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Pricing Interest Rate Derivatives under Volatility Uncertainty

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Abstract

We study the pricing of contracts in fixed income markets in the presence of volatility uncertainty. We consider an arbitrage-free bond market under volatility uncertainty. The uncertainty about the volatility is modeled by a G-Brownian motion, which drives the forward rate dynamics. The absence of arbitrage is ensured by a drift condition. In such a setting we obtain a sublinear pricing measure for additional contracts. Similar to the forward measure approach, we define a forward sublinear expectation to simplify the valuation of cashflows. Under the forward sublinear expectation, we obtain a robust version of the expectations hypothesis and a valuation method for bond options. With these tools, we derive robust pricing rules for the most common interest rate derivatives: fixed coupon bonds, floating rate notes, interest rate swaps, swaptions, caps, and floors. For fixed coupon bonds, floating rate notes, and interest rate swaps, we obtain a single price, which is the same as in traditional models. For swaptions, caps, and floors, we obtain a range of prices, which is bounded by the prices from traditional models with the highest and lowest possible volatility. Due to these pricing formulas, the model naturally exhibits unspanned stochastic volatility.

Keywords: Robust Finance, Model Uncertainty, Fixed Income Markets
JEL Classification: G12, G13
MSC2010: 91G20, 91G30

1 Introduction

The present paper deals with the pricing of interest rate derivatives in the presence of volatility uncertainty. Due to the assumption of a single, known probability measure, traditional models in finance are subject to model uncertainty. Therefore, a new stream of research, called *robust finance*, emerged in the literature, examining financial markets in the presence of a family of probability measures. The most frequently studied type of model uncertainty is volatility uncertainty. The literature on robust finance has led to pricing rules which are robust with respect to the volatility of the underlying. The aim of this paper is to develop robust pricing rules for contracts traded in fixed income markets.

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The initial setting is an arbitrage-free bond market under volatility uncertainty. The uncertainty about the volatility is modeled by a G-Brownian motion. A G-Brownian motion is basically a standard Brownian motion with a volatility that is completely uncertain but bounded by two extreme values. The bond market is modeled in the spirit of Heath, Jarrow, and Morton [27] (HJM). That is, we model the instantaneous forward rate as a diffusion process, driven by a G-Brownian motion. The absence of arbitrage is ensured by a drift condition, which is derived in a companion paper [29]. Additionally, we assume that the diffusion coefficient of the forward rate is deterministic, which leads to analytical pricing formulas. This corresponds to a forward rate model in which the foward rate is Gaussian distributed.

Such a setting leads to a sublinear pricing measure for additional contracts we add to the bond market. Volatility uncertainty is represented by a family of probability measures, which naturally leads to a sublinear expectation. In particular, we model the forward rate directly under the risk-neutral sublinear expectation. Thus, the sublinear expectation can be used to determine prices of contracts. Due to the sublinearity, we obtain a single price for simple contracts and a range of prices for more complex contracts.

To simplify the pricing of contracts, we introduce the so-called *forward sublinear expectation*. The forward sublinear expectation is defined by a G-backward stochastic differential equation and corresponds to the expectation under the forward measure. The forward measure, invented by Geman [25], is used for pricing in the classical situation without volatility uncertainty [11, 26, 31]. Similar to the forward measure, the forward sublinear expectation has the advantage that computing the sublinear expectation of a discounted payoff reduces to computing the forward sublinear expectation of the payoff, discounted with a bond. Under the forward sublinear expectation, the forward rate and the forward price are symmetric G-martingales. Therefore, we obtain a robust version of the expectations hypothesis and a general pricing method for certain bond options.

With these tools, we derive pricing formulas for the most common interest rate derivatives: fixed coupon bonds, floating rate notes, interest rate swaps, swaptions, caps, and floors. Due to the linearity of the payoff, we obtain a single price for fixed coupon bonds, floating rate notes, and interest rate swaps. Although its derivation differs from the traditional one, the pricing formula is exactly the same as the one from traditional models. Due to the nonlinearity of the payoff, we obtain a range of prices for swaptions, caps, and floors. The range is bounded by the prices from classical models with the highest and lowest possible volatility. Therefore, the pricing of interest rate derivatives under volatility uncertainty reduces to computing prices in models without volatility uncertainty.

The pricing formulas show that volatility uncertainty naturally leads to *unspanned stochastic volatility*. The notion unspanned stochastic volatility roughly describes the fact that fixed income markets are incomplete. It means that interest rate derivatives which are exposed to volatility risk are driven by factors which do not affect the bond prices. Collin-Dufresne and Goldstein [16] use data on interest rate swaps, caps, and floors to empirically show that fixed income markets exhibit unspanned stochastic volatility and study which classes of models are suitable to reproduce this phenomenon. Moreover, there are various models that take this fact into account [e.g. 14, 23, 24]. The theoretical definition of Filipović, Larsson, and Statti [23] says that models display unspanned stochastic volatility if there exist contracts that cannot be replicated by a portfolio of bonds. The present model leads to a single price for interest rate swaps, which is independent of the bounds for the volatility, and a range of prices for caps and floors, where the bounds depend on the bounds for the volatility. Thus, our pricing formulas are in line with the empirical results of Collin-Dufresne and Goldstein [16] in the sense that there are variables affecting the prices of caps and floors but not affecting the prices of interest rate swaps. In addition, we show that swaptions, caps, and floors cannot be hedged with bonds. Therefore, the model exhibits unspanned stochastic volatility according to the definition of Filipović, Larsson, and Statti [23].

The literature on model uncertainty and, especially, volatility uncertainty in financial markets or, primarily, asset markets is very extensive. The first to apply the concept of volatility uncertainty to asset markets were Avellaneda, Levy, and Parás [3] and Lyons [34]. Over a decade afterwards, the topic gained a lot of interest. The interesting fact about volatility uncertainty is that it is represented by a nondominated set of probability measures. Hence, traditional results from mathematical finance like the fundamental theorem of asset pricing break down. There are various attempts to extend the theorem to a multiprior setting [6, 8, 10]. In some situations the theorem can be even extended to a model-free setting, that is, without any reference measure at all [1, 12, 40]. Most of these works also deal with the problem of pricing and hedging derivatives in the presence of model uncertainty. The topic was studied separately in a number of articles and settings, i.e., in the presence of volatility uncertainty [47], in the presence of a general set of priors [e.g. 2, 13, 39], and in a model-free setting [e.g. 5, 7, 41]. The most similar setting is the one of Vorbrink [47], since it focuses on volatility uncertainty, which is modeled by a G-Brownian motion. However, the focus, as in most of the studies from above, lies on asset markets.

In addition, there is an increasing number of articles dealing with interest rate models under model uncertainty [4, 20, 21, 22, 28, 29]. Among these, there are also articles focusing on volatility uncertainty [4, 21, 28, 29]. The only one working in a general HJM framework is the accompanying article [29]. The remaining articles on model uncertainty in interest rate models either correspond to short rate models or do not study volatility uncertainty. The main result of the accompanying article [29] is a drift condition, which shows how to obtain an arbitrage-free term structure in the presence of volatility uncertainty. Based on this, the aim of the present paper is to study the pricing of derivatives in fixed income markets.

There are several ways to describe volatility uncertainty from a mathematical point of view. The classical approach is the one of Denis and Martini [19] and Peng [38]. Actually, these are two different approaches, but they are equivalent as it was shown by Denis, Hu, and Peng [18]. The difference is that Denis and Martini [19] start from a probabilistic setting, whereas the calculus of G-Brownian motion from Peng [38] relies on nonlinear partial differential equations. Moreover, there are various extensions and generalizations [35, 36, 37]. Additional results and a different approach to volatility uncertainty were developed by Soner, Touzi, and Zhang [42, 43, 44, 45]. They also related the topic to second-order backward stochastic differential equations, introduced by Cheridito, Soner, Touzi, and Victoir [15]. In addition, there are many attempts to a pathwise stochastic calculus, which works without any reference measure [see 17, and references therein]. In this paper, we use the calculus of G-Brownian motion, since the literature on G-Brownian motion contains a lot of results. In particular, the results of Hu, Ji, Peng, and Song [30] are of fundamental importance for the results derived in this paper.

The remainder of this paper is organized as follows. Section 2 introduces the overall setting of the model. In Section 3, we show that we can use the risk-neutral sublinear expectation as a pricing measure. In Section 4, we define the forward sublinear expectation and derive the related results. Section 5 contains pricing formulas for the most common interest rate derivatives. In Section 6, we show that the model exhibits unspanned stochastic volatility. Section 7 gives a conclusion. The proofs of Section 3, Section 4, and Section 5 are given in Section A, Section B, and Section C of the appendix, respectively.

2 Arbitrage-Free Bond Market

The uncertainty about the volatility is represented by a G-Brownian motion. That is, we consider a G-Brownian motion B on the G-expectation space $(\Omega, L_G^1(\Omega), \hat{\mathbb{E}})$. Ω is the space of all continuous trajectories on the positive real line and $L_G^1(\Omega)$ is the space of random variables for which the G-expectation is defined. The G-expectation $\hat{\mathbb{E}}$ is a sublinear functional on the space of random variables. The letter G refers to the nonlinear generator $G : \mathbb{R} \to \mathbb{R}$, which is given by

$$G(a) = \tfrac{1}{2} \sup_{\sigma \in [\underline{\sigma}^2, \overline{\sigma}^2]} \{\sigma a\}$$

and characterizes the distribution and the uncertainty of the G-Brownian motion. For further details, the reader may refer to the book of Peng [38].

By the results of Denis, Hu, and Peng [18], we know that $\hat{\mathbb{E}}$ can be represented as the upper expectation of a set of beliefs. That means there exists a set of probability measures $\mathcal{P}$ such that

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[X]$$

for all $X \in L_G^1(\Omega)$, where $\mathbb{E}_P$ denotes the expectation under the probability measure P. This implies that we identify random variables in $L_G^1(\Omega)$ if they are equal *quasi-surely*, that is, P-almost surely for all $P \in \mathcal{P}$. In particular, the set $\mathcal{P}$ contains all beliefs about the volatility in which the volatility is bounded by $\overline{\sigma}$ and $\underline{\sigma}$. Loosely speaking, we have $P \in \mathcal{P}$ if

$$B_t = \int_0^t \sigma_s dW_s \quad P\text{-almost surely}$$

for a $[\underline{\sigma}, \overline{\sigma}]$-valued, adapted process σ, where W is a standard Brownian motion.

We model the forward rate as a diffusion process in the spirit of Heath, Jarrow, and Morton [27]. The forward rate with maturity T at time t is denoted by $f_t(T)$ for $t \leq T \leq \tau$, where τ is a fixed, finite time horizon. We assume that the forward rate evolves according to the dynamics

$$f_t(T) = f_0(T) + \int_0^t \alpha_s(T) ds + \int_0^t \beta_s(T) dB_s + \int_0^t \gamma_s(T) d\langle B \rangle_s$$

for some initial, integrable forward curve $f_0 : [0, \tau] \to \mathbb{R}$ and suitable processes α, β, and γ to be specified. The difference to the classical model without volatility uncertainty is that there is an additional drift term depending on the quadratic variation of the G-Brownian motion. We need the quadratic variation process in order to obtain an arbitrage-free model as it is described below. However, due to the uncertainty about the volatility, the quadratic variation is an uncertain process. Thus, we add an additional drift term to the dynamics of the forward rate.

