On the superposition principle for linear and nonlinear Fokker–Planck–Kolmogorov equations on Hilbert spaces

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Chapter 1

Introduction

Consider the $\mathbb{R}^n$-valued stochastic differential equation (in short: SDE) of the form

$$\begin{aligned} \mathrm{d}X(t) &= b\big(t, X(t)\big)\,\mathrm{d}t + \sigma\big(t, X(t)\big)\,\mathrm{d}W(t), \\ X(0) &= x_{0,n} \end{aligned} \qquad (\mathrm{SDE}_n)$$

on $[0,T]$ for some $T > 0$, where $(W(t))_{t\in[0,T]}$ is an n-dimensional Wiener process with respect to a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and $x_{0,n}$ is some vector in $\mathbb{R}^n$. Furthermore, b and σ are Borel measurable mappings from $[0,T]\times\mathbb{R}^n$ to $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$, respectively.

This kind of equation appears in various fields of science, where the effects of random perturbations are decisive for the dynamics of the described phenomenon. Among them are most prominently physics (e.g. Langevin equations), economics (e.g. the Black–Scholes model) and biology (e.g. models on population growth or epidemics).

The study of existence and uniqueness of solutions to Equation (SDE_n) has been a very active topic of mathematical research during the last 75 years. Naturally, this created a variety of approaches with different notions of solution and assumptions on the coefficients b and σ necessary for it. On the one hand, results considering strong solutions to Equation (SDE_n) go back to those with Lipschitz conditions by K. Itô (see e.g. [Itô42; Itô46; Itô51]) as part of his original Itô-calculus. From there, plenty of famous works generalized and improved these assumptions. Some of the most striking ones are e.g. [Ver80] by A. Y. Veretennikov, where b can be bounded and σ some non-degenerate multiplicative noise, or for singular drifts [KR05] by N. V. Krylov and M. Röckner, where equations with locally integrable b and additive noise σ were studied, and [Zha05] by X. Zhang, where b can again be locally integrable, but σ even a continuous uniformly non-degenerate Sobolev diffusion.

Tanaka's example (see e.g. [RW87, Example 16.5, p. 150]), on the other hand, undeniably shows the importance of the study of weak solutions to Equation (SDE_n). Here, in particular the results of A. V. Skorokhod (see [Sko61; Sko62]), where the coefficients b and σ are assumed to be continuous functions of at most linear growth, stand out the most. The famous Yamada–Watanabe theorem (see [YW71]) is of great importance for connecting those two notions by only having to prove pathwise uniqueness to obtain a strong solution from a weak solution.

A variation of the "weak" approach was developed by D. W. Stroock and S.R.S. Varadhan (see [SV69; SV79]) and is famously known as the martingale problem. While this method does not involve the SDE explicitly, it is in fact still an equivalent formulation to the idea of weak solutions (see e.g. [Kur11, p. 114f]). Let us note, that we will be solely interested in the notion of a martingale solution in the original sense of Stroock–Varadhan while working with SDEs throughout this thesis and do not focus on the, in the literature often naturally established, link to weak solutions.

Instead of considering Equation (SDE_n) directly, we set $A := \frac{1}{2}\sigma\sigma^*$ with $A = (a^{ij})_{1\leq i,j\leq n}$ and consider the corresponding Kolmogorov operator L, which is given by

$$L\varphi(t,y) = \sum_{i,j=1}^{n} a^{ij}(t,y)\partial_{y_i}\partial_{y_j}\varphi(y) + \sum_{i=1}^{n} b^i(t,y)\partial_{y_i}\varphi(y)$$

if the functions $\varphi\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ are sufficiently smooth.

The fundamental idea of Stroock–Varadhan's approach is to study well-posedness of the martingale problem, i.e. to prove existence and uniqueness of probability measures P_n on the path space $C([0,T];\mathbb{R}^n)$ satisfying $P_n[x_n(0) = x_{0,n}] = 1$ and ensuring that, for all $f \in C_c^\infty(\mathbb{R}^n)$, the process

$$f(x_n(t)) - f(x_{0,n}) - \int_0^t Lf(s, x_n(s))\,\mathrm{d}s \tag{1}$$

is a martingale with respect to P_n, where x_n is the canonical process on $C([0,T];\mathbb{R}^n)$. Such a measure P_n is then called martingale solution to the martingale problem for L with coefficients A and b starting from $x_{0,n}$.

The operator L is also a cornerstone for another type of equation, namely for Fokker–Planck–Kolmogorov equations (in short: FPKE), named after A. Fokker (see [Fok14]), M. Planck (see [Pla17]) and A. Kolmogorov (see [Kol31; Kol33; Kol37]), which are second order elliptic or parabolic equations for measures. Together with some initial distribution in a suitable sense, the considered equation is a Cauchy problem (in short: CP), written in shorthand notation

$$\begin{aligned} \partial_t \mu_n &= L^* \mu_n, \\ \mu_n{\restriction_{t=0}} &= \nu_n, \end{aligned} \tag{CP_n}$$

where L^* is the formal adjoint (with respect to the spatial variable) of the operator L and ν_n is a Borel probability measure on $\mathbb{R}^n$. A so-called probability solution μ_n of the form $\mu_n = \mu_{t,n}\mathrm{d}t$ to Equation (CP_n) is a family of Borel probability measures $(\mu_{t,n})_{t\in[0,T]}$ for which, for every function $\varphi \in C_c^\infty(\mathbb{R}^n)$, the integral equality

$$\int_{\mathbb{R}^n} \varphi(y)\,\mu_{t,n}(\mathrm{d}y) = \int_{\mathbb{R}^n} \varphi(y)\,\nu_n(\mathrm{d}y) + \int_0^t \int_{\mathbb{R}^n} L\varphi(s,y)\,\mu_{s,n}(\mathrm{d}y)\,\mathrm{d}s \tag{2}$$

is satisfied for all $t \in [0,T]$.

Since the operator L appears both in the martingale problem and Cauchy problem, an obvious question would be if and how probability and martingale solutions are connected. Let us mention that this problem is actually most interesting and useful if we have no information on uniqueness (as we will see below). Hence, we will restrict our considerations within this thesis to a setting with no assumptions on uniqueness, neither for the martingale problem nor for the Cauchy problem.

One direction of the connection is pretty simple and has been known right from the start: Assume we are given a martingale solution P_n, then we know that the process in

Equation (1) is a continuous martingale. Hence, we can take expectations with respect to P_n yielding (under suitable integrability conditions) that

$$\begin{aligned} 0 &= \int f(x_n(t))\,\mathrm{d}P_n - \int f(x_{0,n})\,\mathrm{d}P_n - \int\int_0^t Lf(s,x_n(s))\,\mathrm{d}s\,\mathrm{d}P_n \\ &= \int_{\mathbb{R}^n} f(y)\; P_n\circ x_n(t)^{-1}(\mathrm{d}y) - \int_{\mathbb{R}^n} f(y)\,\varepsilon_{x_{0,n}}(\mathrm{d}y) - \int_0^t\int_{\mathbb{R}^n} Lf(s,y)\; P_n\circ x_n(s)^{-1}(\mathrm{d}y)\,\mathrm{d}s \end{aligned}$$

holds. Consequently, we can choose the measures $\mu_{t,n} := P_n \circ x_n(t)^{-1}$ which satisfy Equation (2) for the initial Dirac measure $\nu_n := \varepsilon_{x_{0,n}}$. This means that our probability solution is directly created by the martingale solution through its 1-marginal laws (also called time-marginals).

Finite-dimensional superposition principle. The other direction was established by the Ambrosio–Figalli–Trevisan superposition principle (in short: superposition principle), which goes back to the work of L. Ambrosio, A. Figalli and D. Trevisan (see especially [Amb08; Fig08; Tre16]).

Under the assumed integrability condition

$$\int_0^T\int_{\mathbb{R}^n} \left(\|A(t,y)\| + \|b(t,y)\|_{\mathbb{R}^n}\right)\mu_{t,n}(\mathrm{d}y)\,\mathrm{d}t < \infty,$$

D. Trevisan proved that existence of a "narrowly continuous" probability solution to an FPKE implies existence of a martingale solution such that its 1-marginals coincide with the given probability solution, i.e. that

$$P_n \circ x_n(t)^{-1} = \mu_{t,n} \tag{SP$_n$}$$

holds for every $t \in [0,T]$. We refer to Chapter 5 of this thesis for more details.

This seminal result was the starting point for several further articles exploring this direction in recent times. For example, see [RXZ19] for a version for nonlocal FPKEs, [BR18a] for a version for nonlinear FPKEs, or [BRS19], in which a weakened integrability condition is imposed.

For us, Trevisan's superposition principle on $\mathbb{R}^n$ in [Tre16] raised the question if we can prove an infinite-dimensional analogue. Actually, one could directly answer this in the affirmative, because there is already a version in [Tre14, Section 7.1] on $\mathbb{R}^\infty$ equipped with the product topology (i.e. solutions of the martingale problem are probability measures on $C([0,T];\mathbb{R}^\infty)$), but we are rather interested in the Hilbert space case with respect to the norm topology, which is in itself a very different approach to the problem. In general, we have to impose stricter (but still commonly used) compactness assumptions to ensure that the constructed martingale solutions are supported on a path space with values in a separable Hilbert space and continuity with respect to the norm topology of e.g. another, larger separable Hilbert space instead of on $C([0,T];\mathbb{R}^\infty)$ with its componentwise continuity.

The mentioned interest is substantial because, first of all, many applications typically have their setting in Hilbert spaces. More importantly, our initial motivation is to study the following related problem for which the superposition principle on $\mathbb{R}^\infty$ is insufficient. In the articles [GRZ09] and [RZZ15], the authors construct a solution to an infinite-dimensional martingale problem on a separable Hilbert space as a weak limit of finite-dimensional solutions. The authors of [BDRS15] (see also [BKRS15, Section 10.4]) construct a solution to a corresponding infinite-dimensional Cauchy problem in a very similar way. In both cases, we do not have information on uniqueness of such solutions. This means, that with a solution to the martingale problem we obtain some solution to the Cauchy problem by simply setting $\mu_t := P \circ x(t)^{-1}$, as in finite dimensions, but we do not know if this is the particular kind of solution, with all of its properties, that has been constructed with the methods in [BDRS15].

Our approach for an answer is to study both problems simultaneously and to crucially make use of the finite-dimensional superposition principle. This way we solve the martingale problem with a solution that is determined by a solution of the FPKE through the 1-marginals. For a start and for simplicity, we impose the combination of both sets of respective assumptions, because we are confident that coefficients in potential applications will satisfy them anyway. We will later see in Chapter 7 that this is e.g. the case for stochastic Navier–Stokes equations, but there still is obvious potential for improvement.

The aim and content of this thesis is twofold, leading even a bit beyond that described problem to two versions of a superposition principle. In the first part we will, on the one hand, prove a joint existence theorem (see Theorem 6.3.1 below) for solutions of Cauchy problems for (linear) FPKEs and martingale problems on a separable Hilbert space $\mathbb{H}$ via superposition as described above. The constructed solutions then satisfy the infinite-dimensional analogue of Equation (SP_n) (see Equation (SP) below). On the other hand, from our method of construction, we directly obtain a restricted version of the superposition principle on $\mathbb{H}$ (see Corollary 6.3.4 below) to a subclass of solutions that can be represented as a limit of some (later specified) weakly convergent sequence. The second part will be of smaller scope taking up the idea of dealing with nonlinearity via "freezing", as presented in [BR18b, Section 2] and also in [BR18a, Section 2]. It contains an adaptation of our restricted superposition principle (see Theorem 10.2.1 below) to the case of Cauchy problems for nonlinear FPKEs and martingale problems related to so-called McKean–Vlasov equations (also called distribution dependent SDEs in the literature).

Before explaining those results in more detail, let us first briefly introduce infinite-dimensional martingale problems and FPKEs, whose mere concepts carry over from the finite-dimensional case pretty much analogously.

Martingale problems on Hilbert spaces. Under the assumption that there exists another separable Hilbert space $\mathbb{X}$ such that the embedding $\mathbb{X} \subseteq \mathbb{H} \simeq \mathbb{H}^* \subseteq \mathbb{X}^*$ is continuous, dense and compact and that $\{e_1, e_2, \ldots\} \subseteq \mathbb{X}$ for an orthonormal basis $\{e_1, e_2, \ldots\}$ of $\mathbb{H}$, we consider the Borel measurable mappings

$$\begin{aligned} \sigma\colon [0,T] \times \mathbb{H} &\longrightarrow L_2(\mathbb{U};\mathbb{H}), \\ b\colon [0,T] \times \mathbb{H} &\longrightarrow \mathbb{X}^*, \end{aligned}$$

where $\mathbb{U}$ is another separable Hilbert space. In this setting, we can study the following martingale problem on $\mathbb{H}$:

$$\begin{gathered} \text{Existence of a martingale solution } P \in \mathcal{P}(C([0,T];\mathbb{X}^*) \cap L^p([0,T];\mathbb{H})) \\ \text{in the sense of Stroock–Varadhan's martingale problem for the coefficients} \\ b \text{ and } \sigma \text{ and with initial value } x_0 \in \mathbb{H}. \end{gathered} \tag{MP}$$

An initial difference between the infinite-dimensional and the finite-dimensional case of being a solution in the sense of Stroock–Varadhan is that, for convenience, we choose to state the martingale property in the "weak formulation" (as e.g. in [GRZ09]; see Definition 3.2.1 below). This means, that for every $\ell \in \operatorname{span}\{e_1, e_2, \dots\}$ the process M_ℓ defined by

$$M_\ell(t,x) := {}_{\mathbb{X}^*}\langle x(t), \ell\rangle_{\mathbb{X}} - \int_0^t {}_{\mathbb{X}^*}\langle b\big(s, x(s)\big), \ell\rangle_{\mathbb{X}} \,\mathrm{d}s, \quad t \in [0,T],$$

has to be a continuous $(\mathcal{F}_t)$-martingale with respect to P, whose quadratic variation process is given by

$$\langle M_\ell\rangle(t,x) := \int_0^t \big\|\sigma^*\big(s, x(s)\big)(\ell)\big\|_{\mathbb{U}}^2 \,\mathrm{d}s, \quad t \in [0,T],$$

where x is the canonical process on $C([0,T];\mathbb{X}^*)$. Later in Corollary 6.3.2 we show that this "weak formulation" in fact implies the infinite-dimensional analogue of Equation (1) stated via the Kolmogorov operator L.

We follow the articles [GRZ09; RZZ15] (including their more general setting) in our considerations and mainly impose assumptions on demicontinuity, coercivity and growth (see Assumptions **(A1)**–**(A3)** in Subsection 3.2.3) on the coefficients. In Chapter 3 of this thesis we will present an elaborate, combined version of the main theorems in [GRZ09] and [RZZ15] proving existence of a martingale solution to the martingale problem (MP).

FPKEs in infinite dimensions. The mere concept of infinite-dimensional (linear) FPKEs also remains basically the same. Now, we identify $\mathbb{H}$ with ℓ^2, denote by $\{e_1, e_2, \dots\}$ the standard orthonormal basis in ℓ^2 and consider the continuous and dense embedding $\ell^2 \subseteq \mathbb{R}^\infty$, where $\mathbb{R}^\infty$ is equipped with the product topology, thus becoming a Polish space. Then we focus on the Borel measurable mappings

$$\begin{aligned} a^{ij} \colon [0,T] \times \mathbb{R}^\infty &\longrightarrow \mathbb{R}, \\ b^{i} \colon [0,T] \times \mathbb{R}^\infty &\longrightarrow \mathbb{R} \end{aligned}$$

for every $i, j \in \mathbb{N}$.

They induce a Kolmogorov operator L that is acting on suitable finitely based functions, i.e. any function $\varphi \colon \mathbb{R}^\infty \longrightarrow \mathbb{R}$ that is, for some $d \in \mathbb{N}$, a function of class C^2 in finitely many variables $y^1, \dots, y^d$, which is given by

$$L\varphi(t,y) = \sum_{i,j=1}^{d} a^{ij}(t,y)\partial_{e_i}\partial_{e_j}\varphi(y) + \sum_{i=1}^{d} b^{i}(t,y)\partial_{e_i}\varphi(y),$$

for $(t,y) \in [0,T] \times \mathbb{R}^\infty$. Then the infinite-dimensional Cauchy problem can be written in shorthand notation as

$$\begin{aligned} \partial_t \mu &= L^*\mu, \\ \mu_{\restriction t=0} &= \nu, \end{aligned} \tag{CP}$$

where L^* is again the formal adjoint (with respect to the spatial variable) of L and ν is a Borel probability measure on $\mathbb{R}^\infty$.

Now, a probability solution μ on $[0,T] \times \mathbb{R}^\infty$ of the form $\mu = \mu_t \mathrm{d}t$ to Equation (CP) (see Definition 4.2.1 below) is a family of Borel probability measures $(\mu_t)_{t\in[0,T]}$ on $\mathbb{R}^\infty$ for which, for every function φ in a suitable space of finitely based functions (depending on the basis vectors $\{e_1, e_2, \dots\}$), we have

$$\int_{\mathbb{R}^\infty} \varphi(y)\,\mu_t(\mathrm{d}y) = \int_{\mathbb{R}^\infty} \varphi(y)\,\nu(\mathrm{d}y) + \int_0^t \int_{\mathbb{R}^\infty} L\varphi(s,y)\,\mu_s(\mathrm{d}y)\,\mathrm{d}s$$

for $\mathrm{d}t$-a.e. $t \in [0,T]$.

We follow the Monograph [BKRS15, Section 10.4] in our considerations and mainly impose assumptions on symmetry and definiteness of the matrices $(a^{ij})_{1\le i,j\le n}$, on a compact function Θ, on a Lyapunov function V, including the Lyapunov condition

$$LV(t,y) \le C_0 V(y) - \Theta(y)$$

(which we assume to hold on finite-dimensional subspaces), and on the growth and continuity of the coefficients $A = (a^{ij})_{1\le i,j<\infty}$ and b (see Assumptions **(H1)**–**(H4)** in Subsection 4.2.3). In Chapter 4 of this thesis we will present an elaborate version of the main theorem in [BKRS15, Section 10.4] proving existence of a probability solution to Equation (CP).

Joint existence theorem via superposition. Under the combined assumptions of Chapters 3 and 4 (see Subsection 6.2.2 below), we prove a joint existence theorem generating solutions P and μ that satisfy $P \circ x(t)^{-1} = \mu_t$ for every $t \in [0,T]$ while keeping their individual properties/estimates proved before during their separate construction. For that, we identify the space $\mathbb{H}$ with ℓ^2 and $\mathbb{X}$ with the weighted ℓ^2-space $\ell^2(\lambda_i)$ for some sequence $(\lambda_i)_{i\in\mathbb{N}}$ converging to ∞. Then we can consider the embedding

$$\ell^2(\lambda_i) \subseteq \ell^2 \subseteq \ell^2\big(\tfrac{1}{\lambda_i}\big) \subseteq \mathbb{R}^\infty$$

and extend our given functions b and σ from Chapter 3 to $\mathbb{R}^\infty$ by 0 to obtain suitable components b^i and a^{ij} as in Chapter 4.

The main theorem of the first part (see Theorem 6.3.1 below) is stated as follows:

Theorem. Under the assumptions from Subsection 6.2.2 there exists a probability solution $\mu = \mu_t\,\mathrm{d}t$ on $[0,T] \times \mathbb{H}$ to the Cauchy problem (CP) in the sense of Definition 4.2.1 and a martingale solution $P \in \mathcal{P}\big(C([0,T];\mathbb{X}^*) \cap L^p([0,T];\mathbb{H})\big)$ to the martingale problem (MP) in the sense of Definition 3.2.1, for which the 1-marginal laws of P coincide with μ_t, i.e.

$$P \circ x(t)^{-1} = \mu_t \tag{SP}$$

holds for every $t \in [0,T]$. In particular, Estimates (3.3.1) and (4.3.2) as well as Equation (4.3.3) hold.

From Equation (SP) we also conclude that the mapping $t \longmapsto \mu_t$ is continuous with respect to the topology generated by finitely based functions (see Corollary 6.3.3 below).

As mentioned before, the separate assumptions for the martingale problem (MP) and Cauchy problem (CP) directly ensure existence for both individual solutions, but without any additional information (e.g. on uniqueness) we a priori cannot specify their connection. Hence, the interesting part of the theorem is in fact the method of proof, which is making use of the finite-dimensional superposition principle to control the 1-marginals in the limit.

Idea of proof of Theorem 6.3.1. Let us explain the idea of proof of Theorem 6.3.1 by first showing a figure that describes the scheme which we will follow.

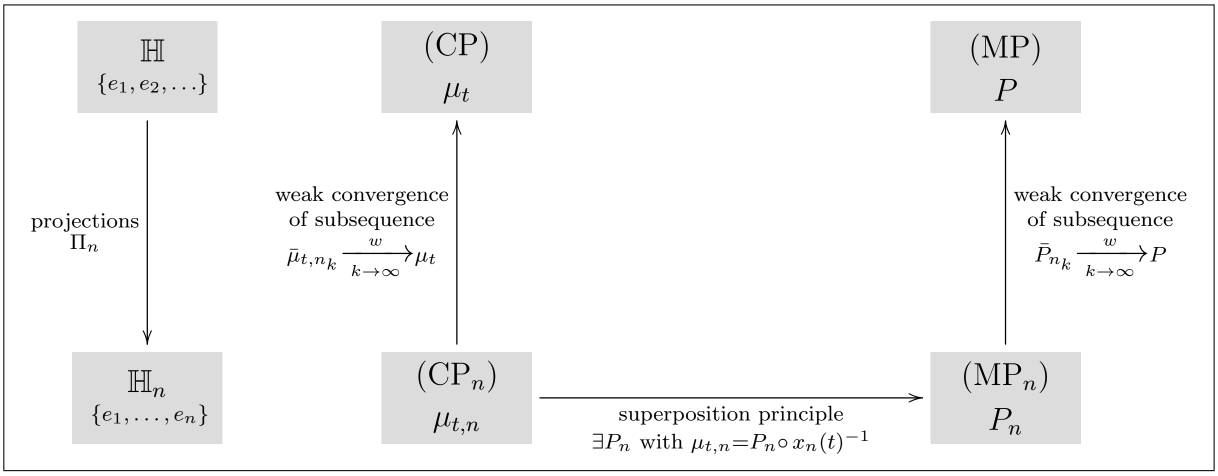

Figure 1.1: Idea and scheme of the proof of the joint existence theorem.

Starting with an infinite-dimensional setting on $\mathbb{H}$, we use Galerkin approximations and, therefore, project everything via mappings Π_n onto the finite-dimensional spaces $\mathbb{H}_n$, that are defined to be the linear span of $\{e_1, \dots, e_n\}$. On $\mathbb{H}_n$ we ensure that, for any $n \in \mathbb{N}$, there exist solutions $\mu_{t,n}$ to the Cauchy problems (CP_n) with coefficients $\Pi_n b$ and $\Pi_n A\,\Pi_n^*$ and that we can apply a version of Trevisan's finite-dimensional superposition principle to them, which we present in Chapter 5 of this thesis in detail.

Then we obtain, on the one hand, a family of probability solutions $(\mu_{t,n})_{n\in\mathbb{N}}$ and, on the other hand, directly for every $n \in \mathbb{N}$ constructed by the superposition principle, a family of martingale solutions $(P_n)_{n\in\mathbb{N}}$, that are connected by Equation (SP_n). By using the same techniques and calculations that we comprehensively study in the existence proofs in Chapters 3 and 4, we prove tightness of both families. From there, we conclude existence of subsequences $(\mu_{t,n_k})_{k\in\mathbb{N}}$ and $(P_{n_k})_{k\in\mathbb{N}}$ (by a diagonal argument on a joint index set) that each converges weakly to a solution of the respective infinite-dimensional equation, i.e. μ_t and P from the figure above. Most importantly, we show that the 1-marginal laws of this infinite-dimensional martingale solution coincide with the probability solution of the infinite-dimensional Cauchy problem, i.e. that Equation (SP) holds.

In short, we note that the concurrent study of weak convergence of finite-dimensional Cauchy and martingale solutions, the correspondence of their infinite-dimensional limits μ and P via Equation (SP) as well as the method by which P is basically "generated" by the family $(\mu_{t,n})_{n\in\mathbb{N}}$ of solutions to (CP_n) via the finite-dimensional superposition principle are the key points of this result.

Superposition principle on Hilbert spaces. From the scheme of proof of Theorem 6.3.1, we directly obtain a corollary (see Corollary 6.3.4 below), which is a restricted superposition principle on $\mathbb{H}$ to a subclass of solutions.

Its conditional formulation is closely related to the original statement of Trevisan's finite-dimensional superposition principle, but too extensive for an introduction. Let us break down its core essence as follows:

If we are given any solution μ to the Cauchy problem (CP), for which there already exists a subsequence $(\mu_{t,n_k})_{k\in\mathbb{N}}$ of finite-dimensional solutions being created by Galerkin approximations and converging weakly to μ as well as the necessary integrability conditions and assumptions for the corresponding martingale problem, we obtain a solution P to the martingale problem (MP) satisfying Equation (SP).

We acknowledge that this subclass of solutions to Equation (CP) for which the restricted superposition principle holds is difficult to identify. However, we can directly see that it is a convex set.

Future research. One next step for future research is to unify the collected assumptions in Subsection 6.2.2. In particular, the coercivity from the part on martingale problems should be replaced by an assumption similar to the Lyapunov condition seen in the part on FPKEs.

Obviously, we are also interested in proving the restricted superposition principle from Corollary 6.3.4 for larger or at least easier to identify subclasses of solutions to an FPKE than a family of solutions that can be represented as limits of some weakly convergent subsequences of certain finite-dimensional solutions. But, since this requires further research, we see our result as a first step and “proof of concept” in that direction.

In Chapter 8 of this thesis, we briefly elaborate on more directions of research that might benefit from our studies. Among them are restricted well-posedness for FPKEs and martingale problems and existence of flows for FPKEs.

Application: d-dimensional stochastic Navier–Stokes equation. In Chapter 7, we discuss one possible application for the methods studied in Theorem 6.3.1, namely d-dimensional stochastic Navier–Stokes equations. We focus on showing the connection between the already existing example for FPKEs in [BDRS15, Example 3.5, p. 17f], which can partly also be found in [BKRS15, Example 10.1.6, p. 411f and Example 10.4.3, p. 425f], and the one for martingale problems in [GRZ09, Chapter 6, p. 1749ff] and [RZZ15, Section 5.1, p. 377f] in order to fit everything into our combined framework. For clarifications on the basic setting we will refer to the classical book [Tem77] by R. Temam (and also partly to the article [FG95]).

In short, we will show that the solution to the FPKE of the d-dimensional stochastic Navier–Stokes equation obtained in [BDRS15, Example 3.5] is indeed identical with the 1-marginals of a solution to the corresponding martingale problem.

Nonlinear superposition principle. In the second part of the thesis we adapt Corollary 6.3.4 to a nonlinear version of the restricted superposition principle on $\mathbb{H}$ (see Theorem 10.2.1 below).

For this, we consider coefficients b and σ that can in addition explicitly depend on measures, i.e.

$$\begin{aligned} b&\colon [0,T]\times\mathbb{H}\times\mathcal{P}(\mathbb{H}) \longrightarrow \mathbb{X}^*,\\ \sigma&\colon [0,T]\times\mathbb{H}\times\mathcal{P}(\mathbb{H}) \longrightarrow L_2(\mathbb{U},\mathbb{H}).\end{aligned}$$

This leads to Cauchy problems for nonlinear FPKEs, where this dependence is on the solution itself, i.e. we study the Kolmogorov operator L_μ given by

$$L_\mu\varphi(t,y) = \sum_{i,j} a^{ij}(t,y,\mu_t)\partial_{e_i}\partial_{e_j}\varphi(y) + \sum_i b^i(t,y,\mu_t)\partial_{e_i}\varphi(y)$$

for sufficiently smooth finitely based functions $\varphi\colon \mathbb{R}^\infty \longrightarrow \mathbb{R}$. For an extensive list of applications of nonlinear FPKEs we refer to [Fra05, p. 8].

Now, the conditional formulation of Corollary 6.3.4 allows us to make use of the idea in [BR18a] and [BR18b, Section 2] on "freezing" of a nonlinear solution. This means, that if we are given a probability solution μ to a nonlinear FPKE

$$\partial_t \mu = L_\mu^* \, \mu,$$

we fix this μ and consider the linear FPKE

$$\partial_t \varrho = L_\mu^* \, \varrho$$

for which μ is a particular solution. This allows us to apply results for linear Cauchy problems, but now with coefficients depending on some fixed measure μ_t. In our case, we will assume that the assumptions on our coefficients are uniform in the measure-component (see Assumptions **(NN)**, **(NA1)**–**(NA3)** and **(NH1)** in Section 10.1 below) and, hence, satisfy all assumptions necessary for Corollary 6.3.4.

What's more, the martingale solution that we obtain for the martingale problem with coefficients $b(\cdot,\cdot,\mu_\cdot)$ and $\sigma(\cdot,\cdot,\mu_\cdot)$ is connected to so-called McKean–Vlasov SDEs (as we explain in Subsection 9.3.2).

McKean–Vlasov equations are distribution dependent stochastic differential equations (sometimes also shortly called DDSDEs in the literature) of the form

$$\mathrm{d}X(t) = b\big(t, X(t), \mathcal{L}_{X(t)}\big)\,\mathrm{d}t + \sigma\big(t, X(t), \mathcal{L}_{X(t)}\big)\,\mathrm{d}W(t),$$

where the coefficients b and σ can explicitly depend on the law of the $X(t)$.

Their name goes back to A. Vlasov (see e.g. the reprinted article [Vla68]), who originally proposed an idea for this kind of equation in 1938, and H. McKean (see [McK66; McK67]), who was the first to study them systematically. Besides many classical results (see e.g. [Fun84], [Szn84], [Sch87]) following up in this direction, there has been growing interest in this field of research recently, resulting in new finite-dimensional works like [Wan18], [MV16], [HW19], [HŠS18], [RST18], [BR18a], [HRW19] and [CG19]. Infinite-dimensional results for McKean–Vlasov equations are, however, less common, but e.g. studied in [AD95], [KX95, Chapter 9] and lately in [BM19] as well as in two master theses at Bielefeld University, from which the one by R. Heinemann, that also covers delay, is in preparation to be published in the near future.

Structure of the thesis.
In Chapter 2, we state some basic notation and essentials used throughout the thesis.

Part I:
Chapter 3 begins with a detailed explanation of the framework (see Sections 3.1 and 3.2 below) for martingale problems on Hilbert spaces including the notion of a martingale solution in Subsection 3.2.1. In Section 3.3 we state the existence theorem (see Theorem 3.3.1 below) for which we need all assumptions imposed before in Subsection 3.2.3. Section 3.4 contains a comprehensive proof for both the infinite-dimensional (see Subsection 3.4.1 below) and the finite-dimensional (see Subsection 3.4.2 below) case. In particular, Lemma 3.4.2 is proved there, which comprises a crucial a priori energy estimate. The main references for this chapter are [GRZ09; RZZ15].

In Chapter 4, we first introduce the necessary framework (see Sections 4.1 and 4.2 below) for infinite-dimensional Cauchy problems including the notion of a probability solution in Subsection 4.2.2. Imposing the assumption from Subsection 4.2.3, we then state two existence theorems (see Theorems 4.3.1 and 4.3.2 below) in Section 4.3. Section 4.4 is devoted to an extensive proof of Theorem 4.3.1, which is preceded by a lemma on countable measure-separating families of functions (see Lemma 4.4.1 below). Section 10.4 of the Monograph [BKRS15] is our reference of choice for this chapter.

Chapter 5 starts with the necessary framework (see Section 5.1.1 below) for FPKEs and martingale problems in finite dimensions. After a historical overview in Section 5.2 discussing results from the articles [Amb08], [Fig08] and [Tre16], we state a finite-dimensional superposition principle (see Theorem 5.3.1 below) in Section 5.3, which was proved in [BRS19] and will be our reference of choice when applying the superposition principle to probability solutions in finite dimensions later in Chapter 6.

In Chapter 6 we combine our considerations from Chapters 3–5. After recalling the essential framework (see Sections 6.1 and 6.2), we collect all necessary assumptions in Subsection 6.2.2. In Section 6.3 we state the main result of this first part of the thesis, i.e. a joint existence theorem (see Theorem 6.3.1 below), as well as further properties of the solutions (see Corollaries 6.3.2 and 6.3.3). In addition, we also state Corollary 6.3.4, a restricted superposition principle on $\mathbb{H}$. We devote Subsection 6.3.1 to following up on questions concerning consistency, which appeared and were mentioned in the prior chapters, in particular those concerning the definition of martingale and probability solutions in finite dimensions. Finally, in Section 6.4 we give a comprehensive proof of Theorem 6.3.1.

Chapter 7 consists of the application of the methods studied in Theorem 6.3.1 to d-dimensional stochastic Navier–Stokes equations. The main references are [BDRS15; GRZ09; RZZ15] and supplementing basics are taken from [Tem77; FG95].

Finally, in Chapter 8 we give a short outlook on future research.

Part II:
Chapter 9 begins in Section 9.1 with laying out the essential framework for the nonlinear case. In Section 9.2 we explain all necessary details for Cauchy problems for nonlinear FPKEs, including the corresponding notion of solution in Definition 9.2.1, which is a generalization of Chapter 4. Subsequently, in Section 9.3 we concentrate on martingale problems for coefficients depending explicitly on a fixed measure and their connection to McKean–Vlasov equations, including the corresponding notion of solution in Definition 9.3.1.

In Chapter 10, after imposing all necessary assumptions in Section 10.1, we state and prove in Section 10.2 the main result of this second part of the thesis, i.e. a nonlinear version of the restricted superposition principle on $\mathbb{H}$ (see Theorem 10.2.1 below).

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Chapter 2

Mathematical preliminaries

Let us begin by stating some basic notation and essentials used throughout this thesis.

For a topological space U the expression $\mathcal{B}(U)$ will denote the Borel-σ-algebra of U. A map $f\colon U_1 \longrightarrow U_2$ between topological spaces U_1 and U_2 is said to be Borel-measurable if it is $\mathcal{B}(U_1)/\mathcal{B}(U_2)$-measurable. Sometimes f will also simply be called Borel function.

Spaces of measures. We will use the following notation for spaces of measures on some measurable space $(S, \mathcal{B}(S))$:

$\mathcal{P}(S)$	space of all Borel probability measures,
$\mathcal{M}_+(S)$	space of all nonnegative finite Borel measures.

Spaces of continuous functions. For an open set $\Gamma \subseteq \mathbb{R}^d$, two topological spaces U_1 and U_2, a metric space V_2 and a normed space X_2, we will use the following notation for spaces of continuous functions:

$C(U_1; U_2)$	cont. functions from U_1 to U_2,
$C_b(U_1; V_2)$	cont. functions from U_1 to V_2 that are bounded,
$C_c(U_1; X_2)$	cont. functions from U_1 to X_2 with compact support,
$C^k(\Gamma; \mathbb{R})$	cont. functions from Γ to $\mathbb{R}$ with k continuous derivatives

and all types of admissible combinations of the above, supplemented most prominently by

$C_c^\infty(\Gamma; \mathbb{R})$	smooth functions from Γ to $\mathbb{R}$ with compact support.

The notation $C(U_1)$ always refers to the case where $U_2 = \mathbb{R}$ (analogously for the other spaces).

For a sequence of Borel measures $(\mu_n)_{n\in\mathbb{N}}$ on a measurable space $(S, \mathcal{B}(S))$ that converges weakly to a Borel measure μ on $(S, \mathcal{B}(S))$, i.e. for every $f \in C_b(S)$ we have

$$\lim_{n\to\infty} \int_S f \,\mathrm{d}\mu_n = \int_S f \,\mathrm{d}\mu,$$

we write $\mu_n \xrightarrow[n\to\infty]{w} \mu$. If this equation holds for every $f \in C_c(S)$, the convergence is called vague and denoted by $\mu_n \xrightarrow[n\to\infty]{v} \mu$.

Furthermore, recall that a family of probability measures M on a topological space U is called tight if for every $\varepsilon > 0$ there exits a compact set $K_\varepsilon \subseteq U$ such that

$$\mu(U \setminus K_\varepsilon) < \varepsilon$$

for every $\mu \in M$.

L^p/ℓ^p-spaces. Let $p \in [1,\infty)$. For a measure space $(S, \mathcal{S}, \mu)$ and a normed space $(X, \mathcal{B}(X))$ with norm $\|\cdot\|_X$, we will use the following notation for L^p-spaces:

$L^p(S; X, \mu)$ space of equivalence classes of μ-measurable functions $f\colon S \longrightarrow X$ such that $\|f\|_{L^p} := \left(\int_S \|f\|_X^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} < \infty.$

The notation $L^p(S; X)$ always refers to the classical Lebesgue measure and $L^p(S, \mu)$ means the case where $X = \mathbb{R}$. If we in particular write $L^p_{\text{loc}}([0,T]\times\mathbb{R}^d, \mu)$, we mean L^p-integrable on compact sets in $[0,T]\times\mathbb{R}^d$.

For $p \in [1,\infty)$, we use the following notation for the special case of sequence spaces:

ℓ^p space of all sequences $(y_n)_{n\in\mathbb{N}}$, $y_n \in \mathbb{R}$, such that $\sum_{n\in\mathbb{N}} |y_n|^p < \infty$,

and for the weighted ℓ^p-space for some arbitrary sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \geq 0$:

$\ell^p(\lambda_n)$ space of all sequences $(y_n)_{n\in\mathbb{N}}$, $y_n \in \mathbb{R}$, such that $\sum_{n\in\mathbb{N}} \lambda_n |y_n|^p < \infty$.

Spaces of finitely based functions. Let $\{e_1, e_2, \ldots\}$ be the standard orthonormal basis in the separable Hilbert space ℓ^2. Consider the continuous and dense embedding

$$\ell^2 \subseteq \mathbb{R}^\infty.$$

Then we define the following classes of so-called finitely based functions given by

$$\begin{aligned} \mathcal{F}C^2(\{e_i\}) &:= \big\{f\colon \mathbb{R}^\infty \longrightarrow \mathbb{R} \,\big|\, f(y) = g\big(y^1,\ldots,y^d\big), d\in\mathbb{N}, g\in C^2(\mathbb{R}^d)\big\}, \\ \mathcal{F}C_c^\infty(\{e_i\}) &:= \big\{f\colon \mathbb{R}^\infty \longrightarrow \mathbb{R} \,\big|\, f(y) = g\big(y^1,\ldots,y^d\big), d\in\mathbb{N}, g\in C_c^\infty(\mathbb{R}^d)\big\}, \\ &\Big(\text{where we can write } f(y) = g\big(\langle y, e_1\rangle_{\ell^2},\ldots,\langle y, e_d\rangle_{\ell^2}\big) \text{ if } y \in \ell^2\Big), \end{aligned} \tag{2.0.1}$$

(see e.g. [MR92, p. 54] or [BKRS15, p. 404f]).

Note that by identifying the finite-dimensional space $\mathbb{H}_n := \operatorname{span}\{e_1,\ldots,e_n\}$ with $\mathbb{R}^n$, a vector $(y^1,\ldots,y^n,0,\ldots) \in \mathbb{H}_n$ can be treated as $(y^1,\ldots,y^n) \in \mathbb{R}^n$. In return, we will tacitly treat any $y \in \mathbb{R}^n$ whenever necessary as an element in $\mathbb{R}^\infty$ by considering $(y^1,\ldots,y^n,0,\ldots)$ in the following.

