)e_i \rangle_{\mathcal{H}^0}| \|e_i\|_{\mathcal{H}_n^1} \\ &\leq \sum_{i=1}^n C_{T,f} \|(I - \Delta)e_i\|_{\mathcal{H}_n^0} \|y_1 - y_2\|_{\mathcal{L}^2(\mathcal{O}_i)} \|e_i\|_{\mathcal{H}_n^1}, \end{aligned}$$

and by Assumption (H2) and (H3)

$$\begin{aligned} &\|\Pi_n (\mathcal{B}(t, y_1) - \mathcal{B}(t, y_2))\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}_n^1)} \\ &\leq \sum_{i=1}^n \|\langle (\Sigma \cdot \nabla)(y_1 - y_2) + (H(t, y_1) - H(t, y_2)), (I - \Delta)e_i \rangle_{\mathcal{H}^0} e_i\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}_n^1)} \\ &\leq \sum_{i=1}^n \|\langle (\Sigma \cdot \nabla)(y_1 - y_2) + (H(t, y_1) - H(t, y_2)), (I - \Delta)e_i \rangle_{\mathcal{H}^0}\|_{\ell^2} \|e_i\|_{\mathcal{H}_n^1} \\ &\leq \sum_{i=1}^n \left(\sup_{x \in \mathbb{D}, t \in [0, T]} \|\Sigma(t, x)\|_{\ell^2} + C_{T,H} \right) \|e_i\|_{\mathcal{H}_n^2} \|e_i\|_{\mathcal{H}_n^1} \|y_1 - y_2\|_{\mathcal{L}^2(\mathcal{O}_i)}. \end{aligned}$$

Therefore, we can employ the theory of stochastic (ordinary) differential equations (cf. [167, Theorem 3.1.1, p. 56]) to find a unique $(\mathcal{F}_t)_t$ -adapted process $y_n(t)$ such that P -a.s. for all $t \in [0, T]$

$$\begin{aligned} (3.4) \quad y_n(t) &= y_n(0) + \int_0^t \Pi_n \mathcal{A}(y_n(s)) ds + \int_0^t \Pi_n f(s, y_n(s)) ds \\ &\quad + \sum_{k=1}^{\infty} \int_0^t \Pi_n \mathcal{B}_k(s, y_n(s)) dW_s^k, \end{aligned}$$

and for any $i \leq n$

$$(3.5) \quad \begin{aligned} \langle y_n(t), e_i \rangle_{\mathcal{H}^1} &= \langle y_n(0), e_i \rangle_{\mathcal{H}^1} + \int_0^t \langle \mathcal{A}(y_n(s)), e_i \rangle_{\mathcal{H}^1} ds + \int_0^t \langle f(s, y_n(s)), e_i \rangle_{\mathcal{H}^1} ds \\ &+ \sum_{k=1}^{\infty} \int_0^t \langle \mathcal{B}_k(s, y_n(s)), e_i \rangle_{\mathcal{H}^1} d\mathcal{W}_s^k. \end{aligned}$$

Our strategy now is as follows:

1. Prove uniform *a priori* estimates for y_n .
2. Use these to prove tightness of the associated laws.
3. Use Skorokhod's embedding theorem to translate the weak convergence from the previous step to P -a.s. convergence of the random variables.
4. Prove uniform moment estimates for the terms of the associated martingale problem.
5. Show convergence in probability of the martingale problems.

We start with the *a priori* estimates.

LEMMA 3.11 (*a priori* estimates). *For any $T > 0$, there exists a constant $C_{T,N,f,H,y_0} > 0$ such that for any $n \in \mathbb{N}$*

$$(3.6) \quad \begin{aligned} \mathbb{E} \left[\sup_{t \in [0,T]} \|y_n(t)\|_{\mathcal{H}^1}^2 \right] + \int_0^T \mathbb{E} [\|y_n(s)\|_{\mathcal{H}^2}^2] ds + \int_0^T \mathbb{E} [\|\nabla |y_n(s)|^2\|_{L^2}^2] ds \\ \leq C_{T,N,f,H,y_0}. \end{aligned}$$

Furthermore, in the periodic case it holds that

$$(3.7) \quad \int_0^t \mathbb{E} [\|y_n(s)\|_{\mathcal{L}^4}^4] ds \leq C_{T,N,f,H,y_0}.$$

PROOF. We use Itô's formula as well as Lemmas 2.3 and 2.5 to find

$$(3.8) \quad \begin{aligned} \|y_n(t)\|_{\mathcal{H}^1}^2 &= \|y_0\|_{\mathcal{H}^1}^2 + 2 \int_0^t \langle \mathcal{A}(y_n(s)), y_n(s) \rangle_{\mathcal{H}^1} ds + 2 \int_0^t \langle f(s, y_n(s)), y_n(s) \rangle_{\mathcal{H}^1} ds \\ &+ M(t) + \int_0^t \|\mathcal{B}(s, y_n(s))\|_{L_2(\ell^2; \mathcal{H}^1)}^2 ds \\ &\leq \|y_0\|_{\mathcal{H}^1}^2 - \int_0^t \|y_n(s)\|_{\mathcal{H}^2}^2 - 2 \int_0^t \left(\|\mathbf{v}_n\| \cdot \|\nabla \mathbf{v}_n\|_{L^2}^2 + \|\mathbf{B}_n\| \cdot \|\nabla \mathbf{B}_n\|_{L^2}^2 \right. \\ &\quad \left. + \|\mathbf{v}_n\| \cdot \|\nabla \mathbf{B}_n\|_{L^2}^2 + \|\mathbf{B}_n\| \cdot \|\nabla \mathbf{v}_n\|_{L^2}^2 \right) ds \\ &+ 2 \int_0^t (2N+1) \|\nabla y_n(s)\|_{\mathcal{H}^0}^2 + \|y_n(s)\|_{\mathcal{H}^0}^2 ds \\ &+ 2 \int_0^t \|f(s, y_n(s))\|_{\mathcal{H}^0} \|y_n(s)\|_{\mathcal{H}^2} ds + M(t) \\ &+ \int_0^t \left(\frac{1}{2} \|y_n(s)\|_{\mathcal{H}^2}^2 + C_{T,H} \|y_n(s)\|_{\mathcal{H}^1}^2 + C \|F_H(s)\|_{L^1(\mathbb{D})} \right) ds, \end{aligned}$$

where the term $M(t)$ is a continuous martingale and has the representation

$$M(t) := 2 \sum_{k=1}^{\infty} \langle \mathcal{B}_k(s, y_n(s)), y_n(s) \rangle_{\mathcal{H}^1} d\mathcal{W}_s^k.$$

Taking expectations in Equation (3.8) and applying Young's inequality in the force term thus gives

$$\begin{aligned}
& \mathbb{E} \|y_n(t)\|_{\mathcal{H}^1}^2 \\
& + 2 \int_0^t \mathbb{E} \left[\|\mathbf{v}_n\| \cdot \|\nabla \mathbf{v}_n\|_{L^2}^2 + \|\mathbf{B}_n\| \cdot \|\nabla \mathbf{B}_n\|_{L^2}^2 + \|\mathbf{v}_n\| \cdot \|\nabla \mathbf{B}_n\|_{L^2}^2 + \|\mathbf{B}_n\| \cdot \|\nabla \mathbf{v}_n\|_{L^2}^2 \right] ds \\
& \leq \mathbb{E} \|y_0\|_{\mathcal{H}^1}^2 - \frac{1}{4} \int_0^t \|y_n(s)\|_{\mathcal{H}^2}^2 ds + 2 \int_0^t ((2N+1) \|\nabla y_n(s)\|_{\mathcal{H}^0}^2 + \|y_n(s)\|_{\mathcal{H}^0}^2) ds \\
& \quad + 4 \int_0^t \|f(s, y_n(s))\|_{\mathcal{H}^0}^2 ds + C_{T,H} \int_0^t (\|y_n(s)\|_{\mathcal{H}^1}^2 + C \|F_H(s)\|_{L^1(\mathbb{D})}) ds
\end{aligned}$$

and therefore we find,

$$\begin{aligned}
& \sup_{t \in [0, T]} \mathbb{E} \|y_n(t)\|_{\mathcal{H}^1}^2 + \frac{1}{4} \int_0^T \|y_n(s)\|_{\mathcal{H}^2}^2 ds \\
& + 2 \int_0^T \mathbb{E} \left[\|\mathbf{v}_n\| \cdot \|\nabla \mathbf{v}_n\|_{L^2}^2 + \|\mathbf{B}_n\| \cdot \|\nabla \mathbf{B}_n\|_{L^2}^2 + \|\mathbf{v}_n\| \cdot \|\nabla \mathbf{B}_n\|_{L^2}^2 + \|\mathbf{B}_n\| \cdot \|\nabla \mathbf{v}_n\|_{L^2}^2 \right] ds \\
& \leq \mathbb{E} \|y_0\|_{\mathcal{H}^1}^2 + C_{T,H,N} \int_0^T \sup_{r \in [0, s]} \mathbb{E} \|y_n(r)\|_{\mathcal{H}^1}^2 ds + C_{T,H,f}.
\end{aligned}$$

Gronwall's lemma then implies

$$\begin{aligned}
(3.9) \quad & \sup_{t \in [0, T]} \mathbb{E} \|y_n(t)\|_{\mathcal{H}^1}^2 + \int_0^T \|y_n(s)\|_{\mathcal{H}^2}^2 ds + \int_0^T \mathbb{E} \left[\|\mathbf{v}_n\| \cdot \|\nabla \mathbf{v}_n\|_{L^2}^2 \right. \\
& \quad \left. + \|\mathbf{B}_n\| \cdot \|\nabla \mathbf{B}_n\|_{L^2}^2 + \|\mathbf{v}_n\| \cdot \|\nabla \mathbf{B}_n\|_{L^2}^2 + \|\mathbf{B}_n\| \cdot \|\nabla \mathbf{v}_n\|_{L^2}^2 \right] ds \\
& \leq C_{T,N,H,f,y_0}.
\end{aligned}$$

To “exchange” expectation and supremum, we use the Burkholder-Davis-Gundy (BDG) inequality (cf. [172], Theorem 1.1). More precisely, dropping the negative terms in (3.8), taking suprema and expectations, applying (3.9) and then using the BDG inequality, we find

$$\begin{aligned}
& \mathbb{E} \left[\sup_{t \in [0, T]} \|y_n(t)\|_{\mathcal{H}^1}^2 \right] \leq \mathbb{E} \left[\sup_{t \in [0, T]} |M(t)| \right] + C_{T,N,f,H,y_0} \leq C_{T,N,f,H,y_0} + 3 \mathbb{E} \left[\langle M \rangle_T^{1/2} \right] \\
& \leq C_{T,N,f,H,y_0} + 6 \mathbb{E} \left[\left(\int_0^T \sum_{k=1}^{\infty} \|\mathcal{B}_k(s, y_n(s))\|_{\mathcal{H}^0}^2 \|y_n(s)\|_{\mathcal{H}^2}^2 ds \right)^{1/2} \right] \\
& \leq C_{T,N,f,H,y_0} + 6 \mathbb{E} \left[\left(\sup_{t \in [0, T]} \|\mathcal{B}(t, y_n(t))\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)}^2 \int_0^T \|y_n(s)\|_{\mathcal{H}^2}^2 ds \right)^{1/2} \right] \\
& \leq C_{T,N,f,H,y_0} + \varepsilon \mathbb{E} \left[\sup_{t \in [0, T]} \|\mathcal{B}(t, y_n(t))\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)}^2 \right] + C_\varepsilon \int_0^T \|y_n(s)\|_{\mathcal{H}^2}^2 ds,
\end{aligned}$$

where we applied Young's inequality in the last step.

Now, an application of Lemma 2.5 as well as (3.9) yields

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|y_n(t)\|_{\mathcal{H}^1}^2 \right] \leq C_{T, N, f, H, y_0, \varepsilon} + \varepsilon C_T \mathbb{E} \left[\sup_{t \in [0, T]} \|y_n(t)\|_{\mathcal{H}^1}^2 \right],$$

which for small enough $\varepsilon > 0$ yields

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|y_n(t)\|_{\mathcal{H}^1}^2 \right] \leq C_{T, N, f, H, y_0}.$$

Combining this with (3.9) yields the desired estimate.

In the periodic case, as \mathcal{E} is also orthogonal in \mathcal{H}^0 , we get an Itô formula for the \mathcal{H}^0 -norms as well and find upon taking expectations and using (2.7) of Lemma 2.3 as well as Equation (3.6),

$$\begin{aligned} \mathbb{E} \|y_n(t)\|_{\mathcal{H}^0}^2 &= \mathbb{E} \|y_0\|_{\mathcal{H}^0}^2 + 2 \int_0^t \mathbb{E} \langle \mathcal{A}(y_n(s)), y_n(s) \rangle_{\mathcal{H}^0} ds \\ &\quad + 2 \int_0^t \mathbb{E} \langle f(s, y_n(s)), y_n(s) \rangle_{\mathcal{H}^0} ds + \int_0^t \mathbb{E} \|\mathcal{B}(s, y_n(s))\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)}^2 ds \\ &\leq \mathbb{E} \|y_0\|_{\mathcal{H}^0}^2 + 2 \int_0^t \mathbb{E} [-\|\nabla y_n(s)\|_{\mathcal{H}^0}^2 - \|y_n(s)\|_{\mathcal{L}^4}^4 + CN \|y_n(s)\|_{\mathcal{H}^0}^2] ds \\ &\quad + 2 \int_0^t \mathbb{E} [C_{T, F} \|y_n(s)\|_{\mathcal{H}^0}^2 + \|F_f(t)\|_{L^1(\mathbb{D})}] ds \\ &\quad + \int_0^t \mathbb{E} \left[\frac{1}{2} \|y_n(s)\|_{\mathcal{H}^1}^2 + C_T \|y_n(s)\|_{\mathcal{H}^0}^2 + \|F_H(t)\|_{L^1(\mathbb{D})} \right] ds \\ &\leq -2 \int_0^t \mathbb{E} [\|y_n(s)\|_{\mathcal{L}^4}^4] ds + C_{T, N, F} \int_0^t \mathbb{E} [\|y_n(s)\|_{\mathcal{H}^0}^2] ds \\ &\quad + C_{T, N, f, H, y_0}, \end{aligned}$$

and an application of Gronwall's lemma implies (3.7). \square

Next we want to prove tightness of the laws of the solutions to the approximate equations (3.4). We recall the notation $\mathcal{X} := C(\mathbb{R}_+; \mathcal{H}_{\text{loc}}^0)$ from Section 2.3.

LEMMA 3.12 (tightness). *Let $\mu_n := P \circ (y_n)^{-1}$ be the law of y_n in $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Then the family $(\mu_n)_{n \in \mathbb{N}}$ is tight on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$.*

PROOF. Let $R > 0$. We set

$$\tau_R^n := \inf\{t \geq 0 \mid \|y_n(t)\|_{\mathcal{H}_n^1} \geq R\}.$$

Then, using the Chebychev inequality as well as the *a priori* estimate (3.6), we find

$$\begin{aligned} (3.10) \quad \sup_{n \in \mathbb{N}} P(\tau_R^n < T) &= \sup_{n \in \mathbb{N}} P \left(\sup_{t \in [0, T]} \|y_n(t)\|_{\mathcal{H}_n^1} \geq R \right) \\ &\leq \sup_{n \in \mathbb{N}} \frac{1}{R^2} \mathbb{E} \left[\sup_{t \in [0, T]} \|y_n(t)\|_{\mathcal{H}_n^1}^2 \right] \leq \frac{C_{T, N, f, H, y_0}}{R^2}. \end{aligned}$$

For any $q \geq 2$ and $s, t \in [0, T]$ and any $e \in \mathcal{E}$ (whose support we denote by \mathcal{O}), we find by using Equations (3.5), (2.9) as well as the BDG inequality (cf. [172], Theorem 1.1)

$$\begin{aligned}
& \mathbb{E} [|\langle y_n(t \wedge \tau_R^n) - y_n(s \wedge \tau_R^n), e \rangle_{\mathcal{H}^1}|^q] \\
& \leq 3^{q-1} \left(\mathbb{E} \left[\left| \int_{s \wedge \tau_R^n}^{t \wedge \tau_R^n} \langle \mathcal{A}(y_n(r)), e \rangle_{\mathcal{H}^1} dr \right|^q \right] + \mathbb{E} \left[\left| \int_{s \wedge \tau_R^n}^{t \wedge \tau_R^n} \langle f(r, y_n(r)), e \rangle_{\mathcal{H}^1} dr \right|^q \right] \right. \\
& \quad \left. + \mathbb{E} \left[\left| \sum_{k=1}^{\infty} \int_{s \wedge \tau_R^n}^{t \wedge \tau_R^n} \langle \mathcal{B}_k(r, y_n(r)), e \rangle_{\mathcal{H}^1} d\mathcal{W}_r^k \right|^q \right] \right) \\
& \leq C_{e,q} \left(\mathbb{E} \left[\left| \int_{s \wedge \tau_R^n}^{t \wedge \tau_R^n} \left(1 + \|y_n\|_{\mathcal{L}^3(\mathcal{O})}^3 \right) dr \right|^q \right] + \mathbb{E} \left[\left| \int_{s \wedge \tau_R^n}^{t \wedge \tau_R^n} \|f(r, y_n(r))\|_{\mathcal{H}^0} dr \right|^q \right] \right. \\
& \quad \left. + \mathbb{E} \left[\left| \int_{s \wedge \tau_R^n}^{t \wedge \tau_R^n} \|\mathcal{B}(r, y_n(r))\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)}^2 dr \right|^{q/2} \right] \right),
\end{aligned}$$

where we transferred all the spatial derivatives onto e via (2.14). Applying the Hölder embedding $\mathcal{L}^6(\mathcal{O}) \subset \mathcal{L}^3(\mathcal{O})$ as well as the Sobolev embedding $\mathcal{H}^1 \subset \mathcal{L}^6$, Assumptions (H1), (H3) and Equation (2.12), we can see that

$$\begin{aligned}
& \mathbb{E} [|\langle y_n(t \wedge \tau_R^n) - y_n(s \wedge \tau_R^n), e \rangle_{\mathcal{H}^1}|^q] \\
& \leq C_{e,q,T} \left(\mathbb{E} \left[\left| \int_{s \wedge \tau_R^n}^{t \wedge \tau_R^n} \left(1 + \sup_{r \in [0, T]} \|y_n(r)\|_{\mathcal{H}^1}^3 \right) dr \right|^q \right] \right. \\
& \quad + \mathbb{E} \left[\left| \int_{s \wedge \tau_R^n}^{t \wedge \tau_R^n} \|y_n(r)\|_{\mathcal{H}^1}^2 + \|F_f(r)\|_{L^1(\mathbb{D})} dr \right|^q \right] \\
& \quad \left. + \mathbb{E} \left[\left| \int_{s \wedge \tau_R^n}^{t \wedge \tau_R^n} \|y_n(r)\|_{\mathcal{H}^1}^2 + \|F_H(r)\|_{L^1(\mathbb{D})} dr \right|^{q/2} \right] \right) \\
& \leq C_{e,q,T} \left\{ \mathbb{E} \left[\left(1 + \sup_{r \in [0, T \wedge \tau_R^n]} \|y_n(r)\|_{\mathcal{H}^1}^{3q} \right) |t - s|^q \right] \right. \\
& \quad + \mathbb{E} \left[\left(\sup_{r \in [0, T \wedge \tau_R^n]} \|y_n(r)\|_{\mathcal{H}^1}^{2q} + \|F_f\|_{L^\infty([0, T]; L^1(\mathbb{D}))}^q \right) |t - s|^q \right] \\
& \quad \left. + \mathbb{E} \left[\left(\sup_{r \in [0, T \wedge \tau_R^n]} \|y_n(r)\|_{\mathcal{H}^1}^q + \|F_H\|_{L^\infty([0, T]; L^1(\mathbb{D}))}^{q/2} \right) |t - s|^{q/2} \right] \right\} \\
& \leq C_{e,q,T,R} \mathbb{E} \left[\left(1 + \sup_{r \in [0, T]} \|y_n(r)\|_{\mathcal{H}^1}^2 + \|F_f\|_{L^\infty([0, T]; L^1(\mathbb{D}))}^q + \|F_H\|_{L^\infty([0, T]; L^1(\mathbb{D}))}^{q/2} \right) |t - s|^{q/2} \right] \\
& \leq C_{e,q,T,R,f,H,N,y_0} |t - s|^{q/2},
\end{aligned}$$

where we have used in the penultimate step that $q \geq 2$ as well as the definition of τ_R^n and hence that $\|y_n(r)\|_{\mathcal{H}^1}^q = \|y_n(r)\|_{\mathcal{H}^1}^{q-2} \cdot \|y_n(r)\|_{\mathcal{H}^1}^2 \leq R^{q-2} \|y_n(r)\|_{\mathcal{H}^1}^2$ and similarly for the other terms. Finally, we have used the *a priori* estimate (3.6) in the last step.

Thus by the Kolmogorov-Čentsov continuity criterion (e.g. in the version of [85, Theorem 3.1, p. 28 f.]) with parameters $\beta = \frac{q/2}{q} = 1/2$, we find that for every $T > 0$ and $\alpha \in (0, \frac{1}{2} - \frac{1}{q})$

$$\mathbb{E} \left[\sup_{s,t \in [0,T], |t-s| \leq \delta} |\langle y_n(t \wedge \tau_R^n) - y_n(s \wedge \tau_R^n), e \rangle_{\mathcal{H}^1}|^q \right] \leq C_{e,q,T,R,f,H,N,y_0} \cdot \delta^\alpha.$$

Therefore, for arbitrary $\varepsilon > 0$ and $R > 0$, we find using (3.10)

$$\begin{aligned} & \sup_{n \in \mathbb{N}} P \left\{ \sup_{s,t \in [0,T], |t-s| \leq \delta} |\langle y_n(t) - y_n(s), e \rangle_{\mathcal{H}^1}| > \varepsilon \right\} \\ & \leq \sup_{n \in \mathbb{N}} P \left\{ \sup_{s,t \in [0,T], |t-s| \leq \delta} |\langle y_n(t) - y_n(s), e \rangle_{\mathcal{H}^1}| > \varepsilon; \tau_R^n \geq T \right\} + \sup_{n \in \mathbb{N}} P \{ \tau_R^n < T \} \\ & \leq \frac{C_{e,q,T,R,f,H,N,y_0} \cdot \delta^\alpha}{\varepsilon^q} + \frac{C_{T,N}}{R^2}. \end{aligned}$$

Letting first $\delta \downarrow 0$ and then $R \rightarrow \infty$, we find

$$(3.11) \quad \limsup_{\delta \downarrow 0} \sup_{n \in \mathbb{N}} P \left\{ \sup_{s,t \in [0,T], |t-s| \leq \delta} |\langle y_n(t) - y_n(s), e \rangle_{\mathcal{H}^1}| > \varepsilon \right\} = 0.$$

Thanks to (3.10) and (3.11), we can now invoke Lemma 2.6 to conclude that $(\mu_n)_{n \in \mathbb{N}}$ is a tight family of probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. \square

The tightness implies the existence of a subsequence (which we again denote by $(\mu_n)_{n \in \mathbb{N}}$) that converges weakly to a measure $\mu \in \mathcal{P}(\mathcal{X})$.

Next we apply Skorokhod's coupling theorem (cf. [125, Theorem 4.30, p. 79]) to infer the existence of a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and \mathcal{X} -valued random variables \tilde{y}^n and \tilde{y} such that

- (I) the law of \tilde{y}^n is the same as that of y_n for all $n \in \mathbb{N}$, i.e. $\tilde{P} \circ \tilde{y}^n{}^{-1} = \mu_n$;
- (II) the convergence $\tilde{y}^n \rightarrow \tilde{y}$ holds in \mathcal{X} , \tilde{P} -a.s., and \tilde{y} has law μ .

By Fatou's lemma and the uniform (in n) *a priori* estimates (3.6), the same estimates also hold for the limiting process: for every $T > 0$

$$(3.12) \quad \begin{aligned} \mathbb{E}^{\tilde{P}} \left[\sup_{t \in [0,T]} \|\tilde{y}(t)\|_{\mathcal{H}^1}^2 \right] & \leq \liminf_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} \left[\sup_{t \in [0,T]} \|\tilde{y}^n(t)\|_{\mathcal{H}^1}^2 \right] \\ & = \liminf_{n \rightarrow \infty} \mathbb{E} \left[\sup_{t \in [0,T]} \|y_n(t)\|_{\mathcal{H}^1}^2 \right] < \infty, \end{aligned}$$

as well as

$$(3.13) \quad \begin{aligned} \int_0^T \mathbb{E}^{\tilde{P}} [\|\tilde{y}(s)\|_{\mathcal{H}^2}^2] ds & \leq \liminf_{n \rightarrow \infty} \int_0^T \mathbb{E}^{\tilde{P}} [\|\tilde{y}^n(s)\|_{\mathcal{H}^2}^2] ds \\ & \leq \liminf_{n \rightarrow \infty} \int_0^T \mathbb{E} [\|y_n(s)\|_{\mathcal{H}^2}^2] ds < \infty, \end{aligned}$$

and

$$\begin{aligned}
& \int_0^T \mathbb{E}^{\tilde{P}} \left[\|\tilde{\mathbf{v}}\| \cdot \|\nabla \tilde{\mathbf{v}}\|_{L^2}^2 + \|\tilde{\mathbf{B}}\| \cdot \|\nabla \tilde{\mathbf{B}}\|_{L^2}^2 + \|\tilde{\mathbf{v}}\| \cdot \|\nabla \tilde{\mathbf{B}}\|_{L^2}^2 + \|\tilde{\mathbf{B}}\| \cdot \|\nabla \tilde{\mathbf{v}}\|_{L^2}^2 \right] ds \\
(3.14) \quad & \leq \liminf_{n \rightarrow \infty} \int_0^T \mathbb{E} \left[\|\mathbf{v}_n\| \cdot \|\nabla \mathbf{v}_n\|_{L^2}^2 + \|\mathbf{B}_n\| \cdot \|\nabla \mathbf{B}_n\|_{L^2}^2 + \|\mathbf{v}_n\| \cdot \|\nabla \mathbf{B}_n\|_{L^2}^2 \right. \\
& \quad \left. + \|\mathbf{B}_n\| \cdot \|\nabla \mathbf{v}_n\|_{L^2}^2 \right] ds \\
& < \infty.
\end{aligned}$$

In the periodic case $\mathbb{D} = \mathbb{T}^3$, we additionally have

$$(3.15) \quad \int_0^t \mathbb{E}^{\tilde{P}} [\|\tilde{y}(s)\|_{\mathcal{L}^4}^4] ds \leq \liminf_{n \rightarrow \infty} \int_0^t \mathbb{E} [\|y_n(s)\|_{\mathcal{L}^4}^4] ds < \infty.$$

Next we want to study the martingale problem associated to our solutions \tilde{y}_n and prove that their limit solves a martingale problem as well, giving existence of a weak solution to the TMHD equations. To this end, take any $\varphi \in C_c^\infty(\mathbb{R})$, $e \in \mathcal{E}$, $t \geq 0$, $y \in \mathcal{X}$. We define the following process:

$$M_e^\varphi(t, y) := I_1^\varphi(t, y) - I_2^\varphi(t, y) - I_3^\varphi(t, y) - I_4^\varphi(t, y) - I_5^\varphi(t, y),$$

where

$$\begin{aligned}
I_1^\varphi(t, y) &:= \varphi(\langle y(t), e \rangle_{\mathcal{H}^1}), \\
I_2^\varphi(t, y) &:= \varphi(\langle y(0), e \rangle_{\mathcal{H}^1}), \\
I_3^\varphi(t, y) &:= \int_0^t \varphi'(\langle y(0), e \rangle_{\mathcal{H}^1}) \cdot \langle \mathcal{A}(y(s)), e \rangle_{\mathcal{H}^1} ds, \\
I_4^\varphi(t, y) &:= \int_0^t \varphi'(\langle y(0), e \rangle_{\mathcal{H}^1}) \cdot \langle f(s, y(s)), e \rangle_{\mathcal{H}^1} ds, \\
I_5^\varphi(t, y) &:= \frac{1}{2} \int_0^t \varphi''(\langle y(0), e \rangle_{\mathcal{H}^1}) \cdot \|\langle \mathcal{B}(s, y(s)), e \rangle_{\mathcal{H}^1}\|_{\ell^2}^2 ds.
\end{aligned}$$

Our aim now is to show that $M_e^\varphi(\cdot, y)$ is a martingale with respect to the stochastic basis $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mu, (\mathcal{B}_t(\mathcal{X}))_{t \geq 0})$, i.e. it solves the martingale problem. This implies the existence of a weak solution.

Since $e \in \mathcal{E} \subset \mathcal{V}$, as noted before, it has compact support, i.e. there exists an $m \in \mathbb{N}$ such that $\text{supp}(e) \subset \mathcal{O} := \overline{B_m(0)} \subset \mathbb{R}^3$.

We will use the following estimate below.

LEMMA 3.13. *For a function $y = (\mathbf{v}, \mathbf{B})$ such that $\|\mathbf{v}\| \|\nabla \mathbf{v}\| \in L^2$, $\|\mathbf{B}\| \|\nabla \mathbf{B}\| \in L^2$, it holds that $y \in \mathcal{L}^{12}$ and*

$$(3.16) \quad \|y\|_{\mathcal{L}^{12}}^4 \leq C (\|\|\mathbf{v}\| \|\nabla \mathbf{v}\|\|_{L^2}^2 + \|\|\mathbf{B}\| \|\nabla \mathbf{B}\|\|_{L^2}^2).$$

PROOF. By the Gagliardo-Nirenberg-Sobolev inequality (cf. Chapter II, Equation (2.5)) for scalar functions $f \in W^1(\mathbb{R}^3)$

$$\|f\|_{L^6} \leq C \|\nabla f\|_{L^2},$$

it follows that

$$\begin{aligned}
\| |y|^2 \|_{\mathcal{L}^6}^2 &\leq C \|\nabla |y|^2\|_{L^2}^2 = \|\nabla (|\mathbf{v}|^2 + |\mathbf{B}|^2)\|_{L^2}^2 = C \left\| \left(\sum_{i=1}^3 |\partial_i (|\mathbf{v}|^2 + |\mathbf{B}|^2)|^2 \right)^{1/2} \right\|_{L^2}^2 \\
&= C \left\| \left(\sum_{i=1}^3 \left| \sum_{j=1}^3 \partial_i (v_j^2 + B_j^2) \right|^2 \right)^{1/2} \right\|_{L^2}^2 = 4C \int \sum_{i=1}^3 \left| \sum_{j=1}^3 (v_j \partial_i v_j + B_j \partial_i B_j) \right|^2 dx \\
&\leq 24C \int \sum_{j=1}^3 \left(|v_j|^2 \sum_{i=1}^3 |\partial_i v_j|^2 + |B_j|^2 \sum_{i=1}^3 |\partial_i B_j|^2 \right) dx \\
&= 24C \int \sum_{j=1}^3 (|v_j|^2 |\nabla v_j|^2 + |B_j|^2 |\nabla B_j|^2) dx \\
&\leq 24C \int \left(\sup_{j \in \{1,2,3\}} |v_j|^2 \sum_{j=1}^3 |\nabla v_j|^2 + \sup_{j \in \{1,2,3\}} |B_j|^2 \sum_{j=1}^3 |\nabla B_j|^2 \right) dx \\
&\leq 24C \int (|\mathbf{v}|^2 |\nabla \mathbf{v}|^2 + |\mathbf{B}|^2 |\nabla \mathbf{B}|^2) dx = 24C (\|\mathbf{v}\|_{L^2} \|\nabla \mathbf{v}\|_{L^2}^2 + \|\mathbf{B}\|_{L^2} \|\nabla \mathbf{B}\|_{L^2}^2),
\end{aligned}$$

which proves the claim. \square

In order to show convergence of the martingale problems, we need to prove uniform moment estimates as well as convergence in probability. The next lemma provides the moment estimates.

