On some Stochastic Control Problems arising in Environmental Economics and Commodity Markets

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Contents

Introduction

Optimal control problems deal with situations in which an agent aims at optimizing a given performance criterion by suitably adjusting certain dynamics. Usually, the variable chosen by the agent in order to fulfill her aim is called control, while the controlled dynamical system is called state-variable.

Optimal control problems can be formulated in deterministic or in stochastic settings with discrete or continuous time variable, and find a wide range of applications in different fields such as Biology, Economics, Engineering, Finance, Physics etc. The interested reader may refer to [86]. To find the optimal control, a popular solution technique is based on the dynamic programming principle: the original optimal control problem is split into simpler subproblems in a recursive way. The basic idea of this approach is to consider a family of control problems parametrized by the initial state values, and to find a relation between the associated subproblems. This solution technique, alternative to the so-called Pontryagin's maximum principle [114], dates back to the works of the mathematician Richard Ernest Bellman in the early 1950s, see [24], who pointed out:

> "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

In the continuous time setting, the dynamic programming approach yields an evolution equation which characterizes the solution to the problem. This equation is also known as the Hamilton-Jacobi-Bellman (HJB) equation, and can be viewed as an extension of former results in classical Physics obtained by William Rowan Hamilton and Carl Gustav Jacob Jacobi in the 19th century (see also Chapter 6 in [64]). In this thesis, we consider two special classes of continuous time stochastic optimal control problems. In particular, we study models dealing with so-called *impulse* and *singular controls* and which are motivated by questions arising in commodity markets and environmental economics.

In many real situations, optimization problems arise in which acting on a system gives rise to both proportional and fixed costs. A typical example is stock management. In general, when ordering a quantity, we pay an ordering cost comprising a fixed cost, which is independent of the quantity ordered, and a proportional cost, which is linearly dependent on the ordered quantity. In the mathematical formulation of such a case, it is then reasonable to think that the economic agent will exert control (i.e. make an order) only at certain discrete dates in order to manage the amount of fixed costs due to the interventions. If we also allow for a random environment, those situations can be suitably modeled in a framework with *stochastic impulse controls* which, mathematically, are collections of an increasing sequence of stopping times (the intervention times) and a sequence of random variables (giving the sizes of interventions). The book [27] provides an early mathematical theory of those *impulse control problems*. Further contributions to this topic are given by [36, 97, 105] in the context of optimal control of exchange and interest rates, [85] as an application to portfolio management with fixed transaction costs, [29, 73] as examples for optimal inventory control, [7] in the context of rational harvesting of renewable resources, and [37] in the context of optimal dividend problems. In a Markovian setting, the HJB equation associated to stochastic impulse control problems reads as a (quasi) variational inequality (QVI), which, roughly speaking, is a differential problem with a nonlocal constraint. The optimal impulse control strategy is then of pure jump type and usually characterized by regions (whose geometry has to be endogenously determined) that divide the state space and in which different control actions are applied. In Chapter 1, we study a *two-player stochastic impulse game* which is motivated by a problem of optimal pollution control: on the one hand, there is a firm which aims at maximizing its profits by expanding its production, and thereby increasing the level of pollutants' emissions. On the other hand, the government aims at minimizing the social costs of pollution, and introduces regulatory constraints on the emissions' level, which then effectively reduce the output of production. Further details will be provided later.

Singular stochastic controls have been designed to model the limiting behavior of a control system in which the control can cause instantaneous displacements in the state variable. When these displacements are "small and very frequent" it is appropriate to consider models involving singularly continuous displacements. A control is now described by the cumulative amount of actions performed up to a certain time, and it is mathematically modeled through a process with paths of bounded variation. Such problems were originally introduced to deal with questions arising in aerospace engineering (see, for example, [20]), but in the latest decades they have experienced applications also in Biology, Economics, Finance, Physics etc. Singular controls can be seen as a limit case of impulse controls by letting the fixed cost component go to zero, and, conversely, any impulse control can be seen as a singular control (see, for example, [28]). From the point of view of the theory of differential equations, the HJB equation associated to a singular stochastic control problem is simpler than that related to impulse problems; in fact the nonlocal constraint is now replaced by a local gradient

constraint. In many problems formulated in a Markovian framework, the HJB equation relates to a so-called free boundary problem and, similarly to the impulse control setting, the state space usually splits into two regions called the *inaction* (or *waiting*) and the *action* regions (in Chapter 2 the action region is referred to *selling* region, whereas in Chapter 4 we use the term *installation* region). In the first region it is optimal not to exert control while in the action region it is. Indeed, the optimal control rule usually prescribes to exert the minimal amount needed to keep the (optimally controlled) state variable in the waiting region. Mathematically, such a policy leads to a so-called Skorokhod reflection problem (see [48] and [88], among others). As a consequence, the optimal control strategy is usually singular with respect to the Lebesgue measure (in the sense that it increases only on a set of times of zero-Lebesgue measure), and might even be discontinuous. Early mathematical contributions to those *singular stochastic control problems* are given by [26, 72, 77] among others, and applications in Economics/Finance are, for example, problems of optimal dividends, irreversible investment, optimal liquidation, optimal management of debt ratio, and optimal harvesting (see, for example, [9, 12, 59, 89], among the references mentioned later). Chapter 2 studies a *two-dimensional singular stochastic control problem* with a so-called *finite-fuel* constraint (i.e. the total amount of control to be used stays bounded) in which the control variable decreases the level of the state variable proportionally to the exerted control. This setting is used to model an optimal extraction problem: a price-maker company extracts an exhaustible commodity from a reservoir, and sells it in the spot market. While extracting, we assume that the company's actions have an impact on the commodity's spot price which is considered as one component of the state variable. Its second component is given by the level of the reservoir. The company then aims at maximizing the total expected profits from selling the commodity, net of the total expected costs of extraction.
In Chapter 4 we consider a singular stochastic control which affects linearly the drift coefficient of one component of the state variable (which evolves as an Itô diffusion). We use this setting for an application to an optimal installation problem of solar panels: a price-maker company can increase its level of installed power (this is one component of the state variable) by installations of solar panels, so to generate electricity and to sell it in the spot market. Hereby, the current level of the company's installed power has an impact on the electricity price (this is the other component of the state variable and has a mean-reverting behavior), and affects its mean-reversion level. Then, the company aims at maximizing the total expected net profits. Further details on Chapter 2 and Chapter 4 are provided later.

The solution of the problem of Chapter 4 relies on the result of Chapter 3 which is of independent interest. There, we obtain so far unproved properties of a ratio involving a class of Hermite and parabolic cylinder functions. In particular, this ratio is shown to be strictly decreasing and bounded by universal constants, and this result is closely related to the so-called Turán types inequalities[1]. The ratio arises, for example, in some problems of stochastic control when working with Ornstein-Uhlenbeck dynamics (see also Remark 6.8 in [22]).

Appendix A recalls the definition and some properties of the Ornstein-Uhlenbeck process, that is used to model the commodity's price in Chapter 2 and the electricity price in Chapter 4. Properties of the increasing eigenfunctions of the infinitesimal generator of the Ornstein-Uhlenbeck process are included. These properties are exploited when constructing an explicit solution to the corresponding HJB equation, and, especially, Appendix A provides a link between Ornstein-Uhlenbeck processes and Hermite functions (parabolic cylinder functions) that is essential for the proof of the main result in Chapter 3. Moreover, Appendix B and Appendix C contain some proofs and auxiliary results that complete the results of Chapter 2 and of Chapter 4.

We now proceed by providing a more detailed outline of Chapter 1, Chapter 2 and Chapter 4. Especially, in the following, we give a more precise review of the studied model (including economic motivation), we discuss the contribution of each chapter to the literature, and we describe the techniques used to solve the considered problems.

On a Strategic Model of Pollution Control (Chapter 1)[2]

In recent years, the growing importance of global environmental issues, such as the global warming, pushed countries or institutions to adopt environmental policies aiming at reducing the level of pollution. Some of these policies are the result of international agreements (such as the Kyoto Protocol of 1997, or the Paris Climate Agreement of 2016); some others are adopted more on a local scale: it is indeed a news of December 2016 that the authorities of Beijing issued a five-day warning and ordered heavy industries to slow or halt their production due to increasing smog.[3]

Environmental problems have attracted the interest of the scientific community as well (see, e.g, [98], and Chapter 9 of [106] for an exhaustive introduction to pollution control policies). Many papers in the mathematical and economic literature take the point of view of a social planner to model the problem of reducing emissions of pollutants arising from the production process of the industrial sector. For example, in [111, 112] a social planner aims at finding a time at which the reduction of the rate of emissions gives rise to the minimal social costs. In [113] the optimal environmental

[1]These are special inequalities that hold for many special functions and polynomials. They have been discovered by Paul Turán (see [126]).

[2]This chapter is based on a joint work with Giorgio Ferrari. Parts of this introduction and of Chapter 1 have been first published in *Ann. Oper. Res.*, volume 275, number 2, pages 297-319 (2019).

[3]See, for example, the article on The Guardian [124].

policy to be adopted is the one that maximizes the economy's instantaneous net payoff, i.e. the sum of the economic damage of pollution and of the economic benefits from production. Finally, [70, 118] consider the planner's problem of choosing the abatement policy, and research and development investment, that minimize the costs of achieving a given target of CO_2 concentration. All those works tackle the resulting mathematical problems with techniques from (stochastic) optimal control theory, and provide policy recommendations.

In Chapter 1, we do not take the point of view of a fictitious social planner, but we propose a *strategic model* of pollution control. An infinitely-lived profit maximizing firm, representative of the productive sector of a country, produces a single good, and faces fixed and proportional costs of capacity expansion. In line with other articles in the environmental economics literature (cf. [112, 113]), we suppose that the output of production is proportional to the level of pollutants' emissions. Those are negatively perceived by the society, and we assume that the social costs of pollution can be measured by a suitable penalty function. A government (or a government environmental agency) intervenes in order to dam the level of emissions, e.g., by introducing regulatory constraints on the emissions' level, which then effectively cap the output of production. We suppose that the interventions of the government have also some negative impact on the social welfare (e.g., they might cause an increase in the level of unemployment or foregone taxes), and we assume that such negative externality can be quantified in terms of instantaneous costs with fixed and proportional components. The government thus aims at minimizing the total costs of pollution and of the interventions on it.

Due to the fixed costs of interventions faced by the firm and the government, it is reasonable to expect that the two agents intervene only at discrete times on the output of production. Between two consecutive intervention times, the latter is assumed to evolve as a general regular one-dimensional Itô-diffusion[4]. We therefore model the previously discussed pollution control problem as a *stochastic impulse nonzero-sum game* between the government and the firm. The policy of each player is characterized by a pair consisting of a sequence of times, and a sequence of sizes of interventions on the output of production, and each player aims at picking a policy that optimizes her own performance criterion, given the policy adopted by the other player. The two players thus interact strategically in order to determine an equilibrium level of the output of production, i.e. of the level of pollutants' emissions.

We assume that the policies of both the government and the firm are of barrier type. Such policies are characterized by four constant trigger values chosen by the agents: on the one hand, whenever the output of production falls below a constant threshold, the firm pushes the output of production to an upper constant level; on the

[4] Uncertain capital depreciation or technological uncertainty might justify the stochastic nature of the output of production (see also [11, 49, 53, 129]).

other hand, whenever the level of emissions reaches an upper threshold, the government provides regulatory constraints which let the output of production jump to a constant lower value. By employing these policies, the two agents keep the output of production (equivalently, the level of pollutants' emissions) within an interval whose size is the result of their strategic interaction. We then construct accordingly a couple of candidate equilibrium policies, and of associated equilibrium values.

In order to choose those four trigger values we require that the agents' performance criteria associated to the previous policies are suitably smooth, as functions of the current output of production level. Namely, each agent imposes that her own candidate equilibrium value is continuously differentiable at her own trigger values. We then move on proving a verification theorem which provides sufficient conditions under which the previous candidate strategies indeed form an equilibrium. In particular, we show that if the solution of a suitable system of four highly nonlinear algebraic equations exists and satisfies a set of appropriate inequalities, then such a solution will trigger an equilibrium. Our results are finally complemented by a numerical study in the case of (uncontrolled) output of production given by a geometric Brownian motion. Also, we discuss the dependency of the (equilibrium) trigger values and of the equilibrium impulses' size on the model parameters. This comparative statics analysis shows interesting new behaviors that we explain as a consequence of the strategic interaction between the firm and the government. As an example, we find, surprisingly, that the higher the fixed costs for the firm, the smaller the sizes of the impulses applied by both the agents on the production process.

The contribution of this chapter is twofold. On the one hand, we propose a general strategic model that highlights the interplay between the productive sector and the government of a country for the management of the pollution which inevitably arises from the production process[5]. On the other hand, from a mathematical point of view, our model is one of the first dealing with a two-player nonzero-sum stochastic impulse game. It is worth noticing that a verification theorem for two-player nonzero-sum stochastic impulse games, in which the uncontrolled process is a multi-dimensional Itô-diffusion, has been recently proved in [2]. There the authors give a set of sufficient conditions under which the solutions (in an appropriate sense) of QVIs identify with equilibrium values of the game. Then, they consider a one-dimensional symmetric game with linear running costs, and obtain equilibrium values and equilibrium policies by finding the solutions of the related system of QVIs, and by verifying their optimality.

Our methodology is different with respect to that of [2]. Here, we obtain candidate equilibrium values without relying on solving the system of QVIs that would be associated to our game. Indeed, our candidate equilibrium values are constructed as

[5]For other works modeling the pollution control problem as a dynamic game one can refer, among others, to the example in Section 4 of [46], and [91, 128].

the performance criteria that the players obtain by applying a potentially suboptimal policy. This construction, which employs probabilistic properties of one-dimensional Itô-diffusions, has been already used in single-agent impulse control problems (see, e.g., [7, 8, 50]), and has the advantage of providing candidate equilibrium values which are automatically continuous functions of the underlying state variable. As a computationally useful byproduct, in our asymmetric setting we only have to find the four equilibrium trigger values, and for that we only need four equations. This is in contrast to the eight equations one would obtain by imposing C^0 and C^1-regularity of the solutions to the system of QVIs (cf. [2]).

An Optimal Extraction Problem with Price Impact (Chapter 2)[6]

The problem of a company that aims at determining the extraction rule of an exhaustible commodity, while maximizing net profits, has been widely studied in the literature. To the best of our knowledge, the first contribution on this topic is the seminal paper [74], in which a deterministic model of optimal extraction has been proposed. Since then, many authors have generalized the setting of [74] by allowing for stochastic commodity prices and for different specifications of the admissible extraction rules (see, e.g., [5, 25, 34, 57, 63, 104, 109, 110] among a huge literature in Economics and applied Mathematics).

In Chapter 2, we consider an optimal extraction problem for an infinitely-lived profit maximizing company. The company extracts an exhaustible commodity from a reservoir with a finite capacity incurring constant proportional costs, and then immediately sells the commodity in the spot market. The admissible extraction rules must not be rates, also lump sum extractions are allowed. Moreover, we assume that the company is a large player in the market, and therefore, its extraction strategies affect the market price of the commodity. This happens in such a way that whenever the company extracts the commodity and sells it in the market, the commodity's price is instantaneously decreased proportionally to the extracted amount.

Our mathematical formulation of the previous problem leads to a *two-dimensional degenerate finite-fuel singular stochastic control problem* (see [35, 79, 80, 82] as early contributions, and [22, 71] for recent applications to optimal liquidation problems). The underlying state variable is a two-dimensional process (X, Y) whose components are the commodity's price and the level of the reservoir (i.e. the amount of commodity still available). The price process is a linearly controlled Itô-diffusion, while the dynamics of the level of the reservoir are purely controlled and do not have any diffusive component. In particular, we assume that, in absence of any interventions, the commodity's

[6] This chapter is joint work with Giorgio Ferrari. Parts of this introduction and of Chapter 2 have been first published in *Appl. Math. Optim.*, DOI: 10.1007/s00245-019-09615-9 (September 2019).

price evolves either as a drifted Brownian motion or as an Ornstein-Uhlenbeck process, and we solve explicitly the optimal extraction problem by following a guess-and-verify approach. This relies on the construction of a classical solution to the associated HJB equation, which, in our problem, takes the form of a variational inequality with state-dependent gradient constraint. To the best of our knowledge, this is the first work that provides the explicit solution to an optimal extraction problem under uncertainty for a price-maker company facing a diffusive commodity's spot price with additive and mean-reverting dynamics.

In the simpler case of a drifted Brownian dynamics for the commodity's price, we find that the optimal extraction rule prescribes at any time to extract just the minimal amount needed to keep the commodity's price below an endogenously determined constant critical level $x^\star$, the free boundary. A lump sum extraction (and therefore a jump in the optimal control) may be observed only at initial time if the initial commodity's price exceeds the level $x^\star$. In such a case, depending on the initial level of the reservoir, it might be optimal either to deplete the reservoir or to extract a block of commodity so that the price is reduced to the desired level $x^\star$.

If the commodity's price has additionally a mean-reverting behavior and evolves as an Ornstein-Uhlenbeck process, the analysis is much more involved and technical than in the Brownian case. This is due to the unhandy and non-explicit form of the eigenfunctions of the infinitesimal generator of the Ornstein-Uhlenbeck process. We show that the optimal extraction rule is triggered by a critical price level that - differently to the Brownian case - is not anymore constant, but it is depending on the current level of the reservoir y. This critical price level - that we call $F^{-1}(y)$ - is the inverse of a positive, strictly decreasing, C^∞-function F that we determine explicitly. It is optimal to extract in such a way that the joint process (X,Y) is kept within the region $\{(x,y) : \; x \leq F^{-1}(y)\}$, and a suitable lump sum extraction should be made only if the initial data lie outside the previous region. The free boundary F has an asymptote at a point x_∞ and it is zero at the point x_0. These two points have a clear interpretation, as they correspond to the critical price levels triggering the optimal extraction rule in a model with infinite fuel and with no market impact, respectively.

In both the Brownian and the Ornstein-Uhlenbeck case, the optimal extraction rule is mathematically given through the solution to a Skorokhod reflection problem with oblique reflection at the free boundary in the direction $(-\alpha,-1)$. Here $\alpha > 0$ is the marginal market impact of the company's actions on the commodity's price. Indeed, if the company extracts an amount, say $d\xi_t$, at time t, then the price is linearly reduced by $\alpha d\xi_t$ and the level of the reservoir by $d\xi_t$. Moreover, we prove that the value function is a classical $C^{2,1}$-solution to the associated HJB equation.

When the price follows an Ornstein-Uhlenbeck dynamics, our proof of the optimality of the constructed candidate value function partly employs arguments developed in the study of an optimal liquidation problem tackled in the recent [22], which shares mathematical similarities with our problem. Indeed, in the case of a "small" marginal cost of extraction, due to the unhandy and implicit form of the increasing eigenfunction of the infinitesimal generator of the Ornstein-Uhlenbeck process, we have not been able to prove via direct means an inequality that the candidate value function needed to satisfy in order to solve the HJB equation. For this reason, in such a case, we adopted ideas from [22] where an interesting reformulation of the original singular control problem as a calculus of variations approach has been developed. However, it is also worth noticing that when the marginal cost of extraction is "large enough", the approach of [22] is not directly applicable since a fundamental assumption in [22] (cf. Assumption 2.2-(C5) therein) is not satisfied. Instead, a direct study of the variational inequality leads to the desired result. This fact suggests that a combined use of the calculus of variations method and of the standard guess-and-verify approach could be successful in intricate problems where neither of the two methods leads to prove optimality of a candidate value function for any choice of the model's parameters. We refer to the proof of Proposition 2.4.10 and to Remark 2.4.11 for details.

As a byproduct of our results, we find that the directional derivative (in the direction $(-\alpha, -1)$) of the optimal extraction problem's value function coincides with the value function of an optimal stopping problem. This fact, which is consistent with the findings of [79, 80], also allows us to explain quantitatively why, in the case of a drifted Brownian dynamics for the commodity's price, the level $x^\star$ triggering the optimal extraction rule is independent of the current level of the reservoir y. Indeed, in such a case, the value function of the optimal stopping problem is independent of y and, therefore, so is also its free boundary $x^\star$.

Thanks to the explicit nature of our results, we can provide a detailed comparative statics analysis. We obtain theoretical results on the dependency of the value function and of the critical price levels $x^\star$, x_∞, and x_0 with respect to some of the model's parameters. In the case of an Ornstein-Uhlenbeck commodity's price, numerical results are also derived to show the dependency of the free boundary curve F with respect to the volatility, the mean reversion level, and the mean-reversion speed.

Optimal Installation of Solar Panels with Price Impact: a Solvable Singular Stochastic Control Problem (Chapter 4)[7]

Chapter 4 proposes a model in which a company can increase its current electricity production by irreversible investments in solar panels, while maximizing net profits.

[7] This chapter is based on a joint work with Tiziano Vargiolu. Parts of this introduction and of Chapter 4 have been published in [84].

Irreversible investment problems have been widely studied in the context of real options and optimal capacity expansion. Related models in the economics literature are, for example, [30] and the monography [47]. Other relevant articles appearing in the mathematical literature are [3, 41, 43, 56, 58, 62, 90, 101, 116, 122], among many others.

We consider an infinitely-lived profit maximizing company which is a large player in the market. The company can install solar panels in order to increase its production level of electricity up to a given maximum level. The electricity generated will immediately be sold in the market, and while installing additional panels, the company incurs constant proportional costs. As it is assumed that the company is a large market player, its activities have an impact on the electricity price. In particular, we assume that the long-term electricity price level is negatively affected by the current level of installed power; that is, the electricity price will tend to move towards a lower price level if the electricity production is increased. Therefore, the company has to install solar panels carefully in order to avoid permanently low electricity prices which clearly decrease the marginal profits from selling electricity in the market.

The model is mathematically formulated as a two-dimensional degenerate singular stochastic control problem (see, for example, [79, 80, 82] as early contributions) whose components are the electricity price (modeled as an Ornstein-Uhlenbeck process) and the current level of installed power which is purely controlled. To the best of our knowledge, the work of this chapter is the first which provides the complete explicit solution to a two-dimensional degenerate singular stochastic control problem in which the drift of one component of the state process (the electricity price) is linearly affected by the monotone process giving the cumulative amount of control (the level of installed power). It is worth noticing that our mathematical formulation shares similarities with the recent article [55] in which a central bank can choose a control of bounded variation for managing the inflation. The methodology and results of [55] are indeed different with respect to ours: an explicit solution is not constructed, but the authors provide a theoretical study of the structure and regularity of the value function. Upon relying on a combination of techniques from viscosity theory and free-boundary analysis, it is shown that the control problem's value function satisfies a second-order smooth-fit principle. The latter is then exploited in order to determine a system of functional equations solved by two monotone curves that split the state space in three connected regions where different control actions should be applied.

Price impact models have gained the interest of many researchers in recent years. Some of these works are also formulated as a singular stochastic control problem and study questions of optimal execution: [21] and [22] take into account a multiplicative and transient price impact, whereas [71] considers an exponential parametrization in a geometric Brownian motion setting allowing for a permanent price impact. Also, a price impact model with singular stochastic controls has been studied by [4], motivated by an irreversible capital accumulation problem with permanent price impact, and in Chapter 2 of this thesis (cf. [60]). In all of the aforementioned papers on price impact models dealing with singular stochastic controls [4, 21, 22, 60, 71], the agents' actions can lead to an immediate jump in the underlying price process, whereas in the setting of Chapter 4, it cannot. Finally, [39, 40] show how to incorporate a market impact due to cross-border trading in electricity markets, and [117] models the price impact of wind electricity production on power prices.

In our model the firm's installation strategy is represented by an increasing control, possibly non-absolutely continuous, and we take into account a running payoff function which depends linearly on the level of installed power and on the electricity price. Following an educated guess for a classical solution to the associated HJB equation, and imposing $C^{2,1}-$regularity of the value function, we show that the optimal installation rule is triggered by a threshold which is a function of the current level of installed power, and we provide a closed-form expression of the value function. The threshold, also called free boundary in the sequel, uniquely solves an ordinary differential equation (ODE) for which we implement a numerical solution. Then, we characterize the geometry of the *waiting* and *installation* regions. We show that the optimal installation strategy is such that the company keeps the state process inside the *waiting* region. In particular, the state process is pushed towards the free boundary by installing a block of solar panels immediately, if the initial electricity price is above the critical threshold (if the maximum level of installed power, that the company is able to reach, is not sufficiently high, the company will immediately install the maximum number of panels). Thereafter, the joint process will be reflected along the free boundary. The construction of the reflected diffusion relies on ideas in [42] that are based on the transformation of probability measures in the spirit of Girsanov. The uniqueness of the optimal diffusion process then follows by the global Lipschitz continuity of our free boundary. Then, as a byproduct, we find that the derivative of the value function (in the direction $(0, 1)$) identifies with the value function of an optimal stopping problem. This fact highlights the (economic) components which are taken into account in the company's decision of acting. Our results are finally complemented by a numerical discussion of the dependency on the model parameters. We find, for example, that a higher mean-reversion level of the fundamental price process leads to a quicker installation of solar panels.

From the modeling point of view, it is common in the literature to represent electricity prices via a mean-reverting behavior, and to include (jump) terms to incorporate seasonal fluctuations and daily spikes, cf. [32, 38, 68, 130] among others. Here, we do not represent the spikes and seasonal fluctuations, with the following justification: the installation time of solar panels usually takes several days or weeks, which makes the company indifferent to daily or weekly spikes. Also, the high lifespan of solar panels and the underlying infinite time horizon setting allow us to neglect the seasonal patterns. We therefore assume that the fundamental electricity price has solely a mean-reverting behavior, and evolves according to an Ornstein-Uhlenbeck process[8]. We are also neglecting the stochastic and seasonal effects of solar production. In fact, solar panels do not obviously produce power during the night, produce less in winter than in summer (these two effects could be covered via a deterministic seasonal component), and also produce less when it is cloudy (this should be modelled with a stochastic process). Since here we are interested in a long-term optimal behaviour, we interpret the average electricity produced in a generic unit of time as proportional to the installed power. All of this can be mathematically justified if we interpret our fundamental price to be, for example, a weekly average price as e.g. in [33, 69], who used exactly this representation to get rid of daily and weekly seasonalities.

[8]We allow for negative prices by modeling the electricity price via an Ornstein-Uhlenbeck process. Indeed, negative electricity prices can be observed in some markets, for example in Germany, cf. [99].

Chapter 1

On a Strategic Model of Pollution Control

1.1 Introduction

We propose a strategic model of pollution control. A firm, representative of the productive sector of a country, aims at maximizing its profits by expanding its production. Assuming that the output of production is proportional to the level of pollutants' emissions, the firm increases the level of pollution. The government of the country aims at minimizing the social costs due to the pollution, and introduces regulatory constraints on the emissions' level, which then effectively cap the output of production. Supposing that the firm and the government face both proportional and fixed costs in order to adopt their policies, we model the previous problem as a stochastic impulse two-person nonzero-sum game. The state variable of the game is the level of the output of production which evolves as a general linearly controlled one-dimensional Itô-diffusion. We construct a pair of candidate equilibrium policies and of corresponding equilibrium values, and we provide a set of sufficient conditions under which they indeed realize an equilibrium. Our results are complemented by a numerical study when the (uncontrolled) output of production evolves as a geometric Brownian motion, and the firm's operating profit and the government's running cost functions are of power type. An analysis of the dependency of the equilibrium policies and values on the model parameters yields interesting new behaviors that we explain as a consequence of the strategic interaction between the firm and the government.

The present chapter is based on [61]. It is organized as follows. In Section 1.2 we introduce the setting and formulate the problem. In Section 1.3.1 we construct candidate equilibrium policies and candidate equilibrium values, whereas in Section 1.3.2 we provide a verification theorem. Finally, in Section 1.4 we provide the numerical solution to an example, and we study the dependency of the equilibrium with respect to the model parameters. Conclusions are finally drawn in Section 1.5.

1.2 Setting and Problem Formulation

We consider a firm (agent 1), and a government (agent 2). The firm produces a single good, and its profits from production are described by a function $\pi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ which is continuous, strictly concave and increasing. We assume that the production process leads to emissions, for example of greenhouse gases such as CO_2, that are proportional to the level of the output (see also [112, 113], among others). These emissions have a negative externality on the social welfare, and the resulting disutility incurred by the society is measured by a cost function $C : \mathbb{R}_+ \mapsto \mathbb{R}_+$ that depends on the rate of emissions. The function C is continuous, strictly convex and increasing.

The production process is assumed to be stochastic, since it may depend on uncertain capital depreciation or other exogenous random factors (see also [11, 30, 53, 129], among others). In particular, let $W = (W_t)_{t\geq 0}$ be a one-dimensional, standard Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions. The output of production at time $t \geq 0$ is denoted by X_t, and it evolves as a linear Itô-diffusion on $(0, \infty)$; that is,

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \qquad X_0 = x > 0, \tag{1.1}$$

for some Borel-measurable functions μ, σ to be specified. Here, μ is the trend of the production, while σ is a measure of the fluctuations around this trend.

To account for the dependency of X on its initial level, from now on we shall write X^x where appropriate, and $\mathbb{P}_x$ to refer to the probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{P}_x(\,\cdot\,) = \mathbb{P}(\,\cdot\,|X_0 = x)$, $x \in (0, \infty)$. Throughout this chapter we will equivalently use the notations $\mathbb{E}[f(X_t^x)]$ and $\mathbb{E}_x[f(X_t)]$, $f : \mathbb{R} \to \mathbb{R}$ Borel-measurable and integrable, to refer to expectations under the measure $\mathbb{P}_x$.

For the coefficients of the SDE (1.1) we make the following assumption, which will hold throughout the chapter.

Assumption 1.2.1. *The functions $\mu : \mathbb{R} \mapsto \mathbb{R}$ and $\sigma : \mathbb{R} \mapsto (0, \infty)$ are such that*

$$|\mu(x) - \mu(y)| \leq K|x - y|, \qquad |\sigma(x) - \sigma(y)| \leq h(|x - y|), \qquad x, y \in (0, \infty), \tag{1.2}$$

for some $K > 0$, and $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ strictly increasing such that $h(0) = 0$ and

$$\int_{(0,\varepsilon)} \frac{du}{h^2(u)} = \infty \quad \text{for every } \varepsilon > 0. \tag{1.3}$$

As a consequence of the above assumption one has that if a solution to (1.1) exists, then it is pathwise unique by the Yamada-Watanabe's Theorem (cf. [81], Proposition

5.2.13 and Remark 5.3.3, among others). Moreover, from (1.2) and (1.3) it follows that for every $x \in (0,\infty)$ there exists $\varepsilon > 0$ such that

$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{1+|\mu(y)|}{\sigma^2(y)}\,dy < +\infty. \tag{1.4}$$

Local integrability condition (1.4) implies that (1.1) has a weak solution (up to a possible explosion time) that is unique in the sense of probability law (cf. [81], Section 5.5C). Therefore, (1.1) has a unique strong solution (possibly up to an explosion time) due to [81], Corollary 5.3.23. Moreover, X is also regular in the sense that any point of the interior of its state space can be reached in finite time with positive probability. In line with applications, we assume that the boundary point $+\infty$ is not attainable for the process X, that is $+\infty$ cannot be reached in finite time with positive probability. One-dimensional diffusions like the geometric Brownian motion and the CIR process (under a suitable restriction on the parameters, i.e. the so-called Novikov's conditions) satisfy the assumptions of our setting.

Remark 1.2.2. *An example of microfoundation for a stochastic dynamics of the output of production is the following (cf. [30]). Assume that at time $t \geq 0$ the output of production X_t is given in terms of the capital stock, K_t, and the output of labor, L_t, by*

$$X_t = \left(K_t^{\rho} L_t^{1-\rho}\right)^{\gamma}, \quad 0 < \rho \leq 1, \ \text{and}\ \gamma > 0. \tag{1.5}$$

Also, suppose that the firm is faced with a constant elasticity demand function

$$P_t = X_t^{\lambda-1}, \quad 0 < \gamma\lambda < 1, \tag{1.6}$$

where P_t is the product price at time $t \geq 0$, and λ is a measure of the firm's monopoly power. Since the input of labor L_t is chosen such that $L_t = \arg\max_L \left\{P_t X_t - wL\right\}$, for some wage $w > 0$, one can obtain from (1.5) *and* (1.6) *that*

$$L_t = \left[\frac{\gamma\lambda}{w}(1-\rho)\right]^{\frac{1}{1-(1-\rho)\gamma\lambda}} K_t^{\frac{\rho\gamma\lambda}{1-(1-\rho)\gamma\lambda}} = \hat{\alpha} K_t^{\frac{\rho\gamma\lambda}{1-(1-\rho)\gamma\lambda}}, \tag{1.7}$$

where $\hat{\alpha} := \left[\frac{\gamma\lambda}{w}(1-\rho)\right]^{\frac{1}{1-(1-\rho)\gamma\lambda}}$. Hence, by plugging (1.7) *into* (1.5) *we have*

$$X_t = \hat{\alpha}^{(1-\rho)\gamma} K_t^{\frac{\gamma\rho}{1-(1-\rho)\gamma\lambda}}. \tag{1.8}$$

If now capital stock is stochastic and depreciates at a rate $\delta > 0$, i.e. $dK_t = -\delta K_t dt + \sigma K_t dW_t$ for some Brownian motion W (see, e.g., [129]), by Itô's formula one finds that X_t evolves as

$$dX_t = \hat{\mu} X_t dt + \hat{\sigma} X_t dW_t,$$

for suitable constants $\hat{\mu}, \hat{\sigma}$.

Both the agents can influence the process of production: on the one hand, the firm can instantaneously increase the level of production, for example by increasing the capital stock. This leads to instantaneous costs for the firm which have both a variable and a fixed component, and that we model through a function $g_1 : \mathbb{R}_+ \mapsto \mathbb{R}_+$ of the size of interventions on the production. In particular we take

$$g_1(\xi) := K_1 + \kappa_1 \xi, \quad \xi \geq 0,$$

with $K_1, \kappa_1 > 0$. On the other hand, the government can introduce regulatory constraints that effectively force the firm to decrease the level of production[1], hence of the emissions. A similar situation has happened in December 2016 in Beijing where authorities issued a five-day warning and ordered heavy industries to slow or halt production in order to reduce the smog in the air. We assume that the instantaneous costs of a similar action incurred by the government can be measured by a function $g_2 : \mathbb{R}_+ \mapsto \mathbb{R}_+$ given by

$$g_2(\eta) := K_2 + \kappa_2 \eta, \quad \eta \geq 0,$$

with $K_2, \kappa_2 > 0$. Such costs might arise because of an increase in the rate of unemployment or forgone taxes due to a possible decrease of the production capacity.

Because of the presence of fixed costs, it is reasonable to expect that the firm (resp. the government) intervenes only at discrete times on the output of production by shifting the current level of output up (resp. down) of some nonzero amount. More formally, the action of any agent is defined as follows.

Definition 1.2.3. *The actions ν_1 and ν_2 of the firm and of the government, respectively, are pairs*

$$\nu_1 := (\tau_{1,1}, \dots, \tau_{1,n}, \dots; \xi_1, \dots, \xi_n, \dots),$$
$$\nu_2 := (\tau_{2,1}, \dots, \tau_{2,n}, \dots; \eta_1, \dots, \eta_n, \dots),$$

where $0 \leq \tau_{i,1} \leq \tau_{i,2} \leq \dots$, for $i = 1, 2$, is an increasing sequence of $\mathbb{F}$-stopping times, ξ_n are positive $\mathcal{F}_{\tau_{1,n}}$-measurable random variables, and η_n are positive $\mathcal{F}_{\tau_{2,n}}$-measurable random variables.

Intervening on the output of production, the two agents modify the dynamics of the production process which then becomes

$$\begin{cases} X_t^{x,\nu_1,\nu_2} = & x + \int_0^t \mu(X_s^{x,\nu_1,\nu_2})ds + \int_0^t \sigma(X_s^{x,\nu_1,\nu_2})dW_s \\ & +\alpha \sum_{k:\tau_{1,k} \leq t} \xi_k \prod_{l \geq 1} \mathbb{1}_{\{\tau_{1,k} \neq \tau_{2,l}\}} - \sum_{k:\tau_{2,k} \leq t} \eta_k, \quad t \geq 0, \\ X_{0-}^{x,\nu_1,\nu_2} = & x > 0, \end{cases} \tag{1.9}$$

[1] Restrictions on the output of production can be achieved by the government in different ways. The interested reader may refer to the classical book [108].

where $\alpha > 0$ measures the effect of an increase in the capital stock on the output of production, and $X^{x,\nu_1,\nu_2}_{t-} := \lim_{\varepsilon \downarrow 0} X^{x,\nu_1,\nu_2}_{t-\varepsilon}$ for any $t \geq 0$.

In (1.9) ξ_k represents the lump-sum increase of the output of production made by the firm at time $\tau_{1,k}$. Moreover, η_k is the impact on production of the regulatory constraints imposed by the government at time $\tau_{2,k}$. If both the agents are willing to intervene on the output of production at the same time, it is reasonable to allow the government to have the priority: the infinite product $\prod_{l \geq 1} \mathbb{1}_{\{\tau_{1,k} \neq \tau_{2,l}\}}$ in (1.9) takes care of that. We write X^{x,ν_1,ν_2} to stress the dependence of the output of production on its initial level, and on the actions ν_1 and ν_2 adopted by the two agents.

Remark 1.2.4. *Following the microfoundation of Remark 1.2.2, suppose that at a certain time $\tau_{1,k}$ the firm increases the capital stock by an amount ξ_k, while the government does not intervene. Then we have by* (1.8) *that*

$$X_{\tau_k} = \hat{\alpha}^{(1-\rho)\gamma} K_{\tau_k}^{\frac{\rho\gamma}{1-(1-\rho)\gamma\lambda}} = \hat{\alpha}^{(1-\rho)\gamma} \Big(K_{\tau_k -} + \xi_k \Big)^{\frac{\rho\gamma}{1-(1-\rho)\gamma\lambda}}.$$

Taking $\gamma > 1$, for $\rho = \frac{1-\gamma\lambda}{\gamma-\gamma\lambda} \in (0,1)$ and λ such that $\gamma\lambda \in (0,1)$, we find

$$X_{\tau_k} = X_{\tau_k -} + \hat{\alpha}^{(1-\rho)\gamma} \xi_k,$$

that is consistent with (1.9) *if we set $\alpha := \hat{\alpha}^{(1-\rho)\gamma}$.*

The firm's total expected profits arising from production, net of present costs, are

$$\mathcal{J}_1(x, \nu_1, \nu_2) := \mathbb{E}_x \Bigg[\int_0^\infty e^{-r_1 t} \pi(X_t^{\nu_1,\nu_2}) dt - \sum_{k \geq 1} e^{-r_1 \tau_{1,k}} g_1(\xi_k) \mathbb{1}_{\{\tau_{1,k} < \infty\}} \Bigg], \tag{1.10}$$

where $r_1 > 0$ is the subjective discount factor of the firm.

Furthermore, the government's total expected costs arising from the emissions of pollutants is

$$\mathcal{J}_2(x, \nu_1, \nu_2) := \mathbb{E}_x \Bigg[\int_0^\infty e^{-r_2 t} C(\beta X_t^{\nu_1,\nu_2}) dt + \sum_{k \geq 1} e^{-r_2 \tau_{2,k}} g_2(\eta_k) \mathbb{1}_{\{\tau_{2,k} < \infty\}} \Bigg], \tag{1.11}$$

for some $r_2 > 0$ and $\beta > 0$. The constant β is the proportional factor between the rate of emissions and the output of production, while r_2 characterizes the time preferences of the government.

Remark 1.2.5. *We notice that the choice of a constant $\beta > 0$ in* (1.11)*, and of a constant $\alpha > 0$ in* (1.9) *is just to simplify exposition. Indeed, the results of this chapter do hold even if we allow for suitable state dependent $\beta(\cdot)$ or $\alpha(\cdot)$.*

The set of admissible actions is given as follows.

Definition 1.2.6. *For any initial level of the production $x > 0$, we say that the actions $\nu_1 := (\tau_{1,1}, \dots, \tau_{1,n}, \dots; \xi_1, \dots, \xi_n, \dots)$ and $\nu_2 := (\tau_{2,1}, \dots, \tau_{2,n}, \dots; \eta_1, \dots, \eta_n, \dots)$ are admissible, and we write $(\nu_1, \nu_2) \in \mathcal{T}(x)$, if the following hold true:*

(i) There exists a unique strong solution to (1.9) *with right-continuous sample paths such that $X_t^{x,\nu_1,\nu_2} \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

(ii) The functionals (1.10) *and* (1.11) *are finite; that is,*

(a) $$\mathbb{E}_x\bigg[\int_0^\infty e^{-r_1 t}\pi(X_t^{\nu_1,\nu_2})dt + \int_0^\infty e^{-r_2 t}C(\beta X_t^{\nu_1,\nu_2})dt\bigg] < \infty,$$

(b) $$\mathbb{E}_x\bigg[\sum_{k\geq 1} e^{-r_1\tau_{1,k}} g_1(\xi_k)\mathbb{1}_{\{\tau_{1,k}<\infty\}} + \sum_{k\geq 1} e^{-r_2\tau_{2,k}} g_2(\eta_k)\mathbb{1}_{\{\tau_{2,k}<\infty\}}\bigg] < \infty.$$

(iii) If $\tau_{i,k} = \tau_{i,k+1}$ for some $i = 1,2$ and $k \geq 1$, then $\tau_{i,k} = \tau_{i,k+1} = \infty$ $\mathbb{P}_x$-a.s.

(iv) One has $\lim_{k\to\infty} \tau_{i,k} = +\infty$ $\mathbb{P}_x$-a.s. for $i = 1,2$.

Notice that requirements (iii) and (iv) prevent each agent to act twice at the same time, and to accumulate her interventions. For future use, we make the following standing assumption.

Assumption 1.2.7. *It holds*

$$\mathbb{E}_x\bigg[\int_0^\infty e^{-r_1 t}\pi(X_t)dt + \int_0^\infty e^{-r_2 t}C(\beta X_t)dt\bigg] < \infty.$$

Remark 1.2.8. *Notice that in the benchmark cases in which the uncontrolled output of production is such that $dX_t = \mu X_t dt + \sigma X_t dW_t$, i.e. $X_t = x\exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\}$, $\mu \in \mathbb{R}$, $\sigma > 0$, and $\pi(x) = x^a$, $a \in (0,1)$, and $C(x) = x^b$, $b > 1$, one has that Assumption 1.2.7 is satisfied by taking*

$$r_1 > \bigg[\mu a - \frac{\sigma^2 a}{2}(1-a)\bigg]^+ \quad \textit{and} \quad r_2 > \bigg[\mu b + \frac{\sigma^2 b}{2}(b-1)\bigg]^+.$$

We now introduce the policies (strategies) of the firm and of the government that they are allowed to follow in order to intervene on the output of production. We assume that these policies are of so-called barrier type that can be informally described as follows (see also [2]).

(i) The firm increases its production instantaneously by exerting an impulse whenever the output of production is such that $X_t \leq b_1^1$, and shifts the process upwards to some b_2^1, where b_j^1, $j = 1,2$, are real constants chosen by the firm such that $b_2^1 > b_1^1$.

(ii) The government introduces regulatory constraints whenever the level of production, hence of emissions, is too large, i.e. $X_t \geq b_2^2$, and induces a shift of the process downwards to some b_1^2, where b_j^2, $j = 1, 2$, are real constants chosen by the government such that $b_2^2 > b_1^2$.

More formally, a policy of an agent is defined as follows.

Definition 1.2.9. *The policies φ_1 and φ_2 of the firm and of the government, respectively, are given by pairs*

$$\varphi_1 := (b_1^1; b_2^1) \in ([0, \infty) \cup \{-\infty\}) \times (0, \infty),$$
$$\varphi_2 := (b_1^2; b_2^2) \in [0, \infty) \times (0, \infty],$$

where $b_1^1 < b_2^1$ and $b_1^2 < b_2^2$.

Notice that the firm does not intervene on the output of production whenever it picks $b_1^1 = -\infty$. Similarly, the government does not intervene if $b_2^2 = \infty$. Therefore, for any $b_2^1, b_1^2 \in (0, \infty)$, we denote the non-intervention policies by $\overline{\varphi}_1 := (-\infty; b_2^1)$ and $\overline{\varphi}_2 := (b_1^2; \infty)$, respectively. The constant barriers b_j^i, $i, j = 1, 2$, of the government (resp. the firm) are decided *ex-ante* and do not dynamically react to the policy followed by the firm (resp. government). Therefore, they trigger *precommitted policies* of the two agents.

In the following, we describe the actions that are induced by the policies. To simplify the notations, the associated action to the policy φ_i of agent i is denoted by $\nu_i(\varphi_i)$, and we write $X_t^{x,\varphi_1,\varphi_2}$ in order to stress the dependency of the policies on the output of production, that is $X_t^{x,\varphi_1,\varphi_2} \equiv X_t^{x,\nu_1(\varphi_1),\nu_2(\varphi_2)}$. Then, for any $x > 0$ given and fixed and $\varphi_i \neq \overline{\varphi}_i$, $i = 1, 2$, we set

$$\nu_1(\varphi_1) := (\tau_{1,1}^{\varphi_1,\varphi_2}, \dots, \tau_{1,n}^{\varphi_1,\varphi_2}, \dots; \xi_1^{\varphi_1,\varphi_2}, \dots, \xi_n^{\varphi_1,\varphi_2}, \dots),$$
$$\nu_2(\varphi_2) := (\tau_{2,1}^{\varphi_1,\varphi_2}, \dots, \tau_{2,n}^{\varphi_1,\varphi_2}, \dots; \eta_1^{\varphi_1,\varphi_2}, \dots, \eta_n^{\varphi_1,\varphi_2}, \dots),$$

where we have introduced:

(a) the sequence of the firm's intervention times $\{\tau_{1,k}^{\varphi_1,\varphi_2}\}_{k\geq 1}$ such that $\tau_{1,k}^{\varphi_1,\varphi_2} := \inf\{t > \tau_{1,k-1}^{\varphi_1,\varphi_2} : X_t^{x,\varphi_1,\varphi_2} \leq b_1^1\}$ with $\tau_{1,0}^{\varphi_1,\varphi_2} := 0$ $\mathbb{P}$-a.s.;

(b) the sequence of the government's intervention times $\{\tau_{2,k}^{\varphi_1,\varphi_2}\}_{k\geq 1}$ such that $\tau_{2,k}^{\varphi_1,\varphi_2} := \inf\{t > \tau_{2,k-1}^{\varphi_1,\varphi_2} : X_t^{x,\varphi_1,\varphi_2} \geq b_2^2\}$ with $\tau_{2,0}^{\varphi_1,\varphi_2} := 0$ $\mathbb{P}$-a.s.;

(c) the sequence of interventions of the firm $\xi_k^{\varphi_1,\varphi_2} := \frac{1}{\alpha}(b_2^1 - X^{x,\varphi_1,\varphi_2}_{\tau_{1,k}^{\varphi_1,\varphi_2}-})$ for all $k \geq 1$;

(d) the sequence of impulses applied by the government $\eta_k^{\varphi_1,\varphi_2} := X^{x,\varphi_1,\varphi_2}_{\tau_{2,k}^{\varphi_1,\varphi_2}-} - b_1^2$ for all $k \geq 1$.

By the definition of $\tau_{1,k}^{\varphi_1,\varphi_2}$ and $\tau_{2,k}^{\varphi_1,\varphi_2}$ one has that the sequence of impulses $\xi_k^{\varphi_1,\varphi_2}$ and $\eta_k^{\varphi_1,\varphi_2}$ are constant-sized (apart the initial impulses, that depend on the initial state x). In particular, $\xi_k^{\varphi_1,\varphi_2} := (b_2^1 - b_1^1)/\alpha$ and $\eta_k^{\varphi_1,\varphi_2} := b_2^2 - b_1^2$ for all $k \geq 2$, and $\xi_1^{\varphi_1,\varphi_2} := (b_2^1 - x \wedge b_1^1)/\alpha$ and $\eta_1^{\varphi_1,\varphi_2} := x \vee b_2^2 - b_1^2$.

Moreover, $\nu_i(\overline{\varphi}_i)$ is associated to the non-intervention action, that is $\tau_{1,k} = \infty$ $\mathbb{P}_x$-a.s. for any $k \geq 1$ if $i = 1$, and $\tau_{2,k} = \infty$ $\mathbb{P}_x$-a.s. for any $k \geq 1$ if $i = 2$.

The agents pick their policies within the following admissible class.

Definition 1.2.10. *We say that the policies $\varphi_1 = (b_1^1; b_2^1)$ and $\varphi_2 = (b_1^2; b_2^2)$ are admissible, and we write $(\varphi_1, \varphi_2) \in \mathcal{S}$, if at least one of the following conditions hold true:*

(i) The firm or the government follows a non-intervention policy, that is $\varphi_i = \overline{\varphi}_i$ for some $i \in \{1, 2\}$.

(ii) One has $b_1^1 < b_2^2$ and $b_2^1, b_1^2 \in (b_1^1, b_2^2)$.

We define the *firm's action region* as $\mathcal{A}_1 := [0, b_1^1]$ and the *government's action region* as $\mathcal{A}_2 := [b_2^2, \infty)$ with the convention that $[0, -\infty] = \emptyset = [\infty, \infty)$. In the rest of this chapter, we will denote by $\mathcal{I}_i := \mathbb{R}_+ \setminus \mathcal{A}_i$ the *inaction region* of agent i.