Sobolev spaces. Let $p \in [1,\infty)$, $k \in \mathbb{N}\cup\{0\}$ and let $\Gamma \subseteq \mathbb{R}^d$ be an open set. Then we denote Sobolev spaces as follows:

$W^{k,p}(\Gamma)$ space of functions $f \in L^p(\Gamma)$ such that all of its mixed partial weak derivatives up to order k are of class $L^p(\Gamma)$, i.e. $W^{k,p}(\Gamma) := \{f \in L^p(\Gamma) \mid D^{\boldsymbol{\alpha}} f \in L^p(\Gamma) \text{ for all } |\boldsymbol{\alpha}| \leq k\}$. The space is equipped with the norm

$$\|u\|_{W^{k,p}} := \sum_{|\boldsymbol{\alpha}| \leq k} \|D^{\boldsymbol{\alpha}} u\|_{L^p}.$$

For $p = 2$, the norm is induced by the inner product

$$\langle u_1, u_2 \rangle_{W^{k,2}} := \sum_{|\boldsymbol{\alpha}| \leq k} \langle D^{\boldsymbol{\alpha}} u_1, D^{\boldsymbol{\alpha}} u_2 \rangle_{L^2}$$

making $W^{k,2}(\Gamma)$ a Hilbert space. We choose to use the common notation $H^{2,k}$ instead of $W^{k,2}$, which in particular helps us to better distinguish from Wiener processes which we usually denote by W.

Hilbert–Schmidt operators and matrices. If A is an operator, then A^* denotes its adjoint operator and $\operatorname{tr} A$ its trace. For two separable Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, let $L_2\big(\mathbb{H}_1; \mathbb{H}_2\big)$ be the space of all Hilbert–Schmidt operators from $\mathbb{H}_1$ to $\mathbb{H}_2$ with inner product $\langle \cdot, \cdot \rangle_{L_2(\mathbb{H}_1;\mathbb{H}_2)}$ and norm $\|\cdot\|_{L_2(\mathbb{H}_1;\mathbb{H}_2)}$. Note that for an operator $L \in L_2\big(\mathbb{H}_1; \mathbb{H}_2\big)$ we have $L^* \in L_2\big(\mathbb{H}_2; \mathbb{H}_1\big)$ and $\|L\|_{L_2(\mathbb{H}_1;\mathbb{H}_2)} = \|L^*\|_{L_2(\mathbb{H}_2;\mathbb{H}_1)}$ for its dual operator. For more information on Hilbert–Schmidt operators we refer to [LR15, Appendix B].

A symmetric matrix $M \in \mathbb{R}^{d \times d}$ is called positive definite if $y^T M y > 0$ for all $y \in \mathbb{R}^d \setminus \{0\}$. M is said to be positive semidefinite or nonnegative definite if $y^T M y \geq 0$ for all $y \in \mathbb{R}^d$.

Miscellaneous. We will use the standard notation for partial derivatives from differential calculus

$$\partial_t f := \frac{\partial f}{\partial t}, \quad \partial_{x_i} f := \frac{\partial f}{\partial x_i}, \quad \partial_{x_i} \partial_{x_j} f := \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

On a normed space $(X, \|\cdot\|_X)$ we denote by $\operatorname{dist}(x, A) := \inf\{\|x - y\|_X \mid y \in A\}$ the distance of a point $x \in X$ to a set $A \subseteq X$. By $B_r(a)$ we denote the open ball of radius r centered at point a. The Kronecker delta will be denoted by

$$\delta_{i,j} := \begin{cases} 1, & i = j, \\ 0, & \text{else.} \end{cases}$$

Finally, by span we denote the linear span of a set of vectors in a vector space.

Part I

On the superposition principle for linear FPKEs

Chapter 3

Martingale problems on Hilbert spaces

The study of martingale problems famously goes back to D. W. Stroock and S.R.S. Varadhan, whose seminal work was initiated in the late 1960s (see e.g. [SV69]) and, after being further developed and considerably extended, eventually published in the book "Multidimensional Diffusion Processes" in 1979 (see [SV79]).

Assume that we have some given initial value $x_0 \in \mathbb{R}^d$, some bounded Borel-measurable function $A = (a^{ij})_{1 \leq i,j \leq d}$ on $[0, \infty) \times \mathbb{R}^d$ taking values in the space of symmetric nonnegative definite real $d \times d$-matrices and some bounded Borel-measurable $\mathbb{R}^d$-valued function b on $[0, \infty) \times \mathbb{R}^d$. Furthermore, consider the operator L, which is given by

$$L\varphi(t, y) = \sum_{i,j=1}^{d} a^{ij}(t, y)\partial_{y_i}\partial_{y_j}\varphi(y) + \sum_{i=1}^{d} b^i(t, y)\partial_{y_i}\varphi(y)$$

if the functions $\varphi\colon \mathbb{R}^d \longrightarrow \mathbb{R}$ are sufficiently smooth. Putting it simple, the fundamental idea of their approach is to study well-posedness (under additional assumptions) of the martingale problem, i.e. to prove existence and uniqueness of probability measures P on the path space $C([0, \infty); \mathbb{R}^d)$ satisfying $P[x(0) = x_0] = 1$ and ensuring that, for all $f \in C_c^\infty(\mathbb{R}^d)$, the process

$$f(x(t)) - f(x_0) - \int_0^t Lf(s, x(s))\,\mathrm{d}s$$

is a martingale with respect to P, where x is the canonical process. Such a measure P is then called martingale solution to the martingale problem for L with coefficients A and b starting from x_0.

This martingale problem a priori does not explicitly involve any stochastic differential equation, but we can directly associate it via weak solutions (see e.g. [Kur11, p. 114f] or [Øks03, Section 8.3, p. 146f], where local boundedness of the coefficients is imposed) to an SDE of the form

$$\begin{aligned} \mathrm{d}X(t) &= b\big(t, X(t)\big)\,\mathrm{d}t + \sigma\big(t, X(t)\big)\,\mathrm{d}W(t), \\ X(0) &= x_0 \end{aligned}$$

by setting $A = \frac{1}{2}\sigma\sigma^T$. Hence, it is reasonable and quite common in the literature to link the martingale solution P directly to this SDE. We refer to [SV79, Chapter 6] and e.g. [KS91, Section 5.4, p. 311ff] for a detailed description of the origin, framework and connections of this theory. However, we do not focus on the explicit step to establish the link to weak solutions within this thesis.

From there, martingale problems became an intensively studied field of research, in particular due to their strong connection to the theory of semigroups and also to PDEs (see e.g. [EK86] or [RW87] and the references therein). The approach was extended to the infinite-dimensional case in the following years (see e.g. [Tud84], [Mét88, Chapter V], [GG94a; GG94b] and the references therein), while initial results on Hilbert spaces can even be tracked back to the Thesis of M. Viot (see [Vio76]) and for Banach spaces to the work of E. Dettweiler (see e.g. [Det89; Det92]) as it is described in [BG99, p. 187].

We are especially interested in results on separable Hilbert spaces with assumptions on continuity, coercivity and growth of the coefficients. Therefore, this chapter will be based on two recent articles that are both considering martingale problems and martingale solutions in infinite dimensions. The first one is “Martingale solutions and Markov selections for stochastic partial differential equations” by B. Goldys, M. Röckner and X. Zhang, which appeared in SPA in 2009 (see [GRZ09]). The second one is “Existence and uniqueness of solutions to stochastic functional differential equations in infinite dimensions” by M. Röckner, R. Zhu and X. Zhu, which appeared in Nonlinear Analysis in 2015 (see [RZZ15]).

We will present a straightforward combination of both articles in a suitable setting for its later use in the superposition principle (see Chapter 6 below) and, besides, provide additional elaboration on key steps of the main proof including some minor modifications and improvements.

This turned out to be most convenient, because while being very similar (or even analogue) in many ideas, results and proofs (actually, [RZZ15] is based on [GRZ09]), there are some small differences that would make it inconvenient to refer to exactly one of them without explaining a lot of details on results remaining true in our specific setting. Moreover, since we will be directly applying and repeating techniques and calculations from the proof of the main theorem (see Theorem 3.3.1 below) in our proof of the main theorem of Chapter 6 (see Section 6.4 below), it is beneficial to already give a presentation in all detail at this point.

Let us quickly describe some of the major differences between [RZZ15] and [GRZ09]. On the one hand, stochastic evolution equations with delay are considered in [RZZ15], which makes proofs and ideas more complicated as well as assumptions on the coefficients way more complex (and in fact more general) than necessary for our setting without delay.

On the other hand, [GRZ09] is a more extensive article also covering Markov selections and families. In particular, the authors consider stochastic evolution equations, but only in the “autonomous case”, i.e. with drift and diffusion coefficients that are not explicitly depending on t as a parameter. Furthermore, an additional property, which is called (M3) in [GRZ09, p. 1730], is imposed in the definition of a martingale solution. This property is dropped as a requirement for being a solution and instead transformed into an a priori estimate in [RZZ15, Lemma 3.1, p. 368] as well as in our proof (see Lemma 3.4.2 below). Apart from that, most assumptions and ideas are in a way closer related to our setting. Therefore, we tend to follow the structure of [GRZ09] more often compared to [RZZ15].

Let us note, that we will not consider any uniqueness results, even though they are covered in [RZZ15, Chapter 4, p. 373ff].

In short, we will state and prove Theorem 3.3.1, a result on existence of martingale solutions on a separable Hilbert space $\mathbb{H}$ under assumptions on continuity, coercivity and growth of our coefficients b and σ. The main idea of proof will be Galerkin approximations, i.e. we will construct a solution as the limit of solutions to finite-dimensional martingale problems obtained by projecting onto finite-dimensional spaces $\mathbb{H}_n$.

3.1 Framework

Let $\mathbb{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and norm $\| \cdot \|_{\mathbb{H}}$. Furthermore, let $\mathbb{X}$ be another separable Hilbert space and let $\mathbb{Y}$ be a separable, reflexive Banach space with norms $\| \cdot \|_{\mathbb{X}}$ and $\| \cdot \|_{\mathbb{Y}}$, respectively, such that we have

$$\mathbb{X} \subseteq \mathbb{Y} \subseteq \mathbb{H}$$

continuously and densely as well as $\mathbb{X} \subseteq \mathbb{Y}$ compactly. By identifying $\mathbb{H}$ and its dual space $\mathbb{H}^*$ via the Riesz isomorphism, we obtain

$$\mathbb{X} \subseteq \mathbb{Y} \subseteq \mathbb{H} \simeq \mathbb{H}^* \subseteq \mathbb{Y}^* \subseteq \mathbb{X}^*,$$

where $\mathbb{Y}^*$ and $\mathbb{X}^*$ are the dual spaces of $\mathbb{Y}$ and $\mathbb{X}$, respectively.

In addition, let $\mathbb{U}$ be another separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathbb{U}}$ and norm $\| \cdot \|_{\mathbb{U}}$ and let $\{u_1, u_2, \dots\}$ be an orthonormal basis of $\mathbb{U}$.

Remark. The assumption that the embedding $\mathbb{X} \subseteq \mathbb{Y}$ is compact implies that $\mathbb{X} \subseteq \mathbb{H}$ is compact and, therefore, the embeddings $\mathbb{Y}^* \subseteq \mathbb{X}^*$ and $\mathbb{H} \subseteq \mathbb{X}^*$ are compact as well. This follows, on the one hand, from the fact that the composition of a compact and a continuous embedding operator is again compact (see e.g. [Zei90, Proposition 21.35, p. 265]) and, on the other hand, from Schauder's theorem (see e.g. [Yos80, p. 282]), which states that if the embedding operator $i_{X,Y}$ of Banach spaces X and Y is compact, then its dual operator i_{Y^*,X^*} is compact as well.

Remark. It follows from Kuratowski's theorem (see e.g. [Kur66, p. 487f] or [Par67, Section I.3, p. 15ff]) that we have $\mathbb{X} \in \mathcal{B}(\mathbb{Y})$, $\mathbb{Y} \in \mathcal{B}(\mathbb{H})$ and $\mathcal{B}(\mathbb{Y}) = \mathcal{B}(\mathbb{H}) \cap \mathbb{Y}$, $\mathcal{B}(\mathbb{X}) = \mathcal{B}(\mathbb{Y}) \cap \mathbb{X}$.

Remark. When later addressing the connection of martingale problems to weak solutions of an SDE (see Equation (3.2.2) below), $\mathbb{U}$ will be the space on which the appropriate cylindrical Wiener process $W(t)$, $t \geq 0$, with respect to a complete filtered probability space $\big(\breve{\Omega}, \breve{\mathcal{F}}, (\breve{\mathcal{F}}_t), \breve{P}\big)$ is defined.

We denote the dual pairing between $\mathbb{X}$ and $\mathbb{X}^*$ by

$${}_{\mathbb{X}^*}\langle z, v \rangle_{\mathbb{X}}$$

for $z \in \mathbb{X}^*$, $v \in \mathbb{X}$. Note that

$${}_{\mathbb{X}^*}\langle z, v \rangle_{\mathbb{X}} = \langle z, v \rangle_{\mathbb{H}}$$

holds if $z \in \mathbb{H}$ and that we always have the Cauchy–Schwarz type inequality

$${}_{\mathbb{X}^*}\langle z, v \rangle_{\mathbb{X}} \leq \|z\|_{\mathbb{X}^*} \|v\|_{\mathbb{X}} \tag{3.1.1}$$

for any $z \in \mathbb{X}^*$, $v \in \mathbb{X}$.

For an orthonormal basis $\{e_1, e_2, \dots\} \subseteq \mathbb{X}$ of $\mathbb{H}$ we define

$$\mathbb{H}_n := \operatorname{span}\{e_1, \dots, e_n\},$$

for every $n \in \mathbb{N}$, as well as $\mathcal{E} := \operatorname{span}\{e_1, e_2, \dots\}$, which is a dense subset of $\mathbb{X}$. Let $\Pi_n \colon \mathbb{X}^* \longrightarrow \mathbb{H}_n$ be defined by

$$\Pi_n z := \sum_{i=1}^{n} {}_{\mathbb{X}^*}\langle z, e_i \rangle_{\mathbb{X}} e_i, \quad z \in \mathbb{X}^*. \tag{3.1.2}$$

Since $\mathbb{X} \subset \mathbb{H}$ is compact, and hence so is $\mathbb{H} \subset \mathbb{X}^*$, we can in fact choose and fix the orthonormal basis $\{e_1, e_2, \dots\} \subseteq \mathbb{X}$ in such a way that

$$\|\Pi_n z\|_{\mathbb{X}^*} \leq \|z\|_{\mathbb{X}^*} \tag{3.1.3}$$

holds for every $n \in \mathbb{N}$ and $z \in \mathbb{X}^*$ (see [AR89, Proposition 3.5, p. 424]). Note that the restriction $\Pi_{n \restriction \mathbb{H}}$ is in fact the orthogonal projection onto $\mathbb{H}_n$ in $\mathbb{H}$, which justifies to refer to Π_n as a projection in the following. In addition, let $\breve{\Pi}_n$ be the orthogonal projection onto $\operatorname{span}\{u_1, \dots, u_n\}$ in $\mathbb{U}$.

Furthermore, denote by $\mathfrak{U}^\varrho$, for $\varrho \geq 1$, the class of functions $\mathcal{N} \colon \mathbb{Y} \longrightarrow [0, \infty]$ with the following properties:

(i) $\mathcal{N}(y) = 0$ implies $y = 0$,

(ii) $\mathcal{N}(cy) \leq c^\varrho \mathcal{N}(y)$ holds for every $c \geq 0$ and $y \in \mathbb{Y}$,

(iii) the set $\{y \in \mathbb{Y} \,|\, \mathcal{N}(y) \leq 1\}$ is compact in $\mathbb{Y}$.

Remark. From properties (i)–(iii) we conclude that, for any $\alpha > 0$, the sublevel sets $\{y \in \mathbb{Y} \,|\, \mathcal{N}(y) \leq \alpha\}$ are compact in $\mathbb{Y}$ since elements in $\{\mathcal{N} \leq \alpha\}$ are also in the set $\alpha^{\frac{1}{\varrho}}\{\mathcal{N} \leq 1\}$, which is compact as an image of a compact set under a continuous mapping. Hence, we in particular know that any function in $\mathfrak{U}^\varrho$ is lower semi-continuous on $\mathbb{Y}$.

Furthermore, we can extend a function $\mathcal{N} \in \mathfrak{U}^\varrho$ to a $\mathcal{B}(\mathbb{X}^*)/\mathcal{B}([0, \infty])$-measurable one on $\mathbb{X}^*$ by setting $\mathcal{N}(y) = \infty$ for $y \in \mathbb{X}^* \setminus \mathbb{Y}$. Then $\int_0^t \mathcal{N}\big(y(s)\big)\,\mathrm{d}s$ is defined for every $y \in C\big([0, \infty); \mathbb{X}^*\big)$. Note that $\mathcal{N}$, as a function on $\mathbb{X}^*$, is still lower semi-continuous since the embedding $\mathbb{Y} \subseteq \mathbb{X}^*$ is continuous and compact.

Let the mapping

$$b \colon [0, \infty) \times \mathbb{Y} \longrightarrow \mathbb{X}^*$$

be $\mathcal{B}([0, \infty)) \otimes \mathcal{B}(\mathbb{Y})/\mathcal{B}(\mathbb{X}^*)$-measurable and let

$$\sigma \colon [0, \infty) \times \mathbb{Y} \longrightarrow L_2(\mathbb{U}; \mathbb{H})$$

be $\mathcal{B}([0, \infty)) \otimes \mathcal{B}(\mathbb{Y})/\mathcal{B}\big(L_2(\mathbb{U}; \mathbb{H})\big)$-measurable. Let

$$\Omega := C\big([0, \infty); \mathbb{X}^*\big)$$

be the space of all continuous functions from $[0, \infty)$ to $\mathbb{X}^*$ equipped with the metric

$$\rho(z_1, z_2) := \sum_{m=1}^{\infty} \frac{1}{2^m} \Big(\sup_{r \in [0, m]} \|z_1(r) - z_2(r)\|_{\mathbb{X}^*} \wedge 1 \Big), \quad z_1, z_2 \in \Omega.$$

By $x\colon \Omega \longrightarrow \mathbb{X}^*$ we denote the canonical process on Ω given by $x(t,\omega) := \omega(t)$. We define the σ-algebra

$$\mathcal{F}_t := \sigma\big(x(s) \,\big|\, s \le t\big)$$

for every $t \ge 0$ and set $\mathcal{F} := \bigvee_{t\ge 0} \mathcal{F}_t$.

Finally, define

$$\mathbb{S} := C\big([0,\infty);\mathbb{X}^*\big) \cap L^p_{\mathrm{loc}}([0,\infty);\mathbb{Y}),$$

where the $p \ge 2$ is later to be specified in our assumptions (see Subsection 3.2.3). Note that $\mathbb{S}$, equipped with the metric given by $(z_1,z_2) \longmapsto \rho(z_1,z_2) + \|z_1 - z_2\|_{L^p}$ for $z_1, z_2 \in \mathbb{S}$, is a Polish space.

3.2 Equation, Solution, Assumptions

Based on the framework from Section 3.1 we can now introduce the martingale problem under consideration, the corresponding notion of a martingale solution and the necessary assumptions on the coefficients for the existence result in Section 3.3.

3.2.1 Martingale solution

Let us formalize the notion of a martingale solution for the martingale problem in the sense of Stroock–Varadhan with coefficients b and σ and initial value x_0, which is suitable in our infinite-dimensional setting.

Definition 3.2.1. *(martingale solution) A probability measure $P \in \mathcal{P}(\Omega)$ is called martingale solution to the martingale problem with coefficients b and σ and initial value $x_0 \in \mathbb{H}$ if the following conditions hold.*

(M1) $P\big[x(0) = x_0\big] = 1$, *and for every $k \in \mathbb{N}$*

$$P\Big[x \in \Omega \,\Big|\, \text{For } \mathrm{d}s\text{-a.e. } s \in [0,k] : x(s) \in \mathbb{Y} \text{ and } \int_0^k \big\|b(s,x(s))\big\|_{\mathbb{X}^*}\,\mathrm{d}s + \int_0^k \big\|\sigma(s,x(s))\big\|^2_{L_2(\mathbb{U};\mathbb{H})}\,\mathrm{d}s < \infty\Big] = 1.$$

(M2) *For every $\ell \in \mathcal{E}$ the process M_ℓ defined by*

$$M_\ell(t,x) := {}_{\mathbb{X}^*}\langle x(t), \ell\rangle_{\mathbb{X}} - \int_0^t {}_{\mathbb{X}^*}\langle b\big(s,x(s)\big), \ell\rangle_{\mathbb{X}}\,\mathrm{d}s, \quad t \ge 0,$$

is a continuous $(\mathcal{F}_t)$-martingale with respect to P, whose quadratic variation process is given by

$$\langle M_\ell\rangle(t,x) := \int_0^t \big\|\sigma^*\big(s,x(s)\big)(\ell)\big\|^2_{\mathbb{U}}\,\mathrm{d}s, \quad t \ge 0.$$

Remark. Assume we consider martingale solutions on the finite-dimensional space $\mathbb{R}^d$. Since Condition **(M2)** holds for all $\ell \in \mathcal{E}$, we in particular have that

$$x(t) - \int_0^t b(s, x(s))\, \mathrm{d}s = \sum_{i=1}^{d} {}_{\mathbb{X}^*}\langle x(t), e_i\rangle_{\mathbb{X}} e_i + \sum_{i=1}^{d} \int_0^t {}_{\mathbb{X}^*}\langle b\big(s, x(s)\big), e_i\rangle_{\mathbb{X}}\, \mathrm{d}s\, e_i$$

is an $\mathbb{R}^d$-valued $(\mathcal{F}_t)$-martingale with respect to P.

3.2.2 Martingale problem and associated SDE

With the notion of solution from Definition 3.2.1 in mind, we can now state what we mean by a martingale problem arising from given coefficients b and σ and an initial value $x_0 \in \mathbb{H}$. In concrete terms, we will consider the following martingale problem:

$$\text{Existence of a martingale solution } P \in \mathcal{P}(\mathbb{S}) \text{ in the sense of Definition 3.2.1 for coefficients } b \text{ and } \sigma \text{ and with initial value } x_0 \in \mathbb{H}, \tag{3.2.1}$$

where $P \in \mathcal{P}(\mathbb{S})$ means that we are explicitly searching for solutions that also require paths from the path space $C\big([0,\infty); \mathbb{X}^*\big)$ to be of class $L^p_{\mathrm{loc}}([0,\infty); \mathbb{Y})$.

Remark. To martingale problem (3.2.1) we can associate the infinite-dimensional stochastic differential equation

$$\begin{aligned} \mathrm{d}X(t) &= b\big(t, X(t)\big)\, \mathrm{d}t + \sigma\big(t, X(t)\big)\, \mathrm{d}W(t), \\ X(0) &= x_0, \end{aligned} \tag{3.2.2}$$

for $t \geq 0$ and some cylindrical Wiener process $W(t)$, $t \geq 0$, on $\mathbb{U}$, via weak solutions in a similar way as in the finite-dimensional case (as explained in [RZZ15, Theorem 2.2, p. 364] by using [Ond05, Theorem 2, p. 1007]). However, in this thesis we will only consider the martingale problem in the original sense of Stroock–Varadhan without explicitly involving this connection to SDEs.

3.2.3 Assumptions

Now let us state our assumptions on the coefficients b and σ necessary for the existence result in Section 3.3 (see Theorem 3.3.1 below).

(N) There exists a function $\mathcal{N} \in \mathfrak{U}^p$ for some $p \geq 2$ such that for every $n \in \mathbb{N}$ there exists a constant $C_n \geq 0$ with

$$\mathcal{N}(v) \leq C_n \|v\|^p_{\mathbb{H}_n},$$

for any $v \in \mathbb{H}_n$.

(A1) (Demicontinuity) For any $v \in \mathbb{X}$, $t \geq 0$ and every sequence $(y_k)_{k\in\mathbb{N}}$ with $y_k \xrightarrow[k\to\infty]{} y$ in $\mathbb{Y}$, we have

$$\lim_{k\to\infty} {}_{\mathbb{X}^*}\langle b(t, y_k), v\rangle_{\mathbb{X}} = {}_{\mathbb{X}^*}\langle b(t, y), v\rangle_{\mathbb{X}}$$

and

$$\lim_{k\to\infty} \big\|\sigma^*(t, y_k)(v) - \sigma^*(t, y)(v)\big\|_{\mathbb{U}} = 0.$$

(A2) (Coercivity) There exists a locally bounded measurable function $\lambda_1 \colon [0,\infty) \longrightarrow [0,\infty)$ such that for all $v \in \mathbb{X}$ and $t \geq 0$

$$ {}_{\mathbb{X}^*}\langle b(t,v), v\rangle_{\mathbb{X}} \leq -\mathcal{N}(v) + \lambda_1(t)(1 + \|v\|_{\mathbb{H}}^2) $$

holds.

(A3) (Growth) There exist locally bounded measurable functions $\lambda_2, \lambda_3, \lambda_4 \colon [0,\infty) \longrightarrow [0,\infty)$ and constants $\gamma' \geq \gamma > 1$ such that for all $y \in \mathbb{Y}$ and $t \geq 0$ we have

$$ \|b(t,y)\|_{\mathbb{X}^*}^{\gamma} \leq \lambda_2(t)\mathcal{N}(y) + \lambda_3(t)(1 + \|y\|_{\mathbb{H}}^{\gamma'}) $$

and

$$ \|\sigma(t,y)\|_{L_2(\mathbb{U};\mathbb{H})}^2 \leq \lambda_4(t)(1 + \|y\|_{\mathbb{H}}^2). $$

Remark. Note that in finite dimensions, the demicontinuity in Assumption **(A1)** actually yields continuity for the mappings $y \longmapsto b(t,y)$ and $y \longmapsto \sigma(t,y)$ for every $t \geq 0$.

3.3 Results

The following theorem on existence of martingale solutions to the martingale problem (3.2.1) is the main result of this chapter and is based on [RZZ15, Theorem 2.1, p. 363f] and [GRZ09, Theorem 4.6, p. 1739]. We will present a comprehensive proof in the next section.

Theorem 3.3.1. *Suppose that the Assumptions* ***(N)*** *and* ***(A1)–(A3)*** *hold. Then there exists a martingale solution $P \in \mathcal{P}(\mathbb{S})$ to the martingale problem* (3.2.1) *with initial value $x_0 \in \mathbb{H}$ in the sense of Definition 3.2.1. Furthermore, for every $q \geq 1$ and $T > 0$ we have*

$$ \mathbb{E}^P\left[\sup_{t\in[0,T]} \|x(t)\|_{\mathbb{H}}^{2q} + \int_0^T \|x(t)\|_{\mathbb{H}}^{2(q-1)}\mathcal{N}(x(t))\,\mathrm{d}t\right] < \infty. \tag{3.3.1} $$

3.4 Proof

Before we can start with the proof, we will first introduce a lemma from [GRZ09] that is the essential tool to prove tightness of a family of probability measures on $C([0,\infty);\mathbb{X}^*)$ in the following.

Lemma 3.4.1 (see [GRZ09], Lemma 4.3, p. 1734)**.** *Let $(P_n)_{n\in\mathbb{N}}$ be a family of probability measures on $\Omega = C([0,\infty);\mathbb{X}^*)$ and $\mathcal{N}$ as in Assumption* ***(N)****. Assume that $\mathbb{X}$ is compactly embedded into $\mathbb{H}$ and that for some $\beta > 0$ and any $T > 0$*

$$ \sup_{n\in\mathbb{N}} \mathbb{E}^{P_n}\left[\sup_{t\in[0,T]} \|x(t)\|_{\mathbb{H}} + \sup_{s,t\in[0,T],\, s\neq t} \frac{\|x(t)-x(s)\|_{\mathbb{X}^*}}{|t-s|^{\beta}} + \int_0^T \mathcal{N}(x(s))\,\mathrm{d}s\right] < \infty. $$

Then $(P_n)_{n\in\mathbb{N}}$ is tight in $\mathbb{S}\ \big(= C\big([0,\infty);\mathbb{X}^*\big) \cap L^p_{loc}([0,\infty);\mathbb{Y})\big)$.

Proof. A proof is given in [GRZ09, Appendix C.1, p. 1759f]. □

Against the first intuition, we will present the infinite-dimensional proof (see Subsection 3.4.1) before the finite-dimensional one (see Subsection 3.4.2). Hence, for now, we just have to assume existence of solutions for our martingale problem in the finite-dimensional case, which is (in itself) a separate matter that we will prove afterwards. This way, we can reuse an overwhelming part of our methods and calculations (in particular those on $\mathbb{H}_n$), which is convenient to reduce repetitions, while still presenting it for completeness.

3.4.1 Infinite-dimensional case

The proof in infinite dimensions is mainly based on Galerkin approximations. Recall that we have defined $\Pi_n \colon \mathbb{X}^* \longrightarrow \mathbb{H}_n$ in Equation (3.1.2) by

$$\Pi_n z = \sum_{i=1}^{n} {}_{\mathbb{X}^*}\langle z, e_i\rangle_{\mathbb{X}} e_i, \quad z \in \mathbb{X}^*.$$

Recall that $\Pi_{n \restriction \mathbb{H}}$ is the orthogonal projection onto $\mathbb{H}_n$ in $\mathbb{H}$ and we have

$$ {}_{\mathbb{X}^*}\langle \Pi_n b(t,y), v\rangle_{\mathbb{X}} = \langle \Pi_n b(t,y), v\rangle_{\mathbb{H}} = {}_{\mathbb{X}^*}\langle b(t,y), v\rangle_{\mathbb{X}}, \tag{3.4.1}$$

for every $y \in \mathbb{Y}$ and $v \in \mathbb{H}_n$. For $v \in \mathbb{H}_n$, the convergence $y_k \xrightarrow[k\to\infty]{} y$ in $\mathbb{Y}$ yields

$$\lim_{k\to\infty} \big\|(\Pi_n\sigma(t,y_k)\breve{\Pi}_n)^*(v) - (\Pi_n\sigma(t,y)\breve{\Pi}_n)^*(v)\|_{\mathbb{U}} = 0, \tag{3.4.2}$$

where we have defined $\breve{\Pi}_n$ to be the orthogonal projection onto $\operatorname{span}\{u_1, \dots, u_n\}$ in $\mathbb{U}$. In addition, the estimates

$$\begin{aligned} \|\Pi_n\sigma(t,y)\breve{\Pi}_n\|^2_{L_2(\mathbb{U};\mathbb{H})} &\leq \|\sigma(t,y)\|^2_{L_2(\mathbb{U};\mathbb{H})}, \\ \|\Pi_n b(t,y)\|^{\gamma}_{\mathbb{X}^*} &\leq \|b(t,y)\|^{\gamma}_{\mathbb{X}^*} \end{aligned} \tag{3.4.3}$$

hold for every $y \in \mathbb{Y}$.

Consequently, Assumptions **(A1)**–**(A3)** remain valid for the coefficients $\Pi_n b$ and $\Pi_n \sigma$ on the finite-dimensional spaces $\mathbb{H}_n$. Then, for each $n \in \mathbb{N}$, we consider the finite-dimensional martingale problem given by

Existence of a martingale solution $P_n \in \mathcal{P}(\Omega_n)$ in the sense of Definition 3.2.1 for coefficients $\Pi_n b$ and $\Pi_n\sigma$ and with initial value $\Pi_n x_0 \in \mathbb{H}_n$, (3.4.4)

where we set

$$\Omega_n := C([0,\infty); \mathbb{H}_n).$$

Furthermore, we denote by x_n the canonical process on Ω_n given by $x_n(t,\omega) := \omega(t)$ and define

$$\mathcal{F}_t^{(n)} := \mathcal{B}(C([0,t];\mathbb{H}_n)) \quad \text{and} \quad \mathcal{F}^{(n)} := \bigvee_{t\geq 0} \mathcal{F}_t^{(n)}.$$

Remark. Let us mention that for

$$W_n(t) := \breve{\Pi}_n W(t) = \sum_{i=1}^{n} \langle W(t), u_i\rangle_{\mathbb{U}} u_i$$

the finite-dimensional stochastic differential equation on $\mathbb{H}_n$ associated to this martingale problem is given by

$$\begin{aligned} \mathrm{d}X_n(t) &= \Pi_n b\big(t, X_n(t)\big)\,\mathrm{d}t + \Pi_n\sigma\big(t, X_n(t)\big)\,\mathrm{d}W_n(t), \\ X_n(0) &= \Pi_n x_0 \end{aligned} \tag{3.4.5}$$

(see e.g. [Kur11, p. 114f]).

In Subsection 3.4.2 below we will prove existence of martingale solutions for the martingale problem with coefficients b and σ in the finite-dimensional case. Assuming this result to be true for now, we can conclude (since Equations (3.4.1)–(3.4.2) yield that Assumptions **(A1)**–**(A3)** remain valid under Π_n) the same for our martingale problem (3.4.4) with coefficients $\Pi_n b$ and $\Pi_n\sigma$, i.e. there exists a probability measure $P_n \in \mathcal{P}(\Omega_n)$ such that Conditions **(M1)** and **(M2)** hold. In order to construct a solution to the martingale problem (3.2.1) in infinite dimensions, we first need the following a priori energy estimate for the canonical processes x_n with respect to P_n.

Lemma 3.4.2 (see [RZZ15], Lemma 3.1, p. 368)**.** *Let the Assumptions **(N)** and **(A1)**–**(A3)** be fulfilled. Then for every $q \geq 1$ and any $t \geq s \geq 0$ there exists a constant $C_{q,t,t-s} > 0$ such that for all $n \in \mathbb{N}$ we have*

$$\begin{aligned} \mathbb{E}^{P_n}\Big[\sup_{r\in[s,t]} \|x_n(r)\|_{\mathbb{H}}^{2q}\Big] + \mathbb{E}^{P_n}\left[\int_s^t \|x_n(r)\|_{\mathbb{H}}^{2(q-1)}\mathcal{N}(x_n(r))\mathrm{d}r\right] \\ \leq C_{q,t,t-s}\left(\mathbb{E}^{P_n}\left[\|x_n(s)\|_{\mathbb{H}}^{2q}\right] + 1\right). \end{aligned} \tag{3.4.6}$$

Remark. This a priori estimate is similar to those used in the variational approach for SPDEs. For example, a mostly analogue lemma and proof can be found in [LR15, Lemma 5.1.5], where SPDEs with locally monotone coefficients are considered.

Proof of Lemma 3.4.2. First, note that by Condition **(M2)** the equality

$$x_n(t) = \Pi_n x_0 + \int_0^t \Pi_n b(r, x_n(r))\,\mathrm{d}r + M_n(t, x_n), \quad t \geq 0, \tag{3.4.7}$$

holds in $\mathbb{H}_n$, where $M_n(t, x_n)$, $t \geq 0$, is an $\mathbb{H}_n$-valued continuous $(\mathcal{F}_t^{(n)})$-martingale with respect to P_n, whose covariation operator process in $\mathbb{H}_n$ is given by

$$\ll M_n \gg (t, x_n) = \int_0^t \Pi_n\sigma(r, x_n(r))\breve{\Pi}_n\,\breve{\Pi}_n^*\sigma^*(r, x_n(r))\Pi_n^*\,\mathrm{d}r, \quad t \geq 0.$$

By using Itô's formula for $\mathbb{H}_n$-valued semimartingales, we obtain for $s \leq t$ the identity

$$\begin{aligned} \|x_n(t)\|_{\mathbb{H}}^{2q} = \|x_n(s)\|_{\mathbb{H}}^{2q} &+ 2q\int_s^t \|x_n(r)\|_{\mathbb{H}}^{2(q-1)}\,{}_{\mathbb{X}^*}\langle \Pi_n b(r, x_n(r)), x_n(r)\rangle_{\mathbb{X}}\,\mathrm{d}r \\ &+ q\int_s^t \|x_n(r)\|_{\mathbb{H}}^{2(q-1)}\|\Pi_n\sigma(r, x_n(r))\breve{\Pi}_n\|_{L_2(\mathbb{U};\mathbb{H})}^2\,\mathrm{d}r \\ &+ 2q(q-1)\int_s^t \|x_n(r)\|_{\mathbb{H}}^{2(q-2)}\|\left(\Pi_n\sigma(r, x_n(r))\breve{\Pi}_n\right)^*(x_n(r))\|_{\mathbb{U}}^2\,\mathrm{d}r \\ &+ M_n^{(q)}(t, x_n) - M_n^{(q)}(s, x_n), \end{aligned} \tag{3.4.8}$$

where $M_n^{(q)}(t, x_n)$ is a continuous real-valued $(\mathcal{F}_t^{(n)})$-martingale with respect to P_n. In fact, we have

$$M_n^{(q)}(t, x_n) = 2q \int_0^t \|x_n(r)\|_{\mathbb{H}}^{2(q-1)} \langle x_n(r), \Pi_n \sigma(r, x_n(r))\, \mathrm{d}W_n(r)\rangle_{\mathbb{H}}$$

and its quadratic variation process is given by

$$\langle M_n^{(q)}\rangle(t, x_n) = 4q^2 \int_0^t \|x_n(r)\|_{\mathbb{H}}^{4(q-1)} \|(\Pi_n \sigma(r, x_n(r))\breve{\Pi}_n)^*(x_n(r))\|_{\mathbb{U}}^2\, \mathrm{d}r.$$

Minding the local boundedness of the functions λ_i, $i = 1, \dots 4$, we set

$$\lambda_*(t) := \sup_{\substack{i=1,\dots 4\\ r\in[0,t]}} |\lambda_i(r)|.$$

By applying Assumptions **(A2)** and **(A3)**, we then obtain

$$\begin{aligned}
\|x_n(t)\|_{\mathbb{H}}^{2q} \le \|x_n(s)\|_{\mathbb{H}}^{2q} &+ 2q \int_s^t \|x_n(r)\|_{\mathbb{H}}^{2(q-1)} \Big(-\mathcal{N}(x_n(r)) + \lambda_1(r)(1 + \|x_n(r)\|_{\mathbb{H}}^2)\Big)\, \mathrm{d}r \\
&+ q \int_s^t \|x_n(r)\|_{\mathbb{H}}^{2(q-1)} \lambda_4(r)(1 + \|x_n(r)\|_{\mathbb{H}}^2)\, \mathrm{d}r \\
&+ 2q(q-1) \int_s^t \|x_n(r)\|_{\mathbb{H}}^{2(q-2)} \|(\Pi_n \sigma(r, x_n(r))\breve{\Pi}_n)^*\|_{L_2(\mathbb{H};\mathbb{U})}^2 \|x_n(r)\|_{\mathbb{H}}^2\, \mathrm{d}r \\
&+ M_n^{(q)}(t, x_n) - M_n^{(q)}(s, x_n) \\
\le \|x_n(s)\|_{\mathbb{H}}^{2q} &- 2q \int_s^t \|x_n(r)\|_{\mathbb{H}}^{2(q-1)} \mathcal{N}(x_n(r))\, \mathrm{d}r + C_q \lambda_*(t) \int_s^t (\|x_n(r)\|_{\mathbb{H}}^{2q} + 1)\, \mathrm{d}r \\
&+ M_n^{(q)}(t, x_n) - M_n^{(q)}(s, x_n),
\end{aligned} \tag{3.4.9}$$

where in the last step the estimate

$$\begin{aligned}
\|x_n(r)\|_{\mathbb{H}}^{2(q-1)}(1 + \|x_n(r)\|_{\mathbb{H}}^2) &= (\|x_n(r)\|_{\mathbb{H}}^{2(q-1)} \cdot 1 + \|x_n(r)\|_{\mathbb{H}}^{2q}) \\
&\le C_q(\|x_n(r)\|_{\mathbb{H}}^{2q} + 1^q)
\end{aligned} \tag{3.4.10}$$

follows from Young's inequality (with coefficients $\frac{2q}{2(q-1)}$ and q) and C_q is a constant (depending on q, but independent of n), that may change from line to line.