LEMMA 3.14 (uniform integrability of M_e^φ). *The following estimate holds*

$$(3.17) \quad \sup_{n \in \mathbb{N}} \mathbb{E}^{\tilde{P}} \left[|M_e^\varphi(t, \tilde{y}_n)|^{4/3} \right] + \mathbb{E}^{\tilde{P}} \left[|M_e^\varphi(t, \tilde{y})|^{4/3} \right] < \infty.$$

PROOF. We show that each of the terms of M_e^φ are bounded. The terms $I_{1,2}^\varphi$ are obviously bounded by a constant C_φ , since $\varphi \in C_c^\infty$.

For I_3^φ we have by Jensen's inequality for the temporal integral, as well as Equations (2.9) and (3.16)

$$\begin{aligned}
\mathbb{E}^{\tilde{P}} \left[|I_3^\varphi(t, \tilde{y}_n)|^{4/3} \right] &\leq T^{4/3-1} \|\varphi'\|_{L^\infty}^{4/3} \int_0^T \mathbb{E}^{\tilde{P}} \left[|\langle \mathcal{A}(\tilde{y}_n(s)), e \rangle_{\mathcal{H}^1}|^{4/3} \right] ds \\
&\leq C_{T,\varphi,e} \int_0^T \mathbb{E}^{\tilde{P}} \left[1 + \|\tilde{y}_n(s)\|_{\mathcal{L}^3(\mathcal{O})}^4 \right] ds \\
&\leq C_{T,\varphi,e} \int_0^T \mathbb{E}^{\tilde{P}} \left[1 + \|\tilde{y}_n(s)\|_{\mathcal{L}^{12}(\mathcal{O})}^4 \right] ds \\
&= C_{T,\varphi,e} \int_0^T \mathbb{E}^{\tilde{P}} \left[1 + \|\tilde{y}_n(s)\|_{L^6(\mathcal{O})}^2 \right] ds \\
&\leq C_{T,\varphi,e} \int_0^T \mathbb{E}^{\tilde{P}} \left[1 + \|\nabla |\tilde{y}_n(s)|^2\|_{L^2(\mathcal{O})}^2 \right] ds \\
&\leq C_{T,\varphi,e} \int_0^T \mathbb{E}^{\tilde{P}} \left[1 + \|\mathbf{v}\|_{L^2} \|\nabla \mathbf{v}\|_{L^2}^2 + \|\mathbf{B}\|_{L^2} \|\nabla \mathbf{B}\|_{L^2}^2 \right] ds.
\end{aligned}$$

This last term is bounded by our *a priori* estimates (3.14).

In the case of periodic domain $\mathbb{D} = \mathbb{T}^3$, we have by Equations (2.9) and (3.15)

$$\begin{aligned} \mathbb{E}^{\tilde{P}} [|I_3^\varphi(t, \tilde{y}_n)|^{4/3}] &\leq T^{4/3-1} \|\varphi\|_{L^\infty}^{4/3} \int_0^T \mathbb{E}^{\tilde{P}} [| \langle \mathcal{A}(\tilde{y}_n(s)), e \rangle_{\mathcal{H}^1} |^{4/3}] ds \\ &\leq C_{T,\varphi,e} \int_0^T \mathbb{E}^{\tilde{P}} [1 + \|\tilde{y}_n(s)\|_{\mathcal{L}^3(\mathbb{T}^3)}^4] ds \\ &\leq C_{T,\varphi,e} \int_0^T \mathbb{E}^{\tilde{P}} [1 + \|\tilde{y}_n(s)\|_{\mathcal{L}^4(\mathbb{T}^3)}^4] ds \\ &\leq C_{T,\varphi,e,N,f,H,y_0}. \end{aligned}$$

The other two terms can be dealt with swiftly: by Jensen's inequality for the convex function $x \mapsto x^{4/3}$, Hölder's inequality for $p = 3/2, q = 3$, Jensen's inequality for the concave function $x \mapsto x^{2/3}$, Assumption (H1) and (3.12) we find

$$\begin{aligned} \mathbb{E}^{\tilde{P}} [|I_4^\varphi(t, \tilde{y}_n)|^{4/3}] &\leq C_{T,\varphi} \|e\|_{\mathcal{H}^2} \int_0^T \mathbb{E}^{\tilde{P}} [1 \cdot \|f(s, \tilde{y}_n(s))\|_{\mathcal{H}^0}^{4/3}] ds \\ &\leq C_{T,\varphi,e} \int_0^T \left(\mathbb{E}^{\tilde{P}} [\|f(s, \tilde{y}_n(s))\|_{\mathcal{H}^0}^2] \right)^{2/3} ds \\ &\leq C_{T,\varphi,e} \left(\int_0^T \left(1 + \mathbb{E}^{\tilde{P}} [\|\tilde{y}_n(s)\|_{\mathcal{H}^0}^2] + \|F_f(s)\|_{L^1(\mathbb{D})} \right) ds \right)^{2/3} \\ &\leq C_{T,\varphi,e} \left(1 + \mathbb{E}^{\tilde{P}} \left[\sup_{s \in [0,T]} \|\tilde{y}_n(s)\|_{\mathcal{H}^1}^2 \right] + \|F_f\|_{L^1([0,T] \times \mathbb{D})} \right)^{2/3} \\ &\leq C_{T,\varphi,e,N,f,H,y_0}, \end{aligned}$$

and by Equation (2.10), we have in a similar manner

$$\begin{aligned} \mathbb{E}^{\tilde{P}} [|I_5^\varphi(t, \tilde{y}_n)|^{4/3}] &\leq C_{T,\varphi} \int_0^T \mathbb{E}^{\tilde{P}} [\|\mathcal{B}(s, \tilde{y}_n(s))\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)}^{2 \cdot 4/3}] ds \\ &\leq C_{T,\varphi,e,H,\Sigma} \int_0^T 1 + \mathbb{E}^{\tilde{P}} [\|\tilde{y}_n(s)\|_{\mathcal{L}^2(\mathcal{O})}^{8/3}] ds \\ &\leq C_{T,\varphi,e,H,\Sigma} \int_0^T 1 + \mathbb{E}^{\tilde{P}} [\|\tilde{y}_n(s)\|_{\mathcal{L}^3(\mathcal{O})}^4] ds \\ &\leq C_{T,\varphi,e,H,\Sigma} \int_0^T 1 + \mathbb{E}^{\tilde{P}} [\|\tilde{y}_n(s)\|_{\mathcal{L}^{12}(\mathcal{O})}^4] ds \\ &\leq C_{T,\varphi,e,H,\Sigma} \int_0^T \mathbb{E}^{\tilde{P}} [1 + \|\mathbf{v}\|_{L^2} \|\nabla \mathbf{v}\|_{L^2}^2 + \|\mathbf{B}\|_{L^2} \|\nabla \mathbf{B}\|_{L^2}^2] ds \\ &\leq C_{T,\varphi,e,H,\Sigma,N,y_0}. \end{aligned}$$

□

Our next and last ingredient for the existence proof of weak solutions is the convergence in probability of the martingales defined above, as in [196, Lemma 3.12, p. 236 f.].

LEMMA 3.15 (convergence in probability). *For every $t > 0$, $M_e^\varphi(t, \tilde{y}_n) \rightarrow M_e^\varphi(t, \tilde{y})$ in probability, i.e. for every $\varepsilon > 0$*

$$(3.18) \quad \lim_{n \rightarrow \infty} \tilde{P} \{ |M_e^\varphi(t, \tilde{y}_n) - M_e^\varphi(t, \tilde{y})| > \varepsilon \} = 0.$$

PROOF. As before, we show convergence of all the terms of M_e^φ and we denote $\mathcal{O} := \text{supp}(e)$. Note that by (II), and since \mathcal{O} is bounded, we find by the definition of the metric on \mathcal{X} that

$$(3.19) \quad \lim_{n \rightarrow \infty} \int_{\mathcal{O}} |\tilde{y}_n(t, x, \tilde{\omega}) - \tilde{y}(t, x, \tilde{\omega})|^2 dx = 0, \quad \tilde{P} - \text{a.a. } \tilde{\omega} \in \tilde{\Omega}.$$

This then implies that for P -a.a. $\tilde{\omega} \in \tilde{\Omega}$ and all $t \in [0, T]$

$$|\langle \tilde{y}_n(t, \cdot, \tilde{\omega}) - \tilde{y}(t, \cdot, \tilde{\omega}), e \rangle_{\mathcal{H}^1}| \leq \|\tilde{y}_n(t, \cdot, \tilde{\omega}) - \tilde{y}(t, \cdot, \tilde{\omega})\|_{\mathcal{L}^2(\mathcal{O})} \|e\|_{\mathcal{H}^2} \rightarrow 0,$$

as $n \rightarrow \infty$. By the continuity and boundedness of φ and Lebesgue's dominated convergence theorem, we find

$$\lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} [|I_1^\varphi(t, \tilde{y}_n) - I_1^\varphi(t, \tilde{y})|] = \lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} [|\varphi(\langle \tilde{y}_n(t), e \rangle_{\mathcal{H}^1}) - \varphi(\langle \tilde{y}(t), e \rangle_{\mathcal{H}^1})|] = 0,$$

and similarly

$$\lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} [|I_2^\varphi(t, \tilde{y}_n) - I_2^\varphi(t, \tilde{y})|] = \lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} [|\varphi(\langle \tilde{y}_n(0), e \rangle_{\mathcal{H}^1}) - \varphi(\langle \tilde{y}(0), e \rangle_{\mathcal{H}^1})|] = 0.$$

For I_3^φ , we define for any $R > 0$ and $n \in \mathbb{N}$ the stopping times

$$\tilde{\tau}_R^n := \inf\{t \geq 0 \mid \|\tilde{y}_n(t)\|_{\mathcal{H}^1} \geq R\}.$$

Then by Chebychev's inequality, (I) and (3.6), we get

$$(3.20) \quad \begin{aligned} \sup_{n \in \mathbb{N}} \tilde{P} \{\tilde{\tau}_R^n \leq T\} &= \sup_{n \in \mathbb{N}} \tilde{P} \{\|\tilde{y}_n(t)\|_{\mathcal{H}^1} \geq R\} \leq \sup_{n \in \mathbb{N}} \frac{1}{R^2} \mathbb{E}^{\tilde{P}} [\|\tilde{y}_n(t)\|_{\mathcal{H}^1}^2] \\ &\leq \sup_{n \in \mathbb{N}} \frac{1}{R^2} \mathbb{E}^P [\|y_n(t)\|_{\mathcal{H}^1}^2] \leq \frac{C_{T,N,f,H,y_0}}{R^2}. \end{aligned}$$

For arbitrary $R > 0$ we thus find

$$\begin{aligned} &\lim_{n \rightarrow \infty} \tilde{P} \{ |I_3^\varphi(t, \tilde{y}_n) - I_3^\varphi(t, \tilde{y})| > \varepsilon \} \\ &\leq \lim_{n \rightarrow \infty} \tilde{P} \{ |I_3^\varphi(t, \tilde{y}_n) - I_3^\varphi(t, \tilde{y})| > \varepsilon, \tilde{\tau}_R^n > T \} \\ &\quad + \lim_{n \rightarrow \infty} \tilde{P} \{ |I_3^\varphi(t, \tilde{y}_n) - I_3^\varphi(t, \tilde{y})| > \varepsilon, \tilde{\tau}_R^n \leq T \} \\ &\leq \lim_{n \rightarrow \infty} \tilde{P} \{ |I_3^\varphi(t, \tilde{y}_n) - I_3^\varphi(t, \tilde{y})| > \varepsilon, \tilde{\tau}_R^n > T \} + \frac{C_{T,N,f,H,y_0}}{R^2}. \end{aligned}$$

Equations (2.9), (2.11) of Lemma 2.4 – and again the continuity and boundedness of φ – imply that for any $t \in [0, T]$ and all $\tilde{\omega} \in \tilde{\Omega}$ with $\tilde{\tau}_R^n(\tilde{\omega}) > T$

$$\begin{aligned} &|\varphi'(\langle \tilde{y}_n(t, \tilde{\omega}), e \rangle_{\mathcal{H}^1}) \langle \mathcal{A}(\tilde{y}_n(t, \tilde{\omega})), e \rangle_{\mathcal{H}^1} - \varphi'(\langle \tilde{y}(t, \tilde{\omega}), e \rangle_{\mathcal{H}^1}) \langle \mathcal{A}(\tilde{y}(t, \tilde{\omega})), e \rangle_{\mathcal{H}^1}| \\ &\leq |\varphi'(\langle \tilde{y}_n(t, \tilde{\omega}), e \rangle_{\mathcal{H}^1}) - \varphi'(\langle \tilde{y}(t, \tilde{\omega}), e \rangle_{\mathcal{H}^1})| |\langle \mathcal{A}(\tilde{y}_n(t, \tilde{\omega})), e \rangle_{\mathcal{H}^1}| \\ &\quad + |\varphi'(\langle \tilde{y}(t, \tilde{\omega}), e \rangle_{\mathcal{H}^1})| |\langle \mathcal{A}(\tilde{y}(t, \tilde{\omega})) - \mathcal{A}(\tilde{y}_n(t, \tilde{\omega})), e \rangle_{\mathcal{H}^1}| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

This – combined with Markov’s inequality and Lebesgue’s dominated convergence theorem – yields the desired convergence:

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \tilde{P} \{ |I_3^\varphi(t, \tilde{y}_n) - I_3^\varphi(t, \tilde{y})| > \varepsilon \} \\
& \leq \lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \tilde{P} \{ |I_3^\varphi(t, \tilde{y}_n) - I_3^\varphi(t, \tilde{y})| > \varepsilon, \tilde{\tau}_R^n > T \} \\
& \leq \lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\varepsilon} \mathbb{E}^{\tilde{P}} [1_{\{\tilde{\tau}_R^n > T\}} |I_3^\varphi(t, \tilde{y}_n) - I_3^\varphi(t, \tilde{y})|] \\
& \leq \lim_{R \rightarrow \infty} \frac{1}{\varepsilon} \mathbb{E}^{\tilde{P}} \left[\lim_{n \rightarrow \infty} \int_0^t 1_{\{\tilde{\tau}_R^n > T\}} \cdot \left| \varphi'(\langle \tilde{y}_n(t), e \rangle_{\mathcal{H}^1}) \langle \mathcal{A}(\tilde{y}_n(t)), e \rangle_{\mathcal{H}^1} \right. \right. \\
& \quad \left. \left. - \varphi'(\langle \tilde{y}(t), e \rangle_{\mathcal{H}^1}) \langle \mathcal{A}(\tilde{y}(t)), e \rangle_{\mathcal{H}^1} \right| ds \right] = 0.
\end{aligned}$$

Similarly, for I_4^φ we get by Assumption (H1) and similar convergence arguments that

$$\lim_{n \rightarrow \infty} \tilde{P} \{ |I_4^\varphi(t, \tilde{y}_n) - I_4^\varphi(t, \tilde{y})| > \varepsilon \} = 0.$$

Finally, for I_5^φ we find that since

$$\begin{aligned}
& \left| \|\langle \mathcal{B}(s, \tilde{y}_n(s)), e \rangle_{\mathcal{H}^1}\|_{\ell^2}^2 - \|\langle \mathcal{B}(s, \tilde{y}(s)), e \rangle_{\mathcal{H}^1}\|_{\ell^2}^2 \right| \\
& \leq \left| \sum_{k=1}^{\infty} |\langle \mathcal{B}_k(s, \tilde{y}_n(s)), e \rangle_{\mathcal{H}^1}|^2 - |\langle \mathcal{B}_k(s, \tilde{y}(s)), e \rangle_{\mathcal{H}^1}|^2 \right| \\
& \leq \left| \sum_{k=1}^{\infty} |\langle \mathcal{B}_k(s, \tilde{y}_n(s)) - \mathcal{B}_k(s, \tilde{y}(s)), e \rangle_{\mathcal{H}^1}| \cdot |\langle \mathcal{B}_k(s, \tilde{y}_n(s)), e \rangle_{\mathcal{H}^1} + \langle \mathcal{B}_k(s, \tilde{y}(s)), e \rangle_{\mathcal{H}^1}| \right| \\
& \leq \|e\|_{\mathcal{H}^2} \left| \sum_{k=1}^{\infty} |\langle \mathcal{B}_k(s, \tilde{y}_n(s)) - \mathcal{B}_k(s, \tilde{y}(s)), e \rangle_{\mathcal{H}^1}| \cdot (\|\mathcal{B}_k(s, \tilde{y}_n(s))\|_{\mathcal{H}^0} + \|\mathcal{B}_k(s, \tilde{y}(s))\|_{\mathcal{H}^0}) \right| \\
& \leq C_e \|\langle \mathcal{B}(s, \tilde{y}_n(s)) - \mathcal{B}(s, \tilde{y}(s)), e \rangle_{\mathcal{H}^1}\|_{\ell^2} \cdot (\|\mathcal{B}(s, \tilde{y}_n(s))\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)} + \|\mathcal{B}(s, \tilde{y}(s))\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)}),
\end{aligned}$$

when combining this with (2.12), the convergence

$$\lim_{n \rightarrow \infty} \tilde{P} \{ |I_5^\varphi(t, \tilde{y}_n) - I_5^\varphi(t, \tilde{y})| > \varepsilon \} = 0,$$

follows. This concludes the proof. \square

We are now in a position to prove the main result of this section.

PROOF OF THEOREM 3.10. Let $t > s$, G be a bounded \mathbb{R} -valued, $\mathcal{B}_s(\mathcal{X})$ -measurable continuous function on \mathcal{X} . Then we have by the generalised Lebesgue theorem

$$\begin{aligned}
& \mathbb{E}^\mu [(M_e^\varphi(t, y) - M_e^\varphi(s, y)) \cdot G(y)] \\
& = \mathbb{E}^{\tilde{P}} [(M_e^\varphi(t, \tilde{y}) - M_e^\varphi(s, \tilde{y})) \cdot G(\tilde{y})] \quad (\mu = \tilde{P} \circ \tilde{y}^{-1}) \\
& = \lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{P}} [(M_e^\varphi(t, \tilde{y}_n) - M_e^\varphi(s, \tilde{y}_n)) \cdot G(\tilde{y}_n)] \quad ((3.17), (3.18)) \\
& = \lim_{n \rightarrow \infty} \mathbb{E}^P [(M_e^\varphi(t, y_n) - M_e^\varphi(s, y_n)) \cdot G(y_n)] \quad (\tilde{P} \circ \tilde{y}_n^{-1} = P \circ y_n^{-1}) \\
& = 0.
\end{aligned}$$

The last step holds due to the martingale property of $M_e^\varphi(s, y_n)$ on the stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$ and since $G(y_n)$ is \mathcal{F}_s -measurable. Therefore, μ solves the martingale problem and by Proposition 3.4 we infer the existence of a weak solution to (2.4). \square

3.4. Proof of Theorem 1.1.

PROOF OF THEOREM 1.1. The only thing left to prove is estimate (1.3). By Itô's formula, taking expectations and using (2.6), Assumption (H1) and Equation (2.12), we find

$$\begin{aligned} \mathbb{E}\|y(t)\|_{\mathcal{H}^0}^2 &= \|y_0\|_{\mathcal{H}^0}^2 + 2 \int_0^t \mathbb{E}\langle \mathcal{A}(y(s)), y(s) \rangle_{\mathcal{H}^0} ds \\ &\quad + 2 \int_0^t \mathbb{E}\langle f(s, y(s)), y(s) \rangle_{\mathcal{H}^0} ds + \int_0^t \mathbb{E}\|\mathcal{B}(s, y(s))\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)}^2 ds \\ &\leq \|y_0\|_{\mathcal{H}^0}^2 - \frac{3}{2} \int_0^t \mathbb{E}\|y(s)\|_{\mathcal{H}^1}^2 ds + C_{f,H} + C_{T,f,H} \int_0^t \mathbb{E}\|y(s)\|_{\mathcal{H}^0}^2 ds, \end{aligned}$$

which by Gronwall's lemma implies

$$\mathbb{E}\|y(t)\|_{\mathcal{H}^0}^2 + \int_0^t \mathbb{E}\|y(s)\|_{\mathcal{H}^1}^2 ds \leq C_{T,f,H} (\|y_0\|_{\mathcal{H}^0}^2 + 1).$$

Applying Burkholder's inequality and performing similar calculations as in the proof of Lemma 3.11, we get the desired estimate. \square

4. Feller Property and Existence of Invariant Measures

In this section we prove further properties of the tamed MHD equations. First, we show that they generate a Feller semigroup under stronger assumptions. Then, we prove that there exists an invariant measure for this Feller semigroup in the case of periodic boundary conditions.

We consider the *time-homogeneous case*, i.e. the functions f, Σ, H of our equations are assumed to be independent of time. Furthermore, we assume Lipschitz conditions on the first-order derivatives of the function H :

(H3)' There exists a constant $C_H > 0$ and a function $F_H \in L^1(\mathbb{D})$ such that for all $x \in \mathbb{D}$, $y, y' \in \mathbb{R}^6$, $j = 1, 2, 3$ the following conditions hold:

$$\begin{aligned} \|\partial_{x^j} H(x, y)\|_{\ell^2}^2 + \|H(x, y)\|_{\ell^2}^2 &\leq C_H |y|^2 + F_H(x), \\ \|\partial_{x^j} H(x, y) - \partial_{x^j} H(x, y')\|_{\ell^2} &\leq C_H |y - y'|, \\ \|\partial_{y^j} H(x, y)\|_{\ell^2} &\leq C_H, \\ \|\partial_{y^j} H(x, y) - \partial_{(y')^j} H(x, y')\|_{\ell^2} &\leq C_H |y - y'|. \end{aligned}$$

For an initial condition $y_0 \in \mathcal{H}^1$, let $y(t; y_0)$ be the unique solution to (2.4) with $y(0; y_0) = y_0$. Then by the uniqueness of solutions, we know that $\{y(t; y_0) \mid y_0 \in \mathcal{H}^1, t \geq 0\}$ is a strong Markov process with state space \mathcal{H}^1 . In proving the Feller property of the associated semigroup, we need the following result.

LEMMA 4.1. For $y_0, y'_0 \in \mathcal{H}^1$, $R > 0$, define the stopping times

$$\begin{aligned} \tau_R^{y_0} &:= \inf\{t \geq 0 \mid \|y(t; y_0)\|_{\mathcal{H}^1} > R\}, \\ \tau_R &:= \tau_R^{y_0, y'_0} := \tau_R^{y_0} \wedge \tau_R^{y'_0}. \end{aligned}$$

Assume (H1), (H2), (H3)'. Then there is a constant $C_{t,R,N,f,H,\Sigma} > 0$ such that

$$\mathbb{E} [\|y(t \wedge \tau_R; y_0) - y(t \wedge \tau_R; y'_0)\|_{\mathcal{H}^1}^2] \leq C_{t,R,N,f,H,\Sigma} \|y_0 - y'_0\|_{\mathcal{H}^1}^2.$$

PROOF. We follow the proof of [196, Lemma 4.1, p. 238 ff.] For notational convenience, we denote $y(t) := y(t; y_0)$, $\tilde{y}(t) := y(t; y'_0)$, $z(t) := y(t) - \tilde{y}(t)$ and $t_R := t \wedge \tau_R$. By Itô's formula, we have

$$\begin{aligned} \|z(t_R)\|_{\mathcal{H}^1}^2 &= \|z(0)\|_{\mathcal{H}^1}^2 + 2 \int_0^{t_R} \langle \mathcal{A}(y(s)) - \mathcal{A}(\tilde{y}(s)), z(s) \rangle_{\mathcal{H}^1} ds \\ &\quad + 2 \int_0^{t_R} \langle f(y(s)) - f(\tilde{y}(s)), z(s) \rangle_{\mathcal{H}^1} ds \\ &\quad + 2 \sum_{k=1}^{\infty} \int_0^{t_R} \langle \mathcal{B}_k(y(s)) - \mathcal{B}_k(\tilde{y}(s)), z(s) \rangle_{\mathcal{H}^1} dW_s^k \\ &\quad + \sum_{k=1}^{\infty} \int_0^{t_R} \|\mathcal{B}_k(y(s)) - \mathcal{B}_k(\tilde{y}(s)), z(s)\|_{\mathcal{H}^1}^2 ds \\ &=: \|z(0)\|_{\mathcal{H}^1}^2 + I_1(t_R) + I_2(t_R) + I_3(t_R) + I_4(t_R). \end{aligned}$$

We denote by z_v the velocity component of z , i.e. $z_v := v - \tilde{v}$, and similarly we write $z_B := B - \tilde{B}$. Then $I_1(t_R)$ has the following form:

$$\begin{aligned} I_1(t_R) &= 2 \int_0^{t_R} \langle \Delta z, z \rangle_{\mathcal{H}^1} - 2 \int_0^{t_R} \langle g_N(|y|^2)y - g_N(|\tilde{y}|^2)\tilde{y}, z \rangle_{\mathcal{H}^1} \\ &\quad + 2 \int_0^{t_R} \left\{ \langle -(\mathbf{v} \cdot \nabla)\mathbf{v} + (\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}}, z_v \rangle_{\mathcal{H}^1} + \langle (\mathbf{B} \cdot \nabla)\mathbf{B} - (\tilde{\mathbf{B}} \cdot \nabla)\tilde{\mathbf{B}}, z_B \rangle_{\mathcal{H}^1} \right. \\ &\quad \left. + \langle -(\mathbf{v} \cdot \nabla)\mathbf{B} + (\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{B}}, z_B \rangle_{\mathcal{H}^1} + \langle (\mathbf{B} \cdot \nabla)\mathbf{v} - (\tilde{\mathbf{B}} \cdot \nabla)\tilde{\mathbf{v}}, z_B \rangle_{\mathcal{H}^1} \right\} ds. \end{aligned}$$

The first term is readily analysed:

$$2 \int_0^{t_R} \langle \Delta z, z \rangle_{\mathcal{H}^1} = -2 \int_0^{t_R} \|z\|_{\mathcal{H}^2}^2 ds + 2 \int_0^{t_R} \|z\|_{\mathcal{H}^1}^2 ds.$$

For the second term, we find by using Young's inequality, $g_N(r) \leq Cr$, Hölder's inequality (with $p = 3$, $q = 3/2$) and the Sobolev embedding $\mathcal{H}^1 \subset \mathcal{L}^6$ that for some $\theta \in \mathbb{R}$

$$\begin{aligned} \langle g_N(|y|^2)y - g_N(|\tilde{y}|^2)\tilde{y}, z \rangle_{\mathcal{H}^1} &= \langle g_N(|y|^2)z, z \rangle_{\mathcal{H}^1} + \langle (g_N(|y|^2) - g_N(|\tilde{y}|^2))\tilde{y}, z \rangle_{\mathcal{H}^1} \\ &= \langle g_N(|y|^2)z, (I - \Delta)z \rangle_{\mathcal{H}^0} + \langle g'_N(\theta) (|y|^2 - |\tilde{y}|^2) \tilde{y}, (I - \Delta)z \rangle_{\mathcal{H}^0} \\ &\leq \|g_N(|y|^2)|z|\|_{L^2} \|z\|_{\mathcal{H}^2} + \|g'_N(\theta) \cdot |z|(|y| + |\tilde{y}|)|\tilde{y}\|_{L^2} \|z\|_{\mathcal{H}^2} \\ &\leq 2\varepsilon \|z\|_{\mathcal{H}^2}^2 + C_{\varepsilon, N} \left(\|g_N(|y|^2)|z|\|_{L^2}^2 + \| |z|(|y|^2 + |\tilde{y}|^2) \|_{L^2}^2 \right) \\ &\leq 2\varepsilon \|z\|_{\mathcal{H}^2}^2 + C_{\varepsilon, N} \| |z|(|y|^2 + |\tilde{y}|^2) \|_{L^2}^2 \\ &\leq 2\varepsilon \|z\|_{\mathcal{H}^2}^2 + C_{\varepsilon, N} \|z\|_{\mathcal{L}^6}^2 (\|y\|_{\mathcal{L}^6}^4 + \|\tilde{y}\|_{\mathcal{L}^6}^4) \\ &\leq 2\varepsilon \|z\|_{\mathcal{H}^2}^2 + C_{\varepsilon, N} \|z\|_{\mathcal{H}^1}^2 (\|y\|_{\mathcal{H}^1}^4 + \|\tilde{y}\|_{\mathcal{H}^1}^4) \\ &\leq 2\varepsilon \|z\|_{\mathcal{H}^2}^2 + C_{\varepsilon, N, R} \|z\|_{\mathcal{H}^1}^2. \end{aligned}$$

The third term consists of four sub-terms, all of which are very similar. We thus only estimate one of them: Young's inequality, the Sobolev embedding $H^1 \subset L^6$ as well as the Gagliardo-Nirenberg-Sobolev inequalities $\|\mathbf{u}\|_{L^\infty} \leq C\|\mathbf{u}\|_{H^2}^{3/4}\|\mathbf{u}\|_{L^2}^{1/4}$ and $\|\nabla\mathbf{u}\|_{L^3} \leq C\|\mathbf{u}\|_{H^2}^{3/4}\|\mathbf{u}\|_{L^2}^{1/4}$, combined with another application of Young's inequality (with $p = 4/3$, $q = 4$) yield

$$\begin{aligned} & \langle (\mathbf{v} \cdot \nabla)\mathbf{v} - (\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}}, \mathbf{z}_v \rangle_{H^1} = \langle (\mathbf{z}_v \cdot \nabla)\mathbf{v}, \mathbf{z}_v \rangle_{H^1} + \langle (\tilde{\mathbf{v}} \cdot \nabla)\mathbf{z}_v, \mathbf{z}_v \rangle_{H^1} \\ & = \langle (\mathbf{z}_v \cdot \nabla)\mathbf{v}, (I - \Delta)\mathbf{z}_v \rangle_{H^0} + \langle (\tilde{\mathbf{v}} \cdot \nabla)\mathbf{z}_v, (I - \Delta)\mathbf{z}_v \rangle_{H^0} \\ & \leq 2\varepsilon\|\mathbf{z}_v\|_{H^2}^2 + C_\varepsilon (\|(\mathbf{z}_v \cdot \nabla)\mathbf{v}\|_{L^2}^2 + \|(\tilde{\mathbf{v}} \cdot \nabla)\mathbf{z}_v\|_{L^2}^2) \\ & \leq 2\varepsilon\|\mathbf{z}_v\|_{H^2}^2 + C_\varepsilon (\|\mathbf{z}_v\|_{L^\infty}^2\|\nabla\mathbf{v}\|_{L^2}^2 + \|\tilde{\mathbf{v}}\|_{L^6}^2\|\nabla\mathbf{z}_v\|_{L^3}^2) \\ & \leq 2\varepsilon\|\mathbf{z}_v\|_{H^2}^2 + C_\varepsilon \left(\|\mathbf{z}_v\|_{H^2}^{3/2}\|\mathbf{z}_v\|_{L^2}^{1/2}\|\mathbf{v}\|_{H^1}^2 + \|\tilde{\mathbf{v}}\|_{H^1}^2\|\mathbf{z}_v\|_{H^2}^{3/2}\|\mathbf{z}_v\|_{L^2}^{1/2} \right) \\ & \leq 4\varepsilon\|\mathbf{z}_v\|_{H^2}^2 + C_{\varepsilon,R}\|\mathbf{z}_v\|_{L^2}^2 \leq 4\varepsilon\|\mathbf{z}_v\|_{H^2}^2 + C_{\varepsilon,R}\|\mathbf{z}_v\|_{H^1}^2. \end{aligned}$$

Analysing the other sub-terms in the same way and putting everything together, we find

$$(4.1) \quad I_1(t_R) \leq -2 \int_0^{t_R} \|z(s)\|_{\mathcal{H}^2}^2 ds + C_{\varepsilon,N,R} \int_0^{t_R} \|z(s)\|_{\mathcal{H}^1}^2 ds + 10\varepsilon \int_0^{t_R} \|z(s)\|_{\mathcal{H}^2}^2 ds.$$

$I_2(t_R)$ is analysed using Young's inequality and Assumption (H1)

$$\begin{aligned} I_2(t_R) &= 2 \int_0^{t_R} \langle f(y(s)) - f(\tilde{y}(s)), z(s) \rangle_{\mathcal{H}^1} ds \\ &\leq \varepsilon \int_0^{t_R} \|z(s)\|_{\mathcal{H}^2}^2 ds + C_\varepsilon \int_0^{t_R} \|f(y(s)) - f(\tilde{y}(s))\|_{\mathcal{H}^0}^2 ds \\ &\leq \varepsilon \int_0^{t_R} \|z(s)\|_{\mathcal{H}^2}^2 ds + C_{\varepsilon,f} \int_0^{t_R} \|z(s)\|_{\mathcal{H}^0}^2 ds. \end{aligned}$$

The term $I_3(t_R)$ is a martingale and thus vanishes after taking expectations.