The forward rate determines the remaining quantities on the bond market. The bond market consists of zero-coupon bonds, which are defined by

$$P_t(T) := \exp\Big(-\int_t^T f_t(s)ds\Big),$$

and the money-market account, which is given by

$$M_t := \exp\Big(\int_0^t r_s ds\Big),$$

where r denotes the short rate process, defined by $r_t := f_t(t)$. The money-market account is used as the numéraire, i.e., we focus on the discounted bonds,

$$\tilde{P}_t(T) := M_t^{-1} P_t(T).$$

Since we want the bond market to be arbitrage-free, we model the forward rate in such a way that it satisfies the drift condition of the companion paper [29]. In particular, we model the forward rate directly under the risk-neutral sublinear expectation. Thus, we assume that the drift terms of the forward rate are given by

$$\alpha_t(T) := 0, \quad \gamma_t(T) := \beta_t(T) b_t(T),$$

where

$$b_t(T) := \int_t^T \beta_t(s)ds.$$

This implies that the discounted bonds are symmetric G-martingales, satisfying

$$\tilde{P}_t(T) = \tilde{P}_0(T) - \int_0^t b_s(T)\tilde{P}_s(T)dB_s,$$

and hence, that the bond market is arbitrage-free [29]. Here we see that we need the second drift term in the forward rate dynamics to obtain an arbitrage-free model.

Moreover, we want to obtain specific pricing formulas for common interest rate derivatives. Therefore, we assume that the diffusion coefficient β is a deterministic function $\beta : [0,\tau]^2 \to \mathbb{R}$, which is Borel measurable and bounded. This is similar to the classical case without volatility uncertainty, in which we obtain analytical pricing formulas by assuming that the diffusion coefficient is deterministic. So our model corresponds to a forward rate model with Gaussian forward rates. The assumption ensures that the forward rate meets all regularity assumptions of the accompanying paper [29]. Additionally, it implies that the process $b(T)$ is bounded and continuous for $T \leq \tau$.

3 Risk-Neutral Valuation

Now we extend the bond market to an additional contract for which we want to find a price. A typical contract in fixed income markets consists of a stream of cashflows. So let us consider a contract X which has a payoff of ξ_i at each time T_i for a fixed tenor structure

$$0 < T_0 < T_1 < ... < T_N = \tau.$$

The price at time t of such a contract is denoted by X_t. As for the bonds, we consider the discounted payoff

$$\tilde{X} := \sum_{i=0}^{N} M_{T_i}^{-1} \xi_i$$

and the discounted price

$$\tilde{X}_t := M_t^{-1} X_t,$$

where we assume that $M_{T_i}^{-1}\xi_i \in L_G^2(\Omega_{T_i})$ for all $i = 0, 1, ..., N$ to have some regularity.

The pricing of contracts in the presence of volatility uncertainty differs from the traditional literature. Classical arbitrage pricing theory suggests that prices are determined by computing the expected discounted payoff under the risk-neutral measure. The important difference in the case of volatility uncertainty is that the risk-neutral sublinear expectation is nonlinear. In particular, this implies

$$\hat{\mathbb{E}}[\tilde{X}] \geq -\hat{\mathbb{E}}[-\tilde{X}], \tag{3.1}$$

that is, the upper expectation does not necessarily coincide with the lower expectation. Thus, we distinguish between symmetric and asymmetric contracts. We consider two contracts, a contract X^S, which has a symmetric payoff, and a contract X^A, which has an asymmetric payoff. Strictly speaking, this means that $\tilde{X}^S$ satisfies (3.1) with equality and for $\tilde{X}^A$ the inequality (3.1) is strict. Of course, $\tilde{X}^S$ and $\tilde{X}^A$ are defined as above by considering different payoffs ξ_i^S and ξ_i^A, respectively. The related prices are denoted by X_t^S, $\tilde{X}_t^S$, X_t^A, and $\tilde{X}_t^A$.

A natural approach to pricing contracts under volatility uncertainty is the following. In the case of a symmetric payoff, we proceed as in the classical case without volatility uncertainty and choose the expected discounted payoff as a price for the contract. In the case of an asymmetric payoff, we use the upper and lower expectation as bounds for the price, which is a typical approach in the literature on model uncertainty. Hence, we assume that

$$\tilde{X}_t^S = \hat{\mathbb{E}}_t[\tilde{X}^S],$$

where $\hat{\mathbb{E}}_t$ denotes the conditional sublinear expectation, and

$$\hat{\mathbb{E}}[\tilde{X}^A] > \tilde{X}_0^A > -\hat{\mathbb{E}}[-\tilde{X}^A].$$

Since X^S has a symmetric payoff, we know by the martingale representation theorem for symmetric G-martingales [42, 46] that there exists a suitable process H such that

$$\tilde{X}_t^S = \hat{\mathbb{E}}[\tilde{X}^S] + \int_0^t H_s dB_s.$$

The reason we only impose assumptions on the price of the asymmetric contract at time 0 is described below. Based on these assumptions, we want to show that the extended bond market is arbitrage-free, since this enables us to use the risk-neutral sublinear expectation to determine prices as described above.

In order to show that the market is arbitrage-free, we introduce the following notions related to the extended bond market. As in the companion paper [29], we allow for trading an infinite number of bonds, which is based on the idea of Björk, Di Masi, Kabanov, and Runggaldier [9]. The symmetric contract can be traded dynamically, but we only allow static trading strategies for the asymmetric contract.

Definition 3.1. *An admissible market strategy* $\pi = (\pi^0, \pi^S, \pi^A)$ *is a triple consisting of a process* $\pi^0 \in \tilde{M}_G^2(0,\tau)$ *such that* $\pi^0 b\tilde{P} \in \tilde{M}_G^2(0,\tau)$*, a process* $\pi^S \in M_G^2(0,\tau)$ *such that* $\pi^S H \in M_G^2(0,\tau)$*, and a constant* $\pi^A \in \mathbb{R}$*. The corresponding portfolio value at terminal time* τ *is given by*

$$\tilde{v}_\tau(\pi) := \int_0^\tau \int_0^T \pi_t^0(T) d\tilde{P}_t(T) dT + \int_0^\tau \pi_t^S d\tilde{X}_t^S + \pi^A(\tilde{X}^A - \tilde{X}_0^A). \tag{3.2}$$

The space $M_G^2(0,\tau)$ is the space of admissible integrands in the calculus of G-Brownian motion. The space $\tilde{M}_G^2(0,T)$ is introduced in the accompanying paper [29] and consists of processes depending on two time indices and for which we can define stochastic integrals with an additional integrator. Hence, the assumptions ensure that the integrals in (3.2) are well-defined. The assumption that the asymmetric contract can only be traded statically might seem restrictive. This is a common assumption in the literature on robust finance, since it is important for excluding arbitrage. In our case, the assumption is also reasonable, since most contracts in fixed income markets are traded over-the-counter. Moreover, we use the quasi-sure definition of arbitrage, which is commonly used in the literature on model uncertainty.

Definition 3.2. *An admissible market strategy* π *is called arbitrage strategy if it holds*

$$\tilde{v}_\tau(\pi) \geq 0 \quad \textit{quasi-surely} \quad \textit{and} \quad P\big(\tilde{v}_\tau(\pi) > 0\big) > 0 \quad \textit{for at least one } P \in \mathcal{P}.$$

We say that the extended bond market is arbitrage-free if there is no arbitrage strategy.

The following proposition shows that we can use the risk-neutral sublinear expectation as a pricing measure. Therefore, the succeeding sections are devoted to pricing derivatives by evaluating the sublinear expectation of discounted cashflows.

Proposition 3.1. *The extended bond market is arbitrage-free.*

The assertion can be verified intuitively by the following observations. The bond market itself is arbitrage-free by the results of the accompanying work [29]. Adding an additional symmetric G-martingale to the market does not change the result. Thus, the market can be enlarged by the symmetric contract without admitting arbitrage opportunities. The pricing of derivatives with an asymmetric payoff was studied by Vorbrink [47] in a similar setting. He showed that prices bounded by the upper and lower expectation do not admit arbitrage. Hence, we can also add the asymmetric contract to the market without admitting arbitrage, since its price is bounded by the upper and lower expectation of its payoff.

4 Forward Sublinear Expectation

In the classical case without volatility uncertainty, interest rate derivatives are priced under the forward measure. The evaluation of the expected discounted payoff of an interest rate derivative can be very elaborate. This is due to the fact that the discount factor, in addition to the payoff, is stochastic. The common way to avoid this issue is the forward measure approach. The forward measure is equivalent to the pricing measure and defined by choosing a particular density process. The density process is defined in such a way that the expectation of the discounted payoff under the risk-neutral measure can be rewritten as the expectation of the payoff under the forward measure discounted by a zero-coupon bond. Thus, by changing the measure, we can replace the stochastic discount factor by the current bond price, which is already determined by the model.

In this model, we define the *forward sublinear expectation* to simplify the pricing of interest rate derivatives. We choose the same density process as the one which is used to define the forward measure. Then we use it to define the forward sublinear expectation, which corresponds to the expectation under the forward measure. For $t \leq T \leq \tau$, we define the density process X^T by

$$X_t^T := \tfrac{\tilde{P}_t(T)}{P_0(T)}.$$

Hence, the density process satisfies the G-stochastic differential equation

$$X_t^T = 1 - \int_0^t b_s(T) X_s^T dB_s.$$

Definition 4.1. *For $T \leq \tau$ and $\xi \in L_G^p(\Omega_T)$, where $p > 1$, we define the T-forward sublinear expectation $\hat{\mathbb{E}}^T$ by*

$$\hat{\mathbb{E}}_t^T[\xi] := Y_t^{T,\xi},$$

where $Y^{T,\xi}$ solves the G-backward stochastic differential equation

$$Y_t^T = \xi - \int_t^T b_s(T) Z_s d\langle B\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t).$$

By the results from Hu, Ji, Peng, and Song [30], we know that the solution is given by

$$Y_t^{T,\xi} = (X_t^T)^{-1} \hat{\mathbb{E}}_t[X_T^T \xi]$$

and that $\hat{\mathbb{E}}_t^T$ defines a consistent sublinear expectation. Thus, we basically arrive at the same expression as in the classical definition of the forward measure. For further details regarding G-backward stochastic differential equations, the reader may refer to the work of Hu, Ji, Peng, and Song [30].

As an immediate consequence we find that the valuation of discounted cashflows reduces to determining the forward sublinear expectation of a cashflow. Furthermore, we find that the forward rate and the forward price process $X^{T,T'}$, defined as

$$X_t^{T,T'} := \tfrac{P_t(T')}{P_t(T)}$$

for $t \leq T < T' \leq \tau$, are symmetric G-martingales under the forward sublinear expectation. In particular, this implies that the upper and lower expectation of the short rate are given by the forward rate.

Proposition 4.1. *(i) For $t \leq T \leq \tau$ and $\xi \in L_G^p(\Omega_T)$, where $p > 1$, we have*

$$M_t \hat{\mathbb{E}}_t[M_T^{-1} \xi] = P_t(T) \hat{\mathbb{E}}_t^T[\xi].$$

(ii) The forward rate is a symmetric G-martingale under $\hat{\mathbb{E}}^T$. In particular, we obtain the robust expectations hypothesis,

$$\hat{\mathbb{E}}_t^T[r_T] = f_t(T) = -\hat{\mathbb{E}}_t^T[-r_T].$$

(iii) The forward price is a symmetric G-martingale under $\hat{\mathbb{E}}^T$ *and satisfies*

$$X_t^{T,T'} = X_0^{T,T'} - \int_0^t \sigma_s(T,T') X_s^{T,T'} dB_s^T,$$

where B^T *is a G-Brownian motion under* $\hat{\mathbb{E}}^T$ *and* $\sigma(T,T')$ *is defined by*

$$\sigma_t(T,T') := b_t(T') - b_t(T).$$

Moreover, we have $X^{T,T'} \in M_G^p(0,T)$ *for all* $p \geq 2$.