Notice that admissible policies (φ_1, φ_2) exist because the constant trigger values b_j^i, $i, j = 1, 2$, of agent i do not depend on the policy employed by agent $j \neq i$. That is, independently of the policy of agent j, agent i will always force the process X to stay in her *inaction region* $\mathcal{I}_i$. A rigorous formalization of (φ_1, φ_2) can be obtained by the arguments employed in Definition 2.2 of [2]. We now show that admissible policies (φ_1, φ_2) in fact imply admissible actions.

Lemma 1.2.11. *Recall Definition 1.2.6. Then for any $x > 0$ and $(\varphi_1, \varphi_2) \in \mathcal{S}$, the actions $(\nu_1(\varphi_1), \nu_2(\varphi_2)) \in \mathcal{T}(x)$.*

Proof. Let $x > 0$ be given and fixed. Existence of a unique strong solution to (1.9) with right-continuous paths can be obtained by arguing as in Lemma 2.3 of [2]. Also, $X_t^{x,\varphi_1,\varphi_2} \in [0, \infty)$ $\mathbb{P}$-a.s. for all $t > 0$ since $b_2^1, b_1^2 \in [0, \infty)$. Hence, Condition (i) of Definition 1.2.6 is satisfied.

Now suppose that $\varphi_i \neq \overline{\varphi}_i$ for any $i = 1, 2$. The fact that $X_t^{x,\varphi_1,\varphi_2} \in [b_1^1, b_2^2]$ $\mathbb{P}$-a.s. for all $t > 0$ and the continuity of π and C in particular imply that $(ii) - (a)$ of Definition 1.2.6 is fulfilled. As for $(ii) - (b)$ note that $\xi_k^{\varphi_1,\varphi_2} \leq b_2^1/\alpha$ $\mathbb{P}_x$-a.s. for all $k \in \mathbb{N}$, and that $\eta_k^{\varphi_1,\varphi_2} \leq \max(b_2^2 - b_1^2, x - b_1^2)$ $\mathbb{P}_x$-a.s. for all $k \in \mathbb{N}$. Hence there exists a positive constant Θ (possibly depending on x) such that $g_1(\xi_k^{\varphi_1,\varphi_2}) + g_2(\eta_k^{\varphi_1,\varphi_2}) \leq \Theta$ $\mathbb{P}_x$-a.s. for all $k \in \mathbb{N}$. In order to prove that $(ii) - (b)$ of Definition 1.2.6 holds true, it thus suffices to show that for any $i = 1, 2$ one has

$$\mathbb{E}_x\left[\sum_{k \geq 1} e^{-r_i \tau_{i,k}^{\varphi_1,\varphi_2}}\right] < \infty.$$

To accomplish that one can adapt to our setting arguments from the proof of Proposition 4.7 in [2]. We provide these arguments here for the sake of completeness. Without loss of generality we consider the case $i = 1$, since the treatment of the case $i = 2$ is analogous. Defining $\tilde{\tau} := \inf\{t > 0 : X_t^{b_2^1,\varphi_1,\varphi_2} \leq b_1^1\}$, and exploiting the time-homogeneity of the production process X and the independence of the Brownian increments, we can write for any $k \geq 1$

$$\mathbb{E}_x\big[e^{-r_1\tau_{1,k}^{\varphi_1,\varphi_2}}\big] = \mathbb{E}_x\big[e^{-r_1\tau_{1,k-1}^{\varphi_1,\varphi_2}}\big]\mathbb{E}\big[e^{-r_1\tilde{\tau}}\big].$$

By iterating the previous argument one finds

$$\mathbb{E}_x\big[e^{-r_1\tau_{1,k}^{\varphi_1,\varphi_2}}\big] = \mathbb{E}_x\big[e^{-r_1\tau_{1,1}^{\varphi_1,\varphi_2}}\big]\big(\mathbb{E}\big[e^{-r_1\tilde{\tau}}\big]\big)^{k-1}.$$

Then summing over k on both sides of the previous equation and applying Fubini-Tonelli's theorem, we obtain

$$\mathbb{E}_x\Bigg[\sum_{k\geq 1} e^{-r_1\tau_{1,k}^{\varphi_1,\varphi_2}}\Bigg] = \mathbb{E}_x\big[e^{-r_1\tau_{1,1}^{\varphi_1,\varphi_2}}\big]\sum_{k\geq 0}\Big(\mathbb{E}\big[e^{-r_1\tilde{\tau}}\big]\Big)^k,$$

and the series on the right-hand-side above converges as $\mathbb{E}[e^{-r_1\tilde{\tau}}] < 1$.

Because $b_1^1 < b_2^2$ by assumption, and $b_2^1, b_1^2 \in (b_1^1, b_2^2)$, condition (iii) and (iv) of Definition 1.2.6 are satisfied.

Finally, if $\varphi_i = \overline{\varphi}_i$ for some $i \in \{1, 2\}$, also the actions $(\nu_1(\varphi_1), \nu_2(\varphi_2)) \in \mathcal{T}(x)$. In fact, conditions $(ii) - (b)$, (iii) and (iv) can be shown to be valid by proceeding as above. Condition $(ii) - (a)$ instead follows by using Assumption 1.2.7 and exploiting the arguments of the proof of Proposition 1.3.2 below (with $\tau_2 = \infty$ therein, that is, when $\varphi_1 \neq \overline{\varphi}_1$ and $\varphi_2 = \overline{\varphi}_2$). □

Given the policy adopted by the other agent, the firm aims at maximizing its profit, whereas the government at minimizing the social costs of pollution. Hence, for any $x > 0$, the two agents aim at finding $(\varphi_1^*, \varphi_2^*) \in \mathcal{S}$ such that

$$\begin{cases} \mathcal{J}_1(x, \nu_1(\varphi_1^*), \nu_2(\varphi_2^*)) \geq \mathcal{J}_1(x, \nu_1(\varphi_1), \nu_2(\varphi_2^*)), & \forall \varphi_1 \text{ such that } (\varphi_1, \varphi_2^*) \in \mathcal{S}, \\ \mathcal{J}_2(x, \nu_1(\varphi_1^*), \nu_2(\varphi_2^*)) \leq \mathcal{J}_2(x, \nu_1(\varphi_1^*), \nu_2(\varphi_2)), & \forall \varphi_2 \text{ such that } (\varphi_1^*, \varphi_2) \in \mathcal{S}. \end{cases} \tag{$\mathcal{P}$}$$

Definition 1.2.12. *Let $x > 0$. If $(\varphi_1^*, \varphi_2^*) \in \mathcal{S}$ satisfying $(\mathcal{P})$ exist, we call them equilibrium policies, and we define the equilibrium values as*

$$V_1(x) := \mathcal{J}_1(x, \nu_1(\varphi_1^*), \nu_2(\varphi_2^*)) \quad \text{and} \quad V_2(x) := \mathcal{J}_2(x, \nu_1(\varphi_1^*), \nu_2(\varphi_2^*)).$$

1.3 Solving the Strategic Pollution Control Problem

In this section, we first construct a pair of admissible candidate equilibrium policies which is such that both agents do not follow a non-intervention policy. Then, under suitable requirements, we show that these policies indeed solve problem $(\mathcal{P})$.

1.3.1 Construction of a Candidate Solution

We conjecture that both agents follow an admissible intervention policy, that is, the equilibrium boundaries $\tilde{b}^i_j$, $i,j=1,2$, are such that $\tilde{b}^1_1 \neq -\infty$ and $\tilde{b}^2_2 \neq \infty$. The associated policies are denoted by $\tilde{\varphi}_i$, $i=1,2$, and the expected payoffs associated to $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ are defined as

$$v_1(x) := \mathcal{J}_1(x, \nu_1(\tilde{\varphi}_1), \nu_2(\tilde{\varphi}_2)) \quad \text{and} \quad v_2(x) := \mathcal{J}_2(x, \nu_1(\tilde{\varphi}_1), \nu_2(\tilde{\varphi}_2)), \quad x>0.$$

Moreover, thanks to Assumption 1.2.7, the performance criteria associated with no interventions are finite and given by

$$G_1(x) := \mathbb{E}_x\left[\int_0^\infty e^{-r_1 s}\pi(X_s)ds\right] \quad \text{and} \quad G_2(x) := \mathbb{E}_x\left[\int_0^\infty e^{-r_2 s}C(\beta X_s)ds\right]. \tag{1.12}$$

For frequent future use, we define the infinitesimal generator $\mathcal{L}$ of the uncontrolled diffusion X^x by

$$\big(\mathcal{L}u\big)(x) := \frac{1}{2}\sigma^2(x)u''(x) + \mu(x)u'(x), \qquad x>0,$$

for any $u \in C^2((0,\infty))$. Then, for fixed $r>0$, under Assumption 1.2.1 there always exist two linearly independent, strictly positive solutions to the ordinary differential equation $\mathcal{L}u = ru$ satisfying a set of boundary conditions based on the boundary behaviour of X^x (see, e.g., pp. 18-19 of [31]). These functions span the set of solutions of $\mathcal{L}u = ru$, and are uniquely defined up to multiplication if one of them is required to be strictly increasing and the other one to be strictly decreasing. We denote the strictly increasing solution by ψ_r and the strictly decreasing one by ϕ_r. From now on we set $\psi_i := \psi_{r_i}$ and $\phi_i := \phi_{r_i}$ for $i=1,2$.

Remark 1.3.1. *The functions G_1 and G_2 are the expected cumulative present value of the flows $\pi(X^x_t)$ and $C(\beta X^x_t)$, respectively. It is well known that G_i, $i=1,2$, can be represented in terms of the fundamental solutions ψ_i and ϕ_i, $i=1,2$. We refer the reader to equation (3.3) in [7], among others.*

For any $i=1,2$ we introduce the strictly decreasing and positive function F_i such that $F_i(x) := \phi_i(x)/\psi_i(x)$. Also, for given $\tilde{b}^i_j$, $i,j=1,2$, such that $0<\tilde{b}^1_1<\tilde{b}^1_2<\tilde{b}^2_2$ and $\tilde{b}^1_1<\tilde{b}^2_1<\tilde{b}^2_2$, we set

$$A_i(x) := \frac{\psi_i(x)}{\psi_i(\tilde{b}^1_1)}\left[\frac{F_i(\tilde{b}^2_2)-F_i(x)}{F_i(\tilde{b}^2_2)-F_i(\tilde{b}^1_1)}\right], \quad B_i(x) := \frac{\psi_i(x)}{\psi_i(\tilde{b}^2_2)}\left[\frac{F_i(x)-F_i(\tilde{b}^1_1)}{F_i(\tilde{b}^2_2)-F_i(\tilde{b}^1_1)}\right] \quad i=1,2. \tag{1.13}$$

We define w_i as the restriction of v_i on $\mathcal{I}_1 \cap \mathcal{I}_2 = (\tilde{b}^1_1, \tilde{b}^2_2)$, i.e. $w_i := v_i|_{\mathcal{I}_1\cap\mathcal{I}_2}$. The next result provides a representation of v_i, $i=1,2$.

Proposition 1.3.2. *Recall* (1.13)*, let* $x > 0$*, and* $\tilde{b}^i_j$*,* $i, j = 1, 2$*, such that* $0 < \tilde{b}^1_1 < \tilde{b}^1_2 < \tilde{b}^2_2$ *and* $\tilde{b}^1_1 < \tilde{b}^2_1 < \tilde{b}^2_2$*. Then, the performance criteria* $v_1(x)$ *and* $v_2(x)$ *associated to the policies* $(\tilde{\varphi}_1, \tilde{\varphi}_2) \in \mathcal{S}$ *can be represented as*

$$v_1(x) = \begin{cases} w_1(\tilde{b}^1_2) - K_1 - \frac{\kappa_1}{\alpha}(\tilde{b}^1_2 - x), & x \leq \tilde{b}^1_1, \\ \big[w_1(\tilde{b}^1_2) - K_1 - \frac{\kappa_1}{\alpha}(\tilde{b}^1_2 - \tilde{b}^1_1) - G_1(\tilde{b}^1_1)\big] A_1(x) & \\ +\big[w_1(\tilde{b}^2_1) - G_1(\tilde{b}^2_2)\big] B_1(x) + G_1(x), & x \in (\tilde{b}^1_1, \tilde{b}^2_2), \\ w_1(\tilde{b}^2_1), & x \geq \tilde{b}^2_2, \end{cases} \tag{1.14}$$

and

$$v_2(x) = \begin{cases} w_2(\tilde{b}^1_2), & x \leq \tilde{b}^1_1 \\ \big[w_2(\tilde{b}^2_1) + K_2 + \kappa_2(\tilde{b}^2_2 - \tilde{b}^2_1) - G_2(\tilde{b}^2_2)\big] B_2(x) & \\ +\big[w_2(\tilde{b}^1_2) - G_2(\tilde{b}^1_1)\big] A_2(x) + G_2(x), & x \in (\tilde{b}^1_1, \tilde{b}^2_2), \\ w_2(\tilde{b}^2_1) + K_2 + \kappa_2(x - \tilde{b}^2_1), & x \geq \tilde{b}^2_2. \end{cases} \tag{1.15}$$

Moreover, under the requirement

$$\big(1 - A_i(\tilde{b}^1_2)\big)\big(1 - B_i(\tilde{b}^2_1)\big) - B_i(\tilde{b}^1_2) A_i(\tilde{b}^2_1) \neq 0, \quad i = 1, 2, \tag{1.16}$$

one has

$$\begin{aligned} w_1(\tilde{b}^1_2) = &\left[1 - A_1(\tilde{b}^1_2) - \frac{B_1(\tilde{b}^1_2) A_1(\tilde{b}^2_1)}{1 - B_1(\tilde{b}^2_1)}\right]^{-1} \left[\frac{G_1(\tilde{b}^2_1) B_1(\tilde{b}^1_2)}{1 - B_1(\tilde{b}^2_1)} + G_1(\tilde{b}^1_2)\right. \\ &- \left(K_1 + \kappa_1(\tilde{b}^1_2 - \tilde{b}^1_1) + G_1(\tilde{b}^1_1)\right)\left(\frac{A_1(\tilde{b}^2_1) B_1(\tilde{b}^1_2)}{1 - B_1(\tilde{b}^2_1)} + A_1(\tilde{b}^1_2)\right) \\ &\left. - G_1(\tilde{b}^2_2)\left(\frac{B_1(\tilde{b}^2_1) B_1(\tilde{b}^1_2)}{1 - B_1(\tilde{b}^2_1)} + B_1(\tilde{b}^1_2)\right)\right], \end{aligned} \tag{1.17}$$

$$\begin{aligned} w_1(\tilde{b}^2_1) = &\big[1 - B_1(\tilde{b}^2_1)\big]^{-1} \big[\big(w_1(\tilde{b}^1_2) - K_1 - \kappa_1(\tilde{b}^1_2 - \tilde{b}^1_1) - G_1(\tilde{b}^1_1)\big) A_1(\tilde{b}^2_1) \\ &- G_1(\tilde{b}^2_2) B(\tilde{b}^2_1) + G_1(\tilde{b}^2_1)\big], \end{aligned} \tag{1.18}$$

and

$$\begin{aligned} w_2(\tilde{b}^1_2) = &\left[\frac{\big(1 - A_2(\tilde{b}^1_2)\big)\big(1 - B_2(\tilde{b}^2_1)\big)}{B_2(\tilde{b}^1_2)} - A_2(\tilde{b}^2_1)\right]^{-1} \times \\ &\left[\frac{G_2(\tilde{b}^1_2)\big(1 - B_2(\tilde{b}^2_1)\big)}{B_2(\tilde{b}^1_2)} + G_2(\tilde{b}^2_1) + K_2 + \kappa_2(\tilde{b}^2_2 - \tilde{b}^2_1) - G_2(\tilde{b}^2_2)\right. \\ &\left. - G_2(\tilde{b}^1_1)\left(A_2(\tilde{b}^1_2)\frac{1 - B_2(\tilde{b}^2_1)}{B_2(\tilde{b}^1_2)} + A_2(\tilde{b}^2_1)\right)\right], \end{aligned} \tag{1.19}$$

$$\begin{aligned} w_2(\tilde{b}^2_1) = &\big[1 - B_2(\tilde{b}^2_1)\big]^{-1} \big[\big(K_2 + \kappa_2(\tilde{b}^2_2 - \tilde{b}^2_1) - G_2(\tilde{b}^2_2)\big) B_2(\tilde{b}^2_1) \\ &+ \big(w_2(\tilde{b}^1_2) - G_2(\tilde{b}^1_1)\big) A_2(\tilde{b}^2_1) + G_2(\tilde{b}^2_1)\big]. \end{aligned} \tag{1.20}$$

Proof. We consider only the case $i = 1$ since the arguments are symmetric for $i = 2$. Let $x > 0$ be given and fixed, and define $\tau_1 := \inf\{t \geq 0 : X_t^x \leq \tilde{b}_1^1\}$ and $\tau_2 := \inf\{t \geq 0 : X_t^x \geq \tilde{b}_2^2\}$. According to the policies $(\tilde{\varphi}_1, \tilde{\varphi}_2)$, the stopping time $\tau_1 \wedge \tau_2$ is the first time at which either the firm or the government intervenes. Then, noticing that X is uncontrolled up to time $\tau_1 \wedge \tau_2$, the payoff of the firm associated to $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ satisfies the functional relation

$$
\begin{aligned}
v_1(x) = \mathbb{E}_x\Bigg[\int_0^{\tau_1 \wedge \tau_2} & e^{-r_1 t}\pi(X_t)dt + e^{-r_1\tau_1}\mathbb{1}_{\{\tau_1<\tau_2\}}\Big(v_1(\tilde{b}_2^1) - K_1 - \frac{\kappa_1}{\alpha}(\tilde{b}_2^1 - X_{\tau_1}^{\tilde{\varphi}_1,\tilde{\varphi}_2})\Big) \\
&+ e^{-r_1\tau_2}\mathbb{1}_{\{\tau_1>\tau_2\}}v_1(\tilde{b}_1^2)\Bigg].
\end{aligned}
\tag{1.21}
$$

Recall that w_i denotes the restriction of v_i on $\mathcal{I}_1 \cap \mathcal{I}_2$. Then, taking $x \in (\tilde{b}_1^1, \tilde{b}_2^2) = \mathcal{I}_1 \cap \mathcal{I}_2$ in (1.21), noticing that $\tilde{b}_2^1$ and $\tilde{b}_1^2$ belong to $\mathcal{I}_1 \cap \mathcal{I}_2$ and recalling (1.12), by the strong Markov property we can write

$$
\begin{aligned}
w_1(x) =& \big(w_1(\tilde{b}_2^1) - K_1 - \frac{\kappa_1}{\alpha}(\tilde{b}_2^1 - \tilde{b}_1^1) - G_1(\tilde{b}_1^1)\big)\mathbb{E}_x\big[e^{-r_1\tau_1}\mathbb{1}_{\{\tau_1<\tau_2\}}\big] \\
&+ \big(w_1(\tilde{b}_1^2) - G_1(\tilde{b}_2^2)\big)\mathbb{E}_x\big[e^{-r_1\tau_2}\mathbb{1}_{\{\tau_1>\tau_2\}}\big] + G_1(x).
\end{aligned}
$$

By using now the formulas for the Laplace transforms of hitting times of a linear diffusion (see, e.g., [45], eq. (4.3)), we find (cf. (1.13))

$$
\mathbb{E}_x\big[e^{-r_1\tau_1}\mathbb{1}_{\{\tau_1<\tau_2\}}\big] = A_1(x), \quad \mathbb{E}_x\big[e^{-r_1\tau_2}\mathbb{1}_{\{\tau_1>\tau_2\}}\big] = B_1(x),
$$

so that

$$
w_1(x) = \big(w_1(\tilde{b}_2^1) - K_1 - \frac{\kappa_1}{\alpha}(\tilde{b}_2^1 - \tilde{b}_1^1) - G_1(\tilde{b}_1^1)\big)A_1(x) + \big(w_1(\tilde{b}_1^2) - G_1(\tilde{b}_2^2)\big)B_1(x) + G_1(x),
$$

for all $x \in (\tilde{b}_1^1, \tilde{b}_2^2)$.

Taking $x \leq \tilde{b}_1^1$ in (1.21) we obtain $\tau_1 = 0$ and then $v_1(x) = w_1(\tilde{b}_2^1) - K_1 - \frac{\kappa_1}{\alpha}(\tilde{b}_2^1 - x)$, while picking $x \geq \tilde{b}_2^2$ we have $\tau_2 = 0$ and thus $v_1(x) = w_1(\tilde{b}_1^2)$. Therefore we can write

$$
v_1(x) = \begin{cases}
w_1(\tilde{b}_2^1) - K_1 - \frac{\kappa_1}{\alpha}(\tilde{b}_2^1 - x), & x \leq \tilde{b}_1^1, \\
\big[w_1(\tilde{b}_2^1) - K_1 - \frac{\kappa_1}{\alpha}(\tilde{b}_2^1 - \tilde{b}_1^1) - G_1(\tilde{b}_1^1)\big]A_1(x) & \\
+\big[w_1(\tilde{b}_1^2) - G_1(\tilde{b}_2^2)\big]B_1(x) + G_1(x), & x \in (\tilde{b}_1^1, \tilde{b}_2^2), \\
w_1(\tilde{b}_1^2), & x \geq \tilde{b}_2^2.
\end{cases}
\tag{1.22}
$$

Let (1.16) hold. Recalling again that $\tilde{b}_2^1, \tilde{b}_1^2 \in (\tilde{b}_1^1, \tilde{b}_2^2)$ by construction, and taking first $x = \tilde{b}_2^1$ and then $x = \tilde{b}_1^2$ in (1.22), we obtain a linear system of two equations for the two unknowns $w_1(\tilde{b}_2^1)$ and $w_1(\tilde{b}_1^2)$. Once solved, this system yields

$$\begin{aligned} w_1(\tilde{b}_2^1) =& \left[1 - A_1(\tilde{b}_2^1) - \frac{B_1(\tilde{b}_2^1)A_1(\tilde{b}_1^2)}{1 - B_1(\tilde{b}_1^2)}\right]^{-1} \left[\frac{G_1(\tilde{b}_1^2)B_1(\tilde{b}_2^1)}{1 - B_1(\tilde{b}_1^2)} + G_1(\tilde{b}_2^1)\right. \\ & - \left(K_1 + \kappa_1(\tilde{b}_2^1 - \tilde{b}_1^1) + G_1(\tilde{b}_1^1)\right)\left(\frac{A_1(\tilde{b}_1^2)B_1(\tilde{b}_2^1)}{1 - B_1(\tilde{b}_1^2)} + A_1(\tilde{b}_2^1)\right) \\ & \left. - G_1(\tilde{b}_2^2)\left(\frac{B_1(\tilde{b}_1^2)B_1(\tilde{b}_2^1)}{1 - B_1(\tilde{b}_1^2)} + B_1(\tilde{b}_2^1)\right)\right], \end{aligned}$$

and

$$\begin{aligned} w_1(\tilde{b}_1^2) =& \left[1 - B_1(\tilde{b}_1^2)\right]^{-1}\left[\left(w_1(\tilde{b}_2^1) - K_1 - \kappa_1(\tilde{b}_2^1 - \tilde{b}_1^1) - G_1(\tilde{b}_1^1)\right)A_1(\tilde{b}_1^2)\right. \\ & \left. - G_1(\tilde{b}_2^2)B(\tilde{b}_1^2) + G_1(\tilde{b}_1^2)\right]. \end{aligned}$$

Notice that the denominators in the definition of $w_1(\tilde{b}_2^1)$ are nonzero. Indeed, $B_1(\tilde{b}_1^2) \neq 1$ since $\tau_2 > 0$ $\mathbb{P}$-a.s. for $x = \tilde{b}_1^2 < \tilde{b}_2^2$, and $(1 - A_1(\tilde{b}_2^1))(1 - B_1(\tilde{b}_1^2)) - B_1(\tilde{b}_2^1)A_1(\tilde{b}_1^2) \neq 0$ by (1.16).

The proof is then completed. □

It is easy to see from (1.14) and (1.15) that v_i, $i = 1, 2$, is by construction a continuous function on $(0, \infty)$. In order to obtain the equilibrium four boundaries $\tilde{b}_j^i$, $i, j = 1, 2$, we first assume that each agent picks her own action boundary $\tilde{b}_i^i$, $i = 1, 2$, such that v_i is also continuously differentiable there. This gives

$$v_1'(\tilde{b}_1^1+) = \frac{\kappa_1}{\alpha}, \tag{1.23}$$

$$v_2'(\tilde{b}_2^2-) = \kappa_2, \tag{1.24}$$

where we have set $v_i'(\cdot\pm) := \lim_{\varepsilon\downarrow 0} v_i'(\cdot \pm \varepsilon)$.

The two equations (1.23) and (1.24) may be interpreted as the so-called *smooth-fit* equations, well known optimality conditions in the literature on singular/impulse control and optimal stopping (see, e.g., [66, 107]). Furthermore, we assume that at each intervention the firm and the government shift the process X to the points that give rise to the maximal net profits and minimal total costs, respectively. This means that $\tilde{b}_2^1, \tilde{b}_1^2 \in (\tilde{b}_1^1, \tilde{b}_2^2)$ are selected such that

$$\tilde{b}_2^1 = \arg\sup_{y \geq \tilde{b}_1^1}\left\{v_1(y) - \frac{\kappa_1}{\alpha}(y - x) - K_1\right\}, \quad x \leq \tilde{b}_1^1,$$

and

$$\tilde{b}_1^2 = \arg\inf_{y \leq \tilde{b}_2^2}\left\{v_2(y) + \kappa_2(x - y) + K_2\right\}, \quad x \geq \tilde{b}_2^2.$$

Consequently,

$$v_1'(\tilde{b}_2^1) = \frac{\kappa_1}{\alpha}, \tag{1.25}$$

$$v_2'(\tilde{b}_1^2) = \kappa_2. \tag{1.26}$$

The *four* equations (1.23)-(1.26) can be used in order to obtain the *four* unknowns $\tilde{b}_1^1, \tilde{b}_2^1, \tilde{b}_1^2, \tilde{b}_2^2$, whenever a solution to such a highly nonlinear system exists.

1.3.2 The Verification Theorem

Here, we prove a verification theorem providing a set of sufficient conditions under which the solution to (1.23)-(1.26) (if it exists) characterizes an equilibrium; that is, $(\tilde{\varphi}_1, \tilde{\varphi}_2) = (\varphi_1^*, \varphi_2^*)$, and $v_1 \equiv V_1$, $v_2 \equiv V_2$ (cf. Definition 1.2.12). For its proof the following assumption is needed.

Assumption 1.3.3.

(i) There exists $\hat{x}_1 > 0$ such that the function $\theta_1 : \mathbb{R}_+ \mapsto \mathbb{R}$ with $\theta_1(x) := \pi(x) + \frac{\kappa_1}{\alpha}(\mu(x) - r_1 x)$ attains a local maximum at $\hat{x}_1$ and is increasing on $(0, \hat{x}_1)$;

(ii) There exists $\hat{x}_2 > 0$ such that the function $\theta_2 : \mathbb{R}_+ \mapsto \mathbb{R}$ with $\theta_2(x) := C(\beta x) + \kappa_2(\mu(x) - r_2 x)$ attains a local minimum at $\hat{x}_2$ and is increasing on $(\hat{x}_2, \infty)$.

Remark 1.3.4. *It is worth noticing that Assumption 1.3.3 is verified by the benchmark cases $\mu(x) = \mu x$, $\mu \in \mathbb{R}$, $\pi(x) = x^a$, $a \in (0,1)$, and $C(x) = x^b$, $b > 1$, with $\hat{x}_1 = \left[\frac{\kappa_1}{a\alpha}(r_1 - \mu)\right]^{\frac{1}{a-1}}$, $\hat{x}_2 = \left[\frac{\kappa_2}{b\beta^b}(r_2 - \mu)\right]^{\frac{1}{b-1}}$ (whenever $r_1 \wedge r_2 > \mu$).*

Theorem 1.3.5 (Verification Theorem)**.** *Let Assumption 1.3.3 hold. Let $\tilde{b}_j^i$, $i, j = 1, 2$, be a solution of* (1.23)-(1.26) *such that $0 < \tilde{b}_1^1 < \tilde{b}_2^1 < \tilde{b}_2^2$, $\tilde{b}_1^1 < \tilde{b}_1^2 < \tilde{b}_2^2$ and satisfying* (1.16)*, recall v_1, v_2 as in* (1.14) *and* (1.15)*, and suppose that*

$$v_1'(x) \geq \frac{\kappa_1}{\alpha}, \qquad \text{for all } x \in (\tilde{b}_1^1, \tilde{b}_2^1], \tag{1.27}$$

$$v_1'(x) < \frac{\kappa_1}{\alpha}, \qquad \text{for all } x \in (\tilde{b}_2^1, \tilde{b}_2^2], \tag{1.28}$$

$$v_2'(x) < \kappa_2, \qquad \text{for all } x \in (\tilde{b}_1^1, \tilde{b}_1^2), \tag{1.29}$$

$$v_2'(x) \geq \kappa_2, \qquad \text{for all } x \in [\tilde{b}_1^2, \tilde{b}_2^2), \tag{1.30}$$

and

$$\tilde{b}_1^1 \leq \hat{x}_1, \tag{1.31}$$

$$\pi(\tilde{b}_1^1) + \frac{c_1}{\alpha}\mu(\tilde{b}_1^1) - r_1 v_1(\tilde{b}_1^1) \leq 0, \tag{1.32}$$

$$\tilde{b}_2^2 \geq \hat{x}_2, \tag{1.33}$$

$$C(\beta \tilde{b}_2^2) + \kappa_2 \mu(\tilde{b}_2^2) - r_2 v_2(\tilde{b}_2^2) \geq 0. \tag{1.34}$$

Then, for $x > 0$, the policies $(\tilde{\varphi}_1, \tilde{\varphi}_2) \in \mathcal{S}$ such that

$$\begin{cases} \tau_{i,k}^{\tilde{\varphi}_1,\tilde{\varphi}_2} = \inf\{t > \tau_{i,k-1}^{\tilde{\varphi}_1,\tilde{\varphi}_2} : X_t^{\tilde{\varphi}_1,\tilde{\varphi}_2} \notin \mathcal{I}_i\}, & k \geq 1,\ \mathbb{P}_x\text{-a.s.}, \\ \tau_{i,0}^{\tilde{\varphi}_1,\tilde{\varphi}_2} = 0, & \mathbb{P}_x\text{-a.s.}, \end{cases} \tag{1.35}$$

for $i = 1, 2$, *and*

$$\tilde{\xi}_k^{\tilde{\varphi}_1,\tilde{\varphi}_2} = \frac{1}{\alpha}\Big(\tilde{b}_2^1 - X^{\tilde{\varphi}_1,\tilde{\varphi}_2}_{\tau^{\tilde{\varphi}_1,\tilde{\varphi}_2}_{1,k}-}\Big), \qquad \tilde{\eta}_k^{\tilde{\varphi}_1,\tilde{\varphi}_2} = X^{\tilde{\varphi}_1,\tilde{\varphi}_2}_{\tau^{\tilde{\varphi}_1,\tilde{\varphi}_2}_{2,k}-} - \tilde{b}_1^2, \qquad k \geq 1, \qquad \mathbb{P}_x\text{-a.s.}, \tag{1.36}$$

form an equilibrium, and v_1 *and* v_2 *are the corresponding equilibrium values; that is,*

$$v_1 = V_1, \quad v_2 = V_2 \quad \text{on } (0, \infty).$$

Proof. The proof is organized in two steps.

Step 1. Here, we discuss the regularity properties of the function v_i, $i = 1, 2$, constructed in Proposition 1.3.2. Note that by (1.14) and (1.15) one can directly check that $v_i \in C((0,\infty))$ for $i = 1, 2$. Moreover, by (1.23) and (1.24) one has $v_1 \in C^1((0, \tilde{b}_2^2))$, $v_2 \in C^1((\tilde{b}_1^1, \infty))$ and it can be checked by direct calculations that $v_1'' \in L^\infty_{\text{loc}}((0, \tilde{b}_2^2))$ and $v_2'' \in L^\infty_{\text{loc}}((\tilde{b}_1^1, \infty))$. Also, for any $x \in (\tilde{b}_1^1, \tilde{b}_2^2)$ we have from (1.14) and (1.15) that $\big(\mathcal{L}v_1 - r_1 v_1\big)(x) + \pi(x) = 0$ and $\big(\mathcal{L}v_2 - r_2 v_2\big)(x) + C(\beta x) = 0$.

Because θ_1 is increasing on $(0, \hat{x}_1)$ (cf. Assumption 1.3.3), and $\tilde{b}_1^1 \leq \hat{x}_1$ by assumption, we obtain from (1.14) that for any $x < \tilde{b}_1^1$ one has

$$\begin{aligned} &\big(\mathcal{L}v_1 - r_1 v_1\big)(x) + \pi(x) = \theta_1(x) - r_1\big(v_1(\tilde{b}_2^1) - K_1 - \frac{\kappa_1}{\alpha}\tilde{b}_2^1\big) \\ &\leq \theta_1(\tilde{b}_1^1) - r_1\big(v_1(\tilde{b}_2^1) - K_1 - \frac{\kappa_1}{\alpha}\tilde{b}_2^1\big) = \pi(\tilde{b}_1^1) + \frac{\kappa_1}{\alpha}\mu(\tilde{b}_1^1) - r_1 v_1(\tilde{b}_1^1) \leq 0, \end{aligned} \tag{1.37}$$

where we have used that $v_1(\tilde{b}_2^1) = v_1(\tilde{b}_1^1) + K_1 + \frac{\kappa_1}{\alpha}(\tilde{b}_2^1 - \tilde{b}_1^1)$, (1.31) and (1.32).

Similarly, one can check that $\big(\mathcal{L}v_2 - r_2 v_2\big)(x) + C(\beta x) \geq 0$ for all $x > \tilde{b}_2^2$ due to (1.33), (1.34), and the fact that θ_2 is increasing on $(\hat{x}_2, \infty)$ (cf. Assumption 1.3.3).

Step 2. Given $x > 0$ we now prove that $(\tilde{\varphi}_1, \tilde{\varphi}_2) \in \mathcal{S}$ are equilibrium policies; that is,

$$\begin{aligned} v_1(x) &\geq \mathcal{J}_1(x, \nu_1(\varphi_1), \nu_2(\tilde{\varphi}_2)), \quad \forall \varphi_1 \text{ s.t. } (\varphi_1, \tilde{\varphi}_2) \in \mathcal{S}, \\ v_2(x) &\leq \mathcal{J}_2(x, \nu_1(\tilde{\varphi}_1), \nu_2(\varphi_2)), \quad \forall \varphi_2 \text{ s.t. } (\tilde{\varphi}_1, \varphi_2) \in \mathcal{S}, \end{aligned}$$

with equalities when we pick $\varphi_1 = \tilde{\varphi}_1$ and $\varphi_2 = \tilde{\varphi}_2$. Without loss of generality we consider $i = 1$, since the arguments for $i = 2$ are analogous.

Let φ_1 be such that $(\varphi_1, \tilde{\varphi}_2) \in \mathcal{S}$, and for $N > 0$ set $\tau_{R,N} := \tau_R \wedge N$, where $\tau_R := \inf\{s > 0 : X_s^{x,\varphi_1,\tilde{\varphi}_2} \notin (-R, R)\}$, with the usual convention $\inf \emptyset = \infty$. Since $X_t^{x,\varphi_1,\tilde{\varphi}_2} \leq \tilde{b}_2^2$ $\mathbb{P}$-a.s. for all $t > 0$, by the regularity of v_1 discussed in *Step 1* we can apply the generalized Itô's formula for semimartingales (see, e.g., [102], Theorems 2.1

and 6.2), so to obtain

$$v_1(x) = \mathbb{E}_x\Bigg[- \int_0^{\tau_{R,N}} e^{-r_1 t}(\mathcal{L}v_1 - r_1 v_1)(X_t^{\varphi_1,\tilde{\varphi}_2})dt + e^{-r_1\tau_{R,N}} v_1(X^{\varphi_1,\tilde{\varphi}_2}_{\tau_{R,N}}) \\ - \sum_{k:\,\tau_{1,k}^{\varphi_1,\tilde{\varphi}_2}<\tau_{R,N}} e^{-r_1\tau_{1,k}^{\varphi_1,\tilde{\varphi}_2}}\big(v_1(X^{\varphi_1,\tilde{\varphi}_2}_{\tau_{1,k}^{\varphi_1,\tilde{\varphi}_2}}) - v_1(X^{\varphi_1,\tilde{\varphi}_2}_{\tau_{1,k}^{\varphi_1,\tilde{\varphi}_2}-})\big) \\ - \sum_{k:\,\tau_{2,k}^{\varphi_1,\tilde{\varphi}_2}<\tau_{R,N}} e^{-r_1\tau_{2,k}^{\varphi_1,\tilde{\varphi}_2}}\big(v_1(X^{\varphi_1,\tilde{\varphi}_2}_{\tau_{2,k}^{\varphi_1,\tilde{\varphi}_2}}) - v_1(X^{\varphi_1,\tilde{\varphi}_2}_{\tau_{2,k}^{\varphi_1,\tilde{\varphi}_2}-})\big)\Bigg]. \tag{1.38}$$

By using again that $X_t^{x,\varphi_1,\tilde{\varphi}_2} \leq \tilde{b}_2^2$ for all $t > 0$ $\mathbb{P}$-a.s., and since $(\mathcal{L}v_1 - r_1 v_1)(x) \leq -\pi(x)$ for a.e. $x < \tilde{b}_2^2$ due to (1.37), we obtain from (1.38) that

$$v_1(x) \geq \mathbb{E}_x\Bigg[\int_0^{\tau_{R,N}} e^{-r_1 t}\pi(X_t^{\varphi_1,\tilde{\varphi}_2})dt - \sum_{k:\,\tau_{1,k}^{\varphi_1,\tilde{\varphi}_2}<\tau_{R,N}} e^{-r_1\tau_{1,k}^{\varphi_1,\tilde{\varphi}_2}}\big(v_1(X^{\varphi_1,\tilde{\varphi}_2}_{\tau_{1,k}^{\varphi_1,\tilde{\varphi}_2}}) - v_1(X^{\varphi_1,\tilde{\varphi}_2}_{\tau_{1,k}^{\varphi_1,\tilde{\varphi}_2}-})\big) \\ - \sum_{k:\,\tau_{2,k}^{\varphi_1,\tilde{\varphi}_2}<\tau_{R,N}} e^{-r_1\tau_{2,k}^{\varphi_1,\tilde{\varphi}_2}}\big(v_1(X^{\varphi_1,\tilde{\varphi}_2}_{\tau_{2,k}^{\varphi_1,\tilde{\varphi}_2}}) - v_1(X^{\varphi_1,\tilde{\varphi}_2}_{\tau_{2,k}^{\varphi_1,\tilde{\varphi}_2}-})\big) + e^{-r_1\tau_{R,N}} v_1(X^{\varphi_1,\tilde{\varphi}_2}_{\tau_{R,N}})\Bigg]. \tag{1.39}$$

In order to take care of the two sums in the expectation above, we define the nonlocal operator

$$\big(\mathcal{M}_1 v_1\big)(x) := \sup_{\xi\geq 0}\big\{v_1(x+\alpha\xi) - g_1(\xi)\big\},$$

and we notice that $\tilde{\xi}_k^{\varphi_1,\tilde{\varphi}_2}$ of (1.36) is such that $\tilde{\xi}_k^{\varphi_1,\tilde{\varphi}_2} = \arg\sup_{\xi\geq 0}\big\{v_1(x+\alpha\xi) - g_1(\xi)\big\}$, for all $k \in \mathbb{N}$, due to (1.27) and (1.28). Hence

$$\big(\mathcal{M}_1 v_1\big)(x) = \begin{cases} v_1(\tilde{b}_2^1) - K_1 - \frac{\kappa_1}{\alpha}(\tilde{b}_2^1 - x), & \text{if } x \leq \tilde{b}_2^1, \\ v_1(x) - K_1, & \text{if } x > \tilde{b}_2^1. \end{cases} \tag{1.40}$$

One can easily see from (1.14) and (1.40) that $v_1(x) \geq \big(\mathcal{M}_1 v_1\big)(x)$ for all $x \in (0, \tilde{b}_1^1] \cup (\tilde{b}_2^1, \infty)$, with equality for $x \leq \tilde{b}_1^1$. Then, noticing that $x \mapsto v_1(x) - \big(\mathcal{M}_1 v_1\big)(x)$ is increasing for any $x \in (\tilde{b}_1^1, \tilde{b}_2^1]$ by (1.27) and (1.40), we conclude that $v_1(x) \geq \big(\mathcal{M}_1 v_1\big)(x)$ for all $x > 0$. Therefore

$$v_1(X^{x,\varphi_1,\tilde{\varphi}_2}_{\tau_{1,k}^{\varphi_1,\tilde{\varphi}_2}-}) \geq \big(\mathcal{M}_1 v_1\big)(X^{x,\varphi_1,\tilde{\varphi}_2}_{\tau_{1,k}^{\varphi_1,\tilde{\varphi}_2}-}) \geq v_1\big(X^{x,\varphi_1,\tilde{\varphi}_2}_{\tau_{1,k}^{\varphi_1,\tilde{\varphi}_2}}\big) - g_1(\xi_k^{\varphi_1,\tilde{\varphi}_2}). \tag{1.41}$$

Moreover, because $X^{x,\varphi_1,\tilde{\varphi}_2}_{\tau_{2,k}^{\varphi_1,\tilde{\varphi}_2}-} \geq \tilde{b}_2^2$ $\mathbb{P}$-a.s. and $X^{x,\varphi_1,\tilde{\varphi}_2}_{\tau_{2,k}^{\varphi_1,\tilde{\varphi}_2}} = \tilde{b}_1^2$ $\mathbb{P}$-a.s., we find by (1.14) that

$$v_1(X^{x,\varphi_1,\tilde{\varphi}_2}_{\tau_{2,k}^{\varphi_1,\tilde{\varphi}_2}-}) = v_1(X^{x,\varphi_1,\tilde{\varphi}_2}_{\tau_{2,k}^{\varphi_1,\tilde{\varphi}_2}}), \tag{1.42}$$

upon noticing that $v_1(\tilde{b}_1^2) = w_1(\tilde{b}_1^2)$ since $\tilde{b}_1^2 \in (\tilde{b}_1^1, \tilde{b}_2^2)$. It thus follows from (1.41) and (1.42) that

$$\begin{aligned} v_1(x) \geq \mathbb{E}_x \bigg[& \int_0^{\tau_{R,N}} e^{-r_1 t} \pi(X_t^{x,\varphi_1,\tilde{\varphi}_2}) dt \\ & - \sum_{k:\tau_{1,k}^{\varphi_1,\tilde{\varphi}_2} < \tau_{R,N}} e^{-r_1 \tau_{1,k}^{\varphi_1,\tilde{\varphi}_2}} g_1(\xi_k^{\varphi_1,\tilde{\varphi}_2}) + e^{-r_1 \tau_{R,N}} v_1(X_{\tau_{R,N}}^{x,\varphi_1,\tilde{\varphi}_2}) \bigg]. \end{aligned} \tag{1.43}$$

Now, v_1 is continuous and $X_t^{x,\varphi_1,\tilde{\varphi}_2} \in [0, \tilde{b}_2^2]$ $\mathbb{P}$-a.s. by admissibility of $(\varphi_1, \tilde{\varphi}_2)$. Hence,

$$e^{-r_1 \tau_{R,N}} v_1(X_{\tau_{R,N}}^{x,\varphi_1,\tilde{\varphi}_2}) \geq -e^{-r_1 \tau_{R,N}} |v_1(X_{\tau_{R,N}}^{x,\varphi_1,\tilde{\varphi}_2})| \geq -e^{-r_1 \tau_{R,N}} \max_{x \in [0,\tilde{b}_2^2]} |v_1(x)|,$$

and from (1.43) we have

$$\begin{aligned} v_1(x) \geq \mathbb{E}_x \bigg[& \int_0^{\tau_{R,N}} e^{-r_1 t} \pi(X_t^{x,\varphi_1,\tilde{\varphi}_2}) dt \\ & - \sum_{k:\tau_{1,k}^{\varphi_1,\tilde{\varphi}_2} < \tau_{R,N}} e^{-r_1 \tau_{1,k}^{\varphi_1,\tilde{\varphi}_2}} g_1(\xi_k^{\varphi_1,\tilde{\varphi}_2}) - e^{-r_1 \tau_{R,N}} \max_{x \in [0,\tilde{b}_2^2]} |v_1(x)| \bigg]. \end{aligned} \tag{1.44}$$

By using the dominated convergence theorem for the last term in (1.44) and the monotone convergence theorem for the integral and the series in (1.44), we let first $R \to \infty$ and then $N \to \infty$, and we find

$$v_1(x) \geq \mathcal{J}_1(x, \nu_1(\varphi_1), \nu_2(\tilde{\varphi}_2)).$$

Finally, by construction we also have $v_1(x) = \mathcal{J}_1(x, \nu_1(\tilde{\varphi}_1), \nu_2(\tilde{\varphi}_2))$.

Because arguments analogous to the ones employed for v_1 we have that $v_2(x) \leq \mathcal{J}_2(x, \nu_1(\tilde{\varphi}_1), \nu_2(\varphi_2))$ for all φ_2 such that $(\tilde{\varphi}_1, \varphi_2) \in \mathcal{S}$, and $v_2(x) = \mathcal{J}_2(x, \nu_1(\tilde{\varphi}_1), \nu_2(\tilde{\varphi}_2))$, we conclude that $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ are equilibrium policies and (v_1, v_2) are the corresponding equilibrium values. □

Remark 1.3.6. *As a byproduct of Theorem 1.3.5 we have that, if* (1.27)-(1.34) *are fulfilled, then v_1 and v_2 satisfy a.e. the system of quasi-variational inequalities*

$$\begin{aligned} \max\{\big(\mathcal{L}v_1 - r_1 v_1\big)(x) + \pi(x),\, \mathcal{M}_1 v_1(x) - v_1(x)\} = 0, &\quad \text{for a.e. } x < \tilde{b}_2^2, \\ \min\{\big(\mathcal{L}v_2 - r_2 v_2\big)(x) + C(\beta x),\, \mathcal{M}_2 v_2(x) - v_2(x)\} = 0, &\quad \text{for a.e. } x > \tilde{b}_1^1, \\ v_1(x) \geq \mathcal{M}_1 v_1(x), &\quad \forall x > 0, \\ v_2(x) \leq \mathcal{M}_2 v_2(x), &\quad \forall x > 0, \\ v_1(x) = v_1\big(\tilde{b}_1^2\big), &\quad \forall x \geq \tilde{b}_2^2, \\ v_2(x) = v_2\big(\tilde{b}_2^1\big), &\quad \forall x \leq \tilde{b}_1^1. \end{aligned} \tag{1.45}$$

A system analogous to (1.45) *has been introduced in the context of nonzero-sum stochastic differential games with impulse controls in [2].*

1.4 A Numerical Example and Comparative Statics

Verification Theorem 1.3.5 involves the highly nonlinear system of *four* algebraic equations (1.23)–(1.26) for the *four* boundaries. We have solved this system numerically in a specific setting by using MATLAB. In particular, for the numerical example we have assumed that the uncontrolled output of production evolves as a geometric Brownian motion, i.e. $\mu(x) = \mu x$ and $\sigma(x) = \sigma x$ for some $\mu \in \mathbb{R}$ and $\sigma > 0$. Moreover, we have taken an operating profit function of Cobb-Douglas type $\pi(x) = x^a$, $a \in (0,1)$, and a social disutility function of the form $C(x) = x^b$, $b > 1$.