For $I_4(t_R)$, we use the following considerations, similar to the ones in the proof of Lemma 2.5:

$$\|\mathcal{B}(y) - \mathcal{B}(\tilde{y})\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^1)}^2 = \|\mathcal{B}(y) - \mathcal{B}(\tilde{y})\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)}^2 + \|\nabla(\mathcal{B}(y) - \mathcal{B}(\tilde{y}))\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)}^2,$$

and the latter term consists (by using the chain rule) of terms of the following form:

$$\begin{aligned} & \partial_{x^j}(\mathcal{B}_k(y) - \mathcal{B}_k(\tilde{y})) = \partial_{x^j}(\mathcal{P}(\Sigma_k(x) \cdot \nabla)z + \mathcal{P}(H_k(x, y) - H_k(x, \tilde{y}))) \\ & = \mathcal{P} \left\{ ((\partial_{x^j}\Sigma_k(x)) \cdot \nabla)z + (\Sigma_k(x) \cdot \nabla)\partial_{x^j}z + (\partial_{x^j}H_k)(x, y) - (\partial_{x^j}H_k)(x, \tilde{y}) \right. \\ & \quad \left. + \sum_{i=1}^6 [(\partial_{y^i}H_k(x, y))\partial_{x^j}y^i - (\partial_{\tilde{y}^i}H_k(x, \tilde{y}))\partial_{x^j}\tilde{y}^i] \right\} \\ & = \mathcal{P} \left\{ ((\partial_{x^j}\Sigma_k(x)) \cdot \nabla)z + (\Sigma_k(x) \cdot \nabla)\partial_{x^j}z + (\partial_{x^j}H_k)(x, y) - (\partial_{x^j}H_k)(x, \tilde{y}) \right. \\ & \quad \left. + \sum_{i=1}^6 [(\partial_{y^i}H_k(x, y))\partial_{x^j}z^i - \{(\partial_{y^i}H_k(x, y)) - (\partial_{\tilde{y}^i}H_k(x, \tilde{y}))\}\partial_{x^j}\tilde{y}^i] \right\}. \end{aligned}$$

Thus we find, using Assumptions (H2) and (H3)', as well as Equation (2.1), the Gagliardo-Nirenberg inequality and Young's inequality, that

$$\begin{aligned}
& \|\mathcal{B}(y) - \mathcal{B}(\tilde{y})\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^1)}^2 \\
&= \|(\Sigma \cdot \nabla) z + H(y) - H(\tilde{y})\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)}^2 + \sum_{k=1}^{\infty} \|\nabla (\mathcal{B}_k(y) - \mathcal{B}_k(\tilde{y}))\|_{\mathcal{H}^0}^2 \\
&\leq 2 \sum_{k=1}^{\infty} \|(\Sigma_k \cdot \nabla) z\|_{\mathcal{H}^0}^2 + \|H_k(y) - H_k(\tilde{y})\|_{\mathcal{H}^0}^2 \\
&\quad + \sum_{k=1}^{\infty} \int_{\mathbb{D}} \sum_{j=1}^3 \left| ((\partial_{x^j} \Sigma_k(x)) \cdot \nabla) z + (\Sigma_k(x) \cdot \nabla) \partial_{x^j} z + (\partial_{x^j} H_k)(x, y) - (\partial_{x^j} H_k)(x, \tilde{y}) \right. \\
&\quad \left. + \sum_{i=1}^6 [(\partial_{y^i} H_k(x, y)) \partial_{x^j} z^i - \{(\partial_{y^i} H_k(x, y)) - (\partial_{\tilde{y}^i} H_k(x, \tilde{y}))\} \partial_{x^j} \tilde{y}^i] \right|^2 dx \\
&\leq 2 \sup_{x \in \mathbb{D}} \|\Sigma_k(x)\|_{\ell^2}^2 \|\nabla z\|_{\mathcal{H}^0}^2 + C_H \|z\|_{\mathcal{H}^0}^2 + 2 \sum_{k=1}^{\infty} \int_{\mathbb{D}} \sum_{j=1}^3 \left| (\Sigma_k(x) \cdot \nabla) \partial_{x^j} z \right|^2 dx \\
&\quad + 2 \sum_{k=1}^{\infty} \int_{\mathbb{D}} \sum_{j=1}^3 \left| ((\partial_{x^j} \Sigma_k(x)) \cdot \nabla) z + (\partial_{x^j} H_k)(x, y) - (\partial_{x^j} H_k)(x, \tilde{y}) \right. \\
&\quad \left. + \sum_{i=1}^6 (\partial_{y^i} H_k(x, y)) \partial_{x^j} z^i - \sum_{i=1}^6 \{(\partial_{y^i} H_k(x, y)) - (\partial_{\tilde{y}^i} H_k(x, \tilde{y}))\} \partial_{x^j} \tilde{y}^i \right|^2 dx \\
&\leq 2 \sup_{x \in \mathbb{D}} \|\Sigma_k(x)\|_{\ell^2}^2 \|z\|_{\mathcal{H}^1}^2 + C_H \|z\|_{\mathcal{H}^0}^2 + 2 \sup_{x \in \mathbb{D}} \|\Sigma_k(x)\|_{\ell^2}^2 \sum_{j,l=1}^3 \|\partial_{x^l} \partial_{x^j} z\|_{\mathcal{L}^2}^2 \\
&\quad + 8 \sup_{x \in \mathbb{D}} \|\nabla \Sigma_k(x)\|_{\ell^2}^2 \|\nabla z\|_{\mathcal{H}^0}^2 + C_H \|z\|_{\mathcal{H}^1}^2 + C_H \int_{\mathbb{D}} \sum_{j=1}^3 \sum_{i=1}^6 |z|^2 |\partial_{x^j} \tilde{y}^i|^2 dx \\
&\leq 2d^2 \sup_{x \in \mathbb{D}} \|\Sigma_k(x)\|_{\ell^2}^2 \|z\|_{\mathcal{H}^2}^2 + C_{\Sigma, H} \|z\|_{\mathcal{H}^1}^2 + C_H \|z\|_{\mathcal{L}^\infty}^2 \|\tilde{y}\|_{\mathcal{H}^1}^2 \\
&\leq \left(\frac{1}{2} + \varepsilon\right) \|z\|_{\mathcal{H}^2}^2 + C_{\Sigma, H} \|z\|_{\mathcal{H}^1}^2 + C_H \|z\|_{\mathcal{H}^0}^2 \|\tilde{y}\|_{\mathcal{H}^1}^8.
\end{aligned}$$

Integrating over time we finally get

$$\begin{aligned}
I_4(t_R) &= \int_0^{t_R} \|\mathcal{B}(y(s)) - \mathcal{B}(\tilde{y}(s))\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^1)}^2 ds \\
&\leq \int_0^{t_R} \left(\frac{1}{2} + \varepsilon\right) \|z\|_{\mathcal{H}^2}^2 + C_{H,R} \|z\|_{\mathcal{H}^0}^2 + C_{\Sigma} \|z\|_{\mathcal{H}^1}^2 ds,
\end{aligned}$$

and thus, adding all the contributions together,

$$\begin{aligned}
& \mathbb{E} [\|z(t_R)\|_{\mathcal{H}^1}^2] \\
&\leq \|z(0)\|_{\mathcal{H}^1}^2 - (3/2 - 12\varepsilon) \mathbb{E} \left[\int_0^{t_R} \|z(s)\|_{\mathcal{H}^2}^2 ds \right] + C_{\varepsilon, R, N, f, H, \Sigma} \mathbb{E} \left[\int_0^{t_R} \|z(s)\|_{\mathcal{H}^1}^2 ds \right].
\end{aligned}$$

Choosing $\varepsilon = \frac{1}{8}$, we find

$$\mathbb{E} [\|z(t \wedge \tau_R)\|_{\mathcal{H}^1}^2] \leq \|z(0)\|_{\mathcal{H}^1}^2 + C_{R,N,f,H,\Sigma} \int_0^t \mathbb{E} [\|z(s \wedge \tau_R)\|_{\mathcal{H}^1}^2 ds].$$

An application of Gronwall's Lemma then yields the desired result. \square

Let $BC_{\text{loc}}(\mathcal{H}^1)$ denote the set of bounded, locally uniformly continuous functions on \mathcal{H}^1 . The supremum norm

$$\|\phi\|_{\infty} := \sup_{y \in \mathcal{H}^1} |\phi(y)|$$

turns this space into a Banach space.

For $t \geq 0$, define the semigroup T_t associated with the Markov process $\{y(t; y_0) \mid y_0 \in \mathcal{H}^1, t \geq 0\}$ by

$$T_t \phi(y_0) := \mathbb{E} [\phi(y(t; y_0))], \quad \phi \in BC_{\text{loc}}(\mathcal{H}^1).$$

Using the previous lemma, we show that this is a Feller semigroup.

THEOREM 4.2 (Feller property). *Under the Assumptions (H1), (H2) and (H3)', for every $t \geq 0$, T_t maps $BC_{\text{loc}}(\mathcal{H}^1)$ into itself, i.e. it is a Feller semigroup on $BC_{\text{loc}}(\mathcal{H}^1)$.*

PROOF. Let $\phi \in BC_{\text{loc}}(\mathcal{H}^1)$ be arbitrary. We need to show that for every $t > 0$, $m \in \mathbb{N}$

$$\lim_{\delta \rightarrow 0} \sup_{y_0, y'_0 \in \mathbb{B}_m, \|y_0 - y'_0\|_{\mathcal{H}^1} \leq \delta} |T_t \phi(y_0) - T_t \phi(y'_0)| = 0,$$

where $\mathbb{B}_m := \{y \in \mathcal{H}^1 \mid \|y\|_{\mathcal{H}^1} \leq m\}$ denotes the \mathcal{H}^1 -ball of radius m .

As before, we define the stopping times

$$\begin{aligned} \tau_R^{y_0} &:= \inf\{t \geq 0 \mid \|y(t; y_0)\|_{\mathcal{H}^1} > R\}, \\ \tau_R &:= \tau_R^{y_0, y'_0} := \tau_R^{y_0} \wedge \tau_R^{y'_0}. \end{aligned}$$

Let $\varepsilon > 0$. We estimate using the definition of the semigroup and the triangle inequality

$$\begin{aligned} (4.2) \quad |T_t \phi(y_0) - T_t \phi(y'_0)| &\leq \mathbb{E} [|\phi(y(t; y_0)) - \phi(y(t \wedge \tau_R; y_0))|] \\ &\quad + \mathbb{E} [|\phi(y(t \wedge \tau_R; y_0)) - \phi(y(t \wedge \tau_R; y'_0))|] \\ &\quad + \mathbb{E} [|\phi(y(t \wedge \tau_R; y'_0)) - \phi(y(t; y'_0))|]. \end{aligned}$$

The first term can be estimated by

$$\begin{aligned} \mathbb{E} [|\phi(y(t; y_0)) - \phi(y(t \wedge \tau_R; y_0))|] &= \mathbb{E} [1_{\{\tau_R < t\}} |\phi(y(t; y_0)) - \phi(y(t \wedge \tau_R; y_0))|] \\ &\leq 2\|\phi\|_{\infty} P\{\tau_R < t\} \leq 2\|\phi\|_{\infty} P\left(\{\tau_R^{y_0} < t\} \cup \{\tau_R^{y'_0} < t\}\right) \\ &\leq 2\|\phi\|_{\infty} \frac{1}{R^2} \mathbb{E} \left[\sup_{s \in [0, t]} \|y(s; y_0)\|_{\mathcal{H}^1}^2 + \sup_{s \in [0, t]} \|y(s; y'_0)\|_{\mathcal{H}^1}^2 \right] \\ &\leq 4\|\phi\|_{\infty} \frac{1}{R^2} \sup_{y_0 \in \mathbb{B}_m} \mathbb{E} \left[\sup_{s \in [0, t]} \|y(s; y_0)\|_{\mathcal{H}^1}^2 \right] \leq 4\|\phi\|_{\infty} \frac{C_{t,N,f,H,m}}{R^2}, \end{aligned}$$

where we have used the *a priori* estimates of (1.3).

The same estimate applies to the third term, so we can find $R > m$ sufficiently large such that for $y_0, y'_0 \in \mathbb{B}_m$ we have

$$(4.3) \quad \mathbb{E} [|\phi(y(t; y_0)) - \phi(y(t \wedge \tau_R; y_0))|] \leq \varepsilon,$$

$$(4.4) \quad \mathbb{E} [|\phi(y(t \wedge \tau_R; y'_0)) - \phi(y(t; y'_0))|] \leq \varepsilon.$$

For the remaining term, we note that since ϕ is uniformly continuous on $\mathbb{B}_R \supset \mathbb{B}_m$, we can choose $\eta > 0$ sufficiently small such that for all $y, y' \in \mathbb{B}_R$ satisfying $\|y - y'\|_{\mathcal{H}^1} \leq \eta$, we find

$$|\phi(y) - \phi(y')| \leq \varepsilon.$$

Thus, for every $y_0, y'_0 \in \mathbb{B}_m$ with $\|y_0 - y'_0\|_{\mathcal{H}^1} \leq \delta := \frac{\sqrt{\varepsilon}\eta}{\sqrt{2\|\phi\|_\infty C_{t,R,N,f,H,\Sigma}}}$ – here, $C_{t,R,N,f,H,\Sigma}$ denotes the constant from Lemma 4.1 – by applying said lemma, we have

$$(4.5) \quad \begin{aligned} & \mathbb{E} [|\phi(y(t \wedge \tau_R; y_0)) - \phi(y(t \wedge \tau_R; y'_0))|] \\ &= \mathbb{E} [1_{\{\|y(t \wedge \tau_R; y_0) - y(t \wedge \tau_R; y'_0)\|_{\mathcal{H}^1} \leq \eta\}} |\phi(y(t \wedge \tau_R; y_0)) - \phi(y(t \wedge \tau_R; y'_0))|] \\ & \quad + \mathbb{E} [1_{\{\|y(t \wedge \tau_R; y_0) - y(t \wedge \tau_R; y'_0)\|_{\mathcal{H}^1} > \eta\}} |\phi(y(t \wedge \tau_R; y_0)) - \phi(y(t \wedge \tau_R; y'_0))|] \\ &\leq \varepsilon + 2\|\phi\|_\infty P \{ \|y(t \wedge \tau_R; y_0) - y(t \wedge \tau_R; y'_0)\|_{\mathcal{H}^1} > \eta \} \\ &\leq \varepsilon + \frac{2\|\phi\|_\infty}{\eta^2} \mathbb{E} [\|y(t \wedge \tau_R; y_0) - y(t \wedge \tau_R; y'_0)\|_{\mathcal{H}^1}^2] \\ &\leq \varepsilon + \frac{2\|\phi\|_\infty C_{t,R,N,f,H,\Sigma}}{\eta^2} \|y_0 - y'_0\|_{\mathcal{H}^1}^2 \leq 2\varepsilon. \end{aligned}$$

The claim now follows from combining Equations (4.2)–(4.5). \square

In the periodic case, we can show existence of an invariant measure for our equations:

THEOREM 4.3 (Invariant measures in the periodic case). *Under the hypotheses (H1), (H2), (H3)', in the periodic case $\mathbb{D} = \mathbb{T}^3$, there exists an invariant measure $\mu \in \mathcal{P}(\mathcal{H}^1)$ associated to $(T_t)_{t \geq 0}$ such that for every $t \geq 0$, $\phi \in BC_{loc}(\mathcal{H}^1)$*

$$\int_{\mathcal{H}^1} T_t \phi(y_0) d\mu(y_0) = \int_{\mathcal{H}^1} \phi(y_0) d\mu(y_0).$$

PROOF. By Itô's formula in \mathcal{H}^0 , we have

$$\begin{aligned} \mathbb{E} [\|y(t)\|_{\mathcal{H}^0}^2] &= \|y(0)\|_{\mathcal{H}^0}^2 + 2 \int_0^t \mathbb{E} [\langle \mathcal{A}(y(s)), y(s) \rangle_{\mathcal{H}^0}] ds \\ & \quad + 2 \int_0^t \mathbb{E} [\langle f(y(s)), y(s) \rangle_{\mathcal{H}^0}] ds + \int_0^t \mathbb{E} [\|\mathcal{B}(y(s))\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^0)}^2] ds. \end{aligned}$$

By Equation (2.7), Young's inequality, Equation (2.12) as well as Assumption (H1), this can be estimated as

$$\begin{aligned}
\mathbb{E} [\|y(t)\|_{\mathcal{H}^0}^2] &\leq \|y(0)\|_{\mathcal{H}^0}^2 - 2 \int_0^t \mathbb{E} [\|\nabla y(s)\|_{\mathcal{H}^0}^2 + \|y(s)\|_{\mathcal{L}^4}^4 - CN\|y(s)\|_{\mathcal{H}^0}^2] \, ds \\
&\quad + \int_0^t \mathbb{E} [\|f(s, y(s))\|_{\mathcal{H}^0}^2 + \|y(s)\|_{\mathcal{H}^0}^2] \, ds \\
&\quad + \int_0^t \mathbb{E} \left[\frac{1}{2} \|y(s)\|_{\mathcal{H}^1}^2 + C_H \|y\|_{\mathcal{H}^0}^2 + \|F_H\|_{L^1(\mathbb{D})} \right] \, ds \\
&\leq \|y(0)\|_{\mathcal{H}^0}^2 - \frac{3}{2} \int_0^t \mathbb{E} [\|y(s)\|_{\mathcal{H}^1}^2] \, ds - 2 \int_0^t \mathbb{E} [\|y(s)\|_{\mathcal{L}^4}^4] \, ds \\
&\quad + C_{f,H,N} \int_0^t \mathbb{E} [\|y(s)\|_{\mathcal{H}^0}^2] \, ds + C_{f,H} \cdot t.
\end{aligned}$$

Since we are in the case of bounded domains, we have by the embedding $\mathcal{L}^4(\mathbb{T}^3) \subset \mathcal{L}^2(\mathbb{T}^3)$ and Young's inequality

$$\|y\|_{\mathcal{H}^0}^2 \leq C_{\mathbb{D}} \|y\|_{\mathcal{L}^4}^2 \leq \varepsilon \|y\|_{\mathcal{L}^4}^4 + C_{\varepsilon, \mathbb{D}}.$$

and hence for sufficiently small $\varepsilon = \varepsilon_{f,N,H} > 0$, after rearranging

$$\begin{aligned}
(4.6) \quad \mathbb{E} [\|y(t)\|_{\mathcal{H}^0}^2] &+ \int_0^t \mathbb{E} [\|y(s)\|_{\mathcal{H}^1}^2] \, ds + \int_0^t \mathbb{E} [\|y(s)\|_{\mathcal{L}^4}^4] \, ds \\
&\leq C_{f,H,N,\mathbb{D}} (\|y(0)\|_{\mathcal{H}^0}^2 + t).
\end{aligned}$$

Now using the \mathcal{H}^1 -Itô formula and the corresponding higher-order estimates (2.8), (2.13), we find by proceeding in the same way as in the above calculations and additionally using (4.6) that

$$\begin{aligned}
\mathbb{E} [\|y(t)\|_{\mathcal{H}^1}^2] &= \|y(0)\|_{\mathcal{H}^1}^2 + 2 \int_0^t \mathbb{E} [\langle \mathcal{A}(y(s)), y(s) \rangle_{\mathcal{H}^1}] \, ds \\
&\quad + 2 \int_0^t \mathbb{E} [\langle f(y(s)), y(s) \rangle_{\mathcal{H}^1}] \, ds + \int_0^t \mathbb{E} [\|\mathcal{B}(y(s))\|_{L_2(\ell^2 \times \ell^2; \mathcal{H}^1)}^2] \, ds \\
&\leq -\frac{1}{2} \int_0^t \mathbb{E} [\|y(s)\|_{\mathcal{H}^2}^2] \, ds + C_{f,H,N} \int_0^t \mathbb{E} [\|y(s)\|_{\mathcal{H}^1}^2] \, ds + C_{f,H,N} \cdot t \\
&\leq -\frac{1}{2} \int_0^t \mathbb{E} [\|y(s)\|_{\mathcal{H}^2}^2] \, ds + C_{f,H,N,\mathbb{D}} \cdot t,
\end{aligned}$$

which implies that

$$(4.7) \quad \frac{1}{t} \int_0^t \mathbb{E} [\|y(s)\|_{\mathcal{H}^2}^2] ds \leq C_{f,H,N,\mathbb{D}}.$$

Note that this inequality is uniform in t and in the initial condition y_0 .

We now conclude as follows: following the notation of G. Da Prato and J. Zabczyk [49], we define the kernels

$$R_T(y_0, \Gamma) := \frac{1}{T} \int_0^T (T_s 1_\Gamma)(y_0) dt, \quad y_0 \in \mathcal{H}^1, \Gamma \in \mathcal{B}(\mathcal{H}^1).$$

Set $E = \mathcal{H}^1$ and let $\nu \in \mathcal{P}(\mathcal{H}^1)$ be an arbitrary probability measure. Define the associated measure

$$R_T^* \nu(\Gamma) = \int_E R_T(y_0, \Gamma) \nu(dy_0).$$

We will show that these measures are tight. To this end, consider the sets $\Gamma_r := \{y_0 \in \mathcal{H}^2 \mid \|y_0\|_{\mathcal{H}^2} \leq r\} \subset \mathcal{H}^2 \subset \mathcal{H}^1$. Since they are bounded subsets of \mathcal{H}^2 and the embedding $\mathcal{H}^2 \subset \mathcal{H}^1$ is compact, they are compact subsets of \mathcal{H}^1 . Now, using (4.7), we show that the measures $R_T^* \nu$ are concentrated on Γ_r for sufficiently large $r > 0$, hence tight: denoting the complement of a set $A \subset E$ by A^c and using Chebychev's inequality, we find

$$\begin{aligned} R_T^* \nu(\Gamma_r^c) &= \int_E R_T(y_0, \Gamma_r^c) \nu(dy_0) = \int_E \frac{1}{T} \int_0^T \mathbb{E} [1_{\Gamma_r^c}(y(t; y_0))] dt \nu(dy_0) \\ &\leq \int_E \frac{1}{T} \int_0^T \mathbb{E} \left[\frac{\|y(t; y_0)\|_{\mathcal{H}^2}^2}{r^2} \right] dt \nu(dy_0) = \frac{1}{r^2} \int_E \frac{1}{T} \int_0^T \mathbb{E} [\|y(t; y_0)\|_{\mathcal{H}^2}^2] dt \nu(dy_0) \\ &\leq \frac{1}{r^2} \int_E C_{f,H,N,\mathbb{D}} \nu(dy_0) = \frac{C_{f,H,N,\mathbb{D}}}{r^2}. \end{aligned}$$

Therefore, the Krylov-Bogoliubov theorem [49, Corollary 3.1.2, p. 22] ensures the existence of an invariant measure. \square

REMARK 4.4. *Note that Z. Brzeźniak and G. Dhariwal [23] prove existence of an invariant measure for a similar system even in the case of the full space, so there might be hope to extend this theorem also in our case.*

Dynamical Systems and Random Attractors

ABSTRACT. In this chapter, we prove the existence of random dynamical systems and random attractors for a large class of locally monotone stochastic partial differential equations perturbed by additive Lévy noise. The main result is applicable to various types of SPDEs such as stochastic Burgers-type equations, stochastic 2D Navier-Stokes equations, the stochastic 3D Leray- α model, stochastic power law fluids, the stochastic Ladyzhenskaya model, stochastic Cahn-Hilliard-type equations, stochastic Kuramoto-Sivashinsky-type equations, stochastic porous media equations and stochastic p -Laplace equations.

1. Introduction

Since the foundational work in [45, 47, 200] the long-time behaviour of SPDEs in terms of the existence of random attractors has been extensively investigated (cf. e.g. [17, 31, 71, 80, 94–97, 99, 105, 106, 138, 156, 158, 202, 221, 245, 255, 256, 263]), resulting in an ever increasing list of specific SPDEs for which the existence of a random attractor has been verified. While the proofs rely on common ideas, the field yet lacks a general, unifying framework overcoming the case-by-case verification. The main aim of this work is to further push in the direction of such a unifying framework by providing a general, abstract result on the existence of random attractors for locally monotone SPDEs.

More precisely, we prove the existence of random dynamical systems and random attractors for SPDEs of the form

$$(1.1) \quad dX_t = A(X_t)dt + dN_t,$$

where N_t is a Lévy type noise satisfying a moment condition (N) and A is locally monotone (cf. (A2) below) with respect to a Gelfand triple $V \subseteq H \subseteq V^*$. The abstract framework introduced here relies on the concept of *locally* monotone operators. This extends previously available results, which were restricted to monotone operators, and constitutes important progress in so far that, in contrast to the monotone framework, it includes SPDEs arising in fluid dynamics as particular examples. Indeed, the generality of this framework is demonstrated by application to a large class of SPDEs, including, stochastic reaction-diffusion equations, stochastic Burgers-type equations, stochastic 2D Navier-Stokes equations, the stochastic Leray- α model, stochastic power law fluids, the stochastic Ladyzhenskaya model, stochastic Cahn-Hilliard-type equations as well as stochastic Kuramoto-Sivashinsky-type equations. This recovers results from the literature as simple applications of the abstract framework introduced here and generalises many known results. In particular, we generalise the results given in [17, 94, 97]. We refer to Section 6 for more details.

The first main result, stated in detail in Theorem 4.1 below, addresses the existence of random dynamical systems associated to (1.1).

THEOREM 1.1 (Theorem 4.1 below). *Assume that A is hemicontinuous, locally monotone, coercive and satisfies a growth condition. Further assume that $V \subseteq H$ is compact and that there exists a hemicontinuous, strictly monotone operator $M : V \rightarrow V^*$ satisfying a growth condition. Then there is a continuous random dynamical system S generated by solutions to (1.1).*

Under a slightly stronger coercivity condition we then prove the existence of a random attractor, leading to the second main result.

THEOREM 1.2 (Theorem 5.1 below). *Assume that A is hemicontinuous, locally monotone, coercive and satisfies a growth condition. Further assume that $V \subseteq H$ is compact and that there exists a hemicontinuous, strictly monotone operator $M : V \rightarrow V^*$ satisfying a growth condition. Then the random dynamical system S is compact and there is a random attractor for S .*

Notably, the Lévy process N_t in (1.1) is only assumed to take values in H which is the natural choice of noise as far as trace-class noise is considered. This is in contrast to a number of works where the noise was assumed to take values in the domain of the operator A , in order to make sense of the transformed equation for $\tilde{Z}_t := X_t - N_t$ which has the form

$$d\tilde{Z}_t = A(\tilde{Z}_t + N_t)dt.$$

It was later noticed in [94] that this assumption can be relaxed to $N_t \in H$ by not subtracting the noise directly, but a form of *nonlinear* Ornstein-Uhlenbeck process instead. More precisely, if the operator A possesses a strongly monotone part M , we construct in Theorem 3.1 a strictly stationary solution u_t of the equation (for sufficiently large $\sigma > 0$)

$$du_t = \sigma M(u_t)dt + dN_t.$$

Here, the smoothing properties of the operator M guarantee that u_t takes values in the space V . This allows us to prove the existence of a random dynamical system, assuming only trace-class noise in H .

The existence of a random attractor is typically proven in two steps: In the first step, uniform bounds on the H -norm of the flow are established, which means that there exists a bounded attracting set. In the second step, the existence of a compact attracting set is shown. In this work, we use the compactness of the embedding $V \subseteq H$ to prove that the cocycle S is compact, which together with the first step implies the existence of a compact attracting set. Notably, the approach introduced here only relies on the standard coercivity assumption of the variational approach to SPDE. This avoids further assumptions typically required in the literature in order to prove higher regularity of solutions. In particular, this avoids to pose stronger regularity assumptions on the noise.

From a technical perspective, we provide general, by now standard and relatively easy to check conditions that guarantee that a given SPDE generates a random dynamical system with an associated random attractor. We allow for additive, trace-class Lévy noise which results in more involved estimates, e.g. when checking exponential integrability properties of the strictly stationary Ornstein-Uhlenbeck process. The step of transforming the SPDE into a random PDE compared to that step of [94] employs a more general deterministic result, namely the well-posedness of deterministic PDE with locally monotone coefficients as proved by W. Liu and M. Röckner in [165]. In the examples, we tried to find optimal conditions on the parameters within the limitations of the framework, especially the condition $\beta(\alpha - 1) \leq 2$ using interpolation inequalities, which was not necessary in the monotone case, as there $\beta = 0$. This requires a more careful analysis of these examples.

1.1. Literature. We now give a brief account on the available literature on random attractors for SPDE. Since this is a very active research field, this attempt has to remain incomplete and we restrict ourselves to those works which appear most relevant to the results of this work.

Random attractors were first studied in [45, 47, 200]. They are a very important concept of capturing the long-time behaviour of random dynamical systems (RDS) and there are many results on existence and properties of random attractors for various SPDEs [31, 71, 80, 96, 99, 105, 106, 138, 156, 158, 202, 221, 245, 255, 256, 263].

Equivalent conditions for the existence of random attractors were given in [46]. Further properties of random attractors that have been studied include measurability [45, 47, 48], upper-semicontinuity [35, 36, 154, 220, 242], regularity [104, 155, 157], and dimension estimates [55, 143, 259]. The problem of unbounded domains has also been addressed, e.g. in [12, 25, 136, 168, 219]. Furthermore, the concept of a weak random attractor has been introduced recently in [222, 223]. Further references are given in the discussion of the examples in Section 6.

Stochastic (partial) differential equations driven by Lévy noise have been studied widely, motivated among other things by applications in finance, statistical mechanics, fluid dynamics. For an overview we refer to [189]. For results on random attractors, cf. [97] and the references therein. Well-posedness for locally monotone SPDEs driven by Lévy noise was first studied by Z. Brzeźniak, W. Liu and J.H. Zhu [26].

1.2. Overview. In Section 2 we state the assumptions on the coefficients and the noise N . In Section 3 we study strictly stationary solutions for strongly monotone SPDEs. The following section, Section 4, is devoted to constructing a stochastic flow via transformation of equation (2.1) into a random PDE. This stochastic flow is then proven to be compact in Section 5. Combining this with the existence of a random bounded absorbing set then immediately imply the existence of a random attractor. Applications to various SPDEs are given in Section 6. Appendix 7 gathers the necessary results on random PDEs with locally monotone coefficients. In Appendix C we recall the basic notions and results concerning stochastic flows, random dynamical systems and random attractors.

Within the project, the authors role was to apply the theory to examples, i.e. write Section 6, as well as the introduction and to simplify the proofs wherever possible. I have also provided a proof of the perfection theorem in Appendix C, Theorem C.12, which was needed for proving the existence of the nonlinear Ornstein-Uhlenbeck process.

2. Main Framework

Let $(H, \langle \cdot, \cdot \rangle_H)$ be a real separable Hilbert space, identified with its dual space H^* by the Riesz isomorphism. Let V be a real reflexive Banach space continuously and densely embedded into H . In particular, there is a constant $\lambda > 0$ such that $\lambda \|v\|_H^2 \leq \|v\|_V^2$ for all $v \in V$. Then we have the following Gelfand triple

$$V \subseteq H \equiv H^* \subseteq V^*.$$

If ${}_{V^*}\langle \cdot, \cdot \rangle_V$ denotes the dualization between V and its dual space V^* , then

$${}_{V^*}\langle u, v \rangle_V = \langle u, v \rangle_H, \quad \forall u \in H, v \in V.$$

As mentioned in the introduction, we consider SPDEs of the form

$$(2.1) \quad dX_t = A(X_t)dt + dN_t,$$

where $A : V \rightarrow V^*$ is $\mathcal{B}(V)/\mathcal{B}(V^*)$ -measurable (we extend A by 0 to H) and $N : \mathbb{R} \times \Omega \rightarrow H$ is a centered, two-sided Lévy process on H .