For every given $n \in \mathbb{N}$ define the stopping time

$$\tau_n^R := \inf\{r \in [0, t] \mid \|x_n(r)\|_{\mathbb{H}} > R\} \wedge t, \quad R > 0.$$

Here, as usual, we set $\inf \emptyset := \infty$. Then we have

$$\lim_{R\to\infty} \tau_n^R = t, \quad P_n\text{-a.s.},\ n \in \mathbb{N}.$$

By the Burkholder–Davis–Gundy inequality, Assumption **(A3)**, Estimate (3.4.10), Young's inequality and the stochastic Fubini theorem, we have

$$
\begin{aligned}
&\mathbb{E}^{P_n}\left[\sup_{r\in[s,t\wedge\tau_n^R]}\left|M_n^{(q)}(r,x_n)-M_n^{(q)}(s,x_n)\right|\right]\\
&\quad\le C_q\,\mathbb{E}^{P_n}\left[\left(\int_s^{t\wedge\tau_n^R}\|x_n(r)\|_{\mathbb{H}}^{4(q-1)}\|(\Pi_n\sigma(r,x_n(r))\breve{\Pi}_n)^*(x_n(r))\|_{\mathbb{U}}^2\,\mathrm{d}r\right)^{\frac12}\right]\\
&\quad\le C_q\,\mathbb{E}^{P_n}\left[\left(\int_s^{t\wedge\tau_n^R}\|x_n(r)\|_{\mathbb{H}}^{2q}\,\|x_n(r)\|_{\mathbb{H}}^{2(q-1)}\,\|\Pi_n\sigma(r,x_n(r))\breve{\Pi}_n\|_{L_2(\mathbb{U};\mathbb{H})}^2\,\mathrm{d}r\right)^{\frac12}\right]\\
&\quad\le C_q\,\mathbb{E}^{P_n}\left[\sup_{r\in[s,t\wedge\tau_n^R]}\|x_n(r)\|_{\mathbb{H}}^{q}\left(\lambda_*(t)\int_s^{t\wedge\tau_n^R}(\|x_n(r)\|_{\mathbb{H}}^{2q}+1)\,\mathrm{d}r\right)^{\frac12}\right]\\
&\quad\le\frac12\,\mathbb{E}^{P_n}\left[\sup_{r\in[s,t\wedge\tau_n^R]}\|x_n(r)\|_{\mathbb{H}}^{2q}\right]+C_q\lambda_*(t)\,\mathbb{E}^{P_n}\left[\int_s^{t\wedge\tau_n^R}(\|x_n(r)\|_{\mathbb{H}}^{2q}+1)\,\mathrm{d}r\right]\\
&\quad\le\frac12\,\mathbb{E}^{P_n}\left[\sup_{r\in[s,t\wedge\tau_n^R]}\|x_n(r)\|_{\mathbb{H}}^{2q}\right]+C_q\lambda_*(t)\int_s^t\mathbb{E}^{P_n}\left[\sup_{\tilde r\in[s,r\wedge\tau_n^R]}\|x_n(\tilde r)\|_{\mathbb{H}}^{2q}\right]+1\,\mathrm{d}r,
\end{aligned}
\tag{3.4.11}
$$

where C_q is again a constant that may change from line to line (depending on q, but independent of n).

Thus, by first taking suprema and then expectations with respect to P_n on both sides of Inequality (3.4.9), we obtain

$$
\begin{aligned}
&\mathbb{E}^{P_n}\left[\sup_{r\in[s,t\wedge\tau_n^R]}\|x_n(r)\|_{\mathbb{H}}^{2q}\right]+2q\mathbb{E}^{P_n}\left[\int_s^{t\wedge\tau_n^R}\|x_n(r)\|_{\mathbb{H}}^{2(q-1)}\mathcal{N}(x_n(r))\,\mathrm{d}r\right]\\
&\qquad\le\mathbb{E}^{P_n}\left[\|x_n(s)\|_{\mathbb{H}}^{2q}\right]+\frac12\mathbb{E}^{P_n}\left[\sup_{r\in[s,t\wedge\tau_n^R]}\|x_n(r)\|_{\mathbb{H}}^{2q}\right]\\
&\qquad\quad+C_{q,t}\int_s^t\mathbb{E}^{P_n}\left[\sup_{\tilde r\in[s,r\wedge\tau_n^R]}\|x_n(\tilde r)\|_{\mathbb{H}}^{2q}\right]+1\,\mathrm{d}r.
\end{aligned}
$$

Subtracting the reappearing term $\frac12\mathbb{E}^{P_n}\left[\sup_{r\in[s,t\wedge\tau_n^R]}\|x_n(r)\|_{\mathbb{H}}^{2q}\right]$, multiplying with factor 2 and omitting (all terms are nonnegative) the second term of the left hand side of this inequality yields

$$
\mathbb{E}^{P_n}\left[\sup_{r\in[s,t\wedge\tau_n^R]}\|x_n(r)\|_{\mathbb{H}}^{2q}\right]\le 2\mathbb{E}^{P_n}\left[\|x_n(s)\|_{\mathbb{H}}^{2q}\right]+2C_{q,t}\int_s^t\mathbb{E}^{P_n}\left[\sup_{\tilde r\in[s,r\wedge\tau_n^R]}\|x_n(\tilde r)\|_{\mathbb{H}}^{2q}\right]+1\,\mathrm{d}r,
$$

which by using Gronwall's inequality gives

$$
\begin{aligned}
\mathbb{E}^{P_n}\left[\sup_{r\in[s,t\wedge\tau_n^R]}\|x_n(r)\|_{\mathbb{H}}^{2q}\right]&\le\mathbb{E}^{P_n}\left[\sup_{r\in[s,t\wedge\tau_n^R]}\|x_n(r)\|_{\mathbb{H}}^{2q}\right]+1\\
&\le\exp\big(2C_{q,t}(t-s)\big)\Big(2\mathbb{E}^{P_n}\left[\|x_n(s)\|_{\mathbb{H}}^{2q}\right]+1\Big).
\end{aligned}
\tag{3.4.12}
$$

On the other hand, by omitting the first term (after subtracting) and using the previous Estimate (3.4.12), we have for the second term on the left hand side

$$\begin{aligned}
&\mathbb{E}^{P_n}\left[\int_s^{t\wedge\tau_n^R} \|x_n(r)\|_{\mathbb{H}}^{2(q-1)} \mathcal{N}(x_n(r))\,\mathrm{d}r\right] \\
&\qquad \le C_q \mathbb{E}^{P_n}\left[\|x_n(s)\|_{\mathbb{H}}^{2q}\right] + C_{q,t}\int_s^t \mathbb{E}^{P_n}\left[\sup_{\tilde r\in[s,r\wedge\tau_n^R]} \|x_n(\tilde r)\|_{\mathbb{H}}^{2q}\right] + 1\,\mathrm{d}r \\
&\qquad \le C_q\Big(\mathbb{E}^{P_n}\left[\|x_n(s)\|_{\mathbb{H}}^{2q}\right] + 1\Big) + C_{q,t}\Big(\mathbb{E}^{P_n}\left[\|x_n(s)\|_{\mathbb{H}}^{2q}\right] + 1\Big)\int_s^t \exp\big(C_{q,t}(r-s)\big)\mathrm{d}r \\
&\qquad \le C_{q,t}\Big(\mathbb{E}^{P_n}\left[\|x_n(s)\|_{\mathbb{H}}^{2q}\right] + 1\Big)\exp\big(C_{q,t}(t-s)\big),
\end{aligned}$$

where C_q and $C_{q,t}$ are constants that may change from line to line (depending on q (and t), but independent of n). Hence, we conclude by combining both estimates

$$\begin{aligned}
\mathbb{E}^{P_n}\left[\sup_{r\in[s,t\wedge\tau_n^R]} \|x_n(r)\|_{\mathbb{H}}^{2q}\right] &+ \mathbb{E}^{P_n}\left[\int_s^{t\wedge\tau_n^R} \|x_n(r)\|_{\mathbb{H}}^{2(q-1)} \mathcal{N}(x_n(r))\,\mathrm{d}r\right] \\
&\le C_{q,t,t-s}\left(\mathbb{E}^{P_n}\left[\|x_n(s)\|_{\mathbb{H}}^{2q}\right] + 1\right)
\end{aligned}$$

and by letting $R\to\infty$, the desired result follows from the monotone convergence theorem. □

By using the existence of martingale solutions P_n for the martingale problem on $\mathbb{H}_n$ constructed with the help of Galerkin approximations, i.e. via the projections Π_n, and the provided framework from the beginning of this section, we can now complete the proof of Theorem 3.3.1 in the infinite-dimensional case.

Proof of Theorem 3.3.1 for $\mathbb{H}$.
We divide this proof into 4 steps.

Step 1: Extend P_n to $\bar P_n$
Note that $\Omega_n = C([0,\infty);\mathbb{H}_n)$ is a closed subset of Ω. We extend P_n to a probability measure $\bar P_n$ on $(\Omega,\mathcal{F})$ by setting

$$\bar P_n[A] := P_n[A\cap\Omega_n], \quad A\in\mathcal{F}.$$

Remark. This implies that for the canonical processes x on Ω and x_n on Ω_n we have, for suitable measurable functions f, the identity

$$\begin{aligned}
\mathbb{E}^{\bar P_n}\big[f(x(t))\big] &= \int_\Omega f(x(t))\,\mathrm{d}\bar P_n = \int_\Omega f(\omega(t))\,\mathrm{d}\bar P_n(\omega) \\
&= \int_{\Omega_n} f(\omega(t))\,\mathrm{d}P_n(\omega) = \int_{\Omega_n} f(x_n(t))\,\mathrm{d}P_n = \mathbb{E}^{P_n}\big[f(x_n(t))\big].
\end{aligned}$$

In particular we can, therefore, replace terms of the type $\mathbb{E}^{P_n}\big[\|x_n(t)\|_{\mathbb{X}^*}\big]$ by $\mathbb{E}^{\bar P_n}\big[\|x(t)\|_{\mathbb{X}^*}\big]$ in the upcoming calculations.

Step 2: Tightness
In the following we show that $(\bar{P}_n)_{n\in\mathbb{N}}$ is tight on $\mathbb{S}\ \big(=C\big([0,\infty);\mathbb{X}^*\big)\cap L^p_{\text{loc}}([0,\infty);\mathbb{Y})\big)$.

We want to use Lemma 3.4.1 here. The critical part is to prove that for some $\beta>0$ and every $T>0$ we have

$$\sup_{n\in\mathbb{N}}\mathbb{E}^{\bar{P}_n}\left[\sup_{s,t\in[0,T],\,s\neq t}\frac{\|x(t)-x(s)\|_{\mathbb{X}^*}}{|t-s|^{\beta}}\right]<\infty.$$

For the other two terms we just apply the estimates obtained in Lemma 3.4.2 (and Jensen's inequality). First of all, note that by the extension of P_n from **Step 1** the equality

$$\sup_{n\in\mathbb{N}}\mathbb{E}^{\bar{P}_n}\left[\sup_{s,t\in[0,T],\,s\neq t}\frac{\|x(t)-x(s)\|_{\mathbb{X}^*}}{|t-s|^{\beta}}\right]=\sup_{n\in\mathbb{N}}\mathbb{E}^{P_n}\left[\sup_{s,t\in[0,T],\,s\neq t}\frac{\|x_n(t)-x_n(s)\|_{\mathbb{X}^*}}{|t-s|^{\beta}}\right]$$

holds. Therefore, we can use the representation of x_n given in Equation (3.4.7).

Furthermore, we have the inequality

$$(1+z)\leq(1+z)^m\leq 2^m(1+z^m),\tag{3.4.13}$$

for any $z\geq 0$ and $m\geq 1$.

By using Hölder's inequality in the first, Inequality (3.4.3) in the second, Assumption **(A3)** in the third, Inequality (3.4.13) and the local boundedness of the functions λ_2 and λ_3 in the fourth as well as Lemma 3.4.2 in the sixth step, we obtain

$$\begin{aligned}
&\mathbb{E}^{P_n}\left[\sup_{s,t\in[0,T],\,s\neq t}\left(\frac{1}{|t-s|^{\gamma-1}}\left\|\int_s^t\Pi_n b(r,x_n(r))\,\mathrm{d}r\right\|_{\mathbb{X}^*}^{\gamma}\right)\right]\\
&\leq\mathbb{E}^{P_n}\left[\int_0^T\left\|\Pi_n b(r,x_n(r))\right\|_{\mathbb{X}^*}^{\gamma}\mathrm{d}r\right]\\
&\leq\mathbb{E}^{P_n}\left[\int_0^T\left\|b(r,x_n(r))\right\|_{\mathbb{X}^*}^{\gamma}\mathrm{d}r\right]\\
&\leq\mathbb{E}^{P_n}\left[\int_0^T\lambda_2(r)\mathcal{N}(x_n(r))+\lambda_3(r)\big(1+\|x_n(r)\|_{\mathbb{H}}^{\gamma'}\big)\,\mathrm{d}r\right]\\
&\leq C_T\mathbb{E}^{P_n}\left[\int_0^T\mathcal{N}(x_n(r))+\big(1+\|x_n(r)\|_{\mathbb{H}}^{2\gamma'}\big)\,\mathrm{d}r\right]\\
&\leq C_T\mathbb{E}^{P_n}\left[\int_0^T\mathcal{N}(x_n(r))\,\mathrm{d}r\right]+C_T\Big(\mathbb{E}^{P_n}\Big[\sup_{r\in[0,T]}\|x_n(r)\|_{\mathbb{H}}^{2\gamma'}\Big]+1\Big)\\
&\leq C_T\Big(\mathbb{E}^{P_n}\left[\|x_n(0)\|_{\mathbb{H}}^2\right]+1\Big)+C_{\gamma',T}\Big(\mathbb{E}^{P_n}\left[\|x_n(0)\|_{\mathbb{H}}^{2\gamma'}\right]+1\Big)\\
&\leq C_T\Big(\|\Pi_n x_0\|_{\mathbb{H}}^2+1\Big)+C_{\gamma',T}\Big(\|\Pi_n x_0\|_{\mathbb{H}}^{2\gamma'}+1\Big)\\
&\leq C_{\gamma',T},
\end{aligned}\tag{3.4.14}$$

where C_T and $C_{\gamma',T}$ are constants (that may change from line to line). Consequently, for every $\beta_1\in\big(0,\frac{\gamma-1}{\gamma}\big)$ we have

$$\mathbb{E}^{P_n}\left[\sup_{s,t\in[0,T],\,s\neq t}\frac{\big\|\int_s^t\Pi_n b(r,x_n(r))\,\mathrm{d}r\big\|_{\mathbb{X}^*}}{|t-s|^{\beta_1}}\right]\leq C_{\gamma,\gamma',T}.\tag{3.4.15}$$

For every $0 \leq s < t \leq T$ and any $q > 1$ we again use the Burkholder–Davis–Gundy inequality, Hölder's inequality, Assumption **(A3)** as well as Equation (3.4.13) and Lemma 3.4.2 to calculate

$$\begin{aligned}\mathbb{E}^{P_n}\Big[\big\|M_n(t,x_n)-M_n(s,x_n)\big\|_{\mathbb{H}}^{2q}\Big] &\leq C_q\mathbb{E}^{P_n}\left[\left(\int_s^t \|\sigma(r,x_n(r))\|^2_{L_2(\mathbb{U};\mathbb{H})}\,\mathrm{d}r\right)^q\right]\\ &\leq C_q|t-s|^{q-1}\,\mathbb{E}^{P_n}\left[\int_s^t \|\sigma(r,x_n(r))\|^{2q}_{L_2(\mathbb{U};\mathbb{H})}\,\mathrm{d}r\right]\\ &\leq C_{q,T}|t-s|^q\left(\mathbb{E}^{P_n}\Big[\sup_{r\in[0,T]}\|x_n(r)\|_{\mathbb{H}}^{2q}\Big]+1\right)\\ &\leq C_{q,T}|t-s|^q\left(\mathbb{E}^{P_n}\left[\|x_n(0)\|_{\mathbb{H}}^{2q}\right]+1\right).\end{aligned} \tag{3.4.16}$$

Hence, by the Garsia–Rodemich–Rumsey inequality (see e.g. [SV79, Corollary 2.1.4, p. 49] or [FV10, Theorem A.1, p. 571]) we get for every $\beta_2 \in \left(0, \frac{q-1}{2q}\right)$ the inequality

$$\mathbb{E}^{P_n}\left[\sup_{s,t\in[0,T],\,s\neq t}\frac{\|M_n(t,x_n)-M_n(s,x_n)\|_{\mathbb{H}}}{|t-s|^{\beta_2}}\right] \leq C_{q,T}. \tag{3.4.17}$$

Now, by combining Estimates (3.4.15) and (3.4.17), we obtain for some $\beta > 0$ (chosen small enough, i.e. $\beta \in \left(0, \frac{q-1}{2q}\wedge\frac{\gamma-1}{\gamma}\right)$ for some $q > 1$) the estimate

$$\sup_{n\in\mathbb{N}}\mathbb{E}^{P_n}\left[\sup_{s,t\in[0,T],\,s\neq t}\frac{\|x_n(t)-x_n(s)\|_{\mathbb{X}^*}}{|t-s|^{\beta}}\right] < \infty,$$

which, as mentioned above, implies that $(\bar{P}_n)_{n\in\mathbb{N}}$ is tight on $\mathbb{S}$.

Step 3: Prokhorov's theorem and Skorokhod's representation theorem
From Prokhorov's theorem (see e.g. [Bil99, Theorems 5.1 and 5.2, p. 59f]) it follows that there exists a weakly convergent subsequence. Hence, we have to select this subsequence here, but, without loss of generality and for simplicity, we keep the notation $(\bar{P}_n)_{n\in\mathbb{N}}$ for the sequence and let $P \in \mathcal{P}(\mathbb{S})$ denote the limit.

Now we can apply Skorokhod's representation theorem to the law of x under $\bar{P}_n$ (see e.g. [Jak97] or [Bil99, Theorem 6.7, p. 70]), i.e. there exist $\mathbb{S}$-valued random variables $\tilde{x}_n$, $n \in \mathbb{N}$, and $\tilde{x}$ on a common probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that we have

(i) $\tilde{x}_n$ (under $\tilde{P}$) has the law $\bar{P}_n$ for each $n \in \mathbb{N}$, i.e. $\tilde{P}\circ\tilde{x}_n^{-1} = \bar{P}_n$,

(ii) $\tilde{x}$ (under $\tilde{P}$) has the law P, i.e. $\tilde{P}\circ\tilde{x}^{-1} = P$,

(iii) $\tilde{x}_n \xrightarrow[n\to\infty]{} \tilde{x}$ in $\mathbb{S}$, $\tilde{P}$-a.s.

Note that the laws $\bar{P}_n\circ x^{-1}$ and $P\circ x^{-1}$ simplify to $\bar{P}_n$ and P, respectively, since x is the canonical process.

Step 4: (M1) and (M2) for P

First we verify Condition (M1) for P.

We have

$$\begin{aligned} P[x(0)=x_0] &\overset{\text{(ii)}}{=} \tilde{P}[\tilde{x}(0)=x_0] \overset{\text{(iii)}}{=} \lim_{n\to\infty} \tilde{P}[\tilde{x}_n(0)=\Pi_n x_0] \overset{\text{(i)}}{=} \lim_{n\to\infty} \bar{P}_n[x(0)=\Pi_n x_0] \\ &= \lim_{n\to\infty} P_n[x_n(0)=\Pi_n x_0] = 1 \end{aligned} \tag{3.4.18}$$

since P_n satisfies Condition **(M1)**.

For any $q \geq 1$ and $0 \leq s < t$, define

$$\xi_q(s,t,x) := \sup_{r\in[s,t]} \|x(r)\|_{\mathbb{H}}^{2q} + \int_s^t \|x(r)\|_{\mathbb{H}}^{2(q-1)} \mathcal{N}(x(r))\,\mathrm{d}r. \tag{3.4.19}$$

Then $x \longmapsto \xi_q(s,t,x)$ is lower semi-continuous on $\mathbb{S}$, which follows from the lower semi-continuity of the mappings $\mathcal{N}$ and $z \longmapsto \|z\|_{\mathbb{H}}$ on $\mathbb{X}^*$ (where $\|\cdot\|_{\mathbb{H}}$ is extended to a function on $\mathbb{X}^*$ by setting $\|z\|_{\mathbb{H}} := \infty$, if $z \in \mathbb{X}^* \setminus \mathbb{H}$, as e.g. in [LR15, Exercise 4.2.3, p. 90f]). By Fatou's lemma and Lemma 3.4.2, we have

$$\begin{aligned} \mathbb{E}^P[\xi_q(0,t,x)] &\overset{\text{(ii)}}{=} \mathbb{E}^{\tilde{P}}[\xi_q(0,t,\tilde{x})] \overset{\text{(iii)}}{\leq} \liminf_{n\to\infty} \mathbb{E}^{\tilde{P}}[\xi_q(0,t,\tilde{x}_n)] \\ &\overset{\text{(i)}}{=} \liminf_{n\to\infty} \mathbb{E}^{\bar{P}_n}[\xi_q(0,t,x)] = \liminf_{n\to\infty} \mathbb{E}^{P_n}[\xi_q(0,t,x_n)] \\ &\leq \liminf_{n\to\infty} C_{q,t}(\mathbb{E}^{P_n}[\|x_n(0)\|_{\mathbb{H}}^{2q}] + 1) \\ &= \liminf_{n\to\infty} C_{q,t}(\|\Pi_n x_0\|_{\mathbb{H}}^{2q} + 1) < \infty. \end{aligned} \tag{3.4.20}$$

Thus, the P-a.s. integrability condition in **(M1)** follows from Assumption **(A3)** since for $k \in \mathbb{N}$ and $x \in \mathbb{S}$ we have

$$\begin{aligned} \int_0^k \big\|b(s,x(s))\big\|_{\mathbb{X}^*}\,\mathrm{d}s &\leq \int_0^k \Big(1 + \underbrace{\lambda_2(s)\mathcal{N}(x(s)) + \lambda_3(s)(1+\|x(s)\|_{\mathbb{H}}^{\gamma'})}_{\geq 0}\Big)^{\frac{1}{\gamma}}\,\mathrm{d}s \\ &\leq k + C_k \int_0^k \mathcal{N}(x(s))\,\mathrm{d}s + C_{k,\gamma'} \int_0^k (1+\|x(s)\|_{\mathbb{H}}^{2\gamma'})\,\mathrm{d}s, \end{aligned} \tag{3.4.21}$$

by using that $(1+z)^{\frac{1}{m}} \leq (1+z)$ for $z \geq 0$ and $m \geq 1$ as well as Equation (3.4.13), and

$$\int_0^k \big\|\sigma(s,x(s))\big\|_{L_2(\mathbb{U};\mathbb{H})}^2\,\mathrm{d}s \leq C_k \int_0^k (1+\|x(s)\|_{\mathbb{H}}^2)\,\mathrm{d}s. \tag{3.4.22}$$

Now we verify (M2) for P.

Fix $\ell \in \mathcal{E}$. We will show that $M_\ell(t,x)$, $t \geq 0$, in Condition **(M2)** is a continuous $(\mathcal{F}_t)$-martingale with respect to P, whose quadratic variation process is given by

$$\langle M_\ell\rangle(t,x) = \int_0^t \|\sigma^*(s,x(s))(\ell)\|_{\mathbb{U}}^2\,\mathrm{d}s, \quad t \geq 0.$$

First of all, since $\tilde{x}_n \xrightarrow[n\to\infty]{} \tilde{x}$ in $\mathbb{S}$ $\tilde{P}$-a.s., Inequality (3.1.1) implies that we have

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{P}}\big[|{}_{\mathbb{X}^*}\langle \tilde{x}_n(t) - \tilde{x}(t), \ell\rangle_{\mathbb{X}}|\big] \leq \|\ell\|_{\mathbb{X}} \lim_{n\to\infty} \mathbb{E}^{\tilde{P}}\big[\|\tilde{x}_n(t) - \tilde{x}(t)\|_{\mathbb{X}^*}\big] = 0 \tag{3.4.23}$$

by dominated convergence, because $\sup_{n\in\mathbb{N}} \mathbb{E}^{\tilde{P}}\big[\|\tilde{x}_n(t)\|_{\mathbb{X}^*}^{2q}\big] < \infty$ holds for any $q \geq 1$. In fact, we have

$$\mathbb{E}^{\tilde{P}}\big[\|\tilde{x}_n(t)\|_{\mathbb{X}^*}^{2q}\big] \leq C\,\mathbb{E}^{\tilde{P}}\big[\|\tilde{x}_n(t)\|_{\mathbb{H}}^{2q}\big] \leq C\,\mathbb{E}^{P_n}\Big[\sup_{r\in[0,t]} \|x_n(r)\|_{\mathbb{H}}^{2q}\Big] \leq C_{q,t}$$

by the continuity of the embedding $\mathbb{H} \subseteq \mathbb{X}^*$ and Lemma 3.4.2, where C and $C_{q,t}$ are constants, that are independent of n.

Now, for the second term of M_ℓ in Condition **(M2)**, define

$$G(t,x) := \int_0^t {}_{\mathbb{X}^*}\langle b(s,x(s)), \ell\rangle_{\mathbb{X}}\,\mathrm{d}s$$

and, for any $R > 0$, define the auxiliary term

$$G_R(t,x) := \int_0^t {}_{\mathbb{X}^*}\langle b(s,x(s)), \ell\rangle_{\mathbb{X}} \cdot \chi_R\big({}_{\mathbb{X}^*}\langle b(s,x(s)), \ell\rangle_{\mathbb{X}}\big)\,\mathrm{d}s,$$

where $\chi_R \in C_c^\infty(\mathbb{R})$ is a cutoff function with

$$\chi_R(r) = \begin{cases} 1, & \text{if } |r| \leq R, \\ 0, & \text{if } |r| > 2R. \end{cases}$$

We want to show that

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{P}}\Big[\big|G(t,\tilde{x}_n) - G(t,\tilde{x})\big|\Big] = 0 \tag{3.4.24}$$

holds. By inserting two auxiliary terms, it remains to prove

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{P}}\Big[\big|G(t,\tilde{x}_n) - G_R(t,\tilde{x}_n)\big| + \big|G_R(t,\tilde{x}_n) - G_R(t,\tilde{x})\big| + \big|G_R(t,\tilde{x}) - G(t,\tilde{x})\big|\Big] = 0.$$

Since $y \longmapsto {}_{\mathbb{X}^*}\langle b(s,y), \ell\rangle_{\mathbb{X}}\ \chi_R\big({}_{\mathbb{X}^*}\langle b(s,y), \ell\rangle_{\mathbb{X}}\big)$ is a bounded continuous function on $\mathbb{Y}$ by Assumption **(A1)** and $\tilde{x}_n \xrightarrow[n\to\infty]{} \tilde{x}$ in $\mathbb{S}$ $\tilde{P}$-a.s., we obtain $\tilde{P}$-a.s. (e.g. by using the continuous mapping theorem and dominated convergence)

$$G_R(t,\tilde{x}_n) \xrightarrow[n\to\infty]{} G_R(t,\tilde{x}).$$

Furthermore, by using Inequality (3.1.1), Hölder's inequality and Estimate (3.4.14), we have

$$\begin{aligned}
\mathbb{E}^{\tilde{P}}\big[|G_R(t,\tilde{x}_n)|^\gamma\big] &\leq \mathbb{E}^{\tilde{P}}\left[\Big(\int_0^t \|b(s,\tilde{x}_n(s))\|_{\mathbb{X}^*}\|\ell\|_{\mathbb{X}}\,\mathrm{d}s\Big)^\gamma\right] \\
&\leq \mathbb{E}^{\tilde{P}}\left[\int_0^t \|b(s,\tilde{x}_n(s))\|_{\mathbb{X}^*}^\gamma\,\mathrm{d}s \cdot \Big(\int_0^t \|\ell\|_{\mathbb{X}}^{\frac{\gamma}{\gamma-1}}\,\mathrm{d}s\Big)^{\frac{\gamma-1}{\gamma}}\right] \\
&= C_{\gamma,t}\,\mathbb{E}^{P_n}\left[\int_0^t \|b(s,x_n(s))\|_{\mathbb{X}^*}^\gamma\,\mathrm{d}s\right] \leq C_{\gamma',\gamma,t}.
\end{aligned}$$

Consequently, $\sup_{n\in\mathbb{N}} \mathbb{E}^{\tilde{P}}\left[|G_R(t,\tilde{x}_n)|^\gamma\right] < \infty$ which implies, since $\gamma > 1$, that

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{P}}\left[|G_R(t,\tilde{x}_n) - G_R(t,\tilde{x})|\right] = 0$$

holds by dominated convergence.

By using Inequality (3.1.1), Hölder's inequality, the stochastic Fubini theorem, the Chebyshev–Markov inequality (with power γ) and the calculations from Estimate (3.4.14), we obtain

$$\begin{aligned}
&\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \mathbb{E}^{\tilde{P}}\left[\left|G(t,\tilde{x}_n) - G_R(t,\tilde{x}_n)\right|\right] \\
&= \lim_{R\to\infty} \sup_{n\in\mathbb{N}} \mathbb{E}^{\tilde{P}}\left[\left|\int_0^t {}_{\mathbb{X}^*}\langle b(s,\tilde{x}_n(s)),\ell\rangle_{\mathbb{X}} \cdot \left(1 - \chi_R\left({}_{\mathbb{X}^*}\langle b(s,\tilde{x}_n(s)),\ell\rangle_{\mathbb{X}}\right)\right) \mathrm{d}s\right|\right] \\
&\leq \lim_{R\to\infty} \sup_{n\in\mathbb{N}} \mathbb{E}^{\tilde{P}}\left[\int_0^t \|b(s,\tilde{x}_n(s))\|_{\mathbb{X}^*} \|\ell\|_{\mathbb{X}} \cdot 1_{\{|{}_{\mathbb{X}^*}\langle b(s,\tilde{x}_n(s)),\ell\rangle_{\mathbb{X}}| \geq R\}} \mathrm{d}s\right] \\
&\leq \|\ell\|_{\mathbb{X}} \lim_{R\to\infty} \sup_{n\in\mathbb{N}} \left(\left(\mathbb{E}^{\tilde{P}}\left[\int_0^t \|b(s,\tilde{x}_n(s))\|_{\mathbb{X}^*}^\gamma \mathrm{d}s\right]\right)^{\frac{1}{\gamma}} \left(\int_0^t \tilde{P}\left[|{}_{\mathbb{X}^*}\langle b(s,\tilde{x}_n(s)),\ell\rangle_{\mathbb{X}}| \geq R\right] \mathrm{d}s\right)^{\frac{\gamma-1}{\gamma}}\right) \\
&\leq \|\ell\|_{\mathbb{X}} \lim_{R\to\infty} \sup_{n\in\mathbb{N}} \left(\left(\mathbb{E}^{\tilde{P}}\left[\int_0^t \|b(s,\tilde{x}_n(s))\|_{\mathbb{X}^*}^\gamma \mathrm{d}s\right]\right)^{\frac{1}{\gamma}} \left(\int_0^t \frac{\|\ell\|_{\mathbb{X}}^\gamma}{R^\gamma} \mathbb{E}^{\tilde{P}}\left[\|b(s,\tilde{x}_n(s))\|_{\mathbb{X}^*}^\gamma\right] \mathrm{d}s\right)^{\frac{\gamma-1}{\gamma}}\right) \\
&\leq \|\ell\|_{\mathbb{X}}^\gamma \lim_{R\to\infty} \sup_{n\in\mathbb{N}} \left(\left(\mathbb{E}^{P_n}\left[\int_0^t \|b(s,x_n(s))\|_{\mathbb{X}^*}^\gamma \mathrm{d}s\right]\right) \frac{1}{R^{\gamma-1}}\right) \qquad (3.4.25) \\
&\leq \|\ell\|_{\mathbb{X}}^\gamma \lim_{R\to\infty} \sup_{n\in\mathbb{N}} \left(C_{\gamma',t} \frac{1}{R^{\gamma-1}}\right) \\
&= 0.
\end{aligned}$$

Similarly, we repeat the estimates from above and use Assumption **(A3)** as well as Equation (3.4.20) to obtain

$$\begin{aligned}
\lim_{R\to\infty} \mathbb{E}^{\tilde{P}}&\left[\left|G_R(t,\tilde{x}) - G(t,\tilde{x})\right|\right] \\
&= \lim_{R\to\infty} \mathbb{E}^{\tilde{P}}\left[\left|\int_0^t {}_{\mathbb{X}^*}\langle b(s,\tilde{x}(s)),\ell\rangle_{\mathbb{X}} \cdot \left(\chi_R\left({}_{\mathbb{X}^*}\langle b(s,\tilde{x}(s)),\ell\rangle_{\mathbb{X}}\right) - 1\right) \mathrm{d}s\right|\right] \\
&\leq \|\ell\|_{\mathbb{X}}^\gamma \lim_{R\to\infty} \left(\left(\mathbb{E}^{\tilde{P}}\left[\int_0^t \|b(s,\tilde{x}(s))\|_{\mathbb{X}^*}^\gamma \mathrm{d}s\right]\right) \frac{1}{R^{\gamma-1}}\right) \\
&\leq \|\ell\|_{\mathbb{X}}^\gamma \lim_{R\to\infty} \left(C_{\gamma',t} \frac{1}{R^{\gamma-1}}\right) = 0.
\end{aligned}$$

Altogether, this proves Equation (3.4.24) by combining the calculations for $G(t,\tilde{x}_n)$, $G_R(t,\tilde{x}_n)$, $G_R(t,\tilde{x})$ and $G(t,\tilde{x})$ from above.

Furthermore, by Inequality (3.1.1) and a calculation analogue to Estimate (3.4.14) (i.e. in particular Assumption **(A3)** and Lemma 3.4.2), we have

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}^{\tilde{P}} \Big[\Big| \int_0^t {}_{\mathbb{X}^*}\langle \Pi_n b(s,\tilde{x}_n(s)), \ell\rangle_{\mathbb{X}} - {}_{\mathbb{X}^*}\langle b(s,\tilde{x}_n(s)), \ell\rangle_{\mathbb{X}} \, \mathrm{d}s \Big| \Big] \\
&\le \lim_{n\to\infty} \mathbb{E}^{\tilde{P}} \Big[\int_0^t \big| {}_{\mathbb{X}^*}\langle b(s,\tilde{x}_n(s)), (\Pi_n - \mathrm{Id})\ell\rangle_{\mathbb{X}} \big| \, \mathrm{d}s \Big] \\
&\le \lim_{n\to\infty} \|(\Pi_n - \mathrm{Id})\ell\|_{\mathbb{X}} \, \mathbb{E}^{P_n} \Big[\int_0^t \|b(s,x_n(s))\|_{\mathbb{X}^*} \, \mathrm{d}s \Big] \\
&= 0.
\end{aligned} \tag{3.4.26}$$

Then, by combining Equations (3.4.23), (3.4.24) and (3.4.26), we obtain that for $t > 0$ the identity

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}^{\tilde{P}} \Big[\big| \langle M_n(t,\tilde{x}_n), \ell\rangle_{\mathbb{H}} - M_\ell(t,\tilde{x}) \big| \Big] \\
&= \lim_{n\to\infty} \mathbb{E}^{\tilde{P}} \Big[\big| {}_{\mathbb{X}^*}\langle M_n(t,\tilde{x}_n), \ell\rangle_{\mathbb{X}} - M_\ell(t,\tilde{x}) \big| \Big] \\
&= \lim_{n\to\infty} \mathbb{E}^{\tilde{P}} \Bigg[\Bigg| {}_{\mathbb{X}^*}\langle \tilde{x}_n(t) - \tilde{x}(t), \ell\rangle_{\mathbb{X}} - {}_{\mathbb{X}^*}\langle \Pi_n x_0 - x_0, \ell\rangle_{\mathbb{X}} \\
&\qquad - \int_0^t {}_{\mathbb{X}^*}\langle \Pi_n b(s,\tilde{x}_n(s)) - b(s,\tilde{x}(s)), \ell\rangle_{\mathbb{X}} \, \mathrm{d}s \Bigg| \Bigg] \\
&= 0
\end{aligned} \tag{3.4.27}$$

holds. Let $0 \le s < t$ and let g be any real-valued bounded $(\mathcal{F}_s)$-measurable continuous function on $\mathbb{S}$. Then we have

$$\begin{aligned}
\mathbb{E}^{P} \Big[\big(M_\ell(t,x) - M_\ell(s,x) \big) g(x) \Big] \\
&= \mathbb{E}^{\tilde{P}} \Big[\big(M_\ell(t,\tilde{x}) - M_\ell(s,\tilde{x}) \big) g(\tilde{x}) \Big] \\
&= \lim_{n\to\infty} \mathbb{E}^{\tilde{P}} \Big[\big(\langle M_n(t,\tilde{x}_n), \ell\rangle_{\mathbb{H}} - \langle M_n(s,\tilde{x}_n), \ell\rangle_{\mathbb{H}} \big) g(\tilde{x}_n) \Big] \\
&= \lim_{n\to\infty} \mathbb{E}^{\bar{P}_n} \Big[\big(\langle M_n(t,x), \ell\rangle_{\mathbb{H}} - \langle M_n(s,x), \ell\rangle_{\mathbb{H}} \big) g(x) \Big] \\
&= 0
\end{aligned} \tag{3.4.28}$$

by using Equation (3.4.27). The arbitrariness of g yields by a monotone class argument that

$$\mathbb{E}^{P} \big[M_\ell(t,x) \big| \mathcal{F}_s \big] = M_\ell(s,x)$$

is fulfilled since $\mathcal{F}_s = \sigma(x(r) \mid r \le s)$. Hence, $M_\ell(t,x)$ is an $(\mathcal{F}_t)$-martingale.

Finally, we will now prove the representation formula of the quadratic variation $\langle M_\ell \rangle$ given in Condition **(M2)** by showing that $M_\ell^2 - \langle M_\ell \rangle$ is a martingale.