Among the possible parameters' values satisfying Assumption 1.2.7, we pick for example those provided in Table 1.1, and we notice that for such a choice the performance

μ	σ	r_1	r_2	α	β	K_1	κ_1	K_2	κ_2	a	b
0.02	0.20	0.10	0.10	1	1	0.5	0.8	0.6	0.3	0.5	2

Table 1.1: Parameters' values for the numerical example.

criteria associated with no interventions (cf. (1.12)) are given by

$$G_1(x) = \frac{1}{r_1 - \frac{\mu}{2} + \frac{\sigma^2}{8}}\sqrt{x} = \frac{1000}{95}\sqrt{x}, \quad \text{and} \quad G_2(x) = \frac{1}{r_2 - 2\mu - \sigma^2}x^2 = 50x^2. \quad (1.46)$$

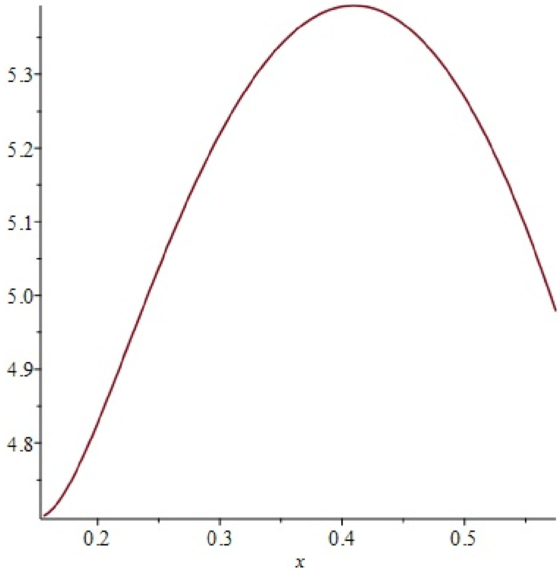
Also, by an application of the Newton method in MATLAB, we find that the numerical solution to (1.23)-(1.26) is given by

$$\tilde{b}_1^1 = 0.1558984470, \quad \tilde{b}_2^1 = 0.3825673799,$$
$$\tilde{b}_1^2 = 0.2359455020, \quad \tilde{b}_2^2 = 0.5746537199,$$

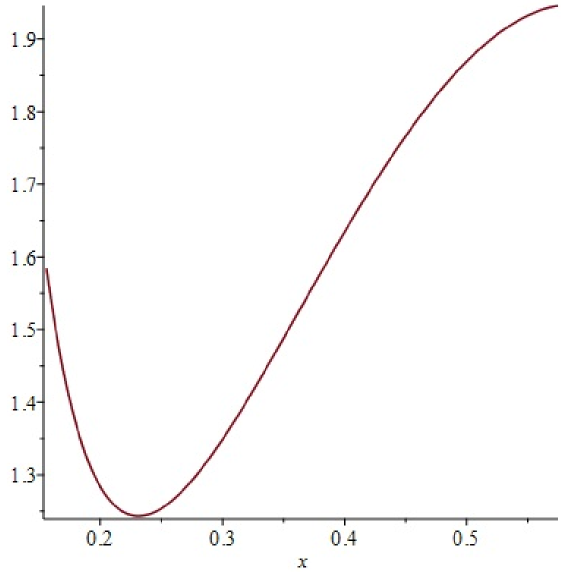
where we have evaluated $w_i(\tilde{b}_2^1)$ and $w_i(\tilde{b}_1^2), i = 1, 2$, by (1.17)-(1.20). One also finds (cf. (1.31)-(1.34))

$$\hat{x}_1 = \left[2\kappa_1(r_1 - \mu)\right]^{-2} = 61.03515625 > \tilde{b}_1^1,$$
$$\pi(\tilde{b}_1^1) + \frac{c_1}{\alpha}\mu(\tilde{b}_1^1) - r_1 v_1(\tilde{b}_1^1) = -0.0727643376 \leq 0,$$
$$\hat{x}_2 = \frac{\kappa_2(r_2 - \mu)}{2} = 0.012 < \tilde{b}_2^2,$$
$$C(\beta\tilde{b}_2^2) + \kappa_2\mu(\tilde{b}_2^2) - r_2 v_2(\tilde{b}_2^2) = 0.1390988361 \geq 0.$$

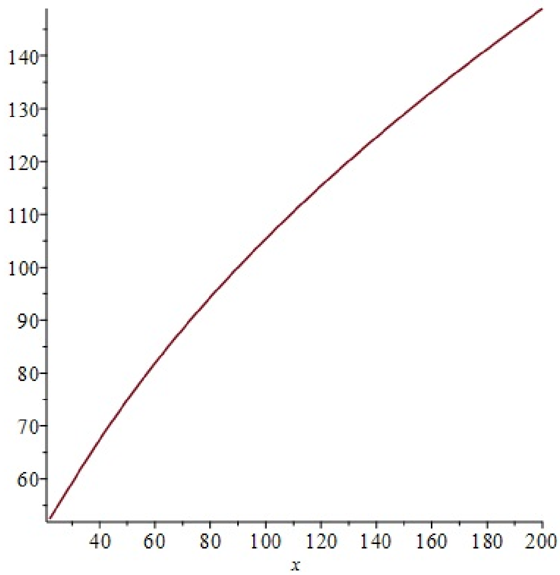
The plots of the equilibrium values and of their derivatives in the joint inaction region $(\tilde{b}_1^1, \tilde{b}_2^2)$ are provided in Figures 1.1(a), 1.1(b), and 1.2(a) and 1.2(b), respectively. In Figures 1.1(c) and 1.1(d) one observes the drawings of the value functions that the firm and the government would have in a non-strategic setting (i.e. if the two agents optimize their own performance criterion in absence of the other agent).



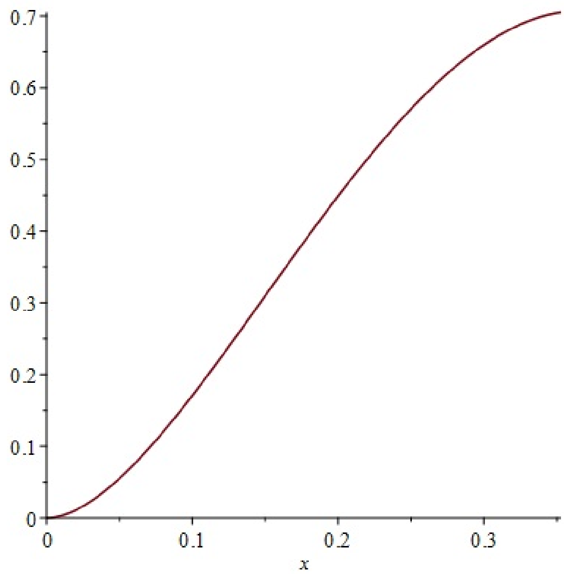
(a) Equilibrium value V_1 in $(\tilde{b}_1^1, \tilde{b}_2^2)$.



(b) Equilibrium value V_2 in $(\tilde{b}_1^1, \tilde{b}_2^2)$.



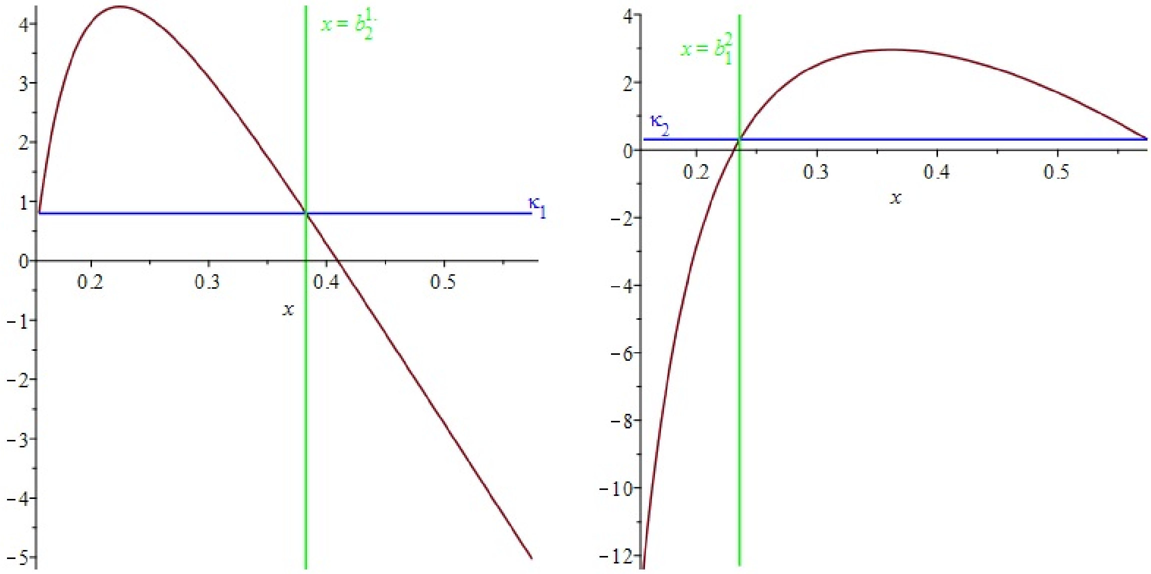
(c) Value function of the firm in the *inaction* region for a non-strategic model.



(d) Value function of the government in the *inaction* region for a non-strategic model.

Figure 1.1: Value functions in the strategic and non-strategic setting.

Comparing Figures 1.1(a) and 1.1(b) with Figures 1.1(c) and 1.1(d), one can notice that the value functions that the two agents would have in a non-strategic setting are monotone with respect to the state variable. On the contrary, the equilibrium values V_1 and V_2 are not monotone functions, and this is clearly a consequence of the strategic interaction between the two agents. From Figures 1.2(a) and 1.2(b) one can also check that conditions (1.27)-(1.30) are satisfied.



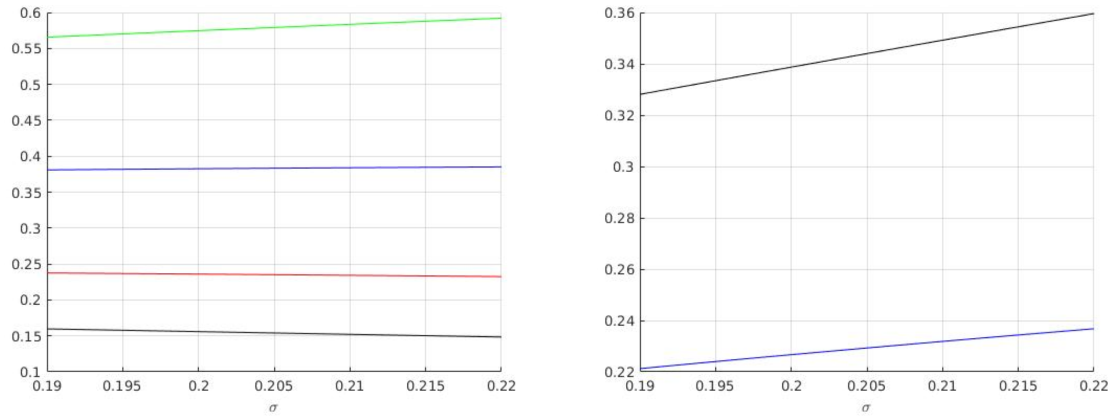
(a) Derivative of V_1.

(b) Derivative of V_2.

Figure 1.2: Derivatives of the equilibrium values.

We now discuss the dependency of our equilibrium policies with respect to the model parameters. The following plots are obtained with MATLAB through an application of the Newton method initialized at the parameters' values specified in Table 1.1.

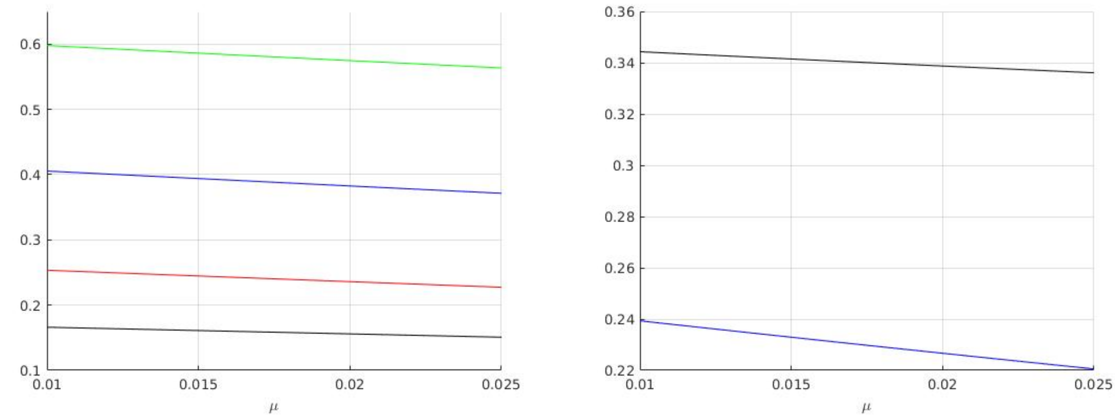
Figure 1.3(a) displays the behavior of the equilibrium boundaries (optimal action thresholds) $\tilde{b}_1^1$ and $\tilde{b}_2^2$ when the volatility σ varies in the range $[0.19, 0.22]$. Furthermore, Figure 1.3(b) shows how the optimal size of interventions, $\tilde{b}_2^1 - \tilde{b}_1^1$ and $\tilde{b}_2^2 - \tilde{b}_1^2$, changes with σ. One can observe that the optimal action threshold of the government increases with σ, whereas the firm's action threshold decreases. This behavior is well-known in the real options literature (see the seminal article by [93]): when uncertainty increases, the agent is more reluctant to act and her inaction region becomes larger. Furthermore, Figure 1.3(b) reveals that the strength of interventions of the firm and of the government increases with increasing volatility. The higher are the fluctuations of the production/pollution process, the more the agents are afraid of a quicker need of a new costly intervention. Hence both the agents increase the size of their impulses in order to postpone their next action.



(a) The equilibrium boundaries $\tilde{b}_1^1$ (black), $\tilde{b}_2^1$ (blue), $\tilde{b}_1^2$ (red), $\tilde{b}_2^2$ (green).

(b) Optimal size of interventions: firm (blue) and government (black).

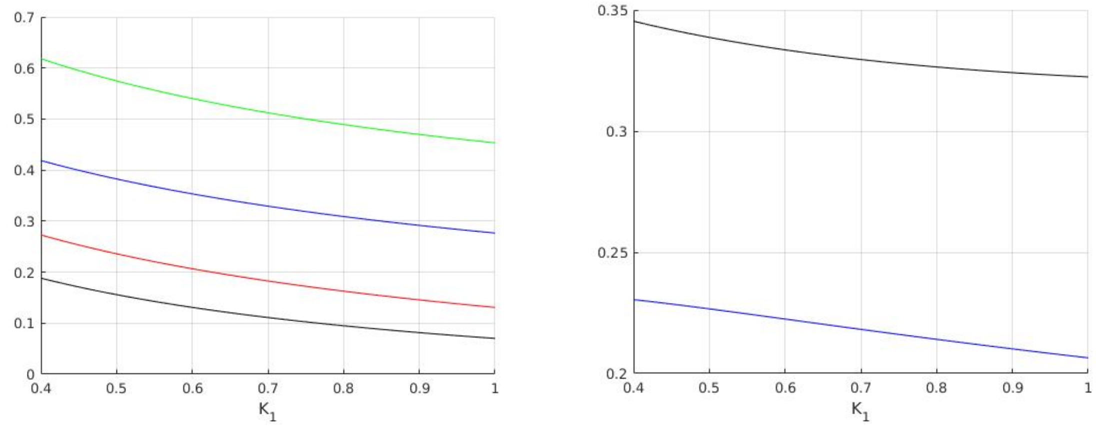
Figure 1.3: Dependency of the equilibrium on the volatility σ.



(a) The equilibrium boundaries $\tilde{b}_1^1$ (black), $\tilde{b}_2^1$ (blue), $\tilde{b}_1^2$ (red), $\tilde{b}_2^2$ (green).

(b) Optimal size of interventions: firm (blue) and government (black).

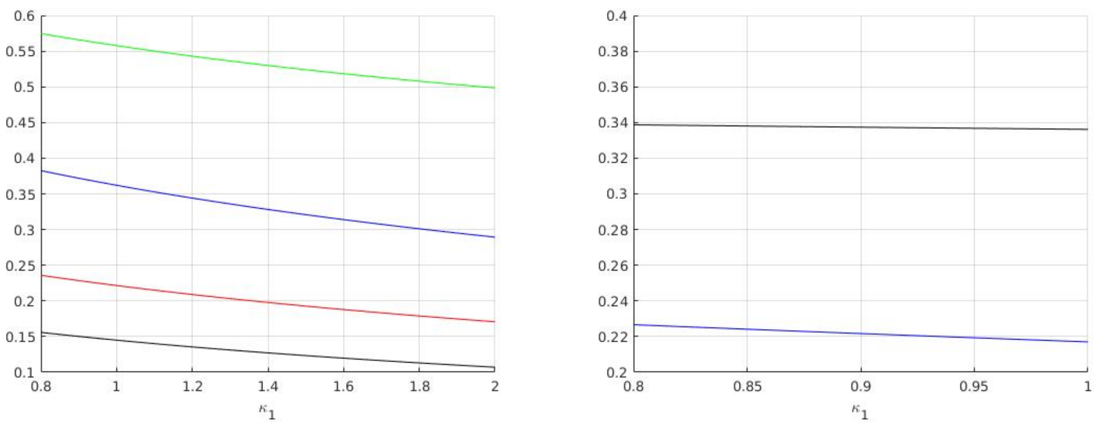
Figure 1.4: Dependency of the equilibrium on the drift μ.



(a) The equilibrium boundaries $\tilde{b}_1^1$ (black), $\tilde{b}_2^1$ (blue), $\tilde{b}_1^2$ (red), $\tilde{b}_2^2$ (green).

(b) Optimal size of interventions: firm (blue) and government (black).

Figure 1.5: Dependency of the equilibrium on the firm's fixed cost K_1.



(a) The equilibrium boundaries $\tilde{b}_1^1$ (black), $\tilde{b}_2^1$ (blue), $\tilde{b}_1^2$ (red), $\tilde{b}_2^2$ (green).

(b) Optimal size of interventions: firm (blue) and government (black).

Figure 1.6: Dependency of the equilibrium on the firm's variable cost κ_1.

We now take $\sigma = 0.2$, and we let μ vary in the interval $[0.01, 0.025]$. Figure 1.4(a) leads us to the following conclusion: as the drift μ increases, the firm's action region becomes smaller. That is, a higher trend of the output of production decreases the firm's willingness to intervene. We can also observe from Figure 1.4(a) that the government's threshold decreases with μ: since the output of production, and therefore the rate of emissions, increases faster, the government tries to dam the increasing social cost by introducing more severe regulatory constraints. Figure 1.4(b) shows that the higher the trend of the output of production is, the lower is the size of interventions $\tilde{b}_2^1 - \tilde{b}_1^1$, i.e. the lower the willingness of the firm to pay for additional capacity. Also, one can observe that the government's size of interventions decrease with increasing μ. We believe that this effect is due to the strategic interactions between the two agents, and it might be justified as follows. The higher μ is, the smaller is the length of the joint inaction region (see Figure 1.4(a)). Hence, the government reduces the size of interventions when μ increases so to likely reduce the firm's incentive to intervene.

Finally, we analyze the dependency of the action thresholds and of the equilibrium impulses' size with respect to the cost components K_1 and κ_1 (see Figures 1.5 and 1.6). Similar behaviors are also observed with respect to K_2 and κ_2. Higher fixed costs lead to decreasing equilibrium boundaries, see Figure 1.5(a), and therefore to a larger inaction region of the firm. As a consequence, the government exploits the firm's reluctance to invest when fixed costs are larger and confines the production process below a lower level. A particular comment is deserved by Figure 1.5(b) where we observe that the sizes of interventions of both agents are decreasing with respect to K_1. This behavior might be explained once more as an effect of the strategic interaction between the two agents. When K_1 increases, the firm reduces the size of its interventions in order to likely avoid a possible further action by the government, and, in turn, a further costly capacity expansion. As a result of the reduction of the joint inaction region (see Figure 1.5(a)), the government also diminishes its size of interventions so to try to prevent the firm from undertaking a further capacity expansion. A similar rationale might also explain the behavior of the equilibrium thresholds and equilibrium impulses' sizes with respect to the variable costs κ_1.

1.5 Conclusions

In this chapter, a government and a firm, representative of the productive sector of a country, are the two players of a stochastic nonzero-sum game of impulse control. The firm faces both proportional and fixed costs to expand its stochastically fluctuating production with the aim of maximizing its expected profits. The government introduces regulatory constraints with the aim of reducing the level of emissions of pollutants and of minimizing the related total expected costs. Assuming that the emissions' level is proportional to the output of production, by issuing environmental policies the government effectively forces the firm to decrease its production.

We have modeled the agent's policies by barrier strategies that are characterized by four constant trigger values, chosen by the agents. We have then constructed a candidate equilibrium in this strategic problem when both agents do not follow a non-intervention policy. Under a set of sufficient conditions, those policies do indeed form an equilibrium. Finally, we have studied numerically the case in which the (uncontrolled) output of production evolves as a geometric Brownian motion, and the firm's operating profit and the government's running cost function are of power type. Within such a setting, a study of the dependency of the equilibrium policies and values on the model parameters have yielded interesting new behaviors that we have explained as a result of the strategic interaction between the firm and the government.

There are many directions in which it would be interesting to extend the present study. As an example, one might consider a two-dimensional formulation of our game in which the state variables are given by the production capacity of the firm and the level of pollution. The firm faces a costly capacity expansion and maximizes its net expected profits. The output of production, however, increases the emissions, which in turn contribute to the accumulation of a pollution stock. The government aims at reducing the level of the pollution stock by issuing costly environmental policies. This would lead to a daunting two-dimensional stochastic game with impulse controls for which a sophisticated theoretical and numerical analysis might be needed.

Chapter 2

An Optimal Extraction Problem with Price Impact

2.1 Introduction

A price-maker company extracts an exhaustible commodity from a reservoir, and sells it in the spot market. In absence of any actions of the company, the commodity's spot price evolves either as a drifted Brownian motion or as an Ornstein-Uhlenbeck process. While extracting, the company's actions have an impact on the commodity's spot price. The company aims at maximizing the total expected profits from selling the commodity, net of the total expected proportional costs of extraction. We model this problem as a two-dimensional degenerate singular stochastic control problem with finite fuel that we solve explicitly. On the one hand, when the (uncontrolled) price is a drifted Brownian motion, it is optimal to extract whenever the current price level is larger or equal than an endogenously determined constant threshold. On the other hand, when the (uncontrolled) price evolves as an Ornstein-Uhlenbeck process, the optimal extraction rule is triggered by a curve depending on the current level of the reservoir. Such a curve is a strictly decreasing C^{∞}-function for which we provide an explicit expression. Finally, our study is complemented by a theoretical and numerical analysis of the dependency of the optimal extraction strategy and value function on the model's parameters.

This chapter is based on the article [60]. In Section 2.2 we introduce the setting and formulate the problem. In Section 2.3 we provide preliminary results and a Verification Theorem. The explicit solution to the optimal extraction problem is then constructed in Sections 2.4.1 and 2.4.2 when the commodity's price is a drifted Brownian motion and an Ornstein-Uhlenbeck process, respectively. A connection to an optimal stopping problem is derived in Section 2.4.2.1. A sensitivity analysis is presented in Section 2.5. Finally, we conclude in Section 2.6.

2.2 Setting and Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, with filtration $\mathbb{F}$ generated by a standard one-dimensional Brownian motion $(W_t)_{t\geq 0}$, and as usual augmented by $\mathbb{P}$-null sets.

We consider a company extracting a commodity from a reservoir with a finite capacity $y \geq 0$, and selling it instantaneously in the spot market. We assume that, in absence of any interventions of the company, the (fundamental) commodity's price $(X_t^x)_{t\geq 0}$ evolves stochastically according to the dynamics

$$dX_t^x = \big(a - bX_t^x\big)dt + \sigma dW_t, \qquad X_0^x = x \in \mathbb{R}, \tag{2.1}$$

for some constants $a \in \mathbb{R}$, $b \geq 0$ and $\sigma > 0$. In the following, we identify the fundamental price when $b = 0$ with a drifted Brownian motion with drift a. On the other hand, when $b > 0$ the price is of Ornstein-Uhlenbeck type, thus having a mean-reverting behavior typically observed in the commodity market (see, e.g., Chapter 2 of [92]). In this latter case, the parameter $\frac{a}{b}$ represents the mean-reversion level, and b is the mean-reversion speed. In our model we do not restrict our attention to positive fundamental prices, since certain commodities have been traded also at negative prices. For example, that happened in Alberta (Canada) in October 2017 and May 2018 where the producers of natural gas faced the tradeoff between paying customers to take gas, or shutting down the wells[1].

The reserve level can be decreased at a constant proportional cost $c > 0$. The extraction does not need to be performed at a rate, and we identify the cumulative amount of commodity that has been extracted up to time $t \geq 0$, ξ_t, as the company's control variable. It is an $\mathbb{F}$-adapted, nonnegative, and increasing càdlàg (right-continuous with left-limits) process $(\xi_t)_{t\geq 0}$ such that $\xi_t \leq y$ a.s. for all $t \geq 0$ and $\xi_{0-} = 0$ a.s. The constraint $\xi_t \leq y$ for all $t \geq 0$ has the clear interpretation that at any time it cannot be extracted more than the initial amount of commodity available in the reservoir. For any given $y \geq 0$, the set of *admissible extraction strategies* is therefore defined as

$$\begin{aligned}\mathcal{A}(y) := \{\xi : \Omega \times [0,\infty) \mapsto [0,\infty) : (\xi_t)_{t\geq 0} \text{ is } \mathbb{F}\text{-adapted, } t \mapsto \xi_t \text{ is increasing, càdlàg,} \\ \text{with } \xi_{0-} = 0 \text{ and } \xi_t \leq y \text{ a.s.}\}.\end{aligned}$$

Clearly, $\mathcal{A}(0) = \{\xi \equiv 0\}$.

The level of the reservoir at time t, Y_t, then evolves as

$$dY_t^{y,\xi} = -d\xi_t, \qquad Y_{0-}^{y,\xi} = y \geq 0,$$

[1] See, e.g., the article on the Financial Post [65], or the news on the website of the U.S. Energy Information Administration [127]

where we have written $Y^{y,\xi}$ in order to stress the dependency of the reservoir's level on the initial amount of commodity y and on the extraction strategy ξ.

While extracting, the company affects the market price of the commodity. In particular, when following an extraction strategy $\xi \in \mathcal{A}(y)$, the market price at time t, X_t, is instantaneously reduced by $\alpha d\xi_t$, for some $\alpha > 0$, and the spot price thus evolves as

$$dX_t^{x,\xi} = \big(a - bX_t^{x,\xi}\big)dt + \sigma dW_t - \alpha d\xi_t, \qquad X_{0-}^{x,\xi} = x \in \mathbb{R}. \tag{2.2}$$

We notice that for any $\xi \in \mathcal{A}(y)$ there exists a unique strong solution to (2.2) by Theorem 6 in Chapter V of [115], and we denote it by $X^{x,\xi}$ in order to keep track of its initial value $x \in \mathbb{R}$, and of the adopted extraction strategy $\xi \in \mathcal{A}(y)$.

Remark 2.2.1. *Notice that when $b = 0$, the impact of the company's extraction on the price is permanent. On the other hand, it is transient (or temporary) in the mean-reverting case $b > 0$ because, in the absence of any interventions from the company, the impact decreases since X reverts back to its mean-reversion level.*

The company aims at maximizing the total expected profits, net of the total expected costs of extraction. That is, for any initial price $x \in \mathbb{R}$ and any initial value of the reserve $y \geq 0$, the company aims at determining $\xi^\star \in \mathcal{A}(y)$ that attains

$$V(x,y) := \mathcal{J}(x,y,\xi^\star) = \sup_{\xi \in \mathcal{A}(y)} \mathcal{J}(x,y,\xi), \tag{2.3}$$

where

$$J(x,y,\xi) := \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}(X_t^{x,\xi} - c)d\xi_t^c + \sum_{t\geq 0:\Delta\xi_t \neq 0} e^{-\rho t}\big[(X_{t-}^{x,\xi} - c)\Delta\xi_t - \frac{1}{2}\alpha(\Delta\xi_t)^2\big]\bigg], \tag{2.4}$$

for any $\xi \in \mathcal{A}(y)$, and for a given discount factor $\rho > 0$. Here, and also in the following, $\Delta\xi_t := \xi_t - \xi_{t-}$, $t \geq 0$, and ξ^c denotes the continuous part of $\xi \in \mathcal{A}(y)$.

Remark 2.2.2. *In* (2.4) *the integral term in the expectation is intended as a standard Lebesgue-Stieltjes integral with respect to the continuous part ξ^c of ξ. The sum takes instead care of the lump sum extractions, and its form might be informally justified by interpreting any lump sum extraction of size $\Delta\xi_t$ at a given time t as a sequence of infinitely many infinitesimal extractions made at the same time t. In this way, setting $\epsilon_t := \frac{\Delta\xi_t}{N}$, the net profit accrued at time t by extracting a large amount $\Delta\xi_t$ of the commodity is*

$$\sum_{j=0}^{N-1} e^{-\rho t}\big(X_{t-}^{x,\xi} - c - j\alpha\epsilon_t\big)\epsilon_t$$
$$\overset{N\to\infty}{\longrightarrow} \int_0^{\Delta\xi_t} e^{-\rho t}\big(X_{t-}^{x,\xi} - c - \alpha u\big)du = e^{-\rho t}\Big[\big(X_{t-}^{x,\xi} - c\big)\Delta\xi_t - \frac{1}{2}\alpha(\Delta\xi_t)^2\Big].$$

This heuristic argument - also discussed at pp. 329–330 of [6] in the context of one-dimensional monotone follower problems - can be rigorously justified, and technical details on the convergence can be found in the recent [23]. We also refer to [75, 132] as other papers on singular stochastic control problems employing such a definition for the integral with respect to the control process.

2.3 Preliminary Results and a Verification Theorem

In this section, we derive the HJB equation associated to V and we provide a verification theorem. We start by proving the following preliminary properties of the value function V.

Proposition 2.3.1. *There exists a constant $K > 0$ such that for all $(x, y) \in \mathbb{R} \times [0, \infty)$ one has*

$$0 \leq V(x, y) \leq Ky(1 + y)\big(1 + |x|\big). \tag{2.5}$$

In particular, $V(x, 0) = 0$. Moreover, V is increasing with respect to x and y.

Proof. The proof is organized in two steps. We first prove that (2.5) holds true, and then we show the monotonicity properties of V.

Step 1. The nonnegativity of V follows by taking the admissible (no-)extraction rule $\xi \equiv 0$ such that $\mathcal{J}(x, y, 0) = 0$ for all $(x, y) \in \mathbb{R} \times [0, \infty)$. The fact that $V(x, 0) = 0$ clearly follows by noticing that $\mathcal{A}(0) = \{\xi \equiv 0\}$ and $\mathcal{J}(x, y, 0) = 0$.

To determine the upper bound in (2.5), let $(x, y) \in \mathbb{R} \times (0, \infty)$ be given and fixed, and for any $\xi \in \mathcal{A}(y)$ we have

$$\begin{aligned}
&\bigg|\mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\big(X_t^{x,\xi} - c\big)d\xi_t^c + \sum_{t \geq 0: \Delta\xi_t \neq 0} e^{-\rho t}\big[(X_{t-}^{x,\xi} - c)\Delta\xi_t - \frac{\alpha}{2}(\Delta\xi_t)^2\big]\bigg]\bigg| \\
\leq &\mathbb{E}\bigg[\int_0^\infty e^{-\rho t}|X_t^{x,\xi}|d\xi_t^c\bigg] + cy + \mathbb{E}\bigg[\sum_{t \geq 0: \Delta\xi_t \neq 0} e^{-\rho t}\big[|X_{t-}^{x,\xi}|\Delta\xi_t + \frac{\alpha}{2}(\Delta\xi_t)^2\big]\bigg],
\end{aligned} \tag{2.6}$$

where we have used the fact that $c\int_0^\infty e^{-\rho t}d\xi_t = c\int_0^\infty \rho e^{-\rho t}\xi_t dt \leq cy$ to obtain the term cy in right-hand side above.

We now aim at estimating the two expectations appearing in right-hand side of (2.6). To accomplish that, denote by $X^{x,0}$ the solution to (2.2) associated to $\xi \equiv 0$ (i.e. the solution to (2.1)). Then, if $b = 0$ one easily finds $X_t^{x,\xi} = X_t^{x,0} - \alpha\xi_t \geq -|X_t^{x,0}| - \alpha y$ a.s., since $\xi_t \leq y$ a.s. If $b > 0$, because $X_t^{x,\xi} \leq X_t^{x,0}$ a.s. for all $t \geq 0$ and $\xi_t \leq y$ a.s., one has

$$\begin{aligned}
X_t^{x,\xi} &= x + \int_0^t \big(a - bX_s^{x,\xi}\big)ds + \sigma W_t - \alpha\xi_t \geq x + \int_0^t \big(a - bX_s^{x,0}\big)ds + \sigma W_t - \alpha y \\
&= X_t^{x,0} - \alpha y \geq -|X_t^{x,0}| - \alpha y.
\end{aligned}$$

Moreover, one clearly has $X_t^{x,\xi} \leq X_t^{x,0} \leq |X_t^{x,0}| + \alpha y$ for $b \geq 0$. Hence, in any case,

$$|X_t^{x,\xi}| \leq |X_t^{x,0}| + \alpha y. \tag{2.7}$$

By an application of Itô's formula we find for $b = 0$ that

$$|e^{-\rho t} X_t^{x,0}| \leq |x| + \rho \int_0^t e^{-\rho u} |X_u^{x,0}| du + |a| \int_0^t e^{-\rho u} du + \left| \int_0^t e^{-\rho u} \sigma dW_u \right|,$$

and for $b > 0$ that

$$|e^{-\rho t} X_t^{x,0}| \leq |x| + \rho \int_0^t e^{-\rho u} |X_u^{x,0}| du + \int_0^t e^{-\rho u} (|a| + b|X_u^{x,0}|) du + \left| \int_0^t e^{-\rho u} \sigma dW_u \right|.$$

The previous two equations imply that, in both cases $b = 0$ and $b > 0$, there exists $C_1 > 0$ such that

$$\mathbb{E}\left[\sup_{t \geq 0} e^{-\rho t} |X_t^{x,0}| \right] \leq |x| + C_1 \left(1 + \int_0^\infty e^{-\rho u} \mathbb{E}\big[|X_u^{x,0}|\big] du \right) + \sigma \mathbb{E}\left[\sup_{t \geq 0} \left| \int_0^t e^{-\rho u} dW_u \right| \right].$$

Then, the Burkholder-Davis-Gundy's inequality (see, e.g., Theorem 3.28 in Chapter 3 of [81]) yields

$$\mathbb{E}\left[\sup_{t \geq 0} e^{-\rho t} |X_t^{x,0}| \right] \leq |x| + C_1 \left(1 + \int_0^\infty e^{-\rho u} \mathbb{E}\big[|X_u^{x,0}|\big] du \right) + C_2 \mathbb{E}\left[\left(\int_0^\infty e^{-2\rho u} du \right)^{\frac{1}{2}} \right].$$

for a constant $C_2 > 0$, and therefore

$$\mathbb{E}\left[\sup_{t \geq 0} e^{-\rho t} |X_t^{x,0}| \right] \leq C_4 \big(1 + |x| \big), \tag{2.8}$$

for some constant $C_4 > 0$, since there exists $C_3 > 0$ such that $\int_0^\infty e^{-\rho u} \mathbb{E}\big[|X_u^{x,0}|\big] du \leq C_3(1 + |x|)$ by Lemma A.1.1 with $q = 1$, $\kappa \equiv b$ and $\kappa\mu \equiv a$.

Now, exploiting (2.7) and (2.8), in both cases $b = 0$ and $b > 0$ we have the following:

(i) For a suitable constant $K_0 > 0$ (independent of x and y)

$$\begin{aligned} \mathbb{E}\left[\int_0^\infty e^{-\rho t} |X_t^{x,\xi}| d\xi_t^c \right] &\leq \mathbb{E}\left[\int_0^\infty e^{-\rho t} |X_t^{x,0}| d\xi_t^c \right] + \alpha y \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} \xi_t^c dt \right] \\ &\leq y \mathbb{E}\left[\sup_{t \geq 0} e^{-\rho t} |X_t^{x,0}| \right] + \alpha y^2 \leq C_4 y \big(1 + |x|\big) + \alpha y^2 \leq K_0 y (1 + y) \big(1 + |x|\big). \end{aligned} \tag{2.9}$$

Here we have used: (2.7) and an integration by parts for the first inequality; the fact that $\xi_t^c \leq y$ a.s. for the second one; equation (2.8) to have the penultimate step.

(ii) Employing again (2.7), the fact that $\sum_{t\geq 0:\Delta\xi_t\neq 0}\Delta\xi_t \leq y$, and (2.8), we also have

$$\mathbb{E}\bigg[\sum_{t\geq 0:\Delta\xi_t\neq 0} e^{-\rho t}\big[|X^{x,\xi}_{t-}|\Delta\xi_t + \frac{\alpha}{2}(\Delta\xi_t)^2\big]\bigg] \leq \frac{3}{2}\alpha y^2 + \mathbb{E}\bigg[\sum_{t\geq 0:\Delta\xi_t\neq 0} e^{-\rho t}|X^{x,0}_t|\Delta\xi_t\bigg]$$
$$\leq \frac{3}{2}\alpha y^2 + y\mathbb{E}\bigg[\sup_{t\geq 0}\big(e^{-\rho t}|X^{x,0}_t|\big)\bigg] \leq \frac{3}{2}\alpha y^2 + C_4 y\big(1+|x|\big) \leq K_1 y(1+y)\big(1+|x|\big), \tag{2.10}$$

for some $K_1 > 0$.

Thus, using (i) and (ii) in (2.6), we conclude that there exists a constant $K > 0$ such that $|\mathcal{J}(x,y,\xi)| \leq Ky(1+y)\big(1+|x|\big)$ for any $\xi \in \mathcal{A}(y)$, and therefore (2.5) holds.

Step 2. To prove that $x \mapsto V(x,y)$ is increasing for any $y \geq 0$, let $x_2 \geq x_1$, and observe that one clearly has $X^{x_2,\xi}_t \geq X^{x_1,\xi}_t$ a.s. for any $t \geq 0$ and $\xi \in \mathcal{A}(y)$. Therefore $\mathcal{J}(x_2,y,\xi) \geq \mathcal{J}(x_1,y,\xi)$ which implies $V(x_2,y) \geq V(x_1,y)$. Finally, letting $y_2 \geq y_1$, we have $\mathcal{A}(y_2) \supseteq \mathcal{A}(y_1)$, and thus $V(x,y_2) \geq V(x,y_1)$ for any $x \in \mathbb{R}$. □

We now move on by providing an heuristic derivation of the dynamic programming equation that we expect that V should satisfy. At initial time the company is faced with two possible actions: extract or wait. On the one hand, suppose that at time zero the company does not extract for a short time period Δt, and then it continues by following the optimal extraction rule (if one exists). Since this action is not necessarily optimal, it is associated to the inequality

$$V(x,y) \geq \mathbb{E}\Big[e^{-\rho\Delta t}V(X^x_{\Delta t-},y)\Big], \quad (x,y) \in \mathbb{R}\times(0,\infty).$$

Then supposing V is $C^{2,1}(\mathbb{R}\times[0,\infty))$, we can apply Itô's formula, divide by Δt, invoke the mean value theorem, let $\Delta t \to 0$, and obtain

$$\mathcal{L}V(x,y) - \rho V(x,y) \leq 0, \quad (x,y) \in \mathbb{R}\times(0,\infty).$$

Here $\mathcal{L}$ is given by the second order differential operator

$$\mathcal{L} := \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} + \begin{cases} (a-bx)\dfrac{\partial}{\partial x}, & \text{if } b > 0, \\ a\dfrac{\partial}{\partial x}, & \text{if } b = 0. \end{cases} \tag{2.11}$$

On the other hand, suppose that the company immediately extracts an amount $\varepsilon > 0$ of the commodity, sells it in the market, and then follows the optimal extraction rule (provided that one exists). With reference to (2.4), this action is associated to the inequality

$$V(x,y) \geq V(x-\alpha\varepsilon, y-\varepsilon) + (x-c)\varepsilon - \frac{1}{2}\alpha\varepsilon^2,$$

which, adding and substracting $V(x-\alpha\varepsilon, y)$, dividing by ε, and letting $\varepsilon \to 0$, yields

$$0 \geq -\alpha V_x(x,y) - V_y(x,y) + x - c.$$

We expect that only one of those two actions can be optimal, and given the Markovian nature of our setting, the previous inequalities suggest that V should identify with an appropriate solution w to the Hamilton-Jacobi-Bellman (HJB) equation

$$\max\Big\{\mathcal{L}w(x,y) - \rho w(x,y), -\alpha w_x(x,y) - w_y(x,y) + x - c\Big\} = 0, \quad (x,y) \in \mathbb{R} \times (0,\infty), \tag{2.12}$$

with boundary condition $w(x,0) = 0$ (cf. Proposition 2.3.1), and satisfying the growth condition in (2.5). Equation (2.12) takes the form of a variational inequality with state-dependent gradient constraint.

With reference to (2.12) we introduce the *waiting region*

$$\mathbb{W} := \{(x,y) \in \mathbb{R}\times(0,\infty) : \mathcal{L}w(x,y) - \rho w(x,y) = 0,\ -\alpha w_x(x,y) - w_y(x,y) + x - c < 0\}, \tag{2.13}$$

in which we expect that it is not optimal to extract the commodity, and the *selling region*

$$\mathbb{S} := \{(x,y) \in \mathbb{R}\times(0,\infty) : \mathcal{L}w(x,y) - \rho w(x,y) \leq 0,\ -\alpha w_x(x,y) - w_y(x,y) + x - c = 0\}, \tag{2.14}$$

where it should be profitable to extract and sell the commodity. In the following, we will denote by $\overline{\mathbb{W}}$ the topological closure of $\mathbb{W}$.

The next theorem shows that a suitable solution to HJB equation (2.12) identifies with the value function, whenever there exists an admissible extraction rule that keeps (with minimal effort) the state process (X,Y) inside $\overline{\mathbb{W}}$.

Theorem 2.3.2 (Verification Theorem). *Suppose there exists a function $w : \mathbb{R} \times [0,\infty) \mapsto \mathbb{R}$ such that $w \in C^{2,1}(\mathbb{R} \times [0,\infty))$, solves HJB equation (2.12) with boundary condition $w(x,0) = 0$, is increasing in y, and satisfies the growth condition*

$$0 \leq w(x,y) \leq Ky(1+y)(1+|x|), \quad (x,y) \in \mathbb{R} \times (0,\infty), \tag{2.15}$$

for some constant $K > 0$. Then $w \geq V$ on $\mathbb{R} \times [0,\infty)$.

Moreover, suppose that for all initial values $(x,y) \in \mathbb{R} \times (0,\infty)$, there exists a process $\xi^\star \in \mathcal{A}(y)$ such that

$$(X_t^{x,\xi^\star}, Y_t^{y,\xi^\star}) \in \overline{\mathbb{W}}, \quad \text{for all } t \geq 0,\ \mathbb{P}\text{-a.s.}, \tag{2.16}$$

$$\xi_t^\star = \int_{[0,t]} \mathbb{1}_{\{(X_s^{x,\xi^\star}, Y_s^{y,\xi^\star}) \in \mathbb{S}\}} d\xi_s^\star, \quad \text{for all } t \geq 0,\ \mathbb{P}\text{-a.s.} \tag{2.17}$$

Then we have $w = V$ on $\mathbb{R} \times [0,\infty)$ and $\xi^\star$ is optimal; that is, $\mathcal{J}(x,y,\xi^\star) = V(x,y)$ for all $(x,y) \in \mathbb{R} \times [0,\infty)$.

Proof. The proof is organized in two steps. Since by assumption $w(x,0) = 0 = V(x,0)$, $x \in \mathbb{R}$, in the following argument we can assume that $y > 0$.

Step 1. Let $(x,y) \in \mathbb{R} \times (0,\infty)$ be given and fixed. Here, we show that $V(x,y) \leq w(x,y)$. Let $\xi \in \mathcal{A}(y)$, and for $N \in \mathbb{N}$ set $\tau_{R,N} := \inf\{s \geq 0 : X^{x,\xi}_s \notin (-R,R)\} \wedge N$. By Itô-Tanaka-Meyer's formula, we find

$$\begin{aligned}
&e^{-\rho\tau_{R,N}} w(X^{x,\xi}_{\tau_{R,N}}, Y^{y,\xi}_{\tau_{R,N}}) - w(x,y) \\
&= \int_0^{\tau_{R,N}} e^{-\rho s}\Big(\mathcal{L}w(X^{x,\xi}_s, Y^{y,\xi}_s) - \rho w(X^{x,\xi}_s, Y^{y,\xi}_s)\Big)ds + \underbrace{\sigma \int_0^{\tau_{R,N}} e^{-\rho s} w_x(X^{x,\xi}_s, Y^{y,\xi}_s) dW_s}_{=:M_{\tau_{R,N}}} \\
&\quad + \sum_{0 \leq s \leq \tau_{R,N}} e^{-\rho s}\big[w(X^{x,\xi}_s, Y^{y,\xi}_s) - w(X^{x,\xi}_{s-}, Y^{y,\xi}_{s-})\big] \\
&\quad + \int_0^{\tau_{R,N}} e^{-\rho s}\big[-\alpha w_x(X^{x,\xi}_s, Y^{y,\xi}_s) - w_y(X^{x,\xi}_s, Y^{y,\xi}_s)\big] d\xi^c_s.
\end{aligned} \tag{2.18}$$

Now,

$$\begin{aligned}
&w(X^{x,\xi}_s, Y^{y,\xi}_s) - w(X^{x,\xi}_{s-}, Y^{y,\xi}_{s-}) = w(X^{x,\xi}_{s-} - \alpha\Delta\xi_s, Y^{y,\xi}_{s-} - \Delta\xi_s) - w(X^{x,\xi}_{s-}, Y^{y,\xi}_{s-}) \\
&= \int_0^{\Delta\xi_s} \frac{\partial w(X^{\xi}_{s-} - \alpha u, Y^{y,\xi}_{s-} - u)}{\partial u} du \\
&= \int_0^{\Delta\xi_s} \Big[-\alpha w_x(X^{x,\xi}_{s-} - \alpha u, Y^{y,\xi}_{s-} - u) - w_y(X^{x,\xi}_{s-} - \alpha u, Y^{y,\xi}_{s-} - u)\Big] du,
\end{aligned}$$

which used in (2.18) gives the equivalence

$$\begin{aligned}
&\int_0^{\tau_{R,N}} e^{-\rho s}\big(X^{x,\xi}_s - c\big) d\xi^c_s + \sum_{0 \leq s \leq \tau_{R,N}} e^{-\rho s} \int_0^{\Delta\xi_s} \big(X^{x,\xi}_{s-} - \alpha u - c\big) du - w(x,y) \\
&= \int_0^{\tau_{R,N}} e^{-\rho s}\Big(\mathcal{L}w(X^{x,\xi}_s, Y^{y,\xi}_s) - \rho w(X^{x,\xi}_s, Y^{y,\xi}_s)\Big)ds + M_{\tau_{R,N}} \\
&\quad + \sum_{0 \leq s \leq \tau_{R,N}} e^{-\rho s} \int_0^{\Delta\xi_s} \Big[-\alpha w_x(X^{x,\xi}_{s-} - \alpha u, Y^{y,\xi}_{s-} - u) - w_y(X^{x,\xi}_{s-} - \alpha u, Y^{y,\xi}_{s-} - u) \\
&\quad + (X^{x,\xi}_{s-} - \alpha u - c)\Big] du - e^{-\rho\tau_{R,N}} w(X^{x,\xi}_{\tau_{R,N}}, Y^{y,\xi}_{\tau_{R,N}}) \\
&\quad + \int_0^{\tau_{R,N}} e^{-\rho s}\Big[-\alpha w_x(X^{x,\xi}_s, Y^{y,\xi}_s) - w_y(X^{x,\xi}_s, Y^{y,\xi}_s) + X^{x,\xi}_s - c\Big] d\xi^c_s.
\end{aligned}$$

Since w satisfies (2.12) and $w \geq 0$, by taking expectations on both sides of the latter equation, and using that $\mathbb{E}[M_{\tau_{R,N}}] = 0$, we have

$$w(x,y) \geq \mathbb{E}\Big[\int_0^{\tau_{R,N}} e^{-\rho s}\big(X^{x,\xi}_s - c\big) d\xi^c_s + \sum_{0 \leq s \leq \tau_{R,N}} e^{-\rho s} \int_0^{\Delta\xi_s} \big(X^{x,\xi}_{s-} - \alpha u - c\big) du\Big].$$

We now want to take limits as $N \uparrow \infty$ and $R \uparrow \infty$ on the right-hand side of the equation above. To this end notice that one has a.s.

$$\begin{aligned} &\Big| \int_0^{\tau_{R,N}} e^{-\rho s}\big(X^{x,\xi}_s - c\big) d\xi^c_s + \sum_{0 \le s \le \tau_{R,N}} e^{-\rho s} \int_0^{\Delta \xi_s} \big(X^{x,\xi}_{s-} - \alpha u - c\big) du \Big| \\ &\le \int_0^{\infty} e^{-\rho s} |X^{x,\xi}_s| d\xi^c_s + cy + \sum_{s \ge 0 : \Delta\xi_s \neq 0} e^{-\rho s}\big(|X^{x,\xi}_{s-}|\Delta\xi_s + \frac{\alpha}{2}(\Delta\xi_s)^2\big), \end{aligned} \tag{2.19}$$

and the right-hand side of (2.19) is integrable by (2.9) and (2.10). Hence, we can invoke the dominated convergence theorem in order to take limits as $R \uparrow \infty$ and then as $N \uparrow \infty$, so as to get

$$\mathcal{J}(x, y, \xi) \le w(x, y).$$

Since $\xi \in \mathcal{A}(y)$ is arbitrary, we have

$$V(x, y) \le w(x, y), \tag{2.20}$$

which yields $V \le w$ by arbitrariness of (x, y) in $\mathbb{R} \times (0, \infty)$.