We assume that N is given by its canonical realization on $\Omega := D(\mathbb{R}; H)$, the space of all càdlàg paths in H endowed with the canonical filtration

$$\mathcal{F}_s^t = \sigma(\omega(u) - \omega(v) | \omega \in \Omega, s \leq u, v \leq t)$$

and Wiener shifts $\{\theta_t\}_{t \in \mathbb{R}}$ (cf. e.g. [7, Appendix A.3], [6, Section 1.4.1]). We impose some moment condition on N which will be specified below. Let \mathbb{P} be the law of N on Ω . Then $(\Omega, \mathcal{F}, \{\mathcal{F}_s^t\}_{t \in [s, \infty)}, \{\theta_t\}_{t \in \mathbb{R}}, \mathbb{P})$ is an ergodic metric dynamical system. We denote the augmented filtration by $\{\bar{\mathcal{F}}_s^t\}_{t \in [s, \infty)}$ and note that $\{\bar{\mathcal{F}}_s^t\}_{t \in [s, \infty)}$ is right-continuous. The extension of \mathbb{P} to $\bar{\mathcal{F}}$ is denoted by $\bar{\mathbb{P}}$ and we define $\bar{\mathcal{F}}_{-\infty}^t := \sigma\left(\bigcup_{-\infty < s \leq t} \bar{\mathcal{F}}_s^t\right)$.

Suppose that for some $\alpha \geq 2$ and $\beta \geq 0$ with $\beta(\alpha - 1) \leq 2$, there exist constants $C, K \geq 0$ and $\gamma > 0$ such that the following conditions hold for all $v, v_1, v_2 \in V$:

(A1) (Hemicontinuity) The map $s \mapsto v^* \langle A(v_1 + sv_2), v \rangle_V$ is continuous on \mathbb{R} .

(A2) (Local monotonicity)

$$2v^* \langle A(v_1) - A(v_2), v_1 - v_2 \rangle_V \leq (C + \eta(v_1) + \rho(v_2)) \|v_1 - v_2\|_H^2,$$

where $\eta, \rho : V \rightarrow \mathbb{R}_+$ are locally bounded measurable functions.

(A3) (Coercivity)

$$2v^* \langle A(v), v \rangle_V \leq -\gamma \|v\|_V^\alpha + K \|v\|_H^2 + C.$$

(A4) (Growth)

$$\|A(v)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \leq C(1 + \|v\|_V^\alpha)(1 + \|v\|_H^\beta).$$

In order to be able to deduce the existence and uniqueness of solutions from the results derived in [26], we note that due to the Lévy-Itô decomposition (cf. e.g. [1, Theorem 4.1]), and since ν is assumed to have first moment, we have \mathbb{P} -a.s.

$$(2.2) \quad N_t = mt + W_t + \int_H z \tilde{N}(t, dz), \quad \forall t \in \mathbb{R},$$

where $m \in H$, W_t is a trace-class Q -Wiener process on H and \tilde{N} is a compensated Poisson random measure on H with intensity measure ν (cf. [1] for Definitions). Now we state the assumptions on the Lévy noise as follows:

(N) The process $(N_t)_{t \in \mathbb{R}}$ is a two-sided Lévy process with values in H and the corresponding Lévy measure has finite moments up to order 4. Furthermore, without loss of generality, we assume $m = 0$.

Throughout this chapter we work with the convention that $C, \tilde{C} \geq 0$ and $c, \tilde{c} > 0$ are generic constants, each of which is not important for its specific value and allowed to change from line to line.

Let us now define what we mean by a solution to (2.1).

DEFINITION 2.1. *A càdlàg, H -valued, $\{\bar{\mathcal{F}}_s^t\}_{t \in [s, \infty)}$ -adapted process $\{S(t, s; \omega)x\}_{t \in [s, \infty)}$ is a solution to (2.1) with initial condition x at time s if for \mathbb{P} -a.a. $\omega \in \Omega$, $S(\cdot, s; \omega)x \in L_{loc}^\alpha([s, \infty); V)$ and*

$$S(t, s; \omega)x = x + \int_s^t A(S(r, s; \omega)x) dr + N_t(\omega) - N_s(\omega), \quad \forall t \geq s.$$

3. Strictly Stationary Solutions for Monotone SPDE

The construction of stochastic flows for locally monotone SPDEs driven by Lévy noise (presented in Section 4 below) is based on strictly stationary solutions for strongly monotone SPDE driven by Lévy noise. The existence and uniqueness of such strictly stationary solutions is proven in this section, which might be of independent interest. This generalises a similar construction presented in [94] for the case of trace-class Wiener noise.

More precisely, in this section we consider strongly monotone SPDE of the form

$$(3.1) \quad dX_t = \sigma M(X_t)dt + dN_t,$$

where $\sigma > 0$, N_t is a two-sided Lévy process (as above) and $M : V \rightarrow V^*$ is measurable. Instead of the local monotonicity condition (A2), we assume that M is strongly monotone, i.e.

(A2') (Strong monotonicity) There exists a constant $c > 0$ such that

$$2_{V^*} \langle M(v_1) - M(v_2), v_1 - v_2 \rangle_V \leq -c \|v_1 - v_2\|_V^\alpha, \quad \forall v_1, v_2 \in V,$$

where α is the same constant as in (A3).

It is easy to see that (A2') implies that (A3) also holds for M .

By the above Lévy-Itô decomposition (2.2), we may rewrite (3.1) as

$$(3.2) \quad dX_t = \sigma M(X_t)dt + dW_t + \int_H z \theta dN(dt, dz)$$

and [26, Theorem 1.2] implies the existence and uniqueness of an $\bar{\mathcal{F}}_s^t$ -adapted variational solution $X(t, s; \omega)x$ for each $x \in H$. Strictly stationary solutions to (3.1) are constructed by letting $s \rightarrow -\infty$ in $X(t, s; \omega)x$ and then selecting a strictly stationary version u from the resulting stationary limit process using Proposition C.12 in Appendix B.

THEOREM 3.1 (Strictly stationary solutions). *Suppose that M satisfies (A1), (A2'), (A4) with $\beta = 0$ and let $X(\cdot, s; \omega)x$ be the solution to (3.2) starting in $x \in H$ at time s . Then*

(i) *There exists an $\bar{\mathcal{F}}_{-\infty}^t$ -adapted, \mathcal{F} -measurable process*

$$u \in L^2(\Omega; D(\mathbb{R}; H)) \cap L^\alpha(\Omega; L_{loc}^\alpha(\mathbb{R}; V))$$

such that

$$\lim_{s \rightarrow -\infty} X(t, s; \cdot)x = u_t$$

in $L^2(\Omega; H)$ for all $t \in \mathbb{R}$, $x \in H$.

(ii) *u solves (3.1) in the following sense:*

$$(3.3) \quad u_t = u_s + \sigma \int_s^t M(u_r)dr + N_t - N_s, \quad \mathbb{P}\text{-a.s.}, \quad t \geq s.$$

(iii) *u can be chosen to be strictly stationary with càdlàg paths and satisfying $u(\omega) \in L_{loc}^\alpha(\mathbb{R}; V)$, for all $\omega \in \Omega$.*

(iv) *Let $2 \leq p \leq 4$, then for each $\delta \geq 0$, $t \in \mathbb{R}$ and large enough $\sigma > \frac{8\delta}{pc\lambda}$, there is a constant $C(\delta, \sigma) > 0$ such that*

$$(3.4) \quad \mathbb{E} \int_{-\infty}^t e^{\delta r} \|u_r\|_V^\alpha \|u_r\|_H^{p-2} dr \leq C(\delta, \sigma) e^{\delta t},$$

where $C(\delta, \sigma) \rightarrow 0$ for $\sigma \rightarrow \infty$.

(v) *There exists a θ -invariant set $\Omega_0 \subseteq \Omega$ of full \mathbb{P} -measure such that for $\omega \in \Omega_0$ and $s, t \in \mathbb{R}$, $s < t$,*

$$\frac{1}{t-s} \int_s^t \|u_r(\omega)\|_V^\alpha dr \rightarrow \mathbb{E}\|u_0\|_V^\alpha \leq C(\sigma), \quad s \rightarrow -\infty,$$

where $C(\sigma) \rightarrow 0$ for $\sigma \rightarrow \infty$.

Let $p \in \mathbb{N}$, $p \geq 2$, then

(vi) *There exists a θ -invariant set $\Omega_0 \subseteq \Omega$ of full \mathbb{P} -measure such that for $\omega \in \Omega_0$*

$$(3.5) \quad \frac{1}{t} \int_0^t \|u_r(\omega)\|_H^p dr \rightarrow \mathbb{E}\|u_0\|_H^p, \quad t \rightarrow \pm\infty.$$

(vii) $\|u_t(\omega)\|_H^p$ has sublinear growth, i.e.

$$\lim_{t \rightarrow \pm\infty} \frac{\|u_t(\omega)\|_H^p}{|t|} = 0.$$

PROOF. As the operator in (3.2) is strongly monotone, some parts of the proof here are similar to the associated statements in [94]. So here we only highlight the differences arising from allowing Lévy noise and otherwise refer to [94].

Let $X(t, s; \omega)x$ denote the variational solution to (3.1) starting at time s in $x \in H$ (cf. [26]).

(i) First we show that there is an $\bar{\mathcal{F}}_{-\infty}^t$ -adapted, \mathcal{F} -measurable process $u : \mathbb{R} \times \Omega \rightarrow H$ such that

$$\lim_{s \rightarrow -\infty} X(t, s; \cdot)x = u_t,$$

in $L^2(\Omega; H)$ for each $t \in \mathbb{R}$, independent of $x \in H$.

Following the same line of argument as in [94, p. 143], using the coercivity, Itô's formula and the comparison lemma [94, Lemma 5.1] for $\alpha > 2$ or Gronwall's lemma for $\alpha = 2$, respectively, we obtain that for all $t \wedge 0 \geq s_2$

$$\begin{aligned} & \mathbb{E} \sup_{r \in [t, \infty)} \|X(r, s_2; \cdot)x - X(r, s_1; \cdot)y\|_H^2 \\ & \leq \begin{cases} \left(\left(\frac{\alpha}{2} - 1 \right) c\sigma \lambda^{\frac{\alpha}{2}} (t - s_2) \right)^{-\frac{2}{\alpha-2}}, & \text{if } \alpha > 2; \\ 2 \left(e^{\frac{c\sigma\lambda}{2}s_1} \|y\|_H^2 + e^{\frac{c\sigma\lambda}{2}s_2} \|x\|_H^2 + C \right) e^{\frac{c\sigma\lambda}{2}s_2} e^{-c\sigma\lambda t}, & \text{if } \alpha = 2. \end{cases} \end{aligned}$$

Hence, $X(\cdot, s; \cdot)x$ is a Cauchy sequence in $L^2(\Omega; D([t, \infty); H))$ and

$$u_t := \lim_{s \rightarrow -\infty} X(t, s; \cdot)x$$

exists as a limit in $L^2(\Omega; H)$ for all $t \in \mathbb{R}$ and u is $\bar{\mathcal{F}}_{-\infty}^t$ -adapted.

Since $X(\cdot, s; \cdot)x$ also converges in $L^2(\Omega; D([t, \infty); H))$, u is càdlàg \mathbb{P} -almost surely. Since u is $\bar{\mathcal{F}}$ -measurable, we can choose an indistinguishable \mathcal{F} -measurable version of u .

(ii) The next step consists of showing that u solves (3.3).

This is achieved using Itô's formula for $\|\cdot\|_H^2$ (with the only difference being an additional term of $\int_H \|z\|_H^2 \nu(dz)$ on the right-hand side), the compactness of the embedding as well as the monotonicity trick and the hemicontinuity (A1). For details, cf. [94, p.144].

(iii) Now we prove the crude stationarity for u . Let us first show $X(t, s; \omega)x = X(0, s - t; \theta_t \omega)x$ for all $t \geq s$, \mathbb{P} -almost surely.

Let $h > 0, t \geq s$ and define $\bar{X}_h(t)(\omega) := X(t-h, s-h; \theta_h \omega)x$. Then for \mathbb{P} -a.a. $\omega \in \Omega$ (with zero set possibly depending on s, h, x)

$$\begin{aligned} \bar{X}_h(t)(\omega) &= X(t-h, s-h; \theta_h \omega)x \\ &= x + \sigma \int_{s-h}^{t-h} M(X(r, s-h; \theta_h \omega)x)dr + N_{t-h}(\theta_h \omega) - N_{s-h}(\theta_h \omega) \\ &= x + \sigma \int_{s-h}^{t-h} M(X(r, s-h; \theta_h \omega)x)dr + N_t(\omega) - N_s(\omega) \\ &= x + \sigma \int_s^t M(\bar{X}_h(r)(\omega))dr + N_t(\omega) - N_s(\omega). \end{aligned}$$

Hence, by uniqueness, $X(t-h, s-h; \theta_h \omega)x = X(t, s; \omega)x$, \mathbb{P} -almost surely. In particular

$$(3.6) \quad X(0, s-t; \theta_t \omega)x = X(t, s; \omega)x,$$

\mathbb{P} -almost surely (with zero set possibly depending on t, s, x).

Now for an arbitrary sequence $s_n \rightarrow -\infty$ there exists a subsequence (again denoted by s_n) such that $X(t, s_n; \cdot)x \rightarrow u_t$ and $X(0, s_n-t; \cdot)x \rightarrow u_0$ \mathbb{P} -almost surely. Thus passing to the limit in (3.6) gives

$$u_0(\theta_t \omega) = u_t(\omega),$$

\mathbb{P} -almost surely (with zero set possibly depending on t).

Since $u \in L^\alpha(\Omega; L_{\text{loc}}^\alpha(\mathbb{R}; V))$, in particular $u(\omega) \in L_{\text{loc}}^\alpha(\mathbb{R}; V)$ for almost all $\omega \in \Omega$ and since u is \mathcal{F} -measurable, we now use Proposition C.12 in Appendix B to deduce the existence of an indistinguishable, \mathcal{F} -measurable, $\bar{\mathcal{F}}_{-\infty}^t$ -adapted, strictly stationary, càdlàg process \tilde{u} such that $\tilde{u}(\omega) \in L_{\text{loc}}^\alpha(\mathbb{R}; V)$ for all $\omega \in \Omega$, i.e. crude stationarity.

(iv) Next we proceed to prove (3.4). Let $\delta \geq 0$ and note that by (A2') and (A4)

$$2\sigma_{V^*} \langle M(v), v \rangle_V + \text{tr}Q \leq -\frac{c\sigma}{2} \|v\|_V^\alpha + C, \quad \forall v \in V.$$

An application of Itô's formula and the product rule yields that

$$\begin{aligned} e^{\delta t_2} \|u_{t_2}\|_H^p &= e^{\delta t_1} \|u_{t_1}\|_H^p \\ &+ \frac{p}{2} \int_{t_1}^{t_2} e^{\delta r} \|u_r\|_H^{p-2} (2\sigma_{V^*} \langle M(u_r), u_r \rangle_V + \text{tr}Q) dr \\ &+ p \int_{t_1}^{t_2} e^{\delta r} \|u_r\|_H^{p-2} \langle u_r, dW_r \rangle_H + p\left(\frac{p}{2} - 1\right) \int_{t_1}^{t_2} e^{\delta r} \|u_r\|_H^{p-4} \|Q^{\frac{1}{2}} u_r\|_H^2 dr \\ &+ p \int_{t_1}^{t_2} \int_H e^{\delta r} \|u_r\|_H^{p-2} \langle u_r, z \rangle_H \tilde{N}(dr, dz) \\ &+ \int_{t_1}^{t_2} \int_H e^{\delta r} (\|u_r + z\|_H^p - \|u_r\|_H^p - p\|u_r\|_H^{p-2} \langle u_r, z \rangle_H) N(dr, dz) \\ &+ \delta \int_{t_1}^{t_2} e^{\delta r} \|u_r\|_H^p dr. \end{aligned}$$

Therefore, by (A3)

$$\begin{aligned}
\mathbb{E}e^{\delta t_2} \|u_{t_2}\|_H^p &\leq \mathbb{E}e^{\delta t_1} \|u_{t_1}\|_H^p \\
&+ \frac{p}{2} \mathbb{E} \int_{t_1}^{t_2} e^{\delta r} \|u_r\|_H^{p-2} \left(-\frac{c\sigma}{2} \|u_r\|_V^\alpha + C \right) dr \\
&+ p \left(\frac{p}{2} - 1 \right) \mathbb{E} \int_{t_1}^{t_2} e^{\delta r} \|u_r\|_H^{p-4} \|Q^{\frac{1}{2}} u_r\|_H^2 dr \\
&+ \mathbb{E} \int_{t_1}^{t_2} \int_H e^{\delta r} \left(\|u_r + z\|_H^p - \|u_r\|_H^p - p \|u_r\|_H^{p-2} \langle u_r, z \rangle_H \right) N(dr, dz) \\
&+ \delta \mathbb{E} \int_{t_1}^{t_2} e^{\delta r} \|u_r\|_H^p dr.
\end{aligned}$$

Noting that

$$| \|x + h\|_H^p - \|x\|_H^p - p \|x\|_H^{p-2} \langle x, h \rangle_H | \leq C_p (\|x\|_H^{p-2} \|h\|_H^2 + \|h\|^p),$$

we obtain by using the moment Assumption (N) that

$$\begin{aligned}
&\mathbb{E} \int_{t_1}^{t_2} \int_H e^{\delta r} \left(\|u_r + z\|_H^p - \|u_r\|_H^p - p \|u_r\|_H^{p-2} \langle u_r, z \rangle_H \right) N(dr, dz) \\
&= \mathbb{E} \int_{t_1}^{t_2} \int_H e^{\delta r} \left(\|u_r + z\|_H^p - \|u_r\|_H^p - p \|u_r\|_H^{p-2} \langle u_r, z \rangle_H \right) \nu(dz) dr \\
&\leq C \mathbb{E} \int_{t_1}^{t_2} \int_H e^{\delta r} \left(\|u_r\|_H^{p-2} \|z\|_H^2 + \|z\|_H^p \right) \nu(dz) dr \\
&\leq C \left(\mathbb{E} \int_{t_1}^{t_2} e^{\delta r} \left(\|u_r\|_H^{p-2} + 1 \right) dr \right),
\end{aligned}$$

and thus

$$\begin{aligned}
\mathbb{E}e^{\delta t_2} \|u_{t_2}\|_H^p &\leq \mathbb{E}e^{\delta t_1} \|u_{t_1}\|_H^p - \frac{pc\sigma}{4} \mathbb{E} \int_{t_1}^{t_2} e^{\delta r} \|u_r\|_H^{p-2} \|u_r\|_V^\alpha dr \\
&+ \left(p \left(\frac{p}{2} - 1 \right) \text{tr} Q + C \right) \mathbb{E} \int_{t_1}^{t_2} e^{\delta r} \|u_r\|_H^{p-2} dr \\
&+ \delta \mathbb{E} \int_{t_1}^{t_2} e^{\delta r} \|u_r\|_H^p dr + C \int_{t_1}^{t_2} e^{\delta r} dr.
\end{aligned}$$

Applying Young's inequality and the embedding $V \subset H$, we get

$$\begin{aligned}
e^{\delta t_2} \mathbb{E} \|u_{t_2}\|_H^p &\leq e^{\delta t_1} \mathbb{E} \|u_{t_1}\|_H^p - \left(\frac{pc\sigma}{4} - 2\delta\lambda^{-1} \right) \mathbb{E} \int_{t_1}^{t_2} e^{\delta r} \|u_r\|_H^{p-2} \|u_r\|_V^\alpha dr \\
&+ C \int_{t_1}^{t_2} e^{\delta r} dr.
\end{aligned}$$

Stationarity of u_t implies

$$(3.7) \quad \left(\frac{pc\sigma}{4} - 2\delta\lambda^{-1} \right) \mathbb{E} \int_{t_1}^{t_2} e^{\delta r} \|u_r\|_H^{p-2} \|u_r\|_V^\alpha dr \leq C \int_{t_1}^{t_2} e^{\delta r} dr$$

and thus (3.4) holds, provided σ is sufficiently large that $\frac{pc\sigma}{4} - 2\delta\lambda^{-1} > 0$.

(v) Applying (3.7) for $\delta = 0$ and $p = 2$ yields

$$\mathbb{E} \int_{t_1}^{t_2} \|u_r\|_V^\alpha dr \leq C(t_2 - t_1).$$

Since u_t is stationary, we have $\mathbb{E}\|u_r\|_V^\alpha = \mathbb{E}\|u_0\|_V^\alpha$. Hence,

$$\mathbb{E}\|u_0\|_V^\alpha \leq C < \infty,$$

and Birkhoff's ergodic theorem implies the claimed convergence.

(vi) The convergence (3.5) follows exactly as in [94, Proof of Theorem 3.3 (i), 146 f.] from the stationarity and Birkhoff's ergodic theorem as well as an application of Itô's formula and the *a priori* bounds arising from [94, Lemma 5.2] in the case $\alpha > 2$ and Gronwall's lemma for $\alpha = 2$, respectively.

(vii) This is proven by invoking the dichotomy of linear growth (cf. [7, Proposition 4.1.3]) in the same way as in [94, Proof of Theorem 3.3 (ii), p. 147]. \square

4. Generation of Random Dynamical Systems

In order to construct a stochastic flow associated to (2.1), we aim to transform (2.1) into a random PDE. However, since we only assume that N_t takes values in H we cannot directly subtract the noise. Motivated by [94] we use the transformation based on a strongly stationary solution to the strictly monotone part of (2.1). More precisely, we impose the following assumption:

(V) There exists an operator $M : V \rightarrow V^*$ satisfying (A1), (A2') and (A4) with $\beta = 0$.

The motivation behind the Assumption (V) is that M is the strongly monotone part of A in (2.1). For example, for many semilinear SPDE such as stochastic reaction-diffusion equations, stochastic Burgers equations and stochastic 2D Navier-Stokes equations, one can take $M = \Delta$ (standard Laplace operator). For quasilinear SPDE like stochastic porous media equations, stochastic p -Laplace equations or stochastic Cahn-Hilliard type equations one can take $M(v) = \Delta(|v|^{r-1}v)$, $M(v) = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$ and $M(v) = -\Delta^2 v$, respectively (see Section 6 for more concrete examples).

Following the arguments given in [94], for $\sigma > 0$ we may consider the \mathcal{F} -measurable, strictly stationary solution u_t (given by Theorem 3.1) to

$$du_t = \sigma M(u_t)dt + dN_t.$$

The key point is that u takes values in V , while N takes values in H . The operator M is used to construct Ornstein-Uhlenbeck type process corresponding to $dX_t = \sigma M(X_t)dt + dN_t$. If N_t takes values in V (cf. [97]), then this regularizing property is not needed and we can just choose $M = -Id_H$. The condition (V) can be removed in this case.

Let $X(t, s; \omega)x$ denote a variational solution to (2.1) starting in x at time s (the existence and uniqueness of this solution is proved in Theorem 4.1).

Defining $\bar{X}(t, s; \omega)x := X(t, s; \omega)x - u_t(\omega)$ we get

$$\begin{aligned} \langle \bar{X}(t, s; \omega)x, v \rangle_H &= \langle x - u_s, v \rangle_H + \int_s^t v^* \langle A(\bar{X}(r, s; \omega)x + u_r), v \rangle_V dr \\ &\quad - \sigma \int_s^t v^* \langle M(u_r), v \rangle_V dr, \quad v \in V, \mathbb{P}\text{-a.s.} \end{aligned}$$

We have used the following stationary conjugation mapping

$$(4.1) \quad T(t, \omega)y := y - u_t(\omega),$$

and the conjugated process $Z(t, s; \omega)x := T(t, \omega)X(t, s; \omega)T^{-1}(s, \omega)x$ satisfies

$$(4.2) \quad Z(t, s; \omega)x = x + \int_s^t (A(Z(r, s; \omega)x + u_r) - \sigma M(u_r)) dr$$

as an equation in V^* . Let

$$A_\omega(r, v) := \begin{cases} A(v + u_r) - \sigma M(u_r), & \text{if } u_r \in V; \\ A(v), & \text{else,} \end{cases}$$

where for the simplicity of notations we suppressed the ω -dependency of u_r .

Since $u_r(\omega) \in V$ for all $\omega \in \Omega$ and a.a. $r \in \mathbb{R}$, from (4.2) we obtain

$$(4.3) \quad Z(t, s; \omega)x = x + \int_s^t A_\omega(r, Z(r, s; \omega)x) dr.$$

In order to define the associated stochastic flow to (2.1), we first solve (4.3) for each $\omega \in \Omega$ and then set

$$(4.4) \quad S(t, s; \omega)x := T(t, \omega)^{-1}Z(t, s; \omega)T(s, \omega)x.$$

This is done in detail in the proof of Theorem 4.1 below. For this purpose and also in order to subsequently prove the compactness of the stochastic flow, we need to impose the following additional assumption:

(A5) The embedding $V \subseteq H$ is compact.

THEOREM 4.1 (Generation of stochastic flows). *Suppose that (A1)–(A5), (V) are satisfied and there exist non-negative constants C and κ such that*

$$(4.5) \quad \begin{aligned} \eta(v_1 + v_2) &\leq C(\eta(v_1) + \eta(v_2)), \\ \rho(v_1 + v_2) &\leq C(\rho(v_1) + \rho(v_2)), \\ \eta(v) + \rho(v) &\leq C(1 + \|v\|_V^\alpha)(1 + \|v\|_H^\kappa), \quad \forall v_1, v_2, v \in V. \end{aligned}$$

Then we have the following:

- (i) *There is a unique solution $Z(t, s; \omega)$ to (4.3). $Z(t, s; \omega)$ and $S(t, s; \omega)$ (defined in (4.4)) are stationary conjugated continuous RDS in H and $S(t, s; \omega)x$ is a solution of (2.1) in the sense of Definition 2.1.*
- (ii) *The maps $t \mapsto Z(t, s; \omega)x$, $S(t, s; \omega)x$ are càdlàg, $x \mapsto Z(t, s; \omega)x$, $S(t, s; \omega)x$ are continuous locally uniformly in s, t and $s \mapsto Z(t, s; \omega)x$, $S(t, s; \omega)x$ are right-continuous.*

PROOF. (i) We consider (4.3) as an ω -wise random PDE. We use this point of view to define the associated stochastic flow.

In order to obtain the existence and uniqueness of solutions to (4.3) for each fixed $(\omega, s) \in \Omega \times \mathbb{R}$, we need to verify the Assumptions (H1)–(H4) (see Appendix 7) for $(t, v) \mapsto A_\omega(t, v)$. We check (H1)–(H4) for $A_\omega(t, v)$ on each bounded interval $[S, T] \subset \mathbb{R}$ and for each fixed $\omega \in \Omega$. For ease of notations we suppress the ω -dependency of the coefficients in the following calculations.

(H1): Follows immediately from (A1) for A .

(H2): Let $v_1, v_2 \in V$, $\omega \in \Omega$ and $t \in \mathbb{R}$ such that $u_t(\omega) \in V$. Then by (A2) and (4.5) we find

$$\begin{aligned} & 2_{V^*} \langle A_\omega(t, v_1) - A_\omega(t, v_2), v_1 - v_2 \rangle_V \\ &= 2_{V^*} \langle A(v_1 + u_t) - A(v_2 + u_t), (v_1 + u_t) - (v_2 + u_t) \rangle_V \\ &\leq (C + \eta(v_1 + u_t) + \rho(v_2 + u_t)) \|v_1 - v_2\|_H^2 \\ &\leq (C + C\eta(u_t) + C\rho(u_t) + C\eta(v_1) + C\rho(v_2)) \|v_1 - v_2\|_H^2. \end{aligned}$$

Note that by (4.5)

$$\eta(u_t) + \rho(u_t) \leq C(1 + \|u_t\|_V^\alpha)(1 + \|u_t\|_H^\kappa).$$

Since $u_t(\omega) \in L_{\text{loc}}^\alpha(\mathbb{R}; V) \cap L_{\text{loc}}^\infty(\mathbb{R}; H)$, we have

$$f(t) := C + C\eta(u_t) + C\rho(u_t) \in L_{\text{loc}}^1(\mathbb{R}),$$

i.e. (H2) holds for A_ω . For $t \in \mathbb{R}$ such that $u_t(\omega) \notin V$ a similar calculation holds.

(H3): For $v \in V$, $\omega \in \Omega$ and $t \in \mathbb{R}$ such that $u_t(\omega) \in V$, by (A3) we can estimate

$$\begin{aligned} (4.6) \quad & 2_{V^*} \langle A_\omega(t, v), v \rangle_V = 2_{V^*} \langle A(v + u_t), v + u_t \rangle_V \\ & \quad - 2_{V^*} \langle A(v + u_t), u_t \rangle_V - 2\sigma_{V^*} \langle M(u_t), v \rangle_V \\ & \leq K \|v + u_t\|_H^2 - \gamma \|v + u_t\|_V^\alpha + C \\ & \quad + 2 \|A(v + u_t)\|_{V^*} \|u_t\|_V - 2\sigma_{V^*} \langle M(u_t), v \rangle_V. \end{aligned}$$

For any $\varepsilon_1, \varepsilon_2 > 0$, by (A4), the condition $(\alpha - 1)\beta \leq 2$ and Young's inequality there exist constants $C_{\varepsilon_1}, C_{\varepsilon_2}$ such that

$$\begin{aligned} & 2 \|A(v + u_t)\|_{V^*} \|u_t\|_V \\ & \leq C(1 + \|v + u_t\|_V^{\alpha-1}) \left(1 + \|v + u_t\|_H^{\frac{\beta(\alpha-1)}{\alpha}}\right) \|u_t\|_V \\ & \leq \varepsilon_1(1 + \|v + u_t\|_V^\alpha) + C_{\varepsilon_1} \left(1 + \|v + u_t\|_H^{\beta(\alpha-1)}\right) \|u_t\|_V^\alpha \\ & \leq \varepsilon_1 \|v + u_t\|_V^\alpha + C_{\varepsilon_1} \|u_t\|_V^\alpha \|v + u_t\|_H^2 + 2C_{\varepsilon_1} \|u_t\|_V^\alpha + \varepsilon_1 \end{aligned}$$

and

$$\begin{aligned} 2\sigma_{V^*} \langle M(u_t), v \rangle_V & \leq C_{\varepsilon_2} \|M(u_t)\|_{V^*}^{\frac{\alpha}{\alpha-1}} + \varepsilon_2 \|v\|_V^\alpha \\ & \leq C_{\varepsilon_2} (C \|u_t\|_V^\alpha + C) + \varepsilon_2 \|v\|_V^\alpha, \end{aligned}$$

where we recall that M satisfies (A4) with $\beta = 0$.

Combining the above estimates with (4.6) we have

$$\begin{aligned} 2_{V^*} \langle A_\omega(t, v), v \rangle_V & \leq (K + C_{\varepsilon_1} \|u_t\|_V^\alpha) \|v + u_t\|_H^2 - (\gamma - \varepsilon_1) \|v + u_t\|_V^\alpha \\ & \quad + 2C_{\varepsilon_1} \|u_t\|_V^\alpha + C + \varepsilon_1 + C_{\varepsilon_2} (C \|u_t\|_V^\alpha + C) + \varepsilon_2 \|v\|_V^\alpha. \end{aligned}$$

Using

$$\|v + u_t\|_V^\alpha \geq 2^{1-\alpha} \|v\|_V^\alpha - \|u_t\|_V^\alpha$$

we obtain (for ε_1 small enough):

$$\begin{aligned} & 2_{V^*} \langle A_\omega(t, v), v \rangle_V \\ & \leq -(\gamma - \varepsilon_1 - 2^{\alpha-1} \varepsilon_2) 2^{1-\alpha} \|v\|_V^\alpha + 2(K + C_{\varepsilon_1} \|u_t\|_V^\alpha) \|v\|_H^2 \\ & \quad + (\gamma - \varepsilon_1 + 2C_{\varepsilon_1} + CC_{\varepsilon_2}) \|u_t\|_V^\alpha + 2(K + C_{\varepsilon_1} \|u_t\|_V^\alpha) \|u_t\|_H^2 + CC_{\varepsilon_2} + C + \varepsilon_1. \end{aligned}$$

Now choosing $\varepsilon_1, \varepsilon_2$ small enough yields

$$(4.7) \quad 2_{V^*} \langle A_\omega(t, v), v \rangle_V \leq -\tilde{\gamma} \|v\|_V^\alpha + g(t) \|v\|_H^2 + \tilde{f}(t),$$

where

$$\begin{aligned} \tilde{\gamma} &:= (\gamma - \varepsilon_1 - 2^{\alpha-1} \varepsilon_2) 2^{1-\alpha} > 0; \\ g(t) &:= 2(K + C_{\varepsilon_1} \|u_t\|_V^\alpha) \in L^1_{\text{loc}}(\mathbb{R}); \\ \tilde{f}(t) &:= (\gamma - \varepsilon_1 + 2C_{\varepsilon_1} + CC_{\varepsilon_2}) \|u_t\|_V^\alpha \\ &\quad + 2(C + C_{\varepsilon_1} \|u_t\|_V^\alpha) \|u_t\|_H^2 + CC_{\varepsilon_2} + C + \varepsilon_1 \in L^1_{\text{loc}}(\mathbb{R}). \end{aligned}$$

Here the local integrability of g and \tilde{f} follows from the local L^α -integrability of u in V and local boundedness of u in H . For $t \in \mathbb{R}$ such that $u_t(\omega) \notin V$ we can use the same calculation to prove (H3).