The first part of Proposition 4.1 follows by a simple calculation. The second part relies on the Girsanov transformation of Hu, Ji, Peng, and Song [30], which shows that the forward rate is a symmetric G-martingale. For the third part, we apply Itô's formula to the forward price and then we use the Girsanov transformation as in the second part.

We call Proposition 4.1 *(ii)* the *robust expectations hypothesis*. The traditional expectations hypothesis states that the forward rate reflects the future expectations of the short rate. In the classical case without volatility uncertainty, we know that the forward rate is a martingale under the forward measure, which was introduced by Geman [25]. Therefore, the expectations hypothesis holds true under the forward measure. In our case, we obtain a much stronger version, a robust expectations hypothesis. This is because the forward rate is a symmetric G-martingale under the forward sublinear expectation. Thus, the forward rate reflects the upper expectation of the short rate and the lower expectation of the short rate. In particular, it implies that the forward rate reflects the expectation of the short rate in each possible scenario for the volatility.

For certain bond options, the problem of finding a price can be reduced to computing the price in the corresponding model without volatility uncertainty. If we want to price a bond option, i.e., a contract which has a payoff given by a function depending on a selection of bonds, we need to evaluate the forward sublinear expectation. In general, this can be elaborate. However, if we impose further assumptions on the payoff function, we can show that the upper, respectively lower, expectation is given by the price from the classical model with the highest, respectively lowest, possible volatility.

Proposition 4.2. *For* $1 \leq n \leq N$ *and* $N' := N - n + 1$, *let* $\varphi : \mathbb{R}^{N'} \to \mathbb{R}$ *be a convex function such that*

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y| \tag{4.1}$$

for a positive integer m *and a constant* $C > 0$ *and, for* $\sigma > 0$ *and* $T < T_n$, *let the function* $u_\sigma : [0,T] \times \mathbb{R}^{N'} \to \mathbb{R}$ *be defined by*

$$u_\sigma(t, x_n, ..., x_N) := \mathbb{E}_P[\varphi(X_T^n, ..., X_T^N)],$$

where the processes X^i, *for* $i = n, ..., N$, *are given by*

$$X_T^i = x_i + \sigma \int_t^T \sigma_s(T, T_i) X_s^i dW_s$$

and W *is a standard Brownian motion under* P. *Then it holds*

$$\hat{\mathbb{E}}_t^T[\varphi(X_T^{T,T_n}, ..., X_T^{T,T_N})] = u_{\overline{\sigma}}(t, X_t^{T,T_n}, ..., X_t^{T,T_N})$$

and

$$-\hat{\mathbb{E}}_t^T[-\varphi(X_T^{T,T_n}, ..., X_T^{T,T_N})] = u_{\underline{\sigma}}(t, X_t^{T,T_n}, ..., X_t^{T,T_N}).$$

The assertion follows by using the nonlinear Feynman-Kac formula of Hu, Ji, Peng, and Song [30] and the classical Feynman-Kac formula. First of all, we apply the nonlinear Feynman-Kac formula, which tells us that the forward sublinear expectation is the solution to a nonlinear partial differential equation. However, due to the convexity of the payoff function, the nonlinear partial differential equation reduces to a linear one. Then we can apply the classical Feynman-Kac formula to obtain the result.

5 Common Interest Rate Derivatives

In this section, we consider typical contracts in fixed income markets and show how to price them with the tools from the preceding sections. In particular, we study linear contracts such as fixed coupon bonds, floating rate notes, and interest rate swaps and nonlinear contracts such as swaptions, caps, and floors. So let us consider a contract X which pays ξ_i at each time T_i. The tenor structure is the one from Section 3. The discounted payoff is then given by

$$\tilde{X} = \sum_{i=0}^{N} M_{T_i}^{-1} \xi_i.$$

In order to price the contract, we are interested in $M_t \hat{\mathbb{E}}_t[\tilde{X}]$ and $-M_t \hat{\mathbb{E}}_t[-\tilde{X}]$ for $t \leq T_0$. These two expressions tell us if the contract has a symmetric or asymmetric payoff and how to price the contract.

5.1 Fixed Coupon Bonds

A fixed coupon bond is a contract which pays a fixed rate of interest $K > 0$ on a nominal value, which is normalized to 1, at each payment date T_i and the nominal value at the last payment date T_N. Hence, the cashflows are given by

$$\xi_i = (T_i - T_{i-1})K$$

for $i = 1, ..., N-1$ and

$$\xi_N = 1 + (T_N - T_{N-1})K.$$

The discounted payoff is given by

$$\tilde{X} = M_{T_N}^{-1} + \sum_{i=1}^{N} M_{T_i}^{-1} (T_i - T_{i-1})K.$$

Fixed coupon bonds can be priced as in the classical case without volatility uncertainty. Due to its simple payoff structure, we find that the contract has a symmetric payoff. Moreover, we obtain the same expression for the price as the one obtained by traditional term structure models.

Proposition 5.1. *Let $\tilde{X}$ be the discounted payoff of a fixed coupon bond. Then it holds*

$$M_t \hat{\mathbb{E}}_t[\tilde{X}] = P_t(T_N) + \sum_{i=1}^{N} P_t(T_i)(T_i - T_{i-1})K = -M_t \hat{\mathbb{E}}_t[-\tilde{X}].$$

The main difference compared to traditional models, in genereal regarding pricing interest rate derivatives, is that the contract cannot be decomposed into single cashflows. The common approach to pricing contracts in fixed income markets is to price every cashflow separately. This works, since the expectation under the risk-neutral measure is linear. In the presence of volatility uncertainty, the pricing measure is sublinear. Thus, such a decomposition only leads to an upper and a lower bound for the pricing measure. However, if the contract has constant cashflows such as a fixed coupon bond, the upper bound coincides with the lower bound. Hence, the contract has a symmetric payoff and we obtain a single price.

5.2 Floating Rate Notes

A floating rate note is a fixed coupon bond in which the fixed rate K is replaced by the simply compounded spot rate $L_{T_{i-1}}(T_i)$, which is reset at each payment date T_i. For $t \leq T \leq \tau$, the simply compounded spot rate $L_t(T)$ is defined by

$$L_t(T) := \tfrac{1}{T-t}(\tfrac{1}{P_t(T)} - 1).$$

The cashflows are then given by

$$\xi_i = (T_i - T_{i-1})L_{T_{i-1}}(T_i)$$

for $i = 1, ..., N-1$ and

$$\xi_N = 1 + (T_N - T_{N-1})L_{T_{N-1}}(T_N).$$

The discounted payoff is given by

$$\tilde{X} = M_{T_N}^{-1} + \sum_{i=1}^{N} M_{T_i}^{-1}(T_i - T_{i-1})L_{T_{i-1}}(T_i).$$

Floating rate notes can also be priced as in the classical case without volatility uncertainty. Although the cashflows are not constant, the contract yet has a symmetric payoff. As in the classical case, the price is simply given by the price of a zero-coupon bond with maturity T_0.

Proposition 5.2. *Let $\tilde{X}$ be the discounted payoff of a floating rate note. Then it holds*

$$M_t\hat{\mathbb{E}}_t[\tilde{X}] = P_t(T_0) = -M_t\hat{\mathbb{E}}_t[-\tilde{X}].$$

The assertion can be proven by following the same steps as in the previous subsection. We use the sublinearity to obtain an upper and a lower bound. Then we show by a few calculations that the cashflows have a symmetric payoff. Thus, the upper and the lower bound coincide, which shows that the contract has a symmetric payoff. In particular, we obtain the same expression for the price as the one from classical models.

5.3 Interest Rate Swaps

An interest rate swap exchanges the floating rate $L_{T_{i-1}}(T_i)$ with a fixed rate K at each payment date T_i. Without loss of generality we consider a payer interest rate swap, that is, we pay the fixed rate K and receive the floating rate $L_{T_{i-1}}(T_i)$. Hence, the cashflows are given by

$$\xi_i = (T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)$$

for $i = 1, ..., N$ and the discounted payoff is given by

$$\tilde{X} = \sum_{i=1}^{N} M_{T_i}^{-1}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big).$$

We obtain the same pricing formula for interest rate swaps as in traditional models. The payoff is symmetric and the price is given by a linear combination of zero-coupon bonds with different maturities. In particular, this implies that the swap rate, i.e., the fixed rate K which makes the value of the contract zero, is uniquely determined and does not differ from the expression obtained by standard models.

Proposition 5.3. *Let $\tilde{X}$ be the discounted payoff of an interest rate swap. Then it holds*

$$M_t\hat{\mathbb{E}}_t[\tilde{X}] = P_t(T_0) - P_t(T_N) - \sum_{i=1}^{N} P_t(T_i)(T_i - T_{i-1})K = -M_t\hat{\mathbb{E}}_t[-\tilde{X}].$$

The assertion basically follows from Proposition 5.1 and Proposition 5.2. The payoff of an interest rate swap can be written as the difference between the payoff of a floating rate note and the payoff of a fixed coupon bond. Since both of these payoffs are symmetric, the payoff of the swap is also symmetric. The price is then given by the difference between the price of a floating rate note and the price of a fixed coupon bond.

5.4 Swaptions

A swaption gives the buyer the right to enter an interest rate swap at the first payment date T_0. Hence, by Proposition 5.3, the cashflows are given by

$$\xi_0 = \Big(1 - P_{T_0}(T_n) - \sum_{i=1}^{N} P_{T_0}(T_i)(T_i - T_{i-1})K\Big)^+$$

and $\xi_i = 0$ for $i = 1, ..., N$. The discounted payoff is given by

$$\tilde{X} = M_{T_0}^{-1}\Big(1 - P_{T_0}(T_n) - \sum_{i=1}^{N} P_{T_0}(T_i)(T_i - T_{i-1})K\Big)^+.$$

Swaptions can be priced by using the pricing formulas from traditional models to compute the upper and lower bound for the price. Due to the nonlinearity of the payoff function, the upper and the lower expectation of the discounted payoff do not necessarily coincide. Thus, the contract has an asymmetric payoff. The related pricing bounds are given by the price from the classical case with the highest and lowest possible volatility.

Theorem 5.1. *Let $\tilde{X}$ be the discounted payoff of a swaption and, for $\sigma > 0$, let the function $u_\sigma : [0, T_0] \times \mathbb{R}^N \to \mathbb{R}$ be defined by*

$$u_\sigma(t, x_1, ..., x_N) := \mathbb{E}_P\Big[\Big(1 - X_{T_0}^N - \sum_{i=1}^{N} X_{T_0}^i(T_i - T_{i-1})K\Big)^+\Big],$$

where the processes X^i, for $i = 1, ..., N$, are given by

$$X^i_{T_0} = x_i - \sigma \int_t^{T_0} \sigma_s(T_0, T_i) X^i_s dW_s$$

and W is a standard Brownian motion under P. Then it holds

$$M_t \hat{\mathbb{E}}_t[\tilde{X}] = P_t(T_0) u_{\overline{\sigma}}\big(t, \tfrac{P_t(T_1)}{P_t(T_0)}, ..., \tfrac{P_t(T_N)}{P_t(T_0)}\big)$$

and

$$-M_t \hat{\mathbb{E}}_t[-\tilde{X}] = P_t(T_0) u_{\underline{\sigma}}\big(t, \tfrac{P_t(T_1)}{P_t(T_0)}, ..., \tfrac{P_t(T_N)}{P_t(T_0)}\big).$$

The theorem is a straightforward application of Proposition 4.2. We can easily show that the payoff function of a swaption is convex. In addition, we can show that the payoff function satisfies the inequality (4.1). Hence, we can apply Proposition 4.2, which proves the assertion.