Step 2. Here, we prove that $V(x, y) \ge w(x, y)$ for any $(x, y) \in \mathbb{R} \times (0, \infty)$. Let $\xi^\star \in \mathcal{A}(y)$ satisfying (2.16) and (2.17), and let $\tau^\star_{R,N} := \inf\{t \ge 0 : X^{x,\xi^\star}_t \notin (-R, R)\} \wedge N$, for $N \in \mathbb{N}$. Then, by employing the same arguments as in *Step 1*, all the inequalities become equalities and we obtain

$$\begin{aligned} &\mathbb{E}\Big[\int_0^{\tau^\star_{R,N}} e^{-\rho s}\big(X^{x,\xi^\star}_s - c\big) d\xi^{\star,c}_s + \sum_{0 \le s \le \tau^\star_{R,N}} e^{-\rho s} \int_0^{\Delta \xi^\star_s} \big(X^{x,\xi^\star}_{s-} - c - \alpha u\big) du \Big] \\ &+ \mathbb{E}\Big[e^{-\rho \tau^\star_{R,N}} w(X^{x,\xi^\star}_{\tau^\star_{R,N}}, Y^{\xi^\star}_{\tau^\star_{R,N}}) \Big] = w(x, y), \end{aligned}$$

where $\xi^{\star,c}$ denotes the continuous part of $\xi^\star$. If now

$$\lim_{N \uparrow \infty} \lim_{R \uparrow \infty} \mathbb{E}\Big[e^{-\rho \tau^\star_{R,N}} w(X^{x,\xi^\star}_{\tau^\star_{R,N}}, Y^{\xi^\star}_{\tau^\star_{R,N}}) \Big] = 0, \tag{2.21}$$

then we can take limits as $R \uparrow \infty$ and $N \uparrow \infty$, and by (2.19) (with $\xi = \xi^\star$) together with (2.9) and (2.10) we find $\mathcal{J}(x, y, \xi^\star) = w(x, y)$. Since clearly $V(x, y) \ge \mathcal{J}(x, y, \xi^\star)$, then $V(x, y) \ge w(x, y)$ for all $(x, y) \in \mathbb{R} \times (0, \infty)$. Hence, using (2.20), $V = w$ on $\mathbb{R} \times (0, \infty)$, and therefore on $\mathbb{R} \times [0, \infty)$ because $V(x, 0) = 0 = w(x, 0)$ for all $x \in \mathbb{R}$.

To complete the proof it thus only remains to prove (2.21), and we accomplish that in the following. Since $y \mapsto w(x, y)$ is increasing by assumption, we have by (2.15) and (2.7) that

$$\begin{aligned} 0 \le e^{-\rho \tau^\star_{R,N}} w(X^{x,\xi^\star}_{\tau^\star_{R,N}}, Y^{\xi^\star}_{\tau^\star_{R,N}}) &\le e^{-\rho \tau^\star_{R,N}} w(X^{x,\xi^\star}_{\tau^\star_{R,N}}, y) \le e^{-\rho \tau^\star_{R,N}} K y(1 + y)\big(1 + |X^{x,\xi^\star}_{\tau^\star_{R,N}}|\big) \\ &\le K y(1 + y)\big[(1 + \alpha y) e^{-\rho \tau^\star_{R,N}} + e^{-\rho \tau^\star_{R,N}} |X^{x,0}_{\tau^\star_{R,N}}|\big] \\ &\le K y(1 + y)\big[(1 + \alpha y) e^{-\rho \tau^\star_{R,N}} + e^{-\frac{\rho}{2} \tau^\star_{R,N}} \sup_{t \ge 0} e^{-\frac{\rho}{2} t} |X^{x,0}_t|\big]. \end{aligned}$$

Taking expectations and employing Hölder's inequality

$$
\begin{aligned}
0 &\le \mathbb{E}\big[e^{-\rho\tau^\star_{R,N}} w(X^{x,\xi^\star}_{\tau^\star_{R,N}}, Y^{\xi^\star}_{\tau^\star_{R,N}})\big] \\
&\le Ky(1+y)\Big[(1+\alpha y)\mathbb{E}\big[e^{-\rho\tau^\star_{R,N}}\big] + \mathbb{E}\Big[e^{-\rho\tau^\star_{R,N}}\Big]^{\frac{1}{2}}\mathbb{E}\Big[\sup_{t\ge 0} e^{-\rho t}|X^{x,0}_t|^2\Big]^{\frac{1}{2}}\Big].
\end{aligned} \tag{2.22}
$$

To take care of the third expectation on right hand side of (2.22), observe that by Itô's formula we have (in both cases $b = 0$ and $b > 0$)

$$
\begin{aligned}
e^{-\rho t}(X^{x,0}_t)^2 \le x^2 &+ \int_0^t e^{-\rho u}\Big[\rho(X^{x,0}_u)^2 + \sigma^2\Big]du \\
&+ \int_0^t 2e^{-\rho u}|X^{x,0}_u|(|a| + b|X^{x,0}_u|)du + 2\sigma \sup_{t\ge 0}\Big|\int_0^t e^{-\rho u}X^{x,0}_u dW_u\Big|.
\end{aligned} \tag{2.23}
$$

Notice that $\int_0^\infty e^{-2\rho u}\mathbb{E}\big[|X^{x,0}_u|^2\big]du \le C_1(1+|x|^2)$, for some constant $C_1 > 0$, which is due to Lemma A.1.1 with $q = 2$, $\kappa \equiv b$ and $\kappa\mu \equiv a$, and therefore an application of the Burkholder-Davis-Gundy's inequality (see, e.g., Theorem 3.28 in [81]) gives

$$
\mathbb{E}\Big[\sup_{t\ge 0}\Big|\int_0^t e^{-\rho u}X^{x,0}_u dW_u\Big|\Big] \le C_2(1+|x|), \tag{2.24}
$$

for a suitable $C_2 > 0$. Then taking expectations in (2.23), employing (2.24), we easily obtain that there exists a constant $C_3 > 0$ such that

$$
\mathbb{E}\big[\sup_{t\ge 0} e^{-\rho t}|X^{x,0}_t|^2\big] \le C_3(1+|x|^2).
$$

Hence, when taking limits as $R \uparrow \infty$ and $N \uparrow \infty$ in (2.22), the right-hand side of (2.22) converges to zero, thus proving (2.21) and completing the proof. □

2.4 Constructing the Optimal Solution

We make the guess that the company extracts and sells the commodity only when the current price is sufficiently large. We therefore expect that for any $y > 0$ there exists a critical price level $G(y)$ (to be endogenously determined) separating the *waiting region* $\mathbb{W}$ and the *selling region* $\mathbb{S}$ (cf. (2.13) and (2.14)). In particular, we suppose that

$$
\begin{aligned}
\mathbb{W} &= \{(x,y) \in \mathbb{R}\times(0,\infty) \,:\, y > 0 \text{ and } x < G(y)\}, \\
\mathbb{S} &= \{(x,y) \in \mathbb{R}\times(0,\infty) \,:\, y > 0 \text{ and } x \ge G(y)\}.
\end{aligned}
$$

According to such a guess, and with reference to (2.12), the candidate value function w should satisfy

$$
\mathcal{L}w(x,y) - \rho w(x,y) = 0, \qquad \text{for all } (x,y) \in \mathbb{W}. \tag{2.25}
$$

It is well known that (2.25) admits two fundamental strictly positive solutions $\varphi(x)$ and $\psi(x)$, with the former one being strictly decreasing and the latter one being strictly increasing. Therefore, any solution to (2.25) can be written as

$$w(x,y) = A(y)\psi(x) + B(y)\varphi(x), \quad (x,y) \in \mathbb{W},$$

for some functions $A(y)$ and $B(y)$ to be found. In both cases $b = 0$ and $b > 0$ (cf. (2.2)), the function φ increases exponentially to $+\infty$ as $x \downarrow -\infty$ (see, e.g., Appendix 1 in [31]). In light of the growth conditions of V proved in Proposition 2.3.1, we therefore guess $B(y) = 0$ so that

$$w(x,y) = A(y)\psi(x) \tag{2.26}$$

for any $(x,y) \in \mathbb{W}$.

For all $(x,y) \in \mathbb{S}$, w should instead satisfy

$$-\alpha w_x(x,y) - w_y(x,y) + x - c = 0, \tag{2.27}$$

implying

$$-\alpha w_{xx}(x,y) - w_{yx}(x,y) + 1 = 0. \tag{2.28}$$

To find $G(y)$ and $A(y)$, $y > 0$, we impose that $w \in C^{2,1}$, and therefore by (2.26), (2.27), and (2.28) we obtain for all $(x,y) \in \overline{\mathbb{W}} \cap \mathbb{S}$, i.e. $x = G(y)$, that

$$-\alpha A(y)\psi'(x) - A'(y)\psi(x) + x - c = 0 \quad \text{at} \quad x = G(y), \tag{2.29}$$

$$-\alpha A(y)\psi''(x) - A'(y)\psi'(x) + 1 = 0 \quad \text{at} \quad x = G(y). \tag{2.30}$$

From (2.29) and (2.30) one can easily derive that $A(y)$ and $G(y)$, $y > 0$, satisfy

$$-\alpha A(y)\left(\psi'(x)^2 - \psi(x)\psi''(x)\right) + (x-c)\psi'(x) - \psi(x) = 0 \quad \text{at} \quad x = G(y). \tag{2.31}$$

In the following we continue our analysis by studying separately the cases $b = 0$ and $b > 0$, corresponding to a fundamental price of the commodity that is a drifted Brownian motion and an Ornstein-Uhlenbeck process, respectively. We will see that the form of the optimal extraction rule substantially differs among these two cases, and we will also provide a quantitative explanation of this by identifying an optimal stopping problem related to our optimal extraction problem (see Section 2.4.2.1 and Remark 2.4.16).

2.4.1 The Case of a Drifted Brownian Motion Fundamental Price

We start with the simpler case $b = 0$, and we therefore study the company's extraction problem (2.3) when the fundamental commodity's price is a drifted Brownian motion.

Dynamics (2.1) with $b = 0$ yield

$$dX_t^{x,\xi} = a dt + \sigma dW_t - \alpha d\xi_t, \qquad X_{0-}^{x,\xi} = x \in \mathbb{R},$$

for any $\xi \in \mathcal{A}(y)$, and consequently (2.25) reads as

$$\frac{\sigma^2}{2} w_{xx}(x,y) + a w_x(x,y) - \rho w(x,y) = 0, \quad (x,y) \in \mathbb{R} \times (0,\infty).$$

The increasing fundamental solution ψ to the latter equation is given by

$$\psi(x) = e^{nx} \qquad \text{with} \qquad n := -\frac{a}{\sigma^2} + \sqrt{\left(\frac{a}{\sigma^2}\right)^2 + 2\frac{\rho}{\sigma^2}} > 0. \tag{2.32}$$

For future use, we notice that n solves $\Psi(n) = 0$ with

$$\Psi(u) := \frac{\sigma^2}{2} u^2 + au - \rho, \quad u \in \mathbb{R}. \tag{2.33}$$

Upon observing that $\psi'(x)^2 - \psi(x)\psi''(x) = 0$ for all $x \in \mathbb{R}$, we see that any explicit dependency on y disappears in (2.31), and we therefore obtain that the critical price $G(y)$ identifies for any $y > 0$ with the constant value

$$x^\star = c + \frac{1}{n}, \tag{2.34}$$

which uniquely solves the equation $(x^\star - c)n - 1 = 0$ (cf. (2.31) and (2.32)).

Moreover, by using either (2.29) or (2.30), and by imposing $A(0) = 0$ (since we must have $V(x,0) = 0$ for all $x \in \mathbb{R}$; cf. Theorem 2.3.2), the function A in (2.26) is given by

$$A(y) := \frac{1}{\alpha n^2} e^{-cn-1}\big(1 - e^{-\alpha n y}\big), \quad y \geq 0.$$

In light of the previous findings, the candidate *waiting region* $\mathbb{W}$ is given by

$$\mathbb{W} = \{(x,y) \in \mathbb{R} \times (0,\infty) \,:\, y > 0 \text{ and } x < x^\star\},$$

and we expect that the *selling region* $\mathbb{S}$ is such that $\mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2$, where

$$\mathbb{S}_1 := \{(x,y) \in \mathbb{R} \times (0,\infty) \,:\, x \geq x^\star \text{ and } y \leq (x - x^\star)/\alpha\},$$
$$\mathbb{S}_2 := \{(x,y) \in \mathbb{R} \times (0,\infty) \,:\, x \geq x^\star \text{ and } y > (x - x^\star)/\alpha\}.$$

In $\mathbb{S}_1$, we believe that it is optimal to deplete the reservoir immediately. In $\mathbb{S}_2$ the company should make a lump sum extraction of size $(x - x^\star)/\alpha$, and then sell the commodity continuously and in such a way that the joint process (X, Y) is kept inside $\overline{\mathbb{W}}$, until there is nothing left in the reservoir. These considerations suggest to introduce the candidate value function

$$w(x,y) := \begin{cases} \frac{1}{\alpha n^2} e^{(x-c)n-1}(1-e^{-\alpha n y}), & \text{if } (x,y) \in \mathbb{W} \cup ((-\infty, x^\star] \times \{0\}), \\ \frac{1}{\alpha n^2}\left(1-e^{-\alpha n (y - \frac{x-x^\star}{\alpha})}\right) & \\ +(x-c)\left(\frac{x-x^\star}{\alpha}\right) - \frac{1}{2\alpha}(x-x^\star)^2 & \text{if } (x,y) \in \mathbb{S}_2, \\ (x-c)y - \frac{1}{2}\alpha y^2, & \text{if } (x,y) \in \mathbb{S}_1 \cup ((x^\star, \infty) \times \{0\}). \end{cases} \tag{2.35}$$

Notice that the first term in the second line of (2.35) is the continuation value starting from the new state $(x^\star, y - \frac{x-x^\star}{\alpha})$, and that w above is continuous by construction. From now on, we will refer to the critical price level $x^\star$ as to the *free boundary*.

The next proposition shows that w actually identifies with the value function V.

Proposition 2.4.1. *The function* $w : \mathbb{R} \times [0,\infty) \mapsto [0,\infty)$ *defined in* (2.35) *is a* $C^{2,1}(\mathbb{R} \times [0,\infty))$ *solution to the HJB equation* (2.12) *such that*

$$0 \le w(x,y) \le Ky(1+y)(1+|x|), \quad (x,y) \in \mathbb{R} \times [0,\infty), \tag{2.36}$$

for a suitable constant $K > 0$.

Moreover, it identifies with the value function V *from* (2.3)*, and the admissible control*

$$\xi^\star_t := y \wedge \sup_{0 \le s \le t} \frac{1}{\alpha}\big[x - x^\star + as + \sigma W_s\big]^+, \quad t \ge 0, \qquad \xi^\star_{0-} = 0, \tag{2.37}$$

with $x^\star$ *as in* (2.34)*, is an optimal extraction strategy.*

Proof. The proof is organized in steps.

Step 1. We start proving that $w \in C^{2,1}(\mathbb{R} \times [0,\infty))$. One can easily check that $w(x,0) = 0$ for any $x \in \mathbb{R}$, and that w is continuous on $\mathbb{R} \times [0,\infty)$ (recall also the comment after (2.35)). Denote by $\text{Int}(\cdot)$ the interior of a set. Then, for all $(x,y) \in \text{Int}(\mathbb{W})$ we derive from (2.35)

$$w_x(x,y) = \frac{1}{\alpha n} e^{(x-c)n-1}(1-e^{-\alpha n y}), \quad w_{xx}(x,y) = \frac{1}{\alpha} e^{(x-c)n-1}(1-e^{-\alpha n y}), \tag{2.38}$$

and

$$w_y(x,y) = \frac{1}{n} e^{(x-c)n-1} e^{-\alpha n y}. \tag{2.39}$$

Also, for all $(x,y) \in \text{Int}(\mathbb{S}_2)$ we find from (2.35) by direct calculations that

$$w_x(x,y) = -\frac{1}{\alpha n} e^{-\alpha n (y - \frac{x-x^\star}{\alpha})} + \frac{x-c}{\alpha}, \quad w_{xx}(x,y) = \frac{1}{\alpha}\left(1 - e^{-\alpha n (y - \frac{x-x^\star}{\alpha})}\right), \tag{2.40}$$

and

$$w_y(x,y) = \frac{1}{n} e^{-\alpha n (y - \frac{x-x^\star}{\alpha})}. \tag{2.41}$$

Finally, for $(x,y) \in \mathrm{Int}(\mathbb{S}_1)$ we have

$$w_x(x,y) = y, \quad w_{xx}(x,y) = 0, \quad w_y(x,y) = x - c - \alpha y. \tag{2.42}$$

From the previous expressions it is now straightforward to check that $w \in C^{2,1}(\mathbb{R} \times [0,\infty))$ upon recalling $x^\star = c + \frac{1}{n}$ (cf. (2.34)).

Step 2. Here we prove that w solves HJB equation (2.12). By construction we have $-\alpha w_x(x,y) - w_y(x,y) + x - c = 0$ for $(x,y) \in \mathbb{S}$, and $\mathcal{L}w(x,y) - \rho w(x,y) = 0$ for $(x,y) \in \mathbb{W}$. Hence it remains to prove that $-\alpha w_x(x,y) - w_y(x,y) + x - c \leq 0$ for $(x,y) \in \mathbb{W}$ and $\mathcal{L}w(x,y) - \rho w(x,y) \leq 0$ for $(x,y) \in \mathbb{S}$. This is accomplished in the following.

On the one hand, letting $(x,y) \in \mathbb{W}$ we obtain from the first equation in (2.38) and (2.39) that

$$-\alpha w_x(x,y) - w_y(x,y) + x - c = -\frac{1}{n} e^{(x-c)n-1} + x - c \leq 0,$$

where the last inequality is due to $e^{(x-c)n-1} \geq (x-c)n$, which derives from the well-known property of the exponential function $e^q \geq q + 1$ for all $q \in \mathbb{R}$.

On the other hand, for $(x,y) \in \mathbb{S}_1$ we find from the third line of (2.35) and (2.42) that

$$\mathcal{L}w(x,y) - \rho w(x,y) = ay - \rho(x-c)y + \frac{\alpha}{2}\rho y^2 =: H_1(x,y).$$

We now want to prove that $H_1(x,y) \leq 0$ for all $(x,y) \in \mathbb{S}_1$. Because $y \leq \frac{x-x^\star}{\alpha}$ with $x^\star = c + \frac{1}{n}$, we find

$$\frac{\partial H_1}{\partial y}(x,y) = a - \rho(x-c) + \alpha\rho y \leq a - \frac{\rho}{n}.$$

In order to study the sign of $\frac{\partial H_1}{\partial y}$, we need to distinguish two cases. If $a \leq 0$, then it follows immediately $\frac{\partial H_1}{\partial y}(x,y) \leq 0$. If $a > 0$, then recall Ψ from (2.33) and notice that because $u \mapsto \Psi(u)$ is increasing on $(-a/\sigma^2, \infty) \supset \mathbb{R}_+$, $\Psi(n) = 0$, and $\Psi(\frac{\rho}{a}) > 0$, one has $\frac{\rho}{a} \geq n$. Hence again $\frac{\partial H_1}{\partial y}(x,y) \leq 0$. Since now $\lim_{y \downarrow 0} H_1(x,y) = 0$ for any $x \geq x^\star$, then we have just proved that $H_1(x,y) \leq 0$ for all $y \leq \frac{x-x^\star}{\alpha}$, and for any $x \geq x^\star$. Hence, $\mathcal{L}w - \rho w \leq 0$ in $\mathbb{S}_1$.

Also, for $(x,y) \in \mathbb{S}_2$, we find

$$\mathcal{L}w(x,y) - \rho w(x,y) = \frac{a}{\alpha}(x - x^\star) - \rho(x-c)\big(\frac{x-x^\star}{\alpha}\big) + \frac{\rho}{2\alpha}(x-x^\star)^2 =: H_2(x).$$

To obtain the first equality in the equation above we have used the second line of (2.35), (2.40), and that n solves $\Psi(n) = 0$ with Ψ as in (2.33). Notice that $H_2(x^\star) = 0$ and $H_2'(x) = \frac{1}{\alpha}\big(a - \rho(x-c)\big)$. If $a \leq 0$, we clearly have that $H_2'(x) \leq 0$, since $x \geq x^\star > c$. If $a > 0$, then $H_2'(x) \leq 0$ if and only if $x \geq c + \frac{a}{\rho}$, but the latter inequality holds for any $x \geq x^\star$ since we have proved above that for $a > 0$ we have $\frac{\rho}{a} \geq n$, and therefore, $x^\star = c + \frac{1}{n} \geq c + \frac{a}{\rho}$. Hence, in any case, $H_2'(x) \leq 0$ for all $x \geq x^\star$, and then $\mathcal{L}w - \rho w \leq 0$ in $\mathbb{S}_2$.

Combining all the previous findings we have that w is a $C^{2,1}(\mathbb{R} \times [0,\infty))$ solution to the HJB equation (2.12).

Step 3. Here we verify that w satisfies all the requirements needed to apply Theorem 2.3.2.

The fact that $y \mapsto w(x,y)$ is increasing in $\mathbb{W}$ and $\mathbb{S}_2$ easily follows from (2.39) and (2.41), respectively. The monotonicity of $w(x,\cdot)$ in $\mathbb{S}_1$ is instead due to (2.42) and to the fact that $y \leq (x - x^\star)/\alpha$ in $\mathbb{S}_1$ and $x^\star > c$.

In order to show the upper bound in (2.36), notice that

$$w(x,y) \leq \frac{1}{\alpha n^2}, \quad \text{for all } (x,y) \in \mathbb{W}, \tag{2.43}$$

since $x < x^\star$. Further, we find for all $(x,y) \in \mathbb{S}_2$ that

$$\begin{aligned} w(x,y) &= \frac{1}{\alpha n^2}\left(1 - e^{-\alpha n (y - \frac{x - x^\star}{\alpha})}\right) + (x-c)\left(\frac{x - x^\star}{\alpha}\right) - \frac{1}{2\alpha}(x - x^\star)^2 \\ &\leq \frac{1}{\alpha n^2} + (x-c)\left(\frac{x - x^\star}{\alpha}\right) \leq \frac{1}{\alpha n^2} + (x-c)y, \end{aligned}$$

where we have used that $y > (x - x^\star)/\alpha$ for all $(x,y) \in \mathbb{S}_2$. Finally, for all $(x,y) \in \mathbb{S}_1$ it is clear that

$$w(x,y) = (x-c)y - \frac{1}{2}\alpha y^2 \leq (x-c)y. \tag{2.44}$$

Hence, from (2.43)-(2.44) we see that w satisfies the required growth condition.

We now show the nonnegativity of w. Since $y \leq (x - x^\star)/\alpha$ in $\mathbb{S}_1$, we find by both (2.34) and (2.42)

$$w_y(x,y) = x - c - \alpha y \geq x^\star - c \geq 0, \quad (x,y) \in \mathbb{S}_1.$$

Clearly, $w_y \geq 0$ on $\mathbb{W} \cup \mathbb{S}_2$ by (2.39) and (2.41). Thus, w_y is nonnegative on $\mathbb{R} \times [0,\infty)$, and this fact, together with $w(\cdot, 0) = 0$, implies that w is nonnegative on $\mathbb{R} \times [0,\infty)$.

Step 4. The control $\xi^\star$ given by (2.37) is admissible, and satisfies (2.16) and (2.17). Since by *Step 1* and *Step 2* w is a $C^{2,1}$-solution to the HJB equation (2.12), and by *Step 3* satisfies all the requirements of Theorem 2.3.2, we conclude that

$$w(x,y) = V(x,y), \quad (x,y) \in \mathbb{R} \times [0,\infty),$$

by Theorem 2.3.2. □

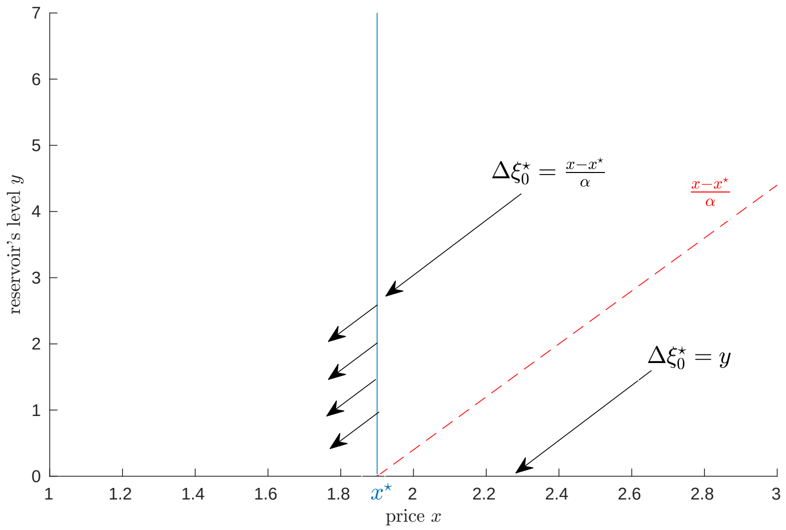


Figure 2.1: A graphical illustration of the optimal extraction rule $\xi^\star$ (cf. (2.37)) and of the free boundary $x^\star$. The plot has been obtained by using $a = 0.4$, $\sigma = 0.8$, $\rho = 3/8$, $c = 0.3$, $\alpha = 0.25$. The optimal extraction rule prescribes the following. In the region $\{(x, y) \in \mathbb{R} \times (0, \infty) : x < x^\star\}$ it is optimal not to extract. If at initial time (x, y) is such that $x > x^\star$ and $y \leq (x - x^\star)/\alpha$, then the reservoir should be immediately depleted. On the other hand, if (x, y) is such that $x > x^\star$ and $y > (x-x^\star)/\alpha$, then one should make a lump sum extraction of size $(x - x^\star)/\alpha$, and then keep on extracting until the commodity is exhausted by just preventing the price to rise above $x^\star$.

Remark 2.4.2. *Notice that, as $\alpha \downarrow 0$, the optimal extraction rule $\xi^\star$ of (2.37) converges to the extraction rule $\widehat{\xi}$ that prescribes to instantaneously deplete the reservoir as soon as the price reaches $x^\star$; i.e., defining, for any given and fixed $(x, y) \in \mathbb{R} \times [0, \infty)$, $\widehat{\tau}(x, y) := \inf\{t \geq 0 : x + at + \sigma W_t \geq x^\star\}$, one has $\widehat{\xi}_t = 0$ for all $t < \widehat{\tau}(x, y)$ and $\widehat{\xi}_t = y$ for all $t \geq \widehat{\tau}(x, y)$. The latter control can be easily checked to be optimal for the extraction problem in which the company does not have market impact (i.e. $\alpha = 0$).*

2.4.2 The Case of a Mean-Reverting Fundamental Price

In this section, we assume $b > 0$, and we study the optimal extraction problem (2.3) when the commodity's price evolves as a linearly controlled Ornstein-Uhlenbeck process

$$dX_t^{x,\xi} = (a - bX_t^{x,\xi})dt + \sigma dW_t - \alpha d\xi_t, \quad X_{0-}^{x,\xi} = x \in \mathbb{R},$$

for any $\xi \in \mathcal{A}(y)$. In the following, we will often refer to Lemma A.1.2 in Appendix A, in which, regarding the notations, we exploit the results with $\kappa \equiv b$ and $\kappa\mu \equiv a$. Here, for instance, (A.5) reads as

$$\psi(x) = e^{\frac{(bx-a)^2}{2\sigma^2 b}} D_{-\frac{\rho}{b}}\Big(- \frac{(bx-a)}{\sigma b}\sqrt{2b}\Big). \tag{2.45}$$

For any $y > 0$, from (2.31) we find a representation of $A(y)$ in terms of $G(y)$; that is,

$$A(y) = \frac{(G(y)-c)\psi'(G(y)) - \psi(G(y))}{\alpha[\psi'(G(y))^2 - \psi''(G(y))\psi(G(y))]}. \tag{2.46}$$

Notice that the denominator of $A(y)$ is nonzero due to Lemma A.1.2-(3).

For our subsequent analysis it is convenient to look at G as a function of the state variable $y \in (0,\infty)$, and, in particular, we conjecture that it is the inverse of an injective nonnegative function F to be endogenously determined together with its domain and its behavior. This is what we are going to do in the following. From now on we set $G \equiv F^{-1}$.

Since we have $V(x,0) = 0$ (cf. Theorem 2.3.2) for any $x \in \mathbb{R}$, we impose $A(0) = 0$. Then, from (2.46) we obtain the boundary condition

$$x_0 := F^{-1}(0) \quad \text{solving} \quad (x_0 - c)\psi'(x_0) - \psi(x_0) = 0. \tag{2.47}$$

In fact, existence and uniqueness of such x_0 is given by the following (more general) result. Its proof can be found in Appendix B.

Lemma 2.4.3. *Recall that $\psi^{(k)}$ denotes the derivative of order k, $k \in \mathbb{N}_0$, of ψ. Then, for any $k \in \mathbb{N}_0$, there exists a unique solution on (c,∞) to the equation*

$$(x-c)\psi^{(k+1)}(x) - \psi^{(k)}(x) = 0.$$

In particular, there exists $x_0 > c$ uniquely solving $(x-c)\psi'(x) - \psi(x) = 0$ and $x_\infty > c$ uniquely solving $(x-c)\psi''(x) - \psi'(x) = 0$.

From (2.29) and (2.30) we have

$$A'(y) = \frac{(F^{-1}(y)-c)\psi''(F^{-1}(y)) - \psi'(F^{-1}(y))}{\psi''(F^{-1}(y))\psi(F^{-1}(y)) - \psi'(F^{-1}(y))^2}, \quad y > 0, \tag{2.48}$$

and the denominator of $A'(y)$ is nonzero due to Lemma A.1.2-(3).

Now, we define the functions $M : \mathbb{R} \mapsto \mathbb{R}$ and $N : \mathbb{R} \mapsto \mathbb{R}$ such that for any $x \in \mathbb{R}$

$$M(x) := \frac{(x-c)\psi'(x) - \psi(x)}{\alpha[\psi'(x)^2 - \psi''(x)\psi(x)]}, \quad N(x) := \frac{(x-c)\psi''(x) - \psi'(x)}{\psi''(x)\psi(x) - \psi'(x)^2}, \tag{2.49}$$

and, by differentiating M and rearranging terms, we obtain

$$M'(x) = \frac{[\psi'''(x)\,[(x-c)\psi'(x) - \psi(x)] - \psi''(x)\,[(x-c)\psi''(x) - \psi'(x)]]\,\psi(x)}{\alpha[\psi'(x)^2 - \psi''(x)\psi(x)]^2}.$$

However, by noticing that $M(x) = A(F(x))$ (cf. (2.46) and (2.49)), the chain rule yields $M'(x) = A'(F(x))F'(x)$, which in turn gives

$$F'(x) = \frac{M'(x)}{N(x)}, \tag{2.50}$$

upon observing that $N(x) = A'(F(x))$ from (2.48) and (2.49).

Recall that by Lemma 2.4.3 there exists a unique $x_\infty > c$ solving $N(x_\infty) = 0$; that is, solving $(x-c)\psi''(x) - \psi'(x) = 0$. Due to (2.50), this point is a vertical asymptote of F', and the next result shows that x_∞ is located to the left of x_0. The proof can be found in Appendix B.

Lemma 2.4.4. *Recall Lemma 2.4.3 and let x_0 and x_∞ be the unique solutions to $M(x) = 0$ (i.e. $(x-c)\psi'(x) - \psi(x) = 0$) and $N(x) = 0$ (i.e. $(x-c)\psi''(x) - \psi'(x) = 0$), respectively. We have $x_\infty < x_0$.*

The following useful corollary immediately follows from the proof of Lemma 2.4.3.

Corollary 2.4.5. *One has*

$$(x-c)\psi'(x) - \psi(x) < 0, \quad \textit{for all } x < x_0,$$

and

$$(x-c)\psi''(x) - \psi'(x) > 0, \quad \textit{for all } x > x_\infty.$$

By integrating (2.50) in the interval $[x, x_0]$, for $x \in (x_\infty, x_0]$, and using the fact that $F(x_0) = 0$ (cf. (2.47)), we obtain

$$F(x) = \int_x^{x_0} \frac{[\psi'''(x)\,[(x-c)\psi'(x) - \psi(x)] - \psi''(x)\,[(x-c)\psi''(x) - \psi'(x)]]\,\psi(x)}{-\alpha[\psi''(z)\psi(z) - \psi'(z)^2][(z-c)\psi''(z) - \psi'(z)]} dz, \tag{2.51}$$

which is well defined, but possibly infinite for $x = x_\infty$. In the following we will refer to F as to the *free boundary.* We now prove properties of F that have been only conjectured so far.

Proposition 2.4.6. *The free boundary F defined in* (2.51) *is strictly decreasing for all $x \in (x_\infty, x_0)$ and belongs to $C^\infty((x_\infty, x_0])$. Moreover,*

$$\lim_{x \downarrow x_\infty} F(x) = \infty = \lim_{x \downarrow x_\infty} F'(x). \tag{2.52}$$

Proof. Step 1. We start by proving the claimed monotonicity. Notice that by (2.51) one has $F'(z) = -\Theta(z)$, where the function $\Theta : (x_\infty, \infty] \mapsto \mathbb{R}$ is given by

$$\Theta(z) := \frac{[\psi'''(z)\,[(z-c)\psi'(z) - \psi(z)] - \psi''(z)\,[(z-c)\psi''(z) - \psi'(z)]]\,\psi(z)}{-\alpha[\psi''(z)\psi(z) - \psi'(z)^2][(z-c)\psi''(z) - \psi'(z)]}.$$

By Lemma A.1.2 one has $\psi''(z)\psi(z) - \psi'(z)^2 > 0$ for any $z \in \mathbb{R}$. Moreover, $\Phi(z) := (z-c)\psi''(z) - \psi'(z) > 0$ for all $z > x_\infty > c$ by Corollary 2.4.5. Therefore the denominator of Θ is strictly negative for any $z \in (x_\infty, x_0)$. Again, an application of Corollary 2.4.5 implies that the numerator of Θ is strictly negative for any $z \in (x_\infty, x_0)$, and therefore $\Theta > 0$ and $F' < 0$. Thus, we conclude that F is strictly decreasing.

Step 2. To prove (2.52), recall that from *Step 1* we have set $\Phi(z) = (z-c)\psi''(z) - \psi'(z) > 0$ for all $z \in (x_\infty, x_0)$, and define for any $z \in (x_\infty, x_0)$

$$h(z) := \frac{\left[\psi'''(z)\left[(z-c)\psi'(z) - \psi(z)\right] - \psi''(z)\left[(z-c)\psi''(z) - \psi'(z)\right]\right]\psi(z)}{-\alpha[\psi''(z)\psi(z) - \psi'(z)^2]},$$

which is continuous and nonnegative by *Step 1*. Notice that $h/\Phi = \Theta$, with Θ as in *Step 1*.

By de l'Hopital's rule,

$$\lim_{z \downarrow x_\infty} \frac{\Phi(z)}{z - x_\infty} = \lim_{z \downarrow x_\infty} \Phi'(z) = (x_\infty - c)\psi'''(x_\infty) =: \ell > 0,$$

so that, for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that if $|z - x_\infty| < \delta_\varepsilon$, then $\left|\frac{\Phi(z)}{z-x_\infty} - \ell\right| < \varepsilon$. Thus, for any $\varepsilon > 0$, we let δ_ε be as above, and we take $x \in (x_\infty, x_\infty + \delta_\varepsilon)$. Then, recalling (2.51), we see that there exists a constant $C > 0$ (possibly depending on x_∞ and x_0, but not on x) such that

$$\begin{aligned} F(x) &= \int_x^{x_0} \Theta(z)dz = \int_x^{x_0} \frac{h(z)}{(z - x_\infty)\frac{\Phi(z)}{(z-x_\infty)}}dz \\ &\geq \int_x^{x_\infty+\delta_\varepsilon} \frac{C}{(\ell + \varepsilon)}\frac{dz}{(z - x_\infty)} + C\int_{x_\infty+\delta_\varepsilon}^{x_0} \frac{dz}{\Phi(z)} \to \infty \end{aligned}$$

as $x \downarrow x_\infty$.

Finally, since the integrand in (2.51) is a C^∞-function on $(x_\infty, x_0]$, it follows that F is so as well. □

Remark 2.4.7. *The critical price levels x_0 and x_∞ have a clear interpretation. x_0 is the free boundary arising in the optimal extraction problem when we set $\alpha = 0$, so that the company's actions have no market impact. x_∞ is the free boundary of the optimal extraction problem when there is an infinite amount of commodity available in the reservoir, i.e. $y = \infty$.*

Given F as above, we now introduce the sets $\mathbb{S}_1$ and $\mathbb{S}_2$ that partition the (candidate) *selling region* $\mathbb{S}$:

$$\mathbb{S}_1 := \{(x,y) \in \mathbb{R} \times (0,\infty) \,:\, x \geq F^{-1}(y) \text{ and } y \leq (x - x_0)/\alpha\},$$
$$\mathbb{S}_2 := \{(x,y) \in \mathbb{R} \times (0,\infty) \,:\, x \geq F^{-1}(y) \text{ and } y > (x - x_0)/\alpha\}.$$

and the (candidate) *waiting region*

$$\mathbb{W} := \{(x,y) \in \mathbb{R} \times (0,\infty) \,:\, x < F^{-1}(y)\}.$$

We now make a guess on the structure of the optimal strategy in terms of the sets $\mathbb{W}$ and $\mathbb{S}_1$ and $\mathbb{S}_2$. If the current price x is sufficiently low, and in particular it is such that $x < F^{-1}(y)$ (i.e. $(x,y) \in \mathbb{W}$), we conjecture that the company does not extract, and the payoff accrued is just the continuation value $A(y)\psi(x)$. Whenever the price attempts to cross the critical level $F^{-1}(y)$, then the company makes infinitesimal extractions that keep the state process (X,Y) inside the region $\{(x,y) \in \mathbb{R}\times(0,\infty) : x \leq F^{-1}(y)\}$ (that is, inside $\overline{\mathbb{W}}$). If the current price x is sufficiently high (i.e. $x > F^{-1}(y)$) and the current level of the reservoir is sufficiently large (i.e. lies in $\mathbb{S}_2$), then the company makes an instantaneous lump sum extraction of suitable amplitude z, and pushes the joint process (X,Y) to the locus of points $\{(x,y) \in \mathbb{R} \times (0,\infty) : y = F(x)\}$, and then continues extracting as before. The associated payoff is then the sum of the continuation value starting from the new state $(x-\alpha z, y-z)$, and the profits accrued from selling z units of the commodity, that is $(x-c)z - \frac{1}{2}\alpha z^2$. If the current capacity level is not large enough (i.e. $y \leq \frac{x-x_0}{\alpha}$, so that $(x,y) \in \mathbb{S}_1$), then the company immediately depletes the reservoir. This action is associated to the net profit $(x-c)y - \frac{1}{2}\alpha y^2$.

In light of the previous conjecture we therefore define our candidate value function as

$$w(x,y) := \begin{cases} A(y)\psi(x), & \text{if } (x,y) \in \mathbb{W} \cup ((-\infty, x_0] \times \{0\}), \\ A\big(F(x-\alpha z)\big)\psi(x-\alpha z) & \\ +(x-c)z - \frac{1}{2}\alpha z^2, & \text{if } (x,y) \in \mathbb{S}_2, \\ (x-c)y - \frac{1}{2}\alpha y^2, & \text{if } (x,y) \in \mathbb{S}_1 \cup ((x_0,\infty) \times \{0\}), \end{cases} \tag{2.53}$$

where, for any $(x,y) \in \mathbb{S}_2$, we denote by $z := z(x,y)$ the unique solution to

$$y - z = F(x - \alpha z). \tag{2.54}$$

In fact, its existence and uniqueness is guaranteed by the next lemma, whose proof is in Appendix B.

Lemma 2.4.8. *For any $(x,y) \in \mathbb{S}_2$, there exists a unique solution $z(x,y)$ to (2.54). Moreover, we have $z(x,y) \in (\frac{x-x_0}{\alpha}, \frac{x-x_\infty}{\alpha} \wedge y]$,*

$$z(x, F(x)) = 0 \quad \textit{for any } x \in (x_\infty, x_0), \tag{2.55}$$

and

$$z(x,y) = \frac{x-x_0}{\alpha}, \quad \textit{for any } (x,y) \in \mathbb{R} \times (0,\infty) \textit{ such that } x \geq x_0 \textit{ and } y = \frac{x-x_0}{\alpha}. \tag{2.56}$$

Next, we verify that w is a classical solution to the HJB equation (2.12). This is accomplished in the next two results.

Lemma 2.4.9. *The function w is $C^{2,1}(\mathbb{R} \times [0, \infty))$.*

Proof. Continuity is clear by construction. We therefore need to evaluate the derivatives of w.

Denoting by $\mathrm{Int}(\cdot)$ the interior of a set, we have by (2.53) that for all $(x, y) \in \mathrm{Int}(\mathbb{W})$

$$w_x(x,y) = A(y)\psi'(x), \quad w_{xx}(x,y) = A(y)\psi''(x), \quad w_y(x,y) = A'(y)\psi(x), \tag{2.57}$$

and that for all $(x, y) \in \mathrm{Int}(\mathbb{S}_1)$

$$w_x(x,y) = y, \quad w_{xx}(x,y) = 0, \quad w_y(x,y) = x - c - \alpha y. \tag{2.58}$$

All the previous equations easily give the continuity of the derivatives in $\mathrm{Int}(\mathbb{W})$ and $\mathrm{Int}(\mathbb{S}_1)$.

To evaluate w_x, w_{xx} and w_y for $(x, y) \in \mathrm{Int}(\mathbb{S}_2)$, we need some more work. From (2.54), we calculate the derivatives of $z = z(x, y)$ with respect to x and y by the help of the implicit function theorem, and we obtain

$$z_x(x,y) = \frac{F'(x - \alpha z)}{\alpha F'(x - \alpha z) - 1}, \tag{2.59}$$

and

$$z_y(x,y) = \frac{1}{1 - \alpha F'(x - \alpha z)}, \tag{2.60}$$

for any $(x, y) \in \mathrm{Int}(\mathbb{S}_2)$. Moreover, recalling that we have set $G \equiv F^{-1}$, and taking $y = F(x - \alpha z)$, we find from (2.29)

$$A'(F(x - \alpha z)) = \frac{x - \alpha z - c}{\psi(x - \alpha z)} - \alpha A(F(x - \alpha z))\frac{\psi'(x - \alpha z)}{\psi(x - \alpha z)}, \tag{2.61}$$

and from (2.30)

$$A'(F(x - \alpha z)) = \frac{1 - \alpha A(F(x - \alpha z))\psi''(x - \alpha z)}{\psi'(x - \alpha z)}. \tag{2.62}$$

By differentiating w with respect to x strictly inside $\mathbb{S}_2$ (cf. the second line of (2.53)), and using (2.59) and (2.61), we obtain

$$w_x(x,y) = A(F(x - \alpha z))\psi'(x - \alpha z) + z. \tag{2.63}$$

Also, by (2.62) and (2.59)

$$w_{xx}(x,y) = A(F(x - \alpha z))\psi''(x - \alpha z). \tag{2.64}$$

Moreover, differentiating with respect to y the second line of (2.53), and using (2.60) and (2.61), yields

$$w_y(x,y) = A'(F(x-\alpha z))\psi(x-\alpha z). \tag{2.65}$$

Equations (2.63)-(2.65) hold for any $(x,y) \in \text{Int}(\mathbb{S}_2)$, and give that $w \in C^{2,1}(\text{Int}(\mathbb{S}_2))$. Moreover, the previous calculations obtained in $\text{Int}(\mathbb{W})$, $\text{Int}(\mathbb{S}_1)$ and $\text{Int}(\mathbb{S}_2)$ reveal that the derivatives of w are also continuous in $\mathbb{R} \times \{0\}$.

Now, let $(x_n, y_n)_n \subseteq \text{Int}(\mathbb{S}_2)$ be any sequence converging to $(x, F(x))$, $x \in (x_\infty, x_0]$. Since $\lim_{n\to\infty} z(x_n, y_n) = 0$ by continuity of z, and because A, ψ, ψ' and ψ'' are also continuous, we conclude from (2.57) and (2.63)–(2.65) that $w \in C^{2,1}(\overline{\mathbb{W}} \cap \overline{\mathbb{S}_2})$, where $\overline{\mathbb{W}}$ and $\overline{\mathbb{S}_2}$ denote the closures of $\mathbb{W}$ and $\mathbb{S}_2$.

In order to prove that $w \in C^{2,1}(\mathbb{S}_1 \cap \overline{\mathbb{S}_2})$, consider a sequence $(x_n, y_n)_n \subseteq \mathbb{S}_2$ converging to $(x, \frac{x-x_0}{\alpha})$, $x \geq x_0$. Again by the continuity of F and exploiting that $F(x_0) = 0$ we get $\lim_{n\to\infty} z(x_n, y_n) = \frac{1}{\alpha}(x - x_0)$. Therefore, we have $w \in C^{2,1}(\mathbb{S}_1 \cap \overline{\mathbb{S}_2})$ by (2.58) and (2.63)–(2.65), and upon employing $A(F(x_0)) = 0$ and $\psi(x_0)A'(F(x_0)) = \frac{\psi(x_0)}{\psi'(x_0)} = x_0 - c$ by (2.62).

Collecting all the previous results, the claim follows. □

Proposition 2.4.10. *The function w as in* (2.53) *is a $C^{2,1}(\mathbb{R} \times [0,\infty))$ solution to the HJB equation* (2.12)*, and it is such that $w(x,0) = 0$.*

Proof. The claimed regularity follows from Lemma 2.4.9, whereas we see from (2.53) that $w(x,0) = 0$ upon recalling that $A(0) = 0$. Hence, we assume in the following that $y > 0$. Moreover, it is important to recall that in (2.29) and (2.30) we have set $G \equiv F^{-1}$.

By construction $\mathcal{L}w(x,y) - \rho w(x,y) = 0$ for all $(x,y) \in \mathbb{W}$. Moreover, $-\alpha w_x(x,y) - w_y(x,y) + (x-c) = 0$ for all $(x,y) \in \mathbb{S}_1$. Also, $-\alpha w_x(x,y) - w_y(x,y) + (x-c) = 0$ for all $(x,y) \in \mathbb{S}_2$ by employing (2.63) and (2.65), and observing that from (2.29) one has

$$-\alpha A(F(x-\alpha z))\psi'(x-\alpha z) - A'(F(x-\alpha z))\psi(x-\alpha z) + (x-\alpha z) - c = 0.$$

Hence, it is left to show that

$$-\alpha w_x(x,y) - w_y(x,y) + x - c \leq 0, \quad \forall (x,y) \in \mathbb{W}, \tag{2.66}$$

$$\mathcal{L}w(x,y) - \rho w(x,y) \leq 0, \quad \forall (x,y) \in \mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2 \tag{2.67}$$

In *Step 1* below we prove that (2.66) holds, whereas the proof of (2.67) is separately performed for $\mathbb{S}_1$ and $\mathbb{S}_2$ in *Step 2* and *Step 3* respectively.

Step 1. Here we prove that (2.66) holds for any $(x,y) \in \mathbb{W}$. Notice that (2.29) gives

$$A'(y) = \frac{F^{-1}(y) - c}{\psi(F^{-1}(y))} - \frac{\alpha A(y)\psi'(F^{-1}(y))}{\psi(F^{-1}(y))}. \tag{2.68}$$

Then, by using the first and the third equation of (2.57), and (2.68), we rewrite the left-hand side of (2.66) (after rearranging terms) as

$$\alpha A(y)\left[\frac{\psi'(F^{-1}(y))\psi(x)}{\psi(F^{-1}(y))} - \psi'(x)\right] - \frac{F^{-1}(y) - c}{\psi(F^{-1}(y))}\psi(x) + x - c = Q(x, F^{-1}(y)),$$

for any $(x, y) \in \mathbb{W}$. Here, we have defined

$$Q(x, q) := \alpha A(F(q))\left[\frac{\psi'(q)\psi(x)}{\psi(q)} - \psi'(x)\right] - \frac{q - c}{\psi(q)}\psi(x) + x - c,$$

for any $(x, q) \in \mathbb{R} \times [x_\infty, x_0]$. Since $Q(q, q) = 0$, in order to have (2.66) it suffices to show that one has (recall that $(x_\infty, x_0]$ is the domain of F)

$$Q_x(x, q) \geq 0, \quad \text{for any } x \leq q, \text{ for all } q \in (x_\infty, x_0].$$

We prove this in the following.

Differentiating Q with respect to x, and using (2.46), gives

$$Q_x(x, q) = \frac{\psi(q) - (q - c)\psi'(q)}{\psi''(q)\psi(q) - \psi'(q)^2}\left[\frac{\psi'(x)\psi'(q)}{\psi(q)} - \psi''(x)\right] - (q - c)\frac{\psi'(x)}{\psi(q)} + 1. \tag{2.69}$$

Take $x \leq x_\infty$ and $q = x_\infty$, and recall that $x_\infty > c$ solves $(x_\infty - c) = \frac{\psi'(x_\infty)}{\psi''(x_\infty)}$. Then, after some simple algebra, we have

$$Q_x(x, x_\infty) = 1 - \frac{\psi''(x)}{\psi''(x_\infty)} \geq 0,$$

where the last inequality is due to the fact that $x \mapsto \psi''(x)$ is strictly increasing.

Moreover, we find

$$Q_x(x, x_0) = 1 - (x_0 - c)\frac{\psi'(x)}{\psi(x_0)} \geq 0, \quad \text{for any } x \leq x_0, \tag{2.70}$$

due to the fact that $x_0 > c$ uniquely solves $(x_0 - c)\psi'(x_0) - \psi(x_0) = 0$ and $x \mapsto 1 - (x_0 - c)\frac{\psi'(x)}{\psi(x_0)} < 0$ is strictly decreasing.