(H4): For $v \in V$, $\omega \in \Omega$ and $t \in \mathbb{R}$ such that $u_t(\omega) \in V$:

$$\begin{aligned} \|A_\omega(t, v)\|_{V^*}^{\frac{\alpha}{\alpha-1}} &= \|A(v + u_t) - \sigma M(u_t)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \\ &\leq C \left(\|A(v + u_t)\|_{V^*}^{\frac{\alpha}{\alpha-1}} + \|M(u_t)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \right) \\ &\leq C (1 + \|v + u_t\|_V^\alpha) \left(1 + \|v + u_t\|_H^\beta \right) + C \|M(u_t)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \\ &\leq C \left(1 + \|v\|_V^\alpha + \|u_t\|_V^\alpha + \|v\|_H^\beta + \|u_t\|_H^\beta + \|v\|_V^\alpha \|v\|_H^\beta \right. \\ &\quad \left. + \|v\|_V^\alpha \|u_t\|_H^\beta + \|u_t\|_V^\alpha \|v\|_H^\beta + \|u_t\|_V^\alpha \|u_t\|_H^\beta \right) \\ &\leq C \left(1 + \|u_t\|_H^\beta \right) \|v\|_V^\alpha + C(1 + \|u_t\|_V^\alpha) \|v\|_H^\beta + C \|v\|_V^\alpha \|v\|_H^\beta \\ &\quad + C \left(1 + \|u_t\|_V^\alpha + \|u_t\|_H^\beta + \|u_t\|_V^\alpha \|u_t\|_H^\beta \right) \\ &\leq (C_1(t) + C_2(t)) \|v\|_V^\alpha \left(1 + \|v\|_H^\beta \right), \end{aligned}$$

where

$$\begin{aligned} C_1(t) &:= C \left(1 + \|u_t\|_V^\alpha + \|u_t\|_H^\beta + \|u_t\|_V^\alpha \|u_t\|_H^\beta \right) \in L^1_{\text{loc}}(\mathbb{R}); \\ C_2(t) &:= C \left(1 + \|u_t\|_H^\beta \right) \in L^\infty_{\text{loc}}(\mathbb{R}). \end{aligned}$$

This yields (H4) on any bounded interval $[S, T] \subseteq \mathbb{R}$.

For $t \in \mathbb{R}$ such that $u_t \notin V$ one can show (H4) by a similar calculation.

Hence, (H1)–(H4) are satisfied for A_ω for each $\omega \in \Omega$ and on each bounded interval $[S, T] \subseteq \mathbb{R}$. By Theorem 7.1 there thus exists a unique solution

$$Z(\cdot, s; \omega)x \in L^\alpha_{\text{loc}}([s, \infty); V) \cap L^\infty_{\text{loc}}([s, \infty); H)$$

to (4.3) for every $(s, \omega, x) \in \mathbb{R} \times \Omega \times H$.

By the uniqueness of solutions for (4.3) we have the flow property

$$Z(t, s; \omega)x = Z(t, r; \omega)Z(r, s; \omega)x.$$

Therefore, by Proposition C.9 the family of maps given by

$$(4.8) \quad S(t, s; \omega) := T(t, \omega) \circ Z(t, s; \omega) \circ T^{-1}(s, \omega)$$

defines a stochastic flow.

Strict stationarity of u_t implies that $A_\omega(t, v) = A_{\theta_t\omega}(0, v)$. By the uniqueness of solutions for (4.3) we deduce that

$$Z(t, s; \omega)x = Z(t - s, 0; \theta_s\omega)x$$

and thus $Z(t, s; \omega)x$ is a cocycle. Since $T(t, \omega)$ is a stationary conjugation, the same holds for $S(t, s; \omega)x$.

Measurability of $\omega \mapsto Z(t, s; \omega)x$ follows as in the proof of [97, Theorem 1.4]. In fact, the same argument proves $\bar{\mathcal{F}}_s^t$ -adaptedness of $\omega \mapsto Z(t, s; \omega)x$. Due to (4.8), in order to deduce measurability and $\bar{\mathcal{F}}_s^t$ -adaptedness of $S(t, s; \omega)x$ we only need to prove local uniform continuity of $x \mapsto Z(t, s; \omega)x$ which is done in (ii) below. Then it is simple to show that $S(t, s; \omega)x$ is a solution to (2.1).

(ii) Since $T(t, \omega)y$ is càdlàg in t locally uniformly in y , $t \mapsto S(t, s; \omega)x$ is càdlàg. Since (H2) holds for A_ω , by Gronwall's lemma (cf. [167, Theorem 5.2.4 (i), Eq. (5.32)]) we have for $s \leq t$,

$$\begin{aligned} & \|Z(t, s; \omega)x - Z(t, s; \omega)y\|_H^2 \\ & \leq \exp \left[\int_s^t (f(r) + \eta(Z(r, s; \omega)x) + \rho(Z(r, s; \omega)y)) dr \right] \|x - y\|_H^2. \end{aligned}$$

By (4.5) for $y \in B(x, r) := \{y \in H \mid \|x - y\|_H \leq r\}$ we have

$$\int_s^t (f(r) + \eta(Z(r, s; \omega)x) + \rho(Z(r, s; \omega)y)) dr \leq C.$$

Thus $x \mapsto Z(t, s; \omega)x$ is continuous locally uniformly in s, t . Moreover, for $s_1 < s_2$ we have

$$\begin{aligned} & \|Z(t, s_1; \omega)x - Z(t, s_2; \omega)x\|_H^2 \\ & = \|Z(t, s_2; \omega)Z(s_2, s_1; \omega)x - Z(t, s_2; \omega)x\|_H^2 \\ & \leq \exp \left[\int_{s_2}^t (f(r) + \eta(Z(r, s_1; \omega)x) + \rho(Z(r, s_2; \omega)x)) dr \right] \|Z(s_2, s_1; \omega)x - x\|_H^2, \end{aligned}$$

which implies right-continuity of $s \mapsto Z(t, s; \omega)x$.

Right continuity of $s \mapsto S(t, s; \omega)x$ and continuity of $x \mapsto S(t, s; \omega)x$ locally uniformly in s, t follow from the corresponding properties of $Z(t, s; \omega)$. \square

5. Existence of a Random Attractor

In the following let \mathcal{D} be the system of all tempered sets. Now we are in a position to state the main result of this chapter.

THEOREM 5.1. *Suppose that (A1)–(A5), (V) and (4.5) hold and let $S(t, s; \omega)$ be the continuous cocycle constructed in Theorem 4.1. Then*

(i) $S(t, s; \omega)$ is a compact cocycle.

For $\alpha = 2$ additionally assume $K < \frac{\gamma\lambda}{4}$ in (A3). Then

(ii) there is a random \mathcal{D} -attractor \mathcal{A} for $S(t, s; \omega)$.

As a first step of the proof of Theorem 5.1 we shall prove bounded absorption. Let $B(x, r) := \{y \in H \mid \|x - y\|_H \leq r\}$.

PROPOSITION 5.2 (Bounded absorption). *Assume (A1)–(A5), (V) and (4.5). If $\alpha = 2$, additionally assume $K < \frac{\gamma\lambda}{4}$ in (A3). Then there is a random bounded \mathcal{D} -absorbing set $\{F(\omega)\}_{\omega \in \Omega}$ for $S(t, s; \omega)$.*

More precisely, there is a measurable function $R : \Omega \rightarrow \mathbb{R}_+ \setminus \{0\}$ such that for all $D \in \mathcal{D}$ there is an absorption time $s_0 = s_0(D; \omega)$ such that

$$(5.1) \quad S(0, s; \omega)D(\theta_s \omega) \subseteq B(0, R(\omega)), \quad \forall s \leq s_0, \quad \mathbb{P}\text{-a.s.}$$

PROOF. By (4.7) we have

$$2_{V^*} \langle A_\omega(t, v), v \rangle_V \leq -(\gamma - \varepsilon_1 - 2^{\alpha-1} \varepsilon_2) 2^{1-\alpha} \|v\|_V^\alpha + 2(K + C_{\varepsilon_1} \|u_t\|_V^\alpha) \|v\|_H^2 + \tilde{f}(t).$$

Note that for $\alpha = 2$ we also have $K < \frac{\gamma\lambda}{4}$, and choosing $\varepsilon_1, \varepsilon_2$ small enough, we conclude

$$2_{V^*} \langle A_\omega(t, v), v \rangle_V \leq c(t, \omega) \|v\|_H^2 + \tilde{f}(t, \omega), \quad \forall v \in V,$$

where $c(t, \omega) := -\tilde{c} + C \|u_t(\omega)\|_V^\alpha$ and

$$\tilde{f}(t, \omega) = C (1 + \|u_t\|_V^\alpha + \|u_t\|_H^2 + \|u_t\|_H^2 \|u_t\|_V^\alpha)$$

for some $C, \tilde{c} > 0$.

Note that \tilde{c} does not depend on σ . For a.e. $t \geq s$ we obtain

$$\begin{aligned} \frac{d}{dt} \|Z(t, s; \omega)x\|_H^2 &= 2_{V^*} \langle A_\omega(t, Z(t, s; \omega)x), Z(t, s; \omega)x \rangle_V \\ &\leq c(t, \omega) \|Z(t, s; \omega)x\|_H^2 + \tilde{f}(t, \omega). \end{aligned}$$

By Theorem 3.1, for sufficiently large σ , there is a subset $\Omega_0 \subseteq \Omega$ of full \mathbb{P} -measure such that

$$\frac{1}{-s} \int_s^0 \|u_\tau(\omega)\|_V^\alpha d\tau \rightarrow \mathbb{E} \|u_0\|_V^\alpha < \frac{\tilde{c}}{2C}, \quad \text{for } s \rightarrow -\infty$$

and $\|u_t(\omega)\|_H^2 \|u_t(\omega)\|_V^\alpha$ is exponentially integrable for all $\omega \in \Omega_0$.

Hence, there is an $s_0(\omega) \leq 0$ such that

$$\frac{1}{-s} \int_s^0 (-\tilde{c} + C \|u_\tau\|_V^\alpha) d\tau \leq -\frac{\tilde{c}}{2},$$

for all $s \leq s_0(\omega)$, $\omega \in \Omega_0$ and some $\tilde{c} > 0$.

Let $D \in \mathcal{D}$, $x_s(\omega) \in D(\theta_s\omega)$. For some $\tilde{s}_0 = \tilde{s}_0(D; \omega)$, by Gronwall's lemma we obtain

$$\begin{aligned}
& \|Z(0, s; \omega)x_s(\omega)\|_H^2 \\
& \leq \|x_s(\omega)\|_H^2 e^{\frac{\tilde{c}s}{2}} + \int_s^{s_0} \tilde{e}^{\frac{\tilde{c}}{2}} f(r, \omega) dr + \int_{s_0}^0 e^{\int_r^0 (-\tilde{c} + C\|u_r\|_V^\alpha) d\tau} \tilde{f}(r, \omega) dr \\
(5.2) \quad & \leq 1 + \int_{-\infty}^{s_0} e^{\frac{\tilde{c}}{2}r} \tilde{f}(r, \omega) dr + \int_{s_0}^0 e^{\int_r^0 (-\tilde{c} + C\|u_r\|_V^\alpha) d\tau} \tilde{f}(r, \omega) dr \\
& =: R(\omega), \quad \forall s \leq \tilde{s}_0, \mathbb{P} - \text{a.s.},
\end{aligned}$$

where the finiteness of the second term follows from the exponential integrability of \tilde{f} .

Since $T(t, \omega) = T(\theta_t\omega)$ is a bounded tempered map, we find bounded absorption for $S(t, s; \omega)$. \square

PROOF OF THEOREM 5.1. (i) Compactness of the cocycles $S(t, s; \omega)$, $Z(t, s; \omega)$ follows as in [95, Theorem 3.1].

(ii) We prove that $Z(t, s; \omega)x$ is \mathcal{D} -asymptotically compact. By Proposition 5.2 there is a random, bounded \mathcal{D} -absorbing set F . Let

$$K(\omega) := \overline{Z(0, -1; \omega)F(\theta_{-1}\omega)}, \quad \forall \omega \in \Omega.$$

Since $F(\theta_{-1}\omega)$ is a bounded set and $Z(t, s; \omega)$ is a compact flow, $K(\omega)$ is compact. Furthermore, $K(\omega)$ is \mathcal{D} -absorbing:

$$\begin{aligned}
Z(0, s; \omega)D(\theta_s\omega) &= Z(0, -1; \omega)Z(-1, s; \omega)D(\theta_s\omega) \\
&\subseteq Z(0, -1; \omega)F(\theta_{-1}\omega) \subseteq K(\omega),
\end{aligned}$$

for all $s \leq s_0$ \mathbb{P} -almost surely. By Theorem C.7 this yields the existence of a random \mathcal{D} -attractor for $Z(t, s; \omega)$ and thus, by Theorem C.10 for $S(t, s; \omega)$. \square

6. Examples

The main results of Theorems 4.1 and 5.1 are applicable to a large class of SPDE, which not only generalises/improves many existing results but also can be used to obtain the existence of random attractors for some new examples. In this section, we mostly present those stochastic equations with a locally monotone operator in the drift, hence the existing results of [94, 95, 97] concerning only monotone operators are not applicable to those examples. We gather the examples considered in these papers at the end of this section.

Here is an overview of the examples considered: In Section 6.1 we study general Burgers-type equations. Sections 6.2 and 6.3 are devoted to Newtonian fluids, in particular we study the 2D Navier-Stokes equations and the 3D Leray- α model. More similar examples where the framework can be applied are summarised in Remark 6.2. We then move on to non-Newtonian fluids in Sections 6.4 and 6.5, where power law fluids and the Ladyzhenskaya model are discussed. Sections 6.6 and 6.7 are concerned with Cahn-Hilliard-type equations in the sense of [184] and general Kuramoto-Sivashinsky-type equations. Finally, in Section 6.8 we show how the aforementioned equations with monotone operators can be embedded into framework presented here.

Notations In this section we use D_i to denote the spatial derivative $\frac{\partial}{\partial x_i}$ and $\Lambda \subseteq \mathbb{R}^d$ is supposed to be an open, bounded domain with smooth boundary and outward pointing unit normal vector n on $\partial\Lambda$. For the Sobolev space $W_0^{1,p}(\Lambda, \mathbb{R}^d)$ ($p \geq 2$) we always use the following (equivalent) Sobolev norm

$$\|u\|_{1,p} := \left(\int_{\Lambda} |\nabla u(x)|^p dx \right)^{\frac{1}{p}}.$$

Most examples below deal with equations for vector-valued quantities. However, in some examples like those of Sections 6.1, 6.6 and 6.7, we are in the scalar-valued case. We use the same notation for L^p and Sobolev spaces in either case, as there is no risk of confusion. Thus, for $p \geq 1$, let L^p denote either the vector-valued L^p -space $L^p(\Lambda, \mathbb{R}^d)$ or the scalar-valued L^p -space $L^p(\Lambda, \mathbb{R})$, with norm $\|\cdot\|_{L^p}$.

For an \mathbb{R}^d -valued function $u : \Lambda \rightarrow \mathbb{R}^d$ we define

$$u \cdot \nabla = \sum_{j=1}^d u_j \partial_j$$

and for an $\mathbb{R}^{d \times d}$ -valued function $M : \Lambda \rightarrow \mathbb{R}^{d \times d}$

$$\operatorname{div}(M) = \left(\sum_{j=1}^d \partial_j M_{i,j} \right)_{i=1}^d.$$

For the reader's convenience, we recall the following Gagliardo-Nirenberg interpolation inequality (cf. e.g. [210, Theorem 2.1.5]).

If $m, n \in \mathbb{N}$ and $q \in [1, \infty]$ such that

$$\frac{1}{q} = \frac{1}{2} + \frac{n}{d} - \frac{m\theta}{d}, \quad \frac{n}{m} \leq \theta \leq 1,$$

then there exists a constant $C > 0$ such that

$$(6.1) \quad \|u\|_{W^{n,q}} \leq C \|u\|_{W^{m,2}}^{\theta} \|u\|_{L^2}^{1-\theta}, \quad u \in W^{m,2}(\Lambda).$$

In particular, if $d = 2$, we have the following well-known estimate on \mathbb{R}^2 (cf. [165, 214]):

$$(6.2) \quad \|u\|_{L^4}^4 \leq C \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2, \quad u \in W_0^{1,2}(\Lambda).$$

6.1. Stochastic Burgers-type and Reaction-Diffusion Equations. We consider the following semilinear stochastic equation

$$(6.3) \quad dX_t = \left(\Delta X_t + \sum_{i=1}^d f_i(X_t) D_i X_t + f_0(X_t) \right) dt + dN_t,$$

for the scalar quantity X on Λ . Let $(N_t)_{t \in \mathbb{R}}$ be an $L^2(\Lambda)$ -valued two-sided Lévy process satisfying (N). Suppose the coefficients satisfy the following conditions:

- (i) f_i is Lipschitz on \mathbb{R} for all $i = 1, \dots, d$;
- (ii) $f_0 \in C^0(\mathbb{R})$ satisfies

$$(6.4) \quad \begin{aligned} |f_0(x)| &\leq C(|x|^r + 1), \quad x \in \mathbb{R}; \\ (f_0(x) - f_0(y))(x - y) &\leq C(1 + |y|^s)(x - y)^2, \quad x, y \in \mathbb{R}. \end{aligned}$$

where C, r, s are some positive constants.

EXAMPLE 1. *Assume*

(1) *If $d = 1, r = 2, s = 2,$*

(2) *If $d = 2, r = 2, s = 2,$ and $f_i, i = 1, 2, 3$ are bounded,*

(3) *If $d = 3, r = 2, s = \frac{4}{3}$ and $f_i, i = 1, 2, 3$ are bounded measurable functions which are independent of $X_t.$*

Furthermore assume that the constant K in the condition (A3) and the domain Λ satisfy $K < \frac{\lambda}{8}.$

Then there is a continuous cocycle and a random attractor associated to (6.3).

PROOF. We consider the following Gelfand triple

$$V := W_0^{1,2}(\Lambda) \subseteq H := L^2(\Lambda) \subseteq V^* = (W_0^{1,2}(\Lambda))^*$$

and define the operator

$$A(u) = \tilde{A}(u) + f_0(u) = \Delta u + \sum_{i=1}^d f_i(u) D_i u + f_0(u), \quad u \in V.$$

One can show that A satisfies (A1)–(A3) with $\alpha = 2$ and $\gamma = \frac{1}{2}$ and a constant K (see [165, Example 3.2]). For (A4) we note that

$$\|\tilde{A}u + f_0(u)\|_{V^*}^2 \leq C \left(\|\tilde{A}u\|_{V^*}^2 + \|f_0(u)\|_{V^*}^2 \right).$$

The first term satisfies

$$\|\tilde{A}u\|_{V^*}^2 \leq C(1 + \|u\|_V^2)(1 + \|v\|_H^\nu),$$

where $\nu = 2$ in case (1) and $\nu = 0$ in case (2). For the second term we note that by applying Hölder's inequality and (6.1)

$$\begin{aligned} |{}_{V^*}\langle f_0(u), v \rangle_V|^2 &\leq \begin{cases} \|v\|_{L^\infty}^2 (1 + \|u\|_{L^2}^4), & d = 1 \\ \|v\|_{L^2}^2 (1 + \|u\|_{L^4}^4), & d = 2 \\ \|v\|_{L^6}^2 (1 + \|u\|_{L^{12/5}}^4), & d = 3 \end{cases} \\ &\leq \|v\|_V^2 (1 + \|u\|_V^2 \|u\|_H^2). \end{aligned}$$

Therefore, (A4) holds with $\alpha = \beta = 2.$

Note that (A5), (V) and (4.5) hold obviously with $M = \Delta,$ therefore, the assertion follows from Theorem 4.1 and Theorem 5.1. \square

REMARK 6.1. (1) *If $d = 1,$ one may take $f_1(x) = x$ such that Theorem 1 can be applied to the classical stochastic Burgers equation (i.e. (6.3) with $f_0 \equiv 0$). Note that we may also allow a polynomial perturbation f_0 in the drift of (6.3). Hence, Theorem 1 also covers stochastic reaction-diffusion-type equations. Due to the restrictions of the variational approach to (S)PDE we can only consider reaction terms of at most quadratic growth. However, as outlined in [94, Remark 4.6], the main ideas apply to SRDE with higher-order reaction terms as well, e.g. using the mild approach to SPDE.*

(2) *The stochastic Burgers equation has been studied intensively over the last decades. W. E, K.M. Khanin, A.E. Mazel and Y.G. Sinai [66] proved the existence of singleton random attractors in 1D for periodic boundary conditions and noise of spatial regularity $C^3.$ R. Iturriaga and Khanin in [120] generalised these periodic results to the multi-dimensional case with spatial C^4 noise. Y. Bakhtin [8] studied the case on $[0, 1]$ with random boundary conditions of Ornstein-Uhlenbeck-type. The case on the whole space driven by a space-time homogeneous Poisson point field was studied by Bakhtin, E. Cator and Khanin [9].*

In [51], G. Da Prato and A. Debussche study the stochastic Burgers equation on an interval with Dirichlet boundary conditions and for cylindrical Wiener noise. They note [51, Remark 2.4] that one can prove existence of a random attractor using essentially the same techniques as [47]. The theorems proved in this chapter extend the above results to the case of more general, rougher noise as well as to the more general class of equations of the form (6.3).

6.2. Stochastic 2D Navier-Stokes Equation and Other Hydrodynamical Models. The next example is stochastic 2D Navier-Stokes equation driven by additive noise. The Navier-Stokes equation is an important model in fluid mechanics to describe the time evolution of incompressible fluids. It can be formulated as follows

$$(6.5) \quad \begin{aligned} \partial_t u(t) &= \nu \Delta u(t) - (u(t) \cdot \nabla) u(t) + \nabla p(t) + f, \\ \operatorname{div}(u) &= 0, \quad u|_{\partial\Lambda} = 0, \quad u(0) = u_0, \end{aligned}$$

where $u(t, x) = (u^1(t, x), u^2(t, x))$ represents the velocity field of the fluid, ν is the viscosity constant, $p(t, x)$ is the pressure and f is a (known) external force field acting on the fluid. The stochastic version was first considered by A. Bensoussan and R. Temam in [13] and has since been studied intensively. Random attractors for additive (as well as linear multiplicative) Wiener noise were first obtained by H. Crauel and F. Flandoli [47].

As usual we define (cf. [214, Theorems 1.4 and 1.6]):

$$(6.6) \quad \begin{aligned} H &= \{u \in L^2(\Lambda; \mathbb{R}^2) : \nabla \cdot u = 0 \text{ in } \Lambda, \quad u \cdot n = 0 \text{ on } \partial\Lambda\}; \\ V &= \{u \in W_0^{1,2}(\Lambda; \mathbb{R}^2) : \nabla \cdot u = 0 \text{ in } \Lambda\}. \end{aligned}$$

The Helmholtz-Leray projection P_H and the Stokes operator L with viscosity constant ν are defined by

$$\begin{aligned} P_H &: L^2(\Lambda, \mathbb{R}^2) \rightarrow H, \quad \text{orthogonal projection;} \\ L &: H^{2,2}(\Lambda, \mathbb{R}^2) \cap V \rightarrow H, \quad Lu = \nu P_H \Delta u. \end{aligned}$$

We thus arrive at the following abstract formulation of the Navier-Stokes equation

$$(6.7) \quad u' = Lu + F(u) + f, \quad u(0) = u_0 \in H,$$

where $f \in H$ (for simplicity we write f for $P_H f$ again) and

$$F : V \times V \rightarrow V^*, \quad F(u, v) := -P_H [(u \cdot \nabla) v], \quad F(u) := F(u, u).$$

It is well known that $F : V \times V \rightarrow V^*$ is well-defined and continuous. Using the Gelfand triple $V \subset H \equiv H^* \subset V^*$, one sees that L extends by continuity to a map $L : V \rightarrow V^*$. Now we consider a random forcing and thus obtain the stochastic 2D Navier-Stokes equation

$$(6.8) \quad dX_t = (LX_t + F(X_t) + f) dt + dN_t,$$

where $(N_t)_{t \in \mathbb{R}}$ is a two-sided trace-class Lévy process in H satisfying (N).

EXAMPLE 2. (*Stochastic 2D Navier-Stokes equation*) *There exists a continuous cocycle and a random attractor associated to (6.8).*

PROOF. According to the result in [165, Example 3.3], (A1)–(A4) hold with $\alpha = \beta = 2$, $\eta \equiv 0$ and $\rho(v) = \|v\|_{L^4}^4$ and $K = 0$. (A5), (V) and (4.5) hold obviously (with $M := L$). Therefore, the assertion follows from Theorem 4.1 and Theorem 5.1. \square

REMARK 6.2. (1) *The above result improves the classical results in [47, Theorem 7.4] and [45, Example 3.1] by allowing more general types of noise. Besides Lévy-type noise being allowed here, even for Wiener-type noise, we don't need impose any further assumptions on the noise except those needed for the well-posedness of the equation.*

(2) As we mentioned in the introduction, many other hydrodynamical systems also satisfy the local monotonicity (A2) and coercivity condition (A3). For example, I.D. Chueshov and A. Millet [40] studied well-posedness and large deviation principles for abstract stochastic semilinear equations (driven by Wiener noise), covering a wide class of fluid dynamical models.

In fact, they consider abstract equations of the form

$$du(t) = (Lu(t) + B(u(t), u(t)) + \mathcal{R}u(t))dt + \sigma(u(t))dW(t).$$

The operator L is a linear unbounded, self-adjoint and negative definite operator with $V = D((-L)^{1/2})$, H is a separable Hilbert space such that the Gelfand triple $V \subseteq H \subseteq V^*$ holds. In [40], the inclusions do not have to be compact, but we have to assume this. $\mathcal{R}: H \rightarrow H$ is a bounded linear operator. The bilinear operator B satisfies certain continuity, symmetry and interpolation/growth conditions, cf. [40, (C1)]. These assumptions imply the conditions of this article:

(A1) is clear by the continuity assumptions on the operators. (A2) has been shown in [40, Eq. (2.8)] for the operator B . For the other two operators this follows immediately. (A3) with $\alpha = 2$, $\gamma = 1$ and $K = \|\mathcal{R}\|$ follows as by assumption $v^*\langle B(v, v), v \rangle_V = 0$, and (A4) with $\beta = 2$ is implied by

$$|v^*\langle B(v, v), u \rangle_V| = |v^*\langle B(v, u), v \rangle_V| \leq C\|u\|_V\|v\|_H\|v\|_V.$$

As we assumed bounded domains, (A5) holds and finally (V) holds for $M = L$. Since $\alpha = 2$, we get the additional constraint $K = \|\mathcal{R}\| < \frac{\lambda}{4}$.

Therefore, Theorem 4.1 and Theorem 5.1 can be applied to show the existence of a continuous cocycle and of a random attractor for all the hydrodynamical models studied in [40] driven by additive Lévy-type noise. These models include stochastic magneto-hydrodynamic equations, the stochastic Boussinesq model for the Bénard convection, the stochastic 2D magnetic Bénard problem and the stochastic 3D Leray- α model driven by additive noise. For brevity we shall restrict our attention to one further example, namely the stochastic 3D Leray- α model.

6.3. Stochastic 3D Leray- α Model. We now apply the main result to the 3D Leray- α model of turbulence, which is a regularization of the 3D Navier-Stokes equation and was first considered by J. Leray [148] in order to prove the existence of a solution to the Navier-Stokes equation in \mathbb{R}^3 . Here we use a special smoothing kernel, which goes back to A. Cheskidov, D.D. Holm, E. Olson and E.S. Titi [39] (cf. [218] for more references). It has been shown there that the 3D Leray- α model compares successfully with experimental data from turbulent channel and pipe flows for a wide range of Reynolds numbers and therefore has the potential to become a good sub-grid-scale large-eddy simulation model for turbulence. The (deterministic) Leray- α model can be formulated as follows:

$$(6.9) \quad \begin{aligned} \partial_t u &= \nu \Delta u - (v \cdot \nabla)u - \nabla p + f, \\ \operatorname{div}(u) &= 0, \quad u|_{\partial\Lambda} = 0, \quad u = v - \varepsilon^2 \Delta v, \end{aligned}$$

where $\nu > 0$ is the viscosity, u is the velocity, p is the pressure and f is a given body-forcing term. Using the same divergence-free Hilbert spaces V and H as in (6.6) (but in the 3D case), one can rewrite the stochastic Leray- α model in the following abstract form:

$$(6.10) \quad dX_t = (LX_t + F(X_t, X_t) + f) dt + dN_t,$$

where $f \in H$, $(N_t)_{t \in \mathbb{R}}$ is a trace-class Lévy process in H satisfying condition (N), and

$$Lu = \nu P_H \Delta u, \quad F(u, v) = -P_H \left[\left((I - \varepsilon^2 \Delta)^{-1} u \cdot \nabla \right) v \right].$$

The stochastic 3D Leray- α model was studied by G. Deugoue and M. Sango in [59] and I.D. Chueshov and A. Millet in [40] for the case of Brownian motion noise. The inviscid case $\nu = 0$ was investigated by D. Barbato, H. Bessaih and B. Ferrario in [10]. The model has also been extended to the case of 3D MHD equations by Deugoué, P.A. Razafimandimby and Sango in [57].

EXAMPLE 3. (*Stochastic 3D Leray- α model*) *There exists a continuous cocycle and a random attractor associated to (6.10).*

PROOF. Conditions (A1)–(A4) have been checked above and in [164, Example 3.6] with $\alpha = 2$, $K = 0$, $\beta = 2$. Condition (V) holds with $M := L$ and (A5) is clear. The assertion now follows from Theorem 4.1 and Theorem 5.1. \square

REMARK 6.3. *To the best of our knowledge, the existence of a random attractor seems to be new for this model.*

6.4. Stochastic Power Law Fluids. The next example is an SPDE model which describes the velocity field of a viscous and incompressible non-Newtonian fluid subject to random forcing in dimension $2 \leq d \leq 4$. The deterministic model has been studied intensively in PDE theory (cf. [84, 121] and the references therein). For a vector field $u : \Lambda \rightarrow \mathbb{R}^d$, we define the rate-of-strain tensor by

$$e(u) : \Lambda \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d; \quad e_{i,j}(u) = \frac{\partial_i u_j + \partial_j u_i}{2}, \quad i, j = 1, \dots, d$$

and we consider the case that the stress tensor has the following polynomial form:

$$\tau(u) : \Lambda \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d; \quad \tau(u) = 2\nu(1 + |e(u)|)^{p-2}e(u),$$

where $\nu > 0$ is the kinematic viscosity and $p > 1$ is a constant, and for $U \in \mathbb{R}^d \otimes \mathbb{R}^d$ we define $|U| = \left(\sum_{i,j=1}^d |U_{ij}|^2\right)^{1/2}$.