5.5 Caps and Floors

A cap gives the buyer the right to exchange the floating rate $L_{T_{i-1}}(T_i)$ with a fixed rate K at each payment date T_i. The cashflows are called caplets and are given by

$$\xi_i = (T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+$$

for $i = 1, ..., N$. The discounted payoff is given by

$$\tilde{X} = \sum_{i=1}^N M^{-1}_{T_i}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+.$$

Similar to swaptions, we can use the pricing formulas from traditional models to compute the upper and the lower bound for the price of a cap. The upper bound and the lower bound are given by the price from the classical case with highest and lowest possible volatility, respectively. These prices are obtained by computing the prices of put options on the forward price.

Theorem 5.2. *Let $\tilde{X}$ be the discounted payoff of a cap and, for $i = 1, ..., N$ and $\sigma > 0$, let the function $u^i_\sigma : [0, T_{i-1}] \times \mathbb{R} \to \mathbb{R}$ be defined by*

$$u^i_\sigma(t, x_i) := \mathbb{E}_P[(K_i - X^i_{T_{i-1}})^+],$$

where the strike price K_i is given by

$$K_i := \tfrac{1}{1+(T_i - T_{i-1})K},$$

the process X^i satisfies

$$X^i_{T_{i-1}} = x_i - \sigma \int_t^{T_{i-1}} \sigma_s(T_{i-1}, T_i) X^i_s dW_s,$$

and W is a standard Brownian motion under P. Then it holds

$$M_t \hat{\mathbb{E}}_t[\tilde{X}] = \sum_{i=1}^N P_t(T_{i-1}) \tfrac{1}{K_i} u^i_{\overline{\sigma}}\big(t, \tfrac{P_t(T_i)}{P_t(T_{i-1})}\big)$$

and

$$-M_t \hat{\mathbb{E}}_t[-\tilde{X}] = \sum_{i=1}^N P_t(T_{i-1}) \tfrac{1}{K_i} u^i_{\underline{\sigma}}\big(t, \tfrac{P_t(T_i)}{P_t(T_{i-1})}\big).$$

The proof of Theorem 5.2 differs from all of the preceding cases. Due to the asymmetric payoff of each cashflow, we cannot price each caplet separately. Moreover, we cannot directly apply Proposition 4.2 as in the proof of Theorem 5.1, since the contract has several payoffs occuring at different times. Thus, we proceed via backward induction. That means, we start to price the last caplet from the sum and then we work recursively backwards until we arrive at the first caplet. In the end, we obtain the pricing formula from classical models with the highest and lowest possible volatility for the upper and lower bound of the price, respectively.

A floor gives the buyer the right to exchange a fixed rate K with the floating rate $L_{T_{i-1}}(T_i)$ at each payment date T_i. The cashflows are called floorlets and are given by

$$\xi_i = (T_i - T_{i-1})\big(K - L_{T_{i-1}}(T_i)\big)^+$$

for $i = 1, ..., N$. The discounted payoff is given by

$$\tilde{X} = \sum_{i=1}^{N} M_{T_i}^{-1}(T_i - T_{i-1})\big(K - L_{T_{i-1}}(T_i)\big)^+.$$

Floors can be priced in the same manner as caps. The only difference is that we need to compute prices of call options instead of put options on the forward price to obtain the pricing bounds.

Theorem 5.3. *Let $\tilde{X}$ be the discounted payoff of a floor and, for $i = 1, ..., N$ and $\sigma > 0$, let the function $v_\sigma^i : [0, T_{i-1}] \times \mathbb{R} \to \mathbb{R}$ be defined by*

$$v_\sigma^i(t, x_i) := \mathbb{E}_P[(X_{T_{i-1}}^i - K_i)^+],$$

where the strike price K_i is given by

$$K_i := \tfrac{1}{1+(T_i - T_{i-1})K},$$

the process X^i satisfies

$$X_{T_{i-1}}^i = x_i - \sigma \int_t^{T_{i-1}} \sigma_s(T_{i-1}, T_i) X_s^i dW_s,$$

and W is a standard Brownian motion under P. Then it holds

$$M_t \hat{\mathbb{E}}_t[\tilde{X}] = \sum_{i=1}^{N} P_t(T_{i-1}) \tfrac{1}{K_i} v_{\overline{\sigma}}^i\big(t, \tfrac{P_t(T_i)}{P_t(T_{i-1})}\big)$$

and

$$-M_t \hat{\mathbb{E}}_t[-\tilde{X}] = \sum_{i=1}^{N} P_t(T_{i-1}) \tfrac{1}{K_i} v_{\underline{\sigma}}^i\big(t, \tfrac{P_t(T_i)}{P_t(T_{i-1})}\big).$$

The proof of Theorem 5.3 works analogously to the proof of Theorem 5.2 with a few subtle changes.

6 Market Incompleteness

Traditional term structure models suggest that fixed income markets are complete. In the classical HJM framework, the drift condition uniquely specifies the drift of the forward rate in terms of the diffusion coefficient. Moreover, additional assumptions on the diffusion coefficient ensure that the equivalent martingale measure is unique. Hence, the bond market is complete, which implies that any contingent claim can be hedged by a portfolio consisting of bonds. Term structures predicted by traditional short rate models, in particular, meet these conditions.

Indeed, fixed income markets seem to be incomplete according to a phenomenon referred to as *unspanned stochastic volatility*. By using data on interest rate swaps, caps, and floors, Collin-Dufresne and Goldstein [16] provide empirical evidence for the fact that certain interest rate derivatives are driven by factors which do not affect the term structure. In particular, this implies that such derivatives are exposed to risks that cannot be hedged by bonds and thus, that fixed income markets are incomplete. Therefore, Collin-Dufresne and Goldstein [16] examine which classes of models are rich enough to exhibit unspanned stochastic volatility. Subsequently, people have been developing models that are able to display unspanned stochastic volatility [e.g. 14, 23, 24]. According to the definition of Filipović, Larsson, and Statti [23], a model exhibits unspanned stochastic volatility if there exist contingent claims that cannot be replicated.

Due to the pricing formulas from the preceding section, the present model naturally displays unspanned stochastic volatility. The reason is that derivatives which are exposed to volatility risk cannot be replicated by trading bonds. The formal argument works as follows. Let us suppose that there is a contract $\tilde{X}$ that has an asymmetric payoff and let us suppose that there exists a hedging strategy for $\tilde{X}$. That means there exists some initial capital $x \in \mathbb{R}$ and an admissible market strategy π such that $\pi^S = 0$, $\pi^A = 0$, and

$$x + \int_0^\tau \int_0^T \pi_t^0(T) d\tilde{P}_t(T) dT = \tilde{X} \quad \text{quasi surely.}$$

However, the upper and lower expectation of the left-hand side are the same, since the double integral can be written as an integral with respect to G-Brownian motion. Hence, this implies that $\tilde{X}$ has a symmetric payoff, which contradicts our assumptions. This shows that derivatives with an asymmetric payoff cannot be hedged by a portfolio of bonds. Since swaptions, caps, and floors have an asymmetric payoff, we know that these contracts cannot be hedged. By the definition of Filipović, Larsson, and Statti [23], this implies that the model exhibits unspanned stochastic volatility. Moreover, the pricing formulas from this section are in line with the empirical findings of Collin-Dufresne and Goldstein [16]. The prices of caps and floors are affected by the bounds for the volatility, which do not affect the prices of swaps.

7 Conclusion

In the present paper, we deal with the pricing of contracts in fixed income markets under volatility uncertainty. The starting point is an arbitrage-free bond market under volatility uncertainty. Such a framework leads to a sublinear pricing measure, which can be used to determine prices of simple derivatives and pricing bounds of more complex derivatives. To simplify the pricing procedure, we introduce the forward sublinear expectation, under which the expectations hypothesis holds in a robust sense. We apply these preliminary results by pricing the most common interest rate derivatives: fixed coupon bonds, floating rate notes, interest rate swaps, swaptions, caps, and floors. We obtain a single price for fixed coupon bonds, floating rate notes, and interest rate swaps, whereas swaptions, caps, and floors lead to a range of prices. The pricing formula for fixed coupon bonds, floating rate notes, and interest rate swaps is the same as the one obtained by traditional models. The range of prices for swaptions, caps, and floors is bounded by the prices from classical models with the highest and lowest possible volatility. This implies that swaptions, caps, and floors cannot be hedged and hence, that volatility uncertainty yields unspanned stochastic volatility. Concluding, our findings show that some results from the traditional literature, such as the expectations hypothesis and the pricing formulas for fixed coupon bonds, floating rate notes, and interest rate swaps, are robust with respect to changes in the volatility and how to robustify results from the traditional literature which are not, such as the pricing formulas for swaptions, caps, and floors.

Appendix

A Proof of Proposition 3.1

Let us assume that there exists an arbitrage strategy π. First of all, we examine the case in which X^A is not traded, i.e., $\pi^A = 0$. By the definition of arbitrage, we know that

$$\tilde{v}_\tau(\pi) \geq 0 \quad \text{quasi-surely} \quad \text{and} \quad P\big(\tilde{v}_\tau(\pi) > 0\big) > 0 \quad \text{for at least one } P \in \mathcal{P}.$$

This implies

$$\hat{\mathbb{E}}[\tilde{v}_\tau(\pi)] > 0. \tag{A.1}$$

Using the sublinearity of $\hat{\mathbb{E}}$, we get

$$\hat{\mathbb{E}}[\tilde{v}_\tau(\pi)] \leq \hat{\mathbb{E}}\Big[\int_0^\tau \int_0^T \pi^0(t,T) d\tilde{P}(t,T) dT\Big] + \hat{\mathbb{E}}\Big[\int_0^\tau \pi_t^S d\tilde{X}_t^S\Big] + \hat{\mathbb{E}}[\pi^A(\tilde{X}^A - \tilde{X}_0^A)].$$

The first summand on the right-hand side vanishes [29]. The same applies to the remaining two, since $\tilde{X}_t^S$ is an integral with respect to G-Brownian motion and $\pi^A = 0$. Thus, we obtain

$$\hat{\mathbb{E}}[\tilde{v}_\tau(\pi)] \leq 0,$$

which is a contradiction to (A.1).

Now we consider the case in which $\pi^A \neq 0$. By the definition of arbitrage, we have

$$\tilde{v}_\tau(\pi) \geq 0 \quad \text{quasi-surely},$$

which is equivalent to

$$-\tilde{v}_\tau(\pi) \leq 0 \quad \text{quasi-surely}.$$

Thus, it holds

$$\hat{\mathbb{E}}[-\tilde{v}_\tau(\pi)] \leq 0. \tag{A.2}$$

Here we can again use the sublinearity of $\hat{\mathbb{E}}$ and the insights from the first step to get

$$\begin{aligned}\hat{\mathbb{E}}[-\tilde{v}_\tau(\pi)] \geq& \hat{\mathbb{E}}[-\pi^A(\tilde{X}^A - \tilde{X}_0^A)] - \hat{\mathbb{E}}\Big[\int_0^\tau \pi_t^S d\tilde{X}_t^S\Big] - \hat{\mathbb{E}}\Big[\int_0^\tau \int_0^T \pi^0(t,T)d\tilde{P}(t,T)dT\Big] \\ =& \hat{\mathbb{E}}[-\pi^A(\tilde{X}^A - \tilde{X}_0^A)].\end{aligned}$$

Using the properties of the sublinear expectation, we obtain

$$\hat{\mathbb{E}}[-\pi^A(\tilde{X}^A - \tilde{X}_0^A)] = (\pi^A)^+(\hat{\mathbb{E}}[-\tilde{X}^A] + \tilde{X}_0^A) + (\pi^A)^-(\hat{\mathbb{E}}[\tilde{X}^A] - \tilde{X}_0^A) > 0,$$

since

$$\hat{\mathbb{E}}[\tilde{X}^A] > \tilde{X}_0^A > -\hat{\mathbb{E}}[-\tilde{X}^A].$$

This is a contradiction to (A.2). Hence, there is no arbitrage strategy.