By the Burkholder–Davis–Gundy inequality and a calculation analogue to Estimate (3.4.16) (i.e. in particular Hölder's inequality, Assumption **(A3)** and Lemma 3.4.2), we have for any $q \ge 1$ the inequality

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \mathbb{E}^{\tilde{P}} \left[|\langle M_n(t,\tilde{x}_n), \ell\rangle_{\mathbb{H}}|^{2q} \right] &\le C_q \sup_{n\in\mathbb{N}} \mathbb{E}^{\tilde{P}} \left[\left(\int_0^t \|\sigma^*(s,\tilde{x}_n(s))(\ell)\|_{\mathbb{U}}^2 \, \mathrm{d}s \right)^q \right] \\
&\le C_{q,t} \sup_{n\in\mathbb{N}} \mathbb{E}^{P_n} \left[\int_0^t \|\sigma^*(s,x_n(s))(\ell)\|_{\mathbb{U}}^{2q} \, \mathrm{d}s \right] < \infty.
\end{aligned} \tag{3.4.29}$$

Hence, by using Equation (3.4.27) we obtain that for $t \geq 0$ the identity

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{P}}\left[\left|\langle M_n(t,\tilde{x}_n),\ell\rangle_{\mathbb{H}} - M_\ell(t,\tilde{x})\right|^2\right] = 0 \tag{3.4.30}$$

holds, i.e. $\lim_{n\to\infty} \mathbb{E}^{\tilde{P}}\left[\left|\langle M_n(t,\tilde{x}_n),\ell\rangle_{\mathbb{H}}\right|^2\right] = \mathbb{E}^{\tilde{P}}\left[\left|M_\ell(t,\tilde{x})\right|^2\right]$. Furthermore, by the dominated convergence theorem, we also have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}^{\tilde{P}}\left[\int_0^t \left\|(\Pi_n\sigma(s,\tilde{x}_n(s))\breve{\Pi}_n)^*(\ell) - \sigma^*(s,\tilde{x}(s))(\ell)\right\|_{\mathbb{U}}^2 \,\mathrm{d}s\right] \\
&\quad\leq \lim_{n\to\infty}\Bigg(\mathbb{E}^{\tilde{P}}\Bigg[\int_0^t \underbrace{\left\|(\Pi_n\sigma(s,\tilde{x}_n(s))\breve{\Pi}_n)^*(\ell) - (\Pi_n\sigma(s,\tilde{x}(s))\breve{\Pi}_n)^*(\ell)\right\|_{\mathbb{U}}^2}_{\xrightarrow[n\longrightarrow\infty]{} 0} \,\mathrm{d}s\Bigg] \\
&\qquad + \mathbb{E}^{\tilde{P}}\Bigg[\int_0^t \underbrace{\left\|(\Pi_n\sigma(s,\tilde{x}(s))\breve{\Pi}_n)^*(\ell) - \sigma^*(s,\tilde{x}(s))(\ell)\right\|_{\mathbb{U}}^2}_{\xrightarrow[n\longrightarrow\infty]{} 0} \,\mathrm{d}s\Bigg]\Bigg) \\
&\quad = 0.
\end{aligned} \tag{3.4.31}$$

In fact, the convergence

$$\left\|(\Pi_n\sigma(s,\tilde{x}(s))\breve{\Pi}_n)^*(\ell) - \sigma^*(s,\tilde{x}(s))(\ell)\right\|_{\mathbb{U}}^2 \xrightarrow[n\longrightarrow\infty]{} 0$$

follows directly from the definition of Π_n and $\breve{\Pi}_n$. Moreover, by Assumption **(A1)** we have $\left\|\sigma^*(s,\tilde{x}_n(s))(\ell) - \sigma^*(s,\tilde{x}(s))(\ell)\right\|_{\mathbb{U}} \xrightarrow[n\longrightarrow\infty]{} 0$ and, hence,

$$\begin{aligned}
&\left\|(\Pi_n\sigma(s,\tilde{x}_n(s))\breve{\Pi}_n)^*(\ell) - (\Pi_n\sigma(s,\tilde{x}(s))\breve{\Pi}_n)^*(\ell)\right\|_{\mathbb{U}}^2 \\
&\quad = \left\|\breve{\Pi}_n\sigma^*(s,\tilde{x}_n(s))\Pi_n(\ell) - \breve{\Pi}_n\sigma^*(s,\tilde{x}(s))\Pi_n(\ell)\right\|_{\mathbb{U}}^2 \\
&\quad \leq \left\|\sigma^*(s,\tilde{x}_n(s))\Pi_n(\ell) - \sigma^*(s,\tilde{x}(s))\Pi_n(\ell)\right\|_{\mathbb{U}}^2 \\
&\quad \leq C\left\|\sigma^*(s,\tilde{x}_n(s))\Pi_n(\ell) - \sigma^*(s,\tilde{x}_n(s))(\ell)\right\|_{\mathbb{U}}^2 + C\left\|\sigma^*(s,\tilde{x}_n(s))(\ell) - \sigma^*(s,\tilde{x}(s))(\ell)\right\|_{\mathbb{U}}^2 \\
&\qquad + C\left\|\sigma^*(s,\tilde{x}(s))(\ell) - \sigma^*(s,\tilde{x}(s))\Pi_n(\ell)\right\|_{\mathbb{U}}^2 \\
&\quad \leq C\underbrace{\left\|\sigma^*(s,\tilde{x}_n(s))\right\|_{L_2(\mathbb{H};\mathbb{U})}^2}_{\leq \lambda_4(s)(1+\|\tilde{x}_n(s)\|_{\mathbb{H}}^2)}\left\|(\Pi_n - \mathrm{Id})(\ell)\right\|_{\mathbb{H}}^2 + C\left\|\sigma^*(s,\tilde{x}_n(s))(\ell) - \sigma^*(s,\tilde{x}(s))(\ell)\right\|_{\mathbb{U}}^2 \\
&\qquad + C\left\|\sigma^*(s,\tilde{x}(s))\right\|_{L_2(\mathbb{H};\mathbb{U})}^2\left\|(\mathrm{Id} - \Pi_n)(\ell)\right\|_{\mathbb{H}}^2 \\
&\quad \xrightarrow[n\longrightarrow\infty]{} 0
\end{aligned}$$

is fulfilled since $\tilde{x}_n \xrightarrow[n\to\infty]{} \tilde{x}$ in $\mathbb{S}$ $\tilde{P}$-a.s. and $\ell \in \mathcal{E}$, where C is a constant. Altogether, Equations (3.4.30) and (3.4.31) imply that

$$\mathbb{E}^P\left[M_\ell^2(t,x) - \int_0^t \|\sigma^*(r,x(r))(\ell)\|_{\mathbb{U}}^2\,\mathrm{d}r\,\middle|\,\mathcal{F}_s\right] = M_\ell^2(s,x) - \int_0^s \|\sigma^*(r,x(r))(\ell)\|_{\mathbb{U}}^2\,\mathrm{d}r$$

holds by applying the same technique as in Equation (3.4.28). Now the result follows, because (by the Doob-Meyer decomposition) the increasing continuous adapted process $\langle M_\ell\rangle$, for which $M_\ell^2 - \langle M_\ell\rangle$ is a martingale, is unique (see e.g. [RY99, Theorem 1.3, p. 120]). □

3.4.2 Finite-dimensional case

As explained before, we will now go back and consider the martingale problem only in finite dimensions. The main idea is to use cutoff functions for the coefficients b and σ in order to apply well-known results on existence of martingale solutions, in particular those from [SV79], where assumptions on boundedness and continuity of b and σ are imposed.

Then, after proving tightness, we will again show that the probability measure P, which has been constructed as a limit, satisfies Conditions **(M1)** and **(M2)**. These calculations are almost completely analogous to those presented before in infinite dimensions. Hence, we will refer to equations from Subsection 3.4.1 frequently in order to shorten the length of this proof and focus on the differences between the finite- and infinite-dimensional case instead.

Proof of Theorem 3.3.1 for $\mathbb{R}^d$.
Let $\mathbb{U} = \mathbb{H} = \mathbb{Y} = \mathbb{X} = \mathbb{R}^d$. In this case we have $\Omega = C([0,\infty);\mathbb{R}^d)$. Define for $m \in \mathbb{N}$, $t \geq 0$ and $y \in \mathbb{R}^d$ the measurable functions b_m and σ_m by

$$\begin{aligned} b_m(t,y) &:= 1_{\{t\leq m\}}\chi_m(y)b(t,y),\\ \sigma_m(t,y) &:= 1_{\{t\leq m\}}\chi_m(y)\sigma(t,y), \end{aligned}$$

where $0 \leq \chi_m \in C(\mathbb{R}^d;\mathbb{R})$ is a decreasing cutoff function with

$$\chi_m(y) = \begin{cases} 1, & \text{for } \|y\|_{\mathbb{R}^d} \leq m,\\ 0, & \text{for } \|y\|_{\mathbb{R}^d} > 2m. \end{cases}$$

Hence, for each $t \geq 0$, the mappings $y \longmapsto b_m(t,y)$ and $y \longmapsto \sigma_m(t,y)$ are continuous, since **(A1)** yields continuity in y for b and σ themselves in finite dimensions.

Furthermore, minding Assumptions **(A3)** and **(N)**, the inequalities

$$\begin{aligned} \|\sigma_m(t,y)\|^2_{L_2(\mathbb{R}^d;\mathbb{R}^d)} &\leq \|\sigma(t,y)\|^2_{L_2(\mathbb{R}^d;\mathbb{R}^d)} \leq \lambda_4(t)(1+\|y\|^2_{\mathbb{R}^d}),\\ \|b_m(t,y)\|^\gamma_{\mathbb{R}^d} &\leq \|b(t,y)\|^\gamma_{\mathbb{R}^d} \leq \lambda_2(t)\underbrace{\mathcal{N}(y)}_{\leq C_d\|y\|^p_{\mathbb{R}^d}} + \lambda_3(t)(1+\|y\|^{\gamma'}_{\mathbb{R}^d}) \end{aligned} \tag{3.4.32}$$

hold by construction for every $(t,y) \in [0,\infty)\times\mathbb{R}^d$. Hence, for $m \in \mathbb{N}$, the functions b_m and σ_m are bounded, since for any ball $B_{2m}(0)$ they are bounded on $[0,m]\times B_{2m}(0)$ and zero on the complement.

Consequently, by applying [SV79, Theorem 6.1.7, p. 144], we conclude that for every $m \in \mathbb{N}$ there exists a probability measure $P_m \in \mathcal{P}(\Omega)$ such that $P_m[x(0) = x_0] = 1$ and

$$M_m(t,x) := x(t) - x(0) - \int_0^t b_m(s,x(s))\,\mathrm{d}s, \quad t \geq 0,$$

is a continuous $\mathbb{R}^d$-valued $(\mathcal{F}_t)$-martingale, whose covariation operator process is given by

$$\ll M_m \gg (t,x) = \int_0^t (\sigma_m)^*(s,x(s))\sigma_m(s,x(s))\,\mathrm{d}s, \quad t \geq 0.$$

Analogue to Lemma 3.4.2:
First of all, we need an analogue to the a priori energy estimate in Lemma 3.4.2.

However, for the coercivity condition, we can only estimate

$$\langle b_m(r,y), y\rangle_{\mathbb{R}^d} = 1_{\{r\le m\}}\chi_m(y)\langle b(r,y), y\rangle_{\mathbb{R}^d} \le \lambda_1(r)(1+\|y\|_{\mathbb{R}^d}^2) \tag{3.4.33}$$

by dropping the negative $\mathcal{N}$-term in Assumption **(A2)**. Therefore, we have to modify our calculations slightly, but we still apply Itô's formula for $\mathbb{R}^d$-valued semimartingales as in Equation (3.4.8). For any $q \ge 1$, the identity

$$\begin{aligned}\|x(t)\|_{\mathbb{R}^d}^{2q} = \|x(0)\|_{\mathbb{R}^d}^{2q} &+ 2q\int_0^t \|x(r)\|_{\mathbb{R}^d}^{2(q-1)} \langle b_m(r,x(r)), x(r)\rangle_{\mathbb{R}^d}\,\mathrm{d}r \\ &+ q\int_0^t \|x(r)\|_{\mathbb{R}^d}^{2(q-1)}\|\sigma_m(r,x(r))\|_{L_2(\mathbb{R}^d;\mathbb{R}^d)}^2\,\mathrm{d}r \\ &+ 2q(q-1)\int_0^t \|x(r)\|_{\mathbb{R}^d}^{2(q-2)}\|\sigma_m^*(r,x(r))\,(x(r))\|_{\mathbb{R}^d}^2\,\mathrm{d}r \\ &+ M_m^{(q)}(t,x)\end{aligned}$$

follows, where $M_m^{(q)}(t,x)$ is a continuous real-valued $(\mathcal{F}_t)$-martingale with respect to P_m, whose quadratic variation process is given by

$$\langle M_m^{(q)}\rangle(t,x) = 4q^2\int_0^t \|x(r)\|_{\mathbb{R}^d}^{4(q-1)}\|\sigma_m^*(r,x(r))(x(r))\|_{\mathbb{R}^d}^2\,\mathrm{d}r.$$

Hence, we conclude by using Inequality (3.4.32) that

$$\begin{aligned}\|x(t)\|_{\mathbb{R}^d}^{2q} \le \|x(0)\|_{\mathbb{R}^d}^{2q} &+ 2q\int_0^t \|x(r)\|_{\mathbb{R}^d}^{2(q-1)} \langle b_m(r,x(r)), x(r)\rangle_{\mathbb{R}^d}\,\mathrm{d}r \\ &+ q\int_0^t \|x(r)\|_{\mathbb{R}^d}^{2(q-1)}\|\sigma(r,x(r))\|_{L_2(\mathbb{R}^d;\mathbb{R}^d)}^2\,\mathrm{d}r \\ &+ 2q(q-1)\int_0^t \|x(r)\|_{\mathbb{R}^d}^{2(q-2)}\|\sigma(r,x(r))\|_{L_2(\mathbb{R}^d;\mathbb{R}^d)}^2\|x(r)\|_{\mathbb{R}^d}^2\,\mathrm{d}r \\ &+ M_m^{(q)}(t,x)\end{aligned}$$

holds and proceed as in Estimate (3.4.9) (by applying Inequality (3.4.33) and Assumption **(A3)**) to obtain

$$\|x(t)\|_{\mathbb{R}^d}^{2q} \le \|x(0)\|_{\mathbb{R}^d}^{2q} + C_q\lambda_*(t)\int_0^t (\|x(r)\|_{\mathbb{R}^d}^{2q}+1)\,\mathrm{d}r + M_m^{(q)}(t,x). \tag{3.4.34}$$

Next, the application of Gronwall's inequality is obviously easier without the $\mathcal{N}$-term. After defining the stopping time

$$\tau^R := \inf\{r\in[0,t] \mid \|x(r)\|_{\mathbb{R}^d} > R\}\wedge t, \quad R>0,$$

a repetition of the calculations from Estimate (3.4.11) yields

$$\begin{aligned}&\mathbb{E}^{P_m}\left[\sup_{r\in[0,t\wedge\tau^R]}\left|M_m^{(q)}(r,x)\right|\right] \\ &\le C_q\mathbb{E}^{P_m}\left[\left(\int_0^{t\wedge\tau^R}\|x(r)\|_{\mathbb{R}^d}^{4(q-1)}\|\sigma_m^*(r,x(r))(x(r))\|_{\mathbb{R}^d}^2\,\mathrm{d}r\right)^{\frac{1}{2}}\right] \\ &\le \frac{1}{2}\mathbb{E}^{P_m}\left[\sup_{r\in[0,t\wedge\tau^R]}\|x(r)\|_{\mathbb{R}^d}^{2q}\right] + C_q\lambda_*(t)\int_0^{t\wedge\tau^R}\mathbb{E}^{P_m}\left[\sup_{\tilde r\in[0,r\wedge\tau^R]}\|x(\tilde r)\|_{\mathbb{R}^d}^{2q}\right]+1\,\mathrm{d}r.\end{aligned}$$

Hence, by again first taking suprema and then expectations with respect to P_m on both sides of Inequality (3.4.34), we obtain

$$\mathbb{E}^{P_m}\left[\sup_{r\in[0,t\wedge\tau^R]}\|x(r)\|_{\mathbb{R}^d}^{2q}\right] \leq 2\mathbb{E}^{P_m}\left[\|x(0)\|_{\mathbb{R}^d}^{2q}\right] + 2C_{q,t}\int_0^t \mathbb{E}^{P_m}\left[\sup_{\tilde{r}\in[0,r\wedge\tau^R]}\|x(\tilde{r})\|_{\mathbb{R}^d}^{2q}\right] + 1\,\mathrm{d}r,$$

which by using Gronwall's inequality as in Estimate (3.4.12) and letting $R \longrightarrow \infty$ gives

$$\mathbb{E}^{P_m}\left[\sup_{r\in[0,t]}\|x(r)\|_{\mathbb{R}^d}^{2q}\right] \leq \exp\big(2C_{q,t}t\big)\Big(2\|x_0\|_{\mathbb{R}^d}^{2q}+1\Big) = C_{q,t} < \infty.$$

Since we have dropped the negative $\mathcal{N}$-term, we use Assumption **(N)** as a finite-dimensional replacement instead in order to control the growth of b_m in Assumption **(A3)**. We then have

$$\mathbb{E}^{P_m}\left[\int_0^t \mathcal{N}(x(r))\,\mathrm{d}r\right] \leq C_d\,\mathbb{E}^{P_m}\left[\int_0^t \|x(r)\|_{\mathbb{R}^d}^{p}\,\mathrm{d}r\right] \leq C_d\,t\,C_{p,t} < \infty.$$

Tightness as in Step 2:
We will show that $(P_m)_{m\in\mathbb{N}}$ is tight on $C\big([0,\infty);\mathbb{R}^d\big)$ analogous to the infinite-dimensional case by using Lemma 3.4.1.

By using Estimate (3.4.32), we can reproduce Estimate (3.4.14) to obtain

$$\mathbb{E}^{P_m}\left[\sup_{s,t\in[0,T],\,s\neq t}\frac{\big\|\int_s^t b_m(r,x(r))\,\mathrm{d}r\big\|_{\mathbb{R}^d}}{|t-s|^{\beta_1}}\right] \leq C_{p,\gamma,\gamma',T}$$

for every $\beta_1 \in \big(0,\frac{\gamma-1}{\gamma}\big)$ as well as Equation (3.4.16), which yields

$$\mathbb{E}^{P_m}\left[\sup_{s,t\in[0,T],\,s\neq t}\frac{\|M_m(t,x)-M_m(s,x)\|_{\mathbb{R}^d}}{|t-s|^{\beta_2}}\right] \leq C_{q,T}$$

for every $\beta_2 \in \big(0,\frac{q-1}{2q}\big)$. Combining both estimates yields that for some $\beta > 0$ (chosen small enough) the estimate

$$\sup_{m\in\mathbb{N}}\mathbb{E}^{P_m}\left[\sup_{s,t\in[0,T],\,s\neq t}\frac{\|x(t)-x(s)\|_{\mathbb{R}^d}}{|t-s|^{\beta}}\right] < \infty$$

holds, which, as we mentioned above, implies tightness of $(P_m)_{m\in\mathbb{N}}$.

Theorems of Prokhorov and Skorokhod as in Step 3:
We follow **Step 3** from the infinite-dimensional proof. First, we again use Prokhorov's theorem and select a subsequence if necessary. Then, we apply Skorokhod's representation theorem to the law of x under P_m, i.e. there exist $C\big([0,\infty);\mathbb{R}^d\big)$-valued random variables $\tilde{x}_m$, $m\in\mathbb{N}$, and $\tilde{x}$ on a common probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ such that we have

(i) $\tilde{x}_m$ (under $\tilde{P}$) has the law P_m for each $m\in\mathbb{N}$, i.e. $\tilde{P}\circ\tilde{x}_m^{-1} = P_m$,

(ii) $\tilde{x}$ (under $\tilde{P}$) has the law P, i.e. $\tilde{P}\circ\tilde{x}^{-1} = P$,

(iii) $\tilde{x}_m \xrightarrow[m\to\infty]{} \tilde{x}$ in $C\big([0,\infty);\mathbb{R}^d\big)$, $\tilde{P}$-a.s.

Note that the laws $P_m \circ x^{-1}$ and $P \circ x^{-1}$ simplify to P_m and P, respectively, since x is the canonical process.

Conditions (M1) and (M2) for P **as in Step 4:**
First, we will verify Condition **(M1)** for P. We have, similar to Equation (3.4.18) in infinite dimensions,

$$P[x(0)=x_0] = \tilde{P}[\tilde{x}(0)=x_0] = \lim_{m\to\infty} \tilde{P}[\tilde{x}_m(0)=x_0] = \lim_{m\to\infty} P_m[x(0)=x_0] = 1$$

since P_m satisfies Condition **(M1)**. Furthermore, for any $q \geq 1$ and $t \geq 0$, define

$$\xi_q(t,x) := \sup_{r\in[0,t]} \|x(r)\|_{\mathbb{R}^d}^{2q} + \int_0^t \mathcal{N}(x(r))\,\mathrm{d}r.$$

As in Equation (3.4.20) we then have

$$\begin{aligned}\mathbb{E}^P[\xi_q(t,x)] = \mathbb{E}^{\tilde{P}}[\xi_q(t,\tilde{x})] &\leq \liminf_{m\to\infty} \mathbb{E}^{\tilde{P}}[\xi_q(t,\tilde{x}_m)] = \liminf_{m\to\infty} \mathbb{E}^{P_m}[\xi_q(t,x)] \\ &\leq \liminf_{m\to\infty} C_{p,q,t} < \infty,\end{aligned}$$

which also yields the required integrability in Condition **(M1)**.

Now we will prove that Condition **(M2)** holds for P. Fix $\ell \in \mathcal{E} = \operatorname{span}\{e_1,\ldots,e_d\}$. We will show that $M_\ell(t,x)$, $t \geq 0$, in Condition **(M2)** is a continuous $(\mathcal{F}_t)$-martingale with respect to P, whose quadratic variation process is given by

$$\langle M_\ell\rangle(t,x) = \int_0^t \|\sigma^*(s,x(s))(\ell)\|_{\mathbb{R}^d}^2\,\mathrm{d}s, \quad t\geq 0.$$

As in Equation (3.4.27) we first prove that for $t>0$

$$\lim_{m\to\infty} \mathbb{E}^{\tilde{P}}\Big[\big|\langle M_m(t,\tilde{x}_m),\ell\rangle_{\mathbb{R}^d} - M_\ell(t,\tilde{x})\big|\Big] = 0 \tag{3.4.35}$$

holds by considering

$$\mathbb{E}^{\tilde{P}}\left[\left|\langle \tilde{x}_m(t)-\tilde{x}(t),\ell\rangle_{\mathbb{R}^d} - \int_0^t \langle b_m(s,\tilde{x}_m(s)) - b(s,\tilde{x}(s)),\ell\rangle_{\mathbb{R}^d}\,\mathrm{d}s\right|\right].$$

On the one hand, by repeating the calculations from Inequality (3.4.23), we have

$$\lim_{m\to\infty} \mathbb{E}^{\tilde{P}}\big[|\langle \tilde{x}_m(t)-\tilde{x}(t),\ell\rangle_{\mathbb{R}^d}|\big] = 0.$$

On the other hand, we split the integral for the second term as follows

$$\begin{aligned}&\mathbb{E}^{\tilde{P}}\left[\left|\int_0^t \langle b_m(s,\tilde{x}_m(s)) - b(s,\tilde{x}(s)),\ell\rangle_{\mathbb{R}^d}\,\mathrm{d}s\right|\right] \\ &\quad= \mathbb{E}^{\tilde{P}}\left[\left|\int_0^t \langle b_m(s,\tilde{x}_m(s)) - b(s,\tilde{x}_m(s)),\ell\rangle_{\mathbb{R}^d} + \langle b(s,\tilde{x}_m(s)) - b(s,\tilde{x}(s)),\ell\rangle_{\mathbb{R}^d}\,\mathrm{d}s\right|\right].\end{aligned}$$

Since, by a calculation similar to Estimate (3.4.14), we have

$$\mathbb{E}^{\tilde{P}}\left[\int_0^t \|b(s,\tilde{x}_m(s))\|_{\mathbb{R}^d}^{\gamma}\,\mathrm{d}s\right] = \mathbb{E}^{P_m}\left[\int_0^t \|b(s,x(s))\|_{\mathbb{R}^d}^{\gamma}\,\mathrm{d}s\right] \le C_{\gamma',t}, \tag{3.4.36}$$

we can proceed similar to Estimate (3.4.25) for the first summand, i.e.

$$\begin{aligned}
&\lim_{m\to\infty} \mathbb{E}^{\tilde{P}}\left[\left|\int_0^t \langle b_m(s,\tilde{x}_m(s)) - b(s,\tilde{x}_m(s)),\ell\rangle_{\mathbb{R}^d}\,\mathrm{d}s\right|\right]\\
&\quad\le \lim_{m\to\infty} \mathbb{E}^{\tilde{P}}\left[\int_0^t \left|\langle b(s,\tilde{x}_m(s)),\ell\rangle_{\mathbb{R}^d}\right| \left|1_{\{s\le m\}}\chi_m(\tilde{x}_m(s)) - 1\right|\mathrm{d}s\right]\\
&\quad\le \|\ell\|_{\mathbb{R}^d} \lim_{m\to\infty} \mathbb{E}^{\tilde{P}}\left[\int_0^t \|b(s,\tilde{x}_m(s))\|_{\mathbb{R}^d}\left(1_{\{s\ge m\}} + 1_{\left\{\|\tilde{x}_m(s)\|_{\mathbb{R}^d}\ge m\right\}}\right)\mathrm{d}s\right]\\
&\quad\le \|\ell\|_{\mathbb{R}^d} \lim_{m\to\infty}\left(\left(\mathbb{E}^{\tilde{P}}\left[\int_0^t \|b(s,\tilde{x}_m(s))\|_{\mathbb{R}^d}^{\gamma}\,\mathrm{d}s\right]\right)^{\frac{1}{\gamma}}\left(\int_0^t \frac{1}{m}\,\mathbb{E}^{\tilde{P}}[s]\,\mathrm{d}s\right)^{\frac{\gamma-1}{\gamma}}\right. \qquad (3.4.37)\\
&\qquad\qquad + \left(\mathbb{E}^{\tilde{P}}\left[\int_0^t \|b(s,\tilde{x}_m(s))\|_{\mathbb{R}^d}^{\gamma}\,\mathrm{d}s\right]\right)^{\frac{1}{\gamma}}\\
&\qquad\qquad\qquad \left.\cdot\left(\int_0^t \frac{1}{m^2}\,\mathbb{E}^{\tilde{P}}\left[\sup_{0\le r\le s}\|\tilde{x}_m(r)\|_{\mathbb{R}^d}^2\right]\mathrm{d}s\right)^{\frac{\gamma-1}{\gamma}}\right)\\
&\quad\le \|\ell\|_{\mathbb{R}^d} \lim_{m\to\infty}\left((C_{\gamma',t})^{\frac{1}{\gamma}}\left(C_t\,\frac{1}{m}\right)^{\frac{\gamma-1}{\gamma}} + (C_{\gamma',t})^{\frac{1}{\gamma}}\left(C_t\,\frac{1}{m^2}\right)^{\frac{\gamma-1}{\gamma}}\right)\\
&\quad= 0.
\end{aligned}$$

The convergence of the second summand, i.e.

$$\lim_{m\to\infty} \mathbb{E}^{\tilde{P}}\left[\left|\int_0^t \langle b(s,\tilde{x}_m(s)) - b(s,\tilde{x}(s)),\ell\rangle_{\mathbb{R}^d}\,\mathrm{d}s\right|\right] = 0$$

follows from Assumption **(A1)** and again Estimate (3.4.36) since $\gamma > 1$.

Consequently, by using a calculation similar to Equation (3.4.28), a monotone class argument yields

$$\mathbb{E}^P\left[M_\ell(t,x)\middle|\mathcal{F}_s\right] = M_\ell(s,x).$$

Finally, we will prove the representation formula of the quadratic variation $\langle M_\ell\rangle$ given in Condition **(M2)**. We proceed as in the infinite-dimensional proof. Repeating the calculations from Estimate (3.4.29) for $q \ge 1$ yields

$$\begin{aligned}
\sup_{m\in\mathbb{N}} \mathbb{E}^{\tilde{P}}\left[|\langle M_m(t,\tilde{x}_m),\ell\rangle_{\mathbb{R}^d}|^{2q}\right] &\le C_q \sup_{m\in\mathbb{N}} \mathbb{E}^{\tilde{P}}\left[\left(\int_0^t \|\sigma_m^*(s,\tilde{x}_m(s))(\ell)\|_{\mathbb{R}^d}^2\,\mathrm{d}s\right)^q\right]\\
&\le C_{q,t} \sup_{m\in\mathbb{N}} \mathbb{E}^{\tilde{P}}\left[\int_0^t \|\sigma^*(s,\tilde{x}_m(s))(\ell)\|_{\mathbb{R}^d}^{2q}\,\mathrm{d}s\right]\\
&= C_{q,t} \sup_{m\in\mathbb{N}} \mathbb{E}^{P_n}\left[\int_0^t \|\sigma^*(s,x(s))(\ell)\|_{\mathbb{R}^d}^{2q}\,\mathrm{d}s\right] < \infty
\end{aligned}$$

giving us an analogue of Equation (3.4.30), i.e.

$$\lim_{m\to\infty} \mathbb{E}^{\tilde{P}}\Big[\big|\langle M_m(t,\tilde{x}_m),\ell\rangle_{\mathbb{R}^d}\big|^2\Big] = \mathbb{E}^{\tilde{P}}\Big[\big|M_\ell(t,\tilde{x})\big|^2\Big],$$

which follows directly from Equation (3.4.35).

Furthermore, as in Equation (3.4.31) we have

$$\lim_{m\to\infty} \mathbb{E}^{\tilde{P}}\left[\int_0^t \big\|\sigma_m^*(s,\tilde{x}_m(s))(\ell) - \sigma^*(s,\tilde{x}(s))(\ell)\big\|_{\mathbb{R}^d}^2 \,\mathrm{d}s\right] = 0.$$

In fact, we split the integrand into

$$\begin{aligned} &\big\|\sigma_m^*(s,\tilde{x}_m(s))(\ell) - \sigma^*(s,\tilde{x}(s))(\ell)\big\|_{\mathbb{R}^d}^2 \\ &\quad \le 2\big\|\sigma_m^*(s,\tilde{x}_m(s))(\ell) - \sigma^*(s,\tilde{x}_m(s))(\ell)\big\|_{\mathbb{R}^d}^2 + 2\big\|\sigma^*(s,\tilde{x}_m(s))(\ell) - \sigma^*(s,\tilde{x}(s))(\ell)\big\|_{\mathbb{R}^d}^2. \end{aligned}$$

Then, for the first term we have

$$\begin{aligned} \big\|\sigma_m^*(s,\tilde{x}_m(s))(\ell) &- \sigma^*(s,\tilde{x}_m(s))(\ell)\big\|_{\mathbb{R}^d}^2 \\ &\le \Big|1_{\{s\le m\}}\chi_m(\tilde{x}_m(s)) - 1\Big| \, \big\|\sigma^*(s,\tilde{x}_m(s))(\ell)\big\|_{\mathbb{R}^d}^2 \\ &\le \Big(1_{\{s\ge m\}} + 1_{\left\{\|\tilde{x}_m(s)\|_{\mathbb{R}^d}\ge m\right\}}\Big) \, \big\|\sigma^*(s,\tilde{x}_m(s))(\ell)\big\|_{\mathbb{R}^d}^2 \end{aligned}$$

and can proceed as in Estimate (3.4.37). For the second term we use Assumption **(A1)**.

Altogether, we have

$$\mathbb{E}^P\left[M_\ell^2(t,x) - \int_0^t \|\sigma^*(r,x(r))(\ell)\|_{\mathbb{R}^d}^2 \,\mathrm{d}r \,\Big|\, \mathcal{F}_s\right] = M_\ell^2(s,x) - \int_0^s \|\sigma^*(r,x(r))(\ell)\|_{\mathbb{R}^d}^2 \,\mathrm{d}r$$

by applying the same technique as before in Equation (3.4.28). Hence, as in the infinite dimensional case, the result follows since we have shown that $M_\ell^2 - \langle M_\ell\rangle$ is a martingale. □

Chapter 4

Linear FPKEs in infinite dimensions

Fokker–Planck–Kolmogorov equations are second order elliptic or parabolic equations for measures, that have been extensively studied by A. Kolmogorov (see [Kol31], [Kol33], [Kol37]) since the 1930s and prior to this (independently) in the physics literature by A. Fokker (see [Fok14]) and M. Planck (see [Pla17]) as well as authors like M. von Smoluchowski (see [Smo16]) or S. Chapman (see [Cha28]).

The close connection to physics is explained by its relevance in various fields of research. This includes most prominently statistical mechanics, where the corresponding equation for densities traditionally arises to describe time evolution of a probability density function (e.g. of the velocity of a small particle) under influence of drift and diffusion forces (in particular Brownian motion).

In the following, we will lay our focus on Cauchy problems for parabolic linear FPKEs (also called weak parabolic equations for measures with initial data) and refer to Section 9.2 for an explanation about the difference between linear and nonlinear FPKEs. The distinction between elliptic and parabolic equations follows as usual from the distinction between elliptic and parabolic operators that appear in the equation (in our case the operator L introduced below). Note that in the parabolic case the coefficients (appearing in L) may explicitly depend on the time parameter t.

Putting it simple, for some $T > 0$, some domain $\Gamma \subseteq \mathbb{R}^d$, some Borel function $A = (a^{ij})_{1 \le i,j \le d}$ on $[0,T] \times \Gamma$ taking values in the space of nonnegative symmetric $\mathbb{R}^{d\times d}$-matrices and some $\mathbb{R}^d$-valued Borel function b on $[0,T] \times \Gamma$, a finite-dimensional parabolic FPKE for Borel measures on $[0,T] \times \Gamma$ is generally of the form

$$\partial_t \mu = L^* \mu,$$

understood in the weak sense (via test functions)

$$\int \partial_t \varphi + L\varphi \,\mathrm{d}\mu = 0,$$

where L^* is the formal adjoint of the operator L, which is given by

$$L\varphi(t,y) = \sum_{i,j=1}^{d} a^{ij}(t,y)\partial_{y_i}\partial_{y_j}\varphi(t,y) + \sum_{i=1}^{d} b^i(t,y)\partial_{y_i}\varphi(t,y)$$

if the functions $\varphi\colon [0,T] \times \Gamma \longrightarrow \mathbb{R}$ are sufficiently smooth. Together with an initial distribution μ_0 in a suitable sense, the considered equation is a Cauchy problem. We refer to Section 4.2 below for an explanation on the precise understanding of this shorthand notation of the equation.

Let us note (as it is explained in [BKRS15, p. ix]) that it is beneficial to consider FPKEs a priori as equations for measures and not for functions. Of course, in finite dimensions, the measure μ can indeed have a density with respect to Lebesgue measure, which induces a corresponding equation for functions. But in infinite dimensions (with the absence of a Lebesgue measure) or in the case of singular or degenerate coefficients this approach via functions reaches its limits, which gives significant importance to the approach via measures.

Consequently, after being studied in finite dimensions, the research on infinite-dimensional FPKEs in this general (measure-based) setting received growing attention in the 1990s (see e.g. [BR94], [BR95], [BKR96]) and intensified since the beginning of the 21st century (see e.g. [BR01], [RS06], [BDR09], [BDR10], [BDR11], [Tre14], [BKRS15] and the references therein). Going back, it was initially motivated by diffusion processes and stochastic analysis in infinite dimensions, where in particular [AH77] and the work of A. Kirillov (e.g. [Kir91; Kir93; Kir94a; Kir94b]) were named by the authors of [BKR09, p. 975] and [BKRS15, p. x] as being formative.

This chapter will be mainly based on Chapter 10 of the AMS Monograph "Fokker–Planck–Kolmogorov equations" by V. Bogachev, N. V. Krylov, M. Röckner and S. Shaposhnikov from 2015 (see [BKRS15]). In Section 10.4, the authors prove two theorems (which originally have been proved in their prior work [BDRS15]) on existence of so-called probability solutions (see Definition 4.2.1 below) to the Cauchy problem for linear FPKEs in infinite dimensions.

The aim of this chapter is to present a slightly modified and more elaborate version of the main theorem from Section 10.4, i.e. Theorem 10.4.1 in [BKRS15], by providing additional details and clarification on key steps of the proof.

Moreover, as in Chapter 3, we will be directly applying and repeating techniques and calculations from the proof of the main theorem (see Theorem 4.3.1 below) later in the proof of the main theorem of Chapter 6 (see Section 6.4 below), which benefits from presenting all details at this point already.

The proof will heavily rely on finite-dimensional a priori estimates, calculations and results on existence of probability solutions to Cauchy problems. Hence, we will be referring to the two articles [BDR08a] and [BDR08b], which are closely related to the ideas and results of [BDRS15] and, therefore, fit perfectly in our setting.

In short, we will prove existence of probability solutions to the Cauchy problem for linear Fokker–Planck–Kolmogorov equations on infinite-dimensional spaces under assumptions on continuity and growth of our coefficients b and A and further assumptions on a so-called Lyapunov function V. As the main idea of the proof, we will use Galerkin approximations, i.e. construct a solution as a limit of solutions to finite-dimensional Cauchy problems by projecting onto finite-dimensional spaces $\mathbb{H}_n$.

4.1 Framework

Let $\mathbb{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and norm $\| \cdot \|_{\mathbb{H}}$. Recall that all infinite-dimensional separable Hilbert spaces are isometrically isomorphic to ℓ^2 (see e.g. [Bre11, Remark 10, p. 144]) and that we can treat ℓ^2 as a subspace of $\mathbb{R}^\infty$, where $\mathbb{R}^\infty$, equipped with the product topology, is a Polish space. This means we consider the continuous and dense embedding

$$\ell^2 \subseteq \mathbb{R}^\infty.$$

As in [BKRS15, Section 10.4], we let $\{e_1, e_2, \dots\}$ be the standard orthonormal basis in ℓ^2. Again, for any $n \in \mathbb{N}$, define $\mathbb{H}_n := \operatorname{span}\{e_1, \dots, e_n\}$. Let Π_n^∞ be the projection onto $\mathbb{H}_n$ in $\mathbb{R}^\infty$ given by

$$\Pi_n^\infty y := \sum_{i=1}^n y^i e_i = (y^1, \dots, y^n, 0, \dots) \Bigg(= \sum_{i=1}^n \langle y, e_i \rangle_{\ell^2} \, e_i, \quad \text{for } y \in \ell^2 \Bigg),$$

for any $y \in \mathbb{R}^\infty$ with $y = (y^i)_{i \in \mathbb{N}}$.

Fix $T > 0$. For every $i, j \in \mathbb{N}$, let the mappings

$$a^{ij} \colon [0,T] \times \mathbb{R}^\infty \longrightarrow \mathbb{R},$$
$$b^i \colon [0,T] \times \mathbb{R}^\infty \longrightarrow \mathbb{R}$$

be $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^\infty)/\mathcal{B}(\mathbb{R})$-measurable. In addition, we set

$$b_n := (b^1, \dots, b^n) \text{ and } A_n := (a^{ij})_{1 \le i,j \le n} \tag{4.1.1}$$

and $A := (a^{ij})_{1 \le i,j < \infty}$.

We can then consider the associated Kolmogorov operator L to our FPKE, acting on functions $\varphi \in \mathcal{F}C^2(\{e_i\})$, i.e. finitely based functions defined in Section 2 (see Equation (2.0.1)), which is given by

$$L\varphi(t,y) = \sum_{i,j=1}^d a^{ij}(t,y) \partial_{e_i} \partial_{e_j} \varphi(y) + \sum_{i=1}^d b^i(t,y) \partial_{e_i} \varphi(y),$$

for $(t,y) \in [0,T] \times \mathbb{R}^\infty$ and some $d \in \mathbb{N}$ depending on φ.

Remark. As usual, L obviously also acts on finitely based functions that are in addition explicitly depending on time, because this time-dependence is "irrelevant" for the partial derivatives appearing in the operator. But we will not need the often used classes of time-dependent test functions and rather mostly apply L to functions $\varphi \in \mathcal{F}C_c^\infty(\{e_i\})$ in the following.

Furthermore, denote by ν_n the projection onto $\mathbb{H}_n$ of a Borel probability measure ν on $\mathbb{R}^\infty$ (which will serve as our initial condition in the following), i.e. $\nu_n := \nu \circ (\Pi_n^\infty)^{-1}$.

Let us introduce the terms compactness and non-degeneracy for real-valued functions, which we will later use in the assumptions (see Subsection 4.2.3 below) for the main theorem.

Definition 4.1.1 (compact function, see e.g. [BKRS15, Definition 2.3.1, p. 62]). *A real-valued function f on a topological space is called compact if the sublevel sets $\{f \leq R\}$ are compact for any $R \in \mathbb{R}$.*

Remark. Let us note that, for real-valued continuous functions f on $\mathbb{R}^n$, this definition is equivalent to the condition $\lim_{\|y\|_{\mathbb{R}^n} \to \infty} f(y) = \infty$. In the following, we will mainly assume Borel functions $f \colon \mathbb{R}^\infty \longrightarrow [0, \infty]$ to be compact, which means that in this case the sublevel sets $\{f \leq R\}$ have to be compact for any $R < \infty$.

Definition 4.1.2 (non-degenerate function, see e.g. [BDR08a, p. 410]). *A compact function $f \in C^2(\mathbb{R}^n)$ is called non-degenerate if there exists a sequence $(c_k)_{k \in \mathbb{N}}$ of numbers with $c_k \xrightarrow[k \to \infty]{} \infty$ such that the level sets $f^{-1}(c_k) = \{y \in \mathbb{R}^n \mid f(y) = c_k\}$ are C^1-surfaces.*

Example (see [BDR08a, p. 410]). If a function f is convex, then it is non-degenerate. If $f(y) = f_0((y, y))$, where $f_0 \in C^2([0, \infty))$ is increasing to ∞, it is also non-degenerate.

Finally, we consider (strongly) Lindelöf spaces, which are generalizing the notion of a compact space by weakening the requirement that the subcover has to be finite to only countability.

Definition 4.1.3 (Lindelöf). *A topological space in which every open cover has a countable subcover is a Lindelöf space.*

Example. Every compact space is a Lindelöf space.

Definition 4.1.4 (strongly Lindelöf). *A topological space is a strongly Lindelöf space if every open subspace is Lindelöf.*

Example. Every Polish space is a strongly Lindelöf space.

4.2 Equation, Solution, Assumptions

Based on the framework from Section 4.1 we can now introduce the considered Cauchy problem, the corresponding notion of a probability solution and the necessary assumption for the existence result in Section 4.3.