In the case of deterministic forcing, the dynamics of power law fluids can be modelled by the following PDE (cf. [121, Chapter 5]):

$$(6.11) \quad \begin{aligned} \partial_t u &= \operatorname{div}(\tau(u)) - (u \cdot \nabla)u - \nabla p + f, \\ \operatorname{div}(u) &= 0, \quad u|_{\partial\Lambda} = 0, \quad u(0) = u_0, \end{aligned}$$

where $u = u(t, x) = (u_i(t, x))_{i=1}^d$ is the velocity field, p is the pressure and f is an external force.

REMARK 6.4. *For $p = 2$, (6.11) describes Newtonian fluids and (6.11) reduces to the classical Navier-Stokes equation (6.5).*

The cases $p \in (1, 2)$ and $p \in (2, \infty)$ are called shear-thinning fluids and shear-thickening fluids, respectively. They have been widely studied in different fields of science and engineering (cf. e.g. [84, 121] and the references therein).

In this section, we only consider the case $p \geq \frac{d+2}{2} \geq 2$, i.e. the shear-thickening case. In the following we consider the Gelfand triple $V \subset H \subset V^*$, where

$$\begin{aligned} V &= \{u \in W_0^{1,p}(\Lambda; \mathbb{R}^d) : \nabla \cdot u = 0 \text{ in } \Lambda\}; \\ H &= \{u \in L^2(\Lambda; \mathbb{R}^d) : \nabla \cdot u = 0 \text{ in } \Lambda, \quad u \cdot n = 0 \text{ on } \partial\Lambda\}. \end{aligned}$$

Let P_H be the orthogonal (Helmholtz-Leray) projection from $L^2(\Lambda, \mathbb{R}^d)$ to H . As in Example 6.2, the operators

$$\mathcal{N} : W^{2,p}(\Lambda; \mathbb{R}^d) \cap V \rightarrow H, \quad \mathcal{N}(u) := P_H [\operatorname{div}(\tau(u))];$$

$$F : (W^{2,p}(\Lambda; \mathbb{R}^d) \cap V) \times (W^{2,p}(\Lambda; \mathbb{R}^d) \cap V) \rightarrow H;$$

$$F(u, v) := -P_H [(u \cdot \nabla)v], \quad F(u) := F(u, u)$$

can be extended to the well defined operators:

$$\mathcal{N} : V \rightarrow V^*; \quad F : V \times V \rightarrow V^*.$$

In particular, one can show that

$$v^* \langle \mathcal{N}(u), v \rangle_V = - \int_{\Lambda} \sum_{i,j=1}^d \tau_{i,j}(u) e_{i,j}(v) dx, \quad u, v \in V;$$

$$v^* \langle F(u, v), w \rangle_V = -v^* \langle F(u, w), v \rangle_V, \quad v^* \langle F(u, v), v \rangle_V = 0, \quad u, v, w \in V.$$

Now (6.11) with random forcing can be reformulated in the following abstract form:

$$(6.12) \quad dX_t = (\mathcal{N}(X_t) + F(X_t) + f)dt + dN_t,$$

with $f \in H$ and N_t being a trace-class Lévy process in H satisfying the condition (N).

EXAMPLE 4. (*Stochastic power law fluids*) Suppose that $2 \leq d \leq 4$ and $p \in [\frac{d+2}{2}, 3]$, then there exists a continuous cocycle and a random attractor associated to (6.12).

PROOF. From [166, Example 3.5] we know that (A1) and (A2) hold with $\rho(v) = C_{\varepsilon} \|v\|_V^{\frac{2p}{2p-d}}$ and $\eta \equiv 0$, and the operator $M := \mathcal{N}$ is in fact strongly monotone. (A3) holds with $\alpha = p$ and $K = 0$. Furthermore, we have

$$\|F(v)\|_{V^*} \leq \|v\|_{L^{\frac{2p}{p-1}}}^2, \quad v \in V.$$

An application of the Gagliardo-Nirenberg interpolation inequality (6.1) yields

$$\|v\|_{L^{\frac{2p}{p-1}}} \leq C \|v\|_V^{\theta} \|v\|_H^{1-\theta},$$

with $\theta = \frac{d}{(d+2)p-2d}$. Note that $2\theta \leq p-1$ if $p \geq \frac{d+2}{2}$, hence the embedding $V \subseteq H$ implies

$$\begin{aligned} \|F(v)\|_{V^*} &\leq C \|v\|_V^{2\theta} \|v\|_H^{2(1-\theta)} \leq C \|v\|_V^{2\theta} \|v\|_H^{(p-1)-2\theta} \|v\|_H^{2(1-\theta)-((p-1)-2\theta)} \\ &\leq \|v\|_V^{p-1} \|v\|_H^{3-p} \quad \Rightarrow \quad \|F(v)\|_{V^*}^{\frac{p}{p-1}} \leq C \|v\|_V^p \|v\|_H^{\frac{(3-p)p}{p-1}}, \end{aligned}$$

which implies $\alpha = p$, $\beta = \frac{(3-p)p}{p-1}$. Since $p \leq 3$, we get $\beta \geq 0$. The condition $\beta(\alpha-1) \leq 2$ is equivalent to $(3-p)p \leq 2$, which is satisfied for $p \geq 2$.

It is also easy to see that

$$\|\mathcal{N}(v)\|_{V^*} \leq C(1 + \|v\|_V^{p-1}), \quad v \in V.$$

Hence the growth condition (A4) holds with the above α and β . (V) and (A5) are clearly satisfied. The assertion now follows from Theorem 4.1 and Theorem 5.1. \square

6.5. Stochastic Ladyzhenskaya Model. The Ladyzhenskaya model is a higher order variant of the power law fluid where the stress tensor has the form

$$\tau(u): \Lambda \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d, \quad \tau(u) = 2\mu_0(1 + |e(u)|^2)^{\frac{p-2}{2}}e(u) - 2\mu_1\Delta e(u) = \tau^{\mathcal{N}}(u) + \tau^{\mathcal{L}}(u).$$

The model was pioneered by O.A. Ladyzhenskaya [142] and further analyzed by various authors (see [254] and the references therein). Compared to the power law fluids considered above, there is an additional fourth order term $\nabla \cdot (-2\mu_1\Delta e(u))$ present in the equation.

The existence of random attractors for this model has been studied for $p \in (1, 2)$, i.e. shear-thinning fluids, by J.Q. Duan and C.D. Zhao in [254] and for $p > 2$ by B.L. Guo and C.X. Guo [108].

In this section we apply the general framework to this model in the case $p \in (1, 3]$, recovering the results of [254] and parts of the results of [108]. This restriction on p allows us to understand the nonlinear term as a perturbation of the linear term. It is necessary again due to the restriction $\beta(\alpha-1) \leq 2$ which restricts the homogeneity in (A4). Furthermore, applying the Gagliardo-Nirenberg inequality (6.1), we find a ‘‘maximal’’ range $(1, p_c] \subset (1, 3]$ of parameters p to which the method presented here applies.

In what follows, the exact form of the powers in the stress tensor does not play any role, i.e. the results apply just as well to the case

$$\tilde{\tau}^{\mathcal{N}}(u) = 2\mu_0(1 + |e(u)|)^{p-2}e(u).$$

Consider the Gelfand triple $V \subset H \subset V^*$, where

$$\begin{aligned} V &= \{u \in W_0^{2,2}(\Lambda; \mathbb{R}^d) : \nabla \cdot u = 0 \text{ in } \Lambda\}; \\ H &= \{u \in L^2(\Lambda; \mathbb{R}^d) : \nabla \cdot u = 0 \text{ in } \Lambda, u \cdot n = 0 \text{ on } \partial\Lambda\}. \end{aligned}$$

Let P_H be the orthogonal (Helmholtz-Leray) projection from $L^2(\Lambda, \mathbb{R}^d)$ to H . Similar to Examples 6.2 and 6.4, the operators

$$\begin{aligned} \mathcal{N} &: C_c^\infty(\Lambda; \mathbb{R}^d) \cap V \rightarrow H, \quad \mathcal{N}(u) := P_H [\operatorname{div}(\tau^{\mathcal{N}}(u))]; \\ \mathcal{L} &: C_c^\infty(\Lambda; \mathbb{R}^d) \cap V \rightarrow H, \quad \mathcal{L}u := P_H [\operatorname{div}(\tau^{\mathcal{L}}(u))]; \\ F &: (C_c^\infty(\Lambda; \mathbb{R}^d) \cap V) \times (C_c^\infty(\Lambda; \mathbb{R}^d) \cap V) \rightarrow H; \\ F(u, v) &:= -P_H [(u \cdot \nabla)v], \quad F(u) := F(u, u); \end{aligned}$$

can be extended to the well defined operators:

$$\mathcal{N} : V \rightarrow V^*; \quad \mathcal{L} : V \rightarrow V^*; \quad F : V \times V \rightarrow V^*.$$

With these preparations, we can write the model in the abstract form

$$(6.13) \quad dX_t = (\mathcal{N}(X_t) + \mathcal{L}X_t + F(X_t) + f)dt + dN_t,$$

where N_t is a two-sided Lévy-process satisfying the condition (N). We then have the following result:

EXAMPLE 5. (*Ladyzhenskaya model*) *Let $d \leq 6$. Then there exists a $p_c = p_c(d) > 2$ such that for $p \in (1, p_c]$ there is a continuous cocycle and a random attractor associated to (6.13).*

PROOF. We note the following properties of $\tau^{\mathcal{N}}$ [121, pp. 198, Lemma 1.19]:

$$(6.14) \quad (\tau_{ij}^{\mathcal{N}}(e(u)) - \tau_{ij}^{\mathcal{N}}(e(v)))(e_{ij}(u) - e_{ij}(v)) \geq 0;$$

$$(6.15) \quad \tau_{ij}^{\mathcal{N}}(e(u))e_{ij}(u) \geq 0;$$

$$(6.16) \quad |\tau_{ij}^{\mathcal{N}}(e(u))| \leq C(1 + |e(u)|)^{p-1}.$$

Furthermore, we need the following higher-order version of Korn's inequality (a proof can be found at the end of this section):

$$(6.17) \quad \|\nabla e(u)\|_{L^2} \geq C\|u\|_{H^{2,2}} \quad \forall u \in W_0^{2,2}(\Lambda; \mathbb{R}^d).$$

The condition (A1) is clear. For (A2) we have to estimate three terms:

- (a) $v^* \langle \mathcal{N}(u) - \mathcal{N}(v), u - v \rangle_V = \langle \tau^{\mathcal{N}}(e(u)) - \tau^{\mathcal{N}}(e(v)), e(u) - e(v) \rangle_H \leq 0$ by (6.14).
(b) In this case we get by (6.17)

$$v^* \langle \mathcal{L}(u - v), u - v \rangle_V = -2\mu_1 \|\nabla e(u - v)\|_{L^2}^2 \leq -C\|u - v\|_{H^{2,2}}^2.$$

- (c) We estimate

$$\begin{aligned} v^* \langle F(u) - F(v), u - v \rangle_V &= v^* \langle F(u - v, v), u - v \rangle_V \\ &\leq C \|\nabla v\|_{L^q} \|u - v\|_{L^{\frac{2q}{q-1}}}^2 \leq C \|\nabla v\|_{L^q} \|u - v\|_V^{2\theta} \|u - v\|_H^{2(1-\theta)} \\ &\leq \varepsilon \|u - v\|_V^2 + C_\varepsilon \|\nabla v\|_{L^q}^\nu \|u - v\|_H^2, \end{aligned}$$

where we applied the Gagliardo-Nirenberg interpolation inequality (6.1) as well as Young's inequality. Here the exponents θ and γ are defined by

$$\theta = \frac{d}{4q}, \quad \nu = \frac{1}{1-\theta}, \quad \nu' = \frac{\nu}{\nu-1} = \frac{1}{\theta}.$$

For the above calculations to work, we need to have

$$\frac{d}{4q} = \theta \in (0, 1) \quad \text{and} \quad q > 1 \Leftrightarrow q \in \left(\frac{d}{4} \vee 1, \infty \right),$$

On the other hand, for the term $\|\nabla v\|_{L^q}$ to be bounded, we need the Sobolev embedding $H^{2,2} \subset H^{1,q}$ which holds only if

$$2 - \frac{d}{2} \geq 1 - \frac{d}{q} \Leftrightarrow q \leq \frac{2d}{d-2}.$$

Furthermore, to check (4.5), we have to interpolate once more:

$$\|\nabla v\|_{L^q} \leq \|v\|_{H^{2,2}}^\theta \|v\|_{L^2}^{1-\theta},$$

which implies

$$\theta = \frac{qd + 2q - 2d}{4q}.$$

The condition $\theta \in [\frac{1}{2}, 1)$ from (6.1) implies $q \geq 2$ and the condition $\nu\theta = \frac{4q}{4q-d}\theta \leq 2$ implies $d \leq 6$.

Thus, in total we have to have

$$q \in \left(\frac{d}{4} \vee 2, \frac{2d}{d-2} \right],$$

which is nonempty for $1 < d < 10$.

Putting the three estimates together we find

$$\begin{aligned} v^* \langle \mathcal{N}(u) + \mathcal{L}u + F(u) - \mathcal{N}(v) - \mathcal{L}v - F(v), u - v \rangle_V \\ \leq -(C - \varepsilon) \|u - v\|_{H^{2,2}}^2 + C_\varepsilon \|\nabla v\|_{L^q}^{\frac{4q}{4q-d}} \|u - v\|_H^2, \end{aligned}$$

i.e. (A2) with $\rho(v) = C_\varepsilon \|\nabla v\|_{L^q}^{\frac{4q}{4q-d}}$ and $\eta = 0$. By the choice of q and the Sobolev embedding theorem, ρ is locally bounded.

For Assumption (A3) we proceed in a similar fashion (by the incompressibility condition, the term involving F is zero):

- (a) ${}_{V^*}\langle \mathcal{N}(v), v \rangle_V = -\langle \tau^{\mathcal{N}}(e(v)), e(v) \rangle_H \leq 0$ by (6.15).
- (b) ${}_{V^*}\langle \mathcal{L}(v), v \rangle_V = -\|\nabla e(v)\|_{L^2}^2 \leq -C_1\|v\|_{H^{2,2}}^2 = -C_1\|v\|_V^2$,

and thus (A3) holds with $\alpha = 2$. Here we have again the case that the constant K in (A3) vanishes, thus the condition $K < \frac{\lambda C_1}{4}$ is trivially satisfied.

Note that up to this point, the parameter p did not appear in any of the calculations. Assumption (A4) requires to calculate three terms again:

- (a) For the term \mathcal{N} , we distinguish two cases:

- (i) Let $1 < p \leq 2$. By (6.16) we find

$$\begin{aligned} |{}_{V^*}\langle \mathcal{N}(v), u \rangle_V| &\leq \int_{\Lambda} |\tau^{\mathcal{N}}(e(v))| |e(u)| dx \leq C \int_{\Lambda} (1 + |e(v)|)^{p-1} |e(u)| dx \\ &\leq C(1 + \|e(v)\|_{L^p}^{p-1}) \|e(u)\|_{L^p} \leq C(1 + \|v\|_V^{p-1}) \|u\|_V \\ &\leq C(1 + \|v\|_V) \|u\|_V. \end{aligned}$$

- (ii) Now let $p > 2$. Again, applying (6.16) we get

$$\begin{aligned} |{}_{V^*}\langle \mathcal{N}(v), u \rangle_V| &\leq \int_{\Lambda} |\tau^{\mathcal{N}}(e(v))| |e(u)| dx \leq C \int_{\Lambda} (1 + |e(v)|)^{p-1} |e(u)| dx \\ &\leq C(1 + \|e(v)\|_{L^p}^{p-1}) \|e(u)\|_{L^p} \leq C(1 + \|v\|_{H^{1,p}}^{p-1}) \|u\|_{H^{1,p}} \\ &\leq C \left(1 + \|v\|_V^{\theta(p-1)} \|v\|_H^{(1-\theta)(p-1)} \right) \|u\|_V \end{aligned}$$

where we used the Sobolev embedding $V = H^{2,2} \subset H^{1,p}$ which holds for $p \leq \frac{2d}{d-2}$ and the Gagliardo-Nirenberg inequality (6.1) with

$$\theta = \frac{dp + 2p - 2d}{4p},$$

which has to be in $[\frac{1}{2}, 1)$. However, since $\alpha = 2$, we need that $\theta(p-1) \leq 1$. As long as $p \leq 2$ this condition is always satisfied. For $p > 2$ this is more difficult. We want to have

$$1 \geq \theta(p-1) \Leftrightarrow 0 \geq p^2 - 3p + \frac{2d}{d+2}.$$

We see that the latter condition is always strictly satisfied for $p = 2$ but never satisfied for $p = 3$. The critical value of p can be calculated as

$$p_c = \frac{3}{2} + \frac{1}{2} \sqrt{\frac{d+18}{d+2}}.$$

As $d > 1$ we find that $p_c < 2.618$. As $d \leq 6$ we find $p_c \geq \frac{3}{2} + \frac{1}{2} \sqrt{\frac{24}{8}} \approx 2.36$.

This leaves us with two conditions for this range of p :

$$2 < p \leq \frac{2d}{d-2} \wedge p_c = p_c.$$

- (b) $|{}_{V^*}\langle \mathcal{L}v, u \rangle_V| = |\langle \nabla e(v), \nabla e(u) \rangle_{L^2}| \leq \|\nabla e(v)\|_{L^2} \|\nabla e(u)\|_{L^2} \leq \|v\|_V \|u\|_V.$

(c) For the last term we find

$$|\langle F(v, v), u \rangle| = |\langle F(v, u), v \rangle| \leq C \|\nabla u\|_{L^q} \|v\|_{L^{\frac{2q}{q-1}}}^2 \leq C \|u\|_V \|v\|_V^{2\theta} \|v\|_H^{2(1-\theta)},$$

where we have taken the biggest possible q , $q = \frac{2d}{d-2}$, and where $\theta = \frac{d-2}{8}$ and since $\alpha = 2$ we again need to have

$$2\theta \leq 1 \Leftrightarrow d \leq 6.$$

The conditions (V) and (A5) are easily seen to be satisfied. \square

PROOF OF (6.17). The classical Korn inequality states that

$$(6.18) \quad \int_{\Lambda} |e(u)|^2 dx \geq C \|u\|_{H^{1,2}}^2 \quad \forall u \in H_0^{1,2}(\Lambda; \mathbb{R}^d).$$

We would like to set $u = \nabla v$ for $v \in H_0^{2,2}(\Lambda; \mathbb{R}^d)$. Note that

$$(\nabla e(v))_k = (\partial_k e_{ij}(v))_{i,j=1}^d = \frac{1}{2} (\partial_i (\partial_k v)_j + \partial_j (\partial_k v)_i) = e(\partial_k v).$$

Now by applying (6.18) to the vector $\partial_k v$ for fixed k , we find

$$\begin{aligned} \int_{\Lambda} |\nabla e(v)|^2 dx &= \sum_k \int_{\Lambda} |\partial_k e(v)|^2 dx = \sum_k \int_{\Lambda} |e(\partial_k v)|^2 dx \\ &\geq C \sum_k \|\partial_k v\|_{H^{1,2}}^2 = C \sum_k \sum_{i,j} \|\partial_i \partial_k v_j\|_{L^2}^2 \\ &= C \sum_k \sum_i \|\partial_i \partial_k v\|_{L^2(\Lambda; \mathbb{R}^d)}^2 = C \|v\|_{H^{2,2}(\Lambda; \mathbb{R}^d)}^2. \end{aligned}$$

\square

6.6. Stochastic Cahn-Hilliard-type Equations. The Cahn-Hilliard equation is a classical model to describe phase separation in a binary alloy. The reader is referred to A. Novick-Cohen [185] for a survey of the classical Cahn-Hilliard equation (see also G. Da Prato, A. Debussche [50] and N. Elezović, A. Mikelić [67] for the stochastic case) and to [184] for Cahn-Hilliard type equations. Let $d \leq 3$. We want to study stochastic Cahn-Hilliard type equations of the following form:

$$(6.19) \quad \begin{aligned} dX &= (-\Delta^2 X + \Delta \varphi(X)) dt + dN_t, \quad X(0) = X_0, \\ \nabla X \cdot n &= \nabla(\Delta X) \cdot n = 0 \quad \text{on } \partial\Lambda, \end{aligned}$$

where X is a scalar function, N_t is an $L^2(\Lambda)$ -valued, two-sided Lévy process satisfying condition (N), and the nonlinearity φ is a function that is specified below. Let

$$V_0 := \{u \in H^{4,2}(\Lambda) : \nabla u \cdot n = \nabla(\Delta u) \cdot n = 0 \text{ on } \partial\Lambda\},$$

where $H^{4,2}(\Lambda)$ denotes the standard Sobolev space on Λ (with values in \mathbb{R}).

We consider the following Gelfand triple

$$V \subset H := L^2(\Lambda) \subset V^*,$$

where

$$V := \text{completion of } V_0 \text{ w.r.t. } \|\cdot\|_{H^{2,2}}.$$

Recall that we use the following (equivalent) Sobolev norm on $H^{2,2}$:

$$\|u\|_{H^{2,2}} := \left(\int_{\Lambda} |\Delta u|^2 dx \right)^{1/2}.$$

Then we get the following result for (6.19).

EXAMPLE 6. (*Stochastic Cahn-Hilliard type equations*) Suppose that $\varphi \in C^1(\mathbb{R})$ and there exist some positive constants C and $p \leq 2$ such that

$$(6.20) \quad \begin{aligned} \varphi'(x) &\geq -C_\varphi, \quad |\varphi(x)| \leq C(1 + |x|^p), \quad x \in \mathbb{R}; \\ |\varphi(x) - \varphi(y)| &\leq C(1 + |x|^{p-1} + |y|^{p-1})|x - y|, \quad x, y \in \mathbb{R}. \end{aligned}$$

Let C_{GN} be the constant from the Gagliardo-Nirenberg interpolation inequality (6.1) for $H^{1,2}(\Lambda) \subset H^{2,2}(\Lambda) \cap L^2(\Lambda)$ and λ the constant from the embedding $V \subset H$. Assume that $C_\varphi < \frac{\sqrt{\lambda}}{2C_{GN}^2}$.

Then there exists a continuous cocycle and a random attractor associated to (6.19).

PROOF. We denote

$$A(u) := -\Delta^2 u + \Delta \varphi(u), \quad u \in H^{4,2}(\Lambda).$$

Note that for $u \in V_0$ by Sobolev's inequality (the embedding $V \subset W^{d,1} \subset L^\infty$ holds by our assumption on the dimension d) we have

$$\begin{aligned} |_{V^*} \langle A(u), v \rangle_V| &= | \langle -\Delta u + \varphi(u), \Delta v \rangle_{L^2} | \leq \|v\|_V (\|u\|_V + \|\varphi(u)\|_{L^2}) \\ &\leq C \|v\|_V (1 + \|u\|_V + \|u\|_{L^\infty}^p) \leq C \|v\|_V (1 + \|u\|_V + \|u\|_V^p), \quad v \in V. \end{aligned}$$

Therefore, by continuity A can be extended to a map from V to V^* . Moreover, this also implies that A is hemicontinuous, i.e. (A1) holds.

The other conditions (A2)–(A4) as well as (4.5) were shown in [166, Example 3.3] with $\alpha = 2$, $\beta = 2(p - 1)$. As we need the exact form of the coercivity condition (A3) to check the condition $K < \frac{\gamma\lambda}{4}$, we repeat its proof. By the interpolation inequality (6.1) and Young's inequality we have for any $v \in V$,

$$\begin{aligned} |_{V^*} \langle \Delta \varphi(v), v \rangle_V| &= - \int_\Lambda \varphi'(v) |\nabla v|^2 dx \leq C_\varphi \|v\|_{H^{1,2}}^2 \leq C_\varphi C_{GN}^2 \|v\|_V \|v\|_H \\ &\leq \frac{1}{2} \|v\|_V^2 + \frac{1}{2} C_\varphi^2 C_{GN}^4 \|v\|_H^2, \end{aligned}$$

i.e. (A3) holds with $\alpha = 2$ and $K = \frac{1}{2} C_\varphi^2 C_{GN}^4$ and $\gamma = \frac{1}{2}$. Thus by our assumption on C_φ , the inequality $K < \frac{\gamma\lambda}{4} = \frac{\lambda}{8}$ holds. The condition (V) is satisfied as the operator $M := -\Delta^2$ is strongly monotone. (A5) and (4.5) are clearly satisfied as well. \square

REMARK 6.5. (1) Note that the technical constraint $\beta(\alpha - 1) = 2(p - 1) \leq 2$ forces $p \leq 2$, so the method does not cover the “classical” Cahn-Hilliard equation for which φ is a double-well potential, $\varphi(u) = u^3 - u$, i.e. $p = 3$.

(2) The results of this article on existence of a random attractor for stochastic Cahn-Hilliard type equations seem not have been established in the literature before.

6.7. Stochastic Kuramoto-Sivashinsky Equation. The Kuramoto-Sivashinsky equation combines features of the Burgers equation with the Cahn-Hilliard type equations studied in the previous section. It was introduced in the works of Y. Kuramoto [139] and D.M. Michelson and G.I. Sivashinsky [177, 204] as a model for flame propagation. The equation in one spatial dimension has the form

$$(6.21) \quad \partial_t u = -\partial_x^4 u - \partial_x^2 u - u \partial_x u.$$

The first two terms on the right-hand side are of Cahn-Hilliard type (with $\varphi(x) = x$), the last term is of Burgers type. We briefly show the existence of a continuous cocycle as well as a random attractor in the periodic case for a slightly generalised model.

EXAMPLE 7 (Stochastic Kuramoto-Sivashinsky equation). Let $\Lambda = (-L, L)$, $L > 0$ and $p \leq 2$. Let $\varphi \in C^1(\mathbb{R})$ satisfy the conditions (6.20) as well as $C_\varphi < \frac{\sqrt{\Lambda}}{2C_{GN}^2}$, where C_{GN} is as in Section 6.6. Furthermore, let N_t be an H -valued two-sided Lévy process satisfying condition (N). The space H is defined below.

Then the equation

$$(6.22) \quad du = \left(-\partial_x^4 u - \partial_x^2 \varphi(u) - u \partial_x u\right) dt + dN_t$$

with boundary conditions

$$\partial_x^i u(-L, t) = \partial_x^i u(L, t), \quad i = 0, \dots, 3$$

and initial condition $u(x, 0) = u_0(x)$, $x \in \Lambda$ generates a continuous cocycle and has a random attractor.

PROOF. Let

$$H = \left\{ u \in L^2(\Lambda) : \int_\Lambda u(x) dx = 0 \right\}, \quad V = H_{per}^2 \cap H.$$

We write

$$A(u) := \mathcal{L}u + \mathcal{N}(u) + \mathcal{B}(u) := -\partial_x^4 u + \partial_x^2 \varphi(u) - u \partial_x u, \quad u \in H^{4,2}(\Lambda).$$

\mathcal{L} , \mathcal{N} have been extended to operators from V to V^* in Section 6.6, where the conditions (A1)–(A4) were checked for them as well. That \mathcal{B} is well-defined can be seen from the following calculations: by (6.1) we find

$$|{}_{V^*} \langle u \partial_x u, v \rangle_V| = \frac{1}{2} |\langle \partial_x(u^2), v \rangle_{L^2}| \leq \frac{1}{2} \|u\|_{L^4}^2 \|\partial_x v\|_{L^2} \leq C \|u\|_V \|u\|_H \|v\|_V.$$

This not only implies the extendability but also gives the remaining contribution to (A1) as well as to (A4) with $\alpha = 2$, $\beta = 2$. For the local monotonicity we note that by the embeddings $H^{2,2} \subseteq H^{1,2} \subseteq W^{1,1} \subseteq L^\infty$, we find

$$2 {}_{V^*} \langle u \partial_x u - v \partial_x v, u - v \rangle_V = 2 \int_\Lambda (u - v)^2 \partial_x v dx \leq C \|\partial_x v\|_{L^\infty} \|u - v\|_H^2$$

which gives (A2) with another locally bounded contribution $\rho_{\mathcal{B}}(v) = \|\partial_x v\|_{L^\infty}$. For (A3) we note that ${}_{V^*} \langle \mathcal{B}(v), v \rangle_V = 0$. Thus the conditions (A1)–(A4) are satisfied with $\alpha = \beta = 2$. The conditions (A5), (V) and (4.5) are again clearly satisfied. \square

REMARK 6.6. *D.S. Yang [238] has studied stochastic Kuramoto-Sivashinsky equation in the case $d = 1$, $\varphi(x) = -x$ with periodic boundary conditions and proved the existence of a random attractor for H -valued trace-class Wiener noise. The above result extends this to a more general class of equations and also to the case of Lévy noise.*

6.8. SPDE with Monotone Coefficients. In [94], the stochastic evolution equation

$$dX_t = A(t, X_t)dt + dW_t + \mu X_t \circ d\beta_t$$

is considered on a Gelfand triple $V \subseteq H \subseteq V^*$, where the Wiener process takes values in H , $\mu \in \mathbb{R}$, β_t is a real-valued Brownian motion and \circ denotes Stratonovich integration. The operator A in this context satisfies (A1), (A2) with $\rho = \eta = 0$, (A3), and (A4) with $\beta = 0$ and coefficients C, γ, K depending on (t, ω) . This case of a “globally” monotone operator (typically just called *monotone operator*) is covered by the theorems in this work, if $\mu = 0$ and the coefficients C, γ, K are independent of (t, ω) and satisfy $K < \frac{\gamma\lambda}{4}$. Note that $\beta(\alpha - 1) = 0 \leq 2$ is satisfied in this case, and so is (4.5).

Accordingly, all examples considered in [94] under these assumptions are covered by the results of this chapter. These examples include the stochastic generalised p -Laplace equations on a Riemannian manifold, stochastic reaction diffusion equations, the stochastic porous media equation as well as the stochastic p -Laplace type equations studied by W.Q. Zhao and Y.R. Li [258] and the degenerate semilinear parabolic equation considered by M.H. Yang and P.E. Kloeden in [239]. For more details the reader is referred to [94] and the references therein.

7. Existence and Uniqueness of Solutions to Locally Monotone PDE

In this section we recall an existence and uniqueness result for locally monotone PDE (cf. [164, 166, 167]). As before, let $V \subseteq H \subseteq V^*$ be a Gelfand triple. We consider the following general nonlinear evolution equation

$$(7.1) \quad \begin{aligned} u'(t) &= A(t, u(t)), \quad \forall 0 < t < T, \\ u(0) &= u_0 \in H, \end{aligned}$$

where $T > 0$, u' is the generalised derivative of u on $(0, T)$ and $A : [0, T] \times V \rightarrow V^*$ is restrictedly measurable, *i.e.* for each dt -version of $u \in L^1([0, T]; V)$, $t \mapsto A(\cdot, u(\cdot))$ is V^* -measurable on $[0, T]$.

Suppose that for some $\alpha > 1, \beta \geq 0$ there exist constants $c > 0, C \geq 0$ and functions $f, g \in L^1([0, T]; \mathbb{R})$ such that the following conditions hold for all $t \in [0, T]$ and $v, v_1, v_2 \in V$:

(H1) (Hemicontinuity) The map $s \mapsto {}_{V^*}\langle A(t, v_1 + sv_2), v \rangle_V$ is continuous on \mathbb{R} .

(H2) (Local monotonicity)

$$2_{{V^*}}\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_V \leq (f(t) + \eta(v_1) + \rho(v_2)) \|v_1 - v_2\|_H^2,$$

where $\eta, \rho : V \rightarrow [0, +\infty)$ are measurable and locally bounded functions.

(H3) (Coercivity)

$$2_{{V^*}}\langle A(t, v), v \rangle_V \leq -c\|v\|_V^\alpha + g(t)\|v\|_H^2 + f(t).$$

(H4) (Growth)

$$\|A(t, v)\|_{{V^*}}^{\frac{\alpha}{\alpha-1}} \leq \left(f(t) + C\|v\|_V^\alpha \right) \left(1 + \|v\|_H^\beta \right).$$

THEOREM 7.1. *Suppose that $V \subseteq H$ is compact and (H1)–(H4) hold. Then for any $u_0 \in H$, (7.1) has a solution u on $[0, T]$, *i.e.**

$$u \in L^\alpha([0, T]; V) \cap C([0, T]; H), \quad u' \in L^{\frac{\alpha}{\alpha-1}}([0, T]; V^*),$$

and

$$\langle u(t), v \rangle_H = \langle u_0, v \rangle_H + \int_0^t {}_{V^*}\langle A(s, u(s)), v \rangle_V ds, \quad \forall t \in [0, T], v \in V.$$

Moreover, if there exist non-negative constants C, γ such that

$$(7.2) \quad \eta(v) + \rho(v) \leq C(1 + \|v\|_V^\alpha)(1 + \|v\|_H^\gamma), \quad v \in V,$$

then the solution of (7.1) is unique.