By differentiating Q_x of (2.69) with respect to q one obtains

$$Q_{xq}(x, q) = \left[\frac{\psi'''(q)\left[(q - c)\psi'(q) - \psi(q)\right] - \psi''(q)\left[(q - c)\psi''(q) - \psi'(q)\right]}{\left(\psi''(q)\psi(q) - \psi'(q)^2\right)^2}\right]\Phi(x, q), \tag{2.71}$$

where we have introduced the function

$$\Phi(x, q) := \psi'(x)\psi'(q) - \psi''(x)\psi(q), \quad \text{for all } (x, q) \in \mathbb{R}^2,$$

that is such that

$$\Phi_q(x, q) = \psi'(x)\psi''(q) - \psi''(x)\psi'(q) > 0, \quad \forall x \leq q, \tag{2.72}$$

since ψ'/ψ'' is decreasing due to Lemma A.1.2 with $k = 1$.

By Corollary 2.4.5 we have that

$$\psi'''(q)\left[(q-c)\psi'(q)-\psi(q)\right]-\psi''(q)\left[(q-c)\psi''(q)-\psi'(q)\right]\leq 0, \tag{2.73}$$

for all $q \in [x_\infty, x_0]$. Hence, the term multiplying Φ in the right-hand side of (2.71) is negative.

In light of (2.72), we know that $\Phi(x,q)$ is increasing in q for $q \geq x$. We now have three possible cases.

(a) If Φ is such that $\Phi(x,q) < 0$ for all $q \in [x_\infty, x_0]$, then by (2.73) (and noticing that the function in (2.73) in fact appears in the numerator of Q_{xq}) we must have $Q_{xq}(x,q) \geq 0$ for all $q \in [x_\infty, x_0]$, so that

$$0 \leq Q_x(x,x_\infty) \leq Q_x(x,q) \leq Q_x(x,x_0), \quad \text{for all } q \in [x_\infty, x_0], \text{ and } x \leq x_\infty. \tag{2.74}$$

(b) If Φ is such that $\Phi(x,q) > 0$ for all $q \in [x_\infty, x_0]$, then by (2.73) we must have $Q_{xq}(x,q) \leq 0$ for all $q \in [x_\infty, x_0]$, so that

$$0 \leq Q_x(x,x_0) \leq Q_x(x,q) \leq Q_x(x,x_\infty), \quad \text{for all } q \in [x_\infty, x_0], \text{ and } x \leq x_\infty.$$

(c) If Φ is such that $\Phi(x,q) \leq 0$ for all $q \in [x_\infty, \bar{q}]$, where $\bar{q} \in [x_\infty, x_0]$, and $\Phi(x,q) > 0$ for all $q \in [\bar{q}, x_0]$, then by (2.73) we must have $Q_{xq}(x,q) \geq 0$ for all $q \in [x_\infty, \bar{q}]$, and $Q_{xq}(x,q) \leq 0$ for all $q \in [\bar{q}, x_0]$, so that

$$Q_x(x,q) \geq \min\{Q_x(x,x_\infty), Q_x(x,x_0)\} \geq 0, \quad \text{for all } q \in [x_\infty, x_0] \text{ and } x \leq x_\infty. \tag{2.75}$$

From (2.74)-(2.75), we then conclude that (2.66) holds for any $(x,y) \in \mathbb{W}$ such that $x \leq x_\infty$.

Now, take $x \in (x_\infty, x_0]$ and let $q \in [x, x_0]$. For $q = x$ we find from (2.69) that

$$Q_x(x,x) = 0. \tag{2.76}$$

Then, proceeding as above, from (2.70) and (2.76), we obtain that $Q_x(x,q) \geq 0$ for all $x \in (x_\infty, x_0]$ with $q \in [x, x_0]$.

Hence, in conclusion, $Q_x(x, F^{-1}(y)) \geq 0$ for all $x \leq F^{-1}(y)$ and $y > 0$, and (2.66) is then established.

Step 2. Here, we show that (2.67) holds in $\mathbb{S}_1$. Setting

$$\bar{x} = \frac{a+\rho c}{\rho + b},$$

by Lemma B.2.1 in Appendix B we have $\bar{x} \leq x_0$, with x_0 solving $(x_0-c)\psi'(x_0)-\psi(x_0) = 0$ (cf. Lemma 2.4.3).

Now, let $(x,y) \in \mathbb{S}_1$ be given and fixed. Thanks to the first and second equation in (2.58) we have

$$\mathcal{L}w(x,y) - \rho w(x,y) = (a-bx)y - \rho\Big[(x-c)y - \frac{1}{2}\alpha y^2\Big] =: \widetilde{Q}(x,y).$$

Clearly $\widetilde{Q}(x,0) = 0$. Also, since $(x,y) \in \mathbb{S}_1$ is such that $y \leq \frac{1}{\alpha}(x - x_0)$ and $x \geq x_0$, we have

$$\widetilde{Q}_y(x,y) = a - bx - \rho(x-c) + \alpha\rho y \leq a - bx - \rho(x_0 - c) \leq a + \rho c - x_0(\rho + b) \leq 0,$$

where the last inequality is due to $x_0 \geq \bar{x}$. Hence $\mathcal{L}w(x,y) - \rho w(x,y) \leq 0$ on $\mathbb{S}_1$.

Step 3. Here we provide the proof of (2.67) in $\mathbb{S}_2$, separately for the two cases: *(i)* $a - bc \leq 0$ and *(ii)* $a - bc > 0$, and different approaches are followed in these two cases (see also Remark 2.4.11).

(i) Assume $a-bc \leq 0$. Let $(x,y) \in \mathbb{S}_2$ be given and fixed, and recall that $x \geq F^{-1}(y)$ and $y > \frac{1}{\alpha}(x - x_0)$ for all $(x,y) \in \mathbb{S}_2$. By employing (2.63) and (2.64), and observing that from (2.25) one has

$$\begin{aligned}&\Big[\frac{\sigma^2}{2}A(F(x-\alpha z))\psi''(x-\alpha z) + \big(a - b(x-\alpha z)\big)A(F(x-\alpha z))\psi'(x-\alpha z)\\ &\quad - \rho A(F(x-\alpha z))\psi(x-\alpha z)\Big]\Big|_{z=z(x,y)} = 0,\end{aligned}$$

we get

$$\begin{aligned}&\mathcal{L}w(x,y) - \rho w(x,y)\\ &= \Big[(a-bx)z - \rho(x-c)z + \frac{1}{2}\rho\alpha z^2 - b\alpha z A(F(x-\alpha z))\psi'(x-\alpha z)\Big]\Big|_{z=z(x,y)}.\end{aligned} \tag{2.77}$$

Since $z > 0$, $A > 0$, and $\psi' > 0$, one has that $\mathcal{L}w(x,y) - \rho w(x,y) \leq \widehat{Q}(x,y)$, where we have set

$$\widehat{Q}(x,y) := \Big[(a-bx)z - \rho(x-c)z + \frac{1}{2}\rho\alpha z^2\Big]\Big|_{z=z(x,y)}.$$

Observe that $\widehat{Q}(F^{-1}(y),y) = 0$ since $z(F^{-1}(y),y) = 0$ (cf. (2.55)). Hence, it suffices to show that $\widehat{Q}_x(x,y) < 0$ for all $(x,y) \in \mathbb{S}_2$. Differentiating $\widehat{Q}$ with respect to x gives

$$\widehat{Q}_x(x,y) = z(x,y)\Big(-b - \rho + \rho\alpha z_x(x,y)\Big) + z_x(x,y)\Big[(a-bx) - \rho(x-c)\Big].$$

Since $z_x > 0$ and $\alpha z_x < 1$ (cf. (2.59) and recall that $F' < 0$), and $x \geq F^{-1}(y) \geq x_\infty$, we find

$$\begin{aligned}\widehat{Q}_x(x,y) &\leq z_x(x,y)\Big[a + \rho c - F^{-1}(y)(\rho + b)\Big]\\ &\leq z_x(x,y)\big[a + \rho c - x_\infty(\rho+b)\big] = z_x(x,y)(\rho+b)\big(\bar{x} - x_\infty\big),\end{aligned}$$

and clearly $\widehat{Q}_x(x,y) \leq 0$ if $a - bc \leq 0$, since the latter implies $\bar{x} \leq c < x_\infty$.

This shows that $\widehat{Q} < 0$ on $\mathbb{S}_2$, and therefore that w solves (2.67) in $\mathbb{S}_2$ if $a - bc \leq 0$.

(ii) Assume that $a - bc > 0$. In this case, as discussed in Remark 2.4.11, we did not succeed proving (2.67) by studying the sign of $\mathcal{L}w - \rho w$ as done in *(i)* above. Therefore, we follow a different approach which is based on that developed in the proof of Lemma 6.7 in [22]. Here we just provide the main ideas, since most of the arguments follow from [22].

Let $(x,y) \in \overline{\mathbb{W}} \cap \mathbb{S}_2$ be given and fixed, and consider an arbitrary $z_o > 0$. From (2.54) we find $z(x + \alpha z_o, y + z_o) = z_o$, and employing the latter we have from (2.53), (2.63) and (2.64) that

$$\begin{aligned}
&\mathcal{L}w(x + \alpha z_o, y + z_o) - \rho w(x + \alpha z_o, y + z_o) \\
&= -\alpha b z_o A(F(x))\psi'(x) + \big(a - b(x + \alpha z_o)\big) z_o - \rho\big((x + \alpha z_o) - c\big) z_o + \frac{1}{2}\rho\alpha z_o^2 =: U(z_o).
\end{aligned}$$

Notice that $U(0) = 0$, hence to show negativity of U it suffices to prove that $U'(z_o) \leq 0$ for all $z_o > 0$. We find

$$\begin{aligned}
U'(z_o) &= -\alpha b A(F(x))\psi'(x) - \alpha b z_o + (a - b(x + \alpha z_o)) - \rho(x + \alpha z_o - c) \\
&= b\left(x - c - \alpha A(F(x))\psi'(x)\right) + (x + \alpha z_o - c)\left[-(b+\rho) + \frac{a - b(x + \alpha z_o)}{(x + \alpha z_o) - c}\right],
\end{aligned}$$

after rearranging terms, and adding and substracting the term $b(x - c)$ to obtain the second equality above. Now, define the function

$$\kappa(x) := -(b + \rho) + \frac{a - bx}{x - c}, \tag{2.78}$$

and notice that

$$\kappa(x_\infty) = (\psi'(x_\infty))^{-1}\left((a - bx_\infty)\psi''(x_\infty) - (b+\rho)\psi'(x_\infty)\right) = -\frac{\sigma^2}{2}\frac{\psi'''(x_\infty)}{\psi'(x_\infty)} < 0,$$

where we have used that x_∞ solves $x_\infty - c = \frac{\psi'(x_\infty)}{\psi''(x_\infty)}$ for the first equality, and Lemma A.1.2-(2) with $k = 1$ for the second equality. Moreover,

$$\kappa'(x) = \frac{bc - a}{(x - c)^2} < 0,$$

since $a > bc$, which then yields $\kappa(x) < 0$ for all $x > x_\infty$. From the monotonicity and the negativity of κ, and the fact that $z_o \mapsto (x + \alpha z_o - c)$ is positive and increasing as $x \geq x_\infty > c$, one obtains that $z_o \mapsto (x + \alpha z_o - c)\kappa(x + \alpha z_o)$ is decreasing. Therefore, one has $U'(z_o) \leq 0$ for all $z_o > 0$ if $U'(0+) \leq 0$.

To prove that the right-derivative $U'(0+)$ is negative, we now explain how to employ in our setting the arguments of the proof of Lemma 6.7 in [22]. First of all, we discuss the standing Assumption 2.2 in [22]. Conditions C2 and C3 are satisfied for $f(x) \equiv x - c$. If $a - bc > 0$, then Condition C5 in Assumption 2.2 of [22] is satisfied for $f(x) \equiv x - c$, $\hat{\sigma} \equiv \sigma$, $\delta \equiv \rho$, $\sigma\rho\hat{\sigma} \equiv a$, and $\beta \equiv b$. Moreover, all the other requirements in Assumption 2.2 of [22] are not needed in our case. Indeed, Condition C6 guarantees the existence and uniqueness of (in our terminology) x_0 and x_∞, that we already have by Lemma 2.4.3; Condition C4 only ensures a growth condition on the value function that we have from Proposition 2.3.1, whereas, in our setting, Condition C1 of [22] just means that the discount factor must be strictly positive.

Then, after reformulating our singular stochastic control problem as a calculus of variations problem where one seeks for a decreasing C^1 function triggering a strategy of reflecting type (see Section 4 in [22]), proceeding as in Section 5 of [22] (see in particular Theorem 5.6 therein), one can prove that our free boundary F^{-1} is a (one-sided) local maximizer of our performance criterion (2.4). Hence, a contradiction argument as that in the proof of Lemma 6.7 in [22] also applies in our case and yields that $U'(0+) \leq 0$. This completes the proof. □

Remark 2.4.11.

1. *As we have seen, the proof of* (2.67) *in* $\mathbb{S}_2$ *when* $a - bc > 0$ *requires a different analysis, and here we try to explain why a more direct approach seems not to lead to the desired result. Assuming* $a - bc > 0$, *if one aims at proving* (2.67) *by studying the sign of* $\mathcal{L}w - \rho w$ *in* $\mathbb{S}_2$, *given that* $z := z(x, y) \geq 0$ *for all* $(x, y) \in \mathbb{S}_2$, *one could try to prove that (cf.* (2.77)*)*

$$L(x, y) := a - bx - \rho(x - c) + \frac{1}{2}\rho\alpha z - b\alpha A(F(x - \alpha z))\psi'(x - \alpha z)$$

is negative for any $(x, y) \in \mathbb{S}_2$. *Calculations, employing* (2.29) *and the definition of* A' *(cf.* (2.48)*), reveal that for any* $y > 0$ *one has* $L(F^{-1}(y), y) = \chi(F^{-1}(y))$, *where, for any* $u \in (x_\infty, x_0]$, *we have set*

$$\chi(u) := (\rho + 2b)(\widehat{x} - u) + b\psi(u)\left[\frac{(u - c)\psi''(u) - \psi'(u)}{\psi''(u)\psi(u) - \psi'(u)^2}\right],$$

with $\widehat{x} := \frac{a+(\rho+b)c}{\rho+2b} < x_\infty$. *By noticing that* $A(F(x - \alpha z))\psi'(x - \alpha z) = w_x(x, y) - z$ *in* $\mathbb{S}_2$ *(cf.* (2.63)*), one has that* L *rewrites as*

$$L(x, y) = a - bx - \rho(x - c) + \frac{1}{2}\rho\alpha z + b\alpha z - b\alpha w_x(x, y),$$

and because $\alpha z_x < 1$ *by* (2.59) *and* $w_{xx} \geq 0$ *by* (2.64), *it is easy to see that* $L_x < 0$ *on* $\mathbb{S}_2$.

Hence, to prove that $L < 0$ *on* $\mathbb{S}_2$ *it would suffice to show that* $\chi < 0$ *on* $(x_\infty, x_0]$. *However, we have not been able to prove this property due to the unhandy implicit* *expression of the function ψ, even if a numerical investigation seems to confirm negativity of χ. For this technical reason in* Step 3-(ii) *of the proof of Proposition 2.4.10 we have hinged on arguments as those originally developed in [22] to address the case $a - bc > 0$.*

2. *It is also worth noticing that the calculus of variations approach of [22] would have not been directly applicable for any choice of the parameters. Indeed, when $a - bc < 0$, the function κ of* (2.78) *is increasing and therefore it has not the monotonicity required in Condition C5 of Assumption 2.2 of [22]. However, under such a parameters' restriction, direct calculations as those developed in* Step 3-(i) *of the proof of Proposition 2.4.10 lead to the desired result. This fact suggests that a combined use of the calculus of variations method and of the more standard direct study of the HJB equation could be successful in complex situations where neither of the two methods seem to leed to the proof of optimality of a candidate value function for any choice of the model's parameters.*

We conclude by showing that w of (2.53) identifies with the value function V. As a byproduct we also provide an optimal extraction rule. We first need the following technical result. Its proof follows by suitably adopting the classical result in [48], upon considering the following joint process (X, ζ) as a (degenerate) diffusion in $\mathbb{R}^2$ with oblique reflection in the direction $(-\alpha, -1)$ at the C^∞-free boundary F (see also [22], Remark 4.2).

Lemma 2.4.12. *Let $(x, y) \in \mathbb{R} \times (0, \infty)$, F be given as in* (2.51)*, $z := z(x, y)$ solving* (2.54)*, and let $\Delta := \Delta(x, y) = y\mathbb{1}_{\{(x,y)\in\mathbb{S}_1\}} + z\mathbb{1}_{\{(x,y)\in\mathbb{S}_2\}}$. Then there exists a (pathwise) unique $\mathbb{F}$-adapted continuous (X, ζ), with ζ increasing, such that*

$$\begin{aligned} X_t &\leq F^{-1}(y - \Delta - \zeta_t), \\ dX_t &= \big(a - bX_t\big)dt + \sigma dW_t - \alpha d\zeta_t, \\ d\zeta_t &= \mathbb{1}_{\{X_t = F^{-1}(y-\Delta-\zeta_t)\}} d\zeta_t, \end{aligned}$$

for any $0 \leq t \leq \tau_\zeta$, with $\tau_\zeta := \inf\{t \geq 0 : \zeta_t \geq y - \Delta\}$, and starting point $(X_0, \zeta_0) = (x - \alpha\Delta, 0)$.

Theorem 2.4.13. *Recall the functions F and w from* (2.51) *and* (2.53)*, respectively. The function w identifies with the value function V from* (2.3)*, and the optimal extraction strategy, denoted by $\xi^\star$, is given by*

$$\xi^\star_t = \begin{cases} \Delta + \zeta_t, & t \in [0, \tau_\zeta), \\ y, & t \geq \tau_\zeta, \end{cases} \tag{2.79}$$

with $\xi^\star_{0-} = 0$, and with Δ, ζ, and τ_ζ as in Lemma 2.4.12.

Proof. We aim at applying Theorem 2.3.2. We already know that $w \in C^{2,1}(\mathbb{R}\times[0,\infty))$ is a solution to the HJB equation (2.12) by Lemma 2.4.9 and Proposition 2.4.10, and that satisfies $w(x,0)=0$ for all $x\in\mathbb{R}$. Moreover, the function w is increasing with respect to y. To see that, notice that one has from (2.48) that $A'(y)>0$, for $y>0$ (since the denominator of (2.48) is positive by Lemma A.1.2-(3) and the numerator is positive as well due to $F^{-1}(y)\geq x_\infty$), and this gives $w_y>0$ on $\mathbb{W}$ and on $\mathbb{S}_2$ (cf. (2.57) and (2.65)). Also, one can easily check from (2.58) that $w_y\geq 0$ on $\mathbb{S}_1$ because $y\leq(x-x_0)/\alpha$ and $x_0>c$.

To prove the upper bound in (2.15), recall that (cf. (2.46))

$$A(y)=\frac{(F^{-1}(y)-c)\psi'(F^{-1}(y))-\psi(F^{-1}(y))}{\alpha[\psi'(F^{-1}(y))^2-\psi''(F^{-1}(y))\psi(F^{-1}(y))]},\quad y\geq 0.$$

Since $x_0\geq F^{-1}(y)\geq x_\infty$ for any $y\geq 0$, by using that ψ, ψ' and ψ'' are continuous we have that there exists a constant $\overline{K}>0$ such that $A(y)\leq\overline{K}$ for all $y\geq 0$. Hence, by (2.53) we have $w(x,y)\leq\overline{K}\psi(F^{-1}(y))\leq\overline{K}\psi(x_0)$ for all $(x,y)\in\mathbb{W}$. Moreover, $0\leq z(x,y)\leq y$ for all $(x,y)\in\mathbb{S}_2$ and thus $(x-c)z-\frac{1}{2}\alpha z\leq(x-c)z\leq(x-c)y$. Since the upper bound in (2.15) is clearly satisfied in $\mathbb{S}_1$, we conclude that there exists a constant $K>0$ such that

$$w(x,y)\leq Ky(1+y)(1+|x|)\quad\text{for all }(x,y)\in\mathbb{R}\times(0,\infty).$$

As for the nonnegativity of w, notice that for all $(x,y)\in\mathbb{S}_1$ we have

$$w(x,y)=(x-c)y-\frac{1}{2}\alpha y^2\geq y\big[x-c-\frac{1}{2}(x-x_0)\big]\geq y\big[\frac{x_\infty-c}{2}+\frac{x_0-c}{2}\big]\geq 0,$$

since $y\leq\frac{x-x_0}{\alpha}$, $x\geq F^{-1}(y)\geq x_\infty$ and $x_0>x_\infty>c$. Moreover, the nonnegativity of ψ and A imply

$$w(x,y)\geq 0,\quad\text{for all }(x,y)\in\mathbb{W},$$

and also, given $(x,y)\in\mathbb{S}_2$, we have

$$\begin{aligned}w(x,y)&=A(F(x-\alpha z))\psi(x-\alpha z)+(x-c)z-\frac{1}{2}\alpha z^2\\&\geq\int_0^z(x-\alpha u-c)du\geq\int_0^z(x_\infty-c)du\geq 0,\end{aligned}$$

since $0\leq z\leq\frac{x-x_\infty}{\alpha}$ and $x_\infty>c$. Therefore $w\geq 0$ on $\mathbb{R}\times[0,\infty)$.

Now, since $\xi^\star$ satisfies (2.16) and (2.17), by Theorem 2.3.2 we therefore conclude that w identifies with V, and that $\xi^\star$ is an optimal extraction strategy. □

Remark 2.4.14. *It is worth noticing that, by adopting the optimal extraction rule $\xi^\star$ as in* (2.79)*, all the commodity is extracted in finite time. In fact, by following arguments as those in Theorem 3.1 of [22], one can show that the time τ_ζ arising in Lemma 2.4.12 has finite moments.*

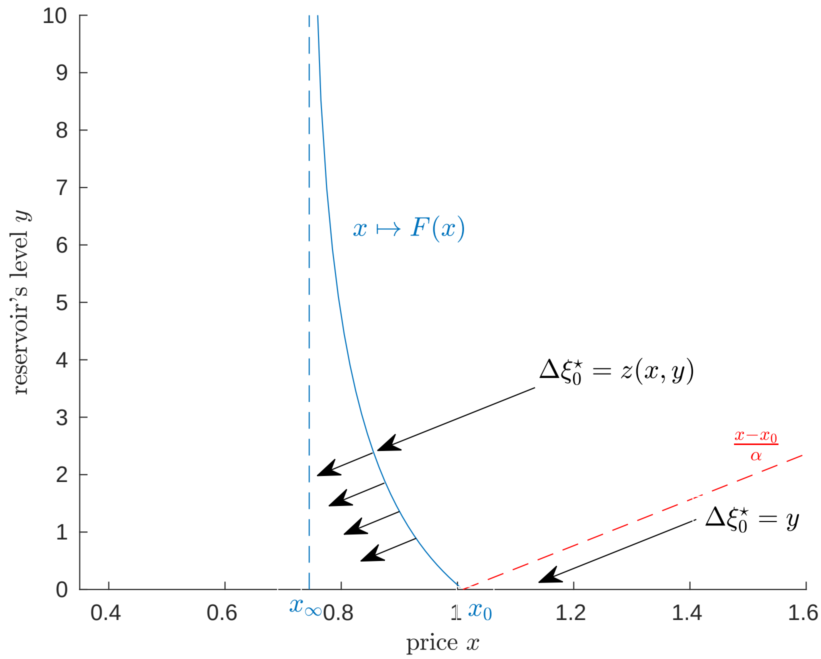


Figure 2.2: A graphical illustration of the optimal extraction rule $\xi^\star$ (cf. (2.79)) and of the free boundary F. The plot has been obtained by using $a = 0.4$, $\sigma = 0.8$, $\rho = 3/8$, $c = 0.3$, $b = 1$, $\alpha = 0.25$, and by numerically evaluating the free boundary of (2.51). The optimal extraction rule prescribes the following. In the region $\{(x,y) \in \mathbb{R} \times (0,\infty) : y < F(x)\}$ it is optimal not to extract. If at initial time (x,y) is such that $x > F^{-1}(y)$ and $y \leq (x - x_0)/\alpha$, then the reservoir should be immediately depleted. On the other hand, if (x,y) is such that $x \geq F^{-1}(y)$ and $y > (x - x_0)/\alpha$, then one should make a lump sum extraction of suitable size $z(x,y)$, and then keep on extracting until the commodity is exhausted by just preventing the (optimally controlled) process (X,Y) to leave the region $\{(x,y) \in \mathbb{R} \times (0,\infty) : y \leq F(x)\}$.

2.4.2.1 A Related Optimal Stopping Problem

In this section, we show that the directional derivative $u := \alpha V_x + V_y$ identifies with the value function of an optimal stopping problem. Such a result is consistent with that obtained - for a different model with Brownian dynamics - in [80], where connections between finite-fuel singular stochastic control problems and questions of optimal stopping have been studied.

Proposition 2.4.15. *The function $u : \mathbb{R} \times (0, \infty) \mapsto \mathbb{R}$ defined by*

$$u(x, y) := \alpha V_x(x, y) + V_y(x, y)$$

admits the probabilistic representation

$$u(x, y) = \sup_{\tau \geq 0} \mathbb{E}\left[e^{-\rho\tau} \left(X^x_\tau - c\right) - \int_0^\tau e^{-\rho s} \alpha b A(y) \psi'(X^x_s) ds\right], \quad (x, y) \in \mathbb{R} \times (0, \infty), \tag{2.80}$$

where the optimization is taken over the set of $\mathbb{F}$-stopping times. Moreover, for F as in (2.51)*, we have that the stopping time*

$$\tau^\star(x; y) = \inf\{t \geq 0 : X^x_t \geq F^{-1}(y)\}, \quad (x, y) \in \mathbb{R} \times (0, \infty),$$

is optimal in (2.80)*.*

Proof. For the rest of this proof, $y \in (0, \infty)$ will be given and fixed. Notice that $u(\cdot, y) \in C^1(\mathbb{R})$ by construction (cf. (2.29) and (2.30)). Moreover, direct calculations on (2.53) show that $u_{xx}(\cdot, y) \in L^\infty_{loc}(\mathbb{R})$. We now show that $u(\cdot, y)$ solves the HJB equation

$$\max\Big\{\mathcal{L}w(x) - \rho w(x) - \alpha b A(y)\psi'(x),\ x - c - w(x)\Big\} = 0, \quad \text{a.e. } x \in \mathbb{R}. \tag{2.81}$$

Recall the selling region $\mathbb{S}$ and the waiting region $\mathbb{W}$. Let $x \in \mathbb{R}$ be such that $(x, y) \in \mathbb{W}$, and notice that by (2.53) we have

$$V_x(x, y) = A(y)\psi'(x), \quad \text{and} \quad V_y(x, y) = A'(y)\psi(x).$$

Then, since $u = \alpha V_x + V_y$,

$$\begin{aligned}
&\mathcal{L}u(x, y) - \rho u(x, y) - \alpha b A(y)\psi'(x) \\
=&\frac{1}{2}\sigma^2 \left(\alpha A(y)\psi'''(x) + A'(y)\psi''(x)\right) + (a - bx)\left(\alpha A(y)\psi''(x) + A'(y)\psi'(x)\right) \\
&- (\rho + b)\alpha A(y)\psi'(x) - \rho A'(y)\psi(x) \\
=&\alpha A(y)\big(\mathcal{L}\psi'(x) - (\rho + b)\psi'(x)\big) + A'(y)\big(\mathcal{L}\psi(x) - \rho\psi(x)\big) = 0,
\end{aligned}$$

upon using that $\psi^{(k)}$ satisfies Lemma A.1.2-(2) with $k = 0, 1$.

Now, let $x \in \mathbb{R}$ be such that $(x, y) \in \mathbb{S}$, so that $u(x, y) = x - c$ (recall (2.27)). If $(x, y) \in \mathbb{S}_1$ then $x \geq x_0$, and using that $\alpha b A(y)\psi'(x) > 0$ we obtain

$$\begin{aligned}
&\mathcal{L}u(x, y) - \rho u(x, y) - \alpha b A(y)\psi'(x) = (a - bx) - \rho(x - c) - \alpha b A(y)\psi'(x) \\
&\leq a - (\rho + b)x + \rho c = (\rho + b)(\bar{x} - x) \leq 0,
\end{aligned}$$

since $x_0 \geq \bar{x}$ by Lemma B.2.1 in Appendix B.

On the other hand, let $x \in \mathbb{R}$ be such that $(x, y) \in \mathbb{S}_2$, set $H(x, y) := \mathcal{L}u(x, y) - \rho u(x, y) - \alpha b A(y)\psi'(x)$, and notice that

$$\frac{\partial H(x, y)}{\partial x} = -(\rho + b) - \alpha b A(y)\psi''(x) < 0,$$

due to the positivity of A and ψ''. Thus, in order to prove that $\mathcal{L}u(x, y) - \rho u(x, y) - \alpha b A(y)\psi'(x) \leq 0$ for all $(x, y) \in \mathbb{S}_2$, it is enough to prove that $H(F^{-1}(y), y) \leq 0$. Set $u := F^{-1}(y)$; then, upon employing the definition of A (cf. (2.46)), we obtain

$$\begin{aligned}
H(u, y) = &\left(\psi(u)\psi''(u) - \psi'(u)^2\right)^{-1} \times \\
&\times \Big[(a - bu - \rho(u - c))\left(\psi(u)\psi''(u) - \psi'(u)^2\right) + b(u - c)\psi'(u)^2 - b\psi(u)\psi'(u)\Big] \\
= &\frac{\sigma^2}{2}\left(\psi(u)\psi''(u) - \psi'(u)^2\right)^{-1} \times \\
&\times \Big[\psi'''(u)\left[(u - c)\psi'(u) - \psi(u)\right] - \psi''(u)\left[(u - c)\psi''(u) - \psi'(u)\right]\Big] < 0,
\end{aligned}$$

where we have applied Lemma A.1.2-(2) with $k = 0$ and $k = 1$ for the last equality, and the last inequality follows from Corollary 2.4.5 since $x_\infty < u \leq x_0$. Hence, $\mathcal{L}u(x, y) - \rho u(x, y) - \alpha b A(y)\psi'(x) \leq 0$ on $\mathbb{S}_2$.

Finally, from Proposition 2.4.10 we have $x - c - u(x, y) \leq 0$ for any $x \in \mathbb{R}$.

The previous inequalities show that $u(\cdot, y)$ identifies with a $W^{2,\infty}_{loc}(\mathbb{R})$-solution to (2.81). Then, a standard verification theorem based on an application of (a generalized version of) Itô's formula, implies that $u(\cdot, y)$ admits representation (2.80) and that the stopping time $\tau^\star(x; y) = \inf\{t \geq 0 : X^x_t \geq F^{-1}(y)\}$ attains the supremum. □

Remark 2.4.16. *A few comments are worth being done.*

1. *With regard to the connection between problems of singular stochastic control and questions of optimal stopping (see, e.g., [51, 52, 78, 80] as early contributions, and the introduction of the recent [46] for a richer literature review), we can interpret the stopping time $\tau^\star(x; y)$ as the optimal time at which an additional unit of the commodity should be extracted. Indeed, the underlying process at that time is such that, in economic terms, equality between the marginal expected optimal profit (i.e. $\alpha V_x + V_y$) and the marginal instantaneous net profit from extraction (i.e. $x - c$) holds.*

2. *If we do not consider price impact in our model (i.e. we take $\alpha = 0$), it can be easily seen that the value function of the resulting optimal extraction problem V is such that*

$$V_y(x, y) = \sup_{\tau \geq 0} \mathbb{E}\Big[e^{-\rho\tau}\left(X^x_\tau - c\right)\Big],$$

a result that is clearly consistent with (2.80)*. The integral term*

$$-\int_0^\tau e^{-\rho s}\alpha b A(y)\psi'(X^x_s)ds$$

appearing in (2.80) *can then be seen as a running cost/penalty whose effect increases with increasing price impact α.*

3. *It can be checked that the arguments of the proof of Proposition 2.4.15 carry over also to the case of a fundamental price given by a drifted Brownian motion, i.e. when $b = 0$ (cf. Section 2.4.1). As one would expect by setting $b = 0$ in the right-hand side of* (2.80)*, in such a case it holds*

$$\alpha V_x(x, y) + V_y(x, y) = \sup_{\tau \geq 0} \mathbb{E}\Big[e^{-\rho\tau}\left(X^x_\tau - c\right)\Big],$$

so that the stopping problem related to the optimal extraction problem does not depend on the current level of the reservoir y. This explains why, in in the drifted Brownian motion case studied in Section 2.4.1, the free boundary $x^\star$ triggering the optimal extraction rule is y-independent.

2.5 Comparative Statics Analysis

In this section, we study the sensitivity of the solution to the extraction problem separately for the case of a fundamental price given by a drifted Brownian motion (Section 2.5.1) and by an Ornstein-Uhlenbeck process (Section 2.5.2). In particular, in Section 2.5.1 we analytically determine the dependency of the free boundary $x^\star$ of (2.34) and of the value function (2.35) on the parameters a and σ. In Section 2.5.2 we study analytically how the value function (2.53) and the critical price levels x_0 and x_∞ from Lemma 2.4.4 depend on a and σ, and, numerically, the sensitivity of the free boundary F with respect to a, σ and b.

2.5.1 Sensitivity Analysis in the Case of a Drifted Brownian Motion Fundamental Price

Here we assume $b = 0$ in (2.2). Thanks to the explicit formula (2.34), studying the sensitivity of the free boundary $x^\star$ with respect to the parameters a and σ is a simple exercise of differentiation.

Proposition 2.5.1. *The free boundary $x^\star$ of* (2.34) *is increasing with respect to both a and σ.*

Proof. We look at the parameter n of (2.32) as a function of a and σ; that is, we set

$$n(a,\sigma) := -\frac{a}{\sigma^2} + \sqrt{\left(\frac{a}{\sigma^2}\right)^2 + 2\frac{\rho}{\sigma^2}}.$$

Then, it is not hard to find by direct calculations that

$$n_a(a,\sigma) = \frac{1}{\sigma^4}\left(\frac{a}{\sqrt{\left(\frac{a}{\sigma^2}\right)^2 + 2\frac{\rho}{\sigma^2}}} - \sigma^2\right), \tag{2.82}$$

and

$$n_\sigma(a,\sigma) = \frac{2}{\sigma^3}\left(a - \frac{\frac{a^2}{\sigma^2} + \rho}{\sqrt{\left(\frac{a}{\sigma^2}\right)^2 + 2\frac{\rho}{\sigma^2}}}\right). \tag{2.83}$$

Clearly, if $a \leq 0$ one has $n_a \leq 0$ and $n_\sigma \leq 0$. Then, suppose $a > 0$ and notice that

$$\sqrt{\left(\frac{a}{\sigma^2}\right)^2 + 2\frac{\rho}{\sigma^2}} \geq \frac{a}{\sigma^2} \quad \text{and} \quad \sqrt{\left(\frac{a}{\sigma^2}\right)^2 + 2\frac{\rho}{\sigma^2}} \leq \frac{a}{\sigma^2} + \frac{\rho}{a}, \tag{2.84}$$

where the second inequality above follows by an application of the binomial formula. By using the first inequality of (2.84) in (2.82), and the second inequality of (2.84) in (2.83), one easily finds that $n_a(a,\sigma) \leq 0$, as well as $n_\sigma(a,\sigma) \leq 0$.

Finally, the claim follows since $x^\star$ is decreasing with respect to n (cf. (2.32)). □

Proposition 2.5.2. *The value function V defined in* (2.3) *is increasing with respect to a and σ.*

Proof. Let $\hat{a} > a$ and $\hat{\sigma} > \sigma$. We show the monotonicity with respect to a and σ separately in two steps.

Step 1. Let $(x,y) \in \mathbb{R} \times (0,\infty)$ be given and fixed. For any $\xi \in \mathcal{A}(y)$, we denote by $\widehat{X}^{x,\xi}_t$ the solution to (2.2) when $b = 0$ and the drift is $\hat{a}$. One clearly has $\widehat{X}^{x,\xi}_t \geq X^{x,\xi}_t$ $\mathbb{P}$-a.s. for any $t \geq 0$. Therefore $\widehat{\mathcal{J}}(x,y,\xi) \geq \mathcal{J}(x,y,\xi)$ for any $\xi \in \mathcal{A}(y)$, where $\hat{\mathcal{J}}$ is given by (2.4) with underlying state $(\widehat{X}^{x,\xi}, Y^{y,\xi})$. Hence, we conclude

$$\widehat{V}(x,y) \geq V(x,y), \quad \forall (x,y) \in \mathbb{R} \times [0,\infty),$$

where $\widehat{V}(x,y) := \sup_{\xi \in \mathcal{A}(y)} \hat{\mathcal{J}}(x,y,\xi)$.

Step 2. To prove the monotonicity of V with respect to σ we adapt to our setting ideas from Theorem 4 in [6]. Let $\widehat{V}$ be the value function when the volatility coefficient in (2.2) is $\hat{\sigma}$. Recall $\mathcal{L}$ as in (2.11), and let $\widehat{\mathcal{L}}$ be as in (2.11) but with volatility coefficient $\widehat{\sigma}$. Then, for all $(x, y) \in \mathbb{R} \times (0, \infty)$ we have

$$\begin{aligned}
&\mathcal{L}\widehat{V}(x,y) - \rho\widehat{V}(x,y) \\
&= \frac{\hat{\sigma}^2}{2}\widehat{V}_{xx}(x,y) + a\widehat{V}_x(x,y) - \rho\widehat{V}(x,y) + \frac{(\sigma^2 - \hat{\sigma}^2)}{2}\widehat{V}_{xx}(x,y) \\
&= \widehat{\mathcal{L}}\widehat{V}(x,y) - \rho\widehat{V}(x,y) + \frac{(\sigma^2 - \hat{\sigma}^2)}{2}\widehat{V}_{xx}(x,y) \leq \frac{(\sigma^2 - \hat{\sigma}^2)}{2}\widehat{V}_{xx}(x,y) \leq 0,
\end{aligned} \tag{2.85}$$

since $\widehat{V}(\cdot, y)$ is convex by the second equations in (2.38) and (2.40), and the second equation of (2.42). Furthermore, since $\widehat{V}$ is the value function of the optimal extraction problem when in (2.2) the volatility is $\hat{\sigma}$, $\widehat{V}$ must satisfy

$$-\alpha\widehat{V}_x(x,y) - \widehat{V}_y(x,y) + (x - c) \leq 0, \tag{2.86}$$

for all $(x, y) \in \mathbb{R} \times (0, \infty)$, and $\widehat{V}(x, 0) = 0$ for all $x \in \mathbb{R}$. Now, arguing as in the first step of the proof of Theorem 2.3.2, by using (2.85) and (2.86), we obtain $\widehat{V} \geq V$, and thus the claimed monotonicity. □

Propositions 2.5.1 and 2.5.2 show that the higher the level of the drift a is, and hence the higher the expected prices are, the later the company starts extracting in order to obtain larger profits. Moreover, higher uncertainty, and hence larger price's fluctuations, are exploited by the company that then sells the commodity at higher prices and increases the resulting profits.

2.5.2 Sensitivity Analysis in the Case of an Ornstein-Uhlenbeck Fundamental Price

We start by studying the sensitivity of x_0 and x_∞ (cf. Lemma 2.4.4) on the model parameters a and σ. In the following, when needed, we write $g(\cdot; a, \sigma)$ in order to emphasize the dependency of a given real-valued function g with respect to a and σ.

Recall that the fundamental increasing solution to the equation $(\mathcal{L} - \rho)u = 0$ is given by (2.45). In the following, when needed, we denote by $\psi^{(k)}(x; a, \sigma)$ the $k-$th derivative with respect to x of ψ. By an application of the dominated convergence theorem one obtains the relation

$$\frac{\partial\psi^{(k)}}{\partial a}(x; a, \sigma) := \psi_a^{(k)}(x; a, \sigma) = -\frac{1}{b}\psi^{(k+1)}(x; a, \sigma), \quad \text{for all } k \in \mathbb{N}_0. \tag{2.87}$$

Analogously, one finds

$$\frac{\partial\psi^{(k)}}{\partial\sigma}(x; a, \sigma) := \psi_\sigma^{(k)}(x; a, \sigma) = \left(\frac{a - bx}{b\sigma}\right)\psi^{(k+1)}(x; a, \sigma) - \frac{k}{\sigma}\psi^{(k)}(x; a, \sigma), \tag{2.88}$$

for all $k \in \mathbb{N}_0$

By employing (2.87), and Lemma A.1.2, one can easily prove the next result.

Lemma 2.5.3. *One has that*

$$\frac{\partial(\psi^{(k)}(x;a,\sigma)/\psi^{(k+1)}(x;a,\sigma))}{\partial a} = \frac{\psi^{(k)}(x;a,\sigma)\psi^{(k+2)}(x;a,\sigma) - \psi^{(k+1)}(x;a,\sigma)^2}{b\psi^{(k+1)}(x;a,\sigma)^2} > 0, \tag{2.89}$$

The proof of the next result can be found in Appendix B. It employs (2.88).

Lemma 2.5.4. *One has that*

$$\begin{aligned}&\frac{\partial(\psi^{(k)}(x;a,\sigma)/\psi^{(k+1)}(x;a,\sigma))}{\partial \sigma}\\ &= \frac{(a-bx)[\psi^{(k+1)}(x;a,\sigma)^2 - \psi^{(k)}(x;a,\sigma)\psi^{(k+2)}(x;a,\sigma)] + b\psi^{(k+1)}(x;a,\sigma)\psi^{(k)}(x;a,\sigma)}{b\sigma\psi^{(k+1)}(x;a,\sigma)^2} > 0.\end{aligned} \tag{2.90}$$

The previous results on the dependency of ψ/ψ_x with respect to a and σ (i.e. (2.89) and (2.90)) allow us to determine the dependency of x_0 and x_∞ on a and σ as well. One may intuitively expect that the company exploits a higher mean reversion level, and thus sells the commodity at higher prices. As an indication of this, we indeed find that x_0, x_∞, and the value function V increase as a increases.

In the following we denote by x_0, x_∞ the unique solutions on (c,∞) to $(x-c)\psi_x(x;a,\sigma) - \psi(x;a,\sigma) = 0$ and $(x-c)\psi_{xx}(x;a,\sigma) - \psi_x(x;a,\sigma) = 0$, respectively. Also, $V(x,y)$ denotes the value function when in (2.2) the mean-reversion level is a/b and the volatility is σ.

Proposition 2.5.5. *Let $\hat{a} > a$, and denote by $\hat{x}_0$ and $\hat{x}_\infty$ the unique solutions on (c,∞) to $(x-c)\psi_x(x;\hat{a},\sigma) - \psi(x;\hat{a},\sigma) = 0$ and $(x-c)\psi_{xx}(x;\hat{a},\sigma) - \psi_x(x;\hat{a},\sigma) = 0$, respectively. Furthermore, we denote by $\widehat{V}(x,y)$, $(x,y) \in \mathbb{R}\times[0,\infty)$, the value function when in (2.2) the mean-reversion level is $\hat{a}/b$ and the volatility is σ. We have*

$$\hat{x}_0 > x_0 \quad \textit{and} \quad \hat{x}_\infty > x_\infty,$$

and

$$\widehat{V}(x,y) \geq V(x,y), \quad \forall (x,y) \in \mathbb{R}\times[0,\infty). \tag{2.91}$$

Proof. For any given $q \in \mathbb{R}$ and $\sigma > 0$, set $H(x;q,\sigma) := (x-c)\psi_x(x;q,\sigma) - \psi(x;q,\sigma)$, $x \in \mathbb{R}$. We have $H_x(x;q,\sigma) > 0$ for all $x > c$. Moreover,

$$H(\hat{x}_0;a,\sigma) = \frac{\psi(\hat{x}_0;\hat{a},\sigma)}{\psi_x(\hat{x}_0;\hat{a},\sigma)}\psi_x(\hat{x}_0;a,\sigma) - \psi(\hat{x}_0;a,\sigma) > 0 = H(x_0;a,\sigma),$$

where we have used that $H(\hat{x}_0;\hat{a},\sigma) = 0$ for the first equality, and Lemma 2.5.3 with $k=0$ for the inequality. Thus, by monotonicity of $H(\cdot;q,\sigma)$ on (c,∞), we have $\hat{x}_0 > x_0$. Analogously, we can prove that $\hat{x}_\infty > x_\infty$ by employing Lemma 2.5.3 with $k=1$.

In order to prove (2.91), we can proceed in the same way as in *Step 1* of the proof of Proposition 2.5.2. □

The next proposition shows that the critical price levels x_0 and x_∞ increase as the price's fluctuations become larger.

Proposition 2.5.6. *Let $\hat{\sigma} > \sigma$, and denote by $\hat{x}_0$ and $\hat{x}_\infty$ the unique solutions on (c, ∞) to $(x-c)\psi_x(x; a, \hat{\sigma}) - \psi(x; a, \hat{\sigma}) = 0$ and $(x-c)\psi_{xx}(x; a, \hat{\sigma}) - \psi_x(x; a, \hat{\sigma}) = 0$, respectively. Furthermore, denote by $\widehat{V}$ the value function when in* (2.2) *the mean-reversion level is a/b and the volatility is $\hat{\sigma}$. We have*

$$\hat{x}_0 > x_0 \quad \text{and} \quad \hat{x}_\infty > x_\infty,$$

and

$$\widehat{V}(x, y) \geq V(x, y), \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R}_+. \tag{2.92}$$

Proof. For any given $q > 0$ and $a \in \mathbb{R}$, set $H(x; a, q) := (x-c)\psi_x(x; a, q) - \psi(x; a, q)$, $x \in \mathbb{R}$. We have $H_x(x; a, q) > 0$ for all $x > c$. Moreover, using that $H(\hat{x}_0; a, \hat{\sigma}) = 0$ we have

$$H(\hat{x}_0; a, \sigma) = \frac{\psi(\hat{x}_0; a, \hat{\sigma})}{\psi_x(\hat{x}_0; a, \hat{\sigma})}\psi_x(\hat{x}_0; a, \sigma) - \psi(\hat{x}_0; a, \sigma) > 0 = H(x_0; a, \sigma),$$

where the inequality is due to Lemma 2.5.4 with $k = 0$. Since $H(\cdot; a, q)$ is increasing for all $x > c$ we have $\hat{x}_0 > x_0$. Analogously, we can prove that $\hat{x}_\infty > x_\infty$ by Lemma 2.5.4 with $k = 1$.

To prove (2.92) we can use the arguments employed in *Step 2* of the proof of Proposition 2.5.2, upon noticing that $\widehat{V}(\cdot, y)$ is convex by the second equations in (2.57) and (2.58), and (2.64) (recall that A is positive and ψ is convex). □

In the following, we assume $y \geq 0$ be given and fixed. The semi-explicit nature of our results allows us to easily study numerically the dependency of $F^{-1}(y)$ with respect to a. This is shown in Figure 2.3. We see that $F^{-1}(y)$ increases as a increases: the higher the level of mean reversion is, the later the company starts extracting in order to obtain larger profits.

Figure 2.4 shows the dependency with respect to σ. We see that $F^{-1}(y)$ increases as σ increases. We thus conclude that higher uncertainty, and hence higher fluctuations around the mean-reversion level, are exploited by the company which then sells the commodity at higher prices and increases its profits.

In Figure 2.5, we can observe the sensitivity $F^{-1}(y)$ with respect to b. Differently to what it is happening when increasing σ and a, now $F^{-1}(y)$ increases as b decreases, and in fact, as $b \downarrow 0$, it converges to $x^\star$, which is the free boundary in the case $b = 0$ (i.e. related to the drifted Brownian motion case). The lower b is, the less transient is the impact of the extraction's policy on the market price of the commodity. This in turn implies a more precautionary behavior of the company.

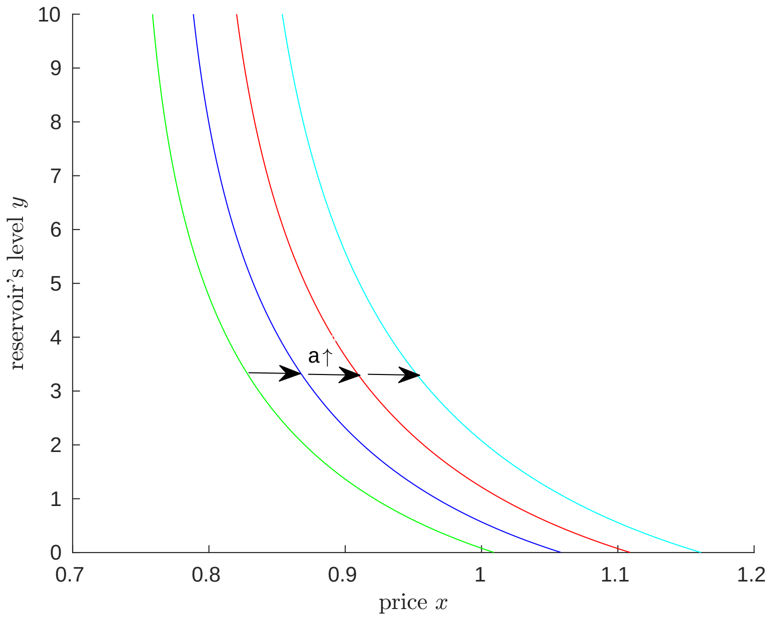


Figure 2.3: A drawing of the free boundary $x \mapsto F(x)$ for $b = 1$, $\sigma = 0.8$, $\rho = 3/8$, $c = 0.3$, $\alpha = 0.25$ and various values for a: $a = 0.4$ (green), $a = 0.5$ (blue), $a = 0.6$ (red), and $a = 0.7$ (cyan).