4.2.1 Equation

Consider the following shorthand notation for a Cauchy problem for a linear Fokker–Planck–Kolmogorov equation given by

$$\begin{aligned} \partial_t \mu &= L^* \mu, \\ \mu_{\restriction t=0} &= \nu \end{aligned} \tag{4.2.1}$$

with respect to a nonnegative finite Borel measure μ of the form $\mu(\mathrm{d}t\, \mathrm{d}y) = \mu_t(\mathrm{d}y)\, \mathrm{d}t$ on $[0, T] \times \mathbb{R}^\infty$, where $(\mu_t)_{t \in [0,T]}$ is a family of Borel probability measures on $\mathbb{R}^\infty$. Furthermore, ν is a Borel probability measure on $\mathbb{R}^\infty$ and L^* is the formal adjoint of the operator L introduced before in Section 4.1.

4.2.2 Notion of solution

Let us formalize the abbreviated notation of Equation (4.2.1) into a definition of a probability solution, which inherently contains the initial condition for the Cauchy problem. This can be done mostly analogously to the case of finite-dimensional FPKEs (see e.g. [BKRS15, Definition 6.1.1 and Proposition 6.1.2, p. 242f]).

Definition 4.2.1 (probability solution). *A finite Borel measure μ on $[0,T]\times\mathbb{R}^\infty$ of the form $\mu(\mathrm{d}t\,\mathrm{d}y)=\mu_t(\mathrm{d}y)\,\mathrm{d}t$, where $(\mu_t)_{t\in[0,T]}$ is a family of Borel probability measures on $\mathbb{R}^\infty$, is called probability solution to Equation* (4.2.1) *if the following conditions hold.*

(i) The functions a^{ij}, b^i are integrable with respect to the measure μ, i.e.

$$a^{ij}, b^i \in L^1([0,T]\times\mathbb{R}^\infty,\mu).$$

(ii) For every function $\varphi\in\mathcal{F}C_c^\infty(\{e_i\})$ we have

$$\int_{\mathbb{R}^\infty}\varphi(y)\,\mu_t(\mathrm{d}y)=\int_{\mathbb{R}^\infty}\varphi(y)\,\nu(\mathrm{d}y)+\int_0^t\int_{\mathbb{R}^\infty}L\varphi(s,y)\,\mu_s(\mathrm{d}y)\,\mathrm{d}s \tag{4.2.2}$$

for $\mathrm{d}t$-a.e. $t\in[0,T]$.

Remark. Note that the integrability condition for the coefficients in infinite dimensions (see (i) in Definition 4.2.1) is stronger than the local integrability condition usually assumed in finite-dimensional definitions (see e.g. [BDR08a, p. 397f] or [BKRS15, Definition 6.1.1, p. 242]).

4.2.3 Assumptions

We impose the following assumptions on the coefficients A and b associated to Equation (4.2.1), which actually had to be modified slightly from [BKRS15] (i.e. Assumptions **(H2)** and **(H3)**) in order to prove Theorem 4.3.1 below:

(H1) For all $n\in\mathbb{N}$, the matrices $A_n=(a^{ij})_{1\le i,j\le n}$ are symmetric and nonnegative definite.

(H2) Let $\Theta\colon\mathbb{R}^\infty\longrightarrow[0,\infty]$ be a compact Borel function, bounded on bounded sets on each space $\mathbb{H}_n$, $n\in\mathbb{N}$, such that, for every $i\in\mathbb{N}$ and $j\le i$,

- the functions $y\mapsto a^{ij}(t,y)$, $t\in[0,T]$, are equicontinuous on every set $\{\Theta\le R\}$ with $R<\infty$ and also on every fixed ball in each $\mathbb{H}_n$,
- for every $t\in[0,T]$ the function $y\mapsto b^i(t,y)$ is continuous on every set $\{\Theta\le R\}$ with $R<\infty$ and also on each $\mathbb{H}_n$.

(H3) There exist numbers $M_0,C_0\ge 0$ and a compact Borel function $V\colon\mathbb{R}^\infty\longrightarrow[1,\infty]$ whose restrictions to $\mathbb{H}_n$ are of class $C^2(\mathbb{H}_n)$ and non-degenerate such that for all $y\in\mathbb{H}_n$, $n\in\mathbb{N}$, $t\in[0,T]$, we have

$$\sum_{i,j=1}^n a^{ij}(t,y)\partial_{e_i}V(y)\partial_{e_j}V(y)\le M_0V(y)^2, \tag{4.2.3}$$

$$LV(t,y)\le C_0V(y)-\Theta(y). \tag{4.2.4}$$

(H4) There exist constants $C_i \geq 0$ and $k_i \geq 0$ such that for all $i \in \mathbb{N}$ and $j \leq i$ we have

$$|a^{ij}(t, y)| + |b^i(t, y)| \leq C_i V(y)^{k_i}\big(1 + \delta_i(\Theta(y))\Theta(y)\big), \tag{4.2.5}$$

for every $(t, y) \in [0, T] \times \mathbb{R}^\infty$, where δ_i is a bounded nonnegative Borel function on $[0, \infty)$ with $\lim\limits_{s\to\infty} \delta_i(s) = 0$.

Remark. We note that these assumptions are not supposed to be perfectly optimal and leave room for improvement. As stated before, we had to modify the assumptions from [BKRS15] on the functions Θ and V slightly to guarantee that the finite-dimensional existence results from [BDR08a], that we intend to use for our proof, will be applicable in this situation.

In any way, further improvements should always be made within the wider scope of simplifying and unifying all collected assumption from Chapters 3–5 (see Subsection 6.2.2) for their later combined use in Chapter 6.

4.3 Results

Let us now state the main result of this chapter on existence of solutions to the Cauchy problem for linear FPKEs in infinite dimensions.

Theorem 4.3.1 (see [BKRS15, Theorem 10.4.1, p. 422])**.** *Assume that conditions **(H1)**–**(H4)** hold. Then, for every Borel probability measure ν on $\mathbb{R}^\infty$ satisfying the condition*

$$W_k := \sup_{n\in\mathbb{N}} \|V(\cdot)^k \circ \Pi_n^\infty\|_{L^1(\nu)} < \infty \tag{4.3.1}$$

for all $k \in \mathbb{N}$, the Cauchy problem (4.2.1) *with initial condition ν has a solution of the form $\mu = \mu_t\,\mathrm{d}t$ with Borel probability measures $(\mu_t)_{t\in[0,T]}$ on $\mathbb{R}^\infty$ such that for all $t \in [0, T]$ and $k \in \mathbb{N}$ we have*

$$\int_{\mathbb{R}^\infty} V(y)^k\,\mu_t(\mathrm{d}y) + k\int_0^t \int_{\mathbb{R}^\infty} V(y)^{k-1}\Theta(y)\,\mu_s(\mathrm{d}y)\,\mathrm{d}s \leq N_k W_k, \tag{4.3.2}$$

where $N_k := M_k e^{M_k} + 1$ and $M_k := k(C_0 + (k-1)M_0)$. In particular,

$$\mu_t(V < \infty) = 1 \tag{4.3.3}$$

holds for all $t \in [0, T]$ and $\mu_t(\Theta < \infty) = 1$ for $\mathrm{d}t$-a.e. $t \in [0, T]$.

A direct consequence is the following theorem, where the inequalities in Assumptions **(H3)** and **(H4)** are changed to make it applicable in the case where V and Θ are e.g. exponentials of quadratic functions (with added constants). Despite not being used in the upcoming chapters, we still want to state it for completeness.

Theorem 4.3.2 (see [BKRS15, Theorem 10.4.2, p. 425])**.** *Suppose that in Theorem 4.3.1 Inequality* (4.2.4) *in Assumption **(H3)** is replaced by*

$$LV(t, y) \leq V(y) - V(y)\Theta(y)$$

and Inequality (4.2.5) *in Assumption* ***(H4)*** *is replaced by*

$$|a^{ij}(t,y)| + |b^i(t,y)| \leq C_i\big(1 + \delta\big(V(y)\Theta(y)\big)V(y)\Theta(y)\big)$$

for $(t,y) \in [0,T] \times \mathbb{R}^\infty$*. Then, for every Borel probability measure* ν *on* $\mathbb{R}^\infty$ *with*

$$W_1 := \sup_{n\in\mathbb{N}} \|V \circ \Pi_n^\infty\|_{L^1(\nu)} < \infty,$$

the Cauchy problem (4.2.1) *with initial distribution* ν *has a solution of the form* $\mu = \mu_t\,\mathrm{d}t$ *with Borel probability measures* $(\mu_t)_{t\in[0,T]}$ *on* $\mathbb{R}^\infty$ *such that for* $t \in [0,T]$ *we have*

$$\int_{\mathbb{R}^\infty} V(y)\,\mu_t(\mathrm{d}y) + \int_0^t \int_{\mathbb{R}^\infty} V(y)\Theta(y)\,\mu_s(\mathrm{d}y)\,\mathrm{d}s \leq 4W_1.$$

Proof of Theorem 4.3.2. We refer to [BKRS15] for an explanation on how to modify the proof of Theorem 4.3.1 in this case. □

4.4 Proof

Before we can actually start with the proof of Theorem 4.3.1 we will first introduce this auxiliary lemma on countable point and measure-separating families of finitely based functions, which is proved by using similar techniques as in [MR92, p. 119].

Lemma 4.4.1. *There exits a countable family* $\mathcal{F}$ *of functions in* $\mathcal{F}C_c^\infty(\{e_i\})$*, which*

i) separates points in $\mathbb{R}^\infty$*,*

ii) separates measures on $\mathcal{B}(\mathbb{R}^\infty)$ *(i.e. for any two measures* $\mu_1, \mu_2 \in \mathcal{B}(\mathbb{R}^\infty)$ *with* $\mu_1 \neq \mu_2$ *there exists* $f \in \mathcal{F}$ *such that* $\int_{\mathbb{R}^\infty} f\,\mathrm{d}\mu_1 \neq \int_{\mathbb{R}^\infty} f\,\mathrm{d}\mu_2$*).*

Proof. Let us begin by first leaving out the countability and showing the following simplified claim instead.

Claim: There exists a family $\tilde{\mathcal{F}} \subseteq \mathcal{F}C_c^\infty(\{e_i\})$ that separates points in $\mathbb{R}^\infty$.

Proof of Claim: Let $y_1, y_2 \in \mathbb{R}^\infty$ with $y_1 \neq y_2$. Consequently, there exists some $d \in \mathbb{N}$ such that their d-th component differs, i.e. $y_1^d \neq y_2^d$. We consider

$$\Pi_d^\infty y_1 = (y_1^1, \ldots, y_1^d) \in \mathbb{R}^d,$$
$$\Pi_d^\infty y_2 = (y_2^1, \ldots, y_2^d) \in \mathbb{R}^d,$$

where Π_d^∞ is the projection onto $\mathbb{R}^d$, and obtain $\Pi_d^\infty y_1 \neq \Pi_d^\infty y_2$. By using the fact that points in $\mathbb{R}^d$ can be separated by functions of class $C_c^\infty(\mathbb{R}^d)$, there exists some $f \in C_c^\infty(\mathbb{R}^d)$ such that $f(\Pi_d^\infty y_1) \neq f(\Pi_d^\infty y_2)$ (choose e.g. f with $f(\Pi_d^\infty y_1) = 1$ and $\mathrm{supp}(f) \subseteq B_{\mathrm{dist}\{\Pi_d^\infty y_1, \Pi_d^\infty y_2\}/2}(\Pi_d^\infty y_1)$). Hence, we can just consider the finitely based function $\xi\colon \mathbb{R}^\infty \longrightarrow \mathbb{R}$ given by

$$\xi(y) := f(\Pi_d^\infty(y)) = f(y^1, \ldots, y^d), \quad y \in \mathbb{R}^\infty,$$

i.e. we constructed a function $\xi \in \mathcal{F}C_c^\infty(\{e_i\})$ that separates the points y_1 and y_2 in $\mathbb{R}^\infty$. The family $\tilde{\mathcal{F}}$ is then chosen to be the collection of all such functions ξ. □

Now let us start with i): By using the claim from above, it remains to show that there exists a subset $\mathcal{F}$ of the family $\tilde{\mathcal{F}}$ that is in fact countable.

For any function $f \in \mathcal{F}C_c^\infty(\{e_i\})$, let $(f,f)\colon \mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow \mathbb{R} \times \mathbb{R}$ be the function given by $(f,f)(y_1,y_2) := (f(y_1), f(y_2))$. Set $D_{\mathbb{R}^\infty} := \{(y_1,y_2) \in \mathbb{R}^\infty \times \mathbb{R}^\infty \mid y_1 = y_2\}$ to be the diagonal of $\mathbb{R}^\infty \times \mathbb{R}^\infty$ and analogously let $D_\mathbb{R}$ be the diagonal of $\mathbb{R} \times \mathbb{R}$. Then, since the functions $f \in \tilde{\mathcal{F}}$ separate points in $\mathbb{R}^\infty$, the equation

$$\big(\mathbb{R}^\infty \times \mathbb{R}^\infty\big) \setminus D_{\mathbb{R}^\infty} = \bigcup_{f \in \tilde{\mathcal{F}}} (f,f)^{-1}(\mathbb{R} \times \mathbb{R} \setminus D_\mathbb{R}) \tag{4.4.1}$$

holds.

Furthermore, note that $\mathbb{R}^\infty \times \mathbb{R}^\infty$ is a Polish space and, hence, a strongly Lindelöf space (see Definition 4.1.4). Since $\big(\mathbb{R}^\infty \times \mathbb{R}^\infty\big) \setminus D_{\mathbb{R}^\infty}$ is an open subset of $\mathbb{R}^\infty \times \mathbb{R}^\infty$, the open cover on the right hand side of Equation (4.4.1) has a countable subcover, i.e. there exists a countable family $\mathcal{F} \subseteq \mathcal{F}C_c^\infty(\{e_i\})$ such that

$$\big(\mathbb{R}^\infty \times \mathbb{R}^\infty\big) \setminus D_{\mathbb{R}^\infty} = \bigcup_{f \in \mathcal{F}} (f,f)^{-1}(\mathbb{R} \times \mathbb{R} \setminus D_\mathbb{R}).$$

The assertion follows.

Finally let us prove ii): Without loss of generality we can assume that $\mathcal{F}$ is a multiplicative system. By a monotone class argument it then remains to prove that $\mathcal{F}$ generates $\mathcal{B}(\mathbb{R}^\infty)$. Since the functions in $\mathcal{F}$ are continuous, we immediately have $\sigma(\mathcal{F}) \subseteq \mathcal{B}(\mathbb{R}^\infty)$.

In addition, we consider the measurable function id mapping from the Polish space $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ to the space $(\mathbb{R}^\infty, \sigma(\mathcal{F}))$ equipped with the countably generated σ-algebra $\sigma(\mathcal{F})$. By Kuratowski's theorem (see e.g. [Kur66, p. 487f] or [Par67, Section I.3, p. 15ff]) it follows that id^{-1} is $\sigma(\mathcal{F})/\mathcal{B}(\mathbb{R}^\infty)$-measurable, which implies that $\mathcal{B}(\mathbb{R}^\infty) \subseteq \sigma(\mathcal{F})$. □

For the proof of Theorem 4.3.1 we will follow [BKRS15] up to some minor modifications.

Proof of Theorem 4.3.1. Let us divide the proof into five steps. Note that we identify $\mathbb{H}_n$ with $\mathbb{R}^n$.

Step 0: Existence of $\mu_{t,n}$ on $\mathbb{H}_n$

In this preliminary step, we will prove existence of Borel probability measures $\mu_{t,n}$ on $\mathbb{H}_n$ such that the measure $\mu_n := \mu_{t,n}\,\mathrm{d}t$ solves the Cauchy problem with coefficients A_n and b_n on $[0,T] \times \mathbb{H}_n$ and initial distribution ν_n. Therefore, for any $m \geq 1$, consider the m-th power of V, i.e. the restriction of the mapping

$$V(\cdot)^m \colon \mathbb{R}^\infty \longrightarrow [1,\infty]$$

to the subspace $\mathbb{H}_n$, as our Lyapunov function. We define

$$M_m := m(C_0 + (m-1)M_0).$$

By using the definition of L in this finite-dimensional case in the first and Assumption

(H3) in the third step, we then have

$$\begin{aligned}LV^m(t,y) &= \sum_{i,j=1}^{n} a^{ij}(t,y)\partial_{e_i}\partial_{e_j}V(y)^m + \sum_{i=1}^{n} b^i(t,y)\partial_{e_i}V(y)^m \\ &= mV(y)^{m-1}\Big(LV(t,y) + (m-1)V(y)^{-1}\sum_{i,j=1}^{n} a^{ij}(t,y)\partial_{e_i}V(y)\partial_{e_j}V(y)\Big) \\ &\le mV(y)^{m-1}\big(C_0V(y) - \Theta(y) + (m-1)M_0V(y)\big) \\ &= M_mV(y)^m - mV(y)^{m-1}\Theta(y)\end{aligned}$$

for $(t,y) \in [0,T] \times \mathbb{H}_n$. Since the function V^m inherits all necessary properties from the Lyapunov function V, we can make use of the existence result given in [BDR08a, Corollary 3.4, p. 415]. Consequently, we obtain the desired probability measures $\mu_{t,n}$ on $\mathbb{H}_n$, where $\mu_{0,n} = \nu_n$, with the property that the function

$$t \mapsto \int_{\mathbb{H}_n} \zeta(y)\,\mu_{t,n}(\mathrm{d}y) \tag{4.4.2}$$

is continuous on $t \in [0,T]$ for every $\zeta \in C_c^\infty(\mathbb{H}_n)$.

Step 1: Extension of $\mu_{t,n}$ on $\mathbb{H}_n$ to $\bar{\mu}_{t,n}$ on $\mathbb{R}^\infty$

Since the measures $\mu_{t,n}$ are yet only defined on the finite-dimensional space $\mathbb{H}_n$, we have to naturally extend them to measures $\bar{\mu}_{t,n}$ on $\mathbb{R}^\infty$ to be precise in the upcoming steps. Since $\mathbb{H}_n$ is a closed subspace of $\mathbb{R}^\infty$, we extend the measures by setting

$$\bar{\mu}_{t,n}(A) := \mu_{t,n}(A \cap \mathbb{H}_n), \quad A \in \mathcal{B}(\mathbb{R}^\infty).$$

This enables us to consistently use both $\mu_{t,n}$ in finite-dimensional and $\bar{\mu}_{t,n}$ in infinite-dimensional calculations in the following.

Step 2: Tightness of $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$

Now let us show that the family of Borel probability measures $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$ is tight on $\mathbb{R}^\infty$ for every fixed $t \in [0,T]$. For $m \ge 1$ define

$$N_m := M_m e^{M_m} + 1.$$

Then, since the function V^m introduced in **Step 0** inherits all necessary properties from the Lyapunov function V (in particular it is ν_n-integrable by assumption since $W_m < \infty$), we have by [BDR08b, Lemma 1, p. 544], for each $m \ge 1$ and for $\mathrm{d}t$-a.e. $t \in [0,T]$, the estimate

$$\begin{aligned}\int_{\mathbb{H}_n} V(y)^m\,\mu_{t,n}(\mathrm{d}y) + m\int_0^t\int_{\mathbb{H}_n} V(y)^{m-1}\Theta(y)\,\mu_{s,n}(\mathrm{d}y)\,\mathrm{d}s& \\ \le N_m \int_{\mathbb{H}_n} V(y)^m\,\nu_n(\mathrm{d}y)& \\ \le N_m W_m.&\end{aligned} \tag{4.4.3}$$

Actually, let us show that Inequality (4.4.3) is even true for every $t \in [0,T]$. In fact (compare e.g. [BDR08a, Proof of Lemma 2.2, p. 402]), since the continuity of $t \mapsto$

$\int_{\mathbb{H}_n} \zeta(y)\,\mu_{t,n}(\mathrm{d}y)$ on $[0,T]$ holds for every $\zeta \in C_c^\infty(\mathbb{H}_n)$, by approximation it also holds for every $\zeta \in C_c(\mathbb{H}_n)$. Then, we obtain the same for every $\zeta \in C_b(\mathbb{H}_n)$ that is constant outside of a compact set because subtracting this constant gives a function in $C_c(\mathbb{H}_n)$ (and $\mu_{t,n}$ being a probability measure allows us to conclude continuity). Furthermore, for any fixed $k \in \mathbb{N}$, Inequality (4.4.3) implies that

$$\int_{\mathbb{H}_n} \min(k, V(y)^m)\,\mu_{t,n}(\mathrm{d}y) + m\int_0^t \int_{\mathbb{H}_n} \min(k, V(y)^{m-1}\Theta(y))\,\mu_{s,n}(\mathrm{d}y)\,\mathrm{d}s \le N_m W_m$$

holds for dt-a.e. $t \in [0,T]$. Since the sublevel set $\{V^m \le k\}$ is compact by Assumption **(H3)** and we have continuity in t of both summands on the left-hand side, it follows that this inequality is fulfilled for every $t \in [0,T]$. By letting $k \to \infty$, the monotone convergence theorem yields the assertion.

Now, by [Bog07, Example 8.6.5, p. 205], we can use Equation (4.4.3) and the compactness of the sublevel sets $\{V^m \le R\}$, $R < \infty$, to conclude that for every fixed $t \in [0,T]$ the family of measures $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$ is tight on $\mathbb{R}^\infty$.

Step 3: Weakly convergent subsequence of $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$
Next, we will deduce existence of a weakly convergent subsequence from the family of Borel probability measures $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$ on $\mathbb{R}^\infty$, converging for every $t \in [0,T]$.

In fact, by using Prokhorov's theorem and a diagonal argument (recall that subsets of tight sets of measures are by definition still tight), there exists a subsequence of $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$ that converges weakly on $\mathbb{R}^\infty$, but only for every rational $t \in [0,T]$. Let us denote this subsequence by $(\bar{\mu}_{t,n_\ell})_{\ell\in\mathbb{N}}$ and its limit, for $t \in [0,T] \cap \mathbb{Q}$, by μ_t in the following. However, in order to obtain a subsequence of $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$ that converges weakly for every $t \in [0,T]$, we first have to prove a series of auxiliary claims making use of pointwise convergence of their respective integrals for functions from a countable measure-separating family.

For our finite-dimensional solutions $\mu_{t,n}$ we know (see e.g. [BDR08a, Lemma 2.1, p. 399] including the explanation about the limit on p. 400) that, for all $\zeta \in C_c^\infty(\mathbb{R}^n)$, the identity

$$\int_{\mathbb{R}^n} \zeta(y)\,\mu_{t,n}(\mathrm{d}y) = \int_0^t \int_{\mathbb{R}^n} L\zeta(s,y)\,\mu_{s,n}(\mathrm{d}y)\,\mathrm{d}s + \int_{\mathbb{R}^n} \zeta(y)\,\nu_n(\mathrm{d}y) \tag{4.4.4}$$

holds for every $t \in [0,T]$.

Claim 1: Equation (4.4.4) remains true for all $\zeta \in C_b^\infty(\mathbb{R}^n)$.

Proof of Claim 1: First of all, consider an arbitrary function $\vartheta \in C_c^\infty(\mathbb{R}^n)$. Now set $m_{(n)} := \max(1, k_1, \dots, k_n)$ and $\delta_{(n)} := \delta_1 + \ldots + \delta_n$. Then by Assumptions **(H1)**, **(H4)** and the boundedness of partial derivatives of $C_c^\infty(\mathbb{R}^n)$-functions we have

$$\begin{aligned} |L\vartheta(t,y)| &\le \sum_{i,j=1}^n |a^{ij}(t,y)||\partial_{e_i}\partial_{e_j}\vartheta(y)| + \sum_{i=1}^n |b^i(t,y)||\partial_{e_i}\vartheta(y)| \\ &\le 2\sum_{i=1}^n \sum_{j\le i} |a^{ij}(t,y)||\partial_{e_i}\partial_{e_j}\vartheta(y)| + \sum_{i=1}^n |b^i(t,y)||\partial_{e_i}\vartheta(y)| \\ &\le K_\vartheta V(y)^{m_{(n)}} + K_\vartheta V(y)^{m_{(n)}}\delta_{(n)}(\Theta(y))\Theta(y), \end{aligned} \tag{4.4.5}$$

for $(t,y) \in [0,T] \times \mathbb{R}^n$, where K_ϑ is some constant that depends on ϑ (and, therefore, also on n).

This is the basis for an approximation argument via C_c^∞-bump functions, i.e. for every $l \geq 0$ we consider cutoff functions $\chi_l \in C_c^\infty(\mathbb{R}^n)$ with

$$\chi_l(y) = \begin{cases} 1, & \text{if } \|y\|_{\mathbb{R}^n} \leq l, \\ 0, & \text{if } \|y\|_{\mathbb{R}^n} > 2l. \end{cases}$$

Hence, for $\zeta \in C_b^\infty(\mathbb{R}^n)$ we have $\chi_l\,\zeta \in C_c^\infty(\mathbb{R}^n)$ and $\chi_l\,\zeta \xrightarrow[l\to\infty]{} \zeta$ pointwise. Now the assertion follows from Lebesgue's dominated convergence theorem, by minding Estimate (4.4.5) and Inequality (4.4.3) for the integral

$$\int_0^t \int_{\mathbb{R}^n} L(\chi_l\,\zeta)(s,y)\,\mu_{s,n}(\mathrm{d}y)\,\mathrm{d}s.$$

In fact, we also use the product rule for the Kolmogorov operator L to conclude that

$$L(\chi_l\,\zeta) = \underbrace{\chi_l L\zeta}_{\longrightarrow L\zeta} + \zeta\,\underbrace{L\chi_l}_{\longrightarrow 0} + 2\sum_{i,j=1}^n a^{ij}\,\underbrace{\partial_{e_i}\chi_l}_{\longrightarrow 0}\,\partial_{e_j}\zeta \xrightarrow[l\to\infty]{} L\zeta$$

holds pointwise. □

We want to make use of Lemma 4.4.1 in the following. It guarantees existence of a countable, measure-separating family $\mathcal{F}$ of functions in $\mathcal{F}C_c^\infty(\{e_i\})$. Hence, it suffices to reduce our calculations to the case where $\zeta \in C_c^\infty(\mathbb{R}^d)$, for some $d \in \mathbb{N}$.

If $n \geq d$, we can treat ζ as a function on $\mathbb{R}^n$, which means that it then belongs to the class $C_b^\infty(\mathbb{R}^n)$. We can repeat the calculations from Equation (4.4.5), but now m and δ only depend on d instead of n. Hence, we have

$$|L\zeta(t,y)| \leq K_\zeta V(y)^m + K_\zeta V(y)^m \delta(\Theta(y))\Theta(y), \tag{4.4.6}$$

for $(t,y) \in [0,T] \times \mathbb{R}^n$, where K_ζ is some constant that depends on ζ (but is independent of n since ζ is a function of $y^1,\dots,y^d$).

Now, for those $\zeta \in C_c^\infty(\mathbb{R}^d)$, let

$$\varphi_n^\zeta(t) := \int_{\mathbb{R}^n} \zeta(y)\,\mu_{t,n}(\mathrm{d}y), \quad t \in [0,T],$$

and

$$G_n^\zeta(t) := \int_0^t \int_{\mathbb{R}^n} L\zeta(s,y)\,\mu_{s,n}(\mathrm{d}y)\,\mathrm{d}s, \quad t \in [0,T].$$

Going back to the index set $(n_\ell)_{\ell\in\mathbb{N}}$ of the subsequence on which $(\bar{\mu}_{t,n_\ell})_{\ell\in\mathbb{N}}$ converges weakly for every rational $t \in [0,T]$, we want to prove that there exits a subsequence of $(\varphi_{n_\ell}^\zeta(t))_{\ell\in\mathbb{N}}$ converging pointwise for $t \in [0,T]$. Since Equation (4.4.4) can be rewritten as

$$\varphi_{n_\ell}^\zeta(t) = G_{n_\ell}^\zeta(t) + \int_{\mathbb{R}^{n_\ell}} \zeta(y)\,\nu_{n_\ell}(\mathrm{d}y)$$

and $\nu_{n_\ell} \xrightarrow[\ell\to\infty]{w} \nu$ by construction, we only have to prove pointwise convergence of a subsequence in $(G_{n_\ell}^\zeta)_{\ell\in\mathbb{N}}$.

Claim 2: The family of functions $(G_n^\zeta)_{n\in\mathbb{N}}$ is equicontinuous on $[0,T]$ and uniformly bounded.

Proof of Claim 2: Let us show, for any $t \in [0,T]$, that for any sequence $(t_l)_{l\in\mathbb{N}}$ with $\lim_{l\to\infty} t_l = t$ we have

$$\limsup_{l\to\infty} \sup_{\substack{n\in\mathbb{N},\\ n\geq d}} |G_n^\zeta(t_l) - G_n^\zeta(t)| = 0.$$

Note that for every $n \geq d$ we have, by using Inequality (4.4.6) in the second and Inequality (4.4.3) in the fourth step, the estimate

$$\begin{aligned}
&|G_n^\zeta(t_l) - G_n^\zeta(t)| \\
&\qquad \leq \int_{t\wedge t_l}^{t\vee t_l} \int_{\mathbb{R}^n} |L\zeta(r,y)|\, \mu_{r,n}(\mathrm{d}y)\,\mathrm{d}r \\
&\qquad \leq \int_{t\wedge t_l}^{t\vee t_l} \int_{\mathbb{R}^n} \Big(K_\zeta V(y)^m + K_\zeta V(y)^m \delta(\Theta(y))\Theta(y) \Big)\, \mu_{r,n}(\mathrm{d}y)\,\mathrm{d}r \\
&\qquad \leq K_\zeta \int_{t\wedge t_l}^{t\vee t_l} \underbrace{\int_{\mathbb{R}^n} V(y)^m\, \mu_{r,n}(\mathrm{d}y)}_{\leq N_m W_m}\,\mathrm{d}r + K_\zeta \underbrace{\int_{t\wedge t_l}^{t\vee t_l} \int_{\mathbb{R}^n} V(y)^m \delta(\Theta(y))\Theta(y)\, \mu_{r,n}(\mathrm{d}y)\,\mathrm{d}r}_{=:I_1(t_l,n)} \\
&\qquad \leq K_\zeta |t - t_l| N_m W_m + K_\zeta I_1(t_l,n).
\end{aligned}$$

We can split the inner integral in $I_1(t_l,n)$, for any $c \geq 1$, into the disjoint sets $\{V^m < c\} \cap \{\Theta < c\}$ and $\{V^m \geq c\} \cup \{\Theta \geq c\}$. Then we obtain

$$I_1(t_l,n) \leq c^2 \sup_{r\in\mathbb{R}} |\delta(r)||t - t_l| + \underbrace{\int_{t\wedge t_l}^{t\vee t_l} \int_{\{V^m\geq c\}\cup\{\Theta\geq c\}} V(y)^m \delta(\Theta(y))\Theta(y)\, \mu_{r,n}(\mathrm{d}y)\,\mathrm{d}r}_{=:I_2(t_l,n,c)}.$$

It remains to show that

$$\lim_{c\to\infty} \limsup_{l\to\infty} \sup_{\substack{n\in\mathbb{N},\\ n\geq d}} I_2(t_l,n,c) = 0$$

holds. By again splitting the inner integral in $I_2(t_l,n,c)$ into the sets $\{V^m \geq c\}$ and $\{\Theta \geq c\}$ as well as using Inequality (4.4.3), we obtain

$$\begin{aligned}
I_2(t_l,n,c) \leq& \int_{t\wedge t_l}^{t\vee t_l} \int_{\{V\geq c^{\frac{1}{m}}\}} V(y)^{m+1}\Theta(y) \frac{\delta(\Theta(y))}{V(y)}\, \mu_{r,n}(\mathrm{d}y)\,\mathrm{d}r \\
&\qquad + \int_{t\wedge t_l}^{t\vee t_l} \int_{\{\Theta\geq c\}} V(y)^m \delta(\Theta(y))\Theta(y)\, \mu_{r,n}(\mathrm{d}y)\,\mathrm{d}r \\
\leq& \sup_{r\in\mathbb{R}} |\delta(r)| c^{-\frac{1}{m}} \underbrace{\int_0^T \int_{\mathbb{R}^n} V(y)^{m+1}\Theta(y)\, \mu_{r,n}(\mathrm{d}y)\,\mathrm{d}r}_{\leq N_{m+2}W_{m+2}} \\
&\qquad + \sup_{r\in[c,\infty)} |\delta(r)| \underbrace{\int_0^T \int_{\mathbb{R}^n} V(y)^m \Theta(y)\, \mu_{r,n}(\mathrm{d}y)\,\mathrm{d}r}_{\leq N_{m+1}W_{m+1}},
\end{aligned}$$

which tends to zero for $c \to \infty$ because, by Assumption **(H4)**, the bounded nonnegative Borel function δ satisfies $\lim_{r\to\infty} \delta(r) = 0$.

The uniform boundedness follows by using Inequality (4.4.6) in the second and Inequality (4.4.3) in the third step, because we obtain

$$\begin{aligned}|G_n^\zeta(t)| &\le \int_0^t \int_{\mathbb{R}^n} |L\zeta(r,y)|\, \mu_{r,n}(\mathrm{d}y)\,\mathrm{d}r \\ &\le \int_0^t \int_{\mathbb{R}^n} \Big(K_\zeta V(y)^m + K_\zeta V(y)^m \delta(\Theta(y))\Theta(y)\Big)\, \mu_{r,n}(\mathrm{d}y)\,\mathrm{d}r \\ &\le C_\zeta (N_m W_m + N_{m+1} W_{m+1})\end{aligned}$$

for any $n \ge d$, where C_ζ is a constant. □

Therefore, for each $\zeta \in \mathcal{F}$, we can apply the Arzelà–Ascoli theorem to $(G_{n_\ell}^\zeta)_{\ell\in\mathbb{N}}$ and obtain existence of a subsequence $(G_{n_\ell}^\zeta)_{\ell\in N_1(\zeta)}$, where we call the set (that is depending on ζ) of remaining indices of that subsequence $N_1(\zeta) \subseteq \mathbb{N}$ in order to simplify the notation, and a function $G^\zeta \in C([0,T])$ such that $\lim_{N_1(\zeta)\ni\ell\to\infty} G_{n_\ell}^\zeta(t) = G^\zeta(t)$ uniformly on $[0,T]$.

Consequently, for every $\zeta \in \mathcal{F}$, the subsequence $\big(\varphi_{n_\ell}^\zeta(t)\big)_{\ell\in N_1(\zeta)}$ converges pointwise for every $t \in [0,T]$ to some limit called $\varphi^\zeta(t)$.

By a diagonal argument, we can in fact choose a further subsequence that is independent of ζ since $\mathcal{F}$ is countable. For simplicity, let us denote this subsequence by $\big(\varphi_{n_\ell}^\zeta(t)\big)_{\ell\in N_2}$, where we call the set of remaining indices $N_2 \subseteq \mathbb{N}$. Hence, we have

$$\varphi^\zeta(t) := \lim_{N_2\ni\ell\to\infty} \varphi_{n_\ell}^\zeta(t) = \lim_{N_2\ni\ell\to\infty} \int_{\mathbb{R}^\infty} \zeta(y)\, \bar{\mu}_{t,n_\ell}(\mathrm{d}y), \quad \zeta \in \mathcal{F}. \tag{4.4.7}$$

Let us now use this pointwise convergence, which holds for every $t \in [0,T]$, in connection with the tightness of the family $(\bar{\mu}_{t,n_\ell})_{\ell\in N_2}$ to conclude weak convergence to a limit measure μ_t, even for the irrational $t \in [0,T]$.

Claim 3: The chosen subsequence $(\bar{\mu}_{t,n_\ell})_{\ell\in N_2}$ of probability measures on $\mathbb{R}^\infty$ converges weakly for any $t \in [0,T] \setminus \mathbb{Q}$ to a limit, that we call μ_t.

Proof of Claim 3: Fix any $t_0 \in [0,T] \setminus \mathbb{Q}$. Then the family $(\bar{\mu}_{t_0,n_\ell})_{\ell\in N_2}$ is tight, i.e. there exists a further subsequence $(\bar{\mu}_{t_0,n_{\ell_l}})_{l\in\mathbb{N}}$ such that $\bar{\mu}_{t_0,n_{\ell_l}} \xrightarrow[l\to\infty]{w} \mu_{t_0}$, for some limit called μ_{t_0}. In particular, by using Equation (4.4.7), we have

$$\varphi^\zeta(t_0) = \lim_{l\to\infty} \int_{\mathbb{R}^\infty} \zeta(y)\, \bar{\mu}_{t_0,n_{\ell_l}}(\mathrm{d}y) = \int_{\mathbb{R}^\infty} \zeta(y)\, \mu_{t_0}(\mathrm{d}y) \tag{4.4.8}$$

for all $\zeta \in \mathcal{F}$, since $\mathcal{F}$ is a subset of the space of bounded and continuous test functions required for weak convergence.

Now, assume that the whole sequence $(\bar{\mu}_{t_0,n_\ell})_{\ell\in N_2}$ does not weakly converge to μ_{t_0}. By again using the tightness of this family of probability measures, there would exist another subsequence $(\bar{\mu}_{t_0,\tilde{n}_{\ell_l}})_{l\in\mathbb{N}}$, which would be weakly convergent to some limit measure called $\tilde{\mu}_{t_0}$ and there would exist some $\zeta_0 \in \mathcal{F}$ with

$$\int_{\mathbb{R}^\infty} \zeta_0(y)\, \mu_{t_0}(\mathrm{d}y) \neq \int_{\mathbb{R}^\infty} \zeta_0(y)\, \tilde{\mu}_{t_0}(\mathrm{d}y)$$

since $\mathcal{F}$ separates measures. But then, by using Equations (4.4.8) and (4.4.7), we would obtain

$$\int_{\mathbb{R}^\infty} \zeta_0(y)\,\mu_{t_0}(\mathrm{d}y) = \varphi^{\zeta_0}(t_0) = \lim_{l\to\infty} \int_{\mathbb{R}^\infty} \zeta_0(y)\,\bar{\mu}_{t_0,n_{\ell_l}}(\mathrm{d}y)$$
$$= \lim_{l\to\infty} \int_{\mathbb{R}^\infty} \zeta_0(y)\,\bar{\mu}_{t_0,\tilde{n}_{\ell_l}}(\mathrm{d}y) = \int_{\mathbb{R}^\infty} \zeta_0(y)\,\tilde{\mu}_{t_0}(\mathrm{d}y),$$

which is a contradiction. □

Hence, we can conclude that we found a subsequence $(\bar{\mu}_{t,n_\ell})_{\ell\in N_2}$, which converges weakly to probability measures μ_t for every $t \in [0,T]$. This proves the assertion stated at the beginning of **Step 3**.

Step 4: μ is solution to Cauchy problem in the sense of Definition 4.2.1
For simplicity of notation, let us refer to the subsequence $(\bar{\mu}_{t,n_\ell})_{\ell\in N_2}$ from **Step 3**, that is weakly convergent for every $t \in [0,T]$, just by $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$ in the following.

First of all, Estimate (4.3.2) follows from Estimate (4.4.3) by using that the compact functions $V \geq 1$ and $\Theta \geq 0$ are lower semicontinuous, hence V^k and $V^{k-1}\Theta$, for any $k \in \mathbb{N}$, are also lower semicontinuous. In fact, we have

$$\begin{aligned}
&\int_{\mathbb{R}^\infty} V(y)^k\,\mu_t(\mathrm{d}y) + k\int_0^t \int_{\mathbb{R}^\infty} V(y)^{k-1}\Theta(y)\,\mu_s(\mathrm{d}y)\,\mathrm{d}s \\
&\quad\leq \liminf_{n\to\infty}\left(\int_{\mathbb{H}_n} V(y)^k\,\mu_{t,n}(\mathrm{d}y) + k\int_0^t\int_{\mathbb{H}_n} V(y)^{k-1}\Theta(y)\,\mu_{s,n}(\mathrm{d}y)\,\mathrm{d}s\right) \\
&\quad\leq N_k W_k
\end{aligned}$$

by Portmanteau's theorem. In particular, Equation (4.3.3) follows.

Now let us prove that the measure $\mu = \mu_t\,\mathrm{d}t$, where $(\mu_t)_{t\in[0,T]}$ is the family of Borel probability measures we obtained in the previous step, is the desired solution.