PROOF. The conclusions follow from a more general result in [166] (see Theorem 1.1 and Remark 1.1(3)) or [167, Theorem 5.2.2]. \square

Appendices

A. L^p Solutions and Integral Equations

In [69] E.B. Fabes, B.F. Jones and N.M. Riviere proved that the weak formulation for the Navier-Stokes equations on the whole space is equivalent to solving a nonlinear integral equation of the form

$$(A.1) \quad u(x, t) + \mathcal{B}(u, u)(x, t) = \int_{\mathbb{R}^d} \Gamma(x - y, t)u_0(y)dy.$$

They used this formulation to prove regularity estimates in mixed space-time L^p spaces. Their results play an important role in showing smoothness for smooth initial data for the weak solution of the tamed Navier-Stokes equations in [197] and in this section we attempt to derive analogous results for the MHD equations.

The main idea of [69] is threefold:

1. Find a divergence-free solution to the heat equation with the initial data of the Navier- Stokes problem via Fourier analysis.
2. Use this solution as a test function in the weak formulation to derive the integral equation.
3. Prove regularity of the solution to the integral equation (which amounts to estimating the nonlinear term $\mathcal{B}(u, u)$).

We follow their steps with the necessary modifications of the MHD case. Fabes, Jones and Riviere consider mixed space-time $L^{p,q}$ -norms on $S_T := \mathbb{R}^d \times [0, T]$ for $p, q \geq 2$, defined by

$$\|u\|_{L^{p,q}(S_T)} := \sum_{j=1}^d \left[\int_0^T \left(\int_{\mathbb{R}^d} |u_j(x, t)|^p dx \right)^{q/p} dt \right]^{1/q},$$

where $\frac{d}{p} + \frac{2}{q} \leq 1$, $d < p < \infty$. The space of functions that have finite $L^{p,q}$ -norm is denoted by $L^{p,q}(S_T)$. As we are only interested in the case $p = q$, we will occasionally assume this for simplicity in the following. All the results that follow, however, are true also in the more general case.

A.1. A Divergence-Free Solution to the Heat Equation on the Whole Space.

The first step consists in constructing a symmetric $d \times d$ matrix-valued function $(t, x) \mapsto (E_{ij}(x, t))_{i,j=1}^d$ with the following properties:

- (i) $\Delta E_{ij}(x, t) - \partial_t E_{ij}(x, t) = 0$ for all $t > 0, x \in \mathbb{R}^d$.
- (ii) $\nabla \cdot E_i(x, t) = 0$ for all $x \in \mathbb{R}^d, t > 0$ where E_i is the i -th row of E_{ij} , i.e. $E_i = (E_{i1}, \dots, E_{id})$
- (iii) For $g \in \mathbb{L}^p(\mathbb{R}^d)$, $1 \leq p < \infty$ (i.e. $g \in L^p(\mathbb{R}^d)$) and $\nabla \cdot g = 0$ in the sense of distributions), the following convergence holds:

$$\int_{\mathbb{R}^d} E(x - y, t)(g(y))dy \rightarrow g(x) \quad \text{in } L^p(\mathbb{R}^d) \quad \text{as } t \downarrow 0.$$

The function is given by

$$(A.2) \quad E_{ij}(x, t) = \delta_{ij}\Gamma(x, t) - R_i R_j \Gamma(x, t),$$

where

$$\Gamma(x, t) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}$$

denotes the Weierstraß kernel and R_j denotes the Riesz transformation,

$$R_j(f)(x) := L^p - \lim_{\varepsilon \rightarrow 0} c_j \int_{|x-y|>\varepsilon} \frac{x_j - y_j}{|x - y|^{n+1}} f(y)dy.$$

Then one can show that $E_{ij}(x, t) \in C^\infty(\mathbb{R}^d \times (0, \infty))$, and for $1 \leq p < \infty$ and $g \in \mathbb{L}^p(\mathbb{R}^d)$,

$$(A.3) \quad E_{ij}(x, t)(g_i)(x) = \int_{\mathbb{R}^d} \Gamma(x - y, t) g_i(y) dy, \quad a.e.$$

We now want to define the nonlinear operator \mathcal{B} . Recall that E_i denotes the i -th row of E_{ij} . We denote by $\langle u(y, s), \nabla E(x - y, t - s) \rangle$ the $d \times d$ -matrix¹

$$(\langle u(y, s), D_{x_k} E_i(x - y, t - s) \rangle)_{i,k=1}^d = \left(\sum_{j=1}^d u_j(y, s) D_{x_k} E_{ij}(x - y, t - s) \right)_{i,k=1}^d,$$

and define the operator $\bar{\mathcal{B}}(u, w)$ by

$$(A.4) \quad \bar{\mathcal{B}}(u, w)(x, t) := \int_0^t \int_{\mathbb{R}^d} \langle u(y, s), \nabla E(x - y, t - s) \rangle \cdot w(y, s) dy ds.$$

Note that even though $E_{ij}(\cdot, 1) \notin L^1(\mathbb{R}^d)$, since its Fourier transform is not continuous at the origin, and L^1 functions have uniformly continuous Fourier transform, cf. [230, Satz V.2.2, p. 212], we still have $D_{x_k} E_{ij} \in L^1(S_T)$. This implies the following:

LEMMA A.1. *Let $u, w \in L^p(S_T)$, $p \geq 2$. Then $\bar{\mathcal{B}}(u, w) \in L^{p/2}(S_T)$.*

PROOF. As this statement is not entirely obvious and we will need it below, we prove it here for the reader's convenience. We want to estimate

$$\begin{aligned} & \sum_i \left[\int_0^T \int_{\mathbb{R}^d} |\bar{\mathcal{B}}_i(u, w)|^{p/2} dx dt \right]^{2/p} \\ &= \sum_i \left[\int_0^T \int_{\mathbb{R}^d} \left| \sum_{j,k} \int_0^t \int_{\mathbb{R}^d} D_{x_k} E_{ij}(x - y, t - s) u_j(y, s) v_k(y, s) dy ds \right|^{p/2} dx dt \right]^{2/p}. \end{aligned}$$

To simplify notations, we let

$$F_{ijk}(y, s) := |D_{x_k} E_{ij}(y, s)| 1_{[0, T]}(s) \quad \text{and} \quad G_{jk}(y, s) := 1_{[0, T]}(s) |u_j(y, s) v_k(y, s)|.$$

We denote by $f \otimes g$ the convolution in space and time, i.e.

$$f \otimes g(x, t) := \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} f(x - y, t - s) g(y, s) dy ds.$$

¹Note that E is a $d \times d$ -matrix, so taking its gradient we get a tensor of rank 3. By multiplying with the vector u , we obtain a tensor of rank 2, i.e. a matrix.

Then, using that $s \in [0, t]$ if and only if $t - s \in [0, t]$, the inner integral of the expression we want to estimate can be written and estimated as

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^d} \left| \sum_{j,k} \int_0^t \int_{\mathbb{R}^d} D_{x_k} E_{ij}(x-y, t-s) u_j(y, s) v_k(y, s) dy ds \right|^{p/2} dx dt \\
&= \int_0^T \int_{\mathbb{R}^d} \left| \sum_{j,k} \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} 1_{[0,t]}(t-s) D_{x_k} E_{ij}(x-y, t-s) 1_{[0,t]}(s) u_j(y, s) v_k(y, s) dy ds \right|^{p/2} dx dt \\
&\leq \int_0^T \int_{\mathbb{R}^d} \left(\sum_{j,k} \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} 1_{[0,T]}(t-s) |D_{x_k} E_{ij}(x-y, t-s) 1_{[0,T]}(s)| |u_j(y, s) v_k(y, s)| dy ds \right)^{p/2} dx dt \\
&= \int_0^T \int_{\mathbb{R}^d} \left(\sum_{j,k} (F_{ijk} \otimes G_{jk})(x, t) \right)^{p/2} dx dt.
\end{aligned}$$

Thus by Young's convolution inequality and the Cauchy-Schwarz-Buniakowski inequality

$$\begin{aligned}
& \sum_i \left[\int_0^T \int_{\mathbb{R}^d} |\bar{\mathcal{B}}_i(u, w)|^{p/2} dx dt \right]^{2/p} \\
&\leq \sum_i \left[\int_0^T \int_{\mathbb{R}^d} \left(\sum_{j,k} (F_{ijk} \otimes G_{jk})(x, t) \right)^{p/2} dx dt \right]^{2/p} \\
&= \sum_i \left\| \sum_{j,k} 1_{[0,T]}(F_{ijk} \otimes G_{jk}) \right\|_{L^{p/2}(\mathbb{R}^{d+1})} \\
&\leq \sum_{i,j,k} \left\| (F_{ijk} \otimes G_{jk}) \right\|_{L^{p/2}(\mathbb{R}^{d+1})} \\
&\leq \sum_{i,j,k} \|F_{ijk}\|_{L^1(\mathbb{R}^{d+1})} \|G_{jk}\|_{L^{p/2}(\mathbb{R}^{d+1})} \\
&= \sum_{i,j,k} \|D_{x_k} E_{ij}\|_{L^1(S_T)} \|u_j v_k\|_{L^{p/2}(S_T)} \\
&= \sum_{i,j,k} \|D_{x_k} E_{ij}\|_{L^1(S_T)} \|u_j\|_{L^p(S_T)} \|v_k\|_{L^p(S_T)} \\
&\leq \left(\sum_{i,j,k} \|D_{x_k} E_{ij}\|_{L^1(S_T)} \right) \left(\sum_{j,k} \|u_j\|_{L^p(S_T)} \|v_k\|_{L^p(S_T)} \right) \\
&= \left(\sum_{i,j,k} \|D_{x_k} E_{ij}\|_{L^1(S_T)} \right) \|u\|_{L^p(S_T)} \|v\|_{L^p(S_T)},
\end{aligned}$$

where we used Hölder's inequality for $p = 1, p' = \infty$ for the sum $\sum_{j,k}$ and the embedding $\ell^\infty(d) \subset \ell^1(d)$. The proof is complete. \square

Given this operator, we define the operator $\mathcal{B}(y_1, y_2)$ by

$$(A.5) \quad \mathcal{B}(y_1, y_2) := \begin{pmatrix} \bar{\mathcal{B}}(v_1, v_2) - \bar{\mathcal{B}}(B_1, B_2) \\ \bar{\mathcal{B}}(v_1, B_2) - \bar{\mathcal{B}}(B_1, v_2) \end{pmatrix}.$$

The operator \mathcal{B} occurs naturally when we (formally) use the function E_i as a test function in the weak formulation. This will be the subject of the next section.

Before moving on with the theory, let us give a useful Sobolev version of the classical Schauder estimates (for lack of a better name) for the heat equation. To this end, we introduce another short-hand notation for the convolution appearing in the definition of $\bar{\mathcal{B}}$, Equation (A.4):

$$(f \bar{\otimes} g)(x, t) := \int_{-\infty}^t \int_{\mathbb{R}^d} f(x - y, t - s) g(y, s) dy ds.$$

LEMMA A.2. *Let $\gamma \in \mathbb{R}$, $p, q \in (1, \infty)$, and $f \in L^q((0, T); W^{\gamma, p})$ be a function. Then the convolution of f with the heat kernel Γ lies in the space $L^q((0, T); W^{\gamma+2, p})$. More precisely,*

$$\|\Gamma \bar{\otimes} f\|_{L^q((0, T); W^{\gamma+2, p})} \leq c \|f\|_{L^q((0, T); W^{\gamma, p})}.$$

These can also be rewritten into estimates with respect to temporal derivatives.

PROOF. In the case $p = q$, the result is classical, cf. e.g. the book of O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'ceva [140, Chapter IV, Equation (3.1), p. 288]. For $p \neq q$, it was proven by N.V. Krylov in [137, Theorem 1.1] using a Banach space version of the Calderón-Zygmund theorem. \square

A.2. Equivalence of Weak Solutions to the MHD Equations and Solutions to the Integral Equation. By $\mathcal{S}(\mathbb{R}^{d+1})$ we denote the Schwartz space of rapidly decreasing functions, and by $\mathcal{S}'(\mathbb{R}^{d+1})$ we denote its dual space, the space of tempered distributions. Our space of test functions for the weak formulation of the MHD equations is

$$\mathcal{D}_T := \{\phi(x, t) \in \mathbb{R}^d \mid \phi_i \in \mathcal{S}(\mathbb{R}^{d+1}), \phi_i \equiv 0 \text{ for } t \geq T, \nabla \cdot \phi(x, t) = 0 \forall x, t\}.$$

DEFINITION A.3 (Weak solution). *A function $y(x, t) = (v(x, t), B(x, t)) \in \mathbb{R}^{2d}$ is a weak solution of the MHD equations with initial value $y_0 = (v_0, B_0)$ if the following conditions hold:*

(i) $v, B \in L^{p, q}(S_T)$, $p, q \geq 2$.

(ii) *For all test functions $\tilde{y} = (\tilde{v}, \tilde{B}) \in \mathcal{D}_T$ we have the following equality:*

$$\begin{aligned} (A.6) \quad & \int_0^T \int_{\mathbb{R}^d} \langle y, \partial_t \tilde{y} + \Delta \tilde{y} \rangle dx + \int_0^T \int_{\mathbb{R}^d} \langle v, (\nabla \tilde{v})(v) - (\nabla \tilde{B})(B) \rangle dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} \langle B, (\nabla \tilde{B})(v) - (\nabla \tilde{v})(B) \rangle dx dt \\ & = - \int_{\mathbb{R}^d} \langle y_0(x), \tilde{y}(x, 0) \rangle dx - \int_0^T \int_{\mathbb{R}^d} \langle f, \tilde{y} \rangle dx dt. \end{aligned}$$

(iii) *For dt-a.e. $t \in [0, T]$, $\nabla \cdot u(\cdot, t) = 0$ in the sense of distributions.*

Note that this definition of weak solution does not directly correspond to the notion of weak solution given in Chapter II, Section 2.2. But we have the following:

LEMMA A.4. *Let $d = 3$, $T > 0$ be arbitrary and let $y = (\mathbf{v}, \mathbf{B})$ be a weak solution in the sense of Chapter II, Definition 2.5. Then y is also a weak solution in the sense of Definition A.3.*

PROOF. We first prove that \mathbf{v}, \mathbf{B} lie in $L^{p,q}(S_T)$ for appropriate p, q . By definition, we have

$$\mathbf{v}, \mathbf{B} \in L^\infty([0, T]; L^4(\mathbb{R}^3; \mathbb{R}^3)) \subset L^q([0, T]; L^4(\mathbb{R}^3; \mathbb{R}^3)).$$

Since we need to have $\frac{3}{p} + \frac{2}{q} \leq 1$, and $p = 4$, this implies $\mathbf{v}, \mathbf{B} \in L^{4,q}(S_T)$ for any $q \geq 4$.

Next we need to prove that the weak formulation of the equation follows for the different set of test functions \mathcal{D}_T . This follows directly from Proposition 2.6. The divergence-freeness is then clear as well. \square

The weak solution satisfies a scalar equation. We cast this scalar equation, by choosing suitable test functions, into an equivalent vector-valued equation. More precisely, we have the following result, corresponding to Theorem (2.1) in [69] for $f = 0$ and Theorem (4.4) for $f \neq 0$.

THEOREM A.5 (Integral Equation). *Let $y_0 \in \mathbb{L}^r(\mathbb{R}^d)$, $1 \leq r < \infty$, $p, q \geq 2$, $p < \infty$. If $y \in L^{p,q}(S_T)$ is a weak solution of the MHD equations with initial value y_0 , then y solves the integral equation*

$$(A.7) \quad y + \mathcal{B}(y, y) = \int_{\mathbb{R}^d} \Gamma(x - z, t) y_0(z) dz + \int_0^t \int_{\mathbb{R}^d} E(x - z, t - s) (f(z, s)) dz ds.$$

PROOF. The idea is to use E_i (the i -th row of the matrix E_{ij}) as a test function in the weak formulation. The problem is that this function is not in \mathcal{D}_T , so it needs to be regularised first.

To this end, take functions $a \in C^\infty(\mathbb{R}^d, \mathbb{R}_+)$, $\psi \in C^\infty(\mathbb{R}, \mathbb{R}_+)$ with

$$a(x) = \begin{cases} 1, & |x| \geq 2, \\ 0, & |x| \leq 1, \end{cases} \quad \psi(t) = \begin{cases} 1, & t \geq 2, \\ 0, & t \leq 1. \end{cases}$$

Let $a_\lambda(x) := a(\lambda x)$, $\psi_\varepsilon(t) := \psi(t/\varepsilon)$. We first regularise in space by defining

$$(A.8) \quad E_{ij}^{(\lambda)} := \mathcal{F}_x^{-1} (a_\lambda \mathcal{F}_x(E_{ij})).$$

Then $E_{ij}^{(\lambda)}(\cdot, t) \in \mathcal{S}(\mathbb{R}^d)$ for all $t > 0$ and is divergence-free: $\sum_j \partial_{x_j} E_{ij}^{(\lambda)}(x, t) = 0$ for all $t > 0, x \in \mathbb{R}^d$ which can be seen by taking Fourier transforms. We fix (x, t) and let E_i^λ be the i -th row of E_{ij}^λ . The space-time regularised i -th row is then defined by

$$(A.9) \quad \phi_{\varepsilon, \lambda}(z, s) := \psi(s + 2) \psi_\varepsilon(t - s) E_i^{(\lambda)}(x - z, t - s).$$

This function is defined on all of \mathbb{R}^{d+1} and moreover $(\phi_{\varepsilon, \lambda}, 0), (0, \phi_{\varepsilon, \lambda}) \in \mathcal{D}_T$. We can thus plug these test functions $\tilde{y}_{\varepsilon, \lambda}$ into (A.6) to find

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} \langle v, (\partial_s + \Delta_z) \phi_{\varepsilon, \lambda} \rangle dz ds + \int_0^T \int_{\mathbb{R}^d} \langle v, (\nabla_z \phi_{\varepsilon, \lambda})(v) \rangle - \langle B, (\nabla_z \phi_{\varepsilon, \lambda})(B) \rangle dz ds \\ & = - \int_{\mathbb{R}^d} \langle v_0(x), \phi_{\varepsilon, \lambda}(x, 0) \rangle dz - \int_0^T \int_{\mathbb{R}^d} \langle f_1, \phi_{\varepsilon, \lambda} \rangle dz ds \end{aligned}$$

and

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} \langle B, (\partial_s + \Delta_z) \phi_{\varepsilon, \lambda} \rangle dz ds + \int_0^T \int_{\mathbb{R}^d} \langle B, (\nabla_z \phi_{\varepsilon, \lambda})(v) \rangle - \langle v, (\nabla_z \phi_{\varepsilon, \lambda})(B) \rangle dz ds \\ & = - \int_{\mathbb{R}^d} \langle B_0(x), \phi_{\varepsilon, \lambda}(x, 0) \rangle dz - \int_0^T \int_{\mathbb{R}^d} \langle f_2, \phi_{\varepsilon, \lambda} \rangle dz ds, \end{aligned}$$

respectively. These two equations can be treated in the same way, so we focus on the first one – the equation for the velocity. The first factor in the definition of $\phi_{\varepsilon, \lambda}$ is identically

one for $s \geq 0$, so we can omit it in the following calculations. To calculate the first term on the left-hand side, note that for $s \geq 0$ by applying the product rule and the chain rule

$$\begin{aligned}
(\Delta_z + \partial_s)\phi_{\varepsilon,\lambda} &= (\Delta_z + \partial_s)\psi_\varepsilon(t-s)E_i^{(\lambda)}(x-z, t-s) \\
&= (\Delta_z - \partial_s)\left(\psi_\varepsilon E_i^{(\lambda)}\right)(x-z, t-s) \\
&= \psi_\varepsilon(t-s)(\Delta_z - \partial_s)\left(E_i^{(\lambda)}\right)(x-z, t-s) \\
&\quad - \frac{1}{\varepsilon}\psi'_\varepsilon(t-s)E_i^{(\lambda)}(x-z, t-s) \\
&= -\frac{1}{\varepsilon}\psi'_\varepsilon(t-s)E_i^{(\lambda)}(x-z, t-s),
\end{aligned}$$

where we used the definition of E_{ij} in the last line. The function $\psi_\varepsilon(t-s)$ is constant for $s \notin [t-2\varepsilon, t-\varepsilon]$, and thus $\psi'_\varepsilon(t-s) \equiv 0$ for $s \notin [t-2\varepsilon, t-\varepsilon]$. Furthermore, $\psi_\varepsilon(t-s) = 0$ for $s \notin [0, t-\varepsilon]$. Thus we find that the left-hand-side of the tested equation equals

$$\begin{aligned}
&-\frac{1}{\varepsilon}\int_{t-2\varepsilon}^{t-\varepsilon}\int_{\mathbb{R}^d}\langle v, E_i^{(\lambda)}(x-z, t-s)\rangle\psi'\left(\frac{t-s}{\varepsilon}\right)dzds \\
&+\int_0^{t-\varepsilon}\int_{\mathbb{R}^d}\psi_\varepsilon(t-s)\left(\langle v, (\nabla E_i^{(\lambda)})(x-z, t-s)(v)\rangle - \langle B, (\nabla E_i^{(\lambda)})(x-z, t-s)(B)\rangle\right)dzds.
\end{aligned}$$

Next we would like to let $\lambda \rightarrow \infty$. Since $E_i^{(\lambda)} \rightarrow E_i$ in L^p , $1 < p < \infty$, in particular for $p = 2$ and $p = 4$, we find that the above converges in² $L^{p,q}(S_T)$ to

$$\begin{aligned}
&-\frac{1}{\varepsilon}\int_{t-2\varepsilon}^{t-\varepsilon}\int_{\mathbb{R}^d}\langle v, E_i(x-z, t-s)\rangle\psi'\left(\frac{t-s}{\varepsilon}\right)dzds \\
&+\int_0^{t-\varepsilon}\int_{\mathbb{R}^d}\psi_\varepsilon(t-s)\left(\langle v, (\nabla E_i)(x-z, t-s)(v)\rangle - \langle B, (\nabla E_i)(x-z, t-s)(B)\rangle\right)dzds
\end{aligned}$$

We first consider the first term. Since $v(\cdot, s) \in L^p(\mathbb{R}^d)$ is weakly divergence-free for a.e. $s \in (0, T)$ by definition, we find that

$$\int_{\mathbb{R}^d}\langle v(z, s), E_i(x-z, t-s)\rangle dzds = \int_{\mathbb{R}^d}\Gamma(x-z, t-s)v_i(z, s)dzds.$$

Thus, by inserting suitable zeroes,

$$\begin{aligned}
&-\frac{1}{\varepsilon}\int_{t-2\varepsilon}^{t-\varepsilon}\psi'\left(\frac{t-s}{\varepsilon}\right)\int_{\mathbb{R}^d}\Gamma(x-z, t-s)v_i(z, s)dzds \\
&= -\frac{1}{\varepsilon}\int_{t-2\varepsilon}^{t-\varepsilon}\psi'\left(\frac{t-s}{\varepsilon}\right)\int_{\mathbb{R}^d}\Gamma(x-z, t-s)v_i(x, s)dzds \\
&\quad - \frac{1}{\varepsilon}\int_{t-2\varepsilon}^{t-\varepsilon}\psi'\left(\frac{t-s}{\varepsilon}\right)\int_{\mathbb{R}^d}\Gamma(x-z, t-s)[v_i(z, s) - v_i(x, s)]dzds \\
&= -v_i(x, t) - \frac{1}{\varepsilon}\int_{t-2\varepsilon}^{t-\varepsilon}\psi'\left(\frac{t-s}{\varepsilon}\right)[v_i(x, s) - v_i(x, t)]ds \\
&\quad - \frac{1}{\varepsilon}\int_{t-2\varepsilon}^{t-\varepsilon}\psi'\left(\frac{t-s}{\varepsilon}\right)\int_{\mathbb{R}^d}\Gamma(x-z, t-s)[v_i(z, s) - v_i(x, s)]dzds.
\end{aligned}$$

²These norms are with respect to the still variable and until this point fixed parameters x and t .

We are left to prove that the last two terms converge to zero in $L^{p/2, q/2}(S_T)$ as $\varepsilon \rightarrow 0$. The idea is that, as ψ_ε is a smooth cutoff function, $\psi'_\varepsilon := \frac{1}{\varepsilon}\psi'\left(\frac{\cdot}{\varepsilon}\right)$ is a mollifier: it is compactly supported on $[\varepsilon, 2\varepsilon]$, has integral one and as we send ε towards zero, it converges to the Dirac distribution, which can be seen as follows: for a Schwartz function $\varphi \in \mathcal{S}(\mathbb{R})$, substitution and integration by parts shows that

$$\begin{aligned} (\psi'_\varepsilon * \varphi)(t) &= \frac{1}{\varepsilon} \int_{\mathbb{R}} \psi'\left(\frac{t-s}{\varepsilon}\right) \varphi(s) ds = \frac{1}{\varepsilon} \int_{t-2\varepsilon}^{t-\varepsilon} \psi'\left(\frac{t-s}{\varepsilon}\right) \varphi(s) ds \\ &= \int_1^2 \psi'(z) \varphi(t - \varepsilon z) dz = [\psi(z) \varphi(t - \varepsilon z)]_1^2 + \varepsilon \int_1^2 \psi(z) \varphi'(t - \varepsilon z) dz \\ &= \varphi(t - 2\varepsilon) + \varepsilon \int_1^2 \psi(z) \varphi'(t - \varepsilon z) dz \rightarrow \varphi(t). \end{aligned}$$

This implies that, as $\varepsilon \rightarrow 0$,

$$-\frac{1}{\varepsilon} \int_{t-2\varepsilon}^{t-\varepsilon} \psi'\left(\frac{t-s}{\varepsilon}\right) \int_{\mathbb{R}^d} \Gamma(x-z, t-s) v_i(z, s) dz ds \rightarrow -v_i(x, t) \quad \text{in } L^{p/2, q/2}(S_T).$$

In a similar way the nonlinear terms converge to

$$\bar{\mathcal{B}}(v, v)(x, t) - \bar{\mathcal{B}}(B, B)(x, t) = \mathcal{B}_1(y, y)(x, t),$$

and by the (weak) divergence-freeness of the initial conditions, we find

$$\lim_{\varepsilon \rightarrow 0} \psi\left(\frac{t}{\varepsilon}\right) \int_{\mathbb{R}^d} \langle v_0(x), E_i^{(\lambda)}(x-z, t) \rangle dz = \int_{\mathbb{R}^d} v_{0,i}(z) \Gamma(x-z, t) dz.$$

The forcing term converges in the same way (but without cancellations due to divergence-freeness) to

$$-\int_0^t \int_{\mathbb{R}^d} \langle f_1, E_i(x-z, t-s) \rangle dz ds = -\int_0^t \int_{\mathbb{R}^d} E(x-z, t-s)(f_1) dz ds.$$

Doing the same for the equation tested against $(0, E_i^{(\lambda)})$ and putting all the terms together, we arrive at the desired equation. \square

REMARK A.6. *We have only shown that y being a weak solution to the MHD equations implies that y solves an integral equation. In fact, one can prove, following the steps in [69] that this is actually an equivalence, i.e. that solutions to the integral equation are also weak solutions.*

A.3. Regularity of Solutions to the Integral Equation.

THEOREM A.7 (Regularity). *Let y be a solution to the equation $y + \mathcal{B}(y, y) = f$ and $y \in L^{p,q}(S_T)$, $\frac{2}{q} + \frac{d}{p} \leq 1$. Let k be a positive integer such that $k + 1 < p, q < \infty$. If*

$$D_x^\alpha D_t^j f \in L^{p/(|\alpha|+2j+1), q/(|\alpha|+2j+1)}(S_T) \quad \text{whenever } |\alpha| + 2j \leq k,$$

then also

$$D_x^\alpha D_t^j y \in L^{p/(|\alpha|+2j+1), q/(|\alpha|+2j+1)}(S_T) \quad \text{for } |\alpha| + 2j \leq k.$$

PROOF. The proof proceeds along the same lines as that of Theorem 3.4 in [69], with the necessary modifications to the MHD case. Since $y = f - \mathcal{B}(y, y)$, we only have to show that $D_x^\alpha D_t^j \mathcal{B}(y, y) \in L^{p/(|\alpha|+2j+1), q/(|\alpha|+2j+1)}(S_T)$.

For $k = 1$, our assumptions read $f \in L^{p,q}(S_T)$ and $D_{x_i} f \in L^{p/2, q/2}(S_T)$ for all i . We split the argument into two parts: that for the v -part of the equation, i.e. for the equation $v + \mathcal{B}_1(y, y) = f_1$ and that for the B -part of the equation, i.e. $B + \mathcal{B}_2(y, y) = f_2$. Since the terms $\bar{\mathcal{B}}_i(u, v)(x, t)$ are of the form (with summation convention)

$$\bar{\mathcal{B}}_i(u, v)(x, t) = (D_{x_k} \Gamma) \bar{\otimes} (u_l [\delta_{il} v_k - R_i R_l v_k])(x, t),$$

and as our assumption on the coefficients p, q implies $1 < \frac{p}{2}, \frac{q}{2} < \infty$, we can apply Lemma A.2 as well as the $L^{p,q}$ -boundedness of the Riesz transform (cf. J.E. Lewis [149, Theorem 4, p. 226]) to find that

$$\begin{aligned} \|D_{x_j} \bar{\mathcal{B}}_i(u, v)\|_{L^{p/2, q/2}(S_T)} &\leq C \|u_l [\delta_{il} v_k - R_i R_l v_k]\|_{L^{p/2, q/2}(S_T)} \\ &\leq C \|u\|_{L^{p,q}(S_T)} (\|\delta_{il} v_k\|_{L^{p,q}(S_T)} + \|R_i R_l v_k\|_{L^{p,q}(S_T)}) \\ &\leq C \|u\|_{L^{p,q}(S_T)} \|v\|_{L^{p,q}(S_T)}, \end{aligned}$$

and therefore $D_{x_i} \mathcal{B}_1(y, y) \in L^{p/2, q/2}(S_T)$. This in turn implies $D_{x_i} v \in L^{p/2, q/2}(S_T)$. The same argument yields $D_{x_i} B \in L^{p/2, q/2}(S_T)$.

For $k > 1$, we use induction over k , assuming that the theorem is true for k . Now assume

$$D_x^\alpha D_t^j f \in L^{p/(|\alpha|+2j+1), q/(|\alpha|+2j+1)}(S_T) \quad \text{for } |\alpha| + 2j \leq k + 1, \quad p, q > k + 2.$$

Derivatives with respect to multi-indices (j, α) with $2j + |\alpha| \leq k$ are covered by the induction hypothesis. We thus only need to consider the case $2j + |\alpha| = k + 1$.