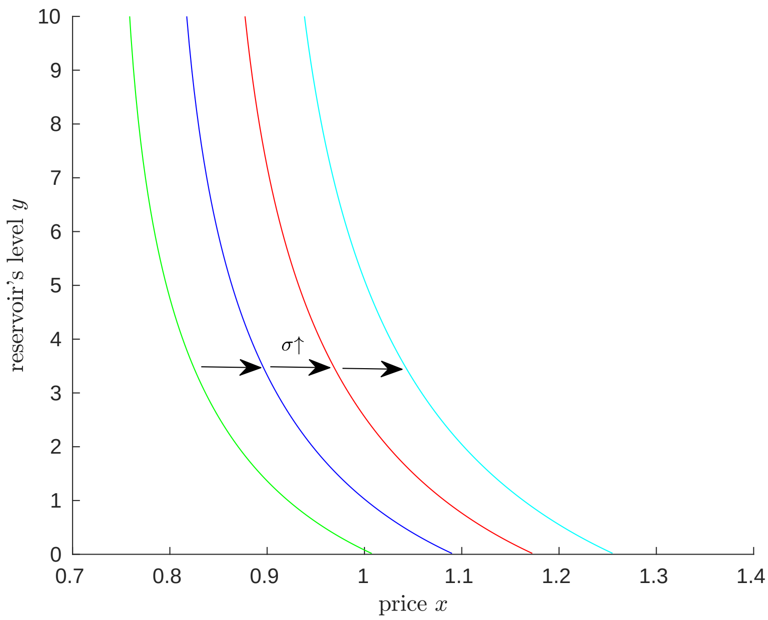


Figure 2.4: A drawing of the free boundary $x \mapsto F(x)$ for $a = 0.4$, $b = 1$, $\rho = 3/8$, $c = 0.3$, $\alpha = 0.25$ and various values for the volatility: $\sigma = 0.8$ (green), $\sigma = 0.9$ (blue), $\sigma = 1$ (red), and $\sigma = 1.1$ (cyan).

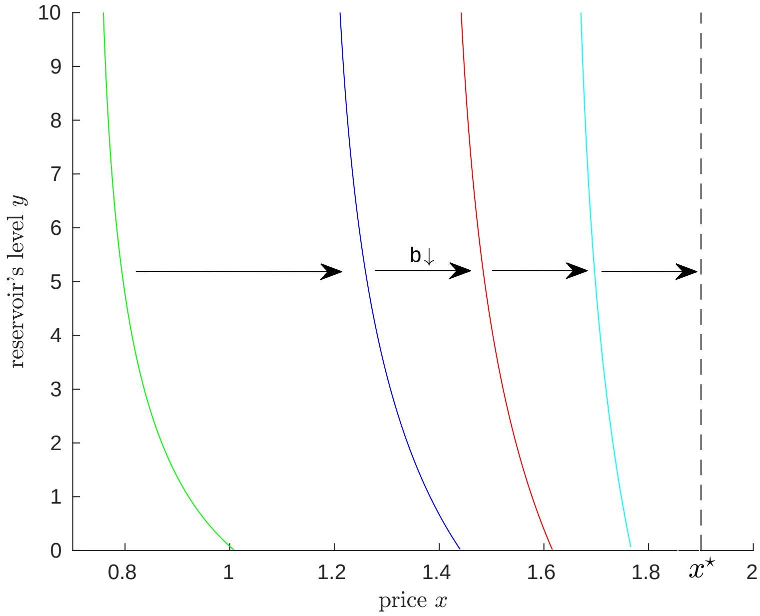


Figure 2.5: A drawing of the free boundary $x \mapsto F(x)$ for $a = 0.4$, $\sigma = 0.8$, $\rho = 3/8$, $c = 0.3$, $\alpha = 0.25$ and various values for the mean reversion speed: $b = 1$ (green), $b = 0.25$ (blue), $b = 0.125$ (red), and $b = 0.05$ (cyan).

As $b \downarrow 0$, the convergence of the free boundary to $x^\star$ can, in fact, be proved formally upon showing that x_0 and x_∞ converge to $x^\star$. To stress the dependency on b, we shall write $x_0(b)$ and $x_\infty(b)$.

Proposition 2.5.7. *We have $x_\infty(b) \to x^\star$ and $x_0(b) \to x^\star$ as $b \downarrow 0$.*

Proof. For $\sigma > 0$ and $a, b \in \mathbb{R}$, consider the ordinary differential equation (ODE)

$$\frac{1}{2}\sigma^2 u''(x) + (a - bx)u'(x) - \rho u(x) = 0, \qquad x \in \mathbb{R}, \tag{2.93}$$

and let $\psi(x; b)$ denote its increasing fundamental solution, where we have stressed the dependency on the parameter b. We reduce (2.93) to an ODE of first order, that is, upon setting $\tilde{u}_1 = u$ and $\tilde{u}_2 = u'$,

$$(\tilde{u}_1, \tilde{u}_2)'(x) = \left(\tilde{u}_2(x), -\frac{2(a - bx)}{\sigma^2}\tilde{u}_2(x) + \frac{2\rho}{\sigma^2}\tilde{u}_1(x)\right) = \mathcal{G}(b, x, \tilde{u}_1(x), \tilde{u}_2(x)),$$

where $\mathcal{G} : \mathbb{R}^4 \mapsto \mathbb{R}^2$ is such that

$$\mathcal{G}(b, x, y, z) = \left(z, -\frac{2(a - bx)}{\sigma^2}z + \frac{2\rho}{\sigma^2}y\right).$$

Clearly, the function $\mathcal{G}$ is continuously differentiable. Therefore, following, for example, Section 1.1 in [44], we obtain that $\psi(x; \cdot)$ is continuous at $b = 0$ for all $x \in \mathbb{R}$. This result, together with the fact that $\psi(x;0) = e^{nx}$ where $n = -\frac{a}{\sigma^2} + \sqrt{\left(\frac{a}{\sigma^2}\right)^2 + 2\frac{\rho}{\sigma^2}}$ (cf. (2.32)), yields $\psi'(x;b)/\psi(x;b) \to n$ and $\psi''(x;b)/\psi'(x;b) \to n$ as $b \downarrow 0$, which in turn implies that $x_\infty(b), x_0(b) \to x^\star$ as $b \downarrow 0$ (cf. (2.34) and Lemma 2.4.3). □

2.6 Conclusions

We have considered a price-maker company that extracts an exhaustible commodity from a reservoir, and sells it in the spot market. While extracting the commodity, the company's actions have an impact on the commodity's spot price. Then, the problem of maximizing the total expected profits from selling the commodity, net of the total expected proportional costs of extraction, has been modeled as a two-dimensional degenerate singular stochastic control problem with finite fuel which we have solved explicitly. Finally, a theoretical and numerical analysis of the dependency of the optimal extraction strategy and of the value function on the model's parameters is provided. It is then complemented with an economical interpretation.

When the (uncontrolled) price is a drifted Brownian motion, it is optimal to extract whenever the current price level exceeds an endogenously determined constant threshold. When the (uncontrolled) price evolves as an Ornstein-Uhlenbeck process, the optimal extraction rule is triggered by a curve which depends on the current level of the reservoir. This curve is a strictly decreasing function for which we have been able to provide an explicit expression. A related optimal stopping problem has given some quantitative explanations why the threshold is independent on the level of the reservoir in both the drifted Brownian motion case and a setting without price impact.

This work could be extended in many ways that are of interest. Regarding, for example, the extraction application, it would be natural to consider the costs as a function of the level of the reservoir that increases, possible to infinity, when the reservoir gets empty. The tools, we have employed in this chapter, cannot be used for such a mathematical formulation because one cannot apply the chain rule any more to find an explicit formula for F' (cf. derivation after Lemma 2.4.3). Instead, one obtains a proper ordinary differential equation for F, and then, the subsequent analysis aiming at verifying the optimality of the constructed problem's solution becomes much more involved.

Chapter 3

Universal Bounds and Monotonicity Properties of Ratios of Hermite and Parabolic Cylinder Functions[1]

3.1 Introduction

Consider the ordinary differential equation (ODE)

$$u''(x) - 2xu'(x) + 2\nu u(x) = 0, \quad \nu < 0,\, x \in \mathbb{R}. \tag{3.1}$$

Following Section 10.2 in [87], the solutions to (3.1) are called Hermite functions and denoted by H_ν. They are closely connected to parabolic cylinder functions. In fact, letting Γ be the Euler's Gamma function, the parabolic cylinder function D_ν, introduced in [131], admits the representation (cf. Section 8.3 in [54])

$$D_\nu(x) := \frac{e^{-\frac{x^2}{4}}}{\Gamma(-\nu)} \int_0^\infty t^{-\nu-1} e^{-\frac{t^2}{2} - xt} dt, \quad x \in \mathbb{R}, \tag{3.2}$$

and satisfies

$$D_\nu(x) = 2^{-\frac{\nu}{2}} e^{-\frac{x^2}{4}} H_\nu\left(\frac{x}{\sqrt{2}}\right), \quad x \in \mathbb{R}. \tag{3.3}$$

In this chapter, we study properties of the ratio $\mathcal{R}_\nu : \mathbb{R} \mapsto \mathbb{R}$, where

$$\mathcal{R}_\nu(x) := \frac{(H_{\nu-1}(x))^2}{H_\nu(x) H_{\nu-2}(x)}, \quad x \in \mathbb{R}, \tag{3.4}$$

[1]This chapter (excluding Conclusions 3.3) has been first published in *Proc. Amer. Math. Soc.*, DOI: https://doi.org/10.1090/proc/14896 (January 2020), published by the American Mathematical Society, © American Mathematical Society.

and thanks to (3.3), our results carry over to the ratio of D_ν as well. In particular, we show that $\mathcal{R}_\nu$ is strictly decreasing, and we derive its best possible upper and lower bounds.

The ratio (3.4) is closely related to the so-called Turán types inequalities. Those inequalities have been discovered in 1941 by P. Turán (published in 1950, see [126]) for Legendre Polynomials P_n, $n \in \mathbb{N}$, and for those functions they read as

$$P_{n-1}(x)P_{n+1}(x) - P_n^2(x) < 0, \quad \text{for all } x \in (-1, 1). \tag{3.5}$$

Notice that the validity of (3.5) was first proved by G. Szegö in 1948 (see [123]). Since then, inequalities of this form have attracted a lot of attention, and have been proved to be valid for other polynomials such as Hermite (obtained from Hermite functions by taking $\nu \in \mathbb{N}$), Jacobi, Laguerre or ultraspherical polynomials (see [67, 123], among others), and for special functions as (modified) Bessel, Gamma, parabolic cylinder or hypergeometric functions (see [10, 15, 16, 17, 18, 125], among many others). Applications of Turán type inequalities can be found in many fields, ranging from biophysics (see [14] and the references therein) to information theory (see [94]) and stochastic control (see [22, 60]).

Properties of ratios of special functions as in (3.4) have also gained interest in recent years. In [120], conjectures about the monotonicity of a ratio associated to exponential series sections are formulated. Those conjectures are then proved in [95, 96] for classical Kummer and Gauss hypergeometric functions, as well as for the so-called q-Kummer confluent hypergeometric and q-hypergeometric functions. Moreover, the monotonicity of a ratio like (3.4) associated to Bessel and modified Bessel functions have been studied by [121], and used for the proofs of Theorem 2.1 and Theorem 3.1 in [15]. Our focus on (3.4) is motivated by an optimal liquidation problem in a financial market (see Remark 6.8 in [22]). Lower and upper bounds for $\mathcal{R}_\nu$ have already been derived by [119], but we are able to show that our bounds are the best possible ones, and this leads to a discrepancy between the results in [119] and ours (see Remark 3.2.4).

In all the aforementioned references on Turán type inequalities (see [10, 15, 16, 17, 18, 67, 95, 96, 123, 125, 126]), the authors use purely analytic approaches to prove their results. Instead, in the next section, we follow a completely different approach that uses probabilistic arguments, and leads to a simple and short proof of our results. In particular, we exploit the relation of Hermite functions to the eigenfunctions of the infinitesimal generator of the Ornstein-Uhlenbeck process. Conclusions are then drawn in Section 3.3.

3.2 Monotonicity of Ratios of Hermite Functions

We use the link between the well known results on Ornstein-Uhlenbeck processes (presented in Appendix A) and the Hermite functions in order to study the monotonicity of the ratio $\mathcal{R}_\nu$ from (3.4).

Theorem 3.2.1. *For all $\nu < 0$, the function $\mathcal{R}_\nu$ as in* (3.4) *is strictly decreasing.*

Proof. To prove the claim, we adopt the setting from Appendix A with $\mu \in \mathbb{R}$, $\sigma > 0$ and $\kappa = 1$. In light of (3.3) and (A.5), we can identify the positive strictly increasing eigenvector of $\mathcal{L} = \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} + (\mu - x)\frac{\partial}{\partial x}$ corresponding to the eigenvalue $-\nu$ with

$$\psi(x) = H_\nu\left(\frac{\mu - x}{\sigma}\right), \quad x \in \mathbb{R}. \tag{3.6}$$

We now complete the proof in two steps. First, *Step 1* proves that the function $\Psi : \mathbb{R} \mapsto \mathbb{R}$ such that

$$\Psi(x) := \frac{\psi'(x)^2}{\psi(x)\psi''(x)}, \quad x \in \mathbb{R},$$

is strictly increasing. Then, *Step 2* makes the conclusion for $\mathcal{R}_\nu$.

Step 1. Let $x, y \in \mathbb{R}$ be such that $y > x$, and, given the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ as in Appendix A, recall X^x that evolves according to the Ornstein-Uhlenbeck dynamics as in (A.1) with $\kappa = 1$. Define the first hitting time of X^x at level y by

$$\tau_y := \inf\{t \geq 0 : X_t^x \geq y\}, \quad \mathbb{P}\text{-a.s.}$$

Direct calculations on (3.6) and the identity (cf., e.g., equation (10.4.4) in [87])

$$H_\nu'(x) = 2\nu H_{\nu-1}(x), \quad x \in \mathbb{R},\ \nu < 0, \tag{3.7}$$

show that the k-th derivative of ψ, denoted by $\psi^{(k)}$, is a strictly increasing positive solution to $(\mathcal{L} + (\nu - k))\,\psi^{(k)} = 0$. Now, since $\psi^{(k)} \in C^2(\mathbb{R})$ for any $k \in \mathbb{N}_0$, Itô's formula (together with a standard localization argument) yields, after taking expectations,

$$\mathbb{E}\left[e^{(\nu-k)\tau_y}\psi^{(k)}\big(X_{\tau_y}^x\big)\right] = \psi^{(k)}(x),$$

and hence

$$\mathbb{E}\left[e^{(\nu-k)\tau_y}\right] = \frac{\psi^{(k)}(x)}{\psi^{(k)}(y)}, \tag{3.8}$$

since $\tau_y < \infty$ $\mathbb{P}$-a.s. as X^x is positively recurrent (cf. Appendix 1.24 in [31]). Now, Hölder's inequality yields

$$\mathbb{E}\left[e^{\nu\tau_y}\right]^{\frac{1}{2}}\,\mathbb{E}\left[e^{(\nu-2)\tau_y}\right]^{\frac{1}{2}} > \mathbb{E}\left[e^{(\nu-1)\tau_y}\right], \tag{3.9}$$

which is strict since the function $f(z) := e^{\nu z}$ is not a multiple of the function $g(z) := e^{(\nu-2)z}$, and the random variable τ_y has a distribution which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. From both (3.8) with $k = 0, 1, 2$ and (3.9), we find

$$\Psi(y) > \Psi(x).$$

Since $y > x$ were arbitrary, we have that $x \mapsto \Psi(x)$ is strictly increasing.

Step 2. Exploiting the identities (3.6) and (3.7), we find that for any $x \in \mathbb{R}$

$$\Psi(x) = \frac{\nu}{\nu - 1} \mathcal{R}_\nu \left(\frac{\mu - x}{\sigma} \right).$$

Therefore, because $\nu < 0$, we conclude by *Step 1* that the function $\mathcal{R}_\nu$ is strictly decreasing. □

The following corollary gives the best possible bounds for $\mathcal{R}_\nu$. These in turn imply the Turán type inequality.

Corollary 3.2.2. *For all $\nu < 0$, the function $\mathcal{R}_\nu$ as in* (3.4) *is such that*

$$1 < \mathcal{R}_\nu(x) < \frac{\nu - 1}{\nu}, \quad \text{for all } x \in \mathbb{R}. \tag{3.10}$$

In particular, the following Turán-type inequality holds:

$$H_{\nu-1}(x)^2 - H_\nu(x) H_{\nu-2}(x) > 0, \quad x \in \mathbb{R}.$$

Proof. Equations (10.6.4) and (10.6.7) in [87] provide the asymptotic behavior of $H_\nu(x)$ for both (large) positive and (large) negative values of x. In particular, it holds that

$$\lim_{x \downarrow -\infty} \mathcal{R}_\nu(x) = \frac{\nu - 1}{\nu}, \qquad \lim_{x \uparrow +\infty} \mathcal{R}_\nu(x) = 1.$$

Thus, (3.10) follows from the strict monotonicity of $x \mapsto \mathcal{R}_\nu(x)$ proved in Theorem 3.2.1. □

Since, by (3.3), we have $\mathcal{R}_\nu \left(\frac{x}{\sqrt{2}} \right) = \frac{D_{\nu-1}(x)^2}{D_\nu(x) D_{\nu-2}(x)}$ for any $x \in \mathbb{R}$, the next proposition easily follows from Theorem 3.2.1 and Corollary 3.2.2.

Proposition 3.2.3. *For all $\nu < 0$, the function $\widetilde{\mathcal{R}}_\nu : \mathbb{R} \mapsto \mathbb{R}$ defined as*

$$\widetilde{\mathcal{R}}_\nu(x) := \frac{D_{\nu-1}(x)^2}{D_\nu(x) D_{\nu-2}(x)}, \quad x \in \mathbb{R},$$

is strictly decreasing. Moreover, it holds

$$1 < \widetilde{\mathcal{R}}_\nu(x) < \frac{\nu - 1}{\nu}, \quad \text{for all } x \in \mathbb{R}.$$

Remark 3.2.4. *It is worth mentioning that lower and upper bounds of the function $\widetilde{\mathcal{R}}_\nu$ associated to the parabolic cylinder function $U_{-\nu-\frac{1}{2}}(x) = D_\nu(x)$ (see (19.3.1) in [1]) have also been derived in [119]. In that paper, the right-hand side of equation (28) (see also Remark 1 in [18]) yields an upper bound for $\widetilde{\mathcal{R}}_\nu$ which is strictly less than the one we have obtained in Proposition 3.2.3. Given that our upper bound is optimal by the proved strict monotonicity of $\widetilde{\mathcal{R}}_\nu$, it seems that there is something fishy in equation (28) of [119]. Also, a simple numerical analysis seems to contradict the upper bound found in [119], cf. Figure 3.1 that has been obtained with MATLAB for the case $\nu = -1.5$.*

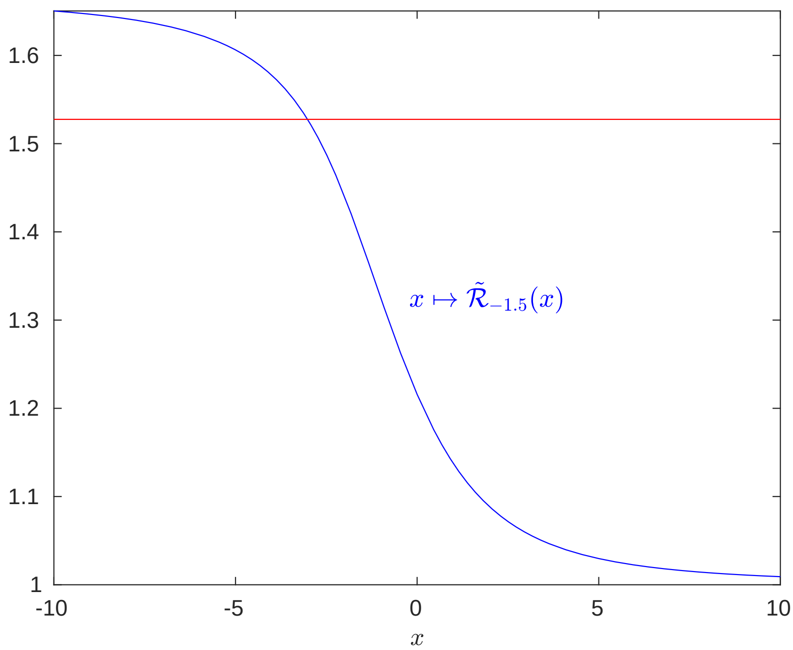


Figure 3.1: A drawing of the function $x \mapsto \widetilde{\mathcal{R}}_{-1.5}(x) = \frac{U_2(x)^2}{U_1(x)U_3(x)}$ (blue line). The red line gives the upper bound for $\widetilde{\mathcal{R}}_{-1.5}$ found in equation (28) of [119], that is $\sqrt{\frac{3.5}{1.5}} \approx 1.53$.

3.3 Conclusions

We have obtained so far unproved properties of a ratio involving a class of Hermite and parabolic cylinder functions. Those ratios are strictly decreasing and bounded by universal constants. Differently to usual analytic approaches, we have employed simple purely probabilistic arguments to derive our results. In particular, we have exploited the relation between Hermite functions (parabolic cylinder functions) and the increasing eigenfunctions of the infinitesimal generator of the Ornstein-Uhlenbeck process. As a byproduct, we have obtained Turán type inequalities.

The results of this chapter can be of interest in several fields. For instance, the ratio, studied here, appears in some problems of stochastic control when dealing with Ornstein-Uhlenbeck processes, and its properties are needed when proving the optimality of a so-called candidate strategy (see, for example, Remark 6.8 in [22], or the proof of Lemma C.2.1 (in Appendix C) that is exploited in Chapter 4 of this thesis).

Chapter 4

Optimal Installation of Solar Panels with Price Impact: a Solvable Singular Stochastic Control Problem

4.1 Introduction

We consider a price-maker company which generates electricity and sells it in the spot market. The company can increase its level of installed power by irreversible installations of solar panels. In absence of the company's economic activities, the spot electricity price evolves as an Ornstein-Uhlenbeck process, and therefore it has a mean-reverting behavior. The current level of the company's installed power has a permanent impact on the electricity price and affects its mean-reversion level. The company aims at maximizing the total expected profits from selling electricity in the market, net of the total expected proportional costs of installation. This problem is modeled as a two-dimensional degenerate singular stochastic control problem in which the installation strategy is identified as the company's control variable. We follow a *guess-and-verify approach* to solve the problem. We find that the optimal installation strategy is triggered by a curve which separates the *waiting region*, where it is not optimal to install additional panels, and the *installation region,* where it is. The curve depends on the current level of the company's installed power, and is the unique strictly increasing function which solves a first-order ordinary differential equation (ODE). Finally, our study is complemented by a numerical analysis of the dependency of the optimal installation strategy on the underlying parameters.

The present chapter is based on the article [84], and it is organized as follows. In Section 4.2 we introduce the setting and formulate the problem. In Section 4.3 we provide preliminary results and a Verification Theorem. Then, in Section 4.4 we derive an expression of the free boundary via an ODE, and an explicit solution is constructed. A connection to an optimal stopping problem is studied in Section 4.5. Finally, Section 4.6 provides a numerical implementation and studies the dependency of the free boundary with respect to the model parameters. Section 4.7 concludes.

4.2 Model and Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space with a filtration $\mathbb{F}$ satisfying the usual conditions, and carrying a standard one-dimensional $\mathbb{F}$-Brownian motion W.

We consider an infinitely-lived company which installs solar panels and sells the electricity produced by those panels instantaneously in the spot market. In absence of the company's economic activities, the fundamental electricity price $(X_t^x)_{t\geq 0}$ evolves stochastically according to an Ornstein-Uhlenbeck dynamics[1]

$$dX_t^x = \kappa\big(\mu - X_t^x\big)dt + \sigma dW_t, \quad X_0^x = x > 0, \tag{4.1}$$

for some constants $\mu \in \mathbb{R}$ and $\kappa, \sigma > 0$.

The level of installed power can be increased at constant proportional cost $c \geq 0$ due to the installation costs of solar panels. It is assumed that the firm cannot reduce the number of solar panels, and thus the installation is irreversible. The current level of installed power is described by the process $(Y_t^{y,I})_{t\geq 0}$, which is given by

$$Y_t^{y,I} = y + I_t, \tag{4.2}$$

where the initial level of installed power is denoted by $y \geq 0$, and I_t is identified as the company's control variable: it is an $\mathbb{F}$-adapted nonnegative and increasing càdlàg process $I = (I_t)_{t\geq 0}$, where I_t represents the total power installed within the interval $[0, t]$. In the following, $(I_t)_{t\geq 0}$ is also referred to as the installation strategy. Moreover, we assume that the level of installed power cannot exceed a given $\bar{y} \in [y, \infty)$ since, for example, only a finite number of solar panels can be installed. The set of admissible installation strategies is therefore defined as

$$\mathcal{I}^{\bar{y}}(y) := \{I : \Omega \times [0,\infty) \mapsto [0,\infty) : (I_t)_{t\geq 0} \text{ is } \mathbb{F}\text{-adapted, } t \mapsto I_t \text{ is increasing, càdlàg,} \\ \text{with } I_{0-} = 0 \leq I_t \leq \bar{y} - y \text{ a.s.}\}.$$

We write $\mathcal{I}^{\bar{y}}(y)$ in order to stress the dependency on both the initial level of installed power y and the maximum possible level $\bar{y}$.

We assume that the current level of electricity production, which is proportional to $Y_t^{y,I}$, affects the electricity market price. In particular, when following an installation

[1] We do not restrict our attention to positive fundamental prices, since negative electricity prices can also be observed, for example, in Germany, cf. [99].

strategy $I \in \mathcal{I}^{\bar{y}}(y)$, the mean level of the market price X is instantaneously reduced at time t by $\beta Y_t^{y,I}$, for some $\beta > 0$, and therefore the spot price $X^{x,y,I}$ evolves as

$$dX_t^{x,y,I} = \kappa\Big((\mu - \beta Y_t^{y,I}) - X_t^{x,y,I}\Big)dt + \sigma dW_t, \quad X_{0-}^{x,y,I} = x > 0. \tag{4.3}$$

The company aims at maximizing the total expected profits from selling electricity in the market, net of the total expected costs of installation. That is, the company aims at determining

$$V(x,y) := \sup_{I \in \mathcal{I}^{\bar{y}}(y)} \mathcal{J}(x,y,I), \quad (x,y) \in \mathbb{R} \times [0,\bar{y}], \tag{4.4}$$

where for any $I \in \mathcal{I}^{\bar{y}}(y)$

$$\mathcal{J}(x,y,I) := \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} X_t^{x,y,I}\left(\alpha Y_t^{y,I}\right) dt - c\int_0^\infty e^{-\rho t} dI_t\bigg], \quad \alpha > 0. \tag{4.5}$$

In (4.5), the parameter α is the proportional factor between the average electricity produced in a generic unit of time and the current level of installed power. Thus, the running gain $X_t^{x,y,I}(\alpha Y_t^{y,I})$ can be viewed as a weekly-averaged revenue deriving from solar production.

For the sake of simplicity, we set $\alpha = 1$ in the following. In fact, the problem of finding an optimal control $I \in \mathcal{I}^{\bar{y}}(y)$ in (4.5) does not change for $\alpha > 0$ upon introducing a new cost factor $\tilde{c} = \frac{c}{\alpha}$.

4.3 A Verification Theorem

The aim of this section is to provide a verification theorem which characterizes the solution to our problem.

The admissible non-installation strategy is denoted by $I^0 \equiv 0$, and we indicate the electricity price process implied by I^0 by $(X_t^{x,y})_{t \geq 0}$, that is $X_t^{x,y} \equiv X_t^{x,y,I^0}$. Then, the expected profits of the firm following the non-installation strategy is described by the function $R : \mathbb{R} \times [0,\bar{y}] \mapsto \mathbb{R}$ such that

$$R(x,y) := \mathcal{J}(x,y,I^0) = \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} X_t^{x,y} y dt\bigg] = \frac{xy}{\rho+\kappa} + \frac{\mu\kappa y}{\rho(\rho+\kappa)} - \frac{\kappa\beta y^2}{\rho(\rho+\kappa)}, \tag{4.6}$$

The following preliminary result provides a growth condition and a monotonicity property of the value function V, and its connection to the function R. The proof of the proposition can be found in Appendix C.

Proposition 4.3.1. *There exist a constant $K > 0$ such that for all $(x,y) \in \mathbb{R} \times [0,\bar{y}]$ one has*

$$|V(x,y)| \leq K\big(1 + |x|\big). \tag{4.7}$$

Moreover, $V(x,\bar{y}) = R(x,\bar{y})$, and V is increasing in x.

In a next step we derive the Hamilton-Jacobi-Bellman (HJB), a particular partial differential equation which characterizes the solution to our problem.

For given and fixed $y \geq 0$, let $\mathcal{L}^y$ be the infinitesimal generator of the diffusion $X^{x,y}$ given by the second order differential operator

$$\mathcal{L}^y u(x,y) := \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} u(x,y) + \kappa\Big((\mu - \beta y) - x\Big)\frac{\partial}{\partial x} u(x,y), \tag{4.8}$$

where $u(\cdot, y) \in C^2(\mathbb{R})$.

The HJB equation, for singular control problems as this one, follows this heuristic argument. At time zero, the firm has two possible options: either it waits for a short time period Δt, in which the firm does not install additional panels and gains running profits from selling y units of electricity in the market, or it can install solar panels immediately in order to increase its level of installed power. After each of these actions the firm behaves optimally. Suppose that the firm follows the first action. Since this action is not necessarily optimal, it is associated to the inequality

$$V(x,y) \geq \mathbb{E}\bigg[\int_0^{\Delta t} e^{-\rho s} X_s^{x,y} y ds + e^{-\rho \Delta t} V(X_{\Delta t}^{x,y}, y)\bigg], \quad (x,y) \in \mathbb{R} \times [0, \bar{y}). \tag{4.9}$$

Employing Itô's formula to the last term of the right-hand side of (4.9), dividing by Δt, and then letting $\Delta t \to 0$, we obtain

$$\mathcal{L}^y V(x,y) - \rho V(x,y) + xy \leq 0, \quad (x,y) \in \mathbb{R} \times [0, \bar{y}).$$

Now, suppose the firm follows the second option, i.e. to increase its level of installed power by $\varepsilon > 0$ units and then to continue optimally. This action is associated to

$$V(x,y) \geq V(x, y + \varepsilon) - c\varepsilon,$$

which in turn, by dividing by ε and letting $\varepsilon \downarrow 0$, implies

$$V_y(x,y) - c \leq 0.$$

The previous observations suggest that V should identify with an appropriate solution w to the HJB equation

$$\max\Big\{\mathcal{L}^y w(x,y) - \rho w(x,y) + xy, w_y(x,y) - c\Big\} = 0, \quad (x,y) \in \mathbb{R} \times [0, \bar{y}), \tag{4.10}$$

with boundary condition

$$w(x, \bar{y}) = R(x, \bar{y}).$$

With reference to (4.10), we introduce the *waiting region*

$$\mathbb{W} := \{(x,y) \in \mathbb{R} \times [0, \bar{y}) : \mathcal{L}^y w(x,y) - \rho w(x,y) + xy = 0,\ w_y(x,y) - c < 0\}, \tag{4.11}$$

where we expect it not to be optimal to install additional solar panels, and the *installation region*

$$\mathbb{I} := \{(x, y) \in \mathbb{R} \times [0, \bar{y}) : \mathcal{L}^y w(x, y) - \rho w(x, y) + xy \leq 0,\ w_y(x, y) - c = 0\}, \quad (4.12)$$

where we expect it to be.

We move on by proving a Verification Theorem. It shows that an appropriate solution to the HJB equation (4.10) identifies with the value function, if an admissible installation strategy exists which keeps the state process (X, Y) inside the waiting region $\overline{\mathbb{W}}$ with minimal effort, i.e. the level of installed power is increased only at the time when (X, Y) enters the installation region $\mathbb{I}$. Here, we have denoted by $\overline{\mathbb{W}}$ the closure of $\mathbb{W}$.

Theorem 4.3.2 (Verification Theorem)**.** *Suppose there exists a function* $w : \mathbb{R} \times [0, \bar{y}] \mapsto \mathbb{R}$ *such that* $w \in C^{2,1}(\mathbb{R} \times [0, \bar{y}])$ *solves the HJB equation* (4.10) *with boundary condition* $w(x, \bar{y}) = R(x, \bar{y})$*, and satisfies the growth condition*

$$|w(x, y)| \leq K\big(1 + |x|\big), \quad (4.13)$$

for a constant $K > 0$*. Then* $w \geq v$ *on* $\mathbb{R} \times [0, \bar{y}]$*.*
Moreover, suppose that for all initial values $(x, y) \in \mathbb{R} \times [0, \bar{y})$*, there exists a process* $I^\star \in \mathcal{I}^{\bar{y}}(y)$ *such that*

$$(X_t^{x,y,I^\star}, Y_t^{y,I^\star}) \in \overline{\mathbb{W}}, \quad \text{for all } t \geq 0,\ \mathbb{P}\text{-a.s.}, \quad (4.14)$$

$$I_t^\star = \int_{0-}^t \mathbb{1}_{\{(X_s^{x,y,I^\star}, Y_s^{y,I^\star}) \in \mathbb{I}\}} dI_s^\star, \quad \text{for all } t \geq 0,\ \mathbb{P}\text{-a.s.} \quad (4.15)$$

Then we have

$$V(x, y) = w(x, y), \quad (x, y) \in \mathbb{R} \times [0, \bar{y}],$$

and $I^\star$ *is optimal; that is,* $V(x, y) = \mathcal{J}(x, y, I^\star)$*.*

Proof. Since we have $w(x, \bar{y}) = R(x, \bar{y}) = V(x, \bar{y})$ by assumption, we let $y < \bar{y}$. In a first step, we prove that $w \geq v$ on $\mathbb{R} \times [0, \bar{y})$, and then in a second step, we show that $w \leq v$ on $\mathbb{R} \times [0, \bar{y})$ and the optimality of $I^\star$ satisfying (4.14) and (4.15).

Step 1. Let $(x, y) \in \mathbb{R} \times [0, \bar{y})$ be given and fixed, and $I \in \mathcal{I}^{\bar{y}}(y)$. For $N > 0$ we set $\tau_{R,N} := \tau_R \wedge N$, where $\tau_R := \inf\{s > 0 : X_s^{x,y,I} \notin (-R, R)\}$. In the following, we write $\Delta I_s := I_s - I_{s-}$, $s \geq 0$, and I^c denotes the continuous part of $I \in \mathcal{I}^{\bar{y}}(y)$. By an application of Itô's formula, we have

$$
\begin{aligned}
& e^{-\rho\tau_{R,N}} w(X^{x,y,I}_{\tau_{R,N}}, Y^{y,I}_{\tau_{R,N}}) - w(x,y) \\
&= \int_0^{\tau_{R,N}} e^{-\rho s}\Big(\mathcal{L}^y w(X^{x,y,I}_s, Y^{y,I}_s) - \rho w(X^{x,y,I}_s, Y^{y,I}_s)\Big)ds + \underbrace{\sigma \int_0^{\tau_{R,N}} e^{-\rho s} w_x(X^{x,y,I}_s, Y^{y,I}_s) dW_s}_{=:M_{\tau_{R,N}}} \\
&\quad + \sum_{0\le s\le \tau_{R,N}} e^{-\rho s}\big[w(X^{x,y,I}_s, Y^{y,I}_s) - w(X^{x,y,I}_s, Y^{y,I}_{s-})\big] + \int_0^{\tau_{R,N}} e^{-\rho s} w_y(X^{x,y,I}_s, Y^{y,I}_s) dI^c_s,
\end{aligned}
\tag{4.16}
$$

upon noticing that $t \mapsto X^{x,y,I}_t$ is continuous almost surely for any $I \in \mathcal{I}^{\bar{y}}(y)$. Now, we find

$$
\begin{aligned}
w(X^{x,y,I}_s, Y^{y,I}_s) - w(X^{x,y,I}_s, Y^{y,I}_{s-}) &= w(X^{x,y,I}_s, Y^{y,I}_{s-} + \Delta I_s) - w(X^{x,y,I}_s, Y^{y,I}_{s-}) \\
&= \int_0^{\Delta I_s} w_y(X^{x,y,I}_s, Y^{y,I}_{s-} + u) du,
\end{aligned}
$$

which substituted back into (4.16) gives the equivalence

$$
\begin{aligned}
& \int_0^{\tau_{R,N}} e^{-\rho s} X^{x,y,I}_s Y^{y,I}_s ds - c \int_0^{\tau_{R,N}} e^{-\rho s} dI_s \\
&= \int_0^{\tau_{R,N}} e^{-\rho s}\Big(\mathcal{L}^y w(X^{x,y,I}_s, Y^{y,I}_s) - \rho w(X^{x,y,I}_s, Y^{y,I}_s) + X^{x,y,I}_s Y^{y,I}_s\Big)ds + M_{\tau_{R,N}} \\
&\quad + \sum_{0\le s\le \tau_{R,N}} e^{-\rho s} \int_0^{\Delta I_s} \big[w_y(X^{x,y,I}_s, Y^{y,I}_{s-} + u) - c\big] du \\
&\quad + \int_0^{\tau_{R,N}} e^{-\rho s}\big[w_y(X^{x,y,I}_s, Y^{y,I}_s) - c\big] dI^c_s + w(x,y) - e^{-\rho\tau_{R,N}} w(X^{x,y,I}_{\tau_{R,N}}, Y^{y,I}_{\tau_{R,N}}),
\end{aligned}
$$

upon adding $\int_0^{\tau_{R,N}} e^{-\rho s} X^{x,y,I}_s Y^{y,I}_s ds - c \int_0^{\tau_{R,N}} e^{-\rho s} dI_s$ on both sides of (4.16). Since w satisfies (4.10) and (4.13), by taking expectations on both sides of the latter equation, and using that $\mathbb{E}[M_{\tau_{R,N}}] = 0$, we have

$$
\mathbb{E}\Big[\int_0^{\tau_{R,N}} e^{-\rho s} X^{x,y,I}_s Y^{y,I}_s ds - c \int_0^{\tau_{R,N}} e^{-\rho s} dI_s\Big] \le w(x,y) + K\mathbb{E}\Big[e^{-\rho\tau_{R,N}}\Big(1 + |X^{x,y,I}_{\tau_{R,N}}|\Big)\Big].
\tag{4.17}
$$

In order to apply the dominated convergence theorem in (4.17), we notice on the one hand that $X^{x,y,I}_t \le X^x_t$ $\mathbb{P}$-a.s. for all $t \ge 0$, and therefore that

$$
\begin{aligned}
X^{x,y,I}_t &= x + \int_0^t \kappa\big((\mu - \beta Y^{y,I}_t) - X^{x,y,I}_s\big)ds + \sigma W_t \\
&\ge x + \int_0^t \kappa\big(\mu - X^x_s\big)ds + \sigma W_t - \kappa\beta\bar{y}t = X^x_t - \kappa\beta\bar{y}t \ge -|X^x_t| - \kappa\beta\bar{y}t,
\end{aligned}
$$

where we have used that $Y_t^{y,I} \leq \bar{y}$ $\mathbb{P}$-a.s. for all $t \geq 0$. Also, one clearly has $X_t^{x,y,I} \leq X_t^x \leq |X_t^x| + \kappa\beta\bar{y}t$. Hence,

$$|X_t^{x,y,I}| \leq |X_t^x| + \kappa\beta\bar{y}t. \tag{4.18}$$

Now, we find that $\mathbb{P}$-a.s.

$$\Big| \int_0^{\tau_{R,N}} e^{-\rho s} X_s^{x,y,I} Y_s^{y,I} ds - c \int_0^{\tau_{R,N}} e^{-\rho s} dI_s \Big| \leq \bar{y} \int_0^{\infty} e^{-\rho s} \Big(|X_s^x| + \kappa\beta\bar{y}s \Big) ds + c\bar{y}, \tag{4.19}$$

and the first expression on the right-hand side of (4.19) is integrable by Lemma A.1.1 with $q = 1$. On the other hand, so to take care of the expectation on the right-hand side of (4.17), we employ again (4.18) to get for some constant $C_1 > 0$

$$\begin{aligned} \mathbb{E}\Big[e^{-\rho\tau_{R,N}}(1 + |X_{\tau_{R,N}}^{x,y,I}|)\Big] &\leq C_1 \mathbb{E}\Big[e^{-\rho\tau_{R,N}}\,(1 + \tau_{R,N})\Big] + \mathbb{E}\Big[e^{-\frac{\rho}{2}\tau_{R,N}} \sup_{t\geq 0} e^{-\frac{\rho}{2}t}|X_t^x|\Big] \\ &\leq C_1 \mathbb{E}\Big[e^{-\rho\tau_{R,N}}\,(1 + \tau_{R,N})\Big] + \mathbb{E}\Big[e^{-\rho\tau_{R,N}}\Big]^{\frac{1}{2}} \mathbb{E}\Big[\sup_{t\geq 0} e^{-\rho t}(X_t^x)^2\Big]^{\frac{1}{2}}, \end{aligned} \tag{4.20}$$

where we have used Hölder's inequality in the last step. As for the last expectation in (4.20), observe that by Itô's formula we find

$$\begin{aligned} e^{-\rho t}(X_t^x)^2 \leq x^2 &+ \int_0^t e^{-\rho u}\Big[\rho(X_u^x)^2 + \sigma^2\Big] du \\ &+ \int_0^t 2e^{-\rho u}|X_u^x|(\kappa(|\mu| + |X_u^x|))du + 2\sigma \sup_{t\geq 0}\Big|\int_0^t e^{-\rho u} X_u^x dW_u\Big|. \end{aligned} \tag{4.21}$$

By an application of the Burkholder-Davis-Gundy inequality (cf. Theorem 3.28 in [81]), we find that

$$\mathbb{E}\Big[\sup_{t\geq 0}\Big|\int_0^t e^{-\rho u}\sigma X_u^x dW_u\Big|\Big] \leq C_2(1 + |x|), \tag{4.22}$$

for some constant $C_2 > 0$. Then, exploiting Lemma A.1.1, we obtain from (4.21) and (4.22)

$$\mathbb{E}\Big[\sup_{t\geq 0} e^{-\rho t}(X_t^x)^2\Big] \leq C_3(1 + x^2), \tag{4.23}$$

for some constant $C_3 > 0$, and therefore, it follows with (4.20)

$$\lim_{N\uparrow\infty}\lim_{R\uparrow\infty} \mathbb{E}\Big[e^{-\rho\tau_{R,N}}(1 + |X_{\tau_{R,N}}^{x,y,I}|)\Big] = 0. \tag{4.24}$$

Hence, we can invoke the dominated convergence theorem in order to take limits as $R \to \infty$ and then as $N \to \infty$, so to get

$$\mathcal{J}(x, y, I) \leq w(x, y). \tag{4.25}$$

Since $I \in \mathcal{I}^{\bar{y}}(y)$ is arbitrary, we have

$$V(x,y) \leq w(x,y), \tag{4.26}$$

which yields $V \leq w$ by arbitrariness of (x,y) in $\mathbb{R} \times [0,\bar{y})$.

Step 2. Let $I^\star \in \mathcal{I}^{\bar{y}}(y)$ satisfying (4.14) and (4.15), and $\tau^\star_{R,N} := \inf\{t \geq 0 : X^{x,y,I^\star}_t \notin (-R,R)\} \wedge N$. Arguing in the same way as in *Step 1* all the inequalities become equalities and we obtain

$$\mathbb{E}\Big[\int_0^{\tau_{R,N}} e^{-\rho s} X^{x,y,I^\star}_s Y^{y,I^\star}_s ds - c\int_0^{\tau_{R,N}} e^{-\rho s} dI^\star_s\Big] + \mathbb{E}\Big[e^{-\rho\tau^\star_{R,N}} w(X^{x,y,I^\star}_{\tau^\star_{R,N}}, I^\star_{\tau^\star_{R,N}})\Big] = w(x,y). \tag{4.27}$$

Now, because $I^\star$ is admissible and upon employing (4.13) and (4.24), we proceed as in *Step 1* , and take limits as $R \uparrow \infty$ and $N \uparrow \infty$ in (4.27), so to find $\mathcal{J}(x,y,I^\star) \geq w(x,y)$. Since clearly $V(x,y) \geq \mathcal{J}(x,y,I^\star)$, then $V(x,y) \geq w(x,y)$ for all $(x,y) \in \mathbb{R} \times [0,\bar{y})$. Hence, using (4.26) $V = w$ on $\mathbb{R} \times [0,\bar{y})$ and $I^\star$ is optimal. □

4.4 Constructing an Optimal Solution to the Installation Problem

In this section, we first construct a candidate value function, and a candidate optimal strategy. Then, we move on by verifying their optimality.

We make the guess that there exists an injective function $F : [0,\bar{y}] \to \mathbb{R}$, called the *free boundary* which separates the *waiting region* $\mathbb{W}$ and the *installation region* $\mathbb{I}$, such that

$$\mathbb{W} = \{(x,y) \in \mathbb{R} \times [0,\bar{y}) : \, x < F(y)\}, \tag{4.28}$$

$$\mathbb{I} = \{(x,y) \in \mathbb{R} \times [0,\bar{y}) : \, x \geq F(y)\}. \tag{4.29}$$

For all $(x,y) \in \mathbb{W}$, the candidate value function w should satisfy (cf. (4.11))

$$\mathcal{L}^y w(x,y) - \rho w(x,y) + xy = 0. \tag{4.30}$$

Recall (4.6). It is straightforward to check that a particular solution to (4.30) is given by the function R. Moreover, it is well known that, for $y \geq 0$ be given and fixed, the homogeneous differential equation

$$\mathcal{L}^y w(x,y) - \rho w(x,y) = 0, \tag{4.31}$$

admits two fundamental strictly positive solutions $\phi(\,\cdot\,;y)$ and $\psi(\,\cdot\,;y)$ with $\phi(\,\cdot\,;y)$ being strictly decreasing and $\psi(\,\cdot\,;y)$ being strictly increasing. Therefore, our candidate value function w takes the form

$$w(x,y) = A(y)\psi(x;y) + B(y)\phi(x;y) + R(x,y), \quad (x,y) \in \mathbb{W}, \tag{4.32}$$

for some functions $A, B : [0, \bar{y}] \mapsto \mathbb{R}$ to be found. Notice that $\phi(x; y)$ grows to $+\infty$ exponentially fast whenever $x \downarrow -\infty$, see Appendix 1 in [31]. In light of both the linear growth of V (cf. Proposition 4.3.1) and the structure of the waiting region $\mathbb{W}$ (cf. (4.28)), we must have $B(y) = 0$ for all $y \in [0, \bar{y}]$. Moreover, letting $\psi(x) \equiv \psi(x; 0)$, we find from Lemma A.1.2-(2) that $\psi^{(k)}$, $k \in \mathbb{N}_0$, satisfies

$$\frac{\sigma^2}{2}\psi^{(k+2)}(x+\beta y) + \kappa\big((\mu - \beta y) - x\big)\psi^{(k+1)}(x+\beta y) - (\rho + k\kappa)\psi^{(k)}(x+\beta y) = 0. \tag{4.33}$$

Thus, we can identify $\psi(x; y) = \psi(x + \beta y)$. Given the previous results, we conjecture that

$$w(x, y) = A(y)\psi\left(x + \beta y\right) + R(x, y), \quad \text{for } (x, y) \in \mathbb{W}. \tag{4.34}$$

We move on to derive equations that characterize the function A and the free boundary F. With reference to (4.12), for all $(x, y) \in \mathbb{I}$, w should instead satisfy

$$w_y(x, y) - c = 0, \tag{4.35}$$

implying

$$w_{yx}(x, y) = 0. \tag{4.36}$$

Now, we impose the so-called *Smooth Fit* condition, i.e. we suppose that $w \in C^{2,1}(\mathbb{R} \times [0, \bar{y}])$, and therefore by (4.34),(4.35) and (4.36), we must have for all $(x, y) \in \overline{\mathbb{W}} \cap \mathbb{I}$ (that is, where $x = F(y)$)

$$A'(y)\psi\big(F(y) + \beta y\big) + \beta A(y)\psi'\big(F(y) + \beta y\big) + R_y(F(y), y) - c = 0, \tag{4.37}$$

$$A'(y)\psi'\big(F(y) + \beta y\big) + \beta A(y)\psi''\big(F(y) + \beta y\big) + R_{yx}(F(y), y) = 0. \tag{4.38}$$

Notice that the derivatives of R can be easily obtained from (4.6), which gives

$$R_y(x, y) = \frac{x}{\rho + \kappa} + \frac{\mu\kappa}{\rho(\rho + \kappa)} - \frac{2\kappa\beta y}{\rho(\rho + \kappa)}, \quad \text{and} \quad R_{xy}(x, y) = (\rho + \kappa)^{-1}.$$

The following lemma provides essential properties of the function A and a lower bound for F that are needed for results of Section 4.4.1 and Section 4.4.2. Its proof can be found in Appendix C.