By Assumption **(H4)** and Estimate (4.3.2) we immediately have $a^{ij}, b^i \in L^1([0,T]\times\mathbb{R}^\infty, \mu)$, since

$$\begin{aligned}
&\int_{[0,T]\times\mathbb{R}^\infty} |a^{ij}(t,y)| + |b^i(t,y)|\,\mu(\mathrm{d}t\,\mathrm{d}y) \\
&\qquad\leq C_k\int_0^T\int_{\mathbb{R}^\infty} V(y)^k\,\mu_t(\mathrm{d}y)\,\mathrm{d}t + C_k\int_0^T\int_{\mathbb{R}^\infty} V(y)^k\Theta(y)\,\mu_t(\mathrm{d}y)\,\mathrm{d}t \\
&\qquad\leq C_k T N_k W_k + C_k N_{k+1} W_{k+1}
\end{aligned}$$

for some constant $C_k \geq 0$, that may change from line to line.

Fix $\zeta \in \mathcal{F}C_c^\infty(\{e_i\})$. According to Equation (4.2.2), the crucial remaining part is to show that for $\bar{\mu}_n := \bar{\mu}_{t,n}\,\mathrm{d}t$

$$\int_0^t\int_{\mathbb{R}^\infty} L\zeta(s,y)\,\bar{\mu}_n(\mathrm{d}s\,\mathrm{d}y) \xrightarrow[n\to\infty]{} \int_0^t\int_{\mathbb{R}^\infty} L\zeta(s,y)\,\mu(\mathrm{d}s\,\mathrm{d}y) \tag{4.4.9}$$

holds for $\mathrm{d}t$-a.e. $t \in [0,T]$.

By definition of L, this reduces to only proving such convergence for functions of the type $(s,y) \mapsto b^i(s,y)\partial_{e_i}\zeta(y)$ and $(s,y) \mapsto a^{ij}(s,y)\partial_{e_j}\partial_{e_i}\zeta(y)$, which we will simply call f in the following.

Claim 4: It suffices to prove Equation (4.4.9) for functions $f_N := \max(\min(f, N), -N)$.

Proof of Claim 4: In order to prove that we can extend the convergence from f_N back to the original functions f, we will show that for every $\varepsilon > 0$ there exists a number $N \in \mathbb{N}$ such that

$$\int_0^t \int_{\mathbb{R}^\infty} 1_{\{|f|>N\}} \, |f(s,y)| \, \bar{\mu}_{s,n}(\mathrm{d}y) \, \mathrm{d}s < \varepsilon.$$

By Assumption **(H4)**, as in the calculations of Inequality (4.4.5), we can estimate $|f|$ by the function $G := V^k(1 + \delta_i(\Theta)\Theta)$, i.e. it suffices to show that we have

$$\int_0^t \int_{\{G \geq N\}} G(y) \, \bar{\mu}_{s,n}(\mathrm{d}y) \, \mathrm{d}s < \varepsilon \tag{4.4.10}$$

for N sufficiently large. Let $\varepsilon > 0$. Take $n_1 \geq 0$ such that $1/n_1 + \delta_i(r) < c\varepsilon$ for all $r \geq n_1$, where the constant $c > 0$ is so small that $cN_{k+1}W_{k+1} < 1/2$. Without loss of generality, we can assume that $\delta_i \leq 1$. Then we split up the integral in Estimate (4.4.10) into

$$\begin{aligned}
&\int_0^t \int_{\{G \geq N\}} G(y) \, \bar{\mu}_{s,n}(\mathrm{d}y) \, \mathrm{d}s \\
&\qquad \leq \int_0^t \int_{\{\Theta \geq n_1\}} G(y) \, \bar{\mu}_{s,n}(\mathrm{d}y) \, \mathrm{d}s + \int_0^t \int_{\{G \geq N, \, \Theta \leq n_1\}} G(y) \, \bar{\mu}_{s,n}(\mathrm{d}y) \, \mathrm{d}s.
\end{aligned}$$

For the first summand we calculate that

$$\begin{aligned}
\int_0^t \int_{\{\Theta \geq n_1\}} G(y) \, \bar{\mu}_{s,n}(\mathrm{d}y) \, \mathrm{d}s &= \int_0^t \int_{\{\Theta \geq n_1\}} \underbrace{\left(\Theta(y)^{-1} + \delta_i(\Theta(y))\right)}_{< c\varepsilon} V(y)^k \Theta(y) \, \bar{\mu}_{s,n}(\mathrm{d}y) \, \mathrm{d}s \\
&\leq c\varepsilon \int_0^t \int_{\mathbb{R}^\infty} V(y)^k \Theta(y) \, \bar{\mu}_{s,n}(\mathrm{d}y) \, \mathrm{d}s \\
&\leq \frac{\varepsilon}{2}
\end{aligned}$$

holds by using Inequality (4.4.3) in the last step. For any $N \geq n_1$ and s we have

$$\begin{aligned}
\int_{\{G \geq N, \, \Theta \leq n_1\}} G(y) \, \bar{\mu}_{s,n}(\mathrm{d}y) &\leq (1 + n_1) \int_{\{V^k \geq N/(1+n_1)\}} V(y)^k \, \bar{\mu}_{s,n}(\mathrm{d}y) \\
&\leq \frac{(1+n_1)^2}{N} N_{2k} W_{2k},
\end{aligned}$$

which can be made smaller than $\varepsilon/2$ uniformly in s for all sufficiently large N to estimate the second summand. □

Claim 5: Equation (4.4.9) holds for functions f_N.

Proof of Claim 5: By Assumption **(H2)** the restriction of such functions f_N to the sets $[0, T] \times \{\Theta \leq R\}$ is continuous in y. Without loss of generality, we can assume that $|f_N| \leq 1$ because we would otherwise divide by N.

If the function f_N was continuous in y on the whole space $\mathbb{R}^\infty$, the claim would follow directly by weak convergence of $\bar{\mu}_{s,n}$ for every fixed s. We can reduce our situation to this by using a continuous extension of f_N to approximate. Let $\varepsilon > 0$. Recall that, by using the equivalence on Polish spaces in Prokhorov's theorem, **Step 3** gives us tightness of the family of measures $\{\bar{\mu}_{s,n} \mid n \in \mathbb{N},\ s \in [0,t]\}$ and that the sublevel sets $\{\Theta \leq R\}$ are compact in $\mathbb{R}^\infty$ for any $R < \infty$ by Assumption **(H2)**. Hence, we can choose R so large that the set $[0,t] \times \{\Theta > R\}$ has measure less than ε with respect to all measures $\bar{\mu}_{s,n}\,\mathrm{d}s$ and $\mu_s\,\mathrm{d}s$. Furthermore, the mapping $s \mapsto f_N(s,\,\cdot\,)$ from $[0,t]$ to $C(\{\Theta \leq R\})$ is Borel-measurable. Then, by Dugundji's theorem (see e.g. [Bor67, Chapter III, Section 7, p. 77ff]), there exists a linear extension operator $E\colon C(\{\Theta \leq R\}) \longrightarrow C_b(\mathbb{R}^\infty)$ such that

$$E\varphi(y) = \varphi(y) \text{ for all } \varphi \in C(\{\Theta \leq R\}),\ y \in \{\Theta \leq R\} \quad \text{and } \|E\varphi\|_\infty = \|\varphi\|_\infty.$$

By setting $g(s,y) := Ef_N(s,\,\cdot\,)(y)$, we obtain a Borel function (since it is Borel-measurable in s and continuous in y, see e.g. [Bog07, Lemma 6.4.6, p. 16]) with $|g| \leq 1$ and $g(s,y) = f_N(s,y)$ for every $y \in \{\Theta \leq R\}$. Hence, we have

$$\int_0^t \int_{\mathbb{R}^\infty} g(s,y)\,\bar{\mu}_{s,n}(\mathrm{d}y)\,\mathrm{d}s \xrightarrow[n\to\infty]{} \int_0^t \int_{\mathbb{R}^\infty} g(s,y)\,\mu_s(\mathrm{d}y)\,\mathrm{d}s$$

and

$$\int_0^t \int_{\mathbb{R}^\infty} |f_N(s,y) - g(s,y)|\,\bar{\mu}_{s,n}(\mathrm{d}y)\,\mathrm{d}s = \int_0^t \int_{\{\Theta > R\}} \underbrace{|f_N(s,y) - g(s,y)|}_{\leq 2}\,\bar{\mu}_{s,n}(\mathrm{d}y)\,\mathrm{d}s < 2\varepsilon,$$
$$\int_0^t \int_{\mathbb{R}^\infty} |f_N(s,y) - g(s,y)|\,\mu_s(\mathrm{d}y)\,\mathrm{d}s = \int_0^t \int_{\{\Theta > R\}} \underbrace{|f_N(s,y) - g(s,y)|}_{\leq 2}\,\mu_s(\mathrm{d}y)\,\mathrm{d}s < 2\varepsilon.$$

□

Consequently, the measure $\mu = \mu_t\,\mathrm{d}t$ satisfies our Cauchy problem (4.2.1) with initial distribution ν.

□

Chapter 5

Superposition principle on $\mathbb{R}^d$

The topic of this chapter is the Ambrosio–Figalli–Trevisan superposition principle, to which we will simply refer as "superposition principle" in the following, i.e. the seminal result arising from the work of L. Ambrosio, A. Figalli and D. Trevisan (see in particular [Amb08; Fig08; Tre16]) that significantly improved the characterization of the connection between finite-dimensional martingale problems in the sense of Stroock–Varadhan and Cauchy problems for Fokker–Planck–Kolmogorov equations. In Section 5.2 we will discuss the individual contributions of those three authors in more detail.

Let us begin by briefly explaining the meaning of the term "superposition principle" in the case of stochastic differential equations and its implications for martingale problems and FPKEs. First of all, the terminology itself originates in the deterministic literature of ordinary differential equations, where superposition for linear systems essentially describes the property that the net response produced by multiple inputs is the sum of the responses that would have been caused by each input individually. The extended usage of this terminology in the setting of SDEs and diffusion processes is explained in [Tre16, p. 7] and [Amb17, p. 431f]) by (simply put) pointing out that the considered probability measures arise from weighted superpositions of deterministic paths. In the introduction of [Tre16, p. 3], D. Trevisan describes the superposition principle for diffusions as a "(non-canonical) way to lift any probability-valued solution of a Fokker–Planck equation to some solution of the corresponding martingale problem", where lifting means that "the 1-marginals of the process which solve the martingale problem coincide with the given solution of the Fokker–Planck equation".

Let us point out, that it is well known (see e.g. [Tre16, p. 7] or [BDRS15, p. 14]) that if we have a martingale solution to a martingale problem, we can directly obtain some probability solution to the associated FPKE.

The superposition principle, however, finally gives us a counterpart establishing the opposite implication. This is of course the significantly more involved direction, because only under (in comparison quite strong) global assumptions it has been shown (see Sections 5.2 and 5.3 below) that existence of a probability solution to an FPKE implies existence of a martingale solution with the characterizing property that its 1-marginals coincide with the given probability solution.

In Chapter 6, we will be directly applying this result to the finite-dimensional solutions of the Cauchy problems created by Galerkin approximations. Therefore, the aim of this chapter is to present the superposition principle in such a way that it is easily applicable in our setting, which has been introduced in the previous chapters.

In short, after a historical overview in Section 5.2 discussing results from the articles [Amb08], [Fig08] and [Tre16], we will state Theorem 5.3.1 in Section 5.3 below, which was proved in [BRS19] and will be our reference of choice when applying the superposition principle to probability solutions in finite dimensions later in Chapter 6.

5.1 Framework, Equation, Solution, Assumptions

Let us now focus on the necessary framework for the considered equation, including in particular the corresponding notion of solution, which will be relevant for the main theorem in Section 5.3 below. Of course, we will base everything on the same setting as in the previous chapter, where in Sections 4.1 and 4.2 an infinite-dimensional version of most of this section can be found. But, in order to be as rigorous as possible (there are some minor apparent differences, that we will discuss later in Chapter 6) and to keep this chapter mostly self-contained, let us quickly recall the crucial parts of the setting for the finite-dimensional space $\mathbb{R}^d$.

5.1.1 Framework

Let $\{e_1, \dots, e_d\}$ be the standard basis of $\mathbb{R}^d$. By $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ we denote the inner product and by $\| \cdot \|_{\mathbb{R}^d}$ the norm on the Euclidean space $\mathbb{R}^d$. Denote by $\| \cdot \|$ the operator norm of an $\mathbb{R}^{d \times d}$-matrix. Assume that the mapping

$$\sigma \colon [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d \times d}$$

is $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{B}\big(\mathbb{R}^{d \times d}\big)$-measurable and that

$$b \colon [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

is $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^d)$-measurable. We set $A := \frac{1}{2}\sigma\sigma^*$ with $A = (a^{ij})_{1 \le i,j \le d}$ for the diffusion matrix and obtain the mappings

$$\begin{aligned} a^{ij} \colon [0,T] \times \mathbb{R}^d &\longrightarrow \mathbb{R}, \\ b^i \colon [0,T] \times \mathbb{R}^d &\longrightarrow \mathbb{R} \end{aligned}$$

as components of A and b.

The Kolmogorov operator associated to the FPKE (and to the martingale problem), acting on functions $\varphi \in C^2(\mathbb{R}^d)$, is given by

$$L\varphi(t,y) = \sum_{i,j=1}^{d} a^{ij}(t,y)\partial_{e_i}\partial_{e_j}\varphi(y) + \sum_{i=1}^{d} b^i(t,y)\partial_{e_i}\varphi(y)$$

for $(t,y) \in [0,T] \times \mathbb{R}^d$. Again, we will mainly use test functions $\varphi \in C_c^\infty(\mathbb{R}^d)$ in the following.

5.1.2 Equation

We consider the following shorthand notation for a Cauchy problem for a linear Fokker–Planck–Kolmogorov equation on $\mathbb{R}^d$ (as in Chapter 4, see Equation (4.2.1), but now only in finite dimensions) given by

$$\begin{aligned} \partial_t \mu &= L^* \mu, \\ \mu_{\restriction t=0} &= \nu, \end{aligned} \tag{5.1.1}$$

with respect to a nonnegative finite Borel measure μ of the form $\mu(\mathrm{d}t\,\mathrm{d}y) = \mu_t(\mathrm{d}y)\,\mathrm{d}t$ on $[0,T]\times\mathbb{R}^d$, where $(\mu_t)_{t\in[0,T]}$ is a family of Borel probability measures on $\mathbb{R}^d$. Furthermore, ν is a Borel probability measure on $\mathbb{R}^d$ and L^* is the formal adjoint of the differential operator L introduced before in Subsection 5.1.1.

5.1.3 Solution

The notion of solution to Equation (5.1.1) can of course be introduced directly as the finite-dimensional analogue of the one given in Definition 4.2.1.

However, let us state the precise formulation that is given in Section 1 of [BRS19] and which we will later use in Section 5.3 below. This will help us (see Lemma 6.3.6 below) to point out the minor apparent differences and to establish consistency between the various results presented in the previous chapters.

Definition 5.1.1 (see [BRS19, p. 1])**.** *A finite Borel measure μ of the form $\mu(\mathrm{d}t\,\mathrm{d}y) = \mu_t(\mathrm{d}y)\,\mathrm{d}t$, where the mapping $t \mapsto \mu_t$ from $[0,T]$ to $\mathcal{P}(\mathbb{R}^d)$ is continuous with respect to the weak topology, is called probability solution to the equation $\partial_t\mu = L^*\mu$ if the following conditions hold.*

(i) The functions a^{ij}, b^i are locally (i.e. on compact sets in $[0,T]\times\mathbb{R}^d$) integrable with respect to the measure μ, i.e.

$$a^{ij}, b^i \in L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d, \mu).$$

(ii) For every function $\varphi \in C_c^\infty(\mathbb{R}^d)$, the integral equality

$$\int_{\mathbb{R}^d} \varphi(y)\,\mu_t(\mathrm{d}y) = \int_{\mathbb{R}^d} \varphi(y)\,\nu(\mathrm{d}y) + \int_0^t \int_{\mathbb{R}^d} L\varphi(s,y)\,\mu_s(\mathrm{d}y)\,\mathrm{d}s$$

is satisfied for all $t \in [0,T]$.

5.1.4 Assumptions

We assume that the following condition holds throughout the whole chapter:

(S1) The diffusion matrix $A = (a^{ij})_{1\le i,j\le d}$ is symmetric and nonnegative definite.

To avoid confusion, we will state additional specific assumptions in the respective sections below (e.g. Assumptions **(S2)** and **(S3)** in Section 5.3).

5.2 The Ambrosio–Figalli–Trevisan superposition principle

Since the superposition principle is the name-giving topic of this thesis, let us shortly review the historical development of the theorem itself by presenting the respective contribution of the three authors Ambrosio, Figalli and Trevisan. This overview is based on [BRS19, Chapter 1] and [Amb08; Fig08; Tre16].

5.2.1 Ambrosio

Continuing the work from [Amb04], L. Ambrosio considered in his article [Amb08] the case where the diffusion matrix is zero, i.e. $A = 0$. Here, the Cauchy problem reduces for a time-dependent vector field b in $\mathbb{R}^d$ to an equation of the form $\partial_t \mu_t + \operatorname{div}(b\,\mu_t) = 0$, also referred to as the "continuity equation". In short, by starting with a solution $\mu_t\,\mathrm{d}t$ of the continuity equation and assuming

$$\int_0^T \int_{\mathbb{R}^d} \frac{\|b(t,y)\|_{\mathbb{R}^d}}{1+\|y\|_{\mathbb{R}^d}}\,\mu_t(\mathrm{d}y)\,\mathrm{d}t < \infty,$$

he proved existence of a measure $\eta \in \mathcal{M}_+\big(\mathbb{R}^d \times C([0,T];\mathbb{R}^d)\big)$ concentrated on the set of pairs (x_0,ω) such that ω is an absolutely continuous solution of the integral equation

$$\omega(t) = x_0 + \int_0^t b(s,\omega(s))\,\mathrm{d}s$$

and the measure μ_t coincides with the image of η under the evaluation mapping $(x_0,\omega) \mapsto \omega(t)$ (see Theorem 3.2 and additionally Definition 3.1 and Remark 3.1 in [Amb08] on pages 9 and 10).

5.2.2 Figalli

Generalizing Ambrosio's result to possibly non-zero diffusion matrices A, A. Figalli considered in his article [Fig08] the case of bounded coefficients.

He proved, by assuming uniform bounds for the coefficients A and b on $[0,T]\times\mathbb{R}^d$, that for every solution $\mu_t \in \mathcal{M}_+(\mathbb{R}^d)$ with $\mu_t(\mathbb{R}^d) \le C$, $t \in [0,T]$, to the Cauchy problem there exists a measurable family of probability measures $(\eta_{x_0})_{x_0\in\mathbb{R}^d}$ such that η_{x_0} is a martingale solution (of the SDE associated to the same diffusion operator L) starting from x_0 (at time 0) for ν-a.e. $x_0 \in \mathbb{R}^d$. Furthermore, the representation formula

$$\int_{\mathbb{R}^d} \varphi(y)\,\mu_t(\mathrm{d}y) = \int_{\mathbb{R}^d\times C([0,T];\mathbb{R}^d)} \varphi(\omega(t))\,\eta_{x_0}(\mathrm{d}\omega)\,\nu(\mathrm{d}x_0)$$

holds (see [Fig08, Theorem 2.6, p. 116]).

5.2.3 Trevisan

Based on techniques from PDE developed in the joint work of Ambrosio and Trevisan (see [AT14]), which had to be adapted from the deterministic to the stochastic theory and in particular to the Euclidean setting, and his PhD thesis from 2014 (see [Tre14]),

D. Trevisan widely extended the results of Figalli in [Fig08] by only imposing low regularity and integrability assumptions on the coefficients for the first time. This work has been published as an article in EJP in 2016 (see [Tre16]).

The main theorem is stated as follows.

Theorem 5.2.1 (Trevisan's superposition principle, see [Tre16, Theorem 2.5])**.** *Let $\mu = (\mu_t)_{t\in[0,T]} \subseteq \mathcal{P}(\mathbb{R}^d)$ be a narrowly continuous solution of Equation* (5.1.1)*. Then there exists η which is a solution to the martingale problem (associated to the same diffusion operator L) such that, for every $t \in [0,T]$, it holds $\eta_t = \mu_t$.*

Here, narrowly continuous means that for $\mu = (\mu_t)_{t\in[0,T]} \subseteq \mathcal{P}(\mathbb{R}^d)$ the map $t \longmapsto \int f \,\mathrm{d}\mu_t$ is continuous for every $f \in C_b(\mathbb{R}^d)$, i.e. that we have continuity with respect to the weak topology as in Definition 5.1.1, and η_t is the marginal law at $t \in [0,T]$, i.e. η_t is the pushforward measure of η under the evaluation map at time $t \in [0,T]$.

In particular, for being a solution to the Cauchy problem (5.1.1) it is by definition assumed in [Tre16] that

$$\int_0^T \int_{\mathbb{R}^d} \big(\|A(t,y)\| + \|b(t,y)\|_{\mathbb{R}^d}\big)\, \mu_t(\mathrm{d}y)\,\mathrm{d}t < \infty \tag{5.2.1}$$

holds.

This integrability condition (5.2.1) on A and b, which as a matter of fact can be seen as an assumption in Theorem 5.2.1, will be the starting point for a generalization in the next section.

5.3 Generalized integrability condition

Let us now present the main result of a further generalization of Theorem 5.2.1 from the article "On the Ambrosio–Figalli–Trevisan superposition principle for probability solutions to Fokker–Planck–Kolmogorov equations" by V. Bogachev, M. Röckner and S. Shaposhnikov (see [BRS19]). In the upcoming chapters, the precise formulation of Theorem 5.3.1 below will be our reference of choice when applying the superposition principle in finite dimensions in our setting.

First of all, the authors of [BRS19] give a rather simple example, where Condition (5.2.1) of Trevisan is not fulfilled.

Example (See [BRS19, p. 3])**.** Consider the one-dimensional case, where $\rho \in C^\infty(\mathbb{R})$, $\rho > 0$, $\int \rho(y)\,\mathrm{d}y = 1$ and $b(y) = \frac{\rho'(y)}{\rho(y)}$. Then $\mu_t(\mathrm{d}y) = \mu(\mathrm{d}y) = \rho\,\mathrm{d}y$ is a stationary solution to the Fokker–Planck–Kolmogorov equation

$$\partial_t \mu = \partial_y \partial_y \mu - \partial_y(b\mu).$$

In particular, $\mu_t = \mu$ satisfies the Cauchy problem with initial data μ. However, it is easy to find a smooth probability density ρ such that

$$\int_{\mathbb{R}} |b(y)|\rho(y)\,\mathrm{d}y = \int_{\mathbb{R}} |\rho'(y)|\,\mathrm{d}y = \infty.$$

We realize that it is therefore preferable to impose an even weaker integrability condition replacing Condition (5.2.1). Hence, let us consider the following assumptions on the coefficients A and b that allow the function $\|A(t,y)\| + |\langle b(t,y), y\rangle_{\mathbb{R}^d}|$ to be of quadratic growth (if no additional information on the solution μ is given).

(S2) For every ball $U \subseteq \mathbb{R}^d$ we have

$$a^{ij}, b^i \in L^1([0,T] \times U, \mu_t \,\mathrm{d}t).$$

(S3) The integrability condition

$$\int_0^T \int_{\mathbb{R}^d} \frac{\|A(t,y)\| + |\langle b(t,y), y\rangle_{\mathbb{R}^d}|}{(1+\|y\|_{\mathbb{R}^d})^2} \,\mu_t(\mathrm{d}y)\,\mathrm{d}t < \infty \tag{5.3.1}$$

holds.

Theorem 5.3.1 (see [BRS19, Theorem 1.1, p. 3])**.** *Suppose that $(\mu_t)_{t\in[0,T]}$ is a solution to the Cauchy problem* (5.1.1) *on $[0,T]$ with initial measure ν and Assumptions **(S1)** – **(S3)** are fulfilled. Then there exists a Borel probability measure P_ν on $C([0,T];\mathbb{R}^d)$ such that the following properties hold:*

(m1) *For all Borel sets $B \subseteq \mathbb{R}^d$ we have $P_\nu\big[\omega \in C([0,T];\mathbb{R}^d) \mid \omega(0) \in B\big] = \nu(B)$.*

(m2) *For every function $f \in C_c^\infty(\mathbb{R}^d)$, the function*

$$(\omega, t) \longmapsto f(\omega(t)) - f(\omega(0)) - \int_0^t Lf(s, \omega(s))\,\mathrm{d}s$$

is a martingale with respect to the measure P_ν and the natural filtration $\mathcal{F}_t = \sigma(\omega(s), s \in [0,t])$.

(m3) *For every function $f \in C_c^\infty(\mathbb{R}^d)$, the equality*

$$\int_{\mathbb{R}^d} f(y)\,\mu_t(\mathrm{d}y) = \int_{C([0,T];\mathbb{R}^d)} f(\omega(t))\,P_\nu(\mathrm{d}\omega)$$

holds for all $t \in [0,T]$.

Proof. We refer to [BRS19], where the proof is given in Chapter 3 starting on page 6. □

Remark.

1. Condition **(m1)** means that $\nu = P_\nu \circ \omega(0)^{-1}$, i.e. ν is the law of $\omega(0)$ under P_ν, while Condition **(m3)** means that $\mu_t = P_\nu \circ \omega(t)^{-1}$, i.e. μ_t is the law of $\omega(t)$ under P_ν.

2. By satisfying Conditions **(m1)** and **(m2)**, P_ν is by definition a martingale solution with respect to an initial measure ν. In addition, Condition **(m3)** is the characterizing property that the 1-marginal laws under the martingale solution coincide with the solution of the FPKE.

We can deduce two obvious corollaries from this theorem by directly ensuring Assumption **(S3)** via simple growth and one-sided estimates on A and b, respectively.

Corollary 5.3.2 (see [BRS19, Corollary 1.2, p. 3]).

i) Assume that the initial measure ν *satisfies* $\log\left(1+\|y\|_{\mathbb{R}^d}^2\right) \in L^1(\mathbb{R}^d,\nu)$ *and we have*

$$\|A(t,y)\| \le C\Big(1+\|y\|_{\mathbb{R}^d}^2 \log\left(1+\|y\|_{\mathbb{R}^d}^2\right)\Big),$$
$$\langle b(t,y),y\rangle_{\mathbb{R}^d} \le C\Big(1+\|y\|_{\mathbb{R}^d}^2 \log\left(1+\|y\|_{\mathbb{R}^d}^2\right)\Big),$$

where C *is a constant. Then the hypotheses of Theorem 5.3.1 are fulfilled, hence its conclusion holds.*

ii) Theorem 5.3.1 also holds if we assume

$$\|A(t,y)\| + |\langle b(t,y),y\rangle_{\mathbb{R}^d}| \le C\big(1+\|y\|_{\mathbb{R}^d}^2\big),$$

where C *is a constant.*

Chapter 6

On the superposition principle on $\mathbb{H}$

After all preparations in the previous Chapters 3, 4 and 5 we finally combine everything in order to present the main theorem of this first part of the thesis. In Section 6.3 we will state Theorem 6.3.1, a joint existence theorem for Cauchy and martingale problems via superposition.

The idea of proof is based on the extensive use of the finite-dimensional superposition principle from Chapter 5 and designed to strongly benefit from the already presented individual existence proofs for both martingale and probability solutions. In short, starting with an infinite-dimensional setting on a separable Hilbert space $\mathbb{H}$, we will use Galerkin approximations and, therefore, project everything via the mappings Π_n onto the finite-dimensional spaces $\mathbb{H}_n$ on which we can apply Theorem 5.3.1. By using tightness of the constructed families of finite-dimensional probability and martingale solutions, we will prove weak convergence of a subsequence (by a diagonal argument on a joint index set) of each of the two families. This way, we will then show existence of solutions to both infinite-dimensional equations and that the 1-marginal laws of the martingale solution coincide with the solution to the Cauchy problem.

Furthermore, from the used scheme of proof, we directly obtain a corollary (see Corollary 6.3.4 below), which is a restricted superposition principle for linear FPKEs and martingale problems on a separable Hilbert space $\mathbb{H}$. Its conditional formulation is closely related to the statement of theorems seen in Chapter 5.

In short, this means that for any given probability solution μ to an infinite-dimensional Cauchy problem, for which there already exists a subsequence of finite-dimensional solutions being created by Galerkin approximations and converging weakly to μ as well as the necessary integrability conditions and assumptions for the corresponding martingale problem, we immediately obtain a martingale solution P to the infinite-dimensional martingale problem satisfying $P \circ x(t)^{-1} = \mu_t$.

Let us point out that in Theorem 7.1 in D. Trevisan's PhD thesis (see [Tre14, Section 7.1, p. 69ff]) a result on $\mathbb{R}^\infty$ is stated, which is a generalization of Theorem 5.2.1. To our best knowledge, the proof only yields existence of martingale solutions on $C([0,T],\mathbb{R}^\infty)$, where $\mathbb{R}^\infty$ is equipped with the product topology. As already explained in the introduction, this setting is very different and should be considered separately from the Hilbert space case. In fact, we will and have to use stricter (but still commonly used) compactness assumptions (see Section 6.2.2 below) to ensure that our constructed solutions remain in a Hilbert space. In particular, we obtain a martingale solution on the path space $C([0,T],\mathbb{X}^*)$ with values even in $\mathbb{H}$, i.e. we have continuity with respect to the norm topology on $\mathbb{X}^*$ instead of the componentwise continuity on $C([0,T],\mathbb{R}^\infty)$.

Before beginning with the proof of Theorem 6.3.1 in Section 6.4, we devote Subsection 6.3.1 to following up on questions concerning consistency, which appeared and were mentioned in the prior chapters, in particular those concerning the definition of martingale and probability solutions in finite dimensions.

Let us begin by collecting the essential framework from Chapters 3, 4 and 5. Note that we will apply our results from Chapter 3 on martingale problems in the simplified case where $\mathbb{H} = \mathbb{Y}$ and on $[0,T]$ instead of $[0,\infty)$.

6.1 Framework

Let $\mathbb{H}$ be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ and norm $\|\cdot\|_{\mathbb{H}}$. As explained in Chapter 4, we use the fact that $\mathbb{H}$ is isometrically isomorphic to ℓ^2 and consider the continuous and dense embedding

$$\ell^2 \subseteq \mathbb{R}^\infty,$$

where $\mathbb{R}^\infty$ is equipped with the product topology and thus a Polish space. Let $\{e_1, e_2, \dots\}$ be the standard orthonormal basis in ℓ^2. In this setting, we can consider the Cauchy problem from Chapter 4.

To study martingale problems on $\mathbb{H}$, we introduce another separable Hilbert space $\mathbb{X}$ for which the embedding

$$\mathbb{X} \subseteq \mathbb{H} \subseteq \mathbb{X}^*$$

is continuous, dense and compact. In addition, we have to ensure, that $\{e_1, e_2, \dots\} \subseteq \mathbb{X}$ holds and we have $\|\Pi_n z\|_{\mathbb{X}^*} \le \|z\|_{\mathbb{X}^*}$ for every $z \in \mathbb{X}^*$, in order to establish the setting of Chapter 3. Here, the projection $\Pi_n \colon \mathbb{X}^* \longrightarrow \mathbb{H}_n$ (as seen before in Equation (3.1.2)) is again defined by

$$\Pi_n z := \sum_{i=1}^{n} {}_{\mathbb{X}^*}\langle z, e_i\rangle_{\mathbb{X}}\, e_i, \quad z \in \mathbb{X}^*.$$

We follow [AR89, Proposition 3.5, p. 424] and [Bre11, Remark 3, p. 136f] and identify $\mathbb{X}$ with the weighted ℓ^2-space $\ell^2(\lambda_i)$ for some sequence $(\lambda_i)_{i\in\mathbb{N}}$ with $\lim_{i\to\infty} \lambda_i = \infty$ and $\lambda_i \ge 0$. By considering its dual $\ell^2\big(\frac{1}{\lambda_i}\big)$ we arrive at the embedding

$$\ell^2(\lambda_i) \subseteq \ell^2 \subseteq \ell^2\big(\tfrac{1}{\lambda_i}\big) \subseteq \mathbb{R}^\infty,$$

where the dual pairing between $\ell^2(\lambda_i)$ and $\ell^2\big(\frac{1}{\lambda_i}\big)$ is given by

$${}_{\mathbb{X}^*}\langle z, v\rangle_{\mathbb{X}} = \sum_{i=1}^{\infty} z^i v^i,$$

for any $z \in \mathbb{X}^*$ and $v \in \mathbb{X}$. Recall that we define $\mathbb{H}_n := \operatorname{span}\{e_1, \dots, e_n\}$, for $n \in \mathbb{N}$, and $\mathcal{E} := \operatorname{span}\{e_1, e_2, \dots\}$.

Remark. It again follows from Kuratowski's theorem (see e.g. [Kur66, p. 487f] or [Par67, Section I.3, p. 15ff]) that we have $\mathbb{X} \in \mathcal{B}(\mathbb{H})$, $\mathbb{H} \in \mathcal{B}(\mathbb{X}^*)$, $\mathbb{X}^* \in \mathcal{B}(\mathbb{R}^\infty)$ and $\mathcal{B}(\mathbb{X}) = \mathcal{B}(\mathbb{H}) \cap \mathbb{X}$, $\mathcal{B}(\mathbb{H}) = \mathcal{B}(\mathbb{X}^*) \cap \mathbb{H}$, $\mathcal{B}(\mathbb{X}^*) = \mathcal{B}(\mathbb{R}^\infty) \cap \mathbb{X}^*$.

Remark. We see that the projection Π_n onto $\mathbb{H}_n$ in $\mathbb{X}^*$ in fact simplifies to

$$\Pi_n z = \sum_{i=1}^{n} {}_{\mathbb{X}^*}\langle z, e_i \rangle_{\mathbb{X}}\, e_i = \sum_{i=1}^{n} z^i e_i = (z^1, \dots, z^n, 0, \dots)$$

for any $z \in \mathbb{X}^*$, which means that it is identical to the restriction to $\mathbb{X}^*$ of the projection Π_n^∞ onto $\mathbb{H}_n$ in $\mathbb{R}^\infty$. Moreover, Π_n^∞ is actually the continuous extension of Π_n to $\mathbb{R}^\infty$.

Fix $T > 0$. Let the mappings

$$\begin{aligned} \sigma &\colon [0,T] \times \mathbb{H} \longrightarrow L_2(\mathbb{U}; \mathbb{H}), \\ b &\colon [0,T] \times \mathbb{H} \longrightarrow \mathbb{X}^* \end{aligned}$$

be Borel-measurable.

In order to obtain components of those coefficients that are defined on $[0,T] \times \mathbb{R}^\infty$, as in Chapter 4, we just extend b and σ by 0 on $\mathbb{R}^\infty \setminus \mathbb{H}$. This means we consider the Borel-measurable mappings

$$\begin{aligned} a^{ij} &\colon [0,T] \times \mathbb{R}^\infty \longrightarrow \mathbb{R}, \\ b^i &\colon [0,T] \times \mathbb{R}^\infty \longrightarrow \mathbb{R}, \end{aligned}$$

that are given by

$$a^{ij}(t,y) := \begin{cases} \frac{1}{2} \langle \sigma(t,y)\sigma(t,y)^* e_i, e_j \rangle_{\mathbb{H}}, & (t,y) \in [0,T] \times \mathbb{H}, \\ 0, & (t,y) \in [0,T] \times \mathbb{R}^\infty \setminus \mathbb{H} \end{cases}$$

and

$$b^i(t,y) := \begin{cases} {}_{\mathbb{X}^*}\langle b(t,y), e_i \rangle_{\mathbb{X}}, & (t,y) \in [0,T] \times \mathbb{H}, \\ 0, & (t,y) \in [0,T] \times \mathbb{R}^\infty \setminus \mathbb{H}. \end{cases}$$

Then we define $A(t,y) := \big(a^{ij}(t,y)\big)_{1 \le i,j < \infty}$ to be our diffusion matrix.

Remark. We note that a^{ij} and b^i, regardless of our choice to simply extend them by 0 on $\mathbb{R}^\infty \setminus \mathbb{H}$, will still be admissible mappings to satisfy all necessary assumption from Chapter 4 (see Subsection 6.2.2 below), because Assumptions **(H1)**–**(H4)** are either imposed on $\mathbb{H}_n$ anyway or remain unchanged as e.g. symmetry or growth.

Remark. Under the projection Π_n the coefficient $\Pi_n b$ reduces for $(t,y) \in [0,T] \times \mathbb{H}$ by construction to

$$\Pi_n b(t,y) = \sum_{i=1}^{n} {}_{\mathbb{X}^*}\langle b(t,y), e_i \rangle_{\mathbb{X}^*}\, e_i = \sum_{i=1}^{n} b^i(t,y)\, e_i = (b^1, \dots, b^n, 0, \dots)(t,y),$$

which precisely gives us the coefficient b_n defined in Chapter 4 (see Equation (4.1.1)).

For the coefficient $\Pi_n\sigma$ we consider the operator $\Pi_n\, A(t,y)\,\Pi_n^*$ and obtain the restriction of A to the finite-dimensional matrix $A_n = (a^{ij})_{1\le i,j\le n}$ from Chapter 4, because we have for $(t,y)\in[0,T]\times\mathbb{H}$

$$\begin{aligned}
\Pi_n\, A(t,y)\,\Pi_n^*(\ell) &= \Pi_n \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a^{ij}(t,y)\langle e_j, \Pi_n^*\ell\rangle_{\mathbb{H}}\, e_i \\
&= \sum_{k=1}^{n}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a^{ij}(t,y)\,\langle \Pi_n e_j, \ell\rangle_{\mathbb{H}}\,\underbrace{\langle e_i, e_k\rangle_{\mathbb{H}}}_{=\delta_{ik}}\, e_k = \sum_{i=1}^{n}\sum_{j=1}^{\infty} a^{ij}(t,y)\,\langle \Pi_n e_j, \ell\rangle_{\mathbb{H}}\, e_i \\
&= \sum_{i=1}^{n}\sum_{j=1}^{\infty} a^{ij}(t,y)\,\sum_{k=1}^{n}\underbrace{\langle e_j, e_k\rangle_{\mathbb{H}}}_{=\delta_{jk}}\,\langle e_k, \ell\rangle_{\mathbb{H}}\, e_i = \sum_{i=1}^{n}\sum_{j=1}^{n} a^{ij}(t,y)\,\langle e_j, \ell\rangle_{\mathbb{H}}\, e_i
\end{aligned}$$

for any $\ell\in\mathbb{H}$.

In addition, let $x_0\in\mathbb{H}$, which means that the Borel probability measure ν will be given by the Dirac measure ε_{x_0}. Then $\nu(\mathbb{H})=1$, i.e. ν is in fact a probability measure on $\mathbb{H}$. For any $n\in\mathbb{N}$, we then set $\nu_n := \varepsilon_{x_0}\circ\Pi_n^{-1}$.

Furthermore, we need the complete remaining framework from Chapters 3 and 4. This means, we refer to Sections 3.1 and 4.1 and refrain from repeating all of it here. However, let us specifically recall that we have defined the spaces

$$\begin{aligned}
\Omega &:= C\big([0,T];\mathbb{X}^*\big),\\
\Omega_n &:= C\big([0,T];\mathbb{H}_n\big),\\
\mathbb{S} &:= \Omega\cap L^p([0,T];\mathbb{H}).
\end{aligned}$$

Furthermore, let x and x_n again denote the canonical processes on Ω and Ω_n, respectively.

6.2 Equations and Assumptions

For the reader's convenience (to avoid excessive turning back of pages) we will now recall the martingale problems and Fokker–Planck–Kolmogorov equations, introduced in the previous Chapters 3 and 4, to which we will refer frequently in the following.

6.2.1 Equations

Consider the following martingale problem on $\mathbb{H}$ given by

$$\begin{gathered}\text{Existence of a martingale solution } P\in\mathcal{P}(\mathbb{S}) \text{ in the sense of Definition 3.2.1}\\ \text{for coefficients } b \text{ and } \sigma \text{ and with initial value } x_0\in\mathbb{H},\end{gathered}\tag{MP}$$

which has been introduced in Section 3.2 (see martingale problem (3.2.1)).