B Proofs of Section 4

B.1 Proof of Proposition 4.1

Part (i) follows by

$$\begin{aligned}M_t\hat{\mathbb{E}}_t[M_T^{-1}\xi] =& P_t(T)M_t\tfrac{P_0(T)}{P_t(T)}\hat{\mathbb{E}}_t[M_T^{-1}\tfrac{P_T(T)}{P_0(T)}\xi] \\ =& P_t(T)(X_t^T)^{-1}\hat{\mathbb{E}}_t[X_T^T\xi] \\ =& P_t(T)\hat{\mathbb{E}}_t^T[\xi].\end{aligned}$$

For part (ii), we define the process B^T, for $t \leq T$, by

$$B_t^T := B_t + \int_0^t b_s(T)d\langle B\rangle_s.$$

By the Girsanov transformation of Hu, Ji, Peng, and Song [30], we know that B^T is a G-Brownian motion under $\hat{\mathbb{E}}^T$. Now we rewrite the dynamics of the forward rate as

$$\begin{aligned}f_t(T) =& f_0(T) + \int_0^t \beta_s(T)dB_s + \int_0^t \beta_s(T)b_s(T)d\langle B\rangle_s \\ =& f_0(T) + \int_0^t \beta_s(T)dB_s^T.\end{aligned}$$

Hence, the forward rate is a symmetric G-martingale under $\hat{\mathbb{E}}^T$.

To obtain part (iii), we apply the Itô formula from Li and Peng [33] to obtain

$$\begin{aligned}X_t^{T,T'} =& \tfrac{\tilde{P}_0(T')}{\tilde{P}_0(T)} - \int_0^t \tfrac{1}{\tilde{P}_s(T)}b_s(T')\tilde{P}_s(T')dB_s + \int_0^t \tfrac{\tilde{P}_s(T')}{\tilde{P}_s(T)^2}b_s(T)\tilde{P}_s(T)dB_s \\ &- \int_0^t \tfrac{1}{\tilde{P}_s(T)^2}b_s(T')b_s(T)\tilde{P}_s(T')\tilde{P}_s(T)d\langle B\rangle_s \\ &+ \int_0^t \tfrac{\tilde{P}_s(T')}{\tilde{P}_s(T)^3}b_s(T)^2\tilde{P}_s(T)^2d\langle B\rangle_s\end{aligned}$$

$$
\begin{aligned}
=&X_0^{T,T'} - \int_0^t \sigma_s(T,T')X_s^{T,T'}dB_s - \int_0^t \sigma_s(T,T')X_s^{T,T'}b_s(T)d\langle B\rangle_s \\
=&X_0^{T,T'} - \int_0^t \sigma_s(T,T')X_s^{T,T'}dB_s^T.
\end{aligned}
$$

Thus, we know that $X^{T,T'}$ is a symmetric G-martingale under $\hat{\mathbb{E}}^T$. Moreover, since the G-stochastic differential equation above is linear in $X^{T,T'}$ and $\sigma(T,T') \in M_G^p(0,T)$ for all $p \geq 2$, we know that $X^{T,T'} \in M_G^p(0,T)$ for all $p \geq 2$ by the results from Li, Lin, and Lin [32].

B.2 Proof of Proposition 4.2

By Definition 4.1, we know that

$$
\hat{\mathbb{E}}_t^T[\varphi(X_T^{T,T_n},...,X_T^{T,T_N})] = Y_t^{T,\varphi(X_T^{T,T_n},...,X_T^{T,T_N})}
$$

solves the G-backward stochastic differential equation

$$
Y_t^{T,\varphi(X_T^{T,T_n},...,X_T^{T,T_N})} = \varphi(X_T^{T,T_n},...,X_T^{T,T_N}) - \int_t^T b_s(T)Z_s d\langle B\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t).
$$

Since B^T is a G-Brownian motion under $\hat{\mathbb{E}}^T$ and K is still a G-martingale under $\hat{\mathbb{E}}^T$, we know that $Y^{T,\varphi(X_T^{T,T_n},...,X_T^{T,T_N})}$ also solves the G-backward stochastic differential equation

$$
Y_t^{T,\varphi(X_T^{T,T_n},...,X_T^{T,T_N})} = \varphi(X_T^{T,T_n},...,X_T^{T,T_N}) - \int_t^T Z_s dB_s^T - (K_T - K_t).
$$

Apart from that, by Proposition 4.1, for $i = n,...,N$, we have

$$
X_T^{T,T_i} = X_t^{T,T_i} - \int_t^T \sigma_s(T,T_i)X_s^{T,T_i}dB_s^T,
$$

where $X_t^{T,T_i} \in L_G^p(\Omega_t)$ for all $p \geq 2$.

In order to solve this G-forward backward stochastic differential equation, we can apply the nonlinear Feynman-Kac formula from Hu, Ji, Peng, and Song [30], since φ satisfies

$$
|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|
$$

for a positive integer m and a constant $C > 0$. Thus, it follows that

$$
\hat{\mathbb{E}}_t^T[\varphi(X_T^{T,T_n},...,X_T^{T,T_N})] = u(t, X_t^{T,T_n},...,X_t^{T,T_N}),
$$

where $u : [0,T] \times \mathbb{R}^{N'} \to \mathbb{R}$ is the unique viscosity solution to the nonlinear partial differential equation

$$
\begin{aligned}
\partial_t u + G\big(H(t,x,D_x^2 u)\big) =&0, \\
u(T,x_n,...,x_N) =&\varphi(x_n,...,x_N),
\end{aligned}
$$

D_x^2 denotes the Hessian, and H is given by

$$H(t,x,D_x^2u) = \sum_{i,j=n}^{N} \sigma_t(T,T_i)x_i\sigma_t(T,T_j)x_j\partial^2_{x_ix_j}u.$$

Now, since

$$u(t,x_n,...,x_N) = \hat{\mathbb{E}}_t^T[\varphi(X_T^{T,T_n},...,X_T^{T,T_N})],$$

where

$$X_T^{T,T_i} = x_i - \int_t^T \sigma_s(T,T_i)X_s^{T,T_i}dB_s^T$$

for $i = n,...,N$, we can use the sublinearity of $\hat{\mathbb{E}}_t^T$ and the convexity of φ to deduce that $u(t,\cdot)$ is convex. Hence, we know that

$$H(t,x,D_x^2u) \geq 0.$$

Therefore, the nonlinear partial differential equation becomes a linear one, i.e., u solves

$$\begin{aligned}\partial_t u + \tfrac{1}{2}\overline{\sigma}^2 H(t,x,D_x^2u) =& 0,\\ u(T,x_n,...,x_N) =& \varphi(x_n,...,x_N).\end{aligned}$$

By applying the classical Feynman-Kac formula, we get

$$u(t,x_n,...,x_N) = u_{\overline{\sigma}}(t,x_n,...,x_N),$$

which proves the first assertion.

To prove the second assertion, we repeat the first step of the proof to obtain that

$$\hat{\mathbb{E}}_t^T[-\varphi(X_T^{T,T_n},...,X_T^{T,T_N})] = u(t,X_t^{T,T_n},...,X_t^{T,T_N}),$$

where $u : [0,T] \times \mathbb{R}^{N'} \to \mathbb{R}$ is the unique viscosity solution to the nonlinear partial differential equation

$$\begin{aligned}\partial_t u + G\big(H(t,x,D_x^2u)\big) =& 0,\\ u(T,x_n,...,x_N) =& -\varphi(x_n,...,x_N).\end{aligned}$$

Moreover, we observe that $u_{\underline{\sigma}}(t,\cdot)$ is convex, since the parameter $\overline{\sigma}$ in the proof of the first assertion is arbitrary. Thus, $-u_{\underline{\sigma}}(t,\cdot)$ is concave and, by the classical Feynman-Kac formula, $-u_{\underline{\sigma}}$ solves

$$\begin{aligned}\partial_t u + \tfrac{1}{2}\underline{\sigma}^2 H(t,x,D_x^2u) =& 0,\\ u(T,x_n,...,x_N) =& -\varphi(x_n,...,x_N).\end{aligned}$$

Therefore, $-u_{\underline{\sigma}}$ also solves the nonlinear partial differential equation from above.

C Proofs of Section 5

C.1 Proof of Proposition 5.1

Using the sublinearity of $\hat{\mathbb{E}}_t$ and Proposition 4.1 (i), we get the upper bound

$$\begin{aligned} M_t\hat{\mathbb{E}}_t[\tilde{X}] \leq & M_t\hat{\mathbb{E}}_t[M_{T_N}^{-1}] + \sum_{i=1}^{N} M_t\hat{\mathbb{E}}_t[M_{T_i}^{-1}(T_i - T_{i-1})K] \\ = & P_t(T_N)\hat{\mathbb{E}}_t^{T_N}[1] + \sum_{i=1}^{N} P_t(T_i)\hat{\mathbb{E}}_t^{T_i}[(T_i - T_{i-1})K] \\ = & P_t(T_N) + \sum_{i=1}^{N} P_t(T_i)(T_i - T_{i-1})K. \end{aligned}$$

Similarly, we obatin

$$\begin{aligned} -M_t\hat{\mathbb{E}}_t[-\tilde{X}] \geq & - M_t\hat{\mathbb{E}}_t[-M_{T_N}^{-1}] - \sum_{i=1}^{N} M_t\hat{\mathbb{E}}_t[-M_{T_i}^{-1}(T_i - T_{i-1})K] \\ = & - P_t(T_N)\hat{\mathbb{E}}_t^{T_N}[-1] - \sum_{i=1}^{N} P_t(T_i)\hat{\mathbb{E}}_t^{T_i}[-(T_i - T_{i-1})K] \\ = & P_t(T_N) + \sum_{i=1}^{N} P_t(T_i)(T_i - T_{i-1})K, \end{aligned}$$

which yields the assertion.