Lemma 4.4.1. *The function A is strictly positive and strictly decreasing. Moreover, A admits the representation*

$$A(y) = \left(\beta\rho(\rho + \kappa)\right)^{-1} \times \frac{(\rho + \kappa)\Big(c\rho + \frac{\kappa\beta}{\rho + \kappa}y - F(y)\Big)\psi'(F(y) + \beta y) + \frac{\sigma^2}{2}\psi''(F(y) + \beta y)}{\psi'(F(y) + \beta y)^2 - \psi''(F(y) + \beta y)\psi(F(y) + \beta y)}, \tag{4.39}$$

and we have

$$F(y) \geq c\rho + \frac{\kappa\beta}{\rho + \kappa}y \geq c\rho, \quad \textit{for all } y \in [0, \bar{y}]. \tag{4.40}$$

4.4.1 The Free Boundary: Existence and Characterization

For the sake of simplicity, we introduce a function $\tilde{F}$ for a substitution, that is, we let

$$\tilde{F}(y) = F(y) + \beta y. \tag{4.41}$$

We aim to prove the existence and a monotonicity property of $\tilde{F}$, satisfying (4.37) and (4.38) (with F being replaced according to (4.41)), so to draw the implications for F after.

We have

$$R_y(F(y), y) = \frac{\rho F(y) + \mu\kappa - 2\kappa\beta y}{\rho(\rho+\kappa)} = \frac{\mu\kappa + \rho\tilde{F}(y) - \beta(\rho+2\kappa)y}{\rho(\rho+\kappa)} = \tilde{R}(\tilde{F}(y), y),$$

where $\tilde{R} : \mathbb{R}^2 \mapsto \mathbb{R}$ is defined as

$$\tilde{R}(x, y) := \frac{\mu\kappa + \rho x - \beta(\rho+2\kappa)y}{\rho(\rho+\kappa)}.$$

Notice that

$$\tilde{R}_x(\tilde{F}(y), y) = (\rho+\kappa)^{-1} = R_{yx}(F(y), y).$$

From now on, we will often use the functions $Q_k : \mathbb{R} \mapsto \mathbb{R}$, $k \in \mathbb{N}_0$, and their first derivatives, given by

$$\begin{aligned} Q_k(z) &:= \psi^{(k)}(z)\psi^{(k+2)}(z) - \psi^{(k+1)}(z)^2, \\ Q_k'(z) &= \psi^{(k)}(z)\psi^{(k+3)}(z) - \psi^{(k+1)}(z)\psi^{(k+2)}(z). \end{aligned} \tag{4.42}$$

Substituting F according to (4.41) in both (4.37) and (4.38), and solving for A and A', gives

$$A(y) = \beta^{-1} \times \frac{\psi'(\tilde{F}(y))\Big(c - \tilde{R}(\tilde{F}(y), y)\Big) + (\rho+\kappa)^{-1}\psi(\tilde{F}(y))}{-Q_0(\tilde{F}(y))}, \tag{4.43}$$

and

$$A'(y) = \frac{\psi''(\tilde{F}(y))\Big(c - \tilde{R}(\tilde{F}(y), y)\Big) + (\rho+\kappa)^{-1}\psi'(\tilde{F}(y))}{Q_0(\tilde{F}(y))}. \tag{4.44}$$

Lemma A.1.2-(3) ensures that Q_k is strictly positive for all $k \in \mathbb{N}_0$, and therefore the denominator on the right-hand side of both (4.43) and (4.44) is nonzero.

In light of the boundary condition $w(x, \bar{y}) = R(x, \bar{y})$ (cf. Theorem 4.3.2), we impose

$$A(\bar{y}) = 0. \tag{4.45}$$

Due to (4.43) and (4.45), we must have that there exists a point $\tilde{x} = \tilde{F}(\bar{y}) \in \mathbb{R}$ solving $H(x) = 0$, where $H : \mathbb{R} \mapsto \mathbb{R}$ is defined as

$$H(x) := \psi'(x)\Big(c - \tilde{R}(x, \bar{y})\Big) + (\rho+\kappa)^{-1}\psi(x). \tag{4.46}$$

Lemma 4.4.2. *There exists a unique solution $\tilde{x} \in \mathbb{R}$ to the equation $H(x) = 0$.*

Proof. We rewrite $H(x) := -(\rho+\kappa)^{-1}\Big(\psi'(x)\Big((\rho+\kappa)\tilde{R}(x,\bar{y}) - c(\rho+\kappa)\Big) - \psi(x)\Big)$. Now, the proof is a slight modification of the proof of Lemma 2.4.3 in Chapter 2 upon adjusting the cost factor by $c(\rho+\kappa) - \frac{\mu\kappa - \beta(\rho+2\kappa)\bar{y}}{\rho}$. □

Differentiating (4.43), we find

$$A'(y) = (\beta(\rho+\kappa))^{-1} \times \frac{P(y, \tilde{F}(y), \tilde{F}'(y))}{Q_0(\tilde{F}(y))^2}, \tag{4.47}$$

where $P : \mathbb{R}^3 \mapsto \mathbb{R}$ is given by

$$\begin{aligned} P(y,z,w) :=& w(\rho+\kappa)\Big(c - \tilde{R}(z,y)\Big)\psi(z)\Big(\psi'''(z)\psi'(z) - \psi''(z)^2\Big) \\ &+ \frac{\beta(\rho+2\kappa)}{\rho}\psi'(z)\Big(\psi'(z)^2 - \psi(z)\psi''(z)\Big) - w\psi(z)\Big(\psi'(z)\psi''(z) - \psi(z)\psi'''(z)\Big) \\ =& -\frac{\beta(\rho+2\kappa)}{\rho}\psi'(z)Q_0(z) + wD(y,z), \end{aligned}$$

with $D : \mathbb{R}^2 \mapsto \mathbb{R}$ defined as

$$D(y,z) = \psi(z)\Big[(\rho+\kappa)(c - \tilde{R}(z,y))Q_1(z) + Q_0'(z)\Big]. \tag{4.48}$$

Now, equating both expressions (4.44) and (4.47), we get

$$P(y, \tilde{F}(y), \tilde{F}'(y)) = \beta Q_0(\tilde{F}(y))\Big((\rho+\kappa)\left(c - \tilde{R}(\tilde{F}(y), y)\right)\psi''(\tilde{F}(y)) + \psi'(\tilde{F}(y))\Big). \tag{4.49}$$

Letting $N : \mathbb{R}^2 \mapsto \mathbb{R}$ be such that

$$N(y,z) = Q_0(z)\left(\frac{\rho+2\kappa}{\rho}\psi'(z) + \Big((\rho+\kappa)\left(c - \tilde{R}(z,y)\right)\psi''(z) + \psi'(z)\Big)\right), \tag{4.50}$$

we obtain from (4.49) the ODE

$$\tilde{F}'(y) = \mathcal{G}(y, \tilde{F}(y)), \tag{4.51}$$

with boundary condition $\tilde{F}(\bar{y}) = \tilde{x}$, cf. Lemma 4.4.2, and where $\mathcal{G} : (\mathbb{R}\times\mathbb{R}) \setminus \{(y,z) \in \mathbb{R}^2 : D(y,z) = 0\} \mapsto \mathbb{R}$ is such that

$$\mathcal{G}(y,z) = \beta \times \frac{N(y,z)}{D(y,z)}. \tag{4.52}$$

The next goal is to prove that the ODE (4.51) admits a unique solution $\tilde{F}$ on $[0,\bar{y}]$ such that $\tilde{F}'(y) \geq \beta$, so to obtain the existence and uniqueness of a strictly increasing free boundary F on $[0,\bar{y}]$ (cf. (4.41)). As a preliminary result we show that the monotonicity property holds at $\bar{y}$, that is $\mathcal{G}(\bar{y},\tilde{x}) > \beta$.

Lemma 4.4.3. *We have $D(\bar{y}, \tilde{F}(\bar{y})) > 0$, and it holds $\tilde{F}'(\bar{y}) > \beta$.*

Proof. Recall the function H from (4.46) which is such that $H(\tilde{F}(\bar{y})) = H(\tilde{x}) = 0$ (cf. Lemma 4.4.2). Therefore, $\bar{y}$ satisfies

$$(\rho + \kappa)\left(c - \tilde{R}(\tilde{F}(\bar{y}), \bar{y})\right) = -\frac{\psi(\tilde{F}(\bar{y}))}{\psi'(\tilde{F}(\bar{y}))}. \tag{4.53}$$

We get from (4.48) and (4.53) that

$$D(\bar{y}, \tilde{F}(\bar{y})) = \frac{Q_0(\tilde{F}(\bar{y}))\psi(\tilde{F}(\bar{y}))\psi''(\tilde{F}(\bar{y}))}{\psi'(\tilde{F}(\bar{y}))} > 0, \tag{4.54}$$

upon recalling that $Q_0 > 0$. Now, Lemma C.2.1 implies $N(\bar{y}, \tilde{F}(\bar{y})) - D(\bar{y}, \tilde{F}(\bar{y})) > 0$. Hence, we find

$$\tilde{F}'(\bar{y}) = \mathcal{G}(\bar{y}, \tilde{F}(\bar{y})) = \beta \times \frac{N(\bar{y}, \tilde{F}(\bar{y}))}{D(\bar{y}, \tilde{F}(\bar{y}))} > \beta. \tag{4.55}$$

□

Now, we state the main result in this subsection. It guarantees the existence and uniqueness of a solution $\tilde{F}$ on $[0, \bar{y}]$ of (4.51) which is such that $\tilde{F}'(y) > \beta$ for all $y \in [0, \bar{y}]$.

Proposition 4.4.4. *There exists a unique solution $\tilde{F}$ on $[0, \bar{y}]$ of the ODE (4.51) with boundary condition $\tilde{F}(\bar{y}) = \tilde{x}$. Moreover, $\tilde{F}'(y) \geq \beta$ for all $y \in [0, \bar{y}]$.*

Proof. The proof is organised in two steps: in a first step, we provide a representation of the function D that is used after. Then, in *Step 2*, we show the existence and uniqueness of a strictly increasing maximal solution $\tilde{F}$ of the ODE (4.51), and prove (by a contradiction) that $\tilde{F}$ in fact exists on the interval $[0, \bar{y}]$.

Step 1. Recall (4.48), and let $\tilde{D} : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be a function which is given by

$$\tilde{D}(y, z) = [(\rho + \kappa)\psi(z)Q_0(z)]^{-1} D(y, z). \tag{4.56}$$

Then, where $\tilde{F}$ exists, we find upon employing (4.43) and (4.44)

$$\tilde{D}(y, \tilde{F}(y)) = -\beta\psi'''(\tilde{F}(y))A(y) - \psi''(\tilde{F}(y))A'(y). \tag{4.57}$$

Now, Lemma A.1.2-(2) gives for any $k \in \mathbb{N}_0$

$$\frac{\sigma^2}{2}\psi^{(k+2)}(x) + \kappa(\mu - x)\psi^{(k+1)}(x) - (\rho + k\kappa)\psi^{(k)}(x) = 0, \quad x \in \mathbb{R}, \tag{4.58}$$

and therefore we have

$$\psi^{(k+2)}(\tilde{F}(y)) = -\frac{2\kappa}{\sigma^2}\left(\mu - \tilde{F}(y)\right)\psi^{(k+1)}(\tilde{F}(y)) + \frac{2(\rho + k\kappa)}{\sigma^2}\psi^{(k)}(\tilde{F}(y)). \tag{4.59}$$

Using (4.57) and the latter equation (4.59) with $k = 0, 1$, we obtain

$$\begin{aligned}\tilde{D}(y, \tilde{F}(y)) =& \frac{2}{\sigma^2}\Big[\kappa\left(\mu - \tilde{F}(y)\right)\left(\beta\psi''(\tilde{F}(y))A(y) + \psi'(\tilde{F}(y))A'(y)\right) \\ &- \rho\big(\beta\psi'(\tilde{F}(y))A(y) + \psi(\tilde{F}(y))A'(y)\big) - \kappa\beta\psi'(\tilde{F}(y))A(y)\Big] \\ =& \frac{2}{\sigma^2}\Big[\tilde{F}(y) - c\rho - \frac{(\rho + 2\kappa)\beta}{\rho + \kappa}y - \kappa\beta\psi'(\tilde{F}(y))A(y)\Big],\end{aligned} \tag{4.60}$$

where we have employed (4.37) and (4.38) (with F being replaced according to (4.41)) for the last equality.

Step 2. Recall (4.51) and (4.52). In the following, we denote by $\mathcal{D}_\mathcal{G}$ the domain of $\mathcal{G}$, that is $\mathcal{D}_\mathcal{G} = (\mathbb{R} \times \mathbb{R}) \setminus \{(y, z) \in \mathbb{R}^2 : D(y, z) = 0\}$. Since $\psi^{(k)}$ is continuously differentiable for any $k \in \mathbb{N}$, the functions N and D are continuously differentiable respectively. Therefore, $\mathcal{G}(y, \cdot)$ is locally Lipschitz-continuous on its domain $\mathcal{D}_\mathcal{G}$ which is an open set. Hence, we find that the ODE (4.51) with the boundary condition $\tilde{F}(\bar{y}) = \tilde{x}$ admits a unique maximal solution $\tilde{F}$ on an interval $I_{\max} = (y_-, y_+)$ with $\bar{y} \in I_{\max}$. Since we want to show the existence and uniqueness of a solution on $[0, \bar{y}]$, it is enough to prove that $y_- < 0$. Following, for example, Theorem 2.10 in [13], $y_- < \bar{y}$ is such that

(i) either $\lim_{y \downarrow y_-} \left(||(y, \tilde{F}(y))||\right)^{-1} = 0$,

(ii) or $\lim_{y \downarrow y_-} \inf_{w \in \partial\mathcal{D}_\mathcal{G}} ||(y, \tilde{F}(y)) - w|| = 0$,

where $\partial\mathcal{D}_\mathcal{G} = \{(y, z) \in \mathbb{R}^2 : D(y, z) = 0\}$ is the boundary of the domain of $\mathcal{G}$, and $||\cdot||$ is a norm in $\mathbb{R}^2$.

Now, suppose that $y_- \geq 0$. Notice that $N(y, \tilde{F}(y)) > D(y, \tilde{F}(y)) > 0$ for all $y \in I_{\max}$ by Lemma 4.4.3 and Lemma C.2.1, and therefore we have $\tilde{F}' > \beta > 0$ on $I_{\max}$. Adjusting slightly the proof of Lemma 4.4.1, we find that $\tilde{F}$ is bounded from below on $(y_-, \bar{y}]$, and together with its monotonicity property, we must have that $\lim_{y \downarrow y_-} \left(||(y, \tilde{F}(y))||\right)^{-1} > K$, for some $K > 0$. Thus, in order to derive a contradiction, it is left to prove that condition (ii) above is not satisfied, so to show $\lim_{y \downarrow y_-} D(y, \tilde{F}(y)) \neq 0$. Again, due to the boundedness of $\tilde{F}$ and the fact that both Q_0 and ψ are strictly positive, we find

$$\psi(\tilde{F}(y))Q_0(\tilde{F}(y)) > K_1, \quad \text{for all } y \in (y_-, \bar{y}],$$

for some $K_1 > 0$. Therefore, upon recalling (4.56), we can complete the proof by showing that $\lim_{y \downarrow y_-} \tilde{D}(y, \tilde{F}(y)) \neq 0$. Lemma 4.4.3 implies

$$\tilde{D}(\bar{y}, \tilde{F}(\bar{y})) > 0. \tag{4.61}$$

Computing the total derivative of $\tilde{D}(y, \tilde{F}(y))$ with respect to $y \in I_{\max}$, upon using (4.60), gives

$$\begin{aligned}\frac{d}{dy}\tilde{D}(y, \tilde{F}(y)) &= \frac{2}{\sigma^2}\left[\tilde{F}'(y)\left(1 - \kappa\beta\psi''(\tilde{F}(y))A(y)\right) - \frac{(\rho+2\kappa)\beta}{\rho+\kappa} - \kappa\beta\psi'(\tilde{F}(y))A'(y)\right] \\ &= \frac{2}{\sigma^2}\left(\tilde{F}'(y) - \beta\right)\left(1 - \kappa\beta\psi''(\tilde{F}(y))A(y)\right),\end{aligned} \tag{4.62}$$

where the last equality holds by an application of (4.38) (again, with F being replaced according to (4.41)). Next, we write the last coefficient in (4.62), that is $1 - \kappa\beta\psi''(\tilde{F}(y))A(y)$, as a function of $G : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ defined as

$$\begin{aligned}G(y, z) &= ((\rho+\kappa)Q_0(z))^{-1} \\ &\quad \times \left[(\rho+2\kappa)\psi(z)\psi''(z) - (\rho+\kappa)\psi'(z)^2 + \kappa(\rho+\kappa)\left(c - \tilde{R}(z, y)\right)\psi'(z)\psi''(z)\right].\end{aligned}$$

Employing (4.43), we get $1 - \kappa\beta\psi''(\tilde{F}(y))A(y) = G(y, \tilde{F}(y))$, and thus we have

$$\frac{d}{dy}\tilde{D}(y, \tilde{F}(y)) = \frac{2}{\sigma^2}\left(\tilde{F}'(y) - \beta\right)G(y, \tilde{F}(y)). \tag{4.63}$$

Now, let $(y^\star, z^\star) \in \mathbb{R} \times \mathbb{R}$ be such that $\tilde{D}(y^\star, z^\star) = 0$. We find from (4.56) that $D(y^\star, z^\star) = 0$. Hence, upon recalling (4.48), it holds

$$(\rho+\kappa)\left(c - \tilde{R}(z^\star, y^\star)\right) = -\frac{Q_0'(z^\star)}{Q_1(z^\star)}. \tag{4.64}$$

Then, exploiting (4.64), we obtain

$$\begin{aligned}G(y^\star, z^\star) = &\left((\rho+\kappa)Q_0(z^\star)Q_1(z^\star)\right)^{-1} \times \Big[(\rho+\kappa)\psi(z^\star)\psi'(z^\star)\psi''(z^\star)\psi'''(z^\star) \\ &- (\rho+2\kappa)\psi(z^\star)\psi''(z^\star)^3 + (\rho+2\kappa)\psi'(z^\star)^2\psi''(z^\star)^2 - (\rho+\kappa)\psi'(z^\star)^3\psi'''(z^\star)\Big] \\ = &-\frac{\sigma^2}{2}\left((\rho+\kappa)Q_1(z^\star)\right)^{-1} Q_2(z^\star) < 0.\end{aligned} \tag{4.65}$$

In (4.65) we have used: (4.58) with $k = 0, 1, 2$ for the last equality, and the fact that Q_1 and Q_2 are strictly positive for the strict inequality.
Recalling that $\tilde{F}' - \beta > 0$ on $I_{\max}$, we conclude from (4.61), (4.63) and (4.65) that $\tilde{D}(y, \tilde{F}(y))$ cannot tend to zero as $y \downarrow y_-$. This completes the proof.

□

Corollary 4.4.5. *The free boundary F as in* (4.28) *and* (4.29) *is well defined. Moreover, it is strictly increasing and given by*

$$F(y) = \tilde{F}(y) - \beta y, \quad \textit{for all } y \in [0, \bar{y}].$$

Proof. The existence and uniqueness is an implication of Proposition 4.4.4. It also ensures that $F'(y) = \tilde{F}'(y) - \beta > 0$ for all $y \in [0, \bar{y}]$. □

4.4.2 The Optimal Strategy and the Value Function: Verification

In the following, the initial price level at which the company starts to install solar panels is denoted by $x_0 := F(0)$, and we define $\bar{x} := F(\bar{y}) = \tilde{x} - \beta\bar{y}$ (cf. (4.41)). Since F is strictly increasing, its inverse function exists on $[x_0, \bar{x}]$ and is denoted by F^{-1}.

We divide the (candidate) *installation region* $\mathbb{I}$ into

$$\mathbb{I}_1 := \{(x, y) \in \mathbb{R} \times [0, \bar{y}) : x \in [F(y), \bar{x})\},$$

and

$$\mathbb{I}_2 := \{(x, y) \in \mathbb{R} \times [0, \bar{y}) : x \geq \bar{x}\}.$$

An optimal installation strategy can be described as follows: in $\mathbb{W}$ (cf. (4.28)), that is, when the current price x is sufficiently low such that $x < F(y)$, the company does not increase the level of installed power. Whenever the price crosses $F(y)$, then the company makes infinitesimal installations so to keep the state process (X, Y) inside $\overline{\mathbb{W}}$. Conversely, if the current price x is sufficiently large such that $x \geq F(y)$ (i.e. in $\mathbb{I}$, cf. (4.29)), then the company makes an instantaneous lump sum installation. In particular, on the one hand, whenever the maximum level of installed power $\bar{y}$, that the firm is able to reach, is sufficiently high (that is $(x, y) \in \mathbb{I}_1$), then the company pushes the state process (X, Y) immediately to the locus of points $\{(x, y) \in \mathbb{R} \times [0, \bar{y}] : x = F(y)\}$ in direction $(0, 1)$, so to increase the level of installed power by $F^{-1}(x) - y$ units. The associated payoff to this action is then the difference of the continuation value starting from the new state $(x, F^{-1}(x))$ and the costs associated to the installation of additional solar panels, that is $c(F^{-1}(x) - y)$. On the other hand, whenever the firm has to restrict its actions due to the upper bound $\bar{y}$ (that is $(x, y) \in \mathbb{I}_2$), then the company immediately installs the maximum number of panels, so to increase the level of installed power up to $\bar{y}$ units, and the associated payoff to such a strategy is $R(x, \bar{y}) - c(\bar{y} - y)$.

In light of the previous discussion, we now define our candidate value function $w : \mathbb{R} \times [0, \bar{y}] \mapsto \mathbb{R}$ as

$$w(x,y) = \begin{cases} A(y)\psi\big(x + \beta y\big) + R(x,y), & \text{if } x \in \mathbb{W} \cup ((-\infty, \bar{x}) \times \{\bar{y}\}), \\ A(F^{-1}(x))\psi\big(x + \beta F^{-1}(x)\big) & \\ +R(x, F^{-1}(x)) - c(F^{-1}(x) - y), & \text{if } (x,y) \in \mathbb{I}_1, \\ R(x, \bar{y}) - c(\bar{y} - y), & \text{if } (x,y) \in \mathbb{I}_2 \cup ([\bar{x}, \infty) \times \{\bar{y}\}). \end{cases} \tag{4.66}$$

The next two results verify that w is a classical solution to the HJB equation (4.10).

Lemma 4.4.6. *The function w is $C^{2,1}(\mathbb{R} \times [0, \bar{y}])$.*

Proof. In the following, we denote by $\mathrm{Int}(\cdot)$ the interior of a set. Clearly, by (4.66) it holds for all $(x, y) \in \mathrm{Int}(\mathbb{W})$ that

$$w_x(x,y) = A(y)\psi'(x+\beta y) + R_x(x,y), \tag{4.67}$$

$$w_{xx}(x,y) = A(y)\psi''(x+\beta y), \tag{4.68}$$

$$w_y(x,y) = A'(y)\psi(x+\beta y) + \beta A(y)\psi'(x+\beta y) + R_y(x,y), \tag{4.69}$$

and moreover,

$$w_x(x,y) = R_x(x,\bar{y}), \quad w_{xx}(x,y) = 0, \quad w_y(x,y) = c, \quad \text{for all } (x,y) \in \mathrm{Int}(\mathbb{I}_2). \tag{4.70}$$

To evaluate w_x, w_{xx} and w_y inside $\mathbb{I}_1$, we need some more work. We find for all $(x, y) \in \mathrm{Int}(\mathbb{I}_1)$

$$\begin{aligned} w_x(x,y) =& A(F^{-1}(x))\psi'\big(x+\beta F^{-1}(x)\big) + R_x\big(x, F^{-1}(x)\big) \\ &+ (F^{-1})'(x)\Big[A'\big(F^{-1}(x)\big)\psi\big(x+\beta F^{-1}(x)\big) + \beta A\big(F^{-1}(x)\big)\psi'\big(x+\beta F^{-1}(x)\big) \\ &+ R_y\big(x, F^{-1}(x)\big) - c\Big], \\ =& A(F^{-1}(x))\psi'\big(x+\beta F^{-1}(x)\big) + R_x\big(x, F^{-1}(x)\big), \end{aligned} \tag{4.71}$$

$$\begin{aligned} w_{xx}(x,y) =& A(F^{-1}(x))\psi''(x+\beta F^{-1}(x)) + (F^{-1})'(x)\Big[A'(F^{-1}(x))\psi'\big(x+\beta F^{-1}(x)\big) \\ &+ \beta A(F^{-1}(x))\psi''\big(x+\beta F^{-1}(x)\big) + R_{yx}(x, F^{-1}(x))\Big] \\ =& A(F^{-1}(x))\psi''(x+\beta F^{-1}(x)), \end{aligned} \tag{4.72}$$

$$w_y(x,y) = c, \tag{4.73}$$

where we have used (4.37) in (4.71), and (4.38) in (4.72). Notice that the functions A, F^{-1}, ψ, ψ', R_y and R_x are continuous. The previous equations and (4.37) easily provide the continuity of the derivatives on $\mathbb{R} \times \{\bar{y}\}$. Letting $(x_n, y_n)_n \subset \mathbb{I}_1$ be any sequence converging to $(F(y), y)$, $y \in [0, \bar{y})$, we find the required continuity results along $\overline{\mathbb{W}} \cap \overline{\mathbb{I}}_1$ upon employing (4.37). Moreover, (4.45) ensures the continuity of w_x and w_{xx} along $\overline{\mathbb{I}}_1 \cap \overline{\mathbb{I}}_2$, and we clearly have the continuity of w_y along $\overline{\mathbb{I}}_1 \cap \overline{\mathbb{I}}_2$. □

Proposition 4.4.7. *The function w from (4.66) is a $C^{2,1}(\mathbb{R} \times [0, \bar{y}])$ solution to*

$$\max\Big\{\mathcal{L}^y w(x,y) - \rho w(x,y) + xy, w_y(x,y) - c\Big\} = 0, \quad \textit{for all } (x,y) \in \mathbb{R} \times [0, \bar{y}), \tag{4.74}$$

such that $w(x, \bar{y}) = R(x, \bar{y})$.

Proof. Lemma 4.4.6 guarantees the claimed regularity of w. Moreover, from (4.66) we see that $w(x,\bar{y}) = R(x,\bar{y})$ since $A(\bar{y}) = 0$, and by construction, we clearly have $\mathcal{L}^y w(x,y) - \rho w(x,y) + xy = 0$ for all $(x,y) \in \mathbb{W}$, and $w_y(x,y) - c = 0$ for all $(x,y) \in \mathbb{I}_1 \cup \mathbb{I}_2$. We prove the inequalities $\mathcal{L}^y w(x,y) - \rho w(x,y) + xy \leq 0$ for all $(x,y) \in \mathbb{I}$, and $w_y(x,y) - c \leq 0$ for all $(x,y) \in \mathbb{W}$, in the following three steps separately. It is worth to bear in mind that $R_x(x,y) = \frac{y}{\rho+\kappa}$ by (4.6).

Step 1. Let $(x,y) \in \mathbb{I}_1$ be fixed. From the second line of (4.66), (4.71) and (4.72), we find

$$\begin{aligned}
&\mathcal{L}^y w(x,y) - \rho w(x,y) + xy \\
=&\mathcal{L}^{F^{-1}(x)} w(x,F^{-1}(x)) - \rho w(x,F^{-1}(x)) + xF^{-1}(x) \\
&+ \kappa\beta w_x(x,F^{-1}(x))(F^{-1}(x) - y) + (c\rho - x)(F^{-1}(x) - y) \\
=&(F^{-1}(x) - y)\left(c\rho + \kappa\beta w_x(x,F^{-1}(x)) - x\right),
\end{aligned} \tag{4.75}$$

where we have employed that $w(x,F^{-1}(x))$ solves

$$\mathcal{L}^{F^{-1}(x)} w(x,F^{-1}(x)) - \rho w(x,F^{-1}(x)) + xF^{-1}(x) = 0.$$

For any $(x,y) \in \mathbb{I}_1$, we have $x \geq F(y)$ implying $F^{-1}(x) \geq y$ because F, and hence F^{-1}, is strictly increasing (cf. Corollary 4.4.5). Thus, in order to show that (4.75) is negative on $\mathbb{I}_1$, it suffices to prove that the function

$$Z(x,F^{-1}(x)) := c\rho + \kappa\beta w_x(x,F^{-1}(x)) - x, \tag{4.76}$$

is negative for any $x \in [x_0,\bar{x}]$. Due to the regularity of w, we can use (4.71), and the fact that $A(F^{-1}(\bar{x})) = A(\bar{y}) = 0$, to obtain

$$Z(\bar{x},F^{-1}(\bar{x})) = c\rho + R_x(\bar{x},\bar{y}) - \bar{x} < 0, \tag{4.77}$$

where the inequality holds by (4.40) with $y = \bar{y}$. Taking the total derivative of $Z(x,F^{-1}(x))$ with respect to x gives

$$\begin{aligned}
\frac{dZ(x,F^{-1}(x))}{dx} =&\kappa\beta w_{xx}(x,F^{-1}(x)) - 1 = \kappa\beta A(F^{-1}(x))\psi''(x+\beta F^{-1}(x)) - 1 \\
=&\left[\rho\left(\psi(x+\beta F^{-1}(x))\psi''(x+\beta F^{-1}(x)) - \psi'(x+\beta F^{-1}(x))^2\right)\right]^{-1} \\
&\times\Bigg[\rho\left(\psi'(x+\beta F^{-1}(x))^2 - \psi(x+\beta F^{-1}(x))\psi''(x+\beta F^{-1}(x))\right) \\
&- \kappa\psi'(x+\beta F^{-1}(x))\psi''(x+\beta F^{-1}(x))\left(c\rho + \frac{\kappa\beta}{\rho+\kappa}F^{-1}(x) - x\right) \\
&- \frac{\sigma^2}{2}\kappa\psi''(x+\beta F^{-1}(x))^2 R_{xy}(x,F^{-1}(x))\Bigg],
\end{aligned} \tag{4.78}$$

where we have employed: $w_{xy}(x, F^{-1}(x)) = 0$, cf. (4.36), for the first equality, and (4.39) for the last equality (after rearranging terms). Now, suppose that there exists a point $x^\star \in [x_0, \bar{x})$ such that $Z(x^\star, F^{-1}(x^\star)) = 0$. It follows from (4.76), together with (4.39) and (4.71), that $(x^\star, F^{-1}(x^\star))$ satisfies

$$\begin{aligned} &c\rho + \frac{\kappa\beta}{\rho+\kappa} F^{-1}(x^\star) - x^\star \\ =&\frac{-\frac{\sigma^2}{2}\kappa\psi'(x^\star + \beta F^{-1}(x^\star))\psi''(x^\star + \beta F^{-1}(x^\star))R_{xy}(x^\star, F^{-1}(x^\star))}{(\rho+\kappa)\psi'(x^\star + \beta F^{-1}(x^\star))^2 - \rho\psi(x^\star + \beta F^{-1}(x^\star))\psi''(x^\star + \beta F^{-1}(x^\star))}. \end{aligned} \tag{4.79}$$

Then, exploiting the latter, one can find with (4.78) that

$$\left.\frac{dZ(x, F^{-1}(x))}{dx}\right|_{x=x^\star} = \frac{\sigma^2}{2} Q_1(x^\star + \beta F^{-1}(x^\star))^{-1} Q_2(x^\star + \beta F^{-1}(x^\star)) > 0, \tag{4.80}$$

after using (4.33) with $k = 0, 1, 2$, and some simple algebra. We conclude from both (4.77) and (4.80) that there cannot exist a point $x^\star \in [x_0, \bar{x})$ such that $Z(x^\star, F^{-1}(x^\star)) = 0$. Therefore, we have $\mathcal{L}^y w(x, y) - \rho w(x, y) + xy \leq 0$ for all $(x, y) \in \mathbb{I}_1$.

Step 2. For all $(x, y) \in \mathbb{I}_2$ we find from the third line of (4.66) and (4.70)

$$\begin{aligned} &\mathcal{L}^y w(x, y) - \rho w(x, y) + xy \\ &= \mathcal{L}^{\bar{y}} R(x, \bar{y}) - \rho R(x, \bar{y}) + x\bar{y} + \kappa\beta R_x(x, \bar{y})(\bar{y} - y) + (c\rho - x)(\bar{y} - y) \\ &= (\bar{y} - y)\left(\frac{\kappa\beta}{\rho+\kappa}\bar{y} + c\rho - x\right) \leq (\bar{y} - y)\left(\frac{\kappa\beta}{\rho+\kappa}\bar{y} + c\rho - \bar{x}\right) \leq 0, \end{aligned}$$

where we have used that $R(x, \bar{y})$ solves $(\mathcal{L}^{\bar{y}} - \rho)R(x, \bar{y}) + x\bar{y} = 0$ for the second equality, $x \geq \bar{x}$ for any $(x, y) \in \mathbb{I}_2$ for the first inequality, and (4.40) with $y = \bar{y}$ and $F(\bar{y}) = \bar{x}$ for the last inequality.

Step 3. Let $(x, y) \in \mathbb{W}$ be fixed. We define

$$S(x, y) := w_y(x, y) - c = A'(y)\psi(x + \beta y) + \beta A(y)\psi'(x + \beta y) + R_y(x, y) - c,$$

where the last equality holds true by (4.69). We clearly have $S(F(y), y) = 0$ from (4.37). Hence, it suffices to show that $S_x(x, y) \geq 0$ because $x < F(y)$ for all $(x, y) \in \mathbb{W}$. Computing the derivative of S with respect to x gives

$$S_x(x, y) = A'(y)\psi'(x + \beta y) + \beta A(y)\psi''(x + \beta y) + R_{xy}(x, y),$$

and from (4.38) we observe that $S_x(F(y), y) = 0$. Moreover, we have

$$S_{xx}(x, y) = A'(y)\psi''(x + \beta y) + \beta A(y)\psi'''(x + \beta y). \tag{4.81}$$

Recall (4.41) and (4.48). Lemma 4.4.3 and Proposition 4.4.4 imply that

$$D(y, F(y) + \beta y) > 0, \quad \text{for all } y \in [0, \bar{y}]. \tag{4.82}$$

Now, exploiting (4.43) and (4.44), we find

$$\begin{aligned} &S_{xx}(F(y), y) \\ &= -\left[(\rho+\kappa)\psi(F(y)+\beta y)Q_0(F(y)+\beta y)\right]^{-1} D(y, F(y)+\beta y) < 0, \quad \text{for all } y \in [0, \bar{y}], \end{aligned} \tag{4.83}$$

where the inequality is due to (4.82) and the fact that Q_0 is (strictly) positive. Since $\frac{\psi'''(\cdot)}{\psi''(\cdot)}$ is increasing by Lemma A.1.2-(3), and $A(y)$ is positive for all $y \in [0, \bar{y}]$ by Lemma 4.4.1, we have for all $x \leq F(y)$

$$A'(y) + \frac{\psi'''(x+\beta y)}{\psi''(x+\beta y)}\beta A(y) < A'(y) + \frac{\psi'''(F(y)+\beta y)}{\psi''(F(y)+\beta y)}\beta A(y) < 0,$$

where we have employed both (4.81) and (4.83) for the last inequality. Thus, we have $S_{xx}(x, y) < 0$, and therefore $S_x(x, y) > 0$ for all $(x, y) \in \mathbb{W}$. This completes the proof. □

We conclude that w identifies with the value function.

Theorem 4.4.8. *Recall w from (4.66), and let $\Delta := (\bar{y} - y)\mathbb{1}_{\{x \geq \bar{x}\}} + (F^{-1}(x) - y)\mathbb{1}_{\{\bar{x} > x > F(y)\}}$. The function w identifies with the value function V from (4.4), and the optimal installation strategy, denoted by $I^\star$, is given by*

$$\begin{cases} I^\star_{0-} = 0, \\ I^\star_t = \begin{cases} \Delta + K_t, & t \in [0, \tau), \\ \Delta + K_\tau, & t \geq \tau, \end{cases} \end{cases} \tag{4.84}$$

where $\tau := \inf\{t \geq 0 : K_t = \bar{y} - (y + \Delta)\}$, and where (X, K) is the unique $\mathbb{F}$-adapted process on $[0, \tau]$ with increasing K and starting point $(X_0, K_0) = (x, 0)$ such that

$$\begin{aligned} &X_t \leq F(y + \Delta + K_t), \\ &dX_t = \kappa\Big((\mu - \beta(y + \Delta + K_t)) - X_t\Big)dt + \sigma dW_t, \\ &dK_t = \mathbb{1}_{\{X_t = F(y+\Delta+K_t)\}} dK_t. \end{aligned} \tag{4.85}$$

Proof. To prove the claim, we aim at applying Theorem 4.3.2. We already know that $w \in C^{2,1}(\mathbb{R} \times [0, \bar{y}])$ is a solution to the HJB equation (4.10) by Proposition 4.4.7. Moreover, the function w satisfies the growth condition in (4.13) upon exploiting the facts that A is continuous, ψ is continuous and increasing, and $|R(x, y)| \leq K\big(1 + |x|\big)$ for any $y \in [0, \bar{y}]$ and some constant $K > 0$.

In a next step, we show the existence of (X, K) satisfying the stochastic differential equation (4.85). To do so, we borrow ideas from [42], cf. Section 5 therein. We let $(\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \mathbb{Q})$ be a filtered probability space with a filtration $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ satisfying the usual conditions, and let B be a $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-Brownian motion under $\mathbb{Q}$. Define the process (X, K) such that

$$dX_t = \kappa\Big((\mu - \beta(y+\Delta)) - X_t\Big)dt + \sigma dB_t, \tag{4.86}$$

$$K_t = \min\Big\{ \sup_{0\leq s\leq t} \{\bar{F}^{-1}(X_s)\}, \bar{y} - (y+\Delta)\Big\}, \tag{4.87}$$

with starting point $(X_0, K_0) = (x, 0)$, and where $\bar{F}^{-1}$ is such that

$$\bar{F}^{-1}(x) := \begin{cases} 0, & \text{if } x < x_0, \\ F^{-1}(x), & \text{if } x \in [x_0, \bar{x}], \\ \bar{y}, & \text{if } x > \bar{x}. \end{cases} \tag{4.88}$$

Notice that the pair (X, K) satisfies

$$X_t \leq F(y + \Delta + K_t),$$
$$dK_t = \mathbb{1}_{\{X_t = F(y+\Delta+K_t)\}} dK_t,$$

for any $t \leq \tau$. Since K is increasing and $K_t \leq \bar{y} - (y + \Delta)$ for any $t \leq \tau$, we apply Girsanov's Theorem (cf. Section 3.5 in [81]), so to obtain an equivalent probability measure $\mathbb{P}$ with respect to $\mathbb{Q}$ such that for any $T > 0$

$$\frac{d\mathbb{P}}{d\mathbb{Q}}\bigg|_{\mathcal{F}^{\mathcal{B}}{}_T} = \exp\left(- \int_0^T \frac{\kappa\beta}{\sigma} K_s dB_s - \frac{1}{2}\int_0^T \left(\frac{\kappa\beta}{\sigma}K_s\right)^2 ds \right),$$

and

$$W_t = B_t + \int_0^t \frac{\kappa\beta}{\sigma} K_s ds,$$

is a standard Brownian motion on $(\Omega, \mathcal{F}^{\mathcal{B}}, (\mathcal{F}^{\mathcal{B}}{}_t)_{t\geq 0}, \mathbb{P})$, where $(\mathcal{F}^{\mathcal{B}}{}_t)_{t\geq 0}$ is the σ-algebra generated by B, and $\mathcal{F}^{\mathcal{B}} = \mathcal{F}^{\mathcal{B}}{}_\infty$. The pair (X, K) constructed in this way is a weak solution to (4.85). We will prove in the following that (X, K) is pathwise unique, hence a strong solution. Recall (4.41). We obtain

$$0 < \left(F^{-1}\right)'(x) \leq \max_{x_0\leq x'\leq \bar{x}} \beta^{-1} \frac{D(F^{-1}(x'), x')}{N(F^{-1}(x'), x') - D(F^{-1}(x'), x')}, \quad \text{for all } x \in [x_0, \bar{x}],$$

where the first inequality is due to the monotonicity of F^{-1} and the last inequality is due to (4.51) and (4.52). The continuity of the functions N and D, and the fact that

$$N(F^{-1}(x), x) - D(F^{-1}(x), x) > 0, \quad \text{for all } x \in [x_0, \bar{x}],$$

which is due to Lemma 4.4.3, Proposition 4.4.4 and Lemma C.2.1, imply $\left(F^{-1}\right)'(x) < \infty$ for all $x \in [x_0, \bar{x}]$. The previous results show that $\bar{F}^{-1}$ is (globally) Lipschitz continuous. Now, fix $\omega \in \Omega$, and let $(\tilde{X}, \tilde{K})$ and $(\hat{X}, \hat{K})$ be two solutions of (4.85). The (global) Lipschitz continuity of $\bar{F}^{-1}$ and the second line of (4.85) imply

$$\begin{aligned} \left|\tilde{K}_t - \hat{K}_t\right| &= \left|\sup_{0\leq s\leq t} \left\{F^{-1}(\tilde{X}_s) - (\bar{y} - (y+\Delta))\right\}^+ - \sup_{0\leq s\leq t} \left\{F^{-1}(\hat{X}_s) - (\bar{y} - (y+\Delta))\right\}^+\right| \\ &\leq \sup_{0\leq s\leq t} \left\{ \left|F^{-1}(\tilde{X}_s) - F^{-1}(\hat{X}_s)\right| \right\} \\ &\leq \sup_{0\leq s\leq t} \bar{K} \left|\tilde{X}_s - \hat{X}_s\right| \leq C_0 \int_0^t \left|\tilde{X}_s - \hat{X}_s\right| + \left|\tilde{K}_s - \hat{K}_s\right| ds, \end{aligned} \tag{4.89}$$

for some constant $C_0 > 0$. Then, again with the second line of (4.85) and (4.89), we find for some constant $C_1 > 0$ the estimate

$$0 \leq \left\|(\tilde{X}_t - \hat{X}_t, \tilde{K}_t - \hat{K}_t)\right\| \leq C_1 \int_0^t \left|\tilde{X}_s - \hat{X}_s\right| + \left|\tilde{K}_s - \hat{K}_s\right| ds, \tag{4.90}$$

where $||\cdot||$ denotes the euclidean norm in $\mathbb{R}^2$. Now, Grönwall's inequality yields

$$\left\|(\tilde{X}_t - \hat{X}_t, \tilde{K}_t - \hat{K}_t)\right\| \leq 0, \tag{4.91}$$

upon recalling that $t \mapsto X_t$ is continuous for any solution of (4.85). Thus, by (4.91), pathwise uniqueness holds, and (4.85) admits a unique strong solution.

Finally, since $I^\star$ from (4.84) satisfies (4.14) and (4.15), we conclude that w identifies with V, and $I^\star$ is an optimal installation strategy by Theorem 4.3.2. □

4.5 A Related Optimal Stopping Problem

In this section, we provide an optimal stopping problem which is related to our optimal control problem. In particular, we show that the function $u := -V_y$ identifies with the value function of an optimal stopping problem. For the subsequent analysis recall that $(X_t^{x,y})_{t\geq 0}$ denotes the electricity price when the company follows the non-installation strategy I^0.

Proposition 4.5.1. *The function $u : \mathbb{R} \times [0, \bar{y}) \mapsto \mathbb{R}$ defined by*

$$u(x, y) := -V_y(x, y),$$

admits the probabilistic representation

$$u(x, y) = \sup_{\tau\geq 0} \mathbb{E}\left[\int_0^\tau e^{-\rho t} \left(\kappa\beta V_x(X_t^{x,y}, y) - X_t^{x,y}\right) dt - ce^{-\rho\tau}\right], \quad (x, y) \in \mathbb{R} \times [0, \bar{y}), \tag{4.92}$$

where the optimization is taken over the set of $\mathcal{F}$-stopping times. Moreover, with F as derived in Section 4.4.1, we have that

$$\tau^\star(x;y) = \inf\{t \geq 0 : X_t^{x,y} \geq F(y)\}, \quad (x,y) \in \mathbb{R} \times [0,\bar{y}),$$

is optimal in (4.92).

Proof. Let $y \in [0,\bar{y})$ be given and fixed. Notice that $u(\cdot,y) \in C^1(\mathbb{R})$ by construction (cf. (4.37) and (4.38)). Direct calculations on (4.66) show that $u_{xx}(\cdot,y) \in L^\infty_{loc}(\mathbb{R})$. Now, we show in *Step 1* that $u(\cdot,y)$ solves the HJB equation

$$\max\Big\{\mathcal{L}^y w(x) - \rho w(x) + \kappa\beta V_x(x,y) - x,\ -c - w(x)\Big\} = 0, \quad \text{a.e. } x \in \mathbb{R}, \tag{4.93}$$

and then, in *Step 2*, we draw the conclusions.

Step 1. Recall the waiting region $\mathbb{W}$ and the installation region $\mathbb{I}$, and let $x \in \mathbb{R}$ be such that $(x,y) \in \mathbb{W}$. Employing (4.69), we have

$$u(x.y) = -V_y(x,y) = -A'(y)\psi(x+\beta y) - \beta A(y)\psi'(x+\beta y) - R_y(x,y),$$

and therefore we obtain with (4.67)

$$\begin{aligned}
&\mathcal{L}^y u(x,y) - \rho u(x,y) + \kappa\beta V_x(x,y) - x\\
=&\frac{1}{2}\sigma^2\left(-A'(y)\psi''(x+\beta y) - \beta A(y)\psi'''(x+\beta y)\right)\\
&+\kappa(\mu - x - \beta y)\left(-A'(y)\psi'(x+\beta y) - \beta A(y)\psi''(x+\beta y) - R_{xy}(x,y)\right)\\
&-\rho\left(-A'(y)\psi(x+\beta y) - \beta A(y)\psi'(x+\beta y) - R_y(x,y)\right)\\
&+\kappa\beta A(y)\psi'(x+\beta y) + \kappa\beta R_x(x,y) - x\\
=&-A'(y)\big(\mathcal{L}^y\psi(x+\beta y) - \rho\psi(x+\beta y)\big) - A(y)\big(\mathcal{L}^y\psi'(x+\beta y) - (\rho+\kappa)\psi'(x+\beta y)\big)\\
=&0,
\end{aligned}$$

upon using (4.33) with $k = 0,1$ for the last equality.

Now, let $x \in \mathbb{R}$ be such that $(x,y) \in \mathbb{I}$, so that $u(x,y) = -c$ (recall (4.70) and (4.73)). On the one hand, if $(x,y) \in \mathbb{I}_1$ then we obtain

$$\mathcal{L}^y u(x,y) - \rho u(x,y) + \kappa\beta V_x(x,y) - x = c\rho - x + \kappa\beta V_x(x,F^{-1}(x)) \leq 0,$$

where we have used (4.71) for the first equality, and the inequality holds true by the proof of Proposition 4.4.7 (cf. *Step 2* therein).

On the other hand, letting $(x,y) \in \mathbb{I}_2$ and recalling (4.70), we have $x \geq F(\bar{y})$ and hence

$$\mathcal{L}^y u(x,y) - \rho u(x,y) + \kappa\beta V_x(x,y) - x = c\rho - x + \frac{\beta\kappa}{\rho+\kappa}\bar{y} \leq \rho c - F(\bar{y}) + \frac{\beta\kappa}{\rho+\kappa}\bar{y} \leq 0,$$

where the last inequality holds true by (4.40) with $y = \bar{y}$.

Finally, from Proposition 4.4.7 we have $-c - u(x, y) \leq 0$ for any $x \in \mathbb{R}$.

Step 2. The inequalities proved in *Step 1* show that $u(\cdot, y)$ identifies with a $W^{2,\infty}_{loc}(\mathbb{R})$-solution to (4.93). Then, employing a standard verification theorem based on an application of (a generalized version of) Itô's formula, we find (4.92) and that the stopping time $\tau^\star(x; y) = \inf\{t \geq 0 : X^{x,y}_t \geq F(y)\}$ is optimal for (4.92). □

Remark 4.5.2. *Two comments are worth being done.*

1. *The related optimal stopping problem* (4.92) *is consistent with the findings in [42], cf. (3.27) therein. In [42], a central bank tries to contain the inflation by acting on the nominal interest rate. Hereby, the authors study a two-dimensional stochastic control problem with Ornstein-Uhlenbeck dynamics which leads to a stochastic differential game.*

2. *With regard to Remark 2.4.16 in Chapter 2, the stopping time $\tau^\star(x; y)$ can be interpreted as the optimal time at which the company should increase its level of installed power by an additional unit. We can observe that the stopping problem* (4.92) *has running cost in terms of $-X^{x,y}_t$ reflecting the forgone gains of not having increased the installed power Y by an additional unit up to time τ, running gains from the indirect change in V which is due to the company's inaction up to time τ and therefore the non-existing (additional) negative price impact on $X^{x,y}_t$, and terminal proportional cost c for acting.*

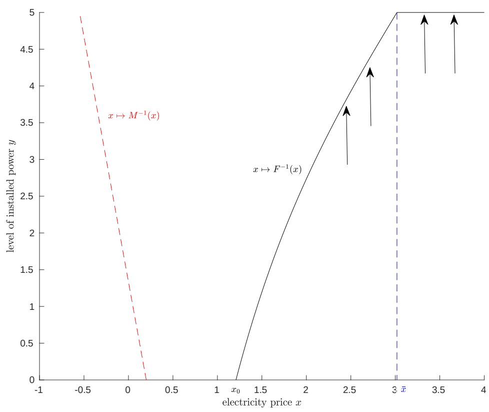
4.6 Numerical Implementation

The ODE (4.51) cannot be solved analytically, but we are able to solve it numerically with MATLAB. Figure 4.1 displays a plot of the inverse of the free boundary with three different values for the drift coefficient μ. In particular we take those parameters' values as given in Table 4.1, and $\mu \in \{0.2; 1.4, 2.25\}$.