For each $n\in\mathbb{N}$, consider the martingale problem on the finite-dimensional space $\mathbb{H}_n$ given by

$$\begin{gathered}\text{Existence of a martingale solution } P_n\in\mathcal{P}(\Omega_n) \text{ in the sense of Definition 3.2.1}\\ \text{for coefficients } \Pi_n b \text{ and } \Pi_n\sigma \text{ and with initial value } \Pi_n x_0\in\mathbb{H}_n,\end{gathered}\tag{MP$_n$}$$

as in the proof of Theorem 3.3.1 (see martingale problem (3.4.4)).

Furthermore, consider the shorthand notation for the Cauchy problem for an infinite-dimensional linear FPKE with coefficients b and A given by

$$\begin{aligned} \partial_t \mu &= L^* \mu, \\ \mu_{\restriction t=0} &= \varepsilon_{x_0}, \end{aligned} \tag{CP}$$

which has been introduced in Section 4.2 (see Equation (4.2.1)). The operator L, acting on functions $\varphi \in \mathcal{F}C^2(\{e_i\})$, is given by

$$L\varphi(t,y) = \sum_{i,j=1}^{d} a^{ij}(t,y)\partial_{e_i}\partial_{e_j}\varphi(y) + \sum_{i=1}^{d} b^i(t,y)\partial_{e_i}\varphi(y),$$

for $(t,y) \in [0,T] \times \mathbb{R}^\infty$ and some $d \in \mathbb{N}$ depending on φ.

For each $n \in \mathbb{N}$, consider the shorthand notation for the Cauchy problem for a finite-dimensional linear FPKE on $\mathbb{H}_n$ with coefficients $\Pi_n b$ and $\Pi_n A \Pi_n^*$ given by

$$\begin{aligned} \partial_t \mu_n &= L^* \mu_n, \\ \mu_{n\restriction t=0} &= \varepsilon_{x_0} \circ \Pi_n^{-1}. \end{aligned} \tag{CP$_n$}$$

The operator L, acting on functions $\varphi \in C^2(\mathbb{H}_n)$, is given by

$$L\varphi(t,y) = \sum_{i,j=1}^{n} a^{ij}(t,y)\partial_{e_i}\partial_{e_j}\varphi(y) + \sum_{i=1}^{n} b^i(t,y)\partial_{e_i}\varphi(y),$$

for $(t,y) \in [0,T] \times \mathbb{H}_n$.

6.2.2 Assumptions

From the previous chapters we impose the following assumptions:

- Chapter 3: Assumptions on $\mathcal{N}$ **(N)**, on demicontinuity **(A1)**, on coercivity **(A2)** and the growth condition **(A3)** with $\mathbb{Y} = \mathbb{H}$ and $t \in [0,T]$.
- Chapter 4: Assumptions on symmetry/definiteness of A_n **(H1)**, on Θ **(H2)**, the Lyapunov condition **(H3)** and the growth condition **(H4)**.

Furthermore, in order to guarantee that our initial measure ε_{x_0} satisfies Condition (4.3.1) in Theorem 4.3.1, we assume that

$$W_k = \sup_{n\in\mathbb{N}} \|V(\cdot)^k \circ \Pi_n\|_{L^1(\nu)} = \sup_{n\in\mathbb{N}} \|V(\cdot)^k \circ \Pi_n\|_{L^1(\varepsilon_{x_0})} = \sup_{n\in\mathbb{N}} |V^k(\Pi_n x_0)| < \infty$$

holds for all $k \in \mathbb{N}$.

Remark. We note that e.g. in [BKRS15, Proposition 7.1.8, p. 293] we can find the idea for a transformation of a given Lyapunov function V to one that already satisfies integrability with respect to the initial measure, i.e. $V \in L^1(\mathbb{R}^n, \nu)$, in the finite-dimensional setting. Adapting this idea would be an option to actually drop the above assumption on W_k.

6.3 Results

The next theorem is the main result of this first part of the thesis. It is a joint existence theorem for linear FPKEs and martingale problems on a separable Hilbert space $\mathbb{H}$ that is based on the finite-dimensional superposition principle. The proof will heavily use ideas, calculations and results from the previous Chapters 3 and 4.

Theorem 6.3.1. *Under the assumptions from Subsection 6.2.2 there exists a probability solution $\mu = \mu_t \,\mathrm{d}t$ on $[0,T] \times \mathbb{H}$ to the Cauchy problem* (CP) *in the sense of Definition 4.2.1 and a martingale solution $P \in \mathcal{P}(\mathbb{S})$ to the martingale problem* (MP) *in the sense of Definition 3.2.1, for which the* 1*-marginal laws of P coincide with μ_t, i.e.*

$$P \circ x(t)^{-1} = \mu_t \tag{6.3.1}$$

holds for every $t \in [0,T]$. In particular, Estimates (3.3.1) *and* (4.3.2) *as well as Equation* (4.3.3) *hold.*

Remark. Let us stress once more the two key points of this result. First, we will concurrently study weak convergence of finite-dimensional Cauchy and martingale solutions as well as the correspondence of their limits μ and P via Equation (6.3.1). Second, we will carry out a method by which P is basically "generated" by the family $(\mu_{t,n})_{n\in\mathbb{N}}$ of solutions to the Cauchy problem (CP$_n$) via the finite-dimensional superposition principle (see Theorem 5.3.1) and by controlling the 1-marginals of $(P_n)_{n\in\mathbb{N}}$.

Obviously, the respective assumptions from Chapters 3 and 4 in Subsection 6.2.2 directly ensure existence for both martingale and probability solutions, but without any additional information (e.g. on uniqueness) we a priori could not specify any such connection.

For completeness, we will show that the weak formulation of Condition **(M2)** in Definition 3.2.1 implies the representation that we have already seen in Condition **(m2)** of Theorem 5.3.1 in finite dimensions in Chapter 5 also in the infinite-dimensional case.

Corollary 6.3.2. *The martingale solution $P \in \mathcal{P}(\mathbb{S})$ to the martingale problem* (MP) *in the sense of Definition 3.2.1 from Theorem 6.3.1 has the property that for every function $f \in \mathcal{F}C_c^\infty(\{e_i\})$ the process*

$$f(x(t)) - f(x_0) - \int_0^t Lf(s, x(s)) \,\mathrm{d}s$$

is an $(\mathcal{F}_t)$-martingale with respect to P.

Proof. Let $f \in \mathcal{F}C_c^\infty(\{e_i\})$ for some $d \in \mathbb{N}$ and $g \in C_c^\infty(\mathbb{R}^d)$, i.e.

$$f(y) = g(y^1, \dots, y^d) = g({}_{\mathbb{X}^*}\langle y, e_1\rangle_{\mathbb{X}}, \dots, {}_{\mathbb{X}^*}\langle y, e_d\rangle_{\mathbb{X}})$$

for $y \in \mathbb{X}^*$. By Condition **(M2)**, we know that

$${}_{\mathbb{X}^*}\langle x(t), e_i\rangle_{\mathbb{X}} = {}_{\mathbb{X}^*}\langle x(0), e_i\rangle_{\mathbb{X}} + \int_0^t {}_{\mathbb{X}^*}\langle b(s, x(s)), e_i\rangle_{\mathbb{X}} \,\mathrm{d}s + M_{e_i}(t, x)$$

is a real-valued semimartingale. Hence, $x(t)^d := \sum_{i=1}^d {}_{\mathbb{X}^*}\langle x(t), e_i\rangle_{\mathbb{X}}\, e_i$ is a d-dimensional semimartingale and we have $f(x(t)) = g(x(t)^d)$.

By Ito's formula for semimartingales on $\mathbb{R}^d$ we know that

$$\begin{aligned} g(x(t)^d) = g(x(0)^d) &+ \sum_{i=1}^{d} \int_0^t {}_{\mathbb{X}^*}\langle b(s,x(s)^d), e_i\rangle_{\mathbb{X}}\, \partial_{e_i} g(x(s)^d)\,\mathrm{d}s \\ &+ \frac{1}{2}\sum_{i,j=1}^{d} \int_0^t \partial_{e_i}\partial_{e_j} g(x(s)^d)\,\mathrm{d}\langle M_{e_i}, M_{e_j}\rangle_s \\ &+ \sum_{i=1}^{d} \int_0^t \partial_{e_i} g(x(s)^d)\,\mathrm{d}M_{e_i}(s) \end{aligned}$$

holds for every $t \in [0,T]$. Since $\sum_{i=1}^{d} \int_0^t \partial_{e_i} g(x(s)^d)\,\mathrm{d}M_{e_i}(s)$ is an $(\mathcal{F}_t)$-martingale, we obtain the same for

$$\begin{aligned} g(x(t)^d) - g(x(0)^d) &- \sum_{i=1}^{d} \int_0^t {}_{\mathbb{X}^*}\langle b(s,x(s)^d), e_i\rangle_{\mathbb{X}}\, \partial_{e_i} g(x(s)^d)\,\mathrm{d}s \\ &- \frac{1}{2}\sum_{i,j=1}^{d} \int_0^t {}_{\mathbb{X}^*}\langle \sigma(s,x(s)^d)\sigma(s,x(s)^d)^* e_i, e_j\rangle_{\mathbb{X}}\, \partial_{e_i}\partial_{e_j} g(x(s)^d)\,\mathrm{d}s \\ = f(x(t)) - f(x_0) - \int_0^t Lf(s,x(s))\,\mathrm{d}s. & \end{aligned}$$

□

The next corollary will highlight the fact that we can directly conclude continuity for the mapping $t \mapsto \mu_t$ with respect to the topology generated by finitely based functions from Equation 6.3.1.

Corollary 6.3.3. *For solutions P and μ constructed in Theorem 6.3.1, Equation* (6.3.1) *implies that the mapping $t \mapsto \mu_t$ from $[0,T]$ to $\mathcal{P}(\mathbb{H})$ is continuous with respect to the topology generated by the class $\mathcal{F}C_c^\infty(\{e_i\})$ of finitely based functions, i.e. that the mapping*

$$t \mapsto \int_{\mathbb{H}} f(y)\,\mu_t(\mathrm{d}y)$$

is continuous for every $f \in \mathcal{F}C_c^\infty(\{e_i\})$.

Proof. From Theorem 6.3.1 we are given measures $\mu_t \in \mathcal{P}(\mathbb{H})$, $t \in [0,T]$, and $P \in \mathcal{P}(\mathbb{S})$ that satisfy Equation (6.3.1).

We know that for $P \in \mathcal{P}(\mathbb{S})$ the canonical process x on $\mathbb{S}$ is in particular a mapping in the path space $C\big([0,T];\mathbb{X}^*\big)$. This means that for any $f \in \mathcal{F}C_c^\infty(\{e_i\})$, and with it some $d \in \mathbb{N}$ and $g \in C_c^\infty(\mathbb{R}^d)$, the mapping

$$t \longmapsto g \circ \Pi_d \circ x(t) = g\big({}_{\mathbb{X}^*}\langle x(t), e_1\rangle_{\mathbb{X}}, \dots, {}_{\mathbb{X}^*}\langle x(t), e_d\rangle_{\mathbb{X}}\big)$$

is continuous. Hence, the mapping $t \mapsto \int_{\mathbb{S}} f(x(t))\,P(\mathrm{d}x)$ is continuous for any $f \in \mathcal{F}C_c^\infty(\{e_i\})$, which yields the assertion. □

We will see later in Section 6.4 that the next corollary is an obvious consequence of the proof of Theorem 6.3.1. By upfront assuming existence of a probability solution to (CP) as well as all desired properties for it (recall that we worked hard to prove them in Chapter 4), we can leave out Assumptions **(H2)**–**(H4)**. Instead, we only ensure the application of the superposition principle directly by assumption, where in particular Assumption **(H1)** is a part of.

Corollary 6.3.4. *Let Assumptions **(N)**, **(A1)**, **(A2)**, **(A3)** and **(H1)** be fulfilled. Assume there exists a probability solution $\mu = \mu_t \, \mathrm{d}t$ on $[0,T] \times \mathbb{H}$ to the Cauchy problem* (CP) *in the sense of Definition 4.2.1 and a subsequence $(\mu_{t,n_k})_{k\in\mathbb{N}}$ on $\mathbb{H}_{n_k}$ of a family of Borel probability measures on $\mathbb{H}_n$ with the following properties:*

- *The measures $(\mu_{t,n_k})_{k\in\mathbb{N}}$ are solutions to the finite-dimensional Cauchy problems* (CP_n) *on $\mathbb{H}_{n_k}$ with the property that the mapping*

$$t \mapsto \int_{\mathbb{H}_{n_k}} \zeta(y) \, \mu_{t,n_k}(\mathrm{d}y)$$

  is continuous on $[0,T]$ for every $\zeta \in C_c^\infty(\mathbb{H}_{n_k})$.

- *For the family $(\bar{\mu}_{t,n_k})_{k\in\mathbb{N}}$ of extended measures to $\mathbb{H}$, we have $\bar{\mu}_{t,n_k} \xrightarrow[k\to\infty]{w} \mu_t$ for every $t \in [0,T]$.*

- *The integrability condition*

$$\int_0^T \int_{\mathbb{H}_{n_k}} \frac{\|\Pi_{n_k} A(t,y) \Pi_{n_k}^*\| + |\langle \Pi_{n_k} b(t,y), y\rangle_{\mathbb{H}_{n_k}}|}{(1 + \|y\|_{\mathbb{H}_{n_k}})^2} \, \mu_{t,n_k}(\mathrm{d}y) \, \mathrm{d}t < \infty$$

  holds for every $k \in \mathbb{N}$.

Then there exists a martingale solution $P \in \mathcal{P}(\mathbb{S})$ to the martingale problem (MP) *in the sense of Definition 3.2.1, for which Equation* (6.3.1) *holds for every $t \in [0,T]$.*

Remark. The statements of Corollaries 6.3.2 and 6.3.3 remain valid in the setting of this corollary.

Remark. Let us note, that it is not sufficient to just restrict the measures μ_t to the finite-dimensional spaces $\mathbb{H}_n$, by e.g. considering the push-forward measures $\mu_t \circ \Pi_n^{-1}$, in order to get a weakly convergent subsequence, because these measures not necessarily form a solution to the Cauchy problems (CP_n) with coefficients $\Pi_n b$ and $\Pi_n A \, \Pi_n^*$. We refer to [BKRS15, Section 10.2, p. 413ff] for more details on this kind of equation.

Before proceeding with the proofs of Theorem 6.3.1 and Corollary 6.3.4 (see Section 6.4 below), we first have to follow up on two auxiliary results concerning the consistency of the definitions of martingale solutions (see Lemma 6.3.5) and probability solutions (see Lemma 6.3.6) used in Chapters 3–5.

6.3.1 Auxiliary consistency results on $\mathbb{H}_n$

Martingale solution:
Comparing the conditions for being a martingale solution in Definition 3.2.1 with those in Theorem 5.3.1, one might rightfully assert that they differ. For consistency in the proof of Theorem 6.3.1, we only need that Conditions **(m1)**–**(m3)** imply Conditions **(M1)**–**(M2)** on the finite-dimensional spaces $\mathbb{H}_n$, $n \in \mathbb{N}$. Actually, Condition **(m1)** would be stated equivalently to Condition **(M1)** in the case where the martingale problem is considered with respect to arbitrary initial measures instead of some Dirac measure.

Lemma 6.3.5. *Let the assumptions from Subsection 6.2.2 be fulfilled. Then, on a finite-dimensional space $\mathbb{H}_n$, a martingale solution in the sense of Theorem 5.3.1 is also a solution in the sense of Definition 3.2.1.*

Proof of Lemma 6.3.5.
(M1): Condition **(M1)** follows directly from Condition **(m1)**, because choosing the one-point Borel set $B := \{\Pi_n x_0\} \subseteq \mathbb{H}_n$ yields

$$\begin{aligned} P_n\big[x_n(0) = \Pi_n x_0\big] &= P_n\big[\omega \in \Omega_n \mid \omega(0) \in \{\Pi_n x_0\}\big] = P_n\big[\omega \in C([0,T];\mathbb{H}_n) \mid \omega(0) \in B\big] \\ &\overset{\textbf{(m1)}}{=} \nu_n(B) = \varepsilon_{x_0} \circ \Pi_n^{-1}(\{\Pi_n x_0\}) = \varepsilon_{x_0}(x_0) = 1. \end{aligned}$$

Furthermore, the P_n-a.s. integrability condition follows from Assumption **(A3)** and Lemma 3.4.2 (as in Equations (3.4.20)–(3.4.22) in Chapter 3).

(M2): Let us identify $\mathbb{H}_n$ with $\mathbb{R}^n$. Comparing both Conditions **(M2)** and **(m2)**, we have to consider

$$M_\ell(t, x_n) = \langle x_n(t), \ell\rangle_{\mathbb{R}^n} - \int_0^t \langle b\big(s, x_n(s)\big), \ell\rangle_{\mathbb{R}^n}\,\mathrm{d}s, \quad t \in [0,T],$$

and prove that the following Claim holds.

Claim: For every $\ell \in \mathcal{E}$ the following properties are satisfied:

(1) M_ℓ is continuous,

(2) M_ℓ is an $(\mathcal{F}_t)$-martingale,

(3) the quadratic variation process of M_ℓ is given by

$$\langle M_\ell\rangle(t, x_n) = \int_0^t \big\|\sigma^*(s, x_n(s))(\ell)\big\|_{\mathbb{R}^n}^2\,\mathrm{d}s, \quad t \in [0,T].$$

Proof of the Claim. Since the continuity is given by construction, we directly have **(1)**.

Now, let us show **(2)**. We will proceed as in [RY99, Chapter VII, §2, p. 293ff], where a proof for locally bounded coefficients a and b is given, which can be adapted to our setting by using in particular Assumption **(A3)** and Lemma 3.4.2.

But first, for any suitable function f and time $t \in [0,T]$, let us introduce the notation

$$\mathbb{M}_t^f := \mathbb{M}_t^f(\omega) := f(\omega(t)) - f(\omega(0)) - \int_0^t Lf(s, \omega(s))\,\mathrm{d}s$$

for the function under consideration from Condition **(m2)**.

Right now, we only know that Condition **(m2)** holds for functions in $C_c^\infty(\mathbb{R}^n)$. First, we will show as in the proof of [RY99, Proposition 2.2, p. 295f] that $\mathbb{M}^f$ is also a martingale for any $f \in C_c^2(\mathbb{R}^n)$.

Step 1: Let $f \in C_c^2(\mathbb{R}^n)$.
Then there exists a compact set $K \subseteq \mathbb{R}^n$ and a sequence $(f_l)_{l\in\mathbb{N}}$ of functions in $C_c^\infty(\mathbb{R}^n)$ vanishing on K^c for which we not only have $f_l \xrightarrow[l\to\infty]{} f$ uniformly on K but also uniform convergence on K for their first and second derivatives.

In order to show that $\mathbb{M}_t^f$ is a martingale, it suffices to prove that $\mathbb{M}_t^{f_l} \xrightarrow[l\to\infty]{L^1} \mathbb{M}_t^f$ holds for every $t \in [0,T]$, because then we have

$$\mathbb{M}_s^f = \lim_{l\to\infty}^{L^1} \mathbb{M}_s^{f_l} = \lim_{l\to\infty}^{L^1} \mathbb{E}^{P_n}\big[\mathbb{M}_t^{f_l} \mid \mathcal{F}_s\big] = \mathbb{E}^{P_n}\big[\mathbb{M}_t^f \mid \mathcal{F}_s\big]$$

and, therefore, P-a.s. convergence of a subsequence to those limits yielding the property directly. Showing that $\mathbb{M}_t^{f_l} \xrightarrow[l\to\infty]{L^1} \mathbb{M}_t^f$ holds reduces to considering the difference

$$\int_0^t Lf_l(s,\omega(s))\,\mathrm{d}s - \int_0^t Lf(s,\omega(s))\,\mathrm{d}s,$$

because f_l, $l \in \mathbb{N}$, and f are continuous on a compact support.

Let c_l, $l \in \mathbb{N}$, be the constants with $c_l \xrightarrow[l\to\infty]{} 0$ that can be chosen to estimate the uniform convergence of the first and second derivatives of f_l to those of f on K. Then, as in Equations (3.4.21)–(3.4.22), we obtain by using Assumption **(A3)**, Lemma 3.4.2 and the equivalence of norms on $\mathbb{R}^n$ that

$$\begin{aligned}
&\mathbb{E}^{P_n}\left[\left|\int_0^t Lf_l(s,\omega(s))\,\mathrm{d}s - \int_0^t Lf(s,\omega(s))\,\mathrm{d}s\right|\right] \\
&\le \mathbb{E}^{P_n}\Bigg[\int_0^t \Big|\sum_{i,j=1}^n a^{ij}(s,x_n(s))\big(\partial_{e_i}\partial_{e_j} f_l(x_n(s)) - \partial_{e_i}\partial_{e_j} f(x_n(s))\big) \\
&\qquad\qquad + \sum_{i=1}^n b^i(s,x_n(s))\big(\partial_{e_i} f_l(x_n(s)) - \partial_{e_i} f(x_n(s))\big)\Big|\,\mathrm{d}s\Bigg] \\
&\le \mathbb{E}^{P_n}\Bigg[c_l \int_0^t \underbrace{\sum_{i,j=1}^n |a^{ij}(s,x_n(s))|}_{\substack{\le n^2 \max\limits_{i,j=1,\dots n} |a^{ij}(s,x_n(s))| \\ \le C\,\|A(s,x_n(s))\|_{L_2(\mathbb{R}^n;\,\mathbb{R}^n)}}} + \underbrace{\sum_{i=1}^n |b^i(s,x_n(s))|}_{\le C\,\|b(s,x_n(s))\|_{\mathbb{R}^n}}\,\mathrm{d}s\Bigg] \\
&\le \mathbb{E}^{P_n}\left[c_l \int_0^t C_t\big(1 + \|x_n(s)\|_{\mathbb{R}^n}^2\big) + C_t\big(1 + \mathcal{N}(x_n(s)) + \|x_n(s)\|_{\mathbb{R}^n}^{2\gamma'}\big)\,\mathrm{d}s\right] \\
&\le c_l\, C_{\gamma',t}\, \mathbb{E}^{P_n}\Big[1 + \|x_n(0)\|_{\mathbb{R}^n}^2 + \|x_n(0)\|_{\mathbb{R}^n}^{2\gamma'}\Big] = c_l\, C_{\gamma',t} \xrightarrow[l\to\infty]{} 0
\end{aligned}$$

holds for some constant $C_{\gamma',t}$ (not depending on l) that may change from line to line.

Next, we will show, again as in [RY99, Proposition 2.2, p. 295f], that for any $f \in C^2(\mathbb{R}^n)$ the function $\mathbb{M}^f$ is a local martingale.

Step 2: Let $f \in C^2(\mathbb{R}^n)$.
Then we can find a sequence $(f_l)_{l\in\mathbb{N}}$ of functions in $C_c^2(\mathbb{R}^n)$ such that $f = f_l$ on an increasing sequence $(K_l)_{l\in\mathbb{N}}$ of compact sets with $K_l \subseteq \mathrm{int}(K_{l+1})$ and $\bigcup_{l\in\mathbb{N}} K_l = \mathbb{R}^n$. Hence, the processes $\mathbb{M}^f$ and $\mathbb{M}^{f_l}$ coincide up to the (increasing to ∞) stopping times describing the first exit of each K_l. Therefore, $\mathbb{M}^f$ is a local martingale.

Next, we conclude as in [RY99, Proposition 2.4, p. 296ff] that M_ℓ is a local martingale for any $\ell \in \mathcal{E}$.

Step 3: M_ℓ is a local martingale
We choose $f \in C^2(\mathbb{R}^n)$ as

$$f(y) := \langle y, \ell\rangle_{\mathbb{R}^n} = \sum_{i=1}^n y^i \ell^i,$$

because then we in particular have

$$\begin{aligned}\int_0^t Lf(s, x_n(s))\,\mathrm{d}s &= \sum_{i=1}^n \int_0^t L(x_n^i(s)\ell^i)\,\mathrm{d}s\\ &= \sum_{i=1}^n \int_0^t b^i(s, x_n(s))\ell^i\,\mathrm{d}s = \int_0^t \langle b(s, x_n(s)), \ell\rangle_{\mathbb{R}^n}\,\mathrm{d}s\end{aligned}$$

and obtain the local martingale property directly.

Step 4: Proof of (3)
Let us briefly address **(3)** before finishing with **(2)**. The idea remains unchanged from the standard proof by considering the function $f(y) := \langle y, \ell\rangle^2_{\mathbb{R}^n}$ and showing that the covariance is given by the integral

$$\begin{aligned}\int_0^t \|\sigma^*(s, x_n(s))(\ell)\|^2_{\mathbb{R}^n}\,\mathrm{d}s &= \int_0^t \langle \sigma^*(s, x_n(s))(\ell), \sigma^*(s, x_n(s))(\ell)\rangle_{\mathbb{R}^n}\,\mathrm{d}s\\ &= \int_0^t \langle \ell, \sigma\sigma^*(s, x_n(s))(\ell)\rangle_{\mathbb{R}^n}\,\mathrm{d}s.\end{aligned}$$

We again refer to the proof of [RY99, Proposition 2.4], where on p. 296ff a version is given, that can be adapted to our setting.

Step 5: M_ℓ is a martingale
To conclude that M_ℓ is a martingale, we just use Assumption **(A3)** and Lemma 3.4.2 (by repeating the calculations from e.g. Equation (3.4.16)) to show that

$$\begin{aligned}\mathbb{E}^{P_n}\left[\langle M_\ell\rangle(t, x_n)\right] &= \mathbb{E}^{P_n}\left[\int_0^t \left\|\sigma^*(s, x_n(s))(\ell)\right\|^2_{\mathbb{R}^n}\,\mathrm{d}s\right]\\ &\le C_T\,T\,\left(\mathbb{E}^{P_n}\left[\|x_n(0)\|^2_{\mathbb{R}^n}\right] + 1\right) < \infty\end{aligned}$$

is finite, where C_T is a constant. □

The proof of this claim finishes the proof of the lemma, because we have ensured that both Conditions **(M1)** and **(M2)** are fulfilled. □

Probability solution:
Let us now focus on the notion of a probability solution in finite dimensions. Considering the Definitions 5.1.1 and [BDR08a, p. 397f] (in whose sense our measures $\mu_{t,n}$ are a priori constructed in Step 0 of the proof of Theorem 4.3.1, see p. 52), we realize that there are two apparent differences.

As already mentioned in the proof of Theorem 4.3.1, Lemma 2.1 in [BDR08a, p. 399] (including the explanation about the limit on p. 400) yields that Equation (4.4.4) holds for every $t \in [0, T]$, which is just as in (ii) of Definition 5.1.1. Hence, it remains to show that the mapping

$$\begin{aligned} [0, T] &\longrightarrow \mathcal{P}(\mathbb{R}^n) \\ t &\longmapsto \mu_{t,n} \end{aligned}$$

is continuous with respect to the weak topology in order to conclude that the solutions constructed in Theorem 4.3.1 satisfy all properties required in Definition 5.1.1.

Lemma 6.3.6. *For every measure $\mu_{t,n}$ constructed in Step 0 of the proof of Theorem 4.3.1, the mapping $t \mapsto \mu_{t,n}$ is continuous with respect to the weak topology.*

Proof. Note that Equation (4.4.2) already gives continuity of

$$t \mapsto \int_{\mathbb{H}_n} \zeta(y)\, \mu_{t,n}(\mathrm{d}y)$$

on $[0, T]$ for every $\zeta \in C_c^\infty(\mathbb{H}_n)$.

Now the assertion follows from a standard approximation argument. The continuity of Equation (4.4.2) for every $\zeta \in C_c^\infty(\mathbb{H}_n)$ implies that we also have it for $\zeta \in C_c(\mathbb{H}_n)$.

From there we also obtain continuity for functions $\zeta \in C_b(\mathbb{H}_n)$, i.e. with respect to the weak topology, by using the same arguments from probability theory used to show that vague convergence implies weak convergence of probability measures. □

Assumption (S3):
Finally let us show, that Assumption **(S3)** already follows from our assumptions from Subsection 6.2.2.

Lemma 6.3.7. *Assumption **(S3)** on integrability for the projected coefficients $\Pi_n b$ and $\Pi_n A\, \Pi_n^*$ on $\mathbb{H}_n$, $n \in \mathbb{N}$, follows from Assumption **(H4)**.*

Proof. Let us show that

$$\int_0^T \int_{\mathbb{R}^n} \frac{\|\Pi_n A(t, y)\Pi_n^*\| + |\langle \Pi_n b(t, y), y\rangle_{\mathbb{R}^n}|}{(1 + \|y\|_{\mathbb{R}^n})^2}\, \mu_{t,n}(\mathrm{d}y)\, \mathrm{d}t < \infty.$$

First, by using the Cauchy–Schwarz inequality, the fact that norms are equivalent on $\mathbb{R}^n$, Assumption **(H4)** and Estimate (4.4.3), we see that for $m := \max\{1, k_1, \ldots, k_n\}$ the estimate

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^n} & \frac{|\langle \Pi_n b(t,y), y\rangle_{\mathbb{R}^n}|}{(1+\|y\|_{\mathbb{R}^n})^2}\, \mu_{t,n}(\mathrm{d}y)\,\mathrm{d}t \\ &\le C \int_0^T \int_{\mathbb{R}^n} \sum_{i=1}^n |b^i(t,y)| \underbrace{\frac{\|y\|_{\mathbb{R}^n}}{(1+\|y\|_{\mathbb{R}^n})^2}}_{\le 1} \mu_{t,n}(\mathrm{d}y)\,\mathrm{d}t \\ &\le C \int_0^T \int_{\mathbb{R}^n} V(y)^m + V(y)^m \Theta(y)\, \mu_{t,n}(\mathrm{d}y)\,\mathrm{d}t \\ &\le CTN_m W_m + CN_{m+1}W_{m+1} < \infty\end{aligned}$$

holds, where C is a constant that may change from line to line. Similarly, the operator norm of the matrix $\Pi_n A(t,y)\Pi_n^*$ can be written as

$$\|\Pi_n A(t,y)\Pi_n^*\| = \max_{j=1,\dots,n} \sum_{i=1}^n |a^{ij}(t,y)|,$$

i.e. we can repeat the above estimate for the first summand by again using in particular Assumption **(H4)** and Estimate (4.4.3). □

Remark. Note that Assumption **(S3)** on $\mathbb{H}_n$ is consistent with both relevant estimates in Assumptions **(A2)** and **(A3)**. But it does not directly follow from them, because from Assumption **(A2)** we only get an upper bound for the inner product with respect to b.

6.4 Proof

Let us briefly recap which scheme we have followed in the proofs of Theorems 3.3.1 and 4.3.1.

Chapter 3: Martingale solution, Theorem 3.3.1
First, we considered (MP$_n$), the finite-dimensional martingale problem on $\mathbb{H}_n$ with coefficients $\Pi_n b$ and $\Pi_n \sigma$ being created by the projections Π_n. By well-known results in finite dimensions, we deduced existence of martingale solutions, i.e. some $P_n \in \mathcal{P}(\Omega_n)$ satisfying Conditions **(M1)** and **(M2)** of Definition 3.2.1, for any $n \in \mathbb{N}$. From there we extended P_n to $\bar{P}_n \in \mathcal{P}(\Omega)$ in Step 1 (see p. 30) and proved tightness of the family $(\bar{P}_n)_{n\in\mathbb{N}}$ in Step 2 (see p. 31). Then in Step 3 (see p. 32), we extracted a subsequence of $(\bar{P}_n)_{n\in\mathbb{N}}$ converging weakly to a probability measure $P \in \mathcal{P}(\mathbb{S})$ that is a solution to the infinite-dimensional martingale problem (MP) with coefficients b and σ according to Step 4 (see p. 33).

Chapter 4: Solution to Cauchy problem, Theorem 4.3.1
In Step 0 (see p. 52), we considered Equation (CP$_n$), the finite-dimensional Cauchy problem on $\mathbb{H}_n$ with coefficients A_n and b_n, which consist of the components a^{ij} and b^i up to n (see Equation (4.1.1)) for any $n \in \mathbb{N}$. We proved existence of solutions $\mu_{t,n}$ to Equation (CP$_n$) by using finite-dimensional results. Then, after extending the family $(\mu_{t,n})_{n\in\mathbb{N}}$ to $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$ on $\mathbb{R}^\infty$ in Step 1 (see p. 53) and proving tightness of $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$ in Step 2 (see p. 53), we extracted a subsequence in Step 3 (see p. 54) that is weakly converging to a

probability measure μ_t. Finally, we proved that $\mu = \mu_t \, \mathrm{d}t$ is a probability solution to the infinite-dimensional Equation (CP) with coefficients A and b in Step 4 (see p. 58).

Before beginning with the proof of Theorem 6.3.1, let us try to simplify the idea and visualize the scheme graphically with the help of the following figure:

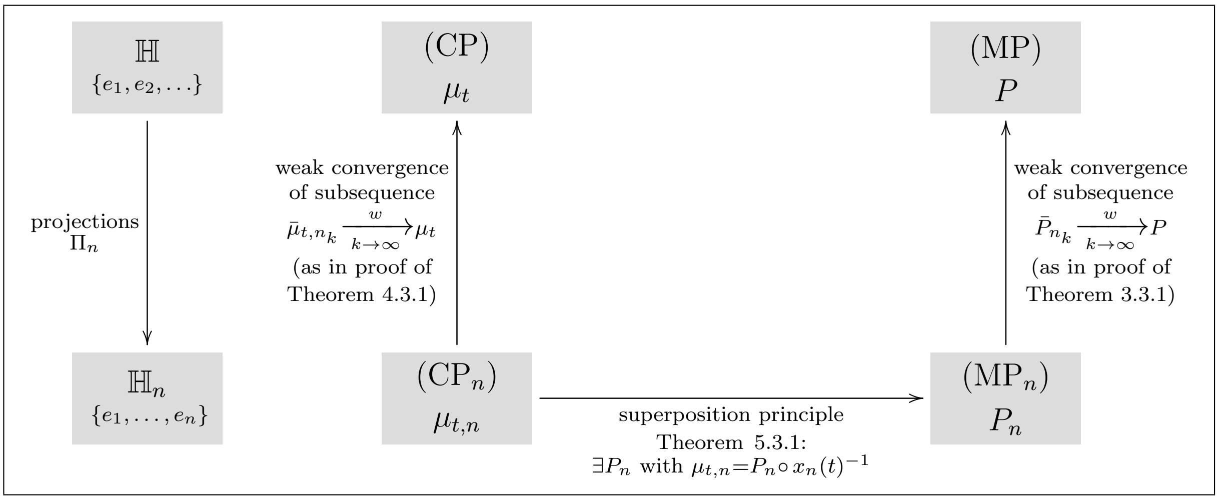

Figure 6.1: Idea and scheme of the proof of Theorem 6.3.1.

Proof of Theorem 6.3.1. Let us divide the proof into nine steps.

Step 1: Starting point

We are given an initial value $x_0 \in \mathbb{H}$ and coefficients b and σ on $[0,T] \times \mathbb{H}$ which directly allow us to study the martingale problem (MP) on $\mathbb{H}$. As described in Section 6.1, we then also consider the components b^i and a^{ij} that are extended to $[0,T] \times \mathbb{R}^\infty$ by 0 for Equation (CP) on $[0,T] \times \mathbb{R}^\infty$ with initial measure ε_{x_0} at the same time. Note that these extensions still satisfy all assumptions that we have imposed in Subsection 6.2.2.

Now, we project the coefficients and initial value/measure down onto $\mathbb{H}_n$, as before via the projections Π_n, to obtain coefficients $\Pi_n b$ and $\Pi_n \sigma$ for the finite-dimensional martingale problem (MP_n) as well as $\Pi_n b$ and $\Pi_n A \, \Pi_n^*$ for the finite-dimensional Cauchy problem (CP_n).

Since we have assumed **(H1)**–**(H4)**, we can conclude existence of solutions $\mu_{t,n}$ to Equation (CP_n) for any $n \in \mathbb{N}$ as in Step 0 from the proof of Theorem 4.3.1 (see p. 52). In fact, the coefficients A_n and b_n of Chapter 4 are exactly $\Pi_n b$ and $\Pi_n A \, \Pi_n^*$ in our setting of this chapter. To those probability measures $\mu_{t,n}$ on $\mathbb{H}_n$ we will now apply the superposition principle from Chapter 5.

Step 2: Application of Theorem 5.3.1

Let us fix some $n \in \mathbb{N}$ for the moment and specify how to exactly apply the finite-dimensional superposition principle on $\mathbb{H}_n$. We start with a solution $\mu_{t,n}$ to Equation (CP_n) with initial distribution $\nu_n = \varepsilon_{x_0} \circ \Pi_n^{-1}$. Now, let us check the necessary assumptions for using Theorem 5.3.1. Assumption **(S1)** follows from Assumption **(H1)**, Assumption **(S2)** is fulfilled, since we have $a^{ij}, b^i \in L^1_{\mathrm{loc}}(\mu_{t,n} \, \mathrm{d}t)$ for our solution $\mu_{t,n}$ by definition, and Assumption **(S3)** for the projected coefficients $\Pi_n b$ and $\Pi_n A \, \Pi_n^*$ follows from Lemma

6.3.7. Furthermore, the continuity in the notion of a probability solution from Chapter 5 holds due to Lemma 6.3.6. Therefore, we can apply Theorem 5.3.1 and conclude that there exists a probability measure P_n on Ω_n such that Conditions **(m1)** and **(m2)** hold. In addition, P_n also satisfies Condition **(m3)**, i.e. for the 1-marginal laws we have

$$P_n \circ x_n(t)^{-1} = \mu_{t,n}$$

for every $t \in [0, T]$.

This means in particular, that P_n is a martingale solution to the martingale problem (MP_n) satisfying Conditions **(M1)** and **(M2)** of Definition 3.2.1, according to Lemma 6.3.5.

Step 3: Tightness of $(\bar{P}_n)_{n\in\mathbb{N}}$
Collect the family $(P_n)_{n\in\mathbb{N}}$ of all probability measures obtained by the application of the superposition principle for each $n \in \mathbb{N}$. Since they are solutions to (MP_n) satisfying Conditions **(M1)** and **(M2)**, we are actually in the same situation as in the proof of Theorem 3.3.1 in Chapter 3. There (see p. 27, right before Lemma 3.4.2), we had to conclude existence of such probability measures (with no additional property except from being a solution to (MP_n)) from our finite-dimensional results. This time, they are just directly "created" by the superposition principle. Hence, since we have assumed **(A1)**–**(A3)**, we can repeat all calculations starting with Lemma 3.4.2 including the extension of P_n to $\bar{P}_n$ and the proof of tightness of the family $(\bar{P}_n)_{n\in\mathbb{N}}$, i.e. Step 2 of the proof of Theorem 3.3.1 (see p. 31).

Step 4: Tightness of $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$
In Chapter 4 we extended the measures $\mu_{t,n}$ on $\mathbb{H}_n$ to $\bar{\mu}_{t,n}$ on $\mathbb{R}^\infty$ and proved tightness of the family of probability measures $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$ for every fixed $t \in [0, T]$, i.e. Step 2 of the proof of Theorem 4.3.1 (see p. 53). These results remain valid because our considerations are unchanged and can, therefore, be carried over directly.

Step 5: Weak convergence on a joint subsequence
For this "diagonal argument", we have to go back and take a close look at the calculations of the proofs of Theorems 3.3.1 and 4.3.1, in order to justify a modification of the index set of the two tight families, which is necessary for selecting a joint index set on which both subsequences converge to a limit helping us to prove that Equation (6.3.1) holds.

First of all, a subset of a tight set of measures is by definition still a tight set and we lose no additional properties by dropping some indices. More importantly, we can verify that both proofs remain unchanged after that point by considering a smaller index set.