C.2 Proof of Proposition 5.2

We obtain an upper bound by using the sublinearity of $\hat{\mathbb{E}}_t$,

$$M_t\hat{\mathbb{E}}_t[\tilde{X}] \leq M_t\hat{\mathbb{E}}_t[M_{T_N}^{-1}] + \sum_{i=1}^{N} M_t\hat{\mathbb{E}}_t[M_{T_i}^{-1}(T_i - T_{i-1})L_{T_{i-1}}(T_i)].$$

By Proposition 4.1 (i) and (iii), we get

$$M_t\hat{\mathbb{E}}_t[M_{T_N}^{-1}] = P_t(T_N)\hat{\mathbb{E}}_t^{T_N}[1] = P_t(T_N)$$

and

$$\begin{aligned} M_t\hat{\mathbb{E}}_t[M_{T_i}^{-1}(T_i - T_{i-1})L_{T_{i-1}}(T_i)] = & M_t\hat{\mathbb{E}}_t\big[M_{T_{i-1}}^{-1}M_{T_{i-1}}\hat{\mathbb{E}}_{T_{i-1}}[M_{T_i}^{-1}](T_i - T_{i-1})L_{T_{i-1}}(T_i)\big] \\ = & M_t\hat{\mathbb{E}}_t\big[M_{T_{i-1}}^{-1}P_{T_{i-1}}(T_i)\hat{\mathbb{E}}_{T_{i-1}}^{T_i}[1]\big(\tfrac{1}{P_{T_{i-1}}(T_i)} - 1\big)\big] \\ = & P_t(T_{i-1})\hat{\mathbb{E}}_t^{T_{i-1}}[1 - P_{T_{i-1}}(T_i)] \\ = & P_t(T_{i-1}) - P_t(T_i). \end{aligned}$$

Thus, we have

$$M_t\hat{\mathbb{E}}_t[\tilde{X}] \leq P_t(T_N) + P_t(T_0) - P_t(T_N) = P_t(T_0).$$

On the other hand, it holds

$$-M_t\hat{\mathbb{E}}_t[-\tilde{X}] \geq -M_t\hat{\mathbb{E}}_t[-M_{T_N}^{-1}] - \sum_{i=1}^{N} M_t\hat{\mathbb{E}}_t[-M_{T_i}^{-1}(T_i - T_{i-1})L_{T_{i-1}}(T_i)].$$

We can do the same steps as above to obtain

$$-M_t\hat{\mathbb{E}}_t[-M_{T_N}^{-1}] = -P_t(T_N)\hat{\mathbb{E}}_t^{T_N}[-1] = P_t(T_N)$$

and

$$M_t\hat{\mathbb{E}}_t[-M_{T_i}^{-1}(T_i - T_{i-1})L_{T_{i-1}}(T_i)] = M_t\hat{\mathbb{E}}_t\big[M_{T_i}^{-1}\big(1 - \tfrac{1}{P_{T_{i-1}}(T_i)}\big)\big] = P_t(T_i) - P_t(T_{i-1}).$$

Therefore we get

$$-M_t\hat{\mathbb{E}}_t[-\tilde{X}] \geq P_t(T_N) + P_t(T_0) - P_t(T_N) = P_t(T_0),$$

which completes the proof.

C.3 Proof of Proposition 5.3

Since the payoff can be decomposed into the payoff of a floating rate note and a fixed coupon bond, we can use the sublinearity of $\hat{\mathbb{E}}_t$, Proposition 5.1, and Proposition 5.2 to get

$$M_t\hat{\mathbb{E}}_t[\tilde{X}] \leq P_t(T_0) - P_t(T_N) - \sum_{i=1}^{N} P_t(T_i)(T_i - T_{i-1})K.$$

By the same argument, it holds

$$-M_t\hat{\mathbb{E}}_t[-\tilde{X}] \geq P_t(T_0) - P_t(T_N) - \sum_{i=1}^{N} P_t(T_i)(T_i - T_{i-1})K.$$

Hence, we obtain the desired result.

C.4 Proof of Theorem 5.1

First of all, by Proposition 4.1 (i), we have

$$M_t\hat{\mathbb{E}}_t[\tilde{X}] = P_t(T_0)\hat{\mathbb{E}}_t^{T_0}\Big[\Big(1 - X_{T_0}^{T_0,T_N} - \sum_{i=1}^{N} X_{T_0}^{T_0,T_i}(T_i - T_{i-1})K\Big)^+\Big]$$

and

$$-M_t\hat{\mathbb{E}}_t[-\tilde{X}] = -P_t(T_0)\hat{\mathbb{E}}_t^{T_0}\Big[-\Big(1 - X_{T_0}^{T_0,T_N} - \sum_{i=1}^{N} X_{T_0}^{T_0,T_i}(T_i - T_{i-1})K\Big)^+\Big].$$

Moreover, if we define the function $\varphi : \mathbb{R}^N \to \mathbb{R}$ by

$$\varphi(x) := \Big(1 - x_N - \sum_{i=1}^{N} x_i(T_i - T_{i-1})K\Big)^+,$$

we can easily show that it is convex and satisfies the estimate (4.1), that is,

$$\begin{aligned}
\varphi\big(\lambda x + (1-\lambda)y\big) =& \Big(1 - \big(\lambda x_N + (1-\lambda)y_N\big) - \sum_{i=1}^{N} \big(\lambda x_i + (1-\lambda)y_i\big)(T_i - T_{i-1})K\Big)^+ \\
\leq& \lambda\Big(1 - x_N - \sum_{i=1}^{N} x_i(T_i - T_{i-1})K\Big)^+ \\
&+ (1-\lambda)\Big(1 - y_N - \sum_{i=1}^{N} y_i(T_i - T_{i-1})K\Big)^+ \\
=& \lambda\varphi(x) + (1-\lambda)\varphi(y)
\end{aligned}$$

for $\lambda \in (0,1)$ and

$$\begin{aligned}
|\varphi(x) - \varphi(y)| =& \Big|\Big(1 - x_N - \sum_{i=1}^{N} x_i(T_i - T_{i-1})K\Big)^+ - \Big(1 - y_N - \sum_{i=1}^{N} y_i(T_i - T_{i-1})K\Big)^+\Big| \\
\leq& \Big|\Big(1 - x_N - \sum_{i=1}^{N} x_i(T_i - T_{i-1})K - \Big(1 - y_N - \sum_{i=1}^{N} y_i(T_i - T_{i-1})K\Big)\Big)^+\Big| \\
\leq& \Big|1 - x_N - \sum_{i=1}^{N} x_i(T_i - T_{i-1})K - \Big(1 - y_N - \sum_{i=1}^{N} y_i(T_i - T_{i-1})K\Big)\Big| \\
=& \Big|(x_N - y_N) + \sum_{i=1}^{N}(x_i - y_i)(T_i - T_{i-1})K\Big| \\
\leq& \sqrt{\sum_{i=1}^{N-1}(T_i - T_{i-1})^2K^2 + \big((T_N - T_{N-1})K + 1\big)^2}|x-y|.
\end{aligned}$$

Thus, the assertion follows by Proposition 4.2.

C.5 Proof of Theorem 5.2

First of all, we compute the upper bound of the cap. The lower bound can be computed analogously, which is described at the end of the proof. Because of the nonlinearity, we cannot compute the upper bound of each caplet separately. Instead, we find the upper bound of the cap by backward induction. We start by computing the upper bound of the last caplet and then we work recursively backwards until we get the upper bound for the cap.

The first step is to split the last caplet from the sum of the cap and to rewrite it as the payoff of a put option on a bond.

$$\begin{aligned}
M_t\hat{\mathbb{E}}_t[\tilde{X}] =& M_t\hat{\mathbb{E}}_t\Big[\sum_{i=1}^{N-1} M_{T_i}^{-1}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+ \\
&+ M_{T_N}^{-1}(T_N - T_{N-1})\big(L_{T_{N-1}}(T_N) - K\big)^+\Big] \\
=& M_t\hat{\mathbb{E}}_t\Big[\sum_{i=1}^{N-1} M_{T_i}^{-1}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+
\end{aligned}$$

$$+ M_{T_{N-1}}^{-1} M_{T_{N-1}} \hat{\mathbb{E}}_{T_{N-1}}[M_{T_N}^{-1}](T_N - T_{N-1})\big(L_{T_{N-1}}(T_N) - K\big)^+\Big]$$

$$= M_t \hat{\mathbb{E}}_t\Big[\sum_{i=1}^{N-1} M_{T_i}^{-1}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+$$

$$+ M_{T_{N-1}}^{-1} P_{T_{N-1}}(T_N)(T_N - T_{N-1})\big(L_{T_{N-1}}(T_N) - K\big)^+\Big]$$

$$= M_t \hat{\mathbb{E}}_t\Big[\sum_{i=1}^{N-1} M_{T_i}^{-1}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+$$

$$+ M_{T_{N-1}}^{-1} \tfrac{1}{K_N}\big(K_N - P_{T_{N-1}}(T_N)\big)^+\Big].$$

Now we extract the second last caplet from the sum to rewrite it as a bond option.

$$M_t \hat{\mathbb{E}}_t[\tilde{X}] = M_t \hat{\mathbb{E}}_t\Big[\sum_{i=1}^{N-2} M_{T_i}^{-1}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+$$

$$+ M_{T_{N-2}}^{-1} M_{T_{N-2}} \hat{\mathbb{E}}_{T_{N-2}}\Big[M_{T_{N-1}}^{-1}\Big((T_{N-1} - T_{N-2})\big(L_{T_{N-2}}(T_{N-1}) - K\big)^+$$

$$+ \tfrac{1}{K_N}\big(K_N - P_{T_{N-1}}(T_N)\big)^+\Big)\Big]\Big]$$

$$= M_t \hat{\mathbb{E}}_t\Big[\sum_{i=1}^{N-2} M_{T_i}^{-1}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+$$

$$+ M_{T_{N-2}}^{-1} P_{T_{N-2}}(T_{N-1}) \hat{\mathbb{E}}_{T_{N-2}}^{T_{N-1}}\big[(T_{N-1} - T_{N-2})\big(L_{T_{N-2}}(T_{N-1}) - K\big)^+$$

$$+ \tfrac{1}{K_N}\big(K_N - P_{T_{N-1}}(T_N)\big)^+\big]\Big]$$

$$= M_t \hat{\mathbb{E}}_t\Big[\sum_{i=1}^{N-2} M_{T_i}^{-1}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+$$

$$+ M_{T_{N-2}}^{-1}\Big(P_{T_{N-2}}(T_{N-1})(T_{N-1} - T_{N-2})\big(L_{T_{N-2}}(T_{N-1}) - K\big)^+$$

$$+ P_{T_{N-2}}(T_{N-1}) \tfrac{1}{K_N} \hat{\mathbb{E}}_{T_{N-2}}^{T_{N-1}}\big[\big(K_N - P_{T_{N-1}}(T_N)\big)^+\big]\Big)\Big]$$

$$= M_t \hat{\mathbb{E}}_t\Big[\sum_{i=1}^{N-2} M_{T_i}^{-1}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+$$

$$+ M_{T_{N-2}}^{-1}\Big(\tfrac{1}{K_{N-1}}\big(K_{N-1} - P_{T_{N-2}}(T_{N-1})\big)^+$$

$$+ P_{T_{N-2}}(T_{N-1}) \tfrac{1}{K_N} \hat{\mathbb{E}}_{T_{N-2}}^{T_{N-1}}\big[\big(K_N - P_{T_{N-1}}(T_N)\big)^+\big]\Big)\Big].$$

Since the payoff function of a put option is convex and Lipschitz continuous, we can apply Proposition 4.2 to get

$$\hat{\mathbb{E}}_{T_{N-2}}^{T_{N-1}}\big[\big(K_N - P_{T_{N-1}}(T_N)\big)^+\big] = u_{\overline{\sigma}}^N\big(T_{N-2}, \tfrac{P_{T_{N-2}}(T_N)}{P_{T_{N-2}}(T_{N-1})}\big).$$

Thus, we have

$$M_t \hat{\mathbb{E}}_t[\tilde{X}] = M_t \hat{\mathbb{E}}_t\Big[\sum_{i=1}^{N-2} M_{T_i}^{-1}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+$$

$$+ M_{T_{N-2}}^{-1}\Big(\tfrac{1}{K_{N-1}}\big(K_{N-1} - P_{T_{N-2}}(T_{N-1})\big)^+$$

$$+ P_{T_{N-2}}(T_{N-1})\tfrac{1}{K_N}u_{\sigma}^N\big(T_{N-2}, \tfrac{P_{T_{N-2}}(T_N)}{P_{T_{N-2}}(T_{N-1})}\big)\Big)\Big].$$