Case 1: $j = 0$. In this case, we apply all but one derivative and see that $D_x^\alpha \mathcal{B}_1(y, y)$ and $D_x^\alpha \mathcal{B}_2(y, y)$ each can be written as a sum of terms of the form $D_{x_m} \bar{\mathcal{B}}(D_x^\beta u_1, D_x^\gamma u_2)$, $u_1, u_2 \in \{v, B\}$ with $|\beta| + |\gamma| = k$. The same reasoning as above, since the (induction) hypothesis implies $1 < \frac{p}{k+2}, \frac{q}{k+2} < \infty$, yields

$$\|D_{x_m} \bar{\mathcal{B}}(D_x^\beta u_1, D_x^\gamma u_2)\|_{L^{p/(k+2), q/(k+2)}(S_T)} \leq C \|\langle D_x^\beta u_1, D_x^\gamma u_2 \rangle_{\mathbb{R}^d}\|_{L^{p/(k+2), q/(k+2)}(S_T)}.$$

If we set $\bar{p} := \frac{p}{|\beta|+1}$, $\bar{q} := \frac{p}{|\gamma|+1}$, we can apply the generalised Hölder inequality with $\bar{r} := \frac{p}{k+2}$ because

$$\frac{1}{\bar{r}} = \frac{k+2}{p} = \frac{|\gamma|+1}{p} + \frac{|\beta|+1}{p} = \frac{1}{\bar{p}} + \frac{1}{\bar{q}}.$$

This implies that

$$\|D_{x_m} \bar{\mathcal{B}}(D_x^\beta u_1, D_x^\gamma u_2)\|_{L^{p/(k+2), q/(k+2)}(S_T)} \leq C \|D_x^\beta u_1\|_{L^{\frac{p}{|\beta|+1}}} \|D_x^\gamma u_2\|_{L^{\frac{p}{|\gamma|+1}}},$$

which is finite by the induction hypothesis.

Case 2: $j > 0$. In this case, we place all the spatial derivatives on the functions u_1, u_2 , so we can write

$$(A.10) \quad D_t^j D_x^\alpha \bar{\mathcal{B}}(u_1, u_2) = \sum_{|\beta|+|\gamma|=\alpha} C_{\beta,\gamma} D_t^j \bar{\mathcal{B}}(D_x^\beta u_1, D_x^\gamma u_2).$$

Since by definition and integration by parts we have

$$\begin{aligned} \bar{\mathcal{B}}(u, w) &= \int_0^t \int_{\mathbb{R}^d} \langle u(y, s), \nabla E(x - y, t - s) \rangle \cdot w(y, s) dy ds \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} 1_{[0,t]}(t - s) D_{x_k} E_{ij}(x - y, t - s) u_j(y, s) w_k(y, s) dy ds \\ &= - \int_{\mathbb{R}} \int_{\mathbb{R}^d} E_{ij}(x - y, s) 1_{[0,t]}(t - s) D_{x_k} [u_j(y, t - s) w_k(y, t - s)] dy ds, \end{aligned}$$

by applying D_t we get two kinds of terms from the product rule:

- (i) If the derivative hits the indicator function, we (formally) get terms of the form

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}^d} \delta_{\{0\}}(t - s) E_{ij}(x - y, t - s) D_{x_k} [u_j(y, s) w_k(y, s)] dy ds \\ &- \int_{\mathbb{R}} \int_{\mathbb{R}^d} \delta_{\{t\}}(t - s) E_{ij}(x - y, t - s) D_{x_k} [u_j(y, s) w_k(y, s)] dy ds \\ &= \int_{\mathbb{R}^d} E_{ij}(x - y, 0) D_{x_k} [u_j(y, t) w_k(y, t)] dy \\ &- \int_{\mathbb{R}^d} E_{ij}(x - y, t) D_{x_k} [u_j(y, t) w_k(y, t)] dy \\ &= D_{x_k} [u_j(x, t) w_k(x, t)] - \int_{\mathbb{R}^d} E_{ij}(x - y, t) D_{x_k} [u_j(y, t) w_k(y, t)] dy, \end{aligned}$$

i.e. in the first term both integrals shrink to a point due to the delta functions and we are left with one spatial derivative of u and w . In the second term, we are left with an unproblematic spatial integral.

- (ii) If the derivative operator hits the function E_{ij} , we use the definition of E_{ij} to find $D_t E_{ij} = \Delta E_{ij}$. So each temporal derivative is transformed into two spatial derivatives. We can then use integration by parts again to transfer all but one of these (spatial) derivatives to u and w , so this term becomes proportional to

$$D_{x_m} \bar{\mathcal{B}}(D_x^{\beta'} u, D_x^{\gamma'} w),$$

where $|\beta'| + |\gamma'| = 1$.

Proceeding inductively, we see that if we apply D_t for $j > 1$ times, we have two cases:

- (i) The time derivative hits the indicator function at least once. In this case we get a term

$$D_t^{r-s} D_x^\nu u(x, t) (D_t^s D_x^\eta w)(x, t),$$

where $s \leq r$ and $|\nu| + |\eta| + 2r = 2(j - 1) + 1 = 2j - 1$. Here we get the factor $j - 1$ because we “lose” one time derivative to the δ -distribution, but we get one more spatial derivative (with the scaling factor 1) due to the derivative from ∇E .

- (ii) All the derivatives hit ∇E . In this case, by continuing as in case 2 above, transferring all but one derivative onto u and w , we get a term

$$D_{x_m} \bar{\mathcal{B}}(D_x^{\beta'} u, D_x^{\gamma'} w),$$

where $|\beta'| + |\gamma'| = 2j - 1$.

With regard to (A.10), we replace u by $D_x^\beta u$ and w by $D_x^\gamma w$, $|\beta| + |\gamma| = |\alpha|$ and apply Lemma A.2 to find

$$\|D_t^j D_x^\alpha \bar{B}(u, w)\|_{L^{\frac{p}{k+2}}(S_T)} \leq C \sum_{|\beta|+|\gamma|+2r=k} \sum_{s \leq r} \|(D_t^{r-s} D_x^\beta u)(D_t^s D_x^\gamma w)\|_{L^{\frac{p}{k+2}}(S_T)}.$$

The summation runs over $|\beta| + |\gamma| + 2r = k$ since in either case we “lose” one (spatial) derivative. By the inductive hypothesis for the induction over k , we have

$$D_t^{r-s} D_x^\beta u \in L^{\frac{p}{|\beta|+2r-2s+1}}(S_T), \quad D_t^s D_x^\gamma w \in L^{\frac{p}{|\gamma|+2s+1}}(S_T).$$

Noting that

$$\frac{|\beta| + 2r - 2s + 1}{p} + \frac{|\gamma| + 2s + 1}{p} = \frac{k + 2}{p},$$

we apply the generalised Hölder inequality to find

$$\begin{aligned} & \|D_t^j D_x^\alpha \bar{B}(u, w)\|_{L^{\frac{p}{k+2}}(S_T)} \\ & \leq C \sum_{|\beta|+|\gamma|+2r=k} \sum_{s \leq r} \|D_t^{r-s} D_x^\beta u\|_{L^{\frac{p}{|\beta|+2r-2s+1}}(S_T)} \|D_t^s D_x^\gamma w\|_{L^{\frac{p}{|\gamma|+2s+1}}(S_T)}, \end{aligned}$$

which is finite. □

B. A Note on Vector Calculus

The condensed notation employed for MHD equations can be confusing at times, so, for the reader's convenience, we would like to clarify some of the notation used throughout this thesis.

B.1. Gradient of a Vector – Navier-Stokes Case. Let $\mathbf{v} = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

be a sufficiently smooth vector field. Then its gradient is defined as the rank 2 tensor (i.e. 3×3 matrix)

$$\nabla \mathbf{v} := \begin{pmatrix} \partial_1 v^1 & \partial_2 v^1 & \partial_3 v^1 \\ \partial_1 v^2 & \partial_2 v^2 & \partial_3 v^2 \\ \partial_1 v^3 & \partial_2 v^3 & \partial_3 v^3 \end{pmatrix}.$$

Formally, one can think of this as being “ $\nabla \otimes \mathbf{v}$ ” (in the sense of the Kronecker product). This then induces the usual Euclidean Hilbert-Schmidt scalar product:

$$\langle \nabla \mathbf{u}, \nabla \mathbf{v} \rangle = \nabla \mathbf{u} \cdot (\nabla \mathbf{v})^T = \sum_{i=1}^3 \langle \nabla u^i, \nabla v^i \rangle = \sum_{i,j=1}^3 \partial_j u^i \partial_j v^i.$$

B.2. Gradient of a Vector – MHD Case. In the MHD case we are dealing with 6-dimensional vector fields

$$y = \begin{pmatrix} \mathbf{v} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix} : \mathbb{R}^3 \rightarrow \mathbb{R}^6,$$

where $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ is endowed with the scalar product

$$\langle y, \tilde{y} \rangle := \left\langle \begin{pmatrix} \mathbf{v} \\ \mathbf{B} \end{pmatrix}, \begin{pmatrix} \tilde{\mathbf{v}} \\ \tilde{\mathbf{B}} \end{pmatrix} \right\rangle := \langle \mathbf{v}, \tilde{\mathbf{v}} \rangle + S \langle \mathbf{B}, \tilde{\mathbf{B}} \rangle.$$

The gradient is then defined as the rank 2 tensor (a 6×3 matrix)

$$\nabla y := \nabla \begin{pmatrix} \mathbf{v} \\ \mathbf{B} \end{pmatrix} := \begin{pmatrix} \nabla \mathbf{v} \\ \nabla \mathbf{B} \end{pmatrix} := \begin{pmatrix} \partial_1 v^1 & \partial_2 v^1 & \partial_3 v^1 \\ \partial_1 v^2 & \partial_2 v^2 & \partial_3 v^2 \\ \partial_1 v^3 & \partial_2 v^3 & \partial_3 v^3 \\ \partial_1 B^1 & \partial_2 B^1 & \partial_3 B^1 \\ \partial_1 B^2 & \partial_2 B^2 & \partial_3 B^2 \\ \partial_1 B^3 & \partial_2 B^3 & \partial_3 B^3 \end{pmatrix} (= “\nabla \otimes y”)$$

This in turn induces the Hilbert-Schmidt scalar product

$$\langle \nabla y, \nabla \tilde{y} \rangle = \nabla \mathbf{v} \cdot (\nabla \tilde{\mathbf{v}})^T + S \nabla \mathbf{B} \cdot (\nabla \tilde{\mathbf{B}})^T = \sum_{i=1}^3 \left(\langle \nabla v^i, \nabla \tilde{v}^i \rangle + S \langle \nabla B^i, \nabla \tilde{B}^i \rangle \right).$$

C. Stochastic Flows and Random Dynamical Systems

We recall the framework of stochastic flows, random dynamical system (RDS) and random attractors. For more details we refer to [7, 45, 47, 200]. Let (H, d) be a complete separable metric space and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ be a metric dynamical system, i.e. $(t, \omega) \mapsto \theta_t(\omega)$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$ -measurable, $\theta_0 = \text{id}$, $\theta_{t+s} = \theta_t \circ \theta_s$ and θ_t is \mathbb{P} -preserving for all $s, t \in \mathbb{R}$.

DEFINITION C.1. *A family of maps $S(t, s; \omega) : H \rightarrow H$, $s \leq t$ is said to be a stochastic flow, if for every $\omega \in \Omega$*

i. $S(s, s; \omega) = \text{id}_H$, for all $s \in \mathbb{R}$.

ii. $S(t, s; \omega)x = S(t, r; \omega)S(r, s; \omega)x$, for all $t \geq r \geq s$, $x \in H$.

A stochastic flow $S(t, s; \omega)$ is called

iii. measurable if $(t, s, \omega, x) \rightarrow S(t, s; \omega)x$ is measurable.

iv. continuous if $x \mapsto S(t, s; \omega)x$ is continuous for all $s \leq t$, $\omega \in \Omega$.

v. a cocycle if $S(t, s; \omega)x = S(t - s, 0; \theta_s \omega)x$, for all $x \in H$, $t \geq s$, $\omega \in \Omega$.

A measurable, cocycle stochastic flow is also called a random dynamical system (RDS).

For a cocycle stochastic flow the notation of the initial time $s \in \mathbb{R}$ is redundant. Therefore, often the notation $\varphi(t, \omega) := S(t, 0; \omega)$ is chosen for cocycles in the literature. Since all the results may be extended to a time-inhomogeneous setup (where $S(t, s; \omega)$ is not a cocycle in general) we prefer to use the notation $S(t, s; \omega)$.

DEFINITION C.2. *A function $f : \mathbb{R} \rightarrow \mathbb{R}_+$ is said to be*

i. tempered if $\lim_{r \rightarrow -\infty} f_r e^{\eta r} = 0$ for all $\eta > 0$;

ii. exponentially integrable if $f \in L^1_{loc}(\mathbb{R}; \mathbb{R}_+)$ and $\int_{-\infty}^t f_r e^{\eta r} dr < \infty$ for all $t \in \mathbb{R}$, $\eta > 0$.

Let us note that the product of two tempered functions is tempered and that the product of a tempered and an exponentially integrable function is exponentially integrable if it is locally integrable.

In the following, let $S(t, s; \omega)$ be a cocycle.

DEFINITION C.3. *A family $\{D(\omega)\}_{\omega \in \Omega}$ of subsets of H is said to be*

i. a random closed set if it is \mathbb{P} -a.s. closed and $\omega \rightarrow d(x, D(\omega))$ is measurable for each $x \in H$. In this case we also call D measurable.

ii. tempered if $t \mapsto \|D(\theta_t \omega)\|_H$ is a tempered function for all $\omega \in \Omega$ (assuming H to be a normed space).

iii. strictly stationary if $D(t, \omega) = D(0, \theta_t \omega)$ for all $\omega \in \Omega$, $t \in \mathbb{R}$.

From now on let \mathcal{D} be a system of families $\{D(\omega)\}_{\omega \in \Omega}$ of subsets of H . For two subsets $A, B \subseteq H$ we define

$$d(A, B) := \begin{cases} \sup_{a \in A} \inf_{b \in B} d(a, b), & \text{if } A \neq \emptyset; \\ \infty, & \text{otherwise.} \end{cases}$$

DEFINITION C.4. A family $\{K(\omega)\}_{\omega \in \Omega}$ of subsets of H is said to be

- i. \mathcal{D} -absorbing, if there exists an absorption time $s_0 = s_0(\omega, D)$ such that

$$S(0, s; \omega)D(\theta_s \omega) \subseteq K(\omega), \quad \forall s \leq s_0$$

for all $D \in \mathcal{D}$ and $\omega \in \Omega_0$, where $\Omega_0 \subseteq \Omega$ is a subset of full \mathbb{P} -measure.

- ii. \mathcal{D} -attracting, if

$$d(S(0, s; \omega)D(\theta_s \omega), K(\omega)) \rightarrow 0, \quad s \rightarrow -\infty$$

for all $D \in \mathcal{D}$ and $\omega \in \Omega_0$, where $\Omega_0 \subseteq \Omega$ is a subset of full \mathbb{P} -measure.

DEFINITION C.5. A cocycle $S(t, s; \omega)$ is called

- i. \mathcal{D} -asymptotically compact if there exists a random, compact, \mathcal{D} -attracting set $\{K(\omega)\}_{\omega \in \Omega}$.
- ii. compact if for all $t > s$, $\omega \in \Omega$ and $B \subseteq H$ bounded, $S(t, s; \omega)B$ is precompact in H .

We define the Ω -limit set by

$$\Omega(D; \omega) := \bigcap_{r < 0} \overline{\bigcup_{\tau < r} S(0, \tau; \omega)D(\theta_\tau \omega)},$$

and one can show that (cf. [47])

$$\Omega(D; \omega) = \{x \in H \mid \exists s_n \rightarrow -\infty, x_n \in D(\theta_{s_n} \omega) \text{ such that } S(0, s_n; \omega)x_n \rightarrow x\}.$$

DEFINITION C.6. Let $S(t, s; \omega)$ be a cocycle. A random closed set $\{\mathcal{A}(\omega)\}_{\omega \in \Omega}$ is called a \mathcal{D} -random attractor for $S(t, s; \omega)$ if it satisfies \mathbb{P} -a.s.

- i. $\mathcal{A}(\omega)$ is nonempty and compact.
- ii. \mathcal{A} is \mathcal{D} -attracting.
- iii. $\mathcal{A}(\omega)$ is invariant under $S(t, s; \omega)$, i.e. for each $s \leq t$

$$S(t, s; \omega)\mathcal{A}(\theta_s \omega) = \mathcal{A}(\theta_t \omega).$$

The following theorem gives a sufficient condition for the existence of a random attractor (cf. e.g. [47]). Let $o \in H$ be an arbitrary point in H .

THEOREM C.7. Let $S(t, s; \omega)$ be a continuous, \mathcal{D} -asymptotically compact cocycle and let K be a corresponding random, compact, \mathcal{D} -attracting set. Then

$$\mathcal{A}(\omega) := \begin{cases} \overline{\bigcup_{D \in \mathcal{D}} \Omega(D; \omega)} & , \text{ if } \omega \in \Omega_0; \\ \{o\} & , \text{ otherwise.} \end{cases}$$

defines a random \mathcal{D} -attractor for $S(t, s; \omega)$ and $\mathcal{A}(\omega) \subseteq K(\omega) \cap \Omega(K; \omega)$ for all $\omega \in \Omega_0$ (where Ω_0 is as in Definition C.4).

Now we introduce the notion of (stationary) conjugation mappings and conjugated stochastic flows (cf. [119, 131]).

DEFINITION C.8. Let (H, d) and (\tilde{H}, \tilde{d}) be two metric spaces.

- i. A family of homeomorphisms $\mathcal{T} = \{T(\omega) : H \rightarrow \tilde{H}\}_{\omega \in \Omega}$ such that the maps $\omega \mapsto T(\omega)x$ and $\omega \mapsto T^{-1}(\omega)y$ are measurable for all $x \in H, y \in \tilde{H}$, is called a stationary conjugation mapping. We set $T(t, \omega) := T(\theta_t \omega)$.
- ii. Let $Z(t, s; \omega), S(t, s; \omega)$ be cocycles. $Z(t, s; \omega)$ and $S(t, s; \omega)$ are said to be stationary conjugated, if there is a stationary conjugation mapping \mathcal{T} such that

$$S(t, s; \omega) = T(t, \omega) \circ Z(t, s; \omega) \circ T^{-1}(s, \omega).$$

It is easy to show that stationary conjugation mappings preserve the stochastic flow and cocycle property.

PROPOSITION C.9. *Let \mathcal{T} be a stationary conjugation mapping and $Z(t, s; \omega)$ be a continuous cocycle. Then*

$$S(t, s; \omega) := T(t, \omega) \circ Z(t, s; \omega) \circ T^{-1}(s, \omega)$$

defines a conjugated continuous cocycle.

The existence of a random attractor is preserved under conjugation.

THEOREM C.10. *Let $S(t, s; \omega)$ and $Z(t, s; \omega)$ be cocycles conjugated by a stationary conjugation mapping \mathcal{T} consisting of uniformly continuous mappings $T(\omega) : H \rightarrow H$. Assume that there is a $\tilde{\mathcal{D}}$ -attractor $\tilde{\mathcal{A}}$ for $Z(t, s; \omega)$ and let*

$$\mathcal{D} := \{ \{T(\omega)\tilde{\mathcal{D}}(\omega)\}_{\omega \in \Omega} \mid \tilde{\mathcal{D}} \in \tilde{\mathcal{D}} \}.$$

Then $\mathcal{A}(\omega) := T(\omega)\tilde{\mathcal{A}}(\omega)$ is a random \mathcal{D} -attractor for $S(t, s; \omega)$.

We will require the following strong notion of stationarity:

DEFINITION C.11. *A map $X : \mathbb{R} \times \Omega \rightarrow H$ is said to satisfy (crude) strict stationarity, if*

$$X(t, \omega) = X(0, \theta_t \omega)$$

for all $\omega \in \Omega$ and $t \in \mathbb{R}$ (for all $t \in \mathbb{R}$, \mathbb{P} -a.s., where the zero-set may depend on t resp.).

As \mathbb{P} is θ -invariant, crude strict stationarity implies stationarity of the law. Objects obtained as limits in $L^2(\Omega)$ or limits in probability usually only satisfy crude strict stationarity. Thus one needs the existence of selections of indistinguishable strictly stationary versions. The following Proposition provides these.

PROPOSITION C.12. *Let $V \subseteq H$ and $X : \mathbb{R} \times \Omega \rightarrow H$ be a process satisfying crude stationarity. Assume that $X \in D(\mathbb{R}; H) \cap L_{loc}^\alpha(\mathbb{R}; V)$ for some $\alpha \geq 1$, \mathbb{P} -a.s. Then there exists a process $\tilde{X} : \mathbb{R} \times \Omega \rightarrow H$ such that*

- i. $\tilde{X}(\cdot)(\omega) \in D(\mathbb{R}; H) \cap L_{loc}^\alpha(\mathbb{R}; V)$ for all $\omega \in \Omega$.*
- ii. X, \tilde{X} are indistinguishable, i.e.*

$$\mathbb{P}[X_t \neq \tilde{X}_t \text{ for some } t \in \mathbb{R}] = 0,$$

with a θ -invariant exceptional set.

- iii. \tilde{X} is strictly stationary.*

PROOF. The proof is based on [147, Proposition 2.8], which in turn is based on [7, Theorem 1.3.2]. Throughout the proof, we denote the Lebesgue measure on \mathbb{R} by λ .

We first note that we may change X on a set of measure zero, so that $X(\cdot)(\omega) \in D(\mathbb{R}; H) \cap L_{loc}^\alpha(\mathbb{R}; V)$ for all $\omega \in \Omega$. Let

$$\Omega_0 = \{ \omega \in \Omega \mid X_t(\omega) = X_0(\theta_t \omega) \text{ for a.a. } t \},$$

$$\Omega_1 = \{ \omega \in \Omega \mid \theta_t \omega \in \Omega_0 \text{ for a.a. } t \}.$$

Step 1: Show $\Omega_0 \in \mathcal{F}$ and $\mathbb{P}(\Omega_0) = 1$. Since X has càdlàg paths, by [127, Remark 1.1.14, p. 5], $X : \mathbb{R} \times \Omega \rightarrow H$ is product-measurable. As $(t, \omega) \mapsto \theta_t \omega$ is measurable by the definition of a metric dynamical system, we find that

$$\begin{aligned} A &:= \{ (t, \omega) \in \mathbb{R} \times \Omega \mid X_t(\omega) \neq X_0(\theta_t \omega) \} \\ &= (X, X_0 \circ \theta)^{-1} ((H \times H) \setminus \Delta) \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \end{aligned}$$

where $\Delta := \{(x, x) \mid x \in H\}$ is the diagonal. By [125, Lemma 1.26], for every $\omega \in \Omega$, the ω -section $A_\omega := \{t \in \mathbb{R} \mid (t, \omega) \in A\}$ is Lebesgue-measurable. Thus, since the maps $I: f \mapsto \int f d\lambda$ and $J: \omega \mapsto 1_{A_\omega}$ are measurable, we find

$$\Omega \setminus \Omega_0 = \{\omega \in \Omega \mid \lambda(A_\omega) > 0\} = (I \circ J)^{-1}((0, \infty)) \in \mathcal{F}.$$

Now, by Fubini's theorem we find

$$\begin{aligned} \int_{\Omega} \lambda(A_\omega) d\mathbb{P}(\omega) &= \int_{\Omega} \int_{\mathbb{R}} 1_A(t, \omega) d\lambda(t) d\mathbb{P}(\omega) \\ &= \int_{\mathbb{R}} \int_{\Omega} 1_A(t, \omega) d\mathbb{P}(\omega) d\lambda(t) = \int_{\mathbb{R}} \mathbb{P}[X_t \neq X_0 \circ \theta_t] d\lambda(t) = 0, \end{aligned}$$

where we used the crude stationarity assumption in the last step. Hence $\lambda(A_\omega) = 0$ for a.a. $\omega \in \Omega$, i.e. there exists an $N' \subset \Omega$ with $\mathbb{P}(N') = 0$ such that $\lambda(A_\omega) = 0$ for all $\omega \in (N')^c$. This means that $\Omega \setminus \Omega_0 \subset N'$ and therefore $\mathbb{P}(\Omega \setminus \Omega_0) = 0$.

Step 2: In a similar way, we establish $\Omega_1 \in \mathcal{F}$ and $\mathbb{P}(\Omega_1) = 1$.

Since by the first step,

$$B := \{(t, \omega) \in \mathbb{R} \times \Omega \mid \theta_t \omega \notin \Omega_0\} = \theta^{-1}(\Omega_0^c) \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{F},$$

we can again consider the measurable ω -sections $B_\omega := \{t \in \mathbb{R} \mid (t, \omega) \in B\} \in \mathcal{B}(\mathbb{R})$ and

$$\Omega \setminus \Omega_1 = \{\omega \in \Omega \mid \lambda(B_\omega) > 0\} \in \mathcal{F}.$$

Therefore, applying Fubini's theorem as well as the θ -invariance of \mathbb{P} gives

$$\begin{aligned} \int_{\Omega} \lambda(B_\omega) d\mathbb{P}(\omega) &= \int_{\Omega} \int_{\mathbb{R}} 1_B(t, \omega) d\lambda(t) d\mathbb{P}(\omega) \\ &= \int_{\mathbb{R}} \int_{\Omega} 1_B(t, \omega) d\mathbb{P}(\omega) d\lambda(t) = \int_{\mathbb{R}} \mathbb{P}[\theta_t \omega \notin \Omega_0] d\lambda(t) \\ &= \int_{\mathbb{R}} \mathbb{P} \circ \theta_t^{-1}(\Omega_0^c) d\lambda(t) = \int_{\mathbb{R}} \mathbb{P}(\Omega_0^c) d\lambda(t) = 0. \end{aligned}$$

Furthermore, Ω_1 is $(\theta_t)_t$ -invariant, which can be seen as follows: let $t \in \mathbb{R}$ be arbitrary. By definition we have

$$\omega \in \Omega_1 \Leftrightarrow \theta_s \omega \in \Omega_0 \quad \forall s \in \Lambda(\omega)^c, \lambda(\Lambda(\omega)) = 0.$$

Now for any $s \in \Lambda(\omega)^c - t$ (which is still a Lebesgue-nullset by translation invariance of the Lebesgue measure), there is an $\tilde{s} \in \Lambda(\omega)^c$ such that $s + t = \tilde{s}$ and hence

$$\theta_s \theta_t \omega = \theta_{t+s} \omega = \theta_{\tilde{s}} \omega \in \Omega_0,$$

which implies that $\theta_t \omega \in \Omega_1$.

Step 3: Definition and well-definedness of \tilde{X} .

Let $x_0 \in H$ be an arbitrary point. Set

$$(C.1) \quad \tilde{X}_t(\omega) := \begin{cases} X_{t-s}(\theta_s \omega), & \omega \in \Omega_1, s \in \mathbb{R} \text{ arbitrary such that } \theta_s \omega \in \Omega_0, \\ x_0, & \omega \in \Omega \setminus \Omega_1. \end{cases}$$

We show that \tilde{X} is well-defined. To this end, let $\omega \in \Omega_1$, $t \in \mathbb{R}$ and $s_i \in \mathbb{R}$ such that $\theta_{s_i} \omega \in \Omega_0$, $i = 1, 2$. Then there exist $\Lambda(\theta_{s_i} \omega) \subset \mathbb{R}$ with $\lambda(\Lambda(\theta_{s_i} \omega)) = 0$ and

$$X_u(\theta_{s_i} \omega) = X_0(\theta_u \circ \theta_{s_i} \omega) = X_0(\theta_{u+s_i}(\omega)) \quad \forall u \in \Lambda(\theta_{s_i} \omega)^c.$$

The set $s_i + \Lambda(\theta_{s_i}\omega)$, $i = 1, 2$ is still a nullset and hence so is $(s_1 + \Lambda(\theta_{s_1}\omega)) \cap (s_2 + \Lambda(\theta_{s_2}\omega))$. Now we let $u_n \in (s_1 + \Lambda(\theta_{s_1}\omega))^c \cup (s_2 + \Lambda(\theta_{s_2}\omega))^c$ with $u_n \downarrow t$. Then

$$\begin{aligned} X_{t-s_1}(\theta_{s_1}\omega) &= \lim_{n \rightarrow \infty} X_{u_n-s_1}(\theta_{s_1}\omega) \quad (X \text{ is right - continuous}) \\ &= \lim_{n \rightarrow \infty} X_0(\theta_{u_n}\omega) \quad (u_n - s_1 \in \Lambda(\theta_{s_1}\omega)) \\ &= \lim_{n \rightarrow \infty} X_{u_n-s_2}(\theta_{s_2}\omega) \quad (u_n - s_2 \in \Lambda(\theta_{s_2}\omega)) \\ &= X_{t-s_2}(\theta_{s_2}\omega) \quad (X \text{ is right - continuous}). \end{aligned}$$

Step 4: \tilde{X} is strictly stationary.

If $\omega \notin \Omega_1$, then $\theta_t\omega \notin \Omega_1$ since otherwise by the θ -invariance of Ω_1 , $\omega = \theta_{-t} \circ \theta_t\omega \in \Omega_1$. Thus, we have for such ω that

$$\tilde{X}_t(\omega) = x_0 = X_0(\theta_t\omega).$$

Now let $\omega \in \Omega_1$ and $t \in \mathbb{R}$. Then there are λ -nullsets $\Lambda(\theta_t\omega), \Lambda(\omega)$ such that

$$\begin{aligned} \tilde{X}_t(\omega) &= X_{t-s}(\theta_s\omega), \quad \forall s \in \Lambda(\theta_t\omega)^c, \\ \tilde{X}_0(\theta_t\omega) &= X_{-r}(\theta_{t+r}\omega), \quad \forall r \in \Lambda(\omega)^c. \end{aligned}$$

Therefore, for $s \in (\Lambda(\theta_t\omega) \cup (t + \Lambda(\omega)))^c$ we have

$$\begin{aligned} \tilde{X}_t(\omega) &= X_{t-s}(\theta_s\omega) \quad (s \in \Lambda(\theta_t\omega)^c) \\ &= X_{-(s-t)}(\theta_{(s-t)+t}\omega) \\ &= \tilde{X}_0(\theta_t\omega) \quad (s - t \in \Lambda(\omega)^c), \end{aligned}$$

i.e. the strict stationarity of \tilde{X} .

Step 5: Regularity of \tilde{X} .

The regularity properties $\tilde{X}(\cdot)(\omega) \in D(\mathbb{R}; H) \cap L_{\text{loc}}^\alpha(\mathbb{R}; V)$ for all $\omega \in \Omega$ follow immediately from the ones for X .

Step 6: \tilde{X} is measurable and \tilde{X}, X are indistinguishable.

We have

$$(C.2) \quad \omega \in \Omega_1 \cap \Omega_0 \Rightarrow \tilde{X}_t(\omega) = X_t(\omega)$$

for all $t \in \mathbb{R}$ by letting $s = 0$ in the definition (C.1). The space $(\Omega, \tilde{\mathcal{F}}, \mathbb{P})$ is complete and since $\mathbb{P}(\Omega_0^c \cup \Omega_1^c) = 0$, we deduce the measurability of \tilde{X} as follows: let $t \in \mathbb{R}$ and $A \in \mathcal{B}(H)$. Then

$$\begin{aligned} \tilde{X}_t^{-1}(A) &= \left(\tilde{X}_t^{-1}(A) \cap (\Omega_0 \cap \Omega_1) \right) \sqcup \left(\tilde{X}_t^{-1}(A) \cap (\Omega_0 \cap \Omega_1)^c \right) \\ &= \left(X_t^{-1}(A) \cap (\Omega_0 \cap \Omega_1) \right) \sqcup \left(\tilde{X}_t^{-1}(A) \cap (\Omega_0 \cap \Omega_1)^c \right) \\ &=: B \sqcup C. \end{aligned}$$

The first term satisfies $B \in \tilde{\mathcal{F}}$ since X is measurable and $\Omega_0, \Omega_1 \in \mathcal{F}$ by Step 1 and 2. The second term C is contained in the set $\Omega_0^c \cup \Omega_1^c$, which is a \mathbb{P} -nullset by Step 1 and 2 and hence $C \in \tilde{\mathcal{F}}$ by completeness. Both imply $\tilde{X}_t^{-1}(A) \in \tilde{\mathcal{F}}$ and thus \tilde{X} is measurable.

Moreover, by (C.2), we have

$$\mathbb{P}[X_t \neq \tilde{X}_t \text{ for some } t] \leq \mathbb{P}(\Omega_0^c \cup \Omega_1^c) = 0.$$

Therefore, X and \tilde{X} are indistinguishable, which concludes the proof. \square

REMARK C.13. From Step 6 of this proof we see that \tilde{X} is $(\tilde{\mathcal{F}}_{-\infty}^t)_{t \in \mathbb{R}}$ -adapted if X is $(\bar{\mathcal{F}}_{-\infty}^t)_{t \in \mathbb{R}}$ -adapted.

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