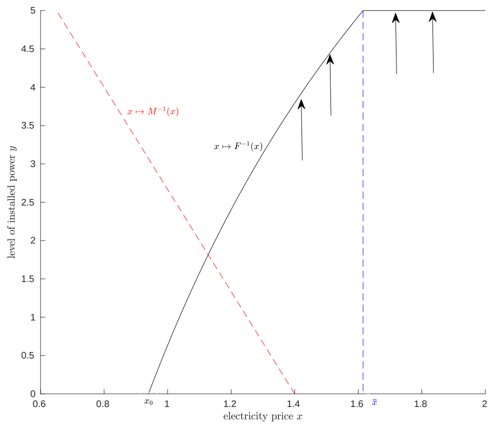
κ	σ	ρ	c	β	$\bar{y}$
0.10	0.50	0.05	0.30	0.15	5

Table 4.1: Parameters' values.

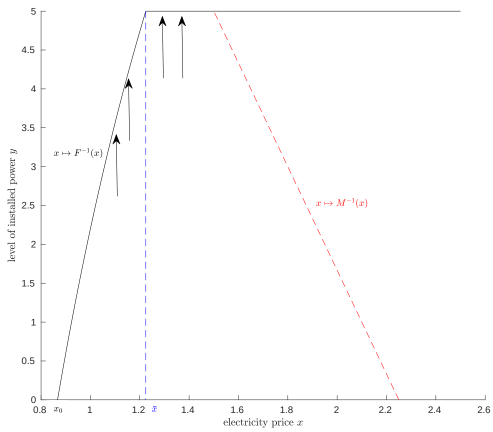
The dashed sloped red line is a plot of the inverse of the function $M : [0, \infty) \mapsto (-\infty, \mu]$ given by $M(y) := \mu - \beta y$ (to which we shall refer as "line of means"). The function M provides the underlying mean reversion level of the process $X^{x,y}$ depending on the level of installed power y. Figures 4.1(a), 4.1(b) and 4.1(c) show three different



(a) The functions F^{-1} and M^{-1} with $\mu = 0.2$.



(b) The functions F^{-1} and M^{-1} with $\mu = 1.4$.



(c) The functions F^{-1} and M^{-1} with $\mu = 2.25$.

Figure 4.1: Plots of the functions $x \mapsto F^{-1}(x)$ and $x \mapsto M^{-1}(x)$ with various values for μ. The optimal installation strategy prescribes the following. In the region $\{(x, y) \in \mathbb{R} \times [0, \bar{y}) : x < F(y)\}$ it is optimal not to install additional solar panels. Conversely, if, at the initial time, (x, y) is such that $x \geq F(y)$ and $y \in [0, \bar{y})$, then the (optimally controlled) process (X, Y) should be pushed in direction $(0, 1)$ as follows: for $x \geq \bar{x}$, the firm should immediately install the maximum number of panels, so to increase the level of installed power by $\bar{y} - y$ units. For (x, y) such that $x \in [F(y), \bar{x})$, the firm should make an initial lump sum installation of size $F^{-1}(x) - y$, and then keep on making infinitesimal installations just preventing the price to exceed $F(y)$ until the maximum quantity of panels is installed.

scenarios. The red line can lie entirely to the left or to the right of F^{-1} (see Figure 4.1(a) and Figure 4.1(c)), or it can intersect F^{-1} (see Figure 4.1(b)). Notice that the position of the current mean reversion level in fact influences the expected time of the next action: if the red line is entirely to the left of F^{-1} (i.e. the current mean reversion level is below $F(y)$ for any $y \in [0, \bar{y}]$), then the electricity price tends to move towards the line of means and therefore to stay below the firm's threshold, at which it starts to undertake the installation of additional solar panels. Conversely, the electricity price tends to move above the firm's threshold $F(y)$ for some $y \in [0, \bar{y}]$, if the red line intersects or lies in the installation region $\mathbb{I}$. Such a case in turn implies that the firm will increase its level of installed power faster. The limiting situation is when the red line is entirely on the right of F^{-1} (i.e. lies entirely in $\mathbb{I}_2$). In this case, the electricity price tends to exceed F independently of the firm's level of installed power. Therefore, the firm will "quickly" install the maximum possible level $\bar{y}$.

The next proposition gives a characterization of when and how the line of means intersects the installation region $\mathbb{I}$, either at the free boundary F or at the locus of points $\{(x, y) : y = \bar{y},\, x \geq \bar{x}\}$.

Proposition 4.6.1. *Given the upper bound $\bar{y}$ for the level of installed power and the corresponding free boundary F, the line of means*

1. *has no intersection with the installation region $\mathbb{I}$ if $F(0) > \mu$;*
2. *intersects the boundary of $\mathbb{I}$ at the free boundary $F(y)$ if $F(0) \leq \mu$ and $\bar{y} \geq y^*$, where*
$$y^* := \big(\beta(\rho + 2\kappa)\big)^{-1} \left((\mu - \rho c)(\rho + \kappa) - \rho \frac{\psi(\mu)}{\psi'(\mu)} \right); \tag{4.94}$$
3. *intersects the boundary of $\mathbb{I}$ at its upper bound $y = \bar{y}$ if $\bar{y} \leq y^*$.*

Proof. For case (1), since the line of means $x = \mu - \beta y$ is decreasing in y and the free boundary F is increasing, there is no intersection if $\mu - \beta \times 0 = \mu < F(0)$.

For cases (2) and (3), let us now assume that $\mu \geq F(0)$, and recall that $\bar{x} = F(\bar{y}) = \tilde{x} - \beta\bar{y}$ where $\tilde{x}$ is such that $H(\tilde{x}) = 0$, with H defined in (4.46). The line of means $x = \mu - \beta y$ and the free boundary $x = F(y)$ have one or zero intersection according to whether $\bar{x} = F(\bar{y}) > \mu - \beta\bar{y}$ or not, respectively, i.e. whether $F(\bar{y}) + \beta\bar{y} = \tilde{x}(\bar{y}) > \mu$, where we have written $\tilde{x}(\bar{y})$ in order to stress the dependency of $\tilde{x}$ on $\bar{y}$. Employing Lemma 4.4.2 and the implicit function theorem, we get

$$\tilde{x}'(\bar{y}) = \frac{\psi'(\tilde{x})\tilde{R}_y(\tilde{x}, \bar{y})}{\psi''(\tilde{x})\left(c - \tilde{R}(\tilde{x}, \bar{y})\right)} = -\frac{\beta(\rho + 2\kappa)\psi'(\tilde{x})}{\rho(\rho + \kappa)\psi''(\tilde{x})\left(c - \tilde{R}(\tilde{x}, \bar{y})\right)} > 0,$$

where the strict inequality holds as $c - \tilde{R}(\tilde{x}, \bar{y}) < 0$ by Lemma 4.4.1 together with (4.43). Therefore, $\tilde{x}(\cdot)$ is increasing. Consequently, if there exists a point y^* such that $\tilde{x}(y^\star) = \mu$, then there is an intersection with F for $\bar{y} \geq y^*$, and there is no intersection with F for $\bar{y} < y^*$. The point y^* is characterized by the fact that the line of means $x = \mu - \beta y$ intersects both the free boundary $x = F(y)$ and the locus of points $\{(x, y) : y = \bar{y}, x \geq \bar{x}\}$ at the same point $(\bar{x}, y^*)$. Thus, in order to find y^*, we must impose simultaneously

$$\begin{cases} \bar{x} = \mu - \beta y^*, \\ \bar{x} = \tilde{x}(y^\star) - \beta y^*, \\ \psi'(\tilde{x}(y^\star))(c - \tilde{R}(\tilde{x}(y^\star), y^*)) + (\rho+\kappa)^{-1}\psi(\tilde{x}(y^\star)) = 0. \end{cases}$$

Therefore, in this case $\tilde{x}(y^\star) = \mu$, and the third equation can be rewritten as

$$\psi'(\mu)\left(c - \frac{(\rho+\kappa)\mu - \beta(\rho+2\kappa)y^*}{\rho(\rho+\kappa)}\right) + (\rho+\kappa)^{-1}\psi(\mu) = 0.$$

The solution is easily obtained as in equation (4.94). □

Remark 4.6.2. *Unfortunately, to discriminate between case (1) and the other two ones in Proposition 4.6.1, one has to solve* (4.51) *numerically in order to check whether* $F(0) > \mu$ *or not. Instead, discriminating between case (2) and case (3) is much easier, as the point* y^* *in equation* (4.94) *is given explicitly in terms of initial parameters and known functions.*

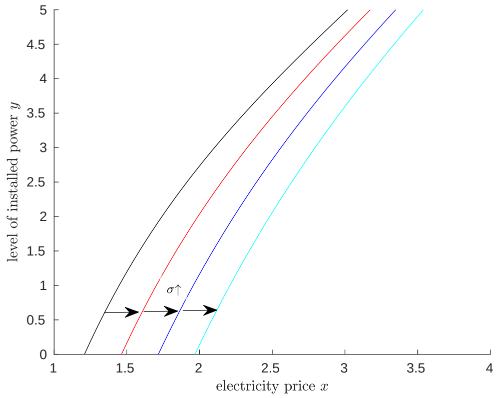
4.6.1 Comparative Statics

In this section, we study the sensitivity of the free boundary on the model parameters numerically. The preliminary parameters' values are given as in Table 4.2, and in the following, we let each of those parameters vary within a particular set. The numerical results can be observed in Figure 4.2 and Figure 4.3.

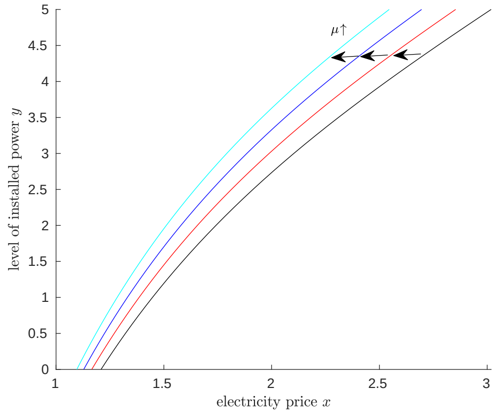
μ	σ	κ	ρ	c	β	$\bar{y}$
0.20	0.50	0.1	0.05	0.30	0.15	5

Table 4.2: Parameters' values for the numerical sensitivity analysis.

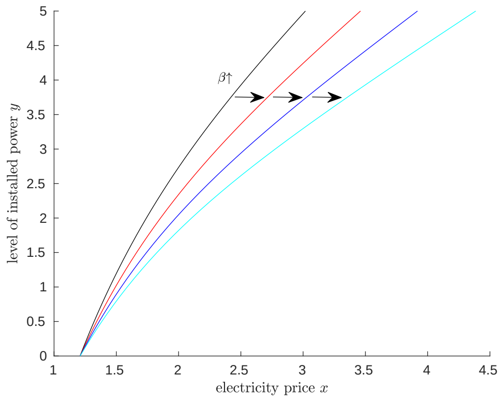
We first study the behavior of the free boundary with respect to the volatility displayed in Figure 4.2(a). Here the volatility parameter σ takes values in $\{0.5; 0.6; 0.7; 0.8\}$, and we can observe that F^{-1} is shifted to the right as σ increases; that is, the installation of additional panels is undertaken at higher prices. The firm might be afraid of receiving negative future prices due to higher uncertainty. This behavior is in line with the real options literature: when uncertainty increases, the agent is more reluctant to act, see for example [93].



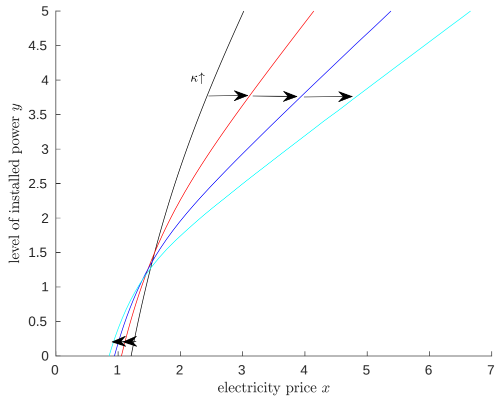
(a) The function F^{-1} with $\sigma = 0.5$ (black), $\sigma = 0.6$ (red), $\sigma = 0.7$ (blue), $\sigma = 0.8$ (cyan).



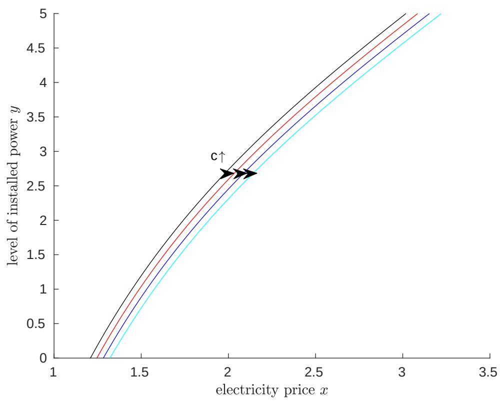
(b) The function F^{-1} with $\mu = 0.2$ (black), $\mu = 0.3$ (red), $\mu = 0.4$ (blue), $\mu = 0.5$ (cyan).



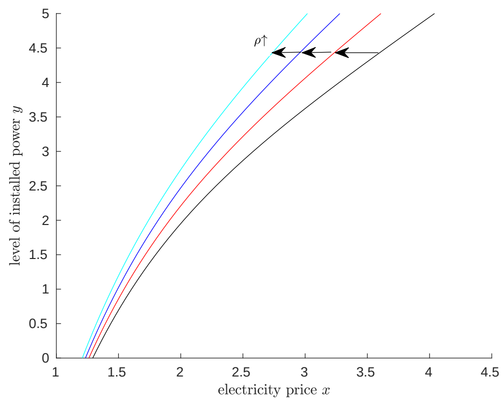
(c) The function F^{-1} with $\beta = 0.15$ (black), $\beta = 0.175$ (red), $\beta = 0.2$ (blue), $\beta = 0.225$ (cyan).



(d) The function F^{-1} with $\kappa = 0.1$ (black), $\kappa = 0.15$ (red), $\kappa = 0.20$ (blue), $\kappa = 0.25$ (cyan).



(e) The function F^{-1} with $c = 0.3$ (black), $c = 0.8$ (red), $c = 1.3$ (blue), $c = 1.8$ (cyan).



(f) The function F^{-1} with $\rho = 0.035$ (black), $\rho = 0.04$ (red), $\rho = 0.045$ (blue), $\rho = 0.05$ (cyan).

Figure 4.2: Sensitivity of the function $x \mapsto F^{-1}(x)$. In each subfigure, the parameter values which are not varied are those provided in Table 4.2.

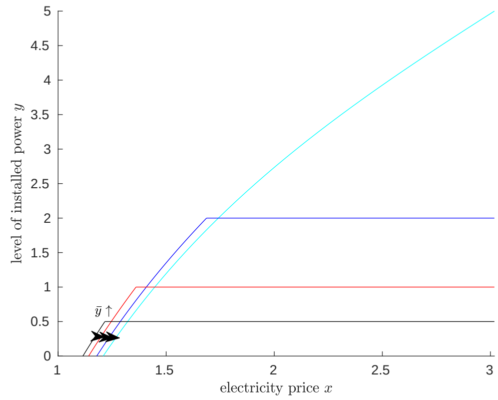


Figure 4.3: Sensitivity of the function $x \mapsto F^{-1}(x)$ with respect to $\bar{y}$. In particular $\bar{y} = 0.5$ (black), $\bar{y} = 1$ (red), $\bar{y} = 2$ (blue), $\bar{y} = 5$ (cyan), and all the other parameter values are those provided in Table 4.2.

Now, we let the mean-reversion level μ vary in $\{0.2; 0.3; 0.4; 0.5\}$. Figure 4.2(b) reveals that the critical threshold F^{-1} moves to the left. A higher value for μ leads the firm to undertake the installation at lower prices. This observation can be explained by the fact that the company is eager to act earlier, the higher the expected future profits.

In Figure 4.2(c), the impact parameter β takes values in $\{0.15; 0.175; 0.2; 0.225\}$, and as a consequence we find that the F^{-1} is shifted to the right as β increases. We explain this observation by the fact that the impact of a higher electricity production on the future electricity prices is higher as β increases. Therefore, the company starts to produce more electricity at higher prices, so to avoid lower (and possibly negative) electricity prices in the short run.

The dependency on κ can be observed in Figure 4.2(d). Here, we let κ taking values in $\{0.1; 0.15; 0.2; 0.25\}$. We find that higher values for the mean reversion speed κ leads the company to start installing solar panels at lower prices, but after some point, the company becomes more reluctant. This behavior can be explained by the fact that two effects play a role: on the one hand, a higher mean reversion speed reduces its ratio with respect to σ, the uncertainty is decreased, and hence a converse behavior with respect to Figure 4.2(a) can be observed. On the other hand, a higher mean reversion speed also intensifies the impact of the company's actions on the price dynamics. Therefore, it behaves as in Figure 4.2(c).

Figure 4.2(e) shows the dependency on the proportional cost of installation c which is valued in $\{0.3; 0.8; 1.3; 1.8\}$. The shift is not parallel as one could suggest from the figure. The function F^{-1} moves to the right, thus the company starts installing solar panels at higher prices. This observation is reasonable since the company waits for higher electricity prices to install additional solar panels whenever the proportional cost of installation increases.

Varying the discount factor ρ in $\{0.035; 0.04; 0.045; 0.05\}$, we find from Figure 4.2(f) that F^{-1} moves to the left, that is, the company starts to install solar panels so to produce more electricity at lower prices. Clearly, a higher discount factor reduces the discounted future profits of the firm. Thus, the firm tends to produce more electricity earlier.

Finally, we let $\bar{y}$ vary in $\{0.5; 1; 2; 5\}$, and we observe that F^{-1} moves to the right as $\bar{y}$ increases. Consequently, the possibility to increase the level of installed power up to a higher level makes the company more reluctant to act.

Remark 4.6.3. *Regarding the sensitivity of the value function V with respect to μ, β and σ, we can analytically obtain the monotonicity properties in the same way as in Proposition 2.5.2 of Chapter 2 upon noticing that V is convex by* (4.68)*, the second equation of* (4.70) *and* (4.72)*. In particular, one has that V is increasing with respect to μ and σ, and decreasing in β. Also, V is clearly decreasing in c upon recalling* (4.5)*. Furthermore, letting $\bar{y}_2 > \bar{y}_1$, we have $\mathcal{I}^{\bar{y}_1}(y) \subset \mathcal{I}^{\bar{y}_2}(y)$ for any $y \in [0, \bar{y}_1]$. Therefore, V is increasing with respect to $\bar{y}$.*

4.7 Conclusions

In this chapter, we have considered a price-maker company which generates electricity and sells it instantaneously in the spot market. The company can increase its level of installed power, which is proportional to its electricity generation, by irreversible installations of solar panels. In absence of any actions of the company, the spot electricity price evolves as an Ornstein-Uhlenbeck process. Including the company's economic activities in the market, the current level of the company's installed power has a permanent impact on the electricity price and affects its mean-reversion level. We have modeled the problem of maximizing the expected net profits as a two-dimensional degenerate singular stochastic control problem in which the installation strategy is identified as the company's control variable. Finally, our study is complemented by a numerical analysis of the dependency of the optimal installation strategy on the model's parameters, and an economical interpretation for those results is provided.

We have followed a guess-and-verify approach to solve the problem. The optimal installation strategy is triggered by a curve which depends on the current level of the company's installed power, and it is the unique strictly increasing function which solves a first-order ODE. It has been shown that this curve coincides with the threshold of a related optimal stopping problem. This stopping problem highlights the (economic) components taken into account by the firm when increasing the level of installed power.

One can think about extensions in many interesting ways. For instance, one could include the possibility of electricity storage. More precisely, the company can decide whether to store the electricity generated by the solar panels, or to sell it in the spot market. Hereby, the storage capacity is finite, and whenever it is fully used, then the company is forced to sell the electricity in the market immediately. The spot electricity price is then solely affected by the amount of electricity which the company sells in the market. The mathematical formulation leads to a daunting three dimensional singular stochastic control problem which requires a solution method that differs from the one used in this chapter. Another extension would be to consider a competition among several market players. In particular, one can study a situation in which there is a fixed number of firms which produce and sell electricity in the market while the total amount of installed power affects the electricity price. This problem would lead to a stochastic game with singular controls.

Appendix A

Facts and Properties of the Ornstein-Uhlenbeck Process

This appendix introduces the so-called Ornstein-Uhlenbeck process and presents some of its well known properties that are exploited in Chapter 2, Chapter 3 and Chapter 4[1].

Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space with a filtration $\mathbb{F}$ satisfying the usual conditions, and carrying a standard one-dimensional $\mathbb{F}$-Brownian motion W.

The Ornstein-Uhlenbeck process has been introduced for the first time in [103], and in modern stochastic analysis it is defined as the unique strong solution to the stochastic differential equation

$$dX_t^x = \kappa\left(\mu - X_t^x\right)dt + \sigma dW_t, \quad X_0^x = x \in \mathbb{R}, \tag{A.1}$$

for $\mu \in \mathbb{R}$ and $\kappa, \sigma > 0$. For any given initial value $x \in \mathbb{R}$, the process $X^x := (X_t^x)_{t\geq 0}$ is Gaussian. In particular, equation (A.1) admits the explicit solution

$$X_t^x = xe^{-\kappa t} + \mu(1 - e^{-\kappa t}) + \int_0^t \sigma e^{\kappa(s-t)} dW_s, \tag{A.2}$$

and it follows from (A.2) that for any $t \geq 0$

$$X_t^x \sim \mathcal{N}\left(xe^{-\kappa t} + \mu\left(1 - e^{-\kappa t}\right), \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right)\right),$$

where $\mathcal{N}(\alpha, \gamma)$ denotes the Gaussian distribution function with mean $\alpha \in \mathbb{R}$ and variance $\gamma > 0$.

The following result provides a useful estimate involving the absolute value of X^x.

Lemma A.1.1. *Let $q \in \{1, 2\}$. We have*

$$\mathbb{E}\left[|X_t^x|^q\right] \leq C(1 + |x|^q), \quad \textit{for some } C > 0. \tag{A.3}$$

[1]Parts of this appendix have been published in [60] and [83].

Proof. Exploiting (A.2), and the fact that

$$\left| \int_0^t e^{\kappa(s-t)} dW_s \right| \leq 1 + \left(\int_0^t e^{\kappa(s-t)} dW_s \right)^2, \quad \mathbb{P}\text{-a.s.},$$

we obtain the result by simple calculations that employ Itô isometry. □

The infinitesimal generator associated to X^x is denoted by $\mathcal{L}$ and, for any $u : \mathbb{R} \mapsto \mathbb{R}$ s.t. $u \in C^2(\mathbb{R})$, it is defined by

$$[\mathcal{L}u](x) := \lim_{t \downarrow 0} \frac{\mathbb{E}\left(u(X_t^x)\right) - u(x)}{t}, \quad x \in \mathbb{R}. \tag{A.4}$$

In particular, by an application of Dynkin's formula (cf. Theorem 7.4.1 in [100]) and of the mean-value theorem, one obtains

$$\mathcal{L}u(x) = \frac{\sigma^2}{2} u''(x) + \kappa(\mu - x)u'(x), \quad x \in \mathbb{R}$$

Given $\rho > 0$, the ODE $(\mathcal{L} - \rho)u = 0$ admits a strictly increasing positive fundamental solution denoted by $\psi(x)$. We recall some important properties of the aforementioned function in the next lemma.

Lemma A.1.2. *The following hold true.*

(1) The strictly increasing positive fundamental solution $\psi(\cdot)$ to the ordinary differential equation $(\mathcal{L} - \rho)u = 0$ is given by

$$\psi(x) = e^{\frac{\kappa(x-\mu)^2}{2\sigma^2}} D_{-\frac{\rho}{\kappa}}\left(- \frac{x - \mu}{\sigma}\sqrt{2\kappa} \right), \tag{A.5}$$

where

$$D_\nu(x) := \frac{e^{-\frac{x^2}{4}}}{\Gamma(-\nu)} \int_0^\infty t^{-\nu-1} e^{-\frac{t^2}{2} - xt} dt, \quad x \in \mathbb{R}, \tag{A.6}$$

is the cylinder function of order $\nu < 0$ and $\Gamma(\,\cdot\,)$ is the Euler's Gamma function. Moreover, ψ is strictly convex.

(2) Denoting by $\psi^{(k)}$ the k-th derivative of ψ, $k \in \mathbb{N}_0$, one has that $\psi^{(k)}$ is strictly convex and it is (up to a positive constant) the positive strictly increasing fundamental solution to $(\mathcal{L} - (\rho + k\kappa))u = 0$.

(3) For any $k \in \mathbb{N}_0$, one has $\psi^{(k)}(x)\psi^{(k+2)}(x) - \psi^{(k+1)}(x)^2 > 0$ for all $x \in \mathbb{R}$.

Proof. (1) We refer the reader to [76], among others. Moreover, the strict convexity of ψ can be checked by direct calculations on (A.5).

(2) Define the function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ by

$$f(t,x) = \frac{1}{\Gamma(\frac{\rho}{\kappa})} t^{\left(\frac{\rho}{\kappa}-1\right)} e^{-\frac{t^2}{2}+t\left(\frac{x-\mu}{\sigma}\right)\sqrt{2\kappa}},$$

that, once differentiated with respect to x, yields

$$f_x(t,x) = \frac{\rho\sqrt{2\kappa}}{\kappa\sigma} \frac{1}{\Gamma(\frac{\rho+\kappa}{\kappa})} t^{\left(\frac{\rho+\kappa}{\kappa}-1\right)} e^{-\frac{t^2}{2}+t\left(\frac{x-\mu}{\sigma}\right)\sqrt{2\kappa}}.$$

Notice that f is the integrand appearing in (A.6) for $\beta = -\frac{\rho}{\kappa}$. Then, differentiating (A.5) with respect to x, and invoking the dominated convergence theorem, we obtain

$$\psi'(x) \propto e^{\frac{\kappa(x-\mu)^2}{2\sigma^2}} D_{-\frac{\rho+\kappa}{\kappa}}\left(-\frac{x-\mu}{\sigma}\sqrt{2\kappa}\right),$$

upon noticing that $f_x(t,x)$ is the integrand of $D_{-\frac{\rho+\kappa}{\kappa}}\left(-\frac{x-\mu}{\sigma}\sqrt{2\kappa}\right)$ (cf. (A.6)).

Hence, ψ' can be identified (modulo a constant) as the positive strictly increasing fundamental solution to $(\mathcal{L} - (\rho+\kappa))u = 0$, and by direct calculations it can be checked that it is strictly convex. By iterating the previous argument, we see that, for any $k \in \mathbb{N}$, the function $\psi^{(k)}$ is strictly convex and identifies with the positive strictly increasing fundamental solution to $(\mathcal{L} - (\rho+k\kappa))u = 0$.

(3) We define the function $f^{(k)} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ by

$$f^{(k)}(t,x) = \frac{\left(\sqrt{2\kappa}/\sigma\right)^{\frac{k}{2}}}{\Gamma(\frac{\rho}{\kappa})^{\frac{1}{2}}} t^{\frac{1}{2}\left(\frac{\rho}{\kappa}+k-1\right)} e^{-\frac{t^2}{4}+\frac{t}{2}\left(\frac{x-\mu}{\sigma}\right)\sqrt{2\kappa}}.$$

By direct calculations, we find

$$\psi^{(k+1)}(x) = \int_0^\infty f^{(k+2)}(t,x) f^{(k)}(t,x)dt, \quad x \in \mathbb{R},$$

that, by the help of Hölder's inequality (which is strict as $f^{(k)}(\cdot,x)$ is not a multiple of $f^{(k+2)}(\cdot,x)$), gives

$$\left(\int_0^\infty f^{(k+2)}(t,x) f^{(k)}(t,x)dt\right)^2 < \int_0^\infty \left(f^{(k+2)}(t,x)\right)^2 dt \int_0^\infty \left(f^{(k)}(t,x)\right)^2 dt.$$

The latter is in fact equivalent to

$$\psi^{(k+2)}(x)\psi^{(k)}(x) - \psi^{(k+1)}(x)^2 > 0.$$

□

Appendix B

Supplemental Material for Chapter 2

In the following, we complete the results of Chapter 2 by presenting the missing proofs and an auxiliary result. We thereby adopt the setting and the notations from that chapter.

B.1 Proofs

Proof of Lemma 2.4.3.

Let $k \in \mathbb{N} \cup \{0\}$ be given and fixed, and define $\Lambda(x) := (x-c)\psi^{(k+1)}(x) - \psi^{(k)}(x)$, $x \in \mathbb{R}$. We then have the following.

(i) For $x \leq c$, it is readily seen that $\Lambda(x) < 0$.

(ii) One has $\Lambda(x) > 0$ for all $x > c + \frac{\psi^{(k)}(c)}{\psi^{(k+1)}(c)}$. To see this, rewrite $\Lambda(x) = \psi^{(k)}(x)\big[(x - c)\frac{\psi^{(k+1)}(x)}{\psi^{(k)}(x)} - 1\big]$, and notice that by Lemma A.1.2

$$\left(\frac{\psi^{(k+1)}(x)}{\psi^{(k)}(x)}\right)' = \frac{\psi^{(k+2)}(x)\psi^{(k)}(x) - (\psi^{(k+1)}(x))^2}{(\psi^{(k)}(x))^2} > 0.$$

Hence, for all $x > c + \frac{\psi^{(k)}(c)}{\psi^{(k+1)}(c)} > c$ one has that $\frac{\psi^{(k+1)}(x)}{\psi^{(k)}(x)} > \frac{\psi^{(k+1)}(c)}{\psi^{(k)}(c)}$, which implies

$$(x-c)\frac{\psi^{(k+1)}(x)}{\psi^{(k)}(x)} - 1 > (x-c)\frac{\psi^{(k+1)}(c)}{\psi^{(k)}(c)} - 1 > 0,$$

for all $x > c + \frac{\psi^{(k)}(c)}{\psi^{(k+1)}(c)}$. The latter clearly gives $\Lambda(x) > 0$ for all $x > c + \frac{\psi^{(k)}(c)}{\psi^{(k+1)}(c)}$.

Since $\Lambda'(x) = (x-c)\psi^{(k+2)}(x) > 0$ for all $x > c$, we conclude from (i) and (ii) that there exists a unique solution on (c, ∞) to the equation $\Lambda(x) = 0$ by continuity of Λ.

Proof of Lemma 2.4.4.

We argue by contradiction, and we suppose $x_\infty \geq x_0$. Then by definition of x_0 and x_∞ we have

$$x_0 - x_\infty = (x_0 - c) - (x_\infty - c) = \frac{\psi(x_0)}{\psi'(x_0)} - \frac{\psi'(x_\infty)}{\psi''(x_\infty)}. \tag{B.1}$$

Since by Lemma A.1.2

$$\Big(\frac{\psi(x)}{\psi'(x)}\Big)' = \frac{\psi'(x)^2 - \psi(x)\psi''(x)}{\psi'(x)^2} < 0, \quad \text{for any } x \in \mathbb{R},$$

we have by (B.1) that

$$x_0 - x_\infty \geq \frac{\psi(x_\infty)}{\psi'(x_\infty)} - \frac{\psi'(x_\infty)}{\psi''(x_\infty)} > 0,$$

again due to Lemma A.1.2. But this contradicts $x_\infty \geq x_0$.

Proof of Lemma 2.4.8.

First of all notice that for the existence of a solution z to (2.54) it is necessary that $y - z \geq 0$ since $F \geq 0$, and that $x - \alpha z \in (x_\infty, x_0]$ since the domain of F is $(x_\infty, x_0]$. Hence, if a solution to (2.54) exists, it must be such that $z(x,y) \in (\frac{x-x_0}{\alpha}, \frac{x-x_\infty}{\alpha} \wedge y]$, for all $(x,y) \in \mathbb{S}_2$.

Let $(x,y) \in \mathbb{S}_2$ with $y > F(x)$ be given and fixed, and define $R(z) = y - z - F(x - \alpha z)$, for $z \in (\frac{x-x_0}{\alpha}, \frac{x-x_\infty}{\alpha} \wedge y)$. Then, one has $R(0) = y - F(x) > 0$ and $\lim_{z \uparrow (\frac{x-x_\infty}{\alpha} \wedge y)} R(z) < 0$. Since $z \mapsto R(z)$ is strictly decreasing (by strict monotonicity of F) it follows that there exists a unique solution to (2.54).

Finally, (2.55) follows by noticing that 0 solves (2.54) when $y = F(x)$ and by uniqueness of the solution. Analogously, (2.56) follows by noticing that $\frac{x-x_0}{\alpha}$ uniquely solves (2.54), since $F(x_0) = 0$.

Proof of Lemma 2.5.4.

The first equality in (2.90) follows from (2.88). In order to prove the last inequality in (2.90), we find by Lemma A.1.2-(2) that

$$\frac{\sigma^2}{2}\psi^{(k+2)}(x;a,\sigma) + (a - bx)\psi^{(k+1)}(x;a,\sigma) - (\rho + kb)\psi^{(k)}(x;a,\sigma) = 0. \tag{B.2}$$

From (B.2), recalling that $\psi^{(k+1)} > 0$, we obtain

$$(a - bx) = -\frac{\sigma^2\psi^{(k+2)}(x;a,\sigma)}{2\psi^{(k+1)}(x;a,\sigma)} + (\rho + kb)\frac{\psi^{(k)}(x;a,\sigma)}{\psi^{(k+1)}(x;a,\sigma)}.$$

and we thus have

$$\begin{aligned}
&(a-bx)\Big[\psi^{(k+1)}(x;a,\sigma)^2-\psi^{(k)}(x;a,\sigma)\psi^{(k+2)}(x;a,\sigma)\Big]+b\psi^{(k+1)}(x;a,\sigma)\psi^{(k)}(x;a,\sigma)\\
=&(\rho+(k+1)b)\psi^{(k)}(x;a,\sigma)\psi^{(k+1)}(x;a,\sigma)-(\rho+kb)\psi^{(k)}(x;a,\sigma)^2\frac{\psi^{(k+2)}(x;a,\sigma)}{\psi^{(k+1)}(x;a,\sigma)}\\
&+\underbrace{\frac{\sigma^2\psi^{(k+2)}(x;a,\sigma)}{2\psi^{(k+1)}(x;a,\sigma)}\Big[\psi^{(k)}(x;a,\sigma)\psi^{(k+2)}(x,a,\sigma)-\psi^{(k+1)}(x;a,\sigma)^2\Big]}_{>0\text{ by Lemma A.1.2}}\\
>&\frac{\psi^{(k)}(x;a,\sigma)}{\psi^{(k+1)}(x;a,\sigma)}\Big[(\rho+(k+1)b)\psi^{(k+1)}(x;a,\sigma)^2-(\rho+kb)\psi^{(k)}(x,a,\sigma)\psi^{(k+2)}(x;a,\sigma)\Big].
\end{aligned}$$

We now aim at establishing that the last term on the right-hand side of the latter equation is positive. With regard to (2.90), this would clearly imply that $\frac{\partial(\psi^{(k)}(x;a,\sigma)/\psi^{(k+1)}(x;a,\sigma))}{\partial\sigma}>0$. From (B.2) we have

$$(\rho+(k+1)b)\psi^{(k+1)}(x;a,\sigma)=\frac{\sigma^2}{2}\psi^{(k+3)}(x;a,\sigma)+(a-bx)\psi^{(k+2)}(x;a,\sigma),$$

which then yields

$$\begin{aligned}
&\frac{\psi^{(k)}(x;a,\sigma)}{\psi^{(k+1)}(x;a,\sigma)}\Big[(\rho+(k+1)b)\psi^{(k+1)}(x;a,\sigma)^2-(\rho+kb)\psi^{(k)}(x;a,\sigma)\psi^{(k+2)}(x;a,\sigma)\Big]\\
=&\frac{\psi^{(k)}(x;\sigma)}{\psi^{(k+1)}(x;a,\sigma)}\Big[\frac{\sigma^2}{2}\psi^{(k+3)}(x;a,\sigma)\psi^{(k+1)}(x;a,\sigma)\\
&+\psi^{(k+2)}(x;a,\sigma)\big((a-bx)\psi^{(k+1)}(x;a,\sigma)-(\rho+kb)\psi^{(k)}(x;a,\sigma)\big)\Big]\\
=&\frac{\sigma^2}{2}\frac{\psi^{(k)}(x;\sigma)}{\psi^{(k+1)}(x;a,\sigma)}\Big[\psi^{(k+3)}(x;a,\sigma)\psi^{(k+1)}(x;a,\sigma)-\psi^{(k+2)}(x;a,\sigma)^2\Big]>0,
\end{aligned}$$

where the last equality follows again by an application of (B.2), and the last inequality by Lemma A.1.2. Hence $\frac{\partial(\psi^{(k)}(x;a,\sigma)/\psi^{(k+1)}(x;a,\sigma))}{\partial\sigma}>0$ and the proof is completed.

B.2 An Auxiliary Result

Lemma B.2.1. *Let x_0 be the solution to* (2.47) *and*

$$\bar{x}:=\frac{a+\rho c}{\rho+b}. \tag{B.3}$$

We have

$$\bar{x}<x_0.$$

Proof. Define $H(x):=(x-c)\psi'(x)-\psi(x)$, $x\in\mathbb{R}$. Since ψ satisfies

$$\frac{\sigma^2}{2}\psi''(x)+(a-bx)\psi'(x)-\rho\psi(x)=0,\quad\text{for all } x\in\mathbb{R},$$

and $\frac{\sigma^2}{2}\psi''(x) > 0$, we find $-\psi(x) < -\frac{(a-bx)}{\rho}\psi'(x)$, $\forall x \in \mathbb{R}$. Thus, we have

$$H(\bar{x}) < (\bar{x} - c)\psi'(\bar{x}) - \frac{(a - b\bar{x})}{\rho}\psi'(\bar{x}) = \Big[(\bar{x} - c)\rho - (a - b\bar{x})\Big]\frac{\psi'(\bar{x})}{\rho} = 0,$$

by the definition of $\bar{x}$. Since $H(x_0) = 0$, $H(x) < 0$ for all $x < x_0$ and $H(x) > 0$ for all $x > x_0$, it must necessarily be $\bar{x} < x_0$. □

Appendix C

Supplemental Material for Chapter 4

This appendix to Chapter 4 provides the missing proofs and an auxiliary result to complete the results of that chapter. Its setting and its notations are adopted.

C.1 Proofs

Proof of Proposition 4.3.1.

The proof employs arguments from the proof of Proposition 2.3.1 in Chapter 2 that are adjusted to the present setting. In a first step we prove that (4.7) holds true, and then, in a second step we show the monotonicity property of V.

Step 1. Let $(x, y) \in \mathbb{R} \times [0, \bar{y}]$ be given and fixed. To prove the lower bound of V, we take the admissible (non-)installation strategy I^0, and since $y \in [0, \bar{y}]$, we obtain

$$V(x, y) \geq R(x, y) > -K_1\big(1 + |x|\big), \tag{C.1}$$

for some $K_1 > 0$.

To determine the upper bound of V, recall the uncontrolled price process X^x from (4.1), and notice that by an application of Itô's formula we find for any $\tilde{\rho} > 0$

$$|e^{-\tilde{\rho}t} X_t^x| \leq |x| + \tilde{\rho} \int_0^t e^{-\tilde{\rho}u} |X_u^x| du + \int_0^t e^{-\tilde{\rho}u} \kappa(|\mu| + |X_u^x|) du + \bigg| \int_0^t e^{-\tilde{\rho}u} \sigma dW_u \bigg|,$$

which in turn implies

$$\mathbb{E}\bigg[\sup_{t\geq 0} e^{-\tilde{\rho}t} |X_t^x|\bigg] \leq |x| + C_1 \bigg(1 + \int_0^\infty e^{-\tilde{\rho}u} \mathbb{E}\big[|X_u^x|\big] du\bigg) + \sigma \mathbb{E}\bigg[\sup_{t\geq 0} \bigg| \int_0^t e^{-\tilde{\rho}u} dW_u \bigg|\bigg], \tag{C.2}$$

for some $C_1 > 0$. An application of the Burkholder-Davis-Gundy inequality (cf. Theorem 3.28 in Chapter 3 of [81]) yields

$$\mathbb{E}\bigg[\sup_{t\geq 0} e^{-\tilde{\rho}t}|X_t^x|\bigg] \leq |x| + C_1\bigg(1+\int_0^\infty e^{-\tilde{\rho}u}\mathbb{E}\big[|X_u^x|\big]du\bigg) + C_2\mathbb{E}\bigg[\bigg(\int_0^\infty e^{-2\tilde{\rho}u}du\bigg)^{\frac{1}{2}}\bigg]. \tag{C.3}$$

for a constant $C_2 > 0$, and therefore, we find upon using Lemma A.1.1 with $q = 1$

$$\mathbb{E}\bigg[\sup_{t\geq 0} e^{-\tilde{\rho}t}|X_t^x|\bigg] \leq C\big(1+|x|\big), \tag{C.4}$$

for some constant $C > 0$.

Now, for any $I \in \mathcal{I}^{\bar{y}}(y)$ we find by (C.4)

$$\begin{aligned}\mathcal{J}(x,y,I) &\leq \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}X_t^{x,y,I}Y_t^{y,I}dt\bigg] \leq \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}X_t^{x}Y_t^{y,I}dt\bigg] \\ &\leq \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\big|X_t^x\big|Y_t^{y,I}dt\bigg] \leq \bar{y}\mathbb{E}\bigg[\int_0^\infty e^{-\frac{\rho}{2}t}|e^{-\frac{\rho}{2}t}X_t^x|dt\bigg] \leq K_2\big(1+|x|\big),\end{aligned} \tag{C.5}$$

for some $K_2 > 0$, and upon observing that $X^{x,y,I} \leq X^x$ $\mathbb{P}$-a.s. for any $I \in \mathcal{I}^{\bar{y}}(y)$. Finally, from (C.1) and (C.5), we have that (4.7) holds with $K = \max(K_1, K_2)$.

Step 2. If $y = \bar{y}$, then the only admissible strategy is I^0, thus $V(x,\bar{y}) = R(x,\bar{y})$. In order to show that $x \mapsto V(x,y)$ is increasing, let $x_2 > x_1$, and notice that one has $X_t^{x_2,y,I} \geq X_t^{x_1,y,I}$ $\mathbb{P}$-a.s. for any $t \geq 0$ and $I \in \mathcal{I}^{\bar{y}}(y)$. Thus $\mathcal{J}(x_2,y,I) \geq \mathcal{J}(x_1,y,I)$ which implies $V(x_2,y) \geq V(x_1,y)$.

Proof of Lemma 4.4.1.

In the following, *Step 1* proves the positivity and the monotonicity property of the function A, while *Step 2* provides both the representation of A and the lower bound of F.

Step 1. Recalling that $R_{yx}(x,y) = (\rho+\kappa)^{-1}$ for all $(x,y) \in \mathbb{R}\times[0,\bar{y}]$, we find from (4.38) that

$$A'(y) = -\beta\frac{\psi''(F(y)+\beta y)}{\psi'(F(y)+\beta y)}A(y) - ((\rho+\kappa)\psi'(F(y)+\beta y))^{-1} = \mathcal{H}(F(y)+\beta y, A(y)), \tag{C.6}$$

where $\mathcal{H} : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ is such that

$$\mathcal{H}(\bar{F},A) = -\beta\frac{\psi''(\bar{F})}{\psi'(\bar{F})}A - \big((\rho+\kappa)\psi'(\bar{F})\big)^{-1} = -\big((\rho+\kappa)\psi'(\bar{F})\big)^{-1}\big(\beta(\rho+\kappa)\psi''(\bar{F})A + 1\big).$$

In light of the boundary condition $w(x,\bar{y}) = R(x,\bar{y})$ (cf. Theorem 4.3.2), we must have that

$$A(\bar{y}) = 0. \tag{C.7}$$

Due to (C.7) and the fact that $\mathcal{H}|_{\mathbb{R}\times[0,\infty)}$ is strictly negative as $\psi^{(k)}$ is strictly positive for any $k \in \mathbb{N}_0$ (cf. Lemma A.1.2-(2)), we conclude that A is both strictly positive and strictly decreasing.

Step 2. Equations (4.37) and (4.38) lead to

$$A(y) = \beta^{-1} \times \frac{\psi'(F(y)+\beta y)\Big(c - R_y(F(y),y)\Big) + (\rho+\kappa)^{-1}\,\psi(F(y)+\beta y)}{\psi'(F(y)+\beta y)^2 - \psi''(F(y)+\beta y)\psi(F(y)+\beta y)}. \tag{C.8}$$

Lemma A.1.2-(3) ensures that the denominator of A is nonzero. Now, the numerator on the right-hand side of (C.8) writes as

$$\begin{aligned}
&(\rho(\rho+\kappa))^{-1}\left[\rho(\rho+\kappa)\psi'(F(y)+\beta y)\,(c - R_y(F(y),y)) + \rho\psi(F(y)+\beta y)\right] \\
&= (\rho(\rho+\kappa))^{-1}\left[(\rho+\kappa)\Big(c\rho + \frac{\kappa\beta}{\rho+\kappa}y - F(y)\Big)\psi'(F(y)+\beta y) + \frac{\sigma^2}{2}\psi''(F(y)+\beta y)\right],
\end{aligned}$$

upon using (4.33) with $k = 0$. Hence,

$$A(y) = (\beta\rho(\rho+\kappa))^{-1} \times \frac{(\rho+\kappa)\Big(c\rho + \frac{\kappa\beta}{\rho+\kappa}y - F(y)\Big)\psi'(F(y)+\beta y) + \frac{\sigma^2}{2}\psi''(F(y)+\beta y)}{\psi'(F(y)+\beta y)^2 - \psi''(F(y)+\beta y)\psi(F(y)+\beta y)}. \tag{C.9}$$

Due to the facts that the denominator on the right-hand side of (C.9) is strictly negative by Lemma A.1.2-(3) and that A is strictly positive by *Step 1*, the numerator on the right-hand side of (C.9) must be strictly negative: this is possible only if

$$c\rho + \frac{\kappa\beta}{\rho+\kappa}y - F(y) < 0,$$

as $\psi^{(k)}$ is strictly positive for any $k \in \mathbb{N}$. Hence, F satisfies

$$F(y) > c\rho + \frac{\kappa\beta}{\rho+\kappa}y \geq c\rho, \quad \text{for all } y \in [0,\bar{y}]. \tag{C.10}$$

C.2 An Auxiliary Result

Lemma C.2.1. *For any $(y,z) \in \mathbb{R}\times\mathbb{R}$ such that $D(y,z) \geq 0$, we have*

$$N(y,z) > D(y,z).$$

Proof. To show the implication, we exploit a result from Chapter 3: we obtain from the proof of Theorem 3.2.1, cf. *Step 1* therein, that the function $\Psi_k : \mathbb{R} \mapsto \mathbb{R}$, $k \in \mathbb{N}_0$, defined by

$$\Psi_k(x) = \frac{\psi^{(k+1)}(x)^2}{\psi^{(k)}(x)\psi^{(k+2)}(x)},$$

is strictly increasing.

Now, recall (4.48), and let $(y,z) \in \mathbb{R} \times \mathbb{R}$ be such that $D(y,z) \geq 0$. The previous inequality implies

$$(\rho + \kappa)\left(c - \tilde{R}(z,y)\right) \geq -\frac{Q_0'(z)}{Q_1(z)}, \tag{C.11}$$

as Q_1 is strictly positive.

In order to proceed, we introduce the function $\Phi : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ such that

$$\Phi(z) := \psi''(z)Q_0(z) - \psi(z)Q_1(z).$$

Exploiting the monotonicity of Ψ_0, we find that Φ is strictly positive. Now, we use both (C.11) and the positivity of Φ to get

$$\begin{aligned} N(y,z) - D(y,z) &= (\rho + \kappa)\left(c - \tilde{R}(z,y)\right)\Phi(z) + \frac{2(\rho+\kappa)}{\rho}\psi'(z)Q_0(z) - \psi(z)Q_0'(z) \\ &\geq -\frac{Q_0'(z)}{Q_1(z)}\Phi(z) + \frac{2(\rho+\kappa)}{\rho}\psi'(z)Q_0(z) - \psi(z)Q_0'(z) \\ &= (\rho Q_1(z))^{-1} Q_0(z)\Big[-\rho\psi''(z)Q_0'(z) + 2(\rho+\kappa)\psi'(z)Q_1(z)\Big], \end{aligned} \tag{C.12}$$

where we have rearranged terms after the equality. To finish the proof, we employ (4.33) with $k = 0,1,2$ for (C.12), so to obtain

$$N(y,z) - D(y,z) \geq \frac{\sigma^2}{2}(\rho Q_1(z))^{-1} Q_0(z)\Big[\psi'''(z)Q_1(z) - \psi'(z)Q_2(z)\Big] > 0, \tag{C.13}$$

where the last inequality holds true upon recalling $Q_k > 0$ and by the fact that $\psi'''(z)Q_1(z) - \psi'(z)Q_2(z) > 0$ since Ψ_1 is strictly increasing. $\square$

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