The next step is to repeat this procedure until we arrive at the last caplet. This means we do the following recursive step for $n = 2, ..., N-1$. We suppose that the upper bound of the cap is given by

$$\begin{aligned}
M_t\hat{\mathbb{E}}_t[\tilde{X}] =& M_t\hat{\mathbb{E}}_t\Big[\sum_{i=1}^{n-1} M_{T_i}^{-1}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+ \\
&+ M_{T_{n-1}}^{-1}\Big(\tfrac{1}{K_n}\big(K_n - P_{T_{n-1}}(T_n)\big)^+ \\
&+ \sum_{i=n+1}^{N} P_{T_{n-1}}(T_{i-1})\tfrac{1}{K_i}u_{\sigma}^i\big(T_{n-1}, \tfrac{P_{T_{n-1}}(T_i)}{P_{T_{n-1}}(T_{i-1})}\big)\Big)\Big].
\end{aligned}$$

Then, as in the first step, we extract the last caplet of the first sum to rewrite it as a bond option.

$$\begin{aligned}
M_t\hat{\mathbb{E}}_t[\tilde{X}] =& M_t\hat{\mathbb{E}}_t\Big[\sum_{i=1}^{n-2} M_{T_i}^{-1}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+ \\
&+ M_{T_{n-2}}^{-1}M_{T_{n-2}}\hat{\mathbb{E}}_{T_{n-2}}\Big[M_{T_{n-1}}^{-1}\Big((T_{n-1} - T_{n-2})\big(L_{T_{n-2}}(T_{n-1}) - K\big)^+ \\
&+ \tfrac{1}{K_n}\big(K_n - P_{T_{n-1}}(T_n)\big)^+ \\
&+ \sum_{i=n+1}^{N} P_{T_{n-1}}(T_{i-1})\tfrac{1}{K_i}u_{\sigma}^i\big(T_{n-1}, \tfrac{P_{T_{n-1}}(T_i)}{P_{T_{n-1}}(T_{i-1})}\big)\Big)\Big]\Big] \\
=& M_t\hat{\mathbb{E}}_t\Big[\sum_{i=1}^{n-2} M_{T_i}^{-1}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+ \\
&+ M_{T_{n-2}}^{-1}P_{T_{n-2}}(T_{n-1})\hat{\mathbb{E}}_{T_{n-2}}^{T_{n-1}}\Big[(T_{n-1} - T_{n-2})\big(L_{T_{n-2}}(T_{n-1}) - K\big)^+ \\
&+ \tfrac{1}{K_n}\big(K_n - P_{T_{n-1}}(T_n)\big)^+ \\
&+ \sum_{i=n+1}^{N} P_{T_{n-1}}(T_{i-1})\tfrac{1}{K_i}u_{\sigma}^i\big(T_{n-1}, \tfrac{P_{T_{n-1}}(T_i)}{P_{T_{n-1}}(T_{i-1})}\big)\Big]\Big] \\
=& M_t\hat{\mathbb{E}}_t\Big[\sum_{i=1}^{n-2} M_{T_i}^{-1}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+ \\
&+ M_{T_{n-2}}^{-1}\Big(P_{T_{n-2}}(T_{n-1})(T_{n-1} - T_{n-2})\big(L_{T_{n-2}}(T_{n-1}) - K\big)^+ \\
&+ P_{T_{n-2}}(T_{n-1})\hat{\mathbb{E}}_{T_{n-2}}^{T_{n-1}}\Big[\tfrac{1}{K_n}\big(K_n - P_{T_{n-1}}(T_n)\big)^+ \\
&+ \sum_{i=n+1}^{N} P_{T_{n-1}}(T_{i-1})\tfrac{1}{K_i}u_{\sigma}^i\big(T_{n-1}, \tfrac{P_{T_{n-1}}(T_i)}{P_{T_{n-1}}(T_{i-1})}\big)\Big]\Big)\Big] \\
=& M_t\hat{\mathbb{E}}_t\Big[\sum_{i=1}^{n-2} M_{T_i}^{-1}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+ \\
&+ M_{T_{n-2}}^{-1}\Big(\tfrac{1}{K_{n-1}}\big(K_{n-1} - P_{T_{n-2}}(T_{n-1})\big)^+ \\
&+ P_{T_{n-2}}(T_{n-1})\hat{\mathbb{E}}_{T_{n-2}}^{T_{n-1}}\Big[\tfrac{1}{K_n}\big(K_n - P_{T_{n-1}}(T_n)\big)^+
\end{aligned}$$

$$+\sum_{i=n+1}^{N} P_{T_{n-1}}(T_{i-1})\tfrac{1}{K_i}u^i_{\overline{\sigma}}\big(T_{n-1},\tfrac{P_{T_{n-1}}(T_i)}{P_{T_{n-1}}(T_{i-1})}\big)\Big]\Big)\Big].$$

Compared to the first step, here we cannot use Proposition 4.2 to evaluate the T_{n-1}-forward expectation in the last expression. Hence, we need to repeat the procedure of the proof of Proposition 4.2. In this case, we need to solve the G-forward backward stochastic differential equation

$$\begin{aligned}
X^{T_{n-1},T_{i-1}}_t =& X^{T_{n-1},T_{i-1}}_{T_{n-2}} - \int_{T_{n-2}}^t \sigma_s(T_{n-1},T_{i-1})X^{T_{n-1},T_{i-1}}_s dB^{T_{n-1}}_s, \quad i=n+1,...,N,\\
X^{T_{i-1},T_i}_t =& X^{T_{i-1},T_i}_{T_{n-2}} - \int_{T_{n-2}}^t \sigma_s(T_{i-1},T_i)X^{T_{i-1},T_i}_s dB^{T_{n-1}}_s\\
&- \int_{T_{n-2}}^t \sigma_s(T_{i-1},T_i)X^{T_{i-1},T_i}_s \sigma_s(T_{n-1},T_{i-1})d\langle B\rangle_s, \quad i=n+1,...,N,\\
Y_t =& \tfrac{1}{K_n}(K_n - X^{T_{n-1},T_n}_{T_{n-1}})^+ + \sum_{i=n+1}^{N} X^{T_{n-1},T_{i-1}}_{T_{n-1}} \tfrac{1}{K_i}u^i_{\overline{\sigma}}(T_{n-1}, X^{T_{i-1},T_i}_{T_{n-1}})\\
&- \int_t^{T_{n-1}} Z_s dB^{T_{n-1}}_s - (K_{T_{n-1}} - K_t),
\end{aligned}$$

where the dynamics of $X^{T_{n-1},T_{i-1}}$ and X^{T_{i-1},T_i} can be obtained as in the proof of Proposition 4.1 (*iii*).

By the nonlinear Feynman-Kac formula from Hu, Ji, Peng, and Song [30] we get

$$Y_{T_{n-2}} = u\big(T_{n-2}, (X^{T_{n-1},T_{i-1}}_{T_{n-2}}, X^{T_{i-1},T_i}_{T_{n-2}})_{i=n+1,\dots,N}\big),$$

where the function $u : [0, T_{n-1}] \times \mathbb{R}^{2(N-n)} \to \mathbb{R}$ is the unique viscosity solution to the nonlinear partial differential equation

$$\begin{aligned}
\partial u_t + G\big(H(D^2_x u, D_x u, t, x)\big) =& 0,\\
u(T_{n-1}, x) =& \varphi(x),
\end{aligned}$$

$x = (\hat{x}_i, \tilde{x}_i)_{i=n+1,\dots,N}$ is a vector in $\mathbb{R}^{2(N-n)}$, D_x denotes the gradient with respect to x, and H and φ are given by

$$\begin{aligned}
H(D^2_x u, D_x u, t, x) :=& \sum_{i,j=n+1}^{N} \sigma_t(T_{n-1},T_{i-1})\hat{x}_i\sigma_t(T_{n-1},T_{j-1})\hat{x}_j\partial^2_{\hat{x}_i\hat{x}_j}u\\
&+ \sum_{i,j=n+1}^{N} \sigma_t(T_{n-1},T_{i-1})\hat{x}_i\sigma_t(T_{j-1},T_j)\tilde{x}_j\partial^2_{\hat{x}_i\tilde{x}_j}u\\
&+ \sum_{i,j=n+1}^{N} \sigma_t(T_{i-1},T_i)\tilde{x}_i\sigma_t(T_{n-1},T_{j-1})\hat{x}_j\partial^2_{\tilde{x}_i\hat{x}_j}u\\
&+ \sum_{i,j=n+1}^{N} \sigma_t(T_{i-1},T_i)\tilde{x}_i\sigma_t(T_{j-1},T_j)\tilde{x}_j\partial^2_{\tilde{x}_i\tilde{x}_j}u\\
&- 2\sum_{i=n+1}^{N} \sigma_t(T_{i-1},T_i)\tilde{x}_i\sigma_t(T_{n-1},T_{i-1})\partial_{\tilde{x}_i}u
\end{aligned}$$

and

$$\varphi(x) := \tfrac{1}{K_n}(K_n - \hat{x}_{n+1})^+ + \sum_{i=n+1}^{N} \hat{x}_i \tfrac{1}{K_i} u^i_{\overline{\sigma}}(T_{n-1}, \tilde{x}_i).$$

The nonlinear Feynman-Kac formula can be applied here, since we have the estimate

$$|\varphi(x) - \varphi(y)| \le C(1 + |x| + |y|)|x - y| \tag{C.1}$$

for a constant $C > 0$. This can be shown by using the fact that, for $i = n + 1, ..., N$, $u^i_{\overline{\sigma}}(T_{n-1}, \cdot)$ is bounded by K_i and Lipschitz continuous with Lipschitz constant 1, which is a consequence of the nonlinear Feynman-Kac formula, i.e.,

$$\begin{aligned}
|\varphi(x) - \varphi(y)| \le & \tfrac{1}{K_n}\big|(K_n - \hat{x}_{n+1})^+ - (K_n - \hat{y}_{n+1})^+\big| \\
& + \sum_{i=n+1}^{N} \tfrac{1}{K_i}\big|\hat{x}_i u^i_{\overline{\sigma}}(T_{n-1}, \tilde{x}_i) - \hat{y}_i u^i_{\overline{\sigma}}(T_{n-1}, \tilde{y}_i)\big| \\
\le & \tfrac{1}{K_n}|\hat{x}_{n+1} - \hat{y}_{n+1}| + \sum_{i=n+1}^{N} \tfrac{1}{K_i}\Big(\big|u^i_{\overline{\sigma}}(T_{n-1}, \tilde{x}_i)\big||\hat{x}_i - \hat{y}_i| \\
& + |\hat{y}_i|\big|u^i_{\overline{\sigma}}(T_{n-1}, \tilde{x}_i) - u^i_{\overline{\sigma}}(T_{n-1}, \tilde{y}_i)\big|\Big) \\
\le & \tfrac{1}{K_n}|\hat{x}_{n+1} - \hat{y}_{n+1}| + \sum_{i=n+1}^{N} |\hat{x}_i - \hat{y}_i| + \tfrac{1}{K_i}|\hat{y}_i||\tilde{x}_i - \tilde{y}_i| \\
\le & (1 + \tfrac{1}{K_n} + \max_i \tfrac{1}{K_i}|y|) \sum_{i=n+1}^{N} |\hat{x}_i - \hat{y}_i| + |\tilde{x}_i - \tilde{y}_i| \\
\le & C(1 + |x| + |y|)|x - y|
\end{aligned}$$

for some suitable constant $C > 0$.