Now, let us precisely explain the choice of the joint index set. Consider the tight family $(\bar{P}_n)_{n\in\mathbb{N}}$ (as in Step 2, see p. 31, Chapter 3). Choose indices for the convergent subsequence and drop all other indices. Let us call this set of remaining indices $N_1 \subseteq \mathbb{N}$. Now consider the family $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$ that is tight on $\mathbb{R}^\infty$ for every fixed $t \in [0, T]$ (as in Step 2, see p. 53, Chapter 4). The family $(\bar{\mu}_{t,n})_{n\in N_1}$ remains tight for every fixed $t \in [0, T]$ if we ignore the dropped indices. Now choose, from the reduced index set N_1, indices for the convergent (for every $t \in [0, T]$) subsequence precisely as in Step 3 of the proof of Theorem 4.3.1 (see p. 54) and drop all others again. Let us call this set of remaining indices $N_2 \subseteq N_1 \subseteq \mathbb{N}$. Go back to the sequence $(\bar{P}_n)$ and drop these indices there as well. The family $(\bar{P}_n)_{n\in N_2}$ still remains tight and this subsequence of course still converges to the same limit.

Hence, we haven chosen a set of indices N_2, for which both tight families have a convergent subsequence with the same indices. Let us, for simplicity, denote these joint subsequences by $(\bar{P}_{n_k})_{k\in\mathbb{N}}$ and $(\bar{\mu}_{t,n_k})_{k\in\mathbb{N}}$ and their limits by P and μ_t, respectively.

Step 6: μ is a solution
Step 4 from Chapter 4 (see p. 58) remains unchanged for proving that $\mu = \mu_t\,\mathrm{d}t$ is a probability solution to the Cauchy problem (CP) on $[0,T]\times\mathbb{R}^\infty$ in the sense of Definition 4.2.1. In particular, Estimates (4.3.2) and Equation (4.3.3) hold.

Step 7: P is a solution
Step 4 from Chapter 3 (see p. 33) also remains unchanged for proving that $P \in \mathcal{P}(\mathbb{S})$ is a solution to the martingale problem (MP) in the sense of Definition 3.2.1 with initial measure ε_{x_0}. In particular, Estimate (3.3.1) holds.

Step 8: 1-marginal laws for the limit
Finally, we have to prove that

$$P \circ x(t)^{-1} = \mu_t$$

holds for every $t \in [0,T]$. In fact, let $f \in \mathcal{F}$, where $\mathcal{F} \subseteq \mathcal{F}C_c^\infty(\{e_i\})$ is a measure-separating family on $\mathbb{R}^\infty$ as in Lemma 4.4.1. Then f is of the form

$$f(y) = g\big(y^1,\dots,y^d\big), \quad y \in \mathbb{R}^\infty,$$

for some $d \in \mathbb{N}$ and $g \in C_c^\infty(\mathbb{R}^d)$.

Note that Condition **(m3)** not only holds for functions in $C_c^\infty(\mathbb{R}^n)$, but also by approximation for functions in $C_b^\infty(\mathbb{R}^n)$. Furthermore, for $n \geq d$, a function in $C_c^\infty(\mathbb{R}^d)$ treated as a function on $\mathbb{R}^n$ is of class $C_b^\infty(\mathbb{R}^n)$.

Since we have $\bar{\mu}_{t,n_k} \xrightarrow[k\longrightarrow\infty]{w} \mu_t$ on $\mathbb{R}^\infty$, we know that

$$\int_{\mathbb{R}^\infty} h(y)\,\mu_t(\mathrm{d}y) = \lim_{k\to\infty}\int_{\mathbb{R}^\infty} h(y)\,\bar{\mu}_{t,n_k}(\mathrm{d}y)$$

is fulfilled for every $h \in C_b(\mathbb{R}^\infty)$, i.e. in particular for the mapping given by $y \longmapsto g\circ\Pi_d^\infty(y) = g\big(y^1,\dots,y^d\big)$. In addition, we have that $\bar{P}_{n_k} \xrightarrow[k\longrightarrow\infty]{w} P$ on Ω, which means that

$$\int_\Omega h(\omega)\,P(\mathrm{d}\omega) = \lim_{k\to\infty}\int_\Omega h(\omega)\,\bar{P}_{n_k}(\mathrm{d}\omega)$$

holds for every $h \in C_b(\Omega)$. Consequently, it is also true for the mapping given by $\omega \longmapsto g\circ\Pi_d\circ x(\cdot,t)(\omega) = g\big(\omega(t)^1,\dots,\omega(t)^d\big)$ for every $t \in [0,T]$.

Then we obtain

$$\begin{aligned}\int_{\mathbb{R}^\infty} f(y)\,\mu_t(\mathrm{d}y) &= \int_{\mathbb{R}^\infty} g\big(y^1,\dots,y^d\big)\,\mu_t(\mathrm{d}y)\\ &= \lim_{k\to\infty}\int_{\mathbb{R}^\infty} g\big(y^1,\dots,y^d\big)\,\bar{\mu}_{t,n_k}(\mathrm{d}y)\\ &= \lim_{k\to\infty}\int_{\mathbb{H}_{n_k}} g\big(y^1,\dots,y^d\big)\,\mu_{t,n_k}(\mathrm{d}y)\\ &= \lim_{k\to\infty}\int_{\Omega_{n_k}} g\big(\omega(t)^1,\dots,\omega(t)^d\big)\,P_{n_k}(\mathrm{d}\omega)\\ &= \lim_{k\to\infty}\int_{\Omega} g\big(\omega(t)^1,\dots,\omega(t)^d\big)\,\bar{P}_{n_k}(\mathrm{d}\omega)\\ &= \int_{\Omega} g\big(\omega(t)^1,\dots,\omega(t)^d\big)\,P(\mathrm{d}\omega)\\ &= \int_{\Omega} f(\omega(t))\,P(\mathrm{d}\omega) = \int_{\mathbb{S}} f(x(t))\,P(\mathrm{d}x)\end{aligned}$$

for any $t \in [0,T]$. Since $f \in \mathcal{F}$ separates measures on $\mathbb{R}^\infty$ (and all of its subsets), the assertion follows.

Step 9: μ_t are probability measures on $\mathbb{H}$

From Estimate (3.3.1) (see also Equation (3.4.19)) we know that

$$\mathbb{E}^P\Big[\sup_{t\in[0,T]}\|x(t)\|_{\mathbb{H}}^{2q}\Big] < \infty$$

holds for every $q \geq 1$, where we made use of the lower semi-continuity of the norm $\|\cdot\|_{\mathbb{H}}$ as an extended function on $\mathbb{X}^*$ and, therefore, of the supremum. Consequently, $P \circ x(t)^{-1}$ is a probability measure on $\mathbb{H}$ for every $t \in [0,T]$, hence by Step 8 so is μ_t.

□

Finally, we show that Corollary 6.3.4 follows directly from the proof of Theorem 6.3.1.

Proof of Corollary 6.3.4. We can just repeat the proof of Theorem 6.3.1, because this time we are simply given an explicit family $(\mu_{t,n_k})_{k\in\mathbb{N}}$ of solutions to the finite-dimensional Cauchy problem (CP$_n$) on $\mathbb{H}_{n_k}$ for which we already know that $(\bar{\mu}_{t,n_k})_{k\in\mathbb{N}}$ converges weakly to the given solution μ of the Cauchy problem (CP).

In particular, all steps from the proof of Theorem 4.3.1 are redundant, because we have directly assumed all desired properties for μ and $\bar{\mu}_{n_k}$. Furthermore, we can apply the finite-dimensional superposition principle, because we have ensured Condition **(S3)** directly by assumption. □

Chapter 7

Application: stochastic Navier–Stokes equations

In this chapter, we will discuss one possible application for the methods studied in Theorem 6.3.1, namely d-dimensional stochastic Navier–Stokes equations. This kind of equation is the most obvious candidate for us, because it has already been discussed in the main references of both Chapters 3 and 4 with their respective focus on martingale and Cauchy problems. Since the referenced articles contain extensive calculations that remain valid in our case, we will focus on showing the connection between them in order to fit everything into our combined framework. Hence, we will on the one hand study the elaboration for FPKEs of [BDRS15, Example 3.5, p. 17f], which can partly also be found in [BKRS15, Example 10.1.6, p. 411f and Example 10.4.3, p. 425f]. On the other hand, we will supplement all necessary details for the “martingale problem”-part from [GRZ09, Chapter 6, p. 1749ff] and [RZZ15, Section 5.1, p. 377f].

We also want to highlight the article [FG95] of F. Flandoli and D. Gatarek, in which martingale solutions for stochastic Navier–Stokes equations on a separable Hilbert space were studied, and the famous book [Tem77] by R. Temam on Navier–Stokes equations at this point, because both serve as foundation concerning the general setting.

Let us begin by stating the equation under consideration. We note, that it should be regarded as a heuristic expression with no need for further interpretation because we are rather interested in the coefficients of the equation that we need for the martingale problem as well as in the specific form of the corresponding Kolmogorov operator L that can be derived from them (recall that Theorem 6.3.1 does not explicitly involve the SDE).

We will adopt predominant portions of notation and approach of [BDRS15, Example 3.5, p. 17f] in the following.

Stochastic Navier–Stokes equations. Let $d \in \mathbb{N}$, $T > 0$ and let $D \subseteq \mathbb{R}^d$ be a bounded domain with smooth boundary. The stochastic Navier–Stokes equation under consideration is formally written as

$$\mathrm{d}u(t,z) = \Pi_{\mathbb{H}}\Big(\Delta_z u(t,z) - \sum_{j=1}^{d} u^j(t,z)\partial_{z_j} u(t,z)\Big)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W(t,z), \tag{SNS}$$

where we have chosen the “force” F for simplicity to be 0, and are hence in the case of the “classical” stochastic Navier–Stokes equation.

Setting. We begin by explaining the occurring framework.
For the space $L^2(D;\mathbb{R}^d)$, we denote the L^2-inner product by $\langle\cdot,\cdot\rangle_{L^2}$ and the corresponding L^2-norm by $\|\cdot\|_{L^2}$. Let

$$\mathcal{V} := C^\infty_{c,\mathrm{div}}(D)^d := \big\{u=(u^1,\dots,u^d)\,\big|\, u^j\in C_c^\infty(D), \operatorname{div} u = 0\big\}$$

be the space of all smooth d-dimensional vector fields on D with compact support in D and divergence free. By $H_0^{2,1}(D)$ we denote the closure of $C_c^\infty(D)$ in the Sobolev space $H^{2,1}(D)$. As in [BDRS15], we define the space

$$V_2 := H^{2,1}_{0,\mathrm{div}}(D)^d := \big\{u=(u^1,\dots,u^d)\,\big|\, u^j\in H_0^{2,1}(D), \operatorname{div} u = 0\big\}$$

of $\mathbb{R}^d$-valued mappings. Then V_2 is equipped with its natural Hilbert norm $\|\cdot\|_{V_2}$ given by

$$\|u\|_{V_2}^2 := \sum_{j=1}^d \|\nabla_z u^j\|_{L^2}^2.$$

Note that V_2 is identical to the closure of $\mathcal{V}$ in $H_0^{2,1}(D)^d$ (see e.g. [Tem77, Theorem 1.6, p. 18 and Remark 2.1(ii), p. 23]). Let $\mathbb{H}$ be the closure of $\mathcal{V}$ in $H_0^{2,0}(D)^d$ as in [GRZ09]. Then $\mathbb{H}$ is clearly a separable Hilbert space and is identical to the closure of V_2 in $L^2(D;\mathbb{R}^d)$ as it is constructed in [BDRS15]. As in [GRZ09], let $\mathbb{X}$ be the closure of $\mathcal{V}$ in the Sobolev space $H_0^{2,2+d}(D)^d$ and let $\mathbb{X}^*$ be its dual.

It is well-known that there exist eigenfunctions $\eta_i\in\mathbb{X}$ of the Laplace operator Δ_z with eigenvalues $-\lambda_i^2$, i.e. $\Delta_z\eta_i = -\lambda_i^2\eta_i$, $\lambda>0$, such that $\{\eta_i \mid i\in\mathbb{N}\}$ is an orthonormal basis in $\mathbb{H}$. Define $\mathbb{H}_n := \operatorname{span}\{\eta_1,\dots\eta_n\}$.

Altogether, we consider the embedding

$$\mathbb{X}\subseteq V_2\subseteq \mathbb{H}\subseteq V_2^*\subseteq\mathbb{X}^*.$$

Furthermore, $\Pi_{\mathbb{H}}$ denotes the orthogonal projection onto $\mathbb{H}$ in $L^2(D;\mathbb{R}^d)$. Let W be a Wiener process of the form

$$W(t,z) = \sum_{i=1}^\infty \sqrt{\alpha_i}\, w_i(t)\eta_i(z),$$

where $\alpha_i\geq 0$, $\sum_{i=1}^\infty \alpha_i<\infty$ and w_i, $i\in\mathbb{N}$, are independent real Wiener processes.

Coefficients. For $t\in[0,T]$ and $v\in\mathcal{V}$ set

$$b(t,v) := \Pi_{\mathbb{H}}\Delta_z v - \Pi_{\mathbb{H}}\sum_{j=1}^d v^j\partial_{z_j} v.$$

Note that the following lemma holds, which can be proved by similar calculations as in [GRZ09, Lemma 6.1, p. 1750].

Lemma 7.0.1 (see [RZZ15, p. 378])**.** *For any $v_1,\tilde v_1, v_2,\tilde v_2\in\mathcal{V}$ we have*

$$\big\|\Pi_{\mathbb{H}}\Delta_z v_1 - \Pi_{\mathbb{H}}\Delta_z v_2\big\|_{\mathbb{X}^*} \leq C\|v_1-v_2\|_{\mathbb{H}},$$

$$\Big\|\Pi_{\mathbb{H}}\sum_{j=1}^d \tilde v_1^j\partial_{z_j}v_1 - \Pi_{\mathbb{H}}\sum_{j=1}^d \tilde v_2^j\partial_{z_j}v_2\Big\|_{\mathbb{X}^*} \leq C\big(\|\tilde v_1\|_{\mathbb{H}}\|v_1-v_2\|_{\mathbb{H}} + \|v_2\|_{\mathbb{H}}\|\tilde v_1-\tilde v_2\|_{\mathbb{H}}\big).$$

Since $\langle \Pi_{\mathbb{H}} w, \eta_i \rangle_{L^2} = \langle w, \eta_i \rangle_{L^2}$ holds for any $w \in L^2(D;\mathbb{R}^d)$, we can, however, directly consider the components b^i of our coefficient b given by

$$b^i(t,v) = \langle \Pi_{\mathbb{H}}(\Delta_z v), \eta_i \rangle_{L^2} - \sum_{j=1}^{d} \langle \Pi_{\mathbb{H}}(v^j \partial_{z_j} v), \eta_i \rangle_{L^2} = \langle v, \Delta_z \eta_i \rangle_{L^2} - \sum_{j=1}^{d} \langle \partial_{z_j} v, v^j \eta_i \rangle_{L^2}.$$

It follows from the last step that those mappings are defined for every $v \in V_2$. Since v is divergence free, we can further rewrite b^i by using integration by parts as

$$b^i(t,v) = \langle v, \Delta_z \eta_i \rangle_{L^2} + \sum_{j=1}^{d} \langle v, v^j \partial_{z_j} \eta_i \rangle_{L^2},$$

which is actually defined for all $v \in \mathbb{H}$. Furthermore, for the coefficient σ, we set

$$\sigma^{ij} :\equiv \begin{cases} \sqrt{2\alpha_i}, & i = j, \\ 0, & i \neq j \end{cases}$$

on $\mathbb{H}$ and obtain for A the components

$$a^{ij} :\equiv \begin{cases} \alpha_i, & i = j, \\ 0, & i \neq j. \end{cases}$$

Hence, the treatment of σ and A as constant coefficients is straightforward and for b we know, that the components b^i are defined on the whole space $\mathbb{H}$ and their representation on V_2 is directly derived from the stochastic Navier–Stokes equation (SNS). Recall that both $\mathbb{H}_n$ and $\mathcal{E} = \operatorname{span}\{\eta_1, \eta_2, \ldots\}$, which is used in the definition of a martingale solution, are subspaces of V_2, simplifying many calculations even more. We realize that we can mimic the proof of Theorem 6.3.1 by constructing a probability solution via Galerkin approximations as in [BDRS15] while simultaneously obtaining finite-dimensional martingale solutions and studying their limit.

Kolmogorov operator L. Now, we can consider the Kolmogorov operator L given by

$$L\varphi(t,u) = \sum_{i=1}^{\infty} \alpha_i \, \partial_{\eta_i} \partial_{\eta_i} \varphi(u) + \sum_{i=1}^{\infty} b^i(t,u) \, \partial_{\eta_i} \varphi(u),$$

for any $\varphi \in \mathcal{F}C_c^\infty(\{\eta_i\})$.

On the assumptions from Subsection 6.2.2. Note that we have only changed two small parts of the assumptions that were already checked in [BDRS15; BKRS15]. The rest of them as well as all assumptions from [GRZ09; RZZ15] remain unchanged.

First of all, since we choose the Lyapunov function $V \colon \mathbb{R}^\infty \longrightarrow [1,\infty]$ as

$$V(u) = \begin{cases} \|u\|_{L^2}^2 + 1, & u \in \mathbb{H}, \\ \infty, & \text{else}, \end{cases}$$

the restriction of V to $\mathbb{H}_n$ is non-degenerate, because it is a convex function.

Second, the function $\Theta\colon \mathbb{R}^\infty \longrightarrow [0,\infty]$, which we have chosen as

$$\Theta(u) = \begin{cases} C_1\|u\|_{V_2}^2, & u \in V_2,\\ \infty, & \text{else,}\end{cases}$$

is bounded on bounded sets on each space $\mathbb{H}_n$. Hence, Assumptions **(N)**, **(A1)**–**(A3)** and **(H1)**–**(H4)** can still be proved as in the references.

Conclusion. The application of our methods from Theorem 6.3.1 extends the individual results of [BDRS15] and [GRZ09] on existence of a probability measure μ solving the Cauchy problem (CP) and a martingale solution P solving the martingale problem (MP) by their connection through Equation (6.3.1). This means that the solution constructed in [BDRS15, Example 3.5, p. 17f] is in fact identical with the 1-marginals of a solution to the corresponding martingale problem. In particular, this implies that the mapping $t \longmapsto \mu_t$ is continuous with respect to the topology generated by finitely based functions.

Chapter 8

Conclusion and perspective

We have seen in Chapter 6 that the idea of proof of Theorem 6.3.1 is essentially the careful combination of three separate (but related) results. By design, this approach creates a "modular" scheme allowing us to replace assumptions and, therefore, leaving room for improvements and simplifications.

As mentioned before, one obvious goal for future research would be to unify the collected assumptions in Subsection 6.2.2. In fact, at the start of this PhD project, the initial leading question raised for martingale problems and SPDEs was, if one could drop Assumption **(A2)** on coercivity in the setting of Chapter 3 and replace it e.g. with a Lyapunov condition similar to Assumption **(H3)** as we have seen in Chapter 4. The corresponding Lyapunov function could, as a further possible generalization of Assumption **(H3)**, even explicitly depend on time. With an application in Theorem 6.3.1 in view, such a simplification would be of high interest and might, therefore, again become a focus of future work. In addition, it would be a quite natural extension and, thus, a beneficial contribution to the research on SPDEs in itself.

Concerning Corollary 6.3.4 we would obviously be interested in proving such a superposition principle on $\mathbb{H}$ for larger classes of solutions to an FPKE than a family of solutions that can be represented as the limit of a weakly convergent subsequence of certain finite-dimensional solutions. As a next step, this means that we should focus on finding specific properties to identify this subclass or even developing assumptions to ensure the existence of such a weakly convergent subsequence for any given probability solution to an infinite-dimensional Cauchy problem.

Continued research in this direction might also impact some more distant but related topics. For example, the idea of Lemma 2.12 in [Tre16] has recently been picked up in the preprint [RRW19] for a finite-dimensional study of so-called restricted well-posedness for FPKEs and martingale problems and their equivalence. Their restriction is with respect to the class of initial conditions as well as to the class of solutions of the FPKE. For a generalization to the infinite-dimensional case it could be a good starting point to restrict the class of solutions of the FPKE even further to those which have been constructed by Galerkin approximations as in Theorem 6.3.1.

Another interesting question would be what consequences follow for a.s. Markov families (as they are e.g. considered in [GRZ09]) and flows for FPKEs. In [Reh19], M. Rehmeier studied existence of flows for FPKEs, but yet only in finite dimensions. The flow property is a weaker property than satisfying the Chapman–Kolmogorov equation, which would yield the Markov property, allowing only to select an a.s. Markov family. Note that an association between martingale and probability solution is important in the considered setting of [Reh19] because there is no uniqueness assumed. A martingale solution that is directly constructed through its 1-marginals from a probability solution to an FPKE, i.e. a smaller class of solutions, might be beneficial to generalize the result to infinite dimensions.

Part II

On the superposition principle for nonlinear FPKEs

Chapter 9

Nonlinear FPKEs and martingale problems associated to McKean–Vlasov equations

As a generalization of Chapter 4, we now consider nonlinear Fokker–Planck–Kolmogorov equations in infinite dimensions. The term "nonlinear" means that the coefficients A and b can depend on the solution itself. This gives an operator L_μ of the form

$$L_\mu \varphi(t, y) = \sum_{i,j} a^{ij}(t, y, \mu_t)\partial_{e_i}\partial_{e_j}\varphi(y) + \sum_i b^i(t, y, \mu_t)\partial_{e_i}\varphi(y),$$

for sufficiently smooth functions $\varphi\colon \mathbb{R}^\infty \longrightarrow \mathbb{R}$, which induces a nonlinear equation

$$\partial_t \mu = L_\mu^* \mu.$$

These nonlinear FPKEs (and their corresponding equations for functions) arise in various fields of research. Among them are e.g. condensed matter and material physics, nonlinear hydrodynamics or surface physics. We refer to [Fra05, p. 8] for an extensive list of applications.

The aim and content of this chapter are two-part preparations allowing us to state and prove the main theorem of this second part of the thesis later in Chapter 10 (see Theorem 10.2.1 below). First, in Section 9.2 below, we explain the necessary framework for nonlinear FPKEs. In contrast to Chapter 4, we will not present an existence result in this case. Instead, we refer to the article [Man15] by O. Manita, which is a further development from prior finite-dimensional results obtained by O. Manita and S. Shaposhnikov (see [MS13]), for conditions ensuring existence of solutions for nonlinear Cauchy problems.

Second, in Section 9.3 below, we work out the other crucial part of a nonlinear version of the superposition principle, i.e. martingale problems for coefficients depending on fixed measures, and focus on their connection to so-called McKean–Vlasov equations. McKean–Vlasov equations are distribution dependent stochastic differential equations, whose name goes back to H. McKean (see [McK66; McK67]) and A. Vlasov (see e.g. the reprinted article [Vla68], whose idea was originally proposed in 1938), of the form

$$\mathrm{d}X(t) = b\big(t, X(t), \mathcal{L}_{X(t)}\big)\,\mathrm{d}t + \sigma\big(t, X(t), \mathcal{L}_{X(t)}\big)\,\mathrm{d}W(t),$$

where the coefficients b and σ can explicitly depend on the law of the process $X(t)$. Sometimes these equations are also just called DDSDEs in the literature. Let us note, that there is growing interest in the field of McKean–Vlasov SDEs with recent finite-dimensional works (see e.g. [Wan18], [MV16], [HW19], [HŠS18], [RST18] and the references therein) as well as growing interest in their connection to FPKEs (see e.g. [BR18a], [HRW19], [CG19] and the references therein). Infinite-dimensional results are however less common, but e.g. studied in [AD95] for semilinear equation with additive noise, in [KX95, Section 9.1] on duals of nuclear spaces and lately in [BM19] as well as, with a setting closely related to ours, in two master theses at Bielefeld University. One of them by R. Heinemann, that also covers delay, is in preparation to be published in the near future.

Let us begin with the essential framework. Note that the only difference to the framework of the linear case in Subsection 6.1 is the explicit dependence of the coefficients b and σ on a measure.

9.1 Framework

As in Subsection 6.1 we start with a separable Hilbert space $\mathbb{H}$, which we identify with ℓ^2, and let $\{e_1, e_2, \dots\}$ be the standard orthonormal basis in ℓ^2. Then all consideration on embeddings remain unchanged. Again, recall that we define $\mathbb{H}_n := \operatorname{span}\{e_1, \dots, e_n\}$, for $n \in \mathbb{N}$, and $\mathcal{E} := \operatorname{span}\{e_1, e_2, \dots\}$.

Now, let us focus on what changed in the nonlinear case. For some fixed $T > 0$, let the mappings

$$\begin{aligned} b &\colon [0,T] \times \mathbb{H} \times \mathcal{P}(\mathbb{H}) \longrightarrow \mathbb{X}^*, \\ \sigma &\colon [0,T] \times \mathbb{H} \times \mathcal{P}(\mathbb{H}) \longrightarrow L_2(\mathbb{U}, \mathbb{H}) \end{aligned}$$

be Borel-measurable.

By extending those mappings by 0 as in Chapter 6, we again obtain mappings

$$\begin{aligned} a^{ij} &\colon [0,T] \times \mathbb{R}^\infty \times \mathcal{P}(\mathbb{R}^\infty) \longrightarrow \mathbb{R}, \\ b^i &\colon [0,T] \times \mathbb{R}^\infty \times \mathcal{P}(\mathbb{R}^\infty) \longrightarrow \mathbb{R} \end{aligned}$$

as coefficients. To be more precise, we now set

$$b^i(t,y,\varrho) := \begin{cases} {}_{\mathbb{X}^*}\langle b(t,y,\varrho), e_i \rangle_{\mathbb{X}}, & (t,y,\varrho) \in [0,T] \times \mathbb{H} \times \mathcal{P}(\mathbb{H}), \\ 0, & \text{else} \end{cases}$$

and

$$a^{ij}(t,y,\varrho) := \begin{cases} \frac{1}{2}\langle \sigma(t,y,\varrho)\sigma(t,y,\varrho)^* e_i, e_j \rangle_{\mathbb{H}}, & (t,y,\varrho) \in [0,T] \times \mathbb{H} \times \mathcal{P}(\mathbb{H}), \\ 0, & \text{else.} \end{cases}$$

Again, let $A(t,y,\varrho) := \big(a^{ij}(t,y,\varrho)\big)_{1 \le i,j < \infty}$ be our diffusion matrix.

Consider the Kolmogorov operator associated to our nonlinear FPKE called L_μ, acting on functions $\varphi \in \mathcal{F}C^2(\{e_i\})$, which is given by

$$L_\mu \varphi(t,y) = \sum_{i,j=1}^d a^{ij}(t,y,\mu_t) \partial_{e_i} \partial_{e_j} \varphi(y) + \sum_{i=1}^d b^i(t,y,\mu_t) \partial_{e_i} \varphi(y)$$

for $(t,y) \in [0,T] \times \mathbb{R}^\infty$ and for some $d \in \mathbb{N}$ depending on φ.

9.2 Nonlinear FPKEs in infinite dimensions

In this section, we will adapt Section 4.2 to the nonlinear case. Note that we keep the suggested setting of [Man15] as an example, but we are in particular working in the special case, where the coefficients depend on the single probability measure μ_t rather than the whole family $(\mu_t)_{t\in[0,T]}$.

9.2.1 Equation

Consider the following shorthand notation for a Cauchy problem for a nonlinear Fokker–Planck–Kolmogorov equation given by

$$\begin{aligned} \partial_t \mu &= L^*_\mu\, \mu, \\ \mu_{\restriction t=0} &= \varepsilon_{x_0}, \end{aligned} \tag{NCP}$$

with respect to a nonnegative finite Borel measure μ of the form $\mu(\mathrm{d}t\,\mathrm{d}y) = \mu_t(\mathrm{d}y)\,\mathrm{d}t$ on $[0,T]\times\mathbb{R}^\infty$, where $(\mu_t)_{t\in[0,T]}$ is a family of Borel probability measures on $\mathbb{R}^\infty$. Furthermore, L^* is the formal adjoint of the operator L defined in Section 9.1 and the initial measure ν is given by the Dirac measure ε_{x_0}.

9.2.2 Solution

The notion of a probability solution in the nonlinear case is completely analogue to Definition 4.2.2. We only have to consider coefficients explicitly depending on a measure.

Definition 9.2.1. *(probability solution, nonlinear) A finite Borel measure μ on $[0,T]\times\mathbb{R}^\infty$ of the form $\mu(\mathrm{d}t\,\mathrm{d}y) = \mu_t(\mathrm{d}y)\,\mathrm{d}t$, where $(\mu_t)_{t\in[0,T]}$ is a family of Borel probability measures on $\mathbb{R}^\infty$, is called probability solution to the Cauchy problem* (NCP) *if*

(i) The functions a^{ij}, b^i are integrable with respect to the measure μ, i.e.

$$a^{ij}(\cdot,\cdot,\mu_\cdot), b^i(\cdot,\cdot,\mu_\cdot) \in L^1([0,T]\times\mathbb{R}^\infty,\mu).$$

(ii) For every function $\varphi \in \mathcal{F}C_c^\infty(\{e_i\})$ we have

$$\int_{\mathbb{R}^\infty} \varphi(y)\,\mu_t(\mathrm{d}y) = \int_{\mathbb{R}^\infty} \varphi(y)\,\nu(\mathrm{d}y) + \int_0^t \int_{\mathbb{R}^\infty} L_\mu\varphi(s,y)\,\mu_s(\mathrm{d}y)\,\mathrm{d}s$$

for every $t\in[0,T]$.

9.3 Martingale problems associated to McKean–Vlasov equations

In this section we will recall Section 3.2 in the special case, where the coefficients b and σ also depend on a fixed measure μ_t that will be a solution to a nonlinear FPKE.

9.3.1 Solution

The notion of a martingale solution remains unchanged from Chapter 3, but we choose to implement Corollary 6.3.2 into Condition **(M2)**. Since we are also working on $[0,T]$ instead of $[0,\infty)$, let us restate the definition before we consider coefficients that will explicitly depend on some fixed measure.

Definition 9.3.1. *A probability measure $P \in \mathcal{P}(\Omega)$ is called martingale solution to the martingale problem with coefficients $\tilde{b}$ and $\tilde{\sigma}$ and initial value $x_0 \in \mathbb{H}$ if the following conditions hold.*

(M1) $P\big[x(0) = x_0\big] = 1$ *and*

$$P\bigg[x \in \Omega \,\bigg|\, \textit{For } \mathrm{d}s\textit{-a.e. } s \in [0,T] : x(s) \in \mathbb{H} \textit{ and } \int_0^T \big\|\tilde{b}(s,x(s))\big\|_{\mathbb{X}^*} \,\mathrm{d}s + \int_0^T \big\|\tilde{\sigma}(s,x(s))\big\|^2_{L_2(\mathbb{U};\mathbb{H})} \,\mathrm{d}s < \infty\bigg] = 1.$$

(M2) *For every function $f \in \mathcal{F}C_c^\infty(\{e_i\})$ the process*

$$\mathbb{M}^f(t,x) := f(x(t)) - f(x_0) - \int_0^t Lf(s,x(s))\,\mathrm{d}s, \quad t \in [0,T],$$

is an $(\mathcal{F}_t)$-martingale with respect to P.

As mentioned before, we will consider martingale problems with "special" functions $\tilde{b}$ and $\tilde{\sigma}$ that explicitly depend on a fixed measure, i.e. $b(\cdot,\cdot,\mu_\cdot)$ and $\sigma(\cdot,\cdot,\mu_\cdot)$, in the following. Let us now elaborate on how this kind of martingale problem is connected to McKean–Vlasov equations.

9.3.2 Equation

Consider the martingale problem with coefficients explicitly depending on fixed measures μ_t, which can be stated as:

$$\text{Existence of a martingale solution } P \in \mathcal{P}(\mathbb{S}) \text{ in the sense of Definition 9.3.1 for the coefficients } b(\cdot,\cdot,\mu_\cdot) \text{ and } \sigma(\cdot,\cdot,\mu_\cdot) \text{ and with initial value } x_0 \in \mathbb{H}. \tag{NMP}$$

Then, for a martingale solution P to martingale problem (NMP) we can conclude, as in Chapter 3 by using [RZZ15, Theorem 2.2, p. 364] and [Ond05, Theorem 2, p. 1007], that, for some cylindrical Wiener process $W(t)$, $t \in [0,T]$, on $\mathbb{U}$ with respect to a complete filtered probability space $\big(\breve{\Omega}, \breve{\mathcal{F}}, (\breve{\mathcal{F}}_t), \breve{P}\big)$, there exists a weak solution to the McKean–Vlasov equation

$$\begin{aligned} \mathrm{d}X(t) &= b\big(t, X(t), \mathcal{L}_{X(t)}\big)\,\mathrm{d}t + \sigma\big(t, X(t), \mathcal{L}_{X(t)}\big)\,\mathrm{d}W(t), \\ X(0) &= x_0, \end{aligned} \tag{MVE}$$

for $t \in [0,T]$, with the law $\breve{P} \circ X^{-1} = P$. In particular, we have for the 1-marginal laws

$$\mathcal{L}_{X(t)} = \breve{P} \circ X(t)^{-1} = \mu_t,$$

for $t \in [0,T]$ (see e.g. [BR18a, p. 5f] for the finite-dimensional analogue).

Chapter 10

Nonlinear Version: Superposition principle on $\mathbb{H}$

In this chapter, we will adapt Corollary 6.3.4 to the nonlinear case and make it the main theorem of this second part of the thesis. By using the framework from Chapter 9, it remains to modify the necessary assumptions.

10.1 Assumptions

The following assumptions on the coefficients b and σ are directly adapted from those in Chapters 3 and 4 (see Subsections 3.2.3 and 4.2.3) by making the estimates uniform in the newly added dependence on measures.

(NN) Assume there exists a function $\mathcal{N} \in \mathfrak{U}^p$ for some $p \geq 2$ such that for every $n \in \mathbb{N}$ there exists a constant $C_n \geq 0$ with

$$\mathcal{N}(y) \leq C_n \|y\|_{\mathbb{H}_n}^p,$$

for any $y \in \mathbb{H}_n$.

(NA1) (Demicontinuity) For any $v \in \mathbb{X}$, $t \in [0,T]$, $\varrho \in \mathcal{P}(\mathbb{H})$ and every sequence $(y_k)_{k\in\mathbb{N}}$ with $y_k \xrightarrow[k\to\infty]{} y$ in $\mathbb{H}$, we have

$$\lim_{k\to\infty} {}_{\mathbb{X}^*}\langle b(t, y_k, \varrho), v\rangle_{\mathbb{X}} = {}_{\mathbb{X}^*}\langle b(t, y, \varrho), v\rangle_{\mathbb{X}}$$

and

$$\lim_{k\to\infty} \left\|\sigma^*(t, y_k, \varrho)(v) - \sigma^*(t, y, \varrho)(v)\right\|_{\mathbb{U}} = 0.$$

(NA2) (Coercivity) There exists a bounded measurable function $\lambda_1 \colon [0,T] \longrightarrow [0,\infty)$ such that for all $y \in \mathbb{X}$, $t \in [0,T]$ and $\varrho \in \mathcal{P}(\mathbb{H})$

$${}_{\mathbb{X}^*}\langle b(t, y, \varrho), y\rangle_{\mathbb{X}} \leq -\mathcal{N}(y) + \lambda_1(t)(1 + \|y\|_{\mathbb{H}}^2)$$

holds.

(NA3) (Growth) There exist bounded measurable functions $\lambda_2, \lambda_3, \lambda_4 \colon [0,T] \longrightarrow [0,\infty)$ and constants $\gamma' \geq \gamma > 1$ such that for all $y \in \mathbb{H}$, $t \in [0,T]$ and $\varrho \in \mathcal{P}(\mathbb{H})$ we have

$$\|b(t,y,\varrho)\|_{\mathbb{X}^*}^{\gamma} \leq \lambda_2(t)\,\mathcal{N}(y) + \lambda_3(t)(1 + \|y\|_{\mathbb{H}}^{\gamma'})$$

and

$$\|\sigma(t,y,\varrho)\|_{L_2(\mathbb{U};\mathbb{H})}^2 \leq \lambda_4(t)(1 + \|y\|_{\mathbb{H}}^2),$$

where $\mathcal{N}$ is defined in Assumption **(NN)**.

(NH1) For all $n \in \mathbb{N}$, $\varrho \in \mathcal{P}(\mathbb{H})$, the matrices $(a^{ij}(\cdot,\cdot,\varrho))_{1\leq i,j\leq n}$ are symmetric and nonnegative definite.

10.2 Result

Let us state the main result of this second part of the thesis, which is an adaption of Corollary 6.3.4. To be more precise, it is a superposition principle for a given probability solution μ on $[0,T] \times \mathbb{H}$ to a nonlinear Cauchy problem yielding existence of a martingale solution whose 1-marginals are equal to μ_t.

Theorem 10.2.1. *Let the assumptions from Section 10.1 be fulfilled. Assume there exists a probability solution $\mu = \mu_t\,\mathrm{d}t$ on $[0,T]\times\mathbb{H}$ to the nonlinear Cauchy problem* (NCP) *in the sense of Definition 9.2.1 and subsequence $(\mu_{t,n_k})_{k\in\mathbb{N}}$ on $\mathbb{H}_{n_k}$ of a family of Borel probability measures on $\mathbb{H}_n$ with the following properties:*

- *The measures $(\mu_{t,n_k})_{k\in\mathbb{N}}$ are solutions to the finite-dimensional nonlinear Cauchy problems with coefficients $\Pi_{n_k} b(\cdot,\cdot,\mu_{\cdot,n_k})$ and $\Pi_{n_k} A(\cdot,\cdot,\mu_{\cdot,n_k})\Pi_{n_k}^*$ on $\mathbb{H}_{n_k}$ with the property that the mapping*

$$t \mapsto \int_{\mathbb{H}_{n_k}} \zeta(y)\,\mu_{t,n_k}(\mathrm{d}y)$$

  is continuous on $[0,T]$ for every $\zeta \in C_c^\infty(\mathbb{H}_{n_k})$.

- *For the family $(\bar{\mu}_{t,n_k})_{k\in\mathbb{N}}$ of extended measures to $\mathbb{H}$, we have $\bar{\mu}_{t,n_k} \xrightarrow[k\to\infty]{w} \mu_t$ for every $t \in [0,T]$.*

- *The integrability condition*

$$\int_0^T \int_{\mathbb{H}_{n_k}} \frac{\|\Pi_{n_k} A(t,y,\mu_{t,n_k})\Pi_{n_k}^*\| + |\langle \Pi_{n_k} b(t,y,\mu_{t,n_k}), y\rangle_{\mathbb{H}_{n_k}}|}{(1 + \|y\|_{\mathbb{H}_{n_k}})^2}\,\mu_{t,n_k}(\mathrm{d}y)\,\mathrm{d}t < \infty$$

  holds for every $k \in \mathbb{N}$.

Then there exists a martingale solution $P \in \mathcal{P}(\mathbb{S})$ to the martingale problem (NMP) *in the sense of Definition 9.3.1, for which*

$$P \circ x(t)^{-1} = \mu_t$$

holds for every $t \in [0,T]$.

Proof of Theorem 10.2.1. Given solutions μ to the nonlinear Cauchy problem (NCP) and μ_{n_k}, $k \in \mathbb{N}$, to the finite-dimensional nonlinear Cauchy problems, we “freeze” all of these measures and consider linear FPKEs of the form

$$\partial_t \varrho = L_{\mu}^{*} \varrho$$

and, for $k \in \mathbb{N}$,

$$\partial_t \varrho = L_{\mu_{n_k}}^{*} \varrho.$$

But then, μ and μ_{n_k}, for $k \in \mathbb{N}$, are again particular solutions of these linear FPKEs. Since our assumptions from Section 10.1 are uniform in the dependence on the measure, all assumptions from Corollary 6.3.4 hold for our new coefficients $b(\cdot, \cdot, \mu_\cdot)$ and $\sigma(\cdot, \cdot, \mu_\cdot)$ in the linear case. Consequently, we can just apply Corollary 6.3.4 and obtain the desired martingale solution P in the sense of Definition 9.3.1. $\square$

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