Regarding the nonlinear partial differential equation, we are only interested in a solution on $[0, T_{n-1}] \times \mathbb{R}_+^{2(N-n)}$, since the processes $X^{T_{n-1},T_{i-1}}$ and X^{T_{i-1},T_i} are positive for $i = n+1, ..., N$. In order to solve this, we simply guess a solution and verify that it solves the nonlinear partial differential equation on $[0, T_{n-1}] \times \mathbb{R}_+^{2(N-n)}$. By the uniqueness we get an expression for the T_{n-1}-forward sublinear expectation from above.

Let us define the function $u^* : [0, T_{n-1}] \times \mathbb{R}^{2(N-n)} \to \mathbb{R}$ by

$$u^*(t, x) := \tfrac{1}{K_n} u^n_{\overline{\sigma}}(t, \hat{x}_{n+1}) + \sum_{i=n+1}^{N} \hat{x}_i \tfrac{1}{K_i} u^i_{\overline{\sigma}}(t, \tilde{x}_i).$$

Now we verify that u^* indeed solves the nonlinear partial differential equation. Since $\partial^2_{\hat{x}_i\hat{x}_j} u^* = 0$ for $i, j \neq n+1$, $\partial^2_{\tilde{x}_i,\hat{x}_j} u^* = \partial^2_{\hat{x}_i\tilde{x}_j} u^* = 0$ for $i \neq j$, and $\partial^2_{\tilde{x}_i\tilde{x}_j} u^* = 0$ for $i \neq j$, we have

$$\begin{aligned}
H(D_x^2 u^*, D_x u^*, t, x) = & \sigma_t(T_{n-1}, T_n)^2 \hat{x}_{n+1}^2 \tfrac{1}{K_n} \partial^2_{\hat{x}_{n+1}\hat{x}_{n+1}} u^n_{\overline{\sigma}} \\
& + 2 \sum_{i=n+1}^{N} \sigma_t(T_{n-1}, T_{i-1}) \hat{x}_i \sigma_t(T_{i-1}, T_i) \tilde{x}_i \tfrac{1}{K_i} \partial_{\tilde{x}_i} u^i_{\overline{\sigma}}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=n+1}^{N}\hat{x}_i\sigma_t(T_{i-1},T_i)^2\tilde{x}_i^2\tfrac{1}{K_i}\partial^2_{\tilde{x}_i\tilde{x}_i}u^i_{\overline{\sigma}}\\
&-2\sum_{i=n+1}^{N}\sigma_t(T_{i-1},T_i)\tilde{x}_i\sigma_t(T_{n-1},T_{i-1})\hat{x}_i\tfrac{1}{K_i}\partial_{\tilde{x}_i}u^i_{\overline{\sigma}}\\
=&\sigma_t(T_{n-1},T_n)^2\hat{x}_{n+1}^2\tfrac{1}{K_n}\partial^2_{\hat{x}_{n+1}\hat{x}_{n+1}}u^n_{\overline{\sigma}}\\
&+\sum_{i=n+1}^{N}\sigma_t(T_{i-1},T_i)^2\tilde{x}_i^2\hat{x}_i\tfrac{1}{K_i}\partial^2_{\tilde{x}_i\tilde{x}_i}u^i_{\overline{\sigma}}.
\end{aligned}
$$

Since we know that $u^i_{\overline{\sigma}}$ is convex in the second variable and $\hat{x}_i\geq 0$ for $i=n+1,...,N$, we get

$$
\begin{aligned}
G\big(H(D_x^2u^*,D_xu^*,t,x)\big)=&\tfrac{1}{2}\overline{\sigma}\sigma_t(T_{n-1},T_n)^2\hat{x}_{n+1}^2\tfrac{1}{K_n}\partial^2_{\hat{x}_{n+1}\hat{x}_{n+1}}u^n_{\overline{\sigma}}\\
&+\sum_{i=n+1}^{N}\tfrac{1}{2}\overline{\sigma}\sigma_t(T_{i-1},T_i)^2\tilde{x}_i^2\hat{x}_i\tfrac{1}{K_i}\partial^2_{\tilde{x}_i\tilde{x}_i}u^i_{\overline{\sigma}}.
\end{aligned}
$$

Thus, it holds

$$
\begin{aligned}
\partial_tu^*+G\big(H(D_x^2u^*,D_xu^*,t,x)\big)=&\tfrac{1}{K_n}\partial_tu^n_{\overline{\sigma}}+\sum_{i=n+1}^{N}\hat{x}_i\tfrac{1}{K_i}\partial_tu^i_{\overline{\sigma}}\\
&+\tfrac{1}{2}\overline{\sigma}\sigma_t(T_{n-1},T_n)^2\hat{x}_{n+1}^2\tfrac{1}{K_n}\partial^2_{\hat{x}_{n+1}\hat{x}_{n+1}}u^n_{\overline{\sigma}}\\
&+\sum_{i=n+1}^{N}\tfrac{1}{2}\overline{\sigma}\sigma_t(T_{i-1},T_i)^2\tilde{x}_i^2\hat{x}_i\tfrac{1}{K_i}\partial^2_{\tilde{x}_i\tilde{x}_i}u^i_{\overline{\sigma}}\\
=&0,
\end{aligned}
$$

since

$$
\partial_tu^n_{\overline{\sigma}}+\tfrac{1}{2}\overline{\sigma}\sigma_t(T_{n-1},T_n)^2\hat{x}_{n+1}^2\partial^2_{\hat{x}_{n+1}\hat{x}_{n+1}}u^n_{\overline{\sigma}}=0
$$

and

$$
\partial_tu^i_{\overline{\sigma}}+\tfrac{1}{2}\overline{\sigma}\sigma_t(T_{i-1},T_i)^2\tilde{x}_i^2\partial^2_{\tilde{x}_i\tilde{x}_i}u^i_{\overline{\sigma}}=0
$$

for $i=n,...,N$ by the classical Feynman-Kac formula. Moreover, it is easy to see that the terminal condition is satisfied.

Hence, we have

$$
Y_{T_{n-2}}=\tfrac{1}{K_n}u^n_{\overline{\sigma}}(T_{n-2},X^{T_{n-1},T_n}_{T_{n-2}})+\sum_{i=n+1}^{N}X^{T_{n-1},T_{i-1}}_{T_{n-2}}\tfrac{1}{K_i}u^i_{\overline{\sigma}}(T_{n-2},X^{T_{i-1},T_i}_{T_{n-2}})
$$

and thus,

$$
\begin{aligned}
M_t\hat{\mathbb{E}}_t[\tilde{X}]=&M_t\hat{\mathbb{E}}_t\Big[\sum_{i=1}^{n-2}M_{T_i}^{-1}(T_i-T_{i-1})\big(L_{T_{i-1}}(T_i)-K\big)^+\\
&+M_{T_{n-2}}^{-1}\Big(\tfrac{1}{K_{n-1}}\big(K_{n-1}-P_{T_{n-2}}(T_{n-1})\big)^+
\end{aligned}
$$

$$
\begin{aligned}
&+ P_{T_{n-2}}(T_{n-1})\Big(\tfrac{1}{K_n}u^n_{\overline{\sigma}}\big(T_{n-2}, \tfrac{P_{T_{n-2}}(T_n)}{P_{T_{n-2}}(T_{n-1})}\big) \\
&+ \sum_{i=n+1}^{N} \tfrac{P_{T_{n-2}}(T_{i-1})}{P_{T_{n-2}}(T_{n-1})}\tfrac{1}{K_i}u^i_{\overline{\sigma}}\big(T_{n-2}, \tfrac{P_{T_{n-2}}(T_i)}{P_{T_{n-2}}(T_{i-1})}\big)\Big)\Big)\Big] \\
=&M_t\hat{\mathbb{E}}_t\Big[\sum_{i=1}^{n-2} M^{-1}_{T_i}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+ \\
&+ M^{-1}_{T_{n-2}}\Big(\tfrac{1}{K_{n-1}}\big(K_{n-1} - P_{T_{n-2}}(T_{n-1})\big)^+ \\
&+ \sum_{i=n}^{N} P_{T_{n-2}}(T_{i-1})\tfrac{1}{K_i}u^i_{\overline{\sigma}}\big(T_{n-2}, \tfrac{P_{T_{n-2}}(T_i)}{P_{T_{n-2}}(T_{i-1})}\big)\Big)\Big],
\end{aligned}
$$

which completes the recursive step.

As it is described above, we repeat this step for $n = 2, ..., N-1$ until we arrive at

$$
\begin{aligned}
M_t\hat{\mathbb{E}}_t[\tilde{X}] =&M_t\hat{\mathbb{E}}_t\Big[M^{-1}_{T_0}\Big(\tfrac{1}{K_1}\big(K_1 - P_{T_0}(T_1)\big)^+ + \sum_{i=2}^{N} P_{T_0}(T_{i-1})\tfrac{1}{K_i}u^i_{\overline{\sigma}}\big(T_0, \tfrac{P_{T_0}(T_i)}{P_{T_0}(T_{i-1})}\big)\Big)\Big] \\
=&P_t(T_0)\hat{\mathbb{E}}^{T_0}_t\Big[\tfrac{1}{K_1}\big(K_1 - P_{T_0}(T_1)\big)^+ + \sum_{i=2}^{N} P_{T_0}(T_{i-1})\tfrac{1}{K_i}u^i_{\overline{\sigma}}\big(T_0, \tfrac{P_{T_0}(T_i)}{P_{T_0}(T_{i-1})}\big)\Big].
\end{aligned}
$$

Here we use the nonlinear Feynman-Kac formula, analogous to the recursive step, to finally obtain

$$
\begin{aligned}
M_t\hat{\mathbb{E}}_t[\tilde{X}] =&P_t(T_0)\Big(\tfrac{1}{K_1}u^1_{\overline{\sigma}}\big(t, \tfrac{P_t(T_1)}{P_t(T_0)}\big) + \sum_{i=2}^{N} \tfrac{P_t(T_{i-1})}{P_t(T_0)}\tfrac{1}{K_i}u^i_{\overline{\sigma}}\big(t, \tfrac{P_t(T_i)}{P_t(T_{i-1})}\big)\Big) \\
=&\sum_{i=1}^{N} P_t(T_{i-1})\tfrac{1}{K_i}u^i_{\overline{\sigma}}\big(t, \tfrac{P_t(T_i)}{P_t(T_{i-1})}\big).
\end{aligned}
$$

In order to prove the second assertion, we perform the same calculations as the ones from above with $M_t\hat{\mathbb{E}}_t[-\tilde{X}]$. We only need to replace the caplets, respectively $u^i_{\overline{\sigma}}$, with negative caplets, respectively $-u^i_{\underline{\sigma}}$ for $i = 1, ..., N$. In the end, this yields

$$
M_t\hat{\mathbb{E}}_t[-\tilde{X}] = -\sum_{i=1}^{N} P_t(T_{i-1})\tfrac{1}{K_i}u^i_{\underline{\sigma}}\big(t, \tfrac{P_t(T_i)}{P_t(T_{i-1})}\big),
$$

which completes the proof.

C.6 Proof of Theorem 5.3

The proof follows exactly the same lines as the proof of Theorem 5.2, where the main difference is that the caplets are replaced by floorlets. Thus, we use the function $\frac{1}{K_i}v^i_{\overline{\sigma}}$, respectively $\frac{1}{K_i}v^i_{\underline{\sigma}}$, for $i = 1, ..., N$ to value the floorlets for the upper, respectively lower, bound of the price. Another difference is that the functions $v^i_{\overline{\sigma}}$ and $v^i_{\underline{\sigma}}$ are not bounded, which is used to obtain the estimate (C.1). However, we can use the fact that $v^i_{\overline{\sigma}}$ and $v^i_{\underline{\sigma}}$ are dominated by a linear function to obtain the same estimate